Expansive geodesic flows on compact manifolds without conjugate points.

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Introduction (in english)

This thesis is divided in two parts which are completely independant. Indeed those two parts are not even even concerned with the same objets, the first being concerned with compact smooth Riemannian manifolds and their geodesic loops while the second studies isometries of some metric spaces which are not uniquely geodesic in general and so somehow rougher than Riemannian manifolds. Those parts can be read in arbitrary order or even entirely skipped.

We give below separate introductions for each part.

Part one

In this part we will be concerned with compact Riemannian manifolds M that satisfy the following properties

- M has no conjugate point, that is geodesics in the universal cover X of M are globally minimizing ("the light in X always takes the shortest path");
- the geodesic flow on the unit tangent bundle SM of M is expansive, meaning that for some $\epsilon > 0$ any geodesic line in M is alone in its ϵ-tubular neighbourhood ("two light rays cannot perpetually stay too close, the closeness in question being uniform").

Then we prove

Proposition. Let M be a compact manifold without conjugate point and expansive geodesic flow. Then

- the number $n(0, R)$ of periodic orbits of length at most R is asymptotic to e^{hR}/hR where $h > 0$ is the entropy of the geodesic flow;
- there exists a unique measure of maximal entropy for the geodesic flow on the unit tangent bundle SM of M.

Let's remind that in [47] R.O. Ruggiero studied such manifolds (that is compact Riemannian manifolds without conjugate points and with expansive geodesic flow) and showed that the dynamics of the geodesic flow on those manifolds has much in common with that on compact negatively curved ones. In particular he proved the

Proposition. Let M be a compact Riemannian manifold without conjugate points and with expansive geodesic flow. Then M is Gromov hyperbolic (as defined in [21]).

Let's remark that Gromov hyperbolicity is a metric large-scale characteristic far weaker than negative curvature for compact manifolds (for example any Riemannian compact surface of genius at least 2 is Gromov hyperbolic). In [47] R.O. Ruggiero also proved the following

Proposition. Let M be a compact Riemannian manifold without conjugate points and with expansive geodesic flow. Then M is a visibility manifold as defined in [16] (we shall also say that is satifies the uniform visibility property).

For a compact non-positively curved manifold M it is proved in [9] that both Gromov hyperbolicity and the uniform visibility property are equivalent to the non-existence of flat totally geodesic plane embedded in M. The link between those two concepts for general manifolds without conjugate points is not completely clear. For example it is stated in [47] that if M is a compact manifold without conjugate points and universal cover X then the visibility property for M is equivalent to X being Gromov hyperbolic. However the proof has a gap as remarked in [49]: the proof that the visibility property of M implies Gromov hyperbolicity still stands but the proof of the converse is flawed.

In [47] R.O. Ruggiero also proves the following propositions which are reminiscent of what happens when X is compact and negatively curved

Proposition. Let M be a compact manifold without conjugate points and with expansive geodesic flow and let X be its universal cover. Then if $c_1 \colon \mathbf{R}_+ \to X$ and $c_2 \colon \mathbf{R}_+ \to X$ are two geodesics in X that satisfy $\sup_{t \geqslant 0} d(c_1(t), c_2(t)) < \infty$ there exists $\beta > 0$ such that $\lim_{t \to +\infty} d(c_1(t + \beta), c_2(t)) = 0$.

Proposition (Uniform contraction of strong stable leaves, see [47])**.** Let M be a compact manifold without conjugate points and with expansive geodesic flow and let X be its universal cover. Then strong stable leaves are uniformly contracted, i.e. if D and ϵ are positive reals then there exists $R > 0$ such that whenever c_1 and c_2 are geodesics on X that satisfy

$$d(c_1(0), c_2(0)) \leqslant D, \qquad \lim_{t \to +\infty} d(c_1(t), c_2(t)) = 0,$$

then

$$\forall t \geqslant R, \ d(c_1(t), c_2(t)) < \epsilon.$$

Since the class of compact manifold without conjugate point and expansive geodesic flow is a natural generalisation of the class of compact negatively curved manifold, it is natural to try to extend some classical results about the later class to the former.

The goal of this part is to prove, using the results from R.O. Ruggiero and in particular the propositions above, that two classical results on the dynamics of the geodesic flow of compact negatively curved manifolds still hold in the more general setting of compact manifolds without conjugate points and with expansive geodesic flow. Those two results are adressed in two consecutive chapters, the second depending on constructions carried out in the first. In the first chapter we prove the following

To this end the properties stated above about the large scale geometry of the universal cover of a compact manifold without conjugate point and with expansive geodesic flow shall not suffice and we will also use extensively the following two propositions of [47], where the second is a rather direct consequence of the first

Proposition (uniform continuity of geodesics on their endpoints, see [47])**.** Let M a compact manifold without conjugate points and with expansive geodesic flow and let X be its universal cover. For $\delta > 0$ there exists $k_0 > 0$ such that if $c_1 \colon [0, a_1] \to X$ and $c_2 \colon [0, a_2] \to X$ are geodesics (where $a_1 > 0$ and $a_2 > 0$) that satisfy

$$\max(d(c_1(0), c_2(0)), d(c_1(a_1), c_2(a_2))) \leqslant 1/k_0,$$

then

$$\forall t \in [0, +\infty[, \ d(c_1(t), c_2(t)) < \delta,$$

where the geodesics c_1 and c_2 are stationarily extended to $\mathbf{R}_+$ (that is the extension of c_1 is constant on $[a_1, +\infty[$ and the extension of c_2 is constant on $[a_2, +\infty[$).

Proposition (endpoints suffice to roughly compare the distance between two geodesics, see [47])**.** Let X be a simply connected manifold without conjugate points, with expansive geodesic flow and compact quotient. There exist constants K, $C > 0$ such that if $c_1 \colon [0, a_1] \to X$ and $c_2 \colon [0, a_2] \to X$ are geodesics in X then

$$\forall t \geqslant 0, \ d(c_1(t), c_2(t)) < K \max(d(c_1(0), c_2(0)), d(c_1(a_1), c_2(a_2))) + C,$$

where the geodesics c_1 and c_2 are stationarily extended to $[0, +\infty[$ (that is the extension of c_1 is constant on $[a_1, +\infty[$ and the extension of c_2 is constant on $[a_2, +\infty[$).

Using those results we extend two classical results about compact negatively curved manifolds to the more general class of compact Riemannian manifolds without conjugate points and with expansive geodesic flow. Each is proved in a separated chapter, the second depending on the first.

Chapter 1: Existence and uniqueness of the measure of maximal entropy

In the first chapter we shall construct a measure of maximal entropy for the geodesic flow and show the uniqueness of such a measure. But let us remind some dynamical facts. To the geodesic flow ϕ defined on the unit tangent bundle SM of M we can associate a non-negative number $h(\phi)$ called its topological entropy that roughly measures the global complexity of the flow on the exponential scale. The Kolmogorov-Sinai entropy $h_\nu(\phi)$ on the other hand is defined for any ϕ-invariant probability measure ν on SM and could be thought as the complexity of the evolution of light when the initial distribution of photons is given by the measure ν. The celebrated variational principle theorem states that the topological entropy $h(\phi)$ is the supremum of the Kolmogorov-Sinai entropy $h_\nu(\phi)$ over all ϕ-invariant probability measures ν. This supremum need not be attained by any measure in general but it is known that such must be the case when the flow ϕ is expansive, and such a measure is then called a measure of maximal entropy.

This chapter is dedicated to constructing a measure of maximal entropy for the geodesic flow (for compact riemannian manifolds without conjugate points and expansive geodesic flow) and to proving the unicity of such a measure.

The construction of such a measure for the geodesic flow on compact negatively curved manifolds was first done by Margulis in [34]. The uniqueness, still in the compact negatively curved case, was proved by Bowen in [8]. Those results (construction and uniqueness) were later extended to compact non-positively rank-one manifolds (which can be thought of as non-positively curved manifolds with “a lot of negative curvature”) by G. Knieper in [28]. We adopted the strategy of G. Knieper for the construction of the measure and to show its uniqueness. Though arguments in [28] use the non-positivity of the curvature even on the local scale, we were able to adapt the arguments to the general setting of compact manifolds without conjugate point and expansive geodesic flow using the theory of Gromov hyperbolic spaces, that of visibility manifolds and the results of [47] cited above.

Chapter 2: The Margulis asymptotic

The second chapter of this part is dedicated to proving that the asymptotics of Margulis for the number of geodesic loops in compact negatively curved manifolds still hold in compact riemannian manifolds without conjugate points and with expansive geodesic flows. First let's remind the Margulis' asymptotics ans associated definitions. If M is some compact Riemannian manifold, let $n(R)$ (resp. $\nu(R)$) be the number of distinct geodesic loops (resp. distinct primitive geodesic loops, that is injective geodesic loops). In his famous thesis (translated and published in [33]), Margulis proved the following

Proposition. If M is a compact Riemannian manifold with negative curvature then

$$\nu(R) \underset{+\infty}{\overset{R}{\sim}} n(R) \underset{+\infty}{\overset{R}{\sim}} \frac{e^{hR}}{hR},$$

where h is the topological entropy of the geodesic flow on the unit tangent bundle SM of M.

We show that the exact same asymptotics hold if M is a compact riemannian manifolds without conjugate points and with expansive geodesic flows. The proof is a simplification and an adaptation of that of Margulis. Our proof is somewhat easier to follow in part because the construction of our measure of maximal entropy (done in the preceding chapter) is different, because many parameters were dropped, but mainly because all arguments involving the transversality or the differentiability of stable or unstable leaves have been left out.

The arguments of Margulis heavily relies on the fact that the geodesic flow is mixing with respect to its measure of maximal entropy. Since we follow the same route, we need to prove that that the geodesic flow on the unit tangent bundle of a compact manifold without conjugate point and with expansive geodesic flow is still mixing with respect to its measure of maximal entropy. To this end we use the argument of M. Babillot in [3] to prove the mixing property for compact rank-one non-positively curved manifolds. More precisely, the proof of the following proposition still holds for compact manifold without conjugate point and with expansive geodesic flow (after proving that the flow is ergodic, which is easily proved using a classical argument of E. Hopf)

Proposition (see [3])**.** Let M be a compact negatively curved Riemannian manifold. Then either the measure of maximal entropy for the geodesic flow on SM is mixing, or the length spectrum of M (that is the set of lengths of geodesics loops of M) is contained in a discrete subgroup of $\mathbf{R}$.

To prove that the flow is mixing there now remains to prove that the length spectrum is not discrete. To achieve this we slightly adapt arguments of [26] and [39]. Complete proofs are given here mainly for the convenience of the reader (and to make sure that not hidden “non-positive curvature” argument have been neglected)

Part two

In this second part we extend a result L. Molnár (see [36]) on the isometries of the space of all self-adjoint bounded linear operators of some Hilbert space when endowed with the Hilbert metric to the case of arbitrary finite dimensional symmetric cones for both the Hilbert and the Thompson metric. But first let's remind some definitions to the reader.

Definition. A convex cone of $\mathbf{R}^n$ is a subset $\mathcal{C} \subset \mathbf{R}^n$ that is convex and invariant by multiplication by positive reals. A convex cone $\mathcal{C} \subset \mathbf{R}^n$ is proper if it contains no complete line and open if is an open subset of $\mathbf{R}^n$. A proper open cone $\mathcal{C}$ of $\mathbf{R}^n$ is self-dual if it is equal to its dual $\mathcal{C}^*$ defined by

$$\mathcal{C}^* = \{x \in \mathbf{R}^n \mid \forall y \in \mathcal{C},\ \langle x, y\rangle > 0\}\,.$$

A cone $\mathcal{C}$ of $\mathbf{R}^n$ is homogeneous is the subgroup of linear automorphisms u of $\mathbf{R}^n$ that satisfy $u(\mathcal{C}) \subset \mathcal{C}$ acts transitively on it.

Definition. Let $\mathcal{C} \subset \mathbf{R}^n$ be a proper open convex cone and define, for $(P, Q) \in \mathcal{C}^2$

$$\begin{aligned} \mathrm{M}(P,Q) &= \inf\{\, t > 0 \mid tQ - P \in \mathcal{C}\,\}\,,\\ d_T(P,Q) &= \log\max\Big(\mathrm{M}(P,Q), \mathrm{M}(Q,P)\Big),\\ d_H(P,Q) &= \log\Big(\mathrm{M}(P,Q)\,\mathrm{M}(Q,P)\Big). \end{aligned}$$

Then $d_T(\cdot,\cdot)$ defines a metric on $\mathcal{C}$ called the Thsompson metric whereas $d_H(\cdot,\cdot)$ defines a metric on the projectivization of $\mathcal{C}$ called the Hilbert metric. Then the set $\mathrm{Aut}(\mathcal{C})$ of linear isomorphisms of $\mathbf{R}^n$ that preserve $\mathcal{C}$ is a group of isometries for the Thompson metric while $\mathrm{Aut}(\mathcal{C})/\mathbf{R}\mathrm{Id}$ (where Id is the identity map) is a group of isometries for the Hilbert metric.

It turns out that those are not the only isometries, except for the Hilbert metric when the cone is Lorentzien (that is a connected component of the cone of light defined by some quadratic form of signature $(n-1,1)$ in $\mathbf{R}^n$). The goal of this part it to give a full description of the isometry group of those two metrics.

It is a classical fact that to any pointed proper open convex self-adjoint homogeneoous cone $(\mathcal{A}, e)$, where $\mathcal{A} \subset \mathbf{R}^n$ and $e \in \mathcal{A}$, we can associate a euclidean Jordan algebra structure J on $\mathbf{R}^n$ (the linear structure of J being that of $\mathbf{R}^n$) with unit e, where euclidean Jordan algebras are defined in the following

Definition. A Jordan algebra is a finite dimensional real-algebra J that is commutative, not necessarily alternative, but that nonetheless satisfies

$$\forall (x,y) \in J^2,\ (xy)x^2 = x(yx^2).$$

A euclidean Jordan algebra is a Jordan algebra J that additionally satisfies

$$\forall (x,y) \in J^2,\ x^2 + y^2 = 0 \Rightarrow x = y = 0.$$

Moreover this association between pointed proper open convex self-adjoint homogeneoous cones and euclidean Jordan algebras is bijective and the cone hence associated associated to a euclidean Jordan algebra J is the connected component of the identity e of J in the set of invertible elements.

It turns out that to a pointed proper open convex self-adjoint homogeneoous cone $\mathcal{C}$ one can attach a Riemannian structure that turns it into a Riemannian symmetric space of non-compact type, which in particular has non-positive curvature. Moreover there exists a natural section $\mathcal{C}_0$ for the projectivization of $\mathcal{C}$ that is a totally geodesic subspace of $\mathcal{C}$ for this Riemannian structure and that contains e. In the following we shall always identify $\mathcal{C}_0$ with the projectivization of $\mathcal{C}$, and therefore isometries for the Hilbert metric with maps $\mathcal{C}_0 \to \mathcal{C}_0$.

Since $\operatorname{Aut}(\mathcal{C})$ acts isometrically and transitively for the Thompson and the Hilbert metric, to investigate isometries for those metrics we need only investigate isometries fixing the identity element $e \in J$. But it turns out that since the geodesic inversions for the Riemannian symmetric structure are isometries for both the Hilbert and the Thompson metric we can show (as did L. Molnár in [36], using an argument presented by Väïsälä in [51] to give a short proof of the Mazur-Ulam theorem) that any isometry for any of those two metrics must preserve constant speed geodesics for the symmetric structure.

So if h is an isometry for the Hilbert or the Thompson metric with $h(e) = e$ then h must preserve constant speed geodesics emanating from e. Using tangent vectors to those geodesics this enables us to construct a sort of differential h_* of h at e (so $h_*\colon J \to J$ if h is an isometry for the Thompson metric and $h_*\colon J_0 \to J_0$ where J_0 is the tangent space to $\mathcal{C}_0$ at e if h is an isometry for the Hilbert metric). However it is not clear that this map h_* is linear.

We are then able to construct a norm preserved by h_* and the Mazur-Ulam theorem then gives us that h_* is a linear automorphism. For technical reasons we then have to split the proof depending on whether we consider the Hilbert or the Thompson metric. In both case we shall need the following We shall also need the following definitions

Definition. Let J be a euclidean algebra. An idempotent of J is an element $p \in J$ that satisfies $p^2 = p$. We say that two idempotents $(p, q) \in J^2$ are orthogonal if $pq = 0$. The rank of a non-zero idempotent is the maximal number s of non-zero mutually orthogonal idempotents $p_1, \ldots, p_s$ that satisfy $p_1 + \cdots p_s = p$. The rank of J is the rank of its unit element.

For the Thompson metric, the metric preserved by h_* is the JB-norm whose isometries were already studied in [24] where it is shown that they are exactly the maps obtained by composing a Jordan isomorphism with the multiplication by some central symmetry (that is an element $b \in J$ that satisfies $b^2 = 1$ and $b(ac) = (ba)c$ for all $(a, c) \in J^2$). It turns out that if b is a central symmetry and Φ is a Jordan isomorphism then the map h associated to $h_* = b\Phi$ is an isometry for the Thompson metric but is an element of $\operatorname{Aut}(\mathcal{C})$ only when $b = 1$. This proves that the set of multiplication by central symmetries is a transversal for the quotient $\operatorname{Iso}_T(\mathcal{C})/\operatorname{Aut}(\mathcal{C})$.

Let us now come to the Hilbert metric and let $h\colon \mathcal{C}_0 \to \mathcal{C}_0$ be an isometry for this metric. As explained above, we associated to h an isometry $h_*\colon J_0 \to J_0$ for some norm

$|\cdot|_\sigma$ and we linearly extend h_* to an isometry $\hat{h}_*\colon J \to J$ for the a semi-metric on J that extends $|\cdot|_\sigma$ (and we still write $|\cdot|_\sigma$ for this semi-metric). Both h_* and $|\cdot|_\sigma$ can also be transported to $J/\mathbf{R}e$ and be denote by $\overline{h}_*$ and $|\cdot|_{\overline{\sigma}}$ the objects obtained in this way. We prove (adapting an argument from [36]) that the classes of $J/\mathbf{R}e$ containing non trivial idempotents are exactly the extreme points of the unit ball of this space for the $|\cdot|_{\overline{\sigma}}$ metric (and those classes are obviously preserved by $\overline{h}_*$). Then we show that if p is a non-trivial idempotent then the rank of the unique idempotent in the class $\overline{h}_*([p])$ is either $\operatorname{rank} p$ or $\operatorname{rank} J - \operatorname{rank} p$ (our proof of this fact uses some quite advanced and very interesting results from [27] but any alternative proof would be welcomed by the author). We then strengthen this result by showing that either the rank of the unique idempotent in the class $\overline{h}_*([p])$ is $\operatorname{rank} p$ for every non-trivial idempotent or is $\operatorname{rank} J - \operatorname{rank} p$ for all of them. This shows, possibly after composing h with the geodesic inversion i_e at e for the Riemannian symmetric structure of $\mathcal{C}_0$, that h_* must preserve idempotents and their rank. An adaptation of a classical exercice in linear algebra then proves that h_* must be a Jordan isomorphism and from this follows that h is the restriction of h_* to $\mathcal{C}$. All in all we proved that either h is (the restriction to $\mathcal{C}_0$ of) a Jordan isomorphism or is (the restriction to $\mathcal{C}_0$ of) the composition of such an isomorphism with the geodesic inversion i_e at e for the Riemannian symmetric structure.

An easy computation shows that i_e is an isometry for the Hilbert metric exactly when J has rank at most 2, that is only if $\mathcal{C}$ is a Lorentzien cone. This shows that $\operatorname{Aut}(\mathcal{C})/\mathbf{R}\mathrm{Id}$ is the isometry group for the Hilbert metric in Lorentzien cones whereas it is a subgroup of index two of it in all the other cases (the geodesic inversion for the Riemannian structure at some chosen point $e \in \mathcal{C}$ being a representative of the other coset).

It is very plausible that our proof can be adapted to infinite dimensional JB-algebras (a natural extension of euclidean Jordan algebras to infinite dimension, see [1]) without much change, but the author did not have time to investigate in this direction.

Introduction (en français)

Cette thèse est divisée en deux parties indépendantes. En effet ces deux parties ne traitent même pas des même objets, la première s'intéressant à une classe particulière de varieté Riemanienne compacte et à leurs lacets géodésiques alors que la seconde s'intéresse aux isométries de certains espaces métriques dans lesquelles les géodésiques ne sont pas localement uniques en général, espaces qui sont donc bien moins «lisses». Ces deux parties peuvent être lues dans n'importe quel ordre ou même entièrement sautées.

Nous donnons ci-dessous une introduction séparée pour chaque partie.

Première partie

Dans cette partie nous étudions les varietés Riemaniennes compacts M qui vérifient de plus

- M est sans points conjugués, autrement dit les géodésiques du revêtement unversel X de M sont globalement minimisantes («la lumière dans X prend toujours le chemin le plus court»);
- Le flot géoédésique sur le fibré tangentiel unitaire SM de M est expansif, c'est à dire il existe $\epsilon > 0$ tel que les lignes géoédésiques de X sont seules dans leur ϵ-voisinage tubulaire («les rayons lumineux dans X ne restent pas perpetullement proche, la proximité étant uniforme»)

Nous démontrons alors la

Proposition. Soit M une variété compacte sans point conjugués de flot géodésique expansif. Alors

- le nombre $n(0, R)$ d'orbites périodiques pour le flot géodésique de longueur au plus R est asymptotique à e^{hR}/hR où $h > 0$ est l'entropie du flot géodésique;
- il existe une unique mesure d'entropie maximale pour le flot géodésique sur le fibré tangent unitaire SM de M.

Rappelons que dans [47] R.O. Ruggiero a étudié ces varietés (c'est à dire les varietés compactes sans point conjugué et à flot géodésique expansif) et a montré que la dynamique du flot géodésique sur ces variétés a beaucoup en commun avec celle des flot géodésiques sur les variétés compactes à courbure strictement négative. En particulier il a démontré

Proposition. Soit M une varieté Riemanienne compacte sans point conjugué et à flot expansif. Alors M est hyperbolique au sens de Gromov.

Remarquons que l'hyperbolicité au sens de Gromov est une caractéristique asymptotique bien plus faible que la stricte négativité de la courbure pour les varietés compactes sans point conjugué (par exemple toutes les métriques sur une surface de genre au moins deux est hyerbolique au sens de Gromov). Dans [47] R.O. Ruggiero a aussi démontre

Proposition. Soit M une varieté compacte sans point conjugué et à flot géodésique expansif. Alors M est une variété visible comme défini dans [16] (on dit aussi que M vérifie la condition de visibilité uniforme).

Pour une varieté compacte à courbure négative ou nulle M il a été démontré dans [9] que l'hyperbolicité au sens de Gromov et la condition de visibilité uniforme sont équivalente à la non-existence de plans plats immergés totallement géodésiques dans M. Le lien entre ces deux concepts dans le cas général des variétés sans points conjugués n'est en revanche par encore totalement clair. Par exemple il est dit dans [47] que si M est une variété compacte sans point conjugué et de revêtement universel X alors la condition de visibilité uniforme est équivalente à l'hyperbolicité au sens de Gromov. Seulement la preuve comporte une erreur comme remarqué dans [49]: la preuve que la condition de visibilité uniforme implique l'hyperbolicité au sens de Gromov n'est pas affectée mais la preuve de l'implication inverse est fausse.

Dans [47] R.O. Ruggiero a aussi démontré les propositions suivantes, qui rappellent le cas des varietés compactes à courbure strictement négative

Proposition. Soit M une variété sans point conjugué et à flot géodésique expansif et soit X son revêtement universel. Alors si $c_1\colon \mathbf{R}_+ \to X$ et $c_2\colon \mathbf{R}_+ \to X$ sont deux géodésiques de X telles que $\sup_{t\geqslant 0} d(c_1(t), c_2(t)) < \infty$ alors il existe $\beta > 0$ tel que $\lim_{t\to+\infty} d(c_1(t+\beta), c_2(t)) = 0$.

Proposition (Contraction uniforme des varietés stables)**.** Soit M une variété compacte sans point conjugué et à flot géodésique expansif et soit X son revêtement universel. Alors les varietés stables de X sont uniformément contractées, c'est à dire que si D et ϵ sont des réels strictement positifs alors il existe $R > 0$ tel que dès que c_1 et c_2 sont des geodesiques de X vérifiant

$$d(c_1(0), c_2(0)) \leqslant D, \qquad \lim_{t\to+\infty} d(c_1(t), c_2(t)) = 0,$$

alors

$$\forall t \geqslant R,\ d(c_1(t), c_2(t)) < \epsilon.$$

Piusque la classe des variétés compactes sans point conjugué et à flot géodésique expansif généralise naturellement celle des varietés compactes à courbure strictement négative, il est naturel d'essayer d'étendre à ce cadre des résultats classiques de ces dernières.

Le but de cette partie est de montrer, en utilisant les résultats de R.O. Ruggiero et en particulier les propositions ci-dessus, que deux résultats classiques sur la dynamique du flot géodésique des varietés compactes à courbure strictement négative s'étendent au cas des variétés compactes sans point conjugués et à flot expansif. Ces deux résultats sont démontrés dans deux chapitres consécutifs, le second utilisant les constructions menées dans le premier.

Pour cela les proprietés ci-dessus à propos de la géométrie asymptotique du revêtement universel d'une variété compacte sans point conjugué ne suffiront pas et nous utiliserons aussi les deux propositions suivantes de [47], la seconde étant une conséquence de la première

Proposition (continuité uniforme des géodésiques en fonction de leurs extrémités)**.** Soit M une variété compacte sans point conjugué et à flot expansif et soit X sont revêtement universel. Alors pour tout $\delta > 0$ il existe $k_0 > 0$ tel que si $c_1\colon [0, a_1] \to X$ et $c_2\colon [0, a_2] \to X$ sont des géodésiques (où $a_1 > 0$ et $a_2 > 0$) vérifiant

$$\max(d(c_1(0), c_2(0)), d(c_1(a_1), c_2(a_2))) \leqslant 1/k_0,$$

alors

$$\forall t \in [0, +\infty[,\ d(c_1(t), c_2(t)) < \delta,$$

où les géodésiques c_1 et c_2 sont étendues de façon stationnaires à $\mathbf{R}_+$ (c'est à dire l'extension de c_1 est constante sur $[a_1, +\infty[$ et celle de c_2 est constante sur $[a_2, +\infty[$).

Proposition (La distance entre extrémités suffisent à estimer grossièrement la distance entre deux géodésiques)**.** Soit X une variété simplement connexe, sans point conjugué, à flot expansif et admettant un quotient compacte. Alors il existe des constantes K, $C > 0$ telles que si $c_1\colon [0, a_1] \to X$ et $c_2\colon [0, a_2] \to X$ sont des géodésiques alors

$$\forall t \geqslant 0,\ d(c_1(t), c_2(t)) < K \max(d(c_1(0), c_2(0)), d(c_1(a_1), c_2(a_2))) + C,$$

où les géodésiques c_1 et c_2 sont étendues de façon stationnaires à $\mathbf{R}_+$ (c'est à dire l'extension de c_1 est constante sur $[a_1, +\infty[$ et celle de c_2 est constante sur $[a_2, +\infty[$).

En utilisant des résultats nous étendons deux résultats classiques des variétés compactes à courbures négatives au cas des variétés compactes sans point conjugué et à flot expansifs. Ils sont chacun démontré dans un chapitre à part, le second dépendant du premier.

Chapter 1: Existence and uniqueness of the measure of maximal entropy

Dans le premier chapitre nous construisons une mesure d'entropie maximale pour le flot géodésique et montrons l'unicité d'une telle mesure. Rappelons tout d'abord certains résultats de dynamique. À un flot géodésique ϕ définie sur le fibré tangent unitaire SM de M nous pouvons associer un nombre positif ou nul $h(\phi)$ appelé entropie de ce flot qui mesure plus ou moins la complexité de la dynamique de ce flot sur une échelle exponentielle. L'entropie de Kolmogorov-Sinai $h_\nu(\phi)$ d'un autre côté est définie pour toute mesure de probabilité ν ϕ-invariante sur SM et peut être imaginée comme la complexité de l'évolution de la lumière si la distribution initiale de la lumière est donnée par la mesure ν. Le célèbre principe variationnel stipule que l'entropy topologique $h(\phi)$ est le supremum de l'entropie de Kolmogorov-Sinai $h_\nu(\phi)$ sur toute les mesure de probailité ϕ-invariant ν. Ce supremum n'est pas atteint en général mais il il a été démontré que tel est le cas dans le cas où le flot est expansif et une telle mesure est alors appelé measure d'entropie maximale pour le flot.

Ce chapitre est consacré à la construction d'une mesure d'entropie maximale pour le flot géodésique (dans le cas d'une variété compacte sans point conjugué et à flot géodésique expansif) et à montrer l'unicité d'une telle mesure.

La construction d'une telle mesure dans le cas des variétés compactes sans point conjugué et à flot expansif a tout d'abord été faite par Margulis dans [34]. L'unicité d'une telle mesure, toujours dans le cas d'une variété compacte sans point conjugué, a été démontrée par Bowen dans [8]. Ces résultats (construction et unicité) ont été étendus au cas des variétés compactes à courbure négative ou nulle de rang un (qui peuvent être vus comme des variétés à courbure négative ou nulle ayant «beaucoup de courbure strictement négative») par G. Knieper dans [28]. Nous avons adopté la stratégie de G. Knieper pour la construction de cette measure et pour montrer son unicité. Bien que les arguments de [28] utilisent la négativité de la courbure même localement, nous avons pu adapter les arguments au cadre général des variétés compactes sans point conjugué et à flot géodésique expansif en utilisant la théorie des espaces hyperboliques au sens de Gromov, celle des espaces visibles, ainsi que les résultats de [47] cités ci-dessus.

Chapter 2: The Margulis asymptotic

Le second chapitre de cette partie est consacré à démontrer que les équivalents de Margulis sur le nombre de lacets géodésiques dans une variété compacte à courbure strictement négative et à flot expansif sont toujours valables dans le cadre plus général des variétés compactes sans point conjugué et à flot expansif. Tout d'abord faisons quelques rappels.Si M est une variété compacte, soit $n(R)$ (resp. $\nu(R)$) le nombre de lacets géodésiques (resp. de lacets géodésiques primitifs, c'est à dire de lacets géodésiques injectifs). Dans sa célèbre thèse (traduite du russe sans [33]), Margulis a démontré

Proposition. If M is a compact Riemannian manifold with negative curvature then

$$\nu(R) \underset{+\infty}{\overset{R}{\sim}} n(R) \underset{+\infty}{\overset{R}{\sim}} \frac{e^{hR}}{hR},$$

where h is the topological entropy of the geodesic flow on the unit tangent bundle SM of M.

Nous montrons que les même équivalents sont toujours valables si M est une variété compacte sans point conjugué et à flot expansif. La preuve est une simplification et une adaptation de celle de Margulis. Notre preuve est plus simple car la construction de la mesure d'entropie maximale est différente, car nous avons pu laisser beaucoup de paramètres de côté, mais surtout car les arguments utilisant la transversalité ou la différentiabilité des variétés stables ou instables ont été supprimés.

L'argument de Margulis utilise le fait que le flot géodésique est mélangeant pour sa mesure d'entropie maximale. Comme nous utilisons la même stratégie, nous devons tout d'abord montrer qu'il en ai de même dans notre cas. Pour cela nous utilisons l'argument utilisé par M. Babillot dans [3] pour montré le caractère mélangeant du flot géodésique sur des variétés compactes à courbure négative ou nulle et de rang un. Plus précisément, la preuve de la proposition suivante est toujours valable sans aucun changement dans le cas d'une variété compacte sans point conjugué et à flot géodésique expansif (après avoir démontré que ce flot est ergodique, ce qui est facile à montrer en utilisant un argument classique de E. Hopf)

Proposition (see [3])**.** Soit M une variété compacte à courbure strictement négative. Alors soit la mesure d'entropie maximale pour le flot géodésique sur SM est mélangeante pour ce flot, soit le spectre des longueurs de M (c'est à dire l'ensemble des longueurs de tous les lacets géodésiques de M) est contenu dans un sous-groupe discret de $\mathbf{R}$.

Pour montrer que le flot est bien mélangeant il suffit maintenant de montrer que le spectre des longueur n'est pas discret. Pour cela nous adaptons légèrement les arguments de [26] et [39]. Des preuves complètes sont données seulement pour le confort du lecteur (et pour s'assurer que aucun «argument caché de courbure négative» n'a été utilisé).

Part two

Dans cette seconde partie nous étendons un résultat de L. Molnár (voir [36]) sur les isométries de l'espace des opérateurs symétriques définis positifs d'un espace de Hilbert complexe pour la métrique de Hilbert ou la métrique de Thompson. Mais tout d'abord rappelons quelques définitions.

Definition. Un cône convexe de $\mathbf{R}^n$ est un sous-ensemble $\mathcal{C} \subset \mathbf{R}^n$ qui est convexe et invariant par multiplication par les scalaires positifs. Un cône convexe $\mathcal{C} \subset \mathbf{R}^n$ est propre si il ne contient aucune droite et ouvert si c'est un sous-ensemble ouvert de $\mathbf{R}^n$. Un cône propre et ouvert $\mathcal{C}$ de $\mathbf{R}^n$ est dit auto-dual si il est égal à son dual $\mathcal{C}^*$ défini par

$$\mathcal{C}^* = \{x \in \mathbf{R}^n \mid \forall y \in \mathcal{C},\ \langle x, y\rangle > 0\}\,.$$

Un cône $\mathcal{C}$ de $\mathbf{R}^n$ est dit homogènesi le groupe des automorphismes linéaires u de $\mathbf{R}^n$ qui vérifient $u(\mathcal{C}) \subset \mathcal{C}$ agit transitivement sur $\mathcal{C}$.

Definition. Soit $\mathcal{C} \subset \mathbf{R}^n$ un cône propre, ouvert et convexe. Pour $(P, Q) \in \mathcal{C}^2$ on définit

$$\begin{aligned} \mathrm{M}(P,Q) &= \inf\{\, t > 0 \mid tQ - P \in \mathcal{C}\,\}\,, \\ d_T(P,Q) &= \log\max\Big(\mathrm{M}(P,Q), \mathrm{M}(Q,P)\Big), \\ d_H(P,Q) &= \log\Big(\mathrm{M}(P,Q)\,\mathrm{M}(Q,P)\Big). \end{aligned}$$

Alors $d_T(\cdot,\cdot)$ définie une métrique sur $\mathcal{C}$ appelée métrique de Thompson alors que $d_H(\cdot,\cdot)$ définie une métrique sur le projectivisé de $\mathcal{C}$ appelée métrique de Hilbert. L'ensemble $\mathrm{Aut}(\mathcal{C})$ des automorphismes linéaires de $\mathbf{R}^n$ qui préservent $\mathcal{C}$ est un groupe d'isométries pour la métriques de Thompson alors que $\mathrm{Aut}(\mathcal{C})/\mathbf{R}\mathrm{Id}$ est un groupe d'isométries pour la métrique de Hilbert.

Il se trouve que les isométries induites par les automorphismes linéaires ne sont pas les seules isométries en général, sauf pour la métrique de Hilbert dans le cas d'un cône Lorentzien (c'est à dire le cône de lumière défini par une forme quadratique de signature $(1, n-1)$ sur $\mathbf{R}^n$). Le but de cette partie est de décrire complètement le groupe d'isométries de chacune de ces deux métriques.

It is a classical fact that to any pointed proper open convex self-adjoint homogeneoous cone $(\mathcal{A}, e)$, where $\mathcal{A} \subset \mathbf{R}^n$ and $e \in \mathcal{A}$, we can associate a euclidean Jordan algebra structure J on $\mathbf{R}^n$ (the linear structure of J being that of $\mathbf{R}^n$) with unit e, where euclidean Jordan algebras are defined in the following

Definition. A Jordan algebra is a finite dimensional real-algebra J that is commutative, not necessarily alternative, but that nonetheless satisfies

$$\forall (x, y) \in J^2,\ (xy)x^2 = x(yx^2).$$

A euclidean Jordan algebra is a Jordan algebra J that additionally satisfies

$$\forall (x, y) \in J^2,\ x^2 + y^2 = 0 \Rightarrow x = y = 0.$$

Moreover this association between pointed proper open convex self-adjoint homogeneoous cones and euclidean Jordan algebras is bijective and the cone hence associated associated to a euclidean Jordan algebra J is the connected component of the identity e of J in the set of invertible elements.

It turns out that to a pointed proper open convex self-adjoint homogeneoous cone $\mathcal{C}$ one can attach a Riemannian structure that turns it into a Riemannian symmetric space of non-compact type, which in particular has non-positive curvature. Moreover there exists a natural section $\mathcal{C}_0$ for the projectivization of $\mathcal{C}$ that is a totally geodesic subspace of $\mathcal{C}$ for this Riemannian structure and that contains e. In the following we shall always identify $\mathcal{C}_0$ with the projectivization of $\mathcal{C}$, and therefore isometries for the Hilbert metric with maps $\mathcal{C}_0 \to \mathcal{C}_0$.

Since $\mathrm{Aut}(\mathcal{C})$ acts isometrically and transitively for the Thompson and the Hilbert metric, to investigate isometries for those metrics we need only investigate isometries fixing the identity element $e \in J$. But it turns out that since the geodesic inversions for the Riemannian symmetric structure are isometries for both the Hilbert and the Thompson metric we can show (as did L. Molnár in [36], using an argument presented by Väisälä in [51] to give a short proof of the Mazur-Ulam theorem) that any isometry for any of those two metrics must preserve constant speed geodesics for the symmetric structure.

So if h is an isometry for the Hilbert or the Thompson metric with $h(e) = e$ then h must preserve constant speed geodesics emanating from e. Using tangent vectors to those geodesics this enables us to construct a sort of differential h_* of h at e (so $h_* \colon J \to J$ if h is an isometry for the Thompson metric and $h_* \colon J_0 \to J_0$ where J_0 is the tangent space to $\mathcal{C}_0$ at e if h is an isometry for the Hilbert metric). However it is not clear that this map h_* is linear.

We are then able to construct a norm preserved by h_* and the Mazur-Ulam theorem then gives us that h_* is a linear automorphism. For technical reasons we then have to split the proof depending on whether we consider the Hilbert or the Thompson metric. In both case we shall need the following We shall also need the following definitions

Definition. Let J be a euclidean algebra. An idempotent of J is an element $p \in J$ that satisfies $p^2 = p$. We say that two idempotents $(p, q) \in J^2$ are orthogonal if $pq = 0$. The rank of a non-zero idempotent is the maximal number s of non-zero mutually orthogonal idempotents $p_1, \ldots, p_s$ that satisfy $p_1 + \cdots p_s = p$. The rank of J is the rank of its unit element.

For the Thompson metric, the metric preserved by h_* is the JB-norm whose isometries were already studied in [24] where it is shown that they are exactly the maps obtained by composing a Jordan isomorphism with the multiplication by some central symmetry (that is an element $b \in J$ that satisfies $b^2 = 1$ and $b(ac) = (ba)c$ for all $(a,c) \in J^2$). It turns out that if b is a central symmetry and Φ is a Jordan isomorphism then the map h associated to $h_* = b\Phi$ is an isometry for the Thompson metric but is an element of $\mathrm{Aut}(\mathcal{C})$ only when $b = 1$. This proves that the set of multiplication by central symmetries is a transversal for the quotient $\mathrm{Iso}_T(\mathcal{C})/\mathrm{Aut}(\mathcal{C})$.

Let us now come to the Hilbert metric and let $h\colon \mathcal{C}_0 \to \mathcal{C}_0$ be an isometry for this metric. As explained above, we associated to h an isometry $h_*\colon J_0 \to J_0$ for some norm $|\cdot|_\sigma$ and we linearly extend h_* to an isometry $\hat{h}_*\colon J \to J$ for the a semi-metric on J that extends $|\cdot|_\sigma$ (and we still write $|\cdot|_\sigma$ for this semi-metric). Both h_* and $|\cdot|_\sigma$ can also be transported to $J/\mathbf{R}e$ and be denote by $\overline{h}_*$ and $|\cdot|_{\overline{\sigma}}$ the objects obtained in this way. We prove (adapting an argument from [36]) that the classes of $J/\mathbf{R}e$ containing non trivial idempotents are exactly the extreme points of the unit ball of this space for the $|\cdot|_{\overline{\sigma}}$ metric (and those classes are obviously preserved by $\overline{h}_*$). Then we show that if p is a non-trivial idempotent then the rank of the unique idempotent in the class $\overline{h}_*([p])$ is either $\operatorname{rank} p$ or $\operatorname{rank} J - \operatorname{rank} p$ (our proof of this fact uses some quite advanced and very interesting results from [27] but any alternative proof would be welcomed by the author). We then strengthen this result by showing that either the rank of the unique idempotent in the class $\overline{h}_*([p])$ is $\operatorname{rank} p$ for every non-trivial idempotent or is $\operatorname{rank} J - \operatorname{rank} p$ for all of them. This shows, possibly after composing h with the geodesic inversion i_e at e for the Riemannian symmetric structure of $\mathcal{C}_0$, that h_* must preserve idempotents and their rank. An adaptation of a classical exercice in linear algebra then proves that h_* must be a Jordan isomorphism and from this follows that h is the restriction of h_* to $\mathcal{C}$. All in all we proved that either h is (the restriction to $\mathcal{C}_0$ of) a Jordan isomorphism or is (the restriction to $\mathcal{C}_0$ of) the composition of such an isomorphism with the geodesic inversion i_e at e for the Riemannian symmetric structure.

An easy computation shows that i_e is an isometry for the Hilbert metric exactly when J has rank at most 2, that is only if $\mathcal{C}$ is a Lorentzien cone. This shows that $\mathrm{Aut}(\mathcal{C})/\mathbf{R}\mathrm{Id}$ is the isometry group for the Hilbert metric in Lorentzien cones whereas it is a subgroup of index two of it in all the other cases (the geodesic inversion for the Riemannian structure at some chosen point $e \in \mathcal{C}$ being a representative of the other coset).

It is very plausible that our proof can be adapted to infinite dimensional JB-algebras (a natural extension of euclidean Jordan algebras to infinite dimension, see [1]) without much change, but the author did not have time to investigate in this direction.

Part I

Expansive geodesic flows on compact manifolds without conjugate points

Part II

On symmetric cones

Bibliography

[1] Erik M. Alfsen, Frederic W. Shultz, and Erling Størmer. "A Gelfand-Neumark theorem for Jordan algebras." In: *Adv. Math.* 28 (1978), pp. 11–56.

[2] E. Andruchow, G. Corach, and D. Stojanoff. "Geometrical significance of the Löwner-Heinz inequality." In: *Proc. Am. Math. Soc.* 128.4 (2000), pp. 1031–1037.

[3] Martine Babillot. "On the mixing property for hyperbolic systems." In: *Isr. J. Math.* 129 (2002), pp. 61–76.

[4] W. Ballmann, M. Brin, and K. Burns. "On surfaces with no conjugate points." In: *J. Differ. Geom.* 25 (1987), pp. 249–273.

[5] Werner Ballmann. *Lectures on spaces of nonpositive curvature. With an appendix by Misha Brin: Ergodicity of geodesic flows.* Basel: Birkhäuser Verlag, 1995, p. 112.

[6] Kamil Bieder. "Wachstumsrate periodischer Bahnen bei geodätischen Anosov-Flüssen". MA thesis. Ruhr Universität Bochum, 2003.

[7] Rufus Bowen. "Entropy-expansive maps." In: *Trans. Am. Math. Soc.* 164 (1972), pp. 323–331.

[8] Rufus Bowen. "Periodic orbits for hyperbolic flows." In: *Am. J. Math.* 94 (1972), pp. 1–30.

[9] Martin R. Bridson. "On the existence of flat planes in spaces of non-positive curvature." In: *Proc. Am. Math. Soc.* 123.1 (1995), pp. 223–235.

[10] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Berlin: Springer, 1999, pp. xxi + 643.

[11] M. Coornaert and G. Knieper. "Growth of conjugacy classes in Gromov hyperbolic groups." In: *Geom. Funct. Anal.* 12.3 (2002), pp. 464–478.

[12] Michel Coornaert. "Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov." In: *Pac. J. Math.* 159.2 (1993), pp. 241–270.

[13] Françoise Dal'Bo. "Remarques sur le spectre des longueurs d'une surface et comptages." In: *Bol. Soc. Bras. Mat., Nova Sér.* 30.2 (1999), pp. 199–221.

[14] Pierre de la Harpe. "On Hilbert's metric for simplices." In: (1993), pp. 97–119.

[15] P. Eberlein. "Geodesic flow in certain manifolds without conjugate points." In: *Trans. Am. Math. Soc.* 167 (1972), pp. 151–170.

[16] P. Eberlein and B. O'Neill. "Visibility manifolds." In: *Pac. J. Math.* 46 (1973), pp. 45–109.

[17] Jost-Hinrich Eschenburg. "Horospheres and the stable part of the geodesic flow." In: *Math. Z.* 153 (1977), pp. 237–251.

[18] Jacques Faraut and Adam Korányi. *Analysis on symmetric cones.* Oxford: Clarendon Press, 1994, pp. xii + 382.

[19] A. Freire and R. Mane. "On the entropy of the geodesic flow in manifolds without conjugate points." In: *Invent. Math.* 69 (1982), pp. 375–392.

[20] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry. 3rd ed.* 3rd ed. Berlin: Springer, 2004, pp. xv + 322.

[21] M. Gromov. *Hyperbolic groups.* Essays in group theory, Publ., Math. Sci. Res. Inst. 8, 75-263 (1987). 1987.

[22] Allen Hatcher. *Algebraic topology.* Cambridge: Cambridge University Press, 2002, pp. xii + 544.

[23] Eberhard Hopf. "Closed surfaces without conjugate points." In: *Proc. Natl. Acad. Sci. USA* 34 (1948), pp. 47–51.

[24] José M. Isidro and Angel Rodríguez Palacios. "Isometries of JB-algebras." In: *Manuscr. Math.* 86.3 (1995), pp. 337–348.

[25] Hyun Jung Kim. "Conformal density of visibility manifold." In: *Bull. Korean Math. Soc.* 38.1 (2001), pp. 211–222.

[26] Inkang Kim. "Marked length rigidity of rank one symmetric spaces and their product." In: *Topology* 40.6 (2001), pp. 1195–1323.

[27] Junghwa Kim and Yongdo Lim. "Jordan automorphic generators of Euclidean Jordan algebras." In: *J. Korean Math. Soc.* 43.3 (2006), pp. 507–528.

[28] Gerhard Knieper. "The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds." In: *Ann. Math. (2)* 148.1 (1998), pp. 291–314.

[29] Bas Lemmens and Mark Roelands. "Les géodésiques uniques de la métrique de Thompson". In: *Ann. Inst. Fourier* 65.3 (2015), pp. 315–348.

[30] Y. Lim. "Riemannian and Finsler structures of symmetric cones". In: *Trends in Mathematics* 4 (2001), pp. 111–118.

[31] Yongdo Lim. "Geometric means on symmetric cones." In: *Arch. Math.* 75.1 (2000), pp. 39–45.

[32] Anthony Manning. "Topological entropy for geodesic flows." In: *Ann. Math. (2)* 110 (1979), pp. 567–573.

[33] G. A. Margulis. *On some aspects of the theory of Anosov systems. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows. Transl. from the Russian by S. V. Vladimirovna.* Berlin: Springer, 2004, p. 139.

[34] G.A. Margulis. "Applications of ergodic theory to the investigation of manifolds of negative curvature." In: *Funct. Anal. Appl.* 3 (1969), pp. 335–336.

[35] S. Mazur and S. Ulam. "Sur les transformations isometriques d'espaces vectoriels, normes." In: *C. R. Acad. Sci., Paris* 194 (1932), pp. 946–948.

[36] Lajos Molnár. "Thompson isometries of the space of invertible positive operators." In: *Proc. Am. Math. Soc.* 137.11 (2009), pp. 3849–3859.

[37] Sheldon E. Newhouse. "Continuity properties of entropy." In: *Ann. Math. (2)* 129.2 (1989), pp. 215–235.

[38] John J. O'Sullivan. "Manifolds without conjugate points." In: *Math. Ann.* 210 (1974), pp. 295–311.

[39] Jean-Pierre Otal. "Sur la géométrie symplectique de l'espace des géodésiques d'une variété à courbure négative. (On the symplectic geometry of the space of geodesics of a manifold with negative curvature)." In: *Rev. Mat. Iberoam.* 8.3 (1992), pp. 441–456.

[40] Gabriel P. Paternain. *Geodesic flows.* Boston, MA: Birkhäuser, 1999, pp. xii + 149.

[41] M. Paternain. "Expansivity and length expansivity for geodesic flows on surfaces." In: *Dynamical systems. Proceedings of the 3rd international school of dynamical systems, Santiago de Chile, 1990.* Harlow, Essex: Longman Scientific & Technical; New York: John Wiley & Sons, Inc., 1993, pp. 195–210.

[42] Miguel Paternain. "Expansive geodesic flows on surfaces." In: *Ergodic Theory Dyn. Syst.* 13.1 (1993), pp. 153–165.

[43] S.J. Patterson. "The limit set of a Fuchsian group." In: *Acta Math.* 136 (1976), pp. 241–273.

[44] Mark Pollicott. "Closed geodesic distribution for manifolds of non-positive curvature." In: *Discrete Contin. Dyn. Syst.* 2.2 (1996), pp. 153–161.

[45] Thomas Roblin. “Ergodicité et équidistribution en courbure négative.” In: *Mém. Soc. Math. Fr., Nouv. Sér.* 95 (2003), p. 96.

[46] Rafael O. Ruggiero. *Dynamics and global geometry of manifolds without conjugate points.* Rio de Janeiro: Sociedade Brasileira de Matemática, 2007, p. 181.

[47] Rafael O. Ruggiero. “Expansive dynamics and hyperbolic geometry.” In: *Bol. Soc. Bras. Mat., Nova Sér.* 25.2 (1994), pp. 139–172.

[48] Rafael O. Ruggiero. “Expansive geodesic flows in manifolds with no conjugate points.” In: *Ergodic Theory Dyn. Syst.* 17.1 (1997), pp. 211–225.

[49] Rafael Oswaldo Ruggiero. “On the divergence of geodesic rays in manifolds without conjugate points, dynamics of the geodesic flow and global geometry.” In: (2003), pp. 231–249.

[50] Dennis Sullivan. “The density at infinity of a discrete group of hyperbolic motions.” In: *Publ. Math., Inst. Hautes Étud. Sci.* 50 (1979), pp. 171–202.

[51] Jussi Väisälä. “A proof of the Mazur-Ulam theorem.” In: *Am. Math. Mon.* 110.7 (2003), pp. 633–635.

[52] Peter Walters. *An introduction to ergodic theory. Paperback ed.* Paperback ed. New York, NY: Springer, 2000, pp. ix + 250.

[53] J.D.Maitland Wright and M.A. Youngson. “On isometries of Jordan algebras.” In: *J. Lond. Math. Soc., II. Ser.* 17 (1978), pp. 340–344.