# Some problems about SLE

Yong Han

UNIVERSITÉ D'ORLÉANS

## *École doctorale Mathématiques, Informatiques, Physique théorique et Ingéniérie des systèmes*

Laboratoire : MAPMO

**THÈSE** présentée par :

**Yong HAN**

soutenue le : **23 Mai 2017**

pour obtenir le grade de : **Docteur de l'université d'Orléans**

Discipline/ Spécialité : **Mathématiques**

# Some problems about SLE

THÈSE DIRIGÉE PAR :

| | |
|---|---|
| Michel ZINSMEISTER | Professeur, Université d'Orléans - Directeur de thèse |

RAPPORTEURS :

| | |
|---|---|
| Weiyuan QIU | Professeur, Université de Fudan |
| Ilia BINDER | Professeur, Université de Toronto |

---

JURY :

| | |
|---|---|
| Yuefei WANG | Professeur, AMSS |
| Jinsong LIU | Professeur, AMSS |
| Guizhen CUI | Professeur, AMSS |
| Yi QI | Professeur, BUAA |
| Michel ZINSMEISTER | Professeur, Université d'Orléans |
| Romain ABRAHAM | Professeur, Université d'Orléans |

**THÈSE DE DOCTORAT DE**
**L'UNIVERSITÉ D'ORLÉANS**

Spécialité

**Mathématiques**

Présentée par

# Yong HAN

Pour obtenir le grade de

**DOCTEUR de l'UNIVERSITÉ D'ORLÉANS**

Sujet de la thèse :

# Some problems about SLE

# Remerciements

This thesis is a co-supervising PhD programme between China and France. Here I want to express my acknowledgements to all these people and institutes without which this work will not be completed successfully.

First, my deepest gratitude goes to Professor Michel Zinsmeister and Yuefei Wang, my supervisors of this thesis. During my thesis, they have gave a lot of instructive advices and useful suggestions on this work. Their insightful thoughts helped me broaden my horizon and their encouragements gave me more enthusiasm and passion on basic research. At the same time I want to thank Professor Weiyuan Qiu and Ilia Binder for the acceptance to be the referees of my thesis. Also many thanks would go to the the members in my jury.

Second, I want to thank my great motherland, it is the scholarship from Chinese Scholarship Council that supported all my expenses in France.

During my stay in Orleans, the members in the laboratory MAPMO have helped me a lot. In particular, I want to give my heartful thanks to Anne Liger, Marie-France Grespier, Marie-Laurence Poncet for their help in many things. At the same time, thanks to my friends in the lab, Remi, Manon, Hieu, Binh, Nhat, Tien, Gregoire for their discussion in research and accompanying in daily life. Also I am in deep debt to my Chinese friends, Xing Feng, Wang Yichang, Zhao Yue, Zhang Henshui, Wei Huaying, Zhang Yunzhi. With them I had enjoyed the traditional Chinese food here and also I didn't feel homesick.

In the last, my thanks would go to my beloved family for their loving considerations and continuous encouragement in me through these years.

# Table des matières

# Chapitre 1

# Introduction

This thesis will focus on three topics related to the SLE($\kappa$) processes. The first part is about the dipolar SLE($\kappa$) process and the conformal restriction measure on the strip ; the second part is about the connectivity property of the Brownian loop measure ; and the third part is about the generalized integral means spectrum of the inner whole plane Loewner processes driven by a Lévy process.

The conformally invariant scaling limits of a series of planar lattice models can be described by the one-parameter family of random fractal curves SLE($\kappa$), which was introduced by Schramm. These models include site percolation on the triangular graph, loop erased random walk, Ising model, harmonic random walk, discrete Gaussian free field, FK-Ising model and uniform spanning tree. Using SLE, mathematicians can get many exponents related to these lattice models, which physicists predicted using non-rigorous methods. For example, Schramm,Werner and Lawler got the intersection exponents of the plane Brownian motion, and then proved Mandelbrot's conjecture about the Hausdorff dimension of Brownian frontier points. Also the arm exponent of these lattice can be obtained by studying some crossing events related to the SLE($\kappa$) process. During the past twenty years, the theory of SLE($\kappa$) process has developed very quickly. And more relations to physical models has been established such as the random surface theory created by Jason Miller and Scott Sheffield and the discrete holomorphic function theory concerning the lattice models. It is of no doubt that SLE process will continue to play the most important role in this area.

Let us now give the structure of this thesis. In chapter one, we will recall some basic facts about the SLE($\kappa$) process. And this include some background related to complex analysis and stochastic analysis. Also the definition of SLE process and some properties will be given.

Chapter two will be devoted to the introduction of the dipolar SLE($\kappa$) process and to the construction of the conformal restriction measure on the strip(which is called the dipolar conformal restriction measure). It is known that chordal SLE($\kappa$) process is generated by a random curve in a simply domain that connects two boundary points, radial SLE($\kappa$) process is a random curve in a simply connected domain that connects an inner point and a boundary point, and the whole plane SLE($\kappa$) process is a random curve in the complex plane from a point to infinity. In this chapter we will see that dipolar SLE($\kappa$) process is a random curve that connects a boundary point to a boundary arc. And the dipolar SLE($\kappa$) also satisfies some nice properties as the chordal SLE($\kappa$) process. Then we will use dipolar SLE process and study the conformal restriction measure in the strip. This is a measure supported on the compact sets of the strip that satisfies conformal restriction properties. To be more precise, let us describe briefly the conformal restriction measure which we study in this thesis : let $\mathcal{S}$ be the strip of width $\pi$ in the plane, i.e,

$$\mathcal{S} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, 0 < y < \pi\}.$$

We are going to study closed random subsets $K$ of $\mathcal{S}$ that have the following form :

(1) $K$ is a connected compact set of $\bar{\mathcal{S}}$ such that $K \cap \mathbb{R} = \{0\}$ and $\mathcal{S} \setminus K$ has two connected components whose boundaries contain $+\infty$ and $-\infty$ respectively.

(2) For any connected compact subset $A$ of $\bar{\mathcal{S}}$ such that $A \cap \mathbb{R}_\pi = \emptyset$, $\mathcal{S} \setminus A$ is simply connected and $d(0, A) > 0$, the law of $\Phi_A(K)$ conditioned on the event $\{K \cap A = \emptyset\}$ is the same as the law of $K$, where $\Phi_A$ is the unique conformal map from $\mathcal{S} \setminus A$ to $\mathcal{S}$ that fixes $+\infty$ and $-\infty$ and $\lim_{z\to+\infty}(\Phi_A(z) - z) = -\lim_{z\to-\infty}(\Phi_A(z) - z)$.

The law of such a set is called a dipolar restriction measure, which is the analogy of the chordal restriction measure defined in [9] and the radial restriction measure defined in [23].

In chapter three we will give an an introduction to the Brownian loop measure and prove a formula predicted by Cardy and Gasma about the total mass that the Brownian loop in the upper half plane disconnects two given points from the boundary. Intuitively the Brownian loop measure is just a measure induced by a planar Brownian path starting from some point in the plane and conditions on returning at the same point. Given two points in the upper half plane, according to the property of the planar Brownian motion, these two points are not on the Brownian loop almost surely. So there are four position relations about the two points and the Brownian loop. We will see that only the case that two points are enclosed by the loop at the same time has the finite mass. In order to give the formula of this finite mass, the SLE bubble measure and its relation to Brownian bubble measure will be recalled. In fact SLE bubble measure is just the measure obtained by passing the two boundary points of chordal SLE to the same boundary point and then rescaling.

In chapter four, we study the generalized spectrum of the inner whole plane Loewner processes driven by Lévy process. Given an univalent function $\phi$ defined on the unit disk with $\phi'(0) = 1$, and $p, q \in \mathbb{R}$, the generalized integral spectrum of $\phi$ is defined as follows.

$$\beta_\phi(p, q) := \limsup_{r\to 1^+} \frac{\log \int_0^{2\pi} |\frac{z}{\phi(z)}|^q |\phi'(re^{i\theta})|^p d\theta}{|\log(r-1)|}.$$

Usually it is difficult to determine the generalized integral spectrum for a simply domain, since it is difficult to get the exact form of a conformal map from the unit disk to the domain. But for some random univalent functions, it is possible to use some methods to compute the **averaged generalized integral means spectrum** which is defined as the following :

$$\overline{\beta}_\phi(p,q) := \limsup_{r\to 1^+} \frac{\log \int_0^{2\pi} \mathbf{E}[|\frac{z}{\phi(z)}|^q |\phi'(re^{i\theta})|^p] d\theta}{|\log(r-1)|}.$$

For the inner whole plane SLE $(f_t)_{t\geq 0}$, $f_0$ is a conformal map on the unit disk with derivative equal to one at 0, we can compute its average generalized integral means spectrum. The situation happens the same when we replace the driving process by a Lévy process. In the fourth chapter we will first give the expressions of $G(z,\bar{z}) = \mathbf{E}[\frac{|z|^q |f'(z)|^p}{|f(z)|^q}]$ for some special values $(p,q)$ and some special Lévy driven processes, and then get the generalized integral means spectrum.

# Chapitre 2

# SLE processes

In this chapter, we recall the basic knowledge about the Schramm Loewner Evolution(SLE). We will recall some basic facts in complex analysis.

We will use the following notations :

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\},$$

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \mathbb{D}_+ := \{z \in \mathbb{C} : |z| > 1\}.$$

## 2.1 Chordal Loewner Process

A **conformal map** on a planar domain is a one-to-one holomorphic function. Recall that a planar domain $D$ is called **simply connected** if any closed curve in $D$ is homotopic to a point. For such domains we have Riemann's conformal mapping theorem :

**Theorem 1** (Riemann mapping theorem)**.** *Given a simply connected domain $D \subset \mathbb{C}$ and a point $z_0 \in D$, then there exists a unique conformal map $\phi$ from $\mathbb{D}$ onto $D$ such that $\phi(0) = z_0$ and $\phi'(z_0) > 0$.*

**Definition 2** (see Chapter 3 in [10])**.** *A connected compact set $A \subset \overline{\mathbb{H}}$ is called a **$\mathbb{H}$-hull** if $\mathbb{H} \setminus A$ is simply conneccted and $A = \overline{A \cap \mathbb{H}}$. By Riemann's mapping theorem, given a $\mathbb{H}$-hull, there exists a unique conformal map $g_A(z)$ from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that $\lim_{z\to\infty}(g_A(z) - z) = 0$. And also we have $a(A) = \lim_{z\to\infty} z(g_A(z) - z) \geq 0$, which is called the **capacity** of $A$. We call $g_A$ the normalised conformal map corresponding to $A$.*

Given a non-selfcrossing curve $\gamma[0, \infty) \to \overline{\mathbb{H}}$ such that $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, then for any given $t \geq 0$, the compact set $K_t$ enclosed by $\gamma[0, t]$ is a $\mathbb{H}$-hull. Denote by $a(t)$ the capacity of $K_t$ and $g_t$ the corresponding normalised conformal map of $K_t$. We have the following Loewner's theorem.

**Lemma 3** (see Chapter 4 in [10])**.** *With above notations,*

*(1) $t \mapsto a(t)$ is strictly increasing and $a(t) \to \infty$ as $t \to \infty$ ;*
*(2) The limit $W_t := \lim_{z \in \mathbb{H} \setminus K_t, z \to \gamma(t)} g_t(z)$ exists and $t \to W_t$ is a continuous function from $\mathbb{R}^+$ to $\mathbb{R}$ ;*
*(3) If $\gamma$ is parameterized in such a way that $a(t) = 2t$, then $g_t$ satisfies the differential equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z. \tag{2.1.1}$$

We call (2.1.1) the chordal **Loewner differential equation**. And this is what Oded Schramm found to define SLE. We have seen that, starting from a non-selfcrossing curve, we can get a continuous function $W_t$ such that the corresponding family of conformal maps satisfies the chordal Loewner equation (2.1.1). Conversely, given a continuous function $W_t$, by solving the differential equation. For given $z$, define

$$\tau(z) := \sup\{t > 0, \min_{0 \le s \le t} |g_s(z) - W(s)| > 0\},$$

and

$$K_t := \{z \in \overline{\mathbb{H}} : \tau(z) \le t\}, H_t := \mathbb{H} \setminus K_t.$$

Then we have

**Lemma 4** (see Chapter 4 in [10])**.** *Suppose $W(t) : [0,T] \to \mathbb{R}$ is a continuous function. And $g_t(z)$ is the solution of the ODE* (2.1.1)*. Define $K_t$ and $H_t$ as above, then*
*(1) For any $t \in [0,T]$, $g_t(z) : H_t \to \mathbb{H}$ is a conformal(holomorphic and injective) map ;*
*(2) For any $t \in [0,T]$, $K_t$ is a $\mathbb{H}$-hull ;*
*(3) At $z = \infty$, $g_t(z)$ has the Laurent expansion $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{|z|^2})$.*
*We call $(g_t : 0 \le t \le T)$ and $(K_t : 0 \le t \le T)$ the **chordal Loewner process(Loewner chain)** driven by $W$.*

*Remark* 5. Usually, for a given continuous function $W$, the Loewner chain driven by $W$ is not generated by a curve.

## 2.2 Radial Loewner Process

**Definition 6.** *If a compact set $K \subset \overline{\mathbb{D}}$ satisfies $0 \notin K$, $K = \overline{K \cap \mathbb{D}}$ and $\mathbb{D} \setminus K$ is simply connected, we call $K$ is a $\mathbb{D}$**-hull**.*

By Riemann mapping theorem, for a $\mathbb{D}$-hull, there exists a unique conformal map $g_K : \mathbb{D} \setminus K \to \mathbb{D}$ such that $g_K(0) = 0, g'_K(0) > 0$. We call $g_K$ the **normalized conformal map** corresponding to $K$. By Schwarz lemma we have $g'_K(0) \ge 1$ and $g'_K(0) = 1$ if and only if $K \subset \partial\mathbb{D}$. Define the capacity of $K$ as follows :

$$\mathrm{cap}(K) := \log g'_K(0).$$

Suppose that $\gamma : [0,T] \to \overline{\mathbb{D}}$ is a non-selfcrossing curve with $\gamma(0) = 1$, $\gamma(0,T) \subset \mathbb{D}$ and $0 \notin \gamma$. Define $K_t$ the hull enclosed by $\gamma[0,t]$, $g_t(z) : \mathbb{D} \setminus K_t \to \mathbb{D}, g_t(0) = 0, g_t'(0) > 0$ is the normalised conformal map.

**Lemma 7** (see Chapter 4 in [10])**.** *With the above notations,*

*(1) $a(t)$ is a strictly increasing continuous non-negative function;*

*(2) The limit $\lambda(t) := \lim_{z\in\mathbb{D}\setminus\gamma[0,t], z\to\gamma(t)} g_t(z) \in \partial\mathbb{D}$ exists, and we can choose a continuous real-valued function $W_t$ such that $\lambda(t) = e^{iW(t)}$.*

*(3) If $\gamma$ is parameterized such that $a(t) = t$, we have the following Loewner differential equation :*

$$\partial_t g_t(z) = g_t(z)\frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)} \quad , \; g_0(z) = z. \tag{2.2.1}$$

We call (2.2.1) the radial **Loewner differential equation**. Conversely, starting from a continuous real-valued function $W_t$, consider the differential equation (2.2.1). For given $z \in \overline{\mathbb{D}} \setminus \{1\}$, define

$$\tau(z) := \sup\{t > 0, \min_{0\le s\le t} |g_s(z) - \exp\{iW(s)\}| > 0\},$$

and

$$K_t := \{z \in \overline{\mathbb{D}} : \tau(z) \le t\}, D_t := \mathbb{D} \setminus K_t.$$

**Lemma 8** (see Chapter 4 in [10])**.** *Suppose $W(t) : [0,T] \to \mathbb{R}$ is a continuous real-valued function. And $g_t(z)$ is the solution of the ODE* (2.2.1)*. Define $K_t$ and $D_t$ as above, then*

*(1) For any $t \in [0,T]$, $g_t(z) : D_t \to \mathbb{D}$ is a conformal map;*

*(2) For any $t \in [0,T]$, $K_t$ is a $\mathbb{D}$-hull;*

*(3) At $z = 0$, $g_t(z)$ has the Taylor expansion $g_t(z) = e^t z + O(|z|^2)$.*

*We call $(g_t : 0 \le t \le T)$ and $(K_t : 0 \le t \le T)$ the **(inner)radial Loewner process(Loewner chain)** driven by $W$.*

*Remark* 9. Denote by $f_t(z) := g_t^{-1}(z)$, then $f_t(z)$ satisfies the so-called reverse Loewner differential equation :

$$\begin{cases} \partial_t f_t(z) = z f_t'(z)\frac{z+\lambda_t}{z-\lambda_t} \\ f_0(z) = z, \text{ for } \forall z \in \mathbb{D}. \end{cases} \tag{2.2.2}$$

*Remark* 10. If we let $\tilde{g}_t(z) = \frac{1}{g_t(\frac{1}{z})}$, where $g_t(z)$ is defined as in Lemma 8, we can get

$$\partial_t \tilde{g}_t(z) = \frac{-1}{g_t^2(\frac{1}{z})}\partial_t g_t(\frac{1}{z})$$

$$= \frac{-1}{g_t^2(\frac{1}{z})} g_t(\frac{1}{z})\frac{\lambda_t + g_t(\frac{1}{z})}{\lambda_t - g_t(\frac{1}{z})}$$

$$= -\tilde{g}_t(z)\frac{\tilde{g}_t(z) + \tilde{\lambda}_t}{\tilde{g}_t(z) - \tilde{\lambda}_t},$$

where $\tilde{\lambda}_t = \frac{1}{\lambda_t}$ and $\tilde{g}_0(z) = z$. Also we have $\tilde{f}_t(z) := \tilde{g}_t^{-1}(z) = \frac{1}{f_t(\frac{1}{z})}$, where $f_t(z) = g_t^{-1}(z)$. And we can check that

$$\begin{cases} \partial_t \tilde{f}_t(z) = z\tilde{f}_t'(z)\frac{z+\tilde{\lambda}_t}{z-\tilde{\lambda}_t} \\ \tilde{f}_0(z) = z \text{ for } \forall z \in \mathbb{D}_+. \end{cases}$$

So the **outer radial Loewner process** driven by $W_t$ is defined as follows :

$$\begin{cases} \partial_t \tilde{g}_t(z) = -\tilde{g}_t(z)\frac{\tilde{g}_t(z)+\tilde{\lambda}_t}{\tilde{g}_t(z)-\tilde{\lambda}_t} \\ \tilde{g}_0(z) = z, \text{ for } \forall z \in \mathbb{D}_+. \end{cases} \quad \begin{cases} \partial_t \tilde{f}_t(z) = z\tilde{f}_t'(z)\frac{z+\tilde{\lambda}_t}{z-\tilde{\lambda}_t} \\ \tilde{f}_0(z) = z \text{ for } \forall z \in \mathbb{D}_+. \end{cases} \tag{2.2.3}$$

Notice that for the outer radial Loewner process, $g_t$ maps the complement of the hull in $\mathbb{D}_+$ onto $\mathbb{D}_+$. Here we can easily define the hulls in $\mathbb{D}_+$.

## 2.3 Whole Plane Loewner process

The proof of Bieberbach's conjecture in fact used the whole plane Loewner theory. In 1923 Loewner proved a part of the conjecture, De Branges proved the conjecture completely with the help of Loewner's method.

**Definition 11.** *If a compact set $K \subset \mathbb{C}$ satisfies $\hat{\mathbb{C}} \setminus K$ is simply connected, we call $K$ is a $\mathbb{C}$-**hull**. Still by Riemann's mapping theorem, there exists a unique conformal map $F_K : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ such that*

$$\lim_{z\to\infty} \frac{F_K(z)}{z} > 0.$$

*In fact, if $0 \in K$, we have $F_K(z) = \frac{1}{f_K(\frac{1}{z})}$, where $f_K(z)$ is the conformal map from $\mathbb{D}$ to $\{\frac{1}{z} : z \in \mathbb{C} \setminus K\}$ with $f_K(0) = 0, f_K'(0) > 0$. We call $g_K := F_K^{-1}$ the normalised conformal map corresponding to $K$. Denote by $\mathrm{cap}(K) = -\log[f_K'(0)] = \log[\lim_{z\to\infty} \frac{F_K(z)}{z}]$, which is called the **capacity** of $K$.*

Similarly, for a non-selfcrossing curve $\gamma : (-\infty, +\infty) \to \mathbb{C}$ with $\gamma(-\infty) = 0, \gamma(+\infty) = \infty$. For any $t \in \mathbb{R}$, denote by $K_t$ the hull enclosed by $\gamma[-\infty, t]$ and denote by $g_t$ the normalised conformal map and $a(t) = \mathrm{cap}(K_t)$. In fact if we define $\tilde{D}_t := \{1/z : z \in D_t\}$, then there exists a unique function $\tilde{g}_t$ from $\tilde{D}_t$ to the unit disk $\mathbb{D}$ such that $\tilde{g}_t(0) > 0$ and $\tilde{g}_t'(0) > 0$. And let $g_t(z) := \frac{1}{\tilde{g}_t(1/z)}$, we can get that

$$g_t(\infty) = \infty, \lim_{z\to\infty} \frac{g_t(z)}{z} = \frac{1}{\tilde{g}_t(0)} > 0.$$

And this is unique. Suppose that $t < s$, we have $D_s \subset D_t$ and then $\tilde{D}_s \subset \tilde{D}_t$. We have $\tilde{g}_t \circ \tilde{g}_s^{-1}(\mathbb{D}) \subset \mathbb{D}$ and then by Schwarz Lemma we have

$$\tilde{g}_t'(0) \le \tilde{g}_s'(0).$$

**Lemma 12** (see Chapter 4 in [10]). *Take the above notation, then*

*(1) $a(t)$ is a strictly increasing function and $a(-\infty) = 0, a(+\infty) = +\infty$ ;*

*(2) The limit $\lambda(t) := \lim_{z \to \gamma(t), z \in \mathbb{C} \setminus K_t} g_t(z)$ exists and there exists a continuous real-valued function $W$ such that $\lambda(t) = e^{iW_t}$ ;*

*(3) If $a(t) = t$, then $g_t(z)$ satisfies the Loewner differential equation*

$$\begin{cases} \partial_t g_t(z) = g_t(z) \frac{\lambda_t + g_t(z)}{\lambda_t - g_t(z)} \\ \lim_{t \to -\infty} e^t g_t(z) = z, \forall z \in \mathbb{C} \setminus \{0\}. \end{cases} \tag{2.3.1}$$

*We call (2.3.1) the (outer) whole plane* ***Loewner differential equation****.*

*(4) $\forall z \in \mathbb{C} \setminus \{0\}$, we have*

$$\lim_{t \to -\infty} e^t g_t(z) = z.$$

Since (4) can not be found in any references, here we give the proof of (4).

*Proof.* Notice that(4) is equivalent to say that $\lim_{t \to -\infty} e^t \frac{1}{\tilde{g}_t(1/z)} = z$ holds for any $z \in \mathbb{C} \setminus \{0\}$, i.e

$$\lim_{t \to -\infty} e^{-t} \tilde{g}_t(z) = z, \quad \forall z \in \mathbb{C}.$$

Here $\tilde{g}_t$ is defined just before the lemma. We define $\tilde{f}_t(z) := \tilde{g}_t^{-1}(z) : \mathbb{D} \to \mathbb{C}$, and let $h_t(z) := e^t \tilde{f}_t(z)$, then $h_t(0) = 0, h'_t(0) = 1$. By Koebe distortion theorem we have

$$\frac{|z|}{(1+|z|)^2} \leq |h_t(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

And then we have

$$e^{-t} \frac{|\tilde{g}_t(z)|}{(1+\tilde{g}_t(z))^2} \leq |z| \leq e^{-t} \frac{|\tilde{g}_t(z)|}{(1-|\tilde{g}_t(z)|)^2}. \tag{2.3.2}$$

And then

$$(1-|\tilde{g}_t(z)|)^2 \leq e^{-t} \frac{|\tilde{g}_t(z)|}{|z|} \leq (1+|\tilde{g}_t(z)|)^2.$$

Since $|\tilde{g}_t(z)| \leq 1$ for any $t$ and $z$. Then we have $\{e^{-t} \frac{\tilde{g}_t(z)}{z} : t \in \mathbb{R}\}$ is a normal family. And we know that if $t_k \to -\infty$ and $e^{-t_k} \frac{\tilde{g}_{t_k}(z)}{z} \to h(z)$, then by (2.3.2) we have that $|h(z)| = 1$ and then $h = 1$ since $h(0) = 1$. □

*Remark* 13. Denote by $f_t(z) := g_t^{-1}(z)$, then $f_t(z)$ satisfies the so-called reverse Loewner differential equation :

$$\begin{cases} \partial_t f_t(z) = z f'_t(z) \frac{z+\lambda_t}{z-\lambda_t} \\ \lim_{t \to -\infty} f_t(e^{-t} z) = z, \quad \forall z \in \mathbb{D}_+. \end{cases} \tag{2.3.3}$$

Conversely, starting from a continuous real-valued function $W_t$, consider the differential equation (2.3.1). For given $z \in \mathbb{C} \setminus \{0\}$, define

$$\tau(z) := \sup\{t : \min_{0\le s\le t} |g_s(z) - \exp\{iW(s)\}| > 0\},$$

and

$$K_t := \{z \in \mathbb{C} : \tau(z) \le t\}, C_t := \hat{\mathbb{C}} \setminus K_t.$$

**Lemma 14** (see Chapter 4 in [10])**.** *Suppose $W : \mathbb{R} \to \mathbb{R}$ is a continuous real-valued function. And $g_t(z)$ is the solution of the ODE* (2.3.1)*. Define $K_t$ and $C_t$ as above, then*

*(1) For any $t \in \mathbb{R}$, $g_t(z) : C_t \to \mathbb{D}_+$ is a conformal map ;*

*(2) For any $t \in \mathbb{R}$, $K_t$ is a $\mathbb{C}$-hull ;*

*(3) At $z = \infty$, $g_t(z)$ has the Laurent expansion $g_t(z) = e^{-t}z + O(\frac{1}{|z|})$.*

*We call $(g_t)$(or $f_t(z) := g_t^{-1}(z)$) and $(K_t)$ the* ***outer whole plane Loewner process(Loewner chain)*** *driven by $W$.*

In fact, we have an **inner version of the whole plane Loewner process**. To express the motivation, we still start from a non-selfcrossing curve $\gamma$ from $\infty$ to 0 such that $\gamma(-\infty) = \infty$ and $\gamma(+\infty) = 0$. $D_t := \hat{\mathbb{C}} \setminus \gamma[-\infty, t]$, then there exists a unique function $g_t : D_t \to \mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. By Koebe's one quarter theorem, we have

$$\frac{1}{4}\frac{1}{d(0, \partial D_t)} \le g_t'(0) \le \frac{1}{d(0, \partial D_t)}.$$

So we have $\lim_{t\to+\infty} g_t'(0) = +\infty$ and $\lim_{t\to-\infty} g_t'(0) = 0$, and we assume $g_t'(0) = e^t$ and $g_t(\gamma(t)) = \lambda_t$. Define $\tilde{\gamma}(t) = \frac{1}{\gamma(t)}$, we get the same case as in Lemma 12. And this time $g_t(z) = \frac{1}{\tilde{g}_t(1/z)}$, where $\tilde{g}_t(z)$ here is like Lemma12 and satisfies

$$\begin{cases} \partial_t \tilde{g}_t(z) = \tilde{g}_t(z)\frac{\tilde{\lambda}_t + \tilde{g}_t(z)}{\tilde{\lambda}_t - \tilde{g}_t(z)} \\ \lim\limits_{t\to-\infty} e^t \tilde{g}_t(z) = z, \quad \forall z \in \mathbb{C} \setminus \{0\}. \end{cases}$$

Where $\tilde{\lambda}_t = \tilde{g}_t(\tilde{\gamma}(t)) = 1/\lambda_t$. Changing to $g_t(z)$, we have

$$\begin{cases} \partial_t g_t(z) = g_t(z)\frac{\lambda_t + g_t(z)}{\lambda_t - g_t(z)} \\ \lim\limits_{t\to-\infty} e^{-t} g_t(z) = z, \quad \forall z \in \mathbb{C}. \end{cases}$$

And $f_t(z) := g_t^{-1}(z)$ satisfies

$$\begin{cases} \partial_t f_t(z) = z f_t'(z)\frac{z+\lambda_t}{z-\lambda_t} \\ \lim\limits_{t\to-\infty} f_t(e^t z) = z, \quad \forall z \in \mathbb{D}. \end{cases}$$

And so we summarize as follows :

**Lemma 15.** *Take above notation,*

$$\begin{cases}\partial_t g_t(z) = g_t(z)\frac{g_t(z)+\lambda_t}{\lambda_t - g_t(z)} \\ \lim\limits_{t\to-\infty} e^{-t} g_t(z) = z, \quad \forall z \in \mathbb{C}.\end{cases} \qquad \begin{cases}\partial_t f_t(z) = z f_t'(z)\frac{z+\lambda_t}{z-\lambda_t} \\ \lim\limits_{t\to-\infty} f_t(e^t z) = z, \quad \forall z \in \mathbb{D}.\end{cases}$$

To continue the procedure, we assume that $\gamma$ is a simple curve from 0 to $\infty$ such that $\gamma(-\infty) = 0$ and $\gamma(+\infty) = \infty$. $D_t := \hat{\mathbb{C}} \setminus \gamma[t, \infty]$, then there exists a unique function $g_t : D_t \to \mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) = e^{-t} > 0$. This time if we let $\tilde{\gamma}(t) = \gamma(-t)$, then $\tilde{\gamma}$ is like Lemma 15. And we can get,

$$\begin{cases}\partial_t g_t(z) = g_t(z)\frac{g_t(z)+\lambda_t}{g_t(z)-\lambda_t} \\ \lim\limits_{t\to+\infty} e^{t} g_t(z) = z, \quad \forall z \in \mathbb{C}.\end{cases} \qquad \begin{cases}\partial_t f_t(z) = z f_t'(z)\frac{\lambda_t+z}{\lambda_t-z} \\ \lim\limits_{t\to+\infty} f_t(e^{-t} z) = z, \quad \forall z \in \mathbb{D}.\end{cases} \tag{2.3.4}$$

where $\lambda_t = g_t(\gamma(t))$. Similarly, starting from a real-valued continuous function $W_t$, let $\lambda_t = \exp\{iW_t\}$. We call the solution of above differential equation (2.3.4) the **inner whole plane Loewner process(chain)** driven by $W_t$.

## 2.4 Approximate Whole Plane Loewner Process by Radial Loewner Process

In this section, we will give a relation between radial Loewner process and whole plane Loewner process, which will be very important in our computation of the spectrum of the random whole plane Loewner chains.

Given a real-valued continuous function $W_t : \mathbb{R} \to \mathbb{R}$, and let $\lambda_t := \exp\{iW_t\}$. For any given $s \in \mathbb{R}$, define conformal maps $(g_t^{(s)})$ as follows : $g_t^{(s)}(z) = e^{-t}z$ if $t \leq -s$ ; if $t \geq -s$, $g_t^{(s)}(z)$ is defined as the solution to the outer radial Loewer differential equation(2.2.3) with initial condition $g_{-s}^{(s)}(z) = e^s z$.

**Lemma 16** (see Chapter 6 in [10])**.** *Take the above notations, then as $s$ tends to $+\infty$, $(g_t^{(s)})$ converges locally uniformly to a limit which we will denote by $(g_t)$. Moreover $(g_t)$ is the outer whole plane Loewner process driven by $W_t$( see Lemma 14).*

We can use above lemma to prove that the inner whole plane Loewner process can also be approximated by the modified inner radial Loewner process. For given $s \in \mathbb{R}$, define conformal maps $(g_t^{(s)})$ as follows : for $t > s$, $g_t^{(s)}(z) = e^{-t}z$ ; for $t \leq s$, $g_t^{(s)}$ is the solution to the **modified inner radial Loewner differential equation** which is defined as follows :

$$\partial_t g_t(z) = g_t(z)\frac{g_t(z) + \lambda(t)}{g_t(z) - \lambda(t)} \tag{2.4.1}$$

with the initial condition $g_s^{(s)}(z) = e^{-s}z$.

**Lemma 17.** *With $g_t^{(s)}$ defined as above, we have*

$$\lim_{s\to+\infty} g_t^{(s)}(z) = g_t(z)$$

*locally uniformly, where $(g_t)$ is the inner whole plane Loewner process (see (2.3.4)) driven by $W_t$.*

*Proof.* Define $\tilde{g}_t^{(s)}(z) = \frac{1}{g_{-t}^{(s)}(\frac{1}{z})}$ and $\tilde{\lambda}(t) = \frac{1}{\lambda(-t)}$. Then $\tilde{g}_t^{(s)}$ satisfies Lemma 16. We have that as $s \to +\infty$, $\tilde{g}_t^{(s)}$ converges to a limit $\tilde{g}_t$, where $\tilde{g}_t$ is the outer whole plane Loewner process driven by $W_t$( see Lemma 14). So $g_t^{(s)}$ converges locally uniformly to a limit $g_t(z) := \frac{1}{\tilde{g}_{-t}(\frac{1}{z})}$. We only need to check that $g_t(z)$ satisfies (2.3.4). □

## 2.5 Driven by jump functions

Suppose that $W_t$ is a function that is right continuous and has left limit at very point, then the corresponding Loewner equation (2.1.1),(2.2.1) and (2.3.1) driven by $W_t$ still have the solution $g_t$ which is conformal. This is because that we only need to consider the equation on the intervals on which $W$ is continuous and then use the conclusions in the previous sections. And also the corresponding conclusions in the previous section still holds with slight modifications.

## 2.6 Stochastic Analysis

In this section we recall some facts from stochastic analysis.

**Definition 18.** *A standard Brownian motion is a stochastic process $(B_t)_{t\geq 0}$ such that*

*(1) $B_0 = 0$ ;*

*(2) For any $0 \leq t_1 < t_2 < t_3 < ... < t_n$,$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, ..., B_{t_n} - B_{t_{n-1}}$ are independent ;*

*(3) For any $0 \leq s \leq t$,$B_t - B_s \sim N(0, t-s)$, where $N(0, t-s)$ denotes the normal distribution with mean $0$ and variance $t-s$ ;*

*(4) Almost surely, the sample path $t \to B_t$ is continuous.*

**Definition 19.** *A Lévy process is a stochastic process $(L_t)_{t\geq 0}$ such that*

*(1) $L_0 = 0$(a.s) ;*

*(2) For any $0 \leq t_1 < t_2 < t_3 < ... < t_n$,$L_{t_2} - L_{t_1}, L_{t_3} - L_{t_2}, ..., L_{t_n} - L_{t_{n-1}}$ are independent ;*

*(3) For any $0 \leq s \leq t$, $L_t - L_s$ has the same law as $L_{t-s}$ ;*

Notice that Brownian motion is a special Lévy process. The essential difference with Brownian motion is that jumps are allowed. The characteristic function of a Lévy process $L_t$ has the form

$$\mathbf{E}[e^{i\xi L_t}] = e^{-t\eta(\xi)}, \tag{2.6.1}$$

where $\eta$ is called the **Lévy symbol** and is a continuous complex function of $\xi \in \mathbb{R}$, satisfying $\eta(0) = 0$ and $\eta(-\xi) = \overline{\eta(\xi)}$. If $\eta(-\xi) = \eta(\xi)$, we call $L_t$ a **symmetric Lévy process**. For Brownian motion, the Lévy symbol is $\eta(\xi) = \frac{\xi^2}{2}$. More generally,the function

$$\eta(\xi) = \frac{|\xi|^\alpha}{2}, \alpha \in (0, 2]$$

is the Lévy symbol of the so-called $\alpha-$stable process.

**Definition 20.** *A stochastic process $(M_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a **martingale** if*

*(1) For any $t \geq 0$, $M_t$ is $\mathcal{F}_t$-measurable and integrable;*
*(2) For any $s \leq t$, $\mathbf{E}[M_t|\mathcal{F}_s] = M_s$ a.s.*

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a non-negative random variable $T$ is called a **stopping time** if for any $t \geq 0$, the event $\{T \leq t\}$ is measurable with respect to $\mathcal{F}_t$.

A stochastic process $(M_t)_{t\geq 0}$ is called a **local martingale** if there exists a sequence of increasing sopping times $T_n \to \infty$ such that for any $n \in \mathbb{N}$,$(M_{t\wedge T_n})_{t\geq 0}$ is a martingale.

Itô calculus is very important in stochastic analysis. Here we will not give the definition of stochastic integral with respect to Brownian motion, even for Lévy process. But we will give Itô's formula which will be used many times in this thesis.

**Theorem 21** (see Proposition 1.6 of Chapter $VII$ in [18])**.** *Suppose that $f(t, x)$ is a function defined on $[0, \infty) \times \mathbb{R}$, and $f$ is $C^1$ with respect to $t$, and $C^2$ with respect to $x$. If a stochastic process $Y_t$ can be written as $dY_t = X_t dB_t + Z_t dt$, then*

$$df(t, Y_t) = \big(\frac{\partial f}{\partial t}(t, Y_t) + \frac{\partial f}{\partial x}(t, Y_t)Z_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, Y_t)\big)dt + \frac{\partial f}{\partial x}(t, Y_t)X_t dB_t.$$

*In particular, if $f(t, Y_t)$ is a local martingale, the drift term in above formula is equal to zero.*

We also have an integral version of Itô's formula. It can stated as the following :

**Theorem 22.** *Suppose that $L_t$ is a Lévy process, then for any $C^2$ function $f$,*

$$M_t^f := f(L_t) - f(L_0) - \int_0^t \Lambda f(L_s)ds$$

*is a local martingale. In particular,if $f(L_t)$ is a local martingale, then $\Lambda f \equiv 0$. Here $\Lambda$ is the **generator** corresponding to the Lévy process $L_t$ which is defined as*

$$\Lambda f(x) = \lim_{t\downarrow 0} \frac{\mathbf{E}^x[f(L_t)] - f(x)}{t}.$$

*Remark* 23. In fact, if $f(t, L_t)$ is a local martingale, then $\partial_t f = -\Lambda f$, here when $\Lambda$ acts on $f$, it is considered as an function of the second variable $x$.

For martingales, the following stopping theorem is very useful to determine the expectation of some random variables.

**Theorem 24** (see section 3 of Chapter $II$ in [18]). *If $(M_t)_{0\le t\le T}$ is a martingale and $\sup_{0\le t\le T} \mathbf{E}[M_t] < \infty$, then $\mathbf{E}[M_0] = \mathbf{E}[M_t]$.*

# 2.7 Chordal Schramm Loewner Evolution

In 1999([20]), Oded Schramm founded that by assigning the driven process to be the one-dimensional standard Brownian motion, the Loewner chains can be used to be the candidate for describing the scaling limits of some lattice models in statistic physics. What he defined is called the **Schramm Loewner evolution (SLE)** process. Now we give the detailed definition of SLE.

**Definition 25** (see [20]). *Given $\kappa > 0$, let $W_t = \sqrt{\kappa}B_t$, we call the random Lowner chain driven by $W_t$ the **chordal** $\mathrm{SLE}(\kappa)$ **process** from $0$ to $\infty$ in $\mathbb{H}$.*

**Proposition 26** (see [19]). *Suppose $\{g_t(z) : t \ge 0\}$ and $\{K_t : t \ge 0\}$ are the chordal SLE process from $0$ to $\infty$ in $\mathbb{H}$. Then*

*(1) (scaling invariance)For any $r > 0$, the process $\hat{g}_t(z) := r^{-\frac{1}{2}} g_{rt}(\sqrt{r}z)$ has the same distribution as $g_t(z)$ considered as stochastic processes indexed by $(t, z)$. In particular, $t \to K_t$ has the same distribution as $t \to r^{-\frac{1}{2}} K_{rt}$ as $\mathbb{H}$-hulls.*

*(2) (Markov property) Suppose that $\tau$ is a stopping time about the filtration generated by $W_t$. Then $\tilde{g}_t(z) := g_{t+\tau} \circ g_\tau^{-1}(z + W_\tau) - W_\tau$ is independent with $\{g_t : 0 \le t \le \tau\}$, and has the same distribution as $g_t(z)$.*

*Remark* 27. By (1), we can define the SLE for any triple $(D, z, w)$ where $D$ is a simply connected domain with $z, w \in \partial D$ by choosing any conformal map $f$ from $\mathbb{H}$ to $D$ that sends $0$ to $z$ and sends $\infty$ to $w$ and then defining the SLE process in $D$ from $z$ to $w$ as the the image of the SLE process in $\mathbb{H}$ from $0$ to $\infty$.

In fact, chordal $\mathrm{SLE}(\kappa)$ process is almost surely generated by a curve. And the property of the curve depends on the parameter $\kappa$.

**Theorem 28** (see [19] and [12]). *Suppose that $(K_t : t \ge 0)$ is the chordal $\mathrm{SLE}(\kappa)$ process from $0$ to $\infty$ in $\mathbb{H}$. Almost surely, there exists a continuous non self-crossing curve $\gamma$ in $\overline{\mathbb{H}}$ such that $\gamma(0) = 0$, $\gamma(\infty) = \infty$ and for any $t \ge 0$, $K_t$ is equal to the complement of the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ in $\overline{\mathbb{H}}$, i.e. the chordal $\mathrm{SLE}(\kappa)$ process is almost surely generated by a curve. We call $\gamma$ the $\mathrm{SLE}(\kappa)$ **trace**.*

**Theorem 29** (see [19]). *Suppose $\gamma$ is the $\mathrm{SLE}(\kappa)$ trace from $0$ to $\infty$ in $\mathbb{H}$, then*

*(a) If $\kappa \in (0, 4]$, almost surely $\gamma$ is a simple curve;*

*(b) If $\kappa \in (4, 8)$, almost surely $\gamma$ is a self-touching curve and encloses $\overline{\mathbb{H}}$, but for any given point $z \in \overline{\mathbb{H}} \setminus \{0\}$, almost surely $z \notin \gamma$;*

*(c) If $\kappa \in [8, \infty)$, almost surely $\gamma[0, \infty] = \overline{\mathbb{H}}$, i.e. $\gamma$ is a space-filling curve.*

For some special values $\kappa$, the SLE($\kappa$) trace satisfies some special properties. The most interesting case is $\kappa = 6$ and $\kappa = \frac{8}{3}$.

**Proposition 30** (see [14][11]). *Suppose that $\gamma$ is the SLE($\kappa$) trace from 0 to $\infty$ in $\mathbb{H}$.*

*(a) If $\kappa = 6$, for any $\mathbb{H}$-hull $K$ with $0 \notin K$, define*

$$\tau_K := \inf\{t \geq 0 : \gamma(t) \in K\},$$

*then $\gamma[0, \tau_K)$ has the same distribution as the SLE(6) in $\mathbb{H} \setminus K$ from 0 to $\infty$ upon hitting $K$. This is called the **local property** of SLE(6).*

*(b) If $\kappa = \frac{8}{3}$, for any $\mathbb{H}$-hull $K$ with $0 \notin K$. Define $\Phi_K(z) = g_K(z) - g_K(0)$, where $g_K(z)$ is the normalised conformal map corresponding to $K$. Then*

$$\mathbf{P}[\gamma \cap K = \emptyset] = \Phi'_K(0)^{\frac{5}{8}}. \tag{2.7.1}$$

*Notice that for two $\mathbb{H}$-hulls $K_1, K_2$ that don't contain 0,*

$$\Phi_{K_1 \cup \Phi^{-1}_{K_1}(K_2)} = \Phi_{K_2} \circ \Phi_{K_1}, \Phi'_{K_1 \cup \Phi^{-1}_{K_1}(K_2)}(0) = \Phi'_{K_2}(0)\Phi'_{K_1}(0).$$

*Then conditioned on $\gamma \cap K = \emptyset$, $\gamma$ has the same distribution as the SLE($\frac{8}{3}$) from 0 to $\infty$ in $\mathbb{H} \setminus K$. This is called the **restriction property** of SLE($\frac{8}{3}$).*

Given $z \in \mathbb{H}$, for $\kappa \in (0, 8)$, we have known that $z \notin \gamma$ a.s. So it makes senses to ask the question about the probability that $z$ lies to the left(right) side of $\gamma$. In fact Oded Schramm has got a formula for this.

**Proposition 31** (see [21]). *Given $z = x + iy \in \mathbb{H}$, $\kappa \in (0, 8)$. Suppose that $\gamma$ is the SLE($\kappa$) trace from 0 to $\infty$ in $\mathbb{H}$. Then the probability that $\gamma$ passes the left of $z$ is*

$$p(z) = C \int_{-\infty}^{\frac{x}{y}} (1 + t^2)^{-\frac{4}{\kappa}} dt, \tag{2.7.2}$$

*where $C = C(\kappa)$ is the constant that make the total integral above equal to 1.*

Notice that for $\kappa = 4$, $p(z) = 1 - \frac{\arg(z)}{\pi}$; for $\kappa = \frac{8}{3}$, $p(z) = \frac{1}{2}(1 + \frac{x}{|z|})$. There are some other properties which we will not list here, they can be found in[19].

*Remark* 32. The properties we have given here are those that will be used in the proof of the main results of this thesis. There are also many other nice properties of SLE($\kappa$) such as the reversibility, duality and Hausdorff dimension etc. We will not give them here.

## 2.8 Radial and Whole Plane SLE

In this section, we will introduce the radial SLE and whole plane SLE process.

**Definition 33.** *Suppose that* $(B_t : t \in \mathbb{R})$ *is a two-sided Brownian motion, we call the inner(outer) Loewner process (chain) driven by* $W_t = \sqrt{\kappa} B_t$ *the* ***inner (outer) radial*** $\mathrm{SLE}(\kappa)$ *process from* $1$ *to* $0(\infty)$ *in* $\mathbb{D}(\mathbb{D}_+)$ *(see section 2.2). And the inner (outer) whole plane Loewner process (chain) driven by* $W_t = \sqrt{\kappa} B_t$ *the* ***inner (outer) whole plane*** $\mathrm{SLE}(\kappa)$ *process from* $0(\infty)$ *to* $\infty(0)$ *in* $\mathbb{C}$ *(see section 2.3).*

*Remark* 34. For any triple $(D, z, w)$, where $D$ is a proper simple connected domain of $\mathbb{C}$ with $z \in D$ and $w \in \partial D$, there exists a unique conformal map $\phi$ from $\mathbb{D}$ onto $D$ with $\phi(0) = z$ and $\phi(1) = w$. We can define the radial $\mathrm{SLE}(\kappa)$ in $D$ from $w$ to $z$ as the image of the radial $\mathrm{SLE}(\kappa)$ in $\mathbb{D}$ from 1 to 0 under the map $\phi$.

In fact, radial $\mathrm{SLE}(\kappa)$ and whole plane $\mathrm{SLE}(\kappa)$ have many of the same properties as the chordal $\mathrm{SLE}(\kappa)$. By the Markov property of Brownian motion, for radial $\mathrm{SLE}(\kappa)$, the Markov property also holds.

**Proposition 35** (see [10][11])**.** *Suppose* $\tau$ *is a stopping time with respect to the filtration generated by* $W_t = \sqrt{\kappa} B_t$ *and* $(g_t)$ *is the radial* $\mathrm{SLE}(\kappa)$ *process from* $1$ *to* $0$ *in* $\mathbb{D}$*. Then* $\tilde{g}_t(z) := g_{t+\tau} \circ g_\tau^{-1}(z\lambda_\tau)/\lambda_\tau$ *has the same distribution as* $g_t(z)$ *and is independent with* $\{g_t : 0 \le t \le \tau\}$*. Here* $\lambda_t = \exp\{iW_t\}$*.*

But for the whole palne $\mathrm{SLE}(\kappa)$ the scaling property holds.

**Proposition 36** (see [10] [11])**.** *Suppose that* $(g_t, K_t : t \in \mathbb{R})$ *is the (outer) whole plane* $\mathrm{SLE}(\kappa)$*, and* $r \in \mathbb{R}$*. then* $(g_{t+r}(e^r z), e^{-r} K_{t+r} : t \in \mathbb{R})$ *has the same distribution as* $(g_t, K_t : t \in \mathbb{R})$*.*

*Remark* 37. So given two different points $z, w \in \mathbb{C}$, we can define the whole plane $\mathrm{SLE}(\kappa)$ in $\mathbb{C}$ from $z$ to $w$ as the image under any conformal map from $\mathbb{C}$ to $\mathbb{C}$ that send $\infty$ to $z$ and $0$ to $w$.

**Theorem 38** (see [10])**.** *Almost surely, the (inner)radial* $\mathrm{SLE}(\kappa)$ *is generated by a curve* $\gamma$ *from* $1$ *to* $0$*, which we call the* ***(inner) radial*** $\mathrm{SLE}(\kappa)$ ***trace****. And*

*(1) If* $\kappa \in (0, 4]$*,* $\gamma$ *is a simple curve a.s;*

*(2) If* $\kappa \in (4, 8)$*,* $\gamma$ *is a selftouching but non-selfcrossing curve a.s. Also* $\overline{\mathbb{D}} = \bigcup_{t \ge 0} K_t$*, where* $K_t$ *is the radial* $\mathrm{SLE}(\kappa)$ *hulls. But for a given* $z \in \overline{\mathbb{D}} \setminus \{1\}$*,* $z \notin \gamma$ *a.s.*

*(3) If* $\kappa \in [8, \infty)$*, then* $\overline{\mathbb{D}} = \gamma[0, \infty]$ *a.s, i,e* $\gamma$ *is a space-filling curve almost surely.*

*Remark* 39. For the whole plane $\mathrm{SLE}(\kappa)$, it is also generated by a curve almost surely and the phase of the curve has the same behavior as the radial $\mathrm{SLE}(\kappa)$.

For special values $\kappa = 6$ and $\kappa = \frac{8}{3}$, the locality and restriction property also hold for radial SLE process.

**Proposition 40** (see Chapter 6 in [10]). *Suppose that $\gamma$ is the radial* SLE($\kappa$) *trace from* 1 *to* 0 *in* $\mathbb{D}$.

*(a) If $\kappa = 6$, for any $\mathbb{D}$-hull $K$ with $1 \notin K$, define*

$$\tau_K := \inf\{t \geq 0 : \gamma(t) \in K\},$$

*then $\gamma[0, \tau_K)$ has the same distribution as the radial* SLE(6) *in $\mathbb{D} \setminus K$ from* 1 *to* 0 *upon hitting $K$. This is called the* ***local property*** *of radial* SLE(6).

*(b) If $\kappa = \frac{8}{3}$, for any $\mathbb{D}$-hull $K$ with $1 \notin K$. Define $\Phi_K(z) = g_K(z)/g_K(1)$, where $g_K(z)$ is the normalised conformal map corresponding to $K$. Then*

$$\mathbf{P}[\gamma \cap K = \emptyset] = |\Phi_K'(0)|^{\frac{5}{48}} |\Phi_K'(1)|^{\frac{5}{8}}. \tag{2.8.1}$$

*Notice that for two $\mathbb{D}$-hulls $K_1, K_2$ that don't contain* 1,

$$\Phi_{K_1 \cup \Phi_{K_1}^{-1}(K_2)} = \Phi_{K_2} \circ \Phi_{K_1},$$

$$\Phi'_{K_1 \cup \Phi_{K_1}^{-1}(K_2)}(0) = \Phi'_{K_2}(0)\Phi'_{K_1}(0), \Phi'_{K_1 \cup \Phi_{K_1}^{-1}(K_2)}(1) = \Phi'_{K_2}(1)\Phi'_{K_1}(1).$$

*Then conditioned on $\gamma \cap K = \emptyset$, $\gamma$ has the same distribution as the* SLE($\frac{8}{3}$) *from* 1 *to* 0 *in $\mathbb{D} \setminus K$. This is called the* ***restriction property*** *of radial* SLE($\frac{8}{3}$).

For whole plane SLE(6), the locality also holds.

**Proposition 41** (see Chapter 6 in [10]). *Suppose $\gamma$ is a (outer)whole plane* SLE(6) *curve from* 0 *and* $\infty$ *and suppose $w \in \mathbb{C} \setminus \{0\}$. Let $t^*$ be the first time $t$ that $\gamma[0, t]$ disconnects $w$ from $\infty$. Then $(\gamma(t) : 0 \leq t \leq t^*)$ has the same distribution of a whole plane* SLE(6) *path from* 0 *to $w$ stopped at the first time that it disconnects $w$ from* $\infty$.

Moreover the frontier points of the whole plane SLE(6) have the same distribution as the frontier points of planar Brownian motion. This is a very deep result that connects SLE and Brownian motion.

**Theorem 42** (see Chapter 6 in [10]). *Let $\gamma$ be a (outer) whole plane* SLE(6) *path from* 0 *to* $\infty$. *Let $D$ be a simply connected domain other than $\mathbb{C}$ containing the origin. Define*

$$\tau_D = \inf\{t : \gamma(t) \in \partial D\}.$$

*Let $B_t$ denote a complex Brownian motion starting at the origin and define*

$$\sigma_D = \inf\{t : B_t \in \partial D\}.$$

*Then $\gamma(\tau_D)$ and $B(\sigma_D)$) have the same distribution, i.e., the measure on $\partial D$ induced by $\gamma(\tau_D)$ is the harmonic measure in $D$ started at* 0.

It is easy to see from the definition that when stopping (outer)whole plane SLE($\kappa$) at some time, the remaining of the trace is the radial SLE($\kappa$) in the complement of the trace.

**Proposition 43.** *Let $(g_t, K_t : t \in \mathbb{R})$ be the whole plane* SLE($\kappa$) *from 0 to $\infty$. Suppose that $t_0 \in \mathbb{R}$ is a stopping time with respect to the filtration generated by $B_t$(Here $W_t = \sqrt{\kappa}B_t$ is the driven process and $B_t$ is a standard two-sided Brownian motion ). Then conditioned on $(K_t, t \le t_0)$, $(K_t : t \ge t_0)$ has the distribution of a radial* SLE($\kappa$) *in $\mathbb{C} \setminus K_{t_0}$ from $\gamma(t_0)$ to $\infty$.*

*Proof.* For any $t \ge 0$, define

$$f_t(z) := \frac{e^{iW_t}}{g_{t+t_0}(z)},$$

Then $f_t(\infty) = 0$ and

$$\partial_t f_t(z) = f_t(z)\frac{e^{iW_t'} + f_t(z)}{e^{iW_t'} - f_t(z)},$$

where $W_t' = W(t_0) - W(t + t_0) = \sqrt{\kappa}B_t$, which has the distribution of a standard one dimensional Brownian motion. □

In fact chordal SLE($\kappa$) and radial SLE($\kappa$) are equivalent in some sense.

**Theorem 44** (see Chapter 6 in [10][13])**.** *Suppose that $(K_t)_{t\ge 0}$ is a chordal* SLE($\kappa$) *process from $-1$ to 1 in $\mathbb{D}$. and $(\tilde{K}_t)_{t\ge 0}$ is a radial* SLE($\kappa$) *process from $-1$ to 0 in $\mathbb{D}$. Define*

$$\tau := \sup\{t > 0 : 0 \notin K_t\}, \tilde{\tau} := \sup\{t > 0 : 1 \notin \tilde{K}_t\}.$$

*Then if $\kappa = 6$, $\{K_t : 0 \le t < \tau\}$ has the same distribution as $\{\tilde{K}_t : 0 \le t < \tilde{\tau}\}$ ; If $\kappa \ne 6$, there exist two sequences of stopping times $(T_n, n \ge 1) \uparrow T$ and $(\tilde{T}_n, n \ge 1) \uparrow \tilde{T}$ such that $\{K_t : 0 \le t < T_n\}$ and $\{\tilde{K}_t : 0 \le t < \tilde{T}_n\}$ are absolutely continuous with each other.*

We have seen in section 2.4 that whole plane Loewner process can be approximated by radial Loewner process. So the whole plane SLE($\kappa$) can also be approximated by radial SLE($\kappa$). We will use this to give a result which will be very important in computing the spectrum of whole plane SLE($\kappa$) process.

The definition of the (outer) radial SLE($\kappa$) can be extended to $t \in \mathbb{R}$ by considering stochastic ODE :

$$\partial_t g_t(z) = g_t(z)\frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, \quad g_0(z) = z, \forall z \in \mathbb{D}_+,$$

where $\lambda(t) = \exp\{i\sqrt{\kappa}B_t\}$ and $B_t$ is a two-sided Brownian motion. Then

**Lemma 45.** *Let $(g_t : t \in \mathbb{R})$ be the (outer) radial* SLE($\kappa$) *defined above, then the map $z \to g_{-t}(z)$ has the same distribution as $z \to g_t^{-1}(z\lambda(t))/\lambda(t)$. And so the solution of the equation*

$$\partial_t f_t(z) = f_t(z)\frac{f_t(z) + \lambda(t)}{f_t(z) - \lambda(t)}, f_0(z) = z, \forall z \in \mathbb{D}_+, \tag{2.8.2}$$

*has the same distribution as $g_t^{-1}(z\lambda(t))/\lambda(t)$.*

**Lemma 46.** *Suppose that $(f_t : t \in \mathbb{R})$ is defined as (2.8.2), then*

$$\lim_{t\to+\infty} e^{-t} f_t(z) \overset{\text{(law)}}{=} F_0(z),$$

*where $F_0(z) = h_0^{-1}(z)$ and $(h_t : t \in \mathbb{R})$ is the outer whole plane* SLE($\kappa$) *process.*

*Proof.* Given $s > 0$, recall the definition of $g_t^{(s)}$ in Lemma 16. For $t \geq -s$, it is the solution of the following differential equation :

$$\partial_t g_t^{(s)} = g_t^{(s)} \frac{\lambda(t) + g_t^{(s)}}{\lambda(t) - g_t^{(s)}}, \quad g_{-s}^{(s)} = e^s z.$$

Then $(g_{-t}^{(s)} : t \leq s)$ satisfies

$$\partial_t g_{-t}^{(s)} = g_{-t}^{(s)} \frac{g_{-t}^{(s)} + \tilde{\lambda}(t)}{g_{-t}^{(s)} - \tilde{\lambda}(t)}, \quad g_{-s}^{(s)} = e^s z.$$

where $\tilde{\lambda}(t) = \lambda(-t) \overset{\text{(law)}}{=} \lambda(t)$. So by the uniqueness of solution the ODE, we have $g_{-t}^{(s)} \overset{\text{(law)}}{=} f_t(g_0^{(s)})$. Let $t = s$ and combing $g_{-s}^{(s)} = e^s z$, we have

$$e^{-t} f_t(z) \overset{\text{(law)}}{=} (g_0^{(t)})^{-1}(z),$$

where $(g_0^{(t)})^{-1}(z)$ is the inverse of $g_0^{(t)}$. By Lemma 16, as $t \to +\infty$, the conformal map $g_0^{(t)}$ converges locally uniformly to $h_0(z)$, where $(h_t : t \in \mathbb{R})$ is the outer whole plane SLE($\kappa$). Therefore the inverse of $g_0^{(t)}$ converges to $h_0^{-1}(z)$. $\square$

Also the (inner) radial SLE($\kappa$) can be extended to $t \in \mathbb{R}$ by considering stochastic ODE :

$$\partial_t g_t(z) = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, \quad g_0(z) = z, \forall z \in \mathbb{D},$$

where $\lambda(t) = \exp\{i\sqrt{\kappa} B_t\}$ and $B_t$ is a two-sided Brownian motion. Then

**Lemma 47.** *Let $(g_t : t \in \mathbb{R})$ be the (inner) radial* SLE($\kappa$) *defined above, then the map $z \to g_{-t}(z)$ has the same distribution as $z \to g_t^{-1}(z\lambda(t))/\lambda(t)$. And so the solution of the equation*

$$\partial_t f_t(z) = f_t(z) \frac{f_t(z) + \lambda(t)}{f_t(z) - \lambda(t)}, f_0(z) = z, \forall z \in \mathbb{D}, \tag{2.8.3}$$

*has the same distribution as $g_t^{-1}(z\lambda(t))/\lambda(t)$.*

**Lemma 48.** *Suppose that $(f_t : t \in \mathbb{R})$ is defined as (2.8.3), then*

$$\lim_{t\to+\infty} e^{t} f_t(z) \overset{\text{(law)}}{=} H_0(z),$$

*where $H_0(z) = h_0^{-1}(z)$ and $(h_t : t \in \mathbb{R})$ is the inner whole plane* SLE($\kappa$) *process.*

*Proof.* Given $s > 0$, recall the definition of $g_t^{(s)}$ in Lemma 17. For $t \geq s$, it is the solution of the following differential equation :

$$\partial_t g_t^{(s)} = g_t^{(s)} \frac{g_t^{(s)} + \lambda(t)}{g_t^{(s)} - \lambda(t)}, \quad g_s^{(s)} = e^{-s} z.$$

So by the uniqueness of solution of the ODE, we have $g_t^{(s)}(z) \overset{\text{(law)}}{=} f_t(g_0^{(s)}(z))$. Therefore $e^t f_t(z) \overset{\text{(law)}}{=} e^t g_t^{(s)}((g_0^{(s)})^{-1})$. Let $t = s$, we get

$$e^t f_t(z) \overset{\text{(law)}}{=} (g_0^{(t)})^{-1}(z)$$

Then combining Lemma 17 and let $t \to +\infty$, we finish the proof. □

## 2.9 Driven by a Lévy process

Since a Lévy process has a modification which is left continuous and has right limit, by section 2.5 we can consider the Loewner process driven by a Lévy process. And by the Markov property of Lévy process, the random Loewner chain we get also satisfies the Markov property( notice that it may not be generated by a continuous curve now since Lévy process may have jumps). In fact, most of the conclusions of SLE process in the previous section still holds for the Loewner chain driven by a general Lévy process. Especially we emphasize that Lemma 48 still holds for Lévy driven processes. This will used in the computation of the avarage spectrum of the inner whole plane Loewner processes driven by a Lévy process.

# Chapitre 3

# Dipolar Conformal Restriction Measure

In this chapter, we will use dipolar SLE($\kappa$) process to construct a conformal restriction measure on the strip $\mathcal{S}$, which is defined as follows :

$$\mathcal{S} = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, 0 < y < \pi\}. \tag{3.0.1}$$

This dipolar conformal restriction is a "dipolar version" of the results derived in the the paper [9] by Lawler,Schramm,and Werner("chordal version") and the paper [23] by Hao Wu("radial version"). The goal is to describe the law of a random set on the strip that satisfies a certain restriction property.

We are going to study closed random subsets $K$ of $\mathcal{S}$ that have the following form :

(1) $K$ is a connected compact set of $\bar{\mathcal{S}}$ such that $K \cap \mathbb{R} = \{0\}$ and $\mathcal{S} \setminus K$ has two connected components that one of their boundaries contains $+\infty$ and the other contains $-\infty$ ;

(2) For any connected compact subset $A$ of $\bar{\mathcal{S}}$ such that $A \cap \mathbb{R}_\pi = \emptyset$, $\mathcal{S} \setminus A$ is simply connected and $d(0, A) > 0$, the law of $\Phi_A(K)$ conditioned on the event $\{K \cap A = \emptyset\}$ is the same as the law of $K$, where $\Phi_A$ is the unique conformal map from $\mathcal{S} \setminus A$ to $\mathcal{S}$ that fixes $+\infty$ and $-\infty$ and $\lim_{z\to+\infty}(\Phi_A(z) - z) = -\lim_{z\to-\infty}(\Phi_A(z) - z)$.

The law of such a set is called a **dipolar restriction measure**, which is the analogy of the chordal restriction measure defined in [9] and the radial restriction measure defined in [23].

The main result of this chapter is the following characterization of all the dipolar restriction measures.

**Theorem 49.** *(1) A dipolar conformal restriction measure is fully characterized by two real parameters $(\alpha, \beta)$ such that*

$$\mathbf{P}[K \cap A = \emptyset] = |\Phi_A'(0)|^\beta \exp\{-\alpha S(A)\}$$

*where $A$ is any connected compact subset of $\bar{\mathcal{S}}$ such that $A \cap \mathbb{R}_\pi = \emptyset$, $\mathcal{S} \setminus A$ is simply connected and $d(0, A) > 0$, and $\Phi_A$ is the conformal map from $\mathcal{S} \setminus A$ to $\mathcal{S}$ that fixes $\pm\infty$ and $S(A)$ is the capacity of $A$(see the definition in section 3.3). We denote this restriction measure by $\mathbf{P}(\alpha, \beta)$.*

*(2) For any $\beta \geq \frac{5}{8}$, the measure $\mathbf{P}(\frac{\beta(1-\beta)}{2\beta+1}, \beta)$ exists and if $K$ is a sample of this measure, then $K \cap \mathbb{R}_\pi$ contains only one point $X + i\pi$ and the random variable has the density function(up to a constant)*

$$\rho(x) = (\cosh \frac{x}{2})^{-\frac{2}{3}(2\beta+1)}.$$

We will prove this theorem step by step in the following sections. It is necessary to give an introduction to the chordal conformal restriction measure and radial conformal restriction measure.

# 3.1 Chordal conformal restriction measure

Let $\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane and $\Omega$ is the collection of subset $K$ of $\mathbb{H}$ that satisfies the following conditions :

(1) $K$ a connected closed set, $K \cap \mathbb{R} = \{0\}$ ;

(2) $\mathbb{H} \setminus K$ has two connected unbounded components.

Let $\mathcal{A}_h$ be the collection of $\mathbb{H}$-hulls and $\mathcal{A}_h^* := \{A \in \mathcal{A}_h : 0 \notin A\}$. Suppose $\mathcal{F}_h$ is the $\sigma$ -algebra on $\Omega$ generated by the class below :

$$\{\{K : K \cap A = \emptyset\} : A \in \mathcal{A}_h^*\}.$$

If a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F}_h)$ satisfies : for any $A \in \mathcal{A}_h^*$, conditioned on $K \cap A = \emptyset$, $\Phi_A(K)$ has the same law as $K$. Here $K$ is a sample of $\mathbf{P}$. We call $\mathbf{P}$ the **conformal restriction measure** on $\mathbb{H}$. Notice that if two probability measures $\mathbf{P}, \mathbf{P}'$ on $(\Omega, \mathcal{F}_h)$ satisfies for any $A \in \mathcal{A}_h^*$, $\mathbf{P}[K \cap A = \emptyset] = \mathbf{P}'[K \cap A = \emptyset]$, then $\mathbf{P} = \mathbf{P}'$.

**Theorem 50** (see [9]). *The conformal restriction measure on $\mathbb{H}$ can be characterized as follows.*

*(1) If $\mathbf{P}$ is a conformal restriction measure on $\mathbb{H}$ , there exists a unique $\beta \in \mathbb{R}$ and $\beta \geq \frac{5}{8}$ such that for any $A \in \mathcal{A}_h^*$ ,*

$$\mathbf{P}[K \cap A = \emptyset] = \Phi'_A(0)^\beta,$$

*where $\Phi_A(z)$ is the conformal map from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ that fixes $0$ and $\infty$. So the conformal restriction can be characterized by one parameter $\beta$, which is denoted by $\mathbf{P}(\beta)$.*

*(2) $\mathbf{P}(\beta)$ exists if and only if $\beta \geq \frac{5}{8}$.*

Notice that by Proposition 30, chordal SLE($\frac{8}{3}$) has the law $\mathbf{P}(\frac{5}{8})$.

## 3.2 Radial conformal restriction measure

Similarly, we have the radial conformal restriction. Still we denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit disk and $\Omega$ the collection of subsets of $\mathbb{D}$ that satisfies the following conditions :

(1) $K$ is a connected closed set, $K \cap \partial\mathbb{D} = \{1\}$ ;

(2) $0 \in K$ and $\mathbb{H} \setminus K$ is connected.

Denote by $\mathcal{A}_d$ the collection of $\mathbb{D}$-hulls and $\mathcal{A}_d^* := \{A \in \mathcal{A}_d : 1 \notin A\}$. Suppose $\mathcal{F}_d$ is the $\sigma$-algebra on $\Omega$ that is generated by the class below :

$$\big\{\{K : K \cap A = \emptyset\} : A \in \mathcal{A}_d^*\big\}.$$

If a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F}_d)$ satisfies for any $A \in \mathcal{A}_d^*$, conditioned on $K \cap A = \emptyset$, $\Phi_A(K)$ has the same law as $K$. Here $K$ is a sample of $\mathbf{P}$ . We call $\mathbf{P}$ a **conformal restriction measure** on $D$ ( or a radial conformal restriction measure). Notice that if two probability measures $\mathbf{P}, \mathbf{P}'$ on $(\Omega, \mathcal{F}_d)$ satisfy that for any $A \in \mathcal{A}_d^*$, $\mathbf{P}[K \cap A = \emptyset] = \mathbf{P}'[K \cap A = \emptyset]$, then $\mathbf{P} = \mathbf{P}'$.

**Theorem 51** (see[23])**.** *The radial conformal restriction measure can be characterized as follows.*

*(1) If $\mathbf{P}$ is a radial conformal restriction measure on $\mathbb{D}$, there are two parameters $(\alpha, \beta)$ such that for $A \in \mathcal{A}_d^*$,*

$$\mathbf{P}[K \cap A = \emptyset] = |\Phi_A'(0)|^{\alpha} |\Phi_A'(1)|^{\beta},$$

*where $\Phi_A$ is the conformal map from $\mathbb{D} \setminus A$ onto $\mathbb{D}$ that fixes $0$ and $1$. Then a radial conformal restriction measure can be characterized by two parametres $(\alpha, \beta)$, which is denoted by $\mathbf{P}(\alpha, \beta)$.*

*(2) $\mathbf{P}(\alpha, \beta)$ exists if and only if $\beta \geq \frac{5}{8}$ and $\alpha \leq \frac{1}{48}((\sqrt{24\beta + 1} - 1)^2 - 4)$.*

By Proposition 40, we know that radial SLE$(\frac{8}{3})$ has the law $\mathbf{P}(\frac{5}{48}, \frac{5}{8})$.

## 3.3 Dipolar Loewner Process

In this section, the dipolar Loewner process will be given(the reader can also refer[24]). Just like the chordal Loewner process, this is a differential equation that describes the evolution of the conformal map corresponding a curve in $\mathcal{S}$.

The upper half plane is denoted by $\mathbb{H}$ , the strip with width $\pi$ is just defined as (3.0.1) and the upper boundary of $\mathcal{S}$ is denoted by $\mathbb{R}_\pi$. We will frequently use the conformal map $\varphi_0(z) := e^z - 1$ from $\mathcal{S}$ to $\mathbb{H}$ that sends $-\infty, 0, +\infty$ to $-1, 0, \infty$ respectively.

**Definition 52.** *A compact connected subset $A \subset \overline{\mathcal{S}}$ is called a* $\mathcal{S}$**-hull (dipolar compact hull)** *if $A = \overline{A \cap \mathcal{S}}$, $A \cap \mathbb{R}_\pi = \emptyset$ and $\mathcal{S} \setminus A$ is simply connected. Denote by $\mathcal{A}_s$ the collection of all $\mathcal{S}$-hulls.*

**Lemma 53.** *For any $A \in \mathcal{A}_s$, there exists a unique conformal map $g_A$ from $\mathcal{S} \setminus A$ onto $\mathcal{S}$ such that $g_A(\pm\infty) = \infty$ and*

$$\lim_{z\to+\infty}[g_A(z) - z] = -\lim_{z\to-\infty}[g_A(z) - z] < \infty.$$

*Proof.* (1)**Existence**. Define $f(z) = e^z$, then $f(A)$ is a $\mathbb{H}$-hull with $0 \notin f(A)$. Then by section 2.1, there exists a unique conformal map $\phi_{f(A)}$ from $\mathbb{H} \setminus f(A)$ onto $\mathbb{H}$ such that

$$\phi_{f(A)}(0) = 0, \quad \lim_{z\to\infty}\frac{\phi_{f(A)}(z)}{z} = 1, \ \ , \phi'_{f(A)}(0) < 1.$$

Let $h(z) := \log\left[\phi_{f(A)}(e^z)\right]$, then

$$h : \mathcal{S} \setminus A \to \mathcal{S},\ h(-\infty) = -\infty,\ h(+\infty) = +\infty,$$

$$h(\mathbb{R}) = \mathbb{R},\ h(\mathbb{R} + i\pi) = \mathbb{R} + i\pi,$$

$$\lim_{z\to+\infty}[h(z) - z] = \lim_{z\to+\infty}\log\frac{\phi_{f(A)}(e^z)}{z} = 0,$$

and

$$\lim_{z\to-\infty}[h(z) - z] = \lim_{z\to-\infty}\log\frac{\phi_{f(A)}(e^z)}{z} = \log\phi'_{f(A)}(0) < 0.$$

Therefore $g_A(z) := h(z) - \frac{1}{2}\log\phi'_{f(A)}(0)$ satisfies

$$\lim_{z\to+\infty}[g_A(z) - z] = -\frac{1}{2}\log\phi'_{f(A)}(0),$$

$$\lim_{z\to-\infty}[g_A(z) - z] = \frac{1}{2}\log\phi'_{f(A)}(0).$$

And $g_A(z)$ fixes $\mathbb{R}$ and $\mathbb{R} + i\pi$.

(2)**Uniqueness**. If $g_1$ and $g_2$ both satisfy the conditions, let $h(z) = g_1 \circ g_2^{-1}(z) : \mathcal{S} \to \mathcal{S}$ and

$$h(\mathbb{R}) = \mathbb{R},\ h(\mathbb{R} + i\pi) = \mathbb{R} + i\pi \quad (*)$$

$$\lim_{z\to+\infty}[h(z) - z] = -\lim_{z\to-\infty}[h(z) - z] \quad (**)$$

By $(*)$ we get that $h(z) = z + c$, where $c$ a real number. By $(**)$, we get $c = 0$. ☐

We call $S(A) := \lim_{z\to+\infty}[g_A(z) - z]$ the **capacity** of $A$ and $g_A$ the **normalised conformal map** with respect to $A$. Notice that $S(A) \geq 0$ and the equality holds if and only if $A \subset \mathbb{R}$. Another useful conformal map about $\mathcal{S}$-hulls is : $\forall A \in \mathcal{A}_s$, there exists a unique conformal map $\Phi_A : \mathcal{S} \setminus A \to \mathcal{S}$ such that

$$\Phi_A(0) = 0, \Phi_A(+\infty) = +\infty, \Phi_A(-\infty) = -\infty$$

Indeed we just let $\Phi_A(z) := g_A(z) - g_A(0)$.

Just like the chordal case, start from a non-selfcrossing curve $\gamma[0, \infty) \subset \overline{\mathcal{S}}$ such that the $\mathcal{S}$-hulls created by $\gamma$ is strictly increasing. In the purpose of simplifying the procedure, we assume that $\gamma$ is a simple curve and $\gamma(0, \infty) \subset \mathcal{S}$. Then for any $t \geq 0$, $K_t := \gamma[0, t]$ is a $\mathcal{S}$-hull. Let $S_t := \mathcal{S} \setminus K_t$. Denote by $S(t)$ the capacity of $\gamma[0, t]$ and $g_t(z)$ the corresponding normalised conformal map. Then we have

**Lemma 54.** *With above notations,*

*(a) $S(t)$ is a strictly increasing continuous function ;*

*(b) The limit $W_t := \lim_{z\in S_t; z\to\gamma(t)} g_t(z)$ exists and is a continuous function ;*

*(c) If the curve is parameterized such that $S(t) = t$, then the conformal maps $g_t(z)$ satisfies the following differential equation :*

$$\partial_t g_t(z) = \coth \frac{g_t(z) - W_t}{2}, \quad g_0(z) = z, \tag{3.3.1}$$

*where* $\coth(z) = (e^z + e^{-z})/(e^z - e^{-z})$.

*We call* (3.3.1) *the* ***dipolar Loewner equation****.*

*Proof.* Recall the proof of Lemma 53, let $f(z) := e^z$ and $\tilde{\gamma}(t) = f(\gamma(t))$. Then $\tilde{\gamma}[0,t]$ is a $\mathbb{H}$-hull and let $\tilde{g}_t$ be the normalised conformal map of $\tilde{\gamma}[0,t]$. Then by Lemma 3 we have

$$g_t(z) = \log \left[\tilde{g}_t(e^z) - \tilde{g}_t(0)\right] - \frac{1}{2}\log g_t'(0), \;\; S(t) = -\frac{1}{2}\log \tilde{g}_t'(0),$$

and the limit $\tilde{W}_t := \lim_{z\in\mathbb{H}\setminus\tilde{\gamma}[0,t], z\to\tilde{\gamma}(t)} \tilde{g}_t(z)$ exists and is continuous. Also by Lemma 3, $\tilde{g}_t'(0)$ is continuous with respect to $t$. Therefore $S(t)$ is continuous. Since $K_t$ is strictly increasing, $S(t)$ is strictly increasing. And the limit

$$\begin{aligned} W_t &= \lim_{z\in S_t; z\to\gamma(t)} g_t(z) = \lim_{z\in S_t; z\to\gamma(t)} \log\left[\tilde{g}_t(e^z) - \tilde{g}_t(0)\right] - \frac{1}{2}\log g_t'(0) \\ &= \lim_{z\in\mathbb{H}\setminus\tilde{\gamma}[0,t], z\to\tilde{\gamma}(t)} \log\left[\tilde{g}_t(z) - \tilde{g}_t(0)\right] - \frac{1}{2}\log g_t'(0) \\ &= \log\left[\tilde{W}_t - \tilde{g}_t(0)\right] - \frac{1}{2}\log g_t'(0). \end{aligned}$$

exists and is continuous with respect to $t$. We are left to prove $(c)$. Notice that for fixed $t \geq 0$, $\mathrm{Im}\,(g_t^{-1}(z) - z)$ is a bounded harmonic function and therefore we have

$$\mathrm{Im}\,(g_t^{-1}(z) - z) = -\frac{1}{\pi}\int_{I_t} \mathrm{Im}\,\frac{e^x}{e^z - e^x}\mathrm{Im}\,(g_t^{-1}(x))dx,$$

where $I_t$ is the interval of the image of $\gamma[0,t]$ under $g_t$. Since a holomorphic function is determined by its imaginary part up to a constant. We have

$$g_t^{-1}(z) - z = -\frac{1}{\pi}\int_{I_t} \frac{e^x}{e^z - e^x}\mathrm{Im}\,(g_t^{-1}(x))dx + C.$$

Since

$$g_t^{-1}(z) - z \to \begin{cases} -t & \text{as } z \to +\infty \\ t & \text{as } z \to -\infty, \end{cases}$$

we have

$$C = -t, \quad 2t = \frac{1}{\pi}\int_{I_t} \mathrm{Im}\,(g_t^{-1}(x))dx.$$

Then

$$g_t^{-1}(z) - z = -\frac{1}{2\pi}\int_{I_t} \frac{e^z + e^x}{e^z - e^x}\mathrm{Im}\,(g_t^{-1}(x))dx. \tag{3.3.2}$$

For fixed $h > 0$, apply above equation to $g_{t,t+h}(z) = g_{t+h} \circ g_t^{-1}(z)$, we have

$$g_t \circ g_{t+h}^{-1}(z) - z = -\frac{1}{2\pi}\int_{I_{t,h}} \frac{e^z + e^x}{e^z - e^x}\mathrm{Im}\,[g_t \circ g_{t+h}^{-1}(x)]dx. \tag{3.3.3}$$

Here $I_{t,h}$ is the interval of the image of $\gamma[t, t+h]$ which contains $W_{t+h}$. Notice that

$$h = \frac{1}{2\pi}\int_{I_{t,h}} \mathrm{Im}\,[g_t \circ g_{t+h}^{-1}(x)]dx.$$

Replace $z$ by $g_{t+h}(z)$ in (3.3.3),

$$g_t(z) - g_{t+h}(z) = -\frac{1}{2\pi}\int_{I_{t,h}} \coth\frac{g_{t+h}(z) - x}{2}\mathrm{Im}\,[g_t \circ g_{t+h}^{-1}(x)]dx.$$

By using the integral middle theorem to both the real parts and imaginary parts of above equation and then divided by $h$ and passing $h$ to 0, we have

$$\partial_t^+ g_t(z) = \coth\frac{g_t(z) - W_t}{2}.$$

Similarly, we can also get the differential equation for the left derivative by letting $h < 0$. $\square$

Conversely, starting from a real-valued continuous function $W(t)$ on $\mathbb{R}^+$, consider the dipolar Loewner equation (3.3.1). For a given $z \in \mathcal{S}$, define

$$\tau(z) := \sup\{t > 0, \min_{0\le s\le t} |g_s(z) - W(s)| > 0\},$$

and

$$K_t := \{z \in \overline{\mathcal{S}} : \tau(z) \le t\}, S_t := \mathcal{S} \setminus K_t.$$

Then we have

**Lemma 55.** *Suppose $W(t) : [0, \infty) \to \mathbb{R}$ is a continuous function. And $g_t(z)$ is the solution of the ODE (3.3.1). Define $K_t$ and $S_t$ as above, then*
*(1) For any $t \ge 0$, $g_t(z) : S_t \to \mathcal{S}$ is a conformal map;*
*(2) For any $t \ge 0$, $K_t$ is a $\mathcal{S}$-hull;*
*(3) At $z = \pm\infty$, $g_t(z)$ has the Laurent expansion $g_t(z) = z \pm t + O(\frac{1}{|z|})$.*
*We call $(g_t : t \ge 0)$ and $(K_t : t \ge 0)$ the* ***dipolar Loewner process(Loewner chain)*** *driven by $W$.*

*Proof.* Notic that the blow up time of (3.3.1) is the first time that $g_t(z) - W(t) = 0, 2\pi i$ and

$$\coth \frac{z}{2} = \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}}.$$

So $e^{z/2} = e^{-z/2} \Leftrightarrow e^z = 1 \Leftrightarrow z = 2k\pi i$. If $z \in \mathbb{R} + i\pi$, we have $\operatorname{Im}(g_t(z)) = \pi$, so the maximal interval that the ODE (3.3.1) lives is $[0, \tau(z))$. Since the vector field $\coth(z/2)$ is holomorphic, we can get $g_t(z)$ is holomorphic.

Fix $T > 0$ and $w \in S$, consider the following initial value problem :

$$\partial_t f_t(w) = -\coth \frac{f_t(w) - W(T-t)}{2}, \quad f_0(w) = w$$

Since

$$-\operatorname{Im} \coth \frac{x + iy}{2} = \frac{2 \sin y}{|e^{z/2} - e^{-z/2}|^2} > 0,$$

this ODE will not blow up in $[0, T]$.

Let $h_t(w) = f_{T-t}(w)$, then $h_t(w)$ satisfies

$$\partial_t h_t(w) = \coth \frac{h_t(w) - W_t}{2}, \quad h_T(w) = w$$

Therefore $g_t(h_0(w) = h_t(w), 0 \le t \le T)$. In particular $g_T(h_0(w)) = w$, this shows that $g_T(z)$ is surjective. By the unique dependence on the initial value of ODE, we can see that $g_T$ is injective. So $g_T(z)$ is a conformal map from $S_T$ onto $S$. By expanding at $\pm\infty$, we can see that for fixed $T$, if $|z|$ is large enough, $g_T(z)$ will not blow up. Therefore $K_T$ is compact and a $\mathcal{S}$-hull. Also by expanding at $\pm\infty$, we have the Laurent expansion. □

*Remark* 56. Usually, given a continuous function, the corresponding dipolar Loewner process(chain) is not generated by a non-selfcrossing curve.

## 3.4 Dipolar Schramm Loewner Evolution

In this section, we will introduce the dipolar Schramm Loewner Evolution and give its properties.

**Definition 57.** *Let $\kappa > 0$ and $(B_t : t \ge 0)$ be a standard one dimensional Brownian motion, we call the dipolar Loewner process (chain) driven by $W_t := \sqrt{\kappa} B_t$ the* ***diolar*** SLE($\kappa$) ***process*** *from 0 to $\mathbb{R}_\pi$.*

*Remark* 58. Choose a conformal map $f$ from $\mathcal{S}$ onto $\mathbb{H}$ that sends $+\infty$ and $-\infty$ to $x_1$ and $x_2$ respectively and keeps 0. Then the image of dipolar SLE($\kappa$) under this conformal map is a SLE($\kappa; \rho, \rho$) process with force points $x_1$ and $x_2$, where $\rho = \frac{1}{2}\kappa - 3$(see [22]). But it is more convenient to deal with dipolar SLE($\kappa$) since there are no force points.

*Remark* 59. For any triple $(D, z, I)$, where $D$ is a proper simple connected domain of $\mathbb{C}$ with $z \in \partial D$ and $I \subset \partial D$ is an arc not containing $z$, there exists a unique conformal map $\phi$ from $\mathcal{S}$ onto $D$ with $\phi(0) = z$ and $\phi(\mathbb{R}_\pi) = I$. We can define the dipolar SLE($\kappa$) in $D$ from $z$ to $I$ as the image of the dipolar SLE($\kappa$) in $\mathcal{S}$ from 0 to $\mathbb{R}_\pi$ under the map $\phi$.

Dipolar SLE($\kappa$) also has the Markov property.

**Proposition 60** ((Markov property))**.** *Suppose $\{g_t(z) : t \geq 0\}$ and $\{K_t : t \geq 0\}$ are the dipolar* SLE($\kappa$) *process from* 0 *to* $\mathbb{R}_\pi$ *in* $\mathcal{S}$. *Suppose that $\tau$ is a stopping time about the filtration generated by $W_t$. Then $\tilde{g}_t(z) := g_{t+\tau} \circ g_\tau^{-1}(z + W_\tau) - W_\tau$ is independent with $\{g_t : 0 \leq t \leq \tau\}$, and has the same distribution as $g_t(z)$.*

In fact dipolar SLE($\kappa$) is equivalent to chordal SLE($\kappa$) in some sense.

**Theorem 61** (see [24])**.** *Let $(K_t)_{t\geq 0}$ be the dipolar* SLE($\kappa$) *process from* 0 *to* $\mathbb{R}_\pi$ *in* $\mathcal{S}$ *and $(\tilde{K}_t)_{t\geq 0}$ be the chordal* SLE($\kappa$) *from* 0 *to* $+\infty$ *in* $\mathcal{S}$. *Define*

$$\tau := \inf\{t > 0 : K_t \cap \mathbb{R}_\pi \neq \emptyset\}, \tilde{\tau} := \inf\{t > 0 : \tilde{K}_t \cap \mathbb{R}_\pi \neq \emptyset\}.$$

*Then if $\kappa = 6$, $\{K_t : 0 \leq t < \tau\}$ has the same distribution as $\{\tilde{K}_t : 0 \leq t < \tilde{\tau}\}$; if $\kappa \neq 6$, there exists two sequences of stopping times $(T_n, n \geq 1) \uparrow \tau$ and $(\tilde{T}_n, n \geq 1) \uparrow \tilde{\tau}$ such that $\{K_t : 0 \leq t < T_n\}$ and $\{\tilde{K}_t : 0 \leq t < \tilde{T}_n\}$ are mutually absolutely continuous.*

*Proof.* Denote by $\phi(z) = e^z - 1 : S_\pi \to \mathbb{H}$, then $\phi(0) = 0$, $\phi(+\infty) = \infty$ and $\phi(-\infty) = -1$. Suppose $g_t(z)$ is the chordal SLE($\kappa$) maps from 0 to $\infty$ in $\mathbb{H}$. For $t < \tilde{\tau}$, $h_t(z) := g_t(e^z - 1) : S_\pi \setminus \tilde{K}_t \to \mathbb{H}$ satisfies $h_t(+\infty) = \infty$. Define

$$\phi_t(z) := \log[g_t(e^z - 1) - g_t(-1)].$$

So $\phi_t(z) : S_\pi \setminus \tilde{K}_t \to S_\pi$ satisfies $\phi_t(+\infty) = +\infty$, $\phi_t(-\infty) = -\infty$ and

$$\lim_{z\to+\infty}[\phi_t(z) - z] = 0, \lim_{\to-\infty}[\phi_t(z) - z] = \log g_t'(-1).$$

Define

$$\Phi_t(z) := \phi_t(z) - \frac{1}{2}\log g_t'(-1).$$

Then $\Phi_t(z) : S_\pi \setminus \tilde{K}_t \to S_\pi$ satisfies $\phi_t(+\infty) = +\infty$, $\phi_t(-\infty) = -\infty$ and

$$\lim_{z\to+\infty}[\Phi_t(z) - z] = -\frac{1}{2}\log g_t'(-1) > 0, \lim_{\to-\infty}[\phi_t(z) - z] = \frac{1}{2}\log g_t'(-1).$$

Since $\partial_t g_t(z) = \frac{-2}{g_t(z) - \sqrt{\kappa}B_t}$, we have

$$\partial_t g_t'(-1) = \frac{-2g_t'(-1)}{(g_t(-1) - \sqrt{\kappa}B_t)^2}.$$

Therefore

$$\partial_t \log g_t'(-1) = \frac{-2}{(g_t(-1) - \sqrt{\kappa}B_t)^2}.$$

$$\log g_t'(-1) = \int_0^t \frac{-2}{(g_s(-1) - \sqrt{\kappa}B_s)^2} ds.$$

$$-\frac{1}{2}\log g_t'(-1) = \int_0^t \frac{1}{(g_s(-1) - \sqrt{\kappa}B_s)^2} ds.$$

Since as $S$-hulls, $S(\tilde{K}_t) = \int_0^t \frac{1}{(g_s(-1)-\sqrt{\kappa}B_s)^2} ds$. Let $u(t) := S(\tilde{K}_t)$, use $u$ as the parameter. Define $\beta(t) := \log[\sqrt{\kappa}B_t - g_t(-1)] - \frac{1}{2}\log g_t'(-1)$, then

$$\begin{aligned}
\partial_t \Phi_t(z) &= \partial_t \phi_t(z) - \frac{1}{2}\partial_t \log g_t'(-1) \\
&= \frac{1}{g_t(e^z - 1) - g_t(-1)}\left[\frac{2}{g_t(e^z-1) - \sqrt{\kappa}B_t} - \frac{2}{g_t(-1) - \sqrt{\kappa}B_t}\right] \\
&+ \frac{1}{(g_t(-1) - \sqrt{\kappa}B_t)^2} \\
&= \frac{g_t(e^z - 1) + \sqrt{\kappa}B_t - 2g_t(-1)}{(g_t(e^z-1) - \sqrt{\kappa}B_t)(g_t(-1) - \sqrt{\kappa}B_t)^2}.
\end{aligned}$$

So

$$\begin{aligned}
\partial_u \Phi_t(z) =& \partial_t \Phi_t(z) \partial_u t = \partial_t \Phi_t(z)(g_t(z) - \sqrt{\kappa}B_t)^2 \\
=& \frac{g_t(e^z - 1) + \sqrt{\kappa}B_t - 2g_t(-1)}{(g_t(e^z - 1) - \sqrt{\kappa}B_t)}.
\end{aligned}$$

Since

$$g_t(e^z - 1) - g_t(-1) = e^{\phi_t(z)} = e^{\Phi_t(z) + \frac{1}{2}\log g_t'(-1)} = e^{\Phi_t(z)}\sqrt{g_t'(-1)}.$$

$$\sqrt{\kappa}B_t - g_t(-1) = e^{\beta(t)}\sqrt{g_t'(-1)}.$$

We can get

$$\partial_u \Phi_t(z) = \frac{e^{\Phi_t(z)} + e^{\beta(t)}}{e^{\Phi_t(z)} - e^{\beta(t)}} = \coth \frac{\Phi_{t(u)} - \beta(t(u))}{2}.$$

Let $W_t := \sqrt{\kappa}B_t$, then

$$
\begin{aligned}
d\beta(t) =& \frac{1}{W_t - g_t(-1)}(dW_t - dg_t(z)) \\
&+ \frac{1}{(W_t - g_t(-1)^2} d\langle W_t - g_t(-1), W_t - g_t(-1)\rangle - \frac{1}{2}\partial_t \log g_t'(-1)dt \\
=& \frac{1}{W_t - g_t(-1)}\sqrt{\kappa}dB_t + \frac{2}{(W_t - g_t(-1)^2}dt \\
&- \frac{\kappa}{2}\frac{1}{(W_t - g_t(-1)^2}dt - \frac{1}{2}\frac{-2}{(W_t - g_t(-1)^2}dt \\
=& \frac{1}{W_t - g_t(-1)}\sqrt{\kappa}dB_t + (3 - \frac{\kappa}{2})\frac{1}{(W_t - g_t(-1)^2}dt.
\end{aligned}
$$

Therefore $d\beta(t(u)) = \sqrt{\kappa}d\tilde{B}_u + (3 - \frac{\kappa}{2})du$, where $\tilde{B}_u$ is a standard one dimensional Brownian motion. So if $\kappa = 6$, $\{K_t : 0 \le t < \tau\}$ and $\{\tilde{K}_t : 0 \le t < \tilde{\tau}\}$ have the same distribution up to a time-change. If $\kappa \ne 6$, by Girsanov's theorem(see [18]) we can get the result. □

*Remark* 62. By the absolutely continuous property in above lemma, we know that dipolar SLE($\kappa$) is also generated by a continuous curve(we call the SLE($\kappa$) **trace**) almost surely and has the same phase transition depending on $\kappa$.

In fact for dipolar SLE($\kappa$) process, we have the following property.

**Proposition 63** (see [24]). *Let $\gamma$ be the diolar* SLE($\kappa$) *trace from* 0 *to* $\mathbb{R}_\pi$ *in* $\mathcal{S}$. *Then*

*(1) If $0 \le \kappa \le 4$, $\gamma$is a simple curve and $\gamma[0,\infty) \subset S_\pi \cup \{0\}$ a.s ;*

*(2) If $4 < \kappa < 8$, almost surely $\gamma$ is a non-selfcroosing curve and for any given $z \in \bar{S_\pi} \setminus \{0\}$ but $z \notin \gamma[0,\infty)$ ;*

*(3) If $\kappa \ge 8$, almost surely $\gamma[0,\infty)$ has Hausdorff dimension 2, i.e. $\gamma$ fills the area it encloses ;*

*(4) Almost surely the limit $m := \lim_{t\to\infty} \gamma(t)$ exists ;*

*(5) Almost surely, $m \in \mathbb{R}_\pi$. If denote by $m = X + i\pi$, then the random variable $X$ has the density function $\rho(x) = (\cosh(\frac{x}{2}))^{-\frac{4}{\kappa}}/c_\kappa$, where $c_\kappa = \int_{-\infty}^{+\infty}(\cosh(\frac{x}{2}))^{-\frac{4}{\kappa}}dx$.*

*Remark* 64. By above proposition if $(K_t)_{t\ge 0}$ is the $\mathbb{H}$-hulls corresponding to dipolar SLE($\kappa$), denote by

$$K_\infty = \underset{t\ge 0}{\cup} K_t,$$

Then $K_\infty$ is bounded. In fact the author in [24] used the boundness of $K_\infty$ to prove above proposition.

Just like the chordal SLE, dipolar SLE(6) satisfies local property and dipolar SLE($\frac{8}{3}$) satisfies restriction property. In [8], the author gave a proof using the conformal field theory to find a martingale to prove the restriction property. Here we use the method in [9] to construct the martingale directly.

It is necessary to give the detailed proof of local property and restriction property here since we can not find any reference about this although it is direct. The main point is to determine how the capacity of a hull changes under a conformal map.

Let $A \in \mathcal{A}_s$ be a $\mathcal{S}$-hull such that $0 \notin A$, and $\Phi_A$ is the unique conformal map $\mathcal{S} \setminus A \to \mathcal{S}$ that fixes $0, +\infty$ and $-\infty$. Suppose that $(K_t : t \geq 0)$ is the dipolar Loewner chain driven by a continuous function $(W_t : t \geq 0)$. Write $\tau_A := \inf\{t \geq 0 : K_t \cap A \neq \emptyset\}$. Then for any $t < \tau_A$, $\Phi_A(K_t)$ is a $\mathcal{S}$-hull. Denote by $\tilde{g}_t$ the normalised conformal map corresponding to $\Phi_A(K_t)$ and $S(t)$ the capacity of $\Phi_A(K_t)$, then $\tilde{g}_t$ satisfies the rescaled dipolar Loewner equation :

$$\partial_t \tilde{g}_t(z) = \partial_t S(t) \coth \frac{\tilde{g}_t(z) - \tilde{W}_t}{2} \quad \tilde{g}_0(z) = z.$$

where $\tilde{W}_t = h_t(W_t)$ is a continuous function. Denote by $h_t(z) := \tilde{g}_t \circ \Phi_A \circ g_t^{-1}(z)$, and $A_t := g_t(A)$, by Schwarz reflection theorem $h_t(z)$ can be analytically extended to a neighbourhood of $W_t$. We have the following :

**Lemma 65.** *Take the notations above, for $0 \leq t < T_A$,*

$$\begin{aligned}
&S(A_t) = S(t) + S(A) - t, \\
&\partial_t S(t) = h_t'(W_t)^2, \\
&\partial_t h_t(W_t) = -3h_t''(W_t), \\
&\partial_t h_t'(W_t) = \frac{1}{2}\frac{(h_t''(W_t))^2}{h_t'(W_t)} - \frac{4}{3}h_t'''(W_t) + \frac{(h_t'(W_t))^3 - h_t'(W_t)}{6}.
\end{aligned}$$

*Proof.* By the definition of $h_t$,

$$\begin{aligned}
\lim_{z\to\pm\infty}(h_t(z) - z) &= \lim_{z\to\pm\infty}(h_t(z) - \Phi_A \circ g_t^{-1}(z)) \\
&+ \lim_{z\to\pm\infty}(\Phi_A \circ g_t^{-1}(z) - g_t^{-1}(z)) + \lim_{z\to\pm\infty}(g_t^{-1}(z) - z) \\
&= \pm S(t) \pm S(A) - g_A(0) \mp t.
\end{aligned}$$

So $h_t(z) + g_A(0)$ is the normalised conformal map corresponding to $A_t$ and therefore $S(A_t) = S(t) + S(A) - t$.

For the second equation take the derivative of $h_t(z)$ with respect to $t$, we get

$$\partial_t h_t(z) = [\partial_t S(t)] \coth \frac{h_t(z) - h_t(W_t)}{2} - h_t'(z) \coth \frac{z - W_t}{2}. \tag{3.4.1}$$

By multiplying $\frac{z-W_t}{2}$ on both side of above equation and let $z \to W_t$, we can get

$$0 = \partial_t S(t) \frac{1}{h_t'(W_t)} - h_t'(W_t).$$

So $\partial_t S(t) = h_t'(W_t)^2$.

The third equation is obtained by passing $z \to W_t$ in the equation (3.4.1).

Take the derivative of $z$ on both sides of (3.4.1) we have

$$\partial_t h_t'(z) = -\frac{1}{2} \frac{h_t'(z)(h_t'(W_t))^2}{\sinh \frac{(h_t(z)-h_t(W_t))^2}{2}} - h_t''(z) \coth \frac{z-W_t}{2} + \frac{1}{2} h_t'(z) \frac{1}{(\sinh \frac{z-W_t}{2})^2}.$$

Then the fourth equation is obtained by passing $z \to W_t$ on the both side of above equation. □

By above lemma, we can get the local property of dipolar SLE(6) and restriction property of dipolar SLE($\frac{8}{3}$).

**Proposition 66.** *Suppose that $\gamma$ is the dipolar* SLE($\kappa$) *trace from* 0 *to* $\mathbb{R}_\pi$ *in* $\mathcal{S}$.

*(a) If $\kappa = 6$, for any S-hull $A$ with $0 \notin A$, define*

$$\tau_A := \inf\{t \geq 0 : \gamma(t) \in A\},$$

*then $\gamma[0, \tau_A)$ has the same distribution as the dipolar* SLE(6) *in $\mathcal{S} \setminus A$ from* 0 *to $\mathbb{R}_\pi$ upon hitting $K$. This is called the* ***local property*** *of dipolar* SLE(6).

*(b) If $\kappa = \frac{8}{3}$, for any $\mathcal{S}$-hull $A$ with $0 \notin A$. Define $\Phi_A(z) = g_A(z) - g_A(0)$, where $g_A(z)$ is the normalised conformal map corresponding to $A$. Then*

$$\mathbf{P}\big[K \cap A = \emptyset\big] = |\Phi_A'(0)|^{\frac{5}{8}} \exp\{-\frac{5}{48} S(A)\}.$$

*where $A_t = g_t(A)$ and $S(A)$ and $S(A_t)$ are the capacities of these two hulls. Notice that for two $\mathcal{S}$-hulls $K_1, K_2$ that don't contain* 0,

$$\Phi_{K_1 \cup \Phi_{K_1}^{-1}(K_2)} = \Phi_{K_2} \circ \Phi_{K_1}, S(K_1 \cup \Phi_{K_1}^{-1}(K_2) = S(K_1) + S(K_2),$$

$$\Phi'_{K_1 \cup \Phi_{K_1}^{-1}(K_2)}(0) = \Phi'_{K_2}(0) \Phi'_{K_1}(0).$$

*Then conditioned on $\gamma \cap A = \emptyset$, $\gamma$ has the same distribution as the dipolar* SLE($\frac{8}{3}$) *from* 0 *to $\mathbb{R}\pi$ in $\mathcal{S} \setminus K$. This is called the* ***restriction property*** *of dipolar* SLE($\frac{8}{3}$).

*Proof.* Suppose that $W_t = \sqrt{\kappa} B_t$ is the driven process of dipolar SLE($\kappa$). Using the same notation as Lemma 65. For $t < \tau_A$, we see that the driven process of the curve $\Phi_A(\gamma)$ is $\tilde{W}_t = h_t(W_t)$. By Itô's formula and combing Lemma 65, we have

$$\begin{aligned} d\tilde{W}_t &= \partial_t h_t(W_t) dt + h_t'(W_t) dW_t + \frac{\kappa}{2} h_t''(W_t) dt \\ &= (\frac{\kappa}{2} - 3) h_t''(W_t) dt + h_t'(W_t) \sqrt{\kappa} dB_t. \end{aligned}$$

So if $\kappa = 6$, by a time change, $\tilde{W}_t$ is scaled a Brownian motion and this proves the locality of dipolar SLE(6).

For $\kappa = \frac{8}{3}$, define

$$Y_t = 1_{\{t<\tau_A\}} h_t'(W_t)^{\frac{5}{8}} \exp\{-\frac{5}{48} S(A_t)\}.$$

Denote by $N_t = \log Y_t = \frac{5}{8} \log h_t'(W_t) - \frac{5}{48} S(A_t)$. By lemma 65, we have

$$dh_t'(W_t) = \partial_t h_t'(W_t)dt + h_t''(W_t)dW_t + \frac{1}{2} h_t'''(W_t) d\langle W, W \rangle_t$$

$$= [\frac{1}{2}\frac{(h_t''(W_t))^2}{h_t'(W_t)} + \frac{(h_t'(W_t))^3 - h_t'(W_t)}{6}]dt + h_t''(W_t)\sqrt{\frac{8}{3}} dB_t,$$

and

$$dS(A_t) = (h_t'(W_t)^2 - 1)dt.$$

So

$$dN_t = \frac{5}{8} d \log h_t'(W_t) - \frac{5}{48} dS(A_t)$$

$$= \frac{5}{8}[-\frac{5}{6}\frac{(h_t''(W_t))^2}{(h_t'(W_t))^2} + \frac{(h_t'(W_t))^2 - 1}{6}]dt$$

$$- \frac{5}{48}(h_t'(W_t)^2 - 1)dt + \frac{5}{8}\frac{h_t''(W_t)}{h_t'(W_t)}\sqrt{\frac{8}{3}} dB_t$$

$$= - \frac{25}{48}\frac{(h_t''(W_t))^2}{(h_t'(W_t))^2} + \frac{5}{8}\frac{h_t''(W_t)}{h_t'(W_t)}\sqrt{\frac{8}{3}} dB_t$$

And then

$$dY_t = Y_t(dN_t + \frac{1}{2} d < N, N >_t) = \sqrt{\frac{25}{24}} Y_t \frac{h_t''(W_t)}{h_t'(W_t)} dB_t.$$

Therefore $Y_t$ is a local martingale. Since $S(A_t) \geq 0$ and $h_t'(W_t) \leq 1$, we get that $Y_t$ is a martingale. Just using the same method as [9](In fact if we regard $h_t'(W_t)$ as the probability that a Brownian motion starting from $W_t$ in $\mathcal{S}$ avoids $A_t$ before exiting $\mathcal{S}$, then for $\tau_A < \infty$, $h_t'(W_t) \to 0$; for $\tau_A = \infty$, $h_t'(W_t) \to 1$ and $S(A_t) \to 0$.), we can get $\lim_{t \to \tau_A} Y_t = 1_{\{\tau_A = \infty\}}$. So by the optional stopping theorem

$$\mathbf{P}[K \cap A = \emptyset] = \mathbf{E}[M_{\tau_A}] = \mathbf{E}[M_0] = |\Phi_A'(0)|^{\frac{5}{8}} \exp\{-\frac{5}{48} S(A)\}.$$

□

*Remark* 67. By above lemma, we have constructed the conformal restriction measure $\mathbf{P}(\frac{5}{48}, \frac{5}{8})$ in Theorem 49.

## 3.5 Brownian Bridge on the strip

In this section the brownian bridge on the strip $\mathcal{S}$ will be given, this is a probability mesure defined on the space of the curves from 0 to $\mathbb{R}_\pi$. The exact definition is as follows :

Suppose that $\mu_\epsilon$ is the law of a planar Brownian motion $(B_t)$ starting from $i\epsilon$ and conditioned to first exit $\mathcal{S}$ from $\mathbb{R}_\pi$. By the optional stopping time theorem, the probability that a one dimension Brownian motion started from $\epsilon$ hits $\pi$ before 0 is $\epsilon/\pi$, so the law $\frac{\pi}{\epsilon}\mu_\epsilon$ converges to a probability measure $\mu^{exc}_{0,\mathbb{R}_\pi}$ in the Prohorov sense, we call this measure the *Brownian bridge measure or Brownian excursion measure* from 0 to $\mathbb{R}_\pi$ on the strip. In fact this measure is just the normalisation of the Brownian excursion measure defined in chapter 5 of the book [10]. The measure is supported on the curves from 0 to $\mathbb{R}_\pi$ in the strip $\mathcal{S}$. We can show that $\mu^{exc}_{0,\mathbb{R}_\pi}$ satisfies the conformal restriction property.

**Proposition 68.** *Suppose that $K$ has the law $\mu^{exc}_{0,\mathbb{R}_\pi}$, then for any $A \in \mathcal{A}_s$ with $0 \notin A$,*

$$\mathbf{P}[K \cap A = \emptyset] = \Phi'_A(0). \tag{3.5.1}$$

*Moreover, by definition $K$ almost surely intersects $\mathbb{R}_\pi$ at some point $X + i\pi$, the law of $X$ has the density function $\rho(x) = \frac{e^x}{(1+e^x)^2}$.*

*Proof.* Let $P(\epsilon)$ be the probability that a Brownian motion started from $i\epsilon$ hits $\mathbb{R}_\pi$ before $\mathbb{R} \cup A$. By the conformal invariance of planar Brownian motion,

$$P(\epsilon) = \mathbf{P}(B_t \text{ started from } \Phi_A(i\epsilon) \text{ hits } \mathbb{R}_\pi \text{ before } \mathbb{R}) = \frac{\operatorname{Im}\Phi_A(i\epsilon)}{\pi}$$

Then we have

$$\mathbf{P}[K \cap A = \emptyset] = \lim_{\epsilon\to 0}\frac{\pi}{\epsilon}P(\epsilon) = \lim_{\epsilon\to 0}\frac{\pi}{\epsilon}\frac{\operatorname{Im}\Phi_A(i\epsilon)}{\pi} = \Phi'_A(0).$$

Notice the Poisson kernel of $\mathcal{S}$ is $H(z, x+i\pi) = -\frac{1}{\pi}\operatorname{Im}\frac{e^x}{e^x+e^z}$, we have

$$\mathbf{P}[B_t \text{ from } i\epsilon \text{ hits } \mathbb{R}_\pi \text{ at } (-\infty, x+i\pi] | B_t \text{ from } i\epsilon \text{ hits } \mathbb{R}_\pi \text{ before } \mathbb{R}]$$

$$= \frac{\pi}{\epsilon}\int_{-\infty}^{x}\frac{-1}{\pi}\operatorname{Im}\frac{e^t}{e^{i\epsilon}+e^t}dt = \frac{1}{\epsilon}\int_{-\infty}^{x}\frac{e^t\sin\epsilon}{e^{2t}+2e^t\cos\epsilon+1}dt.$$

So the hitting point of the Bridge have the distribution function

$$\mathbf{P}[X \le x] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{x}\frac{e^t\sin\epsilon}{e^{2t}+2e^t\cos\epsilon+1}dt = \int_{-\infty}^{x}\frac{e^t}{e^{2t}+2e^t+1}dt.$$

$\square$

## 3.6 Characterization of the conformal restriction measure

In this section we will show that a dipolar conformal restriction can be characterized by two parameters.

Let $\Omega$ be the collection of compact connected subsets $K$ of $\overline{\mathcal{S}}$ such that $K \cap \mathbb{R} = \{0\}$ and $\mathcal{S} \setminus K$ has two connected components that each of them contains $+\infty$ and $-\infty$ on the boundary respectively. Endow $\Omega$ with the $\sigma$-field generated by the family of sets of the type $\{K \in \Omega : K \cap A = \emptyset\}$ where $A \in \mathcal{A}_s^*$. Just like the chordal and radial case this $\sigma$-algebra is the same as the $\sigma$-algebra induced by the Hausdorff metric on $\Omega$. Notice that $\big\{\{K \in \Omega : K \cap A = \emptyset\} : A \in \mathcal{A}_s^*\big\}$ forms an algebra. As the chordal case[9] and radial case [23], by the unique extension theorem of probability measures on algebras, we have :

**Lemma 69.** *If $\mathbf{P}$ and $\mathbf{P}'$ are two probability measures on $\Omega$ such that $\mathbf{P}[K \cap A = \emptyset] = \mathbf{P}'[K \cap A = \emptyset]$ for all $A \in \mathcal{A}_s^*$, then $\mathbf{P} = \mathbf{P}'$.*

Just like the chordal case [9], by endowing $\mathcal{A}_s^*$ with the Hausdorff metric, the function $A \to \mathbf{P}[K \cap A = \emptyset]$ is continuous and this fact will be used in this section.

To prove the existence of the two values $\alpha$ and $\beta$, it is necessary to introduce the *perfect hull* (this notation was first used in [23]) :

Fix $x \in \mathbb{R} \setminus \{0\}$, suppose $(K_t(x))_{t \geq 0}$ is the hulls generated by the driven function $W_t = x - \coth \frac{x}{2} t$ and $g_t$ is the corresponding conformal maps. These hulls $(K_t(x))_{t \geq 0}$ are called the **perfect hulls from** $x$. We can check that $g_t(0) = -\coth \frac{x}{2} t$.

**Lemma 70.** *With the notations above.*

*(1) Define $h_t(z) = g_t(z) - g_t(0) = g_t(z) + \coth \frac{x}{2} t$, then $h_{t+s} = h_t \circ h_s$.*

*(2) Let $K$ be a dipolar restriction sample, then there exists a constant $\nu(x)$ such that*

$$\mathbf{P}\big[K \cap K_t = \emptyset\big] = \exp\{-\nu(x) t\} \ \forall t \geq 0.$$

*Proof.* For fixed $s \geq 0$, by the uniqueness of the solution of the following ODE :

$$\partial_t f_t(z) = \coth \frac{f_t(z) - x}{2} + \coth \frac{x}{2}, \ \ f_0(z) = h_s(z),$$

we can see that both $h_{t+s}$ and $h_t \circ h_s$ satisfies this ODE, and so we have (1). From (1), we have $h_t(K_{s+t}(x) \setminus K_t(x)) = (K_s(x))$ for any $t, s \geq 0$. Then for any $t, s \geq 0$, by the conformal restriction property, we have that

$$\begin{aligned} & \mathbf{P}[K \cap K_{t+s}(x) = \emptyset | K \cap K_t(x) = \emptyset] \\ = \ & \mathbf{P}[K \cap h_t(K_{s+t}(x) \setminus K_t(x)) = \emptyset] = \mathbf{P}[K \cap K_s(x) = \emptyset]. \end{aligned}$$

Thus for any $t, s \geq 0$, we have

$$\mathbf{P}[K \cap K_{t+s}(x) = \emptyset | K \cap K_t(x) = \emptyset] = \mathbf{P}[K \cap K_t(x) = \emptyset] \times \mathbf{P}[K \cap K_s(x) = \emptyset].$$

Combining the fact that $t \to K_t(x)$ is continuous, there exists a constant $\nu(x) \geq 0$ such that

$$\mathbf{P}[K \cap K_t(x) = \emptyset] = \exp\{-t\nu(x)\}.$$

□

In the chordal case, the analogous quantity $\nu(x)$ should be a constant because of the scaling-invariance, but here, just like the radial case, we have a different situation (with one more freedom because of the lack of a restriction condition). In fact, we will show that $\nu(x)$ is a smooth function.

Now we are ready to prove the first part of Theorem 49 that we state as following :

**Proposition 71.** *For any dipolar restriction sample $K$, there exists two constants $\alpha, \beta \in \mathbb{R}$ such that for any $A \in \mathcal{A}_s^*$,*

$$\mathbf{P}[K \cap A = \emptyset] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\} \tag{3.6.1}$$

*where $S(A)$ is the capacity of $A$.*

By lemma 69, for any $\alpha, \beta \in \mathbb{R}$, there exists at most one law that satisfies (3.6.1). The strategy of the proof of above proposition is the same as the strategy used in [23] : first show that (3.6.1) holds for any perfect hulls $K_t(x)$ and then show that the hulls generated by the family of perfect hulls are dense in $\mathcal{A}_s^*$ in the sense of the Hausdorff metric(here the hull generated by two hulls $A_1, A_2$ is defined as $A_1 \cup g_{A_1}^{-1}(A_2)$). And then use the continuity of the map $A \to \mathbf{P}[K \cap A = \emptyset]$.

We will begin by showing that the function $x \to \nu(x)$ is a smooth function on $\mathbb{R} \setminus \{0\}$. And then use the commutation relations derived in Lemma 9 of [23] . It seems that it is easier to work on the upper half plane because we can write down the exact form of the conformal maps from subdomains of $\mathbb{H}$ onto itself.

Let $\varphi_0(z) = e^z - 1$ be the conformal map from $\mathcal{S}$ to $\mathbb{H}$ that sends $+\infty, 0, -\infty$ to $-1, 0, \infty$ respectively. Suppose that $K$ is a sample of the dipolar conformal restriction measure on the strip, then $\tilde{K} := \varphi_0(K)$ is a sample of what we call the dipolar restriction measure on the upper half plane. For $x \in \mathbb{C}$, let $B(x, r)$ denote the ball centered at $x$ with radius $r$.

For $\epsilon > 0$ small enough and $x \in (-1, 0) \cup (0, \infty)$, the map

$$g_{x,\epsilon}(z) := z + \frac{\epsilon^2}{z - x}$$

is a conformal map from $\mathbb{H} \setminus B(x, \epsilon)$ onto $H$ that fix $\infty$. Define

$$f_{x,\epsilon}(z) = \frac{1}{1 - \frac{\epsilon^2}{x(1+x)}}\left(g_{x,\epsilon}(z) + \frac{\epsilon^2}{x}\right) = \frac{1}{1 - \frac{\epsilon^2}{x(1+x)}}\left(z + \frac{\epsilon^2}{z - x} + \frac{\epsilon^2}{x}\right).$$

This is the unique conformal map from $\mathbb{H} \setminus B(x, \epsilon)$ onto $H$ that fixes $-1, 0, \infty$. Denote

$$p_\epsilon(x) = \mathbf{P}\big[\tilde{K} \cap B(x, \epsilon) \neq \emptyset\big].$$

Just as Lemma 5 in [23], we have

**Lemma 72.** *Let $\tilde{K}$ be a dipolar restriction sample in $\mathbb{H}$. For any $x \in (-1,0)\cup(0,\infty)$, the following limits exists*

$$\lambda(x) := \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} p_\epsilon(x).$$

*And further $\lambda(x) \in (0,\infty)$.*

*Proof.* For fixed $x \in (-1,0)\cup(0,\infty)$, let $(\tilde{K}_t(x) : t \geq 0)$ be the perfect hulls from $x$, here we mean $\tilde{K}_t(x) = \varphi_0(K_t(\varphi_0^{-1}(x)))$, where $K_t(\varphi_0^{-1}(x))$ is the perfect hulls in $\mathcal{S}$ from $\varphi_0^{-1}(x) = \log(x+1)$ as defined at the beginning Lemma 70. For small enough $\epsilon > 0$, define $N(\epsilon) = [\epsilon^{-2}]$ which is the integer part of $\epsilon^{-2}$. And $\phi_1 = \phi_2 = ... = \phi_N = f_{x,\epsilon}$. Let $\Phi_\epsilon = \phi_{N(\epsilon)} \circ \phi_{N(\epsilon)-1} \circ ... \circ \phi_1$ be the conformal map from $H := \phi_1^{-1} \circ ... \circ \phi_{N(\epsilon)}^{-1}(\mathbb{H})$ onto $\mathbb{H}$ that fixes $-1, 0, \infty$. Define $A_\epsilon(x) = \overline{\mathbb{H} \setminus H}$. Then the same as, we have $A_\epsilon(x) \to \tilde{K}_{t_x}(x)$ in the Hausdorff sense as $\epsilon \to 0$. Moreover we have that $K_{t_x}(x) \subset A_\epsilon(x)$. Here by compute the capacity we can see that $t_x = \frac{1}{2(1+x)^2}$. In fact

$$S(\varphi^{-1}(B(x,\epsilon)\cap\mathbb{H})) = \frac{1}{2}\log f'_{x,\epsilon}(\infty) - \frac{1}{2}\log f'_{x,\epsilon}(-1) = -\frac{1}{2}\log[1 - \frac{\epsilon^2}{(1+x^2)}],$$

and so

$$t_x = \lim_{\epsilon\to 0} S(A_\epsilon(x)) = \lim_{\epsilon\to 0} -\frac{N(\epsilon)}{2}\log[1 - \frac{\epsilon^2}{(1+x^2)}] = \frac{1}{2(1+x)^2}.$$

Now by the conformal restriction property we have

$$\mathbf{P}[K \cap A_\epsilon(x) = \emptyset] = (1 - p_\epsilon(x))^{N(\epsilon)}.$$

On the other hand, by Lemma 70, we have

$$\begin{aligned} & \mathbf{P}[K \cap A_\epsilon(x) = \emptyset] \to \mathbf{P}[\tilde{K} \cap \tilde{K}_{t_x}(x) = \emptyset] \\ = \; & \mathbf{P}[K \cap K_{t_x}(\varphi_0^{-1}(x)) = \emptyset] = \exp\{-\nu(\log(x+1))t_x\}\text{, as } \epsilon \to 0. \end{aligned}$$

Therefore

$$\lim_{\epsilon\to 0} N(\epsilon)\log(1 - p_\epsilon(x)) = -\frac{1}{2(1+x)^2}\nu(\log(x+1)).$$

This completes the proof. We further get that

$$\lambda(x) = \frac{1}{2(1+x)^2}\nu(\log(x+1)). \tag{3.6.2}$$

□

In fact the function $\lambda(x)$ in the above lemma has a very nice form. In order to determine the exact form of $\lambda(x)$, we need to find some functional relations satisfied by $\lambda(x)$. We will give two functional relations in the following two lemmas : one is the symmetric relation, and the other is the commutation relation.

**Lemma 73** (*symmetric relation*)**.** *For any $x > 0$, the function $\lambda$ satisfies*

$$\lambda(x) = \frac{1}{(1+x)^4}\lambda(-\frac{x}{1+x}). \tag{3.6.3}$$

*Proof.* Write $E(x,\epsilon) := \varphi_0^{-1}(B(x,\epsilon))$. Denote by $I(z) = -\overline{z}$ the reflection of the imaginary axis. Since the law of the conformal restriction measure is invariant under the reflection of the imaginary axis, we have that

$$\mathbf{P}[K \cap E(x,\epsilon) \neq \emptyset] = \mathbf{P}[K \cap I(E(x,\epsilon)) \neq \emptyset]$$

Notice that $\psi := \varphi_0 \circ I \circ \varphi^{-1}(z) = -\frac{\overline{z}}{1+\overline{z}}$, we have

$$\mathbf{P}[\tilde{K} \cap B(x,\epsilon) \neq \emptyset] = \mathbf{P}[\tilde{K} \cap \psi(B(x,\epsilon)) \neq \emptyset].$$

Therefore

$$\begin{aligned}
\lambda(x) = &\lim_{\epsilon\to 0} \frac{1}{\epsilon^2}\mathbf{P}\left[\tilde{K} \cap B(x,\epsilon) \neq \emptyset\right] = \lim_{\epsilon\to 0} \mathbf{P}[\tilde{K} \cap \psi(B(x,\epsilon)) \neq \emptyset] \\
= &\lim_{\epsilon\to 0} |\psi'(x)|^2 \frac{1}{\epsilon^2|\psi'(x)|^2}\mathbf{P}\left[\tilde{K} \cap B(\psi(x), |\psi'(x)|\epsilon) \neq \emptyset\right] \\
= &\frac{1}{(1+x)^4}\lambda(-\frac{x}{1+x}).
\end{aligned}$$

□

Fix $x, y \in (-1,0) \cup (0,\infty)$, define

$$F(x,y) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2}(f_{x,\epsilon}(y) - y), \quad G(x,y) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2}(f'_{x,\epsilon}(y) - 1).$$

By direct computation, we have that

$$F(x,y) = \frac{1}{y-x} + \frac{1}{x} + \frac{y}{x(1+x)}, \quad G(x,y) = \frac{1}{x(1+x)} - \frac{1}{(y-x)^2}.$$

By exactly the same argument as the Lemma 6, Lemma 7, Lemma 8 in [23], we can get the second functional relation satisfied by $\lambda(x)$ which is called the commutation relations. Since the argument is totally the same as, we omit the proof and just give the statement.

**Lemma 74.** *The function $\lambda$ is differentiable in $x \in (-1,0) \cup (0,\infty)$ and satisfies the following commutation relation : for any $x, y \in (-1,0)$ or $x, y \in (0,\infty)$,*

$$\lambda'(y)F(x,y) + 2\lambda(y)G(x,y) = \lambda'(x)F(y,x) + 2\lambda(x)G(y,x). \tag{3.6.4}$$

Combining Lemma 73 and Lemma 74, we can get the exact form of $\lambda(x)$ :

**Lemma 75.** *There exists two constants $c_1$ and $c_2$ such that*

$$\lambda(x) = \frac{c_1(1+x) + c_2 x^2}{x^2(1+x)^2}. \tag{3.6.5}$$

*Proof.* Fix $x$, expand the two sides of (3.6.4) and then let $y \to x$, we can get

$$x^2(1+x)^2\lambda'''(x) + 6x(1+x)(2x+1)\lambda''(x) \\ + 6(1+6x+6x^2)\lambda'(x) + 12(1+2x)\lambda(x) = 0.$$

The solution of above differential equation have the following form :

$$\lambda(x) = \frac{c_1 + c_3 x + c_2 x^2}{x^2(1+x)^2}$$

Since $\lambda(x)$ satisfies (3.6.3), we can get $c_1 = c_3$. $\square$

**Proof of proposition 71** Recall that for the perfect hulls $K_t(x)$ from $x$, the corresponding function $h_t(z)$ satisfies

$$\partial_t h_t(z) = \coth\frac{h_t(z) - x}{2}, \quad h_0(z) = z.$$

From $h_t(0) = 0$ and take the derivative of above differential equation at $z = 0$, we can get

$$h_t'(0) = \exp\{-\frac{1}{4}\frac{1}{(\sinh\frac{x}{2})^2}t\}.$$

Combining $\mathbf{P}\big[K \cap K_t(x) = \emptyset\big] = \exp\{-\nu(x)t\}$ and $S(K_t(x)) = t$. By (3.6.2), we have

$$\nu(x) = 2c_2 + 2c_1\frac{1}{4(\sinh\frac{x}{2})^2}.$$

So if we set $\alpha = 2c_2$ and $\beta = 2c_1$, we can see that

$$\mathbf{P}\big[K \cap K_t(x) = \emptyset\big] = \exp\{-\nu(x)t\} = |h_t'(0)|^\beta \exp\{-\alpha S(K_t(x)))\}.$$

Now we have proved that proposition 71 holds for all the hulls generated by perfect hulls. Then use the same method as Proposition 3.3 in [9] ,the hulls generated by perfect hulls are dense in $\mathcal{A}_s^*$, combing the continuity of $A \to \mathbf{P}[K \cap A = \emptyset]$, we proved proposition 71 for all the hulls in $\mathcal{A}_s^*$.

## 3.7 Construction by Poisson cloud

In this section we will use Brownian bubble measure and SLE($\frac{8}{3}$) to construct $\mathbf{P}(\frac{\beta(1-\beta)}{2\beta+1}, \beta)$. For a conformal map $f$, define the modified Schwarzian derivative as follows :

$$\tilde{S}f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}(\frac{f''(z)}{f'(z)})^2 + \frac{1-f'(z)^2}{2}. \tag{3.7.1}$$

Define

$$\beta = \beta(\kappa) = \frac{6-\kappa}{2\kappa},\ \alpha = \alpha(\kappa) = \frac{(\kappa-2)(6-\kappa)}{8\kappa}.$$

$$\lambda = \lambda(\kappa) = \frac{(6-\kappa)(8-3\kappa)}{2\kappa}.$$

**Lemma 76.** *Suppose that $W_t = \sqrt{\kappa}B_t$ is the driven process of the dipolar* SLE($\kappa$)*, and $K$ is the hull generated by the dipolar* SLE($\kappa$)*. Take the same notations as lemma 65, and $\alpha(\kappa), \beta(\kappa), \lambda(\kappa)$ are as above. Then*

$$Y_t = 1_{\{t<T_A\}} h_t'(W_t)^\beta \exp\{-\alpha S(A_t)\} \exp\{\lambda \int_0^t \tilde{S}h_s(W_s)ds\},$$

*is a local martingale. In particular, when $\kappa \leq \frac{8}{3}$, $Y_t$ is a martingale and*

$$\mathbf{E}[1_{\{\tau_A=\infty\}} \exp\{\lambda \int_0^\infty \tilde{S}h_s(W_s)ds\}] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\}$$

*Proof.* It is proved by the standard stochastic analysis. □

**Lemma 77.** *Suppose that $\mu_{\mathcal{S}}^{bub}(0)$ is the Brownian bubble measure on the strip at* 0 *and $K$ is a sample of this law, then*

$$\mu_{\mathcal{S}}^{bub}(0)[K \cap A \neq \emptyset] = -\frac{1}{6}\tilde{S}\Phi_A(0),\ \ \forall A \in \mathcal{A}_s^*. \tag{3.7.2}$$

*Proof.* Denote $\tilde{A} = \varphi_0(A)$, and $\tilde{A}$ is a compact hull in $\mathbb{H}$. By the definition of the Brownian bubble measure, we have

$$\mu_{\mathcal{S}}^{\text{bub}}(0)[K \cap A \neq \emptyset] = \frac{1}{|\varphi_0'(0)|^2}\mu_{\mathbb{H}}^{\text{bub}}(0)[K \cap \tilde{A} \neq \emptyset] = -\frac{1}{6}Sg_{\tilde{A}}(0).$$

where $Sg_A$ is the Schwarzian derivative. We can check that the normalised conformal map corresponding to $\tilde{A}$ is

$$g_{\tilde{A}}(z) = e^{-S(A)}[e^{g_A(\log(z+1))} - 1].$$

By direct computation, we have $Sg_{\tilde{A}}(0) = \tilde{S}g(0)$. □

Now we can use the above two lemmas to construct the dipolar conformal restriction measure. Just exactly the same argument as [9], we have the following analysis : Suppose that $\kappa \le 8/3$ and take the notations at the beginning of this sction. Consider a Poisson point process $X$ on $\Omega \times [0,\infty)$ with mean (intensity) $\lambda\,\mu \times dt$, where $dt$ is Lebesgue measure. As before, let $\gamma$ denote the SLE$_\kappa$ path, $g_t$ the corresponding conformal maps, and $W_t$ the Loewner driving process. We take $\gamma$ to be independent from $X$.

Let

$$\hat{X} := \big\{ g_t^{-1}(K + W_t) : (K,t) \in X,\, t \in [0,\infty) \big\},$$

and let $\Xi$ be the filling of the union of elements of $\hat{X}$ and $\gamma$, i,e, $\Xi$ is the closure of the domains between $\hat{X}$ and $\gamma$.

Let $A \in \mathcal{A}_s^*$, and let $h_t$ be the normalized conformal map from $\mathcal{S} \setminus g_t(A)$ onto $\mathcal{S}$. By Lemma 77, for any $t > 0$ on the event $\gamma[0,t] \cap A = \emptyset$,

$$\begin{aligned} & \mathbf{P}\big[\{K : g_t^{-1}(K + W_t) \cap A \neq \emptyset\} \bigm| g_t\big] \\ = \ & \mathbf{P}\big[\{K : (K + W_t) \cap g_t(A) \neq \emptyset\} \bigm| g_t\big] \\ = \ & -\tilde{S}h_t(W_t)/6, \end{aligned}$$

where $K$ is independent from $\gamma$ and has law $\mu_{\mathcal{S}}^{\mathrm{bub}}(0)$. Consequently, on the event $\gamma[0,\infty) \cap A = \emptyset$,

$$\mathbf{P}\big[\Xi \cap A = \emptyset \bigm| \gamma\big] = \exp\Big(\lambda \int_0^\infty \frac{\tilde{S}h_t(W_t)}{6}\, dt\Big).$$

By taking expectation and applying Theorem 76 , we get

$$\mathbf{P}\big[\Xi \cap A = \emptyset\big] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\}\,, \tag{3.7.3}$$

Using the same method as [9], we have

**Proposition 78.** *For any $\kappa \in [0, 8/3]$, the law of $\Xi(\kappa)$ is $\mathbf{P}(\alpha(\kappa), \beta(\kappa))$.*

Notice that $\beta(\kappa) \ge \frac{5}{8}$ and $\alpha(\kappa) = \alpha(\beta) = \frac{\beta(1-\beta)}{2\beta+1}$. So we have constructed $\mathbf{P}(\beta, \frac{\beta(1-\beta)}{2\beta+1})$ for $\beta \ge \frac{5}{8}$.

## 3.8 Construction from One-sided restriction measure

In proposition 71, we showed that a dipolar conformal restriction is uniquely characterized by two parameters $(\alpha, \beta)$ such that for any $A \in \mathcal{A}_s^*$, $\mathbf{P}[K \cap A = \emptyset] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\}$. We denote this measure by $\mathbf{P}(\alpha, \beta)$. By Proposition 66 and Proposition 3.5.1, we know that $\mathbf{P}(\frac{5}{48}, \frac{5}{8})$ and $\mathbf{P}(0,1)$ exists. In this section, we will use dipolar SLE$(\kappa;\rho)$ process to construct $\mathbf{P}(\alpha, \beta)$ for other values of $(\alpha, \beta)$.

Dipolar SLE$(\kappa;\rho)$ processes are variants of SLE$(\kappa)$ processes. For what we will use in this chapter, we will here only describe the dipolar SLE$(\kappa;\rho)$ with one force point. It is defined as the solution of the dipolar Lowner equation (3.3.1) where the driven function is replaced by the solution to the following SDE systems :

$$\begin{cases} dW_t = \frac{\rho}{2}\coth\frac{W_t-V_t}{2}dt + \sqrt{\kappa}dB_t \\ dV_t = \coth\frac{V_t-W_t}{2}dt \\ W_0 = 0,\ V_0 = x \in \mathbb{R}\setminus\{0\},\ \frac{W_t-V_t}{W_0-U_0} > 0. \end{cases} \tag{3.8.1}$$

By the same method as the chordal SLE$(\kappa;\rho)$ processes, it can be proven that when $\kappa > 0$ and $\rho > -2$, there is a pathwise unique solution to the above SDE system. And the SLE$(\kappa;\rho)$ is almost surely generated by a continuous curve from 0 to $\mathbb{R}_\pi$.

If we take the limit $x \to 0^+$(repectively $0^-$), the process has a limit which is called the dipolar SLE$(\rho)$ process with force point $0^+$(respectively $0^-$).

Fix $\rho > 0$. Let $g_t(z)$ be the dipolar Loewner chain SLE$(\frac{8}{3};\rho)$ with force point $0^+$ generated by the driven function $(W_t : t \geq 0)$, where $(W_t, V_t)$ is the unique solution of the system in (3.8.1). Recall that dipolar SLE$(\frac{8}{3};\rho)$ is generated by a curve $\gamma$. For any $A \in \mathcal{A}_s^*$, let $\tau_A$ be the first time that $\gamma$ hits $A$. For ant $t < \tau_A$, let $\tilde{g}_t$ be the normalised conformal map of the dipolar hull $\Phi_A(\gamma(t))$. Denote by $h_t(z) = \tilde{g}_t \circ \Phi_A \circ g_t^{-1}(z)$ the conformal map from $\mathcal{S} \setminus g_t(A)$ onto $\mathcal{S}$ that fixes $\pm\infty$. Then we have the following lemma.

**Lemma 79.**

$$M_t = |h_t'(W_t)|^{\frac{5}{8}}|h_t'(V_t)|^{\lambda}\exp\{-\alpha S(A_t)\}Z_t^{\frac{3}{8}\rho}$$

*is a local martingale where*

$$Z_t = \sinh\tfrac{Y_t}{2}/\sinh\tfrac{X_t}{2},$$

$$Y_t = h_t(V_t) - h_t(W_t), \quad X_t = V_t - W_t,$$

$$\alpha = \tfrac{5}{48} + \tfrac{3}{64}\rho(\rho+4), \quad \lambda = \tfrac{1}{32}\rho(3\rho+4),$$

*and $A_t = g_t(A)$ and $S(A_t)$ is the capacity of $A_t$.*

*Proof.* Using Itô formula, combined with lemma 65 we have

$$\begin{aligned} d\log\sinh\frac{Y_t}{2} = {}& \Big[\frac{1}{6}(\coth\frac{Y_t}{2})^2(h_t'(W_t))^2 + \frac{5}{6}\coth\frac{Y_t}{2}h_t''(W_t) \\ & - \frac{\rho}{4}h_t'(W_t)\coth\frac{Y_t}{2}\coth\frac{X_t}{2} + \frac{1}{3}(h_t'(W_t))^2\Big]dt - \frac{1}{2}\coth\frac{Y_t}{2}h_t'(W_t)\sqrt{\frac{8}{3}}dB_t, \end{aligned}$$

$$d\log\sinh\frac{X_t}{2} = \frac{3\rho+2}{12}(\coth\frac{X_t}{2})^2 dt + \frac{1}{3}dt - \frac{1}{2}\coth\frac{X_t}{2}\sqrt{\frac{8}{3}}dB_t,$$

$$d\log h_t'(W_t) = \big(-\frac{5}{6}\frac{(h_t''(W_t))^2}{(h_t'(W_t))^2} + \frac{(h_t'(W_t))^2-1}{6} + \frac{h_t''(W_t)}{h_t'(W_t)}\frac{\rho}{2}\coth\frac{W_t-V_t}{2}\big)dt + \sqrt{\frac{8}{3}}\frac{h_t''(W_t)}{h_t'(W_t)}dB_t,$$

$$d\log h_t'(V_t) = \frac{dh_t'(V_t)}{h_t'(V_t)} = \frac{1}{2}[(\coth\frac{X_t}{2})^2 - 1]dt - \frac{1}{2}[h_t'(W_t)]^2[(\coth\frac{Y_t}{2})^2 - 1]dt,$$

$$d\log S(A_t) = ([h_t'(W_t)]^2 - 1)dt.$$

Let $N_t = \log M_t$, then we have $dM_t = M_t(dN_t + \frac{1}{2}d\langle N, N\rangle_t)$ and we can use the above results to get that the drift term of $dM_t$ is equal to zero and so $M_t$ is a local martingale. □

*Remark* 80. In order to make the computation process more precise, we give the detailed procedure of using Itô's formula to prove that $M_t$ is a local martingale.

$$\begin{aligned} dh_t(V_t) &= \partial_t h_t(V_t)dt + h_t'(V_t)dV_t \\ &= (h_t'(W_t))^2\coth\frac{h_t(V_t)-h_t(W_t)}{2}dt - h_t'(V_t)\coth\frac{V_t-W_t}{2}dt \\ &\quad + h_t'(V_t)\coth\frac{V_t-W_t}{2}dt = (h_t'(W_t))^2\coth\frac{h_t(V_t)-h_t(W_t)}{2}dt. \end{aligned}$$

$$\begin{aligned} dh_t(W_t) &= \partial_t h_t(W_t)dt + h_t'(W_t)[\frac{\rho}{2}\coth\frac{V_t-W_t}{2} + \sqrt{\frac{8}{3}}dB_t] + \frac{h_t''(W_t)}{2}\frac{8}{3}dt \\ &= -3h_t''(W_t)dt + h_t'(W_t)[\frac{\rho}{2}\coth\frac{V_t-W_t}{2} + \sqrt{\frac{8}{3}}dB_t] + \frac{h_t''(W_t)}{2}\frac{8}{3}dt \\ &= [-\frac{5}{3}h_t''(W_t) + \frac{\rho}{2}h_t'(W_t)\coth\frac{V_t-W_t}{2}]dt + \sqrt{\frac{8}{3}}h_t'(W_t)dB_t \end{aligned}$$

Denote by $Y_t = h_t(V_t) - h_t(W_t)$ and $X_t = V_t - W_t$, we have

$$dY_t = [(h_t'(W_t))^2\coth\frac{Y_t}{2} + \frac{5}{3}h_t''(W_t) - \frac{\rho}{2}h_t'(W_t)\coth\frac{X_t}{2}]dt - h_t'(W_t)\sqrt{\frac{8}{3}}dB_t.$$

$$
\begin{aligned}
d\sinh\frac{Y_t}{2} &= \frac{1}{2}\cosh\frac{Y_t}{2}dY_t + \frac{1}{8}\sinh\frac{Y_t}{2}d<Y,Y>_t \\
=&\Big[\frac{1}{2}\frac{(\cosh\frac{Y_t}{2})^2}{\sinh\frac{Y_t}{2}}(h_t'(W_t))^2 + \frac{5}{6}h_t''(W_t)\cosh\frac{Y_t}{2} - \frac{\rho}{4}h_t'(W_t)\cosh\frac{Y_t}{2}\coth\frac{X_t}{2} \\
&+ \frac{1}{3}\sinh\frac{Y_t}{2}(h_t'(W_t))^2\Big]dt - \frac{1}{2}\cosh\frac{Y_t}{2}h_t'(W_t)\sqrt{\frac{8}{3}}dB_t.
\end{aligned}
$$

$$
\begin{aligned}
d\log\sinh\frac{Y_t}{2} &= \Big[\frac{1}{2}(\coth\frac{Y_t}{2})^2(h_t'(W_t))^2 + \frac{5}{6}\coth\frac{Y_t}{2}h_t''(W_t) - \frac{\rho}{4}h_t'(W_t)\coth\frac{Y_t}{2}\coth\frac{X_t}{2} \\
&+ \frac{1}{3}(h_t'(W_t))^2 - \frac{1}{2}\frac{1}{(\sinh\frac{Y_t}{2})^2}\frac{1}{4}\frac{8}{3}(\cosh\frac{Y_t}{2})^2(h_t'(W_t))^2\Big]dt - \frac{1}{2}\coth\frac{Y_t}{2}h_t'(W_t)\sqrt{\frac{8}{3}}dB_t \\
=&\Big[\frac{1}{6}(\coth\frac{Y_t}{2})^2(h_t'(W_t))^2 + \frac{5}{6}\coth\frac{Y_t}{2}h_t''(W_t) - \frac{\rho}{4}h_t'(W_t)\coth\frac{Y_t}{2}\coth\frac{X_t}{2} \\
&+ \frac{1}{3}(h_t'(W_t))^2\Big]dt - \frac{1}{2}\coth\frac{Y_t}{2}h_t'(W_t)\sqrt{\frac{8}{3}}dB_t.
\end{aligned}
$$

Since

$$
dV_t = \coth\frac{V_t - W_t}{2}dt \ \ , dW_t = \frac{\rho}{2}\coth\frac{W_t - V_t}{2}dt + \sqrt{\frac{8}{3}}dB_t,
$$

We have

$$
dX_t = dV_t - dW_t = (1+\frac{\rho}{2})\coth\frac{X_t}{2}dt - \sqrt{\frac{8}{3}}dB_t.
$$

$$
\begin{aligned}
d\sinh\frac{X_t}{2} &= \frac{1}{2}\cosh\frac{X_t}{2}dX_t + \frac{1}{8}\sinh\frac{X_t}{2}d<X,X>_t \\
&= \frac{1}{2}\cosh\frac{X_t}{2}\big((1+\frac{\rho}{2})\coth\frac{X_t}{2}dt - \sqrt{\frac{8}{3}}dB_t\big) + \frac{1}{8}\sinh\frac{X_t}{2}\frac{8}{3}dt \\
&= \frac{1}{2}(1+\frac{\rho}{2})\frac{(\cosh\frac{X_t}{2})^2}{\sinh\frac{X_t}{2}}dt + \frac{1}{3}\sinh\frac{X_t}{2}dt - \frac{1}{2}\cosh\frac{X_t}{2}\sqrt{\frac{8}{3}}dB_t.
\end{aligned}
$$

$$
\begin{aligned}
d\log\sinh\frac{X_t}{2} =&\frac{1}{2}(1+\frac{\rho}{2})(\coth\frac{X_t}{2})^2dt + \frac{1}{3}dt - \frac{\frac{1}{4}(\cosh\frac{X_t}{2})^2}{2(\sinh\frac{X_t}{2})^2}\frac{8}{3}dt - \ \ \frac{1}{2}\coth\frac{X_t}{2}\sqrt{\frac{8}{3}}dB_t \\
=&\frac{3\rho+2}{12}(\coth\frac{X_t}{2})^2dt + \frac{1}{3}dt - \frac{1}{2}\coth\frac{X_t}{2}\sqrt{\frac{8}{3}}dB_t.
\end{aligned}
$$

$$dh_t'(W_t) = \partial_t h_t'(W_t)dt + h_t''(W_t)dW_t + \frac{1}{2}h_t'''(W_t)d < W, W >_t$$

$$= \Big(\frac{1}{2}\frac{(h_t''(W_t))^2}{h_t'(W_t)} - \frac{4}{3}h_t'''(W_t) + \frac{(h_t'(W_t))^3 - h_t'(W_t)}{6}\Big)dt$$

$$+ h_t''(W_t)[\frac{\rho}{2}\coth\frac{W_t - V_t}{2}dt + \sqrt{\frac{8}{3}}dB_t] + \frac{1}{2}h_t'''(W_t)\frac{8}{3}dt$$

$$= \Big(\frac{1}{2}\frac{(h_t''(W_t))^2}{h_t'(W_t)} + \frac{(h_t'(W_t))^3 - h_t'(W_t)}{6} + h_t''(W_t)\frac{\rho}{2}\coth\frac{W_t - V_t}{2}\Big)dt$$

$$+ \sqrt{\frac{8}{3}}h_t''(W_t)dB_t.$$

$$d\log h_t'(W_t) = \frac{1}{h_t'(W_t)}dh_t'(W_t) + \frac{-1}{2(h_t'(W_t))^2}[dh_t'(W_t)]^2$$

$$= \Big(\frac{1}{2}\frac{(h_t''(W_t))^2}{(h_t'(W_t))^2} + \frac{(h_t'(W_t))^2 - 1}{6} + \frac{h_t''(W_t)}{h_t'(W_t)}\frac{\rho}{2}\coth\frac{W_t - V_t}{2}\Big)dt$$

$$+ \sqrt{\frac{8}{3}}\frac{h_t''(W_t)}{h_t'(W_t)}dB_t - \frac{1}{2(h_t'(W_t))^2}\frac{8}{3}[h_t''(W_t)]^2dt$$

$$= \Big(-\frac{5}{6}\frac{(h_t''(W_t))^2}{(h_t'(W_t))^2} + \frac{(h_t'(W_t))^2 - 1}{6} + \frac{h_t''(W_t)}{h_t'(W_t)}\frac{\rho}{2}\coth\frac{W_t - V_t}{2}\Big)dt$$

$$+ \sqrt{\frac{8}{3}}\frac{h_t''(W_t)}{h_t'(W_t)}dB_t$$

$$dh_t'(V_t) = \partial_t h_t'(V_t)dt + h_t''(V_t)dV_t$$

$$= \frac{1}{2}\frac{h_t'(V_t)}{(\sinh\frac{X_t}{2})^2}dt - \frac{1}{2}\frac{h_t'(V_t)[h_t'(W_t)]^2}{[\sinh\frac{Y_t}{2}]^2}dt$$

$$= \frac{1}{2}h_t'(V_t)[(\coth\frac{X_t}{2})^2 - 1]dt - \frac{1}{2}h_t'(V_t)[h_t'(W_t)]^2[(\coth\frac{Y_t}{2})^2 - 1]dt$$

$$d\log h_t'(V_t) = \frac{dh_t'(V_t)}{h_t'(V_t)} = \frac{1}{2}[(\coth\frac{X_t}{2})^2 - 1]dt - \frac{1}{2}[h_t'(W_t)]^2[(\coth\frac{Y_t}{2})^2 - 1]dt$$

$$d\log S(A_t) = d[a(t) + S(A) - t] = ([h_t'(W_t)]^2 - 1)dt.$$

Using above lemma we can construct a "one-sided" restriction measure, we state it as a lemma as follows :

**Lemma 81.** *Suppose that $\gamma$ is the dipolar* $\mathrm{SLE}(\frac{8}{3};\rho)$ *curve with force point* $0^+$ *(respectively $0^-$). Then for any $A \in \mathcal{A}_s^*$ such that $A \cap (0,\infty) = \emptyset$ (respectively $A \cap (-\infty, 0) = \emptyset$),*

$$\mathbf{P}[\gamma \cap A = \emptyset] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\},$$

*where $\alpha = \alpha(\rho)$ is the same as Lemma 79 and*

$$\beta = \beta(\rho) = \frac{5}{8} + \lambda + \frac{3}{8}\rho = \frac{1}{32}(\rho+2)(3\rho+10).$$

*Proof.* Use the same notation as Lemma 79. By the same method as [9], we can see that $0 < h_t'(W_t) \leq 1$ and $h_t'(V_t), Z_t$ are uniformly bounded, hence $M_t$ is a martingale. In fact we can regard $h_t'(W_t)$ as the probability that a Brownian bridge in the strip $\mathcal{S}$ from $W_t$ avoids $g_t(A)$.

So if $\tau_A < \infty$, $h_t'(W_t) \to 0$ as $t \to \tau_A$, and therefore $M_t \to 0$,as $t \to \tau_A$.

If $\tau_A = \infty$, as $t \to \infty$, $h_t'(W_t) \to 1$. Since $V_t \geq W_t$, the brownian Bridge of course will avoid $g_t(A)$ if the Brownian bridge from $W_t$ avoids $g_t(A)$ (here we use the fact that $A \cap (0,\infty) = \emptyset$), hence $h_t'(V_t) \to 1$ as $t \to \infty$. On the other hand, if $\gamma \cap A = \emptyset$, $h_t(z)$ tends to the identity map as $t \to \infty$, hence $S(A_t) \to 0$ as $t \to \infty$. By comparing with the Bessel process, we can see that $Z_t \to 1$ as $t \to \infty$. Thus, almost surely ,

$$\lim_{t\to\tau_A} M_t = 1_{\{\gamma\cap A=\emptyset]\}}.$$

By the optional stopping theorem we have

$$\mathbf{P}[\gamma \cap A = \emptyset] = \mathbf{E}[M_{\tau_A}] = M_0 = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\}.$$

The proof of the case with $0^-$ as a force point is the same. □

*Remark* 82. We call probability measures that satisfies the conditions in Lemma 81 the **one-sided dipolar conformal restriction measure**. In fact we can also get the one-sided restriction measure by adding poisson cloud to $\mathrm{SLE}(\kappa;\rho)$ process for $\kappa \in [0, \frac{8}{3}]$. Using the same notation as lemma 79, we only replace the $\mathrm{SLE}(\frac{8}{3},\rho)$ in Proposition 78 by a $\mathrm{SLE}(\kappa;\rho)$ process. Define

$$a(\kappa,\rho) = \frac{6-\kappa}{2\kappa},\ \ b(\kappa,\rho) = \frac{\rho}{4\kappa}(\rho+4-\kappa), c(\kappa,\rho) = \frac{\rho}{\kappa}.$$

$$\lambda(\kappa) = \frac{(8-3\kappa)(6-\kappa)}{2\kappa},\ \ \alpha(\kappa,\rho) = \frac{(6-\kappa)(\kappa-2)+\rho(\rho+4)}{8\kappa}.$$

**Lemma 83.** *Take notations above,*

$$M_t = 1_{\{t<\tau_A\}}(h_t'(W_t))^a(h_t'(V_t))^b \exp\{-\alpha S(A_t)\}Z_t^c \exp\{\lambda \int_0^t \tilde{S}h_s(W_s)ds\}$$

*is a local martingale.*

The proof of this lemma is just by Ito's calculus. Using the same method as Proposition 78, we can add Brownian bubbles to this SLE($\kappa;\rho$) with force point $0^+$ ($0^-$) to get the one-sided restriction measure with parameters $(a(\kappa,\rho)+b(\kappa,\rho)+c(\kappa,\rho), \alpha(\kappa,\rho))$ for $\kappa \le \frac{8}{3}$.

Now we are ready to construct the dipolar conformal restriction measure from one-sided measure.

**Proposition 84.** *For $\beta > \frac{5}{8}$, let $\rho = \frac{2}{3}(\sqrt{24\beta+1}-1)-2>0$. Let $\gamma^R$ be the dipolar SLE($\frac{8}{3},\rho$) curve with force point $0^-$. Denote by $X$ the terminal point of $\gamma^R$ on $\mathbb{R}_\pi$. Given $\gamma^R$, let $\gamma^L$ be an independent chordal SLE($\frac{8}{3},\rho-2$) curve with force point $0^+$ from $0$ to $X$ in the left connected component of $\mathcal{S}\smallsetminus\gamma^R$. Define $K$ as the closure of union of the domains between $\gamma^R$ and $\gamma^L$. Then the law of $K$ is the dipolar conformal restriction measure $\mathbf{P}(\alpha,\beta)$, where $\alpha$ is the same as Lemma 79, i.e.*

$$\alpha = \alpha(\beta) = \frac{5}{48} + \frac{3}{64}\rho(\rho+4) = \frac{1}{48}((\sqrt{24\beta+1}-1)^2-4).$$

*Proof.* We only need to check that

$$\mathbf{P}[K\cap A=\emptyset] = \Phi_A'(0)^\beta \exp\{-\alpha S(A)\}.$$

Since $\gamma^R$ is the dipolar SLE($\frac{8}{3},\rho$) curve with force point $0^-$, it satisfies the one-sided restriction property by Lemma 81, we know that this is true for $A$ such that $A\cap(-\infty,0)=\emptyset$. Notice that any hull in $\mathcal{A}_s^*$ can be generated by two hulls one of which does not intersect with the positive axis and one of which does not intersect with the negative axis. So we only need to show that this proposition hold for any $A\in\mathcal{A}_s^*$ such that $A\cap(0,\infty)=\emptyset$.

With the same notations as Lemma 79. Since $\rho>0$, $M_t$ is a martingale by the same analysis as Lemma 81. From the proof of lemma 81, we notice that when $\tau_A<\infty$, $M_t\to 0$ as $t\to\tau_A$. When $\tau_A=\infty$, as $t\to\infty$, $h_t'(W_t)\to 1$ $S(A_t)\to 0$ and $Z_t\to 1$. But the situation is different with respect to $h_t'(V_t)$. By one-sided property of the chordal SLE($\frac{8}{3},\rho$) process, we have

$$h_t'(V_t)^\lambda \to \mathbf{P}[\gamma^L\cap A=\emptyset|\gamma^R], \quad \text{as } t\to\infty.$$

Thus by optional stopping theorem,

$$\mathbf{P}[K\cap A=\emptyset] = \mathbf{E}[1_{\{\tau_A=\infty\}}\mathbf{E}[\gamma^L\cap A=\emptyset|\gamma^R]] = \mathbf{E}[M_{\tau_A}] = M_0.$$

□

*Remark* 85. In fact, our construction from SLE$(\frac{8}{3},\rho)$ have an pre-request : we should make sure that dipolar SLE$(\frac{8}{3},\rho)$ intersects with $\mathbb{R}_\pi$. According to [17] and the equivalent relations between dipolar SLE and chordal SLE(see [22]), we can see that only if $\rho \in (-2,2)$ , the dipolar SLE$(\frac{8}{3},\rho)$ can intersect with $\mathbb{R}_\pi$. So we should add $\rho \in (0,2)$ in Proposition 84. Now $\beta \in (\frac{5}{8},2)$. Therefore $\beta \in (\frac{5}{8},2)$ we have constructed $\mathbf{P}(\alpha(\beta),\beta)$ for $\beta \in (\frac{5}{8},2)$ and here $\alpha(\beta) = \frac{1}{48}((\sqrt{24\beta+1}-1)^2-4)$.

*Remark* 86. So far, we can not give the sufficient conditions that $(\alpha,\beta)$ should satisfy for making sure the conformal restriction measure exists like [9] and [23]. And this is one of the continuing work of the author.

# Chapitre 4

# On the Brownian Loop Measure

Using SLE as a tool, many problems relating to the properties of the lattice models have been solved, such as the arm exponents for these models. There are also some variants of SLE (conformal loop ensemble, Brownian loop measure, Brownian bubble measure) that describe the scaling limit of the random loops in these models. Therefore it is natural to use SLE to get properties of these loop measures. One of theses application is to use SLE($\frac{8}{3}$) to study the properties of the Brownian bubble measure and Brownian loop measure. In fact, by rescaling and letting the two end points tends to one common point, one can get the Brownian bubble measure(differ by a constant).

Recently Beliaev and Viklund [4] got a formula for the probability that two given points lies to the left of the SLE($\frac{8}{3}$) curve and used it to study some connectivity functions for SLE($\frac{8}{3}$) bubbles and reconstructed the chordal restriction measure introduced by Lawler, Werner and Schramm [9]. In this chapter, we will follow their work to use the SLE($\frac{8}{3}$) bubble to derive the formula for the total mass of the Brownian loop that disconnects two given points from the boundary. This formula was predicted by Cardy and Gamsa [7], here the formula we get just differ by a constant from theirs.

In the following sections, we will give a brief introduction to the topics that will be used in this paper, which include the Brownian bubble measure, Brownian loop measure, SLE($\kappa$) bubble measure and the relation between these measures. We first state our the main theorem of this chapter as follows.

**Theorem 87.** *Denote by $\mu_{\mathbb{H}}^{loop}$ the Brownian loop measure on the upper half plane and $\gamma$ is a sample of the Brownian loop. Given two points $z = x+iy, w = u+iv \in \mathbb{H}$, let $E(z,w)$ denote the event that $\gamma$ disconnects both $z$ and $w$ from the boundary of $\mathbb{H}$. Then we have*

$$\mu_{\mathbb{H}}^{loop}[E(z,w)] = -\frac{\pi}{5\sqrt{3}} - \frac{1}{10}\eta\, {}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\eta) - \frac{1}{10}\log(\eta(\eta-1)) \tag{4.0.1}$$

$$+\frac{\Gamma(\frac{2}{3})^2}{5\Gamma(\frac{4}{3})}(\eta(\eta-1))^{\frac{1}{3}}\, {}_2F_1(1,\frac{2}{3};\frac{4}{3},\eta). \tag{4.0.2}$$

*where*

$$\eta = \eta(z, w) = -\frac{(x-u)^2 + (y-v)^2}{4yv}, \tag{4.0.3}$$

*and* ${}_3F_2$, ${}_2F_1$ *are the hypergeometric functions.*

(4.0.1) was first given by Cardy using conformal field theory which assumes that $O(n)$ model has the scaling limit. In fact (4.0.1) has a nicer form :

$$\mu^{\text{loop}}[E(z,w)] = -\frac{1}{10}[\log\sigma + (1-\sigma)\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1-\sigma)], \tag{4.0.4}$$

where

$$\sigma = \sigma(z,w) = \frac{|z-w|^2}{|z-\bar{w}|^2} = \frac{(x-u)^2 + (y-v)^2}{(x-u)^2 + (y+v)^2}, \tag{4.0.5}$$

and ${}_3F_2$ is the hypergeometric function.

*Remark* 88. By the conformal invariance of Brownian loop measure (see [15]), for any simply connected domain $D \subset \mathbb{C}$ with $z, w \in D$, we can get the total mass of the Brownian loop in $D$ that disconnect both $z$ and $w$ from $\partial D$ by the conformal map from $D$ to $\mathbb{H}$. In particular, if $D = \mathbb{D}$, we choose the conformal map $\phi(z) = i\frac{1+z}{1-z}$ from $\mathbb{D}$ onto $\mathbb{H}$. Then the total mass of the Brownian loop measure in $\mathbb{D}$ that disconnects $z, w \in \mathbb{D}$ from $\partial\mathbb{D}$ is

$$-\frac{1}{10}[\log\tilde{\sigma} + (1-\tilde{\sigma})\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1-\tilde{\sigma})],$$

where $\tilde{\sigma} = \tilde{\sigma}(z,w) = \frac{|z-w|^2}{|1-z\bar{w}|^2}$.

## 4.1 Brownian loop measure and bubble measure

In this section, we will introduce several measures on the space of continuous curves in the plane. To keep the present chapter short, we will not provide the detailed discussions but instead refer the reader to the fifth chapter of Lawler's book [10] and [15] or the appendix of the thesis.

Let $\mu(z, \cdot; t)$ be the law of a complex Brownian motion $(B_s : 0 \le s \le t)$ starting from $z$. And $\mu(z, \cdot; t)$ can be written as

$$\mu(z, \cdot; t) = \int_{\mathbb{C}} \mu(z, w; t) dw$$

where the above integral can be regarded as the integral of functions which take values in the space of measures. Using the density function of the complex Brownian motion, we can see that the total mass of $\mu(z, w; t)$ is $\frac{1}{2\pi t}\exp\{-\frac{1}{2t}|z-w|^2\}$.

Let $\mu(z, w)$ be the measure defined by $\mu(z, w) = \int_0^\infty \mu(z, w; t)$. This is a $\sigma-$finite infinite measure. If $D \subset \mathbb{C}$ is a simply connected domain with nice boundary and $z, w \in D$, we can define $\mu_D(z, w)$ be the restriction of $\mu(z, w)$ on the space of curves that lie inside $D$. If $z \neq w$, the total measure of $\mu_D(z,w)$ is $\pi G_D(z,w)$, where $G_D(z,w)$ is the Green function on $D$.

If $D$ is a simply connected domain with nice boundary, let $B$ be a complex Brownian motion starting from $z \in D$ and $\tau_D$ the exit time. Denote $\mu_D(z,\partial D)$ the law of $(B_t : 0 \leq t \leq \tau_D)$, we can write

$$\mu_D(z,\partial D) = \int_{\partial D} \mu_D(z,w)dw.$$

Here we can regard $\mu_D(z,w)$ as a measure on the space of curves in $D$ from $z$ to $w \in \partial D$, and the total mass of $\mu_D(z,w)$ is the the poisson kernel $H_D(z,w)$. For $z \in D, w \in \partial D$, $\mu_D(z,w)$ can also be equivalently defined by the limits

$$\mu_D(z,w) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mu_D(z, w + \epsilon \mathbf{n}_w),$$

where $\mathbf{n}_w$ is the inner normal at $w$.

And similarly for $z, w \in \partial D$, we can also define

$$\mu_D(z,w) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon^2} \mu_D(z + \epsilon \mathbf{n}_z, w + \epsilon \mathbf{n}_w)$$

It can be showed that above limits exists in sense of Prohorov convergence(see Chapter 5 of [10]).

Given $z \in \partial D$, the **Brownian bubble measure** $\mu_D^{\text{bub}}(z)$ is defined as the limit

$$\mu_D^{\text{bub}}(z) := \lim_{w \in \partial D, w \to z} \pi \mu_D(z,w).$$

The **Brownian loop measure** is defined as following :

$$\mu_{\mathbb{C}}^{loop} := \int_{\mathbb{C}} \frac{1}{t_\gamma} \mu(z,z) dz = \int_{\mathbb{C}} \int_0^\infty \frac{1}{t_\gamma} \mu(z,z;t) dt dz.$$

Since $\mu(z,z)$ is a measure defined on loops with $z$ as a marked point(called a root), the Brownian loop measure should be understood as the above integral of measures by forgetting the root. For any domain $D$, let $\mu_D^{\text{loop}}$ be the restriction of the Brownian loop measure on the space of loops inside $D$.

For any $a \in \mathbb{R}$, define $\mathbb{H}_a := \{x+iy \in \mathbb{C} : y > a\}$, according to the lowest point of the Brownian loop, the Brownian loop can be decomposed into the following integral of Brownian bubbles(see [15]) :

$$\mu_{\mathbb{C}}^{\text{loop}} = \frac{1}{\pi} \int_{\mathbb{C}} \mu_{\mathbb{H}_y}^{\text{bub}}(x+iy) dx dy. \tag{4.1.1}$$

(4.1.1) will be very important in our computation.

## 4.2 SLE bubble measure

In this section we will define the SLE bubble measure and give the relation between SLE($\frac{8}{3}$) bubble and Brownian bubble measure.

Suppose $\kappa \in (0,4], \epsilon > 0$ and $\gamma^\epsilon$ is the SLE($\kappa$) curve from 0 to $\epsilon$ in the upper half plane. Let $\mu^\epsilon$ denote the law of $\gamma^\epsilon$.

**Lemma 89.** *The limit of the following limit exists :*

$$\mu^{bub}_{SLE(\kappa)}(0) = \lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} \mu^\epsilon. \tag{4.2.1}$$

*We call* $\mu^{bub}_{SLE(\kappa)}(0)$ *the* SLE($\kappa$)***-bubble measure***.

*Proof.* We only need to show that the limit restricted to some generated algebras that consist of finite mass exists. Here we choose the measurable sets $\{\gamma :$ $\gamma$ disconnects $z$ from $\infty\}$ for fixed $z \in \mathbb{H}$. By the definition of the SLE($\kappa$) from 0 to $\epsilon$, we choose the auto-conformal map $F_\epsilon(z) = \frac{\epsilon z}{z+1}$ that sends $\infty$ to $\epsilon$ and fixes 0. We have

$$\mathbf{P}[\gamma \text{ disconnects } z \text{ from } \infty] = p(F_\epsilon^{-1}(z)),$$

where $p(z)$ is the probability that a point $z$ lies to the right side of a chordal SLE($\kappa$) from 0 to $\infty$ in $\mathbb{H}$, which is obtained in (2.7.2).

Therefore

$$\lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} p(F_\epsilon^{-1}(z)) = \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})(\frac{8}{\kappa}-1)} (\frac{x^2+y^2}{y})^{1-\frac{8}{\kappa}} \tag{4.2.2}$$

So for fix $z \in \mathbb{H}$, if we denote $\mu^\epsilon(z)$ the restriction of $\mu^\epsilon$ restricted to the curves that disconnect $z$ from $\infty$, then by the above equation, we know that the limit

$$\mu^{\mathrm{bub}}_{\mathrm{SLE}(\kappa)}(0,z) := \lim_{\epsilon \to 0} \epsilon^{1-\frac{8}{\kappa}} \mu^\epsilon(z)$$

exists and therefore we can define $\mu^{\mathrm{bub}}_{\mathrm{SLE}(\kappa)}(0)$ as the limit of $\mu^{\mathrm{bub}}_{\mathrm{SLE}(\kappa)}(0,z)$ as $z$ tends to zero. $\square$

If $\kappa = \frac{8}{3}$, from (4.2.2), we get that the total mass of the SLE($\frac{8}{3}$)-bubble that disconnects a given point $z = x+iy \in \mathbb{H}$ is $\frac{1}{4}(\frac{y}{x^2+y^2})^2 = \frac{1}{4}(\mathrm{Im}\, \frac{1}{z})^2$ which corresponding to the part (a) of proposition 3.1 in [4]. In fact, [4] also gives the measure of the SLE($\frac{8}{3}$)-bubble that disconnects two points $z, w \in \mathbb{H}$ from $\infty$ which we will state as the following lemma.

**Lemma 90** (see [4])**.** *Let $E(z,w)$ be the event that two points $z, w \in \mathbb{H}$ are disconnected from $\infty$ by a* SLE($\frac{8}{3}$) *curve from 0 to $\epsilon$, then*

$$\mu^\epsilon[E(z,w)] = \frac{1}{4}\mathrm{Im}\,(\frac{1}{z})\mathrm{Im}\,(\frac{1}{w})G(\sigma(z,w))\epsilon^2 + O(\epsilon^3). \tag{4.2.3}$$

*where $\sigma$ is defined as* (4.0.5) *and*

$$G(t) = 1 - t\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t). \tag{4.2.4}$$

*Here ${}_2F_1$ is the hypergeometric function.*

Notice that when $\kappa = \frac{8}{3}$, it holds that $1 - \frac{8}{\kappa} = -2$. so we have

$$\mu^{\text{bub}}_{\text{SLE}}(0)[E(z,w)] = \frac{1}{4}\text{Im}\,(\frac{1}{z})\text{Im}\,(\frac{1}{w}) = \frac{1}{4}\frac{yv}{(x^2+y^2)(u^2+v^2)}G(\sigma(z,w)). \tag{4.2.5}$$

SLE($\frac{8}{3}$)-bubble measure is very closely related to Brownian bubble, in fact they only differ by a constant.

**Lemma 91.**

$$\mu^{bub}_{\mathbb{H}}(0) = \frac{8}{5}\mu^{bub}_{\text{SLE}(\kappa)}(0).$$

*Proof.* By the construction of the Brownian bubble measure at 0 (see the Chapter 5 of Lawler's book [10]), it is the limit of the unique measure on loops in $\mathbb{H}$ rooted at 0 such that the measure that the sample intersect $|z| = r$ is $\frac{1}{r^2}$ for any $r > 0$. So we only need to show that the total mass of the SLE($\frac{8}{3}$)-bubble sample intersecting $|z| = r$ is $\frac{5}{8r^2}$. Define $F_\epsilon(z) = \frac{z}{\epsilon - z}$, the imagine of the circle $|z| = r$ under $F_\epsilon$ is a circle with center $c_0 = -\frac{r^2}{r^2-\epsilon^2}$ and radius $\rho = \frac{\epsilon r}{r^2-\epsilon^2}$. Define the conformal map $\phi_\epsilon(z) = z - c_0 + \frac{\rho^2}{z-c_0}$ which maps $\mathbb{H} \setminus B(c_0,\rho)$ onto $\mathbb{H}$ with the derivative at $\infty$ equaling to 1. By the conformal restriction property of SLE($\frac{8}{3}$), we have

$$\mu^\epsilon[\gamma \cap |z| = r = \emptyset] = \mu^\infty[\gamma \cap B(c_0,\rho) = \emptyset] = \phi'_\epsilon(0)^{\frac{5}{8}}.$$

Therefore we can check that

$$\mu^{\text{bub}}_{\text{SLE}(\kappa)}(0)[\gamma \cap |z| = r \neq \emptyset] = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}(1 - \phi'_\epsilon(0)^{\frac{5}{8}} = \frac{5}{8r^2}.$$

□

## 4.3 Proof of the nicer form

Given two points $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0 \in \mathbb{H}$. By the symmetric property of the the Brownian loop measure, we can assume $y_0 \le v_0$, $u_0 \ge x_0$ without loss of generality. By (4.1.1),

$$\mu^{\text{loop}}_{\mathbb{H}}[E(z_0,w_0)] = \frac{1}{\pi}\int_{\mathbb{H}} \mu^{\text{bub}}_{\mathbb{H}_y}(x+iy)[E(z_0,w_0)]dxdy$$

$$= \frac{1}{\pi}\int_0^{y_0}\int_{\mathbb{R}} \mu^{\text{bub}}_{\mathbb{H}_y}(x+iy)[E(z_0,w_0)]dxdy.$$

Here $E(z_0, w_0)$ denotes the event that the Brownian loop sample in $\mathbb{H}$ disconnects both $z_0$ and $w_0$ from the boundary of $\mathbb{H}$.

By the translation invariance of the Brownian bubble measure, we have

$$\mu_{\mathbb{H}_y}^{\text{bub}}(x+iy)[E(z_0, w_0)] = \mu_{\mathbb{H}}^{\text{bub}}(0)[E(z_0 - z, w_0 - z)].$$

By Lemma 91 we have $\mu_{\mathbb{H}}^{\text{bub}}(0) = \frac{8}{5}\mu_{\text{SLE}(\kappa)}^{\text{bub}}(0)$. Therefore by (4.2.5),

$$\begin{aligned}\mu_{\mathbb{H}}^{\text{loop}}[E(z_0, w_0)] &= \frac{8}{5\pi}\int_0^{y_0}\int_{\mathbb{R}} \mu_{\text{SLE}(\kappa)}^{\text{bub}}(0)[E(z_0 - x - iy, w_0 - x - iy)]dxdy \\ &= \frac{8}{5\pi}\int_0^{y_0}\int_{\mathbb{R}} \frac{1}{4}\text{Im}\Big(\frac{1}{z_0 - x - iy}\Big)\text{Im}\Big(\frac{1}{w_0 - x - iy}\Big) \\ &\qquad G(\sigma(z_0 - x - iy, w_0 - x - iy))dxdy. \qquad (4.3.1)\end{aligned}$$

So in order to prove the theorem, we only need to compute above integral. Define two functions as follows :

$$f(x,y) := \frac{(y_0 - y)(v_0 - y)}{[(x_0 - x)^2 + (y_0 - y)^2] \times [(u_0 - x)^2 + (v_0 - y)^2]}. \qquad (4.3.2)$$

$$\begin{aligned}g(y) := &\frac{(x_0 - u_0)^2 + (y_0 - v_0)^2}{(x_0 - u_0)^2 + (y_0 + v_0 - 2y)^2} \\ &{}_2F_1\Big(1, \frac{4}{3}; \frac{5}{3}; \frac{4(y_0 - y)(v_0 - y)}{(x_0 - u_0)^2 + (y_0 + v_0 - 2y)^2}\Big). \qquad (4.3.3)\end{aligned}$$

**Lemma 92.** *Take the notations as above, for fixed $y > 0$,*

$$\int_{\mathbb{R}} f(x,y)dx = \frac{2(y_0 - y) + v_0 - y_0}{(x_0 - u_0)^2 + \big(2(y_0 - y) + v_0 - y_0\big)^2}\pi. \qquad (4.3.4)$$

*Proof.* For fixed $y > 0$, denote

$$a = y_0 - y, \quad b = v_0 - y, \quad c = u_0 - x_0, \quad d = v_0 - y_0.$$

Then we have

$$f(x,y) = \frac{ab}{[(x_0 - x)^2 + a^2][(u_0 - x)^2 + b^2]}.$$

Doing the standard calculus as follows.

$$
\begin{aligned}
&\int_{\mathbb{R}} f(x,y)dx = \int_{\mathbb{R}} \frac{ab}{[(x_0-x)^2+a^2][(u_0-x)^2+b^2]}dx \\
=&ab\int_{\mathbb{R}} \frac{1}{[x^2+a^2][(x+c)^2+b^2]}dx \\
=&\frac{ab\pi}{ab\big(a^4-2a^2(b^2-c^2)+(b^2+c^2)^2\big)}\Big(b(b^2+c^2-a^2)\arctan[\frac{x}{a}] \\
&+a\big[(a^2+c^2-b^2)\arctan[\frac{c+x}{b}]+bc\log\frac{b^2+(c+x)^2}{a^2+x^2}\big]\Big)|_{-\infty}^{\infty} \\
=&\pi\frac{b(b^2+c^2-a^2)+a(a^2+c^2-b^2)}{a^4-2a^2(b^2-c^2)+(b^2+c^2)^2}.
\end{aligned}
$$

Replace $b$ by $a+d$, we can get

$$\int_{\mathbb{R}} f(x,y)dx = \frac{(2a+d)\pi}{c^2+(2a+d)^2}.$$

And this is what we want. □

By (4.3.1), we have

$$
\begin{aligned}
\mu_{\mathbb{H}}^{\text{loop}}[E(z_0,w_0)] &= \frac{8}{5\pi}\int_0^{y_0}\int_{\mathbb{R}} \frac{1}{4}f(x,y)(1-g(y))dxdy \\
=&\frac{8}{5\pi}\int_0^{y_0}\frac{\pi}{4}\frac{2a+d}{c^2+(2a+d)^2}[1-g(y)]dy = \frac{2}{5}(A-B),
\end{aligned}
\tag{4.3.5}
$$

where

$$A = A(z_0,w_0) = \int_0^{y_0} \frac{2(y_0-y)+v_0-y_0}{(x_0-u_0)^2+\big(2(y_0-y)+v_0-y_0\big)^2}dy, \tag{4.3.6}$$

and

$$B = B(z_0,w_0) = \int_0^{y_0} \frac{2(y_0-y)+v_0-y_0}{(x_0-u_0)^2+\big(2(y_0-y)+v_0-y_0\big)^2}g(y)dy. \tag{4.3.7}$$

**Lemma 93.**

$$A = \frac{1}{4}\log\frac{1}{\sigma},$$

*where $\sigma$ is defined as* (4.0.5).

*Proof.* By (4.3.6) we have

$$A = \int_0^{y_0} \frac{2(y_0 - y) + d}{c^2 + (2(y_0 - y) + d)^2} dy = \int_0^{y_0} \frac{2y + d}{c^2 + (2y + d)^2} dy$$
$$= \frac{1}{2} \int_{d/c}^{\frac{2y_0+d}{c}} \frac{y}{1 + y^2} dy = \frac{1}{4} \log \frac{c^2 + (2y_0 + d)^2}{c^2 + d^2}.$$

In the second equation we used the variable change $y \to y_0 - y$ and in the last equation we used the variable change $y \to \frac{2y+d}{c}$. Notice that

$$\frac{c^2 + (2y_0 + d)^2}{c^2 + d^2} = \frac{(u_0 - x_0)^2 + (y_0 + v_0)^2}{(u_0 - x_0)^2 + (v_0 - y_0)^2} = \frac{1}{\sigma}.$$

□

**Lemma 94.**
$$B = \frac{1}{4}(1 - \sigma)\, {}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1 - \sigma),$$
*where $\sigma$ is defined as* (4.0.5).

*Proof.* By (4.3.7) and the definition of $g(y)$, we have

$$B = \int_0^{y_0} \frac{2(y_0 - y) + d}{c^2 + (2(y_0 - y) + d)^2} \cdot \frac{c^2 + d^2}{c^2 + (2(y_0 - y) + d)^2} \cdot$$
$${}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{4(y_0 - y)((y_0 - y) + d)}{c^2 + (2(y_0 - y) + d)^2}) dy$$
$$= \int_0^{y_0} \frac{2y + d}{c^2 + (2y + d)^2} \cdot \frac{c^2 + d^2}{c^2 + (2y + d)^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{4y(y + d)}{c^2 + (2y + d)^2}) dy$$
$$= \int_{\frac{d}{c}}^{\frac{2y_0+d}{c}} \frac{cy}{c^2 + c^2y^2} \cdot \frac{c^2 + d^2}{c^2 + c^2y^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{c^2y^2 - d^2}{c^2 + c^2y^2}) \cdot \frac{c}{2} dy$$
$$= \frac{1}{2} \frac{c^2 + d^2}{c^2} \int_{\frac{d}{c}}^{\frac{2y_0+d}{c}} \frac{y}{(1 + y^2)^2} \cdot {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{c^2y^2 - d^2}{c^2 + c^2y^2}) dy$$
$$= \frac{1}{4} \int_0^{\frac{4y_0(y_0+d)}{c^2+(2y_0+d)^2}} {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; y) dy$$
$$= \frac{1}{4} \frac{4y_0(y_0 + d)}{c^2 + (2y_0 + d)^2} \cdot {}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; \frac{4y_0(y_0 + d)}{c^2 + (2y_0 + d)^2})$$
$$= \frac{1}{4}(1 - \sigma)\, {}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; 1 - \sigma).$$

Here the second equation used the variable change $y \to y_0 - y$, the third equation used the variable change $y \to \frac{2y+d}{c}$, the fifth equation used the variable change $\frac{c^2y^2-d^2}{c^2+c^2y^2} \to y$ and the sixth equation used the equation about hypergeometric functions below :

$$\int_0^x {}_2F_1(a, b; c, y) dy = x\, {}_3F_2(a, b, 1; c, 2, x).$$

□

Now by (4.3.5) and Lemma 93 and Lemma 94 we get (4.0.4).

## 4.4 Proof of the Gamsa formula

In this section we will prove the the equivalence of formula (4.0.1) and (4.0.4). First we will recall some identities of the hypergeometric functions which will be used in our proof. We will assume that our hypergeometric functions are all well defined. And they satisfies the following identities (see Chapter 8 of [1]) :

$$ {}_2F_1(a,b;c;x) = (1-x)^{-b}\,{}_2F_1(c-a,b;c;\frac{x}{x-1}). \tag{4.4.1} $$

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b+1-c;1-x) $$
$$ + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c+1-a-b;1-x). \tag{4.4.2} $$

$$ {}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x). \tag{4.4.3} $$

Notice that $\eta = \frac{\sigma}{\sigma-1}$ and $\sigma \in (0,1)$. We define a function $\phi$ on $[0,1]$ as following.

$$ \phi(t) = \frac{2\pi}{\sqrt{3}} + \frac{t}{t-1}\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\frac{t}{t-1}) - (1-t)\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-t) $$
$$ - 2\log(1-t) - 2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\sqrt[3]{\frac{t}{(t-1)^2}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3};\frac{t}{t-1}). \tag{4.4.4} $$

To prove that (4.0.1) and (4.0.4) are equivalent, it only needs to show that $\phi(t) \equiv 0$. Notice that $\phi(0) = \frac{2\pi}{\sqrt{3}} - {}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1) = 0$, it is left to show that $\phi'(t) \equiv 0$. Take the following notations.

$$ I(t) := \frac{t}{t-1}\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;\frac{t}{t-1}) - 2\log(1-t), $$

$$ J(t) := -(1-t)\,{}_3F_2(1,\frac{4}{3},1;\frac{5}{3},2;1-t), $$

$$ K(t) := -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\sqrt[3]{\frac{t}{(t-1)^2}}\,{}_2F_1(1,\frac{2}{3};\frac{4}{3};\frac{t}{t-1}). $$

Define a function $f(x) = x\,{}_3F_2(1, \frac{4}{3}, 1; \frac{5}{3}, 2; x)$. It is easy to check that

$$f'(x) = {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; x).$$

So

$$\begin{aligned}\frac{dI(t)}{dt} &= \frac{2}{1-t} + f'(\frac{t}{t-1})\frac{-1}{(1-t)^2} = \frac{2}{1-t} - \frac{1}{(1-t)^2}\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; \frac{t}{t-1}) \\ &= \frac{2}{1-t} - \frac{1}{1-t}\,{}_2F_1(1, \frac{1}{3}; \frac{5}{3}; t). \qquad (4.4.5)\end{aligned}$$

The last equation is due to (4.4.1) by assigning $a = \frac{1}{3}, b = 1, c = \frac{5}{3}$. Similarly we can get that

$$\frac{dJ(t)}{dt} = f'(1-t) = {}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t).$$

Using (4.4.2), let $a = 1, b = \frac{4}{3}, c = \frac{5}{3}$, we have

$${}_2F_1(1, \frac{4}{3}; \frac{5}{3}; 1-t) = -\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; t) + \frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{-\frac{2}{3}}\,{}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{3}; t)$$

By letting $a = \frac{1}{3}, b = \frac{2}{3}, c = \frac{1}{3}$ in (4.4.3), the following holds

$${}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{3}; t) = (1-t)^{-\frac{2}{3}}\,{}_2F_1(0, -\frac{1}{3}; \frac{1}{3}, x) = (1-t)^{-\frac{2}{3}}.$$

Therefore

$$\frac{dJ(t)}{dt} = -\,{}_2F_1(1, \frac{4}{3}; \frac{5}{3}; t) + \frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}(t(1-t))^{-\frac{2}{3}}. \qquad (4.4.6)$$

In the last we deal with the derivative of $K(t)$ with respect to $t$. By letting $a = \frac{1}{3}, b = \frac{2}{3}, c = \frac{4}{3}$ in (4.4.1), we can get

$${}_2F_1(1, \frac{2}{3}; \frac{4}{3}; \frac{t}{t-1}) = (1-t)^{\frac{2}{3}}\,{}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t).$$

Consequently,

$$K(t) = -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{\frac{1}{3}}\,{}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t).$$

And

$$\begin{aligned}\frac{dK(t)}{dt} &= -2\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}\Big[\frac{1}{3}t^{-\frac{2}{3}}\,{}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t) + t^{\frac{1}{3}}\frac{(1-t)^{-\frac{2}{3}} - {}_2F_1(1, \frac{2}{3}; \frac{4}{3}; t)}{3t}\Big] \\ &= -\frac{2}{3}\frac{\Gamma(\frac{2}{3})^2}{\Gamma(\frac{4}{3})}t^{-\frac{2}{3}}(1-t)^{-\frac{2}{3}}. \qquad (4.4.7)\end{aligned}$$

Combining (4.4.5),(4.4.6) and (4.4.7), we have

$$\phi'(t) = \frac{dI(t)}{dt} + \frac{dJ(t)}{dt} + \frac{dK(t)}{t} = \frac{2}{1-t} - \frac{1}{1-t}\,{}_2F_1(1,\frac{1}{3};\frac{5}{3};t) - {}_2F_1(1,\frac{4}{3};\frac{5}{3};t).$$

**Lemma 95.**

$$2 - {}_2F_1(1,\frac{1}{3};\frac{5}{3};t) - (1-t)\,{}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = 0.$$

*Proof.* By definition we have

$${}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{4}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{5}{3})} t^n.$$

Therefore

$$t\,{}_2F_1(1,\frac{4}{3};\frac{5}{3};t) = \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{2}{3})} t^n.$$

Similarly

$${}_2F_1(1,\frac{1}{3};\frac{5}{3};t) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(n+\frac{5}{3})} t^n.$$

By using the relation $\Gamma(x+1) = x\Gamma(x)$, we can see that the coefficient of $t^n$ in the sum is

$$\frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{2}{3})} - \frac{\Gamma(n+\frac{4}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{4}{3})\Gamma(n+\frac{5}{3})} - \frac{\Gamma(n+\frac{1}{3})\Gamma(\frac{5}{3})}{\Gamma(\frac{1}{3})\Gamma(n+\frac{5}{3})} = 0.$$

□

From Lemma 95, we have $\phi'(t) \equiv 0$, and therefore $\phi \equiv \phi(0) = 0$. This completes the proof of the equivalence between (4.0.1) and (4.0.4).

## 4.5 The other cases

Given $z, w \in \mathbb{H}$, and $\gamma$ the sample of the Brownian loop in the upper half plane. According to the property of Brownian path, almost surely, $z, w \notin \gamma$. So except the case that $\gamma$ disconnects both $z$ and $w$ from the boundary, there are three other cases :

(1) $\gamma$ disconnects $z$ from the boundary but does not disconnect $w$ from the boundary ;

(2) $\gamma$ disconnects $w$ from the boundary but does not disconnect $z$ from the boundary ;

(3) $\gamma$ does neither disconnects $z$ from the boundary nor disconnects $w$ from the boundary.

We will show that the total measure of above three cases are infinite. In fact, using the same method as [4], we can show the following lemma.

**Lemma 96.** *Suppose that $\gamma$ is the sample of the* SLE$(\frac{8}{3})$ *from* 0 *to* $\epsilon$ *and denote above three cases by* $E_1(z,w), E_2(z,w)$ *and* $E_3(z,w)$ *respectively. Then*

$$\mathbf{P}[E_1(z,w)] = \frac{1}{4}\epsilon^2\big((\mathrm{Im}\,\frac{1}{z})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big) + O(\epsilon^3), \tag{4.5.1}$$

$$\mathbf{P}[E_2(z,w)] = \frac{1}{4}\epsilon^2\big[(\mathrm{Im}\,\frac{1}{w})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big] + O(\epsilon^3), \tag{4.5.2}$$

$$\mathbf{P}[E_3(z,w)] = 1 - \frac{1}{4}\epsilon^2\big[(\mathrm{Im}\,\frac{1}{w})^2 + (\mathrm{Im}\,\frac{1}{z})^2 - \mathrm{Im}\,\frac{1}{z}\mathrm{Im}\,\frac{1}{w}G(\sigma)\big] + O(\epsilon^3). \tag{4.5.3}$$

The proof of this lemma is the same as in [4]. We only need to prove that for SLE$(\frac{8}{3})$ $\gamma$ from 0 to $\infty$, the following holds.

$$\begin{aligned}&\mathbf{P}[\gamma \text{ passes the left of } z \text{ and the right of } w]\\ =&\frac{1}{4}(1-\frac{x}{|z|})(1+\frac{u}{|w|})(1-\frac{y}{|z|-x}\frac{v}{|w|+u}G(\sigma)).\end{aligned}$$

$$\begin{aligned}&\mathbf{P}[\gamma \text{ passes the left of } w \text{ and the right of } z]\\ =&\frac{1}{4}(1+\frac{x}{|z|})(1-\frac{u}{|w|})(1-\frac{y}{|z|+x}\frac{v}{|w|-u}G(\sigma)).\end{aligned}$$

$$\begin{aligned}&\mathbf{P}[\gamma \text{passes the right of both } z \text{ and } w]\\ =&\frac{1}{4}(1-\frac{x}{|z|})(1-\frac{u}{|w|})(1+\frac{y}{|z|-x}\frac{v}{|w|-u}G(\sigma)).\end{aligned}$$

where $G(\sigma)$ is the same as (4.2.4). Then using the conformal map $F_\epsilon(z) = \frac{\epsilon z}{1+z}$ to convert the SLE$(\frac{8}{3})$ from 0 to $\infty$ into the SLE$(\frac{8}{3})$ from 0 to $\epsilon$. Combing above lemma and the definition of the Brownian bubble measure and lemma 91, we can get

$$\mu_{\mathbb{H}}^{\mathrm{bub}}(0)(E_1(z,w)) = \frac{1}{10}[(\frac{y}{x^2+y^2})^2 - \frac{y}{x^2+y^2}\frac{v}{u^2+v^2}G(\sigma(z,w))].$$

$$\mu_{\mathbb{H}}^{\mathrm{bub}}(0)(E_2(z,w)) = \frac{1}{10}[(\frac{v}{u^2+v^2})^2 - \frac{y}{x^2+y^2}\frac{v}{u^2+v^2}G(\sigma(z,w))].$$

$$\mu_{\mathbb{H}}^{\mathrm{bub}}(0)(E_3(z,w)) = \infty.$$

By relation (4.1.1) and calculating the integral on the upper half plane, we can see that the total mass of the Brownian loop measure on these three sets are infinite. In fact, we can see intuitively that these three cases all contain the loops with arbitrary small diameter, while the event $E(z,w)$ in the main theorem exclude these small loops.

# Chapitre 5

# Integral Means Spectrum of the modified inner Whole plane SLE

In the theory of univalent function, the extremal problems has always been a very important issue. In this chapter we will recall one of the issues in extremal problems—spectrum problem. We will first give some basic definitions of the integral spectrum and the integral means spectrum. And then we will recall some results about the integral means spectrum of the outer whole plane SLE($\kappa$) process. In the last we will recall the so-called generalized integral means spectrum and give the integral means spectrum of the inner whole plane Loewner chain driven by some special Lévy process.

## 5.1 Spectrum problem

Denote by

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \mathbb{D}_+ := \{z \in \overline{\mathcal{C}} : |z| > 1\}.$$

There are two classes $\mathcal{S}$ and $\Sigma$ of univalent functions which are very important. They are defined as follows :

$$\mathcal{S} := \{\phi(z) = z + a_2 z^2 + a_3 z^3 + ... \text{ is univalent on } \mathbb{D}\}, \tag{5.1.1}$$

$$\Sigma := \{\psi(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + ... \text{ is univalent on } \mathbb{D}_+\}, \tag{5.1.2}$$

where "univalent" function we means one-to-one holomorphic function and $a_n$ and $b_n$ are the coefficients of the corresponding Taylor expansion and Laurent expansion.

A natural question is which extremal values $a_n$ and $b_n$ can have among these two classes. First for $\mathcal{S}$, define the Koebe function as follows :

$$k(z) := \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2}, \tag{5.1.3}$$

It can be verified that $\phi \in \mathcal{S}$ and $a_n = n$. In fact Bieberbach conjectured in 1916 that among $\mathcal{S}$, Koebe is the function that makes $|a_n|$ attain the maximal value,i.e. for any $\phi(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{S}$ , $|a_n| \le n$ holds for any $n$. This conjecture was completely proved by De Branges in 1985(see [5]). For $\Sigma$, no corresponding conclusion has been established so far, even the asymptotic behavior of $b_n$ is open. Another extremal problem about univalent function is the spectrum problem. The spectrum has a close relation with the behavior of harmonic measure. Given $p \in \mathbb{R}$, $\phi \in \Sigma$, define $\beta_\phi(p)$ as follows :

$$\beta_\phi(p) := \limsup_{r \to 1^+} \frac{\log \int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta}{|\log |r-1||}.$$

we call $\beta_\phi(p)$ the **integral means spectrum** of $\phi$ or the corresponding simply connected domain $\phi(\mathbb{D}_+)$. Define $B(p) := \sup_{\phi \in \Sigma} \beta_\phi(p)$. Call $B(p)$ the **universal integral means spectrum** . About $B(p)$, there is the following conjecture :

$$B(p) = \begin{cases} \frac{p^2}{4} \text{ if } p^2 < 4 \\ |p| - 1 \text{ if } |p| \ge 2. \end{cases}$$

Notice that if $p = 1$, $\int_0^{2\pi} |\phi'(re^{i\theta})| d\theta$ tends to the length of the boundary $|\partial\phi(\Sigma)|$ as $r \to 1$. Therefore in some sense $\beta_\phi(p)$ reflects the regularity of the boundary of $\phi(\Sigma)$.

For $\phi \in \mathcal{S}$, we can also define the spectrum. Here we use the definition in [6]. Given $p, q \in \mathbb{R}$, $\phi \in \mathcal{S}$. Define $\beta_\phi(p, q)$ as follows :

$$\beta_\phi(p,q) := \limsup_{r \to 1^+} \frac{\log \int_0^{2\pi} |\frac{z}{\phi(z)}|^q |\phi'(re^{i\theta})|^p d\theta}{|\log |r-1||}.$$

Call $\beta_\phi(p, q)$ is the **generalized means integral spectrum** of $\phi$ or the corresponding $\phi(\mathbb{D})$. Notice that for $\phi \in \Sigma$,

$$\beta_\phi(p) = \beta_\psi(p, 2p), \text{ here } \psi(z) = \frac{1}{\phi(1/z)}.$$

Usually, for a given simply connected domain $D$, it is very difficult to compute the integral means spectrum since it is difficult to get the exact form of the conformal map from $\mathbb{D}$ onto $\mathbb{D}$. But for some random simply connected domains(the complement of the whole plane SLE($\kappa$) hulls), some tools can be used to compute the **average integral means spectrum** which is defined as follows :

$$\beta_\phi(p) := \limsup_{r \to 1^+} \frac{\log \int_0^{2\pi} \mathbf{E}[|\phi'(re^{i\theta})|^p] d\theta}{|\log |r-1||}. \tag{5.1.4}$$

$$\beta_\phi(p,q) := \limsup_{r \to 1^+} \frac{\log \int_0^{2\pi} \mathbf{E}[|\frac{z}{\phi(z)}|^q |\phi'(re^{i\theta})|^p] d\theta}{|\log |r-1||}.$$

In the following section, for convenience we will use "spectrum" to denote the average integral means spectrum.

For the outer whole plane SLE($\kappa$) ($g_t : t \in \mathbb{R}$) ( see the definition 33), $f_t(z) := g_t^{-1}(z)$ is the conformal map from $\mathbb{D}_+$ onto $\overline{\mathbb{C}} \setminus K_t$ and $f_t(\infty) = \infty, f_t'(\infty) = e^t$. Therefore $e^{-t} f_t(z) \in \Sigma$. So it is natural to consider the spectrum of $e^{-t} f_t(z)$. By the scaling property of whole plane SLE($\kappa$) (see Proposition 36 ), we only need to consider the spectrum of $f_0$. In the seminal paper [3] and the following paer [2], the authors used the method of finding special solution of some partial differential equation to compute the spectrum of outer whole plane SLE($\kappa$). We stated the result as follows :

**Theorem 97** (see[2] and [3]). *The spectrum of the whole plane* SLE($\kappa$) *at* $t = 0$ *is*

$$\begin{aligned}
\beta_{\mathrm{tip}}(t) &= -t - 1 + \frac{1}{4}\left(4 + \kappa - \sqrt{(4+\kappa)^2 - 8\kappa t}\right), && t \le t_2,\\
\beta_0(t) &= -t + \frac{4+\kappa}{4\kappa}\left(4 + \kappa - \sqrt{(4+\kappa)^2 - 8\kappa t}\right), && t_2 \le t \le t_3,\\
\beta_{\mathrm{lin}}(t) &= t - \frac{(4+\kappa)^2}{16\kappa} && t_3 \le t.
\end{aligned}$$

*where*

$$\begin{aligned}
t_1 &= -\frac{1}{128}(4+\kappa)^2(8+\kappa),\\
t_2 &= -1 - \frac{3\kappa}{8},\\
t_3 &= \frac{3(4+\kappa)^2}{32\kappa}.
\end{aligned}$$

In [6], the authors analyzed the generalized integral means spectrum of inner whole plane process and give the exact form of the spectrum for some special values $(p, q)$ that are on some parabolas.

## 5.2 Spectrum of the Loewner process driven by Lévy process

In this section, we will analyze the spectrum of the inner Loewner process driven by Lévy process. Recall (2.3.4), the inner whole Loewner process driven by the real-valued function $W_t$ is defined as the solution of the following ODE :

$$\begin{cases} \partial_t g_t(z) = g_t(z)\frac{g_t(z)+\lambda_t}{g_t(z)-\lambda_t} \\ \lim\limits_{t\to+\infty} e^t g_t(z) = z, \quad \forall z \in \mathbb{C}. \end{cases} \qquad \begin{cases} \partial_t f_t(z) = z f_t'(z)\frac{\lambda_t+z}{\lambda_t-z} \\ \lim\limits_{t\to+\infty} f_t(e^{-t}z) = z, \quad \forall z \in \mathbb{D}. \end{cases} \tag{5.2.1}$$

Here $\lambda_t = \exp\{iW_t\}$.

In this section, we will assume that $W_t = L_t$ where $L_t$ is a Lévy process. We want to compute the average integral means spectrum of $f_0$, where $f_t$ is defined as

(2.3.4). Recall for Lévy driven Loewner process , we still have Lemma 48 since the proof of Lemma 48 only uses the Markov property of the driven process. Therefore we have

$$\lim_{t\to+\infty} e^t \tilde{f}_t(z) \overset{\text{(law)}}{=} f_0(z), \tag{5.2.2}$$

where $\tilde{f}_t$ is defined as follows :

$$\partial_t \tilde{f}_t(z) = \tilde{f}_t(z)\frac{\tilde{f}_t(z)+\lambda(t)}{\tilde{f}_t(z)-\lambda(t)}, \tilde{f}_0(z) = z, \forall z \in \mathbb{D}, \tag{5.2.3}$$

By the Markov property of Lévy process, we know that for any $s \le t$ :

$$\tilde{f}_t(z) \overset{\text{(law)}}{=} \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)). \tag{5.2.4}$$

Donote by $f := f_0(z)$, according to the definition, we want to compute

$$\mathbb{E}[|f'(z)^p|] = \mathbb{E}[f'(z)^{\frac{p}{2}}\overline{f'(z)}^{\frac{p}{2}}],$$

for $p \in \mathbb{R}$. In fact we can study the two-point function for $z_1, z_2 \in \mathbb{D}$,

$$G(z_1, \overline{z_2}) := \mathbb{E}[z_1^{\frac{q}{2}}\frac{f'(z_1)^{\frac{p}{2}}}{f(z_1)^{\frac{q}{2}}}\overline{z_2}^{\frac{\bar{q}}{2}}\frac{\overline{f'(z_2)}^{\frac{\bar{p}}{2}}}{\overline{f(z_2)}^{\frac{\bar{q}}{2}}}].$$

where $p, q \in \mathbb{C}$.

We define a time-dependent, auxiliary two-point function,

$$\tilde{G}(z_1, \bar{z}_2, t) := \mathbb{E}[z_1^{\frac{q}{2}}\frac{\tilde{f}'_t(z_1)^{\frac{p}{2}}}{\tilde{f}_t(z_1)^{\frac{q}{2}}}\overline{z_2}^{\frac{\bar{q}}{2}}\frac{\overline{\tilde{f}'_t(z_2)}^{\frac{\bar{p}}{2}}}{\overline{\tilde{f}_t(z_2)}^{\frac{\bar{q}}{2}}}],$$

where $\tilde{f}_t$ is defined as (5.2.3).

This time by (5.2.2), the two point function $G(z_1, \bar{z}_2)$ is the limit

$$\lim_{t\to+\infty} e^{\mathrm{Re}\,(p-q)t}\tilde{G}(z_1, \bar{z}_2, t) = G(z_1, \bar{z}_2). \tag{5.2.5}$$

Let us introduce the shorthand notation,

$$X_t(z_1) = \frac{\tilde{f}'_t(z_1)^{\frac{p}{2}}}{\tilde{f}_t(z_1)^{\frac{q}{2}}},\ Y_t(z_2) = \frac{\overline{\tilde{f}'_t(z_2)}^{\frac{\bar{p}}{2}}}{\overline{\tilde{f}_t(z_2)}^{\frac{\bar{q}}{2}}}.$$

Then $\mathcal{M}_s := \mathbb{E}[X_t(z_1)Y_t(z_2)]$ is a local martingale. In fact by the Markov property of Lévy process, we know that for any $s \le t$ :

$$\tilde{f}_t(z) \overset{\text{(law)}}{=} \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)). \tag{5.2.6}$$

Therefore $\mathcal{M}_s = X_s(z_1)Y_s(z_2)G(z_{1,s}, \overline{z_{2,s}}, t-s), \forall s < t$, where

$$z_{1,s} = \frac{\tilde{f}_s(z_1)}{\lambda_s}, \quad z_{2,s} = \frac{\tilde{f}_s(z_2)}{\lambda_s}.$$

Notice that

$$\begin{aligned} dX_s(z_1) =& X_s(z_1)\left[\frac{p}{2} - \frac{q}{2} - \frac{p}{(1-z_{1,s})^2} + \frac{q}{1-z_{1,s}}\right] ds, \\ dY_s(z_2) =& Y_s(z_2)\left[\frac{\bar{p}}{2} - \frac{\bar{q}}{2} - \frac{\bar{p}}{(1-\bar{z}_{2,s})^2} + \frac{\bar{q}}{1-\bar{z}_{2,s}}\right] ds, \\ \frac{dz_{1,s}}{ds} =& z_{1s}\frac{z_{1,s}+1}{z_{1,s}-1}, \quad \frac{d\overline{z_{2,s}}}{ds} = \overline{z_{2,s}}\frac{\overline{z_{2,s}}+1}{\overline{z_{2,s}}-1}. \end{aligned}$$

Since $H(s, L_s) := X_s(z_1)Y_s(z_2)\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)$ is a local martingale for all $s \le t$, by Remark 23, we have

$$-\Lambda H(s, L_s) = \partial_s H(s, L_s),$$

where $\Lambda$ is the generator of the Lévy process $L_t$.

$$\begin{aligned} \partial_s H = H & \left[\frac{p}{2} - \frac{q}{2} - \frac{p}{(1-z_{1,s})^2} + \frac{q}{1-z_{1,s}}\right] \\ &+H\left[\frac{\bar{p}}{2} - \frac{\bar{q}}{2} - \frac{\bar{p}}{(1-\bar{z}_{2,s})^2} + \frac{\bar{q}}{1-\bar{z}_{2,s}}\right] \\ &-X_s(z_1)Y_s(z_2)\partial_\tau \tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s) \\ &+X_s(z_1)Y_s(z_2)\partial_{z_1}\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)z_{1s}\frac{z_{1,s}+1}{z_{1,s}-1} \\ &+X_s(z_1)Y_s(z_2)\partial_{\bar{z}_2}\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)\overline{z_{2,s}}\frac{\overline{z_{2,s}}+1}{\overline{z_{2,s}}-1}, \end{aligned}$$

where $\tau = t - s$. So we have

$$\begin{aligned} &-\Lambda\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s) = \tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)[\frac{p}{2} - \frac{q}{2} - \frac{p}{(1-z_{1,s})^2} + \frac{q}{1-z_{1,s}}] \\ &+\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)[\frac{\bar{p}}{2} - \frac{\bar{q}}{2} - \frac{\bar{p}}{(1-\bar{z}_{2,s})^2} + \frac{\bar{q}}{1-\bar{z}_{2,s}}] \\ &-\partial_\tau\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s) + \partial_{z_1}\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)z_{1s}\frac{z_{1,s}+1}{z_{1,s}-1} \\ &+\partial_{\bar{z}_2}\tilde{G}(z_{1,s}, \overline{z_{2,s}}, t-s)\overline{z_{2,s}}\frac{\overline{z_{2,s}}+1}{\overline{z_{2,s}}-1}. \end{aligned} \tag{5.2.7}$$

Note that by (5.2.5), it holds that

$$\exp\{(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})t\}\tilde{G}(z_1,\overline{z_2},t)(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2}) + \exp\{(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})t\}\partial_t\tilde{G}(z_1,\overline{z_2},t) \to 0 \text{ as } t\to+\infty.$$

So as $t \to +\infty$,

$$\exp\{(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})t\}\partial_t\tilde{G}(z_1,\overline{z_2},t) \to -(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})G(z_1,z_2).$$

Multiplying $\exp\{(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})(t-s)\}$ on the two sides of (5.2.7) and passing $t$ to $\infty$m we can get

$$\begin{aligned}&-\Lambda G(z_1,\overline{z_2}) = G(z_1,\overline{z_2})[\frac{p}{2}-\frac{q}{2}-\frac{p}{(1-z_1)^2}+\frac{q}{1-z_1}]\\ &+G(z_1,\overline{z_2})[\frac{\bar{p}}{2}-\frac{\bar{q}}{2}-\frac{\bar{p}}{(1-\bar{z}_2)^2}+\frac{\bar{q}}{1-\bar{z}_2}]\\ &+(\frac{p-q}{2}+\frac{\bar{p}-\bar{q}}{2})G(z_1,\overline{z_2})+\partial_{z_1}G(z_1,\overline{z_2})z_1\frac{z_1+1}{z_1-1}\\ &+\partial_{\bar{z}_2}G(z_1,\overline{z_2})\overline{z_2}\frac{\overline{z_2}+1}{\overline{z_2}-1}.\end{aligned} \tag{5.2.8}$$

Then $G(z_1,\bar{z}_2)$ satisfies $\mathcal{P}(D)G(z_1,\bar{z}_2)=0$, where

$$\begin{aligned}\mathcal{P}(D) =&\Lambda + z_1\frac{z_1+1}{z_1-1}\partial_{z_1}+\overline{z_2}\frac{\overline{z_2}+1}{\overline{z_2}-1}\partial_{\bar{z}_2}+p-q+\bar{p}-\bar{q}\\ &-\frac{p}{(1-z_1)^2}+\frac{q}{1-z_1}-\frac{\bar{p}}{(1-\bar{z}_2)^2}+\frac{\bar{q}}{1-\bar{z}_2}.\end{aligned} \tag{5.2.9}$$

Recall the definition of $\Lambda$ in Theorem 22 :

$$\Lambda f(x) = \lim_{t\downarrow 0}\frac{\mathbf{E}^x[f(L_t)]-f(x)}{t}.$$

For $k,l\in\mathbb{Z}$,we have

$$\begin{aligned}\Lambda(z^k\bar{z}^l) &= r^{k+l}\Lambda(e^{i\theta(k-l)})\\ &= r^{k+l}\lim_{t\downarrow 0}\frac{\mathbf{E}^\theta[e^{i(k-l)L_t}]-e^{i(k-l)\theta}}{t}\\ &= r^{k+l}\lim_{t\downarrow 0}\frac{e^{i(k-l)\theta}(e^{-t\eta(k-l)}-1)}{t}\\ &= -\eta(k-l)z^k\bar{z}^l,\end{aligned} \tag{5.2.10}$$

where $\eta$ is the Lévy symbol of $L_t$(see (2.6.1)).

### 5.2.1 Drifted Brownian motion

In this section, we consider the special Lévy process $L_t = at + \sqrt{\kappa} B_t$, where $a \in \mathbb{R}, \kappa \geq 0$ and $B_t$ is a standard one-dimensional Brownian motion. By the definition of the L'evy symbol

$$\mathbf{E}[e^{i\xi L_t}] = \mathbf{E}[e^{i\xi(at+\sqrt{\kappa}B_t)}] = e^{ia\xi t - t\frac{\kappa}{2}|\xi|^2} = e^{-t\eta(\xi)}.$$

So

$$\eta(\xi) = \frac{\kappa}{2}|\xi|^2 - ia\xi.$$

By (5.2.10), we have

$$\Lambda(z^k \bar{z}^l) = -\eta(k-l) z^k \bar{z}^l = -\frac{\kappa}{2}(k-l)^2 (z^k \bar{z}^l) + ia(k-l)(z^k \bar{z}^l),$$

So

$$\Lambda = -\frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2 + ia(z\partial_z - \bar{z}\partial_{\bar{z}}).$$

The operator in (5.2.9) becomes

$$\begin{aligned}\mathcal{P}(D) = &-\frac{\kappa}{2}(z_1\partial_{z_1} - \bar{z}_2\partial_{\bar{z}_2})^2 + z_1(\frac{z_1+1}{z_1-1} + ia)\partial_{z_1} \\ &+\overline{z_2}(\frac{\overline{z_2}+1}{\overline{z_2}-1} - ia)\partial_{\bar{z}_2} + p - q + \bar{p} - \bar{q} \\ &-\frac{p}{(1-z_1)^2} + \frac{q}{1-z_1} - \frac{\bar{p}}{(1-\bar{z}_2)^2} + \frac{\bar{q}}{1-\bar{z}_2}. \end{aligned} \tag{5.2.11}$$

We want to find a solution of the PDE

$$\mathcal{P}(D)G(z_1, \bar{z}_2) = 0, \;\; G(0,0) = 1.$$

Suppose that the solution has the form

$$G(z_1, \bar{z}_2) = (1-z_1)^{\alpha}(1-\bar{z}_2)^{\bar{\alpha}} P(z_1\bar{z}_2).$$

Then we can see that

$$\mathcal{P}(D)[(1-z_1)^{\alpha}(1-\bar{z}_2)^{\bar{\alpha}} P(z_1\bar{z}_2)] = (I) + (II) + (III),$$

where

$$(I) = z_1\bar{z}_2(1-z_1)^{\alpha-1}(1-\bar{z}_2)^{\bar{\alpha}-1}(\kappa|\alpha|^2 P(z_1\bar{z}_2) + 2(z_1\bar{z}_2 - 1)P'(z_1\bar{z}_2)),$$

$$(II) = [\mathcal{P}(\partial)(1-z_1)^{\alpha}](1-\bar{z}_2)^{\bar{\alpha}} P(z_1\bar{z}_2),$$

$$(III) = [\mathcal{P}(\bar{\partial})(1-\bar{z}_2)^{\bar{\alpha}}](1-z_1)^{\alpha} P(z_1\bar{z}_2);$$

and

$$\mathcal{P}(\partial) := -\frac{\kappa}{2}(z_1\partial_1)^2 + \left(\frac{z_1+1}{z_1-1} + ia\right) z_1\partial_1 + p - q + \frac{q}{1-z_1} - \frac{p}{(1-z_1)^2},$$

$$\mathcal{P}(\bar\partial) := -\frac{\kappa}{2}(\bar z_2\bar\partial_2)^2 + \left(\frac{\bar z_2+1}{\bar z_2-1} - ia\right) \bar z_2\bar\partial_2 + \bar p - \bar q + \frac{\bar q}{1-\bar z_2} - \frac{\bar p}{(1-\bar z_2)^2}.$$

Notice that $\mathcal{P}(\partial)(1-z_1)^\alpha = 0 \Leftrightarrow \mathcal{P}(\bar\partial)(1-\bar z_2)^{\bar\alpha} = 0$. So if $\mathcal{P}(\partial)(1-z_1)^\alpha = 0$, we have

$$\begin{aligned}
&\mathcal{P}(D)[(1-z_1)^\alpha(1-\bar z_2)^{\bar\alpha}P(z_1\bar z_2)] = 0 \\
&\Leftrightarrow \kappa|\alpha|^2 P(z_1\bar z_2) + 2(z_1\bar z_2 - 1)P'(z_1\bar z_2) = 0 \\
&\Leftrightarrow P(z_1\bar z_2) = (1-z_1\bar z_2)^{-\frac{\kappa|\alpha|^2}{2}}.
\end{aligned}$$

So we need to find $\alpha$ such that $(1-z)^\alpha$ is a solution of $\mathcal{P}(\partial)(1-z)^\alpha = 0$.

By computation, we can see that

$$\mathcal{P}(\partial)(1-z)^\alpha = A(1-z)^\alpha + B(1-z)^{\alpha-1} + C(1-z)^{\alpha-2},$$

where

$$\begin{aligned}
A &= -\frac{\kappa}{2}\alpha^2 + (1+ia)\alpha + p - q, \\
B &= \kappa\alpha^2 - \left(\frac{1}{2}\kappa + 3\right)\alpha - ia\alpha + q, \\
C &= -\frac{\kappa\alpha^2}{2} + \left(2 + \frac{\kappa}{2}\right)\alpha - p.
\end{aligned}$$

Notice that $A+B+C=0$. By choosing $p, q, \alpha$ such that $A=B=C=0$, we get

$$\mathbf{E}\left[ z_1^{\frac{q}{2}} \frac{f'(z_1)^{\frac{p}{2}}}{f(z_1)^{\frac{q}{2}}} \bar z_2^{\frac{\bar q}{2}} \frac{\overline{f'(z_2)}^{\frac{\bar p}{2}}}{\overline{f(z_2)}^{\frac{\bar q}{2}}} \right] = (1-z_1)^\alpha(1-\bar z_2)^{\bar\alpha}(1-z_1\bar z_2)^{-\frac{\kappa|\alpha|^2}{2}}.$$

Since we want to calculate the average spectrum, we need $p, q \in \mathbb{R}$, so we need to get $\alpha$ from $A=B=C=0$.

Assume $\alpha = \alpha_1 + i\alpha_2$. By $C=0$, we get $p = -\frac{\kappa}{2}\alpha^2 + (2+\frac{\kappa}{2})\alpha$. Then by $\operatorname{Im} p = 0$, we get

$$\alpha_2\left(-\kappa\alpha_1 + 2 + \frac{\kappa}{2}\right) = 0.$$

So $\alpha_2 = 0$ or $\alpha_1 = \frac{4+\kappa}{2\kappa}$.

By $B=0$, we get $q = -\kappa\alpha^2 + \left(3+\frac{\kappa}{2}\right)\alpha - ia\alpha$. Then by $\operatorname{Im} q = 0$, we get that

$$2\kappa\alpha_1\alpha_2 - \left(\frac{\kappa}{2}+3\right)\alpha_2 + a\alpha_1 = 0.$$

So if $\alpha_2 = 0$, we have $a = 0$, this is the usual case which has been discussed in[6]. So we assume $a \neq 0$, then

$$\alpha_1 = \frac{\kappa+4}{2\kappa}, \quad \alpha_2 = -\frac{a(\kappa+4)}{\kappa(\kappa+2)},$$

and

$$\alpha = \frac{4+\kappa}{2\kappa}\left(1 - i\frac{2a}{k+2}\right).$$

Then

$$p = p(\kappa, a) = \frac{(\kappa+4)^2}{8\kappa}\left(1 + \frac{4a^2}{(\kappa+2)^2}\right),$$
$$q = q(\kappa, a) = \frac{\kappa+4}{2\kappa}\left(1 + \frac{4a^2}{(\kappa+2)^2}\right).$$

Notice that $\frac{\kappa}{2}|\alpha|^2 = p$. So with these special values we have

$$\mathbf{E}[|z|^q \frac{|f'(z)|^p}{|f(z)|^q}] = \frac{|(1-z)^\alpha|^2}{(1-|z|^2)^{\frac{\kappa|\alpha|^2}{2}}}.$$

Notice that $\operatorname{Re}\alpha > 0$, then we can get that

$$\lim_{r\to 1}\int_{|z|=r} \frac{|(1-z)^\alpha|^2}{(1-|z|^2)^{\frac{\kappa|\alpha|^2}{2}}}|dz| \asymp (1-r^2)^{-\frac{\kappa|\alpha|^2}{2}}.$$

So in our special case the average $(p, q)$-spectrum is $\beta(p, q) = p$. We summary our results as the following theorem :

**Theorem 98.** *Let $f(z) = f_0(z)$ where $(f_t)$ is the whole-plane Loewner process driven by $\lambda(t) = e^{i(at+\sqrt{\kappa}B_t)}$. If $p, q$ take the following values :*

$$\begin{cases} p = p(\kappa, a) = \frac{(\kappa+4)^2}{8\kappa}\left(1 + \frac{4a^2}{(\kappa+2)^2}\right), \\ q = q(\kappa, a) = \frac{\kappa+4}{2\kappa}\left(1 + \frac{4a^2}{(\kappa+2)^2}\right); \end{cases}$$

*then the generalized integral means spectrum $\beta(p, q)$ of $f$ is equal to $p$.*

*Remark* 99. In fact we can regard the Loewner process driven by $L_t = at + \sqrt{\kappa}B_t$ as a stochastic perturbation of the *logarithmic spiral*, which corresponds to the case $\kappa = 0$. Also note that when $\kappa = 0$, $f_0$ is the just the Koebe function, therefore $SLE_\kappa$ might be looked upon as a stochastic perturbation of the Koebe map as $\kappa$ tends to 0.

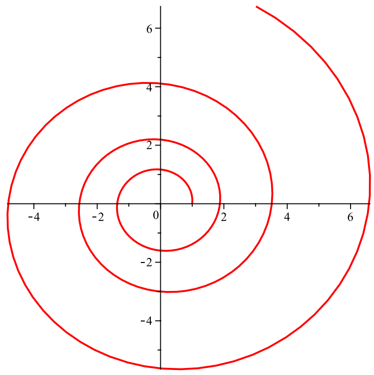


FIGURE 5.2.1 – The logarithmic spiral with $a = 5$.

### 5.2.2 Generalized integral means spectrum of the logarithmic spiral

The logarithmic spiral with parameter $a \in \mathbb{R}$ is the curve parametrized by

$$\gamma(t) = e^{(1+ia)t}, t \geq 0.$$

Let $\Omega_t = \mathbb{C} \setminus \gamma[t, +\infty)$ and $f_t(z) : \mathbb{D} \to \Omega_t$ be the Riemann map, i.e. the conformal map with

$$f_t(0) = 0, f_t'(0) > 0. \tag{5.2.12}$$

By the Koebe-1/4 theorem, $\lim_{t\to-\infty} f_t'(0) = 0$ and $\lim_{t\to\infty} f_t'(0) = \infty$. Then, there exists $t_0$ such that $f_{t_0}'(0) = 1$. Suppose that $f_{t_0}(e^{i\theta_0}) = \gamma(t_0) = e^{(1+ia)t_0}$ for some $\theta_0 \in [0, 2\pi]$.

Consider now the function $\tilde{f}_t$ defined by

$$\tilde{f}_t(z) = e^{(1+ia)t} f_{t_0}(e^{-iat} z).$$

We have $\tilde{f}_t(0) = 0, \tilde{f}_t'(0) = e^t$, and

$$\tilde{f}_t(e^{i(\theta_0+at)}) = e^{(1+ia)(t+t_0)} = \gamma(t + t_0).$$

Hence $\tilde{f}_t : \mathbb{D} \to \Omega_{t+t_0}$ is the Loewner map corresponding to the curve

$$\tilde{\gamma}(t) = e^{(1+ia)(t+t_0)}, t \in \mathbb{R}.$$

Since $\tilde{f}_t(e^{i(\theta_0+at)}) = \tilde{\gamma}(t)$, the associated process is driven by $\tilde{\lambda}(t) = e^{i(\theta_0+at)}$.

Define then the curve

$$\eta(t) = e^{-i\theta_0}\tilde{\gamma}(t), t \in \mathbb{R},$$

and the function

$$h_t(z) = e^{-i\theta_0}\tilde{f}_t(e^{i\theta_0} z).$$

It is clear that $h_t$ is the Loewner map corresponding to $\eta$ and since $h_t(e^{iat}) = \eta(t)$, the associated driven function is $\lambda(t) = e^{iat}$.

Notice that the curve $\eta$ can be obtained by a rotation and a time-translation of the logarithmic spiral $\gamma$. Thus the generalized integral means spectrum should be the same for the maps $h_0$ and $f_0$.

Let us compute the generalized integral means spectrum of $f_0$. We define the function $\Phi$ as follows :

$$\Phi(z) := e^{(1+ia)\frac{2}{1+a^2}\log\left(i\frac{1-z}{1+z}\right)} = \left(i\frac{1-z}{1+z}\right)^{\frac{2(1+ia)}{1+a^2}}, z \in \mathbb{D}.$$

We know that $z \mapsto \log(z)$ maps conformally the upper half plane onto the strip $\mathbb{S}_\pi$, and $z \mapsto e^{(1+ia)z}$ the strip domain $\mathbb{S}_{2\pi/(1+a^2)} := \{x + iy : 0 < y < 2\pi/(1 + a^2)\}$ onto $\mathbb{C}$ with a cut along the whole logarithmic spiral $\gamma(t), t \in \mathbb{R}$. Consequently, $\Phi$ is a conformal map from the unit disk to the complement of the whole logarithmic spiral with $\Phi(1) = 0, \Phi(-1) = \infty$.

Suppose $f_0(z)$ is conformal map corresponding to the whole plane Loewner process driven by $e^{iat}$ at $t = 0$. We call this the half spiral. Near $\infty$, the half spiral and whole spiral have the same spectrum. So we can use $\Phi$ to calculate the spectrum near $\infty$.

Since $|\Phi(z)| = e^{\frac{2}{1+a^2}(\mathrm{Re}\log\frac{i(1-z)}{1+z} - a\mathrm{Im}\log\frac{i(1-z)}{1+z})}$ and $\mathrm{Im}\log\frac{i(1-z)}{1+z} \in [0, 2\pi]$ is bounded. We only need to see $e^{\frac{2}{1+a^2}(\mathrm{Re}\log\frac{i(1-z)}{1+z})}$.

Then

$$|\Phi(z)| \sim |\frac{1-z}{1+z}|^{\frac{2}{1+a^2}}, \; |\Phi'(z)| \sim |1 + z|^{-\frac{3+a^2}{1+a^2}}|1 - z|^{\frac{1-a^2}{1+a^2}}.$$

Thus

$$\frac{|\Phi'(z)|^p}{|\Phi(z)|^q} \sim |1 + z|^{-p\frac{3+a^2}{1+a^2}+\frac{2q}{1+a^2}}|1 - z|^{p\frac{1-a^2}{1+a^2}-\frac{2}{1+a^2}q}.$$

Around $z = -1$, $|1 + z|^2 = (r^2 + 2r\cos\theta + 1)$ behaves like $(\pi - \theta)^2$, so its integral around $\pi$ behaves like $\theta^{-p+\frac{2(q-p)}{1+a^2}+1} \sim (1 - r)^{-p+\frac{2(q-p)}{1+a^2}+1}$. Then we have$\beta_1(p, q) = -p + \frac{2(q-p)}{1+a^2} + 1$.

Suppose

$$\phi(z) = \frac{z}{(1-z)^2} : \mathbb{D} \to \mathbb{C} \setminus (0, -\frac{1}{4}]$$

is the Koebe function with $\phi(0) = 0, \phi(1) = \infty, \phi(-1) = -\frac{1}{4}$.

Let $g$ be the conformal map from $\mathbb{C} \setminus (0, -\frac{1}{4}]$ to $\Omega_0 := \mathbb{C} \setminus \gamma[0, \infty]$ with $g(0) = 0, g(-\frac{1}{4}) = 1$. Then $f_0 = g \circ \phi$. Notice that $g$ and $g'$ is bounded near $-\frac{1}{4}$. So near $-1$,

$$\int \frac{|f_0'(z)|^p}{|f(z)^q|} \sim \int |\phi'(z)|^p.$$

Since the spectrum of a half line near the tip is $-p - 1$. Then we get that near the tip 1, the spectrum of the half spiral should be $\beta_{tip} = -p - 1$.

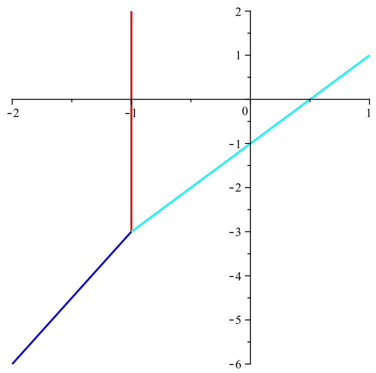

FIGURE 5.2.2 – Phases for the generalized integral means spectrum of the logarithmic spiral.

Since away from $\infty$ and the tip 1, the half spiral is rectifiable, then the spectrum should be $\beta_0 := 0$. We get the following theorem.

**Theorem 100.** *The generalized integral means spectrum of $f_0$ where $f_t$ is the whole-plane Loewner process driven by $\lambda(t) = e^{iat}$ is given by*

$$\beta(p, q) = \max\{\beta_{tip},\ \beta_0,\ \beta_1\}.$$

### 5.2.3 General Lévy process with special symbols

In this section, we will consider general Lévy process but with special symbols. We will use the same method as [16] to compute the generalized integral means spectrum for some special values.

In the following we assume that $p \equiv 2$ and $L_t$ is symmetric(see section 2.6). Suppose $G(z, \bar{z})$ has the form

$$G(z, \bar{z}) = (1 - z)(1 - \bar{z})h(z, \bar{z}).$$

Substituting into (22),

$$\begin{aligned} -\hat{\eta}[(1 - z)(1 - \bar{z})h(z, \bar{z})] + (z + 1)(\bar{z} - 1)z\partial_z h+ \\ (\bar{z} + 1)(z - 1)\bar{z}\partial_{\bar{z}} h + (3 - q)(2z\bar{z} - z - \bar{z})h = 0. \quad (5.2.13) \end{aligned}$$

Assume

$$h(z, \bar{z}) = \sum_{n=0}^{\infty} \theta_n(\chi)(z^n + \bar{z}^n), \ \ \chi := z\bar{z}.$$

By comparing the coefficients before $z^n$, we can get a recursion relation among $\theta_n(\chi)$ :( The symmetric assumption of$L_t$ assures that we only need to compare the coefficient of $z^n$ for $n \in \mathbb{N}$)

$$2\chi(\chi-1)\theta_n'(\chi) - \Big(\eta_n + n + (\eta_n + 2q - n - 6)\chi\Big)\theta_n \tag{5.2.14}$$

$$+\chi\Big(\eta_n + n - 2 + q\Big)\theta_{n+1}(\chi) + (\eta_n - n - 2 + q)\theta_{n-1}(\chi) = 0. \tag{5.2.15}$$

Therefore, we have

(a) If $\eta_1 = 3-q$, only $\theta_0$ is not equal to 0, and we have

$$\theta_0(\chi) = \frac{1}{2(1-\chi)^{3-q}}, \quad G(z,\bar{z}) = \frac{|1-z|^2}{(1-|z|^2)^{3-q}}.$$

So

$$\lim_{r\to 1}\int_{|z|=r} \frac{|1-z|^2}{(1-|z|^2)^{3-q}}|dz| \asymp (1-r^2)^{q-3}.$$

Then the corresponding generalized integral means spectrum is

$$\beta(2,q) = 3-q.$$

(b) If $\eta_1 = 1-q$, only $\theta_0$ and $\theta_1$ don't vanish. And they satisfy the ODE system :

$$\begin{cases} (\chi-1)\theta_0'(\chi) + (3-q)\theta_0(\chi) + (q-2)\theta_1(\chi) = 0 \\ 2\chi(\chi-1)\theta_1'(\chi) + \Big(q-2+(6-q)\chi\Big)\theta_1(\chi) - 2\theta_0(\chi) = 0. \end{cases}$$

Solving this ODE system(combing the initial values $G(0) \equiv 1$), we have

$$\begin{cases} \theta_0(\chi) = \frac{1+\chi}{(1-\chi)^{4-q}} \\ \theta_1(\chi) = -\frac{2}{(2-q)(1-\chi)^{4-q}}. \end{cases}$$

Then

$$\begin{aligned} G(z,\bar{z}) =& |1-z|^2\Big(\frac{1+|z|^2}{(1-|z|^2)^{4-q}} + \frac{2(z+\bar{z})}{(q-2)(1-|z|^2)^{4-q}}\Big) \\ =& \frac{|1-z|^2}{(1-|z|^2)^{4-q}}\Big(1+|z|^2+\frac{2(z+\bar{z})}{q-2}\Big). \end{aligned}$$

Since $\eta_1 = 1-q > 0$, we have $q < 1$ and $1+|z|^2+\frac{2(z+\bar{z})}{q-2}$ is uniformly bounded. Therefore

$$\lim_{r\to 1}\int_{|z|=r} \frac{|1-z|^2}{(1-|z|^2)^{4-q}}\Big(1+|z|^2+\frac{2(z+\bar{z})}{q-2}\Big)|dz| \asymp (1-r^2)^{q-4}.$$

We get the corresponding generalized integral means spectrum is

$$\beta(2,q) = 4-q.$$

We summarize above results as the theorem below :

**Theorem 101.** *Suppose $f = f_0$, here $(f_t)$ is the inner Loewner process driven by $\lambda_t = \exp\{iL_t\}$, where $L_t$ is a symmetric Lévy process. Then*

*(1) If $p = 2$, $\eta_1 = 3 - q$, the average generalized integral means spectrum $f$ is $\beta(2, q) = 3 - q$ ;*

*(2) If $p = 2$, $\eta_1 = 4 - q$, the average generalized integral means spectrum $f$ is $\beta(2, q) = 4 - q$.*

*Remark* 102. Here $p = 2$ is essential, otherwise we will not have the nice form . So far all the spectrum we have got used the special solution of the PDE. And we can not get the special for general $(p, q)$.

# Bibliographie


[1] Richard Beals and Roderick Wong. *Special functions : a graduate text*, volume 126. Cambridge University Press, 2010.

[2] Dmitry Beliaev, Bertrand Duplantier, and Michel Zinsmeister. Integral means spectrum of whole-plane sle. *arXiv preprint arXiv :1605.03112*, 2016.

[3] Dmitry Beliaev and Stanislav Smirnov. Harmonic measure and sle. *Communications in Mathematical Physics*, 290(2) :577–595, 2009.

[4] Dmitry Beliaev and Fredrik Johansson Viklund. Some remarks on sle bubbles and schramm's two-point observable. *Communications in Mathematical Physics*, 320(2) :379–394, 2013.

[5] Louis De Branges. A proof of the bieberbach conjecture. *Acta Mathematica*, 154(1) :137–152, 1985.

[6] Bertrand Duplantier, Xuan Hieu Ho, Thanh Binh Le, and Michel Zinsmeister. Logarithmic coefficients and multifractality of whole-plane sle. *arXiv preprint arXiv :1504.05570*, 2015.

[7] Adam Gamsa and John Cardy. Correlation functions of twist operators applied to single self-avoiding loops. *Journal of Physics A : Mathematical and General*, 39(41) :12983, 2006.

[8] Nam-Gyu Kang and Hee-Joon Tak. Conformal field theory of dipolar sle with the dirichlet boundary condition. *Analysis and Mathematical Physics*, 3(4) :333–373, 2013.

[9] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction : the chordal case. *Journal of the American Mathematical Society*, 16(4) :917–955, 2003.

[10] Gregory F Lawler. *Conformally invariant processes in the plane*, volume 114. American Mathematical Soc., 2008.

[11] Gregory F Lawler, Oded Schramm, and Wendelin Werner. Values of brownian intersection exponents iii : Two-sided exponents. In *Annales de l'Institut Henri*

*Poincare (B) Probability and Statistics*, volume 38, pages 109–123. Elsevier, 2002.

[12] Gregory F Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. In *Selected Works of Oded Schramm*, pages 931–987. Springer, 2011.

[13] Gregory F Lawler, Oded Schramm, and Wendelin Werner. Values of brownian intersection exponents, i : Half-plane exponents. In *Selected Works of Oded Schramm*, pages 859–895. Springer, 2011.

[14] Gregory F Lawler, Oded Schramm, and Wendelin Werner. *Values of Brownian intersection exponents, II : Plane exponents.* Springer, 2011.

[15] Gregory F Lawler and Wendelin Werner. The brownian loop soup. *Probability theory and related fields*, 128(4) :565–588, 2004.

[16] Igor Loutsenko and Oksana Yermolayeva. New exact results in spectra of stochastic loewner evolution. *Journal of Physics A : Mathematical and Theoretical*, 47(16) :165202, 2014.

[17] Jason Miller and Scott Sheffield. Imaginary geometry i : interacting sles. *arXiv preprint arXiv :1201.1496*, 2012.

[18] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 1999.

[19] Steffen Rohde and Oded Schramm. Basic properties of sle. In *Selected Works of Oded Schramm*, pages 989–1030. Springer, 2011.

[20] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118(1) :221–288, 2000.

[21] Oded Schramm. A percolation formula. *Electron. Comm. Probab*, 6 :115–120, 2001.

[22] Oded Schramm and David B Wilson. Sle coordinate changes. *New York J. Math*, 11 :659–669, 2005.

[23] Hao Wu. Conformal restriction : the radial case. *Stochastic Processes and their Applications*, 125(2) :552–570, 2015.

[24] Dapeng Zhan. *Random Loewner chains in Riemann surfaces.* PhD thesis, California Institute of Technology, 2004.

Yong HAN

## Certains problèmes concernant le SLE

**Résumé :**
Cette thèse se concentre sur trois sujets liés aux $\mathrm{SLE}(\kappa)$ processus. La première partie concerne le processus dipolar $\mathrm{SLE}(\kappa)$ et la mesure de restriction conforme à la bande ; La deuxième partie porte sur la propriété de connectivité de la mesure de la boucle brownienne ; Et la troisième partie porte sur le spectre des moyens intégrés généralisés du processus entier intérieur des processus Loewner piloté par un processus Lévy.
**Mots clés :**dipolar SLE ; dipolar restriction mesure ; la mesure de la boucle brownienne ; le spectre des moyens intégrés généralisés.

## Some problems about SLE

**Abstract :**
This thesis focuses on three topics related to the $\mathrm{SLE}(\kappa)$ processes. The first part is about the dipolar $\mathrm{SLE}(\kappa)$ process and the conformal restriction measure on the strip ; the second part is about the connectivity property of the Brownian loop measure ; and the third part is about the generalized integral means spectrum of the inner whole plane Loewner processes driven by a Lévy process.
**Keywords :** dipolar SLE ; dipolar restriction measure ; Brownian loop measure ; average integral means spectrum.

**MAPMO UMR 6628 - CNRS**
**Rue de Chartres - BP6759**
**45067 ORLÉANS CEDEX**