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Thèse

présentée pour l'obtention du titre de

Docteur de l'Université Henri Poincaré, Nancy-I

en Mathématiques

par

**Marie-Amélie Paillusseau-Lawn**

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**Méthodes spinorielles et géométrie para-complexe et  
para-quaternionique en théorie des sous-variétés**

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Thèse soutenue publiquement le 14 Décembre 2006

Membres du jury:

<b>Jens Frehse</b>	Examineur	Professeur, Bonn
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<b>Lionel Bérard-Bergery</b>	Examineur	Professeur, Nancy I
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**Spinorial methods,  
para-complex  
and para-quaternionic geometry  
in the theory of submanifolds**

**Dissertation**

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1. Referent: Professor Dr. W. Ballmann
2. Referent: Professor Dr. V. Cortés

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## Articles publiés ou acceptés

- Affine hyperspheres associated to special para-Kähler manifolds (avec V. Cortés et L.Schäfer)  
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- Decompositions of para-complex vector bundles and para-complex affine immersions (avec L. Schäfer)  
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# Introduction

This thesis is devoted to the theory of immersions, using methods of spin geometry and of para-complex and para-quaternionic geometry, and is subdivided into three different topics. The first two are related to the study of conformal immersions of pseudo-Riemannian surfaces. This is investigated in two ways: On the one hand, the immersions into three-dimensional pseudo-Euclidean spaces and, on the other hand, into the four-dimensional pseudo-sphere  $S^{2,2}$ . The last topic is more general and deals with para-complex vector bundles and para-complex affine immersions.

## 1. Representations of pseudo-Riemannian surfaces in space

The relationship between immersions of Riemannian surfaces in Euclidean three- and four-dimensional spaces and spinors has been studied by many authors ([Ab], [Bär], [KS], [Fr1], [Mo], [Tai], [Vo],...). In fact the spinor representations of surfaces are not only of mathematical interest (see for example their applications in algebraic geometry [P]), but it is also of great importance in many areas of theoretical physics, especially soliton theory ([Tai]) and string theory ([GM], [IT]). This is the subject of the first part of this thesis.

Originally a representation of constant mean curvature surfaces by holomorphic functions was already given by Weierstraß in 1886 (see [We]). It is known that conformal immersions of surfaces in  $\mathbb{R}^3$  can be expressed by a generalized Weierstraß representation in terms of an integral over appropriate (1,0)-forms, which are no more necessarily holomorphic. Due to Eisenhart [Eis], this result was rediscovered by Kenmotsu [Ke] and Konopelchenko [Kon]. Using the fact that a spin bundle associated to a given spin structure on a Riemannian surface  $M$  can be viewed as a square root of the holomorphic cotangent bundle  $T^*M \cong (T^{1,0}M)^*$  (see [At], [Mi]), Kusner and Schmitt derived in [KS] a spinor representation for such surfaces. The integrability condition reduces then to a non-linear Dirac equation: This means, that the existence of a Dirac spinor, i.e. a solution of the Dirac equation

$$D\varphi = H|\varphi|^2\varphi, \tag{0.0.1}$$

where  $H$  is a real-valued function, is locally equivalent to the existence of a conformal immersion  $(M, g) \hookrightarrow \mathbb{R}^3$  with mean curvature  $H$  (see [Bär], [Fr1], [KS], [Tai]). Friedrich gives in [Fr1] such a description in a geometrically invariant fashion: The restriction  $\varphi$  of a parallel spinor field on  $\mathbb{R}^n$  to a Riemannian hypersurface  $M^{n-1}$  is a solution of a generalized Killing equation

$$\nabla_X^{\Sigma M} \varphi = A(X) \cdot \varphi, \tag{0.0.2}$$

where  $\nabla^{\Sigma M}$  is the spin connection on  $M^{n-1}$  and  $A$  is the Weingarten tensor of the immersion. Conversely, Friedrich proves that, in the two-dimensional case, if there exists a generalized Killing spinor field satisfying equation (0.0.2), where  $A$  is an arbitrary field of symmetric endomorphisms of  $TM$ , then  $A$  satisfies the Codazzi-Mainardi and Gauß equations of hypersurface theory and is consequently the Weingarten tensor of an isometric immersion of  $M^2$  in  $\mathbb{R}^3$ . Moreover in this case, a solution  $\varphi$  of the generalized Killing equation is equivalently a solution of the Dirac equation, where  $|\varphi|$  is constant. A similar result holds true for Riemannian surfaces immersed in the sphere  $\mathbb{S}^3$  and the hyperbolic space  $\mathbf{H}^3$  (see [Mo], [Vo]). With the additional condition on the tensor  $A$  to be parallel, the equivalence between an isometric immersion of a three-dimensional manifold into  $\mathbb{R}^4$  and the existence of a generalized Killing spinor on the manifold was also proven in [Mo]. We show that it is equivalent of finding a solution of a Dirac equation on  $M$ , and find analogous results for immersions in  $\mathbb{S}^4$  and  $\mathbf{H}^4$ .

Recently the case of pseudo-Riemannian manifolds of general dimension was examined in [BGM]: it was proven that if  $\varphi$  is solution of a generalized Killing equation with Codazzi tensor  $A$  on a pseudo-Riemannian manifold  $M$ , then the manifold can be embedded as a hypersurface into a Ricci flat manifold equipped with a parallel spinor which restricts to  $\varphi$ . The motivation of chapter 2 was the question if, at least in low dimensions, the result of Friedrich can be generalized to the pseudo-Riemannian case. In fact in the two-dimensional case we prove a similar result for Lorentzian surfaces immersed into the pseudo-Euclidean space  $\mathbb{R}^{2,1}$ . Unfortunately the existence of vectors with negative norms does not allow with this approach to omit the Codazzi condition on the tensor  $A$ . Hence there was a need to change the method.

This motivates the second section of chapter 2. With the methods of para-complex geometry and using real spinor representations we succeed in proving the equivalence between the data of a conformal immersion of a Lorentzian surface in  $\mathbb{R}^{2,1}$  and spinors satisfying a Dirac-type equation on the surface. In fact, Lorentz surfaces can be viewed as real two-dimensional para-complex manifolds, and admit therefore an atlas  $\{U, \phi\}$  such that the coordinate changes are para-holomorphic (see [CMMS]). As in the case of  $(1, 0)$ -forms on complex manifolds, a para-complex  $(1, 0)$ -form  $\omega$  on  $M$  can be written as  $\omega = \phi dz$ , where, having  $e$  as the para-complex unit,  $z = x + ey$  is a para-holomorphic coordinate and  $\phi$  is a para-complex function. We give a Weierstraß representation for arbitrary conformal immersions of Lorentz surfaces in  $\mathbb{R}^{2,1}$  using a triple of para-complex  $(1, 0)$ -forms verifying certain conditions analogous to the complex model. This generalizes a result of Konderak (see [KO]) for Lorentz minimal surfaces. Using the real splitting of the tangent bundle we give in theorem 8 a real version of this result in terms of  $(0+, 1-)$ - and  $(1+, 0-)$ -forms. In analogy to the Riemannian case we consider spin bundles on an oriented and time-oriented Lorentz surface  $M$  as para-complex line bundles  $L$  such that there exists an isomorphism

$$\kappa : L^2 \cong T^*M.$$

Consequently any section of  $L$  may be viewed as a square root of a para-complex  $(1, 0)$ -form on  $M$ . This allows us, with help of the two Weierstrass representations described above, to give, on the one hand, a real, and, on the other hand, a para-complex spinor representation for conformal immersions of  $M$  into the pseudo-Euclidean space  $\mathbb{R}^{2,1}$ . In the

real case we derive a Dirac-type equation for the two spinors related to the representation.

Finally we give a geometrically invariant representation of Lorentzian surfaces in  $\mathbb{R}^{2,1}$  using two non-vanishing spinors  $\varphi_1$  and  $\varphi_2$  satisfying a coupled Dirac equation

$$D\varphi_1 = H\varphi_1, \quad D\varphi_2 = -H\varphi_2, \quad \langle \varphi_1, \varphi_2 \rangle = 1.$$

We show that  $\varphi_1$  and  $\varphi_2$  are equivalently solutions of two generalized Killing equations

$$\nabla_X \varphi_1 = A(X)\varphi_1, \quad \nabla_X \varphi_2 = -A(X)\varphi_2.$$

The Codazzi condition on  $A$  is then no more necessary to prove that this two properties are again equivalent to an isometric immersion  $M \hookrightarrow \mathbb{R}^{2,1}$ , with Weingarten tensor  $A$ .

## 2. (Para-)conformal geometry of pseudo-Riemannian surfaces in $\mathbb{S}^{2,2}$

The second part of this thesis investigates the immersions of surfaces with signature into the pseudo-Riemannian sphere  $\mathbb{S}^{2,2}$ .

In [BFLPP], Burstall, Ferus, Leschke, Pedit and Pinkall apply calculus on the quaternions  $\mathbb{H}$  to the study of Riemannian surfaces conformally immersed in the four sphere  $\mathbb{S}^4 = \mathbb{H}P^1$ . Identifying the Euclidean four-space with the quaternions and the Euclidean three space with the space of imaginary quaternions, they generalize complex Riemannian surface theory to the quaternionic setting. This method then applies to the study of Willmore surfaces. This topic is again of mathematical and physical interest: It is worth pointing out the relation between the Willmore functional and quantum physics, including string theory, where it is well known as Polyakov extrinsic action of two-dimensional gravity. Considering conformal immersions of Riemannian surfaces into the quaternionic projective space  $\mathbb{H}P^1 \cong \mathbb{S}^4$ , they prove that there exists a one-to-one correspondence between such immersions and line subbundles (quaternionic holomorphic curves)  $L$  of the trivial bundle  $M \times \mathbb{H}^2$ . This allows one to define the mean curvature sphere congruence, which can be seen as a complex structure on  $L$ , and the Hopf fields of the immersions. This leads to the definition of the Willmore functional of such surfaces.

We generalize this work to surfaces of arbitrary signature and especially to Lorentzian surfaces. For this purpose we consider the space of para-quaternions  $\tilde{\mathbb{H}}$ . The needed notions of linear algebra and geometry over  $\tilde{\mathbb{H}}$  are introduced in the second section of chapter 1, as they are not extensively discussed in the literature. Roughly speaking, para-quaternionic vector modules are endowed with one complex structure and two para-complex structures satisfying certain commutation relations. We can then generalize the method of [BFLPP] to complex Riemannian surface theory and to para-complex Lorentzian surface theory at the same time. It was proven by Blažić (see [Bl]), that the para-quaternionic projective space  $\tilde{\mathbb{H}}P^1$  is diffeomorphic to the pseudo-sphere  $\mathbb{S}^{2,2}$  of unit vectors in  $\tilde{\mathbb{H}}$ . Hence we consider immersions  $f : M \rightarrow \tilde{\mathbb{H}}P^1$ . Similarly to the quaternionic case, we can identify these immersions with the pull-back of the para-quaternionic tautological bundle  $f^* \tau_{\tilde{\mathbb{H}}P^1} =: L \subset M \times \tilde{\mathbb{H}}^2$ . Hence this yields a one-to-one correspondence between such immersions and para-quaternionic line subbundles of the trivial bundle  $M \times \tilde{\mathbb{H}}^2$ .

Considering then a particular (para-)complex structure on this bundle, the mean curvature pseudo-sphere congruence  $S^\varepsilon$  and the para-quaternionic Hopf fields  $A$  and  $Q$  of the immersion, we can define the energy functional of the surface:

$$E(S^\varepsilon) := \int_M \langle dS^\varepsilon \wedge J^{\varepsilon*} dS^\varepsilon \rangle$$

and the Willmore functional

$$W(L) := \frac{1}{\pi} \int \langle A \wedge J^{\varepsilon*} A \rangle,$$

which is the classical Willmore energy of conformal surfaces. We can then express the energy of a conformal surface of arbitrary signature as the sum of a topological invariant and of the Willmore functional.

### 3. Para-complex vector bundles and para-complex affine immersions

In the third and last part of this thesis we study decompositions of para-complex vector bundles endowed with a connection over a para-complex manifold. This will be the subject of our fourth and last chapter. As in complex geometry, there exists in para-complex geometry the notion of (para-)holomorphic bundles. Abe and Kurosu devoted the first part of their work [AK] to the study of subbundles of complex and holomorphic vector bundles and characterized holomorphic subbundles and subbundles of type  $(1, 1)$  in terms of the associated induced connections and of the second fundamental forms. They generalized in this way some of the results of [D, DV, NPP] and applied their results to complex affine immersions to obtain existence and uniqueness theorems.

We are going to extend these results in the framework of para-complex geometry. This means that we introduce and study para-holomorphic vector bundles and characterize para-holomorphic subbundles and subbundles of type  $(1, 1)$  in terms of the associated induced connections and second fundamental forms. The first part of our study is devoted to connections and morphisms of para-holomorphic vector bundles. In this part we also prove, in proposition 21, an analogue of the well-known theorem of complex geometry ([AHS, K]), which asserts that any connection with vanishing  $(0, 2)$ -curvature on a (para-)complex vector bundle  $E$  induces a unique (para-)holomorphic structure on  $E$ . This generalizes to arbitrary dimensions the result of Erdem [Er3] who has proved this for vector bundles over para-complex curves. Further we recall the fundamental equations for general decompositions of vector bundles with connection and analyze the cases where some of the (sub)bundles are para-holomorphic. We introduce para-complex affine immersions and apply the above results to the decomposition induced by the affine immersions. This will be finally used to obtain existence and uniqueness theorems of para-complex affine immersions. These results have been the subject of a common article with L. Schäfer [LS].

# Introduction (Version Française)

Dans cette thèse nous présentons un travail relatif à la théorie des immersions, utilisant des méthodes issues de la géométrie spinorielle, de la géométrie para-complexe et de la géométrie para-quaternionique. Ce travail se divise en trois parties : Les deux premières sont consacrées à l'étude des immersions conformes de surfaces pseudo-Riemanniennes dans l'espace pseudo-Euclidien de dimension trois d'une part, et dans la pseudo-sphère  $\mathbb{S}^{2,2}$  de dimension quatre d'autre part. La dernière partie, plus générale, a trait aux fibrés vectoriels para-complexes et aux immersions affines para-complexes.

## 1. Représentations des surfaces pseudo-Riemanniennes dans l'espace

La relation entre les immersions de surfaces de Riemann dans les espaces Euclidiens de dimension trois et quatre et les spineurs a été étudiée par un grand nombre d'auteurs ([Ab], [Bär], [KS], [Fr1], [Mo], [Tai], [Vo],...). De fait, la représentation spinorielle des surfaces est un sujet d'intérêt non seulement en mathématiques (citons par exemple ses applications en géométrie algébrique [P]), mais également dans de nombreux domaines de la physique théorique, comme en théorie des solitons ([Tai]) et dans la théorie des cordes ([GM], [IT]). Elle est le sujet principal de la première partie de cette thèse.

Dès 1886, une représentation des surfaces à courbure constante utilisant des fonctions holomorphes idoines fut donnée par Weierstraß (voir [We]). Il est bien connu que les immersions conformes de surfaces dans  $\mathbb{R}^3$  peuvent être exprimées par une représentation de Weierstraß généralisée en termes d'une intégrale de  $(1,0)$ -formes satisfaisant certaines conditions et n'étant plus nécessairement holomorphes. Dû à Eisenhart [Eis], ce résultat fut redécouvert par Kenmotsu [Ke] et Konopelchenko [Kon].

En utilisant le fait qu'un fibré spinoriel associé à une structure spinorielle donnée sur une surface de Riemann  $M$  peut être considéré comme la racine carrée du fibré holomorphe cotangent  $T^*M \cong (T^{1,0}M)^*$  (voir [At], [Mi]), Kusner et Schmitt établissent dans [KS] une représentation spinorielle pour de telles surfaces. La condition d'intégrabilité se ramène à une équation de Dirac non-linéaire : plus explicitement l'existence d'un spineur de Dirac, c'est-à-dire d'une solution de l'équation de Dirac

$$D\varphi = H|\varphi|^2\varphi, \tag{0.0.3}$$

où  $H$  est une fonction réelle, est localement équivalente à l'existence d'une immersion conforme  $(M, g) \hookrightarrow \mathbb{R}^3$  dont la courbure moyenne est  $H$  (voir [Bär], [Fr1], [KS], [Tai]). Friedrich donne dans [Fr1] une description semblable, mais d'une manière géométriquement



invariante: La restriction  $\varphi$  d'un champ de spineurs parallèles sur  $\mathbb{R}^n$  à une hypersurface Riemannienne  $M^{n-1}$  est une solution de l'équation de Killing généralisée

$$\nabla_X^{\Sigma M} \varphi = A(X) \cdot \varphi, \quad (0.0.4)$$

où  $\nabla^{\Sigma M}$  est la connexion spinorielle sur  $M^{n-1}$  et  $A$  est le tenseur de Weingarten de l'immersion. Réciproquement, Friedrich montre que dans le cas de dimension deux, s'il existe un spineur de Killing généralisé satisfaisant l'équation (0.0.2), où  $A$  est un champ quelconque d'endomorphismes symétriques de  $TM$ , alors  $A$  vérifie les équations de Codazzi-Mainardi et de Gauß de la théorie des hypersurfaces. Ce n'est rien d'autre que le tenseur de Weingarten d'une immersion isométrique de  $M^2$  dans  $\mathbb{R}^3$ . En outre, dans ce même cas, la solution  $\varphi$  d'une équation de Killing généralisée est également solution d'une équation de Dirac, où  $|\varphi|$  est constante. Un résultat similaire s'applique aux surfaces de Riemann immergées dans la sphère  $\mathbb{S}^3$  et dans l'espace hyperbolique  $\mathbf{H}^3$  (voir [Mo], [Vo]). De plus il a été montré dans [Mo] que si le tenseur  $A$  est parallèle, alors l'existence d'un spineur de Killing généralisé sur la variété est équivalente à l'existence d'une immersion isométrique dans  $\mathbb{R}^4$ .

Nous montrons que ces deux propriétés sont aussi équivalentes à l'existence d'une solution d'une équation de Dirac sur  $M$ , et nous prouvons des résultats analogues pour les immersions de variétés de dimension trois dans  $\mathbb{S}^4$  et  $\mathbf{H}^4$ .

Récemment le cas des variétés pseudo-Riemanniennes de dimension quelconque a été étudié dans [BGM]: il y est démontré que si  $\varphi$  est une solution de l'équation de Killing généralisée sur une variété pseudo-Riemannienne  $M$ , et si  $A$  est un tenseur de Codazzi, alors la variété peut-être immergée comme hypersurface dans une variété Ricci-plate munie d'un spineur parallèle dont la restriction à l'hypersurface est le spineur  $\varphi$ . Dans le chapitre 2 de cette thèse nous nous intéressons à la question de savoir si, au moins en petites dimensions, le résultat de Friedrich peut être généralisé au cas pseudo-Riemannien. Nous prouvons qu'effectivement, en dimension deux, il existe un résultat similaire pour les surfaces de Lorentz immergées dans l'espace pseudo-Euclidien  $\mathbb{R}^{2,1}$ . Malheureusement l'existence de vecteurs dont la norme est négative ne permet pas, par cette approche, d'omettre la condition de Codazzi sur le tenseur  $A$ . Il était donc nécessaire de changer de méthode.

Cette réflexion motive la deuxième section du chapitre 2. Avec des méthodes de géométrie para-complexe et, en utilisant des représentations spinorielles réelles, nous parvenons à prouver l'équivalence entre une immersion conforme d'une surface de Lorentz dans  $\mathbb{R}^{2,1}$  et l'existence de deux spineurs satisfaisant une équation de type Dirac sur la surface. En effet nous pouvons considérer les surfaces de Lorentz comme des variétés para-complexes de dimension réelle deux. Elles admettent donc un atlas  $\{U, \phi\}$ , tel que les changements de cartes sont para-holomorphes (voir [CMMS]). Comme dans le cas des  $(1,0)$ -formes sur les variétés complexes, une  $(1,0)$ -forme para-complexe  $\omega$  sur  $M$  peut-être écrite sous la forme  $\omega = \phi dz$ , où  $e$  est l'unité para-complexe,  $z = x + ey$  est une coordonnée para-holomorphe et  $\phi$  est une fonction para-complexe. Nous donnons une représentation de Weierstraß para-complexe pour les immersions conformes des surfaces de Lorentz dans  $\mathbb{R}^{2,1}$ , nous servant pour ce faire d'un triplet de  $(1,0)$ -formes para-complexes vérifiant certaines conditions, de manière analogue au cas complexe. Ceci généralise un résultat

de Konderak (voir [KO]) s'appliquant aux surfaces de Lorentz minimales. Utilisant la décomposition réelle du fibré tangent, nous donnons dans le théorème 8 une version réelle de ce résultat en termes de  $(0+, 1-)$ - et de  $(1+, 0-)$ -formes.

En analogie au cas Riemannien nous considérons un fibré spinoriel sur une surface de Lorentz orientée et orientée dans le temps comme un fibré en droite para-complexe  $L$ , tel qu'il existe un isomorphisme

$$\kappa : L^2 \cong T^*M.$$

En conséquence, toute section de  $L$  peut-être interprétée comme une racine carrée d'une  $(1, 0)$ -forme para-complexe sur  $M$ . Ceci nous permet, à l'aide des deux représentations de Weierstrass décrites ci-dessus de donner, d'une part, une représentation spinorielle réelle, et d'autre part une représentation spinorielle para-complexe des immersions conformes de  $M$  dans l'espace pseudo-Euclidien  $\mathbb{R}^{2,1}$ . Dans le cas réel nous établissons une équation de type Dirac pour les deux spineurs associés à la représentation.

Pour finir, nous donnons une représentation géométriquement invariante des surfaces de Lorentz dans  $\mathbb{R}^{2,1}$  à l'aide de deux spineurs  $\varphi_1$  et  $\varphi_2$  vérifiant des équations de Dirac couplées

$$D\varphi_1 = H\varphi_1, \quad D\varphi_2 = -H\varphi_2, \quad \langle \varphi_1, \varphi_2 \rangle = 1.$$

Nous montrons que  $\varphi_1$  et  $\varphi_2$  sont de manière équivalente les solutions de deux équations de Killing généralisées

$$\nabla_X \varphi_1 = A(X)\varphi_1, \quad \nabla_X \varphi_2 = -A(X)\varphi_2.$$

La condition de Codazzi sur le tenseur  $A$  n'est alors plus nécessaire pour démontrer que ces deux propriétés sont équivalente à la donnée d'une immersion isométrique  $M \hookrightarrow \mathbb{R}^{2,1}$ , dont le tenseur de Weingarten est précisément  $A$ .

## 2. Géométrie (para-)conforme des surfaces pseudo-Riemanniennes dans $\mathbb{S}^{2,2}$

La seconde partie de cette thèse est consacrée aux immersions de surfaces à signature quelconque dans la sphère pseudo-Riemannienne  $\mathbb{S}^{2,2}$ .

Dans [BFLPP], Burstall, Ferus, Leschke, Pedit et Pinkall appliquent les méthodes de calcul sur les quaternions  $\mathbb{H}$  à l'étude des surfaces Riemanniennes immergées de manière conforme dans la sphère de dimension quatre  $\mathbb{S}^4 = \mathbb{H}P^1$ . En identifiant l'espace Euclidien de dimension quatre avec les quaternions et l'espace Euclidien de dimension trois avec les quaternions imaginaires, les auteurs généralisent la théorie complexe des surfaces de Riemann au cadre quaternionique. Cette méthode est ensuite appliquée à l'étude des surfaces de Willmore. Remarquons qu'ici encore, ce sujet est d'intérêt autant en mathématique qu'en physique: citons surtout l'importance de la fonctionnelle de Willmore en physique quantique, et plus particulièrement en théorie des cordes, où elle est connue comme action extrinsèque de Polyakov pour la gravité de dimension deux (voir [Y]). Considérant les immersions conformes de surfaces de Riemann dans l'espace projectif quaternionique  $\mathbb{H}P^1 \cong \mathbb{S}^4$ , les mêmes auteurs montrent qu'il existe une correspondance bijective entre de telles immersions et des sous-fibrés en droite  $L$  (ou courbes quaternioniques holomorphes) du fibré trivial  $M \times \mathbb{H}^2$ . Cette relation permet alors de définir la congruence sphérique

associée à l’immersion, qui peut être interprétée comme une structure complexe sur  $L$ , et les champs de Hopf de cette immersion. On peut alors introduire la fonctionnelle de Willmore pour de telles surfaces.

Nous généralisons ce travail à des surfaces de signature quelconque et plus spécialement au cas des surfaces de Lorentz. À cet effet nous considérons l’espace des para-quaternions  $\widetilde{\mathbb{H}}$ . Les notions d’algèbre linéaire et de géométrie sur  $\widetilde{\mathbb{H}}$  n’étant que succinctement présentes dans la littérature, elles seront introduites dans la seconde section du premier chapitre de cette thèse. En bref, les modules para-quaternioniques sont munis d’une structure complexe et de deux structures para-complexes satisfaisant certaines relations de commutativité. Nous pouvons alors généraliser la méthode de [BFLPP] à la fois à la théorie complexe des surfaces de Riemann et à la théorie para-complexe des surfaces de Lorentz. Il a été démontré par Blažić (voir [Bl]), que l’espace projectif para-quaternionique  $\widetilde{\mathbb{H}P^1}$  est diffeomorphe à la pseudo-sphere  $\mathbb{S}^{2,2}$  de vecteurs unitaires dans  $\widetilde{\mathbb{H}}$ . Nous considérons donc des immersions  $f : M \rightarrow \widetilde{\mathbb{H}P^1}$ . En analogie au cas quaternionique, nous pouvons identifier ces immersions avec le pull-back du fibré tautologique para-quaternionique  $f^*\tau_{\widetilde{\mathbb{H}P^1}} =: L \subset M \times \widetilde{\mathbb{H}^2}$ . Cette identification induit une correspondance bijective entre de telles immersions et des sous-fibrés en droite para-quaternionique du fibré trivial  $M \times \widetilde{\mathbb{H}^2}$ . En considérant alors, d’une part, une structure (para-)complexe particulière de ce sous-fibré, la congruence pseudo-sphérique de l’immersion  $S^\varepsilon$ , et, d’autre part, les champs de Hopf para-quaternioniques  $A$  et  $Q$  de cette immersion, nous pouvons alors définir la fonctionnelle d’énergie de la surface:

$$E(S^\varepsilon) := \int_M \langle dS^\varepsilon \wedge J^{\varepsilon*} dS^\varepsilon \rangle$$

et finalement la fonctionnelle de Willmore

$$W(L) := \frac{1}{\pi} \int \langle A \wedge J^{\varepsilon*} A \rangle,$$

qui est l’énergie de Willmore classique pour une surface conforme. Nous avons ainsi exprimé l’énergie d’une surface conforme de signature quelconque comme la somme d’un invariant topologique et de la fonctionnelle de Willmore.

### 3. Fibrés vectoriels para-complexes et immersions affines para-complexes

Dans la troisième et dernière partie de cette thèse, enfin, nous étudions les décompositions de fibrés vectoriels para-complexes d’une variété para-complexe, munis d’une connexion. Ce sujet constitue notre quatrième et dernier chapitre. De manière analogue au cas de la géométrie complexe, il existe en géométrie para-complexe la notion de fibré paraholomorphe. Abe et Kurosu étudient dans la première partie de leur travail [AK] les sous-fibrés de fibrés vectoriels complexes et holomorphes et caractérisent les sous-fibrés holomorphes et les sous-fibrés de type  $(1, 1)$  en termes des connexions associées induites et des secondes formes fondamentales. Ils généralisent ainsi des résultats de [D, DV, NPP] et appliquent ces résultats aux immersions affines complexes, obtenant ainsi des théorèmes d’existence et d’unicité.

Nous élargissons ces résultats au contexte de la géométrie para-complexe. À cet effet nous introduisons et étudions la notion de fibré vectoriel para-holomorphe et nous caractérisons les sous-fibrés para-holomorphes et les sous-fibrés de type  $(1, 1)$  en termes des connexions associées induites et des secondes formes fondamentales. La première section de cette dernière partie de notre travail est consacrée aux connexions et aux morphismes de fibrés vectoriels para-holomorphes. Dans cette partie nous démontrons également dans la proposition 21, un analogue d'un théorème bien connu de la géométrie complexe ([AHS, K]), qui affirme que toute connexion dont la  $(0, 2)$ -courbure s'annule sur un fibré vectoriel (para-)complexe  $E$  induit une unique structure para-holomorphique sur  $E$ . Nous généralisons ainsi à une dimension quelconque un résultat d'Erdem [Er3] démontré dans le cas de fibrés vectoriels sur des courbes para-complexes. Nous rappelons ensuite les équations fondamentales pour des décompositions générales de fibrés vectoriels munis d'une connexion et nous analysons les cas où certains de ces (sous-)fibrés sont para-holomorphes. Nous introduisons alors la notion d'immersion affine para-complexe et appliquons ces résultats à la décomposition induite par une telle immersion. Ceci nous permet finalement d'obtenir des théorèmes d'existence et d'unicité pour des immersions affines para-complexes. Ces résultats sont l'objet d'une publication commune avec L. Schäfer [LS].



# Chapter 1

## Para-complex and para-quaternionic differential geometry

In the first section of this chapter, we recall definitions and basic results about para-complex geometry. We refer to [CFG] for a survey on this topic and to [CMMS]. The second section is devoted to the linear algebra of para-quaternions, which is quite new and not extensively discussed in the literature, and to para-quaternionic differential geometry. Especially we introduce the para-quaternionic projective space, which we need in the third chapter of this thesis. In this interpretation, this was at first studied by Gordejuela (see [Go]). For additional information, we also refer to [Bl] and [Vu].

### 1.1 Para-complex differential geometry

The algebra  $C$  of para-complex numbers is the real algebra generated by 1 and by the para-complex unit  $e$  with  $e^2 = 1$ . For all  $z = x + ey \in C$ ,  $x, y \in \mathbb{R}$  we define the para-complex conjugation  $\bar{\cdot} : C \rightarrow C, x + ey \mapsto x - ey$  and the real and imaginary parts of  $z$

$$\Re(z) := \frac{z + \bar{z}}{2} = x, \quad \Im(z) := \frac{e(z - \bar{z})}{2} = y.$$

We notice that  $C$  is a real Clifford algebra. More precisely, we have

$$C \cong \mathbb{R} \oplus \mathbb{R} \cong Cl_{0,1}.$$

**Definition 1** *A para-complex structure on a real finite dimensional vector space  $V$  is an endomorphism  $J \in \text{End}(V)$  such that  $J^2 = \text{Id}$ ,  $J \neq \pm \text{Id}$  and the two eigenspaces  $V^\pm := \ker(\text{Id} \mp J)$  to the eigenvalues  $\pm 1$  of  $J$  have the same dimension. We call the pair  $(V, J)$  a para-complex vector-space.*

The free  $C$ -module  $C^n$  is a para-complex vector space where its para-complex structure is just the multiplication with  $e$  and the para-complex conjugation of  $C$  extends to  $\bar{\cdot} : C^n \rightarrow C^n, v \mapsto \bar{v}$ . A real scalar product of signature  $(n, n)$  may be defined on  $C^n$  by

$$\langle z, z' \rangle := \Re(z\bar{z}') = \Re(z_1\bar{z}'_1 + \dots + z_n\bar{z}'_n).$$

In the following we will denote by

$$C^{n*} = \{z \in C^n \mid \langle z, z \rangle \neq 0\}$$

the set of non-isotropic elements in  $C^n$  and by  $K^n$  the set of zero divisors. In particular note that in the one-dimensional case

$$C \supset C^* = \{\pm r \exp(e\theta) \mid r \in \mathbb{R}^+, \theta \in \mathbb{R}\} \cup \{\pm re \exp(e\theta) \mid r \in \mathbb{R}^+, \theta \in \mathbb{R}\}.$$

Analogous to the complex case, this can be seen as a para-complex polar decomposition, where  $C^* \simeq \mathbb{R}^+ \times H^1$  and where  $H^1$  are the four hyperbolas  $\{z = x+ey \in C \mid x^2 - y^2 = \pm 1\}$ .

In addition we want to define **square roots** of a para-complex number  $w$  as solutions  $z$  of the equation  $z^2 = w$ , with  $z, w \in C$ . We remark that these are only defined for para-complex numbers  $w$  if  $\Re(w) \geq 0$ . In this case there exist at most four square roots of  $w$ : More precisely  $w$  has exactly four square roots if it is non-isotropic and two square roots if it is isotropic.

**Definition 2** *An almost para-complex structure on a smooth manifold  $M$  is an endomorphism field  $J \in \Gamma(\text{End}(TM))$  such that, for all  $p \in M$ ,  $J_p$  is a para-complex structure on  $T_p M$ . It is called **integrable** if the distributions  $T^\pm M = \ker(\text{Id} \mp J)$  are integrable. An integrable almost para-complex structure on  $M$  is called a **para-complex structure on  $M$**  and a manifold  $M$  endowed with a para-complex structure is called a **para-complex manifold**. The **para-complex dimension** of a para-complex manifold  $M$  is the integer  $n = \dim_C M := \frac{\dim M}{2}$ .*

As in the complex case we can define the **Nijenhuis tensor**  $N_J$  of an almost para-complex structure  $J$  by

$$N_J(X, Y) := [X, Y] + [JX, JY] - J[X, JY] - J[JX, Y],$$

for all vector fields  $X$  and  $Y$  on  $M$ . As shown in [CMMS] we have the

**Proposition 1** *An almost para-complex structure  $J$  is integrable if and only if  $N_J = 0$ .*

The splitting of the tangent bundle of a para-complex, or of an almost para-complex, manifold  $M$  into the eigenspaces  $T^\pm M$  extends to a bi-grading on the exterior algebra:

$$\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+, q-} T^* M \quad (1.1.1)$$

and induces an obvious bi-grading on exterior forms with values in a vector bundle  $E$ .

In particular the corresponding decomposition of differential forms on  $M$  is given by

$$\Omega^k(M) = \bigoplus_{k=p+q} \Omega^{p+, q-}(M). \quad (1.1.2)$$

We consider the de Rham differential  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ . In the case where the almost para-complex structure is integrable we have the splitting  $d = \partial_+ + \partial_-$  with

$$\partial_+ : \Omega^{p+, q-}(M) \rightarrow \Omega^{(p+1)+, q-}(M), \quad \partial_- : \Omega^{p+, q-}(M) \rightarrow \Omega^{p+, (q+1)-}(M).$$

Applying the Frobenius theorem to the distribution  $T^\pm M$  we obtain, on an open neighborhood  $U(p)$  of  $M$ , real functions  $z_\pm^i$ ,  $i = 1, \dots, n$ , which are constant on the leaves of  $T^\mp M$  and for which the differential  $dz_\pm^i$  are linearly independent.  $(z_+^1, \dots, z_+^n, z_-^1, \dots, z_-^n)$  is a system of local coordinates on  $M$ , called **adapted coordinates** (see [CMMS]).

Moreover

$$x_i = \frac{z_+^i + z_-^i}{2}, \quad y_i = \frac{z_+^i - z_-^i}{2}$$

defines a system of local real coordinates on  $U(p)$ .

Similarly to the complex model, we now define local para-holomorphic coordinates, for which the real coordinates  $x_i$  (resp.  $y_i$ ) can be seen as the real (resp. imaginary) part:

**Definition 3** *Let  $(M, J_M)$ ,  $(N, J_N)$  be para-complex manifolds. A smooth map  $\varphi : (M, J_M) \rightarrow (N, J_N)$  is called **para-holomorphic** if  $d\varphi \circ J_M = J_N \circ d\varphi$ . A para-holomorphic map  $f : (M, J) \rightarrow \mathbb{C}$  is called **para-holomorphic function**.*

*A system of local para-holomorphic coordinates is a system of para-holomorphic functions  $z^i$ ,  $i = 1, \dots, n$  defined on an open subset  $U \subset M$  of a para-complex manifold where  $(x^1 = \Re(z^1), \dots, x^n = \Re(z^n), y^1 = \Im(z^1), \dots, y^n = \Im(z^n))$  is a system of real local coordinates.*

The existence of a system of local para-holomorphic coordinates in an open neighborhood  $U$  of any point  $p \in M$  was ensured by [CMMS].

Hence, differently to the complex case there exist, due to the real splitting of the tangent bundle, three different sorts of appropriate local coordinates on  $M$ . The adapted coordinates are very important for the results of the third chapter.

**Definition 4** *Let  $(M, J)$  be a para-complex manifold. A para-complex vector bundle of rank  $r$  is a smooth real vector bundle  $\pi : E \rightarrow M$  of rank  $2r$  where the total space  $E$  is endowed with a fiberwise para-complex structure  $J^E \in \Gamma(\text{End}(E))$ . We will denote it by  $(E, J^E)$ .*

*Given two para-complex vector bundles  $(E, J^E)$ ,  $(F, J^F)$  we define*

$$\begin{aligned} \text{Hom}((E, J^E), (F, J^F)) &:= \{\Phi \in \text{Hom}(E, F) \mid \Phi J^E = J^F \Phi\}, \\ \text{Iso}((E, J^E), (F, J^F)) &:= \text{Iso}(E, F) \cap \text{Hom}((E, J^E), (F, J^F)). \end{aligned}$$

Given a para-complex vector bundle  $(E, J^E)$  over the para-complex manifold  $(M, J)$  the space of one-forms  $\Omega^1(M, E)$  with values in  $E$  has the following decomposition

$$\Omega^1(M, E) = \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E) \tag{1.1.3}$$

where

$$\begin{aligned} \Omega^{1,0}(M, E) &:= \{\omega \in \Omega^1(M, E) \mid J^* \omega = J^E \omega\}, \\ \Omega^{0,1}(M, E) &:= \{\omega \in \Omega^1(M, E) \mid J^* \omega = -J^E \omega\}. \end{aligned}$$



One obtains also a bi-graduation on the para-complex 2-forms with values in  $E$ :

$$\Omega^2(M, E) = \bigoplus_{p+q=2} \Omega^{p,q}(M, E),$$

where the components are defined by

$$\begin{aligned} \Omega^{2,0}(M, E) &= \{K | K_{JX,Y} = J^E K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}, \\ \Omega^{1,1}(M, E) &= \{K | K_{JX,JY} = -K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}, \\ \Omega^{0,2}(M, E) &= \{K | K_{JX,Y} = -J^E K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}. \end{aligned}$$

The corresponding projections with  $K \in \Omega^2(M, E)$  and with  $X, Y \in \Gamma(TM)$  are given by

$$K_{X,Y}^{2,0} = \frac{1}{4}(K_{X,Y} + J^E K_{JX,Y} + J^E K_{X,JY} + K_{JX,JY}), \quad (1.1.4)$$

$$K_{X,Y}^{1,1} = \frac{1}{2}(K_{X,Y} - K_{JX,JY}), \quad (1.1.5)$$

$$K_{X,Y}^{0,2} = \frac{1}{4}(K_{X,Y} - J^E K_{JX,Y} - J^E K_{X,JY} + K_{JX,JY}). \quad (1.1.6)$$

The case  $E = M \times C$  was treated in [CMMS] and led to a graduation of  $C$ -valued differential forms

$$\Omega_C^k(M) := \Omega^k(M, M \times C) = \Omega^k(M, C) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Now we consider a para-complex vector space  $(V, J)$  endowed with a **para-hermitian scalar product**  $g$  on it, i.e.  $g$  is a pseudo-Euclidean scalar product and  $J$  is an anti-isometry for  $g$ :

$$J^* g := g(J\cdot, J\cdot) = -g.$$

A **para-hermitian vector space** is a para-complex vector space endowed with a para-hermitian scalar product. We call  $(J, g)$  a **para-hermitian structure**.

The para-unitary group of a para-complex vector space  $(V, J)$  (see [CMMS]) is then defined by

$$U^\pi(V) = \{A \in GL(V) | [A, J] = 0 \text{ and } A^* g = g\}.$$

Note that if  $V$  has para-complex dimension 1, i.e  $V \simeq C \simeq \mathbb{R}^2$ , then  $U^\pi(V) = \{\pm \exp(e\theta) | \theta \in \mathbb{R}\}$ , where  $e$  is the para-complex unit.

**Definition 5** A **para-hermitian vector bundle**  $(E, J^E, g)$  on a para-complex vector-bundle  $(E, J^E)$  is a para-complex vector bundle  $(E, J^E)$  together with a smooth fiber-wise para-hermitian scalar product  $g$ . We call the pair  $(J^E, g)$  a **para-hermitian structure** on a para-complex vector-bundle  $(E, J^E)$ .

Note that if  $L$  is a para-hermitian line bundle, i.e. a para-hermitian vector bundle of dimension one, then  $L$  has obviously structure group  $Gl(1, C) \cap O(1, 1) = U^\pi(C)$ .

**Definition 6** A para-holomorphic vector bundle is a para-complex vector bundle  $\pi : E \rightarrow M$  whose total space  $E$  is a para-complex manifold, such that the projection  $\pi$  is a para-holomorphic map.

A (local) para-holomorphic section of a para-holomorphic vector bundle  $\pi : E \rightarrow M$  is a (local) section of  $E$  which is a para-holomorphic map. The vector space of para-holomorphic sections of  $E$  will be denoted by  $\mathcal{O}(E)$ .

We consider a fiber  $V = E_x$  for a fixed  $x \in M$  of a para-holomorphic vector bundle  $E$  of real rank  $2r$ , which is a para-complex vector space. Hence we can choose a real basis

$$(e_x^1, \dots, e_x^r, J_x^E e_x^1, \dots, J_x^E e_x^r)$$

at  $x$  which extends to a real point-wise basis

$$(e^1, \dots, e^r, J^E e^1, \dots, J^E e^r)$$

of local para-holomorphic sections of  $E$  on an open set  $U \subset M$  containing  $x$ .

Such a frame will be called **para-holomorphic frame**.

Let  $U_\alpha$  and  $U_\beta$  be two open sets, such that  $E|_{U_\alpha}$  and  $E|_{U_\beta}$  are trivial. We identify  $\mathbb{R}^{2r}$  endowed with the para-complex structure induced by  $(E, \tau^E)$  with  $C^r$  and consider the frame change  $\phi_{\alpha\beta}$  as a map

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(C^r).$$

**Definition 7** A para-holomorphic structure  $\mathcal{E}$  on a para-complex vector bundle  $(E, J^E)$  is a maximal compatible set of such para-holomorphic frames, i.e. a maximal set where the frame change is para-holomorphic.

**Remark 1** A para-holomorphic vector bundle is the same as a para-complex vector bundle  $(E, J^E)$  endowed with a para-holomorphic structure  $\mathcal{E}$ . In the following we will denote it by  $(E, \mathcal{E})$ .

## 1.2 Para-quaternionic differential geometry

### 1.2.1 The para-quaternions

The para-quaternions  $\widetilde{\mathbb{H}}$  are the  $\mathbb{R}$ -algebra generated by 1, i, j, k subject to the relations:

$$i^2 = -1, \quad j^2 = k^2 = 1, \quad ij = -ji = k.$$

Obviously this implies  $kj = -jk = i$  and  $ki = -ik = j$ .

We notice that, like the quaternions, the para-quaternions are a real Clifford algebra. More precisely (following the convention of [LM]) we have

$$\widetilde{\mathbb{H}} = Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2).$$

For a given para-quaternion  $a = a_0 + a_1i + a_2j + a_3k$ ,  $a_0, a_1, a_2, a_3 \in \mathbb{R}$ , we define the para-quaternionic conjugation by

$$\bar{a} := a_0 - a_1i - a_2j - a_3k,$$

the real and imaginary part by

$$\Re(a) := a_0 \in \mathbb{R}, \quad \Im(a) := a_1i + a_2j + a_3k \in \tilde{\mathbb{H}}$$

and the pseudo-Euclidian scalar product on  $\tilde{\mathbb{H}}$  by

$$\langle a, b \rangle_{\mathbb{R}^{2,2}} := \Re(a\bar{b}) = a_0b_0 + a_1b_1 - a_2b_2 - a_3b_3,$$

with  $b = b_0 + b_1i + b_2j + b_3k$  and  $b_0, b_1, b_2, b_3 \in \mathbb{R}$ . Using this scalar product we can identify  $\tilde{\mathbb{H}}$  with the pseudo-Euclidean vector space  $\mathbb{R}^{2,2}$ .

Moreover for all  $a, b \in \Im(\tilde{\mathbb{H}})$  we have:

$$ab = -a_1b_1 + a_2b_2 + a_3b_3 - i(a_2b_3 - a_3b_2) + j(a_3b_1 - a_1b_3) + k(a_1b_2 - a_2b_1),$$

which is equivalent to

$$ab = -\langle a, b \rangle_{\mathbb{R}^{2,2}} + a\tilde{\times}b, \tag{1.2.1}$$

with the pseudo-vector-product  $(a\tilde{\times}b)_i := -\varepsilon_j\varepsilon_k(a_jb_k - a_kb_j)$ ,  $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = -1$ , and  $(i, j, k)$  the cyclic permutation of  $(1, 2, 3)$ .

**Remark 2** *As simple computations show, we get for  $a, b \in \tilde{\mathbb{H}}$ :*

1.  $a\bar{a} = a_0^2 + a_1^2 - a_2^2 - a_3^2 = \Re(a\bar{a})$ ,
2.  $\overline{ab} = \bar{b}\bar{a}$ , which leads to
3.  $|a|^2 := \langle a, a \rangle_{\mathbb{R}} = a\bar{a}$  and  $|ab|^2 = |a|^2|b|^2$ ,
4. for all  $a, b \in \Im(\tilde{\mathbb{H}})$ ,  $a\tilde{\times}b = -b\tilde{\times}a$ ,  $\langle a\tilde{\times}b, a \rangle_{\mathbb{R}^{2,2}} = \langle a\tilde{\times}b, b \rangle_{\mathbb{R}^{2,2}} = 0$ .
5. Let  $[a, b] = ab - ba$ . For  $a, b \in \Im(\tilde{\mathbb{H}})$  we have  $[a, b] = 2a\tilde{\times}b$ .

A para-quaternion  $a \in \tilde{\mathbb{H}}$  is called invertible if  $|a|^2 \neq 0$ . In this case the inverse of a para-quaternion is given by  $a^{-1} := \frac{a}{|a|^2}$ . We denote the group of invertible para-quaternions by

$$\tilde{\mathbb{H}}_* := \{a \in \tilde{\mathbb{H}} \mid |a|^2 \neq 0\}.$$

Moreover, the commutator  $[a, b] = ab - ba$  defines a Lie algebra structure on  $\tilde{\mathbb{H}}$ , such that  $(\tilde{\mathbb{H}}, [\cdot, \cdot]) \cong \mathfrak{gl}(2, \mathbb{R}) = \mathbb{R}1 \oplus \mathfrak{sl}(2, \mathbb{R})$ .

**Lemma 1** *1.  $\mathbb{R}1$  is the center of the Lie algebra  $\tilde{\mathbb{H}}$ ,*

2.  $\mathfrak{S}(\tilde{\mathbb{H}}) \cong \mathfrak{sl}(2, \mathbb{R})$  is the semi-simple part of the Lie-algebra  $\tilde{\mathbb{H}}$ ,
- 3.(i)  $a^2 = -1$  if and only if  $a = \mathfrak{S}(a)$  and  $|a|^2 = 1$ ,

(ii)  $a^2 = 1$  and  $a \neq \pm 1$  if and only if  $|a|^2 = -1$  and  $a = \mathfrak{S}(a)$ .

*Proof:*

1. is obvious, since the real para-quaternions are the only ones which commute with all other elements.
2. Since  $[i, j] = 2k, [j, k] = -2i, [k, i] = 2j$  we have

$$[\tilde{\mathbb{H}}, \tilde{\mathbb{H}}] = [\mathfrak{S}(\tilde{\mathbb{H}}), \mathfrak{S}(\tilde{\mathbb{H}})] = \mathfrak{S}(\tilde{\mathbb{H}}).$$

This yields 2. and we get the following decomposition of the Lie-algebra  $\tilde{\mathbb{H}}$ :

$$\tilde{\mathbb{H}} = \mathbb{R}1 \oplus \mathfrak{S}(\tilde{\mathbb{H}}).$$

3. Now let  $a, b \in \tilde{\mathbb{H}}$ . We have

$$ab = (a_0 + \mathfrak{S}(a))(b_0 + \mathfrak{S}(b)) = a_0b_0 + a_0\mathfrak{S}(b) + b_0\mathfrak{S}(a) + \mathfrak{S}(a)\mathfrak{S}(b).$$

By equation (1.2.1) we have  $(\mathfrak{S}(a))^2 \in \mathbb{R}$  and consequently

$$0 = \mathfrak{S}(a^2) = 2a_0\mathfrak{S}(a) \in \mathbb{R} \Leftrightarrow a_0 = 0 \text{ or } \mathfrak{S}(a) = 0.$$

(i) If  $a^2 = -1$ , then  $a = \mathfrak{S}(a)$  and  $|a|^2 = 1$ , by equation (1.2.1).

(ii) If  $a^2 = 1$  and  $a \neq \pm 1$ , then  $a = \mathfrak{S}(a)$  and  $|a|^2 = -1$ , again by equation (1.2.1).

□

## 1.2.2 Unit para-quaternions

Let us now introduce the Lie group

$$\mathbb{S}^{2,1} := \{a \in \tilde{\mathbb{H}} \mid |a|^2 = 1\} \subset \tilde{\mathbb{H}} \cong \mathbb{R}^{2,2}.$$

of unit para-quaternions.

Obviously  $\mathbb{S}^{2,1} \subset \mathbb{R}^{2,2}$  is a pseudo-sphere of signature (2, 1).

Let

$$a = \underbrace{(a_0 + a_1 i)}_{=:z_1(a)} + \underbrace{(a_2 + a_3 i)}_{=:z_2(a)} j,$$

with  $a \in \tilde{\mathbb{H}}$ . With the help of the map

$$a \mapsto z(a) := \begin{pmatrix} z_1(a) \\ z_2(a) \end{pmatrix} \in \mathbb{C}^2,$$

we can identify  $\tilde{\mathbb{H}}$  with  $\mathbb{C}^2$ .

Moreover we have  $j(a_0 + ia_1) = (a_0 - ia_1)j$ , for  $a_0, b_0 \in \mathbb{R}$ . Now let  $z = z_1 + z_2j \in \mathbb{S}^{2,1}$ ,  $z_1, z_2 \in \mathbb{C}$ . For the  $\mathbb{C}$ -linear map  $A_z : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ ,  $x \mapsto xz$  induced on  $\mathbb{C}^2$  by multiplication from the right on  $\tilde{\mathbb{H}}$  we get in the basis  $\{1, j\}$  (here  $\bar{\cdot}$  denotes the complex conjugation):

$$\begin{aligned} A_z 1 &= z_1 + z_2j \\ A_z j &= j(z_1 + z_2j) = \bar{z}_2 + \bar{z}_1j \Rightarrow A_z = \begin{pmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \end{aligned}$$

This yields the following

**Lemma 2** 1.  $\mathbb{S}^{2,1}$  is isomorphic to the group  $SU(1, 1) \cong Sp(1, \mathbb{R}) = SI(2, \mathbb{R})$ , and the map

$$\rho : \mathbb{S}^{2,1} \rightarrow SU(1, 1), \quad \rho(a) := A_{z(a)}$$

is an isomorphism of Lie groups.

2.  $\mathfrak{S}(\tilde{\mathbb{H}})$  is the Lie-algebra of the Lie-group  $\mathbb{S}^{2,1}$ .

*Proof:*

1.  $\det A_z = z_1\bar{z}_1 - z_2\bar{z}_2 = \langle z, z \rangle_{\mathbb{C}^{1,1}} = a_0^2 + a_1^2 - a_2^2 - a_3^2 = |a|^2 = 1$ , which yields:

$$\mathbb{S}^{2,1} \cong SU(1, 1).$$

2. Let  $t \mapsto c(t)$  be a curve in  $\mathbb{S}^{2,1}$  with  $c(0) = a$  and  $c'(0) = b$ , for  $a, b \in \tilde{\mathbb{H}}$  (then  $c(t)c(\bar{t}) = 1$ ). Hence  $0 = \frac{d}{dt} \Big|_{t=0} (c(t)\overline{c(t)}) = c'(0)\overline{c(0)} + c(0)\overline{c'(0)} = b\bar{a} + a\bar{b}$ . If  $a = 1$ , then  $b = -\bar{b}$ . Therefore  $\mathfrak{S}(\tilde{\mathbb{H}})$  is the Lie-algebra of unit para-quaternions.  $\square$

**Remark 3** We can also define the pseudo-hyperbolic space

$$H^{1,2} := \{\mu \in \tilde{\mathbb{H}} \mid |\mu|^2 = -1\}.$$

$H^{1,2}$  is the pseudo-sphere of signature  $(1, 2)$ . The right-multiplication  $R_j : \tilde{\mathbb{H}} \rightarrow \tilde{\mathbb{H}}$  by  $j$  induces an anti-isometry between  $\mathbb{S}^{2,1}$  and  $H^{1,2}$  which is explicitly given by the map

$$\begin{aligned} A : \quad \mathbb{S}^{2,1} &\rightarrow H^{1,2} \\ a_0 + a_1i + a_2j + a_3k &\mapsto a_2 + a_3i + a_0j + a_1k. \end{aligned}$$

### 1.2.3 Para-quaternionic modules

Let  $V = \tilde{\mathbb{H}}^n \cong \mathbb{R}^{4n}$  be the  $n$ -dimensional standard right-module over the para-quaternions. Let  $h = (h_1, \dots, h_n) \in V$ ,  $h_i \in \tilde{\mathbb{H}}$ . Then the multiplication from the right with para-quaternions is defined by

$$\begin{aligned} V \times \tilde{\mathbb{H}} &\longrightarrow V \\ (h, a) &\mapsto ha = (h_1a, \dots, h_na). \end{aligned}$$

The right-multiplication by  $i, j$  and  $-k$  defines a **para-hypercomplex structure** on  $V$ , i.e endomorphisms  $I, J, K \in \text{End}(\mathbb{R}^{4n})$  such that

$$I^2 = -Id, J^2 = K^2 = Id, IJ = -JI = K.$$

On  $V$  we define an indefinite scalar product of signature  $(2n, 2n)$  by

$$\langle h, h' \rangle := \Re(h\bar{h}') = h_1\bar{h}'_1 + \dots + h_n\bar{h}'_n, \quad (1.2.2)$$

with  $h = (h_1, \dots, h_n)$  and  $h' = (h'_1, \dots, h'_n) \in V$ .

$I$  is an isometry,  $J$  and  $K$  are anti-isometries with respect to this scalar product.

Similarly the left-multiplication by  $i, j, k$  induces endomorphisms  $I', J', K' \in \text{End}(\mathbb{R}^{4n})$  with the same properties.

We denote the set of non-isotropic elements in  $\tilde{\mathbb{H}}^n$  by

$$\tilde{\mathbb{H}}_*^n := \{h \in \tilde{\mathbb{H}}^n \mid \langle h, h \rangle \neq 0\}.$$

If we consider further an endomorphism  $E \in \text{End}(V) := \text{End}_{\tilde{\mathbb{H}}}(V)$  (resp.  $F \in \text{End}_{\tilde{\mathbb{H}}}(V)$ ), such that  $E^2 = -1$  (resp.  $F^2 = 1, F \neq \pm 1$  and  $\dim V^+ = \dim V^-$  where  $V^\pm$  are the real eigenspaces to the eigenvalues  $\pm 1$ ), then one gets a resp. para-complex) structure, which commutes with the para-quaternionic structure. For example, the left multiplication on  $V = \tilde{\mathbb{H}}^n$  with an  $N \in \tilde{\mathbb{H}}$  such that,  $N^2 = -1$  (resp.  $N^2 = 1, N \neq \pm Id$ ) induces such a structure.

**Definition 8** *We call a real vector subspace  $U \subset \tilde{\mathbb{H}}^n$  non-degenerate, if the scalar product  $\langle \cdot, \cdot \rangle$  is not degenerate on  $U$ .*

**Lemma 3** *1. Let  $U \subset \tilde{\mathbb{H}}$  be a real two-dimensional non-degenerate subspace. Then there exist  $N, R \in \tilde{\mathbb{H}}$ , such that :*

- (i) *Either  $N^2 = -1 = R^2$  or  $N^2 = 1 = R^2, N \neq \pm Id \neq R$ ,*
- (ii)  *$NU = U = UR$ ,*
- (iii)  *$U = \{x \in \tilde{\mathbb{H}} \mid NxR = x\}$ ,*
- (iv)  *$U^\perp = \{x \in \tilde{\mathbb{H}} \mid NxR = -x\}$ .*

*The pair  $\pm(N, R)$  is unique.*

*2. Let be  $N, R \in \tilde{\mathbb{H}}, N^2 = -1 = R^2$  (resp.  $N^2 = 1 = R^2, N \neq \pm Id \neq R$ ), then the subspaces*

$$U = \{x \in \tilde{\mathbb{H}} \mid NxR = x\}, U^\perp = \{x \in \tilde{\mathbb{H}} \mid NxR = -x\}$$

*are two-dimensional real non-degenerate orthogonal subspaces of  $\tilde{\mathbb{H}}$ , which satisfy (ii).*

*Proof:*

1. Let  $U \subset \widetilde{\mathbb{H}}$  be a real two-dimensional non-degenerate subspace.  
 Suppose that  $1 \in U$ , then there exists a unit vector  $\nu \in U \subset \widetilde{\mathbb{H}}$  orthogonal to 1.  
 Hence  $\nu \in \mathfrak{S}(\widetilde{\mathbb{H}})$  and  $\{1, \nu\}$  are linear independent over  $\mathbb{R}$ .

Case 1:  $|\nu|^2 = 1$ . With Lemma 2 we have  $\nu^2 = -1$  and left-multiplication with  $\nu$  defines a complex structure on  $U$ . Hence  $\{h, \nu h\}$  is a positive-oriented basis of  $U$  for all  $h \in U$ ,  $h \neq 0$  and, consequently,  $\nu U = U$ . Similarly,  $-\nu$  induces by right-multiplication a complex structure on  $U$  such that  $U(-\nu) = U$  holds. Now we put  $(N, R) := (\nu, -\nu)$ . Obviously, this yields  $U = \{x \in \widetilde{\mathbb{H}} \mid NxR = x\}$ . The uniqueness, up to sign, follows from the fact that  $\{1, N\}$  defines a basis for  $U$ . In fact, suppose that  $N' \neq N$ ,  $N'^2 = -1$  and  $N'U = U$ , then we obtain  $N' = 1x_0 + Nx_1$ ,  $N' \in \mathfrak{S}(\widetilde{\mathbb{H}})$ ,  $x_0, x_1 \in \mathbb{R}$ , which leads to  $N' = Nx_1$  and finally to  $x_1 = \pm 1$ .

If  $U$  is an oriented subspace of  $\widetilde{\mathbb{H}}$ , we can choose the sign of  $\nu$ , such that the complex structure induced by  $N$  on  $U$  coincides with the complex structure given by the orientation. This means that there exists a unique pair  $(N, R)$  satisfying the conditions of the lemma.

Case 2:  $|\nu|^2 = -1$ . With Lemma 2 we get  $\nu^2 = 1$ ,  $\nu \neq \pm 1$ . Let  $w_{\pm} := 1 \pm \nu$ , i.e.  $\nu w_{\pm} = \pm w_{\pm}$ . Denote by  $E^{\pm} := \text{span}\{w_{\pm}\}$  the eigenspaces of the left-multiplication by  $\nu$  to the eigenvalues  $\pm 1$ . Obviously  $\dim E^+ = \dim E^- = 1$ . Therefore  $\nu$  defines by left-multiplication a para-complex structure on  $U$ . Because  $\{1, \nu\}$  is a basis of  $U$ , we derive for all  $x \in U$  that  $x = 1x_0 + \nu x_1$  and consequently  $1x \in U$ ,  $\nu x \in U$ . Hence  $\nu U = U$  holds. Moreover in the basis  $\{1, \nu\}$  the endomorphism defined in this way can be identified with the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Similarly to the first case the pair  $(N, R) := (\nu, \nu)$  satisfies the conditions of the lemma and is unique up to sign.

Suppose now that 1 is not contained in  $U$ . Then choose  $x \in U_*$ , where  $U_* \subset U$  are the invertible elements of  $U$ , and consider the subspace  $U' \subset \widetilde{\mathbb{H}}$ ,  $U' = x^{-1}U \cong U$ . The above proof applies to  $U'$ , since  $1 = xx^{-1} \in U'$ . The pair  $(N, R)$  satisfies the conditions for  $U$  if and only if the pair  $(x^{-1}Nx, R)$  satisfies them for  $U'$ .

2. Let  $\sigma x := NxR$  and let  $V^{\pm}$  be the eigenspaces of  $\sigma$  to the eigenvalues  $\pm 1$ .

Case 1:  $N^2 = R^2 = -1$ . Let  $N = R$ . Because  $N$  is a complex structure,  $x$  and  $Nx$  are linear independent for all  $x \in \widetilde{\mathbb{H}}$ . Obviously 1 and  $N$  are eigenvectors of  $\sigma$  to the eigenvalue  $-1$ . There exists a non-isotropic vector  $\nu$  such that  $\nu \perp \text{span}(1, N)$ . This implies that  $\nu \in \mathfrak{S}(\widetilde{\mathbb{H}})$ . With equation (1.2.1) and remark 2.4 we derive that  $N$  and  $\nu$  anti-commute. Consequently  $\nu$  is an eigenvector of  $\sigma$  to the eigenvalue 1 and therefore  $(\nu, N\nu)$  span  $V^+$ .

Now if  $N \neq \pm R$ , as  $N, R \in \mathbb{S}^{2,1}$ , there exists with lemma 2.1 a  $y \in \mathbb{S}^{2,1}$ , which is unique up to sign and such that  $R = y^{-1}Ny$ . Hence  $\{y, Ny\}$  spans  $V^-$  and  $\{\nu y, N\nu y\}$  spans  $V^+$ .

Case 2:  $N^2 = R^2 = 1$ ,  $N \neq \pm Id \neq R$ . Let  $N = R$ . Because  $N \in \mathfrak{S}(\widetilde{\mathbb{H}})$ , 1 and  $N$  are linear independent and eigenvectors to the eigenvalue 1. If we choose again a non-isotropic vector  $\nu \perp \text{span}(1, N)$ , then  $\nu \in \mathfrak{S}(\widetilde{\mathbb{H}})$ . With the help of remark

2.4 we derive that  $N\nu = N\tilde{\times}\nu$  and  $\nu$  are linear independent eigenvectors to the eigenvalue  $-1$ , i.e.  $\{\nu, N\nu\}$  spans  $V^-$ .

Now let  $N \neq \pm R$ . Then  $N + R$  (resp.  $N - R$ ) is an eigenvector of  $\sigma$  to the eigenvalue  $1$  (resp.  $-1$ ). As the isotropy of  $N + R$  (resp.  $N - R$ ) is equivalent to the fact that  $\langle N, R \rangle = 1$  (resp.  $\langle N, R \rangle = -1$ ),  $N + R$  and  $N - R$  cannot be both isotropic. Suppose that  $N + R$  is non-isotropic. With lemma 3 we see that  $N(N + R)$  is non-isotropic, too and an eigenvector of  $\sigma$  to the eigenvalue  $1$ .  $N(N + R)$  and  $N + R$  are linear independent since for  $\lambda \in \mathbb{R}$

$$N(N + R) + \lambda(N + R) = 0 \quad \Rightarrow \quad N = -\lambda.$$

Therefore  $\{N + R, N(N + R)\}$  span  $V^+$ .

Moreover  $N \in \mathfrak{S}(\tilde{\mathbb{H}}) \cong \mathbb{R}^{1,2}$ , consequently there exists a non-isotropic vector  $\nu \perp N$  such that  $\nu \in \mathfrak{S}(\tilde{\mathbb{H}})$  and  $\nu(N + R)$ ,  $\nu N(N + R)$  are non-isotropic linear independent eigenvectors of  $\sigma$  to the eigenvalue  $-1$ . Then  $\{\nu(N + R), \nu N(N + R)\}$  span  $V^-$ .

Finally we derive, that for all cases  $\dim V^+ = \dim V^- = 2$  and  $V^+$ ,  $V^-$  are orthogonal, which completes the proof of the lemma. □

**Corollary 1** *Let  $U \subset \mathfrak{S}(\tilde{\mathbb{H}})$  be a subspace, then it follows that  $N = -R$  in the first case of Lemma 3 or  $N = R$  in the second case.*

*$N$  is a pseudo-euclidian unit normal vector of  $U$ . In the case  $N^2 = -1$   $N$  is spacelike, in the case  $N^2 = 1$  it is timelike.*

*Proof:*

Let  $(u, v)$  be an orthonormal basis of  $U$ . Because  $u, v \in \mathfrak{S}(\tilde{\mathbb{H}})$  we have with equation (1.2.1):

$$|u\tilde{\times}v|^2 = |u||v| = \pm 1.$$

With the properties of the pseudo-vector-product  $N = -R = u\tilde{\times}v$  satisfies in the case  $N^2 = -1$  and  $N = R = u\tilde{\times}v$  in the case  $N^2 = 1$  the above conditions. □

## 1.2.4 Conformal and para-conformal maps

Let  $V$  and  $W$  be  $n$ -dimensional  $N$ -modules endowed with a scalar product  $\langle \cdot, \cdot \rangle_V$  resp.  $\langle \cdot, \cdot \rangle_W$ .

**Definition 9** *An endomorphism  $F : V \rightarrow W$  is called conformal, if there exists a  $\lambda \in \mathbb{R}^*$ , such that:*

$$\langle Fx, Fy \rangle_W = \lambda \langle x, y \rangle_V$$

for all  $x, y \in V$ .

*We will call  $F$  para-conformal if  $V$  and  $W$  are para-complex modules.*

*A map  $f : M \rightarrow \tilde{M}$  between two (pseudo-)Riemannian manifolds is called (para-) conformal, if the endomorphism  $F := df : T_p M \rightarrow df(T_p M)$  is (para-)conformal for all  $p \in M$ .*



Let  $V = W = (\mathbb{R}^2, J) = \mathbb{C}$  with  $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the complex structure induced by the multiplication with  $i$  and  $F : \mathbb{C} \rightarrow \mathbb{C}$ . Then  $J$  is orthogonal and  $\{v, Jv\}$  is a conformal basis for all  $v \in \mathbb{C}$ ,  $|v| \neq 0$ , i.e.  $\langle v, Jv \rangle = 0$  and  $\langle Jv, Jv \rangle = \lambda \langle v, v \rangle$ ,  $\lambda \in \mathbb{R}^*$ . Then  $\{Fv, JFv\}$  is also a conformal basis. Moreover if  $F$  is conformal, then  $\langle Fx, FJx \rangle = \lambda \langle x, Jx \rangle$ . Hence  $F$  is conformal if and only if  $FJ = \pm JF$ .

Now let  $V = W = (\mathbb{R}^2, E) = C$  be the vectorspace of para-complex numbers with  $E : C \rightarrow C$  the para-complex structure induced by the multiplication with  $e$ . In the basis  $(1, e)$  one can identify the endomorphism  $E$  with the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The center of  $E$  is given by:

$$Z_{GL(\mathbb{R}^2)}(E) = \{A \in GL(\mathbb{R}^2) | AE = EA\} = \{A \in GL(\mathbb{R}^2) | A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}\}.$$

And the maps which anti-commute with  $E$  are given by

$$A_{GL(\mathbb{R}^2)}(E) = \{A \in GL(\mathbb{R}^2) | AE = -EA\} = \{A \in GL(\mathbb{R}^2) | A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, a, b \in \mathbb{R}\}.$$

Moreover the conformal endomorphisms with respect to the pseudo-Euclidean real scalar product of signature  $(1, 1)$  define the **para-conformal group**:

$$\begin{aligned} P_C &:= \{F \in GL(\mathbb{R}^2) | \lambda \langle Fe_i, Fe_j \rangle = \langle e_i, e_j \rangle\} \\ &= \{F \in GL(\mathbb{R}^2) | F = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}\} \\ &\quad \cup \{F \in GL(\mathbb{R}^2) | F = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, a, b \in \mathbb{R}\}. \end{aligned}$$

$F$  is consequently para-conformal if and only if it holds  $FE = EF$  or  $FE = -EF$ .

We now consider immersions  $f : M \rightarrow \mathbb{R}^{2,2} = \tilde{\mathbb{H}}$  of (para-)Riemannian surfaces.

**Definition 10** A Riemannian (resp. Lorentzian) surface is a real two-dimensional connected manifold  $M$  endowed with a complex (resp. para-complex) structure  $J \in \Gamma(\text{End}(TM))$ ,  $J^2 = -Id$  (resp.  $E \in \Gamma(\text{End}(TM))$ ,  $E^2 = Id$  and  $\text{tr}E = 0$ ). In the following text, we will use the notation  $(M, J)$  (resp.  $(M, E)$ ).

The above considerations yield the

**Proposition 2** Let  $f : M \rightarrow \tilde{\mathbb{H}}$  be an immersion.

1. If  $(M, J)$  is a Riemannian surface, then  $f$  is conformal if and only if  $Ndf = dfJ = -dfR$ .
2. If  $(M, E)$  be a Lorentzian surface, then  $f$  para-conformal if and only if  $Ndf = dfE = dfR$ .

*Proof:*

For a Riemannian (resp. Lorentzian) surface  $df(T_pM) \subset \widetilde{\mathbb{H}}$  is a two-dimensional non-degenerate subspace for all  $p \in M$ . Then with Lemma 3 there exist  $N, R \in \widetilde{\mathbb{H}}$  with  $N^2 = -1 = R^2$  (resp.  $N^2 = 1 = R^2$ ), such that  $N(df(T_pM)) = df(T_pM) = (df(T_pM))R$ . In the Riemannian case a complex structure  $J'$  is induced by  $N$  and  $R$  on  $T_pM \simeq df(T_pM)$ . With the above considerations  $f : M \rightarrow \mathbb{R}^{2,2}$  is a conformal map if and only if  $dfJ' = Ndf = -dfR$ . Because  $N$  is an isometry, we derive then:

$$\begin{aligned} \langle J'x, J'y \rangle &= \lambda \langle dfJ'x, dfJ'y \rangle = \lambda \langle Ndfx, Ndfy \rangle = \lambda \langle dfx, dfy \rangle = \langle x, y \rangle \\ \Rightarrow \langle J'x, J'Jx \rangle &= \langle x, Jx \rangle. \end{aligned}$$

Then  $J' \in \text{Conf}(J)$ . Hence this yields  $JJ' = J'J$  and  $J' = \pm J$  follows. In the following we will choose  $N$  such that the complex structure induced by  $N$  on  $T_pM$  coincides with the orientation of  $T_pM$ . And finally we derive:

$$Ndf = dfJ = -dfR.$$

In the Lorentzian case a para-complex structure  $E'$  will be induced from  $N$  on  $T_pM$  by the immersion  $f$ , such that  $dfE' = Ndf$  if and only if  $f : M \rightarrow \mathbb{R}^{2,2}$  is conformal map. We derive then:

$$\begin{aligned} \langle EE'x, x \rangle &= -\langle E'x, Ex \rangle = -\lambda \langle dfE'x, dfEx \rangle \\ &= \lambda \langle dfx, NdfEx \rangle = \langle x, E'Ex \rangle. \end{aligned}$$

Hence  $EE' = E'E$ , which yields  $E = \pm E'$ . Again we can construct  $N$  from Lemma 3 in such a way that the sign of  $E'$  coincides with the one of  $E$ .  $\square$

### 1.2.5 The para-quaternionic projective space

As usual we want to consider the para-quaternionic projective space as the subset of all para-quaternionic lines in  $\widetilde{\mathbb{H}}^{n+1}$ .

The existence of isotropic elements in  $\widetilde{\mathbb{H}}^{n+1}$  leads from a geometrical point of view to difficulties coming from the equivalence classes of this elements. This leads to the following definition: we define for  $u, v \in \widetilde{\mathbb{H}}_*$  the equivalence relation

$$u \sim v \Leftrightarrow v = u\lambda, \lambda \in \widetilde{\mathbb{H}}_*.$$

The para-quaternionic projective space is then the corresponding quotient manifold

**Definition 11**

$$\widetilde{\mathbb{H}}P^n := \widetilde{\mathbb{H}}_*^{n+1} / \widetilde{\mathbb{H}}_*.$$

This definition was at first given by E.Gordejuela [Go] (see Blažić [Bl]).

We want now to introduce affine coordinates for  $\widetilde{\mathbb{H}}P^n$ . Consider the projection

$$\begin{aligned} \pi : \widetilde{\mathbb{H}}_*^{n+1} &\rightarrow \widetilde{\mathbb{H}}P^n \\ x &\mapsto \pi(x) := [x] := x\widetilde{\mathbb{H}}_*. \end{aligned}$$

and let  $\beta \in (\tilde{\mathbb{H}}^{n+1})^* := \text{Hom}_{\tilde{\mathbb{H}}}(\tilde{\mathbb{H}}^{n+1}, \tilde{\mathbb{H}})$ ,  $\beta$  invertible. Then

$$\begin{aligned} \varphi_\beta : U_\beta := \{\pi(x) \in \tilde{\mathbb{H}}P^n \mid \beta(x) \neq 0\} &\rightarrow \{x \in \tilde{\mathbb{H}}_*^{n+1} \mid \beta(x) = 1\} \cong \tilde{\mathbb{H}}_*^n \\ \pi(x) &\mapsto x\langle\beta, x\rangle^{-1} \end{aligned}$$

defines an affine chart of  $\tilde{\mathbb{H}}P^{n+1}$ . If we choose a basis of  $\tilde{\mathbb{H}}_*^{n+1}$  such that  $\beta = (e_{n+1})^*$ , then it holds

$$\varphi_\beta \left( \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \right) = \begin{pmatrix} x_1 x_{n+1}^{-1} \\ \vdots \\ x_n x_{n+1}^{-1} \\ 1 \end{pmatrix} \cong \begin{pmatrix} x_1 x_{n+1}^{-1} \\ \vdots \\ x_n x_{n+1}^{-1} \end{pmatrix} \in \tilde{\mathbb{H}}_*^n.$$

**Proposition 3**

$$\text{Hom}(l, \tilde{\mathbb{H}}_*^{n+1}/l) \cong T_l \tilde{\mathbb{H}}P^n,$$

for all  $l \in \tilde{\mathbb{H}}P^n$ .

*Proof:*

Let  $h := \varphi_\beta \circ \pi : \tilde{\mathbb{H}}_*^{n+1} \rightarrow \tilde{\mathbb{H}}_*^{n+1}$ ,  $x \mapsto x\langle\beta, x\rangle^{-1}$ . Then we have for all  $x \in l$

$$d_x h(v) = v\langle\beta, x\rangle^{-1} - x\langle\beta, x\rangle^{-1}\langle\beta, v\rangle\langle\beta, x\rangle^{-1} \quad (1.2.3)$$

and it follows that  $\ker d_x h = \ker d_x \pi = x\tilde{\mathbb{H}}_* = l$ , such that  $d_x \pi$  induces the isomorphism

$$d_x \pi : \tilde{\mathbb{H}}_*^{n+1}/l \xrightarrow{\sim} T_l \tilde{\mathbb{H}}P^n$$

which depends on  $x$ .

Further with equation (1.2.3) it follows that  $d_{\lambda x} \pi(v\lambda) = d_x \pi(v)$ , for all  $\lambda \in \tilde{\mathbb{H}}_*$ ,  $v \in \tilde{\mathbb{H}}_*^{n+1}$ . Consider now  $F \in \text{Hom}(l, \tilde{\mathbb{H}}_*^{n+1}/l) \cong \tilde{\mathbb{H}}_*^{n+1}/l$ . One has  $d_{\lambda x} \pi(F(x\lambda)) = d_x \pi(F(x))$ , such that  $d_x \pi(F(x))$  does not depend on  $x$ . With

$$\begin{aligned} d_x \pi : \text{Hom}(l, \tilde{\mathbb{H}}_*^{n+1}/l) &\xrightarrow{\sim} T_l \tilde{\mathbb{H}}P^n \\ F &\mapsto d_x \pi(F(x)) = d_x \pi(v) \end{aligned}$$

we have constructed the desired isomorphism, where  $d_x \pi(v)$  is identified with the homomorphism  $F$  which maps  $x \in l$  to  $v \text{ mod } l$ .  $\square$

**Corollary 2** Let  $\tilde{f} : M \rightarrow \tilde{\mathbb{H}}_*^{n+1}$ ,  $f := \pi \tilde{f} : M \rightarrow \tilde{\mathbb{H}}P^n$ .

For all  $p \in M$ ,  $l := f(p)$ ,  $v \in T_p M$ ,

$$df_p : T_p M \rightarrow T_{f(p)} \tilde{\mathbb{H}}P^n \cong \text{Hom}(f(p), \tilde{\mathbb{H}}_*^{n+1}/f(p))$$

is given by

$$df_p(v)(\tilde{f}(p)\lambda) = d\tilde{f}_p(v)\lambda \text{ mod } f(p).$$

*Proof:* We derive with  $l = f(p) = \pi(\tilde{f}(p)) = \tilde{f}(p)\tilde{\mathbb{H}}_*$ :

$$d_p f(v)(\tilde{f}(p)\lambda) = (d_{\tilde{f}(p)}\pi)(d\tilde{f}_p(v))(\tilde{f}(p)\lambda).$$

With proposition 3  $d_p f(v) \in \text{Hom}(f(p), \tilde{\mathbb{H}}^{n+1}/f(p))$  is the homomorphism, which maps  $\tilde{f}(p)\lambda$  to  $d_p \tilde{f}(v)\lambda \bmod f(p)$ .  $\square$

In the following the differential  $d_p f(v)$  will be denoted by  $\delta_x f(v)$  in this interpretation.

We can now obtain a metric  $\langle \cdot, \cdot \rangle_{\tilde{\mathbb{H}}P^n}$  on  $\tilde{\mathbb{H}}P^n$  of signature  $(2n, 2n)$ . Let  $X, Y \in T_l \tilde{\mathbb{H}}P^n$ , then with  $v, w \in \tilde{\mathbb{H}}_*^{n+1}/l$ ,  $x \in \tilde{\mathbb{H}}_*^{n+1}$  Then as  $d_{\lambda x}\pi(\lambda v) = d_x \pi(v)$  it makes sense to define

$$\langle X, Y \rangle_{\tilde{\mathbb{H}}P^n} = \langle d_x \pi(v), d_x \pi(w) \rangle_{\tilde{\mathbb{H}}P^n} := \frac{1}{\langle x, x \rangle} \Re \langle v, w \rangle.$$

where  $\langle \cdot, \cdot \rangle$  is a non-degenerate para-quaternionic hermitian inner product on  $\tilde{\mathbb{H}}^{n+1}$ .

**Proposition 4** (*Blažić [Bl]*) *Let  $\mathbb{S}^{2,2}$  be the pseudo-sphere of dimension four and signature  $(2, 2)$  endowed with the induced metric. Then we have the following isometry between pseudo-Riemannian manifolds:*

$$\mathbb{S}^{2,2} \cong \tilde{\mathbb{H}}P^1.$$



# Chapter 2

## Representation of pseudo-Riemannian surfaces in space

In this chapter we study in the pseudo-Riemannian context and for low dimensions whether given manifolds can be immersed as hypersurfaces of codimension one into manifolds of constant curvature. We relate these to manifolds carrying a spinor satisfying a Dirac equation. In the first section we recall shortly some definitions and basic results about pseudo-Riemannian spin geometry in the context of the theory of hypersurfaces. We refer to [LM] for spin geometry in general and to [Bau] for pseudo-Riemannian spin geometry. Moreover we prove some new results: In particular, we show the equivalence between a solution of the Dirac equation and a solution of the generalized Killing equation  $\nabla_X^{\Sigma M} \varphi = A(X)\varphi$  in the three-dimensional Riemannian case and in the two-dimensional pseudo-Riemannian case. Further we generalize a result of Morel (see [Mo]) to immersions in  $\mathbb{S}^4$  and  $\mathbb{H}^4$ . Finally, generalizing a result of Friedrich ([Fr1]), we prove the equivalence of a solution of the generalized Killing equation on a two-dimensional pseudo-Riemannian surface and a pseudo-Riemannian immersion of this surface in  $\mathbb{R}^{2,1}$ , but only under the condition that  $A$  is a Codazzi tensor. This motivates the second section: In fact with the methods of para-complex geometry and using real spinor representations we succeed in proving the equivalence between the data of a spacelike conformal immersion of a Lorentzian surface in  $\mathbb{R}^{2,1}$  and two spinors satisfying a Dirac-type equation on the surface. Finally we give in this section a geometrically invariant representation of such surfaces using two Dirac spinors.

### 2.1 Pseudo-Riemannian spin geometry

#### 2.1.1 Representations of Clifford algebras

At first we give some algebraic preliminaries. In the first section we use essentially complex representations, which we discuss here briefly, since they are well-known and can easily be found in the literature (compare [Bau, LM, Fr2]).

In parallel we discuss the real case, which we will use in the second section.

Let  $Cl_{p,q} = Cl(\mathbb{R}^{p+q}, q)$  be the Clifford algebra of  $(\mathbb{R}^{p+q}, q)$ , where  $q$  is the bilinear form

$$q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2, \quad x \in \mathbb{R}^{p,q}.$$

Let  $\{e_i\}_{i=1,\dots,p+q}$  be an orthonormal basis of  $\mathbb{R}^{p+q}$  such that:

$$e_i \cdot e_j + e_j \cdot e_i = \begin{cases} -2\delta_{ij}, & i \leq p, \\ 2\delta_{ij}, & i > p. \end{cases}$$

Then  $Cl_{p,q}$  is generated by  $\{e_i\}_{i=1,\dots,p+q}$ .

The Clifford algebra splits in the direct sum of odd and even elements:

$$Cl_{p,q} = Cl_{p,q}^0 \oplus Cl_{p,q}^1.$$

We give the following isomorphism, which is of particular importance for the identification of the spin bundles in the context of immersions of hypersurfaces:

$$Cl_{p,q} \xrightarrow{\cong} Cl_{p+1,q}^0 \tag{2.1.1}$$

$$e_i \mapsto e_{p+1} \cdot e_i, \tag{2.1.2}$$

where  $\mathbb{R}^{p,q} = \text{span}\{e_i | i \neq p+1\}$ .

A pseudo-orthogonal map  $A \in O(p, q)$  can be written with respect to a pseudo-orthonormal basis as a matrix  $\begin{pmatrix} A_p & B \\ C & A_q \end{pmatrix}$ , where  $A_p$  and  $A_q$  are  $p \times p$  (resp  $q \times q$ ) block matrices. We consider in the following the subgroups

$$SO(p, q) = \{A \in O(p, q) | \det(A) > 0\}, \tag{2.1.3}$$

and

$$SO_+(p, q) = \{A \in SO(p, q) | \det(A_p) > 0\}, \tag{2.1.4}$$

which is the connected component of the identity of  $SO(p, q)$ .

The spin group is defined by

$$\text{Spin}(p, q) = \{x_1 \cdot \dots \cdot x_k \in Cl_{p,q}^0 | x_j \in \mathbb{R}^{p+q}, q(x_j) = \pm 1\}.$$

Let  $x \in \mathbb{R}^{p+q}$ . Then the map

$$\lambda : \text{Spin}(p, q) \rightarrow SO(p, q), \quad u \mapsto u \cdot x \cdot u^{-1}$$

defines a two-fold covering of the group  $SO(p, q)$ . We denote by  $\text{Spin}_+(p, q) \subset \text{Spin}(p, q)$  the pre-image of  $SO_+(p, q)$  under  $\lambda$ .

In the basis  $\{e_i\}_{i=1,\dots,p+q}$  the associated real volume element is given by

$$\omega_{p,q} = e_1 \cdot \dots \cdot e_{p+q}.$$

We have

$$\omega_{p,q}^2 = \begin{cases} (-1)^q, & n \equiv 3 \text{ or } 4 \pmod{4}, \\ (-1)^{q+1}, & n \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

The complex volume element is defined by

$$\omega_{p,q}^{\mathbb{C}} = \begin{cases} i^{m+q} e_1 \cdots e_{p+q} & \text{if } n=p+q=2m, \\ i^{m+1+q} e_1 \cdots e_{p+q} & \text{if } n=p+q=2m+1. \end{cases}$$

Obviously  $\omega_{p,q}^{\mathbb{C}^2} = 1$ .

The real spinor module  $\Sigma_{p,q}$  is the restriction of an irreducible real module of the real algebra  $Cl_{p,q}$  to  $\text{Spin}(p,q) \subset Cl_{p,q}^0 \subset Cl_{p,q}$ , whereas the complex spinor module  $\Sigma_n^{\mathbb{C}}$  is given by restricting an irreducible complex module of the complex Clifford algebra  $\mathbb{C}l_n := Cl(\mathbb{C}^{p+q}, q \otimes \mathbb{C}) \cong Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C}$ ,  $p+q=n$  to  $\text{Spin}(n,0)$ .

Generalizing a well-known result (compare with proposition 5.12 of [LM]), we deduce for real spin representations of arbitrary signature the following

**Proposition 5** *Let  $\Delta_{p,q} : \text{Spin}(p,q) \rightarrow \text{GL}(\Sigma_{p,q})$  be the real spin representation of  $\text{Spin}(p,q)$ . There are four different cases:*

1. *If  $n$  is odd,  $q=p+1 \pmod{4}$ , then there exists a unique spin representation and  $\Delta_{p,q}$  is independent of which irreducible representation of  $Cl_{p,q}$  is used.*
2. *If  $n$  is odd,  $q=p-1 \pmod{8}$  or  $n$  even,  $q=p-2 \pmod{8}$ , then  $\Delta_{p,q}$  is the direct sum of two equivalent irreducible representation.*
3. *If  $n$  is odd,  $q=p+3 \pmod{8}$  or  $n$  is even,  $q=p+2 \pmod{8}$ , then  $\Delta_{p,q}$  is irreducible.*
4. *If  $n$  is even,  $p=q \pmod{4}$ , then  $\Delta_{p,q}$  is the direct sum of two inequivalent irreducible representations.*

For complex spin representations  $\Delta_n^{\mathbb{C}} : \text{Spin}(p,q) \rightarrow \text{GL}_{\mathbb{C}}(\Sigma_{p,q})$ ,  $n=p+q$  we have

**Proposition 6**

1. *If  $n$  is odd, then there exists a unique spin representation  $\Delta_n^{\mathbb{C}}$  which is independent of which irreducible representation of  $\mathbb{C}l_n$  is used.*
2. *If  $n$  is even,  $\Delta_n^{\mathbb{C}}$  is the direct sum of two inequivalent irreducible spin representations:*

$$\Delta_n^{\mathbb{C}} = \Delta_n^{\mathbb{C}^+} \oplus \Delta_n^{\mathbb{C}^-}.$$

**Remark 4** *We remark that if  $p+q=n=2m+1$  is odd,  $\omega_{p,q}^{\mathbb{C}}$  acts on  $\Sigma_{p,q}^{\mathbb{C}}$  as the identity. If  $n=2m$  is even,  $\omega_{p,q}^{\mathbb{C}}$  acts on  $\Sigma_{p,q}^{\mathbb{C}^+}$  as  $Id$  and as  $-Id$  on  $\Sigma_{p,q}^{\mathbb{C}^-}$ .*



We now want to consider bilinear forms on complex spinor modules. The following proposition holds (see [Bau]):

**Proposition 7** *The complex spinor module  $\Sigma_{p,q}^{\mathbb{C}}$  is endowed for all  $p, q$ , with a hermitian symmetric bilinear  $\text{Spin}_0$ -invariant form  $\langle \cdot, \cdot \rangle$ , such that*

$$\langle X \cdot \varphi, \psi \rangle = -(-1)^q \langle \varphi, X \cdot \psi \rangle, \quad (2.1.5)$$

for all  $\varphi, \psi \in \Sigma_{p,q}^{\mathbb{C}}$ ,  $X \in \mathbb{R}^{p,q}$ .

**Remark 5**

- (i) *It is not the only admissible bilinear form on  $\Sigma_{p,q}^{\mathbb{C}}$ . A complete classification of admissible bilinear forms on spinor modules was given in [AC].*
- (ii) *If the spinor modul  $\Sigma_{p,q}$  is reducible (i.e  $\Sigma_{p,q} = \Sigma_{p,q}^- \oplus \Sigma_{p,q}^+$ ), then the module  $\Sigma_{p,q}^+$  and  $\Sigma_{p,q}^-$  are either orthogonal or isotropic with respect to the bilinear form  $\langle \cdot, \cdot \rangle$ , i.e for all  $\psi^+ \in \Sigma_{p,q}^+$  and  $\psi^- \in \Sigma_{p,q}^-$ , either  $\beta(\psi^+, \psi^-) = 0$  or  $\beta(\psi^\pm, \psi^\pm) = 0$ .*

## 2.1.2 Pseudo-Riemannian spin geometry

We now give a brief survey of the relevant spin geometric features we use in the following. We refer to [Bau] for more details.

Let  $(M^{p,q}, g)$  be a pseudo-Riemannian manifold. We recall that the tangent bundle splits into the orthogonal direct sum  $TM = \eta^p \oplus \xi^q$  of a  $p$ -dimensional spacelike bundle  $\eta$  and a  $q$ -dimensional timelike bundle  $\xi$  (see [On]). The manifold  $(M^{p,q}, g)$  is called **oriented** (resp. **space-oriented**, resp. **time-oriented**) if the bundle  $TM$  (resp.  $\eta$ , resp.  $\xi$ ) is oriented. If  $(M^{p,q}, g)$  is oriented we can consider the  $\text{SO}(p, q)$ -principal bundle  $P_{\text{SO}}$  of positively oriented orthonormal frames over  $M$ . In the same way if  $M^{p,q}$  is a strongly oriented (i.e oriented and time-oriented) manifold, we can consider the  $\text{SO}_+(p, q)$ -principal bundle  $P_{\text{SO}_+}$  of positively strongly oriented orthonormal frames over  $M$ . In order to deal simultaneously with the pairs  $\text{SO}(p, q)/\text{Spin}(p, q)$  and  $\text{SO}_+(p, q)/\text{Spin}_+(p, q)$  we write  $G(p, q)$  for  $\text{SO}(p, q)$  (resp.  $\text{SO}_+(p, q)$ ) and  $\tilde{G}(p, q)$  for  $\text{Spin}(p, q)$  (resp.  $\text{Spin}_+(p, q)$ ).

**Definition 12** *A spin structure on a (strongly) oriented pseudo-Riemannian manifold is a  $\tilde{G}(p, q)$ -principal bundle  $P_{\tilde{G}}$  together with a two-fold covering*

$$\Lambda : P_{\tilde{G}} \rightarrow P_G,$$

such that the following diagram commutes:

$$\begin{array}{ccc} P_{\tilde{G}} \times_{\lambda} \tilde{G} & \longrightarrow & P_{\tilde{G}} \\ \downarrow \Lambda \times \lambda & & \downarrow \Lambda \\ P_G \times G & \longrightarrow & P_G \end{array} \quad \begin{array}{c} \nearrow \\ M \\ \searrow \end{array}$$

Let  $\{U_\alpha\}$  be an open covering of  $M$  and  $\{\varphi_{\alpha\beta}\}$  resp.  $\{\tilde{\varphi}_{\alpha\beta}\}$  the transition functions of  $P_G$  resp. of a principal  $\tilde{G}(p, q)$ -bundle  $P_{\tilde{G}}$ . Then obviously  $P_{\tilde{G}}$  is a spin structure on  $M$  if and only if  $\lambda(\tilde{\varphi}_{\alpha\beta}) = \varphi_{\alpha\beta}$ .

A criterion for the existence of such structures is given by the following proposition (see [Bau]):

**Proposition 8** *Let  $(M^{p,q}, g)$  be a pseudo-Riemannian connected manifold and  $TM = \eta^p \oplus \xi^q$  a splitting into an orthogonal direct sum of a spacelike bundle  $\eta$  and a timelike bundle  $\xi$ . Then  $(M^{p,q}, g)$  admits a spin structure if and only if*

$$w_2(M) = w_1(\eta)^2, \quad (2.1.6)$$

where  $w_i \in H^i(M, \mathbb{Z}_2)$  denotes the  $i$ -th Stiefel-Whitney-class.

**Remark 6**

- If  $(M, g)$  is strongly oriented, the existence of a spin structure is equivalent to the condition  $w_2(M) = 0$ .
- If  $(M, g)$  is oriented, it is equivalent to the fact that  $w_1(\xi) = w_1(\eta)$ , hence to  $w_1(M) = 0$ . Therefore  $(M, g)$  is a spin manifold if and only if  $w_2(M) = w_1(\xi)w_1(\eta)$ .

*In particular, every time-oriented oriented two-dimensional pseudo-Riemannian manifold admits a spin structure.*

The fiber  $\tilde{G}(p, q)$  of the principal bundle  $P_{\tilde{G}}$  operates on the spinor space  $\Sigma_{p,q}$  via the spin representation

$$\rho : \tilde{G}(p, q) \rightarrow \text{GL}(\Sigma_{p,q})$$

(or  $\rho : \tilde{G}(p, q) \rightarrow \text{GL}_{\mathbb{C}}(\Sigma_{p,q})$  in the complex case). This yields the following

**Definition 13** *The spinor bundle associated to a spin structure on a pseudo-Riemannian manifold  $M$  is the associated vector bundle*

$$\Sigma M = P_{\tilde{G}} \times_{\rho} \Sigma_{p,q}.$$

*Sections  $\psi \in \Gamma(\Sigma M)$  will be referred to as spinors.*

Due to the algebraic preliminaries of the previous paragraph (and in particular due to propositions 5 and 6), we recall that, in even dimension, complex spinor bundles (and, in the second and fourth case of proposition 5, real spinor bundles) split into the positive and negative **half-spinor bundles**

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M.$$

For strongly oriented manifolds, the bilinear form (2.1.5) on  $\Sigma_{p,q}^{\mathbb{C}}$  induces a pseudo-hermitian symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\Sigma M$  such that

$$\langle X \cdot \varphi, \psi \rangle = -(-1)^q \langle \varphi, X \cdot \psi \rangle, \quad (2.1.7)$$

for all  $\varphi, \psi \in \Gamma(\Sigma M)$ ,  $X \in \Gamma(TM)$ .

Let  $\nabla : \Gamma(\Sigma M) \rightarrow \Gamma(T^*M \otimes \Sigma M)$  be covariant derivative on the spin bundle  $\Sigma M$ .

The Dirac operator  $D : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M)$  on a pseudo-Riemannian manifold  $(M, g)$  is given by

$$D\psi = \sum_{i=1}^{p+q} \varepsilon_i e_i \cdot \nabla_{e_i} \psi, \quad \text{with } \varepsilon_i = g(e_i, e_i) \in \{\pm 1\},$$

where  $\{e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}\}$  is an orthonormal basis.

**Remark 7** *The Dirac operator  $D$  is formally self-adjoint with respect to the bilinear form  $\langle \cdot, \cdot \rangle$  defined above if  $q$  is even. In the other cases  $iD$  is formally self-adjoint.*

In this chapter we are more particularly interested in solutions of the Dirac equation

$$D\psi = H|\psi|_g^2 \psi, \quad (2.1.8)$$

where  $H$  is a real valued function. We want to know how this equation transforms under conformal change of the metric. Let  $\tilde{g} = \sigma g$  be a conformally equivalent metric, with  $\sigma$  a positive function. There exists an isomorphism between the principal bundles  $P_G$  and  $\tilde{P}_G$  of orthonormal frames with respect to the metric  $g$  resp.  $\tilde{g}$  given by

$$\begin{aligned} P_G &\rightarrow \tilde{P}_G \\ (s_1, \dots, s_n) &\mapsto (\sigma^{-\frac{1}{2}} s_1, \dots, \sigma^{-\frac{1}{2}} s_n), \end{aligned}$$

where  $(s_1, \dots, s_n)$  are local basis sections of  $P_G$ . This induces an isomorphism between the spin structures  $P_{\tilde{G}}$  and  $\tilde{P}_{\tilde{G}}$  (see [Bau]). Denote by  $\tilde{D}$  the Dirac operator corresponding to the metric  $\tilde{g}$ . Then (see [Bau, BFGK]) we have

$$\tilde{D}\psi = \sigma^{-\frac{n+1}{4}} D(\sigma^{\frac{n-1}{4}} \psi).$$

Now let  $\tilde{\psi} = \sigma^{-\frac{n-1}{4}} \psi$ . We compute

$$\begin{aligned} \tilde{D}\tilde{\psi} &= \sigma^{-\frac{n+1}{4}} D(\sigma^{\frac{n-1}{4}} \tilde{\psi}) = \sigma^{-\frac{n+1}{4}} D\psi = \sigma^{-\frac{n+1}{4}} H|\psi|_g^2 \psi \\ &= \sigma^{\frac{n-1}{2}} H|\tilde{\psi}|_g^2 \tilde{\psi} = \tilde{H}|\tilde{\psi}|_g^2 \tilde{\psi}, \quad \text{with } \tilde{H} = \sigma^{\frac{n-1}{2}} H. \end{aligned}$$

Hence  $\psi$  is a solution of (2.1.8) on  $(M, g)$  if and only if  $\tilde{\psi} = \sigma^{-\frac{n-1}{4}} \psi$  is a solution of the equation  $\tilde{D}\tilde{\psi} = \tilde{H}|\tilde{\psi}|_g^2 \tilde{\psi}$  on  $(M, \tilde{g})$ .

### 2.1.3 Immersions of hypersurfaces of low dimensions via spinors

We now give some results about isometric immersions with codimension 1 of pseudo-Riemannian manifolds. At first we recall the fundamental theorem of hypersurface theory:

**Theorem 4** (cf [On]) *Let  $(M, g)$  be a pseudo-Riemannian manifold with signature  $(p, q)$ ,  $p + q = n$ ,  $\nabla^M$  the Levi-Civita connection on  $M$  and  $A$  be a symmetric tensor such that*

$$\begin{aligned} (\nabla_X^M A)(Y) &= (\nabla_Y^M A)(X) && \text{(Codazzi - Mainardi equation),} \\ R^M(X, Y)Z &= [\langle A(Y), Z \rangle A(X) - \langle A(X), Z \rangle A(Y)] \\ &\quad + \kappa(\langle Y, Z \rangle X - \langle X, Z \rangle Y) && \text{(Gau\ss equation).} \end{aligned}$$

with  $\kappa \in \mathbb{R}$ , for all  $X, Y, Z, W \in T_x M$  and  $x \in M$ .

Then there exists locally a spacelike isometric immersion of  $M$  in  $\mathbb{M}^{p+1, q}$ , where  $A$  is the Weingarten tensor of the immersion and  $\mathbb{M}^{p+1, q}$  is the model space of constant sectional curvature  $\kappa$  and signature  $(p + 1, q)$ .

**Remark 8** *Consequently, in the case of a two-dimensional manifold immersed in  $\mathbb{M}^{p+1, q}$  ( $p+q=2$ ), we have*

$$(\nabla_X^M A)(Y) = (\nabla_Y^M A)(X), \tag{2.1.9}$$

$$R_{1212} = \varepsilon \det A + \varepsilon \kappa, \tag{2.1.10}$$

where  $R_{1212} = g(R(e_1, e_2)e_2, e_1)$  and  $\varepsilon = 1$  in the case of definite signature, and  $\varepsilon = -1$  in the case of signature  $(1, 1)$ .

Let now  $M$  be an oriented pseudo-Riemannian manifold of signature  $(p, q)$ ,  $p + q = n$ , immersed in a pseudo-Riemannian spin manifold  $N$  of signature  $(p + 1, q)$  and let  $\nu$  be the spacelike unit normal vector, i.e  $\langle \nu, \nu \rangle = 1$ . Let  $(e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q})$  be an orthonormal frame of  $TM$ . A local orthonormal frame of  $TN|_M$  is then given by  $\iota : (e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q})$ . This yields a map

$$\iota : (e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}) \rightarrow (\nu, e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q})$$

from frames of  $TM$  to frames of  $TN|_M$ . Due to the isomorphism (2.1.1) the spin structure  $\Lambda : P_{\tilde{G}}(N) \rightarrow P_G(N)$  on  $N$  induces a spin structure  $P_{\tilde{G}}(M) := \Lambda^{-1}(\iota(P_G(M)))$  on  $M$ .

We recall that, using the algebraic preliminaries of last paragraph, we get vector bundle isomorphisms:

$$\begin{cases} \Sigma^+ N|_M \xrightarrow{\cong} \Sigma M & \text{if } n \text{ is odd,} \\ \Sigma N|_M \xrightarrow{\cong} \Sigma M & \text{if } n \text{ is even.} \end{cases}$$

Note however that these isomorphisms do not preserve the Clifford multiplication and the connection.

Further, using the notation " $\cdot$ " (resp. " $\cdot_N$ ") for the Clifford multiplication on  $M$  (resp.  $N$ ), we have

$$X \cdot \varphi = X \cdot_N \nu \cdot_N \varphi,$$

for all  $X \in \Gamma(TM)$ ,  $\varphi \in \Gamma(\Sigma M)$ .

Considering the spin connection induced on the hypersurface  $M$ , we give a formula, which motivates the study of generalized Killing equations: Let  $e_0 := \nu$ ,  $\varepsilon_k = g(e_k, e_k)$  and  $\omega = \omega_{ij}$  be the connection form of the Levi-Civita connection  $\nabla^N$  on  $N$ , then we have for  $\varphi = \Phi|_M$ ,  $X \in TM$ :

$$\begin{aligned}
\nabla_X^{\Sigma N} \varphi &= X(\varphi) + \frac{1}{2} \sum_{0 \leq k < l \leq n} \varepsilon_l \omega_k^l(X) e_k \cdot_N e_l \cdot_N \varphi \\
&= X(\varphi) + \frac{1}{2} \sum_{0 < k \leq n} \varepsilon_0 \omega_k^0(X) e_k \cdot_N \nu \cdot_N \varphi \\
&\quad + \frac{1}{2} \sum_{0 < k < l \leq n} \varepsilon_l \omega_k^l(X) \underbrace{e_k \cdot_N e_l \cdot_N \varphi}_{=e_k \cdot \nu \cdot e_l \cdot \nu \cdot \varphi} \\
&= X(\varphi) + \underbrace{\frac{1}{2} \sum_{0 < k < l \leq n} \varepsilon_l \omega_k^l(X) e_k \cdot e_l \cdot \varphi}_{=\nabla_X^{\Sigma M} \varphi} + \frac{1}{2} \sum_{0 < k \leq n} \omega_k^0(X) e_k \cdot \varphi \\
&= \nabla_X^{\Sigma M} \varphi - \frac{1}{2} \sum_{0 < k \leq n} \varepsilon_k g(A(X), e_k) e_k \cdot \varphi,
\end{aligned}$$

and consequently

$$\nabla_X^{\Sigma N} \varphi = \nabla_X^{\Sigma M} \varphi - \frac{1}{2} A(X) \cdot \varphi, \quad (2.1.11)$$

for all  $X \in \Gamma(TM)$ , where  $A$ , with  $A(X) := \nabla_X \nu$ , is the Weingarten tensor of the immersion.

These considerations lead to the following

**Proposition 9** *Let  $M$  be an oriented pseudo-Riemannian manifold of signature  $(p, q)$ , with  $p+q = n$ , immersed into a pseudo-Riemannian spin manifold  $N$  of signature  $(p+1, q)$ . If  $\Phi \in \Gamma(\Sigma N)$  is a Killing spinor on  $N$ , i.e if*

$$\nabla_X^{\Sigma N} \Phi = \lambda X \cdot_N \Phi,$$

for all  $X \in \Gamma(TM)$ ,  $\lambda \in \mathbb{C}$ , then its restriction  $\varphi = \Phi|_M$  to  $M$  is a solution of the equation

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi + \mu X \cdot \omega_{p,q}^{\mathbb{C}} \cdot \varphi,$$

where  $\mu = -i\lambda$  if  $p+q = n$  is even and  $\mu = \lambda$  if  $p+q$  is odd.

*Proof:* If  $n = 2m$  (resp  $n = 2m+1$ ), we recall that the complex volume element  $\omega_{p+1,q}^{\mathbb{C}} = i^{m+q+1} \omega_{p,q}$  acts as the identity on  $\Sigma N$  (resp  $\Sigma^+ N$ ). Then one has for all  $\varphi \in \Gamma(\Sigma M)$ :

$$\nu \cdot_N \varphi = \nu \cdot_N i^{m+q+1} \omega_{p+1,q} \cdot_N \varphi = i^{m+q+1} \nu \cdot_N \omega_{p,q} \cdot_N \nu \cdot_N \varphi = i^{m+q+1} (-1)^{p+q} \omega_{p,q} \cdot \varphi.$$

Then  $\nu \cdot_N \varphi = i\omega_{p,q}^{\mathbb{C}} \cdot \varphi$ , if  $n$  even and  $\nu \cdot_N \varphi = -\omega_{p,q}^{\mathbb{C}} \cdot \varphi$ , if  $n$  is odd. Consequently

$$\lambda X \cdot_N \varphi = -\lambda X \cdot_N \nu \cdot_N \nu \cdot_N \varphi = \mu X \cdot \omega_{p,q}^{\mathbb{C}} \cdot \varphi,$$

with  $\mu = -i\lambda$  if  $p + q = n$  is even and  $\mu = \lambda$  if  $p + q$  is odd. The above calculation for the induced spin connection terminates the proof.  $\square$

**Lemma 5** *Under the same assumptions as in proposition 9, let  $\Phi \in \Gamma(\Sigma N)$ ,*

$$\nabla_X^{\Sigma N} \Phi = \lambda X \cdot_N \Phi,$$

$\varphi = \Phi|_M$  and  $q$  be an even number. Then

1. If  $\lambda \in \mathbb{R}$ , we have  $|\varphi| = \text{Const}$ .
2. If  $\lambda \in i\mathbb{R}$ , we have  $X|\varphi|^2 = -2i\lambda \langle X \cdot \bar{\varphi}, \varphi \rangle$ , with  $\bar{\varphi} = \varphi^+ - \varphi^-$ , if  $n$  is even, and  $|\varphi| = \text{Const}$ , if  $n$  is odd.

*Proof:*

1. If  $\lambda \in \mathbb{R}$ , we have, as  $q$  is even

$$X|\Phi|^2 = 2\langle \nabla_X^{\Sigma N} \Phi, \Phi \rangle = 2\lambda \langle X \cdot_N \Phi, \Phi \rangle = -2(-1)^q \lambda \langle \Phi, X \cdot_N \Phi \rangle = 0$$

and consequently  $|\varphi| = \text{Const}$ .

2. If  $\lambda \in i\mathbb{R}$ , we have

$$X|\varphi|^2 = 2\mu \langle X \cdot \omega_{p,q}^{\mathbb{C}} \varphi, \varphi \rangle + \langle A(X) \cdot \varphi, \varphi \rangle = 2\mu \langle X \cdot \omega_{p,q}^{\mathbb{C}} \varphi, \varphi \rangle.$$

If  $n$  is odd, as  $\omega_{p,q}^{\mathbb{C}}$  acts on  $\Sigma M$  as the identity, we have

$$X|\varphi|^2 = 2\lambda \langle X \cdot \varphi, \varphi \rangle = 0,$$

and  $|\varphi| = \text{Const}$ .

If  $n$  is even  $\omega_{p,q}^{\mathbb{C}}$  acts on  $\Sigma^{\pm} M$  as  $\pm Id$  and consequently

$$X|\varphi|^2 = -2i\lambda \langle X \cdot \bar{\varphi}, \varphi \rangle,$$

with  $\bar{\varphi} = \varphi^+ - \varphi^-$ .  $\square$

**Definition 14** *A generalized Killing spinor on a pseudo-Riemannian spin manifold  $M$  with spin connection  $\nabla^{\Sigma M}$  is a solution  $\varphi$  of the generalized Killing equation*

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi + \mu X \cdot \omega_{p,q}^{\mathbb{C}} \cdot \varphi, \quad (2.1.12)$$

for all  $X \in TM$ , where  $A$  is a field of  $g$ -symmetric endomorphisms and  $\mu \in \mathbb{C}$ .

**Proposition 10** *Let  $M$  be a pseudo-Riemannian spin manifold and  $\varphi \in \Gamma(\Sigma M)$  be a solution of (2.1.12) on  $M$ . Then  $\varphi$  is a solution of the Dirac equation*

$$D\varphi = H\varphi - n \mu \omega_{p,q}^{\mathbb{C}} \cdot \varphi, \quad (2.1.13)$$

where  $H = -\frac{1}{2} \operatorname{tr}(A)$ .

In the following we call such spinors *generalized Dirac spinors*.

*Proof:*

$$D\varphi = \sum_{i=1}^{p+q} \varepsilon_i e_i \cdot \nabla^{\Sigma M} \varphi = \sum_{i=1}^{p+q} \varepsilon_i e_i \cdot \left( \frac{1}{2} A_i^j e_j + \mu e_i \cdot \omega_{p,q}^{\mathbb{C}} \right) \cdot \varphi,$$

where  $A_i^j := \varepsilon_j g(A(e_i), e_j)$  and  $\varepsilon_j A_i^j$  is symmetric. Then, as  $e_i \cdot e_j$  is antisymmetric, we have

$$D\varphi = -n \mu \omega_{p,q}^{\mathbb{C}} \cdot \varphi + \sum_{i=1}^{p+q} \varepsilon_i \frac{1}{2} A_i^i e_i \cdot e_i \varphi = -n \mu \omega_{p,q}^{\mathbb{C}} \cdot \varphi - \frac{1}{2} \operatorname{tr}(A) \cdot \varphi.$$

□

The spin curvature is defined by

$$R^{\Sigma M}(X, Y) = \nabla_X^{\Sigma M} \nabla_Y^{\Sigma M} \varphi - \nabla_Y^{\Sigma M} \nabla_X^{\Sigma M} \varphi - \nabla_{[X,Y]}^{\Sigma M} \varphi.$$

and can be computed in terms of the curvature tensor  $R^M$  in the following way:

$$R^{\Sigma M}(e_k, e_l) \varphi = \frac{1}{2} \sum_{i \leq j} \varepsilon_i \varepsilon_j \langle R(e_k, e_l) e_i, e_j \rangle e_i \cdot e_j \cdot \varphi. \quad (2.1.14)$$

Let  $\varphi$  be a solution of the generalized Killing equation. A simple calculation shows that the corresponding equation of integrability is given by (compare [Fr1], [Mo] for the Riemannian case):

$$\begin{aligned} R^{\Sigma M}(X, Y) \cdot \varphi &= \frac{1}{2} d^\nabla A(X, Y) \varphi + \frac{1}{4} (A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi \\ &\quad + \mu^2 (Y \cdot X - X \cdot Y) \varphi. \end{aligned} \quad (2.1.15)$$

**Remark 9** *The equations  $R^\Sigma(X, Y) = \frac{1}{4} (A(Y) \cdot A(X) - A(X) \cdot A(Y)) + \mu^2 (Y \cdot X - X \cdot Y)$  and  $d^\nabla A(X, Y) = 0$  are equivalent to the Gauß and Codazzi equation of hypersurface theory.*

### The three-dimensional case

In this paragraph, we show that, on a three-dimensional Riemannian manifold, a solution of the Dirac equation (2.1.13) is equivalent to a solution of the generalized Killing equation (2.1.12) for  $\mu \in \mathbb{R}$  and  $\mu \in i\mathbb{R}$ . Moreover, assuming that the tensor  $A$  is a Codazzi tensor, a solution of the generalized Killing equation for such  $\mu$  is equivalent to an isometric immersion of  $M$  into the simply connected model space  $\mathbb{M}_\kappa^n$ , with constant curvature  $\kappa = 4\mu^2$ . This last result was proven in [Mo] for  $\mu = 0$  and  $A$  parallel. Moreover this generalizes in dimension three a result of Bär, Gauduchon and Moroianu ([BGM]), which proved in

fact that, under the same conditions and for  $\mu = 0$ , a pseudo-Riemannian manifold of general dimension can be embedded as a hypersurface into a Ricci flat manifold. We recall that the model space  $\mathbb{M}_\kappa^n$  admits the maximal number of linearly independent Killing spinors, if the Killing constant is  $\lambda = \pm \frac{\kappa}{2}$  for  $\kappa \geq 0$  and  $\lambda = \pm \frac{i\kappa}{2}$  for  $\kappa < 0$ .

**Theorem 6** *Let  $(M^3, g)$  be a three-dimensional Riemannian spin manifold,  $H : M \rightarrow \mathbb{R}$  a real valued function and  $A$  a field of symmetric endomorphisms on  $TM$ . Then the following statements are equivalent:*

1.  $\varphi$  is a non-vanishing solution of the Dirac equation:

$$D\varphi = H\varphi, \quad |\varphi| = \text{Const.}$$

2.  $\varphi$  is a non-vanishing solution of the generalized Killing-equation

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi,$$

with  $\frac{1}{2} \text{tr}(A) = -H$  and  $A = -\frac{T_\varphi}{\langle \varphi, \varphi \rangle}$ , where

$$T_\varphi(X, Y) = \langle X \cdot \nabla_Y^{\Sigma M} \varphi + Y \cdot \nabla_X^{\Sigma M} \varphi, \varphi \rangle$$

is the energy-momentum tensor.

Moreover if  $A$  is a Codazzi tensor, i.e. if the following condition holds:

$$(\nabla_X^{\Sigma M} A)(Y) - (\nabla_Y^{\Sigma M} A)(X) = 0, \quad (2.1.16)$$

for all vector fields  $X, Y$  on  $M$ , and  $M$  is simply connected, then the first two statements are equivalent to

3. There exists a global isometric immersion  $M^3 \hookrightarrow \mathbb{R}^4$  with mean curvature  $H$  and Weingarten tensor  $A$ .

*Proof:* The proof of "3  $\Rightarrow$  2" (without condition on the tensor  $A$ ) is a direct application of proposition 9 and "2  $\Rightarrow$  1" follows directly from proposition 10.

"1  $\Rightarrow$  2": The three-dimensional complex spinor module is  $\Sigma_3 \cong \mathbb{C}^2$ . The complex spin representation is then real four-dimensional. We now define the map

$$\begin{cases} f : \mathbb{R}^3 \oplus \mathbb{R} \rightarrow \Sigma_3 \\ (v, r) \mapsto v \cdot \varphi + r\varphi \end{cases}$$

where  $\varphi$  is a given non-vanishing spinor.

Obviously  $f$  is an isomorphism. Then for all  $\psi \in \Sigma_3$  there is a unique pair  $(v, r) \in$



$(\mathbb{R}^3 \oplus \mathbb{R}) \cong T_p M^3 \oplus \mathbb{R}$ , such that  $\psi = v \cdot \varphi + r\varphi$ .

Consequently  $\nabla_X^{\Sigma M} \varphi_p \in \Gamma(T_p^* M \otimes \Sigma_3)$  can be expressed as follows:

$$\nabla_X^{\Sigma M} \varphi = B(X) \cdot \varphi + \omega(X)\varphi,$$

for all  $p \in M$  and for all vector fields  $X$ , with  $\omega$  a 1-form and  $B$  a (1,1)-tensor field. Moreover we have

$$X \langle \varphi, \varphi \rangle = \langle \nabla_X^{\Sigma M} \varphi, \varphi \rangle + \langle \varphi, \nabla_X^{\Sigma M} \varphi \rangle = 2 \langle \omega(X)\varphi, \varphi \rangle \Rightarrow \omega(X) = \frac{d(|\varphi|^2)}{2|\varphi|^2}(X).$$

which yields  $\omega(X) = 0$ , as the norm of  $\varphi$  is constant.

We now prove that  $B$  is symmetric. Let  $B = S + T$  with  $S$  the symmetric and  $T$  the antisymmetric part of  $B$ . Let  $\{e_i\}$  be an orthonormal basis of  $TM$  and  $\varphi$  be a solution of the Dirac equation  $D\varphi = H\varphi$ . We have

$$\begin{aligned} D\varphi &= \sum_{i=1}^3 e_i \cdot \nabla_{e_i}^{\Sigma M} \varphi = \sum_{i=1}^3 e_i \cdot B_i^j e_j \cdot \varphi \\ &= \sum_{i=1}^3 T_i^j e_i \cdot e_j \cdot \varphi + \sum_{i=1}^3 S_i^i e_i \cdot e_i \cdot \varphi + \sum_{i \neq j}^3 \underbrace{S_i^j}_{\text{sym.}} \underbrace{e_i \cdot e_j}_{\text{antisym.}} \cdot \varphi \\ &= -2 \sum_{i < j}^3 T_j^i e_i \cdot e_j \cdot \varphi + \sum_{i=1}^3 S_i^i e_i \cdot e_i \cdot \varphi \\ &= -2(T_2^1 e_1 \cdot e_2 + T_3^1 e_1 \cdot e_3 + T_3^2 e_2 \cdot e_3) \cdot \varphi - \text{tr}(B)\varphi = H\varphi \end{aligned}$$

Note that  $\Re \langle (T_2^1 e_3 - T_3^1 e_2 + T_3^2 e_1)\varphi, \varphi \rangle = 0$ . Using the fact that in the three-dimensional case  $e_1 \cdot e_2 = e_3$  we deduce:

$$\begin{aligned} (T_2^1 e_3 - T_3^1 e_2 + T_3^2 e_1)\varphi &= 0 \Leftrightarrow T_2^1 = T_3^1 = T_3^2 = 0, \\ -\text{tr}(B) &= H. \end{aligned}$$

It follows that  $B$  is symmetric, with  $-\text{tr}(B) = H$ .

Further

$$\begin{aligned} T_\varphi(e_i, e_j) &= \langle e_i \cdot \nabla_{e_j}^{\Sigma M} \varphi + e_j \cdot \nabla_{e_i}^{\Sigma M} \varphi, \varphi \rangle = \langle \sum_{k=1}^3 B_j^k e_i \cdot e_k \cdot \varphi + \sum_{k=1}^3 B_i^k e_j \cdot e_k \cdot \varphi, \varphi \rangle \\ &= -\langle B_j^i \varphi + B_i^j \varphi, \varphi \rangle = -2B_j^i |\varphi|^2, \end{aligned}$$

and finally  $B = -\frac{T_\varphi}{2\langle \varphi, \varphi \rangle}$ . We put  $A = 2B$ , which completes the proof.

"2  $\Rightarrow$  3": As  $e_1 \cdot e_2 = e_3$  in the three dimensional case and with  $R_{ijkl} = \langle R(e_i, e_j)e_l, e_k \rangle$ , we have:

$$\begin{aligned} R^{\Sigma M}(e_1, e_2) \cdot \varphi &= \frac{1}{2} \sum_{i \leq j} \langle R(e_1, e_2)e_i, e_j \rangle e_i \cdot e_j \cdot \varphi \\ &= -\frac{1}{2} [R_{1212} \cdot e_1 \cdot e_2 \cdot \varphi - R_{1213} \cdot e_3 \cdot e_1 \cdot \varphi + R_{1223} \cdot e_2 \cdot e_3 \cdot \varphi] \\ &= -\frac{1}{2} [R_{1212} \cdot e_3 \cdot \varphi - R_{1213} \cdot e_2 \cdot \varphi + R_{1223} \cdot e_1] \cdot \varphi. \end{aligned}$$

The same calculation for  $R^{\Sigma M}(e_2, e_3)$  and  $R^{\Sigma M}(e_3, e_1)$  yields

$$R^{\Sigma M}(e_i, e_j) \cdot \varphi = \frac{1}{2}[R_{ijjk} \cdot e_j - R_{ijij} \cdot e_k - R_{ijjk} \cdot e_i] \cdot \varphi, \quad (2.1.17)$$

where  $(i, j, k)$  is any cyclic permutation of  $(1, 2, 3)$ .

Further with a simple calculation we find

$$\begin{aligned} A(e_j) \cdot A(e_i) - A(e_i) \cdot A(e_j) &= 2(A_{ik}A_{jj} - A_{ij}A_{jk})e_i \\ &\quad - 2(A_{ik}A_{ji} - A_{ii}A_{jk})e_j + 2(A_{ij}A_{ji} - A_{ii}A_{jk})e_k. \end{aligned}$$

With the integrability condition (2.1.15) this yields

$$\begin{aligned} (\nabla_{e_j}A)(e_i) - (\nabla_{e_i}A)(e_j) &= (R_{ijjk} - (A_{ik}A_{jj} - A_{ij}A_{jk}))e_i \\ &\quad - (R_{ijik} - (A_{ik}A_{ji} - A_{ii}A_{jk}))e_j + (R_{ijij} - (A_{ij}A_{ji} - A_{ii}A_{jk}))e_k, \end{aligned}$$

which proves that, if  $A$  is a Codazzi tensor, it satisfies the Gauß equations, too. This implication was shown by Morel ([Mo]) in the Riemannian case for a parallel tensor  $A$ . We also refer to [C] for results in general dimension.  $\square$

**Corollary 3** *Let  $(M^3, g)$  be a three-dimensional Riemannian spin manifold,  $H : M \rightarrow \mathbb{R}$  a real valued function and  $A$  a symmetric endomorphism on  $TM$ .*

1. *The following statements are equivalent:*

a.  *$\varphi$  is a non-vanishing solution of the Dirac equation:*

$$D\varphi = H\varphi - \lambda \cdot \omega_3 \cdot \varphi, \quad |\varphi| = \text{Const}, \quad \lambda \in \mathbb{R}.$$

b.  *$\varphi$  is a non-vanishing solution of the generalized Killing equation*

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2}A(X) \cdot \varphi + \lambda X \cdot \omega_3 \cdot \varphi, \quad \lambda \in \mathbb{R},$$

*with  $\frac{1}{2} \text{tr}(A) = -H$  and  $A = -\frac{T_\varphi}{\langle \varphi, \varphi \rangle}$ , where*

$$T_\varphi(X, Y) = \langle X \cdot \nabla_Y^{\Sigma M} \varphi + Y \cdot \nabla_X^{\Sigma M} \varphi, \varphi \rangle$$

*is the energy-momentum tensor.*

*Moreover if  $A$  is a Codazzi tensor,  $M$  is simply connected and  $\lambda = \frac{1}{2}$  both statements are equivalent to*

c. *There exists a global isometric immersion  $M^3 \hookrightarrow \mathbb{S}^4$  into the four-dimensional sphere, with mean curvature  $H$  and Weingarten tensor  $A$ .*

2. *The following statements are equivalent:*

a.  *$\varphi$  is a non-vanishing solution of the Dirac equation:*

$$D\varphi = H\varphi - i\lambda\omega_3 \cdot \varphi, \quad |\varphi| = \text{Const}, \quad \lambda \in \mathbb{R}.$$

b.  $\varphi$  is a non-vanishing solution of the generalized Killing equation

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi + i\lambda X \cdot \omega_3 \cdot \varphi, \quad \lambda \in \mathbb{R},$$

with  $\frac{1}{2} \text{tr}(A) = -H$  and  $A = -\frac{T_\varphi}{\langle \varphi, \varphi \rangle}$ , where

$$T_\varphi(X, Y) = \langle X \cdot \nabla_Y^{\Sigma M} \varphi + Y \cdot \nabla_X^{\Sigma M} \varphi, \varphi \rangle$$

is the energy momentum tensor.

Moreover if  $A$  is a Codazzi tensor,  $M$  is simply connected and  $\lambda = \frac{1}{2}$  both statements are equivalent to

c. There exists a global isometric immersion  $M^3 \hookrightarrow \mathbf{H}^4$  into the four-dimensional hyperbolic space, with mean curvature  $H$  and Weingarten tensor  $A$ .

*Proof:*

1. a)  $\Rightarrow$  b) As  $|\varphi| = \text{Const}$ , using the same argument as in the proof of theorem 6, we have

$$\nabla_X^{\Sigma M} \varphi = B(X) \cdot \varphi = S(X) \cdot \varphi + T(X) \cdot \varphi,$$

where  $B$  is a (1,1)-tensor field and  $S$  (resp.  $T$ ) is the symmetric (resp. antisymmetric) part of  $B$ .

Let  $\varphi$  be a solution of the Dirac equation  $D\varphi = H\varphi - \lambda \cdot \omega_3 \cdot \varphi$ . Then we obtain similarly to the case of theorem 6

$$D\varphi = -2 \sum_{i < j}^3 T_j^i e_i \cdot e_j \cdot \varphi - \text{tr}(B)\varphi = H\varphi - \lambda \cdot \omega_3 \cdot \varphi,$$

which yields

$$\langle D\varphi, \varphi \rangle = -2 \sum_{i < j}^3 T_j^i \underbrace{\langle e_i \cdot e_j \cdot \varphi, \varphi \rangle}_{=0} - \langle \text{tr}(B) \cdot \varphi, \varphi \rangle = \langle H\varphi, \varphi \rangle - \lambda \underbrace{\langle \omega_3 \cdot \varphi, \varphi \rangle}_{=0}.$$

Consequently  $-\text{tr}(B) = H$  and  $-2 \sum_{i < j}^3 T_j^i e_i \cdot e_j \cdot \varphi = \lambda \omega_3 \cdot \varphi$ .

Further we calculate

$$\langle T(e_j) \cdot \varphi, e_i \cdot \varphi \rangle = \sum_k^3 T_j^k \underbrace{\langle e_k \cdot \varphi, e_i \cdot \varphi \rangle}_{=-\langle e_i \cdot e_k \cdot \varphi, \varphi \rangle = 0, i \neq k} = T_j^i |\varphi|^2,$$

$$\text{and } \langle \lambda \omega_3 \cdot e_j \cdot \varphi, e_i \cdot \varphi \rangle = -\langle \lambda \omega_3 \cdot \varphi, e_j \cdot e_i \cdot \varphi \rangle = -2 \sum_{k < l}^3 T_l^k \langle e_k \cdot e_l \cdot \varphi, e_i \cdot e_j \cdot \varphi \rangle.$$

In the three-dimensional case at most three of the four indices differ. Moreover, for  $m \neq n$ ,  $\langle e_m \cdot e_n \cdot \varphi, \varphi \rangle = 0$  holds and as the trace of an antisymmetric tensor

vanishes, we have:  $\langle e_k \cdot e_l \cdot \varphi, e_j \cdot e_i \cdot \varphi \rangle \neq 0 \Leftrightarrow k = i, l = j$  or  $k = j, l = i, i \neq j$ , which yields

$$-\langle \lambda \omega_3 \cdot e_j \cdot \varphi, e_i \cdot \varphi \rangle = 2T_j^i = \langle T(e_j) \cdot \varphi, e_i \cdot \varphi \rangle$$

and finally

$$T(X) \cdot \varphi = \lambda \omega_3 \cdot X \cdot \varphi.$$

Using this equation for the symmetric part of  $B$  we obtain, analogously to the proof of theorem 6:

$$\begin{aligned} T_\varphi(e_i, e_j) &= \langle e_i \cdot \nabla_{e_j}^{\Sigma M} \varphi + e_j \cdot \nabla_{e_i}^{\Sigma M} \varphi, \varphi \rangle \\ &= \left\langle \sum_k^3 S_j^k e_i \cdot e_k \cdot \varphi + \sum_k^3 S_i^k e_j \cdot e_k \cdot \varphi, \varphi \right\rangle + \langle i\lambda \omega_3 \cdot \underbrace{(e_i \cdot e_j + e_j \cdot e_i)}_{=0} \cdot \varphi, \varphi \rangle \\ &= -2S_j^i |\varphi|^2 \quad \Rightarrow S(X) = -\frac{T_\varphi}{2\langle \varphi, \varphi \rangle} := A(X). \end{aligned}$$

b)  $\Rightarrow$  c) Using an analogous calculation to the one of theorem 6 we find with equation (2.1.15)

$$\begin{aligned} (\nabla_{e_j}^{\Sigma M} A)(e_i) - (\nabla_{e_i}^{\Sigma M} A)(e_j) &= (R_{ijjk} - (A_{ik}A_{jj} - A_{ij}A_{jk} - 4\lambda^2))e_i \\ &\quad - (R_{ijik} - (A_{ik}A_{ji} - A_{ii}A_{jk}) - 4\lambda^2)e_j + (R_{ijij} - (A_{ij}A_{ji} - A_{ii}A_{jk}) - 4\lambda^2)e_k, \end{aligned}$$

which yields the result by theorem 4 if  $\lambda = \frac{1}{2}$ .

The rest of the proof is similar to the proof of theorem 6.

2. The proof is identical to the one of the first case.

□

## The two-dimensional case

In this paragraph we generalize the result of Friedrich ([Fr1]) to the pseudo-Riemannian case. The difficulty comes from the existence of isotropic spinors. In fact the half spinor bundles  $\Sigma^\pm M$  are maximal isotropic with respect to the hermitian scalar product on  $\Sigma M$ .

**Theorem 7** *Let  $(M, g)$  be a pseudo-Riemannian surface of signature  $(1, 1)$ ,  $H : M \rightarrow \mathbb{R}$  be a real valued function. Then the following two statements are equivalent:*

1.  $\varphi$  is a non-vanishing non-isotropic solution of the Dirac equation  $D\varphi = H\varphi$  with  $|\varphi| = 1$ ,
2.  $\varphi$  is a non-vanishing non-isotropic solution of the generalized Killing equation

$$\nabla_X^{\Sigma M} \varphi = \frac{1}{2} A(X) \cdot \varphi,$$

where  $A$  is a  $g$ -symmetric endomorphism and  $-\frac{1}{2} \text{tr} A = H$ .

Moreover if  $A$  is a Codazzi tensor and  $M$  simply connected both statements are equivalent to

3. There exists a global isometric spacelike immersion  $M \hookrightarrow \mathbb{R}^{2,1}$  with mean curvature  $H$  and second fundamental form  $A$ .

*Proof:* Again "3  $\Rightarrow$  2" follows from proposition 9 and "2  $\Rightarrow$  1" from proposition 10.

"1  $\Rightarrow$  2". We define  $\beta_\varphi(e_i, e_j) = \langle \nabla_{e_i}^{\Sigma M} \varphi, e_j \cdot \varphi \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the pseudo-hermitian symmetric  $\text{Spin}_0(p, q)$ -invariant bilinear form defined in the last section.

$$\begin{aligned} \beta_\varphi(e_1, e_2) &= \langle \nabla_{e_1}^{\Sigma M} \varphi, e_2 \cdot \varphi \rangle = -\langle \nabla_{e_1}^{\Sigma M} \varphi, e_1^2 \cdot e_2 \cdot \varphi \rangle \\ &= -\langle e_1 \cdot \nabla_{e_1}^{\Sigma M} \varphi, e_1 \cdot e_2 \cdot \varphi \rangle = -\langle D\varphi + e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi, e_1 \cdot e_2 \cdot \varphi \rangle \\ &= -H \langle \varphi, e_1 \cdot e_2 \cdot \varphi \rangle - \langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi, e_1 \cdot e_2 \varphi \rangle. \end{aligned}$$

Moreover  $\langle \varphi, e_1 \cdot e_2 \cdot \varphi \rangle = \langle e_2 \cdot e_1 \cdot \varphi, \varphi \rangle = -\langle e_1 \cdot e_2 \cdot \varphi, \varphi \rangle = -\langle \varphi, e_1 \cdot e_2 \cdot \varphi \rangle = 0$ . Consequently

$$\begin{aligned} \beta_\varphi(e_1, e_2) &= -\langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \varphi, e_1 \cdot e_2 \cdot \varphi \rangle = \langle e_2 \cdot \nabla_{e_2}^{\Sigma M} \cdot \varphi, e_2 \cdot e_1 \cdot \varphi \rangle \\ &= \langle \nabla_{e_2}^{\Sigma M} \cdot \varphi, e_2^2 \cdot e_1 \cdot \varphi \rangle = \beta_\varphi(e_2, e_1), \end{aligned}$$

and  $\beta_\varphi$  is symmetric.

Let us define the endomorphism

$$(B_\varphi)_i^j = g(B_\varphi(e_i), e_j) := \beta_\varphi(e_i, e_j).$$

It is obviously  $g$ -symmetric and  $\text{tr}(B) = g^{ij} B_{ij} = H$ .

Moreover let

$$b_\varphi^\pm(X, Y) = \langle \nabla_X^{\Sigma M} \varphi^\pm, Y \cdot \varphi^\pm \rangle$$

and

$$(B_\varphi^\pm)_i^j = g(B_\varphi^\pm(e_i), e_j) := \beta_\varphi^\pm(e_i, e_j).$$

With the same calculation as above and with  $D\varphi^\pm = H\varphi^\mp$ , we obtain  $\text{tr}(B^\pm) = H\langle \varphi^\mp, \varphi^\pm \rangle$ .

**Claim:**

$$\langle B_\varphi^\pm(X) \cdot \varphi^\pm, e_i \cdot \varphi^\mp \rangle = -3 \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle \quad (2.1.18)$$

*Proof:* Obviously we can suppose that  $\varphi^\pm(p) \neq 0$  in an open neighborhood of  $p$  as  $\langle \varphi^+, \varphi^- \rangle \neq 0$ .

We remark that  $\frac{e_i \cdot \varphi^\pm}{\langle \varphi^+, \varphi^- \rangle}$  is a normalized dual frame of  $\Sigma^\mp M$ . Consequently as  $\langle \nabla_X^{\Sigma M} \varphi^\pm, e_i \cdot \varphi^\mp \rangle = 0$ , because of the isotropy of  $\varphi^\pm$ , we have:

$$\begin{aligned} \nabla_X^{\Sigma M} \varphi &= \sum_1^2 \varepsilon_i \left( \langle \nabla_X^{\Sigma M} \varphi^+, e_i \varphi^+ \rangle \frac{e_i \cdot \varphi^-}{\langle \varphi^+, \varphi^- \rangle} + \langle \nabla_X^{\Sigma M} \varphi^-, e_i \varphi^- \rangle \frac{e_i \cdot \varphi^+}{\langle \varphi^+, \varphi^- \rangle} \right) \\ &= \frac{1}{\langle \varphi^+, \varphi^- \rangle} \sum_1^2 \varepsilon_i (b_{\varphi^+}(X, e_i) e_i \cdot \varphi^- + b_{\varphi^-}(X, e_i) e_i \cdot \varphi^+) \\ &= \frac{1}{\langle \varphi^+, \varphi^- \rangle} (B_\varphi^+(X) \cdot \varphi^- + B_\varphi^-(X) \cdot \varphi^+) \end{aligned}$$

Comparing degrees, this yields

$$\nabla_X^{\Sigma M} \varphi^\pm = \frac{1}{\langle \varphi^+, \varphi^- \rangle} B_\varphi^\pm(X) \cdot \varphi^\mp.$$

Moreover

$$\langle B_\varphi^\pm(X) \cdot \varphi^\pm, e_i \cdot \varphi^\mp \rangle = -2g(B_\varphi^\pm(X), e_i) \langle \varphi^+, \varphi^- \rangle - \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\mp \rangle,$$

but

$$g(B_\varphi^\pm(X), e_i) = b_\varphi^\pm(X, e_i) = \langle \nabla_X^{\Sigma M} \varphi^\pm, e_i \varphi^\pm \rangle = \frac{1}{\langle \varphi^+, \varphi^- \rangle} \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle$$

$$\langle B_\varphi^\pm(X) \cdot \varphi^\pm, e_i \cdot \varphi^\mp \rangle = -3 \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\mp \rangle \quad \square$$

Moreover we have:

$$\langle \nabla_X^{\Sigma M} \varphi, e_i \varphi^\pm \rangle = \langle \nabla_X^{\Sigma M} \varphi^+ + \nabla_X^{\Sigma M} \varphi^-, e_i \varphi^\pm \rangle = \langle \nabla_X^{\Sigma M} \varphi^\pm, e_i \varphi^\pm \rangle = \frac{1}{\langle \varphi^+, \varphi^- \rangle} \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle$$

and

$$\begin{aligned} \langle B_\varphi(X) \cdot \varphi, e_i \cdot \varphi^\pm \rangle &= \langle B_\varphi^+(X) \cdot (\varphi^+ + \varphi^-), e_i \cdot \varphi^\pm \rangle + \langle B_\varphi^-(X) \cdot (\varphi^+ + \varphi^-), e_i \cdot \varphi^\pm \rangle \\ &= \langle B_\varphi^+(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle + \langle B_\varphi^-(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle. \end{aligned}$$

Then with (2.1.18) we have

$$\begin{aligned} \langle B_\varphi(X) \cdot \varphi, e_i \cdot \varphi^\pm \rangle &= \langle \varphi^+, \varphi^- \rangle \langle \nabla_X^{\Sigma M} \varphi^\pm, e_i \varphi^\pm \rangle + \langle B_\varphi^\pm(X) \cdot \varphi^\pm, e_i \cdot \varphi^\mp \rangle \\ &= \langle \varphi^+, \varphi^- \rangle \langle \nabla_X^{\Sigma M} \varphi, e_i \varphi^\pm \rangle - 3 \langle B_\varphi^\pm(X) \cdot \varphi^\mp, e_i \cdot \varphi^\pm \rangle \end{aligned}$$

and finally

$$\langle \nabla_X^{\Sigma M} \varphi, e_i \cdot \varphi^\pm \rangle = -\frac{1}{2 \langle \varphi^+, \varphi^- \rangle} \langle B_\varphi^\pm(X) \cdot \varphi^\pm, e_i \cdot \varphi^\mp \rangle. \quad (2.1.19)$$

Setting  $A := -\frac{1}{2} B_\varphi$  and as  $\|\varphi\| = 1$  and consequently  $\langle \varphi^+, \varphi^- \rangle = 1$ , this finishes the proof, as  $e_i \cdot \varphi^\pm$  is a dual frame of  $\Sigma^\pm M$ .  $\square$

"2  $\Rightarrow$  3" : For the spin curvature in dimension 2 we have with equation (2.1.14):

$$R^{\Sigma M}(e_1, e_2)\varphi = \frac{1}{2} \varepsilon_1 \varepsilon_2 R_{1221} e_1 \cdot e_2 \cdot \varphi = \frac{1}{2} R_{1212} e_1 \cdot e_2 \cdot \varphi.$$

Consequently, using the fact that

$$A(e_2)A(e_1) - A(e_1)A(e_2) = -2 \det(A) e_1 \cdot e_2,$$

the integrability conditions (2.1.15) can be expressed by

$$R_{1212} e_1 \cdot e_2 \cdot \varphi = -\det(A) e_1 \cdot e_2 \cdot \varphi + ((\nabla_{e_2}^{\Sigma M} A)(e_1) - (\nabla_{e_1}^{\Sigma M} A)(e_2)) \cdot \varphi$$

Let now define the vector field

$$B = (\nabla_{e_2}^{\Sigma M} A)(e_1) - (\nabla_{e_1}^{\Sigma M} A)(e_2)$$

and the function

$$f = R_{1212} + \det(A).$$

We recall that the spinor bundle decomposes under the action of the real volume form  $\omega_{1,1}$  into the direct sum

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M,$$

where  $\Sigma^+ M$ , respectively  $\Sigma^- M$ , are the eigenspace to the eigenvalues 1, respectively  $-1$ . Then, for any spinor  $\varphi \in \Gamma(\Sigma)$  we have  $\varphi = \varphi^+ + \varphi^-$ . Consequently we obtain the following system of equations:

$$B \cdot \varphi^\pm = \mp f \varphi^\mp,$$

which leads to

$$\|B\|^2 \varphi^\pm = f^2 \varphi^\pm$$

Moreover  $B$  is non isotropic. In fact if  $\|B\| = 0$ , then  $B \cdot \varphi = 0$ : writting  $B = B_1 e_1 + B_2 e_2$ , we have

$$B_1 e_1 \cdot \varphi = -B_2 e_2 \cdot \varphi \Leftrightarrow B_1 \varphi = B_2 \omega_{1,1} \cdot \varphi \Leftrightarrow B_1 \varphi^\pm = \pm B_2 \varphi^\pm,$$

which yields, as  $\varphi^+$  and  $\varphi^-$  are linearly independant, that  $B_1 = B_2 = 0$ .

$A$  is a Codazzi tensor, then  $B = 0$ . With the above considerations, this is equivalent to  $f = 0$ , which finishes the proof.  $\square$

Obviously this method does not allow to cancel the condition on  $A$  to be Codazzi and we cannot show the equivalence between a solution of the Dirac equation on the surface  $M$  and an immersion of  $M$  into  $\mathbb{R}^{2,1}$ . This motivates a change of the method and the contents of the following section.

## 2.2 A spinor representation for Lorentzian surfaces in $\mathbb{R}^{2,1}$

### 2.2.1 Lorentzian surfaces

In the following we call **Lorentzian surfaces** smooth and orientable two-dimensional manifolds provided with an indefinite metric. Let  $M$  be a strongly oriented smooth two-dimensional manifold with a pseudo-Riemannian metric of signature  $(1,1)$ , i.e. a time-oriented oriented Lorentzian surface. We recall that in this case the existence of spin structures is ensured (see remark 6). Denote by  $P_{\text{Spin}_+}$  a spin structure on  $M$ .

We have

$$\text{Spin}_+(1,1) \subset \mathcal{C}l_{1,1}^0 \cong \mathcal{C}l_{0,1} \cong \mathbb{R} \oplus \mathbb{R} \cong C.$$

Therefore, the spin representation  $\Delta_{1,1}$  splits under the action of the volume form  $\omega_{1,1}$  into the direct sum of two inequivalent representations and it holds for the spinor module  $\Sigma_{1,1} = \Sigma_{1,1}^+ \oplus \Sigma_{1,1}^- \cong \mathbb{R} \oplus \mathbb{R} \cong C$ . We remark that  $\omega_{1,1}$  defines a para-complex structure on  $\Sigma_{1,1}$  and we identify it in the following with the para-complex unit. Therefore the spinor bundle  $\Sigma M = P_{\text{Spin}} \times_{\Delta_{1,1}} \Sigma_{1,1} = P_{\text{Spin}} \times_{\Delta_{1,1}} C$  of  $M$  can be identified with a para-complex line bundle.

Moreover, we have

$$\begin{aligned} \text{SO}_+(1,1) &= \{\exp(e\theta) | \theta \in \mathbb{R}\} \subset H^1, \\ \text{Spin}_+(1,1) &\cong U^\pi(C) = \{\pm \exp(e\theta) | \theta \in \mathbb{R}\} \subset H^1. \end{aligned}$$

The unique two-to-one Spin-covering of  $\text{SO}_+(1,1)$  is given by

$$\begin{aligned} \lambda : C^* \supset \text{Spin}_+(1,1) &\rightarrow \text{SO}_+(1,1) \subset C^*, \\ z &\mapsto z^2. \end{aligned}$$

Let  $L$  be a para-hermitian line bundle over  $M$ . As seen in section 1.1, the transition functions of  $L$  for a certain open covering  $\{U_\alpha\}$  of  $M$  are of the form  $\tilde{\varphi}_{\alpha\beta}(x) = \pm \exp(\omega_{1,1}\theta_{\alpha\beta}(x))$ , where  $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$ ,  $x \in M$ . This means that  $L$  is a  $\text{Spin}_+(1,1)$ -bundle.

Consider now the product bundle  $L^2 := L \otimes_C L$ . This bundle has transition functions given by  $\tilde{\varphi}_{\alpha\beta}^2(x) \in \text{SO}_+(1,1)$  for the same open covering  $\{U_\alpha\}$ . Similarly to the approach of [KS] for Riemannian surfaces the above considerations show, that the classical definition 13 of a spinor bundle reduces to the following

**Definition 15** *A spinor bundle on a strongly oriented Lorentzian surface  $M$  is a para-hermitian line bundle  $L$  endowed with an isomorphism  $\kappa : L \otimes_C L \cong T^*M$ . In the following we will denote it by  $\Sigma M$ .*

A real formulation of definition 15 is given by

**Proposition 11** *A spinor bundle on a strongly oriented Lorentzian surface  $M$  is equivalent to the data of two real line bundles  $L_\pm$  (called **half spinor bundles** and denoted in the following by  $\Sigma^\pm M$ ), with a pairing  $L_+ \otimes_{\mathbb{R}} L_- \rightarrow \mathbb{R}$ , and isomorphisms  $T^\pm M \cong L_\pm \otimes_{\mathbb{R}} L_\pm$ .*

*Proof:* We put  $L_+ \oplus L_- =: L$ . Let  $k_{\alpha\beta}^\pm$  be the transition functions of the bundles  $L_\pm$  with respect to an open covering  $\{U_\alpha\}$ . Then by definition the transition functions of  $L_+ \oplus L_- = L$  are given by  $K_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^+ & 0 \\ 0 & k_{\alpha\beta}^- \end{pmatrix}$ .

Obviously the transition functions of the bundles  $L_+ \otimes_{\mathbb{R}} L_+ \oplus L_- \otimes_{\mathbb{R}} L_-$  and  $L \otimes_C L$  are the same, i.e.  $\tilde{K}_{\alpha\beta} = \begin{pmatrix} (k_{\alpha\beta}^+)^2 & 0 \\ 0 & (k_{\alpha\beta}^-)^2 \end{pmatrix} = K_{\alpha\beta}^2$ .  $\square$

To illustrate this point of view, it is illuminating to consider the Minkowski space  $M = \mathbb{R}^{1,1} = C = \mathbb{R}(1+e) \oplus \mathbb{R}(1-e)$ . We have  $T_p^\pm M = \mathbb{R}(1 \pm e) \cong \mathbb{R}\sqrt{(1 \pm e)} \otimes_{\mathbb{R}} \mathbb{R}\sqrt{(1 \pm e)}$ ,  $p \in M$ .



The pairing

$$\begin{aligned}\mathbb{R}\sqrt{1+e} \otimes_{\mathbb{R}} \mathbb{R}\sqrt{1-e} &\rightarrow \mathbb{R} \\ a\sqrt{1+e} \otimes b\sqrt{1-e} &\mapsto 2ab,\end{aligned}$$

with  $a, b \in \mathbb{R}$ , induces a Clifford multiplication on  $\Sigma_p^\pm M = \mathbb{R}\sqrt{(1 \pm e)}$  by:

$$\begin{aligned}\rho^\pm : T^\pm M \otimes \Sigma^\mp M = \Sigma^\pm M \otimes \Sigma^\pm M \otimes \Sigma^\mp M &\rightarrow \Sigma^\pm M \\ a(1 \pm e) \otimes b\sqrt{1 \mp e} &\mapsto 2ab\sqrt{1 \pm e}\end{aligned}$$

and hence a Clifford multiplication

$$\rho : TM \otimes \Sigma M \rightarrow \Sigma M \quad (2.2.1)$$

on  $\Sigma M = \Sigma^+ M \oplus \Sigma^- M \cong M \times \mathbb{R}^2$ .

Obviously  $(1+e)$ , resp  $(1-e)$  corresponds to the multiplication by  $-2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , resp  $2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ .

Let  $\nabla : \Gamma(\Sigma^\pm) \rightarrow \Gamma(T^*M \otimes \Sigma^\pm)$  be the covariant derivative on the spinor bundle. As  $\{1, e\}$  is an orthonormal basis we have

$$D\psi = \rho(1)\nabla_1\psi - \rho(e)\nabla_e\psi = \frac{1}{2}\rho(1+e)\nabla_{1-e}\psi + \frac{1}{2}\rho(1-e)\nabla_{1+e}\psi,$$

where  $D : \Gamma(\Sigma M) \rightarrow \Gamma(\Sigma M)$  is the Dirac operator on  $\mathbb{R}^{1,1}$  and  $\psi \in \Gamma(\Sigma M)$ . Hence as  $\nabla_{1+e} = 2\frac{\partial}{\partial z_+}$  and  $\nabla_{1-e} = 2\frac{\partial}{\partial z_-}$ , the Dirac operator in the Minkowski space has the form

$$D = 2 \begin{pmatrix} 0 & -\frac{\partial}{\partial z_+} \\ \frac{\partial}{\partial z_-} & 0 \end{pmatrix}. \quad (2.2.2)$$

Remark that for a given  $w \in \text{SO}_+(1,1) \subset C^*$  there exist exactly two square roots  $z \in \text{Spin}_+(1,1)$ . We will denote the one with  $\Im(z) > 0$  by  $z = \sqrt{w}$ . Locally we can consider the  $(1,0)$ -form  $dz$ , where  $z$  is a para-holomorphic coordinate, as a section of  $T^*M$ . There exist four sections  $s$  of  $L$  (see section 1.1) such that  $\kappa(s \otimes s) = dz$ , as  $z$  has to be compatible with the orientation and the time orientation. Without loss of generality we can choose one of these spinors and denote it by  $\varphi = \sqrt{dz}$ . Later we choose a trivialization of  $T^\pm M$ , which induces a trivialization of the spinor bundle. Therefore, we can express any spinor  $s$  in the form  $s = f\varphi$ , for which it holds  $s^2 = f^2 dz$ .

We will use this point of view to derive a spinor representation of Lorentzian surfaces in the Minkowski space  $\mathbb{R}^{2,1}$ .

## 2.2.2 The Weierstraß representation

Using the real splitting (1.1.1) of exterior forms on a para-complex manifold we give a real Weierstraß representation for Lorentzian surfaces. This generalizes a result of Konderak

(see [KO]) for minimal surfaces. We recall that a  $(1+, 0-)$ - (resp. a  $(0+, 1-)$ )-form  $\omega_{\pm}$  on  $M$  can be written as  $\omega_{\pm} = \phi_{\pm} dz_{\pm}$ , where  $z_{\pm}$  are the adapted coordinates introduced in section 1.1 and  $\phi_{\pm}$  are real functions.

Let  $(M, g)$  be a Lorentzian surface with pseudo-Riemannian metric  $g$ . In this chapter, we say that  $M$  is **conformally immersed** in  $\mathbb{R}^{2,1}$  if and only if there exists a smooth map  $F : M \rightarrow \mathbb{R}^{2,1}$ , such that

$$\langle dF(X), dF(Y) \rangle_{\mathbb{R}^{2,1}} = \mu g(X, Y),$$

for all  $X, Y \in TM$ , and where  $\mu$  is a **positive** function. Let  $\{U, \phi\}$  be a local chart on  $M$  and  $(x, y)$  real local coordinates for this chart. Then in this coordinates  $g$  is conformally equivalent to  $dx^2 - dy^2$ , i.e.

$$g|_U = \lambda(dx^2 - dy^2), \quad \lambda > 0$$

and the above definition is equivalent to

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = 0, \quad \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle = -\left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = \lambda > 0. \quad (2.2.3)$$

In local coordinates  $(x_i, x_j)$  we can write  $g = g_{ij} dx^i dx^j$ , with  $i, j, k = 1, 2$ . The Laplace operator on  $M$  is defined for an arbitrary real valued function  $f$  by taking

$$\Delta_g f = g^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} f - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right),$$

where we follow the Einstein summation convention and  $g^{ij}$  is the inverse of the matrix  $g_{ij}$ . Let now  $F : M \rightarrow \mathbb{R}^{2,1}$  be a conformal immersion, then for the local coordinates  $(z_+, z_-)$  we can write  $g = \lambda dz_+ dz_-$ ,  $\lambda > 0$  or in matrix form  $g = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . A simple calculation shows that the Laplacian of a real valued function  $f$  on  $M$  is given by

$$\Delta f = \frac{2\partial^2 f}{\lambda \partial z_+ \partial z_-}, \quad (2.2.4)$$

where  $\lambda$  is the conformal factor of the metric.

Moreover it holds true for the mean curvature  $H = \frac{1}{2} \text{tr } B$  of the surface, where  $B$  is the second fundamental form for  $F$ , that

$$\frac{1}{2} H \nu = \Delta F, \quad (2.2.5)$$

where  $\nu$  is the (spacelike) unit normal vector field of the immersion. Let now  $\omega_+$  and  $\omega_-$  be the triples of forms of the immersion as defined in theorem 8.  $\omega_+ + \omega_- = \frac{\partial F}{\partial z} dz$  and consequently  $\omega_+ = \frac{\partial F}{\partial z_+} dz_+$  and  $\omega_- = \frac{\partial F}{\partial z_-} dz_-$ . Moreover  $\langle \frac{\partial F}{\partial z_+}, \frac{\partial F}{\partial z_-} \rangle = \frac{1}{4} \left( \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \rangle - \langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \rangle \right) = \frac{1}{2} \lambda$ . Then we have

$$H = \frac{2\nu \cdot \partial^+ \omega_-}{\langle \omega_+, \omega_- \rangle}$$

**Theorem 8** *Let  $M$  be a Lorentzian surface. Then the two following conditions are equivalent:*

1. *The map  $F : M \rightarrow \mathbb{R}^{2,1}$  is a conformal immersion.*
2. *There exist a triple  $\omega_+ = (\omega_{1+}, \omega_{2+}, \omega_{3+})$  of  $(1+, 0-)$ -forms and a triple  $\omega_- = (\omega_{1-}, \omega_{2-}, \omega_{3-})$  of  $(0+, 1-)$ -forms on  $M$  such that*

$$(i) \quad \begin{cases} \omega_{1+}^2 + \omega_{2+}^2 - \omega_{3+}^2 = 0, \\ \omega_{1-}^2 + \omega_{2-}^2 - \omega_{3-}^2 = 0, \end{cases} \quad (2.2.6)$$

$$(ii) \quad \omega_{1+}\omega_{1-} + \omega_{2+}\omega_{2-} - \omega_{3+}\omega_{3-} > 0, \quad (2.2.7)$$

(iii) *The forms  $\omega_{i+}$  resp.  $\omega_{i-}$  are  $\partial_+$ -exact resp.  $\partial_-$ -exact.*

*which satisfy the equation*

$$F(q) = \int_p^q (\omega_{1+} + \omega_{1-}, \omega_{2+} + \omega_{2-}, \omega_{3+} + \omega_{3-}) + \text{Constant}.$$

*Proof:* of the theorem 1.  $\Rightarrow$  2.: Consider a conformal immersion  $F = (F_1, F_2, F_3) : M \rightarrow \mathbb{R}^{1,2}$  and let  $\phi_{\pm} = (\phi_{\pm 1}, \phi_{\pm 2}, \phi_{\pm 3})$ ,  $\phi_{\pm i} = \frac{\partial F_i}{\partial z_{\pm}}$ ,  $i \in \{1, 2, 3\}$ . Then  $\omega_{\pm i} := \phi_{\pm i} dz_{\pm}$  are  $(1+, 0-)$ -forms resp.  $(0+, 1-)$ -forms on  $M$ , which obviously verify condition 2.(iii). Moreover we have:

$$\begin{aligned} \phi_1^{\pm 2} + \phi_2^{\pm 2} - \phi_3^{\pm 2} &= \left(\frac{\partial F_1}{\partial z_{\pm}}\right)^2 + \left(\frac{\partial F_2}{\partial z_{\pm}}\right)^2 - \left(\frac{\partial F_3}{\partial z_{\pm}}\right)^2 \\ &= \left(\frac{\partial F_1}{\partial x} \pm \frac{\partial F_1}{\partial y}\right)^2 + \left(\frac{\partial F_2}{\partial x} \pm \frac{\partial F_2}{\partial y}\right)^2 - \left(\frac{\partial F_3}{\partial x} \pm \frac{\partial F_3}{\partial y}\right)^2 \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle \pm 2 \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = \lambda - \lambda + 0 = 0 \end{aligned}$$

which proves 2.(i). Further

$$\begin{aligned} \langle \phi^+, \phi^- \rangle &= \left\langle \frac{\partial F}{\partial z^+}, \frac{\partial F}{\partial z^-} \right\rangle = \left\langle \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \right\rangle \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle - \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = 2\lambda > 0, \end{aligned}$$

which is equivalent to condition 2.(ii).

2.  $\Rightarrow$  1. Condition (iii) yields that  $F$  is well-defined. Moreover with conditions (i) and (ii) we have

$$\begin{aligned} \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle \pm 2 \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle &= 0 \\ \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle - \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle &> 0. \end{aligned}$$

This implies  $\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \rangle + \langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \rangle = 0$  and  $\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle = 0$ . Hence

$$\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \rangle = \langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \rangle = \lambda > 0$$

and  $F$  is a conformal immersion of  $M$  into  $\mathbb{R}^{2,1}$ .  $\square$

**Proposition 12** *A conformal immersion  $F = (F_1, F_2, F_3) : M \rightarrow \mathbb{R}^{2,1}$  is minimal if and only if  $\frac{\partial \phi_-}{\partial z_+} = \frac{\partial \phi_+}{\partial z_-} = 0$ , with  $\phi_{\pm} = \frac{\partial F}{\partial z_{\pm}}$ .*

*Proof:* With equations (2.2.5) and (2.2.4)  $F$  is minimal if and only if

$$0 = \frac{1}{2}H\nu = \Delta F = \frac{2\partial^2 F}{\lambda\partial z_+\partial z_-} = \frac{2\partial^2 F}{\lambda\partial z_-\partial z_+},$$

which yields the result.  $\square$

**Remark 10** *Condition 2.(iii) of theorem 8 is equivalent to the local condition that the forms  $\omega_{i\pm}$  are closed and  $\partial_-\omega_{i+} = -\partial_+\omega_{i-}$ , moreover it implies that the 1-form  $\omega_{i+} + \omega_{i-}$  is exact.*

From this real Weierstraß representation we can derive a para-complex Weierstrass representation in the following way:

**Theorem 9** *Let  $M$  be a Lorentzian surface. Then the following two conditions are equivalent:*

1. *The map  $F : M \rightarrow \mathbb{R}^{2,1}$  is a conformal immersion.*
2. *There exists a triple  $\omega = (\omega_1, \omega_2, \omega_3)$  of  $(1,0)$ -forms on  $M$  satisfying the equation*

$$F(q) = \Re \left( \int_p^q (\omega_1, \omega_2, \omega_3) \right) + \text{Constant},$$

*such that*

$$\omega_1^2 + \omega_2^2 - \omega_3^2 = 0, \tag{2.2.8}$$

$$\omega_1\bar{\omega}_1 + \omega_2\bar{\omega}_2 - \omega_3\bar{\omega}_3 > 0, \tag{2.2.9}$$

$$\text{the 1-forms } \Re(\omega_i) \text{ are exact.} \tag{2.2.10}$$

*Proof:* Considering para-complex  $(1,0)$ -forms  $\omega_i$ , we have  $\omega_i = \tilde{\omega}_i + eJ\tilde{\omega}_i$ , with  $\tilde{\omega}_i \in \Gamma(TM^*)$ . Using now the real splitting (1.1.1),  $\tilde{\omega}_i = \omega_{i+} + \omega_{i-}$  holds, where  $\omega_{i+}$  and  $\omega_{i-}$  are  $(1+,0)$ - resp.  $(0+,1)$ -forms. Consequently

$$\omega_i^2 = ((\omega_{i+} + \omega_{i-}) + e(\omega_{i+} - \omega_{i-}))^2 = 2(\omega_{i+}^2 + \omega_{i-}^2) + 2e(\omega_{i+}^2 - \omega_{i-}^2),$$

and

$$\omega_i \bar{\omega}_i = (\omega_{i+} + \omega_{i-})^2 - (\omega_{i+} - \omega_{i-})^2 = 4\omega_{i+}\omega_{i-}.$$

Simple calculations show that the conditions (2.2.6) resp. (2.2.7) of theorem 8 are equivalent to the conditions (2.2.8) resp. (2.2.9).

Moreover  $\Re(\omega_i) = \tilde{\omega}_i = \omega_{i+} + \omega_{i-}$ . Remark 10 yields then clearly the equivalence between (2.2.10) and part (iii) of theorem 8.  $\square$

This is a generalization of a result of Konderak (see [KO]) for minimal surfaces immersed in  $\mathbb{R}^{2,1}$ . We remark that the minimality of the immersion is just given by the condition on the  $(1,0)$ -forms  $\omega_i$  to be para-holomorphic (i.e locally  $\omega_i = \phi_i dz$ ,  $\phi_i$  para-holomorphic).

### 2.2.3 A Veronese map

Let  $\mathbb{R}P^n = P(\mathbb{R}^{n,1})$  be the real projective space of the pseudo-Euclidean vector space  $\mathbb{R}^{n,1}$ . We introduce the tautological line bundle of  $\mathbb{R}P^n$ :

$$\tau_{\mathbb{R}P^n} = \{(\lambda, v) \in \mathbb{R}P^n \times \mathbb{R}^{n,1} | v \in \lambda\}.$$

Obviously this is a subbundle of the trivial  $(n+1)$ -dimensional bundle  $\mathcal{T}^{n+1} = \mathbb{R}P^n \times \mathbb{R}^{n,1}$ .

We now consider the quadric

$$Q = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1} | x_1^2 - x_2^2 + x_3^2 = 0\}$$

and the maps

$$\begin{aligned} \mathcal{W}_{\pm} : \mathbb{R}^{1,1} &\rightarrow \mathbb{R}^{2,1}. \\ (x_1, x_2) &\mapsto (x_1^2 - x_2^2, \pm(x_1^2 + x_2^2), 2x_1x_2) \end{aligned}$$

Then  $\mathcal{W}_{\pm}$  can be seen as maps into the affine quadric  $Q$ . Obviously  $\mathcal{W}_{\pm}(x) = \mathcal{W}_{\pm}(x')$  is equivalent to  $x' = \pm x$ .

We now define Veronese embeddings by

$$\begin{aligned} \mathcal{V}_{\pm} : \mathbb{R}P^1 &\rightarrow \mathbb{R}P^2 \\ [x_1, x_2] &\mapsto [\mathcal{W}_{\pm}(x_1, x_2)] = [x_1^2 - x_2^2, \pm(x_1^2 + x_2^2), 2x_1x_2]. \end{aligned}$$

**Proposition 13** *The Veronese embeddings  $\mathcal{W}_{\pm}$  induce isomorphisms*

$$\mathcal{V}_{\pm} : \mathbb{R}P^1 \xrightarrow{\sim} [Q]$$

*between the projective space  $\mathbb{R}P^1$  and the projective quadric*

$$[Q] = \{[x_1, x_2, x_3] \in \mathbb{R}P^2 | x_1^2 - x_2^2 + x_3^2 = 0\}.$$

*Proof:* Let  $[y_1, y_2, y_3]$  be a point of the projective quadric. Taking affine charts of  $\mathbb{R}P^1$  and assuming that  $y_3 \neq 0$ , we seek for  $[x_1, x_2]$ , with  $x_1, x_2 \neq 0$ , such that  $[\frac{y_1}{y_3}, \frac{y_2}{y_3}, 1] = [\frac{x_1^2 - x_2^2}{2x_1x_2}, \frac{\pm(x_1^2 + x_2^2)}{2x_1x_2}, 1]$ . Summing up the first and second component gives  $\frac{x_1}{x_2}$  and consequently the surjectivity.  $\square$

**Lemma 10** *The following canonical isomorphism holds:*

$$\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \cong \mathcal{V}_{\pm}^* \tau_{\mathbb{R}P^2}. \quad (2.2.11)$$

*Proof:* We have

$$\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} = \{([z], v \otimes w) \in \mathbb{R}P^1 \times (\mathbb{R}^{2,1})^{\otimes 2} \mid v, w \in [z]\}$$

Moreover

$$\mathcal{V}_{\pm}^* \tau_{\mathbb{R}P^2} = \{([z], v) \in \mathbb{R}P^1 \times Q \mid v \in \mathcal{V}_{\pm}([z]) = [\mathcal{W}_{\pm}(z)]\}.$$

Using the isomorphism  $s \otimes s \rightarrow \mathcal{W}_{\pm}(s)$  we obtain the result.  $\square$

Remark that if  $k_{\alpha\beta}$  are the transition functions of  $\tau_{\mathbb{R}P^1}$  for the covering  $\{U_{\alpha}\}$ , then  $\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}$  and  $\mathcal{V}_{\pm}^* \tau_{\mathbb{R}P^2}$  have the same transition functions  $k_{\alpha\beta}^2$  for this covering.

We now define the map

$$\tilde{\mathcal{V}} : \mathbb{R}P^1 \times \mathbb{R}P^1 \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^2, ([x_1, x_2], [x'_1, x'_2]) \mapsto (\mathcal{V}_+([x_1, x_2]), \mathcal{V}_-([x'_1, x'_2])).$$

Let  $\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n}$  be the vector bundle defined over  $\mathbb{R}P^n \times \mathbb{R}P^n$  such that the fibers are

$$(\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n})_{(p^+, p^-)} := (\tau_{\mathbb{R}P^n})_{p^+} \oplus (\tau_{\mathbb{R}P^n})_{p^-},$$

with  $(p^+, p^-) \in \mathbb{R}P^n \times \mathbb{R}P^n$ .

As it is the Cartesian product of two smooth manifolds,  $\mathbb{R}P^n \times \mathbb{R}P^n$  is a para-complex manifold. In fact, using the identification  $T_{(p^+, p^-)}(\mathbb{R}P^n \times \mathbb{R}P^n) = T_{p^+}\mathbb{R}P^n \oplus T_{p^-}\mathbb{R}P^n$ , we can define a para-complex structure by  $J|_{T_{p^{\pm}}\mathbb{R}P^n} = \pm Id$ . We refer to [CMMS] for more details. Then  $\tau_{\mathbb{R}P^n} \boxplus \tau_{\mathbb{R}P^n}$  has the structure of a para-complex vector bundle over  $\mathbb{R}P^n \times \mathbb{R}P^n$  by defining a para-complex structure which has eigenvalue 1 on the first and  $-1$  on the second summand.

**Corollary 4** *The following canonical isomorphism of para-complex vector spaces holds:*

$$(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \otimes_C (\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \cong \tilde{\mathcal{V}}^*(\tau_{\mathbb{R}P^2} \boxplus \tau_{\mathbb{R}P^2}). \quad (2.2.12)$$

*Proof:* Let  $k_{\alpha\beta}$  be the transition functions of the bundle  $\tau_{\mathbb{R}P^1}$  with respect to an open covering  $\{U_{\alpha}\}$ . Then by definition the transition functions of  $(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1})$  are given by  $K_{\alpha\beta}(p^+, p^-) = \begin{pmatrix} k_{\alpha\beta}(p^+) & 0 \\ 0 & k_{\alpha\beta}(p^-) \end{pmatrix}$ , for  $(p^+, p^-) \in \mathbb{R}P^1 \times \mathbb{R}P^1$ . Moreover from lemma 10 we obtain:

$$\tilde{\mathcal{V}}^*(\tau_{\mathbb{R}P^2} \boxplus \tau_{\mathbb{R}P^2}) \cong \mathcal{V}_+^* \tau_{\mathbb{R}P^2} \boxplus \mathcal{V}_-^* \tau_{\mathbb{R}P^2} \cong \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}.$$

Obviously the transition functions  $\tilde{K}_{\alpha\beta}$  of the bundles  $\tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1} \otimes_{\mathbb{R}} \tau_{\mathbb{R}P^1}$  and  $(\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1}) \otimes_C (\tau_{\mathbb{R}P^1} \boxplus \tau_{\mathbb{R}P^1})$  are the same, i.e.  $\tilde{K}_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^2(p^+) & 0 \\ 0 & k_{\alpha\beta}^2(p^-) \end{pmatrix} = K_{\alpha\beta}^2$ , which proves the lemma.  $\square$

## 2.2.4 The spinor representation

Using theorem 8 and the Veronese map introduced in the last paragraph, we now generalize the results of [KS] to Lorentzian surfaces.

Let  $\omega_{\pm} \in \Gamma(T^*M^{\pm})$ . Locally one can write  $\omega_{\pm} = \phi_{\pm} dz_{\pm}$  where  $\phi_{\pm} \in C^{\infty}(M)$  and the pair  $(z_+, z_-)$  is some adapted local coordinate system on the para-complex surface  $M$ . This yields immediately a local identification of  $C^{\infty}(M) = \Omega^{(1+,0-)}(M) = \Gamma(T^*M^+) = \Omega^{(0+,1-)}(M) = \Gamma(T^*M^-)$ . Let  $M$  be a Lorentzian surface which is conformally immersed in  $\mathbb{R}^{2,1}$ . The condition (2.2.7) of theorem 8 on the isotropic one-forms  $\omega_{i\pm}$  implies that

$$\mathcal{M}_{\pm} := \{x \in M \mid \phi_{i\pm}(x) = 0, \forall i \in \{1, 2, 3\}\} = \emptyset.$$

Therefore we can consider the map

$$\begin{aligned} h : M &\rightarrow \mathbb{R}P_1^2 \times \mathbb{R}P^2 \\ x &\mapsto (h_+(x), h_-(x)) := ([\phi_{1+}(x), \phi_{2+}(x), \phi_{3+}(x)], [\phi_{1-}(x), \phi_{2-}(x), \phi_{3-}(x)]). \end{aligned}$$

Moreover  $h$  can be then considered by condition (2.2.6) as a map into the product of projective quadrics  $[Q] \times [Q] \cong \mathbb{R}P^1 \times \mathbb{R}P^1$ . This allows us to define maps  $f : M \rightarrow \mathbb{R}P^1 \times \mathbb{R}P^1$ , such that  $h = \tilde{\mathcal{V}} \circ f$  and  $f_{\pm} : M \rightarrow \mathbb{R}P^1$ , such that  $h_{\pm} = \tilde{\mathcal{V}}_{\pm} \circ f_{\pm}$ .

Let now define the maps

$$\begin{aligned} k^{\pm} : T^*M^{\pm} &\rightarrow h_{\pm}^*(\tau_{\mathbb{R}P^2}^*) \\ \sum_i^3 a_i \omega_{i\pm}(x) =: \alpha &\mapsto l_a^{\pm}, \end{aligned}$$

where  $l_a^{\pm}$  is the linear functional given by  $l_a^{\pm}(\phi_+(x)) = a \cdot \phi_+(x) = \sum_i^3 a_i \phi_{i\pm}(x) \in \mathbb{R}$ , with  $a = (a_1, a_2, a_3)$  and  $\phi(x) = (\phi_{1+}(x), \phi_{2+}(x), \phi_{3+}(x))$ . We remark that  $l_a$  do not depend on the choice of  $dz_{\pm}$ . We show that  $k$  is an isomorphism: Let  $\alpha = \sum_i^3 b_i \omega_{i\pm}(x)$ , for an other triple  $b = (b_1, b_2, b_3) \neq a$ , then we have

$$0 = \sum_i^3 (a_i - b_i) \omega_{i\pm}(x) = \sum_i^3 (a_i - b_i) \phi_{i\pm}(x) dz_{\pm}$$

and consequently  $(l_a^{\pm} - l_b^{\pm})(\phi^{\pm}(x)) = 0$ , which leads  $l_a = l_b$ .

Hence we have the isomorphism

$$T^*M^{\pm} \cong h_{\pm}^*(\tau_{\mathbb{R}P^2}^*) \cong f_{\pm}^* \mathcal{V}_{\pm}^*(\tau_{\mathbb{R}P^2}^*) \quad (2.2.13)$$

and finally with lemma 10 we find the isomorphisms:

$$\kappa^{\pm} : T^*M^{\pm} \cong f_{\pm}^*(\tau_{\mathbb{R}P^1}^*) \otimes_{\mathbb{R}} f_{\pm}^*(\tau_{\mathbb{R}P^1}^*) \quad (2.2.14)$$

By proposition 11 the above construction gives explicitly two half spinor bundles

$$\Sigma^{\pm} M := f_{\pm}^*(\tau_{\mathbb{R}P^1}^*)$$

on  $M$  and, as  $f_+^*(\tau_{\mathbb{R}P^1}^*) \oplus f_-^*(\tau_{\mathbb{R}P^1}^*) = f^*(\tau_{\mathbb{R}P^1}^* \boxplus \tau_{\mathbb{R}P^1}^*)$ , we have

$$T^*M \cong f^*(\tau_{\mathbb{R}P^1}^* \boxplus \tau_{\mathbb{R}P^1}^*) \otimes_C f^*(\tau_{\mathbb{R}P^1}^* \boxplus \tau_{\mathbb{R}P^1}^*). \quad (2.2.15)$$

Hence

$$\Sigma M := f^*(\tau_{\mathbb{R}P^1}^* \boxplus \tau_{\mathbb{R}P^1}^*)$$

is a spin bundle on  $M$  in the interpretation of definition 15.

The following commutative diagram illustrates the above objects:

$$\begin{array}{ccccc}
TM^* & \xrightarrow{\sim} & h^*(\tau_{\mathbb{R}P_1^2}^* \boxplus \tau_{\mathbb{R}P_1^2}^*) & \xrightarrow{\quad} & \tau_{\mathbb{R}P_1^2}^* \boxplus \tau_{\mathbb{R}P_1^2}^* \\
& & \downarrow & & \downarrow \\
f^*(\tau_{\mathbb{R}P_1^1}^* \boxplus \tau_{\mathbb{R}P_1^1}^*) & \xrightarrow{\quad} & M & \xrightarrow{h} & [Q] \times [Q] \subset \mathbb{R}P_1^2 \times \mathbb{R}P_1^2 \\
& & \downarrow f & \nearrow \tilde{\nu} & \\
\tau_{\mathbb{R}P_1^1}^* \boxplus \tau_{\mathbb{R}P_1^1}^* & \xrightarrow{\quad} & \mathbb{R}P_1^1 \times \mathbb{R}P_1^1 & & 
\end{array}$$

We have then the

**Theorem 11** *Let  $M$  be a strongly oriented Lorentzian surface. Then the following conditions are equivalent*

- 1 *There exists a conformal immersion  $M \rightarrow \mathbb{R}^{2,1}$  with mean curvature  $H$ .*
- 2 *There exists a solution  $\psi = (\psi_1, \psi_2)$  of the Dirac-type equation*

$$\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \langle \psi_1, \psi_2 \rangle.$$

*for some real-valued function  $H$ , necessarily the mean curvature of the surface.*

*Proof:* For pairs of sections  $(s_{1+}, s_{2+}^{\pm})$  and  $(s_{1-}, s_{2-})$  of  $f^*(\tau_{\mathbb{R}P^1})$  we can write

$$\begin{aligned}
\omega_+ &= (\omega_{+1}, \omega_{+2}, \omega_{+3}) = (s_{1+}^2 - s_{2+}^2, s_{1+}^2 + s_{2+}^2, 2s_{1+}s_{2+}), \\
\omega_- &= (\omega_{-1}, \omega_{-2}, \omega_{-3}) = (s_{1-}^2 - s_{2-}^2, -s_{1-}^2 - s_{2-}^2, 2s_{1-}s_{2-}).
\end{aligned}$$

With  $s_i^{\pm 2} = f_{i\pm}^2 dz_{\pm}$ , we have

$$\begin{aligned}
\partial_- \omega_+ &= 2(-f_{1+} \partial_z^- f_{1+} + f_{2+} \partial_z^- f_{2+}, -f_{1+} \partial_z^- f_{1+} - f_{2+} \partial_z^- f_{2+}, -f_{2+} \partial_z^- f_{1+} - f_{2+} \partial_z^- f_{1+}) dz_+ \wedge dz_-, \\
\partial_+ \omega_- &= 2(f_{1-} \partial_z^+ f_{1-} - f_{2-} \partial_z^+ f_{2-}, -f_{1-} \partial_z^+ f_{1-} - f_{2-} \partial_z^+ f_{2-}, -f_{2-} \partial_z^+ f_{1-} - f_{2-} \partial_z^+ f_{1-}) dz_+ \wedge dz_-.
\end{aligned}$$

Then a simple calculation shows that the integrality conditions of theorem 8 for the pair  $(\omega_+, \omega_-)$  are equivalent to the following conditions on  $s_i^{\pm}$ :

$$s_{1+} \partial_- s_{1+} = -s_{2-} \partial_+ s_{2-}, \quad s_{2+} \partial_- s_{2+} = -s_{1-} \partial_+ s_{1-}, \quad (2.2.16)$$

$$s_{1+} \partial_- s_{2+} = s_{1-} \partial_+ s_{2-}, \quad s_{2+} \partial_- s_{1+} = s_{2-} \partial_+ s_{1-}. \quad (2.2.17)$$



We now calculate the mean curvature with respect to  $s_i^\pm$ . The unit normal vector is given by

$$\nu = \frac{\omega_+ \times \omega_-}{\|\omega_+ \times \omega_-\|},$$

where  $\cdot \times \cdot$  is the natural pseudo-vector product in  $\mathbb{R}^{2,1}$  (see [Wei]). We have

$$\begin{aligned} \omega_+ \times \omega_- &= -2(s_{1+}s_{1-} + s_{2+}s_{2-})(s_{1+}s_{2-} + s_{1-}s_{2+}, s_{1-}s_{2+} - s_{2-}s_{1+}, s_{1+}s_{1-} - s_{2+}s_{2-}) \\ \|\omega_+ \times \omega_-\| &= 2(s_{1+}s_{1-} + s_{2+}s_{2-})^2 = -\langle \omega_+, \omega_- \rangle \end{aligned}$$

Then  $\nu = -\frac{(s_{1+}s_{2-} + s_{1-}s_{2+}, s_{1-}s_{2+} - s_{2-}s_{1+}, s_{1+}s_{1-} - s_{2+}s_{2-})}{s_{1+}s_{1-} + s_{2+}s_{2-}}$  and consequently

$$\begin{aligned} H &= \frac{2\langle \nu, \partial_+ \omega_- \rangle}{\langle \omega_+, \omega_- \rangle} \\ &= -\frac{2(s_{1+}s_{2-} + s_{1-}s_{2+}, s_{1-}s_{2+} - s_{2-}s_{1+}, s_{1+}s_{1-} - s_{2+}s_{2-})}{-2(s_{1+}s_{1-} + s_{2+}s_{2-})^3} \cdot 2 \begin{pmatrix} s_{1-}\partial_+s_{1-} - s_{2-}\partial_+s_{2-} \\ -s_{1-}\partial_+s_{1-} - s_{2-}\partial_+s_{2-} \\ s_{1-}\partial_+s_{2-} + s_{2-}\partial_+s_{1-} \end{pmatrix} \\ &= \frac{1}{(s_{1+}s_{1-} + s_{2+}s_{2-})^2} (s_{2-}\partial_+s_{1-} - s_{1-}\partial_+s_{2-}) \end{aligned}$$

Consider now the spinors  $\psi_1 := (s_{1+}, s_{2-})$  and  $\psi_2 := (s_{2+}, s_{1-})$ . Using the equalities (2.2.16) and (2.2.17) we compute

$$\begin{aligned} H\psi_1 &= \frac{2}{(s_{1+}s_{1-} + s_{2+}s_{2-})} (-\partial_+s_{2-}, \partial_-s_{1+}) \\ H\psi_2 &= \frac{2}{(s_{1+}s_{1-} + s_{2+}s_{2-})} (-\partial_-s_{2+}, \partial_+s_{1-}), \end{aligned}$$

which is equivalent to the Dirac-type equation

$$\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \langle \psi_1, \psi_2 \rangle,$$

where  $D$  is the Dirac operator in the sense of equation (2.2.2).  $\square$

## 2.2.5 A geometrically invariant spinor representation

The aim of this section is now to give a geometrically invariant representation of Lorentzian surfaces in  $\mathbb{R}^{2,1}$  by solutions of a coupled Dirac equation similarly to the result of Friedrich [Fr1].

**Theorem 12** *Let  $(M, g)$  be a strongly oriented pseudo-Riemannian surface of signature  $(1, 1)$ ,  $H : M \rightarrow \mathbb{R}$  be a real valued function. Then the following three statements are equivalent:*

1.  $\varphi_1$  and  $\varphi_2$  are non-vanishing non-isotropic solutions of the coupled Dirac equations

$$D\varphi_1 = H\varphi_1, \quad D\varphi_2 = -H\varphi_2 \tag{2.2.18}$$

with  $\langle \varphi_1, \varphi_2 \rangle = 1$ ,

2.  $\varphi_1$  and  $\varphi_2$  are non-vanishing non-isotropic solutions of the generalized Killing equations

$$\nabla_X^{\Sigma M} \varphi_1 = \frac{1}{2}A(X) \cdot \varphi_1, \quad \nabla_X^{\Sigma M} \varphi_2 = -\frac{1}{2}A(X) \cdot \varphi_2, \quad (2.2.19)$$

where  $A$  is a  $g$ -symmetric endomorphism field and  $\frac{1}{2}\text{tr}A = H$ .

3. If  $M$  is simply connected, there exists a global isometric spacelike immersion  $M \hookrightarrow \mathbb{R}^{2,1}$  with mean curvature  $H$  and second fundamental form  $A$ .

*Proof:* "3  $\Rightarrow$  2" Let  $\Phi_1$  be a parallel spinor on  $\mathbb{R}^{2,1}$  and  $\varphi_1 = \Phi_1|_M$  its restriction to  $M$ . From equation (2.1.11) we have that  $\varphi_1$  is a solution of the generalized Killing equation  $\nabla_X^{\Sigma M} \varphi_1 = \frac{1}{2}A(X) \cdot \varphi_1$ , where  $A$  is the Weingarten tensor of the immersion.

**Claim:** The spinor  $\varphi_2 := \nu \tilde{\cdot} \varphi_1$  is solution of the generalized Killing equation

$$\nabla_X^{\Sigma M} \varphi_2 = -\frac{1}{2}A(X) \cdot \varphi_2,$$

where  $\tilde{\cdot}$  denote the Clifford multiplication on  $\mathbb{R}^{2,1}$ .

*Proof:* We have by equation (2.1.11)

$$\begin{aligned} \nabla_X^{\Sigma M} \varphi_2 = \nabla_X^{\Sigma M} (\nu \tilde{\cdot} \varphi_1) &= (\nabla_X^{\Sigma \mathbb{R}^{2,1}} + \frac{1}{2}A(X) \tilde{\cdot} \nu) \tilde{\cdot} \nu \tilde{\cdot} \varphi_1 \\ &= (\nabla_X^{\Sigma \mathbb{R}^{2,1}} \nu) \tilde{\cdot} \varphi_1 + \nu \tilde{\cdot} \nabla_X^{\Sigma \mathbb{R}^{2,1}} \varphi_1 - \frac{1}{2}A(X) \tilde{\cdot} \varphi_1 \\ &= \nu \tilde{\cdot} (\nabla_X^{\Sigma \mathbb{R}^{2,1}} \varphi_1 + \frac{1}{2}A(X) \tilde{\cdot} \nu \tilde{\cdot} \varphi_1) = \nu \tilde{\cdot} \nabla_X^{\Sigma M} \varphi_1. \end{aligned}$$

Hence, as  $\Phi_1$  is parallel, we have

$$\nabla_X^{\Sigma M} \varphi_2 = \frac{1}{2}\nu \tilde{\cdot} A(X) \tilde{\cdot} \nu \tilde{\cdot} \varphi_1 = -\frac{1}{2}A(X) \tilde{\cdot} \nu \tilde{\cdot} \varphi_2 = -\frac{1}{2}A(X) \cdot \varphi_2. \quad \square$$

Moreover we remark that

$$X \langle \varphi_1, \varphi_2 \rangle = \langle \nabla_X^{\Sigma M} \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \nabla_X^{\Sigma M} \varphi_2 \rangle = \langle \varphi_1, \nu \tilde{\cdot} \nabla_X^{\Sigma M} \varphi_1 \rangle = 0,$$

hence  $\langle \varphi_1, \varphi_2 \rangle = \text{Const}$ .  $\square$

"2  $\Rightarrow$  1" follows from proposition 10.

"1  $\Rightarrow$  2". Let  $\varphi_1, \varphi_2$  be two solutions of the system of equations (2.2.18). Similarly to the proof of theorem 7 we define

$$\beta_{\varphi_1}(e_i, e_j) = \langle \nabla_{e_i}^{\Sigma M} \varphi_1, e_j \cdot \varphi_1 \rangle, \quad \beta_{\varphi_2}(e_i, e_j) = \langle \nabla_{e_i}^{\Sigma M} \varphi_2, e_j \cdot \varphi_2 \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the pseudo-hermitian symmetric  $\text{Spin}^+(p, q)$ -invariant bilinear form defined in the last section. Then the same calculation shows that  $\beta_{\varphi_1}$  and  $\beta_{\varphi_2}$  are symmetric. Moreover define the  $g$ -symmetric endomorphisms

$$(B_{\varphi_1})_i^j = g(B_{\varphi_1}(e_i), e_j) := \beta_{\varphi_1}(e_i, e_j) \quad \text{and} \quad (B_{\varphi_2})_i^j = g(B_{\varphi_2}(e_i), e_j) := \beta_{\varphi_2}(e_i, e_j).$$

Clearly  $\frac{\text{tr}(B_{\varphi_1})}{|\varphi_1|^2} = g^{ij}(B_{\varphi_1})_{ij} = -\frac{\text{tr}(B_{\varphi_2})}{|\varphi_2|^2} = H$ .

Then using the method of theorem 7 (see equation (2.1.19)) we show for all  $X \in TM$

$$\nabla_X^{\Sigma M} \varphi_1 = -\frac{1}{2|\varphi_1|^2} B_{\varphi_1}(X) \cdot \varphi_1, \quad \nabla_X^{\Sigma M} \varphi_2 = -\frac{1}{2|\varphi_2|^2} B_{\varphi_2}(X) \cdot \varphi_2.$$

As  $\langle \varphi_1, \varphi_2 \rangle = \text{Const}$ , we have

$$0 = X \langle \varphi_1, \varphi_2 \rangle = \langle \nabla_X^{\Sigma M} \varphi_1, \varphi_2 \rangle + \langle \varphi_1, \nabla_X^{\Sigma M} \varphi_2 \rangle = \left\langle -\frac{B_{\varphi_1}(X)}{2|\varphi_1|^2} - \frac{B_{\varphi_2}(X)}{2|\varphi_2|^2}, \varphi_1, \varphi_2 \right\rangle$$

Let

$$B(X) := \frac{B_{\varphi_1}(X)}{|\varphi_1|^2} + \frac{B_{\varphi_2}(X)}{|\varphi_2|^2}.$$

It is well-defined as the spinors  $\varphi_1, \varphi_2$  are non-trivial at any point.  $B : T(M^{1,1}) \rightarrow T(M^{1,1})$  is obviously  $g$ -symmetric, and  $\text{tr}(B(X)) = H - H = 0$ , i.e. in matrix form  $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , with  $a, b \in \mathbb{R}$ .

This yields

$$0 = \langle B(e_1) \cdot \varphi_1, \varphi_2 \rangle = a \langle e_1 \cdot \varphi_1, \varphi_2 \rangle - b \langle e_2 \cdot \varphi_1, \varphi_2 \rangle$$

and

$$0 = \langle B(e_2) \cdot \varphi_1, \varphi_2 \rangle = b \langle e_1 \cdot \varphi_1, \varphi_2 \rangle + a \langle e_2 \cdot \varphi_1, \varphi_2 \rangle.$$

If  $a \neq 0$  and  $b \neq 0$  we get with a simple calculation that  $\langle e_1 \cdot \varphi_1, \varphi_2 \rangle = \langle e_2 \cdot \varphi_1, \varphi_2 \rangle = 0$ . We remark that  $e_i \cdot \varphi_1$  is a basis of  $\Sigma M$ , then we have

$$\varphi_2 = \langle \varphi_2, e_1 \cdot \varphi_1 \rangle \frac{e_1 \cdot \varphi_1}{|\varphi_1|^2} + \langle \varphi_2, e_2 \cdot \varphi_1 \rangle \frac{e_2 \cdot \varphi_1}{|\varphi_2|^2} = 0.$$

Consequently  $B = 0$  and  $\frac{B_{\varphi_1}(X)}{|\varphi_1|^2} = -\frac{B_{\varphi_2}(X)}{|\varphi_2|^2} =: -A(X)$ , which finishes the proof.

"2  $\Rightarrow$  3" Let  $\varphi_1, \varphi_2$  be solutions of the equations

$$\nabla_X^{\Sigma M} \varphi_1 = \frac{1}{2} A(X) \cdot \varphi_1, \quad \nabla_X^{\Sigma M} \varphi_2 = -\frac{1}{2} A(X) \cdot \varphi_2,$$

Recall that the integrability conditions for these generalized Killing equations (2.2.19) are given by:

$$\begin{aligned} R^{\Sigma M}(X, Y) \cdot \varphi_1 &= d^\nabla A(X, Y) \varphi_1 + (A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi_1 \\ R^{\Sigma M}(X, Y) \cdot \varphi_2 &= -d^\nabla A(X, Y) \varphi_2 + (A(Y) \cdot A(X) - A(X) \cdot A(Y)) \cdot \varphi_2. \end{aligned}$$

Similarly to the proof of theorem 7, defining the vector field

$$B := (\nabla_{e_2}^{\Sigma M} A)(e_1) - (\nabla_{e_1}^{\Sigma M} A)(e_2)$$

and the function

$$f := R_{1212} + \det(A),$$

this is equivalent to the system of equations

$$B \cdot \varphi_1 = f e_1 \cdot e_2 \cdot \varphi_1, \quad B \cdot \varphi_2 = -f e_1 \cdot e_2 \cdot \varphi_2 \quad (2.2.20)$$

and finally to the equations

$$B \cdot \varphi_1^\pm = \mp f \varphi_1^\mp, \quad B \cdot \varphi_2^\pm = \pm f \varphi_1^\mp. \quad (2.2.21)$$

This yields

$$\|B\|^2 \varphi_i^\pm = f^2 \varphi_i^\pm, \quad i = 1, 2 \quad (2.2.22)$$

and consequently  $\|B\|^2 \geq 0$ . Moreover we have

$$\langle B \cdot \varphi_1, B \cdot \varphi_2 \rangle = -f^2 \langle e_1 \cdot e_2 \cdot \varphi_1, e_1 \cdot e_2 \cdot \varphi_2 \rangle = f^2 \langle \varphi_1, \varphi_2 \rangle$$

and

$$\langle B \cdot \varphi_1, B \cdot \varphi_2 \rangle = \langle \varphi_1, B \cdot B \cdot \varphi_2 \rangle = -\|B\|^2 \langle \varphi_1, \varphi_2 \rangle,$$

Then  $\|B\|^2 \leq 0$  holds and finally  $B = 0$ , as  $B$  is non-isotropic (see again the proof of theorem 7.  $\square$ )



# Chapter 3

## (Para-)Conformal geometry of pseudo-Riemannian surfaces in $\mathbb{S}^{2,2}$

It was proven in [BFLPP] that a conformal Riemannian surface  $M$  immersed in the four-sphere  $\mathbb{H}P^1 \cong \mathbb{S}^4$  can be identified with quaternionic line subbundles of  $M \times \mathbb{H}^2$ . In this chapter, we generalize this result to pseudo-Riemannian surfaces  $M$  which are conformally immersed in the para-quaternionic projective space  $\widetilde{\mathbb{H}}P^1 \cong \mathbb{S}^{2,2}$ . We establish in fact a one-to-one correspondence between such immersions and para-quaternionic line subbundles  $L$  of  $M \times \widetilde{\mathbb{H}}$ . As in the Riemannian case, this allows us to define the mean curvature pseudo-sphere congruence, which can be seen as a (para-)complex structure on  $L$ , and the Hopf fields of the immersion. We then generalize the definition of the Willmore functional for such surfaces. This allows us to express the energy of a surface of arbitrary signature as the sum of this functional and of a topological invariant.

### 3.1 Para-quaternionic vector bundles

In the following we always use for the sake of simplicity the notation  $J^\varepsilon$ , where  $J^\varepsilon$  is a complex structure for  $\varepsilon = -1$  and a para-complex structure for  $\varepsilon = 1$ .

**Definition 16** *Let  $M$  be a smooth manifold. A para-quaternionic vector bundle of rank  $n$  is a smooth real vector bundle  $\pi : V \rightarrow M$  of rank  $4n$  with a smooth fiber-preserving right-action of  $\widetilde{\mathbb{H}}$  on  $V$  such that the fibers are para-quaternionic right-modules over  $\widetilde{\mathbb{H}}$ .*

We are more especially interested in para-quaternionic line bundles. Let  $\Sigma$  be the **tautological bundle** over  $\widetilde{\mathbb{H}}P^n$ , i.e

$$\begin{aligned} \Sigma &:= \{(l, v) \in \widetilde{\mathbb{H}}P^n \times \widetilde{\mathbb{H}}^{n+1} \mid v \in l\}, \\ \pi_\Sigma : \Sigma &\rightarrow \widetilde{\mathbb{H}}P^n, (l, v) \mapsto l. \end{aligned}$$

Obviously it is  $\Sigma_l = l$ . Recall, that by proposition 3,  $\text{Hom}(\Sigma, \widetilde{\mathbb{H}}_*^{n+1}/\Sigma) \cong T\widetilde{\mathbb{H}}P^n$ . Let  $f : M \rightarrow \widetilde{\mathbb{H}}P^n$  be a smooth map and consider the pullback

$$L := f^*\Sigma := \{(x, \sigma) \mid \sigma \in \Sigma_{f(x)}\} \subset M \times \Sigma.$$

The fibers  $\Sigma_{f(x)} = f(x)$  over  $x$  are points of the projective space  $\widetilde{\mathbb{H}}P^n$  i.e. one-dimensional subspaces of  $\widetilde{\mathbb{H}}^{n+1}$ . It follows that  $L$  is a line subbundle of  $M \times \widetilde{\mathbb{H}}^{n+1} =: H$ .

Conversely every line subbundle  $L$  of  $H$  defines a map  $f : M \rightarrow \widetilde{\mathbb{H}}P^n$  with  $f(x) := [L_x]$ . Finally we obtain a one-to-one correspondence between smooth maps  $f : M \rightarrow \widetilde{\mathbb{H}}P^n$  and line subbundles of  $H$ .

With corollary 2 this leads to the following

**Definition 17** *Let  $L$  be a line subbundle of  $H$ ,  $p \in M$ ,  $X \in T_pM$  and  $\psi \in \Gamma(L)$ . Moreover let  $\pi_L : H \rightarrow H/L$ , ( $\pi_L \in \Gamma(\text{Hom}(H, H/L))$ ), be the canonical projection. Then we call*

$$\delta = \delta^L \in \Omega^1(\text{Hom}(L, H/L)), \quad \delta_p(X)\psi(p) = \pi_L(d_p\psi(X)) = d_p\psi(X) \bmod L_p$$

the derivative of  $L$ .

With the above identification  $\delta$  corresponds in fact to the derivative of a map  $f : M \rightarrow \widetilde{\mathbb{H}}P^n$  in the interpretation of corollary 2.

From lemma 3 we get the following

**Proposition 14** *Let  $M$  be a Riemannian (resp. Lorentzian) surface,  $f : M \rightarrow \widetilde{\mathbb{H}}P^1$  a conformal immersion and  $L \subset H = M \times \widetilde{\mathbb{H}}^2$  be the corresponding line bundle with derivative  $\delta \in \Omega^1(\text{Hom}(L, H/L))$ . Then there exist unique (para-)complex structures  $J_L^\varepsilon$  on  $L$  and  $J_{H/L}^\varepsilon$  on  $H/L$  such that for all  $x \in M$ :*

$$i.) \quad J_{H/L}^\varepsilon \delta(T_x M) = \delta(T_x M) = \delta(T_x M) J_L^\varepsilon,$$

$$ii.) \quad J_{H/L}^\varepsilon \delta = \delta J_L^\varepsilon.$$

*Proof:* Let  $V, W$  be two one-dimensional para-quaternionic modules.  $\text{Hom}(V, W)$  can be identified with  $\widetilde{\mathbb{H}}$  and is then endowed with the induced para-quaternionic hermitian inner product. Let now  $U \subset \text{Hom}(V, W)$  be a real two-dimensional non-degenerate subspace. There exists by lemma 3 a pair of (para-)complex structures  $(J^\varepsilon, \tilde{J}^\varepsilon)$ , with  $J^\varepsilon \in \text{End}(V)$ ,  $\tilde{J}^\varepsilon \in \text{End}(W)$ , unique up to sign, such that

$$\begin{aligned} \tilde{J}^\varepsilon U &= U = U J^\varepsilon, \\ U &= \{H \in \text{Hom}(V, W) \mid \tilde{J}^\varepsilon H J^\varepsilon = \varepsilon H\}. \end{aligned}$$

This yields the result as  $\delta(T_x M)$  is non-degenerate. □

At this point, we want to make some remarks about differential forms which will be needed in the next section.

We consider one-forms  $\omega_i \in \Omega^1(\text{End}(V))$  with values in the endomorphisms of a vector bundle  $V$  over  $M$ . Let  $(M, J^\varepsilon)$  be a (para-)Riemannian surface and  $J^{\varepsilon*}$  be the (para-)complex structure induced on  $T^*M$  i.e.  $(J^{\varepsilon*}\omega)(X) = \omega(J^\varepsilon X)$ . There exists a bijective

correspondence between 2-forms  $\alpha$  on  $M$  and the quadratic form  $\alpha(X, J^\varepsilon X) =: \alpha(X)$ . With the standard definition of the wedge product we have

$$\omega_1 \wedge \omega_2(X, J^\varepsilon X) = \omega_1(X)\omega_2(J^\varepsilon X) - \omega_1(J^\varepsilon X)\omega_2(X).$$

and consequently

$$\omega_1 \wedge \omega_2 = \omega_1 J^{\varepsilon*} \omega_2 - (J^{\varepsilon*} \omega_1) \omega_2. \quad (3.1.1)$$

Now let  $V = H$ . We want to find a formula for the differential of 1-forms  $\omega \in \Omega^1(\text{End}(H))$  which leave  $L$  invariant, i.e. satisfy  $\omega L \subset L$ . Let  $\psi \in \Gamma(L)$ , then

$$\begin{aligned} \pi_L(d\omega(X_1, X_2)\psi) &= \pi_L(d(\omega\psi)(X_1, X_2) + \omega \wedge d\psi(X_1, X_2)) \\ &= \pi_L[X_1(\omega(X_2)\psi) - X_2(\omega(X_1)\psi) - \omega([X_1, X_2])\psi \\ &\quad + \omega(X_1)d\psi(X_2) - \omega(X_2)d\psi(X_1)]. \end{aligned}$$

Since  $\omega$  stabilizes  $L$ , we have  $\omega([X_1, X_2])\psi \in L$  and  $\pi_L\omega(X_i)d\psi(X_j) = \pi_L\omega(X_i)\delta(X_j)\psi$ , hence

$$\begin{aligned} \pi_L(d\omega(X_1, X_2)\psi) &= \delta(X_1)\omega(X_2)\psi - \delta(X_2)\omega(X_1)\psi + \pi_L\omega(X_1)\delta(X_2)\psi \\ &\quad - \pi_L\omega(X_2)\delta(X_1)\psi \end{aligned}$$

and finally

$$\pi_L(d\omega(X_1, X_2)\psi) = (\delta \wedge \omega + \pi_L\omega \wedge \delta)(X_1, X_2)\psi. \quad (3.1.2)$$

**Definition 18** *A complex (resp. para-complex) para-quaternionic vector bundle is a para-quaternionic vector bundle  $V$  together with an endomorphism field  $J^\varepsilon \in \Gamma(\text{End}(V))$ ,  $p \mapsto J_p^\varepsilon$ , such that  $J^\varepsilon$  is a (para-)complex structure on  $V_p$ , for all  $p \in M$ , acting from the left and commuting with the para-quaternionic structure. We use the notation  $(V, J^\varepsilon)$ .*

Let  $(V, J_V^\varepsilon)$  be a (para-)complex para-quaternionic vector bundle over a (para-)Riemannian surface  $(M, J^\varepsilon)$ . We have the following decomposition:

$$\text{Hom}_{\mathbb{R}}(TM, V) = T^*M \otimes V = K^{1,0}V \oplus K^{0,1}V, \quad (3.1.3)$$

where  $K^{1,0}V$  and  $K^{0,1}V$  are the two eigenspaces of the (para-)complex structure  $J^\varepsilon$ , i.e.

$$\begin{aligned} K^{1,0}V &= \{\omega \in T^*M \otimes V \mid J^{\varepsilon*}\omega = J_V^\varepsilon\omega\}, \\ K^{0,1}V &= \{\omega \in T^*M \otimes V \mid J^{\varepsilon*}\omega = -J_V^\varepsilon\omega\}. \end{aligned}$$

**Remark 11** *Consider a map  $\lambda : M \rightarrow \widetilde{\mathbb{H}}$ , then we have  $d\lambda : TM \rightarrow M \times \widetilde{\mathbb{H}} = \lambda^*T\widetilde{\mathbb{H}}$  and we can use the explicit form of the above decomposition for  $d\lambda$ :*

$$\begin{aligned} d^{1,0}\lambda &= \frac{1}{2}(d\lambda + \varepsilon j^\varepsilon J^{\varepsilon*}d\lambda), \\ d^{0,1}\lambda &= \frac{1}{2}(d\lambda - \varepsilon j^\varepsilon J^{\varepsilon*}d\lambda), \end{aligned}$$

where  $j^\varepsilon : M \rightarrow \text{End}(\widetilde{\mathbb{H}})$  is the left multiplication with  $i$  for  $\varepsilon = -1$  and the left multiplication with  $e$  for  $\varepsilon = 1$ .



This leads to the following definition:

**Definition 19** Let  $(V, J_V^\varepsilon)$  be a (para-)complex para-quaternionic vector bundle over a Riemannian (resp. Lorentzian) surface  $(M, J^\varepsilon)$ . A para-quaternionic linear map

$$D : \Gamma(V) \rightarrow \Gamma(K^{0,1}V)$$

on  $(V, J_V^\varepsilon)$  such that for all  $\psi \in \Gamma(V)$  and all para-quaternionic functions  $\lambda : M \rightarrow \tilde{\mathbb{H}}$

$$D(\psi\lambda) = D\psi\lambda + \psi(d\lambda)^{0,1} = D\psi\lambda + \frac{1}{2}(\psi d\lambda - \varepsilon\psi J_V^\varepsilon J^{\varepsilon*} d\lambda)$$

is called a holomorphic structure on  $(V, J_V^\varepsilon)$  in the case  $\varepsilon = -1$  and a para-holomorphic structure  $(V, J_V^\varepsilon)$  on  $(V, J_V^\varepsilon)$  if  $\varepsilon = 1$ .

A (para-)complex para-quaternionic bundle is called (para-)holomorphic if it admits a (para-)holomorphic structure. Moreover a section  $\psi \in \Gamma(V)$  is called (para-)holomorphic if  $D\psi = 0$ .

**Remark 12** The definition 19 of a para-holomorphic structure is given in the sense of [BFLPP] and differs of the strongest definition we use in chapter 4.

Now consider the product bundle  $H = M \times \tilde{\mathbb{H}}^2$  endowed with a (para-)complex structure  $J_H^\varepsilon : M \rightarrow \text{End}(\tilde{\mathbb{H}}^2) \in \Gamma(\text{End}(H))$ . According to the decomposition (3.1.3), with  $V = H$ , we have for all  $\psi : M \rightarrow \tilde{\mathbb{H}}^2$ , i.e.  $\psi \in \Gamma(H)$

$$d\psi = d^{1,0}\psi + d^{0,1}\psi, \quad (3.1.4)$$

where

$$J^{\varepsilon*} d^{1,0}\psi = J_H^\varepsilon d^{1,0}\psi, \quad J^{\varepsilon*} d^{0,1}\psi = -J_H^\varepsilon d^{0,1}\psi.$$

In other words we have

$$d^{1,0}\psi = \frac{1}{2}(d\psi + \varepsilon J_H^\varepsilon J^{\varepsilon*} d\psi) \quad \text{and} \quad d^{0,1}\psi = \frac{1}{2}(d\psi - \varepsilon J_H^\varepsilon J^{\varepsilon*} d\psi).$$

One observes that

$$d^{0,1} : \Gamma(H) \rightarrow \Gamma(K^{0,1}H)$$

is, by definition 19, a (para-)holomorphic structure on  $(H, J_H^\varepsilon)$ , while

$$d^{1,0} : \Gamma(H) \rightarrow \Gamma(K^{1,0}H)$$

is a (para-)anti-holomorphic structure, i.e. a (para-)holomorphic structure on  $(H, -J_H^\varepsilon)$ .

Further the following decompositions hold:

**Proposition 15** There exists  $\partial : \Gamma(H) \rightarrow \Gamma(K^{1,0}H)$ ,  $\bar{\partial} : \Gamma(H) \rightarrow \Gamma(K^{0,1}H)$  and  $A \in \Gamma(K^{1,0}\text{End}_-(H))$ ,  $Q \in \Gamma(K^{0,1}\text{End}_-(H))$ , where  $\text{End}_-(H)$  are the  $\mathbb{C}$ - (resp.  $C$ -) anti-linear endomorphisms on  $H$ , such that

$$d^{1,0} = \partial + A, \quad d^{0,1} = \bar{\partial} + Q, \quad (3.1.5)$$

where

$$\partial(J_H^\varepsilon)\psi = \bar{\partial}(J_H^\varepsilon)\psi = 0, \quad (3.1.6)$$

for all  $\psi \in \Gamma(H)$ , and where  $A, Q$  anti-commute with the (para-)complex structure  $J_H^\varepsilon$ .

*Proof:* Define

$$\partial\psi := \frac{1}{2}(d^{1,0}\psi + \varepsilon J_H^\varepsilon d^{1,0}(J_H^\varepsilon\psi)), \quad \bar{\partial}\psi := \frac{1}{2}(d^{0,1}\psi + \varepsilon J_H^\varepsilon d^{0,1}(J_H^\varepsilon\psi)).$$

Obviously we have  $J_H^\varepsilon\partial\psi = \partial(J_H^\varepsilon\psi)$ ,  $J_H^\varepsilon\bar{\partial}\psi = \bar{\partial}(J_H^\varepsilon\psi)$ . Let now

$$A = d^{1,0} - \partial, \quad Q = d^{0,1} - \bar{\partial}.$$

We compute then explicitly

$$A\psi = \frac{1}{2}(d^{1,0}\psi - \varepsilon J_H^\varepsilon d^{1,0}(J_H^\varepsilon\psi)), \quad Q\psi = \frac{1}{2}(d^{0,1}\psi - \varepsilon J_H^\varepsilon d^{0,1}(J_H^\varepsilon\psi)). \quad (3.1.7)$$

A simple calculation proves that  $AJ_H^\varepsilon = -J_H^\varepsilon A$  and  $QJ_H^\varepsilon = -J_H^\varepsilon Q$ .

Further, using equation (3.1.7) and by definition of  $d^{1,0}$  and  $d^{0,1}$ , we have

$$A\psi = -\frac{1}{2}\varepsilon J_H^\varepsilon d^{1,0}(J_H^\varepsilon\psi), \quad Q\psi = -\frac{1}{2}\varepsilon J_H^\varepsilon d^{0,1}(J_H^\varepsilon\psi).$$

This means that  $A \in \Gamma(K^{1,0}\text{End}_-(H))$  and  $Q \in \Gamma(K^{0,1}\text{End}_-(H))$  are tensorial. Therefore by definition 19,  $\bar{\partial}$  is a (para-)holomorphic structure, while  $\partial$  is an (para-)anti-holomorphic structure on  $H$ .  $\square$

Moreover using the decompositions (3.1.4) and (3.1.5) we have

$$\begin{aligned} d(J_H^\varepsilon\psi) &= d(J_H^\varepsilon\psi) - J_H^\varepsilon d\psi \\ &= \partial(J_H^\varepsilon\psi) + AJ_H^\varepsilon\psi + \bar{\partial}(J_H^\varepsilon\psi) + QJ_H^\varepsilon\psi - J_H^\varepsilon\partial\psi - J_H^\varepsilon A\psi - J_H^\varepsilon\bar{\partial}\psi - J_H^\varepsilon Q\psi \\ &= -2J_H^\varepsilon(A + Q)\psi, \text{ for all } \psi \in \Gamma(H) \end{aligned}$$

and consequently  $dJ_H^\varepsilon = -2J_H^\varepsilon(A + Q)$ . This yields

$$J_H^\varepsilon dJ_H^\varepsilon = -2\varepsilon(A + Q). \quad (3.1.8)$$

Further  $A$  is by definition of type  $(1, 0)$  and  $Q$  is of type  $(0, 1)$ . Hence it follows

$$dJ_H^\varepsilon\psi = 2(J^{\varepsilon*}Q - J^{\varepsilon*}A)\psi.$$

Then we have

$$J^{\varepsilon*}dJ_H^\varepsilon = 2\varepsilon(Q - A) \quad (3.1.9)$$

and finally, with equations (3.1.8) and (3.1.9), we derive

$$A = -\frac{\varepsilon}{4}(J_H^\varepsilon dJ_H^\varepsilon + J^{\varepsilon*}dJ_H^\varepsilon), \quad Q = \frac{\varepsilon}{4}(J^{\varepsilon*}dJ_H^\varepsilon - J_H^\varepsilon dJ_H^\varepsilon). \quad (3.1.10)$$

## 3.2 Mean curvature pseudo-spheres and Hopf fields

### 3.2.1 Two-pseudo-spheres in $S^{2,2}$

We consider the sets:

$$Z^\varepsilon = \begin{cases} \{S^\varepsilon \in \text{End}(\tilde{\mathbb{H}}^2) | S^{\varepsilon^2} = -\text{Id}\}, & \text{for } \varepsilon = -1, \\ \{S^\varepsilon \in \text{End}(\tilde{\mathbb{H}}^2) | S^{\varepsilon^2} = \text{Id}, S^\varepsilon \neq \text{Id}\}, & \text{for } \varepsilon = 1, \end{cases} \quad (3.2.1)$$

where  $\tilde{\mathbb{H}}^2$  is the two-dimensional standard right-module over  $\tilde{\mathbb{H}}$ .  
Moreover for  $S^\varepsilon \in Z^\varepsilon$  define

$$\Sigma_{S^\varepsilon} = \{p \in \tilde{\mathbb{H}}P^1 | S^\varepsilon p = p\}. \quad (3.2.2)$$

**Definition 20** A two-pseudo-sphere in  $S^{2,2} = \tilde{\mathbb{H}}P^1$  is a two-dimensional surface  $\Sigma \subset \tilde{\mathbb{H}}P^1$  together with an affine chart  $\phi : \tilde{\mathbb{H}}P^1 \setminus \{\infty\} \xrightarrow{\sim} \tilde{\mathbb{H}}_*$ , such that  $\phi(\Sigma) \subset \tilde{\mathbb{H}}_*$  is a two-dimensional (non-degenerate) linear subspace.

**Proposition 16** (i)  $\Sigma_{S^\varepsilon}$  are two-pseudo-spheres for all  $S^\varepsilon \in Z^\varepsilon$ , such that  $\Sigma_{S^\varepsilon} \neq \emptyset$ .

(ii) For each two-pseudo-sphere  $\Sigma \subset \tilde{\mathbb{H}}P^1$  there exists an  $S^\varepsilon$ , unique up to sign, such that  $\Sigma = \Sigma_{S^\varepsilon}$ .

*Proof:* Consider  $\tilde{\mathbb{H}}^2$  as a (para-)complex para-quaternionic right-module, i.e. endowed with an additional (para-)complex structure  $J^\varepsilon \in \text{End}(\tilde{\mathbb{H}}^2)$  acting from the left (see paragraph 1.2.3). Then  $S^\varepsilon$  commutes with the (para-)complex structure  $J^\varepsilon$ .

Due to the assumption  $\Sigma_{S^\varepsilon} \neq \emptyset$  there exists  $v \in \tilde{\mathbb{H}}_*^2$  which satisfies

$$S^\varepsilon v = vN, \text{ for some } N \in \tilde{\mathbb{H}}_*.$$

The condition  $S^{\varepsilon^2} = \varepsilon \text{Id}$  implies  $N^2 = \varepsilon$ .

Now we can choose a basis  $(v, w)$  for  $\tilde{\mathbb{H}}^2$ , such that  $S^\varepsilon w = -vh - wR$ , with  $h, R \in \tilde{\mathbb{H}}$ . It holds  $S^{\varepsilon^2} w = \varepsilon w = -vNh + vhR + wR^2$  and therefore:

$$N^2 = \varepsilon = R^2, \quad Nh = hR.$$

Remark, that in the case where  $\varepsilon = 1$ , we have  $N \neq \pm 1 \neq R$ , as  $J^\varepsilon$  and  $S^\varepsilon$  commute and  $J^\varepsilon$  is an anti-isometry with respect to the pseudo-scalar-product on  $\tilde{\mathbb{H}}^2$ .

We take now the affine chart

$$\begin{aligned} \varphi : \tilde{\mathbb{H}} &\rightarrow \tilde{\mathbb{H}}P^1 \\ x &\mapsto [vx + w]. \end{aligned}$$

Then we have with some  $\gamma \in \tilde{\mathbb{H}}_*$ :

$$\begin{aligned} [vx + w] \in \Sigma_{S^\varepsilon} &\Leftrightarrow S^\varepsilon(vx + w) = (vx + w)\gamma \\ &\Leftrightarrow vNx - vh - wR = (vx + w)\gamma, \end{aligned}$$

which yields  $-R = \gamma$  and  $Nx - h = x\gamma$  and consequently

$$Nx + xR = h.$$

The corresponding homogeneous equation is given by  $Nx + xR = 0$  and with use of lemma 3, as  $N^2 = R^2 = \pm 1$ ,  $N \neq \pm 1 \neq R$ , the space of solutions corresponds to a real two-dimensional non-degenerate subspace of  $\tilde{\mathbb{H}}$ . Recall that  $S^\varepsilon$  determines the pair  $(N, R)$  up to sign.  $\square$

Hence we can see  $Z^\varepsilon$  as the set of two-pseudo-spheres in  $\tilde{\mathbb{H}}P^1$ .

**Proposition 17**  *$Z^\varepsilon$  is an eight-dimensional submanifold of  $\text{End}(\tilde{\mathbb{H}}^2)$ . Further the tangent space of  $Z^\varepsilon$  in  $S^\varepsilon$  is given by*

$$T_{S^\varepsilon}Z^\varepsilon = \{X \in \text{End}(\tilde{\mathbb{H}}^2) | XS^\varepsilon = -S^\varepsilon X\},$$

while the normal space of  $Z^\varepsilon$  in  $S^\varepsilon$  is given by

$$N_{S^\varepsilon}Z^\varepsilon = \{X \in \text{End}(\tilde{\mathbb{H}}^2) | XS^\varepsilon = S^\varepsilon X\}.$$

*Proof:* Consider the following action of  $\text{GL}(\tilde{\mathbb{H}}^2)$  on  $Z^\varepsilon$ :

$$\begin{aligned} \text{GL}(\tilde{\mathbb{H}}^2) \times Z^\varepsilon &\rightarrow Z^\varepsilon \\ (g, S^\varepsilon) &\mapsto gS^\varepsilon g^{-1}. \end{aligned}$$

This action is transitive. Let  $J_0^\varepsilon = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \in Z^\varepsilon$  be the canonical base point.

The stabilizer of  $J_0^\varepsilon$  is given by

$$\text{Stab}(J_0^\varepsilon) = \{g \in \text{GL}(\tilde{\mathbb{H}}^2) | gJ_0^\varepsilon = J_0^\varepsilon g\}.$$

and by the orbit-stabilizer theorem  $Z^\varepsilon = \text{GL}(\tilde{\mathbb{H}}^2)/\text{Stab}(J_0^\varepsilon)$  is a manifold.

Moreover we have

$$\mathfrak{gl}(\tilde{\mathbb{H}}^2) = \mathfrak{a} \oplus \mathfrak{c},$$

where  $\mathfrak{a} \cong T_{J_0^\varepsilon}Z^\varepsilon$  are the matrices which anti-commute, and  $\mathfrak{c}$  the matrices which commute with  $J_0^\varepsilon$ . A simple calculation shows that

$$\mathfrak{a} = \{A \in \text{End}(\tilde{\mathbb{H}}^2) | A = \begin{pmatrix} a & b \\ -\varepsilon b & -a \end{pmatrix}, a, b \in \tilde{\mathbb{H}}\},$$

and consequently  $\dim_{\mathbb{R}} Z^\varepsilon = 8$ .

One sees easily that the two spaces  $\mathfrak{a}$  and  $\mathfrak{c}$  are complementary. Simple calculations show that they are in fact orthogonal with respect to the bi-invariant metric given by the trace form.  $\square$

### 3.2.2 The mean curvature pseudo-spheres

**Definition 21** Let  $(M, J^\varepsilon)$  be a (para-)Riemannian surface. A (para-)holomorphic curve in  $\widetilde{\mathbb{H}P}^n$  is a line subbundle  $L \subset H = M \times \widetilde{\mathbb{H}P}^{n+1}$  together with a (para-)complex structure  $J_L^\varepsilon$  on  $L$  such that

$$J^{\varepsilon*} \delta = \delta J_L^\varepsilon,$$

where  $J^{\varepsilon*}$  is the (para-)complex structure induced by on  $T^*M$  by  $J^\varepsilon$ .

Now consider a (para-)holomorphic curve  $L$  immersed into  $\widetilde{\mathbb{H}P}^1$  with derivative  $\delta \in \Omega^1(L, H/L)$ . Our aim is to show that the (para-)complex structure  $J_L^\varepsilon$  on  $L$  extends to a (para-)complex structure on  $H$ . In fact we have the

**Theorem 13** Let  $L$  be a (para-)holomorphic curve immersed into  $\widetilde{\mathbb{H}P}^1$ . Then there exists a unique (para-)complex structure  $J_H^\varepsilon$  on  $H$  such that

- (i)  $J_H^\varepsilon L = L, \quad d(J_H^\varepsilon)L \subset L,$
- (ii)  $J^{\varepsilon*} \delta = \delta J_H^\varepsilon = J_H^\varepsilon \delta,$
- (iii)  $Q|_L = 0.$

*Proof:* By the definition of a (para-)holomorphic curve and by proposition 14 we derive

$$J^{\varepsilon*} \delta \stackrel{def}{=} \delta J_L^\varepsilon = J_{H/L}^\varepsilon \delta, \tag{3.2.3}$$

where  $J_L^\varepsilon, J_{H/L}^\varepsilon$  are unique (para-)complex structures on  $L$  and  $H/L$  (see proposition 14). At first we want to prove the existence of  $J_H^\varepsilon$ . Let  $H = L \oplus L'$  with some complementary line bundle  $L'$ . Considering the projection  $\pi_L : H \rightarrow H/L$ , we have  $\ker(\pi_L) = L$  which induces the isomorphism  $\pi_L|_{L'} : L' \xrightarrow{\sim} H/L$ . Using this isomorphism, we can define a (para-)complex structure on  $H$  by

$$J_H^\varepsilon := \begin{cases} J_H^\varepsilon|_L := J_L^\varepsilon & \text{on } L \\ J_H^\varepsilon|_{L'} := J_{H/L}^\varepsilon & \text{on } L'. \end{cases}$$

Equation (3.2.3) then leads to (ii). Obviously we have

$$J_H^\varepsilon L = J_L^\varepsilon L = L. \tag{3.2.4}$$

Moreover  $\pi_L J_H^\varepsilon = J_{H/L}^\varepsilon \pi_L$ , therefore by definition of the derivative we derive for all  $\psi \in \Gamma(L)$

$$\begin{aligned} \pi_L d(J_H^\varepsilon) \psi &= \pi_L (d(J_H^\varepsilon \psi) - J_H^\varepsilon d\psi) \\ &= \pi_L d(J_H^\varepsilon|_L \psi) - J_{H/L}^\varepsilon \pi_L d\psi = \delta(J_L^\varepsilon \psi) - J_{H/L}^\varepsilon \delta \psi = 0, \end{aligned}$$

hence

$$d(J_H^\varepsilon)L \subset L. \quad (3.2.5)$$

In addition equations (3.2.4) and (3.2.5) prove (i).

Since  $L'$  is not unique, the extension  $J_H^\varepsilon$  is a priori not unique. Let

$$\tilde{J}_H^\varepsilon = J_H^\varepsilon + R^\varepsilon, \quad R^\varepsilon \in \Gamma(\text{End}(H)).$$

Then  $\tilde{J}_H^\varepsilon \in \Gamma(\text{End}(H))$  is another such (para-)complex structure if and only if

$$\tilde{J}_H^\varepsilon L = L, \quad \pi_L \tilde{J}_H^\varepsilon = J_{H/L}^\varepsilon \pi_L \Leftrightarrow \pi_L R^\varepsilon = 0$$

and

$$\tilde{J}_H^\varepsilon|_L = J_L^\varepsilon \Leftrightarrow (J_H^\varepsilon + R^\varepsilon)|_L = J_L^\varepsilon \Leftrightarrow R^\varepsilon|_L = 0,$$

i.e if and only if

$$R^\varepsilon H \subset L \subset \ker R^\varepsilon. \quad (3.2.6)$$

Hence clearly  $R^\varepsilon \in \text{Hom}(H/L, L)$  and

$$R^\varepsilon \pi_L = R^\varepsilon. \quad (3.2.7)$$

Further (3.2.6) implies  $R^{\varepsilon^2} = 0$  and with  $\tilde{J}_H^{\varepsilon^2} = \varepsilon$  we have

$$R^\varepsilon J_H^\varepsilon + J_H^\varepsilon R^\varepsilon = 0.$$

With equation (3.1.10) we now calculate  $\tilde{Q}$  for the (para-)complex structure  $\tilde{J}_H^\varepsilon$ :

$$\begin{aligned} \tilde{Q} &= \frac{\varepsilon}{4} (J^{\varepsilon*} d(J_H^\varepsilon + R^\varepsilon) - (J_H^\varepsilon + R^\varepsilon) d(J_H^\varepsilon + R^\varepsilon)) \\ &= \frac{\varepsilon}{4} (J^{\varepsilon*} dJ_H^\varepsilon - J_H^\varepsilon dJ_H^\varepsilon) + \frac{\varepsilon}{4} (J^{\varepsilon*} dR^\varepsilon - J_H^\varepsilon dR^\varepsilon - R^\varepsilon dJ_H^\varepsilon - R^\varepsilon dR^\varepsilon) \\ &= Q + \frac{\varepsilon}{4} (J^{\varepsilon*} dR^\varepsilon - J_H^\varepsilon dR^\varepsilon - R^\varepsilon dJ_H^\varepsilon - R^\varepsilon dR^\varepsilon). \end{aligned}$$

Let  $\psi \in \Gamma(L)$ . Then property (3.2.5) leads to  $d(J_H^\varepsilon)\psi \in \Gamma(L)$ , hence  $R^\varepsilon d(J_H^\varepsilon)\psi = 0$ . Moreover, with equation (3.2.6),  $R^\varepsilon\psi = 0$  holds and consequently:

$$(dR^\varepsilon)\psi + R^\varepsilon d\psi = d(R^\varepsilon\psi) = 0 \Rightarrow R^\varepsilon dR^\varepsilon\psi = -R^{\varepsilon^2}d\psi = 0.$$

This yields with relation (3.2.7) and by definition of the derivative

$$\begin{aligned} \tilde{Q}\psi &= Q\psi + \frac{\varepsilon}{4} (J^{\varepsilon*} dR^\varepsilon - J_H^\varepsilon dR^\varepsilon)\psi = Q\psi + \frac{\varepsilon}{4} (J_H^\varepsilon R^\varepsilon - J^{\varepsilon*} R^\varepsilon) d\psi \\ &= Q\psi + \frac{\varepsilon}{4} (J_H^\varepsilon R^\varepsilon \delta - J^{\varepsilon*} R^\varepsilon \delta)\psi. \end{aligned}$$

Moreover by the definition of  $J^{\varepsilon*}$  and  $\delta$  and with equation (3.2.3) we have

$$J^{\varepsilon*} R^\varepsilon \delta\psi = R^\varepsilon J^{\varepsilon*} \delta\psi = R^\varepsilon J_{H/L}^\varepsilon \delta\psi = R^\varepsilon J_H^\varepsilon \delta\psi = -J_H^\varepsilon R^\varepsilon \delta\psi.$$

Consequently

$$\tilde{Q}\psi = Q\psi + \frac{\varepsilon}{2}J_H^\varepsilon R^\varepsilon \delta\psi. \quad (3.2.8)$$

We now want to show the existence of a unique extension with  $Q|_L = 0$ . Let  $J_H^\varepsilon$  be any extension of  $(J^\varepsilon J_{H/L}^\varepsilon)$  and

$$\tilde{J}_H^\varepsilon = J_H^\varepsilon + R^\varepsilon,$$

with

$$R^\varepsilon = -2J_H^\varepsilon Q(X)(\delta(X))^{-1}\pi_L.$$

We remark that  $R^\varepsilon$  is well defined, because it does not depend on  $X$ . In fact it is homogeneous of degree 0 and for any  $\tilde{X} = \cos(\theta)X + \sin(\theta)J^\varepsilon X$  we derive

$$\begin{aligned} Q(\tilde{X})\delta(\tilde{X})^{-1} &= Q(\cos(\theta)X + \sin(\theta)J^\varepsilon X)(\delta(\cos(\theta)X + \sin(\theta)J^\varepsilon X))^{-1} \\ &= Q(X)(\cos(\theta) + \sin(\theta)J_H^\varepsilon)(\delta(X)(\cos(\theta) + \sin(\theta)J_H^\varepsilon))^{-1} = Q(X)\delta(X)^{-1}. \end{aligned}$$

Indeed, for all  $\psi \in \Gamma(L)$  we have

$$\begin{aligned} \tilde{Q}\psi &= Q\psi + \frac{\varepsilon}{2}J_H^\varepsilon R^\varepsilon \delta\psi = Q\psi + \frac{\varepsilon}{2}J_H^\varepsilon R^\varepsilon \pi_L d\psi = Q\psi + \frac{\varepsilon}{2}J_H^\varepsilon R^\varepsilon d\psi \\ &= Q\psi - \varepsilon J_H^\varepsilon{}^2 Q \delta^{-1} \pi_L d\psi = 0. \end{aligned}$$

Moreover it follows immediately from the definition of  $R^\varepsilon$  that

$$L \subset \ker R^\varepsilon$$

and we derive with equation (3.2.5) that

$$QL = \frac{\varepsilon}{4}(J^{\varepsilon*} dJ_H^\varepsilon - J_H^\varepsilon dJ_H^\varepsilon)L \subset L.$$

Hence with equation (3.2.4) it follows

$$R^\varepsilon H \subset L,$$

which finishes the proof of the theorem.  $\square$

For each  $x \in M$ , we have  $L_x \in \tilde{\mathbb{H}}P^1$  and  $J_{H_x}^\varepsilon L_x = L_x$ . By proposition 16  $J_{H_x}^\varepsilon$  defines a 2-pseudo-sphere for all  $x \in M$ . Then  $J_H^\varepsilon$  is a family of two-pseudo-spheres or in other words a sphere congruence in  $\tilde{\mathbb{H}}P^1$ . This justifies the following

**Definition 22** *Under the conditions of theorem 13 the (para-)complex structure  $J_H^\varepsilon$  is called the mean curvature pseudo-sphere (congruence) of  $L$ . In the following it will be denoted by  $S^\varepsilon$ .  $A$  and  $Q$  are called the Hopf fields of  $L$ .*

### 3.2.3 Hopf fields

In this paragraph we prove some results about the mean curvature pseudo-spheres and their corresponding Hopf fields.

**Lemma 14** *Let  $M$  be a (para-)Riemannian surface immersed into  $\widetilde{\mathbb{H}P^1}$ . For the Hopf fields  $A$  and  $Q$  of the associated line bundle  $L$  we have*

$$d(A + Q) = 2(Q \wedge Q + A \wedge A).$$

*Proof:* With equation (3.1.8) we have

$$\begin{aligned} d(A + Q) &= -\frac{\varepsilon}{2}d(S^\varepsilon dS^\varepsilon) = -\frac{\varepsilon}{2}(dS^\varepsilon \wedge dS^\varepsilon) \\ &= -2\varepsilon(S^\varepsilon(A + Q) \wedge S^\varepsilon(A + Q)). \end{aligned}$$

Using the fact that  $A$  and  $Q$  anti-commute with  $S^\varepsilon$  yields

$$d(A + Q) = 2(A \wedge A + A \wedge Q + Q \wedge A + Q \wedge Q).$$

Moreover by definition  $A$  is of type  $(1, 0)$  and  $Q$  of type  $(0, 1)$ , then with equation (3.1.1)

$$A \wedge Q = AJ^{\varepsilon*}Q - J^{\varepsilon*}AQ = -AS^\varepsilon Q - S^\varepsilon AQ = S^\varepsilon AQ - S^\varepsilon AQ = 0.$$

A similar calculation leads to  $Q \wedge A = 0$ , which finishes the proof.  $\square$

We now seek for a condition on  $A$  which is equivalent to the condition (iii) of theorem 13 on  $Q$ , giving the uniqueness of the sphere congruence. This is given by

**Lemma 15** *Let  $L \subset H$  be the line bundle corresponding to an immersed (para-)Riemannian surface and  $S^\varepsilon$  be a (para-)complex structure on  $H$  such that  $S^\varepsilon L = L$  and  $dS^\varepsilon L \subset L$ . Then  $Q|_L = 0$  is equivalent to  $AH \subset L$  holds.*

*Proof:* Since  $dS^\varepsilon L \subset L$  we have by the definition of  $A$  and  $Q$  (cf. equation (3.1.10)) that  $AL \subset L$  and  $QL \subset L$ . Then by lemma 14 we obtain  $d(A + Q) \subset L$ . Hence with formula (3.1.2) it follows:

$$\begin{aligned} 0 = \pi_L d(A + Q) &= \pi_L(dA + dQ) \\ &= \delta \wedge A + \pi_L A \wedge \delta + \delta \wedge Q + \pi_L Q \wedge \delta. \end{aligned}$$

Using again that  $A$  is of type  $(1, 0)$  and  $Q$  of type  $(0, 1)$  yields with equation (3.1.1)

$$\delta \wedge A = \delta J^{\varepsilon*}A - J^{\varepsilon*}\delta A = \delta S^\varepsilon A - \delta S^\varepsilon A = 0$$

Similar calculations give

$$\pi_L Q \wedge \delta = 0, \quad \delta \wedge Q = -2\delta S^\varepsilon Q, \quad \pi_L A \wedge \delta = -2S^\varepsilon \pi_L A \delta.$$



Hence  $S^\varepsilon \delta Q = -S^\varepsilon \pi_L A \delta$  which leads to

$$\delta Q = -\pi_L A \delta.$$

Then  $Q|_L = 0$  if and only if  $\pi_L A \delta L = 0$ . Since  $\delta(X) : L \rightarrow H/L$  is an isomorphism for  $X \neq 0$  we get  $\pi_L A \delta L = 0 \Leftrightarrow \pi_L A|_{H/L} = 0$  and finally with  $AL \subset L$

$$Q|_L = 0 \Leftrightarrow AH \subset L.$$

□

### 3.3 A generalization of the Willmore functional

#### 3.3.1 The Gauss Map

**Definition 23** *Let  $V$  be a para-quaternionic vector space of rank  $n$  and  $A \in \text{End}_{\tilde{\mathbb{H}}}(V)$  a para-quaternionic endomorphism.*

$$\langle A \rangle := \frac{1}{4n} \text{tr}_{\mathbb{R}} \Re(A)$$

*is called the real trace of  $A$ . Moreover  $\langle A, B \rangle := \langle AB \rangle$  defines an indefinite scalar product.*

Obviously  $\langle \text{Id} \rangle = 1$  holds and  $\text{End}_{\pm}(V)$  are perpendicular with respect to the trace inner product. We note that by a property of the trace the scalar product is symmetric.

We now consider the mean curvature pseudo-sphere  $S^\varepsilon$  of a line bundle  $L \subset H$  corresponding to an immersed (para-)Riemannian surface. We have the

**Proposition 18** (i)  $\langle dS^\varepsilon, dS^\varepsilon \rangle = -\varepsilon \langle J^{\varepsilon*} dS^\varepsilon, J^{\varepsilon*} dS^\varepsilon \rangle,$

(ii)  $\langle dS^\varepsilon, J^{\varepsilon*} dS^\varepsilon \rangle = 0.$

*Proof:*

(i) From equation (3.1.8) we get

$$\langle dS^\varepsilon, dS^\varepsilon \rangle = \langle -2S^\varepsilon(A + Q), -2S^\varepsilon(A + Q) \rangle = -4\varepsilon \langle (A + Q), (A + Q) \rangle.$$

Moreover with lemma 15 we can derive for the Hopf fields that  $Q|_L = 0$  and  $AH \subset L$ . Consequently  $QA = 0$ , hence  $\langle Q, A \rangle = \langle A, Q \rangle = 0$  and with equation (3.1.9)

$$\langle dS^\varepsilon, dS^\varepsilon \rangle = -4\varepsilon \langle (Q - A), (Q - A) \rangle = -\varepsilon \langle J^{\varepsilon*} dS^\varepsilon, J^{\varepsilon*} dS^\varepsilon \rangle.$$

(ii) Equations (3.1.8) and (3.1.9) yield again

$$\begin{aligned} \langle dS^\varepsilon, J^{\varepsilon*} dS^\varepsilon \rangle &= \langle -2S^\varepsilon(A + Q), 2\varepsilon(Q - A) \rangle \\ &= -4\varepsilon (\langle S^\varepsilon A, Q \rangle - \langle S^\varepsilon A, A \rangle + \langle S^\varepsilon Q, Q \rangle - \langle S^\varepsilon Q, A \rangle). \end{aligned}$$

From  $QA = 0$  and the symmetry of the inner product we have

$$\begin{aligned}\langle S^\varepsilon A, Q \rangle &= -\langle AS^\varepsilon, Q \rangle = \langle S^\varepsilon Q, A \rangle = 0 \\ \langle S^\varepsilon A, A \rangle &= \langle AS^\varepsilon, A \rangle = -\langle S^\varepsilon A, A \rangle = 0 \\ \langle S^\varepsilon Q, Q \rangle &= \langle QS^\varepsilon, Q \rangle = -\langle S^\varepsilon Q, Q \rangle = 0,\end{aligned}$$

which finishes the proof.  $\square$

From this proposition it follows that the mean curvature pseudo-sphere  $S^\varepsilon$  is (para)-conformal. Therefore we will also call it the (para-)conformal Gauß map.

Finally we prove another lemma which we will need in the following section:

**Lemma 16**

$$\langle dS^\varepsilon \wedge J^{\varepsilon*} dS^\varepsilon \rangle = -4\varepsilon(\langle Q \wedge J^{\varepsilon*} Q \rangle + \langle A \wedge J^{\varepsilon*} A \rangle) \quad (3.3.1)$$

$$\langle dS^\varepsilon \wedge S^\varepsilon dS^\varepsilon \rangle = 4\varepsilon(\langle Q \wedge J^{\varepsilon*} Q \rangle - \langle A \wedge J^{\varepsilon*} A \rangle) \quad (3.3.2)$$

*Proof:* With equation (3.1.9) we have

$$\begin{aligned}\langle dS^\varepsilon \wedge J^{\varepsilon*} dS^\varepsilon \rangle &= \langle 2(J^{\varepsilon*} Q - J^{\varepsilon*} A) \wedge 2\varepsilon(Q - A) \rangle \\ &= 4\varepsilon(\langle J^{\varepsilon*} Q \wedge Q \rangle - \langle J^{\varepsilon*} Q \wedge A \rangle - \langle J^{\varepsilon*} A \wedge Q \rangle + \langle J^{\varepsilon*} A \wedge A \rangle)\end{aligned}$$

As  $A$  is of type  $(1, 0)$  and  $Q$  of type  $(0, 1)$  this yields with equation (3.1.1)

$$J^{\varepsilon*} A \wedge Q = (J^{\varepsilon*} A)(J^{\varepsilon*} Q) - J^{\varepsilon*}(J^{\varepsilon*} A)Q = -J^{\varepsilon*} AS^\varepsilon Q - \varepsilon AQ = J^{\varepsilon*} S^{\varepsilon*} AQ - \varepsilon AQ = 0.$$

Similarly  $J^{\varepsilon*} Q \wedge A = 0$  holds, which yields equation (3.3.1).

With equations (3.1.8) and (3.1.9) we have

$$\langle dS^\varepsilon \wedge S^\varepsilon dS^\varepsilon \rangle = -4\varepsilon(\langle J^{\varepsilon*} Q - J^{\varepsilon*} A \rangle \wedge (Q + A))$$

and a similar calculation yields the result.  $\square$

### 3.3.2 The Willmore functional

Let us consider again the manifolds  $Z^\varepsilon$  of section 3.2.1.

**Definition 24** *Let  $M$  be a (para-)Riemannian surface and  $S^\varepsilon : M \rightarrow Z^\varepsilon$  a map. We define by*

$$E(S^\varepsilon) := \int_M \langle dS^\varepsilon \wedge J^{\varepsilon*} dS^\varepsilon \rangle$$

*the energy functional of the (para-)Riemannian surface. If  $|E(S^\varepsilon)| < \infty$ , we call (para-)harmonic maps the critical points of this functional with respect to variations  $S^{\varepsilon t} := S^\varepsilon + tX$ , where  $X$  has compact support.*

We now want to find the Euler-Lagrange equations for  $E(S)$ .

**Proposition 19**  $S^\varepsilon$  is (para-)harmonic if and only if the  $Z^\varepsilon$ -tangential component  $(dJ^{\varepsilon*}dS^\varepsilon)^T$  vanishes.

*Proof:* Let  $X : M \rightarrow Z^\varepsilon$  be compactly supported and  $S_t^\varepsilon := S^\varepsilon + tX$  be a variation of  $S^\varepsilon$ . Obviously  $X := \frac{d}{dt}S_t^\varepsilon|_{t=0}$  is in  $T_{S_p^\varepsilon}Z^\varepsilon$ , with  $p \in M$ . If  $S^\varepsilon$  is a critical point of  $E$  we have

$$\frac{d}{dt}E(S_t^\varepsilon)|_{t=0} = 0, \quad \text{for all variations } X.$$

Moreover

$$\begin{aligned} \frac{d}{dt}E(S_t^\varepsilon)|_{t=0} &= \frac{d}{dt} \int_M \langle d(S^\varepsilon + tX) \wedge J^{\varepsilon*}d(S^\varepsilon + tX) \rangle|_{t=0} \\ &= \frac{d}{dt} \int_M (\langle dS^\varepsilon \wedge J^{\varepsilon*}dS^\varepsilon \rangle + t\langle dS^\varepsilon \wedge J^{\varepsilon*}dX \rangle \\ &\quad + t\langle dX \wedge J^{\varepsilon*}dS^\varepsilon \rangle + t^2\langle dX \wedge J^{\varepsilon*}dX \rangle)|_{t=0} \\ &= \int_M \langle dS^\varepsilon \wedge J^{\varepsilon*}dX \rangle + \langle dX \wedge J^{\varepsilon*}dS^\varepsilon \rangle. \end{aligned}$$

But with equation (3.1.1) we derive

$$dS^\varepsilon \wedge J^{\varepsilon*}dX = \varepsilon dS^\varepsilon dX - J^{\varepsilon*}dS^\varepsilon J^{\varepsilon*}dX = -J^{\varepsilon*}dS^\varepsilon \wedge dX = dX \wedge J^{\varepsilon*}dS^\varepsilon.$$

This leads by Stokes formula for a critical point

$$\begin{aligned} 0 = \frac{d}{dt}E(S_t^\varepsilon)|_{t=0} &= 2 \int_M \langle dX \wedge J^{\varepsilon*}dS^\varepsilon \rangle = 2 \int_M (\langle d(X \wedge J^{\varepsilon*}dS^\varepsilon) \rangle - \langle X \wedge dJ^{\varepsilon*}dS^\varepsilon \rangle) \\ &= -2 \int_M \langle X \wedge dJ^{\varepsilon*}dS^\varepsilon \rangle = -2 \int_M \langle X, dJ^{\varepsilon*}dS^\varepsilon \rangle. \end{aligned}$$

Hence  $dJ^{\varepsilon*}dS^\varepsilon \in N_{S^\varepsilon}^\varepsilon Z^\varepsilon$ . □

By lemma 16

$$\begin{aligned} E(S^\varepsilon) &= -4\varepsilon \int_M (\langle Q \wedge J^{\varepsilon*}Q \rangle + \langle A \wedge J^{\varepsilon*}A \rangle) \\ &= -8\varepsilon \int_M (\langle A \wedge J^{\varepsilon*}A \rangle - 4\varepsilon \int_M (\langle Q \wedge J^{\varepsilon*}Q - A \wedge J^{\varepsilon*}A \rangle)) \end{aligned} \quad (3.3.3)$$

We are now going to show that the second term  $\int_M (\langle Q \wedge J^{\varepsilon*}Q - A \wedge J^{\varepsilon*}A \rangle)$  is a topological invariant:

**Proposition 20** Let  $\omega_{S^\varepsilon}(X, Y) := \langle X, S^\varepsilon Y \rangle \in \Omega^2(Z^\varepsilon)$ ,  $X, Y \in T_{S^\varepsilon}Z^\varepsilon$ .

Then  $\int_M S^{\varepsilon*}\omega_{S^\varepsilon}$  is a topological invariant i.e. invariant under compactly supported homotopy transformations of  $S^\varepsilon$ .

*Proof:* By the property of the trace and the fact that  $X, Y \in T_{\xi}^{\varepsilon}Z^{\varepsilon}$  we have

$$\omega_{S^{\varepsilon}}(X, Y) = \frac{1}{2}(\langle X, S^{\varepsilon}Y \rangle + \langle S^{\varepsilon}Y, X \rangle) = \frac{1}{2}(\langle X, S^{\varepsilon}Y \rangle - \langle Y, S^{\varepsilon}X \rangle). \quad (3.3.4)$$

We have to show at first that  $\omega_{S^{\varepsilon}}$  is closed. To avoid projections on  $T_{S^{\varepsilon}}Z$  we consider the extension  $\Omega_{S^{\varepsilon}}$  of this two-form on  $\text{End}(\tilde{H}^2)$  and we calculate

$$\begin{aligned} d\Omega_{S^{\varepsilon}}(X, Y, Z) &= \sum_{cycl} X\Omega_{S^{\varepsilon}}(Y, Z) - \sum_{cycl} \Omega_{S^{\varepsilon}}([X, Y], Z) \\ &= \frac{1}{2} \left( \sum_{cycl} X(\langle Y, S^{\varepsilon}Z \rangle - \langle Z, S^{\varepsilon}Y \rangle) - \sum_{cycl} (\langle [X, Y], S^{\varepsilon}Z \rangle - \langle Z, S^{\varepsilon}[X, Y] \rangle) \right) \\ &= \frac{1}{2} \sum_{cycl} ( \langle Y, X(S^{\varepsilon})Z \rangle + \underbrace{\langle Y, S^{\varepsilon}X(Z) \rangle}_{=-\langle X(Z), S^{\varepsilon}Y \rangle} - \langle X(Z), S^{\varepsilon}Y \rangle \\ &\quad - \underbrace{\langle Z, X(S^{\varepsilon})Y \rangle}_{=-\langle Y(X), S^{\varepsilon}Z \rangle} + \langle Y(X), S^{\varepsilon}Z \rangle - \langle Z, S^{\varepsilon}Y(X) \rangle ) \\ &= \frac{1}{2} (\langle Y, X(S^{\varepsilon})Z \rangle + \langle X, Z(S^{\varepsilon})Y \rangle + \langle Z, Y(S^{\varepsilon})X \rangle \\ &\quad - \langle Z, X(S^{\varepsilon})Y \rangle - \langle X, Y(S^{\varepsilon})Z \rangle - \langle Y, Z(S^{\varepsilon})X \rangle) \end{aligned}$$

The six terms are of the form  $\langle A, BC \rangle$ , with  $A, B, C \in T_{S^{\varepsilon}}Z^{\varepsilon}$ . Further we compute

$$\langle A, BC \rangle = \varepsilon \langle S^{\varepsilon 2}A, BC \rangle = \varepsilon \langle S^{\varepsilon}A, BC S^{\varepsilon} \rangle = -\varepsilon \langle AS^{\varepsilon}, S^{\varepsilon}BC \rangle = -\langle A, BC \rangle,$$

which yields  $\langle A, BC \rangle = 0$  and consequently  $d\Omega_{S^{\varepsilon}} = 0$ .

Finally if  $\iota : Z \rightarrow \text{End}(\tilde{H}^2)$  is the canonical inclusion we have

$$d\omega_{S^{\varepsilon}} = d\iota^*\Omega_{S^{\varepsilon}} = \iota^*d\Omega_{S^{\varepsilon}} = 0.$$

Now we show the homotopy invariance of  $\int_M S^{\varepsilon*}\omega_{S^{\varepsilon}}$ . Let

$$\tilde{S}^{\varepsilon} : [0, 1] \times M \rightarrow Z^{\varepsilon} \quad (3.3.5)$$

be a homotopy transformation from  $M_0 = S_0^{\varepsilon}(M)$  to  $M_1 = S_1^{\varepsilon}(M)$ , where we note  $\tilde{S}^{\varepsilon}(t, x) = S_t^{\varepsilon}(x)$ , and let  $\tilde{S}^{\varepsilon}$  be compactly supported (i.e. for any  $t$ ,  $\tilde{S}^{\varepsilon}(t, \cdot)$  is compactly supported). Stokes formula yields

$$\begin{aligned} 0 &= \int_{M \times [0, 1]} \tilde{S}^{\varepsilon*}d\omega_{S^{\varepsilon}} = \int_{M \times [0, 1]} d\tilde{S}^{\varepsilon*}\omega_{S^{\varepsilon}} = \int_{M \times \{1\}} \tilde{S}^{\varepsilon*}\omega_{S^{\varepsilon}} - \int_{M \times \{0\}} \tilde{S}^{\varepsilon*}\omega_{S^{\varepsilon}} \\ &= \int_M S_1^{\varepsilon*}\omega_{S^{\varepsilon}} - \int_M S_0^{\varepsilon*}\omega_{S^{\varepsilon}}. \end{aligned}$$

□

Moreover we have

$$\begin{aligned} S^{\varepsilon*}\omega_{S^{\varepsilon}}(X, Y) &= \omega_{S^{\varepsilon}}(dS^{\varepsilon}(X), dS^{\varepsilon}(Y)) \\ &= \frac{1}{2}(\langle dS^{\varepsilon}(X)S^{\varepsilon}dS^{\varepsilon}(Y) \rangle - \langle dS^{\varepsilon}(Y)S^{\varepsilon}dS^{\varepsilon}(X) \rangle) = \frac{1}{2}\langle dS^{\varepsilon} \wedge S^{\varepsilon}dS^{\varepsilon} \rangle(X, Y) \\ &\stackrel{(3.3.2)}{=} 2\varepsilon(\langle Q \wedge J^{\varepsilon*}Q - A \wedge J^{\varepsilon*}A \rangle)(X, Y) \end{aligned}$$

This leads with equation (3.3.3) to the following

**Definition 25** *Let  $L$  be an immersed (para-)holomorphic curve in  $\widetilde{\mathbb{H}P^1}$  with mean curvature sphere  $S^\varepsilon$ . The energy functional*

$$W(L) := \frac{1}{\pi} \int \langle A \wedge J^{\varepsilon*} A \rangle$$

*is the generalized Willmore functional of the immersion. We call Willmore surfaces critical points of  $W(L)$  with respect to compact supported variations  $S_t^\varepsilon := S^\varepsilon + tX$  of the mean curvature sphere.*

Summarizing, we have expressed the energy of a surface of arbitrary signature as the sum of a topological invariant and of the generalized Willmore functional.

# Chapter 4

## Para-complex affine immersions

In this chapter we study decompositions of para-complex and para-holomorphic vector-bundles, which are endowed with a connection  $\nabla$  over a para-complex manifold. First we prove that any connection with vanishing  $(0, 2)$ -curvature, with respect to the grading defined by the para-complex structure, induces a unique para-holomorphic structure, generalizing in this way a well-known result of complex geometry. Moreover, we obtain results on the connections induced on the subbundles, their second fundamental forms and their curvature tensors. In particular we analyze para-holomorphic decompositions. Then we introduce the notion of para-complex affine immersions: Applying the above results, we obtain existence and uniqueness theorems for para-complex affine immersions. This is a generalization of the results obtained by Abe and Kurosu (see [AK]) to para-complex geometry.

### 4.1 Connections on para-holomorphic vector bundles

**Definition 26** *Let  $(E, \mathcal{E}), (F, \mathcal{F})$  be two para-holomorphic vector bundles. An element  $\varphi \in \Gamma(\text{Hom}((E, J^E), (F, J^F)))$  is called para-holomorphic with respect to  $E$  and  $F$  if it is a para-holomorphic map  $\varphi : (E, J^E) \rightarrow (F, J^F)$  of the para-complex manifolds  $E$  and  $F$ .*

Like in the complex case we have:

**Lemma 17** *Let  $(E, \mathcal{E}), (F, \mathcal{F})$  be two para-holomorphic vector bundles. Then  $\varphi \in \Gamma(\text{Hom}((E, J^E), (F, J^F)))$  is para-holomorphic with respect to  $E$  and  $F$  if and only if  $\varphi\xi \in \mathcal{O}(F|_U)$  for any open set  $U \subset M$  and para-holomorphic section  $\xi \in \mathcal{O}(E|_U)$ .*

**Definition 27**

1. *A connection  $\nabla$  on a para-complex vector bundle  $(E, J^E)$  is called para-complex if it commutes with the para-complex structure on  $E$ , i.e.  $J^E$  is  $\nabla$ -parallel. The set of all such connections will be denoted by  $\mathcal{P}(E, J^E)$ .*

2. Let  $(M, J)$  be a para-complex manifold and  $U \subset M$  be an open set. Let  $\nabla \in \mathcal{P}(E, J^E)$  with  $(E, J^E)$  a para-holomorphic vector bundle  $(E, \mathcal{E})$ . Then  $\nabla$  is adapted if the following equation

$$\nabla_{JX}\xi = J^E\nabla_X\xi$$

is satisfied for all  $X \in \Gamma(TM|_U)$ ,  $\xi \in \mathcal{O}(E|_U)$ . The set of these connections is denoted by  $\mathcal{P}^a(E, \mathcal{E})$ .

**Lemma 18** *Let  $(E, \mathcal{E})$  be a para-holomorphic vector bundle. Then  $\mathcal{P}^a(E, \mathcal{E}) \neq \emptyset$ .*

*Proof:* Let  $(U_\alpha)_{\alpha \in I}$  be a open covering of  $M$ ,  $\lambda_\alpha$  a partition of unity with respect to  $(U_\alpha)_{\alpha \in I}$  and  $s_\alpha \in \mathcal{O}(E|_{U_\alpha})$  a para-holomorphic frame. Then one defines connections on  $E|_{U_\alpha}$  by

$$\nabla^\alpha s_\alpha = 0$$

and a connection

$$\nabla = \sum_{\alpha} \lambda_\alpha \nabla^\alpha.$$

By the definition of a para-holomorphic frame  $J^E$  is  $\nabla$ -parallel and it holds by definition

$$\nabla_{\tau X} s_\alpha = \tau^E \nabla_X s_\alpha.$$

This shows by a short calculation

$$\nabla_{\tau X} \xi = \tau^E \nabla_X \xi$$

for all  $\xi \in \mathcal{O}(E|_U)$  and for all open subsets  $U \subset M$ , i.e.  $\nabla$  defines a connection in  $\mathcal{P}^a(E, \mathcal{E})$ .  $\square$

**Lemma 19** *Given an open set  $U \subset M$  and  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ , then a section  $\xi \in \mathcal{O}(E|_U)$  is para-holomorphic if and only if*

$$\nabla_{JX}\xi = J^E\nabla_X\xi, \tag{4.1.1}$$

for any  $X \in \Gamma(TM|_U)$ .

**Remark 13** *Locally a connection  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$  defines consequently a unique set of para-holomorphic sections. More precisely, we can recover the local para-holomorphic sections defining the para-holomorphic structure from  $\nabla$  by equation (4.1.1).*

**Definition 28** *Let  $K$  be a 1-form with values in  $\text{Hom}((E, J^E), (F, J^F))$ .  $K$  is called para-complex (resp. para-anti-complex) if for any  $X \in TM$*

$$K_{JX} = J^F K_X \text{ (resp. } K_{JX} = -J^F K_X).$$

**Remark 14** *With the above notation  $K$  is para-complex (resp. para-anti-complex) if and only if it is in  $\Omega^{1,0}(\text{Hom}((E, J^E), (F, J^F)))$  (resp. in  $\Omega^{0,1}(\text{Hom}((E, J^E), (F, J^F)))$ ), where the para-complex structure on  $\text{Hom}((E, J^E), (F, J^F))$  is given by*

$$\begin{aligned} \text{Hom}((E, J^E), (F, J^F)) &\rightarrow \text{Hom}((E, J^E), (F, J^F)), \\ v &\mapsto v \circ J^E = J^F \circ v. \end{aligned}$$

Using this notion, the difference  $\nabla - \nabla'$  of two connections  $\nabla, \nabla' \in \mathcal{P}^a(E, \mathcal{E})$  takes values in  $\Omega^{1,0}(\text{Hom}((E, J^E), (E, J^E)))$ . Hence by fixing a connection  $\nabla^0 \in \mathcal{P}^a(E, \mathcal{E})$  the set  $\mathcal{P}^a(E, \mathcal{E})$  has the structure of an affine space

$$\mathcal{P}^a(E, \mathcal{E}) = \{\nabla^0 + K \mid K \in \Omega^{1,0}(\text{Hom}((E, J^E), (E, J^E)))\}.$$

**Lemma 20** *Let  $(E, \mathcal{E}), (F, \mathcal{F})$  be para-holomorphic vector bundles,  $\Phi \in \text{Hom}((E, J^E), (F, J^F))$ . Then*

- (i) *Given two connections  $\nabla \in \mathcal{P}^a(E, \mathcal{E}), \nabla' \in \mathcal{P}^a(F, \mathcal{F})$  such that  $\Phi$  is parallel with respect to the product-connection, i.e.*

$$\nabla'_X \Phi = \Phi \nabla_X, \tag{4.1.2}$$

*then  $\Phi$  is para-holomorphic with respect to  $\mathcal{E}$  and  $\mathcal{F}$ .*

- (ii) *Given  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$  and  $\nabla' \in \mathcal{P}(F, J^F)$ , if  $\Phi$  is surjective and para-holomorphic with respect to  $\mathcal{E}$  and  $\mathcal{F}$  and equation (4.1.2) is satisfied, then  $\nabla' \in \mathcal{P}^a(F, \mathcal{F})$ .*
- (iii) *Conversely, given  $\nabla \in \mathcal{P}(E, J^E)$  and  $\nabla' \in \mathcal{P}^a(F, \mathcal{F})$ , if  $\Phi$  is injective and para-holomorphic with respect to  $\mathcal{E}$  and  $\mathcal{F}$  and equation (4.1.2) is satisfied, then  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ .*

*Proof:* (i) Using equation (4.1.2) and lemma 19 we compute:

$$\nabla'_{JX} \Phi \xi = \Phi \nabla_{JX} \xi = \Phi J^E \nabla_X \xi = J^F \Phi \nabla_X \xi = J^F \nabla'_X \Phi \xi$$

on any open set  $U \subset M$ ,  $\xi \in \mathcal{O}(E|_U)$  and  $X$  a vector field on  $U$ . In other words  $\Phi \xi$  is a para-holomorphic section of  $(F|_U, \mathcal{F})$  and lemma 17 yields the para-holomorphicity of  $\Phi$  with respect to  $\mathcal{E}$  and  $\mathcal{F}$ .

(ii) Let  $x \in M$ , there exists a local para-holomorphic basis  $\mathcal{B}^\mathcal{E} = (e_1, \dots, e_r, J^E e_1, \dots, J^E e_r)$  of  $\mathcal{E}$  on an open set  $U$ , with  $x \in U$ . The surjectivity of  $\Phi$  and  $\Phi J^E = J^F \Phi$  imply the existence of a para-complex basis  $\mathcal{B} = (\Phi e_{k_1}, \dots, \Phi e_{k_l}, \Phi J^E e_{k_{l+1}}, \dots, \Phi J^E e_{k_{2l}})$  of  $F_x$ , with  $\text{rank} F = 2l$ , for  $1 \leq k_1 < \dots < k_{2l} \leq 2r$ . The para-holomorphicity of  $\Phi$  allows to extend  $\mathcal{B}$  on an open set  $V \subset U$  to a basis  $\mathcal{B}^\mathcal{F} \in \mathcal{F}$ . Using equation (4.1.2) we obtain

$$\nabla'_{JX} \Phi e_{k_s} = \Phi \nabla_{JX} e_{k_s} = \Phi J^E \nabla_X e_{k_s} = J^F \Phi \nabla_X e_{k_s} = J^F \nabla'_X \Phi e_{k_s},$$

for  $X$  any vector field over  $U$  and  $s \in \{1, \dots, l\}$ , which proves (ii).

(iii) Let  $U \subset M$  be an open set and  $\xi \in \Gamma(E|_U)$ . With lemma 17  $\xi$  is para-holomorphic if and only if  $\Phi \xi$  is para-holomorphic. Now by lemma 19 it holds  $(\nabla'_{JX} - J^F \nabla') \Phi \xi = 0$ ,



which yields with equation (4.1.2)  $\Phi(\nabla_{JX} - J^E \nabla)\xi = 0$  and from the injectivity of  $\Phi$  we obtain  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ .  $\square$

We now consider the curvature tensor  $R \in \Omega^2(M, \text{End}(E, J^E))$  of a connection  $\nabla \in \mathcal{P}(E, J^E)$  given by

$$R_{X,Y} := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

where  $X, Y \in \Gamma(TM)$ .

As discussed above the space  $\Omega^2(M, \text{End}(E, J^E))$  decomposes into the direct sum

$$\Omega^2(M, \text{End}(E, J^E)) = \Omega^{2,0}(M, \text{End}(E, J^E)) \oplus \Omega^{1,1}(M, \text{End}(E, J^E)) \oplus \Omega^{0,2}(M, \text{End}(E, J^E)),$$

which can be characterized by

$$\begin{aligned} \Omega^{2,0}(M, \text{End}(E, J^E)) &= \{K \mid K_{JX,Y} = J^E K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}, \\ \Omega^{1,1}(M, \text{End}(E, J^E)) &= \{K \mid K_{JX,JY} = -K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}, \\ \Omega^{0,2}(M, \text{End}(E, J^E)) &= \{K \mid K_{JX,Y} = -J^E K_{X,Y} \text{ for } X, Y \in \Gamma(TM)\}. \end{aligned}$$

The corresponding components of  $K \in \Omega^2(M, \text{End}(E, J^E))$  are with  $X, Y \in \Gamma(TM)$

$$K_{X,Y}^{2,0} = \frac{1}{4}(K_{X,Y} + J^E K_{JX,Y} + J^E K_{X,JY} + K_{JX,JY}), \quad (4.1.3)$$

$$K_{X,Y}^{1,1} = \frac{1}{2}(K_{X,Y} - K_{JX,JY}), \quad (4.1.4)$$

$$K_{X,Y}^{0,2} = \frac{1}{4}(K_{X,Y} - J^E K_{JX,Y} - J^E K_{X,JY} + K_{JX,JY}). \quad (4.1.5)$$

Therefore one introduces three classes of connections.

**Definition 29** *A connection  $\nabla \in \mathcal{P}(E, J^E)$  is said to be para-holomorphic (or of type (2, 0)) if  $R_{JX,Y} = J^E R_{X,Y}$ , para-anti-holomorphic (or of type (0, 2)) if  $R_{JX,Y} = -J^E R_{X,Y}$  and of type (1, 1) if  $R_{JX,JY} = -R_{X,Y}$ .*

**Remark 15** *This definition is motivated by the fact that a para-holomorphic connection  $\nabla$  can be restricted to para-holomorphic sections, this means it defines a connection also called  $\nabla$*

$$\nabla : \mathcal{O}(TM) \times \mathcal{O}(E) \rightarrow \mathcal{O}(E).$$

With this notion we prove

**Lemma 21** *If  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$  then the (0, 2)-part of its curvature vanishes.*

*Proof:* Let  $U \subset M$  be an open set and  $\xi \in \mathcal{O}(E|_U)$ . Due to lemma 19 it holds

$$(\nabla_{JX} - J^E \nabla_X)\xi = 0$$

and with this equation we obtain

$$\begin{aligned} 0 &= (\nabla_{JX} - J^E \nabla_X)(\nabla_{JY} - J^E \nabla_Y)\xi \\ &= (\nabla_X \nabla_Y + \nabla_{JX} \nabla_{JY} - J^E \nabla_{JX} \nabla_Y - J^E \nabla_X \nabla_{JY})\xi. \end{aligned}$$

Anti-symmetrizing in  $X, Y$  and using  $N_\tau(X, Y) = 0$  (where  $N_\tau$  is the Nijenhuis-tensor) yields the claimed vanishing of the  $(0, 2)$ -curvature.  $\square$

For vector bundles over real surfaces the following proposition was proven by Erdem [Er3]. We give a more general proof by adapting the methods of complex geometry to our setting, compare for example the work of Atiyah, Hitchin and Singer [AHS] theorem 5.2 or proposition 3.7 in the book of Kobayashi [K].

**Proposition 21** *Let  $\nabla$  be a connection in  $\mathcal{P}(E, J^E)$  with vanishing  $(0, 2)$ -curvature then there exists a unique para-holomorphic vector bundle structure  $\mathcal{E}$  on  $(E, J^E)$  such that  $\nabla \in \mathcal{P}^\alpha(E, \mathcal{E})$ .*

*Proof:* First we introduce an almost para-complex structure on the manifold  $E$ , by giving a decomposition of  $T^*E = T^*E^+ \oplus T^*E^-$  in its eigenspaces.

This will be done in a local trivialization of  $E|_U = U \times \mathbb{R}^{2r}$  over an open subset  $U$ . Using the Frobenius theorem one can choose a set of adapted coordinates  $(z^1, \dots, z^{2n}) = (z_+^1, \dots, z_+^n, z_-^1, \dots, z_-^n)$  of  $U$ , i.e. coordinates  $z_\pm^i$  which are constants on the leaves of  $T^\mp M$  (compare [CMMS] proposition 2). The pull-back of the functions  $z^i$  will be denoted by  $\tilde{z}^i = \pi^*(z^i)$ . On the  $\mathbb{R}^{2r}$ -factor we take linear coordinates, i.e. linear functionals of  $\mathbb{R}^{2r}$   $(w^1, \dots, w^{2r}) = (w_+^1, \dots, w_+^r, w_-^1, \dots, w_-^r)$  corresponding to the eigenspaces of the para-complex structure on  $E$ .

The local expression with respect to the frame  $(w^\alpha)$  of the connection form  $\omega = (\omega_\beta^\alpha)$ ,  $\alpha, \beta = 1, \dots, 2r$  of  $\nabla$  decomposes relative to the decomposition of  $\Omega^1(M, \text{End}(E, J^E))$  into its  $(1, 0)$  and  $(0, 1)$ -part (compare with remark 14):

$$\omega_\beta^\alpha = \omega^{1,0\alpha}_\beta + \omega^{0,1\alpha}_\beta.$$

We now decompose  $T^*E|_U$  into

$$T^*E^+ := \text{span}\{d\tilde{z}^i, dw^\alpha + \sum_{\gamma=1}^{2r} \omega^{0,1\alpha}_\gamma w^\gamma\}, \quad (4.1.6)$$

$$T^*E^- := \text{span}\{d\tilde{z}^{i+n}, dw^{r+\alpha} + \sum_{\gamma=1}^{2r} \omega^{0,1^{r+\alpha}}_\gamma w^\gamma\}, \quad i = 1 \dots, n, \alpha = 1, \dots, r \quad (4.1.7)$$

We are going to show the integrability of this almost para-complex structure. As the reader may observe, one gets another integrable almost para-complex structure by omitting the sum-term in the second factors, which is the standard para-complex structure on the product bundle. Unfortunately,  $\nabla$  is not adapted for this structure.

We check the integrability of the distribution defined in (4.1.6) and (4.1.7), which follows directly from Frobenius theorem, see [Wa] Proposition 2.30.

In our situation this is, we have to show that if  $\eta \in \Omega^1(M, \text{End}(E, J^E))$  is of type  $(1+, 0)$  (resp. type  $(0+, 1-)$ ), then  $d\eta$  has type  $(1+, 1-) + (2+, 0-)$  (resp.  $(1+, 1-) + (0+, 2-)$ ) where the type is taken with respect to the decomposition defined by equations (4.1.6) and (4.1.7).

This will be checked on a basis. Recall that  $\tilde{z}^i$  and  $w^\alpha$  are considered as functions on  $E$ .

$$\begin{aligned}
d(d\tilde{z}^i) &= 0, \quad i = 1, \dots, 2n, \\
d(dw^\alpha + \sum_{\gamma=1}^{2r} \omega^{0,1\alpha}_\gamma w^\gamma) &= \sum_{\gamma=1}^{2r} (d\omega^{0,1\alpha}_\gamma w^\gamma - \omega^{0,1\alpha}_\gamma \wedge dw^\gamma), \quad \alpha = 1, \dots, 2r, \\
&\equiv \sum_{\gamma=1}^{2r} ((d\omega^{0,1\alpha}_\gamma + \omega^{0,1\alpha}_\beta \wedge \omega^{0,1\beta}_\gamma) w^\gamma) \\
&= \sum_{\gamma=1}^{2r} (d\omega^{0,1\alpha}_\gamma)^{1,1} w^\gamma + \underbrace{\sum_{\gamma=1}^{2r} ((d\omega^{0,1\alpha}_\gamma)^{0,2} + \omega^{0,1\alpha}_\beta \wedge \omega^{0,1\beta}_\gamma) w^\gamma}_{=R^{0,2}=0} \equiv 0,
\end{aligned}$$

where  $\equiv$  is meant modulo  $(1+, 1-)$ . We may recall that forms of type  $(1+, 1-)$  and  $(1, 1)$  coincide.

This shows that the decomposition defines the structure of a para-complex manifold on  $E$ . In the above coordinates  $(\tilde{z}^i, w^\alpha)$  the projection

$$\pi : E \rightarrow M$$

is given by

$$\pi(\tilde{z}^i, w^\alpha) = (z^i).$$

This shows that  $\pi$  is para-holomorphic, since  $\pi^* dz^i_\pm = d\tilde{z}^i_\pm$ . For the restriction  $T^*E_0^\pm$  of  $T^*E^\pm$  to the zero section  $s_0 : M \rightarrow E$  of the bundle  $E$ , we find

$$T^*E_0^\pm = \text{span}\{d\tilde{z}^i_\pm, dw^\alpha_\pm\}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, r.$$

This shows that

$$\begin{aligned}
s_0^* d\tilde{z}^i &= dz^i, \quad i = 1, \dots, 2n, \\
s_0^* dw^\alpha &= 0, \quad \alpha = 1, \dots, 2r
\end{aligned}$$

which means that the zero section is para-holomorphic. Therefore the normal bundle is para-holomorphic, too. But  $N$  is canonical isomorphic to  $E$  which implies the para-holomorphicity of  $E$ .

Next we show  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ . Given  $s \in \mathcal{O}(E)$ , then for any  $\eta^\pm \in TE^\pm$  the pull back with  $s$ , i.e.  $s^*\eta^\pm$  is in  $T^*M^\pm$ . We test this on a basis of sections with the local expression of  $s$  given by  $(z^i) \mapsto (z^i, \zeta^\alpha(z^i))$

$$s^*(dz^i) = dz^i, \quad i = 1, \dots, 2n, \quad \alpha = 1, \dots, 2r \quad (4.1.8)$$

$$s^*(dw^\alpha + \sum_{\beta=1}^{2r} \omega^{0,1\alpha}_\beta w^\beta) = d\zeta^\alpha + \sum_{\beta=1}^{2r} \omega^{0,1\alpha}_\beta \zeta^\beta. \quad (4.1.9)$$

Splitting into degrees yields

$$\bar{\partial}\zeta^\alpha + \sum_{\beta=1}^{2r} \omega^{0,1\alpha}_\beta \zeta^\beta = 0, \quad \alpha = 1, \dots, r, \quad (4.1.10)$$

$$\partial\zeta^\alpha + \sum_{\beta=1}^{2r} \omega^{0,1\alpha}_\beta \zeta^\beta = 0, \quad \alpha = r+1, \dots, 2r. \quad (4.1.11)$$

This yields  $\nabla_{\tau_X s} = \tau^E \nabla_X s$ , i.e.  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ .

The uniqueness follows from remark 13.  $\square$

This proposition motivates the following

**Definition 30** *The subset of  $\mathcal{P}(E, J^E)$  consisting of all connections with vanishing  $(0, 2)$ -curvature will be called the set of **adapted connections** and denoted by  $\mathcal{P}^a(E, J^E)$ . The para-holomorphic structure induced on  $(E, J^E)$  by the element  $\nabla \in \mathcal{P}^a(E, J^E)$  will be referred to as  $\mathcal{E}^\nabla$ .*

**Corollary 5**

- (i) *A connection  $\nabla \in \mathcal{P}(E, J^E)$  which is para-holomorphic or of type  $(1, 1)$  induces on  $(E, J^E)$  a unique structure of a para-holomorphic vector bundle  $\mathcal{E}^\nabla$  with  $\nabla \in \mathcal{P}^a(E, \mathcal{E}^\nabla)$ .*
- (ii) *A connection  $\nabla \in \mathcal{P}(E, J^E)$  which is para-anti-holomorphic or of type  $(1, 1)$  induces on  $(E, -J^E)$  a unique structure of a para-holomorphic vector bundle denoted by  $\bar{\mathcal{E}}^\nabla$  with  $\nabla \in \mathcal{P}^a(E, \bar{\mathcal{E}}^\nabla)$ .*

*Proof:* This follows immediately from proposition 21, as the  $(0, 2)$ -curvature with respect to  $J^E$  respectively  $-J^E$  vanishes.  $\square$

**Proposition 22** *[Er3] Let  $(E, \mathcal{E})$  be a para-holomorphic vector bundle endowed with a para-hermitian metric  $h$ , then there exists a unique adapted  $(1, 1)$ -connection  $\nabla^h$  such that  $h$  is  $\nabla^h$ -parallel.*

**Remark 16** *In complex geometry this connection is called **canonical, hermitian or Chern-connection**. We will use this notation with the prefix *para*.*

**Corollary 6** *On a para-holomorphic vector bundle  $(E, \mathcal{E})$  admitting a para-hermitian metric  $h$  there exists a  $(1, 1)$ -connection  $\nabla$  which is adapted to  $\mathcal{E}$ .*

*Proof:* From proposition 22 we obtain a unique adapted  $(1, 1)$ -connection  $\nabla^h$  satisfying  $\nabla^h h = 0$ .  $\square$

For the constraints on para-complex vector-bundles admitting para-hermitian metrics we refer to [Bl].

On the tangent bundle  $TM$  of a para-complex manifold  $(M, J)$  it is natural to consider torsion-free para-complex connections. The set of all such connections will be denoted by  $\mathcal{P}_0(TM, J) \subset \mathcal{P}(TM, J)$ .

Like in the complex case one has the

**Proposition 23** *An almost para-complex manifold  $(M, J)$  is para-complex if and only if it admits a connection  $\nabla^0 \in \mathcal{P}_0(TM, J)$ .*

In fact in [S] was proven that every almost para-complex manifold admits an almost para-complex affine connection with torsion  $N_J = -4T$ , where  $N_J$  is the Nijenhuis-tensor. We denote by  $\mathcal{TM}$  the para-holomorphic structure of the tangent bundle  $(TM, J)$ .

**Proposition 24**  $\mathcal{P}_0(TM, J) \subset \mathcal{P}^a(TM, J)$ .

*Proof:* In fact simple calculations show that a vector field  $Z$  is para-holomorphic if and only if  $[JX, Z] = J[X, Z]$  for all  $X \in \Gamma(TM)$ . Moreover fixing  $\nabla^0 \in \mathcal{P}_0(TM, J)$ , the last equation is equivalent to  $\nabla_{JX}^0 Z = J\nabla_X^0 Z$  for all  $X \in \Gamma(TM)$ . This implies the claim.  $\square$

## 4.2 Decompositions of para-complex vector bundles

Throughout this section we denote by  $E$  a real vector bundle which is a direct sum of two subbundles  $E_1$  and  $E_2$ . As usual  $\iota_i : E_i \rightarrow E$  denote the inclusions of  $E_i$ , with  $i = 1, 2$ , and  $\pi_i : E \rightarrow E_i$  the projections. For these homomorphisms it holds:

$$\pi_i \circ \iota_i = id_{E_i}, \quad \pi_i \circ \iota_j = 0, \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = id_E.$$

We now consider the set of general connections on  $E$  denoted by  $\mathcal{C}(E)$ . In the following  $i$  and  $j$  are elements of  $\{1, 2\}$ . If both appear in the same expression it is meant  $i \neq j$ .

**Definition 31** Let  $\nabla \in \mathcal{C}(E)$  and define

$$\begin{aligned} \nabla_X^i s &:= (\pi_i \circ \nabla_X \circ \iota_i) s = \pi_i \nabla_X \iota_i s = \pi_i \nabla_X (\iota_i s), \\ B_X^i s &:= (\pi_j \circ \nabla_X \circ \iota_i) s = \pi_j \nabla_X \iota_i s = \pi_j \nabla_X (\iota_i s) \text{ for } X \in \Gamma(TM), s \in \Gamma(E). \end{aligned}$$

$\nabla^i \in \mathcal{C}(E_i)$  are called the induced connections on  $E_i$  and  $B^i \in \Omega^1(\text{Hom}(E_i, E_j))$  the second fundamental form of  $\nabla$  on  $E_i$ .

With these notions one has

$$\nabla_X s = \iota_1 \nabla_X^1 \pi_1 s + \iota_2 B_X^1 \pi_1 s + \iota_2 \nabla_X^2 \pi_2 s + \iota_1 B_X^2 \pi_2 s, \quad (4.2.1)$$

$$\nabla_X \iota_i s = \iota_i \nabla_X^i s + \iota_j B_X^i s, \text{ for } X \in \Gamma(TM), \quad (4.2.2)$$

$$\pi_i \nabla_X s = \nabla_X^i \pi_i s + B_X^j \pi_j s, \text{ for } X \in \Gamma(TM). \quad (4.2.3)$$

The second equation is a generalization of the Gauß and Weingarten equations of submanifolds theory. Further we have also a generalization of the corresponding integrability conditions:

**Lemma 22** Let again  $\nabla \in \mathcal{C}(E)$ ,  $\nabla^i \in \mathcal{C}(E_i)$  and  $R$  and  $R^i$  be the corresponding curvature tensors.

1. Then it holds:

$$\pi_i R_{X,Y} \iota_i = R_{X,Y}^i + B_X^j B_Y^i - B_Y^j B_X^i \text{ (Gauß and Ricci)}, \quad (4.2.4)$$

$$\pi_j R_{X,Y} \iota_i = B_X^i \nabla_Y^i + \nabla_X^j B_Y^i - B_Y^i \nabla_X^i - \nabla_Y^j B_X^i - B_{[X,Y]}^i \text{ (Codazzi)}, \quad (4.2.5)$$

for  $X, Y \in \Gamma(TM)$ .

2. Fixing now a torsion-free affine connection  $\nabla^M$  on  $TM$  then equation (4.2.5) is equivalent to

$$\pi_j R_{X,Y} \iota_i = (\hat{\nabla}_X^j B^i)_Y - (\hat{\nabla}_Y^j B^i)_X,$$

where

$$(\hat{\nabla}_X^j B^i)_Y \eta_i := \nabla_X^j (B_Y^i \eta_i) - B_Y^i \nabla_X^j \eta_i - B_{\nabla_X^j Y}^i \eta_i$$

for  $X, Y \in \Gamma(TM)$  and  $\eta_i \in \Gamma(E_i)$ .

Conversely we have the

**Lemma 23** *Let  $E_i, i = 1, 2$  be a vector bundle endowed with connection  $\nabla^i \in \mathcal{C}(E_i)$  and  $B^i \in \Omega^1(\text{Hom}(E_i, E_j)), i \neq j$ . Then*

$$\nabla := \iota_1 \nabla^1 \pi_1 + \iota_2 B^1 \pi_1 + \iota_2 \nabla^2 \pi_2 + \iota_1 B^2 \pi_2 \quad (4.2.6)$$

defines a connection on  $E := E_1 \oplus E_2$  such that  $\nabla^i$  is the connection induced by  $\nabla$  on  $E_i$  and  $B^i$  is the corresponding second fundamental form. In particular, the curvature tensors  $R$  and  $R^i$  of  $\nabla$  resp.  $\nabla^i$  satisfy the Gauß and Ricci and the Codazzi equations of Lemma 22.

Further if  $\nabla^i$  and  $B^i$  satisfy

$$R_{X,Y}^i + B_X^j B_Y^i - B_Y^j B_X^i = 0, \quad (4.2.7)$$

$$B_X^i \nabla_Y^i + \nabla_X^j B_Y^i - B_Y^i \nabla_X^i - \nabla_Y^j B_X^i - B_{[X,Y]}^i = 0, \quad (4.2.8)$$

$X, Y \in \Gamma(TM)$  then  $\nabla$  is flat.

We consider now a second vector bundle  $F = F_1 \oplus F_2$  over the same manifold  $M$ . Let  $\Phi \in \Gamma(\text{Aut}(E, F))$ . We say that  $\Phi$  preserves the decompositions if  $\Phi(E_i) = F_i, i = 1, 2$ . Moreover for such  $\Phi$  we define  $\Phi_i := \pi_i^F \circ \Phi \circ \iota_i^E$ .

**Lemma 24** *Let  $E = E_1 \oplus E_2$  and  $F = F_1 \oplus F_2$  be vector bundles over the same base manifold with a direct sum decomposition, a connection  $\nabla \in \mathcal{C}(E)$  (resp.  $\tilde{\nabla} \in \mathcal{C}(F)$ ), induced connections  $\nabla^i \in \mathcal{C}(E_i)$  (resp.  $\tilde{\nabla}^i \in \mathcal{C}(F_i)$ ) and second fundamental forms  $B^i \in \Omega^1(\text{Hom}(E_i, E_j)), i \neq j$  (resp.  $\tilde{B}^i \in \Omega^1(\text{Hom}(F_i, F_j)), i \neq j$ ). Let  $\Phi \in \Gamma(\text{Aut}(E, F))$  a decomposition preserving map. Then the following two conditions are equivalent:*

$$1. \Phi \nabla_X = \tilde{\nabla}_X \Phi,$$

$$2. \Phi_i \nabla_X^i = \tilde{\nabla}_X^i \Phi_i \quad \text{and} \quad \Phi_j B_X^i = \tilde{B}_X^i \Phi_i,$$

for all  $X \in \Gamma(TM)$ .

**Definition 32** *Let  $(E, J^E)$  be a para-complex vector bundle. A  $J^E$ -invariant subbundle  $E_i$  is called a para-complex subbundle if  $J_i = \pi_i \circ J^E \circ \iota_i$  defines a para-complex structure on  $E_i$ .*

**Remark 17** Let  $(E, J^E)$  be a para-complex vector bundle. A  $J^E$ -invariant subbundle  $E_i$  is a para-complex subbundle iff  $\text{tr}(J_i) = 0$ .

With lemma 24 we get the following

**Corollary 7** Let  $(E, J^E)$  be a para-complex vector bundle over  $(M, J)$  which decomposes into two para-complex subbundles  $E_1$  and  $E_2$ . Moreover let  $\nabla \in \mathcal{C}(E)$  and  $\nabla^i, B^i$  the induced data on  $E_i$ . Then  $\nabla \in \mathcal{P}(E, J^E)$  is equivalent to  $\nabla^i \in \mathcal{P}(E_i, J_i), B^i \in \Omega^1(\text{Hom}((E_i, J_i), (E_j, J_j))), i \neq j$ .

In the following we will always assume that subbundles are para-complex subbundles.

**Definition 33** Let  $(E, \mathcal{E})$  be a para-holomorphic vector bundle and  $(\tilde{E}, J^{\tilde{E}} := J^E|_{\tilde{E}})$  be a para-complex subbundle. We call  $\tilde{E}$  para-holomorphic subbundle of  $(E, \mathcal{E})$  if there exists a para-holomorphic structure  $\tilde{\mathcal{E}}$  on  $\tilde{E}$  such that the inclusion  $\iota_{\tilde{E}}$  is para-holomorphic with respect to  $\tilde{\mathcal{E}}$  and  $\mathcal{E}$ .

The decomposition  $E = E_1 \oplus E_2$  is called para-holomorphic decomposition if both  $E_i, i = 1, 2$  are para-holomorphic.

The structure of the para-holomorphic subbundle  $\tilde{E}$  is uniquely determined by the para-holomorphic structure  $\mathcal{E}$ . Therefore we will denote by  $(\tilde{E}, J^{\tilde{E}})$  a para-holomorphic subbundle of  $(E, \mathcal{E})$ . To a para-complex bundle  $(E, J^E)$  with a connection  $\nabla \in \mathcal{P}^a(E, J^E)$  we associated in section 4.1 the para-holomorphic vector bundle  $\mathcal{E}^\nabla$ . In this situation we call a subbundle  $(S, J^S)$  of  $E$  para-holomorphic if it is para-holomorphic with respect to  $\mathcal{E}^\nabla$ .

We are now going to study decompositions  $E = E_1 \oplus E_2$ , where some of the (sub)bundles are para-holomorphic and extract informations about the data given by the immersion.

**Proposition 25**

1. Let  $(E, J^E)$  be a para-complex vector bundle. Let  $\nabla \in \mathcal{P}^a(E, J^E)$  and  $B^1$  be para-complex, i.e.  $B_{jX}^1 = J^E B_X^1$ . Then  $\nabla^i \in \mathcal{P}^a(E_i, J_i)$  and  $E_1$  is a para-holomorphic subbundle of  $E$ .
2. Let  $(E, \mathcal{E})$  be a para-holomorphic vector bundle,  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$  and  $(E_1, \mathcal{E}_1)$  a para-holomorphic subbundle of  $(E, \mathcal{E})$ . Then  $\nabla^i \in \mathcal{P}^a(E_i, J_i)$  for  $i = 1, 2$ ,  $B^1$  is para-complex and  $\mathcal{E}_1 = \mathcal{E}_1^{\nabla^1}$ .
3. In particular, given  $\nabla \in \mathcal{P}^a(E, J^E)$  and a para-holomorphic decomposition of  $E = E_1 \oplus E_2$ , then the Gauß maps  $B^i$  are para-complex and  $\nabla^i \in \mathcal{P}^a(E_i, J_i)$  for  $i = 1, 2$ .

*Proof:*

1. As  $\nabla \in \mathcal{P}^a(E, J^E) \subset \mathcal{P}(E, J^E)$  we get with corollary 7 that  $\nabla^i \in \mathcal{P}(E_i, J_i)$ . Since  $\nabla \in \mathcal{P}^a(E, J^E)$  the  $(0, 2)$ -curvature of  $\nabla$  vanishes. With equation (4.2.4) we get then

$$\begin{aligned}
(R_{X,Y}^i)^{0,2} &= R_{X,Y}^i - \tau R_{\tau X,Y}^i - \tau R_{X,\tau Y}^i + R_{\tau X,\tau Y}^i \\
&= \pi_i R_{X,Y} \iota_i - B_X^j B_Y^i + B_Y^j B_X^i - \tau(\pi_i R_{\tau X,Y} \iota_i - B_{\tau X}^j B_Y^i + B_Y^j B_{\tau X}^i) \\
&\quad - \tau(\pi_i R_{X,\tau Y} \iota_i - B_X^j B_{\tau Y}^i + B_{\tau Y}^j B_X^i) + \pi_i R_{\tau X,\tau Y} \iota_i - B_{\tau X}^j B_{\tau Y}^i + B_{\tau Y}^j B_{\tau X}^i \\
&= \pi_i \underbrace{(R_{X,Y} - \tau R_{\tau X,Y} - \tau R_{X,\tau Y} + R_{\tau X,\tau Y})}_{=0} \iota_i - B_X^j B_Y^i + B_Y^j B_X^i \\
&\quad + \tau B_{\tau X}^j B_Y^i - \tau B_Y^j B_{\tau X}^i + \tau B_X^j B_{\tau Y}^i - \tau B_{\tau Y}^j B_X^i - B_{\tau X}^j B_{\tau Y}^i + B_{\tau Y}^j B_{\tau X}^i.
\end{aligned}$$

Thus it is clear that if  $B^i$  or  $B^j$  is para-complex  $(R^i)^{0,2}$  vanishes for  $i = 1, 2$ . This implies  $\nabla^i \in \mathcal{P}^a(E_i, J_i)$  for  $i = 1, 2$  because  $B^1$  is para-complex.

We take now an open subset  $U \subset M$ ,  $\eta_1 \in \mathcal{O}(E_1|_U)$  and  $X \in \Gamma(TM|_U)$  and compute

$$\nabla_{JX} \iota_1 \eta_1 = \nabla_{JX}^1 \eta_1 + B_{JX}^1 \eta_1 = J_1 \nabla_X^1 \eta_1 + J_2 B_X^1 \eta_1 = J^E \nabla_X \iota_1 \eta_1.$$

But using lemma 17 and lemma 19 yield that  $\iota_1$  is para-holomorphic with respect to  $\mathcal{E}_1^{\nabla^1}$  and  $\mathcal{E}^\nabla$ .

2. Let  $U \subset M$  be open. Again due to lemma 17 from  $\eta_1 \in \mathcal{O}(E_1|_U)$  we get  $\iota_1 \eta_1 \in \mathcal{O}(E|_U)$ . Using this we calculate

$$\begin{aligned}
\nabla_{JX}^1 \eta_1 + B_{JX}^1 \eta_1 &= \nabla_{JX} \iota_1 \eta_1 = J^E \nabla_X \iota_1 \eta_1 \\
&= J^E \nabla_X^1 \eta_1 + J^E B_X^1 \eta_1 = J_1 \nabla_X^1 \eta_1 + J_2 B_X^1 \eta_1
\end{aligned}$$

for any  $X \in \Gamma(TM|_U)$ . This yields, that  $B^1$  is para-complex,  $\nabla^1 \in \mathcal{P}^a(E_1, \mathcal{E}_1) \subset \mathcal{P}^a(E_1, J_1)$  and  $\mathcal{E}_1 = \mathcal{E}_1^{\nabla^1}$ . As  $B^1$  is para-complex and  $\nabla \in \mathcal{P}^a(E, \mathcal{E})$ , we obtain  $\nabla^2 \in \mathcal{P}^a(E_2, J_2)$ .

3. follows easily from 2. □

**Proposition 26** *Let  $\nabla \in \mathcal{P}^a(E, J^E)$  be para-holomorphic and  $B^1$  be para-complex:*

- (i)  $\nabla^1$  is para-holomorphic if and only if

$$(B_{JX}^2 - J_1 B_X^2) B_Y^1 = 0 \tag{4.2.9}$$

for  $X, Y \in \Gamma(TM)$ .

- (ii)  $\nabla^2$  is para-holomorphic if and only if

$$B_Y^1 (B_{JX}^2 - J_1 B_X^2) = 0 \tag{4.2.10}$$

for  $X, Y \in \Gamma(TM)$ .



*Proof:*  $\nabla \in \mathcal{P}^a(E, J^E)$  is para-holomorphic, then  $R_{JX,Y} = J^E R_{X,Y}$ . Let  $(i, j) \in \{(1, 2); (2, 1)\}$ . Further with equation (4.2.4) it holds

$$\begin{aligned} J^E \pi_i R_{X,Y} \iota_i &= J_i R_{X,Y}^i + J_i B_X^j B_Y^i - J_i B_Y^j B_X^i \text{ and} \\ \pi_i R_{JX,Y} \iota_i &= R_{JX,Y}^i + B_{JX}^j B_Y^i - B_Y^j B_{JX}^i. \end{aligned}$$

Then subtracting these equations we get

$$\begin{aligned} J_i R_{X,Y}^i - R_{JX,Y}^i &= (B_{JX}^j B_Y^i - B_Y^j B_{JX}^i) - (J_i B_X^j B_Y^i - J_i B_Y^j B_X^i) \\ &= \begin{cases} (B_{JX}^j - J_i B_X^j) B_Y^i, & \text{if } B^i \text{ is para-complex,} \\ -B_Y^j (B_{JX}^i - J_i B_X^i), & \text{if } B^j \text{ is para-complex.} \end{cases} \end{aligned}$$

Setting  $i = 1$  yields (i) and setting  $j = 1$  yields (ii).  $\square$

**Corollary 8** *Let  $\nabla \in \mathcal{P}^a(E, J^E)$  be para-holomorphic and  $B^1, B^2$  be para-complex, then  $\nabla^i, i = 1, 2$  are para-holomorphic and the decomposition  $E = E_1 \oplus E_2$  is para-holomorphic.*

*Proof:* Proposition 25.1 shows that  $E_i$  are para-holomorphic and  $\nabla^i \in \mathcal{P}^a(E_i, J_i)$ . Since  $B^1$  and  $B^2$  are para-complex the brackets in equations (4.2.9) and (4.2.10) vanish. This shows that  $\nabla^i$  are para-holomorphic.  $\square$

**Corollary 9** *Let  $\nabla \in \mathcal{P}^a(E, J^E)$  be para-holomorphic and the decomposition  $E = E_1 \oplus E_2$  be also para-holomorphic, then  $\nabla^1$  and  $\nabla^2$  are para-holomorphic.*

*Proof:* From propositions 25.3 follows, that  $B^i$  for  $i = 1, 2$  are para-complex. This and proposition 26 (the last corollary, respectively) show that  $\nabla^i$  are para-holomorphic.  $\square$

**Proposition 27** *Further let  $\nabla \in \mathcal{P}^a(E, J^E)$  be a  $(1, 1)$ -connection and  $B^1$  be para-complex. Then it holds*

1.  $\nabla^1$  is of type  $(1, 1)$  if and only if

$$(J_1 B_{JX}^2 + B_X^2) B_Y^1 - (J_1 B_{JY}^2 + B_Y^2) B_X^1 = 0,$$

for all  $X, Y \in \Gamma(TM)$ .

2.  $\nabla^2$  is of type  $(1, 1)$  if and only if

$$B_X^1 (J_1 B_{JY}^2 + B_Y^2) - B_Y^1 (J_1 B_{JX}^2 + B_X^2) = 0,$$

for all  $X, Y \in \Gamma(TM)$ .

*Proof:*  $\nabla \in \mathcal{P}^a(E, J^E)$  and of type  $(1, 1)$ , i.e.  $R_{JX,JY} = -R_{X,Y}$ . Let  $(i, j) \in \{(1, 2); (2, 1)\}$ . Further with equation (4.2.4) it holds

$$\begin{aligned} \pi_i R_{X,Y} \iota_i &= R_{X,Y}^i + B_X^j B_Y^i - B_Y^j B_X^i \text{ and} \\ \pi_i R_{JX,JY} \iota_i &= R_{JX,JY}^i + B_{JX}^j B_{JY}^i - B_{JY}^j B_{JX}^i. \end{aligned}$$

$$\begin{aligned}
-(R_{X,Y}^i + R_{JX,JY}^i) &= B_X^j B_Y^i - B_Y^j B_X^i + B_{JX}^j B_{JY}^i - B_{JY}^j B_{JX}^i \\
&= \begin{cases} (J_i B_{JX}^j + B_X^j) B_Y^i - (J_i B_{JY}^j + B_Y^j) B_X^i, & \text{if } B^i \text{ is para-complex,} \\ B_X^j (J_j B_{JY}^i + B_Y^i) - B_Y^j (J_j B_{JX}^i + B_X^i), & \text{if } B^j \text{ is para-complex,} \end{cases}
\end{aligned}$$

which yields the proposition.  $\square$

This leads with proposition 25.3 to the following

**Corollary 10** *Let  $\nabla \in \mathcal{P}^a(E, J^E)$  be a  $(1, 1)$ -connection. If the decomposition  $E = E_1 \oplus E_2$  is para-holomorphic,  $\nabla^i$  is of type  $(1, 1)$  if and only if*

$$B_X^j B_Y^i - B_Y^j B_X^i = 0,$$

for all  $X, Y \in \Gamma(TM)$ .

**Lemma 25** *Let  $\nabla \in \mathcal{P}^a(E, J^E)$  be a  $(1, 1)$ -connection. Then*

- (i) *If  $B^1$  is para-complex and  $B^2$  para-anti-complex, then  $\nabla^1$  and  $\nabla^2$  are  $(1, 1)$ -connections and  $(E_2, \bar{\mathcal{E}}_2^{\nabla^2})$  is a para-holomorphic subbundle of  $(E, \bar{\mathcal{E}}^\nabla)$ .*
- (ii) *Conversely if  $(E_1, J_1)$  is a para-holomorphic subbundle of  $(E, \mathcal{E}^\nabla)$  and  $(E_2, -J_2)$  is a para-holomorphic subbundle of  $(E, \bar{\mathcal{E}}^\nabla)$ , then  $\nabla^1$  and  $\nabla^2$  are  $(1, 1)$ -connections,  $B^1$  is para-complex and  $B^2$  is para-anti-complex.*

*Proof:*

- (i) The first part follows easily from proposition 27. The second part follows from proposition 25.1 applied to the para-complex bundle  $(E, -J^E)$ .
- (ii) Proposition 25.2 yields that  $B^1$  is para-complex,  $B^2$  is para-anti-complex and 27 finishes the proof.  $\square$

In the end of this section we consider para-complex and para-holomorphic vector bundles with para-hermitian metrics.

**Lemma 26** *Let  $(E, J^E)$  be a para-complex vector bundle with a para-hermitian metric  $h$  and  $\nabla \in \mathcal{P}(E, J^E)$  a metric connection with respect to  $h$ , i.e.  $\nabla_X h = 0$ .*

- (i) *If the decomposition  $E = E_1 \oplus E_2$  is  $J^E$ -invariant and  $h$ -orthogonal, then it holds*

$$h(B_X^1 \eta_1, \eta_2) + h(\eta_1, B_X^2 \eta_2) = 0 \tag{4.2.11}$$

for  $X \in \Gamma(TM)$  and  $\eta_i \in \Gamma(E_i)$ .

- (ii) *If  $\nabla \in \mathcal{P}^a(E, J^E)$ , then  $\nabla h = 0$  yields that  $\nabla$  is of type  $(1, 1)$ , i.e.  $R_{JX,JY} = -R_{X,Y}$ .*

(iii) If  $\nabla \in \mathcal{P}^a(E, J^E)$  and  $E_1$  is a para-holomorphic subbundle of  $E$ , then

$$B^2 \in \Omega^{0,1}(\text{Hom}((E_2, J_2), (E_1, J_1))). \quad (4.2.12)$$

*Proof:*

(i) It holds for  $\eta_1 \in \Gamma(E_1)$ ,  $\eta_2 \in \Gamma(E_1)$

$$\begin{aligned} 0 &= h(\nabla_X \iota_1 \eta_1, \iota_2 \eta_2) + h(\iota_1 \eta_1, \nabla_X \iota_2 \eta_2) \\ &\stackrel{(4.2.2)}{=} h(\iota_1 \nabla_X^1 \eta_1, \iota_2 \eta_2) + h(\iota_2 B^1 \eta_1, \iota_2 \eta_2) + h(\iota_1 \eta_1, \iota_2 \nabla_X^2 \eta_2) + h(\iota_1 \eta_1, \iota_1 B^2 \eta_2), \end{aligned}$$

which yields the claim with the  $h$ -orthogonality of the decomposition  $E_1 \oplus E_2$ .

(ii) Follows from deriving

$$ZW(h(e, f)) = h(\nabla_Z \nabla_W e, f) + h(\nabla_W e, \nabla_{\bar{Z}} f) + h(\nabla_Z e, \nabla_{\bar{W}} f) + h(e, \nabla_{\bar{Z}} \nabla_{\bar{W}} f)$$

for a  $(1, 0)$  coordinate vector fields  $Z, W$  and  $e, f \in \Gamma(E)$ . Antisymmetrization in  $Z, W$  yields

$$0 = h(R_{Z,W} e, f) + h(e, R_{\bar{Z}, \bar{W}} f)$$

and  $R_{Z,W} e = 0$  follows from  $R_{\bar{Z}, \bar{W}} f = 0$ .

(iii) Follows from equation (4.2.11) and proposition 25.1 by the calculation  $\forall e \in \Gamma(E_1)$  and  $f \in \Gamma(E_2)$

$$\begin{aligned} -h(e, B_{JX}^2 f) &= h(B_{JX}^1 e, f) = h(J^E B_X^1 e, f) = -h(B_X^1 e, J^E f) \\ &= h(e, B_X^2 J^E f) = h(e, J^E B_X^2 f). \end{aligned}$$

□

**Corollary 11** *Let  $(E, J^E, h)$  be a para-hermitian vector bundle,  $\nabla \in \mathcal{P}^a(E, J^E)$  a metric connection with respect to  $h$ ,  $E_1$  a para-holomorphic subbundle of  $E$  and  $(E_2, J_2)$  the orthogonal complement of  $E_1$  with respect to  $h$ .*

*Then the decomposition  $E = E_1 \oplus E_2$  is para-holomorphic if and only if*

$$B^1 = 0 \text{ and } B^2 = 0. \quad (4.2.13)$$

*Proof:* ' $\Rightarrow$ ' Due to the of proposition 25.3 we know that  $B^1$  and  $B^2$  are para-complex. Equation (4.2.12) yields  $B^2 = 0$  and equation (4.2.11)  $B^1 = 0$ .

' $\Leftarrow$ '  $B^1 = 0$  and hence is para-complex. Now, proposition 26 finishes the proof. □

### 4.3 Para-complex affine immersions

In this section, we recall notions of affine immersions and introduce the notion of a para-complex affine immersion with transversal bundle.

Let  $f : M \rightarrow M'$  be a smooth map between two manifolds  $M$  and  $M'$ . Denote the canonical bundle map by  $F : f^*TM' \rightarrow TM'$  and define

$$\begin{aligned} i^f & : TM \rightarrow f^*TM', \\ i^f & := F^{-1}f_*. \end{aligned}$$

In the rest of the section we consider the case, where  $f$  is an immersion.

**Definition 34** We call  $f$  immersion with transversal bundle  $N$  if

$$f^*TM' = i^fTM \oplus N \quad (4.3.1)$$

for some subbundle  $N$  of  $f^*TM'$ . Given an immersion with transversal bundle  $N$ , we denote the canonical maps according to the direct sum (4.3.1) by  $\iota_f, \iota_N$  and  $\pi_f, \pi_N$ , i.e.

$$\pi_f : f^*TM' = i^fTM \oplus N \rightarrow i^fTM, \quad (4.3.2)$$

$$\pi_N : f^*TM' = i^fTM \oplus N \rightarrow N \quad (4.3.3)$$

and

$$\iota_f : i^fTM \rightarrow f^*TM' = i^fTM \oplus N, \quad (4.3.4)$$

$$\iota_N : N \rightarrow f^*TM' = i^fTM \oplus N. \quad (4.3.5)$$

**Definition 35** Let  $M, M'$  be two manifolds endowed with torsion-free affine connections  $\nabla, \nabla'$ ,  $f : M \rightarrow M'$  a smooth map and  $f^*\nabla'$  the pull-back connection of  $\nabla'$ . Then  $f$  is called affine immersion with transversal bundle  $N$ , if the induced connection  $\pi_f \circ f^*\nabla' \circ \iota_f$  on  $i^fTM$  coincides with  $\hat{i}^f \circ \nabla \circ (\hat{i}^f)^{-1}$ , where  $\hat{i}^f = \pi_f \circ i^f \in \Gamma(\text{Iso}(TM, i^fTM))$ . For an affine immersion with transversal bundle  $N$  one defines the affine fundamental form  $B \in \Omega^1(\text{Hom}(TM, N))$  by

$$B := \pi_N (f^*\nabla') \iota_f \hat{i}^f = B^f \hat{i}^f$$

and the shape tensor  $A \in \Omega^1(\text{Hom}(N, TM))$  by

$$A := -(\hat{i}^f)^{-1} \pi_f (f^*\nabla') \iota_N = -(\hat{i}^f)^{-1} B^N.$$

Moreover we denote by  $\nabla^N := \pi_N (f^*\nabla') \iota_N$  the transversal connection.

The following commutative diagram illustrates the above objects:

$$\begin{array}{ccccc} & & i^fTM & & N \\ & & \uparrow & & \uparrow \\ & & \iota_f & & \iota_N \\ & & \swarrow & & \searrow \\ & & i^fTM \oplus N & & \\ & & \downarrow \pi_f & & \downarrow \pi_N \\ & & f^*TM' & \xrightarrow[\sim]{F} & TM' \\ & & \downarrow & & \downarrow \\ & & (M, \nabla) & \xrightarrow{f} & (M', \nabla') \\ & & \uparrow & & \uparrow \\ & & TM & \xrightarrow{f_*} & TM' \\ & & \uparrow i^f & & \\ & & \hat{i}^f = \pi_f \circ i^f & & \end{array}$$

We remark that  $B$  is symmetric by the torsion-freeness of  $\nabla'$ . One can now write the Gauß and Weingarten equation for such immersions

$$(f^*\nabla')_X i^f Y = i^f \nabla_X Y + \iota_N B_X Y \quad (4.3.6)$$

$$(f^*\nabla')_X (\iota_N \xi) = -i^f A_X \xi + \iota_N \nabla_X^N \xi, \quad (4.3.7)$$

for  $X, Y \in \Gamma(TM)$  and  $\xi \in \Gamma(N)$ .

Due to lemma 22 we get with decomposition (4.3.1) the following

**Corollary 12** *Let  $f$  be an affine immersion with transversal bundle  $N$ , then the Gauß, Codazzi and Ricci equations are given by*

$$(\hat{i}^f)^{-1} \pi_f R'_{X,Y} i^f Z = R_{X,Y} Z - A_X B_Y Z + A_Y B_X Z, \quad (4.3.8)$$

$$\pi_N R'_{X,Y} i^f Z = (\hat{\nabla}_X B)_Y Z - (\hat{\nabla}_Y B)_X Z, \quad (4.3.9)$$

$$(\hat{i}^f)^{-1} \pi_f R'_{X,Y} \xi = -(\hat{\nabla}_X A)_Y \xi + (\hat{\nabla}_Y A)_X \xi, \quad (4.3.10)$$

$$\pi_N R'_{X,Y} \xi = R_{X,Y}^N \xi - B_X A_Y \xi + B_Y A_X \xi, \quad (4.3.11)$$

for  $X, Y, Z \in \Gamma(TM)$  and  $\xi \in \Gamma(N)$ , where  $R', R, R^N$  are the curvatures of  $(f^*\nabla')$ ,  $\nabla$ ,  $\nabla^N$  respectively and

$$(\hat{\nabla}_X B)_Y Z := \nabla_X^N (B_Y Z) - B_{\nabla_X Y} Z - B_Y \nabla_X Z,$$

$$(\hat{\nabla}_X A)_Y \xi := \nabla_X (A_Y \xi) - A_{\nabla_X Y} \xi - A_Y \nabla_X^N \xi.$$

Let  $(M, J)$  and  $(M', J')$  be para-complex manifolds. An immersion  $f : M \rightarrow M'$  is said to be **para-holomorphic** if  $f_* J = J' f_*$ . In the remaining part of this section we denote by  $(M, J, \nabla)$  (resp.  $(M', J', \nabla')$ ) para-complex manifolds  $(M, J)$  (resp.  $(M', J')$ ) with a connection  $\nabla \in \mathcal{P}_0(TM, J) \subset \mathcal{P}^a(TM, J)$  (resp.  $\nabla' \in \mathcal{P}_0(TM', J') \subset \mathcal{P}^a(TM', J')$ ). According to section 4.1  $\nabla$  and  $\nabla'$  are adapted to the canonical para-holomorphic vector bundle structures  $\mathcal{P}_M = \mathcal{T}\mathcal{M}^\nabla$  and  $\mathcal{P}_{M'} = \mathcal{T}\mathcal{M}'^{\nabla'}$ .

**Definition 36** *Let  $(M, J)$  be a para-complex manifold. A torsion-free para-complex connection of type  $(1, 1)$  is called **affine para-Kähler connection**.*

We are now going to apply the results of the last two sections to this situation.

**Definition 37** *Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be an affine para-holomorphic immersion with transversal bundle  $N$ , such that  $N$  is an  $f^* J'$ -invariant subbundle of  $f^* TM'$ , then  $f$  is called **para-complex affine immersion with transversal bundle  $N$**  and the induced para-complex structure is denoted by  $J^N = \pi_N(f^* J') \iota_N$ .*

The para-holomorphicity of  $f$  yields the following relation

$$(f^* J') i^f = i^f J. \quad (4.3.12)$$

The pull-back bundle  $f^* TM'$  carries a para-holomorphic structure induced by the para-holomorphic map  $f$  and it holds

$$f^* \nabla' \in \mathcal{P}^a(f^* TM', f^* \mathcal{P}_{M'}).$$

In particular one obtains easily from (4.3.12) the identity

$$f^*\mathcal{P}_{M'} = \mathcal{T}^{f^*\nabla'} = f^*\mathcal{T}\mathcal{M}'^{f^*\nabla'}$$

with  $T = f^*TM'$ .

**Lemma 27** *Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$ . Then it holds*

$$\begin{aligned} \nabla^N \in \mathcal{P}^a(N, J^N), \quad B \in \Omega^{1,0}(\text{Hom}((TM, J), (N, J^N))), \quad \text{i.e. } B \text{ para-complex,} \\ A \in \Omega^1(\text{Hom}((N, J^N), (TM, J))). \end{aligned}$$

*Proof:* We obtain with corollary 7 that  $\nabla^N \in \mathcal{P}(N, J^N)$ ,  $B \in \Omega^1(\text{Hom}((TM, J), (N, J^N)))$  and  $A \in \Omega^1(\text{Hom}((N, J^N), (TM, J)))$ . As  $\nabla' \in \mathcal{P}_0(TM', J') \subset \mathcal{P}^a(TM', J')$ , the second part of proposition 25 finishes the proof.  $\square$

The fact that  $B$  is para-complex and proposition 26 yield

**Proposition 28** *Given a para-complex affine immersion  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  with transversal bundle  $N$  and para-holomorphic connection  $\nabla'$ . Then*

(i)  $\nabla$  is para-holomorphic if and only if

$$(A_{JX} - JA_X)B_Y = 0$$

for  $X, Y \in \Gamma(TM)$ .

(ii)  $\nabla^N$  is para-holomorphic if and only if

$$B_Y(A_{JX} - JA_X) = 0$$

for  $X, Y \in \Gamma(TM)$ .

Moreover, proposition 27 yields

**Proposition 29** *Given a para-complex affine immersion  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  with transversal bundle  $N$ . If  $\nabla'$  is in addition affine para-Kähler, then*

(i)  $\nabla$  is affine para-Kähler if and only if

$$(JA_{JX} + A_X)B_Y - (JA_{JY} + A_Y)B_X = 0,$$

for  $X, Y \in \Gamma(TM)$ .

(ii)  $\nabla^N$  is of type  $(1, 1)$  if and only if

$$B_X(JA_{JY} + A_Y) - B_Y(JA_{JX} + A_X) = 0,$$

for  $X, Y \in \Gamma(TM)$ .

**Definition 38** Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$ .  $f$  is said to be a **para-holomorphic affine immersion**, if  $N$  is a para-holomorphic subbundle of  $(f^*TM, f^*\mathcal{T}\mathcal{M}'^{f^*\nabla'})$ .

We observe that this property of  $f$  is equivalent to the para-holomorphicity of the decomposition (4.3.1).

Therefore we apply proposition 25 to find

**Proposition 30** A para-complex affine immersion  $f$  with transversal bundle  $N$  is a para-holomorphic affine immersion with transversal bundle  $N$  if and only if

$$A \in \Omega^{1,0}(\text{Hom}((N, J^N), (TM, J))). \quad (4.3.13)$$

In addition, from corollary 9 we get

**Proposition 31** Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-holomorphic affine immersion with transversal bundle  $N$ . If  $\nabla'$  is para-holomorphic, then the connections  $\nabla$  and  $\nabla^N$  are para-holomorphic too.

Applying corollary 8 and lemma 27 implies

**Proposition 32** Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$ . Assume, that  $\nabla'$  is para-holomorphic and  $A$  para-complex, then  $\nabla$  and  $\nabla^N$  are para-holomorphic and  $f$  is a para-holomorphic immersion.

**Definition 39** Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be an affine immersion with transversal bundle  $N$ . Then one defines the **first normal space** at  $x \in M$  to be

$$N_1(x) := \text{span}\{B_X Y \mid X, Y \in T_x M\}.$$

With this notion we formulate

**Proposition 33** Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$  and  $\nabla'$  be para-holomorphic.

(i) If  $\nabla$  is para-holomorphic, then one has

$$(R_{JX,Y}^N - J^N R_{X,Y}^N)B_Z = 0$$

with  $X, Y, Z \in \Gamma(TM)$ .

Suppose that  $N_1(x) = N_x$  for all  $x \in M$ , then  $A$  is para-complex,  $\nabla^N$  is para-holomorphic and  $f$  is a para-holomorphic affine immersion.

(ii) If  $\nabla^N$  is para-holomorphic, then one has

$$B_Z(R_{JX,Y} - J^N R_{X,Y}) = 0$$

with  $X, Y, Z \in \Gamma(TM)$ .

Moreover, if for any  $x \in M$  one find a  $Z_x \in T_x M$  such that  $\text{Ker} B_{Z_x} = \{0\}$ , then  $A$  is para-complex and  $f$  is a para-holomorphic affine immersion.

*Proof:*

- (i) Proposition 28 (i) implies using the condition on the first normal space that  $A$  is para-complex. By proposition 30  $f$  is para-holomorphic. Proposition 28 (ii) implies that  $\nabla^N$  is para-holomorphic.

Equation (4.3.11) yields

$$\pi_N R'_{X,Y} \iota_N B_Z = R_{X,Y}^N B_Z - (B_X A_Y - B_Y A_X) B_Z$$

and therefore using the fact that  $R'_{X,Y}$  and  $B$  are para-complex

$$\begin{aligned} 0 &= (R_{JX,Y}^N - J^N R_{X,Y}^N) B_Z - (B_{JX} A_Y - B_Y A_{JX}) B_Z \\ &+ J^N (B_X A_Y - B_Y A_X) B_Z \\ &= (R_{JX,Y}^N - J^N R_{X,Y}^N) B_Z - B_Y (J A_X - A_{JX}) B_Z \\ &= (R_{JX,Y}^N - J^N R_{X,Y}^N) B_Z. \end{aligned}$$

The last equality follows from proposition 28.

Under the condition on the first normal space the last equations yields

$$R_{JX,Y}^N - J^N R_{X,Y}^N = 0.$$

- (ii) The claimed equation follows from an analogous calculation and the rest follows likely.  $\square$

Now we want to consider the special case where the real codimension of the immersion is two.

**Corollary 13** *Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$  of real codimension two and  $\nabla'$  and  $\nabla$  para-holomorphic connections. Then  $\nabla^N$  is para-holomorphic.*

*Proof:* We have to consider two cases. At first let  $x \in M$  be a geodesic point, i.e. the affine second fundamental form  $B_Z$  vanishes at this point for all  $Z \in T_x M$ , then the Ricci-equation (4.3.11) implies

$$R_{JX,Y}^N = J^N R_{X,Y}^N$$

at  $x \in M$  for all  $X, Y \in T_x M$ . If  $x \in M$  is not a geodesic point, we can apply proposition 33, the condition on the normal space holds, as the codimension is two and  $x$  is not geodesic. This shows that  $\nabla^N$  is para-holomorphic.  $\square$

Since the codimension is two we are able to chose a local frame  $\xi$ ,  $J^N \xi \in \Gamma(N|_U)$  of  $N$  over an open set  $U$ . This gives the local formula

$$B_X Y = h^1(X, Y) \xi + h^2(X, Y) J^N \xi$$

with  $X, Y \in \Gamma(TM|_U)$ , where  $h^1$  and  $h^2$  are symmetric bilinear forms. Since  $B$  is para-complex (lemma 27) and symmetric, we see

$$h^1(X, JY) = h^1(JX, Y) = h^2(X, Y)$$



for any  $X, Y \in \Gamma(TM|_U)$  and consequently  $\text{rank } h^1 = \text{rank } h^2$ .

This motivates the definition:

The **type number** of an immersion  $f$  at a point  $x \in M$  is the rank of  $h^1$  at  $x$  and is denoted by  $tn(f(x))$ . We observe that  $tn(f(x))$  does not depend on the choice of the local frame field on  $N$ . Using the proposition 33 we conclude

**Corollary 14** *We make the same assumptions as in corollary 13. If  $tn(f(x)) > 0$  for each  $x \in M$ , then  $A$  is para-complex,  $\nabla^N$  is para-holomorphic and  $f$  is a para-holomorphic affine immersion.*

Application of corollary 10 shows

**Proposition 34** *Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-holomorphic affine immersion with transversal bundle  $N$  and suppose that  $\nabla'$  is affine para-Kähler. Then  $A$  is para-complex and*

(i)  $\nabla$  is affine para-Kähler if and only if

$$A_X B_Y - A_Y B_X = 0$$

for  $X, Y \in \Gamma(TM)$ .

(ii)  $\nabla^N$  is of type  $(1, 1)$  if and only if

$$B_X A_Y - B_Y A_X = 0$$

for  $X, Y \in \Gamma(TM)$ .

If  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  is a para-complex affine immersion with transversal bundle  $N$  such that  $\nabla'$  is affine para-Kähler, then  $f$  induces a para-holomorphic structure  $\mathcal{P}^{f*\nabla'}$  on  $(f^*TM', f^*J')$  and  $\mathcal{A}^{f*\nabla'}$  on  $(f^*TM', -f^*J')$ .

With this notion and lemma 25 we obtain

**Proposition 35** *Let  $f : (M, J, \nabla) \rightarrow (M', J', \nabla')$  be a para-complex affine immersion with transversal bundle  $N$  and  $\nabla'$  be affine para-Kähler. Then*

(i) *If  $A$  is para-anti-complex, then  $\nabla$  is affine para-Kähler and  $\nabla^N$  is of type  $(1, 1)$ . Moreover  $(N, \mathcal{N}^{\nabla^N})$  is a para-holomorphic subbundle of  $(f^*TM', \mathcal{A}^{f*\nabla'})$ .*

(ii) *Conversely if  $(i^f TM, \pi_f(f^*J)\iota_f)$  is a para-holomorphic subbundle of  $(f^*TM', \mathcal{P}^{f*\nabla'})$  and  $(N, -J^N)$  is a para-holomorphic subbundle of  $(f^*TM', \mathcal{A}^{f*\nabla'})$ , then  $\nabla$  is affine para-Kähler and  $\nabla^N$  of type  $(1, 1)$ . Moreover  $A$  is para-anti-complex.*

When  $(N, -J^N)$  is a para-holomorphic subbundle of  $(f^*TM', \mathcal{A}^{f*\nabla'})$ , then the above proposition yields that  $\nabla$  is affine para-Kähler and  $\nabla^N$  is of type  $(1, 1)$ .

We now consider the case  $(M', J', D') = (\mathbb{R}^{2(n+1)}, J'_{can.}, D'_{can.})$ . Then we are in the situation of the para-complex analogue of so called affine Kähler immersions (compare [NPP]):

**Definition 40** *Let  $f : (M, J, \nabla) \rightarrow (\mathbb{R}^{2(n+1)}, J'_{can.}, D'_{can.})$  be a para-complex affine immersion with transversal bundle  $N$  and  $\nabla'$  be affine para-Kähler and let  $(N, -J^N)$  be para-holomorphic subbundle of  $(f^*TM', \mathcal{A}^{f*\nabla'})$ . Then we call  $f$  an affine para-Kähler immersion.*

## 4.4 The fundamental theorems of para-complex affine immersions

In this section we generalize the equivalence and existence theorems of complex affine immersions to para-complex geometry by adapting the methods of [AK], [H] and [O].

We consider the  $n$ -dimensional real affine space  $\mathbb{R}^n$  with its standard basis  $(e_i)_{i=1}^n$  and  $(E_i)_{i=1}^n$  the corresponding frame of parallel vector fields for the standard affine flat connection  $D$ . A map  $f : M \rightarrow \mathbb{R}^n$  from a smooth manifold  $M$  into  $\mathbb{R}^n$  can be expressed as

$$f = \sum_{i=1}^n f^i e_i$$

where the  $f^i$ ,  $i = 1, \dots, n$  are smooth functions. This yields at  $x \in M$  with  $X \in T_x M$

$$f_* X = (df^i)(X)(E_i)_{f(x)}. \quad (4.4.1)$$

We recall the definition of  $i^f : TM \rightarrow f^* T\mathbb{R}^n$  by  $i^f := F_x^{-1}(f_*)_x$ . Further  $f^* E_i$  is given by  $(f^* E_i)_x := F_x^{-1}(E_i)_{f(x)}$  for each  $x \in M$ . Then it holds

$$i_x^f = F_x^{-1}(df^i E_i)_{f(x)} = df^i(f^* E_i)_x, \quad (4.4.2)$$

for each  $x \in M$ .

In the following we consider immersions with target space  $(\mathbb{R}^{2(m+p)}, J', D)$ , where  $J'$  is the standard para-complex structure on  $\mathbb{R}^{2(m+p)}$ .

**Theorem 28** *Let  $(M, J, \nabla)$  be a simply connected  $2m$ -dimensional para-complex manifold with para-complex structure  $J$  and  $\nabla \in \mathcal{P}_0(TM, J)$ .*

*Let  $f, g : (M, J, \nabla) \rightarrow (\mathbb{R}^{2(m+p)}, J', D)$  be para-complex affine immersions with transversal bundle  $N^f$ , resp.  $N^g$ , where  $J^f$  resp.  $J^g$  are the induced para-complex structure on  $N^f$  (resp.  $N^g$ ). We will denote by  $B^f, A^f$  and  $\nabla^{N^f}$  (resp.  $B^g, A^g$  and  $\nabla^{N^g}$ ) the affine second fundamental form, the shape tensor and the transversal connection of the immersion  $f$  (resp.  $g$ ). Moreover suppose that there exists a map  $\Psi \in \Gamma(\text{Iso}((N^f, J^f), (N^g, J^g)))$  such that*

$$B_X^g = \Psi B_X^f, \quad A_X^g \Psi = A_X^f, \quad \nabla_X^{N^g} \Psi = \Psi \nabla_X^{N^f}, \quad (4.4.3)$$

for all  $X \in \Gamma(TM)$ , then there exists a para-complex affine transformation

$$\Phi : (\mathbb{R}^{2(m+p)}, J', D) \rightarrow (\mathbb{R}^{2(m+p)}, J', D)$$

satisfying  $g = \Phi f$  such that the bundle isomorphism which is induced by  $f_*$  on  $N^f$  equals  $\Psi$ .

*Proof:* Let

$$\Phi_T := \hat{i}^g (\hat{i}^f)^{-1} \in \Gamma(\text{Iso}(i^f TM, i^g TM)).$$

One sees easily from this definition

$$\Phi_T \hat{i}^f J(\hat{i}^f)^{-1} = \hat{i}^g J(\hat{i}^g)^{-1} \Phi_T, \quad (4.4.4)$$

$$\Phi_T \hat{i}^f \nabla(\hat{i}^f)^{-1} = \hat{i}^g \nabla(\hat{i}^g)^{-1} \Phi_T. \quad (4.4.5)$$

And similarly with equation (4.4.3)

$$\Phi_T \hat{i}^f A_X^f = \hat{i}^g A_X^g \Psi, \quad (4.4.6)$$

$$\Psi B_X^f (\hat{i}^f)^{-1} = B_X^g (\hat{i}^g)^{-1} \Phi_T \quad (4.4.7)$$

for all  $X \in \Gamma(TM)$ .

We extend now the map  $\Phi_T$  to a map  $\Phi \in \Gamma(\text{Iso}(f^*T\mathbb{R}^{2(m+p)}, g^*T\mathbb{R}^{2(m+p)}))$  defined by

$$\Phi := \iota_g \Phi_T \pi_f + \iota_{Ng} \Psi \pi_{Nf}.$$

With these definitions it holds

$$\Phi(f^*J') = (g^*J')\Phi. \quad (4.4.8)$$

The assumption (4.4.3) and the equations (4.4.5), (4.4.6) and (4.4.7) yield using lemma 24 the relation

$$\Phi(f^*D)_X = (g^*D)_X \Phi \quad (4.4.9)$$

for all  $X \in \Gamma(TM)$ .

Fixing now a point  $x_0 \in M$  we express

$$\Phi_{x_0}((f^*E_i)_{x_0}) = a_i^j (g^*E_j)_{x_0}$$

with  $i, j = 1, \dots, 2(m+p)$ . By parallelity, i.e. equation (4.4.9), we obtain

$$\Phi_x((f^*E_i)_x) = a_i^j (g^*E_j)_x \quad (4.4.10)$$

for all  $x \in M$ .

We use this to define an affine transformation  $\phi : \mathbb{R}^{2(m+p)} \rightarrow \mathbb{R}^{2(m+p)}$  by

$$\phi(f(x_0) + e_i) := g(x_0) + a_i^j e_j$$

and  $\tilde{\phi} : M \rightarrow \mathbb{R}^{2(m+p)}$  by

$$\tilde{\phi}(x) = \phi(f(x)) - g(x) = (a_j^i (f^j(x) - f^k(x_0)) + g^i(x_0) - g^i(x)) e_i$$

for  $x \in M$ . Now equations (4.4.2) and (4.4.10) yield

$$df^i a_i^j (g^*E_j)_x = \Phi_x(df^i(f^*E_i)_x) = \Phi_x i_x^f = i_x^g = dg^i(g^*E_i)_x, \quad (4.4.11)$$

for all  $x \in M$ . From equation (4.4.11) we get

$$d(a_j^i f^j + g^i(x_0) - a_k^i f^k(x_0) - g^i) = a_j^i df^j - dg^i = 0$$

which means that  $\tilde{\phi}$  is constant on  $M$ . But  $\tilde{\phi}(x_0) = 0$  implies the vanishing of  $\tilde{\phi}$  and consequently

$$g = \phi \circ f. \quad (4.4.12)$$

From this equation and equation (4.4.10) we get

$$\Phi_x = G^{-1} \circ (\phi_*)_x \circ F, \quad (4.4.13)$$

for all  $x \in M$ . Further this equation and equation (4.4.8) yield

$$\phi_* J' = J' \phi_*, \quad (4.4.14)$$

i.e.  $\phi$  is a para-complex transformation. Finally equation (4.4.13) shows us, that the bundle isomorphism induced by  $\phi_*$  on  $N^f$  coincides with  $\Psi$ .  $\square$

Now we give the existence theorem of para-complex affine immersions

**Theorem 29** *Let  $(M, J)$  be a para-complex, simply connected  $2m$ -dimensional manifold endowed with a connection  $\nabla \in \mathcal{P}_0(TM, J)$  and given a para-complex vector bundle  $(F, J^F)$  over  $M$  of rank  $2p$  with para-complex structure  $J^F$  endowed with a connection  $\nabla^F \in \mathcal{P}^a(F, J^F)$ .*

*Further suppose that there exist a 1-form*

$$B' \in \Omega^{1,0}(\text{Hom}((TM, J), (F, J^F)))$$

*which is symmetric, i.e.  $B'_X Y = B'_Y X$  for  $X, Y \in \Gamma(TM)$  and a 1-form*

$$A' \in \Omega^1(\text{Hom}((F, J^F), (TM, J))),$$

*such that for all  $X, Y \in \Gamma(TM)$  it holds*

$$R_{X,Y} - A'_X B'_Y + A'_Y B'_X = 0, \quad (4.4.15)$$

$$B'_X \nabla_Y + \nabla_X^F B'_Y - B'_Y \nabla_X - \nabla_Y^F B'_X - B'_{[X,Y]} = 0, \quad (4.4.16)$$

$$A'_X \nabla_Y + \nabla_X A'_Y - A'_Y \nabla_X^F - \nabla_Y A'_X - A'_{[X,Y]} = 0, \quad (4.4.17)$$

$$R_{X,Y}^F - B'_X A'_Y + B'_Y A'_X = 0 \quad (4.4.18)$$

*with  $R$  (resp.  $R^F$ ) the curvature of  $\nabla$  (resp.  $\nabla^F$ ).*

*Then there exists a para-complex affine immersion  $f : (M, J, \nabla) \rightarrow (\mathbb{R}^{2(m+p)}, J', D)$  with transversal bundle  $N$ , affine fundamental form  $B$ , shape tensor  $A$  and transversal connection  $\nabla^N$ , such that*

$$B_X = \Psi B'_X, \quad A_X \Psi = A'_X, \quad \nabla_X^N \Psi = \Psi \nabla_X^F, \quad (4.4.19)$$

*for all  $X \in \Gamma(TM)$ , where  $\Psi \in \Gamma(\text{Iso}((F, J^F), (N, J^N)))$ .*

*Proof:* In order to prove this theorem we consider the vector bundle  $E' := TM \oplus F$  over  $M$  and endow it with the connection

$$\nabla' := \iota'_1 \nabla \pi'_1 + \iota'_2 B' \pi'_2 - \iota'_1 A' \pi'_2 + \iota'_2 \nabla^F \pi'_2, \quad (4.4.20)$$

where  $\iota'_1 : TM \rightarrow E'$  and  $\iota'_2 : F \rightarrow E'$  are the canonical inclusions and  $\pi'_1 : E' \rightarrow TM$  and  $\pi'_2 : E' \rightarrow F$  are the projections. Applying lemma 23 we obtain the flatness of  $\nabla'$ . We define a para-complex structure on  $E'$  by

$$J^{E'} := \iota'_1 J \pi'_1 + \iota'_2 J^F \pi'_2.$$

From equation (4.4.20) and corollary 7 we get

$$\nabla' \in \mathcal{P}(E', J^{E'}), \quad (4.4.21)$$

i.e.  $J^{E'}$  is  $\nabla'$ -parallel.

As the manifold  $M$  is simply-connected,  $\nabla'$  is flat and  $J^{E'}$  is  $\nabla'$ -parallel, we can choose a global para-complex frame  $(\eta'_i)_{i=1}^{2(m+p)} = (\eta'_1, \dots, \eta'_{m+p}, J^{E'} \eta'_1, \dots, J^{E'} \eta'_{m+p})$ . The corresponding coframe will be called  $(\eta^1, \dots, \eta^{2(m+p)})$ . We define one-forms  $\omega^i := \eta^i \circ \iota'_1 = \eta^i|_{TM}$  with  $i = 1, \dots, 2(m+p)$ . The one-forms  $\omega^i$  are closed, as  $\nabla$  and  $\nabla'$  torsion-free and  $B'$  is symmetric:

$$\begin{aligned} d\omega^i(X, Y) &= (\nabla_X \omega^i)Y - (\nabla_Y \omega^i)X \\ &= X\omega^i(Y) - \omega^i(\nabla_X Y) - Y\omega^i(X) + \omega^i(\nabla_Y X) \\ &= X\eta^i(\iota'_1 Y) - \eta^i(\iota'_1 \nabla_X Y) - Y\eta^i(\iota'_1 X) + \eta^i(\iota'_1 \nabla_Y X) \\ &= X\eta^i(\iota'_1 Y) - \eta^i(\iota'_1 \nabla_X Y) - Y\eta^i(\iota'_1 X) + \eta^i(\iota'_1 \nabla_Y X) \\ &= X\eta^i(\iota'_1 Y) - \eta^i(\nabla'_X \iota'_1 Y) - Y\eta^i(\iota'_1 X) + \eta^i(\nabla'_Y \iota'_1 X) \\ &\quad + \eta^i(\iota'_1 B_X Y) - \eta^i(\iota'_1 B_Y X) \\ &= (\nabla'_X \eta^i) \iota'_1 Y - (\nabla'_Y \eta^i) \iota'_1 X = 0. \end{aligned}$$

By simply-connectedness of  $M$  we find functions  $f^i$  satisfying  $df^i = \omega^i$  for  $i = 1, \dots, 2(m+p)$  and define a map  $f : M \rightarrow \mathbb{R}^{2(m+p)}$  via

$$f = \sum_i f^i e_i.$$

Using identity (4.4.1) we derive for  $x \in M$  and  $X \in T_x M$

$$f_*(X) = (df^i)(X)(E_i)_{f(x)} = \omega^i(X)(E_i)_{f(x)}. \quad (4.4.22)$$

Further we introduce  $\Phi \in \Gamma(\text{Iso}(E', f^* \mathbb{R}^{2(m+p)}))$  by

$$\Phi_x(\eta'_i)_x := (f^* E_i)_x, \quad i = 1, \dots, 2(m+p) \text{ for each } x \in M.$$

This definition and equation (4.4.22) yield

$$f_* = F \Phi \iota'_1 \Leftrightarrow i^f = F^{-1} f_* = \Phi \iota'_1. \quad (4.4.23)$$

Consequently  $f$  defines an immersion, since  $F$  and  $\Phi$  are isomorphisms and  $\iota'_1$  is the inclusion.

We obtain

$$(\Phi \circ J^E)(\eta'_k) = \Phi(J^E \eta'_k) = f^*(E_{m+p+k}) = (f^* J')(f^* E_k) = (f^* J')(\Phi(\eta'_k))$$

for  $k = 1, \dots, m + p$ . From this we conclude

$$\Phi J^{E'} = [f^* J'] \Phi, \quad (4.4.24)$$

which implies with equation (4.4.23) that  $f$  is para-holomorphic.

By definition  $\Phi$  sends the  $\nabla^{E'}$ -parallel frame  $(\eta'_1, \dots, \eta'_{m+p}, J^{E'} \eta'_1, \dots, J^{E'} \eta'_{m+p})$  to the  $(f^* D)$ -parallel frame field  $(f^* E_1, \dots, f^* E_{2(m+p)})$ . In other words

$$\Phi \nabla_X^{E'} = (f^* D)_X \Phi, \quad (4.4.25)$$

for all  $X \in \Gamma(TM)$ .

Define now  $N := \Phi(F)$ . Then it holds

$$f^* T\mathbb{R}^{2(m+p)} = \Phi(E') = \Phi(TM) \oplus \Phi(F) = i^f TM \oplus N,$$

as the isomorphism  $\Phi$  maps by equation (4.4.23)  $TM$  to  $i^f TM$ .

$J^N := \pi_N(f^* J') \iota_N$  defines the induced para-complex structure on  $N$  and  $\Psi := \pi_N \Phi \iota'_2$  the induced bundle map. As  $\Phi$  is para-complex (see equation (4.4.24)) and since  $\Phi$  preserves the decomposition it follows

$$\hat{i}^f \in \Gamma(\text{Iso}((TM, J), (i^f TM, \pi_f(f^* J') \iota_f))) \quad (4.4.26)$$

$$\Psi \in \Gamma(\text{Iso}((F, J^F), (N, J^N))). \quad (4.4.27)$$

Let

$$\pi_f(f^* D) \iota_f \in \mathcal{P}(i^f TM, \pi_f(f^* J') \iota_f) \text{ and } \nabla^N \in \mathcal{P}(N, J^N)$$

be the induced connections of  $f^* D$  and

$$B(\hat{i}^f)^{-1} \in \Omega^1(\text{Hom}((i^f TM, \pi_f(f^* J') \iota_f), (N, J^N)))$$

and

$$-\hat{i}^f A \in \Omega^1(\text{Hom}((N, J^N), (i^f TM, \pi_f(f^* J') \iota_f)))$$

be the corresponding second fundamental forms. With the identity (4.4.25) and lemma 24 we obtain, recalling  $\phi_1 = \hat{i}^f, \phi_2 = \Psi, B = B^f \hat{i}^f$  and  $A = -(\hat{i}^f)^{-1} B^N$  and given  $X \in \Gamma(TM)$

$$\hat{i}^f \nabla_X = \pi_f(f^* D)_X \iota_f \hat{i}^f, \quad \Psi \nabla_X^F = \nabla_X^N \Psi, \quad (4.4.28)$$

$$(B_X(\hat{i}^f)^{-1}) \hat{i}^f = \Psi B'_X, \quad -\hat{i}^f A_X \Psi = -\hat{i}^f A'_X. \quad (4.4.29)$$

From these equations one sees that  $f$  is a para-complex affine immersion with transversal bundle  $N$  satisfying (4.4.19).  $\square$

We observe that in the proof of the existence theorem 29, if  $\nabla^F$  is para-holomorphic or of type  $(1, 1)$  then  $\nabla^N$  is para-holomorphic or of type  $(1, 1)$ , too. Moreover the last identity in equation (4.4.19) shows that  $\Psi \in \Gamma(\text{Iso}((F, J^F), (N, J^N)))$  is para-holomorphic with respect to  $\mathcal{F}^{\nabla^F}$  and  $\mathcal{N}^{\nabla^N}$ . Since  $D$  is flat it follows that  $f^* D$  is para-holomorphic and of type  $(1, 1)$ . From these considerations we conclude

**Corollary 15** *If in the situation of theorem 29 the connections  $\nabla$  and  $\nabla^F \in \mathcal{P}^a(F, J^F)$  are para-holomorphic and*

$$A' \in \Omega^{1,0}(\text{Hom}((F, J^F), (TM, J))),$$

*then, there exists a para-holomorphic affine immersion  $f : (M, J, \nabla) \rightarrow (\mathbb{R}^{2(m+p)}, J', D)$  with transversal bundle  $N$  with transversal para-holomorphic connection  $\nabla^N$  and  $\Psi \in \Gamma(\text{Iso}(F, J^F), (N, J^N))$  such that (4.4.19) is satisfied.*

**Corollary 16** *If in the situation of theorem 29 the connection  $\nabla$  is affine para-Kähler,  $\nabla^F \in \mathcal{P}^a(F, J^F)$  is of type  $(1, 1)$  and*

$$A' \in \Omega^{1,0}(\text{Hom}((F, J^F), (TM, J))),$$

*then, there exists a para-complex affine immersion  $f : (M, J, \nabla) \rightarrow (\mathbb{R}^{2(m+p)}, J', D)$  with transversal bundle  $N$  with a transversal connection  $\nabla^N$  which is of type  $(1, 1)$  and  $\Psi \in \Gamma(\text{Iso}(F, J^F), (N, J^N))$  such that (4.4.19) is satisfied.*

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# Anhang

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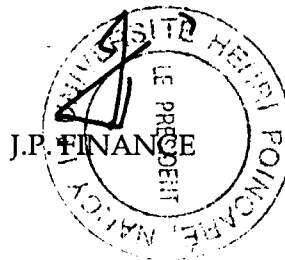
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## Summary

This thesis is devoted to the theory of immersions, using methods of spin geometry, para-complex and para-quaternionic geometry. It is subdivided into three different topics. The first two are related to the study of conformal immersions of pseudo-Riemannian surfaces. On the one hand we study the immersion into three-dimensional pseudo-Euclidean spaces: with the methods of para-complex geometry and using real spinor representations, we prove the equivalence between the data of a conformal immersion of a Lorentzian surface in  $\mathbb{R}^{2,1}$  and spinors satisfying a Dirac-type equation. On the other hand, we consider immersions of such surfaces into the four-dimensional pseudo-sphere  $\mathbb{S}^{2,2}$ : a one-to-one correspondence between such immersions and para-quaternionic line subbundles of the trivial bundle  $M \times \mathbb{H}^2$  is given. Considering a particular (para-)complex structure on this bundle, namely the mean curvature pseudo-sphere congruence, and the para-quaternionic Hopf fields of the immersion, we define the Willmore functional of the surface and can express its energy as the sum of this functional and of a topological invariant. The last topic is more general and deals with para-complex vector bundles and para-complex affine immersions. We introduce para-holomorphic vector bundles and characterize para-holomorphic subbundles and subbundles of type  $(1, 1)$  in terms of the associated induced connections and second fundamental forms. The fundamental equations for general decompositions of vector bundles with connection are studied in the case where some of the (sub)bundle are para-holomorphic in order to prove existence and uniqueness theorems of para-complex affine immersions.

## Mots-clés

Géométrie pseudo-Riemannienne, géométrie spinorielle, géométrie (para-)complexe et (para-)quaternionique, hypersurfaces, surfaces de Lorentz, opérateurs de Dirac, intégrale de Willmore, immersions affines.

## Résumé

Ce travail est relatif à la théorie des immersions et utilise des méthodes issues de la géométrie spinorielle, para-complexe et para-quaternionique. Les deux premières parties sont consacrées aux immersions conformes de surfaces pseudo-Riemanniennes. D'une part, nous étudions ce type d'immersions dans l'espace pseudo-Euclidien de dimension trois. Avec des méthodes de géométrie para-complexe et des représentations spinorielles réelles, l'équivalence entre les données d'une immersion conforme d'une surface de Lorentz dans  $\mathbb{R}^{2,1}$  et de spineurs satisfaisant une équation de type Dirac est prouvée. D'autre part nous considérons des surfaces de Lorentz dans la pseudo-sphère  $\mathbb{S}^{2,2}$ : une bijection entre ces immersions et des sous-fibrés en droite para-quaternioniques du fibré  $M \times \mathbb{H}^2$  est établie. Considérant une structure (para-)complexe particulière de ce fibré, la congruence pseudo-sphérique, et les champs de Hopf para-quaternioniques, nous définissons la fonctionnelle de Willmore de la surface et exprimons son énergie comme la somme de cette fonctionnelle et d'un invariant topologique. La dernière partie, plus générale, traite des fibrés vectoriels et immersions affines para-complexes. Nous introduisons la notion de fibré vectoriel para-holomorphe, et les sous-fibrés para-holomorphes et de type  $(1, 1)$  en termes de connections associées induites et de secondes formes fondamentales. Les équations fondamentales pour des décompositions générales de fibrés vectoriels munis d'une connexion sont étudiées dans le cas où certains des fibrés sont para-holomorphes afin d'obtenir des théorèmes d'existence et d'unicité pour des immersions affines para-complexes.

## Keywords

pseudo-Riemannian geometry, spin geometry, (para-)complex and (para-)quaternionic geometry, hypersurfaces, Lorentz surfaces, Dirac operator, Willmore integral, affine immersions.