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Inégalités de Carleman près du bord, d'une interface et pour des problèmes singuliers

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Première partie

Introduction générale

Chapitre 1

Problématiques liées au contrôle

1.1 Quelques différentes notions de contrôle

Le problème direct pour une équation différentielle, ou pour une équation aux dérivées partielles est le suivant. On se donne un système de la forme

$$\begin{cases} \partial_t u = \mathcal{A}(u, f) \\ u(0) = u_0, \end{cases} \tag{1.1.1}$$

où pour tout $t \geq 0$, $u(t)$ est l'inconnue appartenant à un certain espace d'état H, $\mathcal{A}$ un opérateur qui décrit le système, et f un terme source agissant sur le système. La question directe se traduit par : connaissant f et les données initiales, est-on capable de retrouver l'inconnue u ? Est-elle continue par rapport aux données du problème ? En théorie du contrôle, les problématiques sont différentes. On considère ici la question dite de contrôlabilité. On se donne une donnée intiale u_0 ainsi qu'un couple état cible-temps (u_T, T). Est-on capable de trouver f appartenant à un certain espace de contrôle U (en d'autres termes, d'agir sur l'équation (1.1.1) au travers d'une source) de telle sorte que la solution u issue de la donnée initiale u_0 satisfasse $u(T) = u_T$, c'est-à-dire, *contrôler* la solution pour qu'elle atteigne un état désiré à l'instant désiré ? On renvoie au livre de J.-M. Coron [Cor07] pour une exposition détaillée des notions de contrôle présentées dans cette partie.

Définition 1.1.1. *On dit que le système (1.1.1) est exactement contrôlable en temps $T > 0$, si pour toute donnée initiale $u_0 \in H$ et pour tout état cible $u_T \in H$, il existe un contrôle $f \in L^2((0,T), U)$ tel que*

$$u(T) = u_T.$$

D'autres notions de contrôlabilité existent. Nous pouvons donner par exemple :

Définition 1.1.2. *On dit que le système (1.1.1) est approximativement contrôlable en temps $T > 0$, si pour toute donnée initiale $u_0 \in H$, pour tout état cible $u_T \in H$, et pour toute précision $\varepsilon > 0$, il existe un contrôle $f \in L^2((0,T), U)$ tel que*

$$||u(T) - u_T||_H < \varepsilon.$$

Ou encore, la notion de contrôlabilité aux trajectoires. L'espace des états cibles que l'on souhaite atteindre est celui de l'ensemble des états "naturellement" atteints par l'équation libre.

Définition 1.1.3. *On dit que le système (1.1.1) est contrôlable aux trajectoires en temps $T > 0$, si pour tout couple de données initiales $u_0, v_0 \in H$, il existe un contrôle $f \in L^2((0,T), U)$ tel que $u(T) = v(T)$, où v est solution de l'équation $\partial_t v = \mathcal{A}(v, 0)$, avec $v(0) = v_0$.*

En fait, dans le cas d'un système d'équations différentielles ordinaires linéaires en dimension finie de la forme

$$\begin{cases} \partial_t u = Au + Bf \\ u(0) = u_0, \end{cases} \tag{1.1.2}$$

où A et B sont des matrices de tailles respectives $d \times d$ et $d \times m$, $u(t) \in \mathbb{R}^d$, et $f(t) \in \mathbb{R}^m$, ces trois notions de contrôlabilité sont équivalentes. Ce n'est plus le cas dans le cadre des équations aux dérivées partielles. Nous y reviendrons plus tard. Un critère de contrôlabilité exacte pour le système (1.1.2) existe et fournit une condition nécessaire et suffisante. C'est le critère dit de Kalman.

Théorème 1.1.1 (Critère de Kalman en dimension finie [LaS60, KHN63])**.** *Le système* (1.1.2) *est exactement contrôlable en temps $T > 0$ si et seulement si*

$$\operatorname{Rank}\left(\begin{array}{c|c|c|c} A & BA & \dots & B^{d-1}A \end{array} \right) = \dim H = d. \tag{1.1.3}$$

De ce théorème, on peut déduire la propriété suivante. Si le système est contrôlable en un certain temps $T > 0$, alors il est contrôlable pour tout temps $T' > 0$, car le critère ne dépend pas du temps $T > 0$ choisi. Toutefois, il ne précise pas comment le contrôle dépend du temps. Dans le cadre d'équations aux dérivées partielles, certains systèmes ne sont pas exactement contrôlables pour tout temps $T > 0$ (on parle alors de temps minimal de contrôle). C'est notamment le cas pour les équations de type hyperbolique, car l'influence du contrôle est limitée par la vitesse finie de propagation. De plus, nous verrons que l'on peut quantifier la dépendance en temps des contrôles, par exemple dans le cadre de l'équation de la chaleur.

Ces notions de contrôle présentent un certain défaut : elles dépendent fortement de la donnée initiale, et ne dépendent pas de l'état du système lorsque $0 < t < T$. Il est souvent important de considérer des problèmes avec rétro-action (ou *feedback*), où le contrôle agit sur le système en *répondant* en continu au système, en fonction de l'état de la solution à cet instant. C'est notamment le cas lors de l'étude de problèmes de stabilisation. De tels systèmes peuvent s'écrire sous la forme

$$\begin{cases} \partial_t u = \mathcal{A}(u, f(u)) \\ u(0) = u_0, \end{cases} \tag{1.1.4}$$

Définissant une fonctionnelle (une énergie) dépendant de la solution u de (1.1.4), on s'intéresse alors aux propriétés de décroissance de cette fonctionnelle. Cette notion de stabilisation sera largement étudiée dans ce mémoire, dans le cadre de l'équation des ondes, ainsi que certaines questions de contrôlabilité pour la chaleur.

1.2 Du contrôle vers l'observabilité

Dans ce chapitre, nous présentons comment la notion de contrôlabilité à zéro peut être réduite à l'obtention d'une inégalité fonctionnelle, dite d'observabilité, par une argument de dualité. Ces inégalités jouent un rôle central dans l'étude du contrôle et de la stabilisation des équations aux dérivées partielles. En effet, construire un contrôle est en général un problème difficile. Par contre, beaucoup d'outils existent pour démontrer des inégalités fonctionnelles, telles que les inégalités d'énergie, de Carleman, d'Ingham, ou encore de Gårding.

On se place dans le cadre suivant :

$$\begin{cases} \partial_t u = Au + Bf, \\ u(0) = u_0 \in H, \end{cases} \tag{1.2.1}$$

où A est un opérateur agissant sur un espace de Hilbert H, de domaine $D(A) \subset H$. On suppose que A est le générateur d'un semi-groupe fortement continu, noté $\mathbb{S}(t)$. On supposera que l'opérateur de contrôle B agit sur l'espace des contrôles U, et on supposera de plus que $B \in \mathcal{L}(U, H)$, pour plus de simplicité. On notera que cette condition peut être relaxée, il faut alors remplacer la continuité de l'opérateur B par la condition dite d'admissibilité (voir le livre de Tucsnak et Weiss [TW09] pour plus de détails). C'est notamment utile lorsque l'on souhaite contrôler une équation aux dérivées partielles par le bord du domaine. C'est par le biais de cet opérateur que l'on peut agir sur le système, et en général, celui-ci restreint les possibilités d'action. Un exemple courant d'opérateur B est la multiplication par une fonction indicatrice d'un sous-domaine. Ainsi, notre action sur le système est limitée dans l'espace. On introduit la notion de contrôlabilité à zéro.

Définition 1.2.1. *On dit que le système* (1.2.1) *est contrôlable à zéro en temps $T > 0$, si pour toute donnée initiale $u_0 \in H$, il existe un contrôle $f \in L^2((0,T),U)$ tel que*

$$u(T) = 0.$$

En dimension finie, pour des équations linéaires, toutes les notions de contrôlabilité présentées dans la section 1.1 sont équivalentes. Dans le cadre d'équation aux dérivées partielles linéaires, la contrôlabilité aux trajectoires et la contrôlabilité à zéro sont toujours équivalentes. Dans le cadre d'équations dissipatives, du type équation de la chaleur linéaire par exemple, l'effet régularisant fait qu'il est impossible d'atteindre n'importe quel état de $L^2(\Omega)$. La contrôlabilité à zéro de l'équation de la chaleur n'est donc pas équivalente à la contrôlabilité exacte. Pour des équations réversibles en temps, du type équation d'ondes linéaire par exemple, la contrôlabilité à zéro est équivalente à la contrôlabilité exacte. La contrôlabilité à zéro reste donc une notion centrale.

Grâce aux hyptohèses faites sur $\mathbb{S}(t)$ et B, on peut écrire la solution de (1.2.1) au moyen de la formule de Duhamel

$$u(t) = \mathbb{S}(t)u_0 + \int_0^t \mathbb{S}(t-s)Bf(s)ds. \tag{1.2.2}$$

On introduit le problème adjoint

$$\begin{cases} \partial_t v = -A^* v, \\ v(T) = v_T \in H \end{cases} \tag{1.2.3}$$

Ce problème est bien posé, et on note $\mathbb{S}^*(t)$ le semi-groupe fortement continu généré par l'opérateur A^*. La notion d'observabilité du système (1.2.3) est donnée par la définition suivante.

Définition 1.2.2. *On dit que le système* (1.2.3) *est observable en temps $T > 0$ s'il existe une constante $C_T > 0$ telle que pour tout $v_T \in H$, la solution du problème adjoint* (1.2.3) *vérifie l'inégalité suivante*

$$||\mathbb{S}^*(T)v_T||_H \leq C_T||B^*\mathbb{S}^*(t)v_T||_{L^2((0,T),U)}. \tag{1.2.4}$$

On notera que l'on a $\mathbb{S}^*(T)v_T = v(0)$, où v est la solution du problème adjoint (1.2.3). L'intérêt principal de l'inégalité d'observabilité pour un système adjoint réside dans son lien avec la contrôlabilité à zéro du système initial. Plus précisément, on a le résultat suivant.

Théorème 1.2.1. *La contrôlabilité à zéro au temps $T > 0$ du système* (1.2.1) *est équivalente à l'inégalité* (1.2.4) *avec une constante $C_T > 0$. Dans ce cas, il existe une application linéaire et continue qui à la donnée initiale $u_0 \in H$ du problème* (1.2.1) *associe un contrôle à zéro $f \in L^2((0,T),U)$, qui vérifie*

$$||f||_{L^2((0,T),U)} \leq C_T||u_0||_H. \tag{1.2.5}$$

Une caractéristique remarquable de ce théorème est que la constante impliquée dans l'inégalité d'observabilité est précisément la même que celle impliquée dans l'inégalité (1.2.5). Une preuve de ce résultat est basée sur le lemme d'analyse fonctionnelle abstrait suivant, prouvé par Douglas [Dou66] et Dolecki-Russell [DR77] (voir aussi [Cor07]).

Lemme 1.2.3. *Soit H, H_1 et H_2 trois espaces de Hilbert, et*

$$K_1 : H_1 \longrightarrow H, \quad K_2 : H_2 \longrightarrow H,$$

deux opérateurs linéaires continus. Alors on a équivalence entre :

1. *$K_1(H_1) \subset K_2(H_2)$,*
2. *il existe un operateur continu $\Phi : H_1 \to H_2$ tel que $K_1 = K_2 \circ \Phi$,*
3. *il existe $M \geq 0$ tel que*
$$|||K_1^*|||_{\mathcal{L}(H,H_1)} \leq M|||K_2^*|||_{\mathcal{L}(H,H_2)}. \tag{1.2.6}$$

Si de plus, le point (2) *est vérifié, on peut choisir* $M = |||\Phi|||_{\mathcal{L}(H_1,H_2)}$ *dans* (1.2.6). *Réciproquement, prenant pour* M *dans* (1.2.6) *la meilleure constante possible, alors on peut trouver un opérateur* Φ *pour le point* (2) *tel que* $|||\Phi|||_{\mathcal{L}(H_1,H_2)} = M$.

En effet, si on définit les opérateurs

$$\begin{array}{lll} T_1 : & H \longrightarrow H & \qquad T_2 : \quad L^2((0,T),U) \longrightarrow H \\ & u_0 \longmapsto \mathbb{S}(T)u_0, & \qquad\quad f \longmapsto \int_0^T \mathbb{S}(T-s)Bf(s)ds, \end{array}$$

la contrôlabilité à zéro du problème (1.2.1) est équivalente à montrer, d'après la formule de Duhamel (1.2.2), l'inclusion $T_1(H) \subset T_2(L^2((0,T),U))$. Un calcul de T_1^* et T_2^* montre alors que (1.2.6) est précisément l'inégalité d'observabilité (1.2.4).

Une autre conséquence importante de ce lemme est l'unicité du contrôle f associé à la meilleure constante dans (1.2.5). En effet, prenons la meilleure constante $C_T > 0$ dans l'inégalité (1.2.6). D'après le lemme 1.2.3, on peut construire un opérateur Φ_1 tel que $|||\Phi_1|||_{\mathcal{L}(H,L^2((0,T),U)} = C_T$. Supposons que l'on peut construire une autre application de contrôle Φ_2 telle que $|||\Phi_2|||_{\mathcal{L}(H,L^2((0,T),U)} = C_T$ donnée par le point (2) du lemme 1.2.3. Alors l'application $\tilde{\Phi} := \frac{1}{2}(\Phi_1 + \Phi_2)$ fournirait aussi un contrôle, mais avec une norme plus petite, par stricte convexité des boules dans un espace de Hilbert. Ainsi, on obtient $\tilde{\Phi} = \Phi_1 = \Phi_2$.

Nous citerons aussi la méthode classique, tirée de [Lio88], dite H.U.M. (Hilbert Uniqueness Method), qui permet aussi de montrer le passage de l'observabilité vers la contrôlabilité à zéro. Il faut pour cela faire l'hypothèse d'unicité rétrograde du système adjoint (1.2.3), c'est-à-dire, si la solution v du problème (1.2.3) vérifie $v(0) = 0$, alors $v(t) = 0$ pour tout $t \in (0,T)$. Notons, qu'en pratique, la plupart des problèmes vérifient cette condition, sous des hypothèse raisonnables de régularité des coefficients. Supposons que l'inégalité d'observabilité (1.2.4) est vérifiée. Prenant le produit scalaire du problème adjoint (1.2.3), avec la solution u du problème direct (1.2.1), on a

$$\begin{aligned} -\int_0^T \Big(A^*v, u\Big)_H &= \int_0^T \Big(\partial_t v, u\Big)_H \\ &= -\int_0^T \Big(v, Au + Bf\Big)_H + \Big(v(T), u(T)\Big)_H - \Big(v(0), u(0)\Big)_H \\ &= -\int_0^T \Big(v, Au + Bf\Big)_H + \Big(v(T), u(T)\Big)_H - \Big(v(0), u(0)\Big)_H, \end{aligned}$$

et on obtient ainsi

$$\int_0^T \Big(B^*v, f\Big)_U = \Big(v_T, u(T)\Big)_H - \Big(v(0), u_0\Big)_H. \tag{1.2.7}$$

Considérons l'équation variationnelle suivante

$$a(v_T^1, v_T^2) = \ell(v_T^2), \quad v_T^1, v_T^2 \in H, \tag{1.2.8}$$

avec

$$a(v_T^1, v_T^2) = \int_0^T \Big(B^*\mathbb{S}^*(T-t)v_T^1, B^*\mathbb{S}^*(T-t)v_T^2\Big)dt, \quad \ell(v_T^1) = \Big(\mathbb{S}^*(T)v_T^1, u_0\Big).$$

On peut montrer que la forme bilinéaire a définit un produit scalaire sur H, en utilisant l'hypothèse d'unicité rétrograde du problème adjoint (1.2.3), la continuité de B et l'observabilité (1.2.4). Ainsi, la norme associée définit une norme sur l'espace de Hilbert H, et on notera $\overline{H}$ le complété de H pour cette norme. L'application ℓ étant continue sur H pour cette norme, on peut la prolonger de manière unique en une application (encore notée ℓ) continue sur $\overline{H}$.
Le théorème de représentation de Riesz fournit l'existence et l'unicité d'une solution $\overline{v_T} \in \overline{H}$ à l'équation variationelle

$$a(\overline{v_T}, v_T^2) = \ell(v_T^2), \quad \forall v_T^2 \in \overline{H}.$$

Il suffit alors de prendre $f(t) = \overline{B^*\mathbb{S}^*}(T-t)\overline{v_T}$, où l'on a noté $\overline{B^*\mathbb{S}^*}(T-t)$ l'unique application linéaire continue sur $\overline{H}$ qui coïncide avec $B^*S^*(T-t)$ sur H. On a ainsi,

$$\int_0^T \Big(B^*v, f\Big)_U = \Big(v_0, u_0\Big)_H. \tag{1.2.9}$$

Ceci, combiné à (1.2.7), montre que $\Big(v_T, u(T)\Big)_H = 0$, pour tout $v_T \in H$. Ainsi $u(T) = 0$ et f est bien un contrôle à zéro pour le problème initial (1.2.1).
On peut aussi montrer que la méthode H.U.M. présentée ci-dessus, en plus de fournir l'existence d'un contrôle si l'inégalité (1.2.4) est satisfaite, fournit le contrôle qui minimise la norme $||f||_{L^2((0,T),U)}$.

1.3 Contrôlabilité à zéro de l'équation de la chaleur

Nous avons vu que les problèmes de contrôle sont équivalents à des inégalités dites d'observabilité portant sur les solutions du problème adjoint. Pour la contrôlabilité à zero de la chaleur, nous citerons l'article de Lebeau et Robbiano [LR95] ainsi que l'article de Fursikov et Imanuvilov [FI96]. Dans ce mémoire, pour les résultats de contrôle de l'équation de chaleur, nous suivrons la stratégie de [LR95, JL99, LZ98], qui passe par l'obtention d'une inégalité portant sur les sommes finies de fonctions propres du Laplacien. Cette inégalité, appelée inégalité spectrale, est la conséquence d'inégalités de Carleman pour des opérateurs elliptiques. Néanmoins, d'après la section précédente, la contrôlabilité à zéro de l'équation de la chaleur peut aussi être démontrée par l'obtention directe d'une inégalité d'observabilité pour le problème adjoint, méthode employée dans [FI96]. Elle passe par l'obtention d'une inégalité de Carleman parabolique.

Soit Ω un ouvert borné lisse de $\mathbb{R}^d$ et ω un sous ouvert non-vide de Ω. On s'intéresse à l'équation de la chaleur suivante :

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega f & \text{dans } (0,T)\times\Omega \\ u_{|_{\partial\Omega}} = 0 & \text{sur } (0,T)\times\partial\Omega \\ u_{|_{t=0}} = u_0 & \text{sur } \Omega, \end{cases} \tag{1.3.1}$$

où $\chi \in C_0^\infty(\omega)$, et $f \in L^2((0,T)\times\Omega)$. Le Laplacien $-\Delta$ de Dirichlet étant un opérateur auto-adjoint, positif à résolvante compacte, il admet une décomposition spectrale (λ_j, ϕ_j), $j \in \mathbb{N}$, avec les valeurs propres satisfaisant

$$0 < \lambda_0 \leq \lambda_1 \leq \dots \lambda_k \longrightarrow +\infty, \tag{1.3.2}$$

et les fonctions ϕ_j qui forment une base de Hilbert de $L^2(\Omega)$.

1.3.1 Inégalité spectrale

Un résultat clé dans la construction du contrôle f est l'inégalité suivante portant sur les fonctions propres du Laplacien.

Théorème 1.3.1 ([LR95],[JL99], [LZ98]). *Il existe $C > 0$ tel que pour tout $\Lambda \geq 1$,*

$$||z||_{L^2(\Omega)} \leq e^{C\sqrt{\Lambda}}||z||_{L^2(\omega)}, \quad z \in \mathrm{Vect}\{\phi_k,\ \lambda_k \leq \Lambda\}. \tag{1.3.3}$$

C'est une inégalité d'observabilité dans les espaces engendrés par les sommes finies de fonctions propres. Une fonction de $L^2(\Omega)$ ne pouvant pas être uniquement déterminée par sa valeur sur un sous-domaine, il est naturel que cette inégalité ne soit pas valable pour des séries de fonctions propres (même convergentes), et il est donc aussi naturel que la constante explose en ajoutant des fonctions propres. Il a été démontré dans [JL99] que la dépendance en Λ de la constante dans (1.3.3) est optimale. Plus précisément, on a le résultat suivant.

Proposition 1.3.1. *Soit ω un ouvert non-vide de Ω, tel que $\overline{\omega} \neq \Omega$. Alors il existe deux constantes $C > 0$ et $\Lambda_0 > 0$ telles que pour tout $\Lambda > \Lambda_0$, il existe $z \in \mathrm{Vect}\{\phi_k,\ \lambda_k \leq \Lambda\}$ tel que*

$$||z||_{L^2(\Omega)} \geq Ce^{C\sqrt{\Lambda}}||z||_{L^2(\omega)}.$$

Une preuve du théorème 1.3.1 est basée sur la remarque suivante. Soit $S_0 > 0$. On introduit le cylindre $Z = (0, S_0) \times \Omega$, et pour $0 < \alpha < S_0/2$, on définit $Y = (\alpha, S_0 - \alpha) \times \Omega$. Posons

$$w(s, x) = \sum_{\lambda_k \leq \Lambda} \frac{\sinh(\sqrt{\lambda_k}s)}{\sqrt{\lambda_k}} a_k \phi_k(x). \tag{1.3.4}$$

Ainsi, de telles fonctions w satisfont le problème elliptique suivant

$$\begin{cases} -\partial_s^2 w - \Delta_x w = 0 & \text{dans } (0, S_0) \times \Omega \\ w_{|\partial\Omega} = 0 & \text{sur } (0, S_0) \times \partial\Omega \\ w_{|s=0} = 0 & \text{sur } \Omega \\ \partial_s w_{|s=0} = \sum_{\lambda_k \leq \Lambda} a_k \phi_k & \text{sur } \Omega. \end{cases} \tag{1.3.5}$$

Nous allons utiliser des inégalités d'interpolation de type Hölder comme la proposition 1.3.2.

Le théorème 1.3.1 est une conséquence de l'inégalité suivante, appelée inégalité d'interpolation. Ce type d'inégalités apparaîtra souvent dans ce mémoire. Elles permettent, en autres, de quantifier le prolongement unique de solutions, et de propager de la petitesse d'une zone de l'ouvert considéré à une autre.

Proposition 1.3.2. *Soit ω un ouvert non-vide de Ω. Il existe $C > 0$ et $\delta \in (0, 1)$ tels que pour tout $z \in H^2(Z)$ tel que $z_{|(0,S_0)\times\partial\Omega} = 0$, on a*

$$||z||_{H^1(Y)} \leq C||z||^{1-\delta}_{H^1(Z)} \left(||-\partial_s^2 z - \Delta z||_{L^2(Z)} + |z_{|s=0}|_{H^1(\omega)} + |\partial_s z_{|s=0}|_{L^2(\omega)}\right)^{\delta}.$$

La preuve de ce résultat est basée sur des inégalités de Carleman locales, à l'intérieur et au voisinage des bords $(-S_0, S_0) \times \partial\Omega$ et $\{s = 0\} \times \Omega$. C'est la stratégie que nous adopterons dans les chapitres 5 et 6. On en déduit l'inégalité spectrale (1.3.3) en remarquant que l'on a, pour les fonctions définies en (1.3.4), les relations du système (1.3.5), ainsi que

$$||w||_{H^1(Z)} \lesssim e^{C\sqrt{\Lambda}}||z||_{L^2(\Omega)}, \quad ||z||_{H^1(Y)} \gtrsim ||w||_{L^2(\Omega)}. \tag{1.3.6}$$

Une autre manière d'obtenir des inégalités du type (1.3.3), est l'obtention d'inégalités dites de *doubling*ponctuelles en temps, pour les solution de l'équations de la chaleur. Cette méthode utilise les propriétés dites de log-convexité de l'équation de la chaleur. Elle sera présentée succintement dans la partie Perspectives, et peut présenter un réel avantage, car elle évite de passer par l'opérateur elliptique augmenté d'une variable (1.3.5), ce qui peut poser quelques problèmes comme on le verra dans le chapitre 6.

1.3.2 De l'inégalité spectrale vers la construction du contrôle

On note $\mathbb{S}(t)$, $t \geq 0$, le semi-groupe engendré par le Laplacien de Dirichlet. On notera aussi

$$\Pi_\Lambda : L^2(\Omega) \longrightarrow \mathrm{Vect}\{\phi_k,\ \lambda_k \leq \Lambda\},$$

le projecteur sur l'espace engendré par les sommes finies des fonctions propres associées aux valeurs propres plus petites que la fréquence de coupure Λ. De l'inégalité spectrale, on en déduit le résultat d'observabilité dans les espaces engendrés par les sommes finies de fonctions propres.

Proposition 1.3.3. *Soit $\Lambda > 1$. On a, pour tout $T > 0$,*

$$||\mathbb{S}(T)\Pi_\Lambda u||_{L^2(\Omega)} \leq \frac{1}{\sqrt{T}} e^{C\sqrt{\Lambda}} ||\mathbb{S}(t)\Pi_\Lambda u||_{L^2((0,T)\times\omega)}, \tag{1.3.7}$$

pour tout $u \in L^2(\Omega)$, où $C > 0$ est la même constante que celle de l'inégalité spectrale (1.3.3).

De ce résultat, on peut obtenir, par dualité, en utilisant le théorème 1.2.1, le résultat de contrôlabilité à zéro partiel suivant pour le système (1.3.1).

Proposition 1.3.4. *Soit $\Lambda > 1$. Il existe un contrôle $f \in L^2(\Omega)$ tel que la solution u du problème (1.3.1) vérifie*

$$\Pi_\Lambda u(T) = 0.$$

On a de plus l'estimation suivante

$$||f||_{L^2((0,T)\times\omega)} \leq \frac{1}{\sqrt{T}} e^{C\sqrt{\Lambda}} ||u_0||_{L^2(\Omega)},$$

pour toute donnée initiale $u_0 \in L^2(\Omega)$, et où $C > 0$ est la constante de l'inégalité spectrale (1.3.3).

Un autre résultat important concerne une estimation sur la décroissance naturelle des solutions de la chaleur appartenant à l'orthogonal de $\text{Vect}\{\phi_k,\ \lambda_k \leq \Lambda\}$.

Proposition 1.3.5. *Soit $\Lambda > 0$. On a, pour $t \geq 0$,*

$$||\mathbb{S}(t)(\text{Id} - \Pi_\Lambda)u_0||_{L^2(\Omega)} \leq e^{-\Lambda t} ||u_0||_{L^2(\Omega)},$$

pour tout $u_0 \in L^2(\Omega)$.

Cette décroissance joue un rôle fondamental dans la construction du contrôle à zéro de (1.3.1). L'idée est de prendre une subdivision de l'intervalle $[0, T] = \cup_{j\in\mathbb{N}}[a_j, a_{j+1}]$, avec $a_0 = 0$ et une suite de Λ_j telle que $\Lambda_j \to +\infty$. Sur chaque intervalle de temps $[a_j, a_{j+1}]$, avec j pair, on contrôle à zéro les modes correspondant à une valeur propre plus petite que Λ_j, en utilisant la proposition 1.3.4 (on l'appelle phase active ou de contrôle), et sur chaque intervalle de temps $[a_j, a_{j+1}]$ avec j impair, on laisse le semi-groupe agir via la proposition 1.3.5 (phase passive ou de relaxation), en utilisant la décroissance naturelle. A cause de la phase active, les modes appartenant à $\{\phi_k,\ \lambda_k > \Lambda_j\}$ sont perturbés, mais la dépendance en la fréquence de coupure Λ_j dans la proposition 1.3.4 est plus faible que dans la décroissance naturelle de la proposition 1.3.5, ce qui permet d'ajuster la subdivision de telle sorte que le procédé converge. On peut ainsi montrer le théorème suivant.

Théorème 1.3.2. *Soit $T > 0$. Il existe une constante $C_T > 0$ telle que pour toute donnée initiale $u_0 \in L^2(\Omega)$, il existe un contrôle $f \in L^2((0,T)\times\Omega)$, tel que la solution u de (1.3.1) satisfait $u(T) = 0$ et*

$$||f||_{L^2((0,T)\times\Omega)} \leq C_T ||u_0||_{L^2(\Omega)}.$$

En fait, dans le chapitre 6, dans le cadre de problèmes singuliers, on obtiendra une inégalité spectrale plus faible que (1.3.3) (voir le théorème 3.4.1 dans l'introduction). Néanmoins, pour la construction du contrôle, il suffit d'avoir

$$||z||_{L^2(\Omega)} \leq e^{C\Lambda^\theta} ||z||_{L^2(\omega)}, \quad z \in \text{Vect}\{\phi_k,\ \lambda_k \leq \Lambda\}, \tag{1.3.8}$$

avec $\theta < 1$.

1.3.3 A propos de l'observabilité

Le théorème 1.3.2 est équivalent à l'inégalité d'observabilité du problème adjoint suivant :

$$\begin{cases} \partial_t z + \Delta z = 0 & \text{dans } (0,T)\times\Omega \\ z_{|\partial\Omega} = 0 & \text{sur } (0,T)\times\partial\Omega \\ z_{|t=T} = z_T & \text{sur } \Omega \end{cases} \tag{1.3.9}$$

Théorème 1.3.3. *Il existe une constante $C_T > 0$ telle que*

$$\int_\Omega |z(0)|^2 \leq C_T \int_0^T \int_\omega |z|^2, \tag{1.3.10}$$

pour tout z solution du problème adjoint (1.3.9), avec donnée initiale $v_T \in L^2(\Omega)$.

Depuis l'inégalité spectrale, la méthode de contrôle/relaxation construit directement un contrôle. Une autre méthode proposée par L.Miller [Mil10], consiste à réaliser le même travail, non pas du côté du contrôle, mais du côté de l'observabilité.
On citera aussi la méthode développée par Fursikov et Imanuvilov [FI96], développée indépendamment de [LR95], qui consiste en l'obtention d'une inégalité de Carleman parabolique, qui implique l'inégalité d'observabilité. Une brève introduction à la construction du poids de Carleman est fournie dans le chapitre 2.

1.3.4 A propos de l'inégalité spectrale dans $\mathbb{R}^d$

La constante $C > 0$ dans l'inégalité spectrale 1.3.3 dépend fortement de la zone d'observabilité ω, mais, pour des domaines Ω bornés, l'inégalité est toujours vérifiée pourvu que ω soit un ouvert non-vide. Sur $\mathbb{R}^d$, on renvoie à l'article de J. Le Rousseau et I. Moyano [LRM16]. En notant $B_{\mathbb{R}^d}(y,r)$ la boule ouverte de dimension d centrée en $y \in \mathbb{R}^d$ et de rayon $r > 0$, les auteurs imposent la condition géométrique sur l'ouvert d'observabilité ω suivante

$$\exists \delta, r > 0,\ \forall y \in \mathbb{R}^d,\ \exists y' \in \omega,\ B_{\mathbb{R}^d}(y',r) \subset \omega \text{ et } |y'-y| \leq \delta. \tag{1.3.11}$$

Théorème 1.3.4. *Soit ω un ouvert de $\mathbb{R}^d$ vérifiant la condition (1.3.11). Alors il existe $C > 0$ tel que*

$$||u||_{L^2(\mathbb{R}^d)} \leq e^{C(N+1)} ||u||_{L^2(\omega)}, \tag{1.3.12}$$

pour tout $N \in \mathbb{R}^+$, et pour tout $u \in L^2(\mathbb{R}^d)$ telle que $\operatorname{supp} \hat{u} \subset B_{\mathbb{R}^d}(0,N)$.

Dans [LRM16], les auteurs utilisent une inégalité de Carleman globale sur $\mathbb{R}^d$ (en effet, des inégalités locales ne peuvent être recollées, car $\mathbb{R}^d$ n'est pas un ensemble compact), et utilisent la condition (1.3.11) pour construire la fonction poids. Cette condition géométrique se traduit par le fait que tout point de $\mathbb{R}^d$ doit être à distance fixe δ d'une partie de l'ouvert ω suffisamment grosse (c'est-à-dire qu'elle contient une boule de rayon fixé r). On peut tout de même se poser la question du caractère optimal d'une telle condition géométrique. On apporte ici une réponse partielle.

Soit ω un ouvert de $\mathbb{R}^d$. Supposons qu'il existe une suite $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ telle que pour tout $y' \in \omega$, on a

$$|y_k - y'| > d_k, \tag{1.3.13}$$

avec $d_k \to +\infty$ lorque $k \to +\infty$. Cette condition traduit le fait qu'il existe des parties de $\mathbb{R}^d$ arbitrairement loin de l'ouvert d'observabilité ω. La proposition suivante montre que l'inégalité spectrale donnée par le théorème 1.3.4 ne peut être vraie pour de tels ouverts ω. La preuve de ce résultat est donnée en appendice.

Proposition 1.3.6. *Soit ω vérifiant (8.A.1). Pour tout $k \in \mathbb{N}$, il existe $N_k \in \mathbb{R}^+$ et $f_k \in L^2(\mathbb{R}^d)$ tels que* $\operatorname{supp} \hat{f}_k \subset B(0,N_k)$ *vérifiant*

$$||f_k||_{L^2(\mathbb{R}^d)} \gtrsim e^{k(N_k+1)} ||f_k||_{L^2(\omega)}.$$

1.4 Stabilisation des ondes

Une autre part importante de ce mémoire concerne la stabilisation de l'équation des ondes. Soit Ω un ouvert borné et régulier de $\mathbb{R}^d$ et ω un sous-ouvert de Ω. Pour g une métrique Riemannienne sur $\mathbb{R}^d$, on considère l'équation des ondes amorties suivante

$$\begin{cases} \partial_t^2 u - \Delta_g + a(x)\partial_t u = 0 \quad \mathbb{R}^+ \times \Omega, \\ u_{|_{t=0}} = u_0,\ \partial_t u_{|_{t=0}} = u_1, \\ u_{|_{\partial\Omega}} = 0. \end{cases} \tag{1.4.1}$$

où Δ_g est l'opérateur de Laplace-Beltrami associé à la métrique g, et où a est une fonction bornée positive, satisfaisant $a(x) \geq C > 0$, pour $x \in \omega$. Multipliant (1.4.1) par $\partial_t u$ et intégrant sur Ω, on obtient

$$\frac{1}{2}\frac{d}{dt}\left(||\nabla_g u(t)||^2_{L^2(\Omega)} + ||\partial_t u(t)||^2_{L^2(\Omega)}\right) = -||a^{1/2}(x)\partial_t u(t)||^2_{L^2(\Omega)} \leq 0,$$

où ∇_g représente le gradient associé à la métrique g. Définissant l'énergie du système (1.4.1) par

$$E(u,t) := ||\nabla_g u||^2_{L^2(\Omega)} + ||\partial_t u||^2_{L^2(\Omega)},$$

on obtient la décroissance en temps de $E(u,t)$. En utilisant les propriétés de semi-groupe du générateur des ondes, ainsi qu'un résultat de prolongement unique, on peut montrer que $E(u,t) \to 0$ lorsque $t \to +\infty$. Une question naturelle est donc l'étude de la vitesse de convergence de cette énergie, c'est-à-dire obtenir une estimation du type

$$E(u,t) \leq f(t)G(u_0,u_1),$$

où f est une fonction décroissante qui tend vers zéro en l'infini, et g une fonction. En fait on peut montrer que si $G(u_0,u_1) = E(u,0)$, c'est-à-dire, que si la vitesse de décroissance est uniforme en la donnée initiale, on a toujours

$$E(u,t) \leq Ce^{-Ct}E(u,0), \tag{1.4.2}$$

grâce aux propriétés de semi-groupe des ondes. Depuis les travaux de [RT74] et [BLR92], on sait que la décroissance exponentielle de l'énergie (1.4.2) est reliée à la condition de contrôle géométrique : toute géodésique généralisée doit entrer dans la zone de contrôle ω en temps fini.

1.4.1 Condition de contrôle géométrique

La condition de contrôle géométrique peut s'écrire de la manière suivante.

Définition 1.4.1 (Condition de Contrôle Géométrique (GCC)). *On dit que le terme d'amortissement $a \in C^0(\overline{\Omega})$ satisfait la condition de contrôle géométrique si pour toute géodésique généralisée $\mathbb{R}^+ \ni t \mapsto x(t)$, il existe $t \geq 0$ tel que $a(x(t)) > 0$.*

Les géodésiques généralisées coïncident avec les géodésiques loin du bord, et suivent les lois de l'optique géométrique (Snell-Descartes) quand elles intéragissent avec le bord, pour des conditions de Dirichlet ou de Neumann homogènes. Nous renvoyons aux travaux de Melrose et Sjöstrand [MS78, MS82], et Hörmander [Hör07], section 24.3.
Pour une approche utilisant les mesures de défaut microlocales, introduites par P. Gérard et L. Tartar [Gér91, Tar90], nous citerons [Leb96, BGé97]. Nous expliquons ici brièvement les idées, dans le cas d'une variété Riemannienne sans bord (M,g), pour éviter les complications liées à l'analyse près du bord.

Théorème 1.4.1 (P. Gérard, L. Tartar). *Soit M une variété compacte sans bord. Soit (u_n) une suite bornée dans $L^2(M)$ convergeant faiblement vers 0. Alors il existe une sous-suite (u_{n_k}) et une mesure de Radon positive μ sur $S^*(M)$ telles que*

$$\lim_{k\to\infty}\left(\mathrm{Op}(a)u_{n_k}, u_{n_k}\right)_{L^2} = \langle \mu, a_0\rangle = \int_{S^*(M)} a_0(x,\xi)d\mu(x,\xi),$$

pour tout $a \in \mathcal{S}^0(M)$, dont la partie principale est notée a_0 (homogène de degré 0).

On appelle μ la mesure de défaut microlocale associée à la suite (u_{n_k}). Elle caractérise l'obstruction à la convergence forte, au sens suivant.

Remarque 1.4.2. *Soit (u_n) une suite bornée de $L^2(M)$ convergeant faiblement vers 0, admettant une mesure de défaut microlocale μ. Si $\mu = 0$, alors (u_n) converge fortement vers 0 dans $L^2(M)$.*

Les propriétés centrales de ces mesures de défaut sont les suivantes.

Proposition 1.4.3. *Soit (u_n) une suite bornée de $L^2(M)$ convergeant faiblement vers 0, admettant une mesure de défaut microlocale μ. Soit P un opérateur différentiel d'ordre $m \in \mathbb{N}$, de symbole principal $\sigma(P) = p$. Alors on a équivalence entre*

1. *Pu_n converge vers 0 dans $H^{-m}(M)$*
2. $\mathrm{supp}(\mu) \subset \mathrm{Char}(P)$.

En particulier, si on prend une suite (u_n) de solutions de l'équation des ondes $\partial_t^2 u_n - \Delta_g u_n = 0$, où Δ_g est l'opérateur de Laplace-Beltrami associé à la métrique g, on voit que la mesure de défaut est supportée dans le "cône de lumière" $\{(t, x, \tau, \xi) \in \mathbb{R} \times M \times \mathbb{R} \times T_x^*(M),\ \tau^2 - g_x(\xi, \xi) = 0\}$. On a aussi le résultat suivant.

Proposition 1.4.4. *Soit (u_n) une suite bornée de $L^2(M)$ convergeant faiblement vers 0, de mesure de défaut μ. On suppose de plus que (u_n) est une suite de solutions de l'équation des ondes de la forme*

$$Pu_n := \partial_t^2 u_n - \Delta_g u_n = f_n,$$

avec $f_n \to 0$ dans $H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times M)$. Alors on a

$$H_p \mu = 0,$$

où H_p est l'Hamiltonien associé au symbole $p(t, x, \tau, \xi) = -\tau^2 + g_x(\xi, \xi)$.

Combinant ces deux résultats, on obtient que la mesure de défaut μ est invariante de long des bi-caractéristiques de p. La stratégie de preuve pour la décroissance exponentielle est la suivante, et elle se structure généralement en trois étapes. Dans un premier temps, on montre l'inégalité d'observabilité relaxée suivante, pour T suffisamment grand (en fait plus grand que deux fois le diamètre de M),

$$E(u, T) \leq C \left(\int_0^T \int_M a(x) |\partial_t u|^2 + E_{-1}(u_n, T) \right), \tag{1.4.3}$$

où $E_{-1}(u, t) = ||u(t, .)||^2_{L^2(M)} + ||\partial_t u(t, .)||^2_{H^{-1}(M)}$, et où u est solution du problème

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0, & \mathbb{R} \times M \\ u_{|_{\partial M}} = 0, & \mathbb{R} \times \partial M \\ u_{|_{t=0}} = u_0 \in H_0^1(M),\ \partial_t u_{|_{t=0}} = u_1 \in L^2(M), \end{cases} \tag{1.4.4}$$

Pour montrer cette inégalité, on suppose par l'absurde que (1.4.3) n'est pas vérifiée, c'est-à-dire, on suppose qu'il existe une suite (u_n) de solutions de (1.4.4) telle que

$$E(u_n, T) = 1, \quad \int_0^T \int_M a(x) |\partial_t u_n|^2 + E_{-1}(u, T) \longrightarrow 0. \tag{1.4.5}$$

Avec cette hypothèse, on peut montrer que la suite converge faiblement vers 0 dans $L^2((0, T) \times M)$. On peut ainsi lui associer une mesure de défaut microlocale μ. En utilisant les propositions ci-dessus, on obtient que la mesure est invariante le long des bi-caractéristiques, et on peut montrer que la mesure μ s'annule sur le support de l'amortissement $\{a > 0\}$. De la condition géométrique de contrôle, on déduit que la mesure est identiquement nulle, ce qui prouve la convergence forte de la suite (u_n) vers 0 dans $L^2((0, T) \times M)$, en

contradiction avec (1.4.5). Dans un second temps, on montre qu'il n'y a pas de solutions *invisibles* autres que la solution nulle, c'est-à-dire

$$\mathcal{N}(T) := \{u \text{ solutions de } \partial_t^2 u - \Delta_g u = 0,\ a(x)|\partial_t u|^2 = 0,\ \text{p.p}\} = \{0\}, \tag{1.4.6}$$

pour $T > 0$ suffisamment grand, grâce à un résultat de prolongement unique. Enfin, on montre l'inégalité d'énergie suivante

$$E(u,T) \leq C \int_0^T \int_M a(x)|\partial_t u|^2, \tag{1.4.7}$$

en utilisant le fait que dans l'inégalité (1.4.3), le terme de droite est un terme compact, et donc grâce au résultat d'unicité (1.4.6), on peut le retirer. On obtient finalement (1.4.2) en utilisant la propriété de semi-groupe des ondes avec l'inégalité d'observabilité (1.4.7).

Notons que la décroissance forte (1.4.2) de l'énergie, avec amortissement par le bord peut être aussi démontrée, (sous des conditions géométriques en général plus fortes que GCC), par des méthodes dites de multiplicateurs. Ces méthodes, introduites par [Ho86], permettent d'obtenir des inégalités d'observabilités, qui induisent la stabilisation exponentielle, ainsi que les résultats de contrôle.

1.4.2 Stabilisation faible de l'équation des ondes

Lorsque la condition de contrôle géométrique n'est pas réalisée, on peut espérer une vitesse de convergence plus faible, c'est-à-dire, polynomiale ou logarithmique. On rappelle que l'on s'intéresse à l'equation des ondes amorties suivante

$$\begin{cases} \partial_t^2 u - \Delta_g u + a(x)\partial_t u = 0 \quad \text{dans } \mathbb{R}^+ \times \Omega, \\ u_{|_{t=0}} = u_0,\ \partial_t u_{|_{t=0}} = u_1, \\ u_{|_{\partial\Omega}} = 0. \end{cases} \tag{1.4.8}$$

Dans [Leb96], G. Lebeau montre le résultat suivant de stabilisation pour des données initiales régulières (voir aussi [LR97, Bur98])

$$E(u,t)^{1/2} \leq \frac{C}{\log(2+t)^k} ||(u_0,u_1)||_{D(A^k)} \tag{1.4.9}$$

pour les solutions fortes de (1.4.8). Ici, $||.||_{D(A^k)}$ représente la norme du graphe de l'opérateur itéré A^k. On voit que le taux de décroissance est d'autant meilleur que la donnée initiale est régulière (on renvoie au théorème suivant). Ce type de résultat est quelquefois appelé stabilisation non-uniforme, car il dépend fortement de la régularité de la donnée initiale (en opposition à la discussion menée en (1.4.2)).

Dans ce cadre, aucune condition géométrique n'est imposée. Même s'il existe des géodésiques qui ne rencontrent jamais la zone d'amortissement, les solutions de l'équation des ondes qui garantissent l'obstruction à la convergence exponentielle, ne sont jamais strictement localisées sur la géodésique. Ainsi, la zone $\{a(x) > 0\}$ amortit toujours un peu ces solutions pathologiques et la décroissance donnée par (1.4.9) traduit ce que l'on appelle l'effet tunnel. On notera de plus que Lebeau [Leb96] (pour la stabilisation interne) et Lebeau et Robbiano [LR97] (pour la stabilisation par le bord) montrent dans des configurations géométriques particulières que la vitesse de décroissance logarithmique est atteinte (1.4.9), prouvant l'optimalité de ce taux de décroissance en l'absence de condition géométrique.

La vitesse de décroissance peut être décrite par la position des valeurs propres (complexes) du générateur des ondes amorties. Plus précisement, on réécrit (1.4.1) sous la forme

$$\partial_t U + AU = 0, \quad U_{t=0} = U_0,$$

où $U = {}^t(u, \partial_t u)$, $A = \begin{pmatrix} 0 & -Id \\ -\Delta_g & a(x) \end{pmatrix}$, et $U_0 = {}^t(u_0, u_1)$. On utilisera à maintes reprises le théorème suivant (voir [Bur98, BD08]).

Théorème 1.4.2. *Soit $\mathbb{S}(t) = e^{-tA}$ un semi-groupe fortement continu sur un espace de Hilbert H, associé au générateur $A : H \to H$ de domaine dense $D(A) \hookrightarrow H$. Supposons que les conditions suivantes soient satisfaites*

- $\sup_{t\in\mathbb{R}^+} |||\mathbb{S}(t)|||_{\mathcal{L}(H,H)} < +\infty$;
- *pour tout $\sigma \in \mathbb{R}$, l'opérateur non-borné $i\sigma \,\mathrm{Id} - BA$ est inversible, d'inverse borné ;*
- *il existe $C > 0$ et $\sigma_0 > 0$ tels que*

$$|||(i\sigma \,\mathrm{Id} - A)^{-1}|||_{\mathcal{L}(H,H)} \leq Ce^{C|\sigma|}, \quad |\sigma| \geq \sigma_0. \tag{1.4.10}$$

alors, pour tout $k \in \mathbb{N}$, il existe $C' > 0$ tel que

$$|||\mathbb{S}(t)A^{-k}|||_{\mathcal{L}(H,H)} \leq C' \log(2+t)^{-k}, \quad t \geq 0. \tag{1.4.11}$$

Dans ce théorème, le cas $k = 0$ correspond à la première hypothèse, i.e le caractère borné du semi-groupe et ne donne pas de vitesse de convergence vers 0 des solutions. Toutefois, on peut obtenir, en utilisant un résultat de prolongement unique, que l'énergie tend bien vers 0, mais sans estimation sur la vitesse de convergence. Avec $H = H_0^1 \times L^2(\Omega)$, le résultat de ce théorème peut se traduire par

$$E(u,t)^{1/2} \leq \frac{C'}{\log(2+t)^k} ||A^k U_0||_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{1.4.12}$$

et on voit que le cas $k = 1$ correspond à prendre des données initiales dans le domaine de l'opérateur $D(A) = D(-\Delta_g) \times D((-\Delta_g)^{1/2})$. La première hypothèse du théorème 1.4.2 est en général facile à vérifier pour l'équation des ondes et est en général une simple conséquence des résultats d'existence de solutions fortes. La seconde hypothèse se déduit de résultats de prolongement unique.
L'hypothèse la plus délicate est la troisième (1.4.11). Elle traduit le fait que les valeurs propres complexes de l'opérateur A se situent en dehors d'une zone du type $\{z \in \mathbb{C},\ \mathrm{Re}\, z \in [0, e^{C|z|}C^{-1}]\}$. Montrons ici qu'elle est une conséquence d'inégalités d'observabilité pour des opérateurs elliptiques.

Soit $U = {}^t(v_0, v_1) \in D(A)$ et $F = {}^t(f_0, f_1) \in H_0^1(\Omega) \times L^2(\Omega)$ tels que

$$(i\sigma \,\mathrm{Id} - A)U = F, \quad \sigma \in \mathbb{R}.$$

C'est équivalent à

$$\begin{cases} i\sigma v_0 + v_1 = f_0 \\ i\sigma v_1 - a(x)v_1 + \Delta_g v_0 = f_1, \end{cases} \iff \begin{cases} i\sigma v_0 + v_1 = f_0 \\ -\Delta_g v_0 - \sigma^2 v_0 - ia(x)\sigma v_0 = g, \end{cases}$$

avec $g := (i\sigma - a)f_0 - f_1$. On multiplie par $\overline{v_0}$ la deuxième ligne et on intègre par partie sur Ω, et cela donne

$$||\nabla v_0||^2_{L^2(\Omega)} - \sigma^2 ||v_0||^2_{L^2(\Omega)} - i\sigma \int_\Omega a(x)|v_0|^2 = \int_\Omega g\overline{v_0}.$$

On prend la partie imaginaire, et on utilise le fait que $a \geq C > 0$ sur ω pour obtenir

$$C\sigma \int_\omega |v_0|^2 \leq ||g||_{L^2(\Omega)} ||v_0||_{L^2(\Omega)}. \tag{1.4.13}$$

Si l'on obtient l'inégalité d'observabilité suivante, on en déduit alors facilement $||u_0||_{H^1} \leq Ce^{C|\sigma|}||g||_{L^2}$, ce qui permet de conclure que l'hypothèse (1.4.11) est satisfaite.

Proposition 1.4.5. *Il existe $C > 0$ et $\sigma_0 > 0$ tels que*

$$||v_0||^2_{H^1(\Omega)} \leq Ce^{C|\sigma|} \left(|| - \Delta_g v_0 - \sigma^2 v_0||^2_{L^2(\Omega)} + ||v_0||^2_{L^2(\omega)} \right) \tag{1.4.14}$$

pour tout $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ et pour tout $|\sigma| \geq \sigma_0$.

La proposition 1.4.5 quant à elle s'obtient grâce à des inégalités d'interpolation du même type que (1.3.2), elles-mêmes provenant d'inégalités de Carleman elliptiques adaptées (voir [Leb96, LR97]). Nous montrerons dans ce mémoire à maintes reprises des estimées du même type que celle donnée par la proposition 1.4.5.

Pour démontrer des inégalités d'observabilité de la forme (1.4.14), on peut passer par l'obtention d'inégalités de Carleman pour des opérateurs elliptiques augmentés, comme c'est la cas pour montrer l'inégalité spectrale (voir la proposition 1.3.2). C'est la méthode originale (voir [Leb96, LR97]).
On introduit le cylindre $Z = (0, S_0) \times \Omega$, et pour $0 < \alpha < S_0/2$, on définit $Y = (\alpha, S_0 - \alpha) \times \Omega$. On pose $w_0 = e^{s\sigma} v_0$. On a $e^{s\sigma}(-\Delta v_0 - \sigma^2 v_0) = -\Delta w_0 - \partial_s^2 w_0$, et ainsi, démontrer des inégalités d'observabilité de la forme (1.4.14) est équivalent à montrer des inégalités d'interpolation, comme dans la proposition suivante.

Proposition 1.4.6. *Soit ω un ouvert non vide de Ω. Il existe $C > 0$ et $\delta \in (0, 1)$ tels que*

$$||w_0||_{H^1(Y)} \leq C||w_0||_{H^1(Z)}^{1-\delta} \left(|| - \partial_s^2 w_0 - \Delta w_0||_{L^2(Z)} + ||w_0||_{L^2((\alpha, S_0-\alpha)\times\omega)} \right)^{\delta},$$

pour toute fonction $w_0 \in H^2(Z)$ telle que $w_{0|_{x\in\partial\Omega}} = 0$.

Notons toutefois que dans le cadre de la stabilisation des ondes, le passage à l'opérateur augmenté n'est pas une nécessité. On peut montrer directement des inégalités de la forme (1.4.14), en travaillant sur l'opérateur $-\Delta - \sigma^2$ (méthode qui a été introduite par M. Bellassoued [Bel00, Bel03a, Bel03b]). Dans [Buf17], nous utilisons cette méthode, ainsi que dans la partie III, pour éviter certaines complications liées à l'ajout de la variable supplémentaire s.

Il existe des situations intermédiaires entre celles données par la décroissance exponentielle (liée à la condition de contrôle géométrique), et celles données par la décroissance logarithmique (liée à l'effet tunnel, et sous aucune condition géométrique). Quelques travaux montrent des résultats de décroissance polynomiale tels que [LR05, BH07, Phu07] et sont liés à des propriétés assez fines de stabilité du flot géodésique. Dans ces travaux, les auteurs imposent des géométries particulières, où si l'on perturbe légérement les géodésiques qui ne rentrent pas dans la zone d'amortissement, celles-ci finissent par s'y rendre en temps fini. C'est notamment le cas dans [LR05], où le domaine est un carré et l'amortissement une bande verticale.

Chapitre 2

Inégalités de Carleman

Les inégalités de Carleman sont des inégalités d'énergie *a priori*, sur les solutions d'équations aux dérivées partielles, de la forme

$$||e^{\tau\varphi}u|| \lesssim ||e^{\tau\varphi}Pu||, \tag{2.0.1}$$

où P est un opérateur, u une fonction, φ une fonction appelée *fonction poids*, et $\tau > 0$ un paramètre réel, supposé pris grand. Elles ont été introduites en premier lieu par T. Carleman afin d'obtenir des propriétés de prolongement unique pour des opérateurs elliptiques dans [Car39]. Leur usage quasi-systématique a suivi les travaux de Lars Hörmander [Hör58]. L'intérêt de telles inégalités est la présence du poids φ qui permet de "propager" de l'information depuis les zones où φ est grand vers l'ensemble du domaine, au moyen du grand paramètre $\tau > 0$, que l'on appelle grand paramètre de Carleman. Le lien fort entre inégalités de Carleman et les propriétés de prolongement unique, ainsi que les inégalités d'observabilité en font un outil central en théorie du contrôle [LR95, FI96], stabilisation [LR97] et problèmes inverses [DSFKSU09]. Ces inégalités peuvent être de natures différentes (locales ou globales). Notons qu'une des difficultés principales pour montrer des inégalités globales est la construction de la fonction poids globale. Les inégalités locales peuvent toutefois être recollées (voir par exemple le livre de Hörmander [Hör76], ou la revue [LL12]).

L'obtention d'inégalités locales a été intensivement étudiée à l'intérieur [Hör58], puis au bord pour des conditions diverses (Dirichlet [LR95] ou Neumann [LR97], par exemple). Au voisinage d'une interface, on citera les travaux successifs de [LR10, LR11, LL13] pour des conditions de transmission, ou encore [LLR13] pour des conditions de transmission avec une interface diffusive. Plus généralement, pour des opérateurs elliptiques d'ordre quelconque, au voisinage du bord, sous des conditions de compatibilité de type Lopatinski, on renvoie à [BL15] et [Tat96]. Ces résultats ont été généralisés aux interfaces dans [BL16], et nous proposons une généralisation aux multi-interfaces dans ce mémoire. Dans ces séries d'articles, la frontière (respectivement l'interface) est supposée régulière, car de nombreux arguments pseudo-différentiels sont utilisés, et notamment les caractéristiques complexes des opérateurs sont prises en compte, en utilisant un calcul tangentiel. On renvoie à la section 2.2 pour plus de détails.

Pour des problèmes moins réguliers, il n'existe pas à notre connaissance de résultats généraux. Toutefois plusieurs inégalités de Carleman ont été obtenues. Par exemple, lorsqu'une interface rencontre le bord, on citera [BDL11] où les auteurs considèrent un milieu stratifié, et [BDT13] où certaines hypothèses sur les coefficients sont relâchées. Pour des bords non-réguliers, un résultat principal à été démontré [Bel14] pour des conditions au bord de Dirichlet. Nous apportons une contribution à ce sujet dans la partie III en dimension deux d'espace, où nous démontrons des inégalités de Carleman locales, pour des conditions au bord de type Dirichlet homogène ou Neumann homogène. Le cas d'une interface qui rencontre le bord est aussi considéré. Remarquons que dans le cadre de conditions mixtes (ou problème de Zaremba), une inégalité de Carleman locale a été obtenue dans [CR14].

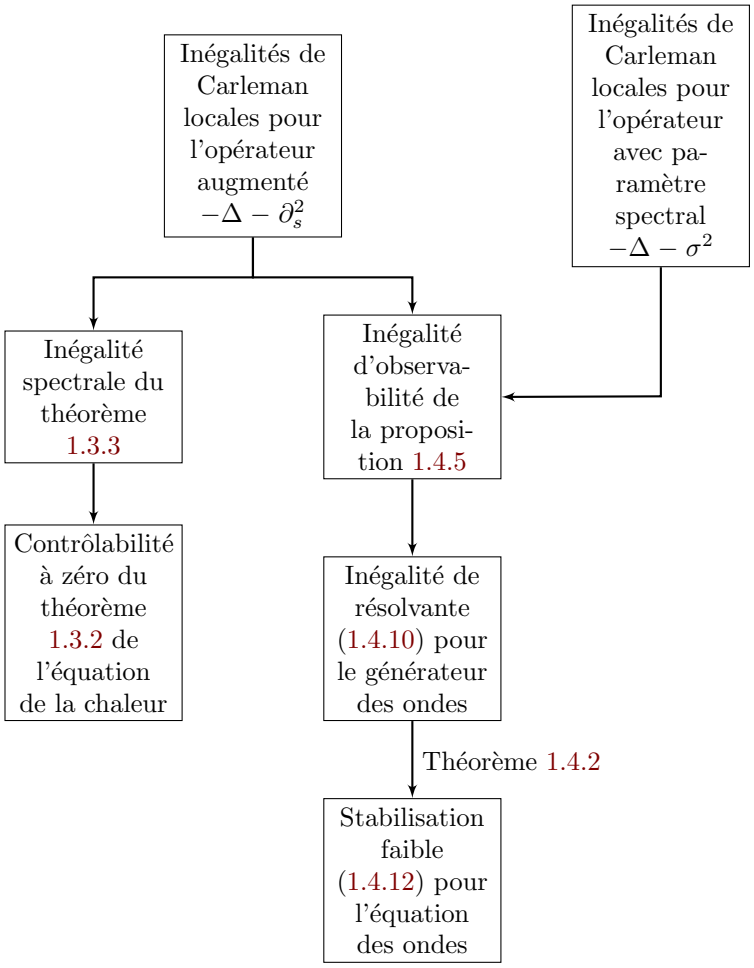


FIGURE 2.1: Exemples d'application des inégalités de Carleman elliptiques en théorie du contrôle.

Pour motiver l'introduction des inégalités de Carleman elliptiques, nous proposons la figure 2.1, qui résume les propriétés et applications en théorie du contrôle, données dans la section précédente. Dans ce qui suit, nous présentons quelques méthodes essentielles pour obtenir des inégalités de Carleman.

2.1 Inégalités de Carleman elliptiques loin du bord

Soit P un opérateur elliptique d'ordre 2. La technique classique pour montrer des inégalités de Carleman est d'introduire l'opérateur conjugué et la fonction conjuguée

$$P_\varphi := e^{\tau\varphi} P e^{-\tau\varphi}, \quad v = e^{\tau\varphi} u.$$

Ainsi, montrer une inégalité du type (2.0.1) est équivalent à montrer l'inégalité suivante

$$||v|| \lesssim ||P_\varphi v||. \tag{2.1.1}$$

La principale difficulté pour montrer de telles inégalités (2.1.1) est le fait que l'opérateur conjugué n'est plus un opérateur elliptique. En effet, supposons pour simplifier que $P = -\Delta$. Ainsi, on a

$$P_\varphi = -\Delta - \tau^2 |\nabla\varphi|^2 + 2\tau \nabla\varphi.\nabla + \tau\Delta\varphi, \tag{2.1.2}$$

et son symbole principal (au sens semi-classique à grand paramètre) est donné par

$$p_\varphi(x,\xi,\tau) := |\xi|^2 - \tau^2|\nabla\varphi|^2 + 2i\nabla\varphi.\xi. \tag{2.1.3}$$

On voit bien que, en dimension supérieure ou égale à 2, P_φ possède un espace caractéristique, c'est-à-dire

$$\mathrm{Char}(p_\varphi) := \big\{(x,\xi,\tau),\ p_\varphi(x,\xi,\tau) = 0\big\} \neq \emptyset.$$

En effet,

$$(x,\xi,\tau) \in \mathrm{Char}(p_\varphi) \quad \Longleftrightarrow \quad |\xi| = |\tau d_x\varphi| \text{ et } \xi.\nabla\varphi = 0.$$

On a ainsi aucune chance de récupérer une estimée elliptique. Le point clé est donc de construire une fonction poids φ satisfaisant la condition de *sous-ellipticité* de Hörmander. Pour cela, on sépare l'opérateur en une partie symétrique et une partie anti-symétrique

$$P_\varphi = \frac{1}{2}\left(P_\varphi + P_\varphi^*\right) + i\frac{1}{2i}\left(P_\varphi - P_\varphi^*\right) := P_2 + iP_1,$$

avec

$$P_2 = \frac{1}{2}\left(P_\varphi + P_\varphi^*\right) = -\Delta - \tau^2|\nabla\varphi|^2, \quad P_1 = \frac{1}{2i}\left(P_\varphi - P_\varphi^*\right) = \frac{1}{i}\left(2\tau\nabla\varphi.\nabla + \tau\Delta\varphi\right),$$

de symboles principaux (au sens semi-classique à grand paramètre)

$$p_2(x,\xi,\tau) = |\xi|^2 - \tau^2|\nabla\varphi|^2, \quad p_1(x,\xi,\tau) = 2\tau\nabla\varphi.\xi.$$

Définition 2.1.1. *Soit V un ouvert borné de $\mathbb{R}^d$. La fonction $\varphi \in C^\infty(\overline{V})$ et P satisfont la condition de sous-ellipticité si $\nabla\varphi \neq 0$ pour tout $x \in \overline{V}$, et si pour tout $(x,\xi,\tau) \in \mathrm{Char}(P_\varphi)$, on a*

$$\{p_2,p_1\}(x,\xi,\tau) > 0, \quad x \in \overline{V},\ \xi \in \mathbb{R}^d,\ \tau > 0,$$

où le crochet de Poisson est donné par

$$\{p_2,p_1\}(x,\xi,\tau) = \partial_\xi p_2(x,\xi,\tau)\partial_x p_1(x,\xi,\tau) - \partial_x p_2(x,\xi,\tau)\partial_\xi p_1(x,\xi,\tau).$$

L'idée est la suivante. Dans les zones microlocales où l'opérateur P_φ n'est pas elliptique, on va chercher à utiliser le commutateur $[P_2,P_1|$ pour obtenir une estimée (on rappelle que le symbole principal du commutateur $i[P_2,P_1]$ est donné par le crochet de Poisson $\{p_2,p_1\}$). Ainsi, on ne récupère pas une inégalité de régularité elliptique car le commutateur est d'ordre 3 (au sens semi-classique), là où P_φ^2 est d'ordre 4. On parle alors de perte de *demi dérivée*. Pour simplifier, nous allons faire la preuve dans le cas où $P = -\Delta$.

Théorème 2.1.1 (Inégalité de Carleman à l'intérieur)**.** *Soit U un ouvert borné de $\mathbb{R}^d$, et soit $\varphi \in C^\infty(\overline{U})$ une fonction poids satisfaisant la condition de sous-ellipticité de la définition (2.1.1). Alors il existe deux constantes $C > 0$ et $\tau_0 > 0$ telles que*

$$\tau^3\int_{\mathbb{R}^d} e^{\tau\varphi}|u|^2 + \tau\int_{\mathbb{R}^d} e^{\tau\varphi}|\nabla u|^2 \leq C\int_{\mathbb{R}^d} e^{\tau\varphi}|\Delta u|^2,$$

pour tout $u \in C_0^\infty(U)$, et tout $\tau \geq \tau_0$.

Une caractéristique intéressante des inégalités de Carleman est l'invariance par l'addition de termes inférieurs. Si on ajoute à l'opérateur $-\Delta$ des potentiels L^∞, l'inégalité du théorème 2.1.1 reste encore vraie, quitte à prendre le grand paramètre τ plus grand, c'est-à-dire, τ_0 plus grand. La grande difficulté des inégalités de Carleman est la construction de fonctions poids adaptées au problème considéré. Dans le cadre décrit ci-dessus, une manière d'obtenir un poids φ satisfaisant la condition de sous-ellipticité de la définition 2.1.1, est le lemme suivant, prouvé par Hörmander dans [Hör76].

Lemme 2.1.2. *Soit $\psi \in C^\infty(\overline{U})$ telle que $\nabla\psi(x) \neq 0$, pour tout $x \in \overline{U}$, et soit $\varphi = e^{\lambda\psi}$ avec $\lambda > 0$. Il existe $\lambda_0 > 0$ tel que φ et P satisfont la condition de sous-ellipticité de la définition 2.1.1 pour tout $\lambda \geq \lambda_0$.*

Ce lemme est appelé procédure de convexification. Dans le théorème 2.1.1, la constante $C > 0$ est indépendante de τ. Si on choisit le poids φ donné par le lemme 2.1.2, la constante $C > 0$ dans le théorème 2.1.1 dépend du paramètre λ. Tracer la dépendance en $\lambda > 0$ a été réalisé par Fursikov et Imanuvilov [FI96]. Deux méthodes pour montrer de telles inégalités existent, les deux étant fondées sur des intégrations par parties. La première consiste à calculer le produit croisé

$$i\int_{\mathbb{R}^d} P_2 v\overline{\tilde{P}_1 v},$$

où $\tilde{P}_1$ est une perturbation de P_1 par un terme d'ordre inférieur, afin d'en tirer une borne inférieure. Cette méthode, quoique directe, est assez calculatoire. Elle est efficace lorsque l'on traite des opérateurs à coefficients de régularité basse, par exemple Lipschitz.

Preuve du théorème 2.1.1 par la première méthode. Posons $v = e^{\tau\varphi}u$, où u est une fonction donnée par le théorème. La première astuce, consiste à modifier légérement le terme en P_1, par l'addition d'un terme d'ordre inférieur, afin de simplifier les calculs. On pose $i\tilde{P}_1 := 2\tau\nabla\varphi.\nabla + 2\tau\Delta\varphi$, et on pose $R = -\tau\Delta\varphi = i(\tilde{P}_1 - P_1)$. On a

$$||Rv||^2_{L^2(\mathbb{R}^d)} + ||P_\varphi v||^2_{L^2(\mathbb{R}^d)} \geq ||P_2 v||^2_{L^2(\mathbb{R}^d)} + ||\tilde{P}_1 v||^2_{L^2(\mathbb{R}^d)} + 2\,\mathrm{Re}\, i\int_{R^d} P_2 v\overline{\tilde{P}_1 v}.$$

Nous allons obtenir une borne inférieure du produit croisé. On rappelle que

$$P_2 = -\Delta - \tau^2|\nabla\varphi|^2, \quad i\tilde{P}_1 = 2\tau\nabla\varphi.\nabla + 2\tau\Delta\varphi. \tag{2.1.4}$$

Nous appellerons I_{ij}, $i,j \in \{1,2\}$, le produit croisé entre le i-ème terme de P_2 avec le j-ième terme de $i\tilde{P}_1$. On calcule alors, par intégrations par parties,

$$\begin{aligned}
I_{12} &= -2\tau\int_{\mathbb{R}^d} \Delta\varphi v\overline{\Delta v} = 2\tau\int_{\mathbb{R}^d} \Delta\varphi|\nabla v|^2 - \tau\int_{\mathbb{R}^d} \Delta^2\varphi|v|^2,\\
I_{11} &= -2\tau\int_{\mathbb{R}^d} \Delta v\nabla\varphi.\overline{\nabla v} = 2\tau\int_{\mathbb{R}^d} \mathrm{d}^2\varphi(\nabla v,\nabla v) - \tau\int_{\mathbb{R}^d} \Delta\varphi|\nabla v|^2,\\
I_{22} &= -2\tau^3\int_{\mathbb{R}^d} |\nabla\varphi|^2\Delta\varphi|v|^2,\\
I_{21} &= \tau^3\int_{\mathbb{R}^d} \left(|\nabla\varphi|^2\Delta\varphi + 2\mathrm{d}^2\varphi(\nabla\varphi,\nabla\varphi)\right)|v|^2.
\end{aligned}$$

Ici, il n'y a pas de termes de frontière, car les fonctions sont supportées loin d'un quelconque bord. Additionnant ces quatre intégrales, on obtient

$$\sum_{i,j=1}^{2} I_{ij} = \tau\int_{\mathbb{R}^d} \left(2\mathrm{d}^2\varphi(\nabla v,\nabla v) + \Delta\varphi|\nabla v|^2\right) + \tau^3\int_{\mathbb{R}^d} \left(-|\nabla\varphi|^2\Delta\varphi + 2\mathrm{d}^2\varphi(\nabla\varphi,\nabla\varphi)\right)|v|^2 - \tau\int_{\mathbb{R}^d} \Delta^2\varphi|v|^2.$$

Utilisant la forme particulière du poids $\varphi = e^{\lambda\psi}$ avec $\psi(x) \neq 0$ sur $\overline{U}$, on a $-|\nabla\varphi|^2\Delta\varphi + 2\mathrm{d}^2\varphi(\nabla\varphi,\nabla\varphi) \geq C|\nabla\varphi|^2 \geq C' > 0$ et $2\mathrm{d}^2\varphi(\nabla v,\nabla v) + \Delta\varphi|\nabla v|^2 \geq C|\nabla v|^2$, en prenant λ suffisamment grand. Ainsi, il existe des constantes $C, C', C'' > 0$ telles que

$$\begin{aligned}
C||P_\varphi v||^2_{L^2(\mathbb{R}^d)} &\gtrsim \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||v||^2_{L^2(\mathbb{R}^d)} - C'\tau||v||^2_{L^2(\mathbb{R}^d)} - C||Rv||^2_{L^2(\mathbb{R}^d)}\\
&\gtrsim \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||v||^2_{L^2(\mathbb{R}^d)} - C'\tau||v||^2_{L^2(\mathbb{R}^d)} - C''\tau^2||v||^2_{L^2(\mathbb{R}^d)}\\
&\gtrsim \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||v||^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

en prenant τ suffisamment grand. Il suffit alors de revenir aux variables initiales $v = e^{\tau\varphi}u$, en remarquant que

$$\begin{aligned}\tau||e^{\tau\varphi}\nabla u||^2_{L^2(\mathbb{R}^d)} + \tau^3||e^{\tau\varphi}u||^2_{L^2(\mathbb{R}^d)} &\leq \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||e^{\tau\varphi}v||^2_{L^2(\mathbb{R}^d)} + \tau^2|||\nabla\varphi|v||^2_{L^2(\mathbb{R}^d)}\\ &\lesssim \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||e^{\tau\varphi}v||^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

pour τ suffisamment grand, ce qui termine la preuve. □

La seconde preuve est basée sur l'inégalité de Gårding et fait apparaître explicitement le crochet de Poisson.

Seconde preuve du théorème 2.1.1. Comme pour la première démonstration, on pose $v = e^{\tau\varphi}u$ et on a

$$||P_\varphi v||^2 = ||P_2 v||^2_{L^2(\mathbb{R}^d)} + ||P_1 v||^2_{L^2(\mathbb{R}^d)} + 2\,\mathrm{Re}\left(P_2 v, iP_1 v\right)_{L^2(\mathbb{R}^d)}.$$

L'opérateur P_2 est symétrique et l'opérateur iP_1 est anti-symétrique, ainsi on a l'égalité

$$2\,\mathrm{Re}\left(P_2 v, iP_1 v\right)_{L^2(\mathbb{R}^d)} = i\Big([P_2, P_1]v, v\Big)_{L^2(\mathbb{R}^d)}.$$

Là où, dans la preuve précédente, on ne calcule que le terme avec le commutateur (bien qu'il soit perturbé par un terme d'ordre inférieur), ici nous allons utiliser aussi le "carré", c'est-à-dire, le terme $||P_2 v||^2_{L^2(\mathbb{R}^d)}$. Soit $\mu > 0$ un paramètre réel. L'égalité $||P_2 v||^2_{L^2(\mathbb{R}^d)} = \left(P_2^2 v, v\right)_{L^2(\mathbb{R}^d)}$ entraine, pour $\tau \geq \mu$,

$$||P_\varphi v||^2_{L^2(\mathbb{R}^d)} \geq \Big(\Big(\frac{\mu}{\tau}P_2^2 + i[P_2, P_1]\Big)v, v\Big)_{L^2(\mathbb{R}^d)} = \frac{1}{\tau}\Big(\big(\mu P_2^2 + i\tau[P_2, P_1]\big)v, v\Big)_{L^2(\mathbb{R}^d)}. \tag{2.1.5}$$

Or, le symbole principal de l'opérateur $\mu P_2^2 + i\tau[P_2, P_1]$ est donné par

$$m(x,\xi,\tau) = \mu p_2^2(x,\xi,\tau) + \tau\{p_2, p_1\}(x,\xi,\tau) \in \mathcal{S}^4_\tau.$$

Les espaces de symboles $\mathcal{S}^m_\tau$ sont définis dans la section 4.3. Pour μ pris suffisamment grand, $m(x,\xi,\tau) > \lambda(x,\xi,\tau)^4$, pour tout $x \in \overline{U}$, tout $\xi \in \mathbb{R}^d$, tout $\tau > 0$, avec $\lambda = (\tau^2 + |\xi|^2)^{1/2}$. En effet, comme le symbole m est homogène de degré 4 en (ξ, τ), on se restreint à prendre des éléments $\rho = (x,\xi,\tau) \in \mathbb{S}_{|(\xi,\tau)|=1} := \{(x,\xi,\tau) \in \overline{U} \times \mathbb{R}^d \times \mathbb{R}^+,\ (\xi,\tau)| = 1\}$. Remarquons que $\mathbb{S}_{|(\xi,\tau)|=1}$ est un ensemble compact.

Soit $\varepsilon > 0$, et soit $A_\varepsilon = \{\rho \in \mathbb{S}_{|(\xi,\tau)|=1}, \{p_2, p_1\} > \varepsilon\}$. Par compacité, utilisant l'hypothèse de sous-ellipticité de la définition 2.1.1, il existe un $\varepsilon > 0$ tel que $p_2(x,\xi,\tau) \geq K > 0$ sur $\mathbb{S}_{|(\xi,\tau)|=1} \setminus A_\varepsilon$. Ainsi, prenant $\mu > 0$ tel que $\mu > \frac{\sup_{\rho\in\mathbb{S}_{|(\xi,\tau)|=1}} |\{p_2,p_1\}(\rho)|}{K}$, on a,

$$m(x,\xi,\tau) \geq C > 0, \quad (x,\xi,\tau) \in \mathbb{S}_{|(\xi,\tau)|=1}.$$

Ainsi, par homogénéité, on a

$$m(x,\xi,\tau) \geq C\lambda^4, \quad (x,\xi,\tau) \in \overline{U} \times \mathbb{R}^d \times \mathbb{R}^+.$$

On peut donc appliquer l'inégalité de Gårding à l'inégalité (2.1.5) pour obtenir

$$C||P_\varphi v||^2_{L^2(\mathbb{R}^d)} \geq \tau^{-1}\sum_{|\alpha|=2} ||D^\alpha v||^2_{L^2(\mathbb{R}^d)} + \tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau^3||v||^2_{L^2(\mathbb{R}^d)}.$$

Il reste à revenir à la variable originale $u = e^{\tau\varphi}v$ comme dans la fin de la première preuve pour obtenir le résultat. □

On remarque qu'ici, on arrive à estimer une norme H^2 sur la fonction u, mais avec un paramètre τ^{-1} devant, ce qui reste néanmoins une perte de demi-dérivée au sens semi-classique. Cette approche est particulièrement adaptée pour des estimations microlocales, c'est-à-dire, s'il faut séparer l'espace des phases $\{(x,\xi,\tau)\}$ en différentes zones.

2.2 Inégalités de Carleman elliptiques près du bord

Près du bord, les inégalités de Carleman sont plus délicates à démontrer, et dépendent assez fortement de la condition au bord imposée.

2.2.1 La méthode de Carleman

Un résultat clé est la proposition suivante, prouvée en premier par Lebeau et Robbiano [LR95], dans le cadre du contrôle de l'équation de chaleur. C'est l'équivalent du théorème 2.1.1, au voisinage d'un bord. La stratégie de preuve est la même que pour la seconde preuve du théorème 2.1.1, mais cette fois-ci, à cause de la présence d'un bord, on utilisera un calcul pseudo-différentiel tangentiel, ainsi qu'une inégalité de Gårding tangentielle. De plus, loin du bord, seul le carré $||P_2u||^2_{L^2}$ est utilisé, alors que dans la preuve de ce résultat près du bord, on utilisera les deux carrés $||P_1u||^2_{L^2}$ et $||P_2u||^2_{L^2}$. Pour la preuve, on renvoie le lecteur à la section 9.E.3 de ce mémoire. On supposera pour simplifier ici que $P = -\Delta$, $\Omega = \mathbb{R}^d_+$, et $\partial\Omega = \{x_d = 0\}$, mais cette preuve d'adapte à un cadre plus général, en utilisant les coordonnées géodésiques normales lorsque l'opérateur est à coefficients variables, et le bord $\partial\Omega$ régulier. Pour V un voisinage de 0, on définit

$$\overline{C}^\infty_0(V^+) := \{u := v_{|_{\mathbb{R}^d_+}},\ v \in C^\infty_0(V)\}.$$

Proposition 2.2.1. *Soit U un voisinage de 0 dans $\mathbb{R}^d$ et soit φ une fonction poids satisfaisant la condition de sous-ellipticité de Hörmander de la définition 2.1.1 dans $\overline{U}^+$, avec $U^+ := U \cap \mathbb{R}^d_+$, et supposons que $|\partial_{x_d}\varphi| \geq C > 0$ dans $\overline{U}$. Alors il existe un voisinage V de 0, deux constantes $\tau_0 > 0$ et $C > 0$ tels que*

$$C\tau^3||v||^2_{L^2(\mathbb{R}^d)} + C\tau||\nabla v||^2_{L^2(\mathbb{R}^d)} + \tau\,\mathrm{Re}\,\mathcal{B}(v) \leq ||P_\varphi v||^2_{L^2(\mathbb{R}^d)}$$

pour tout $v \in \overline{C}^\infty_0(V^+)$ et $\tau \geq \tau_0|\sigma|$, où

$$\mathcal{B}(v) = 2\Big(\partial_{x_d}\varphi D_d v_{|_{x_d=0}}, D_d v_{|_{x_d=0}}\Big)_{L^2(\mathbb{R}^{d-1})} + \Big(A_1 v_{|_{x_d=0}}, D_d v_{|_{x_d=0}}\Big)_{L^2(\mathbb{R}^{d-1})} + \Big(D_d v_{|_{x_d=0}}, A_1' v_{|_{x_d=0}}\Big)_{L^2(\mathbb{R}^{d-1})} + \Big(A_2 v_{|_{x_d=0}}, v_{|_{x_d=0}}\Big)_{L^2(\mathbb{R}^{d-1})}.$$

Les opérateurs A_1, A_1' et A_2 sont des opérateurs différentiels et satisfont

- *A_1, $A_1' \in \mathcal{D}^1_{T,\tau}$ et de plus*
$$a_1 := \sigma(A_1) = \sigma(A_1') = 2\nabla'\varphi.\xi';$$
- *$A_2 \in \mathcal{D}^2_{T,\tau}$ et de plus*
$$a_2 := \sigma(A_2) = 2\partial_{x_d}\varphi\left(\tau^2|\nabla'\varphi|^2) - |\xi'|^2\right).$$

Les espaces $\mathcal{D}^k_{T,\tau}$ sont définis à la section 4.3.2. Brièvement, $A \in \mathcal{D}^k_{T,\tau}$ si A est différentiel en la variable tangentielle x', et si les ordres cumulés de τ et de $\partial_{x'}$ sont au plus égaux à k. Nous pouvons voir rapidement que pour une condition de Dirichlet homogène (comme c'est le cas dans [LR95]), la forme quadratique au bord $\mathcal{B}(v)$ s'écrit

$$\mathcal{B}(v) = 2\Big(\partial_{x_d}\varphi D_d v_{|_{x_d=0}}, D_d v_{|_{x_d=0}}\Big)_{L^2(\mathbb{R}^{d-1})},$$

qui est positive pour $\partial_{x_d}\varphi > 0$ et négative pour $\partial_{x_d}\varphi < 0$. Ainsi, posant $u = e^{\tau\varphi}v$, on obtient le théorème suivant.

Théorème 2.2.1 (Inégalité de Carleman près du bord)**.** *Soit U un ouvert borné de $\mathbb{R}^d$, et soit $\varphi \in C^\infty(\overline{U})$ une fonction poids satisfaisant la condition de sous-ellipticité de la définition (2.1.1) dans $\overline{U^+}$, avec $U^+ := U \cap \mathbb{R}^d_+$.*

Si de plus $\partial_{x_d}\varphi(0) > 0$, alors il existe un voisinage V de 0 dans $\mathbb{R}^d$ tel que $V \subset U$ et deux constantes $C > 0$ et $\tau_0 > 0$ tels que

$$\tau^3 \int_{\mathbb{R}^d} e^{\tau\varphi}|u|^2 + \tau \int_{\mathbb{R}^d} e^{\tau\varphi}|\nabla u|^2 + \tau \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|\partial_{x_d} u_{|_{x_d=0}}|^2 \\ \leq C\left(\int_{\mathbb{R}^d} e^{\tau\varphi}|\Delta u|^2 + \tau^3 \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|u_{|_{x_d=0}}|^2 + \tau \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|\nabla' u_{|_{x_d=0}}|^2 \right),$$

pour tout $u \in \overline{C}_0^\infty(V^+)$, et tout $\tau \geq \tau_0$. Si le signe de $\partial_{x_d}\varphi(0)$ n'est pas préscrit, on a

$$\tau^3 \int_{\mathbb{R}^d} e^{\tau\varphi}|u|^2 + \tau \int_{\mathbb{R}^d} e^{\tau\varphi}|\nabla u|^2 \leq C\Big(\int_{\mathbb{R}^d} e^{\tau\varphi}|\Delta u|^2 \\ + \tau^3 \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|u_{|_{x_d=0}}|^2 + \tau \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|\nabla' u_{|_{x_d=0}}|^2 + \tau \int_{\mathbb{R}^{d-1}} e^{\tau\varphi}|\partial_{x_d} u_{|_{x_d=0}}|^2 \Big),$$

pour tout $u \in \overline{C}_0^\infty(V^+)$, et tout $\tau \geq \tau_0$.

L'inégalité de Carleman de la proposition 2.2.1 est adaptée pour une condition au bord de Dirichlet. Pour d'autre types de conditions au bord, c'est plus délicat. Pour une condition de Neumann homogène par exemple, c'est-à-dire, $\partial_{x_d} u = 0$, on a comme condition au bord sur v une condition de type Robin

$$\partial_{x_d} v = \tau \partial_{x_d}\varphi v,$$

et injectant cette relation dans $\mathcal{B}(v)$ dans la proposition 2.2.1, on obtient un opérateur différentiel agissant sur la trace Dirichlet dont le symbole principal est donné par

$$2\partial_{x_d}\varphi \left(\tau^2 \left((\partial_{x_d}\varphi)^2 + 2|\nabla'\varphi|^2 \right) - 2|\xi'|^2 \right).$$

On voit donc que le signe de la forme quadratique au bord $\mathcal{B}(v)$ n'est pas prescrit dans le cadre d'une condition au bord de type Neumann, car regardant les régimes $|\xi'| \ll \tau$ et $\tau \ll |\xi'|$, le signe du symbole de $\mathcal{B}(v)$ change. Dans la suite, on supposera $\partial_{x_d}\varphi > 0$ (ce qui correspond à une observation depuis l'intérieur jusqu'au bord). Pour traiter cette difficulté, le point clé d'une démonstration dans le cadre de conditions de type Neumann est basé sur un découpage adéquat de l'espace des phases $\{(x, \xi, \tau)\}$. Premièrement, on remarque que sur un voisinage (conique) de l'espace caractéristique de l'opérateur à l'intérieur P_φ, c'est-à-dire, l'endroit où la proposition 2.2.1 est nécessaire, la forme quadratique $\mathcal{B}(v)$ s'avère être positive, ce qui donne une estimée dans cette zone, microlocalement. En dehors de cette zone, où la forme quadratique au bord peut être négative, on utilisera le fait que l'opérateur P_φ est elliptique grâce à la méthode des projecteurs de Calderón, ce qui permet d'obtenir des inégalités au bord pour des opérateurs elliptiques, et ainsi on peut obtenir des estimées sur les traces. Ainsi, les termes de bord seront dominés par le terme source $||P_\varphi v||$ à l'intérieur. Il suffira alors de recoller les différentes inégalités microlocales obtenues. Ceci est possible car le recollement crée des termes issus de commutateurs, et ceux-ci sont d'ordre inférieur, et donc absorbables en prenant le paramètre semi-classique τ grand.

2.2.2 Les projecteurs de Calderón

Nous allons expliquer ici la méthode des projecteurs de Calderón, pour le Laplacien, et sur un demi-espace, pour une exposition simple. Cette méthode s'applique néanmoins pour tout opérateur elliptique d'ordre 2 (on renvoie à la section 4.6.1, où l'on utilisera cette méthode dans un cadre plus général). Considérons $-\Delta$ sur $\Omega = \mathbb{R}^d_+$ et $\partial\Omega = \{x_d = 0\}$. Formellement, on prend une solution $\Delta u = 0$ sur Ω. On écrit

$$-\Delta = -\partial_{x_d}^2 + \mathrm{Op}(|\xi'|^2),$$

et on considère $\hat{u}(\xi', x_d)$ la transformée de Fourier tangentielle de u. Ainsi, l'équation

$$-\partial_{x_d}^2 \hat{u} - |\xi'|^2 \hat{u} = 0 \tag{2.2.1}$$

admet deux solutions, de la forme

$$\hat{u}_1(\xi', x_d) = U_1(\xi')e^{-x_d|\xi'|}, \quad \hat{u}_2(\xi', x_d) = U_2(\xi')e^{x_d|\xi'|}.$$

Toutefois la solution $\hat{u}_2$, pour $x_d > 0$, explose de manière exponentielle lorsque $|\xi'| \to \infty$, donc on peut ne pas la considérer dans $\mathscr{S}'$. Si on regarde ainsi $\hat{u}_1$, on a

$$\partial_{x_d}\hat{u}_1(\xi', x_d) = -|\xi'|U_1(\xi')e^{-x_d|\xi'|} = -|\xi'|\hat{u}_1(\xi', x_d),$$

ce qui implique

$$\partial_{x_d}u(x', 0) = -\operatorname{Op}(|\xi'|)u(x', 0). \tag{2.2.2}$$

On voit ainsi que l'équation à l'intérieur amène automatiquement une relation du type (2.2.2) entre la trace Dirichlet, et la trace Neumann. Remarquons qu'ici, on a pas encore utilisé la condition au bord. Si la condition au bord est "suffisamment bonne", c'est-à-dire, si elle n'est pas colinéaire à la relation (2.2.2), alors on a un système de deux équations à deux inconnues que l'on peut résoudre

$$\begin{cases} \partial_{x_d}u(x', 0) = -\operatorname{Op}(|\xi'|)u(x', 0) \\ \text{condition au bord,} \end{cases}$$

et ainsi obtenir des estimées sur les traces. Ceci est l'idée générale, et on renvoie à la section 4.6.1 pour une exposition de la méthode avec un opérateur elliptique d'ordre 2 avec conditions au bord de type Ventcel.

2.3 Inégalités de Carleman paraboliques

Pour obtenir l'inégalité d'observabilité pour le problème adjoint du contrôle à zéro de la chaleur (cf. théorème 1.3.3), la méthode proposée par A. V. Fursikov et O. Yu. Imanuvilov dans [FI96] consiste en l'obtention d'une inégalité de Carleman parabolique globale sur Ω. Pour des inégalités elliptiques locales sur un ouvert U près du bord, on impose à la fonction poids φ les conditions suivantes

- φ doit satisfaire la condition de sous-ellipticité,
- $\nabla\varphi \neq 0$ sur $\overline{U}$,
- $\partial_\nu\varphi < 0$ sur $\partial\Omega \cap U$.

Pour les inégalités globales, on aimerait imposer les mêmes conditions. Toutefois, il serait impossible de construire une fonction poids telle que

- $\nabla\varphi \neq 0$ sur $\overline{\Omega}$,
- $\partial_\nu\varphi < 0$ sur $\partial\Omega$.

Soit ω l'ouvert de contrôle, sous ouvert de Ω. On impose alors,

1. φ doit satisfaire la condition de sous-ellipticité sur $\Omega \setminus \omega$,
2. $\nabla\varphi \neq 0$ sur $\Omega \setminus \omega$,
3. $\partial_\nu\varphi < 0$ sur $\partial\Omega$.

On rappelle que pour obtenir (1), il suffit de considérer $\varphi = e^{\lambda\psi}$, avec λ assez grand, et $\nabla\psi \neq 0$ sur $\Omega \setminus \omega$ (voir le lemme 2.1.2). L'idée est donc de construire la fonction poids localement sur un voisinage de $\partial\Omega$, et d'en obtenir un prolongement régulier sur Ω. Une des difficultés qui suivent est que l'on ne contrôle pas la position des points critiques de ce prolongement. Il suffit d'utiliser la densité des fonctions de Morse. Leurs points critiques sont en nombre fini, et on peut les déplacer dans la zone ω par le biais de la composition par

un difféomorphisme qui préserve $\partial\Omega$. Pour une telle construction, on renvoie à [Mil63]. Soit φ une fonction régulière satisfaisant les conditions (1), (2) et (3). Soit $T > 0$. On pose

$$\tilde{\varphi}(t,x) = \frac{e^{\tau\varphi(x)} - e^{\tau K}}{t(T-t)}, \quad K > \sup_{\overline{\Omega}} \varphi. \tag{2.3.1}$$

On obtient le théorème suivant

Théorème 2.3.1 ([FI96]). *Il existe $C > 0$ et $\tau_0 > 0$ such that*

$$\tau^3 ||e^{\tau\tilde{\varphi}} u||^2_{L^2((0,T)\times\Omega)} + \tau ||e^{\tau\tilde{\varphi}} \nabla u||^2_{L^2((0,T)\times\Omega)} \leq C \left(||e^{\tau\tilde{\varphi}} (\partial_t + \Delta) u||^2_{L^2((0,T)\times\Omega)} + \tau^3 ||e^{\tau\tilde{\varphi}} u||^2_{L^2((0,T)\times\omega)} \right),$$

pour tout $u \in C^\infty([0,T] \times \overline{\Omega})$ telle que $u_{|_{(0,T)\times\partial\Omega}} = 0$, et pour tout $\tau \geq \tau_0$.

On peut remarquer la présence d'un terme d'observation dans cette inégalité globale, qui est absent dans les versions locales d'inégalités de Carleman telles que le théorème 2.1.1. La dépendance du poids en temps de la forme (2.3.1) est utilisée pour annuler les termes de bord qui pourraient provenir d'intégrations par parties en temps. Du théorème 2.3.1, on en déduit l'inégalité d'observabilité du théorème 1.3.3 en utilisant la croissance de l'énergie de la chaleur rétrograde.

$$\int_\Omega |z(0)|^2 \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_\Omega |v|^2.$$

Les inégalités de Carleman paraboliques de la forme du théorème 2.3.1, sont souvent démontrées globalement, en utilisant la première méthode de preuve décrite plus haut. Notons toutefois que travailler de manière globale pour montrer des inégalités de Carleman (elliptiques ou paraboliques) n'est pas systématique ; on pourra aussi construire un poids global, mais prouver des inégalités de manière locale, et ainsi recoller par compacité. Dans cet esprit, une approche microlocale a été réalisée dans [LL12], dans laquelle la condition de sous-ellipticité apparait comme nécessaire. Une version au bord pour des conditions de Dirichlet a aussi été démontrée.

Chapitre 3

Résultats obtenus dans ce mémoire

3.1 Structure du mémoire

Deux parties principales composent ce mémoire.

La partie II utilise des méthodes essentiellement microlocales pour obtenir des inégalités de Carleman elliptiques aux voisinages d'un bord et d'une interface. Elle est composée de deux chapitres. Le premier concerne la stabilisation faible de l'équation des ondes amorties, pour des conditions au bord de Ventcel. Pour cela, on obtient une inégalité de Carleman au voisinage du bord, pour un opérateur elliptique de la forme $-\Delta - \sigma^2$ (voir la figure 2.1). Le second s'attache à l'obtention d'une inégalité de Carleman dans un cadre général pour des opérateurs elliptiques au voisinage d'une multi-interface. On donnera aussi des applications au contrôle.

Enfin, dans la partie III, on s'intéresse à l'obtention d'inégalités de Carleman pour des problèmes singuliers, dans le cadre de domaines polygonaux. Nous démontrons une inégalité pour l'opérateur augmenté $-\Delta - \partial_s^2$, et une inégalité pour l'opérateur avec paramètre spectral $-\Delta - \sigma^2$. On en déduit les applications au contrôle et à la stabilisation, suivant la stratégie donnée en figure 2.1. Enfin, on démontre les mêmes résultats pour le Laplacien avec un potentiel en carré inverse $-\Delta - \mu|x|^{-2}$, en dimension $d \geq 3$.

En dernière section, nous présentons une méthode pour obtenir des inégalités d'interpolation ponctuelles en temps pour les solutions de la chaleur, qui quantifient le prolongement unique. Ces inégalités sont équivalentes à l'inégalité spectrale. Nous présentons une esquisse d'application à l'équation de Crocco. Cette méthode permet d'éviter l'ajout de la variable s supplémentaire, et nous espérons ainsi pouvoir démontrer une inégalité spectrale en utilisant cette méthode.

3.2 Stabilisation de l'équation des ondes avec condition frontière de Ventcel

On s'intéresse dans ce premier article à l'équation des ondes amorties avec condition au bord de type Ventcel

$$\begin{cases} \partial_t^2 u - \Delta_g u + a\partial_t u = 0 & \text{dans } \mathbb{R}_t \times \Omega_x, \\ \partial_\nu u_{|\partial\Omega} - \delta \Delta_g^T u_{|\partial\Omega} = 0 & \text{sur } \mathbb{R}_t \times \partial\Omega_x, \end{cases}$$

où Δ_g^T est l'opérateur de Laplace-Beltrami de la variété compacte (sans bord) $(\partial\Omega, g^T)$, avec g^T la métrique induite au bord par la métrique Riemannienne de l'intérieur. Ces conditions au bord, caractérisées par la présence d'un opérateur au bord du même ordre que celui à l'intérieur, modélisent des structures dites de couche fine, c'est-à-dire, lorsque le matériau possède une "peau" de rigidité différente. On définit l'énergie

des ondes Ventcel par

$$E_{\text{Ventcel}}(u,t) := \frac{1}{2}\left(||\partial_t u(t)||^2_{L^2(\Omega)} + ||\nabla_g u(t)||^2_{L^2(\Omega)} + \delta|\nabla_g^T u(t)|^2_{L^2(\partial\Omega)}\right). \tag{3.2.1}$$

Considérant les normes apparaissant dans l'énergie (3.2.1), on introduit les espaces de Hilbert, indexés par le petit paramètre $\delta > 0$

$$\mathcal{H}_\delta = \mathcal{V}_\delta \times L^2(\Omega), \quad \delta \in (0,1], \text{ où } \mathcal{V}_\delta = \left\{u \in H^1(\Omega) \ | u_{|\partial\Omega} \in H^1(\partial\Omega)\right\},$$

associés à la norme

$$||u||^2_{\mathcal{V}_\delta} = ||u||^2_{H^1(\Omega)} + \delta||\nabla^T u||^2_{L^2(\partial\Omega)}.$$

On définit la norme de $\mathcal{H}_\delta$ de manière canonique

$$||(u,v)||^2_{\mathcal{H}_\delta} = ||u||^2_{\mathcal{V}_\delta} + ||v||^2_{L^2(\Omega)}.$$

On définit de plus le générateur des ondes

$$A_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & a(x) \end{pmatrix} \tag{3.2.2}$$

sur l'espace $\mathcal{H}_\delta$, de domaine

$$D(A_\delta) := \{(u_0,u_1) \mid u_0 \in H^2(\Omega),\ u_{0|\partial\Omega} \in H^2(\partial\Omega),\ u_1 \in H^1(\Omega),\ u_{1|\partial\Omega} \in H^1(\partial\Omega), \partial_\nu u_{0|\partial\Omega} - \delta\Delta_g^T u_{0|\partial\Omega} = 0\}.$$

Utilisant la méthode présentée en section 1.4, c'est-à-dire au moyen d'une estimée de résolvante, on démontre le résultat de stabilisation suivant.

Théorème 3.2.1 ([Buf17]). *Soit $k \geq 1$. Il existe $C > 0$ tel que pour tout $0 < \delta \leq 1$, on a*

$$E_{Ventcel}(u,t)^{1/2} \leq \frac{C}{(\log(2+t))^k}||A_\delta^k U_0||_{\mathcal{H}_\delta(\Omega)},$$

pour tout u solution de (3.2) avec donné initiale $U_0 = {}^t(u_0, \partial_t u_0)$.

Ce résultat est uniforme par rapport au petit paramètre $0 < \delta < 1$. Ainsi, on retrouve le résultat connu de [LR97] en passant à la limite $\delta \to 0^+$. Une part importante de ce travail consiste en l'obtention d'une inégalité de Carleman (uniforme par rapport au petit paramètre $\delta \in (0,1)$) au voisinage du bord, prenant en compte les conditions de Ventcel pour l'équation elliptique suivante

$$\begin{cases} (-\Delta_g - \sigma^2)u = f & \text{dans } \Omega, \\ \partial_\nu u - \delta\Delta_g^T u = g & \text{sur } \partial\Omega. \end{cases} \tag{3.2.3}$$

Nous obtenons le théorème suivant.

Théorème 3.2.2 ([Buf17]). *Soit $x \in \partial\Omega$ et V un voisinage ouvert de x dans $\overline{\Omega}$. Soit φ une fonction poids satisfaisant la condition de sous-ellipticité d'Hörmander (voir la définition 2.1.1) et les conditions données dans (4.1.24) sur $\overline{V}$. Alors, il existe $\tau_0 > 0$ et $C > 0$ tels que*

$$\tau^3||e^{\tau\varphi}u||^2_{L^2(V)} + \tau||e^{\tau\varphi}\nabla_g u||^2_{L^2(V)} + \tau|e^{\tau\varphi}\partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\Big(||e^{\tau\varphi}f||^2_{L^2(V)} + \tau|e^{\tau\varphi}g|^2_{L^2(V\cap\partial\Omega)} + (\delta^2\tau^5 + \tau^3)|e^{\tau\varphi}u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau|e^{\tau\varphi}\nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)}\Big),$$

et si, de plus, $\partial_\nu\varphi(x) < 0$ sur $\overline{V}$ on a

$$\tau^3||e^{\tau\varphi}u||^2_{L^2(V)} + \tau||e^{\tau\varphi}\nabla_g u||^2_{L^2(V)} + \tau^3|e^{\tau\varphi}u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau|e^{\tau\varphi}\nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau|e^{\tau\varphi}\partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\left(||e^{\tau\varphi}f||^2_{L^2(V)} + \tau|e^{\tau\varphi}g|^2_{L^2(V\cap\partial\Omega)}\right),$$

pour tout $0 < \delta \leq 1$, pour tout $|\sigma| \geq 1$, pour tout $\tau \geq \tau_0|\sigma|$ et pour tout $u \in \overline{C}_0^\infty(V)$, $f \in L^2(\Omega)$ et $g \in L^2(\partial\Omega)$ vérifiant (3.2.3).

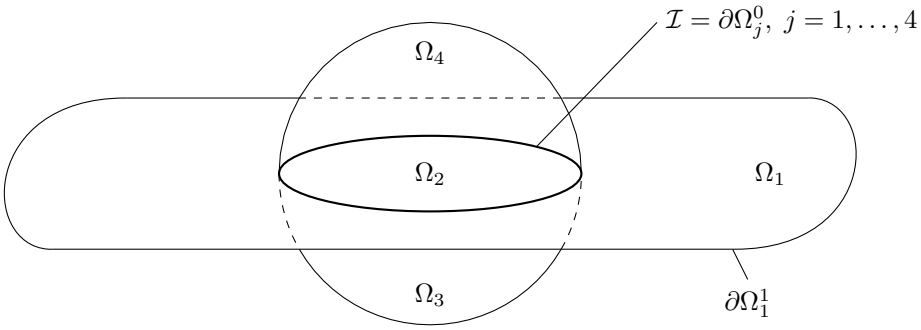


FIGURE 3.1: Une sphère intersecte une surface et crée 4 variétés compactes à bord. L'intersection est une composante connexe de $\partial\Omega_i$, pour tout $i \in \{1, \dots, n\}$, et est appelée multi-interface.

On observe une forte similarité entre le théorème 2.2.1 et celui-ci. La différence réside dans la condition au bord, et dans les conditions imposées sur φ dans en (4.1.24). Une condition supplémentaire (et essentiellement technique) demande

$$|\nabla_g^T \varphi| \leq \nu_0 \inf |\partial_\nu \varphi| \quad \text{sur } V \cap \partial\Omega,$$

pour $\nu_0 > 0$ pris suffisamment petit, c'est-à-dire que la fonction poids doit être relativement plate dans les directions tangentielles par rapport à la direction normale. On retrouve ce type d'hypothèse sur la fonction poids lorsque l'on veut obtenir des inégalités de Carleman au voisinage d'interfaces (voir par exemple [LR11, LLR13, LR10, BL16, LL13]). L'idée derrière cette hypothèse est que l'on ne veut pas trop perturber les dérivées tangentielles sous l'action de la conjugaison. On obtient ce résultat en utilisant la méthode décrite dans la section 2.2.

3.3 Une inégalité de Carleman générale au voisinage d'une multi-interface

Dans ce travail, on généralise les travaux de [BL15] (pour un problème au bord) et [BL16] (pour un problème d'interface), à un problème de multi-interface. Considérons n variétés Ω_k, de dimention $d \geq 2$, $k \in \{1, \dots, n\}$ à bord $\partial\Omega_k$. On suppose qu'elles partagent une partie de leur frontière. On écrit $\partial\Omega_k^i$, $i = 0, \dots, n_k$, les différentes parties connexes de $\partial\Omega_k$ et on suppose que l'une de ces parties connexes est partagée par toutes les variétés. Par exemple

$$\partial\Omega_1^0 = \cdots = \partial\Omega_n^0.$$

On appellera $\mathcal{I}$ la composante connexe commune, et on l'appellera multi-interface (voir la figure 3.1). Le but étant d'obtenir une inégalité de Carleman pour des opérateurs elliptiques à coefficients complexes d'ordre quelconque, sous des conditions de transmission à travers la multi-interface $\mathcal{I}$. Des hypothèses généralisant les conditions de Shapiro-Lopatinksii (ou encore appelées conditions de recouvrement dans la littérature) seront faites.

On considère le problème suivant au voisinage d'une multi-interface

$$\begin{cases} P_k u_k = f_k & \text{dans } \Omega_k, \quad k \in \{1, \dots, n\}, \\ \displaystyle\sum_{k=1}^{n} T_k^\ell(x, D) u_{k|_{\mathcal{I}}} = g_\ell & \text{sur } \mathcal{I}, \qquad \ell \in \{1, \dots, m\}. \end{cases} \tag{3.3.1}$$

où P_k est un opérateur elliptique d'ordre $2m_k$, $m = m_1 + \cdots + m_n$, et T_k^ℓ est un opérateur différentiel au bord, d'ordre plus petit ou égal à $\beta_k^\ell < 2m_k$, $\ell = 1, \dots, m$. On traitera comme exemple d'application le problème

de transmission naturel suivant

$$\begin{cases} -\Delta_{g_k} u_k = f_k, \quad k = 1, \dots, n & \text{dans } \Omega_k \\ u_{1|_{\mathcal{I}_1}} = \cdots = u_{n|_{\mathcal{I}_n}} & \text{sur } \mathcal{I} \\ \sum_{k=1}^n \partial_{\nu_k} u_{k|_{\mathcal{I}}} = 0 & \text{sur } \mathcal{I}. \end{cases} \tag{3.3.2}$$

Pour $k \in \{1, \dots, n\}$, soit φ_k une fonction C^∞ de $\overline{\Omega}_k$, et on considère les opérateurs conjugués

$$P_{k,\varphi_k} := e^{\tau\varphi_k} P_k e^{-\tau\varphi_k}, \quad T^\ell_{k,\varphi_k} := e^{\tau\varphi_k} T^\ell_k e^{-\tau\varphi_k},$$

avec pour symbole principal $p_{k,\varphi}(x,\xi,\tau) = p_k(x, \xi + i\tau d\varphi_k(x))$ (au sens semi-classique). Deux hypothèses principales sur la fonction poids seront faites. La première, est l'hypothèse de sous-ellipticité d'Hörmander similaire à la définition 2.1.1, pour chaque opérateur P_k et fonctions poids associées.

Définition 3.3.1. *Soit $k \in \{1, \dots, n\}$. Soit $x^0 \in \mathcal{I}$, et soit U_k un voisinage ouvert de x^0 dans $\overline{\Omega_k}$. On dit que (P_k, φ_k) satisfait la condition de sous-ellipticité dans $\overline{U}_k$, si $d\varphi_k \neq 0$ dans $\overline{U}_k$, et si*

$$p_{k,\varphi}(x,\xi,\tau) = 0 \Longrightarrow \Big\{ \operatorname{Re} p_{k,\varphi}, \operatorname{Im} p_{k,\varphi} \Big\}(x,\xi,\tau) = \frac{1}{2i} \Big\{ \overline{p}_{k,\varphi}, p_{k,\varphi} \Big\}(x,\xi,\tau) > 0, \tag{3.3.3}$$

pour tout $x \in \overline{U}_k, \tau > 0$, et tout $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

La seconde hypothèse, est l'hypothèse de recouvrement. Elle introduit une notion de compatibilité entre les opérateurs conjugués à l'intérieur P_{k,φ_k}, et les opérateurs conjugués T^{ℓ,φ_k}_k sur la multi-interface $\mathcal{I}$. Elle généralise la condition de Lopatinksii introduite dans [Tat96, BL15] dans le cadre d'un problème au bord, et la condition de transmission dans [BL16] dans le cadre d'un problème de multi-interface. Elle est très proche (et nous décrivons dans le paragraphe 5.2.3 en quel sens) de la condition introduite dans [ADN59, ADN64] pour analyser les propriétés de caractère bien posé au voisinage d'un bord pour des systèmes elliptiques. Dans [ADN59, ADN64], les auteurs recherchent des inégalités d'énergie pour le système classique, tandis qu'ici nous cherchons une inégalité d'énergie pour le système conjugué. Il est donc en un certain sens naturel que le même type de condition de compatibilité apparaisse dans ces deux problèmes. Sous ces deux conditions (sous-ellipticité et recouvrement), on obtient l'inégalité de Carleman suivante.

Théorème 3.3.1. *Soit $x_0 \in \mathcal{I}$ et soit φ une fonction poids dans U telle que $\varphi \in C^0(U)$ et telle que $\varphi_{|_{U_k}} \in C^\infty(U_k)$. De plus, on suppose que (P_k, φ_k) satisfait la condition de sous-ellipticité de la définition 3.3.1 sur un voisinage de x_0 dans Ω_k, pour tout $k \in \{1, \dots, n\}$, et on suppose que $(T^\ell_k, \varphi)_{k\in\{1,\dots,n\}, \ell\in\{1,\dots,m\}}$ recouvrent $(P_k, \varphi)_{k\in\{1,\dots,n\}}$ en $x_0 \in \mathcal{I}$ (voir la définition 5.2.3). Alors il existe un voisinage $\tilde{U}$ de x_0 dans Ω, $C > 0$, et $\tau_0 > 0$ tels que*

$$\sum_{k=1}^n \tau^{-1} ||e^{\tau\varphi_k} u_k||^2_{2m_k,\tau} + \sum_{k=1}^n |e^{\tau\varphi_{|_{\mathcal{I}}}} u_{k|_I}|^2_{2m_k-1,1/2,\tau} \leq C \left(\sum_{k=1}^n ||e^{\tau\varphi_k} P_k u_k||^2_{L^2(\Omega_k)} + \sum_{\ell=1}^m \Big| \sum_{k=1}^n T^\ell_k u_{k|_{\mathcal{I}}} \Big|^2_{\gamma_\ell - 1/2,\tau} \right),$$

pour tout $\tau \geq \tau_0$, pour tout $u \in C^0_0(\tilde{U})$ tel que $u_k := u_{|_{U_k \cap \tilde{U}}} \in \overline{C}^\infty_0(U_k \cap \tilde{U})$.

Dans le cadre du problème de transmission (3.3.2), la condition de recouvrement se traduit de la manière suivante. Il existe $i_0 \in \{1, \dots, n\}$ tel que

$$\partial_\nu \varphi_{i_0} < 0, \text{ et } \partial_\nu \varphi_i > 0, \text{ pour } i \neq i_0, \text{ et de plus satisfait } |\partial_\nu \varphi_{i_1}| \gg |\partial_\nu \varphi_{i_0}| \text{ pour un certain } i_1 \neq i_0. \tag{3.3.4}$$

Dans les applications, cela implique que pour observer une multi-interface, selon le théorème 3.3.1, il faut avoir une région d'observation dans au moins $n-1$ des variétés voisines (les régions d'observation sont les régions où le poids de Carleman est grand). L'optimalité de cette condition est discutée dans des géométries simples.
En dimension un d'espace ($d = 1$), ce genre de configuration est appelé réseau de cordes, et les résultats de contrôle et de stabilisation sont obtenus grâce à la dérivation d'inégalités de Carleman globales (sur le

réseau) où encore à l'aide d'inégalités d'Ingham portant sur les valeurs propres du Laplacien. Pour de telles questions, on renvoie à [BCV11, DZ06, LLS94, Zua13], ainsi qu'aux références contenues dans ceux-ci. Les résultats présentés ici, valables pour $d \geq 2$, permettent d'étendre certains de ces travaux.
De cette inégalité de Carleman, on en déduit des inégalités d'interpolation au voisinage de la multi-interface, et on en déduit les résultats de contrôlabilité à zéro pour l'équation de la chaleur, ainsi que la stabilisation logarithmique pour l'équation des ondes amorties. Plus précisément, on considèrera les configurations géométriques suivantes.

Soit p variétés Riemaniennes compactes (Ω_k, g_k) de dimension $d \geq 2$, $k = 1, \dots, p$, de frontière $\partial\Omega_k$. On pose $\Omega = \cup_{k=1}^{p} \Omega_k$. On supposera que Ω est connexe et que $\partial\Omega_k$ possède n_k composantes connexes, avec $n_k \geq 1$. On suppose de plus que les varitiétés partagent quelques unes de ces composantes connexes du bord avec les autres. Plus précisément, on notera $\mathcal{I}_\ell$, $\ell \in \{1, \dots, L\}$, l'ensemble des multi-interfaces qui sont partagées par au moins deux variétés, et on notera $\mathcal{B}_k$, $k \in \{1, \dots, K\}$, le reste de ces composantes connexes du bord.

Localement, on doit observer une multi-interface à n cotés depuis au moins $(n-1)$ cotés, si l'on souhaite appliquer le théorème 3.3.1. Il est donc naturel que des conditions géométriques entre Ω et l'ouvert de contrôle ω soit imposées.
Soit ω un ouvert de Ω. On écrira $\omega_\ell = \Omega_\ell \cap \omega$, et $\mathcal{L} = \{\ell \in \{1, \dots, p\},\ \omega_\ell \neq \emptyset\}$. Pour $k \in \{1, \dots, p\}$, on définit

$$\mathcal{N}_k : I_\ell \longmapsto \begin{cases} 1 & \text{si } \Omega_k \cap \mathcal{I}_\ell \neq 0 \\ 0 & \text{sinon.} \end{cases}$$

En d'autres termes, cette application retourne 1 si Ω_k partage la multi-interface $\mathcal{I}_\ell$. On pose $\mathcal{V}^\ell := \{k \in \{1, \dots, p\},\ \mathcal{N}_k(\mathcal{I}_\ell) = 1\}$, c'est-à-dire, l'ensemble des variétés qui partagent $\mathcal{I}_\ell$.

Nous allons décrire par récurrence les configurations où l'on peut obtenir des inégalités d'observabilité. On pose dans un premier temps $\mathcal{O}_0 := \mathcal{L}$, et on définit

$$\mathcal{O}_{j+} = \mathcal{O}_j \cup \left(\cup_{\ell=1}^{L} \mathcal{V}_j^\ell\right), \qquad \mathcal{V}_j^\ell := \begin{cases} \tilde{\mathcal{V}}_j^\ell & \text{si } \operatorname{Card} \tilde{\mathcal{V}}_j^\ell \leq 1 \\ \emptyset & \text{sinon,} \end{cases} \quad \text{et où } \tilde{\mathcal{V}}_j^\ell = \mathcal{V}^\ell \setminus \mathcal{O}_j. \tag{3.3.5}$$

Cela signifie qu'à chaque étape, pour $\ell \in \{1, \dots, L\}$, on compte le nombre de cotés de $\mathcal{I}_\ell$ qui appartiennent à $\mathcal{O}_j$. Si ce nombre correspond à une observation depuis $(n-1)$ cotés, on ajoute le coté restant à $\mathcal{O}_j$. Remarquons $\operatorname{Card} \mathcal{V}_j^\ell \leq 1$ à chaque étape, et que si $\mathcal{O}_j = \mathcal{O}_{j+1}$ pour un j donné, alors $\mathcal{O}_{j'} = \mathcal{O}_j$, pour tout $j' \geq j$. On définit ainsi

Définition 3.3.2. *On dira que (Ω, ω) est observable s'il existe $j_0 \in \mathbb{N}$ tel que $\mathcal{O}_{j_0} = \{1, \dots, p\}$ dans la construction décrite en (3.3.5).*

L'algorithme décrit plus haut converge en moins de p itérations, que (Ω, ω) soit observable ou non. Par exemple, dans le dessin de gauche de la figure 3.2, si on suppose $\mathcal{L} = \{2, 4, 5, 6\}$. Alors

1. $\mathcal{O}_0 = \mathcal{L}$, $\cup_{\ell=1}^{L} \tilde{\mathcal{V}}_0^\ell = \{1, 3, 7, 8\}$, $\cup_{\ell=1}^{L} \mathcal{V}_0^\ell = \{1, 8\}$
2. $\mathcal{O}_1 = \{1, 2, 4, 5, 6, 8\}$, $\cup_{\ell=1}^{L} \tilde{\mathcal{V}}_1^\ell = \{3, 7\}$, $\cup_{\ell=1}^{L} \mathcal{V}_1^\ell = \{3\}$
3. $\mathcal{O}_2 = \{1, 2, 3, 4, 5, 6, 8\}$, $\cup_{\ell=1}^{L} \tilde{\mathcal{V}}_2^\ell = \{7\}$, $\cup_{\ell=1}^{L} \mathcal{V}_2^\ell = \{7\}$
4. $\mathcal{O}_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Ainsi (Ω, ω) est observable.

Si on suppose $\mathcal{L} = \{2, 4, 5, 6\}$ dans le dessin de droite de la figure 3.2, alors

1. $\mathcal{O}_0 = \mathcal{L}$, $\cup_{\ell=1}^{L} \tilde{\mathcal{V}}_0^\ell = \{1, 3, 7, 8\}$, $\cup_{\ell=1}^{L} \mathcal{V}_0^\ell = \{8\}$

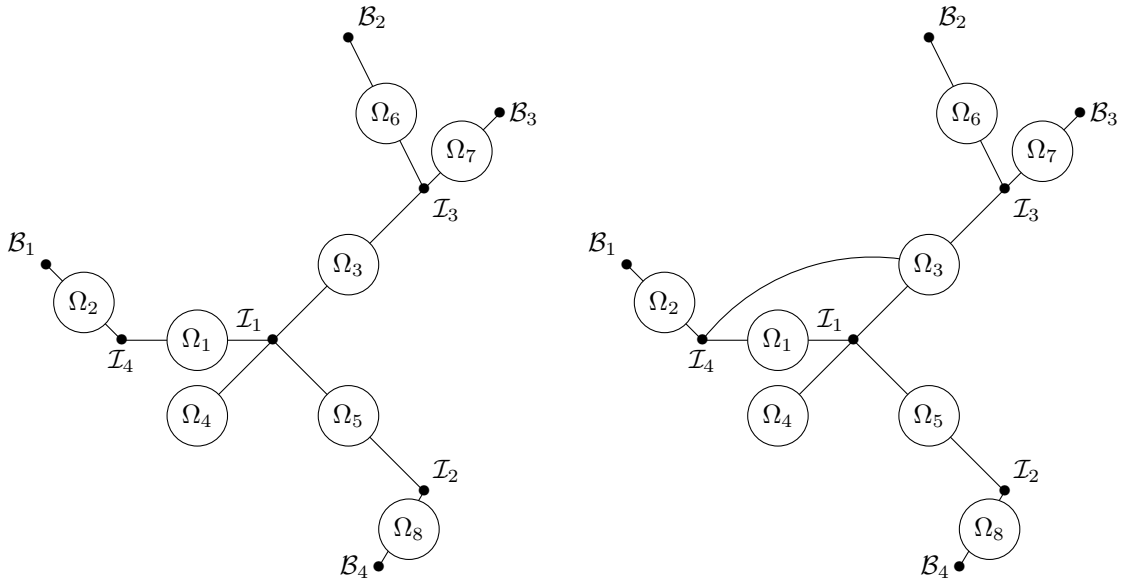


FIGURE 3.2: Sous la forme d'un graphe, les arêtes représentent les variétés Ω_k, et les sommets les multi-interfaces $\mathcal{I}_\ell$, ou les bords $\mathcal{B}_k$.

2. $\mathcal{O}_1 = \{2,4,5,6,8\}$, $\cup_{\ell=1}^{L} \tilde{\mathcal{V}}_1^\ell = \{3,4,7\}$, $\cup_{\ell=1}^{L} \mathcal{V}_1^\ell = \emptyset$
3. $\mathcal{O}_1 = \mathcal{O}_2 \neq \{1,\dots,8\}$.

et (Ω,ω) n'est pas observable.

Sous la condition géométrique d'observabilité de la définition 3.3.2, on en déduit les résultats classiques, comme l'inégalité spectrale pour l'opérateur elliptique associé au problème de transmission (voir le théorème 5.5.5). Elle implique la contrôlabilité à zéro de l'équation de la chaleur avec conditions de transmission (voir le théorème 5.5.6). De la même manière, on en déduit la propriété de stabilisation faible (ou logarithmique) pour l'équation des ondes amorties associée, où la zone d'amortissement vérifie la condition d'observabilité géométrique de la définition 3.3.2 (voir le théorème 5.5.7).

3.4 Quelques problèmes de contrôle dans des configurations singulières

Dans cette troisième partie, nous nous intéressons aux cas de problèmes plus singuliers. Dans les chapitres précédents, l'obtention d'inégalités de Carleman locales près du bord pour des conditions de Neumann, Ventcel, où encore pour les problèmes d'interfaces invoque des arguments microlocaux, et notamment un calcul pseudo-différentiel tangentiel. Dans le cadre d'un bord non-régulier, tel qu'un coin, de tels arguments sont difficiles à mettre en oeuvre, ceci étant dû à la faible régularité de la frontière. Le cadre que nous étudierons est le suivant. Soit Ω un ouvert polygonal et borné de $\mathbb{R}^2$, et soit S une ligne brisée du plan telle que $\Omega \cap S \neq \emptyset$. L'interface S divise Ω en deux composantes connexes Ω^ℓ, $\ell \in \{1,2\}$. Au bord, on note $\partial\Omega_k$ chaque segment du polygone $\partial\Omega$. Même si l'interface S croise le bord en un point où celle-ci est régulière, on considère qu'elle sépare le bord en deux segments distincts (un exemple est donné dans la figure 3.3). Dans ce qui suit, pour une fonction u de Ω, on note u^ℓ sa restriction à Ω^ℓ. Pour une fonction u de Ω, on note $\gamma_k u$ la trace de u sur $\partial\Omega_k^\ell$, et $\gamma_k \partial_\nu u$ la trace de la dérivée normale sur $\partial\Omega_k$ (lorqu'elle existe), où ν est le vecteur normal unitaire sortant. Dans la même idée, on définit $\gamma_S u^\ell$ la trace de u^ℓ sur S et $\gamma_S \partial_{\nu_\ell} u^\ell$ la trace de la

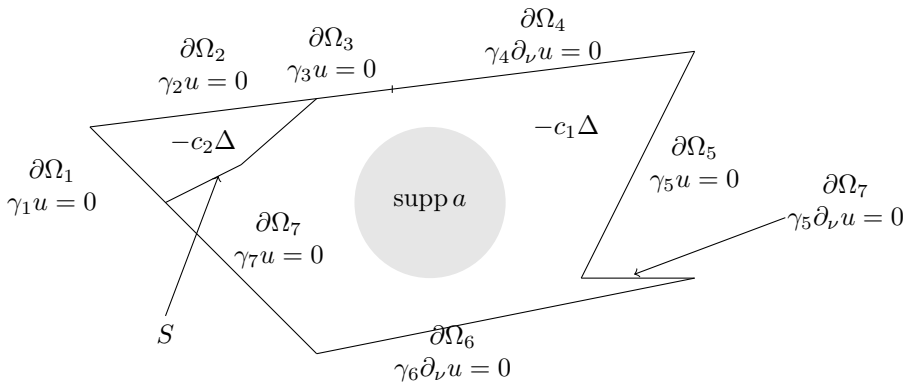


FIGURE 3.3: Un exemple de polygone avec une interface brisée S. Ici, $\mathsf{D} = \{1, 2, 4\}$ et $\mathsf{N} = \{3, 5, 6\}$. En gris est représenté un exemple de support de l'amortissement pour l'équation des ondes, ou de zone de contrôle pour l'équation de la chaleur.

dérivée normale, où ν_ℓ est le vecteur unitaire normal sortant de Ω^ℓ. On imposera des conditions de Dirichlet homogènes sur certains segments du bord, et des conditions de Neumann homogènes sur les autres. On pose ainsi

$$\mathsf{D} := \{i \in \mathbb{N},\ \gamma_i u = 0\}, \quad \text{et } \mathsf{N} := \{i \in \mathbb{N},\ \gamma_i \partial_\nu u = 0\},$$

c'est-à-dire, l'ensemble des indices des parties du bord où les conditions de Dirichlet sont imposées (respectivement les conditions de Neumann). On considèrera les espaces de Sobolev

$$\mathcal{H}^1_{\mathsf{D}} = \{(u^1, u^2),\ u^\ell \in H^1(\Omega_\ell),\ \ell = 1, 2,\ \gamma_i u = 0,\ \forall i \in \mathsf{D}\},$$

et l'opérateur suivant, agissant sur $\mathcal{H}^1_D$,

$$\mathbf{T}u = -\operatorname{div}(a(x)\nabla u), \tag{3.4.1}$$

où $a(x)$ est une fonction constante par morceaux, vérifiant $a(x) = c_k$ sur chaque Ω_k, de domaine

$$D(\mathbf{T}) = \{(u^1, u^2) \in \mathcal{H}^1_{\mathsf{D}}(\Omega),\ \Delta u^\ell \in L^2(\Omega^\ell),\ \ell = 1, 2,\ \gamma_j \partial_\nu u = 0, \forall j \in \mathsf{N},\\ \gamma_S u^1 = \gamma_S u^2,\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0\}.$$

On peut diagonaliser cet opérateur suivant une base de Hilbert, notée (ϕ_j, ν_j). Un coin sera appelé coin *Neumann*, dans les cas suivants :

- un coin du bord où des conditions de Neumann sont imposées de chaque coté ;
- un coin de l'interface à l'intérieur de Ω où elle est brisée ;
- l'intersection de l'interface avec un bord où des conditions de Neumann sont imposées de chaque coté.

Sinon, on parlera de coin *non-Neumann*. On montrera les inégalités spectrales suivantes, dans l'esprit de [LR95, JL99, LZ98].

Théorème 3.4.1. *On suppose que Ω ne contient pas de coin Neumann. Soit ω un ouvert non-vide de Ω, et soit $\eta > 0$. Alors, il existe une constante $C > 0$ telle que*

$$\sum_{\lambda_j \le \Lambda} |a_j|^2 \le C e^{C\Phi_\eta(\Lambda)} \int_\omega \Big| \sum_{\lambda_j \le \Lambda} a_j \phi_j \Big|^2, \quad \Lambda > 0 \tag{3.4.2}$$

pour tout $(a_j) \in \ell^2(\mathbb{N})$, où $\Phi_\eta(\Lambda) = \Lambda^{\frac{7}{8}+\eta}$.

On a aussi le résultat (plus faible) suivant.

Théorème 3.4.2. *On suppose que Ω contient au moins un coin Neumann. Soit ω un ouvert non-vide de Ω et soit $\eta > 0$. Alors, il existe une constante $C > 0$ telle que*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\Phi_\eta(\Lambda)} \int_\omega \Big| \sum_{\lambda_j \leq \Lambda} a_j \phi_j \Big|^2, \quad \Lambda > 0 \tag{3.4.3}$$

pour tout $(a_j) \in \ell^2(\mathbb{N})$, où $\Phi_\eta(\Lambda) = \Lambda^{2+\eta}$.

Remarque 3.4.1. *Comparant avec le théorème 1.3.1, on voit que dans les deux cas (3.4.2) et (3.4.3), on a $\Phi_\eta(\Lambda) > \sqrt{\Lambda}$ pour Λ grand, et donc le résultat obtenu est plus faible que celui obtenu dans le théorème 1.3.1, dans un cadre régulier. Ceci est principalement dû à la méthode utilisée, et nous pensons que ce n'est pas optimal. Toutefois, dans le cadre où nous n'avons pas de coin Neumann, pour $\Lambda > 0$ grand, $\Phi(\Lambda) \ll \Lambda$, et ceci nous permet de retrouver les résultats de contrôlabilité à zéro de l'équation de la chaleur, en utilisant la méthode de contrôle/relaxation [LR95].*

Soit $\omega \Subset \Omega$ un sous ouvert de Ω, et soit χ_ω la fonction caractéristique de ω. On considère le problème de contrôle

$$\begin{cases} \partial_t u + \mathbf{T} u = \chi_\omega h & (t,x) \in (0,T) \times \Omega, \\ \gamma_i u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i, \ i \in \mathsf{D} \\ \gamma_i \partial_\nu u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i, \ i \in \mathsf{N} \\ \gamma_S u^1 = \gamma_S u^2 & (t,x) \in (0,T) \times S, \\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0 & (t,x) \in (0,T) \times S, \\ u(0,x) = u_0(x) \in L^2(\Omega) & x \in \Omega, \end{cases} \tag{3.4.4}$$

et on prouve le théorème de contrôlabilité à zéro suivant.

Théorème 3.4.3. *On suppose que Ω ne contient pas de coin Neumann. Soit ω un ouvert non vide de Ω. Alors, pour tout $u_0 \in L^2(\Omega)$, il existe $h \in L^2((0,T)\times\Omega))$ telle que la solution u de (3.4.4) satisfait $u(T,x) = 0$, pour presque tout $x \in \Omega$.*

Concernant la stabilisation de ondes, nous montrons une inégalité de résolvante, qui à son tour fournit la stabilisation logarithmique. On considère l'équation des ondes amorties suivante.

$$\begin{cases} \partial_t^2 u^\ell + \mathbf{T} u^\ell + b(x) \partial_t u^\ell = 0 & (t,x) \in (0,T) \times \Omega_\ell, \\ \gamma_i u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i, \ i \in \mathsf{D} \\ \gamma_i \partial_\nu u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i, \ i \in \mathsf{N} \\ \gamma_S u^1 = \gamma_S u^2 & (t,x) \in (0,T) \times S, \\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0 & (t,x) \in (0,T) \times S, \\ u(0,x) = u_0(x) & x \in \Omega \\ \partial_t u(0,x) = u_1(x) & x \in \Omega, \end{cases} \tag{3.4.5}$$

où b est une fonction positive, satisfaisant $b \geq C > 0$ sur un sous ouvert $\omega \Subset \Omega$.

On définit l'énergie des ondes (3.4.5) par

$$E(t) = \int_\Omega |\partial_t u|^2 + \int_\Omega |\nabla u|^2,$$

On formule l'équation des ondes (3.4.5) en une écriture de semi-groupe. On définit $\mathcal{H} = H^1_{\mathsf{D}}(\Omega) \times L^2(\Omega)$, et on considère le générateur des ondes suivant $B : D(B) \subset \mathcal{H} \mapsto \mathcal{H}$,

$$B = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathbf{T} & b(x) \end{pmatrix},$$

où $D(B) = D(\mathbf{T}) \times D(\mathbf{T}^{1/2})$ et ainsi l'équation (3.4.5) s'écrit

$$\frac{d}{dt}U + BU = 0, \quad U_{|t=0} = U_0, \tag{3.4.6}$$

avec $U = {}^t(u, \partial_t u)$ et $U_0 = {}^t(u_0, u_1)$. On montre le théorème suivant.

Théorème 3.4.4. *Soit $k \geq 0$. Il existe une constante $C > 0$ telle que*

$$E(u)(t)^{1/2} \leq \frac{C}{(\log(2+t))^k} ||B^k U_0||_{\mathcal{H}(\Omega)},$$

pour tout u, solution de (3.4.5) avec donnée initiale $U_0 = (u_0, u_1) \in D(B)$.

Nous soulignons le fait qu'ici toutes les situations géométriques décrites plus haut, en particulier la présence de coins Neumann, entrent dans le cadre de ce théorème. Le théorème 3.4.4 est la conséquence de l'estimée sur la résolvante qui suit.

Théorème 3.4.5. *Pour tout $\sigma \in \mathbb{R}$, $\sigma \neq 0$, l'opérateur $(i\sigma \,\mathrm{Id} + B)$ est inversible sur $\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)$. De plus, il existe une constante $C > 0$ telle que*

$$|||(i\sigma \,\mathrm{Id} + B)^{-1}|||_{\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)} \leq Ce^{C|\sigma|} \qquad |\sigma| \geq 1.$$

Pour montrer l'inégalité spectrale, notre stratégie est la suivante. Classiquement, on pose $Z = (-S_0, S_0) \times \Omega$ et $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$. Pour l'équation augmentée d'une variable

$$-\partial_s^2 u - \Delta u = f, \tag{3.4.7}$$

au voisinage d'un coin du bord, on a en coordonées polaires $-\Delta = -\partial_r^2 - \frac{\partial_r}{r} - \frac{\Delta_\theta}{r^2}$. Posant $u(x) = v(r, \theta)$ et $f(x) = g(r, \theta)$, (3.4.7) devient

$$-\partial_s^2 v - \partial_r^2 v - \frac{\partial_r}{r} v - \frac{1}{r^2}\Delta_\theta v = g,$$

Posant $w = r^{1/2} v$ et $h = r^{1/2} g$, on obtient

$$-\partial_s^2 w - \partial_r^2 w - \frac{1}{r^2}\left(\Delta_\theta + \frac{1}{4}\right) w = h.$$

On utilise les conditions au bord homogènes pour diagonaliser le laplacien sphérique (on note ν_k les valeurs propres de $-\Delta_\theta$), et on obtient une famille d'équations, indexées par $k \in \mathbb{N}$,

$$Q_k w_k := -\partial_s^2 w_k - \partial_r^2 w_k - \frac{\mu_k}{r^2} w_k = h_k, \tag{3.4.8}$$

où $\mu_k \leq 1/4$. On remarque que $1/4$ est précisément la constante de Hardy critique pour un problème unidimensionnel (voir 6.2.15)

$$\int_0^R \left(|z'|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right) dr \geq C(R) \int_0^R |z|^2 dr. \tag{3.4.9}$$

pour tout $z \in H_0^1(0, R)$. Le cas critique $\mu_k = 1/4$ apparait lorsque la première valeur propre de l'opérateur sphérique vérifie $\nu_0 = 0$, c'est-à-dire, lorsque qu'on effectue l'analyse au voisinage de coins Neumann (voir la définition donnée plus haut).
En s'inspirant du travail réalisé dans [VZ08], où les auteurs démontrent une inégalité de Carleman parabolique pour un opérateur présentant un potentiel $\mu|x|^{-2}$, on introduit une fonction poids de la forme

$$\varphi_\varepsilon(r, s) = r^2 - \varepsilon(s - s_0)^2, \tag{3.4.10}$$

où ε est rélié au grand paramètre τ de Carleman. Dans le cas sous-critique $\mu_k < 1/4$, on montre ainsi l'inégalité de Carleman suivante, uniforme en μ_k.

$$C||e^{\tau\varphi_\varepsilon}Q_k w_k||^2_{L^2} \geq \tau\left(1+|\mu_k|\right)||e^{\tau\varphi_\varepsilon}\frac{w_k}{r}||^2_{L^2} \\ + \tau||e^{\tau\varphi_\varepsilon}\partial_r w_k||^2_{L^2} + \tau^2||e^{\tau\varphi_\varepsilon}w_k||^2_{L^2} + \tau^3||re^{\tau\varphi_\varepsilon}w_k||^2_{L^2} + \tau^{2/3}||e^{\tau\varphi_\varepsilon}\partial_s w_k||^2_{L^2}, \quad (3.4.11)$$

avec $\varepsilon = \varepsilon_0\tau^{-1/3}$, $\tau \geq \tau_0$. Dans le cas critique $\mu_k = 1/4$, on obtient

$$C||e^{\tau\varphi_\varepsilon}Q_k w_k||^2_{L^2(\tilde{Y})} \geq \tau||e^{\tau\varphi_\varepsilon}w_k||^2_{L^2(\tilde{Y})} + \tau^3||re^{\tau\varphi_\varepsilon}w_k||^2_{L^2(\tilde{Y})} + \tau^{1/3}||e^{\tau\varphi_\varepsilon}\partial_s w_k||^2_{L^2(\tilde{Y})} \quad (3.4.12)$$

avec cette fois-ci $\varepsilon = \varepsilon_0\tau^{-2/3}$, $\tau \geq \tau_0$. Notons que dans les deux cas, les normes obtenues au membre de droite correspondent au sens variationnel de l'équation (3.4.8), en utilisant (3.4.9). En sommant sur $k \in \mathbb{N}$ ces inégalités, et en revenant aux coordonées cartésiennes, on obtient ainsi une inégalité de Carleman locale au voisinage d'un point singulier. On considèrera le poids (radial en x)

$$\varphi_\varepsilon(x,s) = |x|^2 - \varepsilon(s-s_0)^2, \quad (3.4.13)$$

Si le point considéré est un coin non-Neumann, on obtient le théorème suivant.

Théorème 3.4.6 (Inégalité de Carleman au voisinage d'un coin non-Neumann)**.** *Soit x_0 un coin non-Neumann de $\partial\Omega$, et soit $s_0 \in (-S_0+\alpha, S_0-\alpha)$. On pose $\mathcal{P} = (s_0, x_0)$, et on suppose $x_0 = 0$ sans perdre de généralité. Soit $\mathcal{V}$ un voisinage ouvert de $\mathcal{P}$ dans Y. Alors il existe $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ tels que*

$$C||e^{\tau\varphi_\varepsilon}\mathcal{T}u||^2_{L^2(\mathcal{V})} \geq \tau^{2/3}||e^{\tau\varphi_\varepsilon}\nabla_{s,x}u||^2_{L^2(\mathcal{V})} + \tau^2||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

où $\varepsilon = \varepsilon_0\tau^{-1/3}$, et pour tout $u \in C^\infty((-S_0,S_0), D(\mathbf{T}))$ tel que $\operatorname{supp} u \subset \mathcal{V}$.

Pour des coins Neumann, on a le résultat suivant.

Théorème 3.4.7 (Inégalité de Carleman au voisinage d'un coin Neumann)**.** *Soit x_0 un coin Neumann de $\partial\Omega$, et $s_0 \in (-S_0+\alpha, S_0-\alpha)$. Soit $\mathcal{P} = (s_0, x_0)$, et on suppose $x_0 = 0$ sans perdre de généralité. Soit $\mathcal{V}$ un voisinage ouvert de $\mathcal{P}$ dans Y. Alors il existe $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ tels que*

$$C||e^{\tau\varphi_\varepsilon}\mathcal{T}u||^2_{L^2(\mathcal{V})} \geq \tau||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

où $\varepsilon = \varepsilon_0\tau^{-2/3}$, et pour tout $u \in C^\infty((-S_0,S_0), D(\mathbf{T}))$, tel que $\operatorname{supp} u \subset \mathcal{V}$.

Toutefois, la dépendance en ε (et donc en τ) de la fonction poids (3.4.10), ne nous permet pas d'obtenir une inégalité d'interpolation directement. Du travail est encore réalisé en section 6.3 pour en déduire une inégalité d'interpolation, plus faible que celles prouvées dans un cadre plus régulièr. Ceci mène aux pertes décrites en remarque 3.4.1 dans la constante $\Phi(\Lambda)$ de (3.4.2).

Le poids de Carleman (3.4.13) étant radial en x, l'inégalité d'interpolation que l'on obtient fournit une observation du voisinage du coin depuis une couronne (qui peut toucher le bord et/ou l'interface). Toutefois, cette couronne se situe dans des régions où le bord $\partial\Omega$ (ou l'interface) est régulière, et donc les intégaités d'interpolation classiques peuvent être appliquées, et ceci nous permet de propager l'information à travers le domaine.

Pour l'estimée de résolvante, on montre une inégalité de la forme de celle de la proposition 1.4.5. On introduit ainsi l'opérateur $\mathbf{T}_\sigma = \mathbf{T} - \sigma^2$. Pour cela, on utilise comme ci-dessus une décomposition polaire localement au voisinage de coins, puis une inégalité de Carleman uni-dimentionelle. Revenant aux coordonnées cartésiennes, on obtiendra finalement le résultat suivant. Soit p un coin, et $R > 0$, on pose $U = \Omega \cap B(p,R)$, où $B(p,R)$ est la boule centrée en p de rayon R, on pose aussi $\omega_R = \{x \in U,\ |x-p| \in (R/2, 3R/4)\}$, et $V = \{x \in U,\ |x-p| < R/2\}$, .

Théorème 3.4.8. *Il existe une constante $C > 0$ telle que*

$$||u||_{H^1(V)} \leq C \left(||\mathbf{T}_\sigma u||_{L^2(U)} + ||u||_{H^1(\omega_R)} \right), \tag{3.4.14}$$

pour tout $u \in D(\mathbf{T})$, et pour tout $|\sigma| \geq 1$.

Une propriété remarquable de ce théorème est que la constante C est uniforme par rapport au paramètre spectral σ (en comparant avec (1.4.5) par exemple). Ceci peut être compris intuitivement par le fait que localement l'ouvert ω_R vérifie la condition de controle géométrique par rapport à V.
Toutefois, si l'on souhaite obtenir une inégalité du type (3.4.14), mais avec ω à la place de ω_R, avec un $\omega \subset \Omega$ arbitraire, il faut propager des inégalités d'interpolation, et on obtient alors

$$||u||_{H^1(\Omega)} \leq C e^{C|\sigma|} \left(||\mathbf{T}_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(\omega)} \right), \tag{3.4.15}$$

Enfin, nous terminerons cette présentation succinte, en remarquant que grâce à la décomposition en harmoniques sphériques, tous nos arguments s'appliquent dans le cadre traité dans [VZ08, Erv08], c'est-à-dire à l'équation suivante.

$$\begin{cases} \partial_t u + Pu = \chi_\omega h & (t,x) \in (0,T) \times \Omega, \\ u(t,x)_{|\partial\Omega} = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \in L^2(\Omega) & x \in \Omega, \end{cases} \tag{3.4.16}$$

où Ω est un ouvert régulier borné de $\mathbb{R}^d$ contenant 0, $d \geq 3$, ω un sous ouvert arbitraire de Ω, et P est l'opérateur

$$P = -\Delta - \frac{\mu}{|x|^2},$$

avec $\mu \leq \mu^* := \frac{(d-2)^2}{4}$. Dans [VZ08, Erv08], les auteurs prouvent un résultat de contrôlabilité à zéro de la même forme que le théorème 3.4.3. Nous ajoutons ici à ce résultat, une inégalité spectrale de la même forme que le théorème 3.4.1. La remarque 3.4.1 concernant la constante $e^{C\Phi_\eta(\Lambda)}$ s'applique aussi.
Notons que K. D. Phung a montré récemment une inégalité spectrale pour l'opérateur P, de la forme

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\sqrt{\Lambda}} \int_\omega \Big| \sum_{\lambda_j \leq \Lambda} a_j \phi_j \Big|^2, \quad \Lambda > 0, \tag{3.4.17}$$

mais pour des valeurs de μ qui restent éloignées de la valeur critique $\mu = \mu^*$. Dans son travail, K.D.Phung utilise des propriétés dites de *doubling* de l'équation de la chaleur, qui lui permettent d'obtenir des inégalités d'interpolation ponctuelles en temps (qui sont équivalentes à l'inégalité spectrale) [Phu17]. Obtenir l'optimalité dans les régimes proches de $\mu = \mu^*$ reste un problème ouvert. Pour plus de détails sur les résultats présents dans ce mémoire, nous renvoyons aux théorèmes 6.1.3 et 6.1.11. Dans le cadre de l'équation des ondes amorties, nous renvoyons aux théorèmes 6.1.6 et 6.1.15, qui sont des résultats nouveaux.

Deuxième partie

Inégalités de Carleman pour des problèmes de frontière et d'interface

Chapitre 4

Stabilisation de l'équation des ondes avec conditions frontière de Ventcel

Ce chapitre a fait l'objet d'un article, publié dans *Journal de Mathématiques Pures et Appliquées.*

Abstract. We consider a damped wave equation on a open subset of $\mathbb{R}^n$ or a smooth Riemannian manifold with boundary, with Ventcel boundary conditions, with a linear damping, acting either in the interior or at the boundary. This equation is a model for a vibrating structure with a layer with higher rigidity of thickness $\delta > 0$. By means of a proper Carleman estimate for second-order elliptic operators near the boundary, we derive a resolvent estimate for the wave semigroup generator along the imaginary axis, which in turn yields the logarithmic decay rate of the energy. This stabilization result is obtained uniformly in δ.
KEYWORDS: Stabilization; wave equation; Ventcel boundary conditions; Carleman estimate; resolvent estimate

AMS 2010 SUBJECT CLASSIFICATION: 35L05, 35L20, 93D15.

4.1 Introduction, statement of the problem, and main results

4.1.1 Introduction

We consider a locally damped wave equation on (Ω, g), a compact Riemannian manifold with smooth boundary $\partial\Omega$, with Ventcel[1] boundary conditions, and we are concerned here with the stabilization of such an equation. This type of boundary conditions is characterized by the presence of a second-order tangential operator at the boundary, for instance the Laplace-Beltrami operator Δ_g^T, reading $\partial_\nu u - \Delta_g^T u = 0$ (ν denotes the outgoing normal unit vector). It generally arises when considering a domain with an thin boundary layer of high rigidity, after some approximations are made. The issue of stabilizing such a wave equation has been the subject of several works, for instance [CKM07, CLT12a, CLT12b, NL10, Hem99, MMS13, KM04]. Here, we consider a linear damping of the form $a(x)\partial_t u$, in the interior of the domain, or $b(x)\partial_t u_{|\partial\Omega}$, within the Ventcel boundary condition at the boundary, where a and b are non-negative functions with a non-empty support of Ω or $\partial\Omega$, respectively. Stabilization is measured by the decay and the convergence to zero of a natural energy function for the solution. Alternative models involving second-order operators in the boundary conditions (or transmission conditions) and studied for similar properties (stabilization, controlability) can be found in [MZ97, KZ06, LLR13].

Since the seminal works of [RT74, BLR92], it is known that a stabilization charaterized by an exponential decay rate for the energy is heavily related to a geometric control condition, GCC for short. Roughly speaking, every generalized geodesic (travelled at speed one), in the sense of [MS78], needs to meet the control region

[1] The name of Alexander Ventcel is often spelled differently, e.g. Wentzell.

in a finite time (see also [BGé97]). In the present case, we do not impose such a strong geometrical condition on the localization of the damping, that is on the support of the functions a and b. Exponential decay rate is thus excluded in general. Here, we follow the approach of [Leb96, LR97, Bel03a] that leads to the proof of a *logarithm* type decay for the energy, which is known to be optimal in the case of Neumann boundary condition (see [LR97]). Setting the damped wave equation in a semigroup form, say $\frac{d}{dt}U + AU = 0$, such a decay can be obtained upon deriving of a resolvent estimate for the semigroup generator of the form $\|(i\sigma\,\mathrm{Id}+A)^{-1}\| \leq C\exp(C|\sigma|)$ for $\sigma \in \mathbb{R}$, with $|\sigma| \geq 1$. Precise statements, including proper operator norms, are given below. Note that the same type of decay is obtained for a damping located in the interior ($a(x)$ non-vanishing on an open subset $\omega_I \Subset \Omega$) or located at the boundary ($b(x)$ non vanishing on an open subset $\omega_B \Subset \partial\Omega$). The resolvent estimate written above can be achieved from Carleman type estimates for a second-order elliptic operator, taking into account the particular boundary condition used in the definition of the damped wave equation problem. Classical boundary conditions, e.g. homogeneous Dirichlet, homogeneous Neumann in the case of an inner damping (a nonvanishing), or Neumann in the case of a boundary damping (b nonvanishing), were treated in the works cited above. The subject of the present article is to consider Ventcel type condition,

$$\partial_\nu u - \delta\Delta_g^T u + b\partial_t u = 0$$

with a parameter $\delta \in (0,1]$. In particular, in the result we obtain, the parameter δ is allowed to tend to 0^+ and we recover the result known for Neumann boundary conditions [LR97]. We refer to Section 4.1.3 for precise statements.

A large part of the present work is devoted to the proof of a local Carleman estimate near the boundary for the following elliptic problem

$$\begin{cases} \Delta_g u + \sigma^2 u = f & \text{in } \Omega \\ \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} = g & \text{in } \partial\Omega. \end{cases}$$

uniformly in σ and δ for $|\sigma| \geq 1$ and $\delta \in (0,1]$. Then, this allows us to derive an interpolation inequality leading to the resolvent estimate for the semigroup generator. Using the analysis of [Bur98, BD08], we can then obtain the logarithm-decay stabilization result. We also show that a similar result can be obtained dynamical Ventcel type boundary conditions.

The proof of the Carleman estimate relies on microlocal techniques at the boundary in the spirit of [LR97, LL12, LR10, LR11, LLR13]. Near the boudary, our analysis is carried out in normal geodesic coordinates, which eases to use pseudodifferential methods.

The outline of this article is the following. Main results are presented in Section 4.1.3. In Section 4.2, we address the well-posedness issues for the damped wave equations we consider as well as the asymptotic behavior of their solutions in the limit $\delta \to 0^+$. Section 4.3 recalls notions around semiclassical calculus and in Section 4.4 we describe the local geometry of the problem near the boundary. In preparation for the derivation of the Carleman estimate, various microlocal regions are introduced in Section 4.5 and microlocal versions of the estimate are obtained in Section 4.6. These estimates are patched together in Section 4.7 yielding the desired local Carleman estimate near the boundary. Finally, in Section 4.8 an interpolation estimate is derived and we achieve the sought resolvent estimate.

4.1.2 Statement of the problem

Let (Ω, g) be a Riemannian manifold of dimension n with smooth boundary $\partial\Omega$. For simplicity, Ω is assumed to be connected. The boundary can be seen as a compact Riemannian manifold of dimension $n-1$ without boundary endowed with the induced metric $g_{|\partial\Omega}$. In local coordinates, the gradient and the divergence on (Ω, g) are given by

$$\nabla_g = \sum_{j=1}^n g^{ij}\partial_{x_i}, \qquad \mathrm{div}_g\, u = \frac{1}{\sqrt{\det(g)}}\sum_{j=1}^n \partial_{x_j}(\sqrt{\det(g)}u_j).$$

where g^{ij} denotes the coefficients of the inverse of the matrix $g = (g_{ij})_{ij}$, with similar formulae for $\nabla_g^T := \nabla_{g_{|\partial\Omega}}$ and $\mathrm{div}_g^T := \mathrm{div}_{g_{|\partial\Omega}}$. The Laplace-Beltrami operator on (Ω, g) then reads

$$\Delta_g = \mathrm{div}_g \nabla_g = \frac{1}{\sqrt{\det(g)}} \sum_{i,j}^{n} \partial_{x_i}(g^{ij}\sqrt{\det(g)}\partial_{x_j}), \tag{4.1.1}$$

with a similar formula for Δ_g^T. We shall denote throughout this paper by ν the outgoing unit normal vector to Ω with respect to the Riemannian metric, and ∂_ν the associated normal derivative. In this setting we consider the following wave equation

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{4.1.2}$$

which corrresponds to a problem with a static boundary condition of Ventcel type. Dynamic boundary conditions can also be considered, namely

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x \quad \partial_t^2 u_{|\partial\Omega} + \frac{1}{\delta}\partial_\nu u_{|\partial\Omega} - \Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x. \tag{4.1.3}$$

In both cases, δ is a small parameter, say $0 < \delta \leq 1$. This kind of boundary conditions may for instance model a thin layer structure surrounding Ω, and the positive parameter δ plays the role of the measure of the thickness of this layer (see Appendix 9.B for a derivation of the model).
We now define usual norms and scalar products on Ω and $\partial\Omega$

$$\Big(u, \tilde{u}\Big)_{L^2(\Omega)} := \int_\Omega u\overline{\tilde{u}}\mathrm{d}x_g, \qquad \Big(v, \tilde{v}\Big)_{L^2(\partial\Omega)} := \int_{\partial\Omega} v\overline{\tilde{v}}\mathrm{d}\sigma_g, \tag{4.1.4}$$

where $\mathrm{d}x_g$ and $\mathrm{d}\sigma_g$ are the volume elements associated with the metrics g and $g_{|\partial\Omega}$. In local coordinates, we have $\mathrm{d}x_g = \sqrt{\det(g)}\mathrm{d}x_1 \dots \mathrm{d}x_n$, and a similar formula for $\mathrm{d}\sigma_g$. We also introduce the following Sobolev H^1 scalar products

$$\Big(u, \tilde{u}\Big)_{H^1(\Omega)} = \Big(u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)},$$
$$\Big(v, \tilde{v}\Big)_{H^1(\partial\Omega)} = \Big(v, \tilde{v}\Big)_{L^2(\partial\Omega)} + \Big(\nabla_g^T v, \nabla_g^T \tilde{v}\Big)_{L^2(\partial\Omega)}. \tag{4.1.5}$$

Throughout this paper, we shall denote by $||.||$ a norm acting on Ω, and by $|.|$ a norm acting on $\partial\Omega$.

Forming the scalar product of (4.1.2) with $\partial_t u$ in space and integrating by parts yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u,t) = 0, \text{ with } E(u,t) := \frac{1}{2}\left(||\partial_t u(t)||^2_{L^2(\Omega)} + ||\nabla_g u(t)||^2_{L^2(\Omega)} + \delta|\nabla_g^T u(t)_{|\partial\Omega}|^2_{L^2(\partial\Omega)}\right), \tag{4.1.6}$$

which corresponds to a conservation of energy $E(u,t)$ of the system.

The purpose of the present article is the study of interior stabilization, namely, the following system

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{4.1.7}$$

where a is a bounded function of Ω satisfying the condition $a \geq C > 0$ on ω_I, where ω_I is a non empty subset of Ω, as well as the problem with damping affecting a subset the boundary

$$\partial_t^2 u - \Delta_g u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_\nu u_{|\partial\Omega} - \delta\Delta_g^T u_{|\partial\Omega} + b\partial_t u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{4.1.8}$$

where $b \in W^{1,\infty}(\partial\Omega)$ satisfying $b \geq C > 0$ on a non-empty subset ω_B of $\partial\Omega$. Computing the evolution of the energy as above, we formally obtain respectively

$$E(u,t) - E(u,0) = -\int_0^t \int_\Omega a|\partial_t u|^2, \quad E(u,t) - E(u,0) = -\int_0^t \int_{\partial\Omega} b|\partial_t u_{|\partial\Omega}|^2,$$

which shows that, in both cases, the energy is a non-increasing function of time. We shall prove that the localized damping effect is actually sufficient to ensure that the energy goes to zero at least logarithmically. In [LR97], in the case where Ω is a ring of $\mathbb{R}^2$, the authors proved that such a logarithmic decay rate is in fact optimal in the case of Neumann boundary conditions. As the decay results here is uniform with respect to the small parameter δ, we expect them to be also optimal.

Below, we shall treat well-posedness and stabilization properties of (4.1.7) and (4.1.8) in the same time (Sections 4.1.3 and 6.1.1).

Static boundary conditions

In fact, we shall consider slightly more general operators at the boundary without adding technicality in the analysis. We shall consider the following system

$$\begin{cases} \partial_t^2 u - \Delta_g u + a\partial_t u = 0 & \text{on } \mathbb{R}_t \times \Omega_x \\ \partial_\nu u_{|\partial\Omega} + \delta \Sigma u_{|\partial\Omega} + b\partial_t u_{|\partial\Omega} = 0 & \text{on } \mathbb{R}_t \times \partial\Omega_x, \end{cases} \tag{4.1.9}$$

where a and b are as above, but at least one is non identically zero, and Σ denotes any positive second-order differential operator on $\partial\Omega$, that vanishes on constant functions, that is

$$\mathbb{C} \subset \ker(\Sigma), \tag{4.1.10}$$

and which furthermore is self-adjoint for the duality bracket $\langle .,. \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}$, where the chosen pivot space is $L^2(\partial\Omega)$ endowed with the inner-product defined by (4.1.4). Note that the definition of $\langle .,. \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}$ depends on the metric g. Hence there is some connection between the operator Σ and g. In particular $\Sigma = -\Delta_g^T$ is a possible choice for Σ. In fact, the choice $\Sigma = -\Delta_g^T$ is natural and motivates the condition (4.1.10) for the type of generalization we consider. Observe that H^{-1} is well defined as derivatives of $L^2(\partial\Omega)$ functions in the distribution sense, since $\partial\Omega$ has no boundary. Thus, the bilinear form

$$\left(u_{|\partial\Omega}, u_{|\partial\Omega}\right)_{L^2(\partial\Omega)} + \langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)} \tag{4.1.11}$$

defines an equivalent norm on $H^1(\partial\Omega)$ to (4.1.5). Furthermore, we define the energy associated to (4.1.9)

$$E_s(u,t) := \frac{1}{2}\left(||\partial_t u(t)||^2_{L^2(\Omega)} + ||\nabla_g u(t)||^2_{L^2(\Omega)} + \delta \left\langle \Sigma u(t)_{|\partial\Omega}, u(t)_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)} \right). \tag{4.1.12}$$

The reader should keep in mind that a prototype of such an operator Σ is $-\Delta_g^T$ defined in (4.1.1), and in this case, the energies E and E_s coincide. To treat the existence and uniqueness properties of evolution system (4.1.9), it is convenient to recast the problem into a semigroup formalism. Considering the norms appearing in the energy of solutions given in (4.1.12), we introduce the natural following spaces

$$\mathcal{H}_\delta = \mathcal{V}_\delta \times L^2(\Omega), \quad \delta \in (0,1], \text{ where } \mathcal{V}_\delta = \left\{ u \in H^1(\Omega) \ | u_{|\partial\Omega} \in H^1(\partial\Omega) \right\},$$

endowed with the norm

$$||u||^2_{\mathcal{V}_\delta} = ||u||^2_{H^1(\Omega)} + \delta \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega),H^1(\Omega)}. \tag{4.1.13}$$

The space $\mathcal{V}_\delta$ together with the norm $||.||_{\mathcal{V}_\delta}$ has a Hilbert space structure. Observe that this norm is equivalent to $\left(||.||^2_{H^1(\Omega)} + \delta |.|^2_{H^1(\partial\Omega)} \right)^{1/2}$. We then define the following norm on $\mathcal{H}_\delta$ as the cannonical norm

$$||(u,v)||^2_{\mathcal{H}_\delta} = ||u||^2_{\mathcal{V}_\delta} + ||v||^2_{L^2(\Omega)}.$$

Each space $\mathcal{H}_\delta$ and $\mathcal{V}_\delta$ indexed by δ is algebraically equal to $\mathcal{H}_{\delta=1}$ and $\mathcal{V}_{\delta=1}$ respectively. Yet, note that this identification does not hold topologically as δ goes to 0. Next, we define the wave operator

$$A_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & a(x) \end{pmatrix} \tag{4.1.14}$$

of domain $D(A_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0|\partial\Omega} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta,\ \partial_\nu u_{0|\partial\Omega} + \delta\Sigma u_{0|\partial\Omega} + b u_{1|\partial\Omega} = 0\}$. The operator A_δ depends on δ through its domain. In this formalism, system (4.1.9) reads as an evolution equation

$$\partial_t U + A_\delta U = 0, \tag{4.1.15}$$

for $U = (u, \partial_t u)$. In the case of (4.1.9), the boundary condition can be understood as an elliptic boundary condition.

Dynamic boundary conditions

In the case of dynamic boundary conditions, we shall consider the following problem

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0 \text{ on } \mathbb{R}_t \times \Omega_x, \quad \partial_t^2 u_{|\partial\Omega} + \frac{1}{\delta}\partial_\nu u_{|\partial\Omega} + \Sigma u_{|\partial\Omega} + \frac{1}{\delta} b\partial_t u_{|\partial\Omega} = 0 \text{ on } \mathbb{R}_t \times \partial\Omega_x, \tag{4.1.16}$$

where a and b are as in (4.1.9). Arguing as in (4.1.6), we define the following energy

$$E_d(u,t) := \frac{1}{2}\left(||\partial_t u(t)||^2_{L^2(\Omega)} + ||\nabla_g u(t)||^2_{L^2(\Omega)} + \delta|\partial_t u_{|\partial\Omega}(t)|^2_{L^2(\partial\Omega)} + \delta \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} \right). \tag{4.1.17}$$

We shall treat system (4.1.16) as a system of equations coupled through the normal derivative term with a transmission condition at the boundary

$$\partial_t^2 u - \Delta_g u + a\partial_t u = 0, \quad \partial_\nu u_{|\partial\Omega} + \delta\partial_t^2 y + \delta\Sigma y + b\partial_t y = 0, \quad u_{|\partial\Omega} = y. \tag{4.1.18}$$

We then define the space of energy

$$\mathcal{K}_\delta := \left\{ (u_0, u_1, y_0, y_1) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\partial\Omega) \times L^2(\partial\Omega) \mid u_{0|\partial\Omega} = y_0 \right\},$$

endowed with the norm $\|(u_0, u_1, y_0, y_1)\|^2_{\mathcal{K}_\delta} = ||u_0||^2_{H^1(\Omega)} + ||u_1||^2_{L^2(\Omega)} + \delta \left\langle \Sigma y_0, y_0 \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \delta |y_1|^2_{L^2(\partial\Omega)}$, yielding a Hilbert space structure. We recast system (4.1.18) into the evolution equation $\partial_t U + B_\delta U = 0$, where $U = (u, \partial_t u, y, \partial_t y)$, and where B_δ is the operator defined on $\mathcal{K}_\delta$

$$B_\delta := \begin{pmatrix} 0 & -\mathrm{Id} & 0 & 0 \\ -\Delta_g & a & 0 & 0 \\ 0 & 0 & -\mathrm{Id} & 0 \\ \frac{1}{\delta}\gamma_1 & 0 & \Sigma & \frac{1}{\delta} b \end{pmatrix},$$

with domain $D(B_\delta) := \left\{ (u_0, u_1, y_0, y_1) \in H^2(\Omega) \times H^1(\Omega) \times H^2(\partial\Omega) \times H^1(\partial\Omega) \mid u_{0|\partial\Omega} = y_0 \right\}$. The operator γ_1 denotes here the trace on $\partial\Omega$ of the normal derivative ∂_ν. In the case of (4.1.16), the boundary condition can be understood as a parabolic boundary condition.

Asymptotic behavior with the parameter δ.

Formally, in the limite $\delta \to 0$, both models (4.1.9) and (4.1.16) lead to the Neumann boundary condition. A precise statement of the convergence in proper spaces is given in Section 4.2.2 in the case of static boundary conditions (4.1.9). The dynamic case is left to the reader. In the case of a parabolic equations across an interface, such an asymptotic result can be found in [LR10].

4.1.3 Main results

Stabilization results on the damped wave equations

The main results of this article are the following stabilization properties.

Theorem 4.1.1. *Let $k \geq 1$. There exists $C > 0$ such that for all $0 < \delta \leq 1$ we have the following energy decay estimate*

$$E_s(u,t)^{1/2} \leq \frac{C}{(\log(2+t))^k} ||A_\delta^k U_0||_{\mathcal{H}_\delta(\Omega)},$$

for all u solutions of (4.1.9) with initial data $U_0 = (u_0, \partial_t u_0)$.

We also have

Theorem 4.1.2. *Let $k \geq 1$. There exists $C > 0$ such that for all $0 < \delta \leq 1$ we have the following energy decay estimate*

$$E_d(u,t)^{1/2} \leq \frac{C}{(\log(2+t))^k} ||B_\delta^k U_0||_{\mathcal{H}_\delta(\Omega)},$$

for all u solutions of (4.1.16) with initial data $U_0 = (u_0, \partial_t u_0, y_0, \partial_t y_0)$.

In connection with condition (4.1.10) the energy introduced in (4.1.12) (resp. (4.1.17)) vanishes on constant functions. The stabilization result of Theorem 4.1.1 (resp. 4.1.2) thus implies a logarithmic convergence of the solution to a constant function.

Observe that the decay rate increases as the regularity of the initial data does. In fact, using semi-group properties one can show that if we simply have $E(u,t) \leq f(t)E(u,0)$ with $f(t) \to 0$ as $t \to +\infty$ then, in fact, the energy decays exponentially. From [BD08, Bur98], it is well known that the stabilization results of Theorem 4.1.1 and 4.1.2 can be reduced to deriving the following resolvent estimate along the imaginary axis.

Theorem 4.1.3. *For all $\sigma \in \mathbb{R}$, $\sigma \neq 0$, the operators $(i\sigma\,\mathrm{Id} + A_\delta)$ and $(i\sigma\,\mathrm{Id} + B_\delta)$ are invertible on $\mathcal{H}_\delta$ and $\mathcal{K}_\delta$ respectively. Moreover, there exists $C¿0$ such that*

$$||(i\sigma\,\mathrm{Id} + A_\delta)^{-1}||_{\mathcal{H}_\delta \to \mathcal{H}_\delta} \leq Ce^{C|\sigma|} \qquad |\sigma| \geq 1, \tag{4.1.19}$$

$$||(i\sigma\,\mathrm{Id} + B_\delta)^{-1}||_{\mathcal{K}_\delta \to \mathcal{K}_\delta} \leq Ce^{C|\sigma|} \qquad |\sigma| \geq 1. \tag{4.1.20}$$

As $D(A_\delta)$ is compactly embedded in $\mathcal{H}_\delta$, the spectrum of A_δ is countable. In the case of an undamped wave equation, i.e $a = 0$ and $b = 0$, the operator A_δ is antisymmetric for the inner-product of $\mathcal{H}_\delta$, and then its eigenvalues are purely imaginary. In the case of stabilization (a or b not identically zero), the only eigenvalue on the imaginary axis is zero. Indeed, let $\sigma \in \mathbb{R}^*$, $\sigma \neq 0$ and consider $U = (u_0, u_1)$ satisfying $(A_\delta + i\sigma)U = 0$. This is equivalent to

$$\begin{cases} u_1 - i\sigma u_0 = 0, & -\Delta_g u_0 - ai\sigma u_0 - \sigma^2 u_0 = 0 \quad \text{in } \Omega, \\ \partial_\nu u_{0|\partial\Omega} + \delta\Sigma u_{0|\partial\Omega} + ib\sigma u_{0|\partial\Omega} = 0 & \text{in } \partial\Omega. \end{cases}$$

Multiplying the second equation by $\overline{u}_0$ and integrating by parts over Ω yields $u_0 = 0$ on ω_I if considering the imaginary part on ω_I, and $u_{0|\partial\Omega} = 0$ on ω_B, thus u_0 satisfies $-\Delta_g u_0 = \sigma^2 u_0$. Thus we can apply Calderón's unique continuation theorem if $\omega_I \neq \emptyset$, and apply Theorem 9.D.1 given in appendix if $\omega_B \neq \emptyset$. The same arguments hold for the operator B_δ. As said above, 0 is an eigenvalue for both operators. To remove this difficulty, we shall work in quotient spaces as described at the end of Sections 4.2.1 and 4.2.1. In these quotient spaces, we can extend the estimates of Theorem 6.1.15 to $\sigma \in \mathbb{R}$, and (6.1.23) and (6.1.24) ensure that all the eigenvalues are not in a closed neighborhood of $i\mathbb{R}$ of the type $\{z := x + iy,\ x \geq 0,\ x \leq e^{-C|y|}\}$ (see for instance [LR97]). This kind of resolvent estimate is heavily related to the Carleman estimate stated in the next section.

Carleman estimate at the boundary

We shall prove Carleman estimates for classes of more general operators in the interior and at the boundary. We thus define the following operators

$$P = -\Delta_g + c(x).\nabla_g + d(x), \qquad S = \Sigma + c^T(x).\nabla_g^T + d^T(x), \tag{4.1.21}$$

where c (resp. c^T) denotes any L^∞ vector field on Ω (resp. $\partial\Omega$), and d (resp d^T) any L^∞ function on Ω (resp. $\partial\Omega$). The estimate we prove in this paper concerns the following system

$$(P-\sigma^2)u = f \text{ on } \Omega, \quad \partial_\nu u + \delta(S-\kappa\sigma^2)u = g \text{ on } \partial\Omega, \tag{4.1.22}$$

where σ is a real number, and κ is equal to 0 or 1. The operators $(P-\sigma^2)$ and $(S-\kappa\sigma^2)$ will be denoted by P_σ and S_σ respectively. We introduce the parameter κ in order to prove a Carleman estimate that allows us to treat both cases of static and dynamic boundary conditions at the same time. More precisely, $\kappa = 0$ corresponds to the static case, and $\kappa = 1$ to the dynamic case. Note that in (4.1.22), we add lower order terms in the interior and at the boundary. Moreover, we consider non-homogeneous equation with f and g as body and surface source terms. To precisely state the result, we need to recallthe notion of sub-ellipticity. For $\tau \geq 1$, we set $P_{\varphi,\sigma} = e^{\tau\varphi}P_\sigma e^{-\tau\varphi}$, where $\varphi \in C^\infty(\mathbb{R}^n)$, and consider $p_{\varphi,\sigma}$ its semi-classical principal symbol. We then have the following definition.

Definition 4.1.4. *Let V be a bounded open subset of Ω and $\varphi \in C^\infty(\overline{V})$. We say that φ satisfies the sub-ellipticity condition on $\overline{V}$ if there exists $\tau_0 > 0$ such that*

$$p_{\varphi,\sigma}(x,\xi,\tau) = 0 \Longrightarrow \frac{1}{2i}\{\overline{p_{\varphi,\sigma}}, p_{\varphi,\sigma}\} > 0, \tag{4.1.23}$$

for all $x \in \overline{V}$, $\xi \in \mathbb{R}^n$, $|\sigma| \geq 1$ and $\tau \geq \tau_0|\sigma|$.

We now consider V a bounded open neighborhood of a point of $\partial\Omega$. We impose additional conditions on φ on V, namely,

$$\nabla_g\varphi \neq 0 \quad \text{on } \overline{V}, \quad \text{and} \quad |\nabla_g^T\varphi| \leq \nu_0 \inf|\partial_\nu\varphi| \quad \text{on } V\cap\partial\Omega, \tag{4.1.24}$$

for a sufficiently small $\nu_0 > 0$. The local Carleman estimate in the neighborhood of the boundary that we shall prove is stated as following

Theorem 4.1.5. *Let $x \in \partial\Omega$ and V be an open neighborhood of x in $\overline{\Omega}$. Let φ be a weight function satisfying the conditions (4.1.23) and (4.1.24) on $\overline{V}$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\begin{aligned}\tau^3||e^{\tau\varphi}u||^2_{L^2(V)} + \tau||e^{\tau\varphi}\nabla_g u||^2_{L^2(V)} + \tau|e^{\tau\varphi}\partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\Big(||e^{\tau\varphi}f||^2_{L^2(V)} \\ + \tau|e^{\tau\varphi}g|^2_{L^2(V\cap\partial\Omega)} + (\delta^2\tau^5+\tau^3)|e^{\tau\varphi}u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau|e^{\tau\varphi}\nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)}\Big),\end{aligned} \tag{4.1.25}$$

and if in addition, $\partial_\nu\varphi(x) < 0$, on $\overline{V}$ we have the stronger estimate

$$\begin{aligned}\tau^3||e^{\tau\varphi}u||^2_{L^2(V)} + \tau||e^{\tau\varphi}\nabla_g u||^2_{L^2(V)} + \tau^3|e^{\tau\varphi}u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} + \tau|e^{\tau\varphi}\nabla_g^T u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \\ + \tau|e^{\tau\varphi}\partial_\nu u_{|\partial\Omega}|^2_{L^2(V\cap\partial\Omega)} \leq C\left(||e^{\tau\varphi}f||^2_{L^2(V)} + \tau|e^{\tau\varphi}g|^2_{L^2(V\cap\partial\Omega)}\right),\end{aligned} \tag{4.1.26}$$

for all $0 < \delta \leq 1$, for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$ and for all $u \in C_0^\infty(V)$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ satisfying (4.1.22).

Observe that the two Carleman estimates are uniform in $\delta > 0$. That will allow us to perform an uniform energy decay estimate with respect to the small parameter δ at the boundary. Furthermore, in the singular limit $\delta \to 0$, we recover the Carleman estimate proved in [LR95].

Remark 4.1.6. *Note that this estimate is invariant by adding lower order terms in σ in the following sense: if we set $L := P - \sigma^2 + r(x)\sigma$ and $L^T := S - \sigma^2 + r^T(x)\sigma$, with r and r^T two L^∞ functions, then we can write*

$$||e^{\tau\varphi}Lu||_{L^2} \leq ||e^{\tau\varphi}(P-\sigma^2)u||_{L^2} + C\sigma||e^{\tau\varphi}u||_{L^2},$$

and the second term can be absorbed by the left hand side of the Carleman estimates of Theorem 4.1.5 by taking τ_0 large. In the same spirit,

$$|e^{\tau\varphi}L^T u|_{L^2} \leq |e^{\tau\varphi}(S-\sigma^2)u|_{L^2} + C\sigma|e^{\tau\varphi}u|_{L^2},$$

and we can absorb the second term by taking τ_0 sufficiently large. This estimate is also invariant by adding lower order operators. If (4.1.25) and (4.1.26) are true for $P = -\Delta_g$, it is also true for P in the form given in (4.1.21), by taking τ_0 large. It will thus be sufficient to derive these estimates keeping only the prinipal part of P.

4.2 Well-posedness and asymptotic behavior

In this section, we survey the well-posedness properties of the damped wave equation with static boundary conditions (4.1.7). We also consider the asymptotic behavior if δ goes to zero. Indeed, formally taking δ equal to zero, system (4.1.7) becomes a damped wave equation with Neumann boundary conditions. We shall make precise in which spaces such convergence can be proven.

4.2.1 Well-posedness properties

The well-posedness properties can be stated for general operators. We set

$$\mathcal{A}_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ P & a \end{pmatrix},$$

with domain $D(\mathcal{A}_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0_{|\partial\Omega}} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta(\partial\Omega),\ \partial_\nu u_0 + \delta S u_0 + b u_1 = 0\}$, for P and S be the operators defined by (4.1.21). In the same idea, we set

$$\mathcal{B}_\delta := \begin{pmatrix} 0 & -\mathrm{Id} & 0 & 0 \\ P & a & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \\ \frac{1}{\delta}\gamma_1 & 0 & S & \frac{1}{\delta} b \end{pmatrix},$$

and observe that $D(B_\delta) = D(\mathcal{B}_\delta)$.

The case of static boundary conditions

Proposition 4.2.1. *There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $F \in \mathcal{H}_\delta$, there exists a unique solution $U = (u, v) \in D(\mathcal{A}_\delta)$ of $(\mathcal{A}_\delta + \lambda\,\mathrm{Id})U = F$. Moreover, there exists $C > 0$ such that*

$$||u||^2_{H^2(\Omega)} + \delta |u_{|\partial\Omega}|^2_{H^2(\partial\Omega)} + ||v||^2_{\mathcal{V}_\delta} \leq C||F||^2_{\mathcal{H}_\delta},$$

for all $\delta \in (0, 1]$, and $\lambda \geq \lambda_0$.

The proof is given in Apendix 9.C.1. We can now state the existence and uniqueness result for the associated evolution equation.

Proposition 4.2.2. *Let $U_0 \in D(\mathcal{A}_\delta)$. Then, there exists a unique U in $C^1([0, +\infty), \mathcal{H}_\delta) \cap C([0, +\infty), D(\mathcal{A}_\delta))$ satisfying the Cauchy problem*

$$\partial_t U + \mathcal{A}_\delta U = 0 \text{ for } t \geq 0, \quad U_{t=0} = U_0.$$

where $D(\mathcal{A}_\delta)$ is endowed with the norm of the graph $||U||^2_{D(\mathcal{A}_\delta)} = ||U||^2_{\mathcal{H}_\delta} + ||\mathcal{A}_\delta U||^2_{\mathcal{H}_\delta}$.

Moreover, we have $||U(t,.)||_{\mathcal{H}_\delta} \leq ||U_0||_{\mathcal{H}_\delta}$ and $||\partial_t U(t,.)||_{\mathcal{H}_\delta} \leq ||\mathcal{A}_\delta U_0||_{\mathcal{H}_\delta}$.

Proof. From the previous propositions, $A_\delta + \lambda_0\,\mathrm{Id}$ is maximal monotoneous on $\mathcal{H}_\delta$. Then, we can apply the Lumer-Philips theorem (see for instance [Paz83], Theorem 4.3) to obtain the result. □

We now focus on the case $\mathcal{A}_\delta = A_\delta$ (see (4.1.14)).

Proposition 4.2.3. *Assume that (4.1.10) holds. Then*

$$\mathrm{Sp}(A_\delta) \cap i\mathbb{R} = \{0\}, \tag{4.2.1}$$

and the subspace E_0 formed by the eigenfunctions of A_δ associated with the eigenvalue 0 is

$$E_0 = \mathbb{C}\ {}^t(1,0). \tag{4.2.2}$$

Proof. By Proposition 4.2.1, the spectrum of A_δ is purely discrete. The fact that $i\sigma$, $\sigma \neq 0$ is not an eigenvalue comes from the discussion below Theorem 6.1.15. It is clear that 0 is an eigenvalue, and that $\mathbb{C}^t(1,0) \subset E_0$. Let $(u_0, u_1) \in D(A_\delta)$ such that $A_\delta(u_0, u_1) = 0$. We obtain $u_1 = 0$, and thus $-\Delta_g u_0 = 0$. By integration by parts we have

$$||\nabla_g u_0||^2_{L^2(\Omega)} + \delta \left\langle \Sigma u_{0|\partial\Omega}, u_{0|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} = 0,$$

and we obtain $\psi = C$ on Ω, which shows equality (4.2.2). □

Actually, if assumption (4.1.10) is not satisfied, then 0 is not an eigenvalue for A_δ. Below, we shall work in quotient spaces $\dot{\mathcal{V}}_\delta = \mathcal{V}_\delta/E_0$ and $\dot{\mathcal{H}}_\delta = \mathcal{H}_\delta/E_0$, where $E_0 := \{(C,0)\ ,\ C \in \mathbb{C}\}$. We set $\dot{A}_\delta$ the operator induced by the projection in the quotient space. We also set: $D(\dot{A}_\delta) := D(A_\delta) \cap \dot{\mathcal{H}}_\delta$. We can endow the space $\dot{\mathcal{V}}_\delta$ with the scalar product

$$\left(u, \tilde{u}\right)_{\dot{\mathcal{V}}_\delta} := \left(\nabla_g u, \nabla_g \tilde{u}\right)_{L^2(\Omega)} + \delta \left\langle \Sigma u, \tilde{u} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)},$$

which defines a norm on $\dot{\mathcal{V}}_\delta$, using the Poincarè inequality. For the sake of simplicity, in the sequel we shall do the following abuse of notation: we shall drop the dots and continue to write $\bullet$ in place of $\dot{\bullet}$, where $\bullet$ is one of the spaces above.

Remark 4.2.4. *Observe moreover that in the case $\mathcal{A}_\delta = A_\delta$, Proposition 4.2.1 holds with $\lambda_0 = 0$ in the above quotient spaces.*

The case of dynamic boundary conditions

We have the counterpart of proposition 4.2.1 for the dynamic case.

Proposition 4.2.5. *There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $F \in \mathcal{K}_\delta$, there exists a unique solution $U = (u_0, u_1, y_0, y_1) \in D(B_\delta)$ of $(\mathcal{B}_\delta + \lambda\,\mathrm{Id})U = F$. Moreover, there exists $C > 0$ such that*

$$||u_0||^2_{H^2(\Omega)} + \delta|y_0|^2_{H^2(\partial\Omega)} + ||u_1||^2_{H^1(\Omega)} + \delta|y_1|^2_{H^1(\partial\Omega)} \leq C||F||^2_{\mathcal{K}_\delta},$$

for all $\delta \in (0,1]$, and $\lambda \geq \lambda_0$.

The proof is given in Appendix 9.C.2. This leads to the following well-posedness result for the damped wave equation (4.1.18) written in a semigroup setting.

Proposition 4.2.6. *Let $U_0 \in D(\mathcal{B}_\delta)$. Then there exists a unique U in $C^1([0,+\infty), \mathcal{K}_\delta) \cap C([0,+\infty), D(\mathcal{B}_\delta))$ satisfying the Cauchy problem*

$$\partial_t U + \mathcal{B}_\delta U = 0, \quad U_{t=0} = U_0$$

where $D(\mathcal{B}_\delta)$ is endowed with the norm of the graph $||U||^2_{D(\mathcal{B}_\delta)} = ||U||^2_{\mathcal{K}_\delta} + ||\mathcal{B}_\delta U||_{\mathcal{K}_\delta}$.

Moreover, we have $||U(t,.)||^2_{\mathcal{K}_\delta} \leq ||U_0||^2_{\mathcal{K}_\delta}$ and $||\partial_t U(t,.)||^2_{\mathcal{K}_\delta} \leq ||\mathcal{B}_\delta U_0||^2_{\mathcal{K}_\delta}$.

We state the following result about the eigenvalues of the operator B_δ.

Proposition 4.2.7. *Assume that (4.1.10) holds. Then*

$$\mathrm{Sp}(B_\delta) \cap i\mathbb{R} = \{0\}, \tag{4.2.3}$$

and the subspace F_0 formed by the eigenfunctions of B_δ associated with the eigenvalue 0 is

$$F_0 = \mathbb{C}\ {}^t(1,0,1,0).$$

Proof. It is sufficient to repeat the proof of Proposition 4.2.3. □

We then quotient the space $\mathcal{K}_\delta$ by F_0, and still denote it $\mathcal{K}_\delta$ by abuse of notation.

4.2.2 Asymptotic behavior as δ goes to zero

In this section, we study the asymptotic behavior as δ goes to zero of the solution u_δ of the damped wave system with static Ventcel boundary condition

$$\begin{aligned} &\partial_t^2 u_\delta - \Delta_g u_\delta + a\partial_t u_\delta = f_\delta \text{ in } \Omega\times\mathbb{R}, \qquad \partial_\nu u_{\delta|_{\partial\Omega}} - \delta\Delta_g^T u_{\delta|_{\partial\Omega}} + b\partial_t u_{\delta|_{\partial\Omega}} = 0 \text{ in } \partial\Omega\times\mathbb{R}, \\ &(u_\delta(0),\partial_t u_\delta(0)) = U_\delta^0 \text{ in } \Omega, \end{aligned} \tag{4.2.4}$$

to the solution of the damped wave equation with homogeneous Neumann boundary condition

$$\partial_t^2 v - \Delta_g v + a\partial_t v = f \text{ in } \Omega, \qquad \partial_\nu v_{|_{\partial\Omega}} + b\partial_t v_{|_{\partial\Omega}} = 0 \text{ in } \partial\Omega, \qquad (v(0),\partial_t v(0)) = V^0 \text{ in } \Omega. \tag{4.2.5}$$

Proposition 4.2.8. *Assume $f_\delta \rightharpoonup f$ in $L^2(0,T;L^2(\Omega))$ and $U_\delta^0 = V^0 \in \mathcal{H}$. We then obtain*

$$u_\delta \rightharpoonup v \text{ in } L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega)),$$

and we have the estimate at the boundary $|\nabla_g^T u_{\delta|_{\partial\Omega}}|_{L^2(\partial\Omega)} = \mathcal{O}(\delta^{-1/2})$.

The proof is given in Appendix 9.E.1.

4.3 Notation and semi-classical operators

In the sections below, we shall use the following notation: $\mathbb{R}^n \ni x = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$, and $\mathbb{R}^n \ni \xi = (\xi', \xi_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ and we shall consider the operators $D = -i\partial_x$ and $D' = -i\partial_{x'}$. For V a neighborhood of a point of the boundary $\partial\Omega$ (resp. a neighborhood of 0 in $\mathbb{R}^n$), we set $V^+ = V\cap\Omega$ (resp. $V^+ = V\cap\mathbb{R}^n$), where $\mathbb{R}^n_+$ is the half-space $\{x\in\mathbb{R}^n,\ x_n > 0\}$.

4.3.1 Semi-classical operators acting on $\mathbb{R}^n$

Here we recall some facts on semi-classical pseudo-differential operators with a large parameter τ, say $\tau \geq \tau_0 \geq 1$. We shall denote by $\mathcal{S}^m_\tau$ the space of smooth functions $a(x,\xi,\tau)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$, with $\tau\geq\tau_0$ as a large parameter, that satisfy the following behavior at infinity: for all multi-indices α, β there exists $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi,\tau)\right| \leq C_{\alpha,\beta}(\tau^2+|\xi|^2)^{(m-|\beta|)/2},$$

for all $(x,\xi',\tau)\in\mathbb{R}^n\times\mathbb{R}^n\times[\tau_0,+\infty)$. For $a\in\mathcal{S}^m_\tau$, we define pseudo-differential operator of order m, denoted by $A = \mathrm{Op}(a)$:

$$Au(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}a(x,\xi,\tau)\hat{u}(\xi)\mathrm{d}\xi, \quad u\in\mathcal{S}(\mathbb{R}^n).$$

One says that a is the symbol of A. We shall denote Ψ^m_τ the set of pseudo-differential operators of order m and denote by $\sigma(A)$ (resp. $\sigma(a)$) the principal symbol of the operator A (resp. the symbol a) and thus $\sigma(D) = \xi$. We shall also denote by $\mathcal{D}^m_\tau$ the space of semi-classical differential operators, i.e the case when the symbol $a(x,\xi,\tau)$ is a polynomial function of order m in (ξ,τ). Throughout the article, we shall use the following order function on the whole phase-space: $\lambda_\tau = (\tau^2+|\xi|^2)^{1/2}$. We recall here the composition formula of pseudo-differential operators. Let $a\in\mathcal{S}^m_\tau$ and $b\in\mathcal{S}^{m'}_\tau$, $m,m'\in\mathbb{R}$, we have

$$\mathrm{Op}(a)\circ\mathrm{Op}(b) = \mathrm{Op}(a\#b),$$

for some $a\#b\in\mathcal{S}^{m+m'}_\tau$, and for all $N\in\mathbb{N}$, there exists $R_N\in\mathcal{S}^{m+m'-N}_\tau$ such that

$$a\#b(x,\xi,\tau) = \sum_{|\alpha|\leq N}\frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a(x,\xi,\tau)\partial_x^\alpha b(x,\xi,\alpha) + R_N(x,\xi,\tau). \tag{4.3.1}$$

For a review on symbolic calculus we refer the reader to [Hör07].

4.3.2 Tangential semi-classical operators

In the section we consider pseudo-differential operators which only acts in the tangential direction x', with parameter x_n. We define $\mathcal{S}^m_{T,\tau}$ as the set of smooth functions $a(x,\xi',\tau)$ defined for τ as a large parameter, say $\tau \geq \tau_0 \geq 1$, satisfying the following behavior at infinity: for all multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$ there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta a(x,\xi',\tau)\right| \leq C_{\alpha,\beta}(\tau^2 + |\xi'|^2)^{(m-|\beta|)/2},$$

for all $(x,\xi',\tau) \in \mathbb{R}^n_+ \times \mathbb{R}^{n-1} \times [\tau_0, +\infty)$. For $a \in \mathcal{S}^m_{T,\tau}$, we define a tangential pseudo-differential operator $B := \mathrm{Op}_T(b)$ of order m by

$$Bu(x) := \frac{1}{(2\pi)^{n-1}} \iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} e^{i(x'-y').\xi'} b(x,\xi',\tau)u(y',x_n)\mathrm{d}y'\mathrm{d}\xi'$$

As in the previous section, we define $\Psi^m_{T,\tau}$ as the set of tangential pseudo-differential operators of order m, and $\mathcal{D}^m_{T,\tau}$ the set of tangential differential operators of order m. We shall denote the tangential order function by $\lambda_{T,\tau} := (\tau^2 + |\xi'|^2)^{1/2}$, and define the following semi-classical Sobolev tangential norms, for fonctions on $\mathbb{R}^{n-1}$ or traces of functions on $\mathbb{R}^n$ at $\{x_n = 0\}$

$$|u|_{m,\tau} := |\,\mathrm{Op}_T(\lambda^m_{T,\tau})u|_{L^2(\mathbb{R}^{n-1})}.$$

We also define the following semi-classical norms on the half space $\mathbb{R}^n_+$

$$||u||^2_{m,\tau} := \sum_{k=0}^m ||D_n^k\,\mathrm{Op}_T(\lambda^{m-k})u||^2_{L^2(\mathbb{R}^n_+)}.$$

Observe that , if $m \in \mathbb{N}$, this semi-classical norm is equivalent to $\sum_{|\alpha|\leq m} \tau^{|\alpha|}||D^{m-|\alpha|}u||_{L^2(\mathbb{R}^n_+)}$, uniformly for $\tau \in [\tau_0, +\infty)$. Below, we shall use several times the following trace lemma.

Lemma 4.3.1. *There exists $C > 0$ such that for all $u \in S(\mathbb{R}^n)$, for $\tau \geq 1$,*

$$|u_{|_{x_n=0}}|_0 \leq C\tau^{-1/2}||u||_{1,\tau}. \tag{4.3.2}$$

The proof is left to the reader.

Remark 4.3.2. *In this paper, we shall use operators whose symbol depends on a additional parameter σ, say $a(x,\xi,\tau,\sigma)$, such that they satisfy*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,\tau,\sigma)\right| \leq C_{\alpha,\beta}(\sigma^2 + \tau^2 + |\xi|^2)^{(m-|\beta|)/2}.$$

However, in the region where $\tau \gtrsim |\sigma|$, we have $a \in \mathcal{S}^m_\tau$, and this property will be used several times.

4.4 Local setting in the neighborhood of the boundary

Here, we consider normal geodesic coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ in a neighborhood V of a point of the boundary. Locally, we have $\Omega = \{x_n > 0\}$ and $\partial\Omega = \{x_n = 0\}$. We recall that from the remark below Theorem 4.1.5, we can consider only the higher order terms in the operator P, since the form of the estimates we want to prove is insensitive to the addition of low order terms. In such local coordinates, the principal part of operator P takes the form (and we still denote by P, by abuse of notation)

$$P = D_n^2 + R(x,D'), \qquad R(x,D') = \sum_{j,k=1}^{n-1} D_j(a_{j,k}(x)D_k),$$

where $a_{j,k} = a_{k,j}$ and, if we denote by $r(x,\xi')$ the homogeneous principal symbol of R,

$$r(x,\xi') = \sum_{j,k=1}^{n-1} a_{j,k}(x)\xi_j\xi_k \in \mathbb{R} \text{ and } \exists C>0,\ \forall (x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1},\ r(x,\xi') \geq C|\xi'|^2. \tag{4.4.1}$$

Note that the principal part of the operator P is chosen to be formally self-adjoint. We also denote the homogeneous principal symbol of P by $p(x,\xi) = \xi_n^2 + r(x,\xi')$. Whenever V is a neighborhood of 0 in $\mathbb{R}^n$, we shall denote by $\overline{C}_0^\infty(V^+)$ the space of restrictions to $\overline{V}^+ := \mathbb{R}^n \cap \{x_n > 0\}$ of C^∞ functions on $\mathbb{R}^n$ compactly supported in V. On the boundary $\{x_n = 0\}$, the operator S is an elliptic second-order differential operator in the x'-direction. If we denote by $s(x',\xi')$ its homogeneous principal symbol, we have

$$s(x',\xi') \in \mathbb{R} \text{ and } \exists C>0,\ \forall (x',\xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, s(x',\xi') \geq C|\xi'|^2.$$

Observe that in these local coordinates, we have $\partial_\nu = -\partial_{x_n}$, where ∂_ν denote the outgoing normal derivative associated with the metric g. In what follows, we shall denote $P_\sigma := P - \sigma^2$ and $S_\sigma := S - \kappa\sigma^2$ for all $\sigma \in \mathbb{R}$, and $\kappa \in \{0,1\}$.

4.4.1 Operator conjugaison by a weight function

As is done classically, we introduce the following conjugated operator

$$P_{\varphi,\sigma} = e^{\tau\varphi} P_\sigma e^{-\tau\varphi},$$

of homogeneous principal symbol $p_{\varphi,\sigma}$, for a smooth function φ to be precisely defined below. We denote also $S_{\varphi,\sigma}$ the conjugated boundary operator

$$S_{\varphi,\sigma} = e^{\tau\varphi_{|x_n=0}} S_\sigma e^{-\tau\varphi_{|x_n=0}},$$

of homogeneous principal symbol $s_{\varphi,\sigma}$. In what follows, we set: $v = e^{\tau\varphi}u$, and we thus have $e^{\tau\varphi}P_\sigma u = P_{\varphi,\sigma}v$ and $e^{\tau\varphi_{|x_n=0}} S_\sigma u_{|x_n=0} = S_{\varphi,\sigma} v_{|x_n=0}$. We have $P = P_2 + iP_1$ by setting

$$P_2 = \frac{1}{2}\left(P_{\varphi,\sigma} + P_{\varphi,\sigma}^*\right), \qquad P_1 = \frac{1}{2i}\left(P_{\varphi,\sigma} - P_{\varphi,\sigma}^*\right).$$

Note that P_2 and P_1 are formally self-adjoints. Observing that $e^{\tau\varphi} D_j e^{-\tau\varphi} = D_j + i\tau\partial_{x_j}\varphi$, we have

$$P_2 = D_n^2 - (\tau\partial_{x_n}\varphi)^2 - \sigma^2 + R(x,D') - r(x,\tau \mathrm{d}_{x'}\varphi) = P_\sigma - p(x,\tau \mathrm{d}_x\varphi), \tag{4.4.2}$$

$$P_1 = \tau D_n \partial_{x_n}\varphi + \tau\partial_{x_n}\varphi D_n + \tau \sum_{j,k=1}^n \left(D_j a_{j,k}(x)\partial_k\varphi + a_{j,k}(x)\partial_j\varphi D_k\right) \tag{4.4.3}$$

$$= 2\tau(\partial_{x_n}\varphi D_n + \tilde{r}(x, \mathrm{d}_{x'}\varphi, D')) \mod \tau\mathcal{D}_T^0, \tag{4.4.4}$$

where $\tilde{r}$ denotes the symmetric bilinear form associated with the quadratic form r, $\tilde{r}(x,\xi,\eta) = \sum_{j,k=1}^n a_{j,k}(x)\xi.\eta$. At the boundary, we also have $S_{\varphi,\sigma} = S_2 + iS_1$, with

$$S_2 = \frac{1}{2}\left(S_{\varphi,\sigma} + S_{\varphi,\sigma}^*\right) \qquad S_1 = \frac{1}{2i}\left(S_{\varphi,\sigma} - S_{\varphi,\sigma}^*\right),$$

that are formally self adjoints and of the form

$$S_2 = s(x,D') - s(x',\tau \mathrm{d}_{x'}\varphi_{|x_n=0}) - \kappa\sigma^2 \mod \mathcal{D}_{T,\tau}^1 \text{ and } S_1 = 2\tau\tilde{s}(x', \mathrm{d}_{x'}\varphi_{|x_n=0}, D') \mod \mathcal{D}_{T,\tau}^1, \tag{4.4.5}$$

where $\tilde{s}$ denotes the symmetric bilinear form associated with the quadratic form s. Their principal symbols are respectively

$$p_2(x,\xi,\tau,\sigma)) = -\sigma^2 + \xi_n^2 + r(x,\xi') - \tau^2(\partial_{x_n}\varphi)^2 - r(x,\tau \mathrm{d}_{x'}\varphi), \tag{4.4.6}$$

$$p_1(x,\xi,\tau) = 2\tau\left(\partial_{x_n}\varphi\xi_n + \tilde{r}(x, \mathrm{d}_{x'}\varphi, \xi')\right), \tag{4.4.7}$$

$$s_2(x',\xi',\tau,\sigma) = s(x,\xi') - s(x',\tau \mathrm{d}_{x'}\varphi_{|x_n=0}) - \kappa\sigma^2, \tag{4.4.8}$$

$$s_1(x',\xi',\tau) = 2\tau\tilde{s}(x', \mathrm{d}_{x'}\varphi_{|x_n=0}, \xi'). \tag{4.4.9}$$

Note that $p_{\varphi,\sigma}(x,\xi,\tau) = p_2(x,\xi,\tau,\sigma) + ip_1(x,\xi,\tau)$. We shall denote the tangential parts of the symbols p_2 and p_1 by

$$\tilde{p}_2(x,\xi',\tau,\sigma) = -\sigma^2 + r(x,\xi') - p(x,\tau \mathrm{d}_x\varphi), \tag{4.4.10}$$
$$\tilde{p}_1(x,\xi',\tau) = 2\tau\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi). \tag{4.4.11}$$

With this notation, if u satisfies $\partial_\nu u_{|_{x_n=0}} + \delta S_\sigma(x',D')u_{|_{x_n=0}} = \tilde{\Theta}$, then v satisfies

$$D_n v_{|_{x_n=0}} = K v_{|_{x_n=0}} + \Theta \tag{4.4.12}$$

where $K \in \delta\mathcal{D}^2_{T,\tau} + \tau\mathcal{D}^0_{T,\tau}$, with principal symbol

$$\begin{aligned} k(x',\xi',\tau,\sigma) &= 2\tau\delta\tilde{s}(x',\xi',\mathrm{d}_{x'}\varphi) - i\left(\tau\partial_{x_n}\varphi + \delta s(x',\xi') - \delta s(x',\tau\mathrm{d}_{x'}\varphi) - \delta\kappa\sigma^2\right) \\ &= \delta s_1(x',\xi',\tau) - i(\tau\partial_{x_n}\varphi + s_2(x',\xi',\tau\sigma)), \end{aligned} \tag{4.4.13}$$

and $\Theta = ie^{\tau\varphi}\tilde{\Theta}$, recalling that $\partial_\nu = -\partial_{x_n}$.

Under the action of conjugaison by the weight function, the resulting operator $P_{\varphi,\sigma}$ is not elliptic. In order to handle the presence of the characteristics set, we shall impose the following condition on the weight function which ensures the positivity of some commutators.

Definition 4.4.1. *Let V be a bounded open set of $\mathbb{R}^n$. We say the weight function $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies the sub-ellipticity property in $\overline{V}$ if $|d_x\varphi| > 0$ in $\overline{V}$ and if there exist $C > 0$ and $\tau_0 > 0$ such that for $|\sigma| \geq 1 > 0$*

$$\forall (x,\xi) \in \overline{V} \times \mathbb{R}^n,\ \forall \tau \geq \tau_0|\sigma|,\quad p_{\varphi,\sigma}(x,\xi,\tau) = 0 \Longrightarrow \{p_2,p_1\}(x,\xi,\tau) \geq C\lambda_\tau^3. \tag{4.4.14}$$

Here, we state the sub-ellipticity condition in normal geodesic coordinates. However, note that this condition is geometrically invariant, and thus this definition is equivalent to (4.1.23) (observe that by the homogeneity of the Poisson bracket, $\{p_2,p_1\} > 0$ implies (4.4.14), see the end of the proof of Proposition 4.4.2). The following proposition provides a construction of a weight function φ that yields sub-ellipticity using a classical convexification procedure. A proof without the parameter σ can be found in [LL12]. With the parameter σ, a proof is given in Appendix 9.E.2.

Proposition 4.4.2. *Let V be a bounded open subset of $\mathbb{R}^n$ and $\psi \in C^\infty(\mathbb{R}^n)$ such that $|d_x\psi| \geq C > 0$ on $\overline{V}$. Then, there exists $\lambda > 0$ sufficiently large, such that the function $\varphi := e^{\lambda\psi}$ satisfies the sub-ellipticity condition on $\overline{V}$ for $\tau \geq \tilde{C}|\sigma|$, where $\tilde{C}$ is a constant satisfying $\tilde{C} \geq \dfrac{1}{\lambda \inf \varphi}$.*

Observe that we impose τ to be larger than σ here. This condition appears naturally in the following proof. In what follows, τ will thus be the principal parameter. Inspecting the proof we actually obtain the stronger property: $p_2 = 0 \Rightarrow \{p_2,p_1\} \geq C\lambda_\tau^3$.

Considering the previous proposition, we shall often write $\tau \geq \tau_0|\sigma|$, where $\tau_0 > 0$ is taken sufficiently large, and we shall use the fact that $\tau + \sigma \approx \tau$ on many occasions in what follows.

4.4.2 Weight function properties

In this section, we first recall the required properties (4.1.23), (4.1.24) for the function φ to be an admissible weight function on $\overline{V}$, where V is an bounded open neighborhood of 0 in $\mathbb{R}^n$. Yet, here we states these conditions in the normal geodesic coordinates introduced above, and we provide a construction for such a function. The weight function to be used, $\varphi \in C^\infty(V)$, is chosen so as to satisfy the following conditions

- $|\nabla_x\varphi| \geq C > 0$;
- For a given $\nu_0 > 0$, we have

$$|\partial_{x_j}\varphi| \leq \nu_0 \inf_{\overline{V}} |\partial_{x_n}\varphi| \quad j = 1,\dots,n-1 \tag{4.4.15}$$

- φ satisfies the sub-ellipticity condition (4.4.14), on $\overline{V}$, which is given in Section 4.4.1.

The value of $\nu_0 > 0$ will be determined in Lemma 4.5.3 and in Lemma 4.6.7 and it is meant to be small. With this parameter, we enforce the weight function to be relatively flat in the tangential directions as compared to its variations in the normal direction. In the applications we have in mind, we shall use weights of the form $e^{\lambda\psi}$. The two first conditions are satisfied if

- $|\nabla\psi| \geq C > 0$;
- For a given $\nu_0 > 0$, we have $|\partial_{x_j}\psi| \leq \nu_0 \inf_{\overline{V}} |\partial_{x_n}\psi| \quad j = 1 \dots n-1$.

If $|\nabla\psi| \geq C > 0$ then for λ sufficiently large, the third condition is satisfied (see Proposition 4.4.2). Observe that if φ is an admissible weight function fulfilling the above conditions, then its normal derivative cannot be zero, implying: $|\partial_{x_n}\varphi| \geq C > 0$ on $\overline{V}$.

4.4.3 A boundary quadratic form

Using integrations by parts and symbolic calculus, we derive a first estimate. It exhibits a quadratic form involving the two traces $u_{|_{x_n=0}}$ and $\partial_{x_n} u_{|_{x_n=0}}$ at the boundary. This estimate is central in what follows. Actually, we shall exploit its structure when considering the phase-space region where the operator $P_{\sigma,\varphi}$ is not elliptic, and use the sub-ellipticity condition (4.4.14) in a crucial way. This estimate is now classical and is proved in [LR95]. A proof with the parameter σ is given in Appendix 9.E.3.

Proposition 4.4.3. *Let V be an open neighborhood of 0 in $\mathbb{R}^n$ and let φ be a weight function satisfying the sub-ellipticity condition (4.4.14) in $\overline{V}^+$, and assume that $|\partial_{x_n}\varphi| \geq C > 0$ on $\overline{V}$. Then there exists $\tau_0 > 0$ and $C' > 0$ such that*

$$C'\tau||v||^2_{1,\tau} + \tau \operatorname{Re}\mathcal{B}(v) \leq ||P_{\varphi,\sigma}v||^2_{0,\tau}$$

for all $v \in \overline{C}^\infty_0(V^+)$, $|\sigma| \geq 1$ and $\tau \geq \tau_0|\sigma|$, where

$$\mathcal{B}(v) = 2\Big(\partial_{x_n}\varphi D_n v_{|_{x_n=0}}, D_n v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(A_1 v_{|_{x_n=0}}, D_n v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(D_n v_{|_{x_n=0}}, A_1' v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(A_2 v_{|_{x_n=0}}, v_{|_{x_n=0}}\Big)_{L^2(\mathbb{R}^{n-1})}.$$

The operators A_1, A_1' and A_2 are differential, and

- A_1, $A_1' \in \mathcal{D}^1_{T,\tau}$ *and satisfy*
$$a_1 := \sigma(A_1) = \sigma(A_1') = 2\tilde{r}(x, \xi', d_{x'}\varphi); \tag{4.4.16}$$
- $A_2 \in \mathcal{D}^2_{T,\tau}$ *and satisfies*
$$a_2 := \sigma(A_2) = 2\partial_{x_n}\varphi\left(\sigma^2 + p(x, \tau d_{x'}\varphi) - r(x, \xi')\right). \tag{4.4.17}$$

4.5 Microlocal regions and roots properties

Here, we consider the principal symbol of the conjugated operator (see (4.4.6)-(4.4.11))

$$\begin{aligned} p_{\varphi,\sigma}(x,\xi,\tau) &= p_2(x,\xi,\tau,\sigma) + ip_1(x,\xi,\tau) \\ &= \tilde{p}_2(x,\xi',\tau,\sigma) + i\tilde{p}_1(x,\xi',\tau) + \xi_n^2 + 2i\tau\partial_{x_n}\varphi\xi_n \\ &= (\xi_n + i\tau\partial_{x_n}\varphi)^2 + (\tau\partial_{x_n}\varphi)^2 + \tilde{p}_2(x,\xi',\tau,\sigma) + i\tilde{p}_1(x,\xi',\tau). \end{aligned}$$

We set: $m = (\tau\partial_{x_n}\varphi)^2 + \tilde{p}_2(x,\xi',\tau,\sigma) + i\tilde{p}_1(x,\xi',\tau)$. Then, we can write $p_{\varphi,\sigma}$ as a factorized second-order polynomial function in the ξ_n variable

$$p_{\varphi,\sigma}(x,\xi',\xi_n,\tau) = (\xi_n + i\alpha + i\tau\partial_{x_n}\varphi)\,(\xi_n - i\alpha + i\tau\partial_{x_n}\varphi)\,,$$

where $\alpha \in \mathbb{C}$ satisfies $\mathrm{Re}(\alpha) \geq 0$ and $\alpha^2 = m$. We can write

$$p_{\varphi,\sigma}(x,\xi',\xi_n,\tau) = \left(\xi_n - \rho^+\right)\left(\xi_n - \rho^-\right),$$

with $\rho^- = -i\tau\partial_{x_n}\varphi - i\alpha$ and $\rho^+ = -i\tau\partial_{x_n}\varphi + i\alpha$. Observe that there exists $C > 0$ such that

$$|\rho^\pm| \leq C\lambda_T, \tau. \tag{4.5.1}$$

We set $\mu(x,\xi',\tau,\sigma) = \tilde{p}_2(x,\xi',\tau,\sigma) + \dfrac{\tilde{p}_1(x,\xi',\tau)^2}{(2\tau\partial_{x_n}\varphi)^2}$. Note that it is a homogeneous function of degree 2 in the (ξ',τ,σ) variable.

Lemma 4.5.1. *We have the following:*

if $\boldsymbol{\mu(x,\xi',\tau,\sigma) < 0}$ *, then $\rho^\pm \notin \mathbb{R}$ and $\mathrm{sign}(\mathrm{Im}(\rho^-)) = \mathrm{sign}(\mathrm{Im}(\rho^+)) = \mathrm{sign}(-\partial_{x_n}\varphi)$;*

if $\boldsymbol{\mu(x,\xi',\tau,\sigma) = 0}$ *, then*

- *if $\partial_{x_n}\varphi > 0$, then $\rho^+ \in \mathbb{R}$ and $\mathrm{Im}(\rho^-) < 0$; thus $(x,\xi',\rho^+,\tau) \in \mathrm{Char}(P_{\varphi,\sigma})$*
- *if $\partial_{x_n}\varphi < 0$, then $\rho^- \in \mathbb{R}$ and $\mathrm{Im}(\rho^+) > 0$; thus $(x,\xi',\rho^-,\tau) \in \mathrm{Char}(P_{\varphi,\sigma})$;*

if $\boldsymbol{\mu(x,\xi',\tau,\sigma) > 0}$ *, then $\rho^\pm \notin \mathbb{R}$, $\mathrm{Im}(\rho^-) < 0$ and $\mathrm{Im}(\rho^+) > 0$;*

The different root configurations are represented in figure 4.1. From Lemma 4.5.2, we have $\mu < 0$ implies $|\xi'| \lesssim \tau$, and that $\mu > 0$ implies $|\xi'| \gtrsim \tau$, and that $\mu = 0$ implies $\tau \lesssim |\xi'| \lesssim \tau$.

Proof. For $z = a + ib \in \mathbb{C}$, $a \neq 0$, we have

$$\mathrm{Re}(z^2) = a^2 - \frac{\mathrm{Im}(z^2)^2}{4a^2}. \tag{4.5.2}$$

Using (4.5.2) with $z = \alpha$, observe that

$$\begin{aligned}\mu = \mathrm{Re}(m) - (\tau\partial_{x_n}\varphi)^2 + \frac{\mathrm{Im}(m)^2}{(2\tau\partial_{x_n}\varphi)^2} &= \mathrm{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2 + \frac{\mathrm{Im}(\alpha^2)^2}{4}\left(\frac{1}{(\tau\partial_{x_n}\varphi)^2} - \frac{1}{\mathrm{Re}(\alpha)^2}\right)\\ &= \left(\mathrm{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2\right)\left(1 + \frac{\mathrm{Im}(\alpha^2)^2}{4\,\mathrm{Re}(\alpha)^2(\tau\partial_{x_n}\varphi)^2}\right).\end{aligned} \tag{4.5.3}$$

Thus $\mu \gtreqless 0$ if and only if $\mathrm{Re}\,\alpha - \tau|\partial_{x_n}\varphi| = |\,\mathrm{Re}\,\alpha| - \tau|\partial_{x_n}\varphi| \gtreqless 0$. This allows us to conclude as $\mathrm{Im}\,\rho^\pm = \pm\,\mathrm{Re}\,\alpha - \tau\partial_{x_n}\varphi$. □

The sign of μ is related to the value of the tangential variables $|\xi'|$ with respect to τ.

Lemma 4.5.2. *For $\delta_0 > 0$ taken sufficiently small, there exists $C > 0$ such that we have the following:*

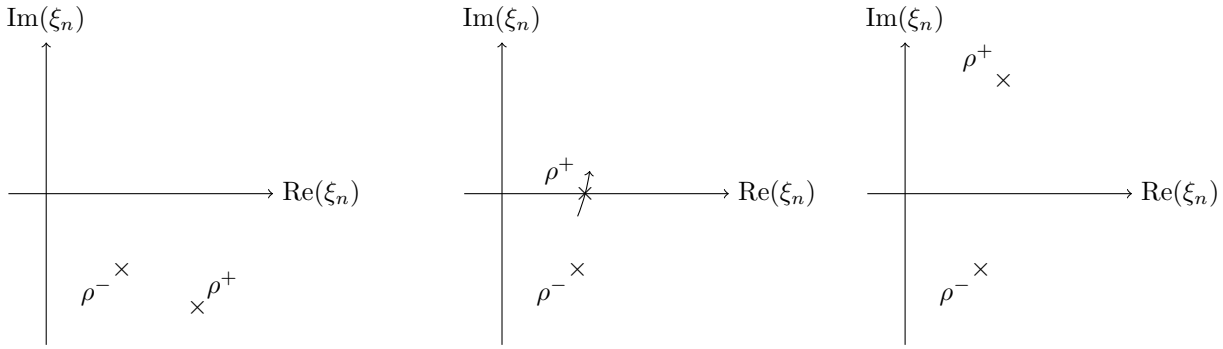
if $\boldsymbol{\mu(x,\xi',\tau,\sigma) \leq \delta_0\lambda_{T,\tau}^2}$ *, then $|\xi'|^2 \leq C\tau^2$;*

if $\boldsymbol{\mu(x,\xi',\tau,\sigma) \geq -\delta_0\lambda_{T,\tau}^2}$ *, then $\frac{\tau^2}{C} \leq |\xi'|^2$.*

Proof. Suppose first that $\mu(x,\xi',\tau,\sigma) \leq \delta_0\lambda_{T,\tau}^2$. This means

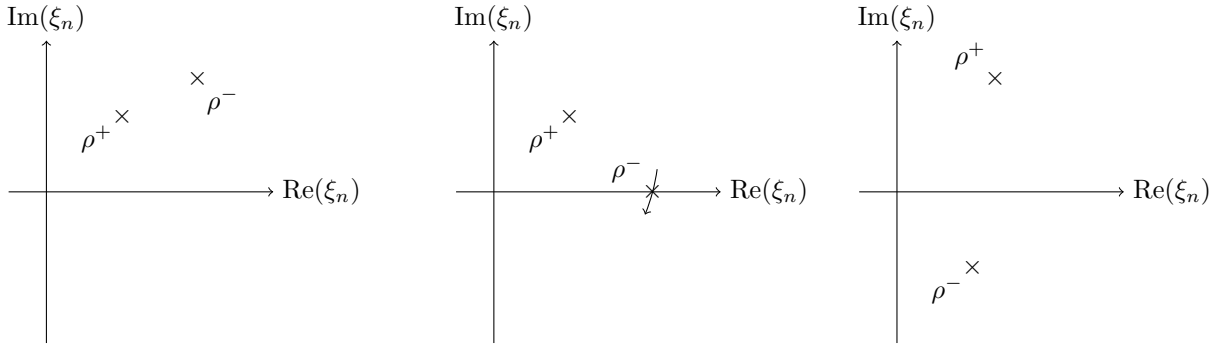
$$r(x,\xi') + \frac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)^2}{(\partial_{x_n}\varphi)^2} \leq \delta_0\lambda_{T,\tau}^2 + p(x,\tau\mathrm{d}_x\varphi) + \sigma^2,$$

The case where $\partial_{x_n}\varphi > 0$:



$\mathcal{E}^-$: $\mu < 0$ | $\mathcal{E}^0$: $\mu = 0$, ρ^+ crosses the real axis | $\mathcal{E}^+$: $\mu > 0$

The case where $\partial_{x_n}\varphi < 0$:



$\mathcal{E}^-$: $\mu < 0$ | $\mu = 0$, ρ^- crosses the real axis | $\mathcal{E}^+$: $\mu > 0$

Figure 4.1: Position of the roots of $p_{\varphi,\sigma}$ as μ varies.

implying for some $C_0 > 0$, and $C_1 > 0$, we have $C_0|\xi'|^2 \leq \delta_0|\xi'|^2 + C_1\tau^2$. Thus, for $\delta_0 < C_0$, we have $|\xi'| \lesssim \tau^2$. Suppose second that $\mu(x,\xi',\tau,\sigma) \geq -\delta_0\lambda_{T,\tau}^2$, meaning

$$p(x,\tau \mathrm{d}_x\varphi) + \sigma^2 \geq r(x,\xi') + \delta_0\lambda_{T,\tau}^2 + \frac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)^2}{(\partial_{x_n}\varphi)^2}.$$

This implies that for some $C_2 > 0$, and $C_3 > 0$ we have $C_2\tau^2 + \sigma^2 \leq C_3|\xi'|^2 + \delta_0\tau^2$. Thus for $\delta_0 < C_2$, we have $\tau^2 \lesssim |\xi'|^2$. □

In the case $\mu = 0$, the operator $P_{\varphi,\sigma}$ is not elliptic, as one of the roots ρ^+ or ρ^- is real. There is a (real) characteristic set. The ellipticity or the non-ellipticity of $P_{\varphi,\sigma}$ can thus be expressed through an algebraic condition on the tangential variables. We introduce the following phase-space regions

$$\mathcal{E}^+ = \left\{(x,\xi',\tau,\sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid \mu(x,\xi',\tau,\sigma) > \eta_1\lambda_{T,\tau}^2\right\}$$

$$\mathcal{E}^- = \left\{(x,\xi',\tau,\sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid \mu(x,\xi',\tau,\sigma) < -\eta_1\lambda_{T,\tau}^2\right\}$$

$$\mathcal{E}^0 = \left\{(x,\xi',\tau,\sigma) \in V^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R} \mid |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma| \mid -2\eta_1\lambda_{T,\tau}^2 < \mu(x,\xi',\tau,\sigma) < 2\eta_1\lambda_{T,\tau}^2\right\},$$

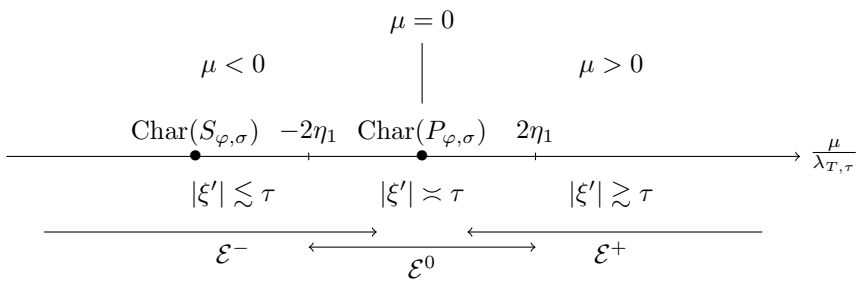


Figure 4.2: Representation of the three microlocal regions.

where $\eta_1 > 0$ will be chosen sufficiently small below (see Proposition 4.6.5 and Lemma 4.6.7). These microlocal regions are sketched in Figure 4.2. We shall thus cut the tangential phase-space into three pieces to isolate the different behaviors of the roots of $P_{\varphi,\sigma}$.

Lemma 4.5.3. *(Localization of the characteristic sets). Let φ be a weight function satisfying the properties of Section 4.4.2. Then, there exist $C > 0$, $C_0 > 0$ and $\tau_0 > 0$ such that for all $|\sigma| \geq 1$ and $\tau \geq \tau_0|\sigma|$ we have*

$$\mathrm{Re}\, s_{\varphi,\sigma}(x',\xi',\tau) = s(x',\xi') - s(x',\tau d_{x'}\varphi_{|_{x_n=0}}) - \kappa\sigma^2 \geq C\lambda_{T,\tau}^2 \qquad \text{if} \quad \mu(x,\xi',\tau,\sigma) \geq -C_0\lambda_{T,\tau}^2.$$

In particular, we have the following inclusions:

$$\mathrm{Char}(S_{\varphi,\sigma}) \subset \mathrm{Char}(\mathrm{Re}(S_{\varphi,\sigma})) \subset \{\mu \leq -C_0\lambda_{T,\tau}^2\} \cap \{x_n = 0\} \subset \mathcal{E}^- \cap \{x_n = 0\},$$

if $0 < \eta_1 \leq C_0$.

Proof. We have, on the one hand

$$\begin{aligned} s(x',\xi') - s(x',\tau \mathrm{d}_{x'}\varphi_{|_{x_n=0}}) - \kappa\sigma^2 &\geq C''|\xi|^2 - C'''\tau^2|\mathrm{d}_{x'}\varphi_{|_{x_n=0}}|^2 - \kappa\sigma^2 \\ &\geq C''|\xi'|^2 - C'''\nu_0^2\tau^2 \inf_{\overline{V}} |\partial_{x_n}\varphi|^2 - \kappa\sigma^2, \end{aligned}$$

and on the other hand, from Lemma 4.5.2, $|\xi'|^2 \gtrsim \lambda_{T,\tau}$, if $\mu \geq -C_0\lambda_{T,\tau}^2$ for $C_0 > 0$ sufficiently small. This yields

$$\begin{aligned} s(x',\xi') - s(x',\tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2 &\geq C'''' \left(|\xi'|^2 + \sigma^2 + \tau^2\right) - C''\nu_0^2\tau^2 \inf |\partial_{x_n}\varphi|^2 - \kappa\sigma^2 \\ &\gtrsim \lambda_{T,\tau}^2, \end{aligned}$$

for all $\tau \geq \tau_0|\sigma|$, by taking ν_0 sufficiently small and τ_0 sufficiently large. The other statements follow. □

Here, we used that the weight function is chosen sufficiently flat in the tangential directions with respect to the normal one in a crucial way: it ensure that the two characteristic sets, that of $P_{\varphi,\sigma}$ and that of $S_{\varphi,\sigma}$ are associated with two different microlocal regions. We shall derive three microlocal estimates corresponding to the previous regions determined by the sign of μ, and prove an uniform Carleman estimate with respect to the small parameter δ, that appears in the boundary condition in (4.1.22). In fact, if we only want estimates with a fixed δ, say equal to one, we can prove such an estimate using only two microlocal regions $\mathcal{E}^-$ and $\mathcal{E}^0 \cup \mathcal{E}^+$. This is due to the fact that the principal symbol of the boundary operator is of order 2 and elliptic in high frequencies. Then, if $\delta = 1$, only second orders terms are relevant. Here, because δ varies in $(0,1]$, we have to treat second- and first-order terms. That is precisely the reason of the apparition of our particular treatment in the zone $\mathcal{E}^+$.

Lemma 4.5.4. *Let $\chi \in \mathcal{S}^0_{T,\tau}$ homogeneous of degree 0, such that $\mathrm{supp}(\chi) \subset \mathcal{E}^+$. Then, $\chi\rho^\pm \in \mathcal{S}^1_{T,\tau}$ and there exists $C > 0$ such that $|\mathrm{Im}(\rho^\pm)| \geq C\lambda_{T,\tau}$ on the support of χ.*

Proof. Let us show first $\chi\rho^{\pm} \in \mathcal{S}^1_{T,\tau}$. As $\tau\partial_{x_n}\varphi \in \mathcal{S}^1_{T,\tau}$, it suffices to prove that $\chi\alpha \in \mathcal{S}^1_{T,\tau}$. We have $\chi\frac{|\alpha^2|}{\lambda^2_{T,\tau}} \in \mathcal{S}^0_{T,\tau}$ as a homogeneous function of degree 0 in the (ξ',τ,σ) variable. In the region $\mathcal{E}^+$, we claim that there exists a neighborhood U of $\overline{\mathbb{R}^-}$ such that $m \notin U$. Indeed, consider $(x,\xi',\tau,\sigma) \in \mathcal{E}^+ \cap (V \times \mathbb{S}_{(\xi',\tau,\sigma)=1})$ where $\mathbb{S}_{(\xi',\tau,\sigma)}$ denotes the unit sphere in the variable (ξ',τ,σ). Suppose that $\mathrm{Im}(m) = 0$. This implies that $\tilde{p}_1(x,\xi',\tau) = 0$ and thus $\mu(x,\xi',\tau,\sigma) = \tilde{p}_2(x,\xi',\tau,\sigma)$. The definition of $\mathcal{E}^+$ yields that $\tilde{p}_2(x,\xi',\tau,\sigma) > 0$ and thus $\mathrm{Re}(m) > 0$. By a compacity argument we have there exists a constant $C > 0$ such that $\mathrm{Re}(m) \geq C > 0$, and then the claim is proved by homogeneity. This allows us to define $\frac{\alpha}{\lambda_{T,\tau}} = F(\frac{m}{\lambda^2_{T,\tau}})$, where F is the complex principal square root. Using Theorem 18.1.10 in [Hör07], we obtain $\frac{\alpha}{\lambda_{T,\tau}} \in \mathcal{S}^0_{T,\tau}$ in a conic neighborhood of the support of χ. Now, we show that $|\,\mathrm{Im}(\rho^{\pm})| \geq C\lambda_{T,\tau} > 0$. We have

$$|\,\mathrm{Im}(\rho^{\pm})| \geq |\,\mathrm{Re}(\alpha)| - \tau|\partial_{x_n}\varphi|. \tag{4.5.4}$$

Using (4.5.3) and observing that on the support of χ, we have $\mu \gtrsim C\lambda^2_{T,\tau}$, we obtain $\mathrm{Re}(\alpha)^2 - (\tau\partial_{x_n}\varphi)^2 \gtrsim \lambda^2_{T,\tau}$, and with (4.5.4) this concludes the proof.

□

4.6 Microlocal Carleman estimates

We recall that the operators $P_{\varphi,\sigma}$ and K are defined in Section 4.4.1. In each region, we define cut-off functions χ depending on (x,ξ',τ,σ), but from Remark 4.3.2, the parameter σ will not be involved in the symbolic calculus.

4.6.1 Estimate in $\mathcal{E}^+$

In this region, we have $\mu > \eta_1$ and thus $|\xi'| \gtrsim \tau$ and both operators $P_{\varphi,\sigma}$ and $S_{\varphi,\sigma}$ are elliptic, and this allows us to estimate v in the interior and at the boundary from a single observation at the boundary. Observe that the estimate here is of better quality than that in the other zones $\mathcal{E}^-$ and $\mathcal{E}^0$.

Proposition 4.6.1. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ satisfying the conditions of section 4.4.2, and $\chi(x,\xi',\tau,\sigma) \in \mathcal{S}^0_{T,\tau}$ be such that $\mathrm{supp}(\chi) \subset \mathcal{E}^+$. Then there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\begin{aligned}\tau^2||\,\mathrm{Op}_T(\chi)v||^2_{1,\tau} + (\tau+\delta^2\tau^3)|\,\mathrm{Op}_T(\chi)v_{|_{x_n=0}}|^2_{1,\tau} + \tau|D_n\,\mathrm{Op}_T(\chi)v_{|_{x_n=0}}|^2_{0,\tau} \leq C\Big(||P_{\varphi,\sigma}v||^2_{0,\tau} \\ + \tau|(D_n-K)v_{|_{x_n=0}}|^2_{0,\tau} + ||v||^2_{1,\tau} + \tau^{-1}|D_nv_{|_{x_n=0}}|^2_{0,\tau} + \tau^{-1}|v_{|_{x_n=0}}|^2_{1,\tau}\Big),\end{aligned} \tag{4.6.1}$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$, for all $v \in \overline{C}^{\infty}_0(V^+)$, and $\delta \in (0,1]$.

Proof. In this microlocal region, we shall apply the Calderón projector method. We shall denote

$$P_{\varphi,\sigma}v = \tilde{f} \text{ in } \mathbb{R}^n_+, \qquad D_nv_{|_{x_n=0}} - Kv_{|_{x_n=0}} = \Theta \text{ in } \{x_n = 0\},$$

for $v \in \overline{C}^{\infty}_0(V^+)$. Let $\chi \in \mathcal{S}^0_{T,\tau}$ as in the statement of the proposition. We set $w := \mathrm{Op}_T(\chi)v$ and $g := \mathrm{Op}_T(\chi)\tilde{f}$. Hence

$$w_1 := P_{\varphi,\sigma}w = \mathrm{Op}_T(\chi)P_{\varphi,\sigma}v + [P_{\varphi,\sigma}, \mathrm{Op}_T(\chi)]v = g + [P_{\varphi,\sigma}, \mathrm{Op}_T(\chi)]v,$$

and on $\{x_n = 0\}$ we have

$$(D_n - K)w_{|_{x_n=0}} =: w_0 = [D_n - K, \mathrm{Op}_T(\chi)]v_{|_{x_n=0}} + \mathrm{Op}_T(\chi)\Theta_{|_{x_n=0}}, \tag{4.6.2}$$

and as $[D_n - K, \mathrm{Op}_T(\chi)] \in \delta\Psi^1_{T,\tau} + \Psi^0_{T,\tau}$ we find

$$|w_0|_0 \lesssim \delta|v_{|_{x_n=0}}|_{1,\tau} + |v_{|_{x_n=0}}|_0 + |\Theta|_0. \tag{4.6.3}$$

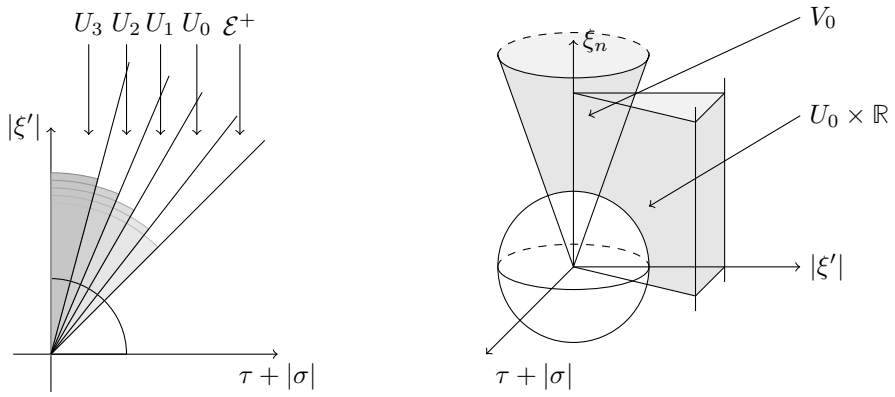


Figure 4.3: Representation of the different conic neighborhoods.

Observing that the commutator $[P_{\varphi,\sigma}, \mathrm{Op}_T(\chi)] \in \Psi^1_\tau$ and does not depend on σ, we obtain

$$||w_1||_{L^2} \lesssim ||g||_{L^2} + ||v||_{1,\tau}. \tag{4.6.4}$$

In what follows, we shall denote by $\underline{w}$ the extension of w by 0 on $\{x_n < 0\}$. We thus obtain the following equality on the whole $\mathbb{R}^n$:

$$P_{\varphi,\sigma}\underline{w} = \underline{w}_1 - i\gamma_1(w)\delta_{x_n=0} - \gamma_0(w)\delta'_{x_n=0} + 2\tau\partial_{x_n}\varphi\gamma_0(w)\delta_{x_n=0} \tag{4.6.5}$$

where $\gamma_0(w) = w_{|x_n=0}$, $\gamma_1(w) = (D_n w)_{|x_n=0}$, and δ is the Dirac measure. Recalling that ρ^+ and ρ^- are the two complex roots of the principal symbol $p_{\varphi,\sigma}$ viewed as a polynomial in the variable ξ_n (with $\mathrm{Im}\,\rho^+ > 0$ and $\mathrm{Im}\,\rho^- < 0$), we find $-2i\tau\partial_{x_n}\varphi = \rho^+(x,\xi',\tau,\sigma) + \rho^-(x,\xi',\tau,\sigma)$. With this relation, (4.6.5) reads

$$P_{\varphi,\sigma}\underline{w} = \underline{w}_1 + W_0\delta_{x_n=0} + W_1\delta'_{x_n=0}, \tag{4.6.6}$$

where

$$W_1 = -\gamma_0(w), \qquad W_0 = i\left(\mathrm{Op}(\rho^+ + \rho^-)\gamma_0(w) - \gamma_1(w)\right), \tag{4.6.7}$$

Let U_0 and U_1 be two conic neighborhoods of $\mathrm{supp}(\chi)$ in $(V \cap \mathbb{R}^n_+) \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$ such that $\overline{U_1} \subset U_0$ and $\overline{U_0} \subset \mathcal{E}^+$. We also define, for $\tau_0 > 0$:

$$V_0 := \left\{(x,\xi,\tau,\sigma) \mid x \in V^+,\ |\sigma| \geq 1,\ \tau \geq \tau_0|\sigma|,\ |\xi_n| \geq C_0|(\xi',\tau)|\right\},$$

$$V_1 := \left\{(x,\xi,\tau,\sigma) \mid x \in V^+, |\sigma| \geq 1, \tau \geq \tau_0|\sigma|,\ |\xi_n| \geq C_1|(\xi',\tau)|\right\},$$

for $0 < C_1 < C_0$ chosen sufficiently large. Note that V_0 and V_1 are conic in (ξ,τ,σ). Let $\hat{\chi}(x,\xi,\tau,\sigma) \in \mathcal{S}^0_\tau$ (see Remark 4.3.2), homogeneous of degree 0, be such that $\hat{\chi}$ is equal to 1 on the conic set $(U_1 \times \mathbb{R}) \cup V_0$, and is equal to 0 outside $(U_0 \times \mathbb{R}) \cup V_1$. Note that it is possible since $((U_1 \times \mathbb{R}) \cup V_0) \cap \mathbb{S}_{|(\xi,\tau,\sigma)|=1} \Subset ((U_0 \times \mathbb{R}) \cup V_1) \cap \mathbb{S}_{|(\xi,\tau,\sigma)|=1}$, where $\mathbb{S}_{|(\xi,\tau,\sigma)|=1}$ denotes the unit sphere on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}$. The microlocal neighborhoods are represented in Figure 4.3.

Observe moreover that on $\mathrm{supp}(\hat{\chi})$, we have $p_{\varphi,\sigma} \neq 0$. Indeed, on the one hand it is true on $U_1 \times \mathbb{R}$ since it is true on $\mathcal{E}^+ \times \mathbb{R}$, and on the other hand it is true on V_1 since $p_{\varphi,\sigma} = 0$ is equivalent to $\xi_n = \rho^+$ or $\xi_n = \rho^-$, and implies from (4.5.1), that there exists a constant $C > 0$ such that $|\xi_n| \leq C\lambda_{T,\tau}$, which can be avoided i n V_1, for C_1 chosen sufficiently large. Thus, we can construct a parametrix $E_N = \mathrm{Op}(e) \in \Psi^{-2}_\tau, N \in \mathbb{N}$, such that

$$e = \sum_{j=0}^{N} e_j, \quad e_0 = \frac{\hat{\chi}}{p_{\varphi,\varphi}}, \quad e_j \in \Psi^{-2-j}_\tau$$

with e_j homogeneous of dergree $-2-j$ and satisfying $E_N P_{\varphi,\sigma} = \mathrm{Op}(\hat{\chi}) + R_N$. where $R_N \in \Psi_\tau^{-N}$. From (4.6.6) we find

$$\underline{w} = E_M \left(W_1 \delta'_{x_n=0} + W_0 \delta_{x_n=0} \right) + g_1, \tag{4.6.8}$$

where $g_1 = E_N(\underline{w}_1) + (\mathrm{Id} - \mathrm{Op}(\hat{\chi}))\,\underline{w} - R_N \underline{w}$. As in $\{x_n > 0\}$ we have $w = \mathrm{Op}(\chi)v$, we observe that $\mathrm{supp}(1-\hat{\chi}) \cap (\mathrm{supp}(\chi) \times \mathbb{R}) = \emptyset$, and we shall make use of the following lemma of [Hör07], Theorem 18.1.35, which proof can be adapted to the semi-classical setting we consider here.

Lemma 4.6.2. *Let $a_{T,m}(x,\xi',\tau,\sigma) \in \mathcal{S}^m_{T,\tau}$ and $b_{m'}(x,\xi,\tau,\sigma) \in \mathcal{S}^{m'}_\tau$ and assume that for some $\delta > 0$ we have $b_{m'}(x,\xi,\tau,\sigma) = 0$ if $\delta|\xi_n| > 1$ and $|(\xi',\tau)| \leq \delta|\xi_n|$. If moreover, $(\mathrm{supp}(a_{T,m} \times \mathbb{R})) \cap \mathrm{supp}(b_{m'}) = \emptyset$, then*

$$\mathrm{Op}_T(a_{T,m}) \circ \mathrm{Op}(b_{m'}) \in \bigcap_{M \in \mathbb{N}} \Psi_\tau^{-M}, \quad \text{and} \quad \mathrm{Op}(b_{m'}) \circ \mathrm{Op}_T(a_{T,m}) \in \bigcap_{M \in \mathbb{N}} \Psi_\tau^{-M}.$$

With Lemma 4.6.2 we find $(\mathrm{Id} - \mathrm{Op}(\hat{\chi})) \circ \mathrm{Op}(\chi) \in \bigcap_M \Psi_\tau^{-M}$, which yields:

$$||g_1||_{2,\tau} \lesssim ||v||_{1,\tau} + ||g||_{0,\tau}. \tag{4.6.9}$$

We have

$$\begin{cases} E_N \left(W_1 \delta'_{x_n=0} + W_0 \delta_{x_n} \right) = T_0 W_0 + T_1 W_1 \\ T_j W := \mathrm{Op}_T(t_j)(x_n) W = \dfrac{1}{(2\pi)^{n-1}} \displaystyle\int_{\mathbb{R}^{n-1}} e^{i(x'-y').\xi'} t_j(x,\xi',\tau,\sigma) W(y') dy' d\xi' \\ t_j = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} e^{ix_n\xi_n} (i\xi_n)^j e(x,\xi,\tau,\sigma) d\xi_n. \end{cases} \tag{4.6.10}$$

Note that from (4.5.1), $e(x,\xi,\tau,\sigma) \lesssim \lambda_\tau^{-2}$ for all $\xi_n \in \mathbb{C}$ satisfying $|\xi_n| \geq R\lambda_{T,\tau}$, with R sufficiently large. Then t_0 defines an absolutely convergent integral, but t_1 has to be taken in the oscillatory integral sense, and

$$t_1 = \frac{1}{2\pi} \partial_{z_n} \int_{\mathbb{R}} e^{iz_n\xi_n} e(x,\xi,\tau,\sigma) d\xi_n \Big|_{z_n = x_n}, \tag{4.6.11}$$

where the derivative is taken in the distribution sense. As for $|\xi_n| \geq C_1 \lambda_{T,\tau}$, $\hat{\chi} = 1$, the symbol $e(x,\xi,\tau,\sigma)$ is holomorphic in the variable ξ_n, we can thus change the contour of integration:

$$t_0 = \frac{1}{2\pi} \int_\beta e^{ix_n\xi_n} e(x,\xi,\tau,\sigma) d\xi_n, \tag{4.6.12}$$

where $\beta = \{\xi_n \in \mathbb{R} \mid \xi_n \in [-R\lambda_{T,\tau}, R\lambda_{T,\tau}]\} \cup \{\xi_n \in \mathbb{C},\ |\xi_n| = R\lambda_{T,\tau},\ \mathrm{Im}\,\xi_n \geq 0\}$, with $R > 0$ chosen sufficiently large to have $\rho^\pm$ in the complex domain delimited by β (such a constant R exists since $\rho^\pm \in \mathcal{S}^1_{T,\tau}$). From (4.6.11), we can also define:

$$t_1 = \frac{1}{2\pi} \int_\beta e^{ix_n\xi_n} i\xi_n e(x,\xi,\tau,\sigma) d\xi_n. \tag{4.6.13}$$

For a review of oscillatory integrals, the reader may refer to [Hör03, Section 7.8]. Observe that $\mathrm{Op}_T(t_j)(x_n)$ is a x_n-family of pseudo-differential operators acting on $\mathbb{R}^{n-1}$. In fact, from (4.6.12) and (4.6.13) we have

$$\left| \partial^l_{x_n} \partial^{\alpha_1}_{x'} \partial^{\alpha_2}_{\xi'} t_j \right| \leq C_{l,\alpha_1,\alpha_2} \lambda_{T,\tau}^{-1-|\alpha_2|+l+j} \qquad j = 0, 1. \tag{4.6.14}$$

Now let U_2 and U_3 be two conic neighborhoods of $\mathrm{supp}(\chi)$ in $(V \cap \mathbb{R}^n_+) \times \mathbb{R}^{n-1} \times \mathbb{R}^+ \times \mathbb{R}$ such that $\overline{U}_3 \subset U_2$ and $\overline{U}_2 \subset U_0$, and we choose $\chi_1(x,\xi',\tau,\sigma) \in \mathcal{S}^0_{T,\tau}$ such that $\chi_1 = 1$ on U_2 and $\chi_1 = 0$ outside U_3 (see figure 4.3). In addition, we set $s_j = \chi_1 t_j$ and $g_2 = \mathrm{Op}_T((1-\chi_1)t_0)(x_n) W_0 + \mathrm{Op}_T((1-\chi_1)t_1)(x_n) W_1$ which yields

$$\underline{w} = \mathrm{Op}(s_0)(x_n) W_0 + \mathrm{Op}(s_1)(x_n) W_1 + g_3, \tag{4.6.15}$$

where $g_3 = g_1 + g_2$. By tangential symbolic calculus (the normal variable ξ_n is not involved in the calculus), as $\mathrm{supp}(1-\chi_1) \cap \mathrm{supp}(\chi) = \emptyset$, using (4.6.14), the trace formula (4.3.2), we obtain

$$|| \operatorname{Op}_T(\lambda_{T,\tau}^l) D_{x_n}^{l'} g_2||_{0,\tau} \lesssim ||v||_{1,\tau} + \tau^{-1}|\gamma_1(v)|_{0,\tau}, \tag{4.6.16}$$

for all $l \in \mathbb{R}, l' \in \mathbb{N}$. This allows us to estimate g_3:

$$||g_3||_{2,\tau} \lesssim ||g||_{0,\tau} + ||v||_{1,\tau} + \tau^{-1}|\gamma_1(v)|_{0,1}. \tag{4.6.17}$$

We now estimate s_j. The symbol $e(x,\xi',\xi_n,\tau,\sigma)$ is holomorphic in ξ_n on the support of χ_1, we then can change the contour of integration in the complex plane

$$s_j = \frac{\chi_1(x,\xi',\tau,\sigma)}{2\pi} \int_{\beta_0} e^{ix_n.\xi_n} e(x,\xi',\xi_n,\tau,\sigma)(i\xi_n)^j d\xi_n \qquad j = 0,1 \tag{4.6.18}$$

where β_0 is a direct contour surrounding ρ^+ in the region where $\mathrm{Im}\,\xi_n \geq c_0\lambda_{T,\tau}^1$, for some $c_0 > 0$. (note that it is possible from Section 4.5). By the residue formula, we have

$$e^{-ix_n\rho^+} s_j = i^{j+1} \frac{(\rho^+)^j \chi_1}{\rho^+ - \rho^-} + m_j, \tag{4.6.19}$$

with $m_j \in \mathcal{S}_{T,\tau}^{-2+j}$. With (4.6.18), we can estimate:

$$|D_{x_n}^l \partial_{x'}^{\alpha_1} \partial_{\xi'}^{\alpha_2} s_j| \leq C_{l,\alpha_1,\alpha_2} e^{-c_0 x_n \lambda_{T,\tau}} (|\xi'| + \tau)^{-1+j+l-|\alpha_2|} \qquad j = 0,1.$$

We thus obtain $e^{c_0 x_n \tau} D_{x_n}^l s_j$ is bounded in $\mathcal{S}_{T,\tau}^{-1+j+l}$ uniformly in $x_n \geq 0$. This yields

$$\begin{aligned}
|| \operatorname{Op}_T(\lambda_{T,\tau}) \operatorname{Op}_T(s_j)(x_n) W_j||_{L^2(\mathbb{R}^n_+)}^2 &\leq C \int_{x_n>0} |\operatorname{Op}_T(s_j) W_j|_{1,\tau}^2(x_n) dx_n \\
&\leq C \int_{x_n>0} e^{-2c_0 x_n \tau} |e^{c_0 x_n \tau} \operatorname{Op}_T(s_j) W_j|_{1,\tau}^2(x_n) dx_n \\
&\leq C|W_j|_{j,\tau}^2 \int_{x_n>0} e^{-2c_0 x_n \tau} dx_n \\
&\leq C\tau^{-1}|W_j|_{j,\tau}^2,
\end{aligned} \tag{4.6.20}$$

and

$$\begin{aligned}
||D_n \operatorname{Op}_T(s_j)(x_n) W_j||_{L^2(\mathbb{R}^n_+)}^2 &\leq C \int_{x_n>0} |D_n \operatorname{Op}_T(s_j) W_j|_0^2(x_n) dx_n \\
&\leq C \int_{x_n>0} e^{-2c_0 x_n \tau} |e^{c_0 x_n \tau} D_n \operatorname{Op}_T(s_j) W_j|_0^2(x_n) dx_n \\
&\leq C|W_j|_{j,\tau}^2 \int_{x_n>0} e^{-2c_0 x_n \tau} dx_n \\
&\leq C\tau^{-1}|W_j|_{j,\tau}^2.
\end{aligned} \tag{4.6.21}$$

Using (4.6.15), (4.6.17), (4.6.20) and (4.6.21), we obtain

$$\begin{aligned}
||w||_{1,\tau} &\lesssim \tau^{-1}||g_3||_{2,\tau} + \tau^{-1/2}|W_1|_{1,\tau} + \tau^{-1/2}|W_0|_0 + \tau^{-2}|D_n v|_{L^2} \\
&\lesssim \tau^{-1}||g||_{0,\tau} + \tau^{-1}||v||_{1,\tau} + \tau^{-1/2}|W_1|_{1,\tau} + \tau^{-1/2}|W_0|_{0,\tau} + \tau^{-2}|\gamma_1(v)|_{L^2}.
\end{aligned} \tag{4.6.22}$$

It remains to determine the two traces W_0 et W_1. Taking the trace $x_n = 0^+$ in (4.6.15) and using the definition of W_0 and W_1 in (4.6.7):

$$\gamma_0(w) = \operatorname{Op}_T(a)\gamma_0(w) + \operatorname{Op}_T(b)\gamma_1(w) + \gamma_0(g_3), \tag{4.6.23}$$

with $a \in \mathcal{S}^0_{T,\tau}$, of principal symbol $\sigma(a) = -\chi_1 \frac{\rho^-}{\rho^+ - \rho^-}|_{x_n=0}$ and $b \in \mathcal{S}^{-1}_{T,\tau}$, of principal symbol $\sigma(b) = \frac{\chi_1}{\rho^+ - \rho^-}|_{x_n=0}$ (see (4.6.19)). Here, $\mathrm{Op}_T(a)$ and $\mathrm{Op}_T(b)$ are the so-called Calderón projectors. Moreover, using again the trace formula (4.3.2), the remainder $\gamma_0(g_3)$ satisfies

$$|\gamma_0(g_3)|_{1,\tau} \leq \tau^{-1/2}||g_3||_{2,\tau} \lesssim \tau^{-1/2}||g||_{0,\tau} + \tau^{-1/2}||v||_{1,\tau} + \tau^{-3/2}|D_n v_{|_{x_n=0}}|_{0,\tau}. \tag{4.6.24}$$

The principal symbol of b satisfies $|\sigma(b)| \geq C\lambda^{-1}_{T,\tau}$ in $\mathrm{supp}(\chi_1)$. Let $\tilde{\chi}(x', \xi', \tau, \sigma) \in \mathcal{S}^0_{T,\tau}$ satisfy the same hypothesis than χ_1, and such that $\chi_1 = 1$ on the support of $\tilde{\chi}$. We can thus construct a parametrix, of symbol denoted by $l_1 \in \mathcal{S}^1_{T,\tau}$, such that

$$\mathrm{Op}_T(l_1)\,\mathrm{Op}_T(b) = \mathrm{Op}_T(\tilde{\chi}) + \tilde{R}, \qquad \tilde{R} \in \mathcal{S}^{-\infty}_{\tau}.$$

Moreover the principal symbol of l_1 is given by $\tilde{\chi}(\rho^+ - \rho^-)$. We now apply this parametrix to (4.6.23), and we find

$$\mathrm{Op}_T(l_1)\,\mathrm{Op}_T(1-a)\gamma_0(w) = \gamma_1(w) + g_4, \tag{4.6.25}$$

where $g_4 = S_1\gamma_1(v) + S_0\gamma_0(v) + \mathrm{Op}_T(l_1)\gamma_0(g_3)$, with $S_0, S_1 \in \mathcal{S}^{-\infty}_{T,\tau}$. Here we used

$$\begin{aligned}
\mathrm{Op}_T(\tilde{\chi})\gamma_1(w) &= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})\gamma_1(w) \\
&= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})D_n\,\mathrm{Op}(\chi)v_{|_{x_n=0}} \\
&= \gamma_1(w) - \mathrm{Op}_T(1-\tilde{\chi})\,\mathrm{Op}(\chi)_{|_{x_n=0}} D_n v_{|_{x_n=0}} - \mathrm{Op}_T(1-\tilde{\chi})\left([D_n, \mathrm{Op}_T(\chi)]v\right)_{|_{x_n=0}},
\end{aligned}$$

and $\mathrm{Op}_T(1-\tilde{\chi}) \circ \mathrm{Op}(\chi)_{|_{x_n=0}} \in \mathcal{S}^{-\infty}_{T,\tau}$, and $\mathrm{Op}_T(1-\tilde{\chi})[D_n, \mathrm{Op}_T(\chi)]_{|_{x_n=0}} \in \mathcal{S}^{-\infty}_{T,\tau}$. From (4.6.24), we have the following estimate on g_4

$$\begin{aligned}
|g_4|_{0,\tau} &\leq |S_1\gamma_1(v)|_{0,\tau} + |S_0\gamma_0(v)|_{0,\tau} + |\gamma_0(g_3)|_{1,\tau}, \\
&\lesssim \tau^{-1/2}||g||_{0,\tau} + \tau^{-1/2}||v||_{1,\tau} + C_N\tau^{-N}|\gamma_0(v)|_{0,\tau} + \tau^{-3/2}|\gamma_1(v)|_{0,\tau}.
\end{aligned} \tag{4.6.26}$$

for τ sufficiently large and N arbitrary. We can thus estimate the Neumann trace from the Dirichlet trace

$$|\gamma_1(w)|_{0,\tau} \lesssim |\gamma_0(w)|_{1,\tau} + |g_4|_{0,\tau}. \tag{4.6.27}$$

Now we use the relation (4.6.2) between the two traces at the boundary, that is, $\gamma_1(w) = K\gamma_0(w) + w_0$. From (4.6.25), we have

$$\mathrm{Op}_T(l_1) \circ \mathrm{Op}_T(1-a_0)\gamma_0(w) - K\gamma_0(w) = w_0 + g_4. \tag{4.6.28}$$

(4.6.28) reads

$$H\gamma_0(w) = w_0 + g_4, \tag{4.6.29}$$

where $H = \mathrm{Op}_T(l_1(1-a)) - K \mod \Psi^0_{T,\tau}$ and the principal symbol of H in the region where $\tilde{\chi}$ is equal to one is given by

$$h = i(\delta s(x', \xi') - \delta s(x', \tau \mathrm{d}_{x'}\varphi) - \delta\kappa\sigma^2 + \tau\partial_{x_n}\varphi) - 2\delta\tilde{s}(x', \xi', \tau\mathrm{d}_{x'}\varphi) + \rho^+.$$

In order to produce an estimate that is uniform in δ, and handle properly the calculus with the large parameter τ, we write δ as the inverse of a large parameter $\delta = \frac{1}{r}$, $r \geq 1$ and we introduce a new symbolic calculus. We define the order function $M^2 = \lambda^2_{T,\tau} + r\lambda_{T,\tau}$, associated with the usual semi-classical metric on the cotangent bundle $T^*\mathbb{R}^{n-1}$: $g = |\mathrm{d}x'|^2 + \frac{|\mathrm{d}\xi'|^2}{\lambda^2_{T,\tau}}$. The following lemma state that symbols can be defined with this order function, viewing the semi-classical calculus in the general Weyl-Hörmander calculus [Hör07, Section 18.4-6] and [Ler10].

Lemma 4.6.3. *The order function M^2 is admissible with respect to the metric g, i.e is slowly varying and temperate.*

A proof can be found in [LLR13], Lemma 4.7, in the semi-classical small parameter setting. Actually, any order function defined by a linear combination of powers of $\lambda_{T,\tau}$ is admissible with respect to the metric g. In this symbol classes, we have $rh \in \mathcal{S}(M^2, g)$. The aim is now to construct a parametrix of rh. We have

$$rh = i(s(x',\xi') - s(x',\tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2) + ir\tau\partial_{x_n}\varphi - 2\tilde{s}(x',\xi',\mathrm{d}_{x'}\varphi) + r\rho^+.$$

Taking the imaginary part, $\mathrm{Im}(rh) = s(x',\xi') - s(x',\tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2 + r\tau\partial_{x_n}\varphi + r\,\mathrm{Im}(\rho^+)$, and remarking that from Lemma 4.5.3, we have $\mathrm{Re}(s_{\varphi,\sigma}) = s(x',\xi') - s(x',\tau \mathrm{d}_{x'}\varphi) - \kappa\sigma^2 \gtrsim \lambda_{T,\tau}^2$, and that from Lemma 4.5.4 $r\,\mathrm{Im}(\rho^+) \gtrsim r\lambda_{T,\tau}^1$, we thus obtain $|rh| \gtrsim \mathrm{Im}\, rh \gtrsim \lambda_{T,\tau}^2 + r\lambda_{T,\tau} = M^2$.

Then rh is an elliptic symbol in the class $\mathcal{S}(M^2, g)$, this allows us to construct a parametrix $L \in \mathcal{S}(M^{-2}, g)$ satisfying $\mathrm{Op}_T(L)rH = \mathrm{Op}(\chi_L) + R_L$ with $R_L \in \mathcal{S}_{T,\tau}^{-\infty}$, for some $\chi_L \in \mathcal{S}_{T,\tau}^0$ equal to 1 in a neighborhood of χ and such that $\tilde{\chi}$ equal to 1 in a neighborhood of χ_L. Applying this parametrix to (4.6.29), we obtain

$$\gamma_0(w) = r\,\mathrm{Op}_T(L)(w_0 + g_4) - R_L\gamma_0(w) + (1 - \mathrm{Op}(\chi_L))\,\mathrm{Op}(\chi_{|x_n=0})\gamma_0(v) \tag{4.6.30}$$

Yet, we use the following lemma

Lemma 4.6.4. *For all $u \in \mathscr{S}(\mathbb{R}^{n-1})$ we have*

$$|\,\mathrm{Op}_T(r\lambda_{T,\tau})\,\mathrm{Op}_T(L)u|_{0,\tau} \lesssim |\,\mathrm{Op}_T(\frac{r}{\lambda_{T,\tau}+r})u|_0. \tag{4.6.31}$$

Proof. There exists $\tilde{u} \in \mathscr{S}(\mathbb{R}^{n-1})$ such that $u = \mathrm{Op}_T(\frac{\lambda_{T,\tau}+r}{r})\tilde{u}$ and it is given by $\mathrm{Op}(\frac{r}{\lambda_{T,\tau}+r})u$. As $r\lambda_{T,\tau} \in \mathcal{S}(r\lambda_{T,\tau}, g)$, $L \in \mathcal{S}(M^{-2}, g)$ and $\lambda_{T,\tau} + r \in \mathcal{S}(\lambda_{T,\tau} + r, g)$ we have $r\lambda_{T,\tau}\#L\#\frac{\lambda_{T,\tau}+r}{r} \in \mathcal{S}(1, g)$ by Theorem 18.5.4 in [Hör07] (stated for the Weyl quantization). Then applying Theorem 18.6.3 in [Hör07], we obtain

$$|\,\mathrm{Op}_T(r\lambda_{T,\tau})\,\mathrm{Op}_T(L)\,\mathrm{Op}_T(\frac{\lambda_{T,\tau}+r}{r})\tilde{u}|_{0,\tau} \lesssim |\tilde{u}|_{0,\tau},$$

that is precisely the result. □

Then using this lemma

$$\begin{aligned} |r\,\mathrm{Op}_T(\lambda_{T,\tau})\,\mathrm{Op}_T(L)(w_0 + g_4)|_{0,\tau} &\lesssim \left|r\,\mathrm{Op}_T\left((\lambda_{T,\tau}^1 + r)^{-1}\right)(w_0 + g_4)\right|_{0,\tau} \\ &\lesssim \frac{r}{\tau + r}\,|w_0 + g_4|_{0,\tau} \end{aligned} \tag{4.6.32}$$

and, with (4.6.3) and (4.6.26)

$$|w_0 + g_4|_{0,\tau} \lesssim \tau^{-1/2}||g||_{0,\tau} + \tau^{-1/2}||v||_{1,\tau} + r^{-1}|\gamma_0(v)|_{1,\tau} + |\gamma_0(v)|_{0,\tau} + |\Theta|_{0,\tau} + \tau^{-3/2}|\gamma_1(v)|_{0,\tau}, \tag{4.6.33}$$

for τ sufficiently large. Observe that in (4.6.30), $(1 - \mathrm{Op}_T(\chi_L))\,\mathrm{Op}_T(\chi) \in \mathcal{S}_{T,\tau}^{-\infty}$ and $R_L \in \mathcal{S}_{T,\tau}^{-\infty}$ and thus, using (4.6.30), (4.6.32) and (4.6.33), we obtain

$$\frac{r+\tau}{r}\tau^{1/2}|\gamma_0(w)|_{1,\tau} \lesssim ||g||_{0,\tau} + ||v||_{1,\tau} + \frac{\tau^{1/2}}{r}|\gamma_0(v)|_{1,\tau} + \tau^{1/2}|\gamma_0(v)|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}|\gamma_1(v)|_{0,\tau}. \tag{4.6.34}$$

From (4.6.22), and using (4.6.7) we have

$$\tau||w||_{1,\tau} \lesssim ||g||_{L^2} + ||v||_{1,\tau} + \tau^{1/2}|\gamma_0(w)|_{1,\tau} + \tau^{1/2}|\gamma_1(w)|_{0,\tau} + \tau^1|\gamma_1(v)|_{0,\tau} \tag{4.6.35}$$

Injecting estimates (4.6.27) and (4.6.34) in (4.6.22) yields

$$\begin{aligned} \tau||w||_{1,\tau} + \frac{\tau^{1/2}(\tau + r)}{r}|\gamma_0(w)|_{1,\tau} + \tau^{1/2}|\gamma_1(w)|_{0,\tau} &\lesssim ||g||_{0,\tau} \\ &+ ||v||_{1,\tau} + \tau^{-1/2}|\gamma_1(v)|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}|\gamma_0(v)|_{1,\tau}, \end{aligned} \tag{4.6.36}$$

by taking τ_0 sufficiently large. Writing $\delta = \frac{1}{r}$ ends the proof. □

4.6.2 Estimate in $\mathcal{E}^0$

We shall derive an estimate in the region where $P_{\varphi,\sigma}$ is not elliptic. This is precisely the region where there is a loss of a half derivative. However, the operator at the boundary $S_{\varphi,\sigma}$ is elliptic here, this allow us to estimate the two traces from a single observation at the boundary.

Proposition 4.6.5. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and let $\nu_0 > 0$, $\eta_1 > 0$ be chosen sufficiently small. Let φ be a weight function satisfying the conditions of Section 4.4.2 in $\overline{V}$, and $\chi_0(x,\xi',\tau,\sigma) \in \mathcal{S}^0_T$ be such that $\mathrm{supp}(\chi) \subset \mathcal{E}^0$. Assume moreover that $\partial_{x_n}\varphi \geq C' > 0$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$\tau||\operatorname{Op}_T(\chi_0)v||^2_{1,\tau} + \delta^2\tau|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|^2_{2,\tau} + \tau|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|^2_{1,\tau} + \tau|D_n\operatorname{Op}(\chi_0)v_{|x_n=0}|^2_{0,\tau} \leq C\left(||P_{\varphi,\sigma}v||^2_{0,\tau} + ||v||^2_{1,\tau} + \delta^2\tau|v_{|x_n=0}|^2_{1,\tau} + \tau|v_{|x_n=0}|^2_0 + \tau|\Theta|^2_{0,\tau}\right). \quad (4.6.37)$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$, for all $u \in \overline{C}^\infty_0(V^+)$ and $\delta \in (0,1]$.

Remark 4.6.6. *Observe that $\mathcal{E}^0$ depends on η_1 and on φ and this is precisely the region where theses parameters will be fixed (see Lemma 4.6.7). Observe also the critical power of τ in the right hand side of the estimate in front of the norm $|v|_{1,\tau}$. This term will not be absorbed directly when we will try to patch the three different estimates. However, this critical term vanishes in the singular limit $\delta = 0$.*

Proof. We set $w = \operatorname{Op}_T(\chi_0)v$. We recall that $\gamma_0(w) = w_{|x_n=0}$ and $\gamma_1(w) = D_n w_{|x_n=0}$. Using Proposition 4.4.3, we have

$$\tau||w||^2_{1,\tau} + \tau\,\mathrm{Re}\,\mathcal{B}(w) \lesssim ||P_{\varphi,\sigma}w||^2_{0,\tau} \quad (4.6.38)$$

where B is the following quadratic form on the boundary:

$$\mathcal{B}(f) = 2\Big(\partial_{x_n}\varphi\gamma_1(f), \gamma_1(f)\Big) + \Big(A_1\gamma_0(f), \gamma_1(f)\Big) + \Big(\gamma_1(f), A_1'\gamma_0(f)\Big) + \Big(A_2\gamma_0(f), \gamma_0(f)\Big), \quad (4.6.39)$$

and $(.,.)$ denotes the scalar product of $L^2(\mathbb{R}^{n-1})$, and the differential operators A_1, A_1' and A_2 are defined by (4.4.16) and (4.4.17). Arguing in the same way as at the beginning of the proof of the estimate in $\mathcal{E}^+$ in Proposition 4.6.1, and using (4.4.12), we have

$$\begin{aligned}\gamma_1(w) &= (\operatorname{Op}_T(\chi_0)D_n v)_{|x_n=0} + ([\operatorname{Op}_T(\chi_0), D_n]v)_{|x_n=0} = \operatorname{Op}_T(\chi_0)(Kv_{|x_n=0} + \Theta) + ([\operatorname{Op}_T(\chi_0), D_n]v)_{|x_n=0} \\ &= K\gamma_0(w) + G_1.\end{aligned}$$

with estimate of the remainder term

$$|G_1|_{0,\tau} \lesssim \delta|\gamma_0(v)|_{1,\tau} + |\gamma_0(v)|_0 + |\Theta|_0, \quad (4.6.40)$$

as $K \in \delta\mathcal{D}^2_{T,\tau} + \tau\mathcal{D}^0$ (see (4.4.13)). It follows that

$$\begin{aligned}\mathcal{B}(w) = \Big(2\partial_{x_n}\varphi K\gamma_0(w), K\gamma_0(w)\Big) + 4\,\mathrm{Re}\,\Big(\partial_{x_n}\varphi K\gamma_0(w), G_1\Big) + \Big(\partial_{x_n}\varphi G_1, G_1\Big) + \Big(A_1\gamma_0(w), K\gamma_0(w)\Big) \\ + \Big(K\gamma_0(w), A_1'\gamma_0(w)\Big) + \Big(A_1\gamma_0(w), G_1\Big) + \Big(G_1, A_1'\gamma_0(w)\Big) + \Big(A_2\gamma_0(w), \gamma_0(w)\Big). \quad (4.6.41)\end{aligned}$$

By the Cauchy-Schwarz inequality, for the terms involving G_1, we have

$$\begin{aligned}\Big|4\,\mathrm{Re}\,\Big(\partial_{x_n}\varphi K\gamma_0(w), G_1\Big) + \Big(\partial_{x_n}\varphi G_1, G_1\Big) + \Big(A_1\gamma_0(w), G_1\Big) + \Big(G_1, A_1'\gamma_0(w)\Big)\Big| \\ \lesssim |K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|^2_{0,\tau}. \quad (4.6.42)\end{aligned}$$

By symbolic calculus, we can write the "principal" terms of (4.6.41) in the following way

$$\begin{aligned}2\Big(\partial_{x_n}\varphi K\gamma_0(w), K\gamma_0(w)\Big) + \Big(A_1\gamma_0(w), K\gamma_0(w)\Big) + \Big(Kw, A_1'\gamma_0(w)\Big) + \Big(A_2\gamma_0(w), \gamma_0(w)\Big) \\ = \Big((\delta^2 B_4 + \delta B_3 + B_2)\gamma_0(w), \gamma_0(w)\Big), \quad (4.6.43)\end{aligned}$$

where the principal symbols of B_4, B_3 and B_2 are respectively

$$\begin{aligned}
b_4 &= 2\partial_{x_n}\varphi_{|x_n=0}|s_{\varphi,\sigma}|^2 \in \mathcal{S}^4_{T,\tau},\\
b_3 &= 4\tau(\partial_{x_n}\varphi_{|x_n=0})^2 s_2(x,\xi',\tau,\sigma) + 4s_1(x,\xi',\tau)\tilde{r}(x',x_n=0,\xi',\mathrm{d}_{x'}\varphi_{|x_n=0}) \in \mathcal{S}^3_{T,\tau},\\
b_2 &= 2\left(\partial_{x_n}\varphi^3\tau^2 + 2\partial_{x_n}\varphi(-r(x,\xi') + p(x,\tau\mathrm{d}_x\varphi) + \sigma^2)\right)_{|x_n=0} \in \mathcal{S}^2_{T,\tau}.
\end{aligned}$$

Now we state positivity result of these symbols.

Lemma 4.6.7. *There exists $C>0$, $\eta_1>0$, satisfying $\eta_1<C_0$, independent of σ, τ and δ such that*

$$b_j(x',\xi',\tau,\sigma) \geq C\lambda^j_{T,\tau}, \qquad j=2,3,4, \tag{4.6.44}$$

for all (x,ξ',τ,σ) satisfying $-2\eta_1\tau^2 \leq \mu(x,\xi',\tau,\sigma) \leq 2\eta_1\tau^2$, where C_0 is the constant given in the second part of Proposition 4.5.3.

Proof of Lemma 4.6.7. The positivity of the symbol b_4 comes precisely from the fact that the boundary operator $S_{\varphi,\sigma}$ is elliptic in this region (see Lemma 4.5.3). Let us prove now the positivity of b_2. Let $(x',x_n=0,\xi',\tau,\sigma)\in\mathcal{E}^0$, i.e be such that $-2\eta_1\lambda^2_{T,\tau} \leq -\mu \leq 2\eta_1\lambda^2_{T,\tau}$. In particular, we have

$$-2\eta_1\lambda^2_{T,\tau} \leq \sigma^2 + p(x',x_n=0,\tau\mathrm{d}_{x'}\varphi) - r(x',x_n=0,\xi'),$$

and thus

$$b_2 \geq 2\partial_{x_n}\varphi^3\tau^2 - 4\eta_1\partial_{x_n}\varphi\lambda^2_{T,\tau}. \tag{4.6.45}$$

Observe that as $(x',x_n=0,\xi',\tau,\sigma)\in\mathcal{E}^0$, there exists $C>0$ such that $\lambda^2_{T,\tau}\leq C\tau^2$ (see Lemma 4.5.2) for η_1 sufficiently small. In addition, if η_1 is chosen sufficiently small, from (4.6.45) we have the positivity of b_2. Let us finally prove the positivity of b_3. From Lemma 4.5.3, we have for η_1 sufficiently small

$$s_2(x',\xi',\tau,\sigma) = s(x',\xi') - s(x',\tau\mathrm{d}_{x'}\varphi) - \kappa\sigma^2 \gtrsim \lambda^2_{T,\tau}.$$

Moreover,

$$4|s_1(x,\xi',\tau)\tilde{r}(x',x_n=0,\xi',\mathrm{d}_{x'}\varphi)| \lesssim \tau|\xi'|^2|\mathrm{d}_{x'}\varphi|^2 \lesssim \nu_0^2\tau|\xi'|^2|\partial_{x_n}\varphi|^2, \tag{4.6.46}$$

since we have (4.4.15), and again for ν_0 sufficiently small, we obtain $b_3\gtrsim\tau\lambda^2_{T,\tau}$ and using again that $\lambda_{T,\tau}\lesssim\tau$ in the region $\mathcal{E}^0$, we conclude the proof. □

We can now apply the microlocal Gårding inequality in (4.6.43), and taking τ sufficiently large we obtain, for an arbitrary $N\in\mathbb{N}$

$$\begin{aligned}
\mathcal{B}(w) \geq\ & C(\delta^2|\gamma_0(w)|^2_{2,\tau} + \delta|\gamma_0(w)|^2_{3/2,\tau} + |\gamma_0(w)|^2_{1,\tau})\\
& - C'\left(|K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|^2_{0,\tau}\right) - C_N|\gamma_0(v)|^2_{-N,\tau}.
\end{aligned} \tag{4.6.47}$$

By the Young inequality, the right hand side of (4.6.47) reads

$$|K\gamma_0(w)|_{0,\tau}|G_1|_{0,\tau} + |\gamma_0(w)|_{1,\tau}|G_1|_{0,\tau} + |G_1|^2_{0,\tau} \lesssim \delta'(\delta^2|\gamma_0(w)|^2_{2,\tau} + \tau^2|\gamma_0(w)|^2_{0,\tau} + |\gamma_0(w)|^2_{1,\tau}) + \delta'^{-1}|G_1|^2_{0,\tau}, \tag{4.6.48}$$

for all $\delta'>0$. From (4.6.38), (4.6.47) and (4.6.48), we obtain

$$\tau||w||^2_{1,\tau} + \tau\left(\delta^2|\gamma_0(w)|^2_{2,\tau} + |\gamma_0(w)|^2_{1,\tau}\right) \lesssim ||P_{\varphi,\sigma}w||^2_{0,\tau} + \tau|G_1|^2_{0,\tau} + C_N\tau|\gamma_0(v)|^2_{-N,\tau}.$$

Using the estimate of $|G_1|_{0,\tau}$ in (4.6.40) we obtain the sought result. □

4.6.3 Estimate in $\mathcal{E}^-$

In this region, we have $\mu < -\eta_1$ implying $|\xi'| \lesssim \tau$, and the operator at the boundary $S_{\varphi,\sigma}$ is not elliptic. However, in the case where $\partial_{x_n}\varphi > 0$, the two roots of $p_{\varphi,\sigma}$ are of negative imaginary parts, and we can estimate the two traces at the boundary with no observation term.

Proposition 4.6.8. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ be a weight function satisfying the conditions of Section 4.4.2 in $\overline{V}$, and $\chi(x,\xi',\tau,\sigma) \in \mathcal{S}^0_T$ be such that $\mathrm{supp}(\chi) \subset \mathcal{E}^-$. Assume moreover that $\partial_{x_n}\varphi \geq C' > 0$ on $\overline{V}$. Then, there exist $\tau_0 > 0$ and $C > 0$ such that*

$$||\operatorname{Op}_T(\chi)v||^2_{2,\tau} + \tau|\operatorname{Op}_T(\chi)v_{|x_n=0}|^2_{1,\tau} + \tau|D_n\operatorname{Op}_T(\chi)v_{|x_n=0}|^2_{0,\tau} \leq C\left(||P_{\varphi,\sigma}v||^2_{0,\tau} + ||v||^2_{1,\tau}\right),$$

for all $|\sigma| \geq 1$, for all $\tau \geq \tau_0|\sigma|$, $u \in \overline{C}^\infty_0(V^+)$ and $\delta \in (0,1]$.

Proof. We follow the proof of the microlocal estimate in the $\mathcal{E}^+$ region to (4.6.12) and (4.6.13). We remark that the integral along the contour β is identically 0, because the integrand is holomorphic and the two poles have a non-positive imaginary part. We then have

$$\underline{w} = E_M(\underline{P_{\varphi,\sigma}w}) + g_1, \tag{4.6.49}$$

where g_1 is a remainder coming from microlocalisations. We recall that it can be estimated by (4.6.9). Thus, (4.6.49) yields

$$||w||_{2,\tau} \lesssim ||P_{\varphi,\sigma}v||_{L^2} + ||v||_{1,\tau}.$$

Taking the γ_0-trace on $\{x_n = 0\}$ in (4.6.49), and thanks to the trace inequality (4.3.2)

$$\tau^{1/2}|w|_{1,\tau} \lesssim ||P_{\varphi,\sigma}v||_{L^2} + ||v||_{1,\tau},$$

and finally taking the γ_1-trace

$$\tau^{1/2}|D_nw|_0 \lesssim ||P_{\varphi,\sigma}v||_{L^2} + ||v||_{1,\tau}.$$

Patching these three estimates, we obtain the sought result. □

4.7 Proof of the local Carleman estimates

4.7.1 A local Carleman estimate from the interior up to the boundary

From the three microlocal estimates of the previous section, we shall derive a local Carleman estimate with observation from the interior, that is, with the condition $\partial_{x_n}\varphi > 0$. With this sign condition, from Propositions 4.6.1, 4.6.5 and 4.6.8, we recall that we have the three following estimates:

$$||\operatorname{Op}_T(\chi_-)v||_{2,\tau} + \tau^{1/2}|\operatorname{Op}_T(\chi_-)v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|D_n\operatorname{Op}_T(\chi_-)v_{|x_n=0}|_{0,\tau} \lesssim ||P_{\varphi,\sigma}v||_{0,\tau} + ||v||_{1,\tau}, \tag{4.7.1}$$

$$\begin{aligned}\tau^{1/2}||\operatorname{Op}_T(\chi_0)v||_{1,\tau} + \delta\tau^{1/2}|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{2,\tau} + \tau^{1/2}\Big(&|\operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{1,\tau}\\ &+ |D_n\operatorname{Op}_T(\chi_0)v_{|x_n=0}|_{0,\tau}\Big) \lesssim ||P_{\varphi,\sigma}v||_{0,\tau} + ||v||_{1,\tau} + \delta\tau^{1/2}|v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|v_{|x_n=0}|_{0,\tau} + \tau^{1/2}|\Theta|_{0,\tau},\end{aligned} \tag{4.7.2}$$

$$\begin{aligned}\tau||\operatorname{Op}_T(\chi_+)v||_{1,\tau} + (\tau^{1/2} + \delta\tau^{3/2})|\operatorname{Op}_T(\chi_+)v_{|x_n=0}|_{1,\tau} + \tau^{1/2}|D_n\operatorname{Op}_T(\chi_+)v_{|x_n=0}|_{0,\tau}&\\ \lesssim ||P_{\varphi,\sigma}v||_{0,\tau} + ||v||_{1,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{-1/2}(|v_{|x_n=0}|_{1,\tau} + |D_nv_{|x_n=0}|_{0,\tau}),&\end{aligned} \tag{4.7.3}$$

where the cut-off functions χ_+, χ_- and χ_0 are respectively supported in $\mathcal{E}^+$, $\mathcal{E}^-$ and $\mathcal{E}^0$, and for η_1 chosen sufficiently small. We recall that $P_{\varphi,\sigma}$ and Θ are defined in Section 4.4.1. We shall denote respectively equalities (4.7.1), (4.7.2) and (4.7.3) by

$$LHS_-(v) \lesssim RHS_-(v), \quad LHS_0(v) \lesssim RHS_0(v), \quad LHS_+(v) \lesssim RHS_+(v). \tag{4.7.4}$$

Due to the critical term $\delta\tau^{1/2}|v|_{x_n=0}|_{1,\tau}$ on the right hand side of (4.7.2) and (4.7.4), we cannot patch directly these estimates and absorb some of the terms on the right hand side by other terms on the left hand side by chosen parameters approprietely. This is the reason for the introduction of an additionnal small parameter $\varepsilon > 0$, independent of δ, τ and σ. Now, we take a partition of unity: $\chi_- + \chi_0 + \chi_+ = 1$ satisfying $\chi_+, \chi_-, \chi_0 \in \mathcal{S}^0_{T,\tau}$ and moreover

$$\text{supp}(\chi_+) \subset \mathcal{E}^+, \qquad \text{supp}(\chi_0) \subset \mathcal{E}^0, \qquad \text{supp}(\chi_-) \subset \mathcal{E}^-.$$

For the construction of χ_+, χ_- and χ_0, take for instance $\zeta_0 \in C_0^\infty(\mathbb{R})$, such that $\zeta_0 = 1$ on $[-\eta_1, \eta_1]$, $\text{supp}(\zeta_0) \subset (-2\eta_1, 2\eta_1)$ and set

$$\chi_0 = \zeta_0(\frac{\mu}{\lambda^2_{T,\tau}}), \quad \chi_- = \mathbf{1}_{(-\infty,0]}(\mu)(1-\chi_0), \quad \chi_+ = \mathbf{1}_{[0,+\infty)}(\mu)(1-\chi_0).$$

Observe that this construction yields indeed tangential symbols in $\mathcal{S}^0_{T,\tau}$ by adapting [Hör07, Theorem 18.1.10].

We then write, for $0 < \varepsilon \leq 1$

$$LHS_-(v) + \varepsilon LHS_0(v) + LHS_+(v) \lesssim LHS_-(v) + \varepsilon RHS_0(v) + LHS_+(v) \tag{4.7.5}$$

Consider the critical term $\varepsilon\delta\tau^{1/2}|v_{|_{x_n=0}}|_{1,\tau}$ on the right hand side of (4.7.5). We write

$$\begin{aligned}\varepsilon\delta\tau^{1/2}|v_{|x_n=0}|_{1,\tau} &\leq \varepsilon\delta\tau^{1/2}\left(|\operatorname{Op}_T(\chi_-)v_{|_{x_n=0}}|_{1,\tau} + |\operatorname{Op}_T(\chi_0)v_{|_{x_n=0}}|_{1,\tau} + |\operatorname{Op}_T(\chi_+)v_{|_{x_n=0}}|_{1,\tau}\right).\\ &\leq \varepsilon LHS_-(v) + \tau^{-1}\varepsilon LHS_0(v) + \varepsilon\tau^{-1}LHS_+(v)\end{aligned}$$

By choosing $\varepsilon = \varepsilon_1 > 0$ sufficiently small and τ sufficiently large, we obtain

$$LHS_-(v) + LHS_0(v) + LHS_+(v) \lesssim ||P_{\varphi,\sigma}v||_{0,\tau} + ||v||_{1,\tau} + \tau^{1/2}|\Theta|_{0,\tau} + \tau^{1/2}|v_{|_{x_n=0}}|_{1,\tau} + |D_n v|_{0,\tau}.$$

We conclude by taking τ sufficiently large, and we obtain the following Carleman estimate in the neighborhood of the boundary

$$\tau^{1/2}||v||_{1,\tau} + \tau^{1/2}|v_{|_{x_n=0}}|_{1,\tau} + \tau^{1/2}|D_n v_{|_{x_n=0}}|_{0,\tau} \lesssim ||P_{\varphi,\sigma}v||_{0,\tau} + \tau^{1/2}|(D_n - K)v_{|_{x_n=0}}|_{0,\tau}.$$

Then coming back to $u = e^{-\tau\varphi}v$ yields

$$\begin{aligned}\tau^{3/2}||e^{\tau\varphi}u||_{L^2} + \tau^{1/2}||e^{\tau\varphi}Du||_{L^2} + \tau^{3/2}|e^{\tau\varphi}u_{|_{x_n=0}}|_{L^2} + \tau^{1/2}|e^{\tau\varphi}Du_{|_{x_n=0}}|_{L^2}&\\ \lesssim ||e^{\tau\varphi}P_\sigma u||_{L^2} + \tau^{1/2}|e^{\tau\varphi}(\partial_{x_n} - \delta S_\sigma)u|_{L^2}.&\end{aligned}$$

This concludes the proof of the second part of Theorem 4.1.5. As the conditions imposed to the weight function in Section 4.4.2 are invariant under change of coordinates, we naturally obtain (4.1.26) in the neighborhood of a point of the boundary $\partial\Omega$.

4.7.2 A local Carleman estimate with a boundary observation

Here, we prove (4.1.25), that is, an estimation with a boundary observation. this will be used below to prove Theorem 6.1.15 in the case $a = 0$ and $b \neq 0$ (see (4.1.9) and (4.1.16)). There, b is a non-negative bounded function satisfying $b \geq C > 0$ on an open subset ω_B of $\partial\Omega$. We then have to observe from the boundary and thus the weight function is chosen such that $\partial_\nu\varphi > 0$ in a neighborhood of a point of ω_B. Hence, it is sufficient to prove a Carleman estimate where there are no assumptions about the sign of $\partial_\nu\varphi$, yet assuming $\partial_\nu\varphi \neq 0$. Observing that Proposition 4.6.1 is independent of the sign of $\partial_\nu\varphi$, it remains to prove an estimate in the region $\mathcal{F} := \mathcal{E}^0 \cup \mathcal{E}^-$.

Proposition 4.7.1. *Let V be an open neighborhood of 0 in $\mathbb{R}^d$, and φ be a weight function satisfying the conditions of Section 4.4.2 in $\overline{V}$ with in particular $|\partial_\nu\varphi| \geq C_0 > 0$, and $\chi \in \mathcal{S}^0_T$ be such that $\operatorname{supp}\chi \subset \mathcal{F}$. Then there exists $C > 0$ and $\tau_0 > 0$ such that*

$$\tau||\operatorname{Op}_T(\chi)v||^2_{1,\tau} + \tau|D_n \operatorname{Op}_T(\chi)v_{|x_n=0}|^2_{0,\tau} \leq C\left(||P_{\varphi,\sigma}v||^2_{0,\tau} + ||v||^2_{1,\tau} + (\delta^2\tau^5+\tau^3)|v_{|x_n=0}|^2_{0,\tau} + \tau|\Theta|^2_{0,\tau}\right). \tag{4.7.6}$$

for all $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$ and $v \in \overline{C}^\infty_0(V^+)$.

Proof.
We set $w := \operatorname{Op}_T(\chi)v$. Using Proposition 4.4.3, there exists $\tau_0 > 0$ such that

$$\tau||w||^2_{1,\tau} + \tau \operatorname{Re}\mathcal{B}(w) \lesssim ||P_{\varphi,\sigma}w||^2_{0,\tau} \lesssim ||P_{\varphi,\sigma}v||^2_{0,\tau} + ||v||^2_{1,\tau}, \tag{4.7.7}$$

for $\tau \geq \tau_0|\sigma|$, where the boundary form $\mathcal{B}$ reads

$$\mathcal{B}(w) = \left(\partial_{x_n}\varphi D_n w_{|x_n=0}, D_n w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left(A_1 w_{|x_n=0}, D_n w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left(D_n w_{|x_n=0}, A_1' w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})} + \left(A_2 w_{|x_n=0}, w_{|x_n=0}\right)_{L^2(\mathbb{R}^{n-1})}.$$

Observe that, by symbolic calculus, for all $N \in \mathbb{N}$,

$$A_j \circ \operatorname{Op}_T(\chi) = \operatorname{Op}_T(c_j) + A_j^N \quad (\text{resp. } A_j' \circ \operatorname{Op}_T(\chi) = \operatorname{Op}_T(c_j') + A_j'^N)$$

with c_j (resp. c_j') $\in \mathcal{S}^j_{T,\tau}$, $\operatorname{supp}(c_j)$ (resp. $\operatorname{supp}(c_j')$) $\subset \operatorname{supp}(\chi)$, A_j^N (resp. $A_j'^N$) $\in \Psi^{-N}_{T,\tau}$. Moreover, as $|\xi'| \lesssim \tau$ in $\operatorname{supp}(\chi)$ by Lemma 4.5.2, we have $c_j, c_j' \in \tau^j\mathcal{S}^0_{T,\tau}$. Hence, we have the estimate

$$|\mathcal{B}(w)| \lesssim |(D_n \operatorname{Op}_T(\chi)v)_{|x_n=0}|^2_{0,\tau} + \tau^2|v_{|x_n=0}|^2_{0,\tau}. \tag{4.7.8}$$

Now we shall use the boundary condition (4.4.12). With the same arguments as above, we have $\operatorname{Op}_T(\chi)K \in (\delta\tau^2+\tau)\psi^0_{T,\tau}$ and thus

$$\begin{aligned}|(D_n \operatorname{Op}_T(\chi)v)_{|x_n=0}|_{0,\tau} &\leq |(\operatorname{Op}_T(\chi)D_n v)_{|x_n=0}|_{0,\tau} + |([D_n, \operatorname{Op}_T(\chi)]v)_{|x_n=0}|_{0,\tau}\\ &\lesssim |(\operatorname{Op}_T(\chi)Kv)_{|x_n=0}|_{0,\tau} + |([D_n, \operatorname{Op}_T(\chi)]v)_{|x_n=0}|_{0,\tau} + |\Theta|_{0,\tau}\\ &\lesssim (\delta\tau^2+\tau)|v_{|x_n=0}|_{0,\tau} + |\Theta|_{0,\tau}.\end{aligned}$$

This, in addition of (4.7.7) and (4.7.8) yields the result. □

We can now conclude the proof of the first part of Theorem 4.1.5 by patching estimate (4.7.6) with estimate (4.6.1), since no condition on the sign of $\partial_{x_n}\varphi$ is needed in the region $\mathcal{E}^+$, and taking τ sufficiently large to absorb low order terms. Note that, as opposed to the proof of Section 4.7.1, there is no need for the introduction of the additional small parameter ε.

4.8 Interpolation and resolvent estimate

4.8.1 Interpolation estimate

Observation from the interior

In this section, following the ideas in [LR95], we shall derive an interpolation inequality from the Carleman estimate (4.1.26). We prove the inequality in the interior, and propagate it up to the boundary. Below, we set $\Omega_\eta = \{x \in \Omega,\ d(x,\partial\Omega) > \eta\}$ and $\Omega^\eta = \{x \in \Omega,\ d(x,\partial\Omega) < \eta\}$.

Theorem 4.8.1. *Let ω_I be an open subset of Ω. There exists $C > 0$ such that*

$$||u||_{\mathcal{V}_\delta} \leq Ce^{C|\sigma|}\left(||P_\sigma u||_{L^2(\Omega)} + |(\partial_\nu + \delta S_\sigma)u|_{L^2(\partial\Omega)} + ||u||_{L^2(\omega_I)}\right),$$

for all $|\sigma| \geq 1$ and $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$. We recall that $||.||_{\mathcal{V}_\delta}$ is the norm defined in (4.1.13).

The proof is splitted in the two following lemmata. First, an interpolation estimate in a neighborhood of the boundary

Lemma 4.8.2. *Let $y_0 \in \partial\Omega$. Then, there exist $\eta > 0$, a neighborhood V of y_0 in $\overline{\Omega}$, $\mu \in (0,1]$, and $C > 0$ such that*

$$||u||_{H^1(V)} + |u_{|\partial\Omega}|_{H^1(V\cap\partial\Omega)} \leq Ce^{C\sigma}\left(||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu} \times \left(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + ||u||_{H^1(\Omega_\eta)}\right)^{\mu}$$

for all $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$, for all $|\sigma| \geq 1$, and for all $\delta > 0$.

Second, an interpolation estimate in the interior, which proof can be found in Appendix 9.E.5. A version without the parameter σ can be found in [LR95].

Lemma 4.8.3. *Let Ω be a connected open set of $\mathbb{R}^n$ and ω_I be an open set compactly embedded in Ω. Then, for all $\eta > 0$, there exists $C > 0$, and $\mu \in (0,1]$ such that*

$$||u||_{H^1(\Omega_\eta)} \leq Ce^{C|\sigma|}||u||^{1-\mu}_{H^1(\Omega)}\left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{L^2(\omega_I)}\right)^{\mu}$$

for all $u \in H^2(\Omega)$, $|\sigma| \geq 1$.

Proof of Lemma 4.8.2 We shall work in normal geodesic coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as in Section 4.4 in a neighborhood U in $\overline{\Omega}$ of $y_0 \in \partial\Omega$. These coordinates are chosen such that $y_0 = 0$. We shall still denote U the corresponding neighborhood of 0 in $\overline{\mathbb{R}^n_+}$. For $\beta > 0$, we define the following anisotropic distance on $\mathbb{R}^n$:

$$d_\beta(x,y) = \left(|x' - y'|^2 + \beta|x_n - y_n|^2\right)^{1/2}. \tag{4.8.1}$$

We choose $r_0 > 0$ such that $x_0 = (0, r_0) \in U$. Then we take W a neighborhood of 0 in $\mathbb{R}^n$ such that $W^+ := W \cap \mathbb{R}^n_+ \subset U$ and such that $d(x_0, W^+) > 0$. We set $\varphi = e^{-\lambda\psi}$ where $\psi(x) = d_\beta(x, x_0)$. Observe that φ is an admissible weight function on W^+ for λ sufficiently large (see Proposition 4.4.2) and for β sufficiently large (see Section 4.4.2). Until the end of the proof, β is kept fixed. We define the following cut-off functions $\chi_0,\ \chi_1 \in C^\infty_0(\mathbb{R}^n)$:

$$\chi_0(x) = \begin{cases} 0 & \text{in} \quad x_n > r_1 \\ 1 & \text{in} \quad x_n \in [0, r_1/2] \end{cases} \quad ; \quad \chi_1(x) = \begin{cases} 0 & \text{if} \quad d_\beta(x,x_0) \leq r_2 \text{ or } d_\beta(x,x_0) > r_5 \\ 1 & \text{in} \quad d_\beta(x,x_0) \in [r_3, r_4], \end{cases}$$

with $0 < r_1 < r_0$ and $0 < r_2 < r_3 < r_4 < r_5$ such that

- r_3 is small enough to have $\overline{B_\beta(x_0, r_3)} \cap \overline{W^+} = \emptyset$, where B_β denotes the open ball associated with the distance d_β;
- r_1 small enough and $r_0 < r_4 < r_5$ are such that $\{x \in \mathbb{R}^n_+ \mid x_n \leq r_1\} \cap \{r_4 \leq d_\beta(x, x_0) \leq r_5\} \subset W^+$.

The geometry of the supports of the cut-off functions is represented in Figure 4.4. As $\partial_{x_n}\varphi \geq C > 0$ on $\operatorname{supp}\chi_0\chi_1$, we can apply the Carleman estimate (4.1.26) on W^+ to $w = \chi_0\chi_1 u$: there exist $\tau_0 > 0$ such that

$$\tau^3||e^{\tau\varphi}w||^2_{L^2(W^+)} + \tau||e^{\tau\varphi}Dw||^2_{L^2(W^+)} + \tau^3|e^{\tau\varphi}w_{|x_n=0}|^2_{L^2(W^0)} + \tau|e^{\tau\varphi}D'w_{|x_n=0}|^2_{L^2(W^0)} \lesssim ||e^{\tau\varphi}P_\sigma w||^2_{L^2(W^+)} + \tau|(e^{\tau\varphi}(\partial_{x_n} - \delta S_\sigma)w)_{|x_n=0}|^2_{L^2(W^0)}, \tag{4.8.2}$$

for all $\tau \geq \tau_0|\sigma|$, where $W^0 = W \cap \{x_n = 0\}$.The right hand side can be estimated as follows

$$\begin{aligned} ||e^{\tau\varphi}P_\sigma w||_{L^2(W^+)} &\lesssim ||e^{\tau\varphi}P_\sigma u||_{L^2(W^+)} + ||e^{\tau\varphi}[P_\sigma, \chi_0\chi_1]u||_{L^2(W^+)} \\ &\lesssim e^{\tau C_3}||P_\sigma u||_{L^2(W^+)} + e^{\tau C_3}||u||_{H^1(W^+\cap\{x_n\in[r_1/2,r_1]\})} + e^{\tau C_1}||u||_{H^1(W^+)}, \end{aligned}$$

with $C_3 > e^{-\lambda(r_0 - r_1)}$ and $C_1 = e^{-\lambda r_4}$. Observe that $C_1 < C_3$. Here, we used that the weight function φ is radial with respect to the distance d_β to x_0 and decreasing as x moves away from x_0, and the commutator

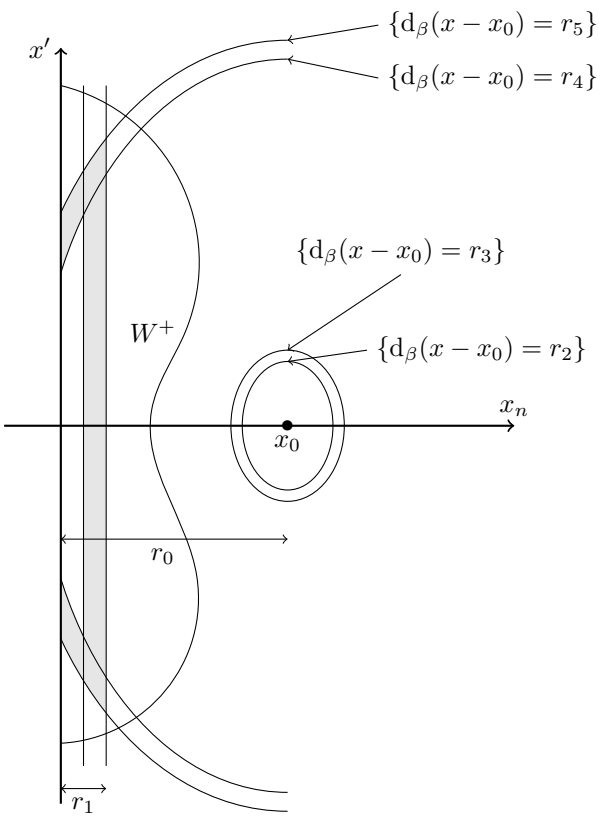


Figure 4.4: Geometry of the cut-off functions near $(0,0)$. The grey zone corresponds to the region where $\chi_0\chi_1$ varies.

$[P_\sigma, \chi_0\chi_1]$ is a differential operator of order 1 supported in the region were $\chi_0\chi_1$ varies (represented in grey in Figure 4.4).
In the same spirit, using that $\delta \leq 1$

$$|(e^{\tau\varphi}(\partial_{x_n}-\delta S_\sigma)w)_{|_{x_n=0}}|_{L^2(W^0)} \lesssim e^{\tau C_3}|(\partial_{x_n}-\delta S_\sigma)u_{|_{x_n=0}}|_{L^2(W^0)} + e^{\tau C_1}\left(|u_{|_{x_n=0}}|_{L^2(W^0)} + |D'u_{|_{x_n=0}}|_{L^2(W^0)}\right).$$

Finally, we can restrict the left hand side of the Carleman estimate to $\tilde{W} := B(0,r_6)\cap\{x_n>0\}$ with $r_6>0$ taken sufficiently small to have $\chi_0\chi_1 = 1$ on $B(0,r_6)$ and this yields, for $\tau \geq 1$,

$$\begin{aligned}\tau^{3/2}||e^{\tau\varphi}\chi_0\chi_1 u||_{L^2(\tilde{W})} + \tau^{1/2}||e^{\tau\varphi}D\chi_0\chi_1 u||_{L^2(\tilde{W})} + \tau^{3/2}|e^{\tau\varphi}\chi_0\chi_1 u_{|_{x_n=0}}|_{L^2(\partial\tilde{W}\cap\{x_n=0\})}\\ + \tau^{1/2}|e^{\tau\varphi}D'\chi_0\chi_1 u_{|_{x_n=0}}|_{L^2(\partial\tilde{W}\cap\{x_n=0\})} \gtrsim e^{\tau C_2}(||u||_{H^1(\tilde{W})} + |u_{|_{x_n=0}}|_{H^1(\partial\tilde{W}\cap\{x_n=0\})}),\end{aligned} \quad (4.8.3)$$

where $C_2 = \inf_{\overline{W}}\varphi$. We finally obtain, coming back to the original coordinates, for some $\eta>0$,

$$\begin{aligned}||u||_{H^1(V)} + |u_{|_{\partial\Omega}}|_{H^1(V\cap\partial\Omega)} \lesssim e^{\tau(C_3-C_2)}\left(||P_\sigma u||_{L^2(\Omega)} + |(\partial_\nu+\delta S_\sigma)u_{|_{\partial\Omega}}|_{L^2(\partial\Omega)} + ||u||_{H^1(\Omega_\eta)}\right)\\ + e^{-\tau(C_2-C_1)}\left(||u||_{H^1(\Omega)} + |u_{|_{\partial\Omega}}|_{H^1(\partial\Omega)}\right),\end{aligned} \quad (4.8.4)$$

where V is an open neighborhood of $y_0\in\partial\Omega$ in $\overline{\Omega}$. Note that we have $0<C_1<C_2<C_3$. Optimizing this inequality by applying the following lemma, which proof is given in Appendix 9.E.6 concludes the proof.

Lemma 4.8.4. *Let $A, B, C \geq 0$ such that $A \leq C$. Suppose there exists $\tilde{\tau}, \beta, \gamma > 0$ such that $A \leq e^{\beta\tau}B + e^{-\gamma\tau}C$ for all $\tau \geq \tilde{\tau}$. Then*

$$A \leq KB^{\mu}C^{1-\mu},$$

with $K = \max((\gamma/\beta)^{\frac{\beta}{\beta+\gamma}} + (\beta/\gamma)^{\frac{\gamma}{\beta+\gamma}}, (\beta/\gamma)^{\frac{\gamma}{\beta+\gamma}} e^{\gamma\tilde{\tau}})$ and $\mu = \frac{\gamma}{\beta+\gamma}$.

□

We end this section by the proof of the theorem.

Proof of Theorem 4.8.1

We shall in fact prove the stronger estimate

$$||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|}\left(||P_{\varphi,\sigma}u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + ||u||_{L^2(\omega_I)}\right). \tag{4.8.5}$$

As $\delta \leq 1$, the result follows. Observe that we can assume that u satisfies

$$\begin{cases} ||P_\sigma u||_{L^2(\Omega)} \leq ||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \\ |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \leq ||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}, \end{cases} \tag{4.8.6}$$

otherwise (4.8.5) follows immediatly. From Lemma 4.8.2 and by a compactness argument, we can find $\eta' > 0$, $\eta'' > 0$, $C > 0$ and $\mu' > 0$ such that

$$\begin{aligned} ||u||_{H^1(\Omega^{\eta'})} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|}\left(||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu'} \left(||P_\sigma u||_{L^2(\Omega)}\right. \\ \left. + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + ||u||_{H^1(\Omega_\eta)}\right)^{\mu'}, \end{aligned} \tag{4.8.7}$$

for all $|\sigma| \geq 1$ and $\delta \in (0, 1]$ and all $0 < \eta \leq \eta''$ (observing that Ω_η increases as η decreases). By Lemma 4.8.3, for all $\eta > 0$, there exists $C' > 0$ such that

$$||u||_{H^1(\Omega_\eta)} \lesssim e^{C'|\sigma|}||u||^{1-\mu}_{H^1(\Omega)}\left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{L^2(\omega_I)}\right)^{\mu}, \tag{4.8.8}$$

for all $|\sigma| \geq 1$ and $\delta \in (0, 1]$. Using (4.8.6) and (4.8.8), we have

$$\begin{aligned} ||u||_{H^1(\Omega_\eta)} + ||P_\sigma u||_{L^2(\Omega)} + |\partial_n u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \lesssim e^{C'|\sigma|}\left(||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu} \\ \times \left(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + ||u||_{L^2(\omega_I)}\right)^{\mu}. \end{aligned} \tag{4.8.9}$$

Injecting estimate (4.8.9) in (4.8.7), we obtain (4.8.5). □

Observation from the boundary

We shall prove here the following theorem (which is the counterpart of Theorem 4.8.1 in the boundary case).

Theorem 4.8.5. *Let $\omega_B \subset \partial\Omega$ be an open subset of the boundary. Then there exists $C > 0$ such that*

$$||u||_{\mathcal{V}_\delta} \leq Ce^{C|\sigma|}\left(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right),$$

for all $|\sigma| \geq 1$, $u \in H^2(\Omega)$ such that $u_{|\partial\Omega} \in H^2(\partial\Omega)$.

Proof. Observe first that we can assume

$$||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} \leq ||u||_{\mathcal{V}_\delta(\Omega)}, \tag{4.8.10}$$

otherwise, the estimate is immediate. We use normal geodesic coordinates (x', x_n) in a neighborhood V of a point $y_0 \in \omega_B$ such that $V \subset \omega_B$. Hence, we consider an open neighborhood W of $(0, 0)$ in $\mathbb{R}^n$ such that $W \cap \{x_n = 0\} \subset V$, and we set $\varphi = e^{\lambda\psi}$, where $\psi(x) = -\mathrm{d}_\beta(x, x_0)$, $x_0 = (0, -r_0)$, $r_0 > 0$ will be fixed below and d_β denotes the anisotropic distance defined in (4.8.1). Observe that for λ sufficiently large and

β sufficiently small, φ fulfills the weight function properties required in Section 4.4.2 in $\overline{W \cap \mathbb{R}^n_+}$. We now define the following C^∞_0 cut-off function

$$\chi(x) = \begin{cases} 1 & \text{if} \quad \mathrm{d}_\beta(x, x_0) < r_1 \\ 0 & \text{if} \quad \mathrm{d}_\beta(x, x_0) > r_2, \end{cases}$$

where the r_j are such that $r_0 < r_1 < r_2$ and $\{x \in \mathbb{R}^n,\ x_n \geq 0\ , \mathrm{d}_\beta(x, x_0) \leq r_2\} \subset \overline{W^+}$. We recall that $W^+ = W \cap \{x_n > 0\}$ and $W^0 = W \cap \{x_n = 0\}$. We can then apply the Carleman estimate (4.1.25) to χu in W^+, as in the present case $\partial_\nu \varphi \geq C > 0$. We have

$$\tau^3 ||e^{\tau\varphi}\chi u||^2_{L^2(W^+)} + \tau ||e^{\tau\varphi}\nabla\chi u||^2_{L^2(W^+)} \lesssim ||e^{\tau\varphi}P_\sigma \chi u||^2_{L^2(W^+)} + \tau |(\partial_{x_n} - \delta S_\sigma)\chi u_{|_{x_n=0}}|_{L^2(W^0)} + \tau^5 |e^{\tau\varphi}\chi u_{|_{x_n=0}}|_{L^2(W^0)}.$$

The right hand side can be estimated as follows (using the fact that commutators are supported in regions where χ varies)

$$||e^{\tau\varphi}P_\sigma(\chi u)||^2_{L^2(W^+)} + \tau |(\partial_{x_n} - \delta S_\sigma)\chi u_{|_{x_n=0}}|_{L^2(W^0)} + \tau^5 |e^{\tau\varphi}\chi u_{|_{x_n=0}}|_{L^2(W^0)} \lesssim e^{\tau C_3}\Big(||P_\sigma u||_{L^2(W^+)} + |(\partial_{x_n} - \delta S_\sigma)u_{|_{x_n=0}}|_{L^2(W^0)} + |u_{|_{x_n=0}}|_{L^2(W^0)}\Big) + e^{\tau C_1}\left(||u||_{H^1(W^+)} + |u_{|_{x_n=0}}|_{H^1(W^0)}\right).$$

with $C_3 > \varphi(0, r_0)$ and $C_1 = \varphi(0, r_1)$. We can restrict the left hand side to $\tilde{W}$, an open subset compactly embedded in $W^+ \cap \{x \in \mathbb{R}^n,\ 0 < \mathrm{d}_\beta(x_{0,x}) < r_1\}$, to obtain

$$||u||_{H^1(\tilde{W})} \leq e^{\tau(C_3 - C_2)}\left(||P_\sigma u||_{L^2(W^+)} + |(\partial_{x_n} - \delta S_\sigma)u_{|_{x_n=0}}|_{L^2(\omega_B)} + |u_{|_{x_n=0}}|_{L^2(w_B)}\right) + e^{-\tau(C_2 - C_1)}\left(||u||_{H^1(W^+)} + |u_{|_{x_n=0}}|_{H^1(\omega_B)}\right).$$

where $C_2 = e^{\lambda \mathrm{d}_\beta(x_0, \tilde{W})}$. Observe that $C_1 < C_2 < C_3$. Coming back to the original coordinates in the neighborhood of y_0, and optimizing this estimate using Lemma 4.8.4, we obtain that there exist $1 \geq \mu > 0$ and $C > 0$ such that

$$||u||_{L^2(O)} \leq C e^{C|\sigma|}(||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)})^{1-\mu} \times \left(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right)^\mu,$$

where O is an open subset compactly embedded in Ω. Then, we apply Theorem 4.8.1, and we find

$$||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)} \lesssim e^{C|\sigma|}\Big(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + e^{C'|\sigma|}\left(||u||_{H^1(\Omega)} + |u_{|\partial\Omega}|_{H^1(\partial\Omega)}\right)^{1-\mu}\left(||P_\sigma u||_{L^2(\Omega)} + |\partial_\nu u_{|\partial\Omega} + \delta S_\sigma u_{|\partial\Omega}|_{L^2(\partial\Omega)} + |u_{|\partial\Omega}|_{L^2(\omega_B)}\right)^\mu\Big). \quad (4.8.11)$$

This, and assumption (4.8.10) ends the proof. □

4.8.2 A resolvent estimate

The boundary static case

We use the notation of Section 6.1.1. We recall that A_δ is the operator defined by:

$$A_\delta := \begin{pmatrix} 0 & -\mathrm{Id} \\ -\Delta_g & a(x) \end{pmatrix}$$

where a is a function satisfying $a \geq C_I > 0$ on an open subset ω_I of Ω, of domain

$$D(A_\delta) := \{(u_0, u_1) \mid u_0 \in H^2(\Omega),\ u_{0_{|\partial\Omega}} \in H^2(\partial\Omega),\ u_1 \in \mathcal{V}_\delta(\partial\Omega),\ \partial_\nu u_0 + \delta\Sigma u_0 + b u_1 = 0\},$$

where b is a bounded function satisfying $b \geq C_B > 0$ on an open subset ω_B of $\partial\Omega$. In fact, we can allow a or b be identically zero, that is $\omega_I = \emptyset$ of $\omega_B = \emptyset$. We shall prove the first part of Theorem 6.1.15. Let $U = (u_0, u_1) \in D(A_\delta)$ be such that $(i\sigma\,\mathrm{Id} + A_\delta)U = F := (f_0, f_1) \in \mathcal{H}_\delta$. This gives

$$i\sigma u_0 - u_1 = f_0, \quad (i\sigma + a(x))u_1 - \Delta_g u_0 = f_1,$$

which is equivalent to

$$u_1 = -f_0 + i\sigma u_0, \quad -\Delta_g u_0 - \sigma^2 u_0 + i\sigma a(x)u_0 = f_1 + a(x)f_0 + i\sigma f_0. \tag{4.8.12}$$

This yields that u_0 satisfies

$$\begin{cases} (-\Delta_g - \sigma^2)u_0 + i\sigma a(x)u_0 = \tilde{f} & \text{in } \Omega \\ \partial_\nu u_0 + \delta\Sigma u_0 + i\sigma b(x)u_0 = \tilde{g} & \text{in } \partial\Omega \end{cases} \tag{4.8.13}$$

where $\tilde{f} = f_1 + a(x)f_0 + i\sigma f_0$ and $\tilde{g} = bf_{0|\partial\Omega}$. Observe that in this case, in the definition of S_σ (see the beginning of Section 4.1.3), κ is equal to 0. We multiply the first equation of (4.8.13) by $\overline{u_0}$ and integrate over Ω, and this yields

$$\begin{aligned} ||\nabla_g u_0||^2_{L^2(\Omega)} + \delta\,\langle \Sigma u_0, u_0\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)} - \sigma^2||u_0||^2_{L^2(\Omega)} + i\sigma\Big(au_0, u_0\Big)_{L^2(\Omega)} \\ + i\sigma\Big(bu_{0|\partial\Omega}, u_{0|\partial\Omega}\Big)_{L^2(\partial\Omega)} = \Big(\tilde{f}, u_0\Big)_{L^2(\Omega)} + \Big(\tilde{g}, u_{0|\partial\Omega}\Big)_{L^2(\partial\Omega)}. \end{aligned} \tag{4.8.14}$$

Taking the imaginary part, we obtain

$$\begin{aligned} C_I|\sigma|\,||u_0||^2_{L^2(\omega_I)} + C_B|\sigma|\,|u_{0|\partial\Omega}|^2_{L^2(\partial\Omega)} \leq & |\sigma|\Big(au_0, u_0\Big)_{L^2(\Omega)} + |\sigma|\Big(bu_{0|\partial\Omega}, u_{0|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\ = & \left|\mathrm{Im}\,\Big(\tilde{f}, u_0\Big)_{L^2(\Omega)}\right| + \left|\mathrm{Im}\,\Big(\tilde{g}, u_0\Big)_{L^2(\partial\Omega)}\right|. \end{aligned} \tag{4.8.15}$$

Now we apply Theorems 4.8.1 and 4.8.5, (since $i\sigma a$ and $i\sigma b$ are low order terms (see Remark 4.1.6), the Carleman estimates of Theorem 4.1.5 holds with P_σ and S_σ replaced by $-\Delta_g - \sigma^2 + i\sigma a$ and $\delta\Sigma + i\sigma b$, and Theorems 4.8.1 and 4.8.5 applies for u_0), and there exists $C > 0$ such that

$$||u_0||_{\mathcal{V}_\delta(\Omega)} \leq Ce^{C|\sigma|}\left(||\tilde{f}||_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)} + ||u_0||_{L^2(\omega_I)} + |u_{0|\partial\Omega}|_{L^2(\omega_B)}\right) \tag{4.8.16}$$

for all $|\sigma| \geq 1$. Using (4.8.15) and (4.8.16), we have for $|\sigma| \geq 1$,

$$(C_I + C_B)||u_0||^2_{\mathcal{V}_\delta} \lesssim e^{C|\sigma|}\left(||\tilde{f}||^2_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)} + \left|\mathrm{Im}\,\Big(\tilde{f}, u_0\Big)_{L^2(\Omega)}\right| + \left|\mathrm{Im}\,\Big(\tilde{g}, u_{0|\partial\Omega}\Big)_{L^2(\partial\Omega)}\right|\right).$$

The young inequality yields

$$\begin{aligned} \left|\Big(\tilde{g}, u_{0|\partial\Omega}\Big)_{L^2(\partial\Omega)}\right| + \left|\Big(\tilde{f}, u_0\Big)_{L^2(\Omega)}\right| \lesssim & \varepsilon^{-1}(C_I + C_B)^{-1}e^{C|\sigma|}(||\tilde{f}||^2_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)}) \\ & + \varepsilon(C_I + C_B)e^{-C|\sigma|}(||u_0||^2_{L^2(\Omega)} + |u_{0|\partial\Omega}|^2_{L^2(\partial\Omega)}), \end{aligned}$$

for all $\delta' > 0$. Using the trace theorem $|u_{0|\partial\Omega}|_{L^2(\partial\Omega)} \lesssim ||u_0||_{H^1}$, we obtain for $\varepsilon > 0$ sufficiently small,

$$||u_0||_{\mathcal{V}_\delta(\Omega)} \lesssim e^{C|\sigma|}\left(||\tilde{f}||_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)}\right),$$

for $|\sigma| \geq 1$. Using now the definition of $\tilde{f}$ and $\tilde{g}$ below (4.8.12), we have

$$\begin{aligned} e^{C|\sigma|}\left(||\tilde{f}||_{L^2(\Omega)} + |\tilde{g}|^2_{L^2(\partial\Omega)}\right) & \lesssim e^{C|\sigma|}\left(||f_1||_{L^2(\Omega)} + (1 + |\sigma|)||f_0||_{L^2(\Omega)} + |f_{0|\partial\Omega}|_{L^2(\partial\Omega)}\right) \\ & \lesssim e^{C'|\sigma|}||(f_0, f_1)||_{\mathcal{H}_\delta}, \end{aligned}$$

and this ends the proof of (6.1.23) in Theorem 6.1.15.

The boundary dynamic case

We now treat the dynamic boundary case, i.e we prove a resolvent estimate for the operator

$$B_\delta := \begin{pmatrix} 0 & -\mathrm{Id} & 0 & 0 \\ -\Delta_g & a & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \\ \frac{1}{\delta}\gamma_1 & 0 & \Sigma & \frac{1}{\delta}b \end{pmatrix},$$

defined precisely in Section 4.1.2. Note that a and b are as above, that is, one of them can be identically zero. We shall see that everything we did in the previous case can be done, i.e prove and use the same Carleman estimate. Indeed, let us consider $U := (u_0, u_1, y_0, y_1) \in D(B_\delta)$ and $F := (f_0, f_1, g_0, g_1) \in \mathcal{K}_\delta$, such that

$$(i\sigma\,\mathrm{Id} + B_\delta)U = F$$

This equality is equivalent to the following system

$$\left\{ \begin{array}{rcl} u_1 &=& i\sigma u_0 - f_0 \\ -\Delta_g u_0 - \sigma^2 u_0 + i\sigma a u_0 &=& \tilde{f} \\ y_1 &=& i\sigma y_0 - g_0 \\ \gamma_1(u_0) + \delta\Sigma y_0 - \delta\sigma^2 y_0 + i\sigma b y_0 &=& \tilde{g}, \end{array} \right. \tag{4.8.17}$$

where $\tilde{f} = f_1 + (i\sigma + a)f_0$ and $\tilde{g} = \delta g_1 + (\delta i\sigma + b)g_0$. As above, we can multiply the second line by $\overline{u_0}$, integrate by parts, use the transmission condition $u_{0|\partial\Omega} = y_0$, and take the imaginary part to obtain

$$\sigma\Big(au_0, u_0\Big)_{L^2(\Omega)} + \sigma\Big(bu_{0|\partial\Omega}, u_{0|\partial\Omega}\Big)_{L^2(\Omega)} = \mathrm{Im}\left(\tilde{f}, u_0\right)_{L^2(\Omega)} + \mathrm{Im}\left(\tilde{g}, u_{0|\partial\Omega}\right)_{L^2(\partial\Omega)}.$$

It is then sufficient to derive a local Carleman estimate for the solution u_0 of the following problem

$$\left\{ \begin{array}{l} (-\Delta_g - \sigma^2)u_0 = \tilde{f} \\ \partial_\nu u_0 + \delta(S - \sigma^2)u_0 = \tilde{g}, \end{array} \right.$$

which corresponds to Theorem 4.1.5 in the case $\kappa = 1$. This allows us to repeat what is done above, and obtain

$$||u_0||_{\mathcal{V}_\delta} \lesssim e^{C|\sigma|}\left(||\tilde{f}||_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)}\right) \tag{4.8.18}$$

Using the definitions of $\tilde{f}$ and $\tilde{g}$, and the fact that $\delta \geq 1$, we have

$$||\tilde{f}||_{L^2(\Omega)} + |\tilde{g}|_{L^2(\partial\Omega)} \lesssim ||f_0||_{H^1(\Omega)} + ||f_1||_{L^2(\Omega)} + \delta^{1/2}\left(|g_0|_{H^1(\partial\Omega)} + |g_1|_{L^2(\Omega)}\right) \lesssim ||(f_0, f_1, g_0, g_1)||_{\mathcal{K}_\delta}, \tag{4.8.19}$$

as $g_0 = f_{0|\partial\Omega}$. From (4.8.17), we obtain

$$\begin{aligned} &||u_0||_{H^1(\Omega)} + ||u_1||_{L^2(\Omega)} + \delta^{1/2}\left(|y_0|_{H^1(\partial\Omega)} + |y_1|_{L^2(\partial\Omega)}\right) \\ &\lesssim (1 + |\sigma|)||u_0||_{H^1(\Omega)} + \ ||f_0||_{L^2(\Omega)} + \delta^{1/2}\left((1 + |\sigma|)|u_{0|\partial\Omega}|_{H^1(\partial\Omega)} + \ |g_0|_{L^2(\partial\Omega)}\right) \\ &\lesssim (1 + |\sigma|)||u_0||_{\mathcal{V}_\delta} + ||f_0||_{L^2(\Omega)} + \delta^{1/2}|g_0|_{L^2(\partial\Omega)}. \end{aligned}$$

This, associated with (4.8.18) and (4.8.19) ends the proof of Theorem 6.1.15. □

Chapitre 5

Une inégalité de Carleman au voisinage d'une multi-interface et applications au contrôle

Abstract. We prove a Carleman estimate in a neighborhood of a multi-interface, that is, near a point where n manifolds intersect, under compatibility assumptions between the Carleman weight $e^{\tau\varphi}$, the operators at the multi-interface, and the elliptic operators in the interior and the usual sub-ellipticity condition. The compatibility condition at the multi-interface is a version of the known covering condition for systems in the literature. This Carleman estimate is a generalization of the results obtained in [BL15, BL16]. Applications in control theory for second-order transmission problems are also considered, and this provides a generalization of the known results for one-dimensional networks of strings [BCV11, DZ06, LLS94, Zua13] to higher dimensions.

5.1 Introduction

We consider n smooth $d-$dimensional manifolds Ω_k, $d \geq 2$, $i \in \{1, \dots, n\}$ with boundary $\partial\Omega_k$. We assume that they share parts of their boundary. More precisely, we denote by $\partial\Omega_k^i$, $i = 0, \dots, n_k$, the different connected components of $\partial\Omega_k$. We assume that one of these connected components is shared by all manifolds. For instance, we assume that we have

$$\partial\Omega_1^0 = \cdots = \partial\Omega_n^0.$$

We set $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$, and we call $\mathcal{I}$ the common connected component of their boundaries. Note that Ω is not a manifold because of its topology near $\mathcal{I}$. We shall refer to $\mathcal{I}$ as an interface between the manifolds. It is itself a $d-1$ dimensional manifold. An example of such configuration is given in Figure 5.1. On each Ω_k, we consider an elliptic differential operator P_k of order $2m_k$. In local coordinates $(x_1, \dots, x_d)$ for Ω_k near a point $x^0 \in \mathcal{I}$, where Ω_k is given by $\{x_d \geq 0\}$, the operator P_k reads

$$P_k(x, D) = \sum_{|\alpha| \leq 2m_k} a_\alpha^k(x) D^\alpha,$$

where the coefficients $a_\alpha^k(x)$ are C^∞ complex valued functions, $D_j = -i\partial_{x_j}$ and $D = (D_1, \dots, D_d)$, and $\alpha \in \mathbb{N}^d$ is a multi-index. With their principal symbols given by

$$p_k(x, \xi) = \sum_{|\alpha| = 2m_k} a_\alpha^k(x) \xi^\alpha, \ \xi \in \mathbb{R}^d, x \in \Omega_k, \tag{5.1.1}$$

the ellipticity assumption reads $|p_k(x, \xi)| \geq C|\xi|^{2m_k}$, with $C > 0$. We set $m = m_1 + \cdots + m_n$. For each manifold Ω_k, we also consider a set of boundary operators T_k^ℓ, $l = 1, \dots, m$, of order less than or equal to

$\beta_k^\ell < 2m_k$, that take the following form in local coordinates

$$T_k^\ell(x,D) = \sum_{|\alpha|\le\beta_k^\ell} t_{\alpha,k}^\ell(x) D^\alpha, \quad \ell \in \{1,\dots,m\},\ k \in \{1,\dots,n\},$$

where the coefficients $t_{\alpha,k}^\ell(x)$ are complex-valued functions defined in a neighborhood of $\mathcal{I}$ in Ω_k. We denote by

$$t_k^\ell(x,\xi) = \sum_{|\alpha|=\beta_k^\ell} t_{\alpha,k}^\ell \xi^\alpha,$$

the principal symbol of the operator T_k^ℓ, for $k \in \{1,\dots,n\}$ and $\ell \in \{1,\dots,m\}$. Note that we may have $t_k^\ell = 0$ if the actual order of T_ℓ^j is less than β_k^ℓ. We assume that the orders satisfy the relations

$$\gamma_\ell := 2m_k - \beta_k^\ell = 2m_j - \beta_j^\ell, \quad \forall k,j \in \{1,\dots,n\}, \ell \in \{1,\dots,m\}. \tag{5.1.2}$$

We shall consider the following elliptic system, coupled through the interface,

$$\begin{cases} P_k u_k = f_k & \text{in } \Omega_k, \quad k \in \{1,\dots,n\}, \\ \displaystyle\sum_{k=1}^n T_k^\ell(x,D) u_{k|_{\mathcal{I}}} = g_\ell & \text{on } \mathcal{I}, \quad \ell \in \{1,\dots,m\}. \end{cases} \tag{5.1.3}$$

The main purpose of this article is the proof of a Carleman estimate for solutions of such system in a neighborhood of a point x^0 of the interface $\mathcal{I}$.
In [Car39], Carleman introduced weighted inequalities to prove uniqueness of the Cauchy problem associated with an elliptic operator. The method has been developped later by Hörmander [Hör58]. Carleman estimates for general elliptic operators has been derived away from boundaries in [Hör76]. Recently, for boundaries and interfaces problems, Carleman estimates has been obtained in [BL15, BL16, Tat96], under Lopatinskii-type assumptions.. The purpose of this article is to generalize these results to systems such as (5.1.3).
The Carleman estimate we prove, in a neighborhood of a point of the mult-interface $\mathcal{I}$, has the following form

$$\tau^{-1}\sum_{k=1}^n ||e^{\tau\varphi_k} u_k|| + \sum_{k=1}^n |e^{\tau\varphi_{|\mathcal{I}}} \operatorname{tr} u_k|^2 \le C\Big(\sum_{k=1}^n ||e^{\tau\varphi_k} P_k u_k|| + \sum_{\ell=1}^m \Big|\sum_{k=1}^n T_k^\ell u_{k|_{\mathcal{I}}}\Big|^2\Big).$$

where φ_k is a weight function on Ω_k, τ is the usual Carleman large parameter, and $\operatorname{tr} u_k$ denotes the successive normal derivatives of u_k at the multi-interface $\mathcal{I}$, that is $\operatorname{tr} u_k = (u_{k|_{\mathcal{I}}}, \partial_\nu u_{k|_{\mathcal{I}}}, \dots, \partial_\nu^{2m_k-1} u_{k|_{\mathcal{I}}})$.
As usual with Carleman estimates, in the interior, some compatibilty conditions between the operators P_k and the weight φ_k are introduced, the so-called *sub-ellipticity* condition (see Section 5.2.1). At the multi-interface, we also impose some compatibility conditions between the weights φ_k, the boundary operators T_k^ℓ and the interior operators P_k. These conditions are a natural generalization of the well-known Lopatinskii condition (see Section 5.2.2) .
Carleman estimates are a powerful tool to prove control and stabilization properties. In the last section of this article, we provide an application of the Carleman estimate we obtain to a natural transmission problem. Under assumption on the control region, we prove the null-controllability result for the associated parabolic problem, and a stabilization result for the associated hyperbolic one. This generalizes what has been done for one-dimensional networks of strings in [BCV11, DZ06, LLS94, Zua13], and the references therein, to higher dimensions.

5.2 Statement of the problem and geometrical configurations

Let $x^0 \in \mathcal{I}$ be a point of the interface. In each manifold Ω_k, we consider an open bounded neighborhood $\tilde{U}_k$ of x^0 in Ω_k where local coordinates as above can be used. More precisely, for $k = 1,\dots,n$, we use local charts:

$$\phi_k : \Omega_k \supset \tilde{U}_k \longrightarrow V_k^+ = V_k \cap \{x_d \ge 0\}, \quad \phi_k \text{ diffeomorphism, for all } k \in \{1,\dots,n\}, \tag{5.2.1}$$

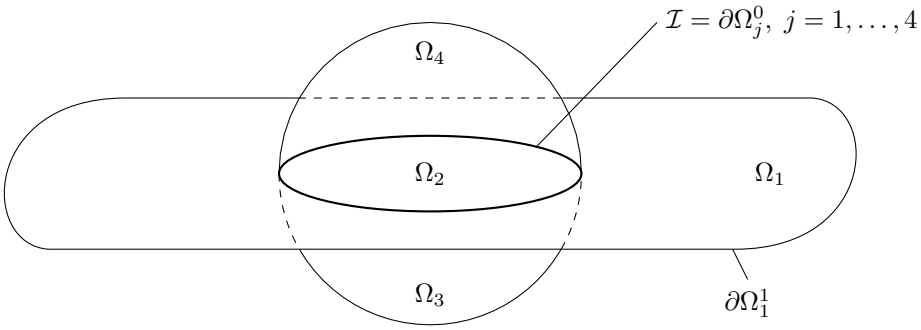


Figure 5.1: A sphere meets a surface and creates 4 smooth manifolds with boundaries. The shared interface is a connected component of $\partial\Omega_i$, for all $i \in \{1, \dots, n\}$.

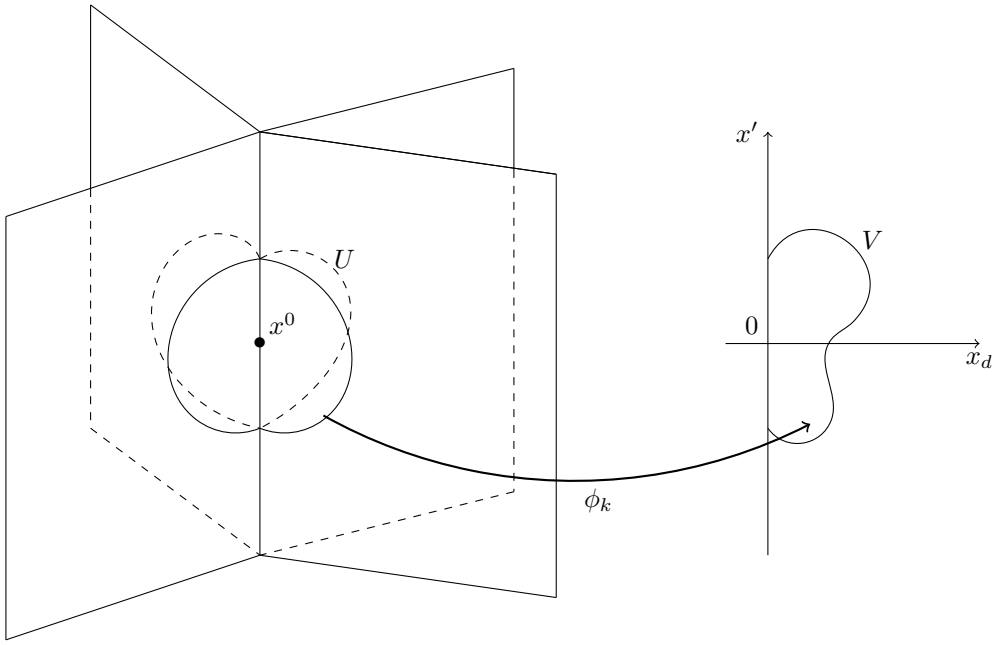


Figure 5.2: Local configuration and local charts in the neighborhood of the interface.

with V_k an open set of $\mathbb{R}^d$, satisfying $\phi_k(x^0) = 0$ and $\phi_k(\mathcal{I} \cap \tilde{U}_k) = \{x_d = 0\} \cap V_k$ for all $k \in \{1, \dots, n\}$. Setting $V = \cap_{k=1}^n V_k$ on $\mathbb{R}^d$ and $V^+ = \{x_d \geq 0\} \cap V$, we consider $U_k = \varphi_k^{-1}(V^+)$, and $U = \cup_{k=1}^n U_k$ in Ω so that we have

$$\phi_k : \Omega_k \supset U_k \longrightarrow V^+, \quad \phi_k \text{ diffeomorphism, for all } k \in \{1, \dots, n\},$$

and $\phi_{k|_{\mathcal{I}}} = \phi_{\ell|_{\mathcal{I}}}$, as sketched in Figure 5.2.

Throughout this article, two assumptions shall play a central role. The first one is the sub-ellipticity assumption, that is classical when working with Carleman estimates, and the second one is the covering condition (also known as the Shapiro-Lopatinskii condition, or complementing condition).

5.2.1 The sub-ellipticity condition

For two functions $f(x, \xi)$ and $g(x, \xi)$ in $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we recall that their Poisson bracket is given by

$$\{f, g\}(x, \xi) = \partial_\xi f(x, \xi) \partial_x g(x, \xi) - \partial_x f(x, \xi) \partial_\xi g(x, \xi).$$

From symbolic calculus, $\{f,g\}$ is the principal symbol of the commutator $i[\mathrm{Op}(f),\mathrm{Op}(g)]$. For $k\in\{1,\dots,n\}$, we let φ_k be a real C^∞ function on $\overline{\Omega}_k$, and we consider the conjugated operators

$$P_{k,\varphi_k}=e^{\tau\varphi_k}P_k e^{-\tau\varphi_k}, \tag{5.2.2}$$

with principal symbol $p_{k,\varphi}(x,\xi,\tau)=p_k(x,\xi+i\tau d\varphi_k(x))$ in a semi-classical sense. In general, P_{k,φ_k} is not elliptic. The sub-ellipticity condition states that $[P^*_{k,\varphi_k},P_{k,\varphi_k}]$ is elliptic positive where P_{k,φ_k} fails to be elliptic. More precisely, we write the following definition.

Definition 5.2.1. *Let $k\in\{1,\dots,n\}$. We say that (P_k,φ_k) satisfies the sub-ellipticity condition on $\overline{U}_k$ if $d\varphi_k\neq 0$ on $\overline{U}_k$ and if*

$$p_{k,\varphi}(x,\xi,\tau)=0\Longrightarrow\Big\{\operatorname{Re}p_{k,\varphi},\operatorname{Im}p_{k,\varphi}\Big\}(x,\xi,\tau)=\frac{1}{2i}\Big\{\overline{p}_{k,\varphi},p_{k,\varphi}\Big\}(x,\xi,\tau)>0, \tag{5.2.3}$$

for all $x\in\overline{U}_k,\tau>0$, and all $\xi\in\mathbb{R}^d$, $\xi\neq 0$.

For the case of elliptic operators of order two, it is well-known that the following lemma gives a construction of such a weight function.

Lemma 5.2.2. *Let U be an open subset of $\mathbb{R}^d$, M be an elliptic operator of order two, and ψ be a smooth function on $\overline{U}$ such that $\nabla\psi\neq 0$ on $\overline{U}$. We set $\varphi_\lambda=e^{\lambda\psi}$, for $\lambda>0$. Then, there exists $\lambda_0>0$ such that (M,φ_λ) satisfies the sub-ellipticity condition of Definition 5.2.1, for all $\lambda\geq\lambda_0$.*

The proof of this lemma can be found for instance in [Hör63]. Note that there exist operators such that the sub-ellipticity condition cannot be achieved, independently of the choice of φ. This is for instance the case for the Bi-Laplace operator. Indeed, in this case, we have $p_{k,\varphi}=q^2_{k,\varphi}$, where q_k is the principal symbol of the Laplace operator. The Poisson bracket is then given by

$$\frac{1}{2i}\Big\{\overline{p}_{k,\varphi},p_{k,\varphi}\Big\}=\frac{1}{2i}\left(\partial_\xi\overline{p}_{k,\varphi}\partial_x p_{k,\varphi}-\partial_x\overline{p}_{k,\varphi}\partial_\xi p_{k,\varphi}\right)=\frac{2|q_{k,\varphi}|^2}{i}\left(\partial_\xi\overline{q}_{k,\varphi}\partial_x q_{k,\varphi}-\partial_x\overline{q}_{k,\varphi}\partial_\xi q_{k,\varphi}\right),$$

and we see that this cannot be positive on the characteristic set of $p_{k,\varphi}$.
In addition to the sub-ellipticity condition of the couples (P_k,φ_k), when considering systems as (5.1.3) with general operators and boundary conditions, we have to impose some compatibility properties between the operators in the interior and at the boundary, even for well-posedness properties. For Carleman estimates, this is also the case, for the conjugated operator.

5.2.2 The covering condition

For $x\in\mathcal{I}$, we shall denote by $T_x(\mathcal{I})$ and $T^*_x(\mathcal{I})$ respectively the tangent and cotangent spaces to $\mathcal{I}$ above x, and $T^*(\mathcal{I})=\{(x,\nu),\ x\in\mathcal{I},\ \nu\in T^*_x(\mathcal{I})\}$. We moreover define the conormal spaces, for $x\in\mathcal{I}$,

$$N^*_{x,k}(\mathcal{I})=\{\nu\in T^*_x(\Omega_k),\ \nu(Y)=0,\ \forall Y\in T_x(\mathcal{I})\},$$

and

$$N^*_k(\mathcal{I})=\{(x,\nu),\ x\in\mathcal{I},\ \nu\in N^*_{x,k}(\mathcal{I})\}.$$

For all $k\in\{1,\dots,n\}$, we define a boundary quadruple $\rho_k=(x,Y,\nu_k,\tau)$, where

$$x\in\mathcal{I}\cap\overline{U},\ Y\in T^*_x(\mathcal{I}),\ \nu_k\in N^*_{x,k}(\mathcal{I}),\ \text{and }\tau>0. \tag{5.2.4}$$

Moreover, we impose ν_k to be an inward pointing conormal vector, that is, $\nu=(0,\dots,0,t)$ with $t>0$ in the local chart (U_k,ϕ_k) given in (5.2.1). We shall denote by $\mathcal{L}^{*,+}_k(\mathcal{I})$ the set of such ρ_k. We define for a fixed $x\in\mathcal{I}$,

$$\mathcal{L}^*_x(\mathcal{I}):=\{(x,Y,\nu_1,\dots,\nu_n,\tau),\ \forall k\in\{1,\dots,n\},\ \rho_k=(x,Y,\nu_k,\tau)\in\mathcal{L}^{*,+}_k(\mathcal{I})\}, \tag{5.2.5}$$

and we also consider

$$\mathcal{L}^*(\mathcal{I}) := \{(x, Y, \nu_1, \dots, \nu_n, \tau),\ (Y, \nu_1, \dots, \nu_n, \tau) \in \mathcal{L}^*_x(\mathcal{I}),\ x \in \mathcal{I}\}. \tag{5.2.6}$$

We introduce a weight function φ defined on U, which is smooth in each Ω_k and continuous across the interface $\mathcal{I}$. We shall denote $\varphi_k = \varphi_{|\Omega_k}$. For $\rho_k \in \mathcal{L}^{*,+}_k(\mathcal{I})$ we introduce, for all $\lambda \in \mathbb{C}$,

$$\tilde{p}_{k,\varphi}(\rho_k, \lambda) = p_k(x, Y + \lambda\nu_k + i\tau d\varphi_k(x)).$$

Considering the following factorization, as a polynomial in $\lambda \in \mathbb{C}$,

$$\tilde{p}_{k,\varphi}(\rho_k, \lambda) = c_k \tilde{p}^-_{k,\varphi}(\rho_k, \lambda)\tilde{p}^0_{k,\varphi}(\rho_k, \lambda)\tilde{p}^+_{k,\varphi}(\rho_k, \lambda), \tag{5.2.7}$$

with $\tilde{p}^0_{k,\varphi}(\rho_k, \lambda) = \Pi_{\mathrm{Im}\,\sigma^j_k(\rho_k)=0}(\lambda - \sigma^j_k(\rho_k))^{\mu^j_k}$ and $\tilde{p}^\pm_{k,\varphi}(\rho_k, \lambda) = \Pi_{\pm\,\mathrm{Im}\,\sigma^j_k(\rho_k)>0}(\lambda - \sigma^j_k(\rho_k))^{\mu^j_k}$, where $(\sigma^j_k)_j$ are the complex roots of $\tilde{p}_{k,\varphi}$ as a polynomial in λ, with multiplicity μ^j_k. It should be kept in mind that this factorization depends heavily on the choice of ρ_k. We set

$$\tilde{\kappa}_{k,\varphi}(\rho_k, \lambda) := \tilde{p}^+_{k,\varphi}(\rho_k, \lambda)\tilde{p}^0(\rho_k, \lambda).$$

We also define at the boundary, for all λ,

$$t^\ell_{k,\varphi}(\rho_k, \lambda) = t^\ell_k(x, Y + \lambda\nu_k + i\tau d\varphi_k(x)).$$

Definition 5.2.3. *We say that $(T^\ell_k, \varphi)_{k\in\{1,\dots,n\},\ell\in\{1,\dots,m\}}$ covers $(P_k, \varphi)_{k\in\{1,\dots,n\}}$ at $\rho^0 \in \mathcal{L}^*(\mathcal{I})$ if for all polynomials $f_k \in \mathbb{C}[\lambda]$ there exists $c_1, \dots, c_m \in \mathbb{C}$ and polynomials $q_k \in \mathbb{C}[\lambda]$ such that, for all $k \in \{1, \dots, n\}$,*

$$f_k(\lambda) = \sum_{l=1}^m c_\ell t^\ell_{k,\varphi}(\rho^0_k, \lambda) + q_k(\lambda)\kappa_{k,\varphi}(\rho^0_k, \lambda).$$

We shall say that $(T^\ell_k, \varphi)_{k,\ell}$ covers $(P_k, \varphi)_k$ at x if this holds for all $\rho^0 \in \mathcal{L}^*_x(\mathcal{I})$, and we shall say that $(T^\ell_k, \varphi)_{k,\ell}$ covers $(P_k, \varphi)_k$ if this holds for all $\rho^0 \in \mathcal{L}^*(\mathcal{I})$

Observe that in the above definition, we can restrict the degree of the polynomials f_k to be less than or equal to $2m_k - 1$, using the Euclidean division. If we write $m^+_k = \mathrm{d}^\circ\kappa_{k,\varphi}$ and $m^-_k = 2m_k - m^+_k$, then $\mathrm{d}^\circ q_k$ is at most equal to $2m_k - m^+_k - 1 = m^-_k - 1$. We can reformulate the above definition as follows. Introducing, for $k \in \{1, \dots, n\}$

$$e^\ell_k(\rho^0_k, \lambda) = \begin{cases} t^\ell_{k,\varphi}(\rho^0_k, \lambda) & \text{if } \ell \in \{1, \dots, m\} \\ \lambda^{\ell-(m+1)}\kappa_{k,\varphi}(\rho^0_k, \lambda) & \text{if } \ell \in \{m+1, \dots, m+m^-_k\}, \end{cases} \tag{5.2.8}$$

and the linear map

$$\Phi : \mathbb{C}^m \times \mathbb{C}^{m^-_1} \times \cdots \times \mathbb{C}^{m^-_n} \to \mathbb{C}_{2m_1-1}[\lambda] \times \cdots \times \mathbb{C}_{2m_n-1}[\lambda]$$
$$\left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ \vdots \\ q_{1,m^-_k} \end{pmatrix}, \dots, \begin{pmatrix} q_{n,1} \\ \vdots \\ q_{n,m^-_n} \end{pmatrix}\right) \mapsto \left(\sum_{\ell=1}^m c_\ell e^\ell_1 + \sum_{\ell=m+1}^{m+m^-_1} q_{1,\ell-m}e^\ell_1, \dots, \sum_{\ell=1}^m c_\ell e^\ell_n + \sum_{\ell=m+1}^{m+m^-_n} q_{n,\ell-m}e^\ell_n\right),$$

Definition 5.2.3 is precisely equivalent to the surjectivity of the above map Φ.

Remark 5.2.4. *Like the sub-ellipticity condition, the covering condition is unvariant under a change of coordinates. In particular, this definition does not depend on the sizes of the conormal vectors ν_k.*

5.2.3 On the well-posedness of systems satisfying the covering condition

The well-posedness of elliptic systems satisfying the covering condition is a known fact for a non-conjugated system (that is, in the case $\tau\varphi = 0$). We refer for instance to [ADN59, ADN64]. In their setting, they consider a more general system. It can be related to our problem as explained as follows. Near a point of $\mathcal{I}$, we can choose a coordinates system for each Ω_k and thus the problem can be reformulated in $\mathbb{R}^d_+$, as in Section5.2. This choice of coordinates has no impact on the covering condition as it is invariant under a change of coordinates. In [ADN59, ADN64], the authors consider systems of mixed-order elliptic operators in the half-space, that are said to be properly elliptic, i.e the roots σ_i of $\det(A(x, \Theta_1 + z\Theta_2))$, viewed as a polynomial in $z \in \mathbb{C}$ satsifies

$$\#\{\mathrm{Im}(\sigma_i) > 0\} = \#\{\mathrm{Im}\,\sigma_i < 0\}, \quad \text{for } \Theta_1,\ \Theta_2 \in \mathbb{R}^d, \text{ linearly independant.}$$

In our case, the considered matrix A is diagonal, thus the determinant is the product of the symbols $p_k(x, \Theta_1 + z\Theta_2)$. If the system is properly elliptic, we obtain that $m = m_1^+ + \cdots + m_n^+ = m_1^- + \cdots + m_n^-$. Hence,

$$\dim(\mathbb{C}^m \times \mathbb{C}^{m_1^-} \times \cdots \times \mathbb{C}^{m_n^-}) = \dim(\mathbb{C}_{2m_1-1}[\lambda] \times \cdots \times \mathbb{C}_{2m_n-1}[\lambda]),$$

and then Φ is onto if and only if Φ is one-to-one. The covering condition stated in [ADN59, ADN64] precisely implies that the map Φ is one-to-one, and the covering condition in the present work implies that the map Φ is onto. Thus, in the setting of properly elliptic diagonal system of operators, those two conditions are consistent. However, in the present article, we ask for the *conjugated* operator to satisfy the covering condition, and moreover, as we shall see, the conjugated system may not be elliptic anymore. We give the following definition.

Definition 5.2.5. *Let (T_k^ℓ, P_k) be a system of operators such as (5.1.3) satisfying*

$$(T_k^\ell, 0)_{k,\ell} \text{ covers } (P_k, 0)_k. \tag{5.2.9}$$

We say that the weight function φ preserves the covering condition for (T_k^ℓ, P_k) if we also have

$$(T_k^\ell, \varphi)_{k,\ell} \text{ covers } (P_k, \varphi)_k.$$

An interesting question then arises: given a system (T_k^ℓ, P_k) that satisfies (5.2.9), can we always find a weight function that preserve the covering condition for (T_k^ℓ, P_k) ?

5.2.4 Statement of the main result

The main result of this article is the following Carleman estimate in the neighborhood of the interface. We define the following space of smooth functions defined in a neighborhood of the point of interest at the interface.

$$\overline{C}_0^\infty(U_k) = \{w \in C^\infty(\Omega_k),\ \mathrm{supp}(w) \subset U_k \text{ is compact in } \Omega_k\}. \tag{5.2.10}$$

Theorem 5.2.6. *Let $x_0 \in \mathcal{I}$ and let φ be a weight function on U such that $\varphi \in C^0(U)$ and such that $\varphi_{|_{U_k}} \in C^\infty(U_k)$. In addition, assume that (P_k, φ_k) satisfies the sub-ellipticity condition of Definition 5.2.1 in a neighborhood of x_0 in Ω_k, for all $k \in \{1, \dots, n\}$, and assume that $(T_k^\ell, \varphi)_{k\in\{1,\dots,n\},\ell\in\{1,\dots,m\}}$ covers $(P_k, \varphi)_{k\in\{1,\dots,n\}}$ at $x_0 \in \mathcal{I}$ (see Definition 5.2.3). Then, there exists a neighborhood $\tilde{U}$ of x_0 in Ω, $C > 0$, and $\tau_0 > 0$ such that*

$$\tau^{-1}\sum_{k=1}^n ||e^{\tau\varphi_k} u_k||^2_{2m_k,\tau} + \sum_{k=1}^n |e^{\tau\varphi_{|\mathcal{I}}}\,\mathrm{tr}\, u_k|^2_{2m_k-1,1/2,\tau}$$
$$\leq C\left(\sum_{k=1}^n ||e^{\tau\varphi_k} P_k u_k||^2_{L^2(\Omega_k)} + \sum_{\ell=1}^m \Big|\sum_{k=1}^n T_k^\ell u_{k|\mathcal{I}}\Big|^2_{\gamma_\ell-1/2,\tau}\right), \tag{5.2.11}$$

for all $\tau \geq \tau_0$, for all $u \in C_0^\infty(\tilde{U})$ such that $u_k := u_{|_{U_k\cap\tilde{U}}} \in \overline{C}_0^\infty(U_k \cap \tilde{U})$.

This Carleman estimate is local in a neighborhood of a point of the interface. Local Carleman estimates can be generally patched together to obtain a global one. For such issue, we refer to Section 5.5.1. The various norms are presented in Section 5.4.1. Let us simply mention that

1. $||.||_{s,\tau}$ is equivalent to $\tau^s||.||_{L^2} + ||.||_{H^s}$ in some Ω_k;
2. $|.|_{s,\tau}$ is equivalent to $\tau^s|.|_{L^2(\mathcal{I})} + |.|_{H^s(\mathcal{I})}$. Norms on the interface are denoted by $|.|$ for easier reading;
3. on $\mathcal{I}$, for $r \in \mathbb{N}$ and $s \in \mathbb{R}$, we mean

$$|\operatorname{tr} w|_{r,s,\tau} = \sum_{\ell=0}^{r} |\partial_\nu^\ell w_{|\mathcal{I}}|_{r+s-\ell,\tau},$$

using the notation of 2).

5.2.5 Conditions in a neighborhood of the interface

Let $x_0 \in \mathcal{I}$. We use the local chart introduce in (5.2.1) and the neighborhood $V^+ = \{x_d \geq 0\} \cap V$ in $\mathbb{R}^d$. We shall often use the notation $x = (x', x_d)$ and $\xi = (\xi', \xi_d)$. We also introduce

$$\overline{C}_0^\infty(V^+) = \{u_{|\overline{V^+}},\ u \in C_0^\infty(V)\}, \tag{5.2.12}$$

the set of restrictions to V^+ of smooth and compactly supported functions in V. Note that Definition (5.2.12) is consistent with (5.2.10).

In this coordinate system, the operators P_k reads

$$P_k = \sum_{|\alpha| \leq 2m_k} a_\alpha^k(x) D^\alpha, \quad k = 1, \dots, n,$$

for x in V^+, and near the boundary $V \cap \{x_d = 0\}$, the boundary operators read

$$T_k^\ell = \sum_{|\alpha| \leq \beta_k^\ell} t_{\alpha_k}^\ell D^\alpha, \quad k = 1, \dots, n,\ \ell = 1, \dots, m,$$

with $\beta_k^\ell \in \{1, \dots, 2m_k - 1\}$ and $m = m_1 + \cdots + m_n$. We call $p_k(x, \xi)$ and $t_k^\ell(x, \xi)$ their principal symbols. We also introduce the symbols

$$p_{k,\varphi}(x, \xi, \tau) = p_k(x, \xi + i\tau \mathrm{d}\varphi_k), \quad t_{k,\varphi}^\ell(x, \xi, \tau) = t_k^\ell(x, \xi + i\tau \mathrm{d}\varphi_k),$$

which are respectively the principal symbols of the conjugated operators $P_{k,\varphi} := e^{\tau\varphi_k} P_k e^{-\tau\varphi_k}$ and $T_{k,\varphi}^\ell := e^{\tau\varphi_k} T_k^\ell e^{-\tau\varphi_k}$. In this coordinate system, we make the following identification

$$\mathcal{L}_k^{*,+}(\mathcal{I}) \cong \{\rho = (x, \xi, \tau) \in (\{x_d = 0\} \cap \overline{V}) \times \mathbb{R}_+^d \times \mathbb{R}^+\}.$$

We furthermore allow one to identify all the $\mathcal{L}_k^{*,+}(\mathcal{I})$ to one another. Moreover, as Definition 5.2.3 is coordinate invariant (see Remark 5.2.4) we can identify $\mathcal{L}^*(\mathcal{I})$ with $\mathcal{L}_k^{*,+}(\mathcal{I})$ for all $k \in \{1, \dots, n\}$. We also set the cotangent bundle over $\mathcal{I}$ with parameter τ as

$$\mathcal{L}^T(\mathcal{I}) \cong \{\rho' = (x, \xi', \tau) \in (\{x_d = 0\} \cap \overline{V}) \times \mathbb{R}^{d-1} \times \mathbb{R}^+\}.$$

For $\rho' \in \overline{V^+} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$, we consider the complex roots $\sigma_{k,1}(\rho'), \dots, \sigma_{k,\ell_k}(\rho')$ of multiplicity $\mu_{k,j}$ (satisfying $\mu_{k,1} + \cdots + \mu_{k,\ell_k} = 2m_k$) of the symbol $p_{k,\varphi}(\rho', \xi_d)$, viewed as a polynomial in ξ_d. Let $\rho_0' \in \overline{V^+} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ be fixed in what follows. From Lemma A.2 in [Bellassoued- Le Rousseau], there exists a conic neighborhood $\mathcal{V}$ of ρ_0' in $\overline{V^+} \times \mathbb{R}^d \times \mathbb{R}^+$ and three smooth and homogeneous polynomials $p_k^\pm(\rho', \xi_d)$, $p_k^0(\rho', \xi_d)$ in ξ_d, such that

1. each polynomial is of constant degree for $\rho' \in \mathcal{V}$;
2. we have the following factorization
$$p_{k,\varphi}(\rho', \xi_d) = a(\rho') p^+_{k,\varphi}(\rho', \xi_d) p^-_{k,\varphi}(\rho', \xi_d) p^0_{k,\varphi}(\rho', \xi_d), \quad \rho' \in \mathcal{V}, \tag{5.2.13}$$
where $a(\rho')$ is the leading coefficient;
3. the roots of $p^+_{k,\varphi}(\rho', \xi_d)$ and $p^-_{k,\varphi}(\rho', \xi_d)$ have positive and negative imaginary parts respectively for all $\rho' \in \mathcal{V}$;
4. at $\rho' = \rho'_0$ we have
$$p^\pm_{k,\varphi}(\rho'_0, \xi_d) = \prod_{\pm \operatorname{Im} \sigma_{k,j}(\rho'_0) > 0} (\xi_d - \sigma_{k,j}(\rho'_0))^{\mu_{k,j}}, \quad p^0_{k,\varphi}(\rho'_0, \xi_d) = \prod_{\operatorname{Im} \sigma_{k,j}(\rho'_0) = 0} (\xi_d - \sigma_{k,j}(\rho'_0))^{\mu_{k,j}}.$$

Note that the decomposition thus depends on ρ'_0. Note also that for $\rho' \in \mathcal{V}$, the sign of the imaginary part of the roots of $p^0_{k,\varphi}(\rho', \xi_d)$ is not prescribed. However, at $\rho' = \rho'_0$ those imaginary parts vanish.

We set $\kappa_{k,\varphi}(\rho', \xi_d) = p^+_{k,\varphi}(\rho', \xi_d) p^0_{k,\varphi}(\rho', \xi_d)$. In fact, while the roots $\sigma_{k,j}(\rho')$ are only continuous, since they can cross each other, we shall work with the polynomials given in (5.2.13); they are smooth, which permits the use of semi-classical calculus. We can reformulate the covering condition at ρ'_0 as follows: for all family of polynomials $(f_k)_{k\in\{1,\dots,n\}} \in \mathbb{C}[\xi_d]^n$, there exists $(c_1, \dots, c_m) \in \mathbb{C}^m$ and a family of polynomials $(q_k)_{k\in\{1,\dots,n\}} \in \mathbb{C}[\xi_d]^n$ such that

$$f_k(\xi_d) = \sum_{\ell=1}^{m} c_\ell t^\ell_{k,\varphi}(\rho'_0, \xi_d) + q_k(\xi_d) \kappa_{k,\varphi}(\rho'_0, \xi_d), \quad \forall k \in \{1, \dots, n\}.$$

In this setting, the covering condition can also be stated in a more convenient way, such as follows. Setting $m^-_k = \mathrm{d}^\circ p^-_{k,\varphi}$, we write

$$\kappa_{k,\varphi}(\rho', \xi_d) = \sum_{i=0}^{2m_k - m^-_k} \kappa^i_{k,\varphi}(\rho') \xi^i_d,$$

where $\kappa^i_{k,\varphi}$ is homogeneous of degree $2m_k - m^-_k - i$ in (ξ', τ). Similarly, we write

$$t^\ell_{k,\varphi}(\rho', \xi_d) = \sum_{i=0}^{\beta^\ell_k} t^{\ell,i}_{k,\varphi}(\rho') \xi^i_d, \tag{5.2.14}$$

where $t^{\ell,i}_{k,\varphi}$ is homogeneous of degree $\beta^\ell_k - i$ in (ξ', τ). Arguing as in (5.2.8), we introduce

$$e^\ell_k(\rho', \xi_d) = \begin{cases} t^\ell_{k,\varphi}(\rho', \xi_d) & \text{if } \ell \in \{1, \dots, m\} \\ \xi_d^{\ell-(m+1)} \kappa_{k,\varphi}(\rho', \xi_d) & \text{if } \ell \in \{m+1, \dots, m+m^-_k\}, \end{cases} \tag{5.2.15}$$

and the covering condition is equivalent to the surjectivity at $\rho' = \rho_0$ of the map

$$\Phi : \mathbb{C}^m \times \mathbb{C}^{m^-_1} \times \cdots \times \mathbb{C}^{m^-_n} \longrightarrow \mathbb{C}_{2m_1-1}[\xi_d] \times \cdots \times \mathbb{C}_{2m_n-1}[\xi_d]$$
$$\left(\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \begin{pmatrix} q_{1,1} \\ \vdots \\ q_{1,m^-_k} \end{pmatrix}, \dots, \begin{pmatrix} q_{n,1} \\ \vdots \\ q_{n,m^-_k} \end{pmatrix} \right) \longmapsto \left(\sum_{\ell=1}^{m} c_\ell e^\ell_1 + \sum_{\ell=m+1}^{m+m^-_1} q_{1,\ell-m} e^\ell_1, \dots, \sum_{\ell=1}^{m} c_\ell e^\ell_n + \sum_{\ell=m+1}^{m+m^-_n} q_{n,\ell-m} e^\ell_n \right).$$

If we write $e^\ell_k(\rho', \xi_d) = \sum_{i=0}^{2m-1} e^{\ell,i}_{k,\varphi}(\rho') \xi^i_d$, the coefficients $e^{\ell,i}_{k,\varphi}$ (that are homogeneous in (ξ', τ)) are given by

- if $\ell \in \{1,\dots,m\}$, $e^{\ell,i}_{k,\varphi}(\rho') = \begin{cases} t^{\ell,i}_{k,\varphi}(\rho') & \text{if } i \in \{0,\dots,\beta^\ell_k\} \\ 0 & \text{otherwise,} \end{cases}$
- if $\ell \in \{m+1,\dots,m+m^-_k\}$, $e^{\ell,i}_{k,\varphi}(\rho') = \begin{cases} \kappa^{i-\ell+m+1}_{k,\varphi}(\rho') & \text{if } i \in \ell-(m+1)+\{0,\dots,2m_k-m^-_k\} \\ 0 & \text{otherwise.} \end{cases}$

if $m^-_k > 0$. Then, for $k \in \{1,\dots,n\}$, we set the $2m_k \times m$ matrix

$$M^1_k(\rho') = \left(e^{\ell,i-1}_{k,\varphi}(\rho')\right)_{(i,\ell)\in\{1,\dots,2m_k\}\times\{1,\dots,m\}}.$$

If moreover $m^-_k > 0$, we set the $2m_k \times m^-_k$ matrix

$$M^2_k(\rho') = \left(e^{m+\ell,i-1}_{k,\varphi}(\rho')\right)_{(i,\ell)\in\{1,\dots,2m_k\}\times\{1,\dots,m^-_k\}}.$$

With M^2_k and M^1_k, we define the $2m \times (m + m^-_1 + \cdots + m^-_n)$ matrix

$$M(\rho') = \begin{pmatrix} M^1_1(\rho') & M^2_1(\rho') & 0 & \dots & 0 \\ M^1_2(\rho') & 0 & M^2_2(\rho') & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ M^1_n(\rho') & 0 & \dots & \dots & M^2_n(\rho') \end{pmatrix}, \tag{5.2.16}$$

where the column associated with $M^2_k(\rho')$ only occurs if $m^-_k > 0$. For convenience, we shall say that $M^2_k(\rho') = \emptyset$ if $m^-_k = 0$. With this matrix, the covering condition at $\rho' = \rho_0$ is equivalent to

$$\operatorname{Rank} M(\rho'_0) = 2m. \tag{5.2.17}$$

Remark 5.2.7. *Considering the size of $M(\rho')$, a necessary condition for the covering condition to holds is $m + m^-_1 + \cdots + m^-_n \geq 2m$.*

Lemma 5.2.8. *If the covering condition is satisfied in a point $\rho'_0 \in \mathcal{L}^T(\mathcal{I})$, then there exists a conic neighborhood $\mathcal{V}$ of ρ'_0 in $\mathcal{L}^T(\mathcal{I})$ such that the rank condition (5.2.17) is fulfilled for every $\rho' \in \mathcal{V}$.*

Proof. If $\operatorname{Rank} M(\rho'_0) \geq 2m$, then there exists a $2m \times 2m$ submatrix $S(\rho'_0)$ such that $\det S(\rho'_0) \neq 0$. As the symbols of M^1_k and of M^2_k are continuous and homogeneous in (ξ', τ) of order $\beta^j_k - (i+1)$ and $2m_k - m^-_k + j - i - 1 - m$ respectively, where i and j index the rows and the columns. As a result, $\det S(\rho_0)$ is also continuous and homogeneous in (ξ', τ). We restrict ourselves to the compact set $\rho' \in \mathcal{L}^T(\mathcal{I}) \cap \mathbb{S}_{(\xi',\tau)=1}$. By continuity, there exists a neighborhood $\mathcal{V}_0$ of ρ'_0 in $\mathcal{L}^T(\mathcal{I}) \cap \mathbb{S}_{(\xi',\tau)=1}$ such that $\det S(\rho') \neq 0$ for all $\rho' \in \mathcal{V}_0$. The result follows by homogeneity. □

Remark 5.2.9. *Note that we proved that if the covering condition is satisfied in some point $\rho'_0 \in \mathcal{L}^T(\mathcal{I})$ then there exists a conic neighborhood $\mathcal{V}$ of ρ'_0 such that (5.2.17) holds, but we did not proved that the covering condition is satisfied for any $\rho' \in \mathcal{V}$. In what follows, we shall only need Lemma 5.2.8.*

5.3 The case of second-order operators

5.3.1 The transmission problem

Here, we address the important case of second-order elliptic operators with real principal symbol, for $k = 1,\dots,n$, namely Laplace-Beltrami operators associated with some metrics g_k on each manifold. We consider the following transmission problem

$$\begin{cases} A_k u_k = 0, \quad k = 1,\dots,n & \text{in the interior of } \Omega_k \\ u_{1|_{\mathcal{I}_1}} = \cdots = u_{n|_{\mathcal{I}_n}} & \text{on } \mathcal{I} \\ \sum_{k=1}^n \partial_{\nu_k} u_{k|_{\mathcal{I}}} = 0 & \text{on } \mathcal{I}, \end{cases} \tag{5.3.1}$$

where $A_k = -\Delta_{g_k}$+lower order terms, where(Ω_k, g_k) are Riemannian manifolds with boundary $\mathcal{I}_k$ satisfying $\mathcal{I} := \mathcal{I}_1 = \cdots = \mathcal{I}_n$, where Δ_{g_k} is the Laplace-Beltrami operator associated with the metric g_k, and ν_k denotes the unit outward pointing normal vector of $\mathcal{I}_k$ with respect to the metric g_k. Following Section 5.2, in $\Omega = \cup_{k=1}^n \Omega_k$, in a neighborhood of a point $y \in \mathcal{I}$ we can use coordinates $x' = (x_1, \dots, x_{d-1})$ on $\mathcal{I}$ and coordinates $x = (x', x_d)$ on Ω_k such that locally $\mathcal{I} = \{x_d = 0\}$ and $\Omega_k = \{x_d \geq 0\}$. In this set of coordinates, System (5.3.1) reads

$$\begin{cases} P_k(x,D)u_k = 0, \quad k = 1, \dots, n & \text{in } \Omega_k \cap \{x_d > 0\} \\ u_{1|_{\mathcal{I}}} = \cdots = u_{n|_{\mathcal{I}}} & \text{on } \mathcal{I} \\ \sum_{k=1}^n \partial_{\nu_k} u_{k|\mathcal{I}} = 0 & \text{on } \mathcal{I}, \end{cases} \tag{5.3.2}$$

where the principal part of $P_k(x,D)$ is of the form $\sum_{1\leq i,j\leq d} D_i a_{i,j}^k D_j$, where $a_{i,j}^k$ are coefficients dependent on the metric g_k. These transmission conditions can be rewritten as the following equations

$$u_{1|_{\mathcal{I}}} - u_{2|_{\mathcal{I}}} = 0,\ u_{2|_{\mathcal{I}}} - u_{3|_{\mathcal{I}}} = 0,\ \dots, u_{n-1|_{\mathcal{I}}} - u_{n|_{\mathcal{I}}} = 0,\ \sum_{k=1}^n \sum_{j=1}^d a_{d,j}^k(x) \partial_{x_j} u_{k|_I} = 0.$$

Here $m = n$. With the notation of System (5.1.3), we consider the following set of operators

$$\begin{cases} T_k^\ell &= \delta_{k,\ell} - \delta_{k,\ell+1}, \qquad k \in \{1, \dots, n\},\ \ell \in \{1, \dots, n-1\}, \\ T_k^n &= \sum_{j=1}^d a_{d,j}^k(x) D_j, \quad k \in \{1, \dots, n\}. \end{cases} \tag{5.3.3}$$

This type of transmission condition across an interface has a natural physical meaning. It simply implies continuity of the solution at $\mathcal{I}$ and that the fluxes add up to zero. We shall conjugate the operators with a function φ, which is chosen smooth in each $\mathcal{I}$, and only continuous across the interface. We shall denote $\varphi_{|\Omega_k} = \varphi_k$. In this example, the principal symbols of the conjugated operators at the boundary read

$$t_{k,\varphi}^j(x,\xi,\tau) = t_k^j(x,\xi), \quad k = 1, \dots, n,\ j = 1, \dots, n-1,$$

as these operators do not involve any derivative, and

$$\begin{aligned} t_{k,\varphi}^n(x,\xi,\tau) &= \sum_{j=1}^d a_{d,j}^k(x)(\xi_j + i\tau \partial_{x_j}\varphi_k) \\ &= a_{d,d}^k(x)\left(\xi_d + \zeta_k\right), \end{aligned}$$

where

$$\zeta_k = i\tau \partial_{x_d}\varphi_k(x) + \sum_{j=1}^{d-1} \frac{a_{d,j}^k(x)}{a_{d,d}^k(x)}(\xi_j + i\tau\partial_{x_j}\varphi_k). \tag{5.3.4}$$

Observe that $\operatorname{Im}\zeta_k$ is a function of $\mathrm{d}\varphi_k$ near $\mathcal{I}$, but not of φ_k itself. As the operators P_k are elliptic, we have $a_{d,d} \neq 0$. We also have, as $a_{i,j} = a_{j,i}$,

$$\begin{aligned} p_{k,\varphi}(x,\xi,\tau) &= \sum_{j,\ell=1}^d a_{j,\ell}^k(x)(\xi_j + i\tau\partial_{x_j}\varphi_k)(\xi_\ell + i\tau\partial_{x_\ell}\varphi_k) \\ &= a_{d,d}^k(x)(\xi_d + i\tau\partial_{x_d}\varphi)^2 + 2(\xi_d + i\tau\partial_{x_d}\varphi_k)\sum_{\ell=1}^{d-1} a_{i,d}^k(x)(\xi_\ell + i\tau\partial_{x_\ell}\varphi_k) \\ &+ \sum_{j,\ell=1}^{d-1} a_{j,\ell}^k(x)(\xi_j + i\tau\partial_{x_j}\varphi_k)(\xi_\ell + i\tau\partial_{x_\ell}\varphi_k) \\ &= a_{d,d}^k(x)\left((\xi_d + \zeta_k(x,\xi',\tau))^2 + q_k(x,\xi',\tau)\right) \\ &= a_{d,d}^k(x)\left(\xi_d + \zeta_k(x,\xi',\tau) + i\alpha_k(x,\xi',\tau)\right)\left(\xi_d + \zeta_k(x,\xi',\tau) - i\alpha_k(x,\xi',\tau)\right), \end{aligned}$$

where $q_k(x,\xi',\tau)$ is homogeneous of order 2 in (ξ',τ), and $\alpha_k(x,\xi',\tau)\in\mathbb{C}$ is chosen so that $\operatorname{Re}\alpha_k\geq 0$ and $\alpha_k^2(x,\xi',\tau)=q_k(x,\xi',\tau)$. Defining

$$\sigma_k^+(x,\xi',\tau)=-\zeta_k(x,\xi',\tau)+i\alpha_k(x,\xi',\tau),\quad \sigma_k^-=-\zeta_k(x,\xi',\tau)-i\alpha_k(x,\xi',\tau), \tag{5.3.5}$$

we can factorize the principal symbols of the operators

$$p_{k,\varphi}(x,\xi,\tau)=a_{d,d}^k(x)(\xi_d-\sigma_k^-(x,\xi',\tau))(\xi_d-\sigma_k^+(x,\xi,\tau)).$$

Observe that $\operatorname{Im}\sigma_k^+\geq\operatorname{Im}\sigma_k^-$ always holds. Several cases can occur, depending on the sign of the imaginary part of the roots of the $p_{k,\varphi}$, viewed as polynomials in ξ_d. To discuss the sign of the imaginary parts of the roots $\sigma_k^\pm$, we introduce the following sets

$$\begin{aligned}\Sigma^+=\Sigma^+(\rho')&=\{k\in\{1,\dots,n\},\ \operatorname{Im}\sigma_k^+,\operatorname{Im}\sigma_k^-\geq 0\},\\ \Sigma^-=\Sigma^-(\rho')&=\{k\in\{1,\dots,n\},\ \operatorname{Im}\sigma_k^+,\operatorname{Im}\sigma_k^-<0\},\\ \Sigma^0=\Sigma^0(\rho')&=\{k\in\{1,\dots,n\},\ \operatorname{Im}\sigma_k^+\geq 0,\ \operatorname{Im}\sigma_k^-<0\}.\end{aligned} \tag{5.3.6}$$

We recall that the covering condition holds if the matrix M built in (5.2.16) satisfies the rank condition (5.2.17). In this particular case, we have $m=n$. The $2\times n$ sub-matrices M_k^1 are only related to operators at the boundary, and do not depend on the sign of the imaginary parts of the roots of the operators in the interior. We detail here how to construct those matrices in the present example. The coefficients $t_{k,\varphi}^{l,i}$ of M_k^1 correspond to the coefficients of the principal symbols of the $t_{k,\varphi}^\ell$. The operators that acts on u_1 for $\ell\in\{1,\dots,m\}$ are respectively (see (5.3.3))

$$T_{1,\varphi}^1=1,\ T_{1,\varphi}^2=0,\ \dots,\ T_{1,\varphi}^{n-1}=0,\ \text{and}\ T_1^n=\sum_{j=1}^n a_{d,j}^1(x)(D_j+i\tau\partial_{x_j}\varphi_1(x)),$$

and the coefficients of their principal symbols written in the form (5.2.14) are given by

$$t_1^{1,0}=1,\ t_1^{1,1}=0,\ \dots,\ t_1^{n-1,0}=0,\ t_1^{n-1,1}=0,\ \text{and}\ t_1^{n,0}=a_{d,d}^1\zeta_1,\ t_1^{n,1}=a_{d,d}^1(x),$$

with ζ_1 defined in (5.3.4) for $k=1$. Then, we obtain

$$M_1^1=\begin{pmatrix}1&0&\dots&0&a_{d,d}^1\zeta_1\\0&0&\dots&0&a_{d,d}^1\end{pmatrix}. \tag{5.3.7}$$

Arguing the same way we have, for $k\in\{2,\dots,n-1\}$, we find

$$M_k^1=\begin{pmatrix}&&&&(k-1)^{\text{th}}&k^{\text{th}}&&&\\0&\dots&0&-1&1&0&\dots&0&a_{d,d}^k\zeta_k\\0&\dots&0&0&0&0&\dots&0&a_{d,d}^k\end{pmatrix}$$

and

$$M_n^1=\begin{pmatrix}0&0&\dots&0&-1&a_{d,d}^n\zeta_n\\0&0&\dots&0&0&a_{d,d}^n\end{pmatrix}.$$

We also detail the structure of the matrices M_k^2 in different situations depending on the cardinal of the sets given in (5.3.6).

- For $k\in\Sigma^0$, we have $m_k^-=1$ and then $M_k^2={}^t(-\sigma_k^+,1)$.
- For $k\in\Sigma^-$, we have $m_k^-=2$ and then $M_k^2=\mathrm{Id}_2$.
- For $k\in\Sigma^+$, we have $m_k^-=0$ and then $M_k^2=\emptyset$, with the meaning given below (5.2.16).

We now investigate what configurations yield (5.2.17).

First case. We first focus on the case where $\operatorname{Card}\Sigma^+ \geq 2$. Without any loss of generality, we may assume that $\{1,2\} \subset \Sigma^+$, that is $\operatorname{Im}\sigma_1^\pm \geq 0$ and $\operatorname{Im}\sigma_2^\pm \geq 0$. In this case the matrices M_1^2 and M_2^2 are empty, and the matrix M reads

$$M = \left(\begin{array}{cccc:c} 1 & 0 & 0 & \cdots & a_{d,d}^1\zeta_1 & \\ 0 & 0 & 0 & \cdots & a_{d,d}^1 & \\ -1 & 1 & 0 & \cdots & a_{d,d}^2\zeta_2 & N \\ 0 & 0 & 0 & \cdots & a_{d,d}^2 & \\ \hdashline & & * & & & * \end{array}\right)$$

with either $N = 0$ or $N = \emptyset$ (in the sense given after (5.2.16)). We then see that the second and the fourth rows are linearly dependent. As the number of lines of M is equal to $2m$, this implies that the rank condition (5.2.17) fails to hold.

Second case. Here, we assume that $\Sigma^+ = \emptyset$. Consider $\ell \in \{1,\dots,n\}$. Without any loss of generality, we may assume that $\{1,\dots,\ell\} \subset \Sigma^0$ and $\{\ell+1,\dots,n\} \subset \Sigma^-$. The matrix M then reads

$$M = \begin{pmatrix} \mathcal{A}_{2\ell}^1 & \mathcal{B}_\ell & 0 \\ \mathcal{A}_{2(n-\ell)}^2 & 0 & \mathrm{Id}_{2(n-\ell)} \end{pmatrix},$$

where,

$$\mathcal{B}_\ell = \begin{pmatrix} -\sigma_1^+ & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & -\sigma_2^+ & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 0 & -\sigma_\ell^+ \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_{2\ell,\ell}(\mathbb{C}),$$

and $\mathrm{Id}_{2(n-\ell)}$ denotes the identy matrix of dimension $2(n-\ell)$. If $\ell = n$ we set $\mathrm{Id}_0 = \emptyset$ in the sense given below (5.2.16). Moreover, for $\ell \neq 1, n$, we have

$$\mathcal{A}_{2\ell}^1 = \begin{pmatrix} & & (\ell-1)^{\text{th}} & \ell^{\text{th}} & & & \\ 1 & 0 & 0 & 0 & \dots & 0 & a_{d,d}^1\zeta_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{d,d}^1 \\ -1 & 1 & 0 & 0 & \dots & 0 & a_{d,d}^2\zeta_2 \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{d,d}^2 \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & -1 & 1 & \dots & 0 & a_{d,d}^\ell\zeta_\ell \\ 0 & \dots & 0 & 0 & \dots & 0 & a_{d,d}^\ell \end{pmatrix} \in \mathcal{M}_{2\ell,n}(\mathbb{C}),$$

and

$$\mathcal{A}^2_{2(n-\ell)} = \begin{pmatrix} & & (\ell)^{\text{th}} & (\ell+1)^{\text{th}} & & & \\ 0 & \dots & -1 & 1 & \dots & 0 & a^{\ell+1}_{d,d}\zeta_{\ell+1} \\ 0 & \dots & 0 & 0 & \dots & 0 & a^{\ell+1}_{d,d} \\ 0 & \dots & 0 & -1 & \dots & 0 & a^{\ell+2}_{d,d}\zeta_{\ell+2} \\ 0 & \dots & 0 & 0 & \dots & 0 & a^{\ell+2}_{d,d} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 & a^{n}_{d,d}\zeta_{n} \\ 0 & \dots & 0 & 0 & \dots & 0 & a^{n}_{d,d} \end{pmatrix} \in \mathcal{M}_{2(n-\ell),n}(\mathbb{C}).$$

For $\ell = 1$, we have

$$\mathcal{A}^1_{2\ell} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}\zeta_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d} \end{pmatrix} \in \mathcal{M}_{2,n}(\mathbb{C}),$$

$$\mathcal{A}^2_{2(n-\ell)} = \begin{pmatrix} -1 & 1 & \dots & 0 & a^{\ell+1}_{d,d}\zeta_{\ell+1} \\ 0 & 0 & \dots & 0 & a^{\ell+1}_{d,d} \\ 0 & -1 & \dots & 0 & a^{\ell+2}_{d,d}\zeta_{\ell+2} \\ 0 & 0 & \dots & 0 & a^{\ell+2}_{d,d} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & a^{n}_{d,d}\zeta_{n} \\ 0 & 0 & \dots & 0 & a^{n}_{d,d} \end{pmatrix} \in \mathcal{M}_{2(n-1),n}(\mathbb{C}).$$

Finally, for $\ell = n$, we have

$$\mathcal{A}^1_{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}\zeta_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d} \\ -1 & 1 & 0 & 0 & \dots & 0 & a^2_{d,d}\zeta_2 \\ 0 & 0 & 0 & 0 & \dots & 0 & a^2_{d,d} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 & a^{n}_{d,d}\zeta_{n} \\ 0 & \dots & 0 & 0 & \dots & 0 & a^{n}_{d,d} \end{pmatrix} \in \mathcal{M}_{2n,n}(\mathbb{C}), \quad \mathcal{A}^2_{2(n-\ell)} = \emptyset.$$

Note that we have

$$\text{Rank}(M) = \text{Rank}(T) + 2(n-\ell), \quad \text{with } T = \text{Rank}\begin{pmatrix} \mathcal{A}_{2\ell} & \mathcal{B}_{\ell}. \end{pmatrix} \tag{5.3.8}$$

By operations on the lines and columns of the $2\ell \times (n+\ell)$ matrix T we can obtain:

- If $\ell < n$, it is of maximal rank. Indeed, from (5.3.8), this is equivalent to having $\text{Rank}(M) = \text{Rank}\begin{pmatrix} \mathcal{A}^1_{2\ell} & \mathcal{B}_{\ell} \end{pmatrix} = 2\ell$.

Yet, by reorganizing columns, we have

$$\begin{aligned}
\mathrm{Rank}\begin{pmatrix}\mathcal{A}^1_{2\ell} & \mathcal{B}_\ell\end{pmatrix} &= \mathrm{Rank}\begin{pmatrix}
1 & -\sigma_1^+ & 0 & 0 & \dots & 0 & 0 & a^1_{d,d}\zeta_1\\
0 & 1 & 0 & 0 & \dots & 0 & 0 & a^1_{d,d}\\
-1 & 0 & 1 & -\sigma_2^+ & \dots & 0 & 0 & a^2_{d,d}\zeta_2\\
0 & 0 & 0 & 1 & \dots & 0 & 0 & a^2_{d,d}\\
\vdots & \vdots & & & \ddots & & & \vdots\\
0 & 0 & 0 & 0 & \dots & 1 & -\sigma_\ell^+ & a^\ell_{d,d}\zeta_\ell\\
0 & 0 & 0 & 0 & \dots & 0 & 1 & a^\ell_{d,d}
\end{pmatrix}\\
&= \mathrm{Rank}\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0 & a^1_{d,d}(\zeta_1+\sigma_1^+)\\
0 & 1 & 0 & 0 & \dots & 0 & 0 & a^1_{d,d}\\
-1 & 0 & 1 & 0 & \dots & 0 & 0 & a^2_{d,d}(\zeta_2+\sigma_2^+)\\
0 & 0 & 0 & 1 & \dots & 0 & 0 & a^2_{d,d}\\
\vdots & \vdots & & & \ddots & & & \vdots\\
0 & 0 & 0 & 0 & \dots & 1 & 0 & a^\ell_{d,d}(\zeta_\ell+\sigma_\ell^+)\\
0 & 0 & 0 & 0 & \dots & 0 & 1 & a^\ell_{d,d}
\end{pmatrix}\\
&\geq \mathrm{Rank}\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 1 & 0 & 0 & \dots & 0 & 0\\
-1 & 0 & 1 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & \vdots & & & \ddots & & \\
0 & 0 & 0 & 0 & \dots & 1 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix} = 2\ell,
\end{aligned}$$

as the final matrix is in $\mathcal{M}_{2\ell}(\mathbb{C})$.

- If $\ell = n$, it is of maximal rank if and only if

$$\sum_{j=1}^{n} a^j_{d,d}(\zeta_j + \sigma_j^+) \neq 0. \tag{5.3.9}$$

Indeed, from (5.3.8), this is equivalent to have $\mathrm{Rank}\begin{pmatrix}\mathcal{A}_{2n} & \mathcal{B}_n\end{pmatrix} = 2n$. Yet, we have

$$\begin{aligned}
&\mathrm{Rank}\begin{pmatrix}\mathcal{A}^1_{2n} & \mathcal{B}_n\end{pmatrix}\\
&= \mathrm{Rank}\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}\zeta_1 & -\sigma_1^+ & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d} & 1 & 0 & \dots & 0 & 0\\
-1 & 1 & 0 & 0 & \dots & 0 & a^2_{d,d}\zeta_2 & 0 & -\sigma_2^+ & \dots & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & a^2_{d,d} & 0 & 1 & \dots & 0 & 0\\
\vdots & & & & \ddots & \vdots & \vdots & \vdots & & & \ddots & \\
0 & \dots & -0 & 0 & \dots & -1 & a^n_{d,d}\zeta_n & 0 & \dots & 0 & 0 & -\sigma_n^+\\
0 & \dots & 0 & 0 & \dots & 0 & a^n_{d,d} & 0 & \dots & 0 & 0 & 1
\end{pmatrix}\\
&= \mathrm{Rank}\begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}(\zeta_1+\sigma_1^+) & -\sigma_1^+ & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\
-1 & 1 & 0 & 0 & \dots & 0 & a^2_{d,d}(\zeta_2+\sigma_2^+) & 0 & -\sigma_2^+ & \dots & 0 & 0\\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0\\
\vdots & & & & \ddots & \vdots & \vdots & \vdots & & & \ddots & \\
0 & \dots & -0 & 0 & \dots & -1 & a^n_{d,d}(\zeta_n+\sigma_n^+) & 0 & \dots & 0 & 0 & -\sigma_n^+\\
0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{pmatrix},
\end{aligned}$$

and then

$$\operatorname{Rank}\begin{pmatrix}\mathcal{A}^1_{2n} & \mathcal{B}_n\end{pmatrix} = \operatorname{Rank}\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}(\zeta_1+\sigma_1^+) \\ -1 & 1 & 0 & 0 & \dots & 0 & a^2_{d,d}(\zeta_2+\sigma_2^+) \\ 0 & -1 & 1 & 0 & \dots & 0 & a^3_{d,d}(\zeta_3+\sigma_3^+) \\ 0 & 0 & -1 & 1 & \dots & 0 & a^4_{d,d}(\zeta_4+\sigma_4^+) \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & a^{n-1}_{d,d}(\zeta_{n-1}+\sigma_{n-1}^+) \\ 0 & 0 & 0 & 0 & \dots & -1 & a^n_{d,d}(\zeta_n+\sigma_n^+) \end{pmatrix} + n.$$

We see that (5.3.9) is a necessary and sufficient condition to obtain the rank condition (5.2.17). Let us prove that (5.3.9) is always satisfied. If not, we have

$$\sum_{j=1}^n a^j_{d,d}\operatorname{Im}\zeta_j = -\sum_{j=1}^n a^j_{d,d}\operatorname{Im}\sigma_j^+.$$

Yet, $\operatorname{Im}\sigma_j^+ = -\operatorname{Im}\zeta_j + \operatorname{Re}\alpha_j$. This implies $\sum_{j=1}^n a^j_{d,d}\operatorname{Re}\alpha_j = 0$. As $\operatorname{Re}\alpha_j \geq 0$ and $a^j_{d,d} > 0$ for all $j \in \{1,\dots,n\}$, then $\operatorname{Re}\alpha_j = 0$. Hence, both roots σ_j^+ and σ_j^- have the same imaginary part, for all $j \in \{1,\dots,n\}$, in contradiction with having $\{1,\dots,n\} \subset \Sigma^0$.

As a result, the covering condition holds in this case.

Third case. In this case, we assume that $\operatorname{Card}\Sigma^+ = 1$ and $\operatorname{Card}\Sigma^0 = n-1$. In such a situation, the matrix has only $2n-1$ columns, and from Remark 5.2.7, the covering condition fails to hold.

Fourth case. We finally assume $\operatorname{Card}\Sigma^+ = 1$ and $\operatorname{Card}\Sigma^0 < n-1$ (or equivalently $\operatorname{Card}\Sigma^- \geq 1$). Let $\ell = \operatorname{Card}\Sigma^0$ and assume, without any loss of generality, $\{1,\dots,\ell\} = \Sigma^0$, $\{\ell+1,\dots,n-1\} = \Sigma^-$, and $\{n\} = \Sigma^+$. Then the matrix M reads

$$M = \begin{pmatrix}\mathcal{A} & \mathcal{B}\end{pmatrix}$$

where

$$\mathcal{A} = \begin{pmatrix} M_1^1 \\ \vdots \\ M_n^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d}\zeta_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & a^1_{d,d} \\ -1 & 1 & 0 & 0 & \dots & 0 & a^2_{d,d}\zeta_2 \\ 0 & 0 & 0 & 0 & \dots & 0 & a^2_{d,d} \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 & a^n_{d,d}\zeta_n \\ 0 & \dots & 0 & 0 & \dots & 0 & a^n_{d,d} \end{pmatrix} \in \mathcal{M}_{2n,n}(\mathbb{C}),$$

and

$$\mathcal{B} = \begin{pmatrix} M_1^2 & & \\ & \ddots & \\ & & M_{n-1}^2 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1^+ & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & \ddots & 0 & 0 & & \vdots & \vdots \\ 0 & \ddots & -\sigma_\ell^+ & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & & \vdots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{2n,2n-\ell-2}(\mathbb{C}),$$

$(\ell)^{\text{th}}$

as $M_n^2 = \emptyset$. We shall make operations on lines and columns to compute the rank of M. First, note that using the last line of M, we obtain $\operatorname{Rank} M = \operatorname{Rank} \begin{pmatrix} \mathcal{A}' & \mathcal{B} \end{pmatrix}$, with

$$\mathcal{A}' = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then using the even lines of $\mathcal{B}$, as the associated lines of $\mathcal{A}'$ are only made of zeros, we obtain $\operatorname{Rank} M = \operatorname{Rank} \begin{pmatrix} \mathcal{A}' & \mathcal{B}' \end{pmatrix}$, with

$$\mathcal{B}' = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & \ddots & 0 & 0 & & \vdots & \vdots \\ 0 & \ddots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & & \vdots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we see that the rank of $\begin{pmatrix} \mathcal{A}' & \mathcal{B}' \end{pmatrix}$, is equal to $2n$.

Conclusion. As the four cases we considered are exhaustive, we find that we have to construct a weight function φ such that the first and third cases described above do not occur, meaning that we require

$$\operatorname{Card} \Sigma^+(\rho') \leq 1, \quad \forall \rho' \in \mathcal{I}, \tag{5.3.10}$$

and moreover

$$\operatorname{Card} \Sigma(\rho') = 1 \Longrightarrow \operatorname{Card} \Sigma^0(\rho') < n-1 \text{ (or equivalently } \operatorname{Card} \Sigma^-(\rho') \geq 1). \tag{5.3.11}$$

Let $i_0 \in \{1, \dots, n\}$ be fixed. First note that if we have, for all $i \neq i_0$,

$$-\tau(\partial_{x_d}\varphi_i + \sum_{j=1}^{d-1} a_{d,j}^i / a_{d,d}^i \partial_{x_j}\varphi_i) < 0, \tag{5.3.12}$$

then (5.3.10) is satisfied as $\operatorname{Im} \sigma_k^- = -\tau(\partial_{x_d}\varphi_i + \sum_{j=1}^{d-1} a_{d,j}^i / a_{d,d}^i \partial_{x_j}\varphi_i) - \operatorname{Re} \alpha_i$ with $\operatorname{Re} \alpha_i \geq 0$. Second, observe that $\operatorname{Im} \sigma_{i_0}^- \geq 0$ implies

$$-\tau(\partial_{x_d}\varphi_{i_0} + \sum_{j=1}^{d-1} a_{d,j}^{i_0} / a_{d,d}^{i_0} \partial_{x_j}\varphi_{i_0}) \geq \operatorname{Re} \alpha_{i_0} \geq 0.$$

Hence, if we construct a weight function φ satisfying

$$-\tau(\partial_{x_d}\varphi_{i_0}+\sum_{j=1}^{d-1}a_{d,j}^{i_0}/a_{d,d}^{i_0}\partial_{x_j}\varphi_{i_0})\geq \operatorname{Re}\alpha_{i_0}$$
$$\Longrightarrow\begin{cases}-\tau(\partial_{x_d}\varphi_{i_1}+\sum_{j=1}^{d-1}a_{d,j}^{i_1}/a_{d,d}^{i_1}\partial_{x_j}\varphi_{i_1})<-\operatorname{Re}\alpha_{i_1},\\ -\tau(\partial_{x_d}\varphi_{i}+\sum_{j=1}^{d-1}a_{d,j}^{i}/a_{d,d}^{i}\partial_{x_j}\varphi_{i})<\operatorname{Re}\alpha_{i},\quad \forall i\neq i_0,i_1,\end{cases}\tag{5.3.13}$$

for some $i_1\neq i_0$, then both (5.3.10) and (5.3.11) hold. Observe that (5.3.13) can be achieved by requiring (5.3.12) and additionaly

$$(\operatorname{Re}\alpha_{i_1})\frac{\partial_{x_d}\varphi_{i_0}+\sum_{j=1}^{d-1}a_{d,j}^{i_0}/a_{d,d}^{i_0}\partial_{x_j}\varphi_{i_0}}{\partial_{x_d}\varphi_{i_1}+\sum_{j=1}^{d-1}a_{d,j}^{i_1}/a_{d,d}^{i_1}\partial_{x_j}\varphi_{i_1}}<\operatorname{Re}\alpha_{i_0}.\tag{5.3.14}$$

In order to fix ideas, assume that $\varphi_k=\varphi_k(x_d)$ does not depend on the variable x'. Then, conditions (5.3.12) and (5.3.14) read

$$\forall i\neq i_0,\ \partial_{x_d}\varphi_i>0,\ \text{and}\quad \frac{\partial_{x_d}\varphi_{i_0}}{\partial_{x_d}\varphi_{i_1}}<K,$$

where K is a positive constant that only depends on the coefficients $a_{j,\ell}^{i_0}$ and $a_{j,\ell}^{i_1}$. Hence, if we choose a weight function φ such that, near the interface,

$$\partial_{x_d}\varphi_{i_0}<0,\ \text{and}\ \partial_{x_d}\varphi_i>0,\ \text{for}\ i\neq i_0,\ \text{and moreover satisfying}\ \partial_{x_d}\varphi_{i_1}\gg\partial_{x_d}\varphi_{i_0}\ \text{for some}\ i_1\neq i_0,\tag{5.3.15}$$

then the covering condition is satisfied. Allowing back a dependance with respect to the variable x' is possible, for instance, if we impose $|\nabla'\varphi_k|\ll|\partial_{x_d}\varphi_k|$, for all $k\neq i_0$.

Remark 5.3.1. *In applications, this means that if we want to observe the interface, according to Theorem 5.2.6, the observation has to be made from at least $n-1$ sides of $\mathcal{I}$ (we recall that in Carleman estimates and applications, the observation region corresponds to regions where the weight function is the largest).*

5.3.2 On the optimality of the observation regions for the transmission problem

In this section, we discuss the optimality of the above conditions on the weight function. First, observe that in the case of two operators ($n=2$), we precisely recover the condition of [BL16], which is proven to be optimal in the case of operators with constant coefficients in [LL13]. Second, in some particular geometric configurations, note that if we observe from only $n-2$ sides of $\mathcal{I}$, then the unique continuation property may fail, which prevents the possibility of proving a Carleman estimate. We consider $\Omega_k=(\mathbb{R}/2\pi\mathbb{Z})\times(0,\pi)$, for $k\in\{1,\dots,n\}$, $n\in\mathbb{N}^*$, $n\geq 3$, and $\mathcal{I}=(\mathbb{R}/2\pi\mathbb{Z})\times\{0\}$. Take $C_k=\begin{pmatrix}1&0\\0&c_k\end{pmatrix}$, such that $c_k^{-1/2}\in\mathbb{N}^*$, and consider the diagonal operator

$$\mathcal{Q}=\begin{pmatrix}-\operatorname{div}(C_1\nabla)&0&\dots&0\\0&-\operatorname{div}(C_2\nabla)&0&0\\\vdots&&\ddots&\\0&0&\dots&-\operatorname{div}(C_n\nabla)\end{pmatrix}\tag{5.3.16}$$

with domain

$$D(\mathcal{Q})=\Big\{(u_1,\dots,u_n)\in\bigoplus_{k=1}^n H^2(\Omega_k),\ u_1=\dots=u_n\ \text{on}\ \mathcal{I},\ \sum_{k=1}^n c_k\partial_{x_2}u_{k|_{x_2=0}}=0\ \text{on}\ \mathcal{I}\Big\}.$$

We have the following proposition on the eigenfunctions of $\mathcal{Q}$.

Proposition 5.3.2. *For all $j \in \mathbb{N}$ and $\ell \in \{2, \dots, n-1\}$, the function*

$$\phi_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{ijx_1}\sin(c_\ell^{-1/2} j x_2) \\ e^{ijx_1}\sin(c_{\ell+1}^{-1/2} j x_2) \\ \vdots \\ e^{ijx_1}\sin(c_{n-1}^{-1/2} j x_2) \\ -c_n^{1/2}\sum_{k=\ell}^{n-1} c_k^{1/2} e^{ijx_1}\sin(c_n^{-1/2} j x_2), \end{pmatrix}$$

where the first $\ell - 1$ componants are equal to zero, is an eigenfunction for System (5.3.16) associated with the eigenvalues j^2.

If $\ell \in \{1, \dots, n-1\}$, the eigenfunction has $\ell - 1$ vanishing components. In the particular case $\ell = n-1$, the eigenfunction has $n-2$ vanishing components. Thus, Proposition 5.3.2 shows that the unique continuation property fails to hold from less than $n-1$ sides of the interface, due to the presence of such *invisible* modes. Since a Carleman estimate near the interface as in Theorem 5.2.6 implies the unique continuation property, the derivation of the Carleman estimate cannot be achieved in this geometrical configuration. Using Remark 5.3.1, this means that in (5.3.15), having $\partial_{x_d}\varphi_{i_0} > 0$, for $i \in J \subset \{1, \dots, n\}$ with Card $J \geq n-1$, is necessary. However, in the example (5.3.16), this is mainly due to the symmetry of the domains Ω_k. The investigation of general configurations where the unique continuation property holds true remains an interesting question.

5.4 Carleman estimate in a neighborhood of the interface

5.4.1 Semi-classical calculus

Classes of symbols

We recall some classes of pseudo-differential operators that we use in what follows. This allows us to introduce some important notation. As above, we write $\rho = (x, \xi, \tau) \in \overline{U} \times \mathbb{R}^d \times \mathbb{R}^+$ and $\rho' = (x, \xi', \tau) \in \overline{U} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$.

Definition 5.4.1. *The symbol class $\mathcal{S}_\tau^m$ is the space of $a(\rho) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ that depends on the large parameter $\tau \geq 1$ satisfying for all multi-indices $\alpha, \beta \in \mathbb{N}^d$*

$$|\partial_x^\alpha \partial_\xi^\beta a(\rho)| \leq C_{\alpha,\beta} \lambda^{m-|\beta|}, \quad \text{for all } x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d,\ \tau \geq 1,$$

where λ is the order function $\lambda = (\tau^2 + |\xi^2|)^{1/2}$. The principal symbol of such a, denoted by $\sigma(a)(\rho)$ is the class of a in the quotient space $\mathcal{S}_\tau^m / \mathcal{S}_\tau^{m-1}$.

We also introduce the tangential symbol class $\mathcal{S}_{T,\tau}^m$ as the space of $a(\rho') \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d-1})$ that depends on the large parameter $\tau \geq 1$ satisfying for all multi-indices $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{d-1}$

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(\rho)| \leq C_{\alpha,\beta} \lambda_T^{m-|\beta|}, \quad \text{for all } x \in \mathbb{R}^d,\ \xi' \in \mathbb{R}^{d-1},\ \tau \geq 1,$$

where λ_T is the order function $\lambda_T = (\tau^2 + |\xi'^2|)^{1/2}$. The principal symbol of such a, also denoted by $\sigma(a)(\rho)$ is the class of a in the quotient space $\mathcal{S}_{T,\tau}^m / \mathcal{S}_{T,\tau}^{m-1}$.

We shall consider symbols that are polynomials in the ξ_d-variable with coefficients in the class of tangential symbols. More precisely, we consider $\mathcal{S}_\tau^{m,r}$ the space of $a(\rho) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ that depends on the large parameter $\tau \geq 1$, and such that

$$a(\rho) = \sum_{j=0}^{m} a_j(\rho') \xi_d^j, \quad a_j \in \mathcal{S}_{T,\tau}^{m-j+r}. \tag{5.4.1}$$

The principal symbol of such a, denoted by $\sigma(a)(\rho)$ is defined by

$$\sigma(a)(\rho) = \sum_{j=1}^{m} \sigma(a_j)(\rho')\xi_d^j.$$

It can be understood as a representative of a in $\mathcal{S}_\tau^{m,r}/\mathcal{S}_\tau^{m,r-1}$.

We also introduce poly-homogeneous symbols.

Definition 5.4.2. *We say that $a \in \mathcal{S}^m_{\tau,\mathrm{cl}}$ (resp. $\mathcal{S}^m_{T,\tau,\mathrm{cl}}$) if there exist $a_j \in \mathcal{S}_\tau^{m-j}$ (resp. $\mathcal{S}_{T,\tau}^{m-j}$), $j \in \mathbb{N}$, homogeneous of degree $m-j$ in (ξ,τ) (resp.(ξ',τ) such that, for all $j_0 \in \mathbb{N}$,*

$$a - \sum_{j=0}^{j_0} a_j \in \mathcal{S}_\tau^{m-j_{0-1}}, \ (\textit{resp. } \mathcal{S}_{T,\tau}^{m-j_0-1}).$$

We say that $a \in \mathcal{S}^{m,r}_{\tau,\mathrm{cl}}$ if

$$a(\rho) = \sum_{j=0}^{m} a_j(\rho')\xi_d^j, \quad \textit{with } a_j^m \in \mathcal{S}^{m-j+r}_{T,\tau,\mathrm{cl}}.$$

Semi-classical pseudo-differential operators

For u in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$, we denote by $\hat{u}$ the Fourier transform, that is,

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)dx.$$

Definition 5.4.3. *For $a \in \mathcal{S}^m_\tau$ (resp. $\mathcal{S}^m_{\tau,\mathrm{cl}}$) we define the pseudo-differential operator*

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi,\tau)\hat{u}(\xi)d\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi,\tau)u(y)dyd\xi,$$

for all $u \in \mathscr{S}(\mathbb{R}^d)$, and we write $\mathrm{Op}(a) \in \Psi^m_\tau$ (resp. $\Psi^m_{\tau,\mathrm{cl}}$). The principal symbol of the pseudo-differential operator $\mathrm{Op}(a)$ is $\sigma(\mathrm{Op}(a)) = \sigma(a)$ (see Definition 5.4.1 and 5.4.2). We also define tangential pseudo-differential associated with a symbol $b \in \mathcal{S}^m_{T,\tau}$ (resp. $\mathcal{S}^m_{T,\tau,\mathrm{cl}}$) by

$$\mathrm{Op}_T(b)u(x) = \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} e^{i(x'-y')\cdot\xi'} b(x,\xi',\tau)u(y',x_n)dy'd\xi',$$

for $u \in \mathscr{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d_+$, and we write $\mathrm{Op}_T(b) \in \Psi^m_{T,\tau}$ (resp. $\Psi^m_{T,\tau,\mathrm{cl}}$). The principal symbol of the pseudo-differential operator $\mathrm{Op}_T(b)$ is $\sigma(\mathrm{Op}_T(b)) = \sigma(b)$ (see Definition 5.4.1 and 5.4.2).

Sobolev norms

For $s \in \mathbb{N}^*$ and $\tau \geq 1$ we introduce the following parameter dependent Sobolev norms

$$||.||^2_{s,\tau} = \sum_{k=0}^{n}\sum_{t=0}^{s} \tau^{2(m-t)}||.||^2_{H^t(\Omega_k)}, \quad \text{and } |.|^2_{s,\tau} = \sum_{t=0}^{s} \tau^{2(s-t)}|.|^2_{H^t(\mathcal{I})}, \tag{5.4.2}$$

where $||.||_{H^t(\Omega_k)}$ and $|.|_{H^t(\mathcal{I})}$ denotes the standard Sobolev spaces in Ω_k and on the interface $\mathcal{I}$ respectively. We define the Sobolev spaces $H^m_\tau(\Omega_k)$ and $H^m_\tau(\mathcal{I})$ as the completion of $C^\infty(\Omega_k)$ and $C^\infty(\mathcal{I})$, the spaces of smooth functions, with respect to the norms (5.4.2). Algebraically, the spaces $H^m_\tau(\Omega_k)$ and $H^m_\tau(\mathcal{I})$ are equal to the standard Sobolev spaces. However, they are not endowed with equivalent norm in the semi-classical

limit $\tau \to +\infty$. As we deal with operators with orders $2m_k \geq 2$, we shall need to estimate traces of the solution and its iterated normal derivatives at $\mathcal{I}$. Hence, we introduce the Hilbert sum

$$H_\tau^{m,s} = \bigoplus_{j=0}^{m} H^{m-j+s}(\mathcal{I}),$$

endowed with the norm

$$|U|_{m,s,\tau}^2 = \sum_{j=0}^{m} |u_j|_{m-j+s,\tau}^2, \quad \text{with } U = (u_0, \dots, u_m). \tag{5.4.3}$$

We shall also apply this norm to the following vectors composed of the various traces of iterated normal derivatives of a function $u \in C^\infty(\Omega_k)$ as

$$\mathrm{tr}^m(u) = (u_{|_\mathcal{I}}, D_\nu u_{|_\mathcal{I}}, \dots, D_\nu^m u_{|_\mathcal{I}}), \quad \text{where } D_\nu = -i\partial_\nu \text{ and } \nu \in N_k^*(\mathcal{I}),$$

(for the definition of $N_k^*(\mathcal{I})$ we refer to the beginning of Section 5.2.2). However, for the sake of simplicity, we shall write $|\operatorname{tr} u|_{m,s,\tau}$ instead of $|\operatorname{tr}^m(u)|_{m,s,\tau}$, whenever no ambiguity may occur.

We introduce the space of smooth functions in the half space, with rapid decay at infinity

$$\mathscr{S}(\overline{\mathbb{R}}_+^d) = \{u_{|_{x_d>0}},\ u \in \mathscr{S}(\mathbb{R}^d)\},$$

that is the space of restrictions to the half space of functions in the Schwartz space. Pseudo-differential operators that behave polynomially in the ξ_d direction such as (5.4.1) may act on functions of $\mathscr{S}(\overline{\mathbb{R}}_+^d)$. Hence, we introduce the norms in $\mathbb{R}_+^d$

$$||u||_{m,s,\tau}^2 = ||\operatorname{Op}_T(\lambda_T^s)u||_{m,\tau}^2, \quad u \in \mathscr{S}(\overline{\mathbb{R}}_+^d),$$

with $||.||_{m,\tau}$ introduced in (5.4.2), and the norm at the interface $\mathcal{I} = \{x_d = 0\} \simeq \mathbb{R}^{d-1}$

$$|U|_{m,s,\tau}^2 = \sum_{j=0}^{m} |\operatorname{Op}_T(\lambda_T^s)u_j|_{m-j,\tau}^2, \quad U = (u_0, \dots, u_m) \in \left(\mathscr{S}(\mathbb{R}^{d-1})\right)^{m+1},$$

with $|.|_{m-j,\tau}$ given by (5.4.2). In what follows, we shall denote the inner product of $L^2(\mathbb{R}_+^d)$ and $L^2(\mathbb{R}^{d-1})$ by

$$\Big(f, g\Big)_+ = \int_{\mathbb{R}_+^d} f\overline{g}, \quad \Big(u, v\Big)_\partial = \int_{\mathbb{R}^{d-1}} u\overline{v},$$

respectively, for $f, g \in L^2(\mathbb{R}_+^d)$ and $u, v \in L^2(\mathbb{R}^{d-1}) \simeq L^2(\mathcal{I})$. We shall also write

$$||f||_+^2 := \int_{\mathbb{R}_+^d} |f|^2.$$

We state the following continuity results for the classes of pseudo-differential operators introduced in Section 5.4.1.

Proposition 5.4.4. *Let $a(\rho) \in \mathcal{S}_\tau^{m,s}$. Then, for every $m' \in \mathbb{N}$ and $s' \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$||\operatorname{Op}(a)u||_{m',s',\tau} \leq C||u||_{m'+m,s'+s,\tau},$$

for all $u \in \mathscr{S}(\overline{\mathbb{R}}_+^d)$.

5.4.2 Sub-elliptic estimate near the interface

The following estimation is classical and based on integration by parts. It describes how the sub-ellipticity property yields an estimation, here with the case of a quadratic form $\mathrm{Re}\left(Av, iBv\right)_+$.

Lemma 5.4.5. *Let U be an open subset of $\overline{V}^+$. Let $a \in \mathcal{S}_\tau^{m,0}$ and $b \in \mathcal{S}_\tau^{m-1,1}$ be real symbols homogeneous of degree m in (ξ, τ). Set*

$$Q_{a,b}(v) = 2\,\mathrm{Re}\left(Av, iBv\right)_+, \quad A = \mathrm{Op}(a),\ B = \mathrm{Op}(b). \tag{5.4.4}$$

Assume that

$$a(\rho) = b(\rho) = 0 \Longrightarrow \{a, b\} > 0$$

for $x \in \overline{U}$, for $(\xi, \tau) \neq 0$. Then there exists $C, C', \tau_0 > 0$ such that

$$C||v||^2_{m,\tau} \leq C'\left(||Av||^2_+ + ||Bv||^2_+ + |\operatorname{tr} v|^2_{m-1,1/2,\tau}\right) + \tau\big(Q_{a,b}(v) - \mathrm{Re}\,\mathcal{B}_{a,b}(v)\big),$$

for $\tau \geq \tau_0$, for all $v \in C^\infty(\overline{\mathbb{R}}^d_+)$ with $\operatorname{supp} v \subset U$, and where $\mathcal{B}_{a,b}$ is a quadratic form at the interface $\{x_d = 0\}$, satisfying

$$|\mathcal{B}_{a,b}(v)| \leq C'|\operatorname{tr} v|^2_{m-1,1/2,\tau}.$$

A proof of this result can be found in [BL15].

5.4.3 An elliptic estimate near the interface

We consider, on $\mathbb{R}^d_+$,

$$P_{k,\varphi} = e^{\tau\varphi_k} P_k e^{-\tau\varphi_k} \in \Psi_\tau^{2m_k,0},$$

of principal symbol $p_{k,\varphi}(\rho) = p_k(x, \xi + i\tau \mathrm{d}\varphi_k(x)) \in \mathcal{S}_\tau^{2m_k,0}$, where p_k denotes the principal symbol of P_k. We also define at the boundary

$$T^j_{k,\varphi} = e^{\tau\varphi_k} T^j_k e^{-\tau\varphi_k} \in \Psi_\tau^{\beta^j_k,0},$$

of principal symbol $t^j_{k,\varphi}(\rho) = t^j_k(x, \xi + i\tau \mathrm{d}\varphi_k(x)) \in \mathcal{S}_\tau^{\beta^j_k,0}$, where t^j_k denotes the principal symbol of T^j_k. Considering an operator with principal symbol with roots that have negative imaginary parts, we may obtain an estimate of all the relevant traces at the boundary (or the interface) without the use of any boundary condition. This is the result of the following lemma, which proof can be found in [BL15].

Lemma 5.4.6. *Let $h(\rho', \xi_d) \in \mathcal{S}_\tau^{k,0}$, with $k \geq 1$, be a monic polynomial function in ξ_d with homogeneous coefficients in (ξ', τ) and $H = \mathrm{Op}(h)$. Let U be a conic open subset of $\overline{V}^+ \times \mathbb{R}^{d-1} \times \mathbb{R}^+$. Assume that all the roots of h have negative imaginary part for all $\rho' \in U$. Let $\chi(\rho') \in \mathcal{S}^0_{\tau,T}$, homogeneous, be such that $\operatorname{supp}\chi \subset U$, and let $N \in \mathbb{N}$. There exist $C > 0$ and $\tau_0 > 0$ such that*

$$||\operatorname{Op}_T(\chi)w||^2_{k,\tau} + |\operatorname{tr}(\operatorname{Op}_T(\chi)w)|^2_{k-1,1/2,\tau} \leq C\left(||H\operatorname{Op}_T(\chi)w||^2_+ + ||w||^2_{k,-N,\tau} + |\operatorname{tr}(w)|^2_{k-1,-N,\tau}\right),$$

for all $w \in \mathscr{S}(\overline{\mathbb{R}}^d_+)$ and $\tau \geq \tau_0$.

Observe that this lemma provides an estimate on all the traces at the boundary, up to microlocal remainder terms, which will be absorbed when patching all the microlocal estimates on the whole phase-space, by taking $\tau > 0$ sufficiently large.

Below, we consider operators that admit roots with non-negative imaginary parts. The covering condition imposed at the multi-interface $\mathcal{I}$ on the set of operators (P_k, T^j_k, φ) is then a key element to obtain an estimation.

5.4.4 Interface quadratic forms

Definition 5.4.7. *For $w = (w_1, \dots, w_n) \in (\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$, $N \in \mathbb{N}$ we define the following quadratic forms*

$$\mathcal{G}(w) := \sum_{s=1}^{N} \left(\sum_{k=1}^{n} A_k^s w_{k|_{x_d=0}}, \sum_{k=1}^{n} B_k^s w_{k|_{x_d=0}} \right)_\partial ,$$

with $A_k^s = \mathrm{Op}(a_k^s)$, and $B_k^s = \mathrm{Op}(b_k^s)$. We say that $\mathcal{G}$ is an interface quadratic form of type $(2m_1-1, \dots, 2m_n-1, \sigma)$ if, for each $s \in \{1, \dots N\}$ we have $a_k^s, b_k^s \in \mathcal{S}^{2m_k-1,\sigma_k}_{\tau,\mathrm{cl}}$, for all $k \in \{1, \dots, n\}$, with $\sigma_1 + \cdots + \sigma_n = 2\sigma$. Moreover, for $z = (z_1, \dots, z_n), \tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_n) \in \mathbb{C}^{2m_1} \times \cdots \times \mathbb{C}^{2m_n}$, where $z_k = (z_k^0, z_k^1, \dots, z_k^{2m_k-1})$ and $\tilde{z}_k = (\tilde{z}_k^0, \tilde{z}_k^1, \dots, \tilde{z}_k^{2m_k-1})$, we define

$$\Sigma_{\mathcal{G}}(\rho', z, \tilde{z}) = \sum_{s=1}^{N} \sum_{k=1}^{n} \Sigma_{a_k^s}(\rho', z_k) \overline{\sum_{k=1}^{n} \Sigma_{b_k^s}(\rho', \tilde{z}_k)},$$

where, for a symbol $a(\rho) = \sum_{j=0}^{p} a_j(\rho')\xi_d^j \in \mathcal{S}^{p,s}_{\tau,\mathrm{cl}}$, we set

$$\Sigma_a(\rho', z) = \sum_{s=0}^{p} a_j(\rho') z_j, \quad \text{for } z = (z_0, \dots, z_p) \in \mathbb{C}^{p+1}.$$

Definition 5.4.8. *Let W be an open conic set in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ and let $\mathcal{G}$ be an interface quadratic form of type $(2m_1 - 1, \dots, 2m_n - 1, \sigma)$ associated with the bilinear symbol $\Sigma_{\mathcal{G}}(\rho', z, \tilde{z})$. We say that $\mathcal{G}$ is positive definite in W if there exists $C > 0$ and $R > 0$ such that*

$$\mathrm{Re}\, \Sigma_{\mathcal{G}}(\rho'_{|_{x_d=0}}, z, z) \geq C \sum_{k=1}^{n} \sum_{j=0}^{2m_k-1} \lambda_T^{2(2m_k-1-j+\sigma_k)} |z_k^j|^2,$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^{2m}$, $z_k = (z_k^0, \dots, z_k^{2m_k-1}) \in \mathbb{C}^{2m_k}$, $\tau \geq 0$, $\lambda_T \geq R$, $\rho'_{|_{x_d=0}} \in W$.

We have the following Gårding-type inequality. The proof follows [BL16].

Lemma 5.4.9. *Let W be an open conic set in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ and let $\mathcal{G}$ be an interface quadratic form of type $(2m_1 - 1, \dots, 2m_n - 1, \sigma)$ that is positive definite in W. Let $\chi \in \mathcal{S}^0_{T,\tau}$ be a homogeneous symbol satisfying $\mathrm{supp}\, \chi_{|_{x_d=0}} \subset W$ and let $M \in \mathbb{N}$. There exists $C > 0$, $C' > 0$ and $\tau_0 > 0$ such that*

$$\mathrm{Re}\, \mathcal{G}(\mathrm{Op}_T(\chi)u) \geq C \sum_{k=1}^{n} |\,\mathrm{tr}\, \mathrm{Op}_T(\chi)u|^2_{2m_k-1,\sigma_k,\tau} - C' \sum_{k=1}^{n} |\,\mathrm{tr}\, u_k|^2_{2m_k-1,\sigma_k-M,\tau}, \tag{5.4.5}$$

for $u = (u_1, \dots, u_k) \in (\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$, $\tau \geq \tau_0$.

Proof. Using Definition 5.4.7, $\mathcal{G}$ reads

$$\mathcal{G}(u) = \sum_{s=1}^{N} \left(\sum_{k=1}^{n} A_k^s u_{k|_{x_d=0}}, \sum_{k=1}^{n} B_k^s u_{k|_{x_d=0}} \right)_\partial , \quad \text{with } A_k^s, B_k^s \in \Psi^{2m_k-1,\sigma_k}_{\tau,\mathrm{cl}}.$$

Using tangential pseudo-differential calculus, with classical symbols, there exist $a_{k,j}^s, b_{k,j}^s \in \mathcal{S}^0_{T,\tau,\mathrm{cl}}$ such that

$$A_k^s = \sum_{j=0}^{2m_k-1} \mathrm{Op}_T(a_{k,j}^s)\, \mathrm{Op}_T(\lambda_T^{2m_k-1-j+\sigma_k}) D_d^j, \quad B_k^s = \sum_{j=0}^{2m_k-1} \mathrm{Op}_T(b_{k,j}^s)\, \mathrm{Op}_T(\lambda_T^{2m_k-1-j+\sigma_k}) D_d^j.$$

Denoting by $a\#b$, $a \in \mathcal{S}^m_{T,\tau,\mathrm{cl}}$, $b \in \mathcal{S}^{m'}_{T,\tau,\mathrm{cl}}$ the product of symbols, that is, for all $M \in \mathbb{N}$,

$$a\#b(\rho') = \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} a(\rho') \partial^\alpha_{x'} b(\rho') + r_M, \quad r_M \in \mathcal{S}^{m+m'-M}_{T,\tau,\mathrm{cl}},$$

and a^* the adjoint symbol, that is, for all $M \in \mathbb{N}$,

$$a^*(\rho') = \sum_{|\alpha|<M} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi} \partial^\alpha_{x} \overline{a}(\rho') + r_M, \quad r_M \in \mathcal{S}^{m-M}_{T,\tau,\mathrm{cl}},$$

the quadratic form $\mathcal{G}$ can be written as

$$\begin{aligned}
&\mathcal{G}(u)\\
&= \sum_{s=1}^{N} \left(\sum_{k=1}^{n} \sum_{j=0}^{2m_k-1} \mathrm{Op}_T(a_{k,j}) \, \mathrm{Op}_T(\lambda_T^{2m_k-1-j+\sigma_k}) D_d^j u_{k|_{x_d=0}}, \sum_{k=1}^{n} \sum_{j=0}^{2m_k-1} \mathrm{Op}_T(b_{k,j}) \, \mathrm{Op}_T(\lambda_T^{2m_k-1-j+\sigma_k}) D_d^j u_{k|_{x_d=0}} \right)_\partial \\
&= \sum_{k,k'=1}^{n} \sum_{j=0}^{2m_k-1} \sum_{j'=0}^{2m_{k'}-1} \left(\mathrm{Op}_T(g^{k',k'}_{j,j}) \, \mathrm{Op}_T(\lambda_T^{2m_k-1-j+\sigma_k}) D_d^j u_{k|_{x_d=0}}, \mathrm{Op}_T(\lambda_T^{2m_{k'}-1-j'+\sigma_{k'}}) D_d^{j'} u_{k'|_{x_d=0}} \right)_\partial
\end{aligned}$$

where $g^{k,k'}_{j,j} \in \mathcal{S}^0_{T,\tau\,\mathrm{cl}}$ is given by

$$g^{k,k'}_{j,j'} = \sum_{s=1}^{N} (b^s_{k,j})^* \# a^s_{k',j'}.$$

We recast this equality into a system formalism. We then consider the following matrix symbol

$$G(\rho') = \begin{pmatrix} g^{1,1}(\rho') & \dots & g^{1,n}(\rho') \\ \vdots & & \vdots \\ g^{n,1}(\rho') & \dots & g^{n,n}(\rho') \end{pmatrix}, \quad g^{i,i'} = \left(g^{i,i'}_{\ell,h}(\rho') \right)_{\substack{0 \le \ell \le m_i - 1 \\ 0 \le h \le m_{i'} - 1}}.$$

It is then sufficient to follow the proof of Lemma 3.3 in [BL16]. □

With the covering condition, we have the following positivity result, at the symbol level. In the following section, this positivity property will yield an actual estimate at the interface by means of Lemma 5.4.9.

Proposition 5.4.10. *Assume the covering condition of Definition 5.2.3 holds at $\rho'_0 \in \mathcal{L}^T(\mathcal{I})$. There exist a conical neighborhood U of ρ'_0 in $\overline{V^+} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$, $C > 0$ such that*

$$\sum_{j=1}^{m} \lambda_T^{2(\gamma_j - 1/2)} \Big| \sum_{k=1}^{n} \Sigma_{t^j_k}(\rho', z_k) \Big|^2 + \sum_{k=1}^{n} \sum_{j=m+1}^{m'_k} \lambda_T^{2(m+m^-_k-j+1/2)} |\Sigma_{e^j_{k,\varphi}}(\rho', z_k)|^2 \ge C \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \lambda_T^{2(2m_k-1/2-j)} |z^j_k|^2,$$

for all $\rho' \in U$, for all $z_k = (z^0_k, \dots, z^{2m_k-1}_k) \in \mathbb{C}^{m_k}$.

Proof. As the covering condition is fulfilled at $\rho'_0 \in \mathcal{L}^T(\mathcal{I})$, there exist an open conic neighborhood U of ρ'_0 in $\overline{V^+} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ such that the rank condition (5.2.17) is satisfied at ρ', for every $\rho' \in U$. Yet, consider the compact set $\mathcal{K} := \overline{U} \cap \mathbb{S}_{|(\xi',\tau)|=1}$, where $\mathbb{S}_{|(\xi',\tau)|=1} := \{(x, \xi', \tau) \in \mathbb{R}^d_+ \times \mathbb{R}^{d-1} \times \mathbb{R}^+, \ |\xi'|^2 + \tau^2 = 1\}$. Let $\tilde{\rho}' \in \mathcal{K}$. in view of (5.2.17), we have that $M(\tilde{\rho}')$ is of maximal rank, that is $M(\tilde{\rho}') = 2m$. Taking $z = (z_1, \dots, z_n)$ with $z_k = (z^0_k, \dots, z^{2m_k-1}_k) \in \mathbb{C}^{2m_k}$, we have

$$\left({}^t M(\tilde{\rho}) z, {}^t M(\tilde{\rho}) z \right) \gtrsim ||z||^2_{\mathbb{C}^{2m}}, \tag{5.4.6}$$

Recalling the form of M in (5.2.16), we have

$$\Big(\overline{M(\tilde\rho')}\ {}^tM(\tilde\rho')z,z\Big)=|\sum_{k=1}^n {}^tM_k^1(\tilde\rho')z_k|^2+\sum_{k=1}^n|{}^tM_k^2(\tilde\rho')z_k|^2$$
$$=\sum_{j=1}^m\big|\sum_{k=1}^n\Sigma_{t^j_{k,\varphi}}(\tilde\rho',z_k)\big|^2+\sum_{k=1}^n\sum_{j=m+1}^{m+m_k^-}|\Sigma_{e^j_{k,\varphi}}(\tilde\rho',z_k)|^2,$$

and with (5.4.6), we have

$$\sum_{j=1}^m\big|\sum_{k=1}^n\Sigma_{t^j_{k,\varphi}}(\tilde\rho',z_k)\big|^2+\sum_{k=1}^n\sum_{j=m+1}^{m+m_k^-}|\Sigma_{e^j_{k,\varphi}}(\tilde\rho',z_k)|^2\gtrsim||z||^2_{\mathbb{C}^{2m}}.\tag{5.4.7}$$

Then, by continuity, (5.4.7) is true in a neighborhood of $\tilde\rho'$, and with a compactness argument, is true for all $\rho'\in\mathcal{K}$. Yet, we define

$$H_{\lambda_T}(\rho')=(x,\lambda_T\xi',\lambda_T\tau),$$

and thus for every $\tilde\rho'\in\mathcal{K}$, we have $\tilde\rho'=H_{\lambda_T}^{-1}\rho'$ with $\rho':=H_{\lambda_T}\tilde\rho'$. We set $\tilde z_k^j=\lambda^{2m_k-1/2-j}z_k^j$, $\tilde z_k=(\tilde z_k^0,\dots,z_k^{2m_k-1})$ and $\tilde z=(\tilde z_1,\dots,\tilde z_n)$. By homogeneity of the symbols, (5.4.7) yields

$$\sum_{j=1}^m\big|\sum_{k=1}^n\Sigma_{t^j_{k,\varphi}}(\tilde\rho',\tilde z_k)\big|^2+\sum_{k=1}^n\sum_{j=m+1}^{m+m_k^-}|\Sigma_{e^j_{k,\varphi}}(\tilde\rho',\tilde z_k)|^2\gtrsim||\tilde z||^2_{\mathbb{C}^{2m}},$$

which reads

$$\sum_{j=1}^m\big|\sum_{k=1}^n\lambda_T^{2m_k+1/2-\beta_k^j}\Sigma_{t^j_{k,\varphi}}(\rho',z_k)\big|^2+\sum_{k=1}^n\sum_{j=m+1}^{m+m_k^-}|\lambda_T^{m+m_k^-+1/2-j}\Sigma_{e^j_{k,\varphi}}(\tilde\rho',z_k)|^2\gtrsim\sum_{k=1}^n\sum_{j=0}^{2m_k-1}|\lambda^{2m_k-1/2-j}z_k^j|^2.$$

Using assumption (5.1.2) yields the result. □

5.4.5 Estimate with the covering condition

The following proposition shows how the covering condition allows one to estimate all the traces at the interface. The estimate is microlocal near a point where the covering condition holds.

Proposition 5.4.11. *Let $\rho_0'=(x_0,\xi_0,\tau_0)$, $x_0\in\mathcal{I}$, such that the covering condition of Definition 5.2.3 holds at ρ_0'. Then, there exists a conic neighborhood $\mathcal{V}$ of ρ_0' in $\overline{V}^+\times\mathbb{R}^{d-1}\times\mathbb{R}^+$ such that for $\chi\in\mathcal{S}^0_{T,\tau}$ homogeneous in (ξ',τ) with $\operatorname{supp}\chi\subset\mathcal{V}$, there exist $C>0$ and $\tau_1>0$ such that*

$$C\sum_{k=1}^n|\operatorname{tr}\operatorname{Op}_T(\chi)v_k|^2_{2m_k-1,1/2,\tau}\le\sum_{j=1}^m|\sum_{k=1}^n T^j_{k,\varphi}v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}+\sum_{k=1}^n||P_{k,\varphi}v_k||^2_+$$
$$+\sum_{k=1}^n\big(||v_k||^2_{2m_k,-1,\tau}+|\operatorname{tr}v_k|^2_{2m_k-1,-1/2,\tau}\big),\tag{5.4.8}$$

for all $\tau\ge\tau_1$, $v=(v_1,\dots,v_k)\in(\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$.

Proof. Consider the factorization described in Section 5.2.5

$$p_{k,\varphi}(\rho',\xi_d)=p^-_{k,\varphi}(\rho',\xi_d)\kappa(\rho',\xi_d),\tag{5.4.9}$$

for ρ' in a conic neighborhood $\mathcal{V}_0$ of ρ', with $p^-_{k,\varphi}$, $\kappa_{k,\varphi}$ polynomials with homogeneous coefficients, $p^-_{k,\varphi}$ being monic. Using Proposition 5.4.10, there exists a conic neighborhood $\mathcal{V}_1$ of ρ'_0, $\mathcal{V}_1 \subset \mathcal{V}_0$, such that

$$\sum_{j=1}^{m} \lambda_T^{2(\gamma_j-1/2)} \Big|\sum_{k=1}^{n} \Sigma_{t^j_{k,\varphi}}(\rho', z_k)\Big|^2 + \sum_{k=1}^{n}\sum_{j=m+1}^{m'_k} \lambda_T^{2(m+m^-_k-j+1/2)} |\Sigma_{e^j_{k,\varphi}}(\rho', z_k)|^2 \\ \geq C \sum_{k=1}^{n}\sum_{j=0}^{2m_k-1} \lambda_T^{2(2m_k-1/2-j)} |z^j_k|^2, \quad (5.4.10)$$

holds for all $\rho' \in \mathcal{V}_1$ and for all $z = (z_1, \dots, z_n) \in \mathbb{C}^{2m}$, with $z_k = (z^0_k, \dots, z^{2m_k-1}_k) \in \mathbb{C}^{2m_k}$. Now we consider a conic neighborhood $\mathcal{V}$ of ρ'_0 satisfying $\mathcal{V} \subset \mathcal{V}_1$, and $\chi \in \mathcal{S}^0_{T,\tau}$ homogeneous such that $\operatorname{supp}\chi \subset \mathcal{V}$, and $\tilde\chi \in \mathcal{S}^0_{T,\tau}$ homogeneous such that $\operatorname{supp}\tilde\chi \subset \mathcal{V}_1$, and $\tilde\chi = 1$ in a neighborhood of $\mathcal{V}$. Consider a smooth extension $\tilde p^-_{k,\varphi}$ outside $\mathcal{V}_0$. Note that by symbolic calculus, we have $\operatorname{Op}_T(\chi)P_{k,\varphi} = \operatorname{Op}(\tilde p^-_{k,\varphi})\operatorname{Op}_T(\chi)\operatorname{Op}(\tilde\chi\kappa_{k,\varphi}) + R_k$, and $R_k \in \Psi^{2m_k,-1}_{\tau,\mathrm{cl}}$. Take also $v = (v_1, \dots, v_n) \in (\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$. By definition, $p^-_{k,\varphi}$ has only roots with negative imaginary parts. Thus we can apply Lemma 5.4.6 to the function $w_k := \operatorname{Op}(\tilde\chi\kappa_{k,\varphi})v_k \in \mathscr{S}(\overline{\mathbb{R}}^d_+)$, and we obtain, for all $k \in \{1, \dots, n\}$, for any $N \in \mathbb{N}$,

$$||\operatorname{Op}_T(\chi)w_k||^2_{m^-_k,\tau} + |\operatorname{tr}(\operatorname{Op}_T(\chi)w_k)|_{m^-_k-1,1/2,\tau} \quad (5.4.11)$$

$$\lesssim ||\operatorname{Op}(p^-_{k,\varphi})\operatorname{Op}_T(\chi)w_k||^2_+ + ||w_k||^2_{m^-_k,-N,\tau} + |\operatorname{tr} w_k|^2_{m^-_k-1,-N,\tau}$$

$$\lesssim ||P_{k,\varphi}v_k||^2_+ + ||v_k||^2_{2m_k,-1\tau} + |\operatorname{tr} v_k|^2_{2m_k-1,-N,\tau}. \quad (5.4.12)$$

By definition, we have $|\operatorname{tr}(\operatorname{Op}_T(\chi)w_k)|^2_{m^-_k-1,1/2,\tau} = \sum_{j=0}^{m^-_k-1} |D^j_d \operatorname{Op}_T(\chi)w_k|^2_{m^-_k-1/2-j,\tau}$, thus (5.4.12) gives

$$\sum_{j=0}^{m^-_k-1} |D^j_d \operatorname{Op}_T(\chi)w_k|^2_{m^-_k-1/2-j,\tau} \lesssim ||P_{k,\varphi}v_k||^2_+ + ||v_k||^2_{2m_k-1,\tau} + |\operatorname{tr} v_k|^2_{2m_k-1,-N,\tau}. \quad (5.4.13)$$

Yet, we recall that for $j \in \{0, \dots, m^-_k - 1\}$, from (5.2.15), we defined $e^{j+m+1}_k(\rho', \xi_d) = \xi^j_d \kappa_{k,\varphi}(\rho', \xi_d)$, for $\rho' \in \mathcal{V}_0$. Hence, by symbolic calculus, for $j \in \{0, \dots, m^-_k - 1\}$

$$D^j_d \operatorname{Op}_T(\chi)\operatorname{Op}(\tilde\chi\kappa_{k,\varphi}) = \operatorname{Op}(\tilde\chi e^{j+m+1}_k)\operatorname{Op}_T(\chi) + R'_k,$$

where the remainder $R'_k \in \Psi^{2m_k-m^-_k+j,-1}_\tau$. From (5.4.13), this yields

$$\sum_{k=1}^{n}\sum_{m+1}^{m'_k} |\operatorname{Op}(\tilde\chi e^j_k)\operatorname{Op}_T(\chi)v_k|^2_{m^-_k-1/2-j,\tau} \lesssim \sum_{k=1}^{n} ||P_{k,\varphi}v_k||^2_+ + ||v_k||^2_{2m_k,-1,\tau} + |\operatorname{tr} v_k|^2_{2m_k-1,-1/2,\tau}. \quad (5.4.14)$$

From (5.4.10), as $\tilde\chi = 1$ on $\mathcal{V}$, we have for all $\rho' \in \mathcal{V}$,

$$\sum_{j=1}^{m} \lambda_T^{2(\gamma_j-1/2)} \Big|\sum_{k=1}^{n} \Sigma_{t^j_{k,\varphi}}(\rho', z_k)\Big|^2 + \sum_{k=1}^{n}\sum_{j=m+1}^{m'_k} \lambda_T^{2(m+m^-_k-j+1/2)} |\tilde\chi(\rho')\Sigma_{e^j_{k,\varphi}}(\rho', z_k)|^2 \\ \geq C \sum_{k=1}^{n}\sum_{j=0}^{2m_k-1} \lambda_T^{2(2m_k-1/2-j)} |z^j_k|^2. \quad (5.4.15)$$

Yet, consider the following quadratic form, for $u = (u_1, \dots, u_n) \in (\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$,

$$\mathcal{G}(u) = \sum_{j=1}^{m} \Big|\sum_{k=1}^{n} T^j_{k,\varphi} u_{k|_{x_d=0}}\Big|^2_{\gamma_j-1/2,\tau} + \sum_{k=1}^{n}\sum_{j=m+1}^{m'_k} |\operatorname{Op}(\tilde\chi e^j_k)u_{k|_{x_d=0}}|^2_{m+m^-_k+1/2-j},$$

which symbol is given by

$$\Sigma_{\mathcal{G}}(\rho',z,\tilde{z})=\sum_{j=1}^{m}\lambda_T^{2(\gamma_j-1/2)}\left(\sum_{k=1}^{n}\Sigma_{t^j_{k,\varphi}}(\rho',z_k)\right)\overline{\left(\sum_{k=1}^{n}\Sigma_{t^j_{k,\varphi}}(\rho',\tilde{z}_k)\right)}$$
$$+\sum_{k=1}^{n}\sum_{j=m+1}^{m+m_k^-}|\tilde{\chi}(\rho')|^2\lambda_T^{2(m+m_k^-+1/2-j}\Sigma_{e^j_k}(\rho',z_k)\overline{\Sigma_{e^j_k}(\rho',\tilde{z}_k)},$$

with $z=(z_1,\dots,z_n)$, where $z_k\in\mathbb{C}^{2m_k}$. With (5.4.15), and the Gårding inequality of Lemma 5.4.9, for $N\in\mathbb{N}$, there exists $\tau_0,C,C'>0$ such that

$$\mathcal{G}(\mathrm{Op}_T(\chi)v)\geq C\sum_{k=1}^{n}|\operatorname{tr}(\mathrm{Op}(\chi))v_k|^2_{2m_k-1,1/2,\tau}-C'\sum_{k=1}^{n}|\operatorname{tr}v_k|^2_{2m_k-1,-N,\tau},\tag{5.4.16}$$

for $\tau\geq\tau_0$. Hence, inserting (5.4.16) in (5.4.14), we obtain

$$\sum_{k=1}^{n}|\operatorname{tr}(\mathrm{Op}(\chi)v_k)|^2_{2m_k-1,1/2,\tau}\lesssim\sum_{j=1}^{m}|\sum_{k=1}^{n}T^j_{k,\varphi}\,\mathrm{Op}_T(\chi)v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}$$
$$+\sum_{k=1}^{n}(||P_{k,\varphi}v_k||^2_{+}+||v_k||^2_{2m_k,-1,\tau}+|\operatorname{tr}v_k|^2_{2m_k-1,-1/2,\tau}).\tag{5.4.17}$$

It remains to see that $T^j_{k,\varphi}\,\mathrm{Op}_T(\chi)=\mathrm{Op}_T(\chi)T^j_{k,\varphi}+[T^j_{k,\varphi},\mathrm{Op}_T(\chi)]$ to obtain

$$\sum_{j=1}^{m}|\sum_{k=1}^{n}T^j_{k,\varphi}\,\mathrm{Op}_T(\chi)v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}\lesssim\sum_{j=1}^{m}|\sum_{k=1}^{n}T^j_{k,\varphi}v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}+\sum_{j=1}^{m}\sum_{k=1}^{n}|\operatorname{tr}v_k|^2_{\beta^j_k,2m_k-\beta^j_k-3/2,\tau}.$$

Using the definition of the norms on the traces (5.4.3), we have

$$\begin{aligned}|\operatorname{tr}v_k|^2_{\beta^j_k,2m_k-\beta^j_k-3/2,\tau}&=\sum_{\ell=0}^{\beta^j_k}|D^\ell v_{k|_{x_d=0}}|^2_{2m_k-\ell-3/2,\tau}\leq\sum_{\ell=0}^{2m_k-1}|D^\ell v_{k|_{x_d=0}}|^2_{2m_k-\ell-3/2,\tau}\\&=|\operatorname{tr}v_k|^2_{2m_k-1,-1/2,\tau},\end{aligned}$$

and the proof is complete. □

5.4.6 Microlocal estimate with operators satisfying both sub-ellipticity and covering condition

Using both the covering condition (by means of Proposition 5.4.11) and the sub-ellipticity condition (by means of Lemma 5.4.5), we are ready to prove the following microlocal Carleman estimate, in a conic neighborhood of a point in $\mathcal{L}^T(\mathcal{I})$.

Theorem 5.4.12. *Let $x_0\in\mathcal{I}\cap\overline{V}$. Assume that (P_k,φ_k) satisfies the sub-ellipticity condition in a neighborhood of x_0 in $\overline{V}^+$, and that (T^j_k,φ) covers (P_k,φ) at $\rho'\in\mathcal{L}^T_{x_0}(\mathcal{I})$. Then, there exists $\mathcal{V}$ a conic neighborhood of ρ' in $\overline{V}^+\times\mathbb{R}^{d-1}\times\mathbb{R}^+$ such that for $\chi\in\mathcal{S}^0_{T,\tau}$ homogeneous of degree 0 satisfying $\operatorname{supp}\chi\subset\mathcal{V}$, there exist $C,\tau_0>0$ such that*

$$\tau^{-1}\sum_{k=1}^{n}||\,\mathrm{Op}_T(\chi)v_k||^2_{2m_k,\tau}+\sum_{k=1}^{n}|\operatorname{tr}(\mathrm{Op}(\chi)v)|^2_{2m_k-1,1/2,\tau}\leq C\Big(\sum_{k=1}^{n}||P_{k,\varphi}v_k||^2_{+}$$
$$+\sum_{j=1}^{m}|\sum_{k=1}^{n}T^j_{k,\varphi}v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}+\sum_{k=1}^{n}||v_k||^2_{2m_k,-1,\tau}+\sum_{k=1}^{n}|\operatorname{tr}v_k|^2_{2m_k-1,-1/2,\tau}\Big),\tag{5.4.18}$$

for $\tau \geq \tau_0$, for $v = (v_1, \dots, v_n) \in (\mathscr{S}(\overline{\mathbb{R}}^d_+))^n$.

Proof. Consider a neighborhood U of x_0 in $\overline{V}^+$ such that the sub-ellipticity condition holds in U. First we write

$$P_{k,\varphi} = P_{k,2} + iP_{k,1}, \quad \text{with } P_{k,2} = \frac{1}{2}\left(P_{k,2} + P^*_{k,2}\right),\ P_{k,1} = \frac{1}{2i}\left(P_{k,2} - P^*_{k,2}\right). \tag{5.4.19}$$

We shall denote by $p_{k,\varphi} \in \mathcal{S}^{m_k,0}_\tau$, $p_{k,2} \in \mathcal{S}^{m_k,0}_\tau$, $p_{k,1} \in \mathcal{S}^{m_k-1,1}_\tau$ their principal symbols, respectively. Let $\mathcal{U}$ be a conical neighborhood of ρ'_0 given by Proposition 5.4.11. We have from (5.4.19)

$$\begin{aligned}
||v_k||^2_{2m_k,-1,\tau} + ||P_{k,\varphi}v_k||^2_+ &\gtrsim ||\operatorname{Op}_T(\chi)v_k||^2_{2m_k,-1,\tau} + ||P_{k,\varphi}\operatorname{Op}_T(\chi)v_k||^2_+ \\
&\gtrsim ||\operatorname{Op}(p_{k,2})\operatorname{Op}_T(\chi)v_k||^2_+ + ||\operatorname{Op}(p_{k,1})\operatorname{Op}_T(\chi)v_k||^2_+ + Q_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k) \\
&\gtrsim \tau^{-1}||\operatorname{Op}(p_{k,2})\operatorname{Op}_T(\chi)v_k||^2_+ + \tau^{-1}||\operatorname{Op}(p_{k,1})\operatorname{Op}_T(\chi)v_k||^2_+ \\
&\quad + Q_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k),
\end{aligned} \tag{5.4.20}$$

where $Q_{p_{k,2},p_{k,1}}$ is the quadratic form defined in (5.4.4). We have from Lemma 5.4.5, as the sub-ellipticity condition holds,

$$\begin{aligned}
C\tau^{-1}||\operatorname{Op}_T(\chi)v_k||^2_{2m_k,\tau} \leq C'\tau^{-1}\big(&||\operatorname{Op}(p_{k,2})\operatorname{Op}_T(\chi)v_k||^2_+ + ||\operatorname{Op}(p_{k,1})\operatorname{Op}_T(\chi)v_k||^2_+ \\
&+ |\operatorname{tr}\operatorname{Op}_T(\chi)v_k|^2_{2m_k-1,1/2,\tau}\big) + \big(Q_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k) - \operatorname{Re}\mathcal{B}_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k)\big),
\end{aligned} \tag{5.4.21}$$

where $\mathcal{B}_{p_{k,2},p_{k,1}}$ satisfies

$$\mathcal{B}_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k) \leq C|\operatorname{tr}(\operatorname{Op}_T(\chi)v_k)|^2_{2m_k-1,1/2,\tau}.$$

From Proposition 5.4.11, with the covering condition, we have

$$\begin{aligned}
C\sum_{k=1}^n |\operatorname{tr}\operatorname{Op}_T(\chi)v_k|^2_{2m_k-1,1/2,\tau} \leq \sum_{j=1}^m |\sum_{k=1}^n T^j_{k,\varphi}v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau} + \sum_{k=1}^n ||P_{k,\varphi}v_k||^2_+ \\
+ \sum_{k=1}^n \big(||v_k||^2_{2m_k,-1,\tau} + |\operatorname{tr} v_k|^2_{2m_k-1,-1/2,\tau}\big).
\end{aligned} \tag{5.4.22}$$

Combining estimates (5.4.21) and (5.4.22), we obtain, with C' taken sufficiently large

$$\begin{aligned}
&C\tau^{-1}\sum_{k=1}^n ||\operatorname{Op}_T(\chi)v_k||^2_{2m_k,\tau} + C\sum_{k=1}^n |\operatorname{tr}\operatorname{Op}_T(\chi)v_k|^2_{2m_k-1,1/2,\tau} \\
&\leq C'\Big(\tau^{-1}\sum_{k=1}^n ||\operatorname{Op}(p_{k,2})\operatorname{Op}_T(\chi)v_k||^2_+ + \tau^{-1}\sum_{k=1}^n ||\operatorname{Op}(p_{k,1})\operatorname{Op}_T(\chi)v_k||^2_+ + \sum_{j=1}^m |\sum_{k=1}^n T^j_{k,\varphi}v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau} \\
&\quad + \sum_{k=1}^n ||P_{k,\varphi}v_k||^2_+ + \sum_{k=1}^n (||v_k||^2_{2m_k,-1,\tau} + |\operatorname{tr} v_k|^2_{2m_k-1,-1/2,\tau})\Big) + \sum_{k=1}^n Q_{p_{k,2},p_{k,1}}(\operatorname{Op}_T(\chi)v_k).
\end{aligned} \tag{5.4.23}$$

It remains to use (5.4.20) to obtain the sought result. □

5.4.7 Proof of the main result.

Let $x_0 \in \mathcal{I}$ and $\overline{V}$ be like in Theorem 5.4.12. By assumption the covering condition holds for all $(x_0, Y, \nu_1, \dots, \nu_n, \tau) \in \mathcal{L}^T_{x_0}(\mathcal{I})$. Using the local coordinates introduced in Section 5.2, we can only consider $\rho = (x, \xi', \xi_d, \tau)$, with $(x, \xi', \tau) \in \mathbb{S}_{|(\xi',\tau)|=1}$. According to Theorem 5.4.12, for all $\rho' = (x, \xi', \tau)$, there exists a conic neighborhood $\mathcal{U}_{\rho'} = O_{\rho'} \times \mathcal{C}_{\rho'}$ of ρ' in $\overline{V} \times \mathbb{R}^{d-1} \times \mathbb{R}^+$ such that the Carleman estimate (5.4.18) holds in $\mathcal{U}_{\rho'}$. By a compactness argument, we can extract from the family of the $\mathcal{U}_{\rho'}$ a finite number of conical subsets $\mathcal{U}_j := O_j \times \mathcal{C}_j$,

$j \in J$ such that $\mathbb{S}_{|(\xi',\tau)|=1} \subset \cup_{j\in J}\mathcal{C}_j$. First we set $W = \cap_{j\in J} O_j$ and $\mathcal{V}_j = W \times \mathcal{C}_j$. We construct a partition of unity $(\chi_j)_{j\in J}$, $\chi_j \in \mathcal{S}^0_{T,\tau}$, homogeneous of degree 0 in $(\xi',\tau) \geq C > 0$ such that

$$\sum_{j\in J} \chi_j(\rho') = 1, \text{ for } (\xi',\tau) \geq C > 0, \quad \operatorname{supp}(\chi_j) \subset \mathcal{V}_j.$$

Applying Theorem 5.4.12, we have

$$\tau^{-1}\sum_{j\in J}\sum_{k=1}^{n} ||\operatorname{Op}_T(\chi_j)v_k||^2_{2m_k\tau} + \sum_{j\in J}\sum_{k=1}^{n} |\operatorname{tr}(\operatorname{Op}(\chi_j)v_k)|^2_{2m_k-1,1/2,\tau} \leq C\Big(\sum_{k=1}^{n} ||P_{k,\varphi}v_k||^2_+ + \sum_{j=1}^{m}|\sum_{k=1}^{n} T^j_k v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau} + \sum_{k=1}^{n}||v_k||^2_{2m_k,-1,\tau} + \sum_{k=1}^{n}|\operatorname{tr} v_k|^2_{2m_k-1,-1/2,\tau}\Big), \quad (5.4.24)$$

for all $v_k \in \overline{C}^{+\infty}_0(W)$, $k = 1,\dots,n$. From the fact that $\tilde{\chi} := 1 - \sum_{j\in J}\chi_j$ belongs to $\mathcal{S}^{-N}_{T,\tau}$ in $W \times \mathbb{R}^{d-1} \times \mathbb{R}^+$, for all $N \in \mathbb{N}$, we have that

$$\tau^{-1}\sum_{k=1}^{n}||v_k||^2_{2m_k\tau} + \sum_{k=1}^{n}|\operatorname{tr}(v_k)|^2_{2m_k-1,1/2,\tau} \leq \tau^{-1}\sum_{j\in J}\sum_{k=1}^{n}||\operatorname{Op}_T(\chi_j)v_k||^2_{2m_k,\tau} + \sum_{j\in J}\sum_{k=1}^{n}|\operatorname{tr}(\operatorname{Op}(\chi_j)v_k)|^2_{2m_k-1,1/2,\tau} + \sum_{k=1}^{n}||v_k||^2_{2m_k-1,1/2-N} + \sum_{k=1}^{n}|\operatorname{tr}(v_k)|^2_{2m_k-1,1/2-N,\tau}, \quad (5.4.25)$$

as $\operatorname{supp}(w_k) \subset W$. Combining (5.4.24) and (5.4.25), we obtain by taking τ large,

$$\tau^{-1}\sum_{k=1}^{n}||v_k||^2_{2m_k-1,1/2,\tau} + \sum_{k=1}^{n}|\operatorname{tr}(v_k)|^2_{2m_k-1,1/2,\tau} \leq C\Big(\sum_{k=1}^{n}||P_{k,\varphi}v_k||^2_+ + \sum_{j=1}^{m}|\sum_{k=1}^{n}T^j_k v_{k|_{x_d=0}}|^2_{\gamma_j-1/2,\tau}\Big).$$

Going back with the original variable $u_k = e^{\tau\varphi_k}v_k$ yields the result, taking into account of the change of variable described in (5.2.1). With the notation introduced there, the open set $\tilde{U}$ can be chosen as $\tilde{U} = \cap_{k=1}^n \phi_k^{-1}(W)$. □

5.5 Applications to control theory for the transmission problem

In this section, we shall place ourselves in the following setting. We consider p smooth compact Riemannian manifolds (Ω_k, g_k) of dimension $d \geq 2$, $k = 1,\dots,p$, with boundary $\partial\Omega_k$. We set $\Omega = \cup_{k=1}^p \Omega_k$. We assume that Ω is connected and that $\partial\Omega_k$ has n_k connected component, with $n_k \geq 1$. We further assume that the manifolds shares some of the connected component of their boundary with others. More precisely, we shall denote by $\mathcal{I}_\ell$, $\ell \in \{1,\dots,L\}$ the set of interfaces that are shared by at least two manifolds, and we shall denote by $\mathcal{B}_k$, $k \in \{1,\dots,K\}$ the remaining connected components of the boundaries. As in Section 5.3, we shall consider the Laplace-Beltrami operator on each Ω_k and a transmission problem as in (5.3.1) at each interface $\mathcal{I}_\ell$, $\ell \in \{1,\dots,L\}$. The case $d = 1$, corrseponds to the case of elliptic problems on a connected graph. However, as we prove our Carleman estimates only for $d \geq 2$, we do not consider the case $d = 1$ is what follows. For such issues, we refer to [BCV11, DZ06, LLS94, Zua13] and the references therein.

5.5.1 Observable domains

We propose a method to describe the global geometrical configuration we can deal with. Indeed, due to the discussion made at the end of Section 5.3.1, locally, we have to observe any n-interface from at least $(n-1)$-sides.

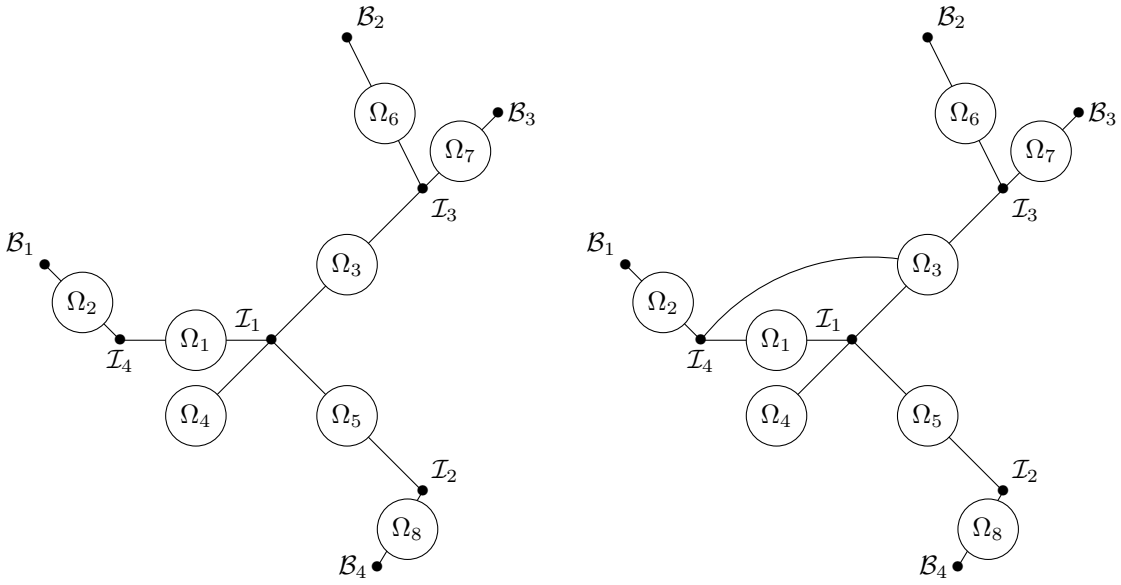


Figure 5.3: We recast geometrical configurations under a graph form: the edges represent the manifolds Ω_k, and the vertices the interfaces $\mathcal{I}_\ell$, or the boundaries $\mathcal{B}_k$.

Let ω be an open subset of Ω. We shall write $\omega_\ell = \Omega_\ell \cap \omega$, and $\mathcal{L} = \{\ell \in \{1,\dots,p\},\ \omega_\ell \neq \emptyset\}$. For $k \in \{1,\dots,p\}$, we define the following map

$$\mathcal{N}_k : I_\ell \longmapsto \begin{cases} 1 & \text{if } \Omega_k \cap \mathcal{I}_\ell \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.1}$$

In other words, this map returns 1 if $\mathcal{I}_\ell$ is an interface shared by Ω_k. We set $\mathcal{V}^\ell := \{k \in \{1,\dots,p\},\ \mathcal{N}_k(\mathcal{I}_\ell) = 1\}$, that is, the set of manifolds that share the interface $\mathcal{I}_\ell$.
Conversely, we define $\mathcal{W}_k := \{\ell \in \{1,\dots,L\},\ \mathcal{N}_k(\mathcal{I}_\ell) = 1\}$, that is, the set of indices of interfaces that are shared by the manifold Ω_k.

We consider the following construction, with $\mathcal{O}_0 := \mathcal{L}$,

$$\mathcal{O}_j = \mathcal{O}_{j-1} \cup \left(\cup_{\ell=1}^L \mathcal{V}_j^\ell\right), \qquad \mathcal{V}_j^\ell := \begin{cases} \tilde{\mathcal{V}}_j^\ell & \text{if } \operatorname{Card} \tilde{\mathcal{V}}_j^\ell \leq 1 \\ \emptyset & \text{otherwise,} \end{cases} \quad \text{and where } \tilde{\mathcal{V}}_j^\ell = \mathcal{V}^\ell \setminus \mathcal{O}_j. \tag{5.5.2}$$

In other words, at each step, for all $\ell \in \{1,\dots,L\}$, we count the number of sides of $\mathcal{I}_\ell$ that belongs to $\mathcal{O}_j$. If this number corresponds to an observation from at least $(n-1)$-sides, we add to $\mathcal{O}_j$ the remaining side. Observe that $\operatorname{Card} \mathcal{V}_j^\ell \leq 1$ at each step. Note also that if $\mathcal{O}_j = \mathcal{O}_{j+1}$ for some j, then $\mathcal{O}_{j'} = \mathcal{O}_j$, for all $j' \geq j$. With this construction, we have the following definition.

Definition 5.5.1. *We say that (Ω, ω) is observable, if there exists $j_0 \in \mathbb{N}$ such that $\mathcal{O}_{j_0} = \{1,\dots,p\}$ in the iterated construction described in (5.5.2).*

In the example given on the left-hand-side of Figure 5.3, assume $\mathcal{L} = \{2,4,5,6\}$. Then

1. $\mathcal{O}_0 = \mathcal{L}$, $\cup_{\ell=1}^L \tilde{\mathcal{V}}_0^\ell = \{1,3,7,8\}$, $\cup_{\ell=1}^L \mathcal{V}_0^\ell = \{1,8\}$
2. $\mathcal{O}_1 = \{1,2,4,5,6,8\}$, $\cup_{\ell=1}^L \tilde{\mathcal{V}}_1^\ell = \{3,7\}$, $\cup_{\ell=1}^L \mathcal{V}_1^\ell = \{3\}$
3. $\mathcal{O}_2 = \{1,2,3,4,5,6,8\}$, $\cup_{\ell=1}^L \tilde{\mathcal{V}}_2^\ell = \{7\}$, $\cup_{\ell=1}^L \mathcal{V}_2^\ell = \{7\}$

4. $\mathcal{O}_3 = \{1,2,3,4,5,6,7,8\}$.

As a result, according to Definition 5.5.1, we have that (Ω,ω) is observable. In the example given on the right hand side of Figure 5.3, assume $\mathcal{L} = \{2,4,5,6\}$. Then

1. $\mathcal{O}_0 = \mathcal{L}$, $\cup_{\ell=1}^L \tilde{\mathcal{V}}_0^\ell = \{1,3,7,8\}$, $\cup_{\ell=1}^L \mathcal{V}_0^\ell = \{8\}$
2. $\mathcal{O}_1 = \{2,4,5,6,8\}$, $\cup_{\ell=1}^L \tilde{\mathcal{V}}_1^\ell = \{3,4,7\}$, $\cup_{\ell=1}^L \mathcal{V}_1^\ell = \emptyset$
3. $\mathcal{O}_1 = \mathcal{O}_2 \neq \{1,\dots,8\}$.

Hence, (Ω,ω) is not observable in this particular case.

5.5.2 Setting

Consider $Z := (0,X_0)\times\Omega$, where $X_0 > 0$ (resp. $Z_k := (0,X_0)\times\Omega_k$). Let $\alpha \in (0,X_0/2)$, and define $Y := (\alpha, X_0-\alpha)\times\Omega$ (resp. $Y_k := (\alpha,X_0-\alpha)\times\Omega_k$). For $z=(z_0,x)\in Z$ (resp. Y), we shall mean $z_0 \in (0,X_0)$ (resp. $z_0\in(\alpha,X_0-\alpha)$) and $x\in\Omega$. For $\ell\in\{1,\dots,L\}$ (see the beginning of this section for the notation), we also set $S_\ell = (0,X_0)\times\mathcal{I}_\ell$ and $S_\ell^\alpha = (\alpha,X_0-\alpha)\times\mathcal{I}_\ell$. For $k\in\{1,\dots,K\}$, we consider $R_k = (0,X_0)\times\mathcal{B}_k$ and $R_k^\alpha = (\alpha,X_0-\alpha)\times\mathcal{B}_k$. We finally set $\mathfrak{W}_k = \{i\in\{1,\dots,p\}, \Omega_i\cap\mathcal{B}_k\neq\emptyset\}$, that is the set of the indices of the manifold that has $\mathcal{B}_k$ as a boundary. We consider the following transmission problem for a general domain Ω, with Dirichlet boundary condition:

$$\begin{cases} -\partial_{x_0}^2 u_k - \Delta_{g_k} u_k = f_k & \text{in } Z_k, \quad k=1,\dots,p \\ u_{i|_{S_\ell}} = u_{j|_{S_\ell}} & \text{on } S_\ell, \quad \forall \ell\in\{1,\dots,L\}, \quad \forall i,j\in\mathcal{V}^\ell, \\ \sum_{k\in\mathcal{V}^\ell} \partial_{\nu_k} u_{k|_{S_\ell}} = 0 & \text{on } S_\ell, \quad \forall \ell\in\{1,\dots,L\}, \\ u_{i|_{\mathcal{B}_j}} = 0 & \text{on } R_k, \quad \forall j\in\{1,\dots,K\}, \quad i\in\mathfrak{W}_j, \end{cases} \tag{5.5.3}$$

where Δ_{g_k} stands for the Laplace-Beltrami operator on Ω_k. Other boundary conditions can also be considered satisfying the Lopatinskii condition. For the sake of simplicity, we only consider Dirichlet boundary conditions here. We define $u(x) = u_k(x)$, and $f(x) = f_k(x)$ if $x\in Z_k$. Similarlly, for v defined on Ω (resp. Z or Y), we set v_k on Ω_k (resp. Z_k or Y_k) as $v_k(x) = v(x)$ for $x\in\Omega_k$ (resp. $v_k(s,x) = v(s,x)$ for $(s,x)\in Z_k$ or Y_k). We also define the Sobolev spaces, for $m\in\mathbb{N}$,

$$H^m(\Omega) = \{u,\ u_k\in H^m(\Omega_k)\},$$

and on Z (resp. Y), we define

$$H^m(Z) = \{u(x) := u_k(x),\ x\in Z_k,\ u_k\in H^m(Z_k)\}, \quad H^m(Y) = \{u(x) := u_k(x),\ x\in Y_k,\ u_k\in H^m(Y_k)\}.$$

5.5.3 An interpolation inequality

By standard arguments, [Rob95], [LR95],[LR97], local Carleman estimates are used to prove interpolation inequalities.

Theorem 5.5.2. *Let ω be a subset of Ω such that (Ω,ω) is observable, in the sense of Definition 5.5.1. There exist $C>0$ and $\delta>0$ such that*

$$||u||_{H^1(Y)} \leq C||u||_{H^1(Z)}^{1-\delta}\Big(||f||_{L^2(Z)} + |u_{|_{x_0=0}}|_{H^1(\omega)} + |\partial_{x_0}u_{|_{x_0=0}}|_{L^2(\omega)}\Big)^\delta, \tag{5.5.4}$$

for all $u\in H^1(Y)$ solution of (5.5.3), with $f\in L^2(Z)$.

It is classical that the proof of Theorem 5.5.2 can be reduced to proving the following lemma, which is a local version of Theorem 5.5.2, in a neighborhood of a multi-interface. Local versions of Theorem 5.5.2 in the interior, or at the boundary for Dirichlet boundary conditions can be found for instance in [LR95]. For $\ell\in\{1,\dots,L\}$ and $k\in\mathcal{V}^\ell$, define $Y_k^\eta = \{x\in Y_k,\ \mathrm{d}(x,\cup_{\ell=1}^L S_\ell^\alpha\cup_{k=1}^K R_k^\alpha) > \eta\}$. Here, d denotes the Riemannian distance.

Lemma 5.5.3. *Let $\ell \in \{1, \dots, L\}$. Let $y \in S_\ell^\alpha$, and let $\eta > 0$. For all $k_0 \in \mathcal{V}^\ell$, there exist a neighborhood V of y in Z_α, $C > 0$ and $\delta \in (0,1)$ such that*

$$||u||_{H^1(V)} \leq C||u||^{1-\delta}_{H^1(Z)} \Big(\sum_{k \in \mathcal{V}^\ell} ||f_k||_{L^2(Z_k)} + \sum_{\substack{k \in \mathcal{V}^\ell \\ k \neq k_0}} ||u||_{H^1(Y_k^\eta)} \Big)^\delta,$$

for all $u \in H^1(Z)$ solution of (5.5.3), with $f \in L^2(Z)$.

This result corresponds to an observation from $(n-1)$-sides of the interface S_ℓ^α, where $n = \operatorname{Card} \mathcal{V}^\ell$, and quantifies the unique continuation property.

Proof of Lemma 5.5.3. This lemma is a consequence of the Carleman estimate. We shall place ourselves in a system of coordinates as given in Section 5.2, in a neighborhood $\mathcal{U}$ of a point $z^0 = (z_0^0, x^0) \in S_\ell^\alpha$, with $x_d^0 = 0$. We shall write $\mathcal{U}^+ = \mathcal{U} \cap \mathbb{R}^{d+1}_+$. We define two anisotropic distances in $\mathbb{R}^{d+1}$, for all $z^1, z^2 \in \mathbb{R}^{d+1}$,

$$\mathrm{d}_\gamma(z^1, z^2)^2 = \gamma |(z_0^1, (z^1)') - (z_0^2, (z^2)')|^2 + |z_d^1 - z_d^2|^2, \tag{5.5.5}$$

$$\mathrm{d}_\gamma^\beta(z^1, z^2)^2 = \gamma |(z_0^1, (z^1)') - (z_0^2, (z^2)')|^2 + \beta |z_d^1 - z_d^2|^2. \tag{5.5.6}$$

For $k \in \mathcal{V}^\ell \setminus \{k_0\}$, we define with $z^1 \in \mathbb{R}^{d+1}$, the weight function $\varphi_k(z^1, z) = e^{\lambda \mathrm{d}_\gamma(z^1, z)^2}$, and for $k = k_0$ we define $\varphi_{k_0}(\overline{z^1}, z) = e^{\lambda \mathrm{d}_\gamma^\beta(\overline{z^1}, z)^2}$, with $\overline{z^1} = (z_0^1, (z^1)', -z_d^1)$, that is, the symmetric of z^1 with respect to the axis $\{x_d = 0\}$. Note first that for $\lambda > 0$ sufficiently large, this family of weight functions satisfies the sub-ellipticity property (see Lemma 5.2.2) in $\overline{\mathcal{U}^+}$. For $z^1 \in \mathbb{R}^{d+1}_+$, it satisfies also conditions (5.3.15). Indeed, we have

$$\partial_{x_d} \varphi_{k|_{x_d=0}} = 2\lambda x_d^1 \varphi_k(z^1, z)_{|_{x_d=0}} > 0, \text{ for } k \neq k_0 \text{ and }, \quad \partial_{x_d} \varphi_{k_0|_{x_d=0}} = -2\lambda x_d^1 \varphi_k(z^1, z)_{|_{x_d=0}} < 0,$$

and moreover, by choosing $\gamma > 0$ and $\beta > 0$ sufficiently small, we can ensure that the covering condition is fulfilled. It is now classical that, using the Carleman estimate of Theorem 5.2.6, we can obtain the interpolation inequality of Lemma 5.5.3. We choose $r_0 > 0$ such that $z^1 = (0, r_0) \in \mathbb{R}^{d+1}_+$. We define the following cut-off functions $\chi_0,\ \chi_1 \in C_0^\infty(\mathbb{R}^{d+1})$:

$$\chi_0(x) = \begin{cases} 0 & \text{in} \quad x_d > r_1 \\ 1 & \text{in} \quad x_d \in [0, r_1/2] \end{cases} \quad ; \quad \chi_1(x) = \begin{cases} 0 & \text{if} \quad d_\gamma(z, z^1) \leq r_2 \text{ or } d_\gamma(y, y_0) > r_5 \\ 1 & \text{in} \quad d_\gamma(z, z^1) \in [r_3, r_4], \end{cases}$$

and

$$\overline{\chi}_1(x) = \begin{cases} 0 & \text{if} \quad d^\beta(z, z^1) \leq r_2 \text{ or } d^\beta(z, z^1) > r_5 \\ 1 & \text{in} \quad d^\beta(z, z^1) \in [r_3, r_4], \end{cases}$$

with $0 < r_1 < r_0$ and $0 < r_2 < r_3 < r_4 < r_5$ such that

- r_3 is small enough to have $\overline{B_\gamma(z^1, r_3)} \cap \overline{\mathcal{U}^+} = \emptyset$, where B_γ denotes the open ball associated with the distance d_γ;
- r_1 small enough and $r_0 < r_4 < r_5$ are such that $\{z = (z_0, x) \in \mathbb{R}^{d+1}_+ \mid x_d \leq r_1\} \cap \{r_4 \leq d_\gamma(z, z^1) \leq r_5\} \subset \mathcal{U}^+$.

We can apply the Carleman estimate of Theorem 5.2.6 on $\mathcal{U}^+$ to $w_k = \chi_0 \chi_1 u_k$, for $k \neq k_0$, and $w_{k_0} = \overline{\chi}_1 u_{k_0}$: there exist $\tau_0 > 0$ such that

$$\sum_{k \in \mathcal{V}^\ell} \tau^{-1} ||e^{\tau \varphi_k} w_k||^2_{2,\tau} \leq C \Big(\sum_{k \in \mathcal{V}^\ell} ||e^{\tau \varphi_k} (\partial_{y_0}^2 + \Delta_{g_k}) w_k||^2_{L^2(Z_k)} + \Big| e^{\tau \varphi_{k|_{y_d=0}}} \sum_{k \in \mathcal{V}^\ell} \partial_{y_d} w_{k|_{\{y_d=0\}}} \Big|^2_{1/2,\tau} \Big),$$

for all $\tau \geq \tau_0$. For $k \neq k_0$, the right hand side can be estimated as follows

$$\begin{aligned} ||e^{\tau \varphi_k} (\partial_{y_0}^2 + \Delta_{g_k}) w||_{L^2(Z_k)} &\lesssim ||e^{\tau \varphi} (\partial_{y_0}^2 + \Delta_{g_k}) u||_{L^2(Z_k)} + ||e^{\tau \varphi} [(\partial_{y_0}^2 + \Delta_{g_k}), \chi_0 \chi_1] u||_{L^2(Z_k)} \\ &\lesssim e^{\tau C_3} ||(\partial_{y_0}^2 + \Delta_{g_k}) u||_{L^2(Z_k)} + e^{\tau C_3} ||u||_{H^1(\mathcal{U}^+ \cap \{y_d \in [r_1/2, r_1]\})} + e^{\tau C_1} ||u||_{H^1(Z_k)}, \end{aligned}$$

with $C_3 > e^{-\lambda(r_0-r_1)}$ and $C_1 = e^{-\lambda r_4}$. Note that we have $C_1 < C_3$. Here, we use that the weight function φ is radial with respect to the distance d_γ to z^1 and decreasing as z moves away from z^1, and the commutator $[(\partial_{y_0}^2 + \Delta_{g_k}), \chi_0\chi_1]$ is a differential operator of order 1 supported in the region were $\chi_0\chi_1$ varies. For $k = k_0$, we have

$$\begin{aligned}||e^{\tau\varphi_{k_0}}(\partial_{y_0}^2 + \Delta_{g_{k_0}})w||_{L^2(Z_{k_0})} &\lesssim ||e^{\tau\varphi}(\partial_{y_0}^2 + \Delta_{g_{k_0}})u||_{L^2(Z_{k_0})} + ||e^{\tau\varphi}[(\partial_{y_0}^2 + \Delta_{g_{k_0}}), \chi_0\chi_1]u||_{L^2(Z_{k_0})} \\ &\lesssim e^{\tau C_3}||(\partial_{y_0}^2 + \Delta_{g_{k_0}})u||_{L^2(Z_{k_0})} + e^{\tau C_1}||u||_{H^1(Z_{k_0})},\end{aligned}$$

In the same spirit,

$$\Big|e^{\tau\varphi_{k|_{y_d=0}}} \sum_{k\in\mathcal{V}^\ell} \partial_{y_d} w_{k|_{\{y_d=0\}}}\Big|_{1/2,\tau} \lesssim e^{\tau C_3}|\sum_{k\in\mathcal{V}^\ell} \partial_{y_d} u_{k|_{\{y_d=0\}}}|_{1/2,\tau} + e^{\tau C_1}||u_k||_{H^1(Z_k)}$$

Finally, we can restrict the left hand side of the Carleman estimate to $\tilde{W} := B(0, r_6) \cap \{y_d > 0\}$ with $r_6 > 0$ taken sufficiently small to have $\chi_0\chi_1 = 1$ on $B(0, r_6)$ and this yields, for $\tau \geq 1$,

$$\sum_{k\in\mathcal{V}^\ell} \tau^{-1}||e^{\tau\varphi_k} w_k||_{2,\tau} \gtrsim e^{\tau C_2}||u||_{H^2(\tilde{W})},$$

where $C_2 > \inf_k \inf_{\overline{W}} \varphi_k$, and $\tau \geq \tau_0$, with τ_0 taken sufficiently large. Note that we have $0 < C_1 < C_2 < C_3$. We finally obtain, coming back to the original coordinates, for some $\eta > 0$,

$$\begin{aligned}||u||_{H^1(V)} \lesssim e^{\tau(C_3-C_2)}\Big(\sum_{k\in\mathcal{V}^\ell} ||(\partial_{x_0}^2 + \Delta_{g_k})u_k||_{L^2(Z_k)} + \sum_{k\in\mathcal{V}^\ell, k\neq k_0} ||u||_{H^1(Y^\eta_{\ell,k})} + ||\sum_{k\in\mathcal{V}^\ell} \partial_{\nu_k} u_{k|_{S_\ell}}||_{L^2(S_\ell)}\Big) \\ + e^{-\tau(C_2-C_1)}||u||_{H^1(Z)},\end{aligned}$$

where V is an open neighborhood of y_0 in $\overline{Z}$. It remains to optimize this inequality with respect to τ to obtain the result. □

5.5.4 A spectral inequality

Define $\mathbb{L}^2 := \bigoplus_{k=1}^p L^2(\Omega_k)$, and more generally $\mathbb{H}^s := \bigoplus_{k=1}^p H^s(\Omega_k)$. We consider the operator defined on Ω by

$$\mathcal{A}(u_1, \dots, u_p) = (-\Delta_{g_1} u_1, \dots, -\Delta_{g_p} u_p), \tag{5.5.7}$$

with domain

$$\begin{aligned}D(\mathcal{A}) = \{u = (u_1, \dots, u_p) \in \mathbb{H}^1,\ \mathcal{A}(u) \in \mathbb{L}^2,\ \forall \ell \in \{1, \dots, L\},\ u_{1|_{\mathcal{I}_\ell}} = \cdots = u_{p|_{\mathcal{I}_\ell}} = 0,\ \sum_{k\in\mathcal{V}^\ell} \partial_{\nu_k} u_{k|_{\mathcal{I}_\ell}} = 0 \\ \text{and } u_{i|_{\mathcal{B}_j}} = 0,\ \forall j \in \{1, \dots, K\},\ \forall i \in \mathfrak{W}_j\}\end{aligned} \tag{5.5.8}$$

where ν_k is the outward unit vector on $\mathcal{I}_\ell$ in the sense of the metric g_k. This operator admits an orthonormal basis of eigenfunctions.

Proposition 5.5.4. *There exists a Hilbert basis of $\mathbb{L}^2$ composed by eigenfunctions $\phi^k = (\phi_1^k, \dots, \phi_p^k) \in \mathbb{L}^2$ of the operator $\mathcal{A}$, associated with eigenvalues $0 < \gamma_0 \leq \gamma_1 \leq \dots$ with $\lim_{j\to\infty} \gamma_j = +\infty$.*

The following spectral inequality that measures the loss of orthogonality of the eigenfunctions ϕ_j, $j \in \mathbb{N}$, when they are restricted to an open subset $\omega \subset \Omega$ such that $\overline{\omega} \neq \Omega$. It also quantifies how linear combinations of these eigenfunctions can be observed from a subdomain. It is a natural consequence of the interpolation inequality of Theorem 5.5.2 (see [LL12]).

Theorem 5.5.5 (Spectral inequality)**.** *Let $\omega \subset \Omega$ be an open subset such that (ω, Ω) is observable in the sense of Definition 5.5.1. There exists $C > 0$ such that for all $\gamma \geq 0$ we have*

$$||w||_{L^2(\Omega)} \leq e^{C\sqrt{\gamma}}||w||_{L^2(\omega)}, \qquad w \in \text{span}(\phi_j;\ \gamma_j \leq \gamma). \tag{5.5.9}$$

By the Lebeau-Robbiano method (see for instance the survey [LL12] for an exposition of the method, and the references therein), from the above spectral inequality, we can construct a control for the following heat equation. Let ω be an open subset of Ω.

$$\begin{cases} \partial_t u_k - \Delta_{g_k} u_k = \chi_k f_k & \text{in } (0,T)\times\Omega_k,\ k\in\{1,\dots,p\} \\ u_{i|_{\mathcal{I}_\ell}} = u_{j|_{\mathcal{I}_\ell}} & \text{at } (0,T)\times\mathcal{I}_\ell, \ell\in\{1,\dots,L\},\ i,j\in\mathcal{V}^\ell \\ \sum_{k\in\mathcal{V}^\ell}^n \partial_{\nu_k} u_{k|_{\mathcal{I}_\ell}} = 0 & \text{at } (0,T)\times\mathcal{I}_\ell, \ell\in\{1,\dots,L\} \\ u_{i|_{\mathcal{B}_j}} = 0 & \text{at } (0,T)\times\mathcal{B}_j, j\in\{1,\dots,K\}, i\in\mathfrak{W}_j \\ (u_{1|_{t=0}},\dots,u_{p|_{t=0}}) = (u_1^0,\dots,u_p^0) & \text{in } \{0\}\times\Omega, \end{cases} \tag{5.5.10}$$

where χ_k is a cut-off function such that $\operatorname{supp}\chi_k \Subset \omega$. We state the null-controllability result of the heat equation 5.5.10.

Theorem 5.5.6. *Let $\omega\subset\Omega$ such that (ω,Ω) is observable in the sense of Definition 5.5.1. For any $T>0$, there exists a constant $C>0$ such that, for all initial data $(u_1^0,\dots,u_p^0)\in\mathbb{L}^2$, there exists a control $f:=(f_1,\dots,f_p)\in L^2((0,T),\mathbb{L}^2)$, such that the solution $u=(u_1,\dots,u_n)$ of (5.5.10) satisfies $u(T)=0$. Moreover the controls satisfy the bound*

$$||f||_{L^2((0,T),\mathbb{L}^2)} \le C\sum_{k=1}^p ||u_k^0||_{L^2(\Omega_k)}.$$

5.5.5 Stabilization of the wave equation

Let $\omega\subset\Omega$ be an open subset. We consider the following damped wave equation.

$$\begin{cases} \partial_t^2 u_k - \Delta_{g_k} u_k + a_k(x)\partial_t u_k = 0 & \text{in } (0,T)\times\Omega_k,\ k\in\{1,\dots,p\} \\ u_{i|_{\mathcal{I}_\ell}} = u_{j|_{\mathcal{I}_\ell}} & \text{at } (0,T)\times\mathcal{I}_\ell, \ell\in\{1,\dots,L\},\ i,j\in\mathcal{V}^\ell \\ \sum_{k\in\mathcal{V}^\ell}^n \partial_{\nu_k} u_{k|_{\mathcal{I}_\ell}} = 0 & \text{at } (0,T)\times\mathcal{I}_\ell, \ell\in\{1,\dots,L\} \\ u_{i|_{\mathcal{B}_j}} = 0 & \text{at } (0,T)\times\mathcal{B}_j, j\in\{1,\dots,K\}, i\in\mathfrak{W}_k \\ (u_{1|_{t=0}},\dots,u_{p|_{t=0}}) = (u_1^0,\dots,u_p^0) & \text{in } \{0\}\times\Omega, \\ (\partial_t u_{1|_{t=0}},\dots,\partial_t u_{p|_{t=0}}) = (v_1^0,\dots,v_p^0) & \text{in } \{0\}\times\Omega, \end{cases} \tag{5.5.11}$$

where a_k is a non-negative compactly supported function satisfying $a_k\ge C>0$ on ω, a non-empty subset of Ω. Observe that in the above wave equation, the damping terms are only directly effective on parts of Ω that intersect ω. We shall see that it is sufficient to ensure stabilization properties. We define the energy of the waves, solution to System (5.5.11) by

$$E(U,t) := \sum_{k=1}^p \left(||u_k||^2_{H^1(\Omega_k)} + ||\partial_t u_k||^2_{L^2(\Omega_k)}\right).$$

We can recast System (5.5.11) into a semi-group formalism

$$\dot U + BU = 0, \tag{5.5.12}$$

where $B=\begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_p \end{pmatrix}$, with $B_k=\begin{pmatrix} 0 & -1 \\ -\Delta_{g_k} & a_k(x) \end{pmatrix}$. We define the norm

$$||(u_1,v_1,\dots,u_p,v_p)||^2_{\mathbb{W}} = ||(u_1,\dots,u_p)||^2_{\mathbb{H}^1} + ||(v_1,\dots,v_p)||^2_{\mathbb{L}^2}.$$

We have the following result, that states that the energy of the strong solutions of (5.5.11) decays with a logarithmic speed.

Theorem 5.5.7. *Assume that (Ω, ω) is observable in the sense of Definition 5.5.1. There exists $C > 0$ such that*

$$E(U,t)^{1/2} \leq \log(2+t)^{-k} ||B^k U_0||_{\mathbb{W}},$$

for all solutions $U = (u_1, \partial_t u_1, \dots, u_p, \partial_t u_p)$ of (5.5.12), with initial data $U_0 = (u_1^0, v_1, \dots, u_p^0, v_p)$.

It is well known [Bur98, BD08] that Theorem 5.5.7 follows from the following resolvent estimate. We define $\mathbb{H}_0^1$ the space of functions of $\mathbb{H}^1$ that satisfies at the boundary

$$u_{i|_{\mathcal{B}_j}} = 0 \text{ at } \mathcal{B}_j, j \in \{1, \dots, K\}, i \in \mathfrak{W}_k.$$

Proposition 5.5.8. *Let ω be a nonempty open subset of Ω and α be such that $\alpha > 0$ on ω. Then, the unbounded operator $i\sigma \operatorname{Id} - B$ is invertible on $\mathbb{H} = \mathbb{H}_0^1 \times \mathbb{L}^2$ for all $\sigma \in \mathbb{R}$ and there exist $K > 0$ and $\sigma_0 > 0$ such that*

$$||(i\sigma \operatorname{Id} - B)^{-1}||_{\mathcal{L}(\mathbb{H},\mathbb{H})} \leq K e^{K|\sigma|}, \qquad \sigma \in \mathbb{R},\ |\sigma| \geq \sigma_0.$$

This is a consequence of the interpolation inequality of Theorem 5.5.2 (see for instance [LR97]).

Troisième partie

Contrôlabilité dans des configurations singulières

Chapitre 6

Problèmes singuliers : potentiels carrés inverses et domaines polygonaux

Abstract. We address the question of controllability and stabilization of equations in singular configurations. More precisely, the purpose is to prove resolvent estimates for the wave equation with a inverse square potential, and a spectral inequality for the Laplace operator with inverse square potential. Both sub-critical and critical case are considered. We address similar questions in two-dimensional bounded domains, with boundaries that contains corners. The case of transmission problems with interfaces that meets a boundary is also considered. The first part is devoted to introducing all these geometrical configurations and reducing them to a class of one-dimensionnal problems. By means of a proper Carleman estimates, we shall deduce interpolation inequalities, which in turn yield control and stabilization properties.

6.1 Introduction

6.1.1 The Laplace operator with singular inverse square potential

Let $d \geq 3$, and let Ω be a bounded smooth open subset of $\mathbb{R}^d$. To ease notation, we assume without any loss of generality that $0 \in \Omega$. Let $\mu \in \mathbb{R}$, we consider the following operator with inverse square potential

$$P = -\Delta - \frac{\mu}{|x|^2}, \tag{6.1.1}$$

with domain

$$D(P) = \{u \in \mathcal{H}_\mu,\ Pu \in L^2(\Omega)\}, \tag{6.1.2}$$

(see (6.1.8) for the definition of the Hilbert space $\mathcal{H}_\mu$), and consider the following equation with Dirichlet boundary condition

$$\begin{cases} Pu = f, & f \in L^2(\Omega), \\ u_{|\partial\Omega} = 0. \end{cases} \tag{6.1.3}$$

One of the purpose of this article is the study of some spectral properties of the operator P.

Sobolev spaces and values of the parameter

It is known since the works of [BG84], [VZ00] that the well-posedness of equations where such operators occur is related to the critical value of the parameter μ

$$\mu^* = \frac{(d-2)^2}{4}. \tag{6.1.4}$$

This is precisely the critical constant in the following Hardy inequality

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{u^2}{|x|^2} \right) \geq C ||u||^2_{L^2(\Omega)}. \tag{6.1.5}$$

We shall give a proof of (6.1.5) in a one-dimensional setting in Lemma 6.2.9. In fact, for a *subcritical* value of $\mu < \mu^*$, Equation (6.1.3) is well-posed in $H^1_0(\Omega)$. Indeed, using the Hardy inequality (6.1.5),

$$\begin{aligned} \int_\Omega |\nabla u|^2 - \mu \int_\Omega \frac{|u|^2}{|x|^2} &= \frac{\mu}{\mu^*} \Big(\int_\Omega |\nabla u|^2 - \mu^* \int_\Omega \frac{u^2}{|x|^2} \Big) + \left(1 - \frac{\mu}{\mu^*} \right) ||\nabla u||^2_{L^2(\Omega)} \\ &\geq C \frac{\mu}{\mu^*} ||u||^2_{L^2(\Omega)} + \left(1 - \frac{\mu}{\mu^*} \right) ||\nabla u||^2_{L^2(\Omega)}, \end{aligned} \tag{6.1.6}$$

we see that the quadratic form $\int_\Omega |\nabla u|^2 - \mu \int_\Omega \frac{|u|^2}{|x|^2}$ associated with the variational form of (6.1.3) defines a norm on $H^1_0(\Omega)$, equivalent to the usual one $\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2$. For the *critical* value $\mu = \mu^*$, the quantity $\int_\Omega |\nabla u|^2 - \mu^* \int_\Omega \frac{|u|^2}{|x|^2}$ is no longer coercive in $H^1_0(\Omega)$. However, from [VZ00], the following so-called *improved Hardy-Poincaré inequality*

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2} \right) \geq C_q ||u||^2_{W^{1,q}_0} \tag{6.1.7}$$

holds for every $1 \leq q < 2$. We denote by $\mathcal{H}_\mu$ the Hilbert space obtained by the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$||u||^2_{\mathcal{H}_\mu} = \int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right). \tag{6.1.8}$$

In the *sub-critical case* $\mu < \mu^*$, from (6.1.6), one has $\mathcal{H}_\mu(\Omega) = H^1_0(\Omega)$, but in the *critical case* $\mu = \mu^*$, from the analysis carried out in [VZ00], $\mathcal{H}_{\mu^*}(\Omega)$ is strictly larger than $H^1_0(\Omega)$.

Well-posedness properties for the elliptic problem

The improved Hardy inequality (6.1.7) and the definition of the space $\mathcal{H}_\mu$ allow one to obtain the well-posedness for the variational equation (see [VZ00]). Let $\mu \leq \mu^*$.

Proposition 6.1.1. *For all $f \in L^2(\Omega)$, there exists an unique weak solution $u \in \mathcal{H}_\mu$ of (6.1.3).*

We moreover have the following spectral decomposition

Proposition 6.1.2. *There exists a countable basis of eigenfunctions $(\phi_j)_j \in \mathcal{H}_\mu$ of P associated with eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty.$$

On the controllability of the associated parabolic equation

Let $\omega \subset \Omega$ be a non-empty open subset of Ω, and let χ_ω be the characteristic function of ω. For controllability results of the following parabolic problem

$$\begin{cases} \partial_t u + Pu = \chi_\omega h & (t,x) \in (0,T) \times \Omega, \\ u(t,x)_{|\partial\Omega} = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \in L^2(\Omega) & x \in \Omega, \end{cases} \tag{6.1.9}$$

where h is the control function, and the observability of the adjoint equation

$$\begin{cases} \partial_t z - Pz = 0 & (t,x) \in (0,T) \times \Omega, \\ z(t,x)_{|\partial\Omega} = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ z(T,x) = z_T(x) \in L^2(\Omega) & x \in \Omega, \end{cases} \tag{6.1.10}$$

we refer to the former work of [VZ08] and [Erv08]. In [VZ08, Erv08], the authors proved the following null-controllability result.

Theorem 6.1.3. *Let ω be any open subset of Ω, and let $\mu \leq \mu^*$. Then for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2((0,T)\times\Omega)$ such that the solution of (6.1.9) satisfies $u(T,x) = 0$, for almost each $x \in \Omega$.*

In [VZ08], a technical geometrical assumption was needed. This assumption was later relaxed in [Erv08]. It is well-known that the null-controllability of (6.1.9) is equivalent to an observability estimate for (6.1.10) (we refer to [VZ08], [Erv08]).

Theorem 6.1.4. *Let ω be any open subset of Ω, and let $\mu \leq \mu^*$. There exists $C_T > 0$ (depending on μ) such that for all $z_T \in L^2(\Omega)$, the solution of (6.1.10) satisfies*

$$\int_\Omega |z(0,x)|^2 \leq C_T \int_0^T \int_\omega |z(t,x)|^2.$$

On the one hand, in both articles [VZ08, Erv08], the proof of Theorem 6.1.4 comes from a proper parabolic Carleman estimate, in the spirit of [FI96]. On the other hand, since the seminal work of [LR95], the null-controllability of the heat equation appears as a consequence of a spectral inequality for the elliptic operator. In the present article, we prove such a spectral inequality for P, in both critical $\mu = \mu^*$ and sub-critical case $\mu < \mu^*$.

We also refer to [Van11] for control and inverse problems related to this topic.

On the stabilization and controllability of the associated wave equation

We present some existing results related to the subject. In [VZ09], the authors considered the following boundary control problem

$$\begin{cases} \partial_t^2 u + Pu = 0 & (t,x) \in (0,T)\times\Omega, \\ u(t,x)_{|\partial\Omega} = h(t,x) & (t,x) \in (0,T)\times\Gamma_0, \\ u(t,x)_{|\partial\Omega} = 0 & (t,x) \in (0,T)\times\partial\Omega\setminus\Gamma_0, \\ u(0,x) = u_0(x) & x \in \Omega \\ \partial_t u(0,x) = u_1(x) & x \in \Omega, \end{cases} \tag{6.1.11}$$

where h is a function acting as a control at the boundary. Under geometrical conditions (a multiplier condition) on Γ_0, that is

$$\Gamma_0 = \{x \in \partial\Omega,\ x.\nu(x) \geq 0\},$$

the authors proved the following exact controllability result.

Theorem 6.1.5. *For $\mu \leq \mu^*$, for all $T > 2\,\mathrm{diam}(\Omega)$, for all initial data $(u_0,u_1) \in L^2(\Omega)\times\mathcal{H}'_\mu$, for all target state $(\bar{u}_0,\bar{u}_1) \in L^2(\Omega)\times\mathcal{H}'_\mu$, there exists $h \in L^2((0,T)\times\Gamma_0)$ such that the weak solution u of (6.1.11) satisfies*

$$u(T,x) = \bar{u}_0(x), \quad \partial_t u(T,x) = u_1(x), \quad a.e\ in\ \Omega.$$

In our setting, we are interested in proving stabilization results for strong solutions of the wave equation, using an internal damping. A boundary damping may also be considered using classical methods. Hence, we consider the following damped wave equation

$$\begin{cases} \partial_t^2 u + Pu + b(x)\partial_t u = 0 & (t,x) \in (0,T)\times\Omega, \\ u(t,x)_{|\partial\Omega} = 0 & (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega \\ \partial_t u(0,x) = u_1(x) & x \in \Omega, \end{cases} \tag{6.1.12}$$

where b is a non-negative function, satisfying $b \geq C > 0$ on some open subset $\omega \subset \Omega$. The energy of solutions of (6.1.12) is given by

$$E_\mu(t) = \int_\Omega |\partial_t u|^2 + \int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right), \tag{6.1.13}$$

We recast the wave equation (6.1.12) into a semi-group formalism. We define the Hilbert space $\mathcal{H} = \mathcal{H}_\mu \times L^2(\Omega)$. We consider the following wave operator $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$,

$$A = \begin{pmatrix} 0 & -\mathrm{Id} \\ P & b(x) \end{pmatrix},$$

with $D(A) = D(P) \times D(P^{1/2})$, and then Equation (6.1.12) reads

$$\frac{d}{dt}U + AU = 0, \quad U_{|_{t=0}} = U_0, \tag{6.1.14}$$

with $U = {}^t(u, \partial_t u)$ and $U_0 = {}^t(u_0, u_1)$. In this article we shall prove the following stabilization result for strong solutions.

Theorem 6.1.6. *Let $k \geq 1$. There exists $C > 0$ such that*

$$E_\mu(u)(t)^{1/2} \leq \frac{C}{(\log(2+t))^k} ||A^k U_0||_{\mathcal{H}(\Omega)},$$

for all u solutions of (6.1.12) with initial data $U_0 = {}^t(u_0, u_1) \in D(A^k)$.

Note that here no assumption is made on the geometry of the open set ω.

6.1.2 Laplace operator on polygonal domains with interfaces

Let Ω be an open polygonal bounded domain in $\mathbb{R}^2$, and let S be a broken line of the plane such that $\Omega \cap S \neq \emptyset$. The interface S divides Ω into two connected components Ω^ℓ, $\ell \in \{1, 2\}$. At the boundary, we shall denote by $\partial\Omega_k$ each sides of the polygon $\partial\Omega$. Even if the interface crosses the boundary at a point where it is smooth, we shall consider that it separate the boundary into two different pieces (see an example in Figure 6.1). Let thus $\mathsf{S} = \{1, \dots, p\}$ denote all the indices associated with the sides of $\partial\Omega$.

In what follows, for a function v on Ω, we denote by v^ℓ its restriction to Ω^ℓ. For a function v^ℓ on Ω^ℓ, we shall denote by $\gamma_k v$ the trace of v on $\partial\Omega_k^\ell$, and denote by $\gamma_k \partial_\nu v$ the trace of its normal derivative on $\partial\Omega_k$ (when such trace exists), where ν denotes the unit outward pointing normal vector. In the same spirit, we define $\gamma_S v^\ell$ as the trace of v^ℓ on S and $\gamma_S \partial_{\nu_\ell} v^\ell$ the trace of its normal derivative, where ν_ℓ denotes the unit normal vector on S pointing outside Ω^ℓ. We shall impose homogeneous Dirichlet boundary conditions on some sides and homogeneous Neumann boundary conditions on others. We shall denote by D and N a partition of S, that is $\mathsf{D} \cup \mathsf{N} = \mathsf{S}$, $\mathsf{D} \cap \mathsf{N} = \emptyset$. D provides the indices of the parts of boundaries where homogeneous Dirichlet boundary conditions holds, and N provides the indices of the parts of boundaries where homogeneous Neumann boundary conditions holds. We consider the Sobolev space

$$\mathcal{H}^1_{\mathsf{D}} = \{u,\ u^\ell \in H^1(\Omega),\ \gamma_i u = 0,\ \forall i \in \mathsf{D}\}.$$

We consider the following operator, acting on $\mathcal{H}^1_D$,

$$\mathbf{T} = -\operatorname{div} a(x)\nabla, \tag{6.1.15}$$

where $a(x)$ is a piecewise constant function satisfying $a(x) = c_k$ on each Ω_k, with domain

$$D(\mathbf{T}) = \big\{u \in \mathcal{H}^1_{\mathsf{D}}(\Omega),\ \Delta u^\ell \in L^2(\Omega^\ell),\ \gamma_j \partial_\nu u = 0, \forall j \in \mathsf{N}, \gamma_S u^1 = \gamma_S u^2,\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0\big\}.$$

In what follows, for the sake of simplicity, we shall assume that

$$\mathsf{D} \neq \emptyset. \tag{6.1.16}$$

A consequence of this assumption is given in the next proposition.

Proposition 6.1.7. *There exists a countable basis of eigenfunctions $(\phi_j)_j \in \mathcal{H}^1_{\mathsf{D}}$ of $\mathbf{T}$ associated with eigenvalues*

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty.$$

The fact that we assumed $\mathsf{D} \neq 0$ implies that 0 is not an eigenvalue of this problem. In fact, for the null-controllability of parabolic equations, this does not change anything, but this simplify the exposition for the stabilization of the wave equation. If Assumption (6.1.16) was not made, we would have to work in quotient spaces, in the spirit of [LR97].

Definition 6.1.8 (Neumann corner)**.** *We set*

$$\mathcal{E} := \{p \in \partial\Omega,\ p \text{ is the edge of the boundary}\} \cup \{p \in S,\ p \text{ is a point where } S \text{ is not smooth}\} \cup (S \cap \partial\Omega),$$

that is the set of points where singular configurations occurs. We shall say that p is a Neumann corner in the following configurations:

- *$p \in S \cap \operatorname{Int}\Omega$, p is a point where the interface S is not smooth;*
- *$p \in \partial\Omega$ such that $\partial\Omega$ is not smooth at p, and Neumann boundary conditions are imposed on each sides of p, ;*
- *$p \in S \cap \partial\Omega$, and Neumann boundary conditions are imposed at the boundary at each sides of p.*

On the controllability of the associated parabolic equation

Many Carleman estimates have been derived for non-smooth situations. We first mention out the former work [DOP02], where the authors proved a parabolic estimate in the case where the domain Ω present an smooth interface. However, a condition on the observation region was made in relation to the coefficients of the operators. In [LR11], the authors proved a parabolic estimate without any assumption on the observation region. We then refer to [LL13] for the derivation of a Carleman estimate near an interface for totally anisotropic operators. More recently, estimates for general interior and interface conditions have been derived in [BL16], provided that some generalized Lopatinskii conditions hold. In the case where interfaces meet the boundary, we first refer to [BDL11]. In this article, the authors proved an elliptic estimate in the case of a stratified media. We then refer to [BDT13] where the authors relaxed the assumptions on the coefficients. Concerning non-smooth boundaries, we refer to [Bel14] where Dirichlet boundary conditions are considered, and where the authors proved a spectral inequality in the spirit of Theorem 6.1.11 by means of a global Carleman estimate. Furthermore, they obtained the optimal constant. Finally, let us refer to [CR14] where the authors proved a elliptic Carleman estimate for the mixed boundary condtion (also called the Zaremba problem).

Let ω be an open subset of Ω, and χ_Ω be its characteristic function. We address the question of the null-controllability of the following heat equation.

$$\begin{cases} \partial_t u^\ell + \mathbf{T} u^\ell = \chi_\omega h & (t,x) \in (0,T) \times \Omega_\ell, \\ \gamma_i u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i,\ i \in \mathsf{D} \\ \gamma_i \partial_\nu u(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega_i,\ i \in \mathsf{N} \\ \gamma_S u^1 = \gamma_S u^2 & (t,x) \in (0,T) \times S, \\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0 & (t,x) \in (0,T) \times S, \\ u(0,x) = u_0(x) \in L^2(\Omega) & x \in \Omega, \end{cases} \tag{6.1.17}$$

where h is the control function and $T > 0$. In this article, we prove the following theorem.

Theorem 6.1.9. *Assume that $\mathcal{E}$ does not contain any Neumann corner. Then, for all $u_0 \in L^2(\Omega)$, there exists $h \in L^2((0,T) \times \Omega)$ such that the solution u of (6.1.17) satisfies*

$$u(T,x) = 0, \quad \text{for almost all } x \in \Omega.$$

The proof of Theorem 6.1.9 is a consequence of the spectral inequality (6.1.22).

On the stabilization of the associated wave equation

We consider the following damped wave equation

$$\begin{cases} \partial_t^2 u^\ell + \mathbf{T} u^\ell + b(x)\partial_t u^\ell = 0 & (t,x) \in \mathbb{R}^+ \times \Omega_\ell, \\ \gamma_i u(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \partial\Omega_i,\ i \in \mathsf{D} \\ \gamma_i \partial_\nu u(t,x) = 0 & (t,x) \in \mathbb{R}^+ \times \partial\Omega_i,\ i \in \mathsf{N} \\ \gamma_S u^1 = \gamma_S u^2 & (t,x) \in \mathbb{R}^+ \times S, \\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0 & (t,x) \in \mathbb{R}^+ \times S, \\ u(0,x) = u_0(x) & x \in \Omega \\ \partial_t u(0,x) = u_1(x) & x \in \Omega, \end{cases} \tag{6.1.18}$$

where b is a non-negative function, satisfying $b \geq C > 0$ on some open subset $\omega \subset \Omega$. The energy of solutions of (6.1.18) is given by

$$E(t) = \int_\Omega |\partial_t u|^2 + \int_\Omega |\nabla u|^2, \tag{6.1.19}$$

We recast the wave equation (6.1.18) into a semi-group formalism. We define the Hilbert space $\mathcal{H} = H^1_{\mathsf{D}}(\Omega) \times L^2(\Omega)$. We consider the following wave operator $B : D(B) \subset \mathcal{H} \mapsto \mathcal{H}$,

$$B = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathbf{T} & b(x) \end{pmatrix},$$

with $D(B) = D(\mathbf{T}) \times D(\mathbf{T}^{1/2})$ and then Equation (6.1.18) reads

$$\frac{d}{dt} U + BU = 0, \quad U_{|_{t=0}} = U_0, \tag{6.1.20}$$

with $U = {}^t(u, \partial_t u)$ and $U_0 = {}^t(u_0, u_1)$. In this article we shall prove the following stabilization result on strong solutions.

Theorem 6.1.10. *Let $k \geq 1$. There exists $C > 0$ such that*

$$E(u)(t)^{1/2} \leq \frac{C}{(\log(2+t))^k} ||B^k U_0||_{\mathcal{H}(\Omega)},$$

for all u solutions of (6.1.18) with initial data $U_0 = (u_0, u_1) \in D(B)$.

6.1.3 Main results

The central result of this paper are the following spectral and resolvent inequalities.

Spectral inequalities in the case of an inverse square potential

Let Ω be an open bounded subset of $\mathbb{R}^d$ as in Section 6.1.1. Let (λ_j, ϕ_j) be a Hilbert basis of eigenfunctions of the operator $P = -\Delta - \frac{\mu}{|x|^2}$.

Theorem 6.1.11. *Let $\mu \leq \mu^*$. Let $\omega \subset \Omega$ be a non empty open subset of $\Omega \subset \mathbb{R}^d$. Let $\eta > 0$. Then there exist $C > 0$ and $\Lambda_0 > 0$ such that*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\Phi^\mu_\eta(\Lambda)} \int_\omega \Big| \sum_{\lambda_j \leq \Lambda} a_j \phi_j \Big|^2, \quad \Lambda > \Lambda_0 \tag{6.1.21}$$

for all $(a_j) \in \ell^2(\mathbb{N})$, with

$$\Phi^\mu_\eta(\Lambda) = \begin{cases} \Lambda^{\frac{7}{8}+\eta} & \text{if } \mu < \mu^* \\ \Lambda^{2+\eta} & \text{if } \mu = \mu^*. \end{cases}$$

Remark 6.1.12. *Note that for the Laplace operator with bounded potentials, the constant in (6.1.21) is of the form $Ce^{C\sqrt{\Lambda}}$ and it is proven to be sharp as far as the parameter Λ is concerned in [JL99]. Due to the presence of the singular potential, the method we employ here exhibits a loss. Optimality is an open question.*

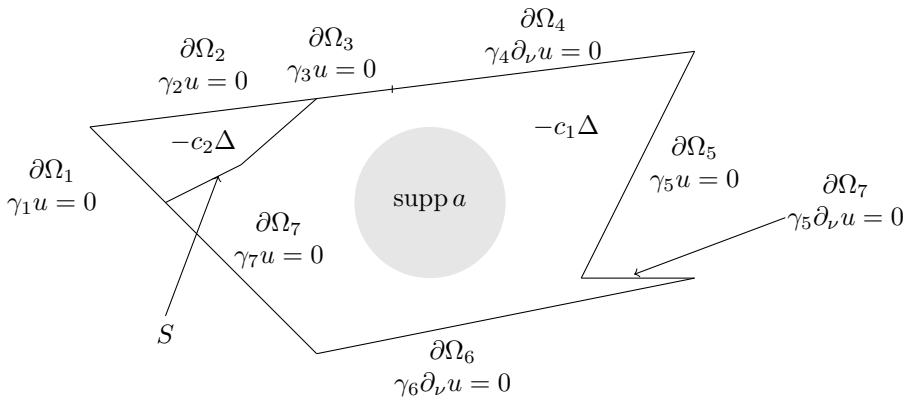


Figure 6.1: An example of polygon with broken interface S. Here, $\mathsf{D} = \{1, 2, 4\}$ and $\mathsf{N} = \{3, 5, 6\}$. In gray is represented an example of compactly supported damping for the wave equation.

Spectral inequalities for the Laplace operator on a polygon

Let Ω be a polygonal domain of $\mathbb{R}^2$ as described in Section 6.1.2. Let (λ_j, ϕ_j) be an Hilbert basis of eigenfunctions of the operator $\mathbf{T} = -\operatorname{div} a(x)\nabla$.

Theorem 6.1.13. *Let $\omega \subset \Omega$ be a non empty open subset of Ω. Let $\eta > 0$. Then, there exist $C > 0$ and $\lambda_0 > 0$ such that*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\Phi_\eta(\Lambda)} \int_\omega \Big| \sum_{\lambda_j \leq \Lambda} a_j \phi_j \Big|^2, \quad \Lambda > \Lambda_0 \tag{6.1.22}$$

for all $(a_j) \in \ell^2(\mathbb{N})$, with

$$\Phi_\eta(\Lambda) = \begin{cases} \Lambda^{\frac{7}{8}+\eta} & \text{if } \mathcal{E} \text{ does not contains any Neumann corner (see Definition 6.1.8)} \\ \Lambda^{2+\eta} & \text{otherwise.} \end{cases}$$

Remark 6.1.14. *Remark 6.1.12 also holds in this case. When considering smooth boundaries, the optimal constant is given by $\sqrt{\Lambda}$.*

Theorems 6.1.11 and 6.1.13 shall be proved by using estimates on the elliptic operator $Q := -\partial_s^2 - \Delta - \mu|x|^{-2}$ (resp. $\mathcal{T} := -\partial_s^2 - \mathbf{T}$), that is, for operators with an additional variable s. We shall refer to these operators as to "augmented" operators. A proper Carleman estimate is needed near the singularity $\{x = 0\}$ (resp. near the singular points of $\{x \in \mathcal{E}\}$ (see Definition 6.1.8)). Because of the singular configuration, the proved Carleman estimates are weaker than usually, and this leads to the losses in the spectral inequalities. However, for applications to control of the associated parabolic equation, in the sub-critical case $\mu < \mu^*$ for the inverse square potential problem, or if $\mathcal{E}$ does not contain any Neumann corner in the polygonal problem, we have $\Phi(\Lambda) < \Lambda$, and we can recover Theorem 6.1.3 and Theorem 6.1.4, using classical method of control/relaxation developed in [LR95]. However, in the critical case $\mu = \mu^*$, or for a Neumann problem, the power of Λ is greater than one. This does not allow one to obtain control results. We mention that in the case of a polygonal domain with mixed Dirichlet and Neumann boundary conditions, the null-controllabilty result of Theorem 6.1.9 of the associated parabolic equation is an original result. We recall that the power Λ^s, with $s = 1$ reflects the decay rate of the heat equation.

Resolvent estimates for waves

For the generator A (resp. B) of the damped wave equation in the case of a singular potential (resp. in a polygonal domain), we prove the following resolvent estimates.

Theorem 6.1.15. *For all $\sigma \in \mathbb{R}$, the operator $(i\sigma\,\mathrm{Id}+A)$ (resp. $(i\sigma\,\mathrm{Id}+B)$) is invertible on $\mathcal{H}_\mu \times L^2(\Omega)$ (resp. $\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)$). Moreover, there exists Cż0 such that*

$$|||(i\sigma\,\mathrm{Id}+A)^{-1}|||_{\mathcal{H}_\mu\times L^2(\Omega)} \leq Ce^{C|\sigma|} \qquad |\sigma| \geq 1, \tag{6.1.23}$$

$$(resp.\ |||(i\sigma\,\mathrm{Id}+B)^{-1}|||_{\mathcal{H}^1_{\mathsf{D}}\times L^2(\Omega)} \leq Ce^{C|\sigma|} \qquad |\sigma| \geq 1). \tag{6.1.24}$$

Theorems 6.1.6 and 6.1.10 are then corollaries by well-known techniques (see [LR97, Bur98, BD08]).

6.2 Carleman estimates for the augmented operators

6.2.1 Estimates in the case of a singular potential in arbitrary dimensions

Here, Ω is a smooth open set of $\mathbb{R}^d$. As it is done classically, to prove spectral inequalities, we introduce an augmented elliptic operator. Let $S_0 > 0$ and $\alpha \in (0, S_0/2)$. Let $Z = (-S_0, S_0) \times \Omega$ and let $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$. We define the elliptic operator with potential

$$Q = -\partial_s^2 + P, \tag{6.2.1}$$

where P is given by (6.1.1). Note that when adding the additional variable s, we obtain a 1-dimensional singularity along $(-S_0, S_0) \times \{x = 0\}$. This is the main difficulty we have to handle, and this justify our choice of the following weight function. In a neighborhood of a point $(s_0, x = 0)$, we shall consider

$$\varphi_\varepsilon(s, x) = |x|^2 - \varepsilon|s - s_0|^2. \tag{6.2.2}$$

Remark that on smooth domains, the weight function is usually chosen to satisfy the sub-ellipticity condition, and also $\nabla\varphi \neq 0$ on the closure of the considered open set. In the present case, the weight function has a vanishing gradient at $(s = s_0, x = 0)$. This particular choice is made to handle the singularity $\frac{\mu}{|x|^2}$ (see the proof of Theorem 6.2.11). First, we state the Carleman estimate for the operator Q defined in (6.2.1), in the sub-critical case $\mu < \mu^*$, with weight φ_ε defined in (6.2.2).

Theorem 6.2.1 (Carleman estimate in the sub-critical case)**.** *Let $s_0 \in (-S_0 + \alpha, S_0 - \alpha)$. Assume $\mu < \mu^*$ and let $\mathcal{V}$ be an open neighborhood of $(s_0, 0)$ in $\overline{Y}$. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$C||e^{\tau\varphi_\varepsilon}Qu||^2_{L^2(\mathcal{V})} \geq \tau^{2/3}||e^{\tau\varphi_\varepsilon}\nabla_{s,x}u||^2_{L^2(\mathcal{V})} + \tau^2||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

for $\varepsilon = \varepsilon_0\tau^{-1/3}$, and for all $u \in C^\infty((-S_0, S_0), D(P))$, and $\operatorname{supp} u \subset \mathcal{V}$.

Second, we state the counterpart result we shall prove in the critical case.

Theorem 6.2.2 (Carleman estimate in the critical case)**.** *Let $s_0 \in (-S_0 + \alpha, S_0 - \alpha)$. Assume $\mu = \mu^*$ and let $\mathcal{V}$ be an open neighborhood of $(s_0, 0)$ in $\overline{Y}$. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$C||e^{\tau\varphi_\varepsilon}Qu||^2_{L^2(\mathcal{V})} \geq \tau||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

for $\varepsilon = \varepsilon_0\tau^{-2/3}$, and for $u \in C^\infty((-S_0, S_0), D(P))$, and $\operatorname{supp} u \subset \mathcal{V}$.

Remark 6.2.3. *Note that we do not estimate the H^1-norm anymore, in Theorem 6.2.2. Moreover, the estimate of the L^2 norm is weaker if compared with Theorem 6.2.1. However, it corresponds to the variational setting described in Section 6.1.1.*

Remark 6.2.4. *The subtle difference in the weight function between Theorem 6.2.1 and Theorem 6.2.2 (remark that the value of ε as a function of τ change) is responsible for the differences pointed out in Remark 6.2.3. The proof of these Carleman estimates shall be made by using spherical coordinates, and decomposition into spherical harmonics. They are given in Section 6.2.3.*

6.2.2 Estimate for the augmented Laplace operator in singular domains

Here, Ω is a polygonal domain of $\mathbb{R}^2$ as described in Section 6.1.2. Let $S_0 > 0$ and $\alpha \in (0, S_0/2)$. Set $Z = (-S_0, S_0) \times \Omega$ and $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$. We define the elliptic operator

$$\mathcal{T} = -\partial_s^2 + \mathbf{T}. \tag{6.2.3}$$

Note that when adding the additional variable s, the geometry turns a singular point x_0 at the boundary into a wedge along $(-S_0, S_0) \times \{x_0\}$. First, we state the Carleman estimate we shall prove for the operator $\mathcal{T}$ defined in (6.2.3), with weight φ_ε defined in (6.2.2). We recall that Neumann corners are introduced in Definition 6.1.8.

Theorem 6.2.5 (Carleman estimate for a non-Neumann corner)**.** *Let x_0 be a non-Neumann corner of Ω, and $s_0 \in (-S_0 + \alpha, S_0 - \alpha)$. Let $\mathcal{P} = (s_0, x_0)$, and we assume $x_0 = 0$ without loss of generality. Let $\mathcal{V}$ be an open neighborhood of $\mathcal{P}$ in Y. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$C||e^{\tau\varphi_\varepsilon}\mathcal{T}u||^2_{L^2(\mathcal{V})} \geq \tau^{2/3}||e^{\tau\varphi_\varepsilon}\nabla_{s,x}u||^2_{L^2(\mathcal{V})} + \tau^2||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

for $\varepsilon = \varepsilon_0\tau^{-1/3}$, and for all $u \in C^\infty((-S_0, S_0), D(\mathbf{T}))$ and $\operatorname{supp} u \subset \mathcal{V}$.

Second, we state the counterpart result, in the case of a Neumann corner.

Theorem 6.2.6 (Carleman estimate for a Neumann corner)**.** *Let x_0 be a Neumann corner of Ω, and $s_0 \in (-S_0 + \alpha, S_0 - \alpha)$. Let $\mathcal{P} = (s_0, x_0)$, and we assume $x_0 = 0$ without loss of generality. Let $\mathcal{V}$ be an open neighborhood of $\mathcal{P}$ in Y. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$C||e^{\tau\varphi_\varepsilon}\mathcal{T}u||^2_{L^2(\mathcal{V})} \geq \tau||e^{\tau\varphi_\varepsilon}u||^2_{L^2(\mathcal{V})} + \tau^3||e^{\tau\varphi_\varepsilon}|x|u||^2_{L^2(\mathcal{V})},$$

for $\varepsilon = \varepsilon_0\tau^{-2/3}$, and for all $u \in C^\infty((-S_0, S_0), D(\mathbf{T}))$, and $\operatorname{supp} u \subset \mathcal{V}$.

Observe that Theorems 6.2.1 and 6.2.5 are similar in form. The same observation can be made for Theorems 6.2.2 and 6.2.6. Remarks 6.2.3 and 6.2.4 also apply in this situation. Note that the boundary conditions are prescribed to $u(s, .)$ through the properties of the domain of T.

6.2.3 Local reduction in spherical harmonics for the augmented operators

In this section, we consider $d \geq 2$, but if $d = 2$ we shall assume that $\mu = 0$, and we shall reduce the problems to a one-dimensional equation, for which we prove inequalities. Consider the Laplace operator in spherical coordinates $x = r\theta$,

$$\Delta = \partial_r^2 + (d-1)\frac{\partial_r}{r} + \frac{1}{r^2}\Delta_\theta, \tag{6.2.4}$$

where Δ_θ denotes the Laplace-Beltrami operator on the unit sphere. Then Q reads

$$Q = -\partial_s^2 - \partial_r^2 - (d-1)\frac{\partial_r}{r} - \frac{1}{r^2}(\Delta_\theta + \mu).$$

In the case where 0 is a point of the interior, the operator Δ_θ admits the following orthogonal basis of eigenfunctions associated with eigenvalues

$$(\psi_j)_{j\geq 0} \in L^2(\mathbb{S}^{d-1}), \quad 0 = \nu_0 < \nu_1 \leq \cdots \leq \nu_n \leq \cdots \to +\infty. \tag{6.2.5}$$

In the case where 0 is the vertex of a corner, two cases need to be discussed. If the corner impose some Dirichlet boundary condition, then 0 cannot be an eigenvalue, and (6.2.5) actually reads

$$(\psi_j)_{j\geq 0} \in L^2(\mathbb{S}^{d-1}), \quad 0 < \nu_0 \leq \nu_1 \leq \cdots \leq \nu_n \leq \cdots \to +\infty. \tag{6.2.6}$$

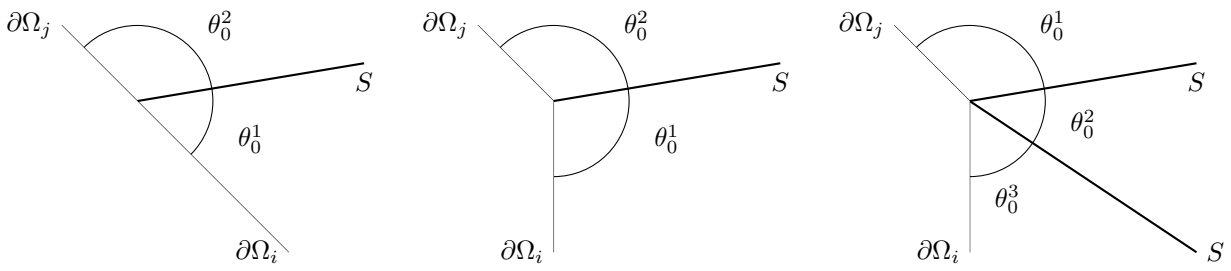


Figure 6.2: Local configuration near a point where the interface meets the boundary.

This is to be connected to Definition 6.1.8. In the case of a point of the boundary, if the corner does not impose any Dirichlet boundary condition then we also have (6.2.5). Note that the eigenvalues $(\nu_j)_{j\in\mathbb{N}}$ should not be confused with the eigenvalues that appear in the statement of the spectral inequalities in Sections 6.1.3 and 6.1.3.
In the case where 0 is a point where the interface is smooth and meets the boundary, we shall denote by θ_0^1 and θ_0^2 the measure of the two adjacent angles formed by the intersection (see examples in Figure 6.2).

We consider the operator

$$L_\theta = \mathrm{div}_\theta\, a(\theta)\nabla_\theta,$$

acting on functions defined for $\theta \in (0, \theta_0^1 + \theta_0^2)$, where $a(\theta)$ is a piecewise constant function satisfying $a(\theta) = \mathbf{1}_{[0,\theta_0^1]}c_1 + \mathbf{1}_{[\theta_0^1,\theta_0^1+\theta_0^2]}c_2$, $c_\ell > 0$, with domain that depend on boundary conditions that are imposed on each sides of the interface

$$\begin{aligned} D(L_\theta) = \big\{\psi,\ \psi_{|_{[0,\theta_0^1]}} \in H^1(0,\theta_0^1),\ \psi_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}} \in H^1(\theta_0^1,\theta_0^1+\theta_0^2),\ \Delta_\theta\psi_{|_{[0,\theta_0^1]}} \in L^2(0,\theta_0^1),\\ \Delta_\theta\psi_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}} \in L^2(\theta_0^1,\theta_0^1+\theta_0^2),\ B_1\psi = 0,\ B_2\psi = 0\ \psi(\theta_0^2) = 0 \text{ or } \psi'(\theta_0^2) = 0,\\ \psi_{|_{[0,\theta_0^1]}}(\theta_0^1) = \psi_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}}(\theta_0^1),\ c_1\psi'_{|_{[0,\theta_0^1]}}(\theta_0^1) - c_2\psi'_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}}(\theta_0^1) = 0\big\}, \end{aligned}$$

where B_1, B_2 denote either Dirichlet or Neumann traces at $\theta = 0$ and $\theta = \theta_0^1 + \theta_0^2$. In the cases where at least a Dirichlet boundary condition is imposed, we obtain (6.2.6) as a spectral basis for L_θ. Otherwise, we have (6.2.5).

In the case where 0 is a point of the interface where it is non-smooth, we shall denote by θ_0^1 and θ_0^2 the measure of the two adjacent angles formed by the intersection (here $\theta_0^1 + \theta_0^2 = 2\pi$), and we shall consider the operator

$$L_\theta = \mathrm{div}_\theta\, a(\theta)\nabla_\theta,$$

acting on functions of the circle $\mathbb{S}^1$, where $a(\theta)$ is a piecewise constant function satisfying $a(\theta) = \mathbf{1}_{[0,\theta_0^1]}c_1 + \mathbf{1}_{[\theta_0^1,2\pi]}c_2$, $c_\ell > 0$. An example is given in Figure 6.3. The domain of L_θ is given by

$$\begin{aligned} D(L_\theta) = \{\psi,\ \psi_{|_{[0,\theta_0^1]}} \in H^1(0,\theta_0^1),\ \psi_{|_{[\theta_0^1,2\pi]}} \in H^1(\theta_0^1,2\pi),\ \Delta_\theta\psi_{|_{[0,\theta_0^1]}} \in L^2(0,\theta_0^1),\ \Delta_\theta\psi_{|_{[\theta_0^1,2\pi]}} \in L^2(\theta_0^1,2\pi),\\ \psi_{|_{[0,\theta_0^1]}}(\theta_0^1) = \psi_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}}(\theta_0^1),\ c_1\psi'_{|_{[0,\theta_0^1]}}(\theta_0^1) - c_2\psi'_{|_{[\theta_0^1,\theta_0^1+\theta_0^2]}}(\theta_0^1) = 0\\ \psi_{|_{[0,\theta_0^1]}}(0) = \psi_{|_{[\theta_0^1,2\pi]}}(2\pi),\ c_1\psi'_{|_{[0,\theta_0^1]}}(0) - c_2\psi'_{|_{[\theta_0^1,2\pi]}}(2\pi) = 0\}. \end{aligned}$$

It is clear that 0 is an eigenvalue for this operator, associated with constant eigenfunctions. Then, we are in the situation described in (6.2.5).

Finally, we can consider the case where 0 is a point where the interface meet the boundary (at a corner or not), and where the interface itself is not smooth. This case is described in Figure 6.2 (right hand side figure).

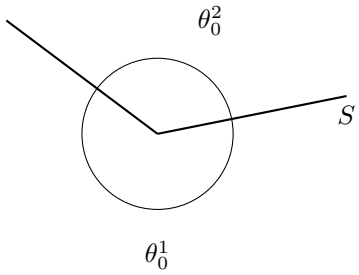


Figure 6.3: Different local configurations near a point where the interface is not smooth.

We shall carry out the analysis for the case of an inverse square potential, but the same can be carried out for the various situations described above for the polygonal problem.

Let $u \in C^\infty((-S_0, S_0), D(Q))$, such that $Qu = f \in L^2(Z)$, and let $R > 0$ such that $B(0, R) \subset \Omega$. We introduce the radial cut-off function that will localize the solution near the singularity

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < R/2 \\ 0 & \text{if } |x| > 3R/4 \end{cases}, \quad \chi \in C_0^\infty(\mathbb{R}^d). \tag{6.2.7}$$

We set $\tilde{u} = \chi u$. Hence, $\tilde{u}$ satisfies

$$Q\tilde{u} = \tilde{f} \in L^2(Z),$$

where $\tilde{f} = \chi f + [Q, \chi]u$. Now, setting $v(s, r, \theta) = \tilde{u}(s, r\theta)$ and $g(s, r, \theta) = \tilde{f}(s, r\theta)$, we obtain

$$-\partial_s^2 v - \partial_r^2 v - (d-1)\frac{\partial_r}{r} v - \frac{1}{r^2}\left(\Delta_\theta + \mu\right) v = g,$$

and the space $L^2((-S_0, S_0) \times B(0, R))$ is then transformed into the weighted L^2-space given by

$$\int_{-S_0}^{S_0} \int_0^R \int_{\mathbb{S}^{d-1}} |v(s, r, \theta)|^2 r^{d-1} ds dr d\theta < \infty, \quad \text{and} \quad \int_{-S_0}^{S_0} \int_0^R \int_{\mathbb{S}^{d-1}} |g(s, r, \theta)|^2 r^{d-1} ds dr d\theta < \infty.$$

In order to work in non-weighted Sobolev spaces, we introduce $w = r^{(d-1)/2} v$, and $h = r^{(d-1)/2} g$ so that

$$\int_{-S_0}^{S_0} \int_0^R \int_{\mathbb{S}^{d-1}} |w(s, r, \theta)|^2 ds dr d\theta < \infty, \quad \text{and} \quad \int_{-S_0}^{S_0} \int_0^R \int_{\mathbb{S}^{d-1}} |h(s, r, \theta)|^2 ds dr d\theta < \infty.$$

Moreover we have the relation

$$-\partial_s^2 w - \partial_r^2 w - \frac{1}{r^2}\left(\Delta_\theta + \mu - \frac{(d-1)(d-3)}{4}\right) w = h.$$

Splitting the variables, and using (6.2.5) or (6.2.6), we can write

$$w(s, r, \theta) = \sum_{k \geq 0} w_k(s, r)\psi_k(\theta), \quad h(s, r, \theta) = \sum_{k \geq 0} h_k(s, r)\psi_k(\theta), \tag{6.2.8}$$

and thus we obtain the countable family of equations in the variable s and r,

$$-\partial_s^2 w_k - \partial_r^2 w_k - \frac{1}{r^2}\left(-\nu_k + \mu - \frac{(d-1)(d-3)}{4}\right) w_k = h_k, \quad k \in \mathbb{N},$$

with

$$\sum_{k \in \mathbb{N}} \int_{-S_0}^{S_0} \int_0^R w_k(s, r)^2 dr ds < \infty, \quad \text{and} \quad \sum_{k \in \mathbb{N}} \int_{-S_0}^{S_0} \int_0^R h_k(s, r)^2 dr ds < \infty. \tag{6.2.9}$$

Setting $\mu_k = \mu - \nu_k - \frac{(d-1)(d-3)}{4}$, we obtain

$$-\partial_s^2 w_k - \partial_r^2 w_k - \frac{\mu_k}{r^2} w_k = h_k. \tag{6.2.10}$$

In all these situations, we can see that

$$\mu_k = \mu - \frac{(d-1)(d-3)}{4} - \nu_k = \mu - \mu^* + \frac{(d-2)^2 - (d-1)(d-3)}{4} - \nu_k = \mu - \mu^* + \frac{1}{4} - \nu_k.$$

Note that $1/4$ is the critical constant in the one-dimensional Hardy inequality given below in (6.2.15). In the cases described above, we may have $\nu_0 = 0$.

This occurs for an inverse square potential as considered here. Hence, in the sub-critical case $\mu < \mu^*$, the associated one-dimensional problem is also a subcritical case, as we have $\mu_k < 1/4$, for all $k \in \mathbb{N}$. On the contrary, in the critical case $\mu = \mu^*$, the equation given by (6.2.10) is a critical case, since $\mu_0 = 1/4$.

In the case of a polygonal problem, the considered problem is (6.2.10) with $d = 2$ and $\mu = \mu^* = 0$, and this yields

$$\mu_k = \frac{1}{4} - \nu_k.$$

In the case of a Neumann corner, $\nu_0 = 0$, and then $\mu_0 = 1/4$ which is a one-dimensional critical case. If the considered corner is not a Neumann corner, then $\mu_k < 1/4$, for all $k \in \mathbb{N}$.

We have by construction that $w_k(s, r) = 0$, for $r > 3R/4$. At the boundary $\{r = 0\}$, we have the following lemma.

Lemma 6.2.7. *For functions w_k defined in (6.2.8), $k \in \mathbb{N}$, we have $\lim_{r\to 0} w_k(r, s) = 0$.*

Proof.
We first focus on the polygonal case, that is, $d = 2$. From the fact that $u \in C^\infty((-S_0, S_0), H^1(\Omega))$, we obtain that

$$\sup_s \int_0^R \int_{\mathbb{S}^1} |\partial_r v(s, r, \theta)|^2 r dr d\theta + \sup_s \int_0^R \int_{\mathbb{S}^1} \Big|\frac{\partial_\theta v(s, r, \theta)}{r}\Big|^2 r dr d\theta < \infty.$$

Using decomposition (6.2.8), we have

$$\sup_s \int_0^R |\partial_r v_k(s, r)|^2 r dr + \nu_k \sup_s \int_0^R \Big|\frac{v_k(s, r)}{r}\Big|^2 r dr < \infty. \tag{6.2.11}$$

For $k \in \mathbb{N}$ such that $\nu_k \neq 0$, we can see that

$$\sup_s \int_0^R |\partial_r w_k(s, r)|^2 dr \lesssim \sup_s \int_0^R |\partial_r v_k(s, r)|^2 r dr + \sup_s \int_0^R \Big|\frac{v_k(s, r)}{r}\Big|^2 r dr < \infty,$$

and then $w_k \in H^1((-S_0, S_0) \times (0, R))$. In the case $\nu_k = 0$, we shall prove $\nabla w_k \in L^q((-S_0, S_0) \times (0, R))$, for all $q \in (1, 2)$. Indeed, we have

$$\int_0^R |\partial_r w_k(s, r)|^q dr \lesssim \int_0^R \Big|\frac{v_k(s, r)}{r^{1/2}}\Big|^q dr + \int_0^R |r^{1/2} \partial_r v_k(s, r)|^q dr. \tag{6.2.12}$$

The last term is bounded, since

$$\int_0^R |r^{1/2} \partial_r v_k(s, r)|^q dr \leq \int_0^R |r^{1/2} \partial_r v_k(s, r)|^2 ds dr.$$

Using the Hölder inequality, we have, for $p > q$,

$$\int_0^R |\frac{v_k(s,r)}{r^{1/2}}|^q dr \leq \left(\int_0^R |v_k(s,r)|^p r dr\right)^{\frac{q}{p}} \left(\int_0^R r^{(\frac{q}{2}-\frac{q}{p})(\frac{p}{p-q})} dr\right)^{\frac{p-q}{p}}$$

The last term is bounded for p taken sufficiently large. From two dimensional Sobolev embbedings, note that one can have, for all $p \geq 1$,

$$\int_0^R |v_k(s,r)|^p r dr \lesssim \left(\int_0^R |\partial_r v_k(s,r)|^2 r dr\right)^{\frac{p}{2}}, \tag{6.2.13}$$

and then, we obtain $\sup_s \int_0^R |\partial_r w_k(s,r)|^q dr < \infty$. As a result, the limit at $r = 0$ exists, and we write $\lim_{r\to 0} w_k(r,s) = C(s)$. It implies by definition that $v_k(r,s) \sim r^{-1/2}C(s)$ for $r > 0$ small, and (6.2.11) yields $C(s) = 0$. This ends the proof for the polygonal case. We can carry out the same analysis for the inverse square potential case, using the Hardy inequality (6.1.7). □

We now state some one-dimensional useful results. The first lemma, which can be found in [VZ08], allows one to handle the traces at $r = 0$ that appears when performing integration by parts.

Lemma 6.2.8. *Let $z \in H_0^1(0,R)$. Then*

$$\lim_{r\to 0} \frac{z(r)^2}{r} = 0. \tag{6.2.14}$$

Proof. We have, using the Cauchy-Schwarz inequality,

$$z(r)^2 = \left(\int_0^r z'(s)ds\right)^2 \leq \int_0^r z'(s)^2 ds \int_0^r 1 ds \leq r\int_0^r z'(s)^2 ds,$$

which yields the result, using the Lebesgue dominated convergence theorem. □

We shall use the following one-dimensionnal Hardy inequality, and it is a key argument in what follows.

Lemma 6.2.9 (One-dimensional Hardy Inequality)**.** *Let $R > 0$. Then, there exists $C > 0$ such that*

$$\int_0^R |z|^2 dr \leq C\int_0^R \left(|z'|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right) dr. \tag{6.2.15}$$

for all $z \in H_0^1(0,R)$.

Proof. We first write

$$\begin{aligned}
\int_0^R |z'|^2 dr &= \int_0^R \left|\frac{d}{dr}\left(\frac{z}{r^{1/2}}\right) r^{1/2} + \frac{z}{2r}\right|^2 dr = \int_0^R \left(\frac{d}{dr}\left(\frac{z}{r^{1/2}}\right)\right)^2 r dr + \int_0^R \frac{d}{dr}\left(\frac{z}{r^{1/2}}\right)\frac{z}{r^{1/2}} + \frac{1}{4}\int_0^R \frac{z^2}{r^2} dr \\
&= \int_0^R \left(\frac{d}{dr}\left(\frac{z}{r^{1/2}}\right)\right)^2 r dr + \frac{1}{2}\left[\left(\frac{z}{r^{1/2}}\right)^2\right]_0^R + \frac{1}{4}\int_0^R \frac{z^2}{r^2} dr \\
&= \int_0^R \left(\frac{d}{dr}\left(\frac{z}{r^{1/2}}\right)\right)^2 r dr + \frac{1}{4}\int_0^R \frac{z^2}{r^2} dr,
\end{aligned}$$

where we used Lemma 6.2.8. Setting $\zeta = r^{-1/2}z$, it remains to prove the following weighted Poincaré inequality

$$\int_0^R |\zeta'|^2 r dr \geq C\int_0^R |\zeta|^2 r dr.$$

Note that using Cauchy-Schwarz inequality, we have

$$\zeta(r)^2 = \left(\int_r^R \zeta'(t)dt\right)^2 \leq \left|\int_r^R \zeta'(t)^2 t dt\right| \left|\int_r^R \frac{1}{t}dt\right|$$

We set $Z(r) = \int_r^R \zeta'(t)^2 t dt$ and $F(r) = \int_r^R \frac{1}{t}dt$. Note that $rF(r)$ is bounded. Hence,

$$\begin{aligned}\int_0^R \zeta(r)^2 r dr &\leq \int_0^R Z(r)|F(r)|rdr \leq C\int_0^R Z(r)dr \leq C\left[Z(r)r\right]_0^R + C\left|\int_0^R Z'(r)rdr\right| \\ &\leq C\int_0^R \zeta'^2 r dr,\end{aligned}$$

which yields the result. □

We also prove the following improved Hardy inequality (see [VZ00]).

Lemma 6.2.10 (One-dimensional improved Hardy inequality)**.** *There exists $C \geq 0$ such that*

$$||z||^2_{W^1_q(0,R)} \leq C\int_0^R \left(|z'|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right)dr \tag{6.2.16}$$

for all $z \in H^1_0(0,R)$, for all $q \in [1,2)$.

Proof. We shall sometimes use the same arguments than those used in the proof of Lemma 6.2.7. We set $\zeta = r^{-1/2}z$. Then,

$$\int_0^R |z'|^2 - \frac{1}{4}\int_0^R \frac{|z|^2}{r^2}dr = \int_0^R r|\zeta'|^2 dr,$$

arguing in the same way as the beginning of the proof of Lemma 6.2.9. As $z' = r^{1/2}\zeta' + \frac{r^{-1/2}}{2}\zeta$, we write

$$\int_0^R |z'|^q dr \lesssim \int_0^R |\zeta'|^q r^{\frac{q}{2}}dr + \int_0^R |\zeta|^q r^{-\frac{q}{2}}.$$

For the first integral, with $q \in (1,2]$, we write

$$||r^{1/2}\zeta'||^q_{L^q(0,R)} \lesssim ||r^{1/2}\zeta'||^q_{L^2(0,R)},$$

and for the second integral, using the Hölder inequality, with q, p such that $q \in (1,2)$, and $p > q$,

$$||\zeta^q r^{-\frac{q}{2}}||_{L^1(0,R)} \leq ||\zeta^q r^{\frac{q}{p}}||_{L^{\frac{p}{q}}(0,R)} ||r^{-\frac{q}{2}-\frac{q}{p}}||_{L^{\frac{p}{p-q}}(0,R)}$$

We see that $||r^{-\frac{q}{2}-\frac{q}{p}}||_{L^{\frac{p}{p-q}}(0,R)} < \infty$, for $p > 4q/2 - q$. Indeed,

$$\int_0^R r^{\left(\frac{q}{2}-\frac{q}{p}\right)\left(\frac{p}{p-q}\right)} < +\infty \Leftrightarrow \left(\frac{q}{2}-\frac{q}{p}\right)\left(\frac{p}{p-q}\right) > -1 \Leftrightarrow p > \frac{4q}{2-q}. \tag{6.2.17}$$

We set $B_2(0,R)$ the unit ball in dimension 2 of radius R. From the Sobolev embbeding $H^1_0(B_2(0,R)) \hookrightarrow L^p(B_2(0,R))$, for all $p \geq 1$, in polar coordinates, we have

$$\left(\int_0^R \zeta^p r dr\right)^{\frac{1}{p}} \lesssim \left(\int_0^R |\zeta'|^2 r dr\right)^{\frac{1}{2}},$$

and then

$$||\zeta^q r^{\frac{q}{p}}||^{\frac{q}{p}}_{L^{\frac{p}{q}}(0,R)} = \left(\int_0^R \zeta^p r dr\right)^{\frac{q}{p}} \lesssim ||\zeta' r^{1/2}||^q_{L^2(0,R)}.$$

Summing up, we obtained

$$||z'||_{L^q(0,R)} \lesssim ||\zeta' r^{1/2}||_{L^2(0,R)} = \left(\int_0^R \left(|z'|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right) dr\right)^{\frac{1}{2}}, \tag{6.2.18}$$

which is the sought result, since the left hand side of (6.2.18) defines a norm on $W^{1,p}_0(0,R)$. □

We define $\mathcal{H}_{\mu_k}(0,R)$ as the completion of $H^1_0(0,R)$ with respect to the norm

$$||u||^2_{\mathcal{H}_{\mu_k}(0,R)} = \int_0^R \left(|\partial_r u|^2 - \mu_k \frac{|u|^2}{r^2}\right) dr.$$

Note that this quantity defines a norm on $H^1_0(0,R)$ using Lemma 6.2.10. In the sub-critical case $\mu_k < 1/4$, we have $\mathcal{H}_{\mu_k} \simeq H^1_0(0,R)$, but in the critical case $\mu_k = 1/4$, the space $\mathcal{H}_{1/4}(0,R)$ is strictly larger than $H^1_0(0,R)$ (see also Section 6.1.1). Indeed, it is sufficient to set $\tilde{\chi} \in C^\infty(0,R)$ be such that $\tilde{\chi}(r) = 1$ for $r < R/4$ and $\tilde{\chi} = 0$ for $r \in (R/2, R)$, and consider the function $\underline{\zeta}(r) = \sqrt{r}\tilde{\chi}(r)$. Indeed, we have

$$\lim_{r\to 0} \underline{\zeta} = \lim_{r\to R} \underline{\zeta} = 0, \quad ||\underline{\zeta}||^2_{\mathcal{H}_{1/4}(0,R)} < +\infty, \text{ and } \partial_r \underline{\zeta} \notin L^2(0,R).$$

6.2.4 Carleman estimate in spherical harmonics

Let $R > 0$. Let $\tilde{Z} = (-S_0, S_0) \times (0,R)$ and let $\tilde{Y} = (-S_0 + \alpha, S_0 - \alpha) \times (0,R)$. We set $\mathcal{I}_\alpha = (-S_0 + \alpha, S_0 - \alpha) \times \{0\}$. For $\mu_k \leq 1/4$, we define

$$Q_k = -\partial_s^2 - \partial_r^2 - \frac{\mu_k}{r^2}.$$

Let $y_0 \in \mathcal{I}_\alpha$, and let $\tilde{\mathcal{V}}$ be a neighborhood of $y_0 = (s_0, 0)$ in $\overline{\tilde{Y}}$. Assume without any loss of generality that $\tilde{\mathcal{V}}$ is of the form

$$\tilde{\mathcal{V}} = (-\tilde{s}_0, \tilde{s}_0) \times (0,R).$$

We then shall consider the following problem

$$\begin{cases} Q_k z = z^0 \in L^2(\tilde{Z}), & \text{in } \tilde{Z}, \\ z(s,0) = 0 & \text{at } \mathcal{I}_\alpha, \\ \text{supp}\, z \subset \tilde{\mathcal{V}}, \end{cases} \tag{6.2.19}$$

Note that the functions w_k defined by (6.2.8) satisfy Problem (6.2.19). We shall consider a weight function $\varphi \in C^\infty(\overline{\tilde{Z}})$ satisfying

$$\varphi_\varepsilon(r,s) = r^2 - \varepsilon(s - s_0)^2, \tag{6.2.20}$$

We now are ready to state the one-dimensional Carleman estimates.

Theorem 6.2.11. *Assume $\mu_k < 1/4$. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$\begin{aligned} C||e^{\tau\varphi_\varepsilon} Q_k z||^2_{L^2(\tilde{Y})} \geq \tau\,(1 + |\mu_k|)\,||e^{\tau\varphi_\varepsilon}\frac{z}{r}||^2_{L^2(\tilde{Y})} \\ + \tau||e^{\tau\varphi_\varepsilon}\partial_r z||^2_{L^2(\tilde{Y})} + \tau^2||e^{\tau\varphi_\varepsilon} z||^2_{L^2(\tilde{Y})} + \tau^3||r e^{\tau\varphi_\varepsilon} z||^2_{L^2(\tilde{Y})} + \tau^{2/3}||e^{\tau\varphi_\varepsilon}\partial_s z||^2_{L^2(\tilde{Y})}, \end{aligned} \tag{6.2.21}$$

holds for all $z \in C^\infty_0((-S_0, S_0), H^1_0(0,R))$, solution of Problem (6.2.19), for $\tau \geq \tau_0$ and for $\varepsilon = \varepsilon_0 \tau^{-1/3}$.

Theorem 6.2.12. *Assume $\mu_k = 1/4$. There exist $\varepsilon_0 > 0$, $\tau_0 > 0$, $C > 0$ such that*

$$C||e^{\tau\varphi_\varepsilon} Q_k z||^2_{L^2(\tilde{Y})} \geq \tau||e^{\tau\varphi_\varepsilon} z||^2_{L^2(\tilde{Y})} + \tau^3||re^{\tau\varphi_\varepsilon} z||^2_{L^2(\tilde{Y})} + \tau^{1/3}||e^{\tau\varphi_\varepsilon}\partial_s z||^2_{L^2(\tilde{Y})} \tag{6.2.22}$$

holds for all $z \in C_0^\infty((-S_0, S_0), \mathcal{H}_{1/4}(0,R))$, solution of Problem (6.2.19), for $\tau \geq \tau_0$ and for $\varepsilon = \varepsilon_0\tau^{-2/3}$.

Proof of Theorems 6.2.1 and 6.2.5 (resp. Theorems 6.2.2 and 6.2.6). Applying Theorem 6.2.11 (resp. Theorem 6.2.12) to each w_k, and summing over k yields the result, recalling that $w = r^{(d-1)/2}v$. □

Remark 6.2.13. *It should be noted that in the proof of Theorem 6.2.12, when additional regularity is needed, we can perform the following regularization procedure. First, we prove estimates (6.2.21) and (6.2.22) for the following operator*

$$Q_k^{1/N} = -\partial_s^2 - \partial_r^2 - \frac{\mu_k}{r^2 + \frac{1}{N}}.$$

For this operator, from classical elliptic estimates, solutions of Problem (6.2.19) are H^2. Then, we pass to the limit $N \to \infty$ using Lemma 6.2.17. We justify in what follows this procedure.

Let $\mu_k = 1/4$ and let z_N be the solution of

$$\begin{cases} Q_k^{1/N} z_N = z^0 \in L^2(\tilde{Z}), & \text{in } \tilde{Z}, \\ z(s,0) = 0 & \text{at } \mathcal{I}_\alpha, \\ \operatorname{supp} z \subset \tilde{\mathcal{V}}. \end{cases} \tag{6.2.23}$$

We define the following associated quadratic forms.

$$a(z,\tilde{z}) := \int_{\tilde{\mathcal{V}}} \partial_s z \partial_s \overline{\tilde{z}} + \int_{\tilde{\mathcal{V}}} \left(\partial_r z \partial_r \overline{\tilde{z}} - \frac{1}{4}\frac{z\overline{\tilde{z}}}{r^2}\right) = \int_{\tilde{\mathcal{V}}} \partial_s z \partial_s \overline{\tilde{z}} + \left(z, \tilde{z}\right)_{\mathcal{H}_{\frac{1}{4}}},$$

and

$$a_N(z,\tilde{z}) := \int_{\tilde{\mathcal{V}}} \partial_s z \partial_s \overline{\tilde{z}} + \int_{\tilde{\mathcal{V}}} \left(\partial_r z \partial_r \overline{\tilde{z}} - \frac{1}{4}\frac{z\overline{\tilde{z}}}{r^2 + \frac{1}{N}}\right).$$

The variational problems of (6.2.19) and (6.2.23) are given by

$$a(z,\tilde{z}) = \left(z^0, \tilde{z}\right)_{L^2(\tilde{\mathcal{V}})}, \quad \forall \tilde{z} \in L^2((-\tilde{s}_0, \tilde{s}_0), \mathcal{H}_{1/4}(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R)), \ \operatorname{supp} \tilde{z} \subset \tilde{\mathcal{V}}, \tag{6.2.24}$$

and

$$a_N(z,\tilde{z}) = \left(z^0, \tilde{z}\right)_{L^2(\tilde{\mathcal{V}})}, \quad \forall \tilde{z} \in L^2((-\tilde{s}_0, \tilde{s}_0), H_0^1(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R)), \ \operatorname{supp} \tilde{z} \subset \tilde{\mathcal{V}}, \tag{6.2.25}$$

respectively.

Lemma 6.2.14. *There exists $C > 0$ and $C_N > 0$ satisfying $C_N \to 0$ as $N \to +\infty$, such that*

$$a_N(z,z) \geq C_N||z||^2_{H^1(\tilde{\mathcal{V}})} + C||z||^2_{L^2(\tilde{\mathcal{V}})}.$$

Proof.
Using Hardy inequality (6.2.15), we have

$$\begin{aligned}
a_N(z,z) &= \int_{\tilde{\mathcal{V}}} |\partial_s z|^2 + \int_{\tilde{\mathcal{V}}} \left(|\partial_r z|^2 - \frac{1}{4}\frac{|z|^2}{r^2 + \frac{1}{N}}\right) \\
&\geq \int_{\tilde{\mathcal{V}}} |\partial_s z|^2 + \frac{1}{1+NR^2}\int_{\tilde{\mathcal{V}}} |\partial_r z|^2 + C\frac{NR^2}{1+NR^2}\int_{\tilde{\mathcal{V}}} |z|^2 + \int_{\tilde{\mathcal{V}}} \left(\frac{1}{4}\frac{NR^2}{1+NR^2}\frac{|z|^2}{r^2} - \frac{1}{4}\frac{|z|^2}{r^2 + \frac{1}{N}}\right) \\
&\geq \int_{\tilde{\mathcal{V}}} |\partial_s z|^2 + \frac{1}{1+NR^2}\int_{\tilde{\mathcal{V}}} |\partial_r z|^2 + C\frac{NR^2}{1+NR^2}\int_{\tilde{\mathcal{V}}} |z|^2 + \int_{\tilde{\mathcal{V}}} \frac{\frac{1}{4}R^2 - \frac{1}{4}r^2 + NR^2r^2\left(\frac{1}{4} - \frac{1}{4}\right)}{r^2\left(1+NR^2\right)\left(r^2 + \frac{1}{N}\right)}|z|^2 \\
&\geq \int_{\tilde{\mathcal{V}}} |\partial_s z|^2 + \frac{1}{1+NR^2}\int_{\tilde{\mathcal{V}}} |\partial_r z|^2 + C\frac{NR^2}{1+NR^2}\int_{\tilde{\mathcal{V}}} |z|^2,
\end{aligned}$$

since $\frac{\frac{1}{4}R^2-\frac{1}{4}r^2+NR^2r^2\left(\frac{1}{4}-\frac{1}{4}\right)}{r^2(1+NR^2)\left(r^2+\frac{1}{N}\right)}$ is non-negative, and this yields the result. □

Lemma 6.2.15. *There exists a unique variational solution $z \in H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))\cap L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R))$ to (6.2.24). Moreover, there exists $C>0$ such that*

$$||z||_{H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))}+||z||_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R))}\leq C||z^0||_{L^2(\tilde{\mathcal{V}})}.$$

Proof. The space $\mathcal{H}_{\frac{1}{4}}(0,R)$ endowed with the norm $||.||_{\mathcal{H}_{\frac{1}{4}}}$ is a Hilbert space (compactly embedded in L^2) from the Hardy inequalities (6.2.15) and (6.2.16). It is sufficient to apply the Lax-Milgram theorem to the following variational equation

$$a(z,w)=\left(z^0,w\right)_{L^2},\quad w\in H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))\cap L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R)).$$

□

Lemma 6.2.16. *There exists a unique variational solution*

$$z_N\in H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))\cap L^2((-\tilde{s}_0,\tilde{s}_0),H_0^1(0,R))$$

to equation (6.2.25). Moreover, there exists $C>0$, independant on N, such that

$$||z_N||^2_{H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))}+\int_{\tilde{\mathcal{V}}}\left(|\partial_r z_N|^2-\frac{1}{4}\frac{|z_N|^2}{r^2+\frac{1}{N}}\right)\leq C||z^0||^2_{L^2(\tilde{\mathcal{V}})}.$$

Proof. For the first part of the lemma, we apply the Lax-Milgram theorem to (6.2.25), using Lemma 6.2.14. For the second part, from the variational equation (6.2.25) with $w=z_N$, we have, using the Young inequality and the Poincaré inequality, for all $\varepsilon>0$,

$$\begin{aligned}\varepsilon^{-1}||z^0||^2_{L^2(\tilde{\mathcal{V}})}+\varepsilon||z_N||^2_{L^2(\tilde{\mathcal{V}})}&\geq\int_{\tilde{\mathcal{V}}}|\partial_s z_N|^2+\int_{\tilde{\mathcal{V}}}\left(|\partial_r z_N|^2-\frac{1}{4}\frac{|z_N|^2}{r^2+\frac{1}{N}}\right)\\&\geq C\int_{\tilde{\mathcal{V}}}|\partial_s z_N|^2+C\int_{\tilde{\mathcal{V}}}|z_N|^2+\frac{1}{2}\int_{\tilde{\mathcal{V}}}\left(|\partial_r z_N|^2-\frac{1}{4}\frac{|z_N|^2}{r^2+\frac{1}{N}}\right).\end{aligned}$$

Taking $\varepsilon>0$ sufficiently small yields the result. □

The following lemma justifies the regularization procedure, in the critical case $\mu_k=1/4$.

Lemma 6.2.17. *Let $\mu_k=1/4$. Let z be the variational solution of (6.2.24) and z_N be the variational solution of (6.2.25). Then, we have*

$$z_N\longrightarrow z \text{ in } L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R))\cap H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R)),$$

as N goes to infinity.

Proof. Let z_N and z^0 satisfying (6.2.23). By Lemma 6.2.16, we have

$$||z_N||_{H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R))}+||z_N||_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R))}\leq\int_{\tilde{\mathcal{V}}}|\partial_s z_N|^2+\int_{\tilde{\mathcal{V}}}\left(|\partial_r z_N|^2-\frac{1}{4}\frac{|z_N|^2}{r^2+\frac{1}{N}}\right)\leq C||z^0||^2_{L^2(\tilde{\mathcal{V}})}.$$

We can extract a subsequence, (denoted again z_N) such that

$$z_N\rightharpoonup\tilde{z}\text{ in }L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{\frac{1}{4}}(0,R))\text{ and in }H^1((-\tilde{s}_0,\tilde{s}_0),L^2(0,R)),\tag{6.2.26}$$

and $\operatorname{supp}\tilde{z}\subset\tilde{\mathcal{V}}$. Let $0<r_0<R$. For all $w\in C_0^\infty((-\tilde{s}_0,\tilde{s}_0)\times(r_0,R))$, we write

$$\Big(\partial_s z_N, \partial_s w\Big)_{L^2(\tilde{\mathcal{V}})} + \int_{\tilde{\mathcal{V}}} \left(\partial_r z_N \partial_r \overline{w} - \frac{1}{4}\frac{z_N \overline{w}}{r^2}\right) + \frac{1}{4}\int_{\tilde{\mathcal{V}}} \left(\frac{1}{r^2} - \frac{1}{r^2+\frac{1}{N}}\right) z_N \overline{w} = \int_{\tilde{\mathcal{V}}} z^0 \overline{w}.$$

We have, on the one hand

$$\frac{1}{r^2} - \frac{1}{r^2+\frac{1}{N}} = \frac{1}{N}\frac{1}{r^2\left(r^2+\frac{1}{N}\right)} \leq \frac{1}{N r_0^4},$$

on the support of w. As a result, using the boundedness of $||z_N||_{L^2(\tilde{\mathcal{V}})}$,

$$\Big|\int_{\tilde{\mathcal{V}}} \left(\frac{1}{r^2} - \frac{1}{r^2+\frac{1}{N}}\right) z_N \overline{w}\Big| \leq \frac{1}{N r_0^4} ||z_N||_{L^2(\tilde{\mathcal{V}})} ||w||_{L^2(\tilde{\mathcal{V}})} \longrightarrow 0, \text{ as } N \to \infty.$$

On the other hand,

$$\int_{\tilde{\mathcal{V}}} \partial_s z_N \partial_s w + \int_{\tilde{\mathcal{V}}} \left(\partial_r z_N \partial \overline{w} - \frac{1}{4}\frac{z_N \overline{w}}{r^2}\right) \longrightarrow \int_{\tilde{\mathcal{V}}} \partial_s \tilde{z} \partial_s w + \int_{\tilde{\mathcal{V}}} \left(\partial_r \tilde{z} \partial \overline{w} - \frac{1}{4}\frac{\tilde{z}\overline{w}}{r^2}\right), \text{ as } N \to \infty \tag{6.2.27}$$

using the weak convergence (6.2.26). Moreover, $\tilde{z} \in L^2((-\tilde{s}_0, \tilde{s}_0), \mathcal{H}_{\frac{1}{4}}(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R))$. Hence, the right member of (6.2.27) is continuous with respect to $w \in C_0^\infty((-\tilde{s}_0, \tilde{s}_0) \times (0,R))$ for the topology of $L^2((-\tilde{s}_0, \tilde{s}_0), \mathcal{H}_{\frac{1}{4}}(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R))$. Hence, we can extend (6.2.27) by density to all $w \in L^2((-\tilde{s}_0, \tilde{s}_0), \mathcal{H}_{\frac{1}{4}}(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R))$. Using Lemma 6.2.15, we obtain $\tilde{z} = z$, and then $z_N \rightharpoonup z$. We now have to obtain the strong convergence. Let us write $K = L^2((-\tilde{s}_0, \tilde{s}_0), \mathcal{H}_{\frac{1}{4}}(0,R)) \cap H^1((-\tilde{s}_0, \tilde{s}_0), L^2(0,R))$. From classical results (see [Bre83]), it is sufficient to show that

$$\limsup_{N\to\infty} ||z_N||_K \leq ||z||_K.$$

Yet, we have

$$||z_N||_K^2 = a(z_N, z_N) \leq a_N(z_N, z_N) = \Big(z^0, z_N\Big)_{L^2(\tilde{\mathcal{V}})} \underset{N\to+\infty}{\longrightarrow} \Big(z^0, z\Big)_{L^2(\tilde{\mathcal{V}})} = a(z,z) = ||z||_K^2.$$

Taking the lim sup in the above inequality yields the sought result. □

Before proving Theorems 6.2.11 and 6.2.12, we provide a heuristic justification of our choice of weight function. As usual with Carleman estimates, it is related to the well-known sub-ellipticity condition. We compute the congugated operator $Q_{k,\varphi_\varepsilon} := e^{\tau\varphi_\varepsilon} Q_k e^{-\tau\varphi_\varepsilon}$, and we set $\zeta = e^{\tau\varphi} z$.

$$\begin{aligned}
Q_{k,\varphi_\varepsilon} &= -(e^{\tau\varphi_\varepsilon}\partial_s e^{-\tau\varphi_\varepsilon})^2 - (e^{\tau\varphi_\varepsilon}\partial_r e^{-\tau\varphi_\varepsilon})^2 - \frac{\mu_k}{r^2} \\
&= -\partial_s^2 - \partial_r^2 - \tau^2((\partial_s\varphi_\varepsilon)^2 + (\partial_r\varphi_\varepsilon)^2) + 2\tau((\partial_r\varphi_\varepsilon)\partial_r + (\partial_s\varphi_\varepsilon)\partial_s) + \tau(\partial_r^2\varphi_\varepsilon) + \tau(\partial_s^2\varphi_\varepsilon) - \frac{\mu_k}{r^2} \\
&= A_k^r + A_k^s + B_k^r + B_k^s,
\end{aligned}$$

with $A_k^s = -\partial_s^2 - \tau^2(\partial_s\varphi_\varepsilon)^2$, $A_k^r = -\partial_r^2 - \tau^2(\partial_r\varphi_\varepsilon)^2 - \mu_k/r^2$, $B_k^s = 2\tau(\partial_s\varphi_\varepsilon)\partial_s + \tau(\partial_s^2\varphi_\varepsilon)$, and $B_k^r = 2\tau(\partial_r\varphi_\varepsilon)\partial_r + \tau(\partial_r^2\varphi_\varepsilon)$.

Formally, as we have to deal with coefficients that may blow-up, the principal symbols of the above operators are given by

$$\begin{aligned}
a_k^s(r,s,\rho,\sigma,\tau) &= \sigma^2 - \tau^2(\partial_s\varphi_\varepsilon)^2, & b_k^s(r,s,\rho,\sigma,\tau) &= 2i\tau(\partial_s\varphi_\varepsilon)\sigma, \\
a_k^r(r,s,\rho,\sigma,\tau) &= \rho^2 - \tau^2(\partial_r\varphi_\varepsilon)^2 - \mu_k/r^2, & b_k^r(r,s,\rho,\sigma,\tau) &= 2i\tau(\partial_r\varphi_\varepsilon)\rho,
\end{aligned}$$

where $\rho, \sigma \in \mathbb{R}$ denote the dual variables of r, s, respectively. We compute the following Poisson bracket

$$\begin{aligned}
\{a_k^r + a_k^s, b_k^r + b_k^s\}(r,s,\rho,\sigma,\tau) &= \partial_{(\rho,\sigma)}(a_k^r + a_k^s)\partial_{(r,s)}(b_k^r + b_k^s) - \partial_{(\rho,\sigma)}(b_k^r + b_k^s)\partial_{(r,s)}(a_k^r + a_k^s) \\
&= 4i\tau\left(d^2\varphi_\varepsilon((\rho,\sigma),(\rho,\sigma)) + \tau^2 d^2\varphi_\varepsilon(\nabla\varphi_\varepsilon, \nabla\varphi_\varepsilon) - \partial_r\varphi_\varepsilon\frac{\mu_k}{r^3}\right).
\end{aligned} \tag{6.2.28}$$

Remark 6.2.18. *At this step, as we shall need some positivity of the Poisson bracket, we see that the choice $\partial_r \varphi_\varepsilon \simeq r$ allows one to reduce the power of the singular behavior of the last term in (6.2.28).*

We shall assume that $s_0 = 0$, up to performing a harmless translation. As we need some convexity of the level sets of the weight function with respect to $\{r = 0\}$, we introduce the weight function (6.2.20), with the parameter ε to be fixed later. With φ_ε of this form we obtain

$$\frac{1}{i}\{a_k^r + a_k^s, b_k^r + b_k^s\}(r,s,\rho,\sigma,\tau)\} = 4\tau\left(\left(2\rho^2 - 2\frac{\mu_k}{r^2}\right) + 8\tau^2 r^2\right) - 8\tau\varepsilon\sigma^2 - 32\tau^3\varepsilon^3 s^2. \tag{6.2.29}$$

From Lemma 6.2.9, in the sub-critical case $\mu_k < \frac{1}{4}$, we formally obtain

$$\rho^2 - \frac{\mu_k}{r^2} \gtrsim \rho^2 + \frac{1}{r^2}.$$

This yields $4\tau\left(\left(2\rho^2 - 2\frac{\mu_k}{r^2}\right) + 8\tau^2 r^2\right) \gtrsim \tau\rho^2 + \frac{\tau}{r^2} + \tau^3 r^2$. Using the following inequality

$$\tau^2 = \tau^2 \frac{1}{\sqrt{\tau r^2}}\sqrt{\tau r^2} \leq \frac{\tau^2}{2}\left(\frac{1}{\tau r^2} + \tau r^2\right),$$

we obtain

$$4\tau\left(\left(2\rho^2 - 2\frac{\mu_k}{r^2}\right) + 8\tau^2 r^2\right) \gtrsim \tau\rho^2 + \frac{\tau}{r^2} + \tau^3 r^2 + \tau^2$$

Then the choice $\varepsilon = \varepsilon_0 \tau^{-1/3}$ makes sense to handle the last term in (6.2.29), as s is bounded, by taking $\varepsilon_0 > 0$ small. As a result, there exists $C > 0$ such that

$$\frac{1}{i}\{a_k^r + a_k^s, b_k^r + b_k^s\}(r,s,\rho,\sigma,\tau)\} \geq C\left(\tau\rho^2 + \tau^2 + \frac{\tau}{r^2} + \tau^3 r^2\right) - 4\varepsilon_0\tau^{2/3}\sigma^2. \tag{6.2.30}$$

Yet, it remains to handle the last negative term $-4\tau^{2/3}\varepsilon_0\sigma^2$. Note that

$$a_k^r + a_k^s = \sigma^2 + \rho^2 - 4\tau^2\left(r^2 + \varepsilon^2 s^2\right) - \frac{\mu_k}{r^2},$$

and we see that on the characteristic set $\{a_k^r + a_k^s = 0\}$, we have

$$\tau^{2/3}\sigma^2 = 4\tau^2\varepsilon_0^2 s^2 - \tau^{2/3}\rho^2 - \tau^{2/3}\frac{\mu_k}{r^2} + 4\tau^{8/3} r^2, \tag{6.2.31}$$

and from (6.2.29) and (6.2.30), by taking τ large, and ε_0 small, we see that we obtain the sub-ellipticity condition, for the sub-critical case

$$a_k^r + a_k^s = 0 \Longrightarrow \frac{1}{i}\{a_k^r + a_k^s, b_k^r + b_k^s\}(r,s,\rho,\sigma,\tau) > 0, \tag{6.2.32}$$

using again that s is bounded. In the critical case $\mu_k = 1/4$, the Hardy inequality of Lemma 6.2.9 shall only yield, again formally,

$$\rho^2 - \frac{\mu_k}{r^2} \gtrsim 1. \tag{6.2.33}$$

It follows that

$$4\tau\left(\left(2\rho^2 - 2\frac{\mu_k}{r^2}\right) + 8\tau^2 r^2\right) \gtrsim \tau + \tau^3 r^2, \tag{6.2.34}$$

and this time the choice $\varepsilon = \varepsilon_0 \tau^{-2/3}$is suitable to handle the last term in (6.2.29) by taking $\varepsilon_0 > 0$ small. Yet, it remains to handle the last negative term $-4\tau^{1/3}\varepsilon_0\sigma^2$. On the characteristic set $\{a_k^r + a_k^s = 0\}$, we have

$$4\tau\varepsilon\sigma^2 = 4\tau^{1/3}\varepsilon_0\sigma^2 = 4\tau^{1/3}\varepsilon_0\left(4\tau^2\left(r^2 + \varepsilon_0^2\tau^{-4/3}s^2\right) + \frac{\mu_k}{r^2} - \rho^2\right).$$

This, in addition to (6.2.34) yields

$$\begin{aligned}\frac{1}{i}\{a_k^r + a_k^s, b_k^r + b_k^s\}(r,s,\rho,\sigma,\tau)\} &\geq C\left(\tau + \tau^3 r^2\right) - 4\tau^{1/3}\varepsilon_0\left(4\tau^2\left(r^2 + \varepsilon_0^2\tau^{-4/3}s^2\right) + \frac{\mu_k}{r^2} - \rho^2\right).\\ &\geq C\left(\tau + \tau^3 r^2\right) - 16\tau^{7/3}\varepsilon_0\left(r^2 + \varepsilon_0^2\tau^{-4/3}s^2\right), \qquad (6.2.35)\end{aligned}$$

using again Hardy inequality, which is given at the symbolic level in (6.2.33). Taking $\varepsilon_0 > 0$ sufficiently small, we can absorb the last term and we obtain the sub-ellipticity property (6.2.32) for the critical case.

Although the choice of our weight function is a bit different than usually, in the sense that its levels sets depend on the large parameter τ, it is suitable to ensure the needed sub-ellipticity property, as we shall see in the proofs of Theorems 6.2.11 and 6.2.12.

The first computations toward the proof of Theorem 6.2.12 are made in the following lemmata. The first shall be used for the sub-critical case. We define $\tilde{Y}_\delta := \{y \in \tilde{Y},\ d(y,\{r=0\}) > \delta\}$.

Lemma 6.2.19. *Let $\mu_k < 1/4$. There exists $C > 0$ such that for $\zeta \in H_0^1(\tilde{\mathcal{V}})$ such that $Q_{k,\varphi_\varepsilon}\zeta \in L^2(\tilde{Z})$, we have*

$$\mathrm{Re}\left(A_k^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)} \geq 4\tau\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}_\delta} r^2|\zeta|^2 - 4\tau\mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} + \mathcal{B}_0(\zeta), \qquad (6.2.36)$$

with

$$\mathcal{B}_0(\zeta) = 2\tau\int_{r=\delta}\left(r|\partial_r\zeta|^2\right)_{|r=\delta} + 2\tau\,\mathrm{Re}\int_{r=\delta}\left(\partial_r\zeta\overline{\zeta}\right)_{|r=\delta} + 8\tau^3\int_{r=\delta}\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\mu_k\tau\int_{r=\delta}\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}.$$

Proof.
We have

$$\begin{aligned}\mathrm{Re}\left(A_k^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)} &= -2\tau\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_r^2\zeta\overline{\partial_r\varphi_\varepsilon\partial_r\zeta} - \tau\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_r^2\zeta\overline{\partial_r^2\varphi_\varepsilon\zeta} - 2\tau^3\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_r\varphi_\varepsilon^3\partial_r\zeta\overline{\zeta}\\ &\quad - \tau^3\,\mathrm{Re}\int_{\tilde{Y}_\delta}|\zeta|^2\partial_r\varphi_\varepsilon^2\partial_r^2\varphi_\varepsilon - 2\tau\mu_k\,\mathrm{Re}\int_{\tilde{Y}_\delta}\frac{1}{r}\partial_r\varphi_\varepsilon\partial_r\zeta\overline{\zeta} - \tau\mu_k\,\mathrm{Re}\int_{\tilde{Y}_\delta}\frac{1}{r^2}\partial_r^2\varphi_\varepsilon|\zeta|^2\\ &= 4\tau\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}_\delta}|\zeta|^2r^2 - 4\tau\mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} + 2\tau\int_{r=\delta}\left(r|\partial_r\zeta|^2\right)_{|r=\delta}\\ &\quad + 2\tau\,\mathrm{Re}\int_{r=\delta}\left(\partial_r\zeta\overline{\zeta}\right)_{|r=\delta} + 8\tau^3\int_{r=\delta}\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\mu_k\tau\int_{r=\delta}\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}\\ &= 4\tau\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}_\delta}r^2|\zeta|^2 - 4\tau\mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} + \mathcal{B}_0(\zeta),\end{aligned}$$

with

$$\mathcal{B}_0(\zeta) = 2\tau\int_{r=\delta}\left(r|\partial_r\zeta|^2\right)_{|r=\delta} + 2\tau\,\mathrm{Re}\int_{r=\delta}\left(\partial_r\zeta\overline{\zeta}\right)_{|r=\delta} + 8\tau^3\int_{r=\delta}\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\mu_k\tau\int_{r=\delta}\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}.$$

This ends the proof. □

We set

$$A_{k,N}^r = -\partial_r^2 - \tau^2(\partial_r\varphi_\varepsilon)^2 - \frac{\mu_k}{r^2 + \frac{1}{N}}, \quad \text{and } Q_{k,\varphi_\varepsilon,N} = A_{k,N}^r + A_k^s + B_k^r + B_k^s.$$

that is, the regularized versions of A_k^r and $Q_{k,\varphi_\varepsilon}$.

Lemma 6.2.20. *Let $\mu_k = 1/4$. There exists $C > 0$ such that for $\zeta \in H_0^1(\tilde{\mathcal{V}})$ such that $Q_{k,\varphi_\varepsilon,N}\zeta \in L^2(\tilde{Z})$, we have*

$$\mathrm{Re}\left(A_{k,N}^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y})} \geq 4\tau\int_{\tilde{Y}}|\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}} r^2|\zeta|^2 - 4\tau\mu_k\int_{\tilde{Y}}\frac{r^2|\zeta|^2}{(r^2 + \frac{1}{N})^2}. \qquad (6.2.37)$$

Proof.
Note that here, $\zeta \in H^2(\tilde{\mathcal{V}}) \cap H_0^1(\tilde{\mathcal{V}})$. First, from (6.2.20), we have $\partial_r \varphi_\varepsilon = 2r$, $\partial_r^2 \varphi_\varepsilon = 2$, and $\partial_r^3 \varphi_\varepsilon = 0$, and we compute by integration by parts

$$\begin{aligned}
\operatorname{Re}\left(A_{k,N}^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y})} =& -2\tau \operatorname{Re}\int_{\tilde{Y}} \partial_r^2\zeta\overline{\partial_r\varphi_\varepsilon\partial_r\zeta} - \tau \operatorname{Re}\int_{\tilde{Y}} \partial_r^2\zeta\overline{\partial_r^2\varphi_\varepsilon\zeta} \\
&- 2\tau^3 \operatorname{Re}\int_{\tilde{Y}} \partial_r\varphi_\varepsilon^3\partial_r\zeta\overline{\zeta} - \tau^3 \operatorname{Re}\int_{\tilde{Y}} |\zeta|^2\partial_r\varphi_\varepsilon^2\partial_r^2\varphi_\varepsilon \\
&- 2\tau\mu_k \operatorname{Re}\int_{\tilde{Y}} \frac{1}{r^2+\frac{1}{N}}\partial_r\varphi_\varepsilon\partial_r\zeta\overline{\zeta} - \tau\mu_k \operatorname{Re}\int_{\tilde{Y}} \frac{1}{r^2+\frac{1}{N}}\partial_r^2\varphi_\varepsilon|\zeta|^2 \\
=& 4\tau\int_{\tilde{Y}} |\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}} |\zeta|^2r^2 - 4\tau\mu_k\int_{\tilde{Y}} \frac{r^2|\zeta|^2}{(r^2+\frac{1}{N})^2} + 2\tau\int_{r=0}\left(r|\partial_r\zeta|^2\right)_{|r=0} \\
&+ 2\tau \operatorname{Re}\int_{r=0}\left(\partial_r\zeta\overline{\zeta}\right)_{|r=0} + 8\tau^3\int_{r=0}\left(r^3|\zeta|^2\right)_{|r=0} + 2\mu_k\tau\int_{r=0}\left(\frac{r|\zeta|^2}{r^2+\frac{1}{N}}\right)_{|r=0} \\
=& 4\tau\int_{\tilde{Y}} |\partial_r\zeta|^2 + 16\tau^3\int_{\tilde{Y}} r^2|\zeta|^2 - 4\tau\mu_k\int_{\tilde{Y}} \frac{r^2|\zeta|^2}{(r^2+\frac{1}{N})^2}, \qquad (6.2.38)
\end{aligned}$$

as all the boundary integrals vanish using that $\zeta_{|r=0} = 0$ and the trace $\partial_r\zeta_{|r=0}$ exists, since

$$\partial_r\zeta \in L^2((-\tilde{s}_0, \tilde{s}_0), H^1(0,R)).$$

□

At this point the reader should note that the precise choice of φ_ε is suitable to handle the singularity in the last term of (6.2.38) in the limit $N \to +\infty$, especially when μ_k is positive (see discussion below Theorem 6.2.12).

Using Lemma 6.2.19, we shall prove the one-dimensional Carleman estimate in the sub-critical case $\mu < 1/4$.

Proof of Theorem 6.2.11.
We shall perform the computations on $\tilde{\mathcal{V}}_\delta := \tilde{Y}_\delta \cap \tilde{\mathcal{V}}$, noting that $\zeta \in H^2(\tilde{\mathcal{V}}_\delta)$.

$$\begin{aligned}
||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} &= ||(A_k^r + A_k^s)\zeta||^2_{L^2} + ||(B_k^r + B_k^s)\zeta||^2_{L^2} + 2\operatorname{Re} I_k \\
&\geq ||(A_k^r + A_k^s)\zeta||^2_{L^2} + 2\operatorname{Re} I_k,
\end{aligned}$$

where

$$\begin{aligned}
I_k := \left((A_k^s + A_k^r)\zeta, (B_k^s + B_k^r)\zeta\right)_{L^2(\tilde{Y}_\delta)} =& \left(A_k^s\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} + \left(A_k^r\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} \\
&+ \left(A_k^s\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)} + \left(A_k^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)}.
\end{aligned}$$

Yet, from (6.2.20), we have $\partial_s\varphi_\varepsilon = -2\varepsilon s$ and $\partial_s^2\varphi_\varepsilon = -2\varepsilon$, and by computations similar to that of the proof of Lemma 6.2.19, we obtain

$$\begin{aligned}
\operatorname{Re}\left(A_k^s\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} =& 2\tau\int_{\tilde{Y}_\delta} \partial_s^2\varphi_\varepsilon|\partial_s\zeta|^2 + \tau\int_{\tilde{Y}_\delta} \partial_s^3\varphi_\varepsilon\partial_s\zeta\zeta + 2\tau^3\int_{\tilde{Y}_\delta} \partial_s\varphi_\varepsilon^2\partial_s^2\varphi_\varepsilon|\zeta|^2 \\
=& -4\tau\varepsilon\int_{\tilde{Y}_\delta} |\partial_s\zeta|^2 - 16\tau^3\varepsilon^3\int_{\tilde{Y}_\delta} s^2|\zeta|^2. \qquad (6.2.39)
\end{aligned}$$

Note that there is no boundary term arising here, as ζ is compactly supported in this direction. We have to handle these two terms as they are of negative sign. We choose $\varepsilon > 0$ such that $\varepsilon = \varepsilon_0\tau^{-1/3}$ where $\varepsilon_0 > 0$ shall be chosen later and it is meant to be small (see discussion below Theorem 6.2.12). This yields

$$16\tau^3\varepsilon^3\int_{\tilde{Y}_\delta} s^2|\zeta|^2 \leq C\tau^2\varepsilon_0^3||\zeta||^2_{L^2(\tilde{Y}_\delta)}. \qquad (6.2.40)$$

In (6.2.39), we now focus on the first term $4\tau\varepsilon\int_{\tilde{Y}_\delta}|\partial_s\zeta|^2$. We shall use the following identity

$$\partial_s^2\zeta = -\left(A_k^s + A_k^r\right)\zeta - \left(\partial_r^2 + \tau^2\left((\partial_s\varphi_\varepsilon)^2 + (\partial_r\varphi_\varepsilon)^2\right) + \frac{\mu_k}{r^2}\right)\zeta. \tag{6.2.41}$$

From (6.2.41), we can write

$$\begin{aligned}-4\tau\varepsilon\int_{\tilde{Y}_\delta}|\partial_s\zeta|^2 &= 4\varepsilon_0\tau^{2/3}\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_s^2\zeta\overline{\zeta}\\ &= -4\varepsilon_0\tau^{2/3}\,\mathrm{Re}\int_{\tilde{Y}_\delta}\overline{\zeta}\left(A_k^r + A_k^s\right)\zeta - 4\varepsilon_0\tau^{2/3}\,\mathrm{Re}\int_{\tilde{Y}_\delta}\overline{\zeta}\left(\partial_r^2 + \tau^2\left((\partial_s\varphi_\varepsilon)^2 + (\partial_r\varphi_\varepsilon)^2\right) + \frac{\mu_k}{r^2}\right)\zeta,\end{aligned} \tag{6.2.42}$$

and by integration by parts

$$-\,\mathrm{Re}\int_{\tilde{Y}_\delta}\overline{\zeta}\left(\partial_r^2 + \tau^2\left(\partial_s\varphi_\varepsilon^2 + \partial_r\varphi_\varepsilon^2\right) + \frac{\mu_k}{r^2}\right)\zeta = \int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + \mathcal{B}_1(\zeta) - \int_{\tilde{Y}_\delta}\left(\tau^2(4r^2 + 4\varepsilon^2 s^2) + \frac{\mu_k}{r^2}\right)|\zeta|^2, \tag{6.2.43}$$

with $\mathcal{B}_1(\zeta) = \mathrm{Re}\int_{r=\delta}\zeta\partial_r\overline{\zeta}$. Hence, from (6.2.42) and (6.2.43), using the Young inequality, we obtain by taking τ large,

$$\begin{aligned}-4\tau\varepsilon\int_{\tilde{Y}_\delta}|\partial_s\zeta|^2 \geq -2\varepsilon_0||(A_k^r + A_k^s)\zeta||^2_{L^2(\tilde{Y}_\delta)} - C\varepsilon_0^3\tau^2||s\zeta||^2_{L^2(\tilde{Y}_\delta)} + 4\varepsilon_0\tau^{2/3}\mathcal{B}_1(\zeta)\\ - 16\varepsilon_0\tau^{8/3}||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + 4\varepsilon_0\tau^{2/3}\Big(||\partial_r\zeta||^2_{L^2(\tilde{Y}_\delta)} - \mu_k||\frac{\zeta}{r}||^2_{L^2(\tilde{Y}_\delta)}\Big).\end{aligned} \tag{6.2.44}$$

The last term in (6.2.44) is non-negative. Indeed, for $\mu_k < 0$ this is immediate, and for $\mu_k \in [0, 1/4)$, this is a consequence of the Hardy inequality (6.2.15). As a result, since s is bounded, we have

$$\begin{aligned}C'\,\mathrm{Re}\left(A_k^s\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} \geq -2\varepsilon_0\left(||(A_k^r + A_k^s)\zeta||^2_{L^2(\tilde{Y}_\delta)} + \tau^{4/3}||\zeta||^2_{L^2(\tilde{Y}_\delta)}\right) - C\varepsilon_0^3\tau^2||\zeta||^2_{L^2(\tilde{Y}_\delta)}\\ - 16\varepsilon_0\tau^{8/3}||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + 4\varepsilon_0\tau^{2/3}\mathcal{B}_1(\zeta).\end{aligned} \tag{6.2.45}$$

Combining estimate (6.2.36) with estimates (6.2.39), (6.2.40) and (6.2.44), we obtain by taking the large parameter τ sufficiently large, and $\varepsilon_0 > 0$ sufficiently small

$$\begin{aligned}C\varepsilon_0||(A_k^r + A_k^s)\zeta||^2_{L^2(\tilde{Y}_\delta)} + C'\left(A_k^r\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)} + C'\left(A_k^s\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} \geq -\varepsilon_0^3\tau^2||\zeta||^2_{L^2(\tilde{Y}_\delta)}\\ + \tau\Big(||\partial_r\zeta||^2_{L^2(\tilde{Y}_\delta)} - \mu_k||\frac{\zeta}{r}||^2_{L^2(\tilde{Y}_\delta)}\Big) + \tau^3||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \mathcal{B}_0(\zeta) + 4\varepsilon_0\tau^{2/3}\mathcal{B}_1(\zeta).\end{aligned} \tag{6.2.46}$$

It remains to compute, again by integration by parts and using $\partial^2_{sr}\varphi_\varepsilon = 0$,

$$\begin{aligned}\mathrm{Re}\left(A_k^r\zeta, B_k^s\zeta\right)_{L^2(\tilde{Y}_\delta)} &= -2\tau\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_s\varphi_\varepsilon\partial_r^2\zeta\overline{\partial_s\zeta} - \tau\,\mathrm{Re}\int_{\tilde{Y}_\delta}\partial_s^2\varphi_\varepsilon\partial_r^2\zeta\overline{\zeta} - 2\tau^3\,\mathrm{Re}\int\partial_s\varphi_\varepsilon\partial_r\varphi_\varepsilon^2\partial_s\zeta\overline{\zeta}\\ &\quad - \tau^3\int_{\tilde{Y}_\delta}\partial_r\varphi_\varepsilon^2\partial_s^2\varphi_\varepsilon|\zeta|^2 - 2\tau\mu_k\,\mathrm{Re}\int_{\tilde{Y}_\delta}\frac{\partial_s\varphi_\varepsilon}{r^2}\partial_s\zeta\overline{\zeta} - \tau\mu_k\,\mathrm{Re}\int_{\tilde{Y}_\delta}\frac{\partial_s^2\varphi_\varepsilon}{r^2}|\zeta|^2\\ &= 2\varepsilon\tau\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 - 4\varepsilon\tau\,\mathrm{Re}\left(s\partial_r\zeta\overline{\partial_s\zeta}\right)_{|r=\delta} - 2\tau\varepsilon\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 - 2\tau\varepsilon\,\mathrm{Re}\left(s\zeta\overline{\partial_r\zeta}\right)_{|r=\delta}\\ &\quad + 8\tau^3\varepsilon\int_{\tilde{Y}_\delta}r^2|\zeta|^2 - 8\tau^3\varepsilon\int_{\tilde{Y}_\delta}^2|\zeta|^2 - 2\varepsilon\tau\mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} + 2\varepsilon\tau\mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2}\\ &= \mathcal{B}_2(\zeta),\end{aligned} \tag{6.2.47}$$

where

$$\mathcal{B}_2(\zeta) = -4\varepsilon\tau \int_{r=\delta} s(\partial_r\zeta\overline{\partial_s\zeta})_{|_{r=\delta}} - \tau\varepsilon \int_{r=\delta} s\zeta\overline{\partial_r\zeta} = 0,$$

using boundary conditions. We also compute,

$$\begin{aligned}\left(A_k^s\zeta, B_k^r\zeta\right)_{L^2(\tilde{Y}_\delta)} =& -2\tau \int_{\tilde{Y}_\delta} \partial_r\varphi_\varepsilon\partial_s^2\zeta\overline{\partial_r\zeta} - \tau\int_{\tilde{Y}_\delta}\partial_r^2\varphi_\varepsilon\partial_s^2\zeta\overline{\zeta} - 2\tau^3\int_{\tilde{Y}_\delta}\partial_s\varphi_\varepsilon^2\partial_r\varphi_\varepsilon\partial_r\zeta\overline{\zeta} - \tau^3\int_{\tilde{Y}_\delta}\partial_s\varphi_\varepsilon^2\partial_r^2\varphi_\varepsilon|\zeta|^2\\ =& -2\tau\int_{\tilde{Y}_\delta}|\partial_s\zeta|^2 - 2\tau\int_{r=\delta}\left(r|\partial_s\zeta|^2\right)_{|_{r=\delta}} + 2\tau\int_{\tilde{Y}_\delta}|\partial_s\zeta|^2 - 8\varepsilon\tau^3\int_{\tilde{Y}_\delta}|\zeta|^2\\ & - 8\varepsilon\tau^3\int_{r=\delta}\left(r|\zeta|^2\right)_{|_{r=\delta}} + 8\varepsilon\tau^3\int_{\tilde{Y}_\delta}|\zeta|^2\\ =&\mathcal{B}_3(\zeta),\end{aligned} \tag{6.2.48}$$

where

$$\mathcal{B}_3(\zeta) = -2\tau\int_{r=\delta}\left(r|\partial_s\zeta|^2\right)_{|_{r=\delta}} - 8\varepsilon\tau^3\int_{r=\delta}\left(r|\zeta|^2\right)_{|_{r=\delta}}.$$

Summing up, from (6.2.46), (6.2.47), (6.2.48), and using

$$||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq ||(A_k^r + A_k^s)\zeta||^2_{L^2} + 2\operatorname{Re} I_k,$$

we obtain,

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq -\varepsilon_0^3\tau^2||\zeta||^2_{L^2(\tilde{Y}_\delta)} + \tau\left(||\partial_r\zeta||^2_{L^2(\tilde{Y}_\delta)} - \mu_k||\frac{\zeta}{r}||^2_{L^2(\tilde{Y}_\delta)}\right) + \tau^3||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \sum_{i=0}^3\mathcal{B}_i(\zeta). \tag{6.2.49}$$

For all $\mu_k < 1/4$, note that one can have

$$\begin{aligned}2\tau^2\left(\frac{1}{4}-\mu_k\right)^{1/2}\int_{\tilde{Y}_\delta}|\zeta|^2 &\leq 2\tau^2\left(\int_{\tilde{Y}_\delta}|r\zeta|^2\right)^{1/2}\left(\left(\frac{1}{4}-\mu_k\right)\int_{\tilde{Y}_\delta}|r^{-1}\zeta|^2\right)^{1/2}\\ &\leq \tau^3||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \left(\frac{1}{4}-\mu_k\right)\tau||r^{-1}\zeta||^2_{L^2(\tilde{Y}_\delta)}.\end{aligned} \tag{6.2.50}$$

- If $\mu_k \in [-1/8, 1/4)$, we have, using Hardy inequality (6.2.15),

$$\begin{aligned}\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 - \mu_k\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} &= 2(\mu_k+\frac{1}{4})\left(\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 - \frac{1}{4}\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2}\right) + 2\left(\frac{1}{4}-\mu_k\right)\left(\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + \frac{1}{4}\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2}\right)\\ &\geq 2C'\left(\mu_k+\frac{1}{4}\right)\int_{\tilde{Y}_\delta}|\zeta|^2 + 2\left(\frac{1}{4}-\mu_k\right)\left(\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + \frac{1}{4}\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2}\right).\end{aligned} \tag{6.2.51}$$

  Then (6.2.52) reads

$$\begin{aligned}C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq& -\varepsilon_0^3\tau^2||\zeta||^2_{L^2(\tilde{Y}_\delta)} + \tau^3||r\zeta||^2_{L^2(\tilde{Y}_\delta)}\\ &+ 2C'\left(\mu_k+\frac{1}{4}\right)\int_{\tilde{Y}_\delta}|\zeta|^2 + 2\left(\frac{1}{4}-\mu_k\right)\left(\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + \frac{1}{4}\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2}\right) + \sum_{i=0}^3\mathcal{B}_i(\zeta).\end{aligned} \tag{6.2.52}$$

  Then, from (6.2.52) and (6.2.50), we have for $\mu_k \in [0, 1/4)$, by taking ε_0 sufficiently small,

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq \tau^2\int_{\tilde{Y}_\delta}|\zeta|^2 + \tau\int_{\tilde{Y}_\delta}|\partial_r\zeta|^2 + \tau\int_{\tilde{Y}_\delta}\frac{|\zeta|^2}{r^2} + \tau^3||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \sum_{i=0}^3\mathcal{B}_i(\zeta). \tag{6.2.53}$$

- If $\mu_k < -1/8$, from (6.2.49), we can obtain directly, using the interpolation inequality (6.2.50) ,

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq \tau^2 \int_{\tilde{Y}_\delta} |\zeta|^2 + \tau \int_{\tilde{Y}_\delta} |\partial_r \zeta|^2 + \tau(1+|\mu_k|)\int_{\tilde{Y}_\delta} \frac{|\zeta|^2}{r^2} + \tau^3 ||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \sum_{i=0}^{3} \mathcal{B}_i(\zeta). \quad (6.2.54)$$

Hence, summing up (6.2.67) and (6.2.55), we have, for $\mu_k < 1/4$,

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq \tau^2 \int_{\tilde{Y}_\delta} |\zeta|^2 + \tau \int_{\tilde{Y}_\delta} |\partial_r \zeta|^2 + \tau(1+|\mu_k|)\int_{\tilde{Y}_\delta} \frac{|\zeta|^2}{r^2} + \tau^3 ||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \sum_{i=0}^{3} \mathcal{B}_i(\zeta). \quad (6.2.55)$$

Note that, using (6.2.44), we can also obtain

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y}_\delta)} \geq \tau^2 \int_{\tilde{Y}_\delta} |\zeta|^2 + \tau \int_{\tilde{Y}_\delta} |\partial_r \zeta|^2 + \tau(1+|\mu_k|)\int_{\tilde{Y}_\delta} \frac{|\zeta|^2}{r^2} + \tau^3 ||r\zeta||^2_{L^2(\tilde{Y}_\delta)} + \tau^{2/3}\int_{\tilde{Y}_\delta} |\partial_s \zeta|^2 + \sum_{i=0}^{3} \mathcal{B}_i(\zeta). \quad (6.2.56)$$

We now focus on boundary terms. We recall that

$$\sum_{i=0}^{3} \mathcal{B}_i(\zeta) = 2\tau \int_{r=\delta} \left(r|\partial_r\zeta|^2\right)_{|r=\delta} + 2\tau \,\mathrm{Re}\int_{r=\delta} \left(\partial_r\zeta\overline{\zeta}\right)_{|r=\delta} + 8\tau^3 \int_{r=\delta} \left(r^3|\zeta|^2\right)_{|r=\delta} + 2\mu_k\tau \int_{r=\delta} \left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}$$
$$+ \mathrm{Re}\int_{r=\delta} \zeta\partial_r\overline{\zeta} - 4\varepsilon\tau\int_{r=\delta} s(\partial_r\zeta\overline{\partial_s\zeta})_{|r=\delta} - \tau\varepsilon\int_{r=\delta} s\zeta\overline{\partial_r\zeta} - 2\tau\int_{r=\delta}\left(r|\partial_s\zeta|^2\right)_{|r=\delta} - 8\varepsilon\tau^3\int_{r=\delta}\left(r|\zeta|^2\right)_{|r=\delta}.$$

Using the Young inequality, one can have

$$\sum_{i=0}^{3} \mathcal{B}_i(\zeta) = C\tau\int_{r=\delta}\left(r|\partial_r\zeta|^2\right)_{|r=\delta} + \mathcal{B}_4(\zeta) \geq \mathcal{B}_4(\zeta),$$

with $\lim_{\delta\to 0}\mathcal{B}_4(\zeta) = 0$, using boundary conditions and Lemma 6.2.8. Hence, we can pass to the limit $\delta \to 0$ in (6.2.8) as all the interior terms are well defined, and we obtain

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y})} \geq \tau^2\int_{\tilde{Y}}|\zeta|^2 + \tau\int_{\tilde{Y}}|\partial_r\zeta|^2 + \tau(1+|\mu_k|)\int_{\tilde{Y}}\frac{|\zeta|^2}{r^2} + \tau^3||r\zeta||^2_{L^2(\tilde{Y})} + \tau^{2/3}\int_{\tilde{Y}}|\partial_s\zeta|^2.$$

Using the change of variables, $\zeta = e^{\tau\varphi_\varepsilon}z$, we obtain

$$\tau||e^{\tau\varphi_\varepsilon}\partial_r z||^2_{L^2(\tilde{Y})} \lesssim \tau||\partial_r\zeta||^2_{L^2(\tilde{Y})} + \tau^3||r\zeta||^2_{L^2(\tilde{Y})},$$
$$\tau^{2/3}||e^{\tau\varphi_\varepsilon}\partial_s z||^2_{L^2(\tilde{Y})} \lesssim \tau^{2/3}||\partial_s\zeta||^2_{L^2(\tilde{Y})} + \tau^{2/3}||\tau\partial_s\varphi_\varepsilon\zeta||^2_{L^2(\tilde{Y})}.$$

As $\tau\partial_s\varphi_\varepsilon = -2\tau^{1/3}\varepsilon_0 s$, the result follows. □

Making use of Lemma 6.2.20, we obtain the proof of the sub-critical case $\mu_k < 1/4$.

Proof of Theorem 6.2.12. We mainly follow the proof of Theorem 6.2.11, with slight modifications. First, we shall use the regularization procedure of Lemma 6.2.17. Second, in the critical case, the Hardy inequality (6.2.15) does not yield a H^1 estimate anymore. We then shall derive a $\mathcal{H}_{\mu_k}$ inequality. We recall that $\zeta \in H^2(\tilde{\mathcal{V}}) \cap H^1_0(\tilde{\mathcal{V}})$ at first, before passing to the limit $N \to +\infty$. We have

$$||Q_{k,\varphi_\varepsilon,N}\zeta||^2_{L^2(\tilde{Y})} = ||(A^r_{k,N} + A^s_k)\zeta||^2_{L^2} + ||(B^r_k + B^s_k)\zeta||^2_{L^2} + 2\,\mathrm{Re}\, I_{k,N}$$
$$\geq ||(A^r_{k,N} + A^s_k)\zeta||^2_{L^2} + 2\,\mathrm{Re}\, I_{k,N},$$

where

$$I_{k,N} := \Big((A_k^s + A_{k,N}^r)\zeta, (B_k^s + B_k^r)\zeta\Big)_{L^2(\tilde{Y})} = \Big(A_{k,N}^s\zeta, B_k^s\zeta\Big)_{L^2(\tilde{Y})} + \Big(A_k^r\zeta, B_k^s\zeta\Big)_{L^2(\tilde{Y})} + \Big(A_k^s\zeta, B_k^r\zeta\Big)_{L^2(\tilde{Y})} + \Big(A_{k,N}^r\zeta, B_k^r\zeta\Big)_{L^2(\tilde{Y})}.$$

From (6.2.37), we recall that

$$\mathrm{Re}\,\Big(A_k^s\zeta, B_k^s\zeta\Big)_{L^2(\tilde{Y})} = -4\tau\varepsilon\int_{\tilde{Y}}|\partial_s\zeta|^2 - 16\tau^3\varepsilon^3\int_{\tilde{Y}} s^2|\zeta|^2. \tag{6.2.57}$$

As in the proof of Theorem 6.2.11, we have to handle these two non-positive terms. We choose $\varepsilon > 0$ such that $\varepsilon = \varepsilon_0\tau^{-2/3}$ where $\varepsilon_0 > 0$ will be chosen below and it is meant to be small. This yields, as s is bounded,

$$16\tau^3\varepsilon^3\int_{\tilde{Y}} s^2|\zeta|^2 \leq 16\tau\varepsilon_0^3||\zeta||^2_{L^2(\tilde{Y})}. \tag{6.2.58}$$

We now focus in (6.2.57) on the first term $4\tau\varepsilon\int_{\tilde{Y}}|\partial_s\zeta|^2$. Again, we use the following identity

$$\partial_s^2\zeta = -\left(A_k^s + A_{k,N}^r\right)\zeta - \left(\partial_r^2 + \tau^2\left((\partial_s\varphi_\varepsilon)^2 + (\partial_r\varphi_\varepsilon)^2\right) + \frac{1}{4(r^2+\frac{1}{N})}\right)\zeta. \tag{6.2.59}$$

This allows us to write, with an integration by parts,

$$\begin{aligned} -4\tau\varepsilon\int_{\tilde{Y}}|\partial_s\zeta|^2 &= 4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\partial_s^2\zeta \\ &= -4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\left(A_{k,N}^r + A_k^s\right)\zeta - 4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\left(\partial_r^2 + \tau^2\left(\partial_s\varphi_\varepsilon^2 + \partial_r\varphi_\varepsilon^2\right) + \frac{1}{4(r^2+\frac{1}{N})}\right)\zeta, \end{aligned} \tag{6.2.60}$$

and by an additional integration by parts, we write

$$\begin{aligned} &-4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\left(\partial_r^2 + \tau^2\left(\partial_s\varphi_\varepsilon^2 + \partial_r\varphi_\varepsilon^2\right) + \frac{1}{4(r^2+\frac{1}{N})}\right)\zeta \\ &= 4\varepsilon_0\tau^{1/3}\int_{\tilde{Y}}|\partial_r\zeta|^2 - 4\varepsilon_0\tau^{1/3}\int_{\tilde{Y}}\left(\tau^2(4r^2+4\varepsilon^2s^2) + \frac{1}{4(r^2+\frac{1}{N})}\right)|\zeta|^2 \\ &\geq -16\varepsilon_0\tau^{7/3}\int_{\tilde{Y}}(r^2+\varepsilon^2s^2)|\zeta|^2 + 4\varepsilon_0\tau^{1/3}\int_{\tilde{Y}}\left(|\partial_r\zeta|^2 - \frac{|\zeta|^2}{4(r^2+\frac{1}{N})}\right), \end{aligned} \tag{6.2.61}$$

using boundary condition. Summing up, by taking τ large, there exists $C > 0$ such that

$$\begin{aligned} \mathrm{Re}\,\Big(A_k^s\zeta, B_k^s\zeta\Big)_{L^2(\tilde{Y})} &= -4\tau\varepsilon\int_{\tilde{Y}}|\partial_s\zeta|^2 - 16\tau^3\varepsilon^3\int_{\tilde{Y}} s^2|\zeta|^2 \\ &\geq -4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\left(A_{k,N}^r + A_k^s\right)\zeta - 16\varepsilon_0\tau^{7/3}\int_{\tilde{Y}} r^2|\zeta|^2 - C\tau\varepsilon_0^3\int_{\tilde{Y}}|\zeta|^2 \\ &\quad + 4\varepsilon_0\tau^{1/3}\int_{\tilde{Y}}\left(|\partial_r\zeta|^2 - \frac{|\zeta|^2}{4(r^2+\frac{1}{N})}\right) \end{aligned} \tag{6.2.62}$$

Using the Young inequality, we obtain

$$-4\varepsilon_0\tau^{1/3}\,\mathrm{Re}\int_{\tilde{Y}}\overline{\zeta}\left(A_{k,N}^r + A_k^s\right)\zeta \geq -2\varepsilon_0||(A_{k,N}^r + A_k^s)\zeta||^2_{L^2(\tilde{Y})} - 2\varepsilon_0\tau^{2/3}||\zeta||^2_{L^2(\tilde{Y})}. \tag{6.2.63}$$

Combining estimates (6.2.57), (6.2.58), (6.2.62) and (6.2.63), and Lemma 6.2.20, we obtain by taking the large parameter τ sufficiently large, and $\varepsilon_0 > 0$ sufficiently small we obtain by taking the large parameter τ sufficiently large, and $\varepsilon_0 > 0$ sufficiently small

$$C\varepsilon_0||(A^r_{k,N}+A^s_k)\zeta||^2_{L^2(\tilde{Y})}+C'\Big(A^r_{k,N}\zeta,B^r_k\zeta\Big)_{L^2(\tilde{Y})}+C'\Big(A^s_k\zeta,B^s_k\zeta\Big)_{L^2(\tilde{Y})}\geq-\varepsilon_0^3\tau||\zeta||^2_{L^2(\tilde{Y})}$$
$$+\tau^{1/3}\Big(||\partial_r\zeta||^2_{L^2(\tilde{Y})}-\frac{1}{4}||\frac{\zeta}{(r^2+\frac{1}{N})^{1/2}}||^2_{L^2(\tilde{Y})}\Big)+\tau\Big(||\partial_r\zeta||^2_{L^2(\tilde{Y})}-\frac{1}{4}||\frac{r\zeta}{(r^2+\frac{1}{N})}||^2_{L^2(\tilde{Y})}\Big)+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}.$$

Note that from $-\frac{1}{(r^2+\frac{1}{N})^{1/2}}\geq-\frac{1}{r}$ and $-\frac{r}{(r+\frac{1}{N})^2}\geq\frac{1}{r}$, this reads

$$C\varepsilon_0||(A^r_{k,N}+A^s_k)\zeta||^2_{L^2(\tilde{Y})}+C'\Big(A^r_{k,N}\zeta,B^r_k\zeta\Big)_{L^2(\tilde{Y})}+C'\Big(A^s_k\zeta,B^s_k\zeta\Big)_{L^2(\tilde{Y})}\geq-\varepsilon_0^3\tau||\zeta||^2_{L^2(\tilde{Y})}$$
$$+\tau\Big(||\partial_r\zeta||^2_{L^2(\tilde{Y})}-\frac{1}{4}||\frac{\zeta}{r}||^2_{L^2(\tilde{Y})}\Big)+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}. \quad (6.2.64)$$

The same computations as in the proof of Theorem 6.2.11 give

$$\Big(A^r_{k,N}\zeta,B^s_k\zeta\Big)_{L^2(\tilde{Y})}=0,\quad\text{and}\ \Big(A^s_k\zeta,B^r_k\zeta\Big)_{L^2(\tilde{Y})}=0. \quad (6.2.65)$$

Summing up, from (6.2.64), (6.2.65), and using

$$||Q_{k,\varphi_\varepsilon,N}\zeta||^2_{L^2(\tilde{Y})}\geq||(A^r_{k,N}+A^s_k)\zeta||^2_{L^2}+2I_{k,N},$$

we obtain

$$C||Q_{k,\varphi_\varepsilon,N}\zeta||^2_{L^2(\tilde{Y})}\geq-\varepsilon_0^3\tau||\zeta||^2_{L^2(\tilde{Y})}+\tau||\zeta||^2_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{1/4}(0,R))}+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}. \quad (6.2.66)$$

Note that we have $Q_{k,\varphi_\varepsilon,N}\zeta=e^{\tau\varphi_\varepsilon}z^0$, and then the left member of (6.2.66) does not depend on N. From Lemma 6.2.17, we can pass to the limit $N\to+\infty$, in (6.2.66), and we obtain

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y})}\geq-\varepsilon_0^3\tau||\zeta||^2_{L^2(\tilde{Y})}+\tau||\zeta||^2_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{1/4}(0,R))}+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}. \quad (6.2.67)$$

Using the Hardy inequality (6.2.15), we obtain by taking $\varepsilon_0 > 0$ sufficiently small,

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y})}\geq\tau||\zeta||^2_{L^2(\tilde{Y})}+\tau||\zeta||^2_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{1/4}(0,R))}+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}.$$

Note that, using again (6.2.60) and (6.2.61), we can also obtain

$$C||Q_{k,\varphi_\varepsilon}\zeta||^2_{L^2(\tilde{Y})}\geq\tau||\zeta||^2_{L^2(\tilde{Y})}+\tau||\zeta||^2_{L^2((-\tilde{s}_0,\tilde{s}_0),\mathcal{H}_{1/4}(0,R))}+\tau^3||r\zeta||^2_{L^2(\tilde{Y})}+\tau^{1/3}||\partial_s\zeta||^2_{L^2(\tilde{Y})}.$$

We end the proof exactly as for the proof of Theorem 6.2.11. □

6.3 Interpolation estimates for augmented operators and proof of the spectral inequalities

The section is devoted to proving an interpolation estimate in the neighborhood of a singular point in Y. In the case of operators with inverse square potential located at the origin, if the estimation is performed away from the origin, the potential can be considered as a low-order term. Then classical estimates appy. In the case of a polygonal problem, near points of the boundary or the interface where it is smooth, we also have classical estimates.

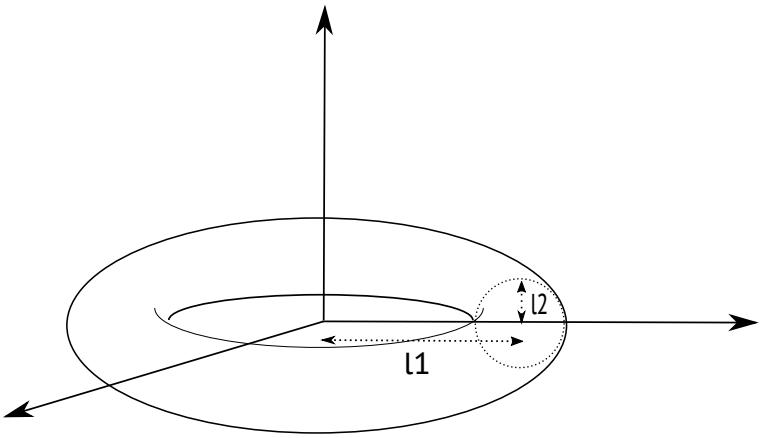


Figure 6.4: The full torus $\tilde{\mathbb{T}}(\ell_1, \ell_2)$.

6.3.1 Inverse square potential case in arbitrary dimensions

We recall that we denote $Z = (-S_0, S_0) \times \Omega$ and $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$. In this section, Ω is a smooth bounded domain of $\mathbb{R}^d$, $d \geq 3$. We recall that $Q = -\partial_s^2 + P$.

Classical estimates

For $\varepsilon > 0$, we set $\underline{Z}_\varepsilon := \{y = (s, x) \in Z,\ |x| > \varepsilon\}$.

Lemma 6.3.1. *Let V and U be two open subsets of $\overline{Y}$ such that $V \cap \{x = 0\} = \emptyset$. There exist $\delta \in (0, 1)$, $\varepsilon > 0$ and $C_\varepsilon > 0$ such that*

$$||u||_{H^1(V \cap Z)} \leq C_\varepsilon \left(||Qu||_{L^2(Z)} + ||u||_{H^1(U)} + |u_{|_{\partial\Omega \cap V}}|_{H^1(\partial\Omega \cap V)} \right)^\delta ||u||^{1-\delta}_{H^1(\underline{Z}_\varepsilon)}. \tag{6.3.1}$$

for all $u \in H^1(\underline{Z}_\varepsilon)$, such that $Qu \in L^2(Z)$.

We have the counterpart in a neighborhood of the boundary $\{s = -S_0\}$ away from $x = 0$.

Lemma 6.3.2. *Let $x_0 \in \omega \subset \Omega$, with $x_0 \neq 0$. There exist V a neighborhood of the point $(-S_0, x_0)$ in $\mathbb{R} \times \mathbb{R}^d$, $\delta \in (0, 1)$, $\varepsilon > 0$ and $C_\varepsilon > 0$ such that*

$$||u||_{H^1(V \cap Z)} \leq C_\varepsilon \left(||Qu||_{L^2(Z)} + |u_{|_{s=-S_0}}|_{L^2(\omega)} + |\partial_s u_{|_{s=-S_0}}|_{L^2(\omega)} \right)^\delta ||u||^{1-\delta}_{H^1(\underline{Z}_\varepsilon)}. \tag{6.3.2}$$

for all $u \in H^1(\underline{Z}_\varepsilon)$, such that $Qu \in L^2(Z)$.

Estimates near the singularity

We have the following central lemma, which proof is given in appendix. We define $Y_\delta := \{y \in Y,\ \frac{1}{2}\delta < d(y, \{x = 0\} < \delta\}$. Let $0 < \ell_1 < \ell_2$. We consider the full torus $\tilde{\mathbb{T}}(\ell_1, \ell_2)$ (see Figure 6.4).

Lemma 6.3.3. *Let $\eta > 0$ and $\delta > 0$. There exist $\ell_1, \ell_2 > 0$ such that $\tilde{\mathbb{T}}(\ell_1, \ell_2) \subset Y$, and constants $N, C, C' > 0$ such that*

$$||u||_{H^1(Y_\delta)} \leq C\delta^{-N} \left(||Qu||_{L^2(Z)} + ||u||_{H^1(\tilde{\mathbb{T}}(\ell_1, \ell_2))} \right)^{C'\delta^{1+\eta}} ||u||^{1-C'\delta^{1+\eta}}_{H^1(\underline{Z}_{\frac{\delta}{4}})}, \tag{6.3.3}$$

for all $u \in H^1(\underline{Z}_{\frac{\delta}{4}})$ such that $Qu \in L^2(Z)$.

The proof of this lemma is provided in Section 10.F.

The main difficulty to prove estimates in the spirit of Lemmata 6.3.1 and 6.3.2 in a neighborhood of a point of $\{x = 0\}$ comes from the particular form of the weight function in the Carleman estimates of Theorem 6.2.11 and Theorem 6.2.12. Whenever the large parameter $\tau > 0$ is fixed, the weight function has proper level sets, that is, some convexity with respect to the boundary. However, these level sets depend of τ themselves, and this is an obstruction to an optimization procedure.

To handle these difficulties, we shall only prove interpolation inequalities for functions that are of the form (6.3.4) below, and we shall directly connect the large parameter τ to the spectral parameter Λ. We shall use the Carleman estimates of Theorems 6.2.1 and 6.2.2 to observe a neighborhood of the singular point from an annulus, that depends on Λ. To keep the dependency of the constants with respect to Λ, we shall use Lemma 6.3.3.

Let $\psi_j(x)$, $j \geq 0$ be the eigenfunctions associated with eigenvalues $\lambda_j > 0$ of the operator P. We set

$$w(s,x) = \sum_{\lambda_j \leq \Lambda} \frac{\sinh(\sqrt{\lambda_j}(s+S_0))}{\sqrt{\lambda_j}} a_j \psi_j(x). \tag{6.3.4}$$

where a_j are complex numbers such that $\sum_{j\geq 0} |a_j|^2 < \infty$, and we also set

$$z(x) = \sum_{\lambda_j \leq \Lambda} a_j \psi_j(x). \tag{6.3.5}$$

For $\mathcal{Z} = (-s_0, s_0) \times \tilde{\mathcal{Z}}$ be an open subset of Z, we define

$$||u||^2_{W_\mu(\mathcal{Z})} = \int_{\mathcal{Z}} |\partial_s u|^2 ds dx + \int_{-s_0}^{s_0} ||u(s,.)||^2_{\mathcal{H}_\mu(\tilde{\mathcal{Z}})} ds, \tag{6.3.6}$$

for $u \in L^2((-s_0,s_0), \mathcal{H}_\mu(\tilde{\mathcal{Z}})) \cap H^1((-s_0,s_0), L^2(\tilde{\mathcal{Z}}))$. Using classical estimates (see [VZ00]), note that one can have the following lemma.

Lemma 6.3.4. *Let $\mu < \mu^*$. There exists $C > 0$ such that*

$$||w||_{H^1(Z)} \leq C||w||_{\mathcal{H}_\mu(Z)}.$$

for w defined in (6.3.4).
Let $\mu = \mu^*$. There exists $C > 0$ such that, for $0 < \varepsilon < 1$, we have

$$||w||^2_{H^1(\underline{Z}_\varepsilon)} \leq \frac{C}{\varepsilon^2} ||w||^2_{W_{\mu^*}(Z)},$$

for w defined in (6.3.4).

Proof. The case $\mu < \mu^*$ is proven in Section 6.1.1. We thus only consider the case $\mu = \mu^*$. Remark that it is sufficient to prove that

$$||u||^2_{\mathcal{H}_{\mu^*}(\Omega)} := \int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2}\right) \geq \frac{C}{\varepsilon^2} \int_{\Omega \setminus B(0,\varepsilon)} (|\nabla u|^2 + |u|^2),$$

holds for all $u \in H^1_0(\Omega)$, as $H^1_0(\Omega)$ is dense in $\mathcal{H}_{\mu^*}$, by definition. Following [VZ00], consider $\chi \in C^\infty_0(\Omega)$, satisfying $\chi = 1$ in $B(0, \varepsilon/2)$, $\chi = 0$ in $\Omega \setminus B(0,\varepsilon)$, $|\nabla \chi| \leq C\varepsilon^{-1}$, and $|\Delta \phi| \leq C\varepsilon^{-2}$. Then, we have

$$\begin{aligned}
\int_\Omega \nabla(\chi u).\nabla((1-\chi)u) &= \int_\Omega \chi(1-\chi)|\nabla u|^2 - \int_\Omega |\nabla\chi|^2 |u|^2 + \int_\Omega (1-2\chi) u \nabla u \cdot \nabla \chi \\
&= \int_\Omega \chi(1-\chi)|\nabla u|^2 - \int_\Omega |\nabla\chi|^2 |u|^2 - \frac{1}{2}\int_\Omega |u|^2 \operatorname{div}((1-2\chi)\nabla\chi) \\
&\geq -\frac{C}{\varepsilon^2}\int_\Omega |u|^2.
\end{aligned}$$

We also have

$$\int_\Omega \frac{|(1-\chi)u|^2}{|x|^2} + \int_\Omega \frac{\chi(1-\chi)|u|^2}{|x|^2} \leq \frac{C}{\varepsilon^2}\int_\Omega |u|^2.$$

Note that

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2}\right) = \int_\Omega \left(|\nabla(\chi u)|^2 - \mu^* \frac{|\chi u|^2}{|x|^2}\right) + \int_\Omega \left(|\nabla((1-\chi)u)|^2 - \mu^* \frac{|(1-\chi)u|^2}{|x|^2}\right) + 2\int_\Omega \left(\nabla(\chi u).\nabla((1-\chi)u) - \mu^* \frac{\chi(1-\chi)|u|^2}{|x|^2}\right).$$

All the integrals are well defined as here $u \in H_0^1(\Omega)$. The first term is non-negative, using Hardy inequality (6.1.5). Using the above estimates, this yields

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2}\right) \geq \int_{\Omega\setminus B(0,\varepsilon)} |\nabla u|^2 - \frac{C}{\varepsilon^2}\int_\Omega |u|^2.$$

Again, using the Hardy inequality (6.1.5), up to change the constants, one can have

$$C(1+\frac{1}{\varepsilon^2})\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2}\right) \geq \int_{\Omega\setminus B(0,\varepsilon)} |\nabla u|^2 + \int_\Omega |u|^2.$$

This ends the proof. □

As a result, for functions w defined in (6.3.4), Lemma 6.3.3 yields the following corollary. Remark that $Qw = 0$.

Corollary 6.3.5. *Let $\mu \leq \mu^*$. Let $\eta > 0$ and $\delta > 0$. There exist $\ell_1, \ell_2 > 0$ such that $\tilde{\mathbb{T}}(\ell_1, \ell_2) \subset Y$, and constants $N, C, C' > 0$ such that*

$$||w||_{H^1(Y_\delta)} \leq C\delta^{-N} ||w||^{C'\delta^{1+\eta}}_{H^1(\tilde{\mathbb{T}}(\ell_1,\ell_2))} ||w||^{1-C'\delta^{1+\eta}}_{W_\mu(Z)}, \tag{6.3.7}$$

for functions w defined in (6.3.4).

We have the following corollary of Lemma 6.3.1.

Corollary 6.3.6. *Let $\mu \leq \mu^*$. Let V and U be two open subsets of $\overline{Y}$ with such that $\{x = 0\} \cap V = \emptyset$. There exist $\delta \in (0,1)$, and $C > 0$ such that*

$$||w||_{H^1(V\cap Z)} \leq C\left(||w||_{H^1(U)} + |w_{|\partial\Omega\cap V}|_{H^1(\Omega\cap V)}\right)^\delta ||w||^{1-\delta}_{H^1(Z)}. \tag{6.3.8}$$

for functions w defined in (6.3.4).

Lemma 6.3.4 with Lemma 6.3.2 yields the following corollary.

Corollary 6.3.7. *Let $\mu \leq \mu^*$. Let $x_0 \in \omega \subset \Omega$, with $x_0 \neq 0$. There exist V a neighborhood of the point $(-S_0, x_0)$ in $\mathbb{R}\times\mathbb{R}^d$, $\delta \in (0,1)$, and $C > 0$ such that*

$$||w||_{H^1(V\cap Z)} \leq C\left(||Qw||_{L^2(Z)} + |w_{|s=-S_0}|_{L^2(\omega)} + |\partial_s w_{|s=-S_0}|_{L^2(\omega)}\right)^\delta ||w||^{1-\delta}_{W_\mu(Z)}. \tag{6.3.9}$$

for functions w defined in (6.3.4).

We now state the weak interpolation estimate in a neighborhood of a point $(s_0, x = 0) \in (-S_0+\alpha, S_0-\alpha)\times\{x=0\}$. We define

$$\mathcal{G}_\eta(\Lambda) = \Lambda^{-\frac{\eta+1}{8}}, \quad \mathcal{G}^*_\eta(\Lambda) = \Lambda^{-\frac{\eta+1}{2}}. \tag{6.3.10}$$

Proposition 6.3.8. *Let $\mu < \mu^*$. There exists U a neighborhood of y_0 in Y such that for $\tilde{Y}$ open subset of Y, for all $\eta > 0$, there exist $C, \delta > 0$ such that*

$$||w||_{L^2(U)} \leq Ce^{C\Lambda^{3/4}} ||w||^{\delta \mathcal{G}_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta \mathcal{G}_\eta(\Lambda)}_{W_\mu(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1, \tag{6.3.11}$$

with $\mathcal{G}_\eta$ is given by (6.3.10), and w and z are given by (6.3.4) and (6.3.5).

We have the counterpart in the critical case.

Proposition 6.3.9. *Let $\mu = \mu^*$. Let $\tilde{Y}$ be an open subset of $Y \setminus \{x = 0\}$. There exists U a neighborhood of y_0 in Y such that for $\tilde{Y}$ open subset of Y, for all $\eta > 0$, there exist $C, \delta > 0$ such that*

$$||w||_{L^2(U)} \leq Ce^{C\Lambda^{3/2}} ||w||^{\delta \mathcal{G}^*_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta \mathcal{G}^*_\eta(\Lambda)}_{W_\mu(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1, \tag{6.3.12}$$

with $\mathcal{G}^*_\eta$ is given by (6.3.10), and w and z are given by (6.3.4) and (6.3.5).

Note that, up to changing the constants, Propositions 6.3.8 and 6.3.9 can be proved for $\Lambda \geq \Lambda_0$, with $\Lambda_0 > 0$ large.

We define

$$\mathcal{F}^\mu_\eta(\Lambda) = \begin{cases} \mathcal{G}_\eta(\Lambda), & \text{if } \mu < \mu^* \\ \mathcal{G}^*_\eta(\Lambda), & \text{if } \mu = \mu^*. \end{cases}, \quad \text{and } \tilde{\mathcal{F}}^\mu(\Lambda) = \begin{cases} \Lambda^{3/4}, & \text{if } \mu < \mu^* \\ \Lambda^{3/2}, & \text{if } \mu = \mu^*. \end{cases}$$

By a compactness argument, one can prove the following corollary.

Corollary 6.3.10. *Let $\mu \leq \mu^*$, and let ω be an open subset of Ω. For all η, there exist $C > 0$ and $\delta > 0$ such that*

$$||w||_{L^2(Y)} \leq Ce^{C\tilde{\mathcal{F}}^\mu_\eta(\Lambda)} |\partial_s w_{|\omega \times \{s=-S_0\}}|^{\delta \mathcal{F}^\mu_\eta(\Lambda)}_{L^2(\omega)} ||w||^{1-\delta \mathcal{F}^\mu_\eta(\Lambda)}_{W_\mu(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \tag{6.3.13}$$

for w defined in (6.3.4).

Proof. First, remark that for a function w defined by (6.3.35), we have $w_{|s=-S_0} = 0$. Let $\mathcal{N}$ be a neighborhood of $\{x = 0\}$ (independant of Λ) in $\mathbb{R}^{d+1}$. By a compactness argument, and applying Proposition 6.3.8 in the sub-critical case or Proposition 6.3.9 in the critical one, we obtain

$$||w||_{L^2(\mathcal{N} \cap Y)} \leq Ce^{C\tilde{\mathcal{F}}^\mu_\eta(\Lambda)} ||w||^{\delta \mathcal{F}^\mu_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta \mathcal{F}^\mu_\eta(\Lambda)}_{W_\mu(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}. \tag{6.3.14}$$

Combining this with (6.3.8)-(6.3.9), this yields

$$||w||_{L^2(Y)} \leq Ce^{C\tilde{\mathcal{F}}^\mu_\eta(\Lambda)} |\partial_s w_{|\omega \times \{s=-S_0\}}|^{\delta \mathcal{F}^\mu_\eta(\Lambda)}_{L^2(\omega)} ||w||^{1-\delta \mathcal{F}^\mu_\eta(\Lambda)}_{W_\mu(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \tag{6.3.15}$$

which is the sought result. □

Proof of Propositions 6.3.8 and 6.3.9. We can assume without any loss of generality that $y_0 = 0$. Let V be a neighborhood of 0 in Y. Consider a cut-off function $\chi(s, r) = \chi_1(s)\chi_2(x) \in C^\infty_0(\mathbb{R}^{d+1})$ such that

$$\chi_1(s) = \begin{cases} 1 & \text{for } s \leq r_1 \\ 0 & \text{for } s \geq r_2, \end{cases} \qquad \chi_2(x) = \begin{cases} 1 & \text{for } |x| \leq r_1 \\ 0 & \text{for } |x| \geq r_2, \end{cases} \tag{6.3.16}$$

with $0 < r_1 < r_2$ such that $B(0, r_2) \subset V$. Then, in the sub-critical case, we apply the Carleman estimate of Theorem 6.2.1 to $\tilde{w} := \chi w$ to obtain

$$\tau^{2/3} ||e^{\tau\varphi_\varepsilon} \nabla_{s,x} \tilde{w}||_{L^2(Y)} + \tau^2 ||e^{\tau\varphi_\varepsilon} \tilde{w}||_{L^2(Y)} \lesssim ||e^{\tau\varphi_\varepsilon} Q\tilde{w}||^2_{L^2(Z)}, \tag{6.3.17}$$

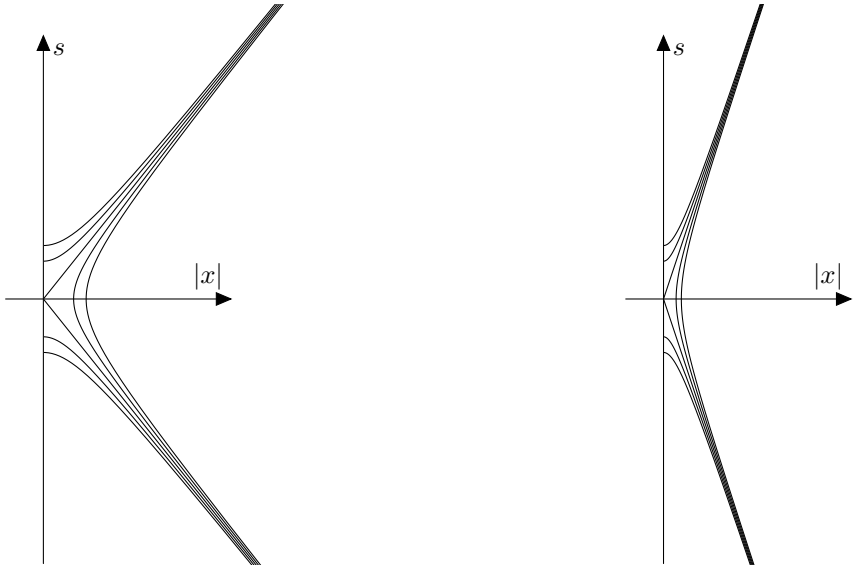


Figure 6.5: Level sets of the weight φ_ε in a neighborhood of $(0,0)$ as τ varies. Left figure: τ small. Right figure τ large.

with weight function $\varphi_\varepsilon(s,x) = |x|^2 - \varepsilon s^2$. The level sets of the weight function φ_ε are sketched in Figure 6.5. The right hand side of (6.3.17) can be estimated as follows

$$||e^{\tau\varphi_\varepsilon(s,x)}Q\tilde{w}||_{L^2(Z)} \lesssim ||e^{\tau\varphi_\varepsilon(s,x)}\chi Qw||_{L^2(Z)} + ||e^{\tau\varphi_\varepsilon(s,x)}[Q,\chi]w||_{L^2(Z)}.$$

In the critical case, we apply Theorem 6.2.2 to $\tilde{w} := \chi w$ to obtain

$$\tau||e^{\tau\varphi_\varepsilon}\tilde{w}||_{L^2(Y)} \lesssim ||e^{\tau\varphi_\varepsilon(s,x)}\chi Qw||_{L^2(Z)} + ||e^{\tau\varphi_\varepsilon(s,x)}[Q,\chi]w||_{L^2(Z)}, \tag{6.3.18}$$

where the commutator $[Q,\chi]$ is a differential operator of order one supported in the region where χ varies. Let $K > 0$ be a constant that shall be fixed below, and let us split $H := \text{supp}[Q,\chi] = A_1^\varepsilon \cup A_2^\varepsilon$ with

$$A_1^\varepsilon := \{(s,x) \in H,\ \varphi_\varepsilon(s,x) = |x|^2 - \varepsilon s^2 < -\varepsilon K\},\ A_2^\varepsilon := \{(s,x) \in H,\ \varphi_\varepsilon(s,x) = |x|^2 - \varepsilon s^2 \geq -\varepsilon K\}. \tag{6.3.19}$$

The regions A_1^ε and A_2^ε are represented in Figure 6.6. The notation A_i^ε is used to emphasize that the above splitting depends on the parameter $\varepsilon > 0$. Observe that for $\varepsilon > 0$ small, $A_1^\varepsilon \subset \{|x| \leq r_1\}$, and consequently

$$\chi(s,r) = \chi_1(s) \text{ in } A_1^\varepsilon. \tag{6.3.20}$$

We recall that in the Carleman estimate, ε depend on τ, with $\varepsilon \to 0$ as $\tau \to +\infty$. Then, using (6.3.20), we have, for τ large,

$$||e^{\tau\varphi_\varepsilon}[Q,\chi]w||_{L^2(Z)} \lesssim ||e^{\tau\varphi_\varepsilon}w||_{L^2(A_1^\varepsilon)} + ||e^{\tau\varphi_\varepsilon}\partial_s w||_{L^2(A_1^\varepsilon)} + ||e^{\tau\varphi_\varepsilon}w||_{H^1(A_\varepsilon^2)}. \tag{6.3.21}$$

Let $\mathcal{U}$ be a neighborhood of 0 such that $s^2 \leq k$ on $\mathcal{U}$, $\chi = 1$ on $\mathcal{U}$, and such that $\mathcal{U} \subset V$, with $k > 0$ to be fixed below. Observe that $\varphi_\varepsilon > -\varepsilon k$ on $\mathcal{U}$.

Then, from (6.3.17)-(6.3.21), we have

$$e^{-k\varepsilon\tau}||w||_{L^2(\mathcal{U})} \lesssim e^{C\tau}||Qw||_{L^2(Y)} + e^{-K\varepsilon\tau}\left(||w||_{L^2(A_1^\varepsilon)} + ||\partial_s w||_{L^2(A_1^\varepsilon)}\right) + e^{C\tau}||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.22}$$

with $C > 0$ is a large constant. At this step, we can not optimize this estimate with respect to τ as in [Rob95], since the domains A_i^ε themselves depend on τ. We shall use the particular form of w to overcome

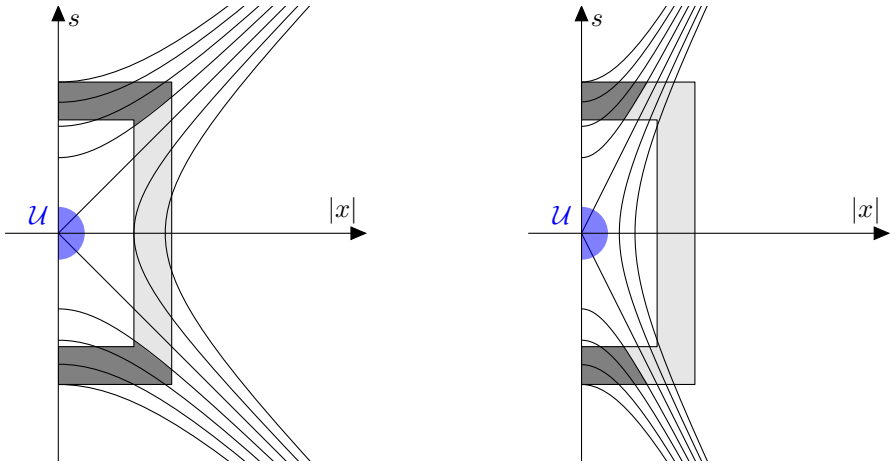


Figure 6.6: The gray region represents the set H. For two values of ε, the darkest region represents A_1^ε, and the brightest A_2^ε. Left figure: ε large. Right figure: ε small.

this difficulty. Note that first, we have $Qw = 0$.

• In the subcritical case, we set $\tau = \eta_1 \Lambda^{3/4}$ with $\eta_1 > 0$ shall be fixed below. We obtain $\varepsilon \simeq \tau^{-1/3} \simeq \eta_1^{-1/3}\Lambda^{-1/4}$ and the reader should keep in mind that now A_2^ε depends on the spectral parameter Λ. Then (6.3.49) reads

$$||w||_{L^2(\mathcal{U})} \lesssim e^{(k-K)\eta_1\sqrt{\Lambda}} \left(||w||_{L^2(A_1^\varepsilon)} + ||\partial_s w||_{L^2(A_2^\varepsilon)} \right) + e^{C\eta_1 \Lambda^{3/4}} ||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.23}$$

Using the form of w (see (6.3.4)), there exists $C_1 > 0$ such that

$$||w||_{L^2(Z)} \lesssim e^{C_1\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad ||\partial_s w||_{L^2(Z)} \lesssim e^{C_1\sqrt{\Lambda}} ||z||_{L^2(\Omega)},$$

and then choosing $k < K$ and η_1 sufficiently large, there exists $C_2, C_3 > 0$ such that (6.3.23) reads

$$||w||_{L^2(\mathcal{U})} \lesssim e^{-C_3\sqrt{\Lambda}} ||z||_{L^2(\Omega)} + e^{C_2\Lambda^{3/4}} ||w||_{H^1(A_2^\varepsilon)}. \tag{6.3.24}$$

Remark that

$$d(\{x = 0\}, A_2^\varepsilon) \gtrsim \Lambda^{-1/8}.$$

Indeed, for all $(s, x) \in A_\varepsilon^2$, we have

$$\begin{cases} |x|^2 + s^2 > 2r_1^2 \\ |x|^2 - \varepsilon s^2 \geq -\varepsilon K. \end{cases}$$

This implies

$$(1 + \varepsilon)|x|^2 \geq \varepsilon(2r_1^2 - K),$$

and we obtain, for $K > 0$ taken smaller than $2r_1^2$ that

$$d(\{x = 0\}, A_2^\varepsilon) \gtrsim \varepsilon^{1/2} \gtrsim \tau^{-1/6} \gtrsim \Lambda^{-1/8}.$$

To handle the last term in the right hand side of (6.3.24), we apply (6.3.7) to w in (6.3.24) with $\delta = C\Lambda^{-1/8}$, $C > 0$. Then, there exist $N > 0$, ℓ_1, ℓ_2 such that, for Λ large, we have

$$||w||_{H^1(Y_\delta)} \leq C\Lambda^N ||w||^{(C'\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(\tilde{\mathbb{T}}(\ell_1,\ell_2))} ||w||^{1-(C'\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(W_\mu(Z))}. \tag{6.3.25}$$

We recall that the set Y_δ is defined at the beginning of this section. Now, $\tilde{\mathbb{T}}(\ell_1,\ell_2)$ is sufficiently away from $\{x=0\}$, and then, using (6.3.8) with $Qw=0$, there exists $\delta_1\in(0,1)$ such that

$$||w||_{H^1(Y_\delta)} \leq C\Lambda^N \left(||w||^{\delta_1}_{H^1(\tilde{Y})}||w||^{1-\delta_1}_{W_\mu(Z)}\right)^{(C\Lambda)^{\frac{-1-\eta}{8}}} ||w||^{1-(C\Lambda)^{\frac{-1-\eta}{8}}}_{W_\mu(Z)}. \tag{6.3.26}$$

This yields

$$||w||_{L^2(Y_\delta)} \leq C\Lambda^N \left(||w||_{H^1(\tilde{Y})}\right)^{(\delta_1 C\Lambda)^{\frac{-1-\eta}{8}}} ||w||^{1-\delta_1(C\Lambda)^{\frac{-1-\eta}{8}}}_{W_\mu(Z)}. \tag{6.3.27}$$

Plugging this in (6.3.24) yields, for Λ large (constants may change),

$$||w||_{L^2(\mathcal{U})} \leq Ce^{C\Lambda^{3/4}}||w||^{(\delta_1 C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(\tilde{Y})} ||w||^{1-\delta_1(C\Lambda)^{\frac{-1-\eta}{8}}}_{W_\mu(Z)} + C'e^{-C'\sqrt{\Lambda}}||z||_{L^2(\Omega)}$$

which is in the form given in Proposition 6.3.8.

- For the critical case $\mu=\mu^*$, we set $\tau=\eta_1\Lambda^{3/2}$. This yields $\varepsilon\simeq\tau^{-2/3}\simeq\Lambda^{-1}$. We obtain

$$||w||_{L^2(\mathcal{U})} \lesssim e^{(k-K)\eta_1\sqrt{\Lambda}}\left(||w||_{L^2(A_1^\varepsilon)}+||\partial_s w||_{L^2(A_2^\varepsilon)}\right) + e^{C\eta_1\Lambda^{3/2}}||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.28}$$

we apply Lemma 6.3.3 to w in (6.3.28) with $\delta=C\Lambda^{-1/2}$, $C>0$ since in this case

$$d(\{x=0\},A_2^\varepsilon)\simeq\Lambda^{-1/2}, \tag{6.3.29}$$

and performing the same computations as above, we obtain for Λ large,

$$||w||_{L^2(Y_\delta)} \leq Ce^{C\Lambda^{3/2}}||w||^{(C\Lambda)^{\frac{-1-\eta}{2}}}_{H^1(\tilde{\mathbb{T}}(\ell_1,\ell_2))}||w||^{1-(C\Lambda)^{\frac{-1-\eta}{2}}}_{W_\mu(Z)}. \tag{6.3.30}$$

We finish the proof as in the sub-critical case. This ends the proof of Proposition 6.3.9. □

6.3.2 Polygonal case

We recall that we have set $Z=(-S_0,S_0)\times\Omega$ and $Y=(-S_0+\alpha,S_0-\alpha)\times\Omega$. In this section, Ω is a two dimensional polygonal domain, as described in Section 6.1.2.

Classical estimates

Near the boundary $\{s=-S_0\}$, we have the following result.

Lemma 6.3.11. *Let $x_0\in\omega\subset\Omega$. There exist V a neighborhood of the point $(-S_0,x_0)$ in Z, $\delta\in(0,1)$, and $C>0$ such that*

$$||u||_{H^1(V)} \leq C\left(||\mathcal{T}u||_{L^2(Z)}+|u_{|_{s=-S_0}}|_{L^2(\omega)}+|\partial_s u_{|_{s=-S_0}}|_{L^2(\omega)}\right)^\delta ||u||^{1-\delta}_{H^1(Z)}. \tag{6.3.31}$$

for all $u\in L^2(\Omega)$ such that $u^i\in H^1(\Omega^i)$, $i=1,2$, and $\mathcal{T}u\in L^2(\Omega)$.

The estimate near a point of the boundary $\partial\Omega\times(-S_0+\alpha,S_0-\alpha)$ where it is smooth, for Dirichlet boundary conditions reads as follows [LR95].

Lemma 6.3.12. *Let V be an open subset of $\overline{Y}$ such that $(-S_0,S_0)\times\partial\Omega$ is smooth in a neighborhood of $\overline{V}$ (if $\overline{V}$ meets this boundary), and V does not meet the interface $(-S_0,S_0)\times S$. Let U be an open set of $\overline{Y}$. There exist $\delta\in(0,1)$ and $C>0$ such that*

$$||u||_{H^1(V)} \leq C\left(||\mathcal{T}u||_{L^2(Z)}+||u||_{H^1(U)}+|u_{|_{\{\partial\Omega\cap V}}|_{H^1(\partial\Omega\cap V)}\right)^\delta ||u||^{1-\delta}_{H^1(Z)}. \tag{6.3.32}$$

for all $u\in L^2(\Omega)$ such that $u^i\in H^1(\Omega^i)$, $i=1,2$, and $\mathcal{T}u\in L^2(\Omega)$.

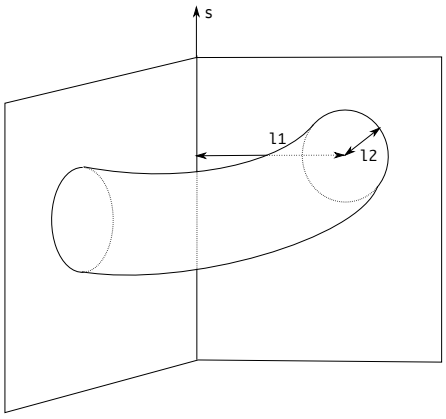


Figure 6.7: The torus $\mathbb{T}(\ell_1;\ell_2)$.

For Neumann boundary conditions near a smooth part of the boundary, we have the following lemma [LR97].

Lemma 6.3.13. *Let V be an open subset of $\overline{Y}$ such that $(-S_0,S_0)\times\partial\Omega$ is smooth in a neighborhood of $\overline{V}$ (if $\overline{V}$ meets this boundary), and V does not meet the interface $(-S_0,S_0)\times S$. Let $U\subset Y$, be an open set such that $0\notin U$. Let U be an open set of $\overline{Y}$. There exist $\delta\in(0,1)$ and $C>0$ such that*

$$||u||_{H^1(V\cap Z)}\leq C\left(||\mathcal{T}u||_{L^2(Z)}+||u||_{H^1(U)}+|\partial_\nu u_{|_{\{x\in\partial\Omega\}\cap V}}|_{L^2(\partial\Omega\cap V)}\right)^{\delta}||u||^{1-\delta}_{H^1(Z)}, \tag{6.3.33}$$

for all $u\in L^2(\Omega)$ such that $u^i\in H^1(\Omega^i)$, $i=1,2$, and $\mathcal{T}u\in L^2(\Omega)$.

In a neighborhood of a point of a smooth interface, we have the following result [LR10].

Lemma 6.3.14. *Let V be an open subset of $\overline{Y}$ such that $V\Subset Z$ and such that $S\times(-S_0,S_0)$ is smooth in a neighborhood of $\overline{V}$ (if $\overline{V}$ meets this interface). Let U be an open set of $\Omega^i\times(-S_0+\alpha,S_0-\alpha)$, for an arbitrary $i\in\{1,2\}$. There exist $C>0$ and $\delta>0$ such that*

$$\begin{aligned}||u||_{H^1(V)}\leq C\Big(&||\mathcal{T}u||_{L^2(\Omega)}+|c_1\gamma_S\partial_{\nu_1}u^1+c_2\gamma_S\partial_{\nu_2}u^2|_{L^2(S\cap U)}\\&+|\gamma_S u^1-\gamma_S u^2|_{H^1(S\cap U)}+||u||_{H^1(U)}\Big)^{\delta}||u||^{1-\delta}_{H^1(Z)},\end{aligned}$$

for all $u\in L^2(\Omega)$ such that $u^i\in H^1(\Omega^i)$, $i=1,2$, and $\mathcal{T}u\in L^2(\Omega)$.

Note that in the four estimates we wrote here, we used the norm $||u||_{H^1(Z)}$ as the functions are H^1 here. Compare with the statements given in Section 6.3.1.

Estimates near corners

Let $0<\ell_1<\ell_2$, and let $p\in\mathcal{E}\times(-S_0+\alpha,S_0-\alpha)$. We shall assume without any loss of generality that $p=0$, up to performing a translation. In coordinates $(s,x)=(s,x_1,x_2)$, we shall consider the disk $\mathbb{D}(\ell_1,\ell_2)=B((0,\ell_1,0),\ell_2)\cap\{x_2=0\}$. We then define $\mathbb{T}(\ell_1,\ell_2)=\tilde{\mathbb{T}}(\ell_1,\ell_2)\cap Y$, where $\tilde{\mathbb{T}}(\ell_1,\ell_2)$ is the revolution of the disk $\mathbb{D}(\ell_1,\ell_2)$ around the axis $\{(x_1,x_2)=(0,0)\}$ (see Figure 6.7).

Let $Y_\delta=\{y=(s,x)\in Y,\ \frac{1}{2}\delta<d(x,0)<\delta\}$.

Lemma 6.3.15. *Let $\eta > 0$ and $\delta > 0$. There exist $\ell_1, \ell_2 > 0$ such that $\tilde{\mathbb{T}}(\ell_1, \ell_2) \subset Y$, and constants $N, C, C' > 0$ such that*

$$||u||_{H^1(Y_\delta)} \leq C\delta^{-N} \left(||\mathcal{T}u||_{L^2(Z)} + ||u||_{H^1(\mathbb{T}(\ell_1,\ell_2))} \right)^{C'\delta^{1+\eta}} ||u||^{1-C'\delta^{1+\eta}}_{H^1(Z)}, \tag{6.3.34}$$

for all $u \in H^1(Z)$ such that $\mathcal{T}u \in L^2(Z)$.

This lemma can be proved by following the proof of Lemma 6.3.3.

Let $\psi_j(x)$, $j \geq 0$ be the eigenfunctions associated with eigenvalues $\lambda_j > 0$ of the operator $\mathbf{T}$. We set

$$w(s,x) = \sum_{\lambda_j \leq \Lambda} \frac{\sinh(\sqrt{\lambda_j}(s+S_0))}{\sqrt{\lambda_j}} a_j \psi_j(x). \tag{6.3.35}$$

where a_j are complex numbers such that $\sum_{j\geq 0} |a_j|^2 < \infty$, and we also set

$$z(x) = \sum_{\lambda_j \leq \Lambda} a_j \psi_j(x). \tag{6.3.36}$$

We recall that $\mathcal{G}_\eta$ and $\mathcal{G}^*_\eta$ are defined in (6.3.10).

We now state the weak interpolation estimate in a neighborhood of a singular point $p \in \mathcal{E} \times (-S_0 + \alpha, S_0 - \alpha)$.

Proposition 6.3.16. *Let p be a non-Neumann corner. There exists U a neighborhood of p in Y such that for $\tilde{Y}$ an open subset of Y, for all $\eta > 0$, there exist $C, \delta > 0$ such that*

$$||w||_{L^2(U)} \leq Ce^{C\Lambda^{3/4}} ||w||^{\delta\mathcal{G}_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta\mathcal{G}_\eta(\Lambda)}_{H^1(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1, \tag{6.3.37}$$

where w and z are given by (6.3.35) and (6.3.36).

We have the counterpart in the Neumann corner case.

Proposition 6.3.17. *Let p be a Neumann corner. There exists U a neighborhood of p in Y such that for $\tilde{Y}$ an open subset of Y, for all $\eta > 0$, there exist $C, \delta > 0$ such that*

$$||w||_{L^2(U)} \leq Ce^{C\Lambda^{3/2}} ||w||^{\delta\mathcal{G}^*_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta\mathcal{G}^*_\eta(\Lambda)}_{H^1(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1, \tag{6.3.38}$$

where w and z are given by (6.3.35) and (6.3.36).

We define

$$\mathcal{F}_\eta(\Lambda) = \begin{cases} \mathcal{G}_\eta(\Lambda), & \text{in a non-Neumann case} \\ \mathcal{G}^*_\eta(\Lambda), & \text{in a Neumann case.} \end{cases}, \quad \text{and } \tilde{\mathcal{F}}(\Lambda) = \begin{cases} \Lambda^{3/4}, & \text{in a non-Neumann case} \\ \Lambda^{3/2}, & \text{in a Neumann case.} \end{cases}$$

Again, by a compactness argument, one can prove the following corollary.

Corollary 6.3.18. *Let ω be an open subset of Ω, for all η, there exist $C, C' > 0$ and $\delta > 0$ such that*

$$||w||_{L^2(Y)} \leq Ce^{C\tilde{\mathcal{F}}_\eta(\Lambda)} |\partial_s w_{|_{\omega\times\{s=-S_0\}}}|^{\delta\mathcal{F}^\mu_\eta(\Lambda)}_{L^2(\omega)} ||w||^{1-\delta\mathcal{F}^\mu_\eta(\Lambda)}_{H^1(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1 \tag{6.3.39}$$

Proof. Note that for a function w defined by (6.3.35), we have $w_{|_{s=-S_0}} = 0$ and $\partial_s w_{|_{s=-S_0}} = z$. By a compactness argument, and applying either Proposition 6.3.16 in the non-Neumann case or Proposition 6.3.17 in the Neumann one, there exists $\mathcal{N}$ a neighborhood of $\{x = 0\}$ in $\mathbb{R}^3$, such that

$$||w||_{L^2(\mathcal{N}\cap Y)} \leq Ce^{C\tilde{\mathcal{F}}(\Lambda)} ||w||^{\delta\mathcal{F}^\mu_\eta(\Lambda)}_{H^1(\tilde{Y})} ||w||^{1-\delta\mathcal{F}^\mu_\eta(\Lambda)}_{H^1(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}, \quad \Lambda \geq 1. \tag{6.3.40}$$

Combining this with Lemmata 6.3.11-6.3.14, this yields

$$||w||_{L^2(Y)} \leq Ce^{C\tilde{\mathcal{F}}(\Lambda)}|\partial_s w_{|\omega\times\{s=-S_0\}}|_{L^2(\omega)}^{\delta\mathcal{F}_\eta^\mu(\Lambda)}||w||_{H^1(Z)}^{1-\delta\mathcal{F}_\eta^\mu(\Lambda)} + Ce^{-C'\sqrt{\Lambda}}||z||_{L^2(\Omega)}, \quad \Lambda \geq 1, \tag{6.3.41}$$

which is the sought result. □

Proof of Propositions 6.3.16 and 6.3.17. First, note that, for w of the form (6.3.35), we have

$$\mathcal{T}w = 0. \tag{6.3.42}$$

The proofs will mainly follow the same ideas than the proofs of Propositions 6.3.8 and 6.3.9. We can assume that $p = 0$ without any loss of generality. Let V be a neighborhood of 0 in Y. Consider a cut-off function $\chi(s,r) = \chi_1(s)\chi_2(x) \in C_0^\infty(\mathbb{R}^3)$ such that

$$\chi_1(s) = \begin{cases} 1 & \text{for } s \leq r_1 \\ 0 & \text{for } s \geq r_2, \end{cases} \qquad \chi_2(x) = \begin{cases} 1 & \text{for } |x| \leq r_1 \\ 0 & \text{for } |x| \geq r_2, \end{cases} \tag{6.3.43}$$

with $0 < r_1 < r_2$ such that $B(0,r_2) \subset V$. Then, in the non-Neumann case, we apply the Carleman estimate of Theorem 6.2.5 to $\tilde{w} := \chi w$ to obtain

$$\tau^{2/3}||e^{\tau\varphi_\varepsilon}\nabla_{s,x}\tilde{w}||_{L^2(Y)} + \tau^2||e^{\tau\varphi_\varepsilon}\tilde{w}||_{L^2(Y)} \lesssim ||e^{\tau\varphi_\varepsilon}\mathcal{T}\tilde{w}||_{L^2(Z)}^2. \tag{6.3.44}$$

The level sets of the weight function φ_ε are sketched in Figure 6.5. The right hand side of (6.3.44) can be estimated as follows

$$||e^{\tau\varphi_\varepsilon(s,x)}\mathcal{T}\tilde{w}||_{L^2(Z)} \lesssim ||e^{\tau\varphi_\varepsilon(s,x)}\chi\mathcal{T}w||_{L^2(Z)} + ||e^{\tau\varphi_\varepsilon(s,x)}[\mathcal{T},\chi]w||_{L^2(Z)} \lesssim ||e^{\tau\varphi_\varepsilon(s,x)}[\mathcal{T},\chi]w||_{L^2(Z)},$$

using (6.3.42). In the Neumann case, we apply Theorem 6.2.6 to $\tilde{w} := \chi w$ to obtain

$$\tau||e^{\tau\varphi_\varepsilon}\tilde{w}||_{L^2(Y)} \lesssim ||e^{\tau\varphi_\varepsilon(s,x)}[\mathcal{T},\chi]w||_{L^2(Z)}, \tag{6.3.45}$$

using again (6.3.42). The commutator $[\mathcal{T},\chi]$ is a differential operator of order one supported in the region where χ varies. Let $K > 0$ be a constant that shall be fixed below, and let us split $H := \text{supp}[\mathcal{T},\chi] = A_1^\varepsilon \cup A_2^\varepsilon$ with

$$A_1^\varepsilon := \{(s,x) \in H,\ \varphi_\varepsilon(s,x) = |x|^2 - \varepsilon s^2 < -\varepsilon K\}, \quad A_2^\varepsilon := \{(s,x) \in H,\ \varphi_\varepsilon(s,x) = |x|^2 - \varepsilon s^2 \geq -\varepsilon K\}. \tag{6.3.46}$$

The regions A_1^ε and A_2^ε are represented in Figure 6.6. The notation A_i^ε is used to emphasize that the above splitting depends on the parameter $\varepsilon > 0$. Observe that for $\varepsilon > 0$ small, $A_1^\varepsilon \subset \{|x| \leq r_1\}$, and consequently

$$\chi(s,r) = \chi_1(s) \text{ in } A_1^\varepsilon. \tag{6.3.47}$$

We recall that in the Carleman estimate, ε depend on τ, such that $\varepsilon \to 0$ as $\tau \to +\infty$. Then, using (6.3.47), we have, for τ large,

$$||e^{\tau\varphi_\varepsilon}[\mathcal{T},\chi]w||_{L^2(Z)} \lesssim ||e^{\tau\varphi_\varepsilon}w||_{L^2(A_1^\varepsilon)} + ||e^{\tau\varphi_\varepsilon}\partial_s w||_{L^2(A_1^\varepsilon)} + ||e^{\tau\varphi_\varepsilon}w||_{H^1(A_\varepsilon^2)}. \tag{6.3.48}$$

Let $\mathcal{U}$ be a neighborhood of 0 such that $s^2 \leq k$ on $\mathcal{U}$, $\chi = 1$ on $\mathcal{U}$,with $k > 0$ to be fixed below, and such that $\mathcal{U} \subset V$. Observe that $\varphi_\varepsilon > -\varepsilon k$ on $\mathcal{U}$. Then, from (6.3.44)-(6.3.48), we have

$$e^{-k\varepsilon\tau}||w||_{L^2(\mathcal{U})} \lesssim e^{-K\varepsilon\tau}\left(||w||_{L^2(A_1^\varepsilon)} + ||\partial_s w||_{L^2(A_1^\varepsilon)}\right) + e^{C\tau}||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.49}$$

with $C > 0$ is a large constant.

• We first assume that $p = 0$ is a non-Neumann corner. As in the proofs of Propositions 6.3.8 and 6.3.9, we set $\tau = \eta_1 \Lambda^{3/4}$, with $\eta_1 > 0$ shall be fixed below. We obtain $\varepsilon \simeq \tau^{-1/3} \simeq \eta_1^{-1/3} \Lambda^{-1/4}$ and the reader should keep in mind that now A_2^ε depends on the spectral parameter Λ. Then (6.3.49) yields

$$||w||_{L^2(\mathcal{U})} \lesssim e^{(k-K)\eta_1\sqrt{\Lambda}} ||w||_{H^1(A_1^\varepsilon)} + e^{C\eta_1 \Lambda^{3/4}} ||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.50}$$

Using the form of w (see (6.3.35)), there exists $C_1 > 0$ such that

$$||w||_{H^1(Z)} \lesssim e^{C_1\sqrt{\Lambda}} ||z||_{L^2(\Omega)},$$

and then choosing $k < K$ and η_1 sufficiently large, there exists $C_2, C_3 > 0$ such that (6.3.55) reads

$$||w||_{L^2(\mathcal{U})} \lesssim e^{-C_3\sqrt{\Lambda}} ||z||_{L^2(\Omega)} + e^{C_2 \Lambda^{3/4}} ||w||_{H^1(A_2^\varepsilon)}. \tag{6.3.51}$$

Note that we have, as in the proofs of Propositions 6.3.8 and 6.3.9

$$d(\{x = 0\}, A_2^\varepsilon) \gtrsim \Lambda^{-1/8}.$$

To handle the last term in the right hand side of (6.3.51), we apply Lemma 6.3.15 to w in (6.3.51) with $\delta = C\Lambda^{-1/8}$, $C > 0$. Then, there exist $0 < \ell_1 < \ell_2$ and $N \in \mathbb{N}$ such that, for Λ large, we have

$$||w||_{H^1(Y_\delta)} \leq C\Lambda^{-N} ||w||^{(C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(\mathbb{T}(\ell_1,\ell_2))} ||w||^{1-(C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(Z)}. \tag{6.3.52}$$

We recall that the set Y_δ is defined at the beginning of this section. Now, $\mathbb{T}(\ell_1, \ell_2)$ is far away from $\{x = 0\}$, but still has a non-empty intersection with $\partial\Omega \times (-S_0 + \alpha, S_0 - \alpha)$ and/or $S \times (-S_0 + \alpha, S_0 - \alpha)$. However, this intersection is located in smooth regions, that is, where Lemmata **??**, 6.3.11, 6.3.12, 6.3.13 and 6.3.14 apply. Then, there exists $\delta_1 \in (0, 1)$ such that

$$||w||_{H^1(Y_\delta)} \leq C\Lambda^{-N} \left(||w||^{\delta_1}_{H^1(\tilde{Y})} ||w||^{1-\delta_1}_{H^1(Z)} \right)^{(C\Lambda)^{\frac{-1-\eta}{8}}} ||w||^{1-(C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(Z)}, \tag{6.3.53}$$

where $\tilde{Y}$ has an empty intersection with the boundaries and the interfaces. This implies

$$||w||_{L^2(Y_\delta)} \leq C\Lambda^{-N} ||w||^{(\delta_1 C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(\tilde{Y})} ||w||^{1-\delta_1(C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(Z)}. \tag{6.3.54}$$

Plugging this in (6.3.51) yields (constants may change)

$$||w||_{L^2(\mathcal{U})} \leq Ce^{C\Lambda^{3/4}} ||w||^{(\delta_1 C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(\tilde{Y})} ||w_k||^{1-\delta_1(C\Lambda)^{\frac{-1-\eta}{8}}}_{H^1(Z)} + Ce^{-C'\sqrt{\Lambda}} ||z||_{L^2(\Omega)}$$

which is in the form given in Proposition 6.3.16.

• For the Neumann case, we set $\tau = \eta_1 \Lambda^{3/2}$. This yields $\varepsilon \simeq \tau^{-2/3} \simeq \Lambda^{-1}$. We obtain

$$||w||_{L^2(\mathcal{U})} \lesssim e^{(k-K)\eta_1\sqrt{\Lambda}} ||w||_{H^1(A_1^\varepsilon)} + e^{C\eta_1 \Lambda^{3/2}} ||w||_{H^1(A_2^\varepsilon)}, \tag{6.3.55}$$

we apply Lemma 6.3.15 to w in (6.3.51) with $\delta = C\Lambda^{-1/2}$, $C > 0$ since in this case

$$d(\{x = 0\}, A_2^\varepsilon) \simeq \Lambda^{-1/2}, \tag{6.3.56}$$

and performing the same computations as above, we obtain for Λ large,

$$||w||_{L^2(Y_\delta)} \leq Ce^{C\Lambda^{3/2}} ||w||^{(C\Lambda)^{\frac{-1-\eta}{2}}}_{H^1(\mathbb{T}(\ell_1,\ell_2))} ||w||^{1-(C\Lambda)^{\frac{-1-\eta}{2}}}_{H^1(Z)}. \tag{6.3.57}$$

We finish the proof as in the Neumann case. This ends the proof of Proposition 6.3.17. □

6.3.3 Proof of the spectral inequalities

Proof of Theorems 6.1.11 and 6.1.13.

First remark that

$$\partial_s w_{|s=-S_0} = \left(\sum_{\lambda_j \leq \Lambda} a_j \cosh(\sqrt{\lambda_j} s) \phi_j \right)_{|s=-S_0} = z.$$

Second, we have

$$||w||^2_{L^2(Y)} = \sum_{k \geq 0} |a_j|^2 \int_{-S_0+\alpha}^{S_0-\alpha} \sum_{\lambda_j \leq \Lambda} a_j \sinh(\sqrt{\lambda_j} s) \geq \kappa ||z||^2_{L^2(\Omega)}. \tag{6.3.58}$$

In the case of an inverse square potential, applying Corollary 6.3.10, we have

$$\kappa ||z||_{L^2(\Omega)} \leq ||w||_{H^1(Y)} \leq C e^{C\tilde{\mathcal{F}}^\mu(\Lambda)} ||z||^{\delta \mathcal{F}^\mu_\eta(\Lambda)}_{L^2(\omega)} ||w||^{1-\delta \mathcal{F}^\mu_\eta(\Lambda)}_{W_\mu(Z)} + C e^{-C\sqrt{\Lambda}} ||z||_{L^2(\Omega)}.$$

Note that we have $||w||_{W_\mu(Z)} \leq C e^{C\sqrt{\Lambda}} ||z||_{L^2(\Omega)}$ for Λ large. Hence, with possibly different constant, we have

$$||z||_{L^2(\Omega)} \leq \left(C e^{C\tilde{\mathcal{F}}^\mu(\Lambda)} \right)^{(\delta \mathcal{F}^\mu_\eta(\Lambda))^{-1}} ||z||_{L^2(\omega)},$$

In the polygonal case, we apply Corollary 6.3.18 to obtain

$$\kappa ||z||_{L^2(\Omega)} \leq ||w||_{H^1(Y)} \leq C e^{C\tilde{\mathcal{F}}(\Lambda)} ||z||^{\delta \mathcal{F}_\eta(\Lambda)}_{L^2(\omega)} ||w||^{1-\delta \mathcal{F}_\eta(\Lambda)}_{H^1(Z)} + C e^{-C\sqrt{\Lambda}} ||z||_{L^2(\Omega)}.$$

In this case, we have $||w||_{H^1(Z)} \leq C e^{C\sqrt{\Lambda}} ||z||_{L^2(\Omega)}$ for Λ large. As a result, we have (with possibly a different constant $C > 0$)

$$||z||_{L^2(\Omega)} \leq \left(C e^{C\tilde{\mathcal{F}}(\Lambda)} \right)^{(\delta \mathcal{F}_\eta(\Lambda))^{-1}} ||z||_{L^2(\omega)},$$

We conclude using the orthogonormality of the eigenfunctions

$$||z||^2_{L^2(\Omega)} = \sum_{\lambda_j \leq \Lambda} |a_j|^2.$$

□

6.4 Estimates for the operator with the spectral parameter

In this section, we shall prove Theorem 6.1.15. For $\sigma \in \mathbb{R}$, we shall denote by

$$Q_\sigma = Q - \sigma^2, \quad \text{and } \mathbf{T}_\sigma = \mathbf{T} - \sigma^2.$$

6.4.1 Decomposition into spherical harmonics

In this section, we follow Section 6.2.3, but the end shall have a different treatement. We consider $d \geq 2$, but if $d = 2$ we shall assume that $\mu = 0$. We shall carry out the analysis for Q_σ here, but the same can be carried out for $\mathbf{T}_\sigma$. Consider the Laplace operator in spherical coordinates $x = r\theta$,

$$\Delta = \partial_r^2 + (d-1)\frac{\partial_r}{r} + \frac{1}{r^2}\Delta_\theta, \tag{6.4.1}$$

where Δ_θ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$, $d \geq 2$. Then Q_σ reads

$$Q_\sigma = -\partial_r^2 - (d-1)\frac{\partial_r}{r} - \frac{1}{r^2}(\Delta_\theta + \mu) - \sigma^2,$$

In the case where 0 is a point of the interior, the operator Δ_θ admits the following orthogonal basis of eigenfunctions associated with eigenvalues

$$(\psi_j)_{j\geq 0} \in L^2(\mathbb{S}^{d-1}), \quad 0 = \nu_0 < \nu_1 \leq \cdots \leq \nu_n \leq \cdots \to +\infty. \tag{6.4.2}$$

In the case where 0 is the vertex of a corner, two cases need to be discussed. For a non-Neumann corner (see Definition 6.1.8), we have

$$(\psi_j)_{j\geq 0} \in L^2(\mathbb{S}^{d-1}), \quad 0 < \nu_0 \leq \nu_1 \leq \cdots \leq \nu_n \leq \cdots \to +\infty. \tag{6.4.3}$$

In the case of a Neumann corner, we have (6.4.2).
The same analysis on the different local geometrical configurations than the one given in Section 6.2.3 can be carried out. What follows is stated for the inverse square potential problem, but can be stated in the same way for the polygonal problem.

Let $u \in D(Q)$ such that $Q_\sigma u = f \in L^2(\Omega)$, and let $R > 0$ such that $B(0,R) \subset \Omega$. We introduce the radial cut-off function that shall localize the solution near the singularity

$$\chi(x) = \begin{cases} 1 & |x| < R/2 \\ 0 & |x| > 3R/4 \end{cases}, \quad \chi \in C_0^\infty(\mathbb{R}^d). \tag{6.4.4}$$

We set $\tilde{u} = \chi u$. Hence, $\tilde{u}$ satisfies

$$Q_\sigma \tilde{u} = \tilde{f} \in L^2(\Omega), \tag{6.4.5}$$

where $\tilde{f} = f + [Q_\sigma, \chi]u$. Now, setting $v(r,\theta) = \tilde{u}(r\theta)$ and $g(r,\theta) = \tilde{f}(r\theta)$, we obtain

$$-\partial_r^2 v - (d-1)\frac{\partial_r}{r} v - \frac{1}{r^2}\left(\Delta_\theta + \mu\right) v - \sigma^2 v = g,$$

and the L^2 space turns to be the weighted L^2 space given by

$$\int_0^R \int_{\mathbb{S}^{d-1}} |v(r,\theta)|^2 r^{d-1} dr d\theta < \infty, \quad \int_0^R \int_{\mathbb{S}^{d-1}} |g(r,\theta)|^2 r^{d-1} dr d\theta < \infty.$$

Splitting the variables, and using (6.4.2)-(6.4.3), we can write

$$v(r,\theta) = \sum_{k\geq 0} v_k(r)\psi_k(\theta), \quad g(r,\theta) = \sum_{k\geq 0} g_k(r)\psi_k(\theta), \tag{6.4.6}$$

we obtain the countable family of equations

$$-\partial_r^2 v_k - (d-1)\frac{\partial_r}{r} v_k - \frac{1}{r^2}\left(-\nu_k + \mu\right) v_k - \sigma^2 v_k = g_k, \tag{6.4.7}$$

We shall separate the analysis into two cases. The critical case,

$$\mu = \mu^* \text{ (resp. the Neumann corner, that is, } d = 2,\ \mu = 0,\ \nu_0 = 0), \tag{6.4.8}$$

and the sub-critical one

$$\mu < \mu^* \text{ (resp. the non-Neumann corner, that is, } d = 2,\ \mu = 0,\ \nu_0 > 0). \tag{6.4.9}$$

For $k \in \mathbb{N}$, we introduce $w_k = r^{(d-1)/2} v_k$, and $h_k = r^{(d-1)/2} g_k$, and then

$$\int_0^R |w_k(r)|^2 dr < \infty, \quad \int_0^R |h_k(r)|^2 dr < \infty.$$

and moreover we have the relation

$$-\partial_r^2 w_k - \frac{1}{r^2}\left(-\nu_k + \mu - \frac{(d-1)(d-3)}{4}\right) w_k - \sigma^2 w_k = h_k.$$

and $w_k(s,0) = 0$, and $w = 0$ in a neighborhood of $\{r = R\}$ (see Lemma 6.2.7). Setting $\mu_k = \mu - \nu_k - \frac{(d-1)(d-3)}{4}$, we obtain

$$-\partial_r^2 w_k - \frac{\mu_k}{r^2} w_k - \sigma^2 w_k = h_k. \tag{6.4.10}$$

In this configuration, we can see that

$$\mu_k = \mu - \nu_k - \frac{(d-1)(d-3)}{4} = \mu - \mu^* + \frac{(d-2)^2 - (d-1)(d-3)}{4} - \nu_k = \mu - \mu^* + \frac{1}{4} - \nu_k \leq \frac{1}{4}.$$

Uniform one-dimensional Carleman estimate

We define

$$Q_{k,\sigma} = -\partial_r^2 - \sigma^2 - \frac{\mu_k}{r^2}, \tag{6.4.11}$$

and we consider the following problem

$$\begin{cases} Q_{k,\sigma} z = z^0 \in L^2(Z), & \text{in } (0,R) \\ z(s,0) = 0 & \text{at } \{0\}, \\ z(r) = 0 & \text{in a neighborhood of } \{R\}. \end{cases} \tag{6.4.12}$$

The analysis of equation (6.4.12) can be carried out like the one carried out for (6.2.19), using a Fourier transform with respect to the variable s. We shall consider the following weight function $\varphi \in C^\infty([0,R])$

$$\varphi(r) = r^2. \tag{6.4.13}$$

We now are ready to state the one-dimensional Carleman estimate, in the sub-critical case $\mu_k < 1/4$.

Proposition 6.4.1. *Assume $\mu_k < 1/4$. There exists $C > 0$ such that*

$$\tau \int_0^R e^{\tau\varphi} |\partial_r z|^2 + \tau(1 + |\mu_k|) \int_0^R e^{\tau\varphi} \frac{|z|^2}{r^2} + \tau^3 \int_0^R e^{\tau\varphi} r^2 |z|^2 \leq C \int_0^R e^{\tau\varphi} |Q_{k,\sigma} z|^2,$$

for all $\tau \geq 1$, for all $\sigma \geq 1$, and for all $z \in \mathcal{H}_{\mu_k}(0,R)$, solution of (6.4.12).

Proof. As usual with Carleman estimates, we shall consider the conjugated operator

$$\begin{aligned} Q_{\sigma,\varphi} = e^{\tau\varphi} Q_{\sigma,k} e^{-\tau\varphi} &= -\partial_r^2 - \tau^2 \varphi'^2 + 2\tau\varphi'\partial_r + \tau\varphi'' - \sigma^2 - \frac{\mu_k}{r^2} \\ &= P_2 + P_1, \end{aligned}$$

where $P_2 = -\partial_r^2 - \tau^2\varphi'^2 - \sigma^2 - r^{-2}\mu_k$, and $P_1 = \tau\varphi'' + 2\tau\varphi'\partial_r$, and we consider $\zeta = e^{\tau\varphi} z$. Using the form of the weight function (6.4.13), we have $\varphi' = 2r$ and $\varphi'' = 2$. We then have

$$||Q_{\sigma,\varphi}\zeta||^2_{L^2(0,R)} = ||P_2\zeta||^2_{L^2(0,R)} + ||P_1\zeta||^2_{L^2(0,R)} + 2\,\mathrm{Re}\left(P_2\zeta, P_1\zeta\right)_{L^2(0,R)} \geq 2\,\mathrm{Re}\left(P_2\zeta, P_1\zeta\right)_{L^2(0,R)}$$

We recall that in the sub-critical case, $\zeta \in H_0^1(0,R) \cap H^2(0,R)$. Yet, for all $\delta > 0$, we compute the above

scalar product, using integration by parts

$$\begin{aligned}\mathrm{Re}\left(P_2\zeta, P_1\zeta\right)_{L^2(\delta,R)} &= -2\tau\,\mathrm{Re}\int_\delta^R \overline{\zeta}\partial_r^2\zeta - 4\tau\,\mathrm{Re}\int_\delta^R r\partial_r^2\zeta\partial_r\overline{\zeta} - 8\tau^3\,\mathrm{Re}\int_\delta^R r^3\zeta\partial_r\overline{\zeta} - 8\tau^3\int_\delta^R r^2|\zeta|^2\\ &\quad - 2\sigma^2\tau\int_\delta^R |\zeta|^2 - 4\sigma^2\tau\,\mathrm{Re}\int_\delta^R r\zeta\partial_r\overline{\zeta} + 2\mu_k\tau\int_\delta^R\frac{|\zeta|^2}{r^2} - 4\tau\mu_k\,\mathrm{Re}\int_\delta^R\frac{\zeta}{r}\partial_r\overline{\zeta}\\ &= 4\tau\int_\delta^R|\partial_r\zeta|^2 + 2\tau\,\mathrm{Re}\left(\partial_r\zeta\zeta\right)_{|r=\delta} + 4\tau^3\int_\delta^R r^2|\zeta|^2 + 4\tau^3\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\tau\sigma^2\left(r|\zeta|^2\right)_{|r=\delta}\\ &\quad + 2\tau\left(r|\partial_r\zeta|^2\right)_{|r=\delta} - 4\tau\mu_k\int_\delta^R\frac{|\zeta|^2}{r^2} + 2\mu_k\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}.\end{aligned}$$

Setting

$$\mathcal{B}(\zeta) := 2\tau\left(r|\partial_r\zeta|^2\right)_{|r=\delta} + 2\tau\,\mathrm{Re}\left(\partial_r\zeta\zeta\right)_{|r=\delta} + 4\tau^3\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\tau\sigma^2\left(r|\zeta|^2\right)_{|r=\delta} + 2\mu_k\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta},$$

we can obtain, using the Young inequality

$$\mathcal{B}(\zeta) \geq C\tau\left(r|\partial_r\zeta|^2\right)_{|r=\delta} + 4\tau^3\left(r^3|\zeta|^2\right)_{|r=\delta} + 2\tau\sigma^2\left(r|\zeta|^2\right)_{|r=\delta} - C'|\mu_k\left(\frac{|\zeta|^2}{r}\right)_{|r=\delta}. \tag{6.4.14}$$

$$\mathrm{Re}\left(P_2\zeta, P_1\zeta\right)_{L^2(\delta,R)} \geq 4\tau\int_\delta^R\left(|\partial_r\zeta|^2 - \mu_k\frac{|\zeta|^2}{r^2}\right) + 4\tau^3\int_\delta^R r^2\zeta^2 + \mathcal{B}(\zeta).$$

We recall that we have $||Q_{\sigma,\varphi}\zeta||^2_{L^2(\delta,R)} \geq 2\,\mathrm{Re}\left(P_2\zeta, P_1\zeta\right)_{L^2(\delta,R)}$. We shall separate two cases.

- For $\mu_k \in [-1/8, 1/4)$, using Hardy inequality (6.2.15), we obtain

$$\begin{aligned}\int_\delta^R|\partial_r\zeta|^2 - \mu_k\int_\delta^R\frac{|\zeta|^2}{r^2} &= 2(\mu_k+\frac{1}{4})\left(\int_\delta^R|\partial_r\zeta|^2 - \frac{1}{4}\int_\delta^R\frac{|\zeta|^2}{r^2}\right) + 2\left(\frac{1}{4}-\mu_k\right)\left(\int_\delta^R|\partial_r\zeta|^2 + \frac{1}{4}\int_\delta^R\frac{|\zeta|^2}{r^2}\right)\\ &\geq 2C(\mu_k+\frac{1}{4})\int_\delta^R|\zeta|^2 + 2\left(\frac{1}{4}-\mu_k\right)\left(\int_\delta^R|\partial_r\zeta|^2 + \frac{1}{4}\int_\delta^R\frac{|\zeta|^2}{r^2}\right),\end{aligned}$$

  and then there exists a constant $C > 0$ such that

$$C\Big(P_2\zeta, P_1\zeta\Big)_{L^2(\delta,R)} \geq \tau^3\int_\delta^R r|\zeta|^2 + \tau\int_\delta^R|\zeta|^2 + \tau\left(\frac{1}{4}-\mu_k\right)\left(\int_\delta^R|\partial_r\zeta|^2 + \int_\delta^R\frac{|\zeta|^2}{r^2}\right) + \mathcal{B}(\zeta).$$

- For $\mu_k < -1/8$, we directly have

$$C\Big(P_2\zeta, P_1\zeta\Big)_{L^2(\delta,R)} \geq \tau^3\int_\delta^R r|\zeta|^2 + \tau\int_\delta^R|\zeta|^2 + \tau\int_\delta^R|\partial_r\zeta|^2 + (1+|\mu_k|)\int_\delta^R\frac{|\zeta|^2}{r^2} + \mathcal{B}(\zeta).$$

In both cases, we obtain

$$C\Big(P_2\zeta, P_1\zeta\Big)_{L^2(\delta,R)} \geq \tau^3\int_\delta^R r|\zeta|^2 + \tau\int_\delta^R|\zeta|^2 + \tau\int_\delta^R|\partial_r\zeta|^2 + \tau(1+|\mu_k|)\int_\delta^R\frac{|\zeta|^2}{r^2} + \mathcal{B}(\zeta).$$

As all the volumic terms pass to the limit $\delta \to 0$, we have, using Lemma (6.2.8) and (6.4.14),

$$C||Q_{\sigma,\varphi}\zeta||^2_{L^2(0,R)} \geq \tau^3\int_\delta^R r|\zeta|^2 + \tau\int_\delta^R|\zeta|^2 + \tau\int_\delta^R|\partial_r\zeta|^2 + \tau(1+|\mu_k|)\int_\delta^R\frac{|\zeta|^2}{r^2}.$$

We now use $z = e^{-\tau\varphi}\zeta$, to obtain

$$\tau||e^{\tau\varphi_\varepsilon}\partial_r z||^2_{L^2(0,R)} \lesssim \tau||\partial_r\zeta||^2_{L^2(Y)} + \tau^3||r\zeta||^2_{L^2(\tilde{Y})},$$

and this yields to result. □

In the sub-critical case $\mu_k < 1/4$, we can come back to the original function $v_k = r^{-\frac{d-1}{2}}w_k$ (see Section 6.4.6), using the following corollary.

Corollary 6.4.2. *Let $v_k = r^{-\frac{d-1}{2}}z$, $g_k = r^{-\frac{d-1}{2}}z^0$, where z and z^0 satisfy (6.4.12), with $\mu_k < 1/4$. Then, there exists $C > 0$ such that*

$$\int_0^R |v_k(r)|^2 r^{d-1}dr + \int_0^R |\partial_r v_k(r)|^2 r^{d-1}dr + (1+|\mu_k|)\int_0^R \frac{|v_k(r)|^2}{r^2}r^{d-1}dr \leq C\int_0^R |g_k(r)|^2 r^{d-1}dr,$$

for all $|\sigma| \geq 1$.

Proof.
Again, we apply the Carleman estimate of Proposition 6.4.1 to z. Then, fixing τ and bounding the weight function $e^{\tau\varphi}$, there exists $C > 0$ such that

$$\int_0^R |\partial_r z|^2 + (1+|\mu_k|)\int_0^R \frac{|z|^2}{r^2} + \int_0^R r^2|z|^2 \leq C\int_0^R |z^0|^2,$$

Then, it is sufficient to observe that

$$|\partial_r v_k(r)|^2 r^{d-1} \lesssim \frac{|z(r)|^2}{r^2} + |\partial_r z(r)|^2,$$

and this ends the proof. □

In the critical case, in order to justify the integration by parts, we can perform the same regularization procedure of Lemma 6.2.17, as described in the proofs of Theorem 6.2.12. We set

$$Q^N_{k,\sigma} = -\partial_r^2 - \sigma^2 - \mu_k\frac{1}{r^2+\frac{1}{N}}.$$

Proposition 6.4.3. *Let $\mu_k = 1/4$. There exists $C > 0$ such that*

$$\tau\int_0^R e^{\tau\varphi}|z|^2 + \tau\int_0^R e^{\tau\varphi}\left(|\partial_r z|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right) + \tau^3\int_0^R e^{\tau\varphi}r^2|z|^2 \leq C\int_0^R e^{\tau\varphi}|Q^N_{k,\sigma}z|^2,$$

for all $\tau \geq 1$, for all $\sigma \geq 1$, and for all $z \in \mathcal{H}_{\frac{1}{4}}(0,R)$, solution of (6.4.12).

Proof.
We set

$$Q^N_{\sigma,\varphi} = -\partial_r^2 - \tau^2\varphi'^2 + 2\tau\varphi'\partial_r + \tau\varphi'' - \sigma^2 - \frac{1}{4}\frac{2}{r^2+\frac{1}{N}} = P_{2,N} + P_1,$$

where $P_2 = -\partial_r^2 - \tau^2\varphi'^2 - \sigma^2 - \frac{1}{4}\frac{1}{r^2+\frac{1}{N}}$, and $P_1 = \tau\varphi'' + 2\tau\varphi'\partial_r$, and we consider $\zeta = e^{\tau\varphi}z$. Using the form of the weight function (6.4.13), we have $\varphi' = 2r$ and $\varphi'' = 2$. We then have

$$||Q^N_{\sigma,\varphi}\zeta||^2_{L^2(0,R)} = ||P_{2,N}\zeta||^2_{L^2(0,R)} + ||P_1\zeta||^2_{L^2(0,R)} + 2\,\mathrm{Re}\left(P_{2,N}\zeta, P_1\zeta\right)_{L^2(0,R)} \geq 2\,\mathrm{Re}\left(P_{2,N}\zeta, P_1\zeta\right)_{L^2(0,R)} \tag{6.4.15}$$

We recall that for the regularized problem, we have $\zeta \in H_0^1(0,R)\cap H^2(0,R)$. Yet, for all $\delta>0$, we compute the above scalar product, using integration by parts

$$\begin{aligned}\mathrm{Re}\left(P_{2,N}\zeta, P_1\zeta\right)_{L^2(0,R)} &= -2\tau\,\mathrm{Re}\int_0^R \overline{\zeta}\partial_r^2\zeta - 4\tau\,\mathrm{Re}\int_0^R r\partial_r^2\zeta\partial_r\overline{\zeta} - 8\tau^3\,\mathrm{Re}\int_0^R r^3\zeta\partial_r\overline{\zeta} - 8\tau^3\int_0^R r^2|\zeta|^2\\ &\quad - 2\sigma^2\tau\int_0^R|\zeta|^2 - 4\sigma^2\tau\,\mathrm{Re}\int_0^R r\zeta\partial_r\overline{\zeta} + \frac{1}{2}\tau\int_0^R\frac{|\zeta|^2}{r^2+\frac{1}{N}} - \tau\,\mathrm{Re}\int_0^R\frac{r\zeta}{r^2+\frac{1}{N}}\partial_r\overline{\zeta}\\ &= 4\tau\int_0^R|\partial_r\zeta|^2 + 2\tau\,\mathrm{Re}\left(\partial_r\zeta\zeta\right)_{|r=0} + 4\tau^3\int_0^R r^2|\zeta|^2 + 4\tau^3\left(r^3|\zeta|^2\right)_{|r=0} + 2\tau\sigma^2\left(r|\zeta|^2\right)_{|r=0}\\ &\quad + 2\tau\left(r|\partial_r\zeta|^2\right)_{|r=0} - \tau\int_0^R\frac{|\zeta|^2}{r^2+\frac{1}{N}} + \frac{1}{2}\left(\frac{r|\zeta|^2}{r^2+\frac{1}{N}}\right)_{|r=0}\\ &= 4\tau\int_0^R|\partial_r\zeta|^2 + 4\tau^3\int_0^R r^2|\zeta|^2 - \tau\int_0^R\frac{|\zeta|^2}{r^2+\frac{1}{N}}\end{aligned}$$

using boundary conditions. Note that $-\frac{1}{r^2+\frac{1}{N}}\geq -\frac{1}{r^2}$. Using this, in addition to (6.4.15) and the Hardy inequality (6.2.15), we have

$$C||Q^N_{\sigma,\varphi}\zeta||^2_{L^2(0,R)} \geq 4\tau\int_0^R\left(|\partial_r\zeta|^2 + \frac{1}{4}\frac{|\zeta|^2}{r^2}\right) + \tau^3\int_0^R r^2|\zeta|^2 + \tau\int_0^R|\zeta|^2. \tag{6.4.16}$$

We now use $z=e^{-\tau\varphi}\zeta$, to obtain

$$\tau||e^{\tau\varphi_\varepsilon}\partial_r z||^2_{L^2(0,R)} \lesssim \tau||\partial_r z||^2_{L^2(Y)} + \tau^3||rz||^2_{L^2(\tilde{Y})},$$

and this yields to result. □

In the critical case $\mu_0=1/4$, we can come back to the original function $v_0 = r^{-\frac{d-1}{2}}w_0$ (see Section 6.4.6), using the following corollary.

Corollary 6.4.4. *Let $v_0 = r^{-\frac{d-1}{2}}z$, $g_0 = r^{-\frac{d-1}{2}}z^0$, where z and z^0 satisfy (6.4.12), with $\mu_0=1/4$. Then, there exists $C>0$ such that*

$$\int_0^R|v_0(r)|^2r^{d-1}dr + \int_0^R\left(|\partial_r v_0(r)|^2 - \mu^*\frac{|v_0(r)|^2}{r^2}\right)r^{d-1}dr \leq C\int_0^R|g_0(r)|^2r^{d-1}dr, \quad \mu^* = \frac{(d-2)^2}{4},$$

for all $|\sigma|\geq 1$.

Proof. First, we apply the Carleman estimate of Proposition 6.4.3 to z. We recall that first for the regularized problem, $z\in H_0^1(0,R)\cap H^2(0,R)$. Then, fixing τ and bounding the weight function $e^{\tau\varphi}$, there exists $C>0$ such that

$$\int_0^R\left(|\partial_r z|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right)dr + \int_0^R r^2z^2 \leq C\int_0^R|Q^N_{k,\sigma}z|^2. \tag{6.4.17}$$

We compute

$$\begin{aligned}\int_0^R|\partial_r z|^2dr &= \int_0^R\left(|\partial_r v_0|^2 + (\frac{d-1}{2})^2\frac{|v_0|^2}{r^2}\right)r^{d-1}dr + (d-1)\int_0^R r^{d-2}v_0\partial_r v_0dr\\ &= \int_0^R\left(|\partial_r v_0|^2 + (\frac{d-1}{2})^2\frac{|v_0|^2}{r^2}\right)r^{d-1}dr + \frac{d-1}{2}\int_0^R r^{d-2}\partial_r|v_0|^2dr\\ &= \int_0^R\left(|\partial_r v_0|^2 + (\frac{d-1}{2})^2\right)\frac{|v_0|^2}{r^2}r^{d-1}dr - \frac{(d-1)(d-2)}{2}\int_0^R r^{d-3}|v_0|^2dr.\end{aligned}$$

Note that the boundary terms that appears when performing integration by parts vanish, using Lemma 6.2.8. Then, we have

$$\int_0^R \left(|\partial_r z|^2 - \frac{1}{4}\frac{|z|^2}{r^2}\right) dr = \int_0^R |\partial_r v_0|^2 r^{d-1} dr + \int_0^R \left(\frac{(d-1)^2}{4} - \frac{(d-1)(d-2)}{2} - \frac{1}{4}\right)\frac{|v_0|^2}{r^2} r^{d-1} dr$$
$$= \int_0^R \left(|\partial_r v_0|^2 - \frac{(d-2)^2}{4}\right)\frac{|v_0|^2}{r^2} r^{d-1} dr. \tag{6.4.18}$$

Note that the left member of (6.4.18) pass to the limit $N \to +\infty$, using an adapted version of Lemma 6.2.17. As a result, the right member pass to the limit too. This ends the proof. □

6.4.2 Interpolation estimates

In this section, we shall use the following notation. In the case of an inverse square potential problem, we shall denote by $\mathfrak{P}_\sigma = P_\sigma$, and in the case of a polygonal problem $\mathfrak{P}_\sigma = \mathbf{T}_\sigma$. The equation in the case of an inverse square potential problem is

$$\begin{cases} \mathfrak{P}_\sigma u = f & \text{in } \Omega \\ u_{|\partial\Omega} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.4.19}$$

and in the polygonal problem is

$$\begin{cases} \mathfrak{P}_\sigma u = f & \text{in } \Omega \\ \gamma_i u = 0 & \text{on } \partial\Omega_i,\ i \in \mathsf{D} \\ \gamma_i \partial_\nu u = 0 & \text{on } \partial\Omega_i,\ i \in \mathsf{N} \\ \gamma_S u^1 = \gamma_S u^2 & \text{on } S, \\ c_1 \gamma_S \partial_{\nu_1} u^1 + c_2 \gamma_S \partial_{\nu_2} u^2 = 0 & \text{on } S. \end{cases} \tag{6.4.20}$$

Following (6.4.5), we localize near a singularity (that is, near 0, in the case of an inverse square potential, and near $p \in \mathcal{E}$ in a case of a polygon), and we can assume without any loss of generality that $p = 0$. We introduce the radial cut-off function

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < R/2 \\ 0 & \text{if } |x| > 3R/4 \end{cases}, \quad \chi \in C_0^\infty(\mathbb{R}^d). \tag{6.4.21}$$

Setting $\tilde{u} = \chi u \in D(\mathfrak{P}_\sigma)$. Hence, $\tilde{u}$ satisfies

$$\mathfrak{P}_\sigma \tilde{u} = \tilde{f} \in L^2(\Omega), \quad \operatorname{supp} \tilde{u} \subset B(0, R), \tag{6.4.22}$$

where $\tilde{f} = f + [\mathfrak{P}, \chi]u$. Note that, in the polygonal problem, near p the boundary conditions are preserved as χ is radial. We define $U = \Omega \cap B(0, R)$. We shall denote

$$||u||^2_{\mathcal{H}_\mu(U)} = \int_U |u|^2 + \int_U \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}\right), \tag{6.4.23}$$

with the convention that if Ω is a polygon (d=2), $\mu = 0$, and then $||.||_{\mathcal{H}_\mu}$ is precisely the H^1 norm.

Theorem 6.4.5. *There exists $C > 0$ such that*

$$||\tilde{u}||^2_{\mathcal{H}_\mu(U)} \leq C||\tilde{f}||^2_{L^2(U)},$$

with $\tilde{u}$ and $\tilde{f}$ satisfying (6.4.22) for all $|\sigma| \geq 1$.

Remark 6.4.6. *At first glance, it is a bit surprising that the constant involved in Theorem 6.4.5, does not depend on the spectral parameter σ. We can explain it by observing that the Carleman weight in the estimates of Propositions 6.4.1 and 6.4.3 correspond to an observation of the singular point from an annulus, and thus the Geometric Control Condition holds.*

Proof. For the case $\mu_k < 1/4$, we apply Corollary 6.4.2 to obtain

$$||v_k||^2_{L^2((0,R),r^{d-1}dr)} + \int_0^R |\partial_r v_k(r)|^2 r^{d-1} dr + (1+|\mu_k|)\int_0^R \frac{|v_k|^2}{r^2} r^{d-1} dr \leq C||g_k||^2_{L^2((0,R),r^{d-1}dr)}. \quad (6.4.24)$$

From Corollary 6.4.4, if $\mu_0 = 1/4$ we obtain

$$||v_0||^2_{L^2((0,R),r^{d-1}dr)} + \int_0^R \left(|\partial_r v_0(r)|^2 - \mu^* \frac{|v_0|^2}{r^2}\right) r^{d-1} dr \leq C||g_0||^2_{L^2((0,R),r^{d-1}dr)}. \quad (6.4.25)$$

In the case (6.4.2), $\mu_k < 1/4$ for all $k \in \mathbb{N}$, and then, using the orthogonality of the eigenfunctions ψ_k given by (6.4.2), we have from (6.4.24)

$$||v||_{H^1(\Omega, r^{d-1}drd\theta)} \leq ||g||_{L^2(\Omega, r^{d-1}drd\theta)}.$$

In the case (6.4.3), we have $\mu_0 = 1/4$ and $\mu_k < 1/4$, for all $k \in \mathbb{N}^*$. Then, we weaken (6.4.24) by writing

$$||v_k||^2_{L^2((0,R),r^{d-1}dr)} + \int_0^R |\partial_r v_k(r)|^2 r^{d-1} dr + (1+|\mu_k|-\mu^*)\int_0^R \frac{|v_k|^2}{r^2} r^{d-1} dr \leq C||g_k||^2_{L^2((0,R),r^{d-1}dr)},$$

for all $k \in \mathbb{N}^*$. Then, using the orthogonality of the eigenfunctions ψ_k given by (6.4.3), we obtain

$$\int_\Omega |v|^2 r^{d-1} drd\theta + \int_\Omega \left(|\nabla v|^2 - \mu^* \frac{|v|^2}{r^2}\right) r^{d-1} drd\theta \leq ||g||_{L^2(\Omega, r^{d-1}dr)}.$$

Theorem 6.4.5 follows by coming back to the Cartesian variables. □

As a corollary, we obtain the following observation estimate. We set $\omega_R = \{x \in U,\ |x| \in (R/2, 3R/4)\}$, and $V = \{x \in U,\ |x| < R/2\}$.

Corollary 6.4.7. *There exists $C > 0$ such that*

$$||u||_{\mathcal{H}_\mu(V)} \leq C\left(||\mathfrak{P}_\sigma u||_{L^2(U)} + ||u||_{H^1(\omega_R)}\right), \quad (6.4.26)$$

for u solution of (6.4.19) (resp. (6.4.20)), and for all $|\sigma| \geq 1$.

Proof. We apply Theorem 6.4.5 to $v = u\chi$ where χ is the cut-off function introduced in (6.4.21). Note that $\mathfrak{P}_\sigma \chi = \chi \mathfrak{P}_\sigma - [\mathfrak{P}_\sigma, \chi]$, where $[\mathfrak{P}_\sigma, \chi]$ is a differential operator of order 1 that does not depend on σ, supported in the region where χ varies, that is $\text{supp}[\mathfrak{P}_\sigma, \chi] \subset \omega_R$. As a result,

$$||\mathfrak{P}_\sigma v||_{L^2(U)} \lesssim ||\mathfrak{P}_\sigma u_0||_{L^2(U)} + ||u||_{H^1(\omega_R)}.$$

Concerning the left member of Theorem 6.4.5, we write, using Lemma 6.3.4,

$$||u||_{\mathcal{H}_\mu(V)} \leq ||\tilde{u}||_{\mathcal{H}_\mu(U)} + ||\tilde{u}||_{\mathcal{H}_\mu(V\setminus U)} \leq ||\tilde{u}||_{\mathcal{H}_\mu(U)} + C_R||\tilde{u}||_{H^1(\omega_R)},$$

and this yields the sought result. □

In the case of the inverse square potential, we define $\Omega^\eta := \{x \in \Omega,\ d(x,0) > \eta\}$. Note that from Lemma 6.3.4, for all $\eta > 0$, there exists $C_\eta > 0$ such that for all $z \in H^1_0(\Omega)$, we have

$$||z||_{H^1(\Omega^\eta)} \leq C_\eta ||u||_{\mathcal{H}_\mu(\Omega)}, \quad \mu \leq \mu^*.$$

As a result, away from the origin, we can propagate the usual H^1 norm, as it is done in [Rob95, LR95]. Hence, by a compactness argument and elliptic regularity, we can have the following global estimate. We recall that by convention, if $d = 2$ we set $\mu = 0$, and then $\mathcal{H}_\mu = H^1$.

Theorem 6.4.8. *For all ω open subset compactly embedded in Ω, there exist $C > 0$ such that*

$$||z||_{\mathcal{H}_\mu(\Omega)} \leq Ce^{C|\sigma|}\left(||z||_{L^2(\omega)} + ||\mathfrak{P}_\sigma z||_{L^2(\Omega)}\right),$$

for all $z \in \mathcal{H}_\mu(\Omega)$ such that $\mathfrak{P}_\sigma z \in L^2(\Omega)$, and for all $|\sigma| \geq 1$.

6.4.3 Proof of the resolvent estimates

In this section, we provide a proof for Theorem 6.1.15. The following lemma states that the derivation of the resolvent estimate of Theorem 6.1.15 is reduced to an observability inequality. Let $\sigma \in \mathbb{R}$, $|\sigma| \geq 1$ be the spectral parameter, and let $U = (u_0, u_1) \in D(A)$ (resp. $D(B)$), $F = (f_0, f_1) \in \mathcal{H}_\mu \times L^2(\Omega)$ (resp. $\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)$) such that

$$(i\sigma + A)U = F \quad \text{resp. } (i\sigma + B)U = F. \tag{6.4.27}$$

We recall that in the case of an inverse square potential, we write $\mathfrak{P}_\sigma = Q_\sigma$, and in the case of a polygonal domain, we have $\mathfrak{P}_\sigma = \mathbf{T}_\sigma$.

Lemma 6.4.9. *Assume that there exists a function $\Theta(\sigma) > 0$ such that for all $v \in D(P)$ (resp. $D(\mathbf{T})$) and $h \in L^2(\Omega)$ satisfiying*

$$\mathfrak{P}_\sigma v = h, \tag{6.4.28}$$

we have the estimate

$$||v||^2_{\mathcal{H}_\mu(\Omega)} \leq \Theta(\sigma)\left(||v||^2_{L^2(\omega)} + ||h||^2_{L^2(\Omega)}\right). \tag{6.4.29}$$

Then there exist $C > 0$ and $\sigma_0 > 0$ such that

$$||U||_{\mathcal{H}_\mu \times L^2(\Omega)} \leq C\Theta(\sigma)||F||_{\mathcal{H}_\mu \times L^2(\Omega)}, \text{ resp. } ||U||_{\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)} \leq C\Theta(\sigma)||F||_{\mathcal{H}^1_{\mathsf{D}} \times L^2(\Omega)}, \quad |\sigma| \geq \sigma_0, \tag{6.4.30}$$

for all $U = (u, v) \in D(A)$ (resp. $D(B)$) and $F = (f, h) \in \mathcal{H}_\mu \times L^2(\Omega)$ (resp. $\mathcal{H}^1_{\mathsf{D}} \times L^2(\omega)$), satisfying

$$(i\sigma + A)U = F \quad \text{resp. } (i\sigma + B)U = F. \tag{6.4.31}$$

Proof.
The analysis for the case of a polygonal domain is classical, as we deal with usual H^1 spaces. It follows the following proof, that we perform in the case with an inverse square potential. First note that if (6.4.29) holds, then the following semi-classical estimate holds

$$\sigma^2||v||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right) \leq \Theta(\sigma)\left(||v||^2_{L^2(\omega)} + ||h||^2_{L^2(\Omega)}\right). \tag{6.4.32}$$

Indeed, multiplying (6.4.28) by $\bar{v}$ and integrating by parts yields

$$\int_\Omega \left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right) - \sigma^2||v||^2_{L^2(\Omega)} + i\sigma||a^{1/2}v||^2_{L^2(\Omega)} = \int_\Omega h\bar{v}. \tag{6.4.33}$$

In particular, taking $|\sigma|$ sufficiently large, we obtain

$$\begin{aligned}\sigma^2||v||^2_{L^2(\Omega)} &\lesssim \int_\Omega \left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right) + (1+\sigma)||v||^2_{L^2(\Omega)} + ||h||^2_{L^2(\Omega)} \\ &\lesssim \int_\Omega \left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right) + ||h||^2_{L^2(\Omega)}.\end{aligned} \tag{6.4.34}$$

Combining (6.4.34) and (6.4.29), yields (6.4.32). Moreover, for all $w \in D(Q)$, $h \in L^2(\Omega)$ such that

$$-\Delta w - \frac{\mu}{|x|^2}w - \sigma^2 + ia(x)\sigma w = h,$$

we also have, using (**??**),

$$\sigma^2||w||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla w|^2 - \mu\frac{|w|^2}{|x|^2}\right) \leq \Theta(\sigma)\left(||w||^2_{L^2(\omega)} + ||h||^2_{L^2(\Omega)}\right). \tag{6.4.35}$$

by taking σ_0 sufficiently large, as $ia(x)\sigma$ can be considered as a low order term. Let $U = (u_0, u_1) \in D(A)$ and $F = (f_0, f_1)$ such that $(A + i\sigma\,\mathrm{Id})U = F$. This is equivalent to the following system

$$\begin{cases} i\sigma u_0 - u_1 = f_0 & \text{in } \Omega, \\ -\Delta u_0 - \frac{\mu}{|x|^2} u_0 + (a(x) + i\sigma)u_1 = f_1 & \text{in } \Omega, \\ u_{0|_{\partial\Omega}} = 0 & \text{at } \partial\Omega \end{cases}$$

which reads

$$\begin{cases} u_1 = i\sigma u_0 - f_0 & \text{in } \Omega, \\ -\Delta u_0 - \frac{\mu}{|x|^2} u_0 - \sigma^2 u_0 + i\sigma a(x) u_0 = g := f_1 - (a(x) + i\sigma) f_0 & \text{in } \Omega, \\ u_{0|_{\partial\Omega}} = 0 & \text{at } \partial\Omega. \end{cases} \tag{6.4.36}$$

Multiplying the second line of (6.4.36) by $\overline{u_0}$ and integrating by parts over Ω yields

$$\int_\Omega \left(|\nabla u_0|^2 - \mu \frac{|u_0|^2}{|x|^2} \right) - \sigma^2 \int_\Omega |u_0|^2 + i\sigma \int_\Omega a(x)|u_0|^2 = \int_\Omega g\overline{u_0}. \tag{6.4.37}$$

Taking the imaginary part, and using $a \geq \alpha > 0$ on ω, we have

$$\sigma\alpha \int_\omega |u_0|^2 \leq \mathrm{Im} \int_\Omega g\overline{u_0}.$$

Applying (6.4.35) to u_0, we obtain

$$\begin{aligned} \sigma^2 ||u_0||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla u_0|^2 - \mu \frac{|u_0|^2}{|x|^2} \right) &\leq \Theta(\sigma) \left(||u_0||^2_{L^2(\omega)} + ||g||^2_{L^2(\Omega)} \right) \\ &\lesssim \Theta(\sigma) \left(\frac{1}{\sigma} \mathrm{Im} \int_\Omega g\overline{u_0} + ||g||^2_{L^2(\Omega)} \right). \end{aligned}$$

Setting $\tilde{\Theta}(\sigma) = 2\frac{\Theta(\sigma)}{\sigma}$, we obtain

$$\begin{aligned} \sigma^2 ||u_0||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla u_0|^2 - \mu \frac{|u_0|^2}{|x|^2} \right) &\lesssim \Theta(\sigma) \left(\frac{1}{\sigma} \left(\tilde{\Theta}(\sigma) ||g||^2_{L^2(\Omega)} + \tilde{\Theta}(\sigma)^{-1} ||u_0||^2_{L^2(\Omega)} \right) + ||g||^2_{L^2(\Omega)} \right) \\ &\lesssim \frac{\Theta(\sigma)^2}{\sigma^2} ||g||^2_{L^2(\Omega)} \\ &\lesssim \Theta(\sigma)^2 \left(\frac{1}{\sigma^2} ||f_1||^2_{L^2(\Omega)} + ||f_0||^2_{L^2(\Omega)} \right). \end{aligned} \tag{6.4.38}$$

Combining (6.4.38) with (6.4.36) proves (6.4.30). □

Proof of Theorem 6.1.15. It is sufficient to combine Theorem 6.4.8 with Lemma 6.4.9. □

Perspectives

As we mention out in the previous chapter, the classical method consisting by adding an additional variable to the elliptic operator may bring some complications, especially when singular configurations are considered. Indeed, the weight function φ we considered brought the proper properties to handle the singular potentials, but when asking convexity of level sets of the weight with respect to the s-axis, our choice change the usual constants in the spectral inequalities.
However, there exists some methods to prove spectral inequalities, without the additional variable. We present in this section the existing results. Let Ω be a smooth bounded domain, and $-\Delta$ be the Dirichlet Laplace operator. We shall denote by (λ_j, ϕ_j) the Hilbert basis of $L^2(\Omega)$ composed by eigenfunctions of $-\Delta$. First, we provide the following theorem, proved in [AEWZ14, PWX17].

Theorem 7.0.10. *Let ω be a non-empty open subset of Ω. Then the following assertions are equivalent, and true:*

1. *there exists $C > 0$ such that for all $\Lambda \geq 0$, and for all sequence $(a_j) \in \ell^2(\mathbb{N})$, we have*
$$\sum_{\lambda_j < \Lambda} |a_j|^2 \leq C e^{C(1+\sqrt{\Lambda})} \int_\omega \Big| \sum_{\lambda_j < \Lambda} a_j \phi_j \Big|^2.$$
2. *there exists $C > 0$ and $\theta \in (0,1)$, such that for all $t > 0$ and for all $u \in L^2(\Omega)$ we have*
$$||e^{t\Delta} u||_{L^2(\Omega)} \leq e^{C(1+\frac{1}{t\theta})} ||u||^\theta_{L^2(\Omega)} ||e^{t\Delta} u||^{1-\theta}_{L^2(\omega)}. \tag{7.0.39}$$
3. *there exists $C > 0$ such that for all $\theta \in (0,1)$, for all $t > 0$ and for all $u \in L^2(\Omega)$ we have*
$$||e^{t\Delta} u||_{L^2(\Omega)} \leq e^{C(1+\frac{1}{t\theta})} ||u||^\theta_{L^2(\Omega)} ||e^{t\Delta} u||^{1-\theta}_{L^2(\omega)}. \tag{7.0.40}$$

In particular, we see that, if we are able to obtain localized one-point in time interpolation inequalities such as (7.0.39), then the classical spectral inequality holds. Proving (7.0.39) for singular configurations is the strategy we would like to follow. Let us explain in smooth configuration the ideas. In the next section, we explain how to obtain one-point in time estimates, following [AN63, Pay75, Isa17, Ves09].

7.1 Introduction and classical log-convexity for caloric functions

Let Ω be a bounded smooth domain of $\mathbb{R}^d$, $d \geq 1$, and we consider the classical heat equation.

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T) \times \Omega \\ u_{|\partial\Omega} = 0 & \text{in } (0,T) \times \partial\Omega \\ u_{|t=0} = u_0. \end{cases} \tag{7.1.1}$$

We define the functional

$$y(t) = ||u(t)||^2_{L^2(\Omega)}. \tag{7.1.2}$$

We have

$$y' = 2\Big(\partial_t u, u\Big)_{L^2(\Omega)} = 2\Big(\Delta u, u\Big)_{L^2(\Omega)} = -2||\nabla u||^2_{L^2(\Omega)}.$$

Multiplying (7.1.1) by Δu, we obtain

$$\frac{d}{2dt} ||\nabla u||^2_{L^2(\Omega)} = -||\Delta u||^2_{L^2(\Omega)},$$

and then $y'' = 4||\Delta u||^2_{L^2(\Omega)}$. As a result, we have

$$\begin{aligned} \log(y)'' = \frac{y'' y - y'^2}{y^2} &= \frac{4}{y^2} \left(||\Delta u||^2_{L^2(\Omega)} ||u||^2_{L^2(\Omega)} - ||\nabla u||^4_{L^2(\Omega)} \right) \\ &\geq 0, \end{aligned}$$

using the Cauchy-Schwarz inequality. As a result, $t \mapsto y(t)$ is a log-convex function, and then, for all $0 \leq t_1 < t_2 < t_3 \leq T$, we have

$$y(t_2) \leq y(t_1)^{1-\lambda} y(t_3)^{\lambda}, \tag{7.1.3}$$

with $\lambda = \frac{t_2 - t_1}{t_3 - t_1}$. The purpose of what follows is to mimmick what we have done above, but with a weighted version of (7.1.2), in order to obtain a localized version of (7.1.3), on a non-empty open subset ω of Ω (compare (7.1.3) with (7.0.39)).

7.2 Weighted log-convexity

We refer to [EKPV08a, EKPV16, EKPV08b, BP17] for expositions of the method presented in what follows. We consider weight functions of the form

$$\varphi(x,t) = \frac{\phi(x)}{h(t)}, \quad t \in (0,T),\ T > 0,$$

with $h(t) = T - t + \tilde{h}$, $\tilde{h} > 0$ and with ϕ being a smooth function. Note that working for t close to T and $\tilde{h}$ small, the function of time $\frac{1}{h(t)}$ may be considered as a semi-classical large parameter (depending in time). Remark also that $\partial_t h(t) = -1$ and then

$$\partial_t \frac{1}{h(t)^m} = \frac{m}{h(t)^{m+1}}, \quad m \in \mathbb{N}. \tag{7.2.1}$$

As a result, derivatives in times increase the order of the operators. We introduce the following quantity

$$y(t) = \int_\Omega |u|^2 e^{\varphi},$$

We set

$$f(t,x) = u(t,x) e^{\varphi(t,x)}, \quad \Delta_\varphi := e^{\varphi} \Delta e^{-\varphi} = \Delta + \frac{|\nabla \phi|^2}{h(t)^2} - 2\frac{1}{h(t)} \nabla\phi.\nabla - \frac{1}{h(t)} \Delta\phi.$$

With this notation, the conjugated heat operator reads

$$e^{\varphi}(\partial_t - \Delta) e^{-\varphi} = \partial_t - \partial_t\varphi - \Delta_\varphi = \partial_t - \mathcal{S} - \mathcal{A},$$

with

$$\mathcal{A} = -2\frac{1}{h(t)} \nabla\varphi.\nabla - \frac{1}{h(t)} \Delta\varphi, \quad \mathcal{S} = \Delta + \frac{|\nabla\varphi|^2}{h(t)^2} + \frac{\phi}{h(t)^2}.$$

The reader should note that the operators $\mathcal{A}$ and $\mathcal{S}$ are almost the same as P_1 and P_2 in Section 2.1.

7.3 Commutator method for log-convexity with boundary

We shall denote by $\Big(.,.\Big)$ the standard scalar product on $L^2(\Omega)$, and by $||.||$ the associated norm. Note that we have the following formal integration by part rules

$$\Big(\mathcal{A}f, g\Big) = -\Big(\mathcal{A}g, f\Big) - \frac{2}{h(t)}\Big(f, \partial_\nu\phi g\Big)_\partial, \quad \Big(\mathcal{S}f, g\Big) = \Big(\mathcal{S}g, f\Big) + \Big(\partial_\nu f, g\Big)_\partial - \Big(f, \partial_\nu g\Big)_\partial, \tag{7.3.1}$$

where $\Big(.,.\Big)_\partial$ denote that standard product on $L^2(\partial\Omega)$ at the boundary. Note that in the previous identities, we did not considered the low-order terms coming from the commutators. We deduce

$$\Big(\mathcal{A}f, f\Big) = -\frac{1}{h(t)}\Big(\partial_\nu\phi f, f\Big)_\partial. \tag{7.3.2}$$

Yet, y satisfies the ordinary differential equation

$$\begin{aligned}\frac{1}{2}y' = \Big(f', f\Big) &= \Big(f' - \mathcal{A}f, f\Big) - \frac{1}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \\ &= \Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big) + \Big(\mathcal{S}f, f\Big) - \frac{1}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \\ &= -\mathcal{N}y + \Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big) - \frac{1}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial,\end{aligned}$$

where

$$\mathcal{N} = -\frac{\Big(\mathcal{S}f, f\Big)}{y}. \tag{7.3.3}$$

We want to derive an ordinary differential equation satisfied by $\mathcal{N}$. We have

$$y' = \Big(f' + \mathcal{S}f - \mathcal{A}f, f\Big) + \Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big) - \frac{4}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial,$$

and also

$$\Big(\mathcal{S}f, f\Big) = \frac{1}{2}\Big(f' + \mathcal{S}f - \mathcal{A}f\Big) - \frac{1}{2}\Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big),$$

which yields

$$y'\Big(\mathcal{S}f, f\Big) = \frac{1}{2}\Big(f' + \mathcal{S}f - \mathcal{A}f, f\Big)^2 - \frac{1}{2}\Big(f' + \mathcal{S}f + \mathcal{A}f, f\Big)^2 - \frac{4}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \Big(\mathcal{S}f, f\Big).$$

From

$$\frac{d}{dt}\Big(\mathcal{S}f, f\Big) = \Big(\mathcal{S}'f, f\Big) + 2\Big(f', \mathcal{S}f\Big) + \Big(\partial_\nu f', f\Big)_\partial - \Big(f', \partial_\nu f\Big)_\partial,$$

and

$$\Big([\mathcal{S}, \mathcal{A}]f, f\Big) = 2\Big(\mathcal{A}f, \mathcal{S}f\Big) - \Big(\partial_\nu f, \mathcal{A}f\Big)_\partial + \Big(\partial_\nu \mathcal{A}f, f\Big)_\partial + \frac{2}{h(t)}\Big(\partial_\nu \phi \mathcal{S}f, f\Big)_\partial,$$

we obtain

$$\frac{d}{dt}\Big(\mathcal{S}f, f\Big)y = \Big(\mathcal{S}'f + [\mathcal{S}, \mathcal{A}]f, f\Big)y + 2\Big(f' - \mathcal{A}f, \mathcal{S}f\Big)y + \tilde{\mathcal{B}}(f)y,$$

where

$$\tilde{\mathcal{B}}(f) = \Big(\partial_\nu f, \mathcal{A}f\Big)_\partial + \Big(\partial_\nu f', f\Big)_\partial - \Big(f', \partial_\nu f\Big)_\partial - \Big(\partial_\nu \mathcal{A}f, f\Big)_\partial - \frac{2}{h(t)}\Big(\partial_\nu \phi \mathcal{S}f, f\Big)_\partial.$$

We finally obtain

$$\frac{d}{dt}\mathcal{N} = \frac{y'\Big(\mathcal{S}f, f\Big) - y\frac{d}{dt}\Big(\mathcal{S}f, f\Big)}{y^2} \tag{7.3.4}$$

$$= \frac{1}{y^2}\Big(\frac{1}{2}\Big(f' + \mathcal{S}f - \mathcal{A}f, f\Big)^2 - \frac{1}{2}\Big(f' + \mathcal{S}f + \mathcal{A}f, f\Big)^2 - \frac{2}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \Big(\mathcal{S}f, f\Big) \tag{7.3.5}$$

$$- \frac{1}{y}\Big(\Big(\mathcal{S}'f + [\mathcal{S}, \mathcal{A}]f, f\Big) + 2\Big(f' - \mathcal{A}f, \mathcal{S}f\Big) + \tilde{\mathcal{B}}(f)\Big) \tag{7.3.6}$$

$$\leq -\frac{1}{y}\Big(\Big(\mathcal{S}'f + [\mathcal{S}, \mathcal{A}]f, f\Big) - ||f' - \mathcal{S}f - \mathcal{A}f||^2\Big) - \frac{\tilde{\mathcal{B}}(f)}{y} - \frac{4}{y^2 h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \Big(\mathcal{S}f, f\Big), \tag{7.3.7}$$

using the polarization formula. Summing up, we have the following system

$$\begin{cases} \frac{1}{2}y'(t) = -\mathcal{N}y + \Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big) - \frac{2}{h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \\ \mathcal{N}'(t) \leq -\frac{1}{y}\Big(\Big(\mathcal{S}'f + [\mathcal{S}, \mathcal{A}]f, f\Big) - ||f' - \mathcal{S}f - \mathcal{A}f||^2\Big) - \frac{\tilde{\mathcal{B}}(f)}{y} - \frac{4}{y^2 h(t)}\Big(\partial_\nu \phi f, f\Big)_\partial \Big(\mathcal{S}f, f\Big). \end{cases} \tag{7.3.8}$$

We first see that, if u is chosen so that it satisfies the heat equation (7.1.1), and the weight ϕ is chosen so that $\partial_\nu \phi_{|\partial\Omega} \leq 0$, then the boundary conditions yields the following system of ordinary differential equations

$$\begin{cases} \frac{1}{2}y'(t) = -\mathcal{N}y \\ \mathcal{N}'(t) \leq -\frac{1}{y}\Big(\mathcal{S}'f + [\mathcal{S},\mathcal{A}]f, f\Big). \end{cases}$$

We have the following lemma, that makes the connection between System (7.3.8) and the log-convexity property. In particular, we see that, if there exists some γ such that

$$-\frac{1}{y}\Big(\mathcal{S}'f + [\mathcal{S},\mathcal{A}]f, f\Big) \leq -\frac{1}{h(t)^\gamma}\Big(\mathcal{S}f, f\Big), \tag{7.3.9}$$

then the log-convexity property holds, as it is stated in the following lemma.

Lemma 7.3.1. *Assume that*

$$\begin{cases} \frac{1}{2}y'(t) = -\mathcal{N}y(t) \\ \mathcal{N}'(t) \leq \frac{1}{h(t)^\gamma}\mathcal{N}(t), \end{cases} \quad 0 < t < T, \tag{7.3.10}$$

holds. Then, for all $0 \leq t_1 < t_2 < t_3 \leq T$, we have

$$y(t_2) \leq y(t_3)^{\frac{1}{1+\mu_\gamma}} y(t_1)^{1-\frac{1}{1+\mu_\gamma}}, \tag{7.3.11}$$

where μ_γ is given by $\mu_\gamma = \frac{\int_{t_2}^{t_3} \frac{1}{h(t)^\gamma}}{\int_{t_1}^{t_2} \frac{1}{h(t)^\gamma}}$.

Remark 7.3.2. *From Lemma 7.3.1, we see that finding a weight function ϕ such that (7.3.9) holds is a sufficient condition to obtain the log-convexity property. However, some additional conditions have to be made to obtain a localized version of (7.3.11) in a subset $\omega \subset \Omega$, making use of the weight function (see Proposition 7.3.4 below).*

Remark 7.3.3. *We also point out that System (7.3.10) is precisely equivalent to*

$$\frac{d}{dt}\left(h(t)^\gamma \frac{d}{dt}\log(y(t))\right) \geq 0, \tag{7.3.12}$$

see the beginning of the proof below. We then point out the similarity with the non-weighted case of Section 7.1.

Proof.

We first claim that System (7.3.10) yields

$$\frac{d}{dt}\left(h(t)^\gamma \frac{d}{dt}\log(y(t))\right) \geq 0. \tag{7.3.13}$$

Indeed,

$$\frac{d}{dt}\left(h(t)^\gamma \frac{d}{dt}\log(y(t))\right) = \frac{d}{dt}\left(h(t)^\gamma \frac{y'(t)}{y(t)}\right) = \frac{d}{dt}\left(-2h(t)^\gamma \mathcal{N}(t)\right) = 2\gamma h(t)^{\gamma-1}\mathcal{N}(t) - 2h(t)^\gamma \mathcal{N}'(t) \geq 0,$$

using the second line of System (7.3.10). We see that (7.3.13) is a weak form of log-convexity. Let $0 < t_1 < t_2 < t_3$. We first integrate (7.3.13) between t and t_2, for $t \in (t_1, t_2)$ to obtain

$$-\frac{d}{dt}\log(y(t)) \geq 2\frac{h(t_2)^\gamma}{h(t)^\gamma}\mathcal{N}(t_2).$$

Now integrating between t_1 and t_2 yields

$$\log(y(t_1)) - \log(y(t_2)) \geq 2 \int_{t_1}^{t_2} \frac{h(t_2)^\gamma}{h(t)^\gamma} \mathcal{N}(t_2). \tag{7.3.14}$$

Second, we integrate (7.3.13) between t_2 and t, for $t \in (t_2, t_3)$. We obtain

$$\frac{d}{dt} \log(y(t)) \geq -2 \frac{h(t_2)^\gamma}{h(t)^\gamma} \mathcal{N}(t_2).$$

Now integrating between t_2 and t_3 yields

$$\log(y(t_2)) - \log(y(t_3)) \leq 2 \int_{t_2}^{t_3} \frac{h(t_2)^\gamma}{h(t)^\gamma} \mathcal{N}(t_2). \tag{7.3.15}$$

Combining estimates (7.3.14) and (7.3.15), we obtain

$$\left(\int_{t_1}^{t_2} \frac{1}{h(t)^\gamma} \right)^{-1} (\log(y(t_1)) - \log(y(t_2))) \geq \left(\int_{t_2}^{t_3} \frac{1}{h(t)^\gamma} \right)^{-1} (\log(y(t_2)) - \log(y(t_3))). \tag{7.3.16}$$

As a result, we deduce the following interpolation inequality in one-point in time

$$y(t_2) \leq y(t_3) \left(\frac{y(t_1)}{y(t_2)} \right)^{\mu_\gamma}, \quad \mu_\gamma = \frac{\int_{t_2}^{t^3} h(t)^{-\gamma} dt}{\int_{t_1}^{t_3} h(t)^{-\gamma} dt}, \tag{7.3.17}$$

which yields the result. □

Now from (7.3.11), with additional conditions on φ (mainly $\varphi \leq 0$), we explain how to obtain a localized version.

Proposition 7.3.4. *Assume that $\varphi \leq 0$ and assume that φ reaches its maximum in the interior of Ω. We further assume that $\max \phi = 0$. Let ω be a neighborhood of $\{x \in \Omega,\ \phi = 0\}$. A that*

$$y(t_2) \leq y(t_3)^{\frac{1}{1+\mu_\gamma}} y(t_1)^{1-\frac{1}{1+\mu_\gamma}}, \tag{7.3.18}$$

holds with μ_γ given by $\mu_\gamma = \frac{\int_{t_2}^{t_3} \frac{1}{h(t)^\gamma}}{\int_{t_1}^{t_2} \frac{1}{h(t)^\gamma}}$, with $\gamma \in [0, 2)$. Then, there exists $\nu \in (0, 1)$ such that for all $T > 0$, there exists some constant $C > 0$

$$\int_\Omega |u(T)|^2 \leq C \left(e^{\frac{C}{T}} \int_\Omega |u(0)|^2 \right)^{1-\nu} \left(\int_\omega |u(T)|^2 \right)^\nu, \tag{7.3.19}$$

Proof. First note that (7.3.18) is equivalent to

$$y(t_2)^{1+\mu_\gamma} \leq y(t_3) y(t_1)^{\mu_\gamma}. \tag{7.3.20}$$

Let $\ell, \tilde{h} > 0$ be such that $\tilde{\ell} := \ell \tilde{h} < T/2$. We first choose $t_3 = T$, $t_2 = T - \tilde{\ell}$ and $t_1 = T - 2\tilde{\ell}$. Thus, the left hand side of (7.3.20) reads

$$y(t_2) \geq e^{2\frac{\inf_\Omega \phi}{(\ell+1)\tilde{h}}} ||u(T - \tilde{\ell})||^2.$$

Concerning the right hand side, we have,

$$\begin{aligned} y(t_3) &= \int_{\Omega\setminus\omega} e^{2\frac{\phi}{\tilde{h}}} |u(T)|^2 + \int_\omega e^{2\frac{\phi}{\tilde{h}}} |u(T)|^2 \leq e^{2\frac{\max_{\Omega\setminus\omega} \phi}{\tilde{h}}} \int_{\Omega\setminus\omega} |u(T)|^2 + e^{2\frac{\max_\omega \phi}{\tilde{h}}} \int_\omega |u(T)|^2 \\ &\leq e^{2\frac{\max_{\Omega\setminus\omega} \phi}{\tilde{h}}} \int_{\Omega\setminus\omega} |u(T)|^2 + \int_\omega |u(T)|^2, \end{aligned}$$

and

$$y(t_1) \leq e^{2\frac{\max_\Omega \phi}{\tilde{h}}} \int_\Omega |u(T-2\tilde{\ell})|^2 \leq \int_\Omega |u(T-2\tilde{\ell})|^2,$$

using the assumptions on ϕ. We also estimate μ_γ. For $\gamma \neq 1$, we have

$$\int_{t_1}^{t_2} h(t)^{-\gamma} = \frac{1}{1-\gamma}(h(t_2)^{1-\gamma} - h(t_1)^{1-\gamma}) = \frac{\tilde{h}}{1-\gamma}((\ell+1)^{1-\gamma} - (\ell+2)^{1-\gamma}), \tag{7.3.21}$$

and for $\gamma = 1$, $\int_{t_1}^{t_2} h(t)^{-\gamma} = \log\left(\frac{\ell+1}{\ell+2}\right)$. By the same computations, we finally obtain

$$\mu_\gamma = \frac{1-(\ell+1)^{1-\gamma}}{(\ell+1)^{1-\gamma} - (2\ell+1)^{1-\gamma}}, \text{ if } \gamma \neq 1, \quad \text{and } \mu_\gamma = \frac{\log(\ell+1)}{\log\left(\frac{2\ell+1}{\ell+1}\right)}, \text{ if } \gamma \neq 1.$$

Summing up, with (7.3.20), we have

$$||u(T-\tilde{\ell})||^{2(1+\mu_\gamma)}_{L^2(\Omega)} \leq e^{-2(1+\mu_\gamma)\frac{\inf_\Omega \phi}{(\ell+1)\tilde{h}} + 2\frac{\max_{\Omega\setminus\omega}\phi}{\tilde{h}}} \int_{\Omega\setminus\omega} |u(T)|^2 \left(\int_\Omega |u(T-2\tilde{\ell})|^2\right)^{\mu_\gamma}$$
$$+ e^{-2(1+\mu_\gamma)\frac{\inf_\Omega \phi}{(\ell+1)\tilde{h}}} \int_\omega |u(T)|^2 \left(\int_\Omega |u(T-2\tilde{\ell})|^2\right)^{\mu_\gamma}.$$

As the weight ϕ is non-positive, we see that we can always find a ℓ such that $-2(1+\mu_\gamma)\frac{\inf_\Omega \phi}{(\ell+1)} + 2\max_{\Omega\setminus\omega}\phi < 0$, and independantly on the value of ℓ we have $-2(1+\mu_\gamma)\frac{\inf_\Omega \phi}{(\ell+1)} > 0$. Then optimizing this inequality with respect to $\tilde{h}$, and using the natural decay of the L^2 norm of solutions of the heat equation yields the result. □

7.4 Application to the Crocco equation

7.4.1 Settings

In this section, we try to apply the method presented above to a degenerating operator at the boundary, we explain what does work for now and what does not, and what remains to do. We consider the degenerate operator

$$\mathcal{P} = \partial_x x^\alpha \partial_x,$$

and we want to prove log convexity properties for solutions to the following degenerating equation

$$\begin{cases} \partial_t u - \mathcal{P}u = 0 & \text{in } \mathbb{R}^+ \times (0,1), \\ \begin{cases} u(t,0) = 0, & \text{if } \alpha \in [0,1), \\ (x^\alpha \partial_x u(t,x))_{|_{x=0}} = 0, & \text{if } \alpha \in [1,2) \end{cases} & \text{on } \mathbb{R}^+ \\ u(t,1) = 0 & \text{on } \mathbb{R}^+, \\ u(0,x) = u_0(x) \in L^2(0,1) & \text{in } (0,1), \end{cases} \tag{7.4.1}$$

with a localized term $\omega \subset [1/2,1]$, in the spirit of (7.0.39). Following [CMV08], for $\alpha \in [0,1)$, we introduce the Hilbert space

$$H^1_\alpha := \{u \in L^2(0,1),\ u \text{ is absolutely continuous in } [0,1],\ x^{\frac{\alpha}{2}}\partial_x u \in L^2(0,1),\ \text{and } u(0) = u(1) = 0\},$$

and for $\alpha \in [1,2)$, we define

$$H^1_\alpha := \{u \in L^2(0,1),\ u \text{ is absolutely continuous in } [0,1],\ x^{\frac{\alpha}{2}}\partial_x u \in L^2(0,1),\ \text{and } u(1) = 0\}.$$

The null-controllability of (7.4.1) (and observability) has been already proved in [CMV08], using a proper parabolic Carleman estimate. However, this method does not provide estimations such as spectral inequality (or equivalently one-point in time interpolation inequality, from Theorem 7.0.10).
As it has been observed in Part III, for singular operators, considering an augmented operator $\partial_s^2 + \mathcal{P}$ bring some complications to prove elliptic Carleman estimates. We want to apply the log-convexity approach to avoid these difficulties.
Following the previous section, we set

$$f(t,x) = u(t,x)e^{\varphi(t,x)}.$$

where ϕ is a smooth weight function, and where u is a solution of System (7.4.1). We introduce the following quantity

$$y(t) = \int_0^1 |f(t)|^2 dx. \tag{7.4.2}$$

We shall take

$$\varphi(x,t) = \frac{\phi(x)}{h(t)^\theta}, \quad \text{with } \phi(x) = \frac{x^{2-\alpha}-1}{a}, \tag{7.4.3}$$

where $a > 0$ is a parameter, and for $0 < t < T$, $h(t) = T - t + \tilde{h}$ with $\tilde{h} > 0$. Note that with this choice, $\varphi \leq 0$ on $(0,1)$ and $\sup_{(0,1)} \varphi = 0$. The function f satisfies the following equation.

$$\begin{cases} \mathcal{Q}_\phi f = 0 & \text{in } (0,T)\times(0,1), \\ \begin{cases} f(t,0) = 0, & \text{if } \alpha \in [0,1), \\ (x^\alpha \partial_x f(t,x))_{|_{x=0^+}} = (x^\alpha \partial_x \varphi f(t,x))_{|_{x=0^+}} & \text{if } \alpha \in [1,2) \end{cases} & \text{on } (0,T) \\ f(t,1) = 0 & \text{on } (0,T), \\ f(0,x) = f_0(x) \in L^2(0,1) & \text{in } (0,1). \end{cases} \tag{7.4.4}$$

where the conjugated operators $\mathcal{Q}_\phi$ is given by

$$\begin{aligned} \mathcal{Q}_\phi &= (\partial_t - \theta\frac{\phi}{h^{\theta+1}}) - (\partial_x - \frac{\partial_x\phi}{h^\theta})x^\alpha(\partial_x - \frac{\partial_x\phi}{h^\theta}) \\ &= \partial_t - \mathcal{P} + \frac{2-\alpha}{ah^\theta} + 2\frac{2-\alpha}{ah^\theta}x\partial_x - \frac{1}{a^2h^{2\theta}}(2-\alpha)^2x^{2-\alpha} - \theta\frac{x^{2-\alpha}-1}{ah^{\theta+1}}. \end{aligned}$$

We shall write $\mathcal{Q}_\phi = \partial_t - \mathcal{S} - \mathcal{A}$, with

$$\mathcal{S} = \mathcal{P} + \frac{1}{a^2h^{2\theta}}(2-\alpha)^2x^{2-\alpha} + \theta\frac{x^{2-\alpha}-1}{ah^{\theta+1}}, \quad \mathcal{A} = -\frac{2-\alpha}{ah^\theta} - 2\frac{2-\alpha}{ah^\theta}x\partial_x.$$

We set

$$\mathcal{N} = -\frac{\Big(\mathcal{S}f, f\Big)}{y},$$

where $\Big(.,.\Big)$ denotes the inner product on $L^2(0,1)$. We have the following integration by part rules

$$\begin{aligned} \Big(\mathcal{A}f, g\Big) &= -\Big(\mathcal{A}g, f\Big) - \frac{2(2-\alpha)}{ah^\theta}\left[xfg\right]_{x=0}^{x=1}, \\ \Big(\mathcal{S}f, g\Big) &= \Big(\mathcal{S}g, f\Big) + \left[x^\alpha\partial_x fg\right]_{x=0}^{x=1} - \left[x^\alpha f\partial_x g\right]_{x=0}^{x=1}. \end{aligned}$$

We shall denote the boundary terms by

$$\mathcal{B}_\mathcal{S}(f,g) = -\left[x^\alpha\partial_x fg\right]_{x=0}^{x=1} - \left[x^\alpha f\partial_x g\right]_{x=0}^{x=1}, \quad \mathcal{B}_\mathcal{A}(f,g) = -\left[xfg\right]_{x=0}^{x=1}.$$

Since $\mathcal{A}$ is skew-symmetric, we have the relation, for all smooth function f,

$$\Big(\mathcal{A}f, f\Big) = \frac{(2-\alpha)}{ah^\theta}\mathcal{B}_\mathcal{A}(f,f). \tag{7.4.5}$$

To handle boundary conditions, we shall need the following lemma.

Lemma 7.4.1. *Let $\alpha \in [1,2)$. Then, for all $w \in H^1_\alpha(0,1)$, we have*

$$\lim_{x\to 0^+} x^{1/2} w(x) = 0, \tag{7.4.6}$$

The proof can be found in [CMV08]. As a result, for f solution of (7.4.4), we have $\mathcal{B}_{\mathcal{A}}(f,f) = 0$.

7.4.2 Derivation of a system of ordinary differential equations

Lemma 7.4.2. *For f solution of problem (7.4.4), we have the following ordinary differential equations*

$$\begin{cases} \frac{1}{2}y'(t) = -\mathcal{N}y(t), \\ \frac{d}{dt}\mathcal{N} \leq -\frac{1}{y}\left(\left((\mathcal{S}' + [\mathcal{S},\mathcal{A}])f, f\right) + \mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f) + \frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f) + \mathcal{B}_{\mathcal{S}}(f',f)\right). \end{cases} \tag{7.4.7}$$

Proof.
We have, for f solution of (7.4.4),

$$\begin{aligned} \frac{1}{2}y' = \Big(f', f\Big) &= \Big(f' - \mathcal{A}f, f\Big) + \frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,f) \\ &= \Big(f' - \mathcal{S}f - \mathcal{A}f, f\Big) + \Big(\mathcal{S}f, f\Big) + \frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,f) \\ &= -\mathcal{N}y. \end{aligned}$$

We want to derive an ordinary differential equation satisfied by $\mathcal{N}$. We have on the one hand,

$$y'\Big(\mathcal{S}f, f\Big) = 2\Big(\mathcal{S}f, f\Big)^2,$$

and on the other hand

$$\frac{d}{dt}\Big(\mathcal{S}f, f\Big) = \Big(\mathcal{S}'f, f\Big) + \Big(\mathcal{S}f', f\Big) + \Big(\mathcal{S}f, f'\Big) = \Big(\mathcal{S}'f, f\Big) + 2\Big(\mathcal{S}f, f'\Big) + \mathcal{B}_{\mathcal{S}}(f', f).$$

Using $f' = \mathcal{S}f + \mathcal{A}f$, we have

$$\frac{d}{dt}\Big(\mathcal{S}f, f\Big) = \Big(\mathcal{S}'f, f\Big) + 2||\mathcal{S}f||^2_{L^2(0,1)} + 2\Big(\mathcal{S}f, \mathcal{A}f\Big) + \mathcal{B}_{\mathcal{S}}(f', f).$$

Yet,

$$2\Big(\mathcal{A}f, \mathcal{S}f\Big) = \Big(\mathcal{S}f, \mathcal{A}f\Big) + \Big(\mathcal{A}f, \mathcal{S}f\Big) = \Big(f, \mathcal{S}\mathcal{A}f\Big) - \Big(f, \mathcal{A}\mathcal{S}f\Big) + \mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f) + 2\frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f),$$

and we obtain

$$\frac{d}{dt}\Big(\mathcal{S}f, f\Big) = \Big((\mathcal{S}' + [\mathcal{S},\mathcal{A}])f, f\Big) + 2||\mathcal{S}f||^2_{L^2(0,1)} + \mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f) + 2\frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f) + \mathcal{B}_{\mathcal{S}}(f',f).$$

Summing up,

$$\begin{aligned} \frac{d}{dt}\mathcal{N} =& \frac{y'\Big(\mathcal{S}f, f\Big) - y\frac{d}{dt}\Big(\mathcal{S}f, f\Big)}{y^2} \\ =& \frac{2}{y^2}\Big(\mathcal{S}f, f\Big)^2 - \frac{1}{y}\left(\Big((\mathcal{S}' + [\mathcal{S},\mathcal{A}])f, f\Big) + 2||\mathcal{S}f||^2_{L^2(0,1)} + \mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f) + 2\frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f) + \mathcal{B}_{\mathcal{S}}(f',f)\right) \\ \leq& -\frac{1}{y}\left(\Big((\mathcal{S}' + [\mathcal{S},\mathcal{A}])f, f\Big) + \mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f) + 2\frac{(2-\alpha)}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f) + \mathcal{B}_{\mathcal{S}}(f',f)\right), \end{aligned} \tag{7.4.8}$$

using the Cauchy-Schwarz inequality. □
Computing the boundary terms, one can prove the following lemma.

Lemma 7.4.3. *For f solution of (7.4.4), with $\alpha \in (0,2)$, we have*

$$2\frac{2-\alpha}{ah^\theta}\mathcal{B}_{\mathcal{A}}(f,\mathcal{S}f)+\mathcal{B}_{\mathcal{S}}(f,\mathcal{A}f)+\mathcal{B}_{\mathcal{S}}(f',f)\geq -2\frac{2-\alpha}{ah^\theta}\left(x^{\alpha+1}|\partial_x f|^2\right)_{|x=1}. \tag{7.4.9}$$

Lemma 7.4.4. *Let $\mathcal{A}$ and $\mathcal{S}$ as above. We choose $\theta = 4$. Then we have*

$$\Big((-\mathcal{S}'-[\mathcal{S},\mathcal{A}])f,f\Big)<0.$$

Proof. We first compute the commutator. We have

$$[\mathcal{S},\mathcal{A}]=[\mathcal{P}+\frac{1}{a^2h^{2\theta}}(2-\alpha)^2x^{2-\alpha}+\theta\frac{x^{2-\alpha}-1}{ah^{\theta+1}},-2\frac{2-\alpha}{ah^\theta}x\partial_x], \tag{7.4.10}$$

since the term $-\frac{2-\alpha}{ah^\theta}$ commutes with any operator in the space variable. We begin with

$$\begin{aligned}
[\mathcal{P},-2\frac{2-\alpha}{ah^\theta}x\partial_x] &= -2\frac{2-\alpha}{ah^\theta}\left(\partial_x x^\alpha\partial_x x\partial_x - x\partial_x^2x^\alpha\partial_x\right)\\
&= -2\frac{2-\alpha}{ah^\theta}\left(x\partial_x x^{\alpha-1}\partial_x x\partial_x + x^{\alpha-1}\partial_x x\partial_x - x\partial_x^2x^\alpha\partial_x\right)\\
&= -2\frac{2-\alpha}{ah^\theta}\left(x\partial_x^2x^{\alpha-1}x\partial_x + x^{\alpha-1}\partial_x x\partial_x - (\alpha-1)x\partial_x x^{\alpha-1}\partial_x - x\partial_x^2x^\alpha\partial_x\right)\\
&= -2\frac{2-\alpha}{ah^\theta}\left(x^{\alpha-1}\partial_x x\partial_x - (\alpha-1)x\partial_x x^{\alpha-1}\partial_x\right)\\
&= -2\frac{2-\alpha}{ah^\theta}\left(\partial_x x^\alpha\partial_x + (1-\alpha)x^{\alpha-1}x^{-\alpha}x^\alpha\partial_x - (\alpha-1)\left(\partial_x x^\alpha\partial_x - x^{\alpha-1}\partial_x\right)\right)\\
&= -2\frac{(2-\alpha)^2}{ah^\theta}\partial_x x^\alpha\partial_x.
\end{aligned} \tag{7.4.11}$$

We also compute

$$[\frac{(2-\alpha)^2}{a^2h^{2\theta}}x^{2-\alpha},-2\frac{(2-\alpha)}{ah^\theta}x\partial_x]=-\frac{2(2-\alpha)^3}{a^3h^{3\theta}}\left(x^{3-\alpha}\partial_x - x\partial_x x^{2-\alpha}\right)=\frac{2}{a^3h^{3\theta}}(2-\alpha)^4x^{2-\alpha}, \tag{7.4.12}$$

and

$$[\frac{\theta}{ah^{\theta+1}}\left(x^{2-\alpha}-1\right),-2\frac{(2-\alpha)}{ah^\theta}x\partial_x]=-2\theta\frac{(2-\alpha)}{a^2h^{2\theta+1}}\left(x^{3-\alpha}\partial_x - x\partial_x x^{2-\alpha}\right)=2\theta\frac{(2-\alpha)^2}{a^2h^{2\theta+1}}x^{2-\alpha}. \tag{7.4.13}$$

On the other hand,

$$\mathcal{S}'=2\theta\frac{1}{a^2h^{2\theta+1}}(2-\alpha)^2x^{2-\alpha}+\theta(\theta+1)\frac{x^{2-\alpha}-1}{ah^{\theta+2}}. \tag{7.4.14}$$

We finally obtain, summing up (7.4.11)-(7.4.14),

$$\begin{aligned}
-\mathcal{S}'-[\mathcal{S},\mathcal{A}] = 2\frac{(2-\alpha)^2}{ah^\theta}\partial_x x^\alpha\partial_x - \left(2\frac{(2-\alpha)^4}{a^3h^{3\theta}}+2\theta\frac{(2-\alpha)^2}{a^2\theta^{2\theta+1}}+2\theta\frac{(2-\alpha)^2}{a^2h^{2\theta+1}}+\theta(\theta+1)\frac{1}{ah^{\theta+1}}\right)x^{2-\alpha}\\
+\theta(\theta+1)\frac{1}{ah^{\theta+2}}.
\end{aligned} \tag{7.4.15}$$

To handle the last term, we have, for all $\varepsilon>0$,

$$\frac{1}{h(t)^{\theta+2}}\int_\Omega|f|^2\leq\frac{\varepsilon}{h^\theta}\int_\Omega x^{\alpha-2}|\partial_x f|^2+\frac{1}{\varepsilon h^{2+\theta/2}}\int_\Omega x^{2-\alpha}|f|^2. \tag{7.4.16}$$

Make use of the Hardy inequalities found in [CMV08], choosing $\theta=4$, we can obtain that these two terms can be absorbed in (7.4.15) by taking ε sufficiently small, and a sufficiently small (again see [CMV08]). We finally obtain $\Big((-\mathcal{S}'-[\mathcal{S},\mathcal{A}])f,f\Big)<0$, which is the sought result. □

Using Lemma 7.4.2, Lemma 7.4.3, and Lemma 7.4.4, we proved the following proposition.

Proposition 7.4.5. *For f solution of problem (7.4.4), we have the following ordinary differential equations*

$$\begin{cases} \dfrac{1}{2}y'(t) = \mathcal{N}y(t), \\ \dfrac{d}{dt}\mathcal{N} \leq \dfrac{1}{y}\left(2\dfrac{2-\alpha}{ah^4}\left(|\partial_x f|^2\right)_{|_{x=1}}\right). \end{cases} \tag{7.4.17}$$

We see that in the second line of (7.4.17), we have a remainder term, localized at the boundary $\{x = 1\}$, that is; away form the singularities. We solve these equations in the same way as Lemma 7.3.1. We set $F(t) = \frac{1}{y(t)}\left(2\frac{2-\alpha}{ah^4}\left(x^{\alpha+1}|\partial_x f(t)|^2\right)_{|_{x=1}}\right)$. This yields, for all $0 \leq t_1 < t_2 < t_3$,

$$y(t_2)^{1+M} \leq y(t_3)y(t_1)^M e^G, \tag{7.4.18}$$

with $G = 8(t_3 - t_1)\int_{t_1}^{t_3} F(t)dt$, and $M = \frac{t_3-t_2}{t_2-t_1}$. At this step, we can not localize the integral $y(t_3)$ on a subset $\omega \subset [1/2, 1]$ by following the proof of Proposition 7.3.4. The main issue is, by taking $t_3 = T$, $t_2 = T - \ell\tilde{h}$ and $t_1 = T - 2\ell\tilde{h}$,

$$G \leq C\ell\tilde{h}\int_{T-2\ell\tilde{h}}^{T} \frac{1}{(T-t+\tilde{h})^4}\frac{(|\partial_x f(t)|^2)_{|_{x=1}}}{\int_\Omega |f(t)|^2}, \tag{7.4.19}$$

and we see that G depends heavily on $\tilde{h}$. Usually, (see [BP17], proof of Theorem 4.1), we use the weight e^φ in the relation $f = e^\varphi u$ to handle remainder terms such as G to prove that it is uniformly bounded with respect to the small parameter $\tilde{h} > 0$. This can be done by using the fact that the remainders are located on region where the weight in space is small. In our context, on the region where $F(t)$ is localized, that is $\{x = 1\}$, the weight function $\phi(x) = x^{2-\alpha}$ is large, and then the term G seems to be not uniformly bounded with respect to $\tilde{h}$.

It remains then to handle this difficulty to prove a spectral inequality for the operator $\mathcal{P} = \partial_x x^\alpha \partial_x$, with $\alpha \in [0, 2)$. This is for now an open problem. One strategy we would like to follow, is to slightly modify the weight function. For instance, we would like to patch together a weight φ of the form (7.4.3), that is adapted to handle the singularity $\{x = 0\}$, and a proper classical weight in the interior, such as the weights constructed in [EFV06, BP17, Phu17].

Annexes

Annexe de l'introduction

8.A Remarque sur l'optimalité de la condition géométrique pour l'inégalité spectrale sur tout l'espace

Preuve de la proposition 1.3.6. On rappelle ici que l'on suppose que l'ouvert de contrôle ω satisfait la condition suivante.

Il existe une suite $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ telle que pour tout $y' \in \omega$, on a

$$|y_k - y'| > d_k, \tag{8.A.1}$$

avec $d_k \to +\infty$ lorque $k \to +\infty$. Comme ω satisfait la condition (8.A.1), pour tout $k \in \mathbb{N}$, on peut prendre une suite $x_k \in \mathbb{R}^d$ telle que $|x_k - x| \geq k$ pour tout $x \in \omega$, et $x_0 = 0$. Dans un premier temps, notons qu'on a l'estimation de queue de Gaussienne suivante ([LRM16]), pour $\alpha \geq 1$,

$$\int_{|x|\geq\alpha} e^{-|x|^2} dx \leq C(d) e^{-\frac{\alpha^2}{2}}, \tag{8.A.2}$$

où $C(d) > 0$ désigne une constante dépendant de la dimension. On considère alors le noyau de la chaleur centré en x_k

$$\phi_s^k(x) = (4\pi s)^{-d/2} e^{-\frac{|x-x_k|^2}{4s}}, \quad x \in \mathbb{R}^d,$$

de transformée de Fourier

$$\hat{\phi}_s^k(\xi) = e^{-ix_k.\xi} e^{-s|\xi|^2}, \quad \xi \in \mathbb{R}^d,$$

On définit, pour $M_k > 0$, la fonction tronquée en fréquences

$$\hat{f}_k(\xi) = e^{-ix_k.\xi} e^{-\frac{|\xi|^2}{M_k}} \chi_{\{|\xi|\leq N_k\}}(\xi), \quad \xi \in \mathbb{R}^d,\ N_k \in \mathbb{N},$$

où χ_I est la fonction caractéristique de l'intervalle I. On considèrera sa transformée de Fourier inverse $f_k(x) = \mathcal{F}^{-1}(\hat{f}_k)(x) \in L^2(\mathbb{R}^d)$, Notons que f_k satisfait la propriété de support supp $\hat{f}_k \subset B(0, N_k)$. Dans un premier temps, on obtient une borne inférieure de la norme $L^2(\mathbb{R}^d)$ de f_k :

$$\begin{aligned}
||f_k||^2_{L^2(\mathbb{R}^d)} = (2\pi)^{-d} ||\hat{f}_k||^2_{L^2} &= (2\pi)^{-d} \int_{B(0,N_k)} e^{-\frac{2|\xi|^2}{M_k}} d\xi = \phi^0_{2/M_k}(0) - (2\pi)^{-d} \int_{\{|\xi|\geq N_k\}} e^{-\frac{2|\xi|^2}{M_k}} d\xi \\
&= \phi^0_{2/M_k}(0) - \frac{M_k^{d/2}}{(2\sqrt{2}\pi)^d} \int_{\{|\xi|\geq\sqrt{\frac{2}{M_k}}N_k\}} e^{-|\xi|^2} d\xi \\
&\geq \frac{M_k^{d/2}}{(8\pi)^{d/2}} - C(d) \frac{M_k^{d/2}}{(2\sqrt{2}\pi)^d} e^{-\frac{2N_k^2}{M_k}}.
\end{aligned} \tag{8.A.3}$$

Dans un second temps, on obtient une borne supérieure de la norme de f_k sur ω. Notons que

$$\begin{aligned} f_k(x) &= (2\pi)^{-d}\int_{B(0,N_k)} e^{-i(x_k-x).\xi}e^{-\frac{|\xi|^2}{M_k}}d\xi = (2\pi)^{-d}\left(\int_{\mathbb{R}^d} e^{-i(x_k-x).\xi}e^{-\frac{|\xi|^2}{M_k}}d\xi - \int_{|\xi|\geq N_k} e^{-i(x_k-x).\xi}e^{-\frac{|\xi|^2}{M_k}}d\xi\right) \\ &= \phi^k_{\frac{1}{M_k}}(x) - R_k(x), \end{aligned}$$

où $R_k(x) = \int_{|\xi|\geq N_k} e^{-i(x_k-x).\xi}e^{-\frac{|\xi|^2}{M_k}}d\xi$. Or, d'après (8.A.1), on a

$$\begin{aligned} ||\phi^k_{\frac{1}{M_k}}||^2_{L^2(\omega)} &= \left(\frac{4\pi}{M_k}\right)^{-d}\int_\omega e^{-\frac{M_k|x-x_k|^2}{2}}dx \leq \left(\frac{4\pi}{M_k}\right)^{-d}\int_{|x-x_k|\geq k} e^{-\frac{M_k|x-x_k|^2}{2}}dx \\ &\leq \left(\frac{4\pi}{M_k}\right)^{-d}\left(\frac{2}{M_k}\right)^{\frac{d}{2}}\int_{|y|\geq k\sqrt{\frac{M_k}{2}}} e^{-y^2}dy \\ &\leq C(d)\frac{(2M_k)^{d/2}}{(4\pi)^d}e^{-k^2\frac{M_k}{4}}, \end{aligned}$$

en utilisant (8.A.2). Enfin, on estime le reste R_k :

$$\begin{aligned} ||R_k||^2_{L^2(\omega)} \leq ||R_k||^2_{L^2(\mathbb{R}^d)} &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-2\frac{|\xi|^2}{M_k}}\chi_{|\xi|\geq N_k}(\xi)d\xi = (2\pi)^{-d}\left(\frac{M_k}{2}\right)^{d/2}\int_{|\eta|\geq N_k\sqrt{\frac{2}{M_k}}} e^{-|\eta|^2}d\eta \\ &\leq C(d)\frac{M_k^{d/2}}{(2\sqrt{2}\pi)^d}e^{-N_k^2\frac{1}{M_k}}, \end{aligned}$$

en utilisant encore l'inégalité (8.A.2). On obtient finalement

$$||f_k||^2_{L^2(\omega)} \leq C(d)\left(\frac{(2M_k)^{d/2}}{(4\pi)^d}e^{-k^2\frac{M_k}{4}} + \frac{M_k^{d/2}}{(2\sqrt{2}\pi)^d}e^{-N_k^2\frac{1}{M_k}}\right). \tag{8.A.4}$$

En combinant (8.A.4) et (8.A.3), on trouve

$$\frac{||f_k||^2_{L^2(\mathbb{R}^d)}}{||f_k||^2_{L^2(\omega)}} \geq \frac{\left((2\pi)^{-d/2} - C(d)(2\sqrt{2}\pi)^d e^{-2\frac{N_k^2}{M_k}}\right)}{C(d)\left(\frac{2^{d/2}}{(4\pi)^d}e^{-k^2\frac{M_k}{4}} + \frac{1}{(2\sqrt{2}\pi)^d}e^{-N_k^2\frac{2}{M_k}}\right)} \geq e^{k^2\frac{M_k}{4}}\frac{\left((2\pi)^{-d/2} - C(d)(2\sqrt{2}\pi)^d e^{-2\frac{N_k^2}{M_k}}\right)}{C(d)\left(\frac{2^{d/2}}{(4\pi)^d} + \frac{1}{(2\sqrt{2}\pi)^d}e^{-N_k^2\frac{2}{M_k}}\right)}.$$

En fixant $M_k = \frac{2N_k}{k}$, avec $N_k > 0$ suffisamment grand, on a

$$||f_k||_{L^2(\mathbb{R}^d)} \gtrsim e^{kN_k}||f_k||_{L^2(\omega)},$$

qui est le résultat annoncé. □

Annexes du chapitre 4

9.B Heuristic derivation of the model

In this section, we derive the Ventcel boundary conditions from a transmission problem in an open subset of $\mathbb{R}^n$. Transmission conditions occurs at the interface of a thin layer that surrounds the boundary $\partial\Omega$. The Ventcel boundary condition arises after some approximation. For the sake of simplicity, we consider the case $\Omega = \mathbb{R}^n_+$ with boundary $\partial\Omega = \{x_n = 0\}$. We set $\Omega^\delta = \{x \in \mathbb{R}^n \mid x_n \in (-\delta, 0)\}$, where $\delta > 0$, that describe a layer at the boundary $\partial\Omega$. We also set $\partial\Omega^\delta = \{x_n = -\delta\}$. We then consider the following elliptic problem

$$(-\partial^2_{x_n} - \Delta_g^T)u_1 = f_1 \text{ on } \Omega, \quad (-\partial_{x_n} c \partial_{x_n} - \Delta_g^T)u_2 = f_2 \text{ on } \Omega^\delta, \tag{9.B.1}$$

where c is a smooth function on Ω^δ, with homogeneous Neumann boundary condition at $\{x_n = -\delta\}$ and transmission condition at $\{x_n = 0\}$

$$\partial_{x_n} u_{2|_{x_n=-\delta}} = 0, \quad u_{1|_{x_n=0}} = u_{2|_{x_n=0}}, \quad \partial_{x_n} u_{1|_{x_n=0}} = c(x', 0)\partial_{x_n} u_{2|_{x_n=0}}. \tag{9.B.2}$$

Multiplying the second equation of (9.B.1) by $\frac{1}{\delta}$, and integrating with respect to the variable x_n from $-\delta$ to 0 we obtain

$$-\frac{1}{\delta}(c\partial_{x_n} u_2)_{|_{x_n=0^-}} - \frac{1}{\delta}\int_{-\delta}^0 \Delta_g^T u_2 dx_n = \frac{1}{\delta}\int_{-\delta}^0 f_2 dx := g. \tag{9.B.3}$$

Using the transmission condition and setting $v = \frac{1}{\delta}\int_{-\delta}^0 u_2 dx_n$, that is averaging u_2 over Ω^δ in the x_n-direction, (9.B.3) reads

$$-\frac{1}{\delta}\partial_{x_n} u_{1|_{x_n=0}} - \Delta_g^T v = g.$$

If we make the approximation $v \approx u_{1|_{x_n=0}}$ which is meaningfull for $\delta > 0$ small as $u_{1|_{x_n=0^+}} = u_{2|_{x_n=0^-}}$ (we identify the function u_2 and its average on a small domain), we obtain the following transmission condition at $\{x_n = 0^+\}$

$$-\frac{1}{\delta}\partial_{x_n} u_{1|_{x_n=0^+}} - \Delta_g' u_{1|_{x_n=0^+}} = g.$$

The transmission condition at the interface for (9.B.1) thus yields a Ventcel-type boundary condition for u_1 if we consider the problem from the side of Ω. The approximation $v \approx u_{1|_{x_n=0^+}}$ is only reasonable for a small value of δ. The Ventcel boundary condition can be seen as a good model for a thin layer structure at the boundary.

9.C Well-posedness results

9.C.1 Proof of Proposition 4.2.1.

We first prove that the operator is coercive. For $\lambda \in \mathbb{R}$, the equation $(A_\delta + \lambda I)U = F$, with $U = (u, v) \in D(A_\delta)$ and $F = (f, g) \in \mathcal{H}_\delta$, reads

$$\left\{ \begin{array}{rcl} v & = & \lambda u - f \\ Pu + (a+\lambda)\lambda u & = & h, \end{array} \right. \tag{9.C.1}$$

where $h = g + (a+\lambda)f$, with boundary conditions

$$\partial_\nu u + \delta S u + \lambda b u = \tilde{h} \text{ on } \partial\Omega$$

where $\tilde{h} = bf_{|\partial\Omega}$. We recall that P and S are defined in (4.1.21). Taking the L^2-inner product of the left hand side of the second line of (9.C.1) with some $\tilde{u} \in \mathcal{V}_\delta$, we obtain

$$\begin{aligned}
& \Big((P+\lambda(a+\lambda))u, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(-\Delta_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)} - \Big(\partial_\nu u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} \\
& + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} \\
= \;& \Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)} + \delta \left\langle \Sigma u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\
& - \Big(\tilde{h}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \delta\Big(d^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} \\
& + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} + \Big(\lambda b u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\
=: \;& a_\delta(u, \tilde{u}) - \Big(\tilde{h}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)}.
\end{aligned}$$

This leads to the following variational problem, for all $\varphi \in \mathcal{V}_\delta$:

$$a_\delta(u, \varphi) = l(\varphi), \tag{9.C.2}$$

where $l(\varphi) = \int_\Omega h\varphi + \int_{\partial\Omega} \tilde{h}\varphi_{|\partial\Omega}$. Let us prove the coercivity of a_δ for λ large. Recalling that $|.|^2_{L^2(\partial\Omega)} + \langle \Sigma ., . \rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)}$ provides an norm equivalent to $|.|_{H^1(\partial\Omega)}$, by the Young inequality, there exists $C_0 > 0$ such that

$$\begin{aligned}
& \left| \Big(c\nabla_g u, u\Big)_{L^2(\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, u_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \Big(du, u\Big)_{L^2(\Omega)} + \delta\Big(d^T u_{|\partial\Omega}, u_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \right| \\
& \quad \leq ||c||_{L^\infty} \Big(\varepsilon^2 ||\nabla_g u||^2_{L^2(\Omega)} + \varepsilon^{-2} ||u||^2_{L^2(\Omega)}\Big) + \delta C_0 |c^T|_{L^\infty} \Big(\varepsilon^2 \left\langle \Sigma u_{|\partial\Omega}, u_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} \\
& \qquad + \varepsilon^{-2} |u_{|\partial\Omega}|^2_{L^2(\partial\Omega)}\Big) + ||d||_{L^\infty} ||u||^2_{L^2(\Omega)} + \delta |d^T|_{L^\infty} |u_{|\partial\Omega}|^2_{L^2(\partial\Omega)},
\end{aligned}$$

for all $\varepsilon > 0$, and for all $u \in \mathcal{V}_\delta$. In particular, we choose ε sufficiently small such that

$$1 - \varepsilon^2 ||c||_{L^\infty} \geq \frac{1}{2} \quad \text{and} \quad 1 - C_0 \varepsilon^2 |c^T|_{L^\infty} \geq \frac{1}{2}, \tag{9.C.3}$$

and we keep the value of ε fixed in what follows. We shall need the following trace lemma.

Lemma 9.C.1. *For all $u \in H^1(\Omega)$ we have*

$$|u|^2_{L^2(\partial\Omega)} \leq \frac{1}{\varepsilon'^2} ||u||^2_{L^2(\Omega)} + \varepsilon'^2 ||\nabla_g u||^2_{L^2(\Omega)},$$

for all $\varepsilon' \in (0, 1]$.

Proof. By locally straightening the boundary, it is sufficient to prove the inequality in the case where $\Omega = \mathbb{R}^n_+$ and $\partial\Omega = \mathbb{R}^n$. We then have for $v \in \overline{C}^\infty_0(\mathbb{R}^n_+)$,

$$u(x', x_n)^2 = -\int_0^{+\infty} \partial_{x_n}(u(x', x_n)^2)\, dx_n = -2\int_0^{+\infty} u\partial_{x_n} u\, dx_n.$$

Applying the Young inequality yields the result. □

For ε fixed by (9.C.3), we can now apply Lemma 9.C.1 to have

$$(C_0\varepsilon^{-2}|c^T|_{L^\infty}+|d^T|_{L^\infty})|u|^2_{L^2(\partial\Omega)}\leq (C_0\varepsilon^{-2}|c^T|_{L^\infty}+|d^T|_{L^\infty})(\varepsilon'^2||\nabla_g u||^2_{L^2(\Omega)}+\frac{1}{\varepsilon'^2}||u||^2_{L^2(\Omega)}).$$

for all $\varepsilon'>0$. In particular, we can take ε' such that $\frac{1}{2}-\varepsilon'^2\left(C_0\varepsilon^2|c^T|_{L^\infty}+|d^T|_{L^\infty}\right)\geq\frac{1}{4}$. Hence, there exists $C_1>0$ such that

$$\begin{aligned}a_\delta(u,u)\geq&||\nabla_g u||^2_{L^2(\Omega)}(1-\varepsilon^2||c||_{L^\infty}-\delta\varepsilon'^2(C_0\varepsilon^{-2}|c^T|_{L^\infty}+|d^T|_{L^\infty}))\\&+||u||^2_{L^2(\Omega)}(\lambda^2-||d||_{L^\infty}-\varepsilon^{-2}-\delta\varepsilon'^{-2}(C_0\varepsilon^{-2}|c^T|_{L^\infty}+|d^T|_{L^\infty}))\\&+\delta\left\langle\Sigma u_{|\partial\Omega},u_{|\partial\Omega}\right\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}(1-C_0\varepsilon^2||c^T||_{L^\infty})\\\geq&\frac{1}{4}||\nabla_g u||^2_{L^2(\Omega)}+\frac{\delta}{2}\left\langle\Sigma u_{|\partial\Omega},u_{|\partial\Omega}\right\rangle_{H^{-1}(\partial\Omega),H^1(\partial\Omega)}+||u||^2_{L^2(\Omega)}(\lambda^2-C_1).\end{aligned}$$

Therefore, taking λ sufficiently large yield the coercivity of the bilinear form a_δ, uniformly in δ. Let us prove that l is continuous. We have

$$|l(\varphi)|\leq|\int_\Omega(g+(a+\lambda)f)\varphi|+|\int_{\partial\Omega}bf_{|\partial\Omega}\varphi_{|\partial\Omega}|\lesssim\left(||f||_{L^2(\Omega)}+||g||_{L^2(\Omega)}+|b|_{L^\infty(\partial\Omega)}||f||_{H^1(\Omega)}\right)||\varphi||_{\mathcal{V}_\delta}.$$

We can then apply the Lax-Milgram theorem to obtain existence and uniqueness of a weak solution $u\in\mathcal{V}_\delta$ of the variational formulation (9.C.2), and we have the bound

$$||u||_{\mathcal{V}_\delta}\lesssim||f||_{L^2(\Omega)}+||g||_{L^2(\Omega)}+|b|_{L^\infty(\partial\Omega)}||f||_{H^1(\Omega)}. \tag{9.C.4}$$

In fact, H^2 regularity holds in the interior (see [Bre83]) by standard elliptic theory. It now suffices to prove that for any $x_0\in\partial\Omega$ there exists a neighborhood V of x_0 in $\overline{\Omega}$ and $\theta\in\overline{C^\infty}(\Omega)$ with $\operatorname{supp}\theta\subset V$ such that $\theta u\in D(\mathcal{A}_\delta)$. In addition, we can impose $\partial_\nu\theta_{|\partial\Omega}=0$. We set $w=\theta u$. From (9.C.2), w satisfy for all $\varphi\in\mathcal{V}_\delta(\Omega\cap V)$:

$$a_\delta(w,\varphi)=\int_{\Omega\cap V}t_1\varphi+\int_{\partial\Omega\cap V}t_2\varphi. \tag{9.C.5}$$

where $t_1:=\theta h-\nabla_g\theta\cdot\nabla_g u-\operatorname{div}_g(u\nabla_g\theta)+c\nabla_g\theta u\in L^2(\Omega\cap V)$, and $t_2:=\theta_{|\partial\Omega}\tilde{h}+\delta[S,\theta_{|\partial\Omega}]u_{|\partial\Omega}\in L^2(\partial\Omega\cap V)$. We choose V as in Section 4.4 and use the local normal geodesic coordinates $x=(x_1,\dots,x_n)$ described therein. In this coordinates $\partial\Omega=\{x_n=0\}$ and $\Omega=\mathbb{R}^n_+$. We set $V^0=V\cap\{x_n=0\}$. Below, we shall denote $\mathcal{V}(V):=\{u\in H^1(V),\ u_{|_{x_n=0}}\in H^1(V^0)\}$. With θ as above, (9.C.5) reads

$$\int_V A\nabla w.\nabla\psi+\delta\int_{V^0}A'\nabla^T w_{|_{V^0}}.\nabla^T\psi_{|_{V^0}}=\int_V\tilde{t}_1\psi+\delta\int_{V_0}\tilde{t}_2\psi_{|_{V^0}},\quad\psi\in\mathcal{V}(V), \tag{9.C.6}$$

where $K,K',k,k'\neq0$ are bounded functions, and, because of the form of $\tilde{h}$, $\tilde{t}_2$ can be written

$$\tilde{t}_2=z_1+\delta z_2\text{ with }|z_1|_{L^2(\partial\Omega)}\lesssim|b|_{W^{1,\infty}(\partial\Omega)}||f||_{H^1(\Omega)},\text{ and }z_2\in\delta L^2(\Omega). \tag{9.C.7}$$

where $A(x)$ (resp. $A'(x)$) is the matrix corresponding to the metric g (resp. $g_{|\partial\Omega}$) satisfying the following ellipticity condition : $\exists C>0$ independent of x, $\forall\xi\in\mathbb{R}^n$ (resp. $\mathbb{R}^{n-1}$), we have

$$|\xi|^2\leq CA\xi.\xi\quad(\text{resp.}|\xi'|^2\leq CA'\xi'.\xi'),$$

where we denoted $\xi=(\xi',\xi_n)$. Let e_k be an element of the cannonical basis of $\mathbb{R}^n$, $k\neq n$, and set $h=|h|e_k$, and $D_hu(x)=\frac{1}{|h|}(u(x+h)-u(x))$, and $|h|$ sufficiently small so that $\operatorname{supp}(w(\cdot+h))\subset V$. We choose $\psi=D_{-h}D_hw$ in (9.C.5). We obtain

$$\int_V D_h(A\nabla w)\cdot D_h(\nabla w)+\delta\int_{V_0}D_h(A'\nabla^T w_{|_{V^0}})\cdot D_h\nabla^T w_{|_{V^0}}=\int_V\tilde{t}_1D_{-h}D_hw+\int_{V_0}\tilde{t}_2D_{-h}D_hw_{|_{V^0}}. \tag{9.C.8}$$

We shall need the following estimation :

$$||D_h v||_{L^2(\Omega)} \leq ||\nabla v||_{L^2(\Omega)}. \tag{9.C.9}$$

A proof can be found in [Bre83, Lemma IX.6]. Using (9.C.9), the right hand side of (9.C.8) reads

$$\int_V \tilde{t}_1 D_{-h} D_h w \leq ||\tilde{t}_1||_{L^2(V)} ||\nabla D_h w||_{L^2(V)} \leq ||\tilde{t}_1||_{L^2(V)} ||D_h w||_{H^1(V)},$$

and with (9.C.7) and trace formula

$$\begin{aligned}\int_{V_0} \tilde{t}_2 D_{-h} D_h w &\lesssim |z_1|_{H^{1/2}(V_0)} |D_{-h} D_h w_{|_{V^0}}|_{H^{-1/2}(V_0)} + \delta |z_2|_{L^2(V^0)} |D_{-h} D_h w_{|_{V^0}}|_{L^2(V_0)} \\ &\lesssim |b|_{W^{1,\infty}(\partial\Omega)} ||f||_{H^1(\Omega)} ||D_h w||_{H^1(V)} + \delta |\tilde{t}_2|_{L^2(V^0)} |D_h w_{|_{V^0}}|_{H^1(V^0)}.\end{aligned}$$

Observe that we have the following Leibniz rule

$$D_h(B\nabla\zeta) = B(\cdot + h) D_h \nabla\zeta + (D_h B)\nabla\zeta,$$

for all matrices B and functions ζ. Then, (9.C.8) yields

$$\tilde{a}_\delta(D_h w, D_h w) \lesssim \left(||\tilde{t}_1||_{L^2(V)} + \delta^{1/2} |\tilde{t}_2|_{L^2(V_0)} + |b|_{W^{1,\infty}(\partial\Omega)} ||f||_{H^1(\Omega)} \right) ||D_h w||_{\mathcal{V}_\delta},$$

where $\tilde{a}_\delta$ is the bilinear form defined by

$$\tilde{a}_\delta(u, v) = \int_U A(x + h)\nabla u \cdot \nabla v \, dx + \delta \int_{V_0} A'(x + h) \nabla^T u_{|_{V_0}} \cdot \nabla^T v_{|_{V_0}} \, dx'.$$

Observe that $\tilde{a}_\delta$ is coercive with the same argumentation used to prove that a_δ is coercive. Then, we can derive the inequality

$$||D_h w||^2_{H^1(V)} + \delta |D_h \nabla^T w|^2_{L^2(U^0)} \lesssim T ||D_h w||_{\mathcal{V}_\delta}.$$

with $T = ||\tilde{t}_1||_{L^2(V)} + |\tilde{t}_2|_{L^2(V_0)} + |b|_{W^{1,\infty}(\partial\Omega)} ||f||_{H^1(\Omega)}$, and then

$$||D_h w||_{H^1(V)} + \delta^{1/2} |D_h \nabla^T w|_{L^2(V^0)} \lesssim T. \tag{9.C.10}$$

Now, for $\psi \in C_0^\infty(V)$, $\tilde{\psi} \in C_0^\infty(V_0)$, and for all $j \in \{1, \dots, n\}$,$l \in \{1, \dots, n-1\}$, we have from (9.C.10)

$$|\int_V w D_{-h} \partial_{x_j} \psi| = |\int_V D_h \partial_{x_j} w \psi| \lesssim T ||\psi||_{L^2(V)} \tag{9.C.11}$$

$$|\int_{V^0} w_{|_{V^0}} D_{-h} \partial_{x_l} \tilde{\psi}| = |\int_{V^0} D_h \partial_{x_l} w_{|_{V^0}} \tilde{\psi}| \lesssim \delta^{-1/2} T |\tilde{\psi}|_{L^2(V^0)}. \tag{9.C.12}$$

Taking the limit $h \longrightarrow 0$, we obtain $w_{|_{V^0}} \in H^2(V_0)$ and $\partial^2_{x_j x_k} w \in L^2(V)$, and $\forall j \in \{1, \dots, n\}$, $k \in \{1, \dots, n-1\}$.

It remains to show that $\partial^2_{x_n} w \in L^2(V)$. As we are working with normal geodesic coordinates, the coefficient A of the n-th raw and n-th column is $a_{nn} = 1$. Then (9.C.6) reads, for $\psi \in C_0^\infty(V)$,

$$\int_V \partial_{x_n} w \partial_{x_n} \psi = \int_V \tilde{t_1} \psi - \sum_{(k,l) \neq (n,n)} \int_V a_{kl} \partial_{x_k} w \partial_{x_l} \psi.$$

and with (9.C.11), this yields $|\int_V \partial_n w \partial_n \tilde{\psi}| \leq CT ||\tilde{\psi}||_{L^2}$. Moreover, $T \lesssim |b|_{W^{1,\infty}(\partial\Omega)} ||f||_{H^1} + ||f||_{L^2(\Omega)} + ||g||_{L^2(\Omega)}$, since we have (9.C.4), we have proved

$$||u||^2_{H^2(\Omega)} + \delta |u_{|\partial\Omega}|^2_{H^2(\partial\Omega)} \lesssim |b|_{W^{1,\infty}(\partial\Omega)} ||f||^2_{H^1(\Omega)} + ||f||_{L^2(\Omega)} + ||g||^2_{L^2(\Omega)}.$$

Using (9.C.1), we obtain the sought result. □

9.C.2 Proof of Proposition 4.2.5

The proof is essentially the same as in Section 9.C.1. Here, we only prove the existence and uniqueness of the announced variationnal problem in $\mathcal{K}_\delta$. The elliptic regularity can then be proved in the same way as in section in 9.C.1 by the Nirenberg translation method. Let $U = (u_0, u_1, y_0, y_1) \in D(\mathcal{B}_\delta)$ and $F = (f_0, f_1, g_0, g_1) \in \mathcal{K}_\delta$. Then $(\mathcal{B}_\delta + \lambda \mathrm{Id})\, U = F$ reads

$$\left\{\begin{array}{l} -u_1 + \lambda u_0 = f_0 \\ Pu_0 + (a+\lambda)u_1 = f_1 \\ -y_1 + \lambda y_0 = g_0 \\ \frac{1}{\delta}\partial_\nu u_{0|\partial\Omega} + Sy_0 + (\frac{1}{\delta}b + \lambda)y_1 = g_1 \end{array}\right. \iff \left\{\begin{array}{l} u_1 = \lambda u_0 - f_0 \\ Pu_0 + \lambda(a+\lambda)u_0 = h_0 \\ y_1 = \lambda y_0 - g_0 \\ \partial_\nu u_{0|\partial\Omega} + \delta S y_0 + \lambda(b+\delta\lambda)y_0 = h_1, \end{array}\right.$$

where $h_0 := f_1 + (a+\lambda)f_0$ and $h_1 := \delta g_1 + (b+\delta\lambda)g_0$. Thus, it is sufficient to prove the existence and uniqueness of a solution $(u, y) \in H^1(\Omega) \times H^1(\partial\Omega)$ such that $u_{|\partial\Omega} = y$ of the following system

$$\left\{\begin{array}{l} Pu + \lambda(a+\lambda)u = h_0 \\ \partial_\nu u_{|\partial\Omega} + \delta S y + \lambda(b+\delta\lambda)y = h_1. \end{array}\right. \tag{9.C.13}$$

Taking the inner product of the first equation with $\tilde{u}$ yields, with integration by parts,

$$\begin{aligned} &\Big((P + \lambda(a+\lambda))\, u, \tilde{u}\Big)_{L^2(\Omega)} \\ =&\Big(-\Delta_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} + \lambda\Big((a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} \\ =&\Big(\nabla_g u, \nabla_g \tilde{u}\Big)_{L^2(\Omega)} + \Big(\lambda(a+\lambda)u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(c\nabla_g u, \tilde{u}\Big)_{L^2(\Omega)} + \Big(du, \tilde{u}\Big)_{L^2(\Omega)} \\ &+ \delta \left\langle \Sigma u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega} \right\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + \delta\lambda\Big((b+\lambda)u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\ &+ \delta\Big(d^T u_{|\partial\Omega}, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} - \Big(h_1, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \\ :=&b_\delta(u, \tilde{u}) - \Big(h_1, \tilde{u}_{|\partial\Omega}\Big)_{L^2(\partial\Omega)}. \end{aligned}$$

This leads to the variationnal problem

$$b_\delta(u, \varphi) = l(\varphi),$$

where $l(\varphi) = \int_\Omega h_0\varphi + \int_{\partial\Omega} h_1 \varphi_{|\partial\Omega}$. Note that $\langle \Sigma u, u\rangle_{H^{-1}(\partial\Omega), H^1(\partial\Omega)} + |u|^2_{L^2(\partial\Omega)}$ provides an equivalent norm to $|u|^2_{H^1(\partial\Omega)}$. We now claim that the bilinear form b_δ is coercive. Indeed, observe that as a and b are non-negative functions, we have $(au, u)_{L^2(\Omega)} \geq 0$ and $(bu_{|\partial\Omega}, u_{|\partial\Omega})_{L^2(\partial\Omega)} \geq 0$. We can now apply the Young inequality to obtain

$$\begin{aligned} \left| \Big(c\nabla_g u, u\Big)_{L^2(\Omega)} + \delta\Big(c^T \nabla_g^T u_{|\partial\Omega}, u_{|\partial\Omega}\Big)_{L^2(\partial\Omega)} \right| \leq& ||c||_{L^\infty} \left(\varepsilon^2 ||\nabla_g u||^2_{L^2(\Omega)} + \varepsilon^{-2} ||u||^2_{L^2(\Omega)} \right) \\ &+ \delta |c^T|_{L^\infty} \left(\varepsilon^2 C |\nabla_g^T u_{|\partial\Omega}|^2_{L^2(\partial\Omega)} + \varepsilon^{-2} |u_{|\partial\Omega}|^2_{L^2(\Omega)} \right). \end{aligned} \tag{9.C.14}$$

for all $\varepsilon > 0$. Taking ε sufficiently small to have $1 - ||c||_{L^\infty}\delta' \geq \frac{1}{2}$ and $1 - C|c^T|_{L^\infty}\delta' \geq \frac{1}{2}$, and now taking λ sufficiently large, we prove the coercivity of the bilinear form b_δ. It remains to prove that the linear form l is continuous on $\mathcal{V}_\delta$. We have $|\int_\Omega h_0\varphi| \leq ||h_0||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}$ and because of the form of h_1, we have $|\int_{\partial\Omega} h_1\varphi_{|\partial\Omega}| \leq |g_0|_{L^2(\Omega)}||\varphi||_{H^1(\Omega)} + \delta|g_1|_{L^2(\Omega)}|\varphi|_{L^2(\partial\Omega)}$. These two estimates yields the continuity of l.

□

9.D Unique continuation property from the boundary

Theorem 9.D.1. *Let $u \in H^2(\Omega)$ be such that*

$$|Pu(x)| \leq |u(x)| + |\nabla_g u(x)| \text{ a.e on } \Omega, \text{ and } \partial_\nu u_{|\partial\Omega} + S u_{|\partial\Omega} = 0 \text{ on } \partial\Omega. \tag{9.D.1}$$

Assume moreover that $u_{|\partial\Omega} = 0$ *on an open subset* ω_B *of* $\partial\Omega$. *Then* $u = 0$ *on* Ω.

Proof. We use the normal normal geodesic coordinates introduced in Section 4.4 in a neighborhood of $y_0 \in \omega_B$. In these coordinates, $y_0 = 0$ and $\partial\Omega = \{x_n = 0\}$. Consider U an open neighborhood of 0 in $\mathbb{R}^n$ such that $U \cap \{x_n = 0\} \subset \omega_B$. Let $x_0 = (x' = 0, -r_0)$, with $r_0 > 0$ to be chosen below. Let $\psi(x) = |x_0 - x|^2$, and define $\varphi(x) = e^{-\lambda\psi(x)}$. Let $r_1 > 0$ and $r_2 > 0$ be such that

- $\{x = (x', x_n) \mid x_n > 0, |x_0 - x| < r_2\} \subset U^+$,
- $r_0 < r_1 < r_2$,

where $U^+ := U \cap \mathbb{R}^n_+$. Observe that for $\lambda > 0$ chosen sufficiently large, φ satisfies the sub-ellipticity condition (4.4.14). Define the cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$:

$$\begin{cases} \chi(x) = 1 & \text{if } |x_0 - x| < r_1 \\ \chi(x) = 0 & \text{if } |x_0 - x| > r_2. \end{cases}$$

From Proposition 4.4.3, there exists $\tau_0 > 0$ such that for $\tau \geq \tau_0$

$$\tau^3||e^{\tau\varphi}\chi u||^2_{L^2(U^+)} + \tau||e^{\tau\varphi}D\chi u||^2_{L^2(U^+)} \lesssim ||e^{\tau\varphi}P\chi u||^2_{L^2(U^+)} + \tau|e^{\tau\varphi}\partial_{x_n}\chi u_{|_{x_n=0}}|^2_{L^2(U_0)} + \tau^3|e^{\tau\varphi}\chi u_{|_{x_n=0}}|^2_{L^2(U_0)}, \quad (9.D.2)$$

for all $u \in \overline{C}_0^\infty(U^+)$, where $V_0 := V \cap \{x_n = 0\}$. Note that $|e^{\tau\varphi}\chi u_{|_{x_n=0}}|^2_{L^2(U_0)} = 0$ as $U \subset \omega_B$. Observing that $P\chi u = \chi Pu + [P, \chi]u$ and $D\chi u = \chi Du + [D, \chi]u$, with assumption (9.D.1), estimate (9.D.2) reads

$$\tau^3||e^{\tau\varphi}\chi u||^2_{L^2(V^+)} + \tau||e^{\tau\varphi}\chi Du||^2_{L^2(V^+)} \lesssim ||e^{\tau\varphi}\chi u|| + ||e^{\tau\varphi}\chi Du||^2_{L^2(V^+)} + ||e^{\tau\varphi}[P, \chi]u||^2_{L^2(V^+)} + \tau||e^{\tau\varphi}[D, \chi]u||^2_{L^2(V^+)} + \tau|e^{\tau\varphi}\partial_{x_n}\chi u_{|_{x_n=0}}|^2_{L^2(V_0)}.$$

Taking τ sufficiently large, we may ignore the first two terms on the right hand side. Now, observing that the commutators $[P, \chi]$ and $[D, \chi]$ are supported in the region where χ varies, we obtain

$$||e^{\tau\varphi}[P, \chi]u||^2_{L^2(V^+)} + ||e^{\tau\varphi}[D, \chi]u||^2_{L^2(V^+)} \lesssim e^{\tau C_1}||u||^2_{H^1(V^+)},$$

with $C_1 = e^{-\lambda r_1}$. Furthermore, note that

$$|e^{\tau\varphi}(\partial_{x_n}\chi u)_{|_{x_n=0}}|^2_{L^2(V_0)} \leq |e^{\tau\varphi}[\partial_{x_n}, \chi]_{|_{x_n=0}}u_{|_{x_n=0}}|^2_{L^2(V_0)} + |e^{\tau\varphi}\chi_{|_{x_n=0}}Su_{|_{x_n=0}}|^2_{L^2(V_0)} = 0,$$

by assumption. We then can restrict the left hand side of (9.D.2) to

$$W := \{x = (x', x_n) \mid x_n > 0, |x_0 - x| < r_1'\}$$

, where r_1' is such that $r_1' < r_1$ and such that $W \neq \emptyset$. This finally yields

$$e^{C_2\tau}||u||^2_{H^1(W^+)} \lesssim e^{\tau C_1}||u||^2_{H^1(V^+)}, \quad (9.D.3)$$

with $C_2 = e^{-\lambda r_1'}$. As $C_1 < C_2$, letting $\tau \longrightarrow +\infty$ in (9.D.3) yields $u = 0$ in W. Coming back to original coordinates, we have found a non-empty open subset $O \subset \Omega$ such that $u_{|_O} = 0$. We can conclude by applying Caldern's unique continuation theorem for elliptic operators of order two. □

9.E Proof of technical results

9.E.1 Proof of Proposition 4.2.8

Below, we shall denote by $\mathcal{H}$ and $\mathcal{V}$ the spaces $\mathcal{V}_{\delta=1}$ and $\mathcal{H}_{\delta=1}$. We recall that $\mathcal{V}_\delta$ and $\mathcal{V}$ (resp. $\mathcal{H}_\delta$ and $\mathcal{H}$) are homeomorphic for each δ. However, only the injection $\mathcal{V}_\delta \hookrightarrow \mathcal{V}$ (resp. $\mathcal{H}_\delta \hookrightarrow \mathcal{H}$) is continuous uniformly

in δ. In particular, $\mathcal{V}'_\delta$ and $\mathcal{V}'$ are homeomorphic and only the injection $\mathcal{V}' \hookrightarrow \mathcal{V}'_\delta$ is continuous uniformly in δ. Observe that $f_\delta \rightharpoonup f$ in $L^2(0,T;L^2(\Omega))$ implies $||f_\delta||_{L^2(0,T;L^2(\Omega))} \leq C$ uniformly in δ. Consider first $U^0_\delta \in D(A_\delta)$. Multiplying (4.2.4) by $\partial_t u_\delta$ and integrating by parts over Ω yields

$$\frac{d}{dt}\left(||\partial_t u_\delta||^2_{L^2(\Omega)} + ||u_\delta||^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}|^2_{L^2(\partial\Omega)}\right) + \int_\Omega a|\partial_t u_\delta|^2 + \int_{\partial\Omega} b|\partial_t u_{\delta|\partial\Omega}|^2 = \int_\Omega f_\delta u_\delta.$$

Integrate in time between 0 and t, and using Proposition 4.2.2,

$$||\partial_t u_\delta(t)||^2_{L^2(\Omega)} + ||u_\delta(t)||^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}(t)|^2_{L^2(\partial\Omega)} + |a^{1/2}|\partial_t u_\delta|^2_{L^2(0,T;L^2(\Omega))} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}|^2_{L^2(0,T;L^2(\partial\Omega))} \leq ||U^0_\delta||^2_{\mathcal{H}} + \int_0^T ||f_\delta||^2_{L^2(\Omega)} \leq C, \quad (9.E.1)$$

from the assumptions on f_δ (constants may change from one line to an other). Now, consider $w \in L^2(0,T,\mathcal{V})$. Multiplying (4.2.4) with $U^0_\delta \in D(A_\delta)$ by w and integrating by parts over $(0,T)\times\Omega$ yields the following variational formulation

$$\int_0^T \left(\partial_t^2 u_\delta, w\right)_{L^2(\Omega)} + \int_0^T \left(\nabla_g u_\delta, \nabla_g w\right)_{L^2(\Omega)} + \delta\int_0^T \left(\nabla_g^T u_{\delta|\partial\Omega}, \nabla_g^T w_{|\partial\Omega}\right)_{L^2(\partial\Omega)} + \int_0^T \left(a\partial_t u_\delta, w\right)_{L^2(\Omega)} + \int_0^T \left(b\partial_t u_{\delta|\partial\Omega}, w_{|\partial\Omega}\right)_{L^2(\partial\Omega)} = \int_0^T \left(f_\delta, w\right)_{L^2(\partial\Omega)}. \quad (9.E.2)$$

With (9.E.1), we have

$$\left|\int_0^T \left(\partial_t^2 u_\delta, w\right)_{L^2(\Omega)}\right| \lesssim \int_0^T \left(||u_\delta||_{\mathcal{V}_\delta} + ||f_\delta||_{L^2(\Omega)} + ||a^{1/2}\partial_t u_\delta||_{L^2(\Omega)} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}|\right)||w||_{\mathcal{V}} \lesssim (||U^0_\delta||_{\mathcal{H}} + ||f_\delta||_{L^2(0,T,L^2(\Omega))})||w||_{\mathcal{V}}.$$

Note that by a density argument that this estimate is still valid for $U^0_\delta \in \mathcal{H}$. We thus obtain for $U^0_\delta \in \mathcal{H}$, that $w \mapsto \int_0^T (\partial_t^2 u_\delta, w)_{L^2(\Omega)} \in \mathcal{V}'$ and thus $\partial_t^2 u_\delta \in L^2(0,T;\mathcal{V}')$. Hence,

$$||\partial_t^2 u_\delta||^2_{L^2(0,T;\mathcal{V}')} + ||\partial_t u_\delta||^2_{L^2(\Omega)} + ||u_\delta||^2_{H^1(\Omega)} + \delta|\nabla_g^T u_{\delta|\partial\Omega}|^2_{L^2(\partial\Omega)} + ||a^{1/2}\partial_t u_\delta||_{L^2(\Omega)} + |b^{1/2}\partial_t u_{\delta|\partial\Omega}| \leq C.$$

This allows to consider a subsequence u_{δ_k} and a function u such that

$$u_{\delta_k} \rightharpoonup u \text{ in } L^2(0,T;H^1(\Omega)), \quad u_{\delta_k} \rightharpoonup u \text{ in } H^1(0,T;L^2(\Omega)),$$
$$u_{\delta_k} \rightharpoonup u \text{ in } H^2(0,T;\mathcal{V}'), \quad b^{1/2}\partial_t u_{\delta_k|\partial\Omega} \rightharpoonup b^{1/2}\partial_t u_{|\partial\Omega} \text{ in } L^2(0,T;L^2(\partial\Omega).$$

We also can extract a subsequence to u_{δ_k}, denoted again by u_{δ_k} such that $\delta_k^{1/2}\nabla_g^T u_{\delta_k}$ is weakly convergent in $L^2(\partial\Omega)$. In particular, we have $|\nabla_g^T u_{\delta_k}|_{L^2(\Omega)} = \mathcal{O}(\delta^{-1/2})$. Taking the limit in (9.E.2), we thus have

$$\int_0^T \left\langle \partial_t^2 u, w\right\rangle_{\mathcal{V}',\mathcal{V}} + \int_0^T \left(\nabla_g u, \nabla_g w\right)_{L^2(\Omega)} + \int_0^T \left(a\partial_t u, w\right)_{L^2(\Omega)} + \int_0^T \left(b\partial_t u_{|\partial\Omega}, w_{|\partial\Omega}\right)_{L^2(\partial\Omega)} = \int_0^T \left(f, w\right)_{L^2(\Omega)}, \quad (9.E.3)$$

yielding $|\int_0^T \left\langle \partial_t^2 u, w\right\rangle_{\mathcal{V}',\mathcal{V}}| \lesssim ||w||_{H^1(\Omega)}$, for all $w \in \mathcal{V}$. As $\mathcal{V}$ is dense in $H^1(\Omega)$ we obtain that $\partial_t^2 u$ can be extended as a linear form defined on $H^1(\Omega)$. We obtain the variational formulation associated with the problem (4.2.5). We can deduce by existence and uniqueness of the solution of (9.E.3) that the limit does not depend on the chosen subsequence. This ends the proof. □

9.E.2 Proof of Proposition 4.4.2

Observe that the parameter σ does not appear in the Poisson bracket $\{p_2, p_1\}$. We set $\beta(x) = \lambda\tau\varphi(x)$ and $\zeta = \beta^{-1}\xi$. After computations of the Poisson bracket, we obtain

$$\{p_2, p_1\}(x, \xi, \tau) = \beta^3(x)(\lambda\mathcal{R}_1 + \mathcal{R}_2),$$

where $\mathcal{R}_1 = 4p(x, \mathrm{d}_x\psi(x))^2 + (\partial_x\psi\partial_\xi p(x, \zeta))^2$ and

$$\begin{aligned}\mathcal{R}_2 = {} & 2\partial_\xi p(x,\zeta)\partial_x\tilde{p}(x,\zeta,\mathrm{d}_x\psi(x)) + \mathrm{d}_x^2\psi(x)(\partial_\xi p(x,\zeta),\partial_\xi p(x,\zeta)) \\ & - \partial_x p(x,\zeta)\partial_\xi p(x,\mathrm{d}_x\psi(x)) + \partial_x p(x,\mathrm{d}_x\psi(x))\partial_\xi p(x,\mathrm{d}_x\psi(x)) + \mathrm{d}_x^2\psi(x)(\partial_\xi p(x,\mathrm{d}_x\psi(x)),\partial_\xi p(x,\mathrm{d}_x\psi(x))).\end{aligned} \tag{9.E.4}$$

On the one hand, as $\mathrm{d}_x\psi \neq 0$, there exists $C_0 > 0$ such that $\mathcal{R}_1 \geq C_0$, and, on the other hand, we observe that $p_{\varphi,\sigma}(x, \xi, \tau) = 0$, in particular $p_2(x, \xi, \tau, \sigma) = 0$, implies $p(x, \xi) = \sigma^2 + p(x, \tau\mathrm{d}_x\varphi)$, which yields $1 + \frac{\sigma^2}{\beta^2} \lesssim |\zeta|^2 \lesssim 1 + \frac{\sigma^2}{\beta^2}$. Hence, there exists a constant $C_1 > 0$ such that $|\mathcal{R}_2| \leq C_1\left(1 + \frac{\sigma^2}{\beta^2}\right)$. Now we choose λ, to be kept fixed in what follows, such that $\lambda C_0 - 2C_1 \geq 1$, and if τ is such that $\dfrac{|\sigma|}{\beta} \leq 1$, we obtain

$$\lambda\mathcal{R}_1 + \mathcal{R}_2 \geq \lambda C_0 - 2C_1 \geq 1 \tag{9.E.5}$$

and thus $p_{\varphi,\sigma} = 0 \Longrightarrow \{p_2, p_1\} > 0$. In particular, it holds on the compact set $\{p_{\varphi,\sigma} = 0\} \cap \{\lambda_\tau = 1\}$. We then conclude by homogeneity. Note that the condition $\tau \geq \tilde{c}|\sigma|$ with $\tilde{c} \geq \frac{1}{\lambda \inf\varphi}$ implies $\frac{|\alpha|}{\beta} \leq 1$. □

9.E.3 Proof of Proposition 4.4.3

We have

$$||P_{\varphi,\sigma}v||^2_{L^2(\mathbb{R}^n_+)} = ||P_2v||^2_{L^2(\mathbb{R}^n_+)} + ||P_1v||^2_{L^2(\mathbb{R}^n_+)} + i\left(\Big(P_1v, P_2v\Big)_{L^2(\mathbb{R}^n_+)} - \Big(P_2v, P_1v\Big)_{L^2(\mathbb{R}^n_+)}\right).$$

Using the forms of P_1 and P_2 in (4.4.2) and (4.4.3) we obtain by integration by parts

$$\begin{aligned} i\left(\Big(P_1v, P_2v\Big)_{L^2(\mathbb{R}^n_+)} - \Big(P_2v, P_1v\Big)_{L^2(\mathbb{R}^n_+)}\right) &= \mathrm{Re}\,\Big(i[P_2, P_1]v, v\Big)_{L^2(\mathbb{R}^n_+)} \\ &+ \mathrm{Re}\,\Big(P_1v_{|x_n=0}, D_nv_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} + \mathrm{Re}\,\Big((D_nP_1 - 2\tau(\partial_{x_n}\varphi)P_2)v_{|x_n=0}, v_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})}. \end{aligned}$$

Using (4.4.2) and (4.4.4), the operator $D_nP_1 - 2\tau(\partial_{x_n}\varphi)P_2$ reads

$$\begin{aligned} D_nP_1 - 2\tau(\partial_{x_n}\varphi)P_2 = {} & 2\tilde{r}(x, \tau\mathrm{d}_{x'}\varphi, D')D_n \\ & - 2\tau\partial_{x_n}\varphi\left(r(x, D') - p(x, \tau\mathrm{d}_x\varphi) - \sigma^2\right) \mod \tau(\mathcal{D}^0D_n + \mathcal{D}^1_\tau). \end{aligned}$$

Hence

$$i\left(\Big(P_1v, P_2v\Big)_{L^2(\mathbb{R}^n_+)} - \Big(P_2v, P_1v\Big)_{L^2(\mathbb{R}^n_+)}\right) = \mathrm{Re}\,\Big(i[P_2, P_1]v, v\Big)_{L^2(\mathbb{R}^n_+)} + \tau\,\mathrm{Re}\,\tilde{\mathcal{B}}(v), \tag{9.E.6}$$

where

$$\begin{aligned} \mathrm{Re}\,\tilde{\mathcal{B}}(v) = {} & 2\Big(\partial_{x_n}\varphi D_nv_{|x_n=0}, D_nv_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} \\ & + 2\Big(\tilde{r}(x, \mathrm{d}_{x'}\varphi, D')v_{|x_n=0}, D_nv_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} + 2\Big(\tilde{r}(x, \mathrm{d}_{x'}\varphi, D')D_nv_{|x_n=0}, v_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} \\ & - 2\Big(\partial_{x_n}\varphi\left(r(x, D') - p(x, \tau\mathrm{d}_x\varphi) - \sigma^2\right)v_{|x_n=0}, v_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} + \Big(C_0v_{|x_n=0}, D_nv_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})} \\ & + \Big((\tilde{C}_0D_n + C_1)v_{|x_n=0}, v_{|x_n=0}\Big)_{L^2(\mathbb{R}^{n-1})}, \end{aligned} \tag{9.E.7}$$

with symbols C_0, $\tilde{C}_0 \in \mathcal{D}^0_{T,\tau} = \mathcal{D}^0_\tau$ and $C_1 \in \mathcal{D}^1_{T,\tau}$. Now, we treat the commutator term in (9.E.6). Its principal symbol is $\{p_2, p_1\}$, a polynomial function of degree 3 in (ξ, τ), and from (4.4.7), we know that p_1 reads $p_1(x, \xi, \tau) = \tau q_1(x, \xi)$, where q_1 is a polynomial of order one in ξ. Thus, the Poisson bracket reads

$$\{p_2, p_1\} = \tau \xi_n^2 \tilde{b}_0(x, \xi', \tau) + \tau \xi_n \tilde{b}_1(x, \xi', \tau) + \tau \tilde{b}_2(x, \xi', \tau), \tag{9.E.8}$$

where $\tilde{b}_j$ is a polynomial function of order j in (ξ', τ), $j = 0, 1, 2$. As we imposed $\partial_{x_n}\varphi \geq C' > 0$, using (4.4.6) and (4.4.7), we have

$$\xi_n^2 = p_2(x, \xi, \tau, \sigma) + \sigma^2 - r(x, \xi') + \tau^2 \partial_{x_n}\varphi^2 + r(x, \tau \mathrm{d}_{x'}\varphi), \tag{9.E.9}$$

$$\xi_n = \frac{1}{\partial_{x_n}\varphi}((2\tau)^{-1} p_1(x, \xi, \tau) - \tilde{r}(x, \mathrm{d}_{x'}\varphi, \xi')). \tag{9.E.10}$$

With (9.E.9) and (9.E.10), the bracket (9.E.8) reads

$$\{p_2, p_1\} = \tau b_0(x) p_2(x, \xi, \tau, \sigma) + b_1(x, \xi', \tau) p_1(x, \xi, \tau) + \tau b_2(x, \xi', \tau, \sigma), \tag{9.E.11}$$

where b_j are polynomial functions of degree j in (ξ', τ), $j = 0, 1$, and b_2 is a polynomial function of degree 2 in (ξ', τ, σ). We can then write

$$i[P_2, P_1] = \tau b_0 P_2 + \mathrm{Op}_T(b_1) P_1 + \tau \,\mathrm{Op}(c_1') + \tau \,\mathrm{Op}_T(b_2),$$

where $\mathrm{Op}(c_1') \in \mathcal{D}^1_\tau$. This yields

$$\begin{aligned} ||P_{\varphi,\sigma} v||^2_{L^2(\mathbb{R}^n_+)} = ||P_2 v||^2_{L^2(\mathbb{R}^n_+)} + ||P_1 v||^2_{L^2(\mathbb{R}^n_+)} + \mathrm{Re}\left(\tau \,\mathrm{Op}_T(b_2) v, v\right)_{L^2(\mathbb{R}^n_+)} + \tau \,\mathrm{Re}\, \tilde{\mathcal{B}}(v) \\ + \mathrm{Re}\left(\left(\tau b_0 P_2 + \mathrm{Op}_T(b_1) P_1 + \tau \,\mathrm{Op}(c_1')\right) v, v \right)_{L^2(\mathbb{R}^n_+)}. \end{aligned} \tag{9.E.12}$$

From (9.*E*.11) and the sub-ellipticity property, we observe

$$p_2(x, \xi, \tau, \sigma) = p_1(x, \xi, \tau) = 0 \;\Rightarrow\; \tau b_2(x, \xi', \tau, \sigma) \gtrsim \lambda_\tau^3 \gtrsim \lambda_{T,\tau}^3. \tag{9.E.13}$$

We define the following quantity

$$\gamma(x, \xi', \tau, \sigma) = \left(\frac{\tilde{p}_1(x, \xi', \tau)}{2\tau}\right)^2 + (\partial_{x_n}\varphi)^2 \tilde{p}_2(x, \xi', \tau, \sigma), \qquad \mathrm{Op}_T(\gamma) \in \mathcal{D}^2_{T,\tau}, \tag{9.E.14}$$

where $\tilde{p}_2$ and $\tilde{p}_1$ are defined in (4.4.10) and (4.4.11). Observing that $p_1(x, \xi, \tau) = 0$ is equivalent to $\xi_n = -(\partial_{x_n}\varphi)^{-1}\tilde{r}(x, \xi, \mathrm{d}_{x'}\varphi)$, we obtain, for $x \in \overline{V}$, $\xi \in \mathbb{R}^n$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$,

$$\begin{cases} \gamma(x, \xi', \tau, \sigma) = 0 \\ \xi_n = -(\partial_{x_n}\varphi)^{-1}\tilde{r}(x, \xi, \mathrm{d}_{x'}\varphi) \end{cases} \iff p_2(x, \xi, \tau, \sigma) = p_1(x, \xi, \tau) = 0 \Longrightarrow \tau b_2(x, \xi', \tau, \sigma) \gtrsim \lambda_{T,\tau}^3.$$

and finally for $x \in \overline{V}$, $\xi' \in \mathbb{R}^{n-1}$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$,

$$\gamma(x, \xi', \tau, \sigma) = 0 \Longrightarrow \tau b_2(x, \xi', \tau, \sigma) \gtrsim \lambda_{T,\tau}^3. \tag{9.E.15}$$

as both sides of the implication do not involve the variable ξ_n. Moreover, if $\gamma = 0$, taking τ sufficiently large with respect to $|\sigma| \geq 1$, we have $\tau^2 \lesssim |\xi'|^2 \lesssim \tau^2$ and (9.E.15) yields

$$\gamma(x, \xi', \tau, \sigma) = 0 \Longrightarrow b_2(x, \xi', \tau, \sigma) \gtrsim \lambda_{T,\tau}^2. \tag{9.E.16}$$

We may now state the following positivity result :

Lemma 9.E.1. *There exists $m_0 > 0$, $\tau_0 > 0$ and $C > 0$ such that*

$$m\lambda_{T,\tau}^{-2}\gamma(x,\xi',\tau,\sigma)^2 + b_2(x,\xi',\tau,\sigma) \geq C\lambda_{T,\tau}^2$$

for all $(x,\xi') \in \overline{V}^+ \times \mathbb{R}^{n-1}$, $|\sigma| \geq 1$, $\tau \geq \tau_0|\sigma|$ and $m \geq m_0$.

The proof is given in Appendix 9.E.4. We can apply the Gårding inequality in the tangential directions to obtain, for $m = m_0$ to remain fixed in what follows,

$$m\,\mathrm{Re}\left(\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2)v, v\right)_{L^2(\mathbb{R}^{n-1})} + \mathrm{Re}\left(\mathrm{Op}_T(b_2)v, v\right)_{L^2(\mathbb{R}^{n-1})} \gtrsim |v|_{1,\tau}^2. \tag{9.E.17}$$

Then, by symbolic calculus

$$\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2) = \mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma)\,\mathrm{Op}_T(\gamma) \mod \Psi_{T,\tau}^1,$$

and thus there exist $c_0 \in \mathcal{S}_{T,\tau}^0$ and $c_1 \in \mathcal{S}_{T,\tau}^1$ such that

$$\left(m\,\mathrm{Op}_T(\lambda_{T,\tau}^{-2}\gamma^2)v, v\right)_{L^2(\mathbb{R}^{n-1})} = \left(\mathrm{Op}_T(\gamma)v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^{n-1})} + \left(\mathrm{Op}_T(c_1)v, v\right)_{L^2(\mathbb{R}^{n-1})}. \tag{9.E.18}$$

In terms of the symbols p_2 and p_1, from (9.E.14), γ reads

$$\begin{aligned}
\gamma(x,\xi',\tau,\sigma) &= (\partial_{x_n}\varphi)^2\tilde{p}_2(x,\xi',\tau,\sigma) + \left(\frac{\tilde{p}_1(x,\xi',\tau)}{2\tau}\right)^2 \\
&= (\partial_{x_n}\varphi)^2\left(p_2(x,\xi,\tau,\sigma) - \xi_n^2\right) + \left(\frac{\tau^{-1}}{2}p_1(x,\xi,\tau) - \xi_n\partial_{x_n}\varphi\right)^2 \\
&= (\partial_{x_n}\varphi)^2 p_2(x,\xi,\tau,\sigma) + \tau^{-1}p_1(x,\xi,\tau)\left(\frac{\tau^{-1}}{4}p_1(x,\xi,\tau) - \xi_n\partial_{x_n}\varphi\right) \\
&= (\partial_{x_n}\varphi)^2 p_2(x,\xi,\tau,\sigma) + \tau^{-1}r_1(x,\xi)p_1(x,\xi,\tau),
\end{aligned}$$

where $r_1(x,\xi) = \dfrac{1}{2}\left(\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi) - \xi_n\partial_{x_n}\varphi\right) \in \mathcal{S}_\tau^1$. We then have

$$\mathrm{Op}_T(\gamma) = \tau^{-1}\,\mathrm{Op}(r_1)P_1 + (\partial_{x_n}\varphi)^2 P_2 + B_1, \tag{9.E.19}$$

with $B_1 \in \mathcal{D}_\tau^1$. Going back to (9.E.17), and using (9.E.18) and (9.E.19), and integrating in the x_n-variable between 0 and $+\infty$

$$\begin{aligned}
&\mathrm{Re}\left(\left((\partial_{x_n}\varphi)^2 P_2 + \tau^{-1}\,\mathrm{Op}(r_1)P_1 + B_1\right)\right)v, \mathrm{Op}_T(c_0)v\Big)_{L^2(\mathbb{R}^n_+)} \\
&\qquad + \mathrm{Re}\left(\mathrm{Op}_T(c_1)v, v\right)_{L^2(\mathbb{R}^n_+)} + \mathrm{Re}\left(\mathrm{Op}_T(b_2)v, v\right)_{L^2(\mathbb{R}^n_+)} \gtrsim ||\,\mathrm{Op}_T(\lambda_{T,\tau}^1)v||_{L^2(\mathbb{R}^n_+)}^2.
\end{aligned} \tag{9.E.20}$$

Thus, (9.E.12) reads

$$\begin{aligned}
||P_{\varphi,\sigma}v||_{L^2(\mathbb{R}^n_+)}^2 \geq\; & C\tau||\,\mathrm{Op}_T(\lambda_{T,\tau}^1)v||_{L^2(\mathbb{R}^n_+)} + ||P_2v||_{L^2(\mathbb{R}^n_+)}^2 + ||P_1v||_{L^2(\mathbb{R}^n_+)}^2 + \tau\,\mathrm{Re}\,\tilde{\mathcal{B}}(v) \\
&+ \mathrm{Re}\left(\left(\tau b_0 P_2 + \mathrm{Op}_T(b_1)P_1 + \tau\,\mathrm{Op}(c_1')\right)v, v\right)_{L^2(\mathbb{R}^n_+)} - \mathrm{Re}\Big(\tau\Big(\mathrm{Op}_T(c_1)v, v\Big)_{L^2(\mathbb{R}^n_+)} \\
&\qquad + \Big(\left(\tau(\partial_{x_n}\varphi)^2 P_2 + \mathrm{Op}(r_1)P_1 + \tau B_1\right)v, \mathrm{Op}_T(c_0)v\Big)_{L^2(\mathbb{R}^n_+)}\Big).
\end{aligned} \tag{9.E.21}$$

Yet, using the definition of r_1 and integrating by parts with respect to x_n,

$$\begin{aligned}
2\,\mathrm{Re}\left(\mathrm{Op}(r_1)P_1v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} &= \mathrm{Re}\left(\tilde{r}(x,\mathrm{d}_{x'}\varphi, D')P_1v, \mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} \\
&- \mathrm{Re}\left(P_1v, D_n\partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} + \mathrm{Im}\left(P_1v_{|_{x_n=0}}, \partial_{x_n}\varphi\,\mathrm{Op}_T(c_0)v_{|_{x_n=0}}\right)_{L^2(\mathbb{R}^{n-1})}.
\end{aligned} \tag{9.E.22}$$

We set $\mathcal{B}(v) = \tilde{\mathcal{B}}(v) + \frac{1}{2}\tau^{-1}\operatorname{Im}\left(P_1 v_{|_{x_n=0}}, \partial_{x_n}\varphi \operatorname{Op}_T(c_0) v_{|_{x_n=0}}\right)_{L^2(\mathbb{R}^{n-1})}$, and this is precisely the boundary quadratic form stated in the proposition. The end of the proof is devoted to the handling of the remainder terms. Using the Young inequality, we obtain

$$\begin{aligned}\Big|\operatorname{Re}\left(\tau b_0 P_2 v + \operatorname{Op}_T(b_1)P_1 v, v\right)_{L^2(\mathbb{R}^n_+)}\\ - \operatorname{Re}\left(\tau(\partial_{x_n}\varphi)^2 P_2 v + \frac{1}{2}\tilde{r}(x, D', \mathrm{d}_{x'}\varphi)P_1 v + \tau B_1 v, \operatorname{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)} + \frac{1}{2}\operatorname{Re}\left(P_1 v, D_n \partial_{x_n}\varphi \operatorname{Op}_T(c_0)v\right)_{L^2(\mathbb{R}^n_+)}\\ + \tau \operatorname{Re}\left(\left(\operatorname{Op}(c_1') - \operatorname{Op}_T(c_1)\right)v, v\right)_{L^2(\mathbb{R}^n_+)}\Big| \lesssim \tau^{-1/2}\left(||P_2 v||^2_{L^2(R^n_+)} + ||P_1 v||^2_{L^2(\mathbb{R}^n_+)}\right)\\ + \tau^{1/2}\left(||\operatorname{Op}_T(\lambda^1_{T,\tau})v||^2_{L^2(\mathbb{R}^n_+)} + ||D_n v||^2_{L^2(\mathbb{R}^n_+)}\right). \quad (9.E.23)\end{aligned}$$

Injecting this estimate in (9.E.21), and taking τ sufficiently large, we obtain

$$||P_{\varphi,\sigma} v||^2_{L^2(\mathbb{R}^n_+)} \geq C\left(\tau||\operatorname{Op}_T(\lambda^1_{T,\tau})v||^2_{L^2(\mathbb{R}^n_+)} + ||P_1 v||^2_{L^2(\mathbb{R}^n_+)}\right) + \tau \operatorname{Re}\mathcal{B}(v) - C'\tau^{1/2}||D_n v||^2_{L^2(\mathbb{R}^n_+)}.$$

Moreover, we have $\xi_n = \dfrac{\tau^{-1}p_1(x,\xi,\tau)}{2\partial_{x_n}\varphi} - \dfrac{\tilde{r}(x,\xi',\mathrm{d}_{x'}\varphi)}{\partial_{x_n}\varphi}$, and this yields

$$\tau||D_n v||^2_{L^2(\mathbb{R}^n_+)} \lesssim \tau^{-1}||P_1 v||^2_{L^2(\mathbb{R}^n_+)} + \tau||\operatorname{Op}_T(\lambda^1_{T,\tau})v||^2_{L^2(\mathbb{R}^n_+)},$$

which gives, for τ sufficiently large, the sought result

$$||P_{\varphi,\sigma} v||^2_{L^2(\mathbb{R}^n_+)} \geq C\left(\tau||\operatorname{Op}_T(\lambda_{T,\tau})v||^2_{L^2(\mathbb{R}^n_+)} + \tau||D_n v||^2_{L^2(\mathbb{R}^n_+)}\right) + \tau \operatorname{Re}\mathcal{B}(v).$$

□

9.E.4 Proof of Lemma 9.E.1

We set $A_m(x,\xi',\tau,\sigma) = m\lambda_{T,\tau}^{-2}\gamma^2(x,\xi',\tau,\sigma) + b_2(x,\xi',\tau,\sigma)$ with $m > 1$. Observe that A_m is homogeneous of order two in the variable (ξ',τ,σ). We may restrict our analysis to the compact set $L = \overline{V} \times \mathcal{K}$ where $\mathcal{K} := \{(\xi',\tau,\sigma) \in \mathbb{R}^{n-1}\times\mathbb{R}^+\times\mathbb{R}^+,\ |(\xi',\tau,\sigma)| = 1\}$. On L, we have $\gamma^2 \geq 0$, and from (9.E.16), having $\gamma = 0$ implies $b_2 \geq C > 0$. Observe that A_m is of the form $mf(y) + g(y)$, with y laying in L, and f and g continuous on L. In addition, $f(y) = 0$ implies $g(y) > 0$. Now consider the compact set $E := \{y \in L,\ f(y) = 0\}$. By a continuity argument there exists an open neighborhood F of E such that $\inf_F g > 0$. Then on $L \setminus F$ we have

$$mf(y) + g(y) \geq m\inf f + \inf g > 0$$

by choosing m sufficiently large. This yields $A_m \geq C > 0$ on L. We then conclude by homogeneity.

□

9.E.5 Proof of Lemma 4.8.3

By a compactness argument, it suffices to prove the estimate with $||u||_{H^1(B(x_0,r_0))}$ in the left hand side, for any x_0 and $r_0 > 0$ such that $B(x_0,r_0) \Subset \Omega$, where $B(x_0,r_0)$ denotes the open ball centered at x_0 with radius r_0.

First, remark that we can assume that u satisfies

$$||P_\sigma u||_{L^2(\Omega)} \leq ||u||_{H^1(\Omega)}, \quad (9.E.24)$$

otherwise the result is clear. As Ω is connected, we can choose $y_0 \in \omega_I$ and find a continuous path Γ such that

$$\Gamma(0) = x_0 \quad \text{and } \Gamma(1) = y_0.$$

Define $r := \frac{1}{6}\min(r_1, r_2)$ where $r_1 := \mathrm{d}(\partial\Omega, \Gamma)$ and $r_2 := \mathrm{d}(y_0, \partial\omega_I)$. Now we can define a sequence $(t_j)_j$ by

$$t_0 = 0 \quad \text{and } t_{j+1} = \inf E_j, \quad j \geq 0,$$

where $E_j = \{t > t_j \mid \Gamma(t) \notin B(\Gamma(t_j), r)\}$. This sequence is finite by compactness, and then we can define $J := \min\{j \in \mathbb{N},\ E_j = \emptyset\}$ and $t_J = 1$. Then we consider $(x^j)_j$ such that $x^j = \Gamma(t_j)$. Let us assume for a moment that we have the following inequality for some $\mu > 0$ and $C > 0$

$$||u||_{H^1(B(x^j,r))} \lesssim e^{C|\sigma|}||u||^{1-\mu}_{H^1(\Omega)} \left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^{j+1},r))}\right)^\mu \quad 0 \leq j \leq J-1. \tag{9.E.25}$$

This, with (9.E.24) yields

$$||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^j,r))} \lesssim e^{C|\sigma|}||u||^{1-\mu}_{H^1(\Omega)} \left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^{j+1},r))}\right)^\mu, \quad 0 \leq j \leq J-1,$$

and by induction we have, for some $\mu' > 0$ and $C' > 0$

$$||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x_0,r))} \lesssim e^{C'|\sigma|}||u||^{1-\mu'}_{H^1(\Omega)} \left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(y_0,r))}\right)^{\mu'}, \quad 0 \leq j \leq J-1.$$

As P is elliptic we have the estimate

$$||u||_{H^1(B(y_0,r))} \lesssim ||Pu||_{L^2(\omega_I)} + ||u||_{L^2(\omega_I)} \leq ||P_\sigma u||_{L^2(\Omega)} + (1+\sigma)||u||_{L^2(\omega_I)},$$

and this gives the result. Let us now prove (9.E.25). We recall that we have the following Carleman estimate away from the boundary (see for instance [Hör09]). Let V be an open bounded subset of $\mathbb{R}^n$, and φ be a weight function satisfying the sub-ellipticity condition in $\overline{V}$. Then, we have

$$\tau^3||e^{\tau\varphi}u||^2_{L^2(V)} + \tau||e^{\tau\varphi}Du||^2_{L^2(V)} \lesssim ||e^{\tau\varphi}P_\sigma u||_{L^2(V)}, \tag{9.E.26}$$

for all $u \in C_0^\infty(V)$ and τ sufficiently large with respect to σ. We shall prove here the following inequality

$$||u||_{H^1(B(x^j,3r))} \lesssim e^{C|\sigma|}||u||^{1-\mu}_{H^1(\Omega)} \left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^j,r))}\right)^\mu.$$

Let us set $\varphi = e^{-\lambda\psi}$ with $\psi(x) = |x - x^j|^2$ and $V = B(x^j, 4r) \setminus B(x^j, r/10)$. Define the cut-off function :

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x - x^j| < r/4 \text{ or } |x - x^j| > 15r/4 \\ 1 & \text{if} \quad 3r/4 < |x - x^j| < 13r/4. \end{cases}$$

Observe that the weight function φ satisfies the sub-ellipticity condition in $\overline{V}$ for λ sufficiently large. We can thus apply (9.E.26) to χu and this yields

$$\tau^3||e^{\tau\varphi}\chi u||^2_{L^2(V)} + \tau||e^{\tau\varphi}D\chi u||^2_{L^2(V)} \lesssim ||e^{\tau\varphi}P_\sigma\chi u||_{L^2(V)}. \tag{9.E.27}$$

On the right hand side of (9.E.27), we have

$$\begin{aligned} ||e^{\tau\varphi}P_\sigma\chi u||_{L^2(V)} &\lesssim ||e^{\tau\varphi}\chi P_\sigma u||_{L^2(\Omega)} + ||e^{\tau\varphi}[P_\sigma, \chi]u||_{L^2} \\ &\lesssim e^{\tau C_3}\left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^j,r))}\right) + e^{\tau C_1}||u||_{H^1(\Omega)}, \end{aligned}$$

where $C_3 = e^{-\lambda r/4}$ and $C_1 = e^{-\lambda 13r/4}$. Here, we used that $[P_\sigma, \chi]$ is an differential operator of order 1, supported in the region where χ varies, and that φ decreases as $|x - x^j|$ increases. We can restrict the left hand side of (9.E.27) to $B(x^j, 3r) \setminus B(x^j, r) =: \tilde{V}$, where $\chi = 1$. This yields

$$||e^{\tau\varphi}\chi u||_{L^2(\tilde{V})} + ||e^{\tau\varphi}D\chi u||_{L^2(\tilde{V})} \gtrsim e^{\tau C_2}||u||_{H^1(\tilde{V})},$$

where $C_2 = \varphi(3r)$. Hence

$$\begin{aligned} ||u||_{H^1(B(x^j,3r))} &\lesssim ||u||_{H^1(B(x^j,r))} + ||u||_{H^1(\tilde{V})} \\ &\lesssim e^{\tau(C_3 - C_2)}\left(||P_\sigma u||_{L^2(\Omega)} + ||u||_{H^1(B(x^j,r))}\right) + e^{-\tau(C_2 - C_1)}||u||_{H^1(\Omega)}. \end{aligned} \tag{9.E.28}$$

Observe that $C_1 < C_2 < C_3$. It remain to apply Lemma 4.8.4 to conclude the proof. □

9.E.6 Proof of Lemma 4.8.4

First observe that $C = 0$ implies $A = 0$, and $B = 0$ also implies $A = 0$ by letting $\tau \to +\infty$. We set $f(\tau) = e^{\beta\tau}B + e^{-\gamma\tau}C$ for all $\tau \in \mathbb{R}$. This function reach its minimum at τ_1 satisfying $e^{\tau_1} = \left(\frac{\gamma C}{\beta B}\right)^{\frac{1}{\beta+\gamma}}$. First assume that $\tau_1 \geq \tilde{\tau}$. Then

$$\begin{aligned} A \leq e^{\beta\tau_1}B + e^{-\gamma\tau_1}C &\leq \left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}} C^{\frac{\beta}{\beta+\gamma}} B^{1-\frac{\beta}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{-\gamma}{\beta+\gamma}} B^{\frac{\gamma}{\beta+\gamma}} C^{1-\frac{\gamma}{\beta+\gamma}} \\ &\leq \left(\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta+\gamma}} + \left(\frac{\gamma}{\beta}\right)^{\frac{-\gamma}{\beta+\gamma}}\right) B^{\frac{\gamma}{\beta+\gamma}} C^{\frac{\beta}{\gamma+\beta}}. \end{aligned}$$

Second, assume that $\tau_1 < \tilde{\tau}$. We then have $\gamma e^{-\gamma\tilde{\tau}}C \leq \beta e^{\beta\tilde{\tau}}B$. Hence,

$$A \leq C \leq C^{\frac{\beta}{\beta+\gamma}} C^{\frac{\gamma}{\beta+\gamma}} \leq \left(\frac{\beta}{\gamma}\right)^{\frac{\gamma}{\beta+\gamma}} e^{\gamma\tilde{\tau}} B^{\frac{\gamma}{\beta+\gamma}} C^{\frac{\beta}{\beta+\gamma}},$$

which gives the result. □

Annexes du chapitre 6

10.F Proof of technical lemmas

We recall that we have the following notation. Let $S_0 > 0$ and $\alpha \in (0, S_0/2)$. Let $Z = (-S_0, S_0) \times \Omega$ and let $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$ in the case where Ω is a smooth bounded domain of $\mathbb{R}^d$ when we shall consider an inverse square potential. In the case where Ω is a polygonal domain of $\mathbb{R}^2$ as described in Section 6.1.2, we have the following notation. Let $Z = (-S_0, S_0) \times \Omega$ and let $Y = (-S_0 + \alpha, S_0 - \alpha) \times \Omega$. Before giving a proof for Lemma 6.3.3, we state several lemmata that shall play a central role, which can be found in [BD10]. We recall that

$$Q = -\partial_s^2 + P,$$

as defined in (6.2.1), and

$$\mathcal{T} = -\partial_s^2 + T,$$

as defined in (6.2.3). In the interior, we have the following interpolation estimate of Hölder-type.

Lemma 10.F.1. *Let $q \in Y$, let $0 < r_0 < r_1 < r_2 < \cdots < r_6$, such that $B(q, r_6) \subset Y \setminus \{x = 0\}$ and let $\alpha > 0$ such that $\alpha > r_0^{-2}$. Then there exists a constant $C > 0$ such that for all $u \in C_0^\infty(B(q, r_6))$ we have*

$$||u||_{H^1(B(q,r_3))} \leq C \left(||Qu||_{L^2(B(q,r_6))} + ||u||_{H^1(B(q,r_2))} \right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(B(q,r_6))}$$

with

$$\Theta = \frac{g(r_3) - g(r_4)}{g(r_1) - g(r_3)}, \quad \text{where } g(r) = e^{-\alpha r^2}.$$

The proof is classical, and comes from a Carleman estimate with weight $\tilde{g}(y) = e^{-\alpha |y-q|^2}$. We provide the proof for the sake of completeness.

Proof. Let us show that the weight function $\tilde{g}$ satisfies the sub-ellipticity condition in the annulus $C(q, r_0, r_6) := \{y \in Y,\ r_0 < ||y| - q| < r_6\}$. The principal symbol $q_{\tilde{g}}$ of the conjugated operator $Q_{\tilde{g}} = e^{\tau \tilde{g}} Q e^{-\tau \tilde{g}}$ far away from $\{x = 0\}$ (thus the potential $\mu |x|^{-2}$ can be seen as a low-order term) is given by

$$\begin{aligned} q_{\tilde{g}}(y, \xi, \tau) &= |\xi|^2 - \tau^2 |\nabla \tilde{g}|^2 + 2i\tau \nabla \tilde{g}.\xi \\ &:= q_2 + i q_1, \end{aligned}$$

with $q_2(y, \xi, \tau) = |\xi|^2 - \tau^2 |\nabla \tilde{g}|^2 = |\xi|^2 - 4\alpha^2 \tau^2 \tilde{g}^2 |y - q|^2$ and $q_1(y, \xi, \tau) = 2\tau \nabla \tilde{g}.\xi = -4\alpha\tau \tilde{g}(y - q).\xi$. Computing the Poisson bracket yields

$$\begin{aligned} \{q_2, q_1\} &= 4\tau (\mathrm{d}^2 \tilde{g}(\xi, \xi) + \tau^2 \mathrm{d}^2 \tilde{g}(\nabla \tilde{g}, \nabla \tilde{g})) \\ &= 4\tau \left(-2\alpha \tilde{g} |\xi|^2 + 4\alpha^2 \tilde{g} \left((y - q).\xi \right)^2 - 8\tau^2 \alpha^3 \tilde{g}^2 |y - q|^2 + 16 \tau^2 \alpha^4 \tilde{g}^3 |y - q|^4 \right). \end{aligned}$$

At this point, we see that on the characteristic set $q_{\tilde{g}} = 0$ we have $(y - q).\xi = 0$ and $|\xi|^2 = 4\alpha^2 \tau^2 \tilde{g}^2 |y - q|^2$, and then

$$\begin{aligned} \{q_2, q_1\} &= 64 \tilde{g}^3 \tau^3 \alpha^3 |y - q|^2 \left(\alpha |y - q|^2 - 1 \right) \\ &\geq 64 \tilde{g}^3 \tau^3 \alpha^3 r_0^2 \left(\alpha r_0^2 - 1 \right), \end{aligned}$$

and we see that the Poisson bracket is positive on the characteristic set as soon as $\alpha > r_0^{-2}$. Then the following Carleman estimate holds on the annulus $C(q, r_1, r_5)$: for all $\zeta \in C_0^\infty(C(q, r_1, r_5))$, we have

$$\tau^3 ||e^{\tau \tilde{g}} \zeta||^2_{L^2(C(q,r_1,r_5))} + \tau ||e^{\tau \tilde{g}} \nabla \zeta||^2_{L^2(C(q,r_1,r_5))} \lesssim ||e^{\tau \tilde{g}} Q\zeta||^2_{L^2(C(q,r_1,r_5))}. \tag{10.F.1}$$

Consider a cut-off function $\chi \in C_0^\infty(C(q, r_0, r_6))$ such that

$$\chi(y) = \begin{cases} 1 & \text{if } y \in C(q, r_2, r_4) \\ 0 & \text{if } y \in C(q, r_0, r_1) \cup C(q, r_5, r_6), \end{cases}$$

and take a function $u \in H^1(Y)$ such that $Qu \in L^2(Z)$. Then, we apply the Carleman estimate (10.F.1) to χu to obtain

$$\begin{aligned} \tau^3 ||e^{\tau \tilde{g}} u||^2_{L^2(C(q,r_2,r_3))} + \tau ||e^{\tau \tilde{g}} \nabla u||^2_{L^2(C(q,r_2,r_3))} \lesssim ||e^{\tau \tilde{g}} Qu||^2_{L^2(C(q,r_1,r_5))} \\ + ||e^{\tau \tilde{g}} u||^2_{H^1(C(q,r_1,r_2))} + ||e^{\tau \tilde{g}} u||^2_{H^1(C(q,r_4,r_5))}. \end{aligned}$$

Using the weight function, we have

$$e^{\tau g(r_3)} ||u||^2_{H^1(C(q,r_2,r_3))} \lesssim e^{\tau g(r_1)} \left(||Qu||^2_{L^2(B(q,r_6))} + ||u||_{H^1(B(q,r_2))} \right) + e^{\tau g(r_4)} ||u||_{H^1(C(q,r_4,r_5))}.$$

Since the estimate is also true on $B(q, r_2)$, and $g(r_3) < g(r_1)$, we obtain

$$||u||_{H^1(B(r_3))} \lesssim e^{\tau(g(r_1)-g(r_3))} \left(||Qu||_{L^2(B(q,r_6))} + ||u||_{H^1(B(q,r_2))} \right) + e^{\tau(g(r_4)-g(r_3))} ||u||_{H^1(C(q,r_4,r_5))}. \tag{10.F.2}$$

We set $A = g(r_1) - g(r_3) > 0$ and $B = g(r_4) - g(r_3) < 0$. We can optimize (10.F.2) with respect to τ as in [Rob95] to obtain

$$||u||_{H^1(B(r_3))} \lesssim \left(||Qu||_{L^2(B(q,r_6))} + ||u||_{H^1(B(q,r_2))} \right)^{1-\delta} ||u||^\delta_{H^1(C(q,r_4,r_5))}, \tag{10.F.3}$$

where $\delta = -\frac{B}{A-B}$. It remains to see that $\delta = \frac{\Theta}{\Theta+1}$ with $\Theta = \frac{B}{A}$ to end the proof. □

We shall use the following lemma, in case of a polygon problem near a corner of the boundary. The same Lemma can be obtained near of singular point of the interface using the same computations. It is the counterpart of Lemma 10.F.1. We recall that for $0 < \ell_1 < \ell_2$, and let $p = (s_p, p) \in \mathcal{E} \times (-S_0 + \alpha, S_0 - \alpha)$. We shall assume without any loss of generality that $p = 0$, up to performing a translation. In coordinates $(s, x) = (s, x_1, x_2)$, we shall consider the disk $\mathbb{D}(\ell_1, \ell_2) = B((0, \ell_1, 0), \ell_2) \cap \{x_2 = 0\}$. We then define $\mathbb{T}(\ell_1, \ell_2) = \tilde{\mathbb{T}}(\ell_1, \ell_2) \cap Y$, where $\tilde{\mathbb{T}}(\ell_1, \ell_2)$ is the revolution of the disk $\mathbb{D}(\ell_1, \ell_2)$ around the axis $\{(x_1, x_2) = (0, 0)\}$ (see Figure 6.7).

Lemma 10.F.2. *Let $0 < r_0 < r_1 < r_2 < \cdots < r_6$, and $q > r_6 > 0$ sufficiently small such that $\mathbb{T}(q, r_6)$ do not contain any other wedges. Let $\alpha > 0$ such that $\alpha > r_0^{-2}$. Then there exists a constant $C > 0$ such that we have*

$$||u||_{H^1(\mathbb{T}(q,r_3))} \leq C \left(||\mathcal{T}u||_{L^2(\mathbb{T}(q,r_6))} + ||u||_{H^1(\mathbb{T}(q,r_2))} \right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(\mathbb{T}(q,r_6))} \tag{10.F.4}$$

for all $u \in H^1(Y)$ satisfying $\mathcal{T}u \in L^2(Z)$ and homogeneous Dirichlet or Neumann boundary conditions on $\mathbb{T}(q, r_6) \cap \partial\Omega \times (-S_0 + \alpha, S_0 - \alpha)$, and with

$$\Theta = \frac{g(r_3) - g(r_4)}{g(r_1) - g(r_3)}, \quad \text{where } g(r) = e^{-\alpha r^2}.$$

Proof. We take $\tilde{g}(s, x) = e^{-\alpha(|x|^2 - \ell_1^2 + s^2)}$. From the computations that are led in the proof of Lemma 10.F.1, such weight function also satisfies the sub-ellipticity property in $W := \mathbb{T}(q, r_6) \setminus \mathbb{T}(q, r_0)$, provided that $\alpha > r_0^{-2}$. By a compactness argument, it remains to prove that (10.F.4) holds in a neighborhood of the

boundary $\mathbb{T}(q,r_6)\cap\partial\Omega\times(-S_0+\alpha,S_0-\alpha)$. Let us prove that a Carleman estimate of the form (10.F.1) holds in this configuration, even in a neighborhood of the boundary. We shall write $\mathfrak{D}:=\partial_s+\Delta$, and the conjugated operator reads

$$\mathfrak{D}_{\tilde g}:=e^{\tau\tilde g}\mathfrak{D}e^{-\tau\tilde g}=P_1+P_2+R,$$

where

$$P_1=\mathfrak{D}+\tau^2|\nabla_{s,x}\tilde g|^2,\quad P_2=-2\tau\mathfrak{D}\tilde g-2\tau\nabla_{s,x}\tilde g.\nabla_{s,x},\quad R=\tau\mathfrak{D}\tilde g.$$

We take a function $w=v_{|W}$ where $v\in C_0^\infty(\tilde{\mathbb{T}}(q,r_6)\setminus\tilde{\mathbb{T}}(q,r_0)$. We have

$$2\,\mathrm{Re}\int_W P_1w\overline{P_2w}\le 2\int_W|\mathfrak{D}_{\tilde g}w|^2+2\int_W|Rw|^2.$$

We shall compute the scalar product $2\int_W P_1wP_2w$, and obtain a bound from below. We denote by I_{ij} the product between the i-th term of P_1 and the j-th term of P_2 and we denote by $\partial W=\overline{W}\cap\partial\Omega\times(S_0-\alpha,S_0+\alpha)$. We have

$$\begin{aligned}I_{11}&=-2\tau\,\mathrm{Re}\int_W\mathfrak{D}w(\mathfrak{D}\tilde g)\overline{w}=2\tau\int_W\mathfrak{D}\tilde g|\nabla_{s,x}w|^2+2\tau\,\mathrm{Re}\int_W\nabla_{s,x}\mathfrak{D}\tilde g.\nabla_{s,x}w\overline{w}-2\tau\,\mathrm{Re}\int_{\partial W}\mathfrak{D}\tilde g\partial_\nu w\overline{w}\\&=2\tau\int_W\mathfrak{D}\tilde g|\nabla_{s,x}w|^2-\tau\int_W\mathfrak{D}^2\tilde g|w|^2-2\tau\,\mathrm{Re}\int_{\partial W}\mathfrak{D}\tilde g\partial_\nu w_{|\partial W}\overline{w}_{|\partial W}+\tau\int_{\partial W}\partial_\nu\mathfrak{D}\tilde g|w_{|\partial W}|^2\\&=2\tau\int_W\mathfrak{D}\tilde g|\nabla_{s,x}w|^2-\tau\int_W\mathfrak{D}^2\tilde g|w|^2-2\tau\,\mathrm{Re}\int_{\partial W}\mathfrak{D}\tilde g_{|\partial W}\partial_\nu w_{|\partial W}\overline{w}_{|\partial W},\end{aligned}$$

since $\tilde g$ the normal derivative of $\tilde g$ vanish ($\tilde g$ is a radial function). We also have

$$\begin{aligned}I_{12}&=-2\tau\,\mathrm{Re}\int_W\mathfrak{D}w\nabla_{s,x}\tilde g.\overline{\nabla_{s,x}w}=2\tau\,\mathrm{Re}\int_W\nabla_{s,x}w.\nabla_{s,x}(\nabla_{s,x}\tilde g.\overline{\nabla_{s,x}w})\\&=2\tau\int_W\nabla^2_{s,x}\tilde g(\nabla_{s,x}w,\nabla_{s,x}w)+2\tau\int_W\nabla^2_{s,x}w(\nabla_{s,x}w,\nabla_{s,x}\tilde g)\\&=2\tau\int_W\nabla^2_{s,x}\tilde g(\nabla_{s,x}w,\nabla_{s,x}w)-\tau\int_W\mathfrak{D}\tilde g|\nabla_{s,x}w|^2.\end{aligned}$$

Note that there are no boundary terms arising here, using again the fact that $\tilde g$ is a radial function. Next,

$$I_{21}=-2\tau^3\int_W|\nabla_{s,x}\tilde g|^2\mathfrak{D}\tilde g|w|^2,$$

and

$$I_{22}=-2\tau^3\,\mathrm{Re}\int_W|\nabla_{s,x}\tilde g|^2\nabla_{s,x}\tilde g.\nabla_{s,x}w\overline{w}=\tau^3\int_W\left(|\nabla_{s,x}\tilde g|^2\mathfrak{D}\tilde g+2\nabla^2_{s,x}\tilde g(\nabla_{s,x}\tilde g,\nabla_{s,x}\tilde g)\right)|w|^2.$$

Again, no boundary terms are arising here from the same reason. As $\tilde g$ satisfies the sub-ellipticity condition, we can obtain from these computations, by taking τ large

$$\tau^3\int_W|w|^2+\tau\int_W|\nabla_{s,x}w|^2\lesssim\int_W|\mathfrak{D}w|^2+2\tau\int_{\partial W}\mathfrak{D}\tilde g_{|\partial W}\partial_\nu w_{|\partial W}w_{|\partial W}.$$

Now setting $u=e^{-\tau\tilde g}w$, we have that w has exactly the same boundary conditions than u, since the normal derivative of the weight vanish. As a result, the following Carleman estimate holds

$$\tau^3\int_W e^{\tau\tilde g}|u|^2+\tau\int_W e^{\tau\tilde g}|\nabla_{s,x}u|^2\lesssim\int_W e^{\tau\tilde g}|\mathfrak{D}u|^2.$$

The end of the proof follows exactly the proof of Lemma 10.F.1. □

In the case of a inverse square potential, we shall use Lemma 10.F.1 with Lemma 10.F.3 to prove Lemma 6.3.3. To prove Lemma 6.3.15 in the case of a polygon problem, we have to use Lemma 10.F.4 and Lemma 10.F.3. The following Lemma 10.F.3 allows to propagate information using the homogeneity. Note that both operators Q and $\mathcal{T}$ are homogeneous with respect to 0, that is, if we set $(\tilde{s},\tilde{x}) = k(s,x)$ we have

$$-\partial_{\tilde{s}}^2 - \Delta_{\tilde{x}} - \mu\frac{1}{|\tilde{x}|^2} = \frac{1}{k}\left(-\partial_s^2 - \Delta_x - \mu\frac{1}{|x|^2}\right), \quad -\partial_{\tilde{s}}^2 - \Delta_{\tilde{x}} = \frac{1}{k}\left(-\partial_s^2 - \Delta_x\right).$$

Lemma 10.F.3. *Let $\tilde{q} \in Y$ and let $0 < \tilde{r}_1 < \cdots < \tilde{r}_6$. For $k \in (0,1)$, we define $r_i = k\tilde{r}_i$ and $q = k\tilde{q}$. Assume that $B(\tilde{q},\tilde{r}_6) \subset Y \setminus \{x = 0\}$ and $B(q,r_6) \subset Y \setminus \{x = 0\}$. If*

$$||u||_{H^1(B(\tilde{q},\tilde{r}_3))} \leq C\left(||Qu||_{L^2(B(\tilde{q},\tilde{r}_6))} + ||u||_{H^1(B(\tilde{q},\tilde{r}_2))}\right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(B(\tilde{q},\tilde{r}_6))}$$

then u also satisfies

$$||u||_{H^1(B(q,r_3))} \leq \frac{C}{k}\left(||Qu||_{L^2(B(q,r_6))} + ||u||_{H^1(B(q,r_2))}\right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(B(q,r_6))} \tag{10.F.5}$$

Remark 10.F.4. *It should be noted that in Lemma 10.F.3, the estimate (10.F.5) holds with the same constant Θ.*

Proof. Noting that, if we write $\tilde{u}(\tilde{y}) = u(y)$, for $y = q + k(\tilde{y} - \tilde{q})$, we obtain

$$k^{d/2}||\tilde{u}||_{H^1(B(\tilde{q},\tilde{r}_i))} \leq ||u||_{H^1(B(q,r_i))} \leq k^{d/2-1}||\tilde{u}||_{H^1(B(\tilde{q},\tilde{r}_i))},$$

and, using the homogeneity of the operator Q, we have

$$||Qu||_{L^2(B(q,r_i))} = \gamma^{d/2-2}||Q\tilde{u}||_{L^2(B(\tilde{q},\tilde{r}_i))},$$

the result follows. □

We shall only carry out a proof for Lemma 6.3.3, but the same can be carried out for Lemma 6.3.15, using Lemma 10.F.4 instead of Lemma 10.F.1.

Proof of Lemma 6.3.3.

We shall consider a sequence of balls and apply by induction Lemma 10.F.3 in order to have an observation as close of $\{x = 0\}$ as we want from the interior of Y. This is the purpose of the construction below that follows from [BD10]. Take $y_0 \in \{x = 0\}$. We can assume without any loss of generality that $y_0 = 0$. We consider the cone $\mathcal{C}$ with apex $y_0 = 0$,

$$\mathcal{C} := \{y \in Z,\ |y| < R_0,\ y.\xi > |y|\cos(\theta_0)\},$$

where ξ an unitary vector, normal to the surface $\{x = 0\}$, and $R_0, \theta_0 > 0$ are such that $\mathcal{C} \subset Y$ (this implies in particular, $\theta_0 < \pi_2$). It should be noted that the measure of the angle here θ_0 is not related to the measures of angles of the polygons considered in Section 6.2.3. We start with $q_0 = (R_0/2)\xi$. We define a 7-uplet of radii $0 < r_0^0 < r_1^0 < r_2^0 := \rho_0 < r_3^0 < r_4^0 < r_5^0 < r_6^0$, with $r_6^0 = \sin(\theta_0)\frac{R_0}{2}$. Let $k > 0$ be fixed below. We also fix $r_3^0 := kr_2^0 + (1-k)\frac{R_0}{2}$. Then, we consider the following sequences

$$q_{j+1} = kq_j, \quad \rho_{j+1} = k\rho_j, \quad r_i^{j+1} = kr_i^j,\ i \in \{0,\ldots,6\}. \tag{10.F.6}$$

Then, we have $r_3^j = kr_2^j + (1-k)|q_j| = r_2^{j+1} + (1-k)|q_j|$. Note that we thus obtain

$$B(q_{j+1},\rho_{j+1}) \subset B(q_j, r_3^j). \tag{10.F.7}$$

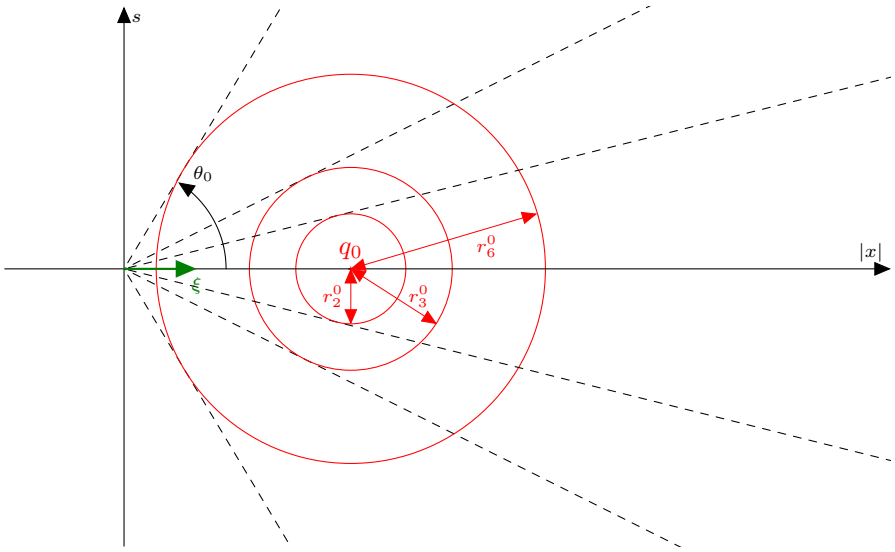


FIGURE 10.8: The initialization of the sequences of balls, depending on s and $|x|$. The scale between the balls at step j and the balls at step $j+1$ is k.

Indeed, let $x \in B(q_{j+1}, \rho_{j+1})$. We have

$$|x - q_j| \leq |x - q_{j+1}| + |q_j - q_{j+1}| \leq r_2^{j+1} + (1-k)|q_j| = r_3^j.$$

Now, fixing ρ_0 We also define $m_i^j = r_i^j / r_0^j$, $i \in \{0, \dots, 6\}$. Note that m_i^j is independent of j, that is, $m_i^j =: m_i$. With all this choices, the value of k is now fixed

$$k = \frac{r_3^0 - \frac{R_0}{2}}{r_2^0 - \frac{R_0}{2}} = \frac{m_6 - m_3 \sin(\theta_0)}{m_6 - m_2 \sin(\theta_0)} \in (0,1).$$

We apply Lemma 10.F.1 to obtain

$$||u||_{H^1(B(q_0, r_3^0))} \leq C \left(||Qu||_{L^2(B(q_0, r_6^0))} + ||u||_{H^1(B(q_0, r_2^0))} \right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(B(q_0, r_6^0))},$$

with

$$\Theta = \frac{e^{-\beta m_3^2} - e^{-\beta m_4^2}}{e^{-\beta r_1^2} - e^{-\beta m_3^Z}}, \qquad \text{with } \beta = \tilde{\alpha} r_0^0 > 1.$$

From (10.F.7), we obtain

$$||u||_{H^1(B(q_1, \rho_1))} \leq C \left(||Qu||_{L^2(B(q_0, r_6^0))} + ||u||_{H^1(B(q_0, r_2^0))} \right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(B(q_0, r_6^0))}.$$

Applying Lemma 10.F.3, we obtain

$$||u||_{H^1(B(q_j, \rho_j))} \leq \frac{C}{k^{j-1}} \left(||Qu||_{L^2(Y)} + ||u||_{H^1(B(q_{j-1}, r_2^{j-1}))} \right)^{\frac{\Theta}{1+\Theta}} ||u||^{\frac{1}{1+\Theta}}_{H^1(\underline{Z}_{\frac{\delta}{4}})}.$$

From Remark 10.F.4 the constants Θ and C are independant of j here. We recall that we defined $\underline{Z}_{\frac{\delta}{4}} := \{z = (s,x) \in Z,\ d(x,0) > \frac{\delta}{4}\}$. We set $\varepsilon_j := d(B(q_j, \rho_j))$. We can apply Lemma 3.2 in [BD10] to obtain

$$||u||_{H^1(B(q_j, \rho_j))} \leq \frac{C}{k^{\frac{j-1}{1-\nu}}} \left(||Qu||_{L^2(Z)} + ||u||_{H^1(B(q_0, r_2^0))} \right)^{\nu^j} ||u||^{1-\nu^j}_{H^1(\underline{Z}_{\frac{\varepsilon_j}{2}})}, \tag{10.F.8}$$

with possibly a different constant $C > 0$, independent of j and

$$\nu = \frac{e^{\beta(m_4^2 - m_3^2)} - 1}{e^{\beta(m_4^2 - m_1^2)} - 1}.$$

At this step we leave the proof of [BD10] by noticing that $d(B(q_j, \rho_j), \{x = 0\}) \simeq k^j$ (see (10.F.6)) and we stop the inductive construction at $j = j_0$ when $k^{j_0} \leq \delta$. This yields

$$j_0 \leq \frac{\log(\delta)}{\log(k)}$$

and then $\nu^{j_0} \geq \nu^{\frac{\log(\delta)}{\log k}} = \delta^{\frac{\log \nu}{\log k}} =: f(\delta)$. Note that estimate (10.F.8) holds also for $f(\delta) = \delta^{\frac{\log(\nu)}{\log(k)}}$ in place of ν^{j_0}, that is,

$$||u||_{H^1(B(q_{j_0}, \rho_{j_0}))} \leq \frac{C}{k^{\frac{j_0 - 1}{1-\nu}}} \left(||Qu||_{L^2(Z)} + ||u||_{H^1(B(q_0, r_2^0))} \right)^{f(\delta)} ||u||^{1-f(\delta)}_{H^1(\underline{Z}_{\frac{\delta}{4}})}, \tag{10.F.9}$$

We recall that we have set

$$Y_\delta = \{y \in Y,\ \frac{1}{2}\delta < d(y, \{x = 0\}) < \delta\}.$$

Then, by a compactness argument, we have proven that there exist $N \in \mathbb{N}$ and a subset $\hat{Y} \Subset Y \setminus \{x = 0\}$ such that

$$||u||_{H^1(Y_\delta)} \leq C\delta^{-N} \left(||Qu||_{L^2(Y)} + ||u||_{H^1(\hat{Y})} \right)^{f(\delta)} ||u||^{1-f(\delta)}_{H^1(\underline{Z}_{\frac{\delta}{4}})}, \tag{10.F.10}$$

for all $u \in H^1(\underline{Z}_{\frac{\delta}{4}})$, $Qu \in L^2(Y)$. Yet, it remains to compute the constant

$$f(\delta) = \delta^{\frac{\log(\nu)}{\log(k)}}.$$

As it is done in [BD10], we can optimize the constants used in the construction of the above procedure. For $\delta_0 > 0$ and $\ell \in (0, 1)$, we set

$$\begin{cases} \beta = (1 + \ell^2)^{1/2}, \\ m := (1 + \ell^2)^{1/4}, \\ m_1 =: m, \\ m_2 = m(1 + \ell^2\delta_0), \\ m_3 = m(1 + \ell\delta_0 + \ell^2\delta_0), \\ m_4 = m(1 + \delta_0 + \ell^2\delta_0), \\ m_6 = m(1 + \delta_0 + 2\ell^2\delta_0). \end{cases}$$

For ℓ near 0, we have on the one hand

$$\begin{aligned} k &= \frac{1 + \delta_0 + 2\ell^2\delta_0 - \big(1 + \ell\delta_0 + \ell^2\delta_0\big)\sin(\theta_0)}{1 + \delta_0 + 2\ell^2\delta_0 - (1 + \ell^2\delta_0)\sin(\theta_0)} = 1 - \frac{\ell\delta_0 \sin(\theta_0)}{1 + \delta_0 + 2\ell^2\delta_0 - (1 + \ell^2\delta_0)\sin(\theta_0)} \\ &= 1 - \frac{\ell\delta_0 \sin(\theta_0)}{1 + \delta_0 - \sin(\theta_0)} + O_{\delta_0}(\ell^2), \end{aligned}$$

and then $\log(k) = -\frac{\ell\delta_0 \sin(\theta_0)}{1+\delta_0+\sin(\theta_0)} + O_{\delta_0}(\ell^2)$. On the other hand

$$\begin{aligned} \nu &= \frac{e^{\beta(m_4 - m_3)} - 1}{e^{\beta(m_4 - m_1)} - 1} = 1 + \frac{e^{\beta(m_4 - m_3)} - e^{\beta(m_4 - m_1)}}{e^{\beta(m_4 - m_1)} - 1} \\ &= 1 - \frac{e^{(1+\ell^2)\left((1+\delta_0+\ell^2\delta_0)^2 - 1\right)} - e^{(1+\ell^2)\left(1+\delta_0+\ell^2\delta_0\right)^2 - \left(1+\ell\delta_0+\ell^2\delta_0\right)^2}}{e^{\beta(m_4 - m_1)} - 1}, \end{aligned}$$

and then $\log(\nu) = -2\delta_0\ell\frac{e^{2\delta_0+\delta_0^2}}{e^{2\delta_0+\delta_0^2}-1} + O_{\delta_0}(\ell^2)$. As a result, passing to the limit $\ell \to 0$, and taking the infimum over δ yields

$$\inf_{k,\nu}\frac{\log(\nu)}{\log(k)} = 2\inf_{\delta>0}\frac{1+\delta_0-\sin(\theta_0)}{\sin(\theta_0)(1-e^{-(2\delta_0+\delta_0^2)})} = 2\inf_{x>0}\frac{\sqrt{1+x}-\sin(\theta_0)}{\sin(\theta_0)(1-e^{-x})}.$$

It is then sufficient to see that in our case, we can take θ_0 as close as $\frac{\pi}{2}$ as we want (we recall that θ_0 is the half-angle of the construction procedure) and then, for all $\eta > 0$, we can choose a construction procedure such that $\frac{\log\nu}{\log k} = 1+\eta$. All the constants involved in the sought result heavily depend on η, but not on δ.□

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Rémi BUFFE

Inégalités de Carleman près du bord, d'une interface et pour des problèmes singuliers

Résumé :
Dans la première partie de ce mémoire, on s'attache à l'obtention d'inégalités de Carleman elliptiques pour des opérateurs d'ordre deux au bord pour des conditions dites de Ventcel. Dans une seconde partie, on démontre une inégalité adaptée aux multi-interfaces, pour des opérateurs elliptiques d'ordre quelconque, sous la condition classique de sous-ellipticité de Hörmander, ainsi que sous une condition de compatibilité entre les opérateurs sur la multi-interface et l'intérieur, dite de recouvrement. Cette condition généralise la condition de Lopatinskii. Enfin, dans une troisième partie, on s'intéresse à la contrôlabilté de l'équation de la chaleur et la stabilisation faible de l'équation des ondes dans des domaines polygonaux.

Mots clés : Equations aux dérivées partielles, Inégalités de Carleman, Contrôle de l'équation de la chaleur, Ventcel, Stabilisation de l'équation des ondes, Inégalité spectrale, Domaines polygonaux, Multi-interfaces.

Carleman estimates near boundaries, interfaces and for singular problems

Abstract :
In the first part of this thesis, we derive elliptic Carleman estimates for second-order operators with Ventcel boundary conditions. In the second part, we prove a proper estimate near multi-interfaces for elliptic operators of any order, under the classical sub-ellipticity condition of Hörmander and under a compatibility condition between the operators in the interior and at the multi-interface, called the covering condition. This condition is a generalization of the well-known Lopatinskii condition. Finally, in the third part, we focus on controllability properties of the heat equation, and stabilization properties of the wave equation for polygonal domains, with mixed boundary conditions.

Keywords : Partial differential equations, Carleman estimates, Control of the heat equation, Ventcel, Stabilization of the wave equation, Spectral inequality, Polygonal domains, Multi-interfaces.

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