Representations of fundamental groups in hyperbolic geometry

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Représentations de groupes fondamentaux en géométrie hyperbolique

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Contents

Introduction

Exhibiting additional structures on manifolds provides information about them which are all the more remarkable as the manifolds are general. In particular, classification theorems of manifolds of some type, as Poincaré–Koebe uniformization theorem of Riemann surfaces or Thurston's geometrization conjecture of manifolds of three dimensions, are far-reaching results in modern mathematics. In the history of those results, what were at first considered as exceptional and singular examples proved to be rules. One may read the introduction of [Ota96] to appreciate the critical role of Riley's example of a hyperbolic structure on the complement of the figure-eight knot. Besides general classification results, specific examples have interests on their own, even though they may not fit in a larger picture.

Interesting additional structures are often of geometric nature. Let X be a manifold on which a group G acts by homeomorphisms, transitively and analytically, that is to say, two group elements whose actions coincide on an open subset must be equal. A (G, X)-structure on a manifold M is an atlas of charts with values in the *model space* X and whose transition mappings are restrictions of elements of G. A (G, X)-structure on a connected manifold M gives rise to a *developing map* $D : \tilde{M} \to X$, where $\tilde{M}$ is a universal cover of M, and a *holonomy representation* $\rho : \pi_1(M) \to G$ satisfying

$$D(\gamma \cdot y) = \rho(\gamma) \cdot D(y)$$

for every element γ in $\pi_1(M)$ and every point y in $\tilde{M}$. The pair (D, ρ) is unique, up to the joint action of G on D by post-composition and on ρ by conjugation. The developing map D of a (G, X)-structure is always a local homeomorphism. Whenever it is a covering map, the structure is called complete. In that case, $\rho(\pi_1(M))$ is a discrete subgroup of G, acting freely and properly discontinuously on X, and M is homeomorphic to the quotient manifold $\rho(\pi_1(M))\backslash X$, so that one says that M is uniformized by X. Note that, for any representation $\rho : \pi_1(M) \to G$, there is at most one complete (G, X)-structure on M with holonomy ρ.

For instance, if X is equipped with a Riemannian metric and that G is the group of isometries of X, then a (G, X)-structure on a manifold M is complete if M, equipped with the metric induced by that of X, is complete, which is automatically true when M is compact.

Given a representation $\rho : \pi_1(M) \to G$, one may ask whether M carries a (G, X)-structure with holonomy representation ρ. Answering this question provides a dictionary between algebra and geometry. For example, if M is a closed connected orientable surface of genus at least 2, then it is well known that the representations $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}}) \simeq \mathrm{PSL}_2(\mathbb{R})$ arising as holonomy representations of complete real hyperbolic structures on M — $(\mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}}), \mathbb{H}^2_{\mathbb{R}})$-structures — form two connected components of the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{R}))$. The quotient of each component by the action of $\mathrm{PSL}_2(\mathbb{R})$ by conjugation is naturally isomorphic to the Teichmüller space of M. Besides, by a theorem of Gallo–Kapovich–Marden [GKM00], every non-elementary representation in $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}_2(\mathbb{C}))$ is the holonomy of a complex projective structure — an $(\mathrm{Aut}(\mathbb{P}^1), \mathbb{P}^1)$-structure where $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$ — either unbranched or with a single branched point. Therefore, even though the existence of a representation $\rho : \pi_1(M) \to G$ does not imply that M carries a complete (G, X)-structure, such representations still provide relevant information about M.

The present thesis is situated in this general context. Chapter 1 presents a strategy to try to determine the representations of finitely generated free groups into any lattice in real Lie groups. Chapter 2 reviews a construction of a complex hyperbolic surface, that is the quotient of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ by a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, examines its properties carefully and yields infinitely many non-conjugate representations into $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, of fundamental groups of closed hyperbolic 3-dimensional manifolds, obtained as surface bundles over the circle.

Deforming representations

Thurston's hyperbolization theorem for surface bundles over the circle states that such bundles with pseudo-Anosov monodromy are hyperbolic. The virtual Haken conjecture (see [Ber15]) gave the reciprocal, up to a finite covering though. A key ingredient for the proof of the latter was the surface subgroup conjecture, stating that fundamental groups of closed hyperbolic manifolds contain many quasi-Fuchsian surface subgroups.

Theorem (Kahn–Marković [KM12b]). *Let M be a closed hyperbolic manifold of 3 dimensions, of the form $\Lambda \backslash \mathbb{H}^3_{\mathbb{R}}$ where Λ is a uniform lattice in $\mathrm{Isom}^+(\mathbb{H}^3_{\mathbb{R}})$. For any $C > 1$, there exist a hyperbolic closed surface S of the form $\pi \backslash \mathbb{H}^2_{\mathbb{R}}$, where π is a uniform lattice in $\mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$, and a C-quasi-conformal mapping $f : \partial_\infty \mathbb{H}^3_{\mathbb{R}} \to \partial_\infty \mathbb{H}^3_{\mathbb{R}}$ such that $f \circ \pi \circ f^{-1}$ is a subgroup of Λ, after identifying $\mathbb{H}^2_{\mathbb{R}}$ with a hyperbolic plane in $\mathbb{H}^3_{\mathbb{R}}$.*

This may be interpreted in the following ways.

- The hyperbolic closed surface S may be deformed into an *immersed surface* in M — in the sense of Tan [Tan94] — such that the induced morphism between fundamental groups is injective. The lattice Λ hence contains a surface subgroup.
- The Fuchsian representation $\pi \to \mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$ is close to a quasi-Fuchsian one $\pi \to \mathrm{Isom}^+(\mathbb{H}^3_{\mathbb{R}})$ whose image is contained in Λ.

The proof uses extensively the decomposition of hyperbolic closed surfaces into pairs of pants and the associated (complex) Fenchel–Nielsen parameters. It appeals to mixing properties of the geodesic flow on the bundle of all oriented orthonormal frames of M in order to find a lot of real hyperbolic pairs of pants with large and identical length parameters, immersed into M almost isometrically, and to show that a finite number of them may be glued together, with twist parameters close to 1, so as to form a closed surface. The construction was inspired by previous partial results of Lewis Bowen who, be it said in passing, used horocyclic flow instead of geodesic flow [Bow05].

The authors chose these very specific Fenchel–Nielsen parameters which allowed them to show that the obtained immersed surface is quasi-Fuchsian. Nevertheless the construction thus focuses on a particular kind of geometric structures on pairs of pants and on surfaces, so-called *skew pants* [KM12b, Ham15] or (R, ε)-*flat* [Ber13]. They actually focus rather on the geometric shape of the hyperbolic pair of pants than its holonomy representation.

Besides the theorem does not give any control on the genus of the surface S. The asymptotic behavior, relative to the genus, of the number of such immersed surfaces in M has been studied in another work by the same authors [KM12a].

In a previous attempt to prove the surface subgroup conjecture, Bowen obtained the following results [Bow09]. Let G be a locally compact topological group with a left-invariant metric d, Λ a lattice in G and ρ a representation in G of the *free group* π generated by a symmetric ($S^{-1} = S$) finite alphabet S. Following the terminology of the author, for any $\varepsilon > 0$, a mapping $\sigma : \pi \to G$ is an ε-*perturbation* of ρ if

$$\forall\, w \in \pi \quad \forall\, s \in S \quad d(\sigma(ws), \sigma(w)\rho(s)) \leq \varepsilon.$$

Such a mapping σ is said to be *virtually a homomorphism*, if π contains a finite-index subgroup π' such that

$$\forall\, w' \in \pi' \quad \forall\, w \in \pi \quad \sigma(w'w) = \sigma(w')\sigma(w)$$

and σ is said to be virtually a homomorphism into Λ if $\sigma(\pi')$ is in addition contained in Λ.

Theorem (Bowen). *If the lattice Λ is uniform, then for any $\varepsilon > 0$, any representation $\rho : \pi \to G$ admits an ε-perturbation that is virtually a homomorphism into Λ.*

Theorem (Bowen). *If G is the group $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ and $\rho : \pi \to G$ is an injective representation onto a convex cocompact subgroup, then for any $\varepsilon > 0$ there exists an ε-perturbation of $\rho : \pi \to G$ that is virtually a homomorphism into Λ.*

Both statements lead to the same conclusion. The difference lies in that the former is true in a very general context with the restriction that the lattice Λ must be uniform, whereas the latter is also true for non-uniform lattices but in a less general context.

Even though only free group representations are considered, these two results and especially the first one suggest that there could be an approach to the problem of determining the representations into a lattice that would deal with any kind of representations or, at least, with a large class of them. Besides, although the proof of the theorem of Kahn–Marković seems limited in that it relies on the geodesic flow on the frame bundle, this flow is simply the restriction to some one-parameter subgroup of an action of $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ on the frame bundle, which is nothing but the action of $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ on the quotient $\Lambda \backslash \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ by multiplication on the right.

Furthermore, the following result extends the theorem of Kahn–Marković, though using the same kind of arguments, and reinforces the idea of a general approach.

Theorem (Hamenstädt [Ham15]). *Let Λ be a uniform lattice in a simple rank one Lie group of non-compact type, distinct from $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ for all positive even integers n. Then Λ contains surface subgroups.*

In chapter 1, a strategy is proposed in that perspective. Although it is successful only for representations of free groups, it has the advantages of getting rid of technical limitations, of being applicable in a very general context and of adopting a unified treatment. Should the strategy succeed for surface groups, then it would lead to simplifications of some arguments for the theorems of Kahn-Marković and Hamenstädt.

The following propositions are the result of a attempt to simplify and generalize techniques at the heart the above results (see [Bow05, part II], [KM12b, lemmas 4.5, 4.6, 4.7], [Ber13, proposition 4.7], [Ham15, proposition 4.3]). The authors essentially show, by appealing to mixing properties of geodesic or horocyclic flow, that, for $\varepsilon > 0$ small enough and $R > 0$ large enough, any real hyperbolic pair of pants with identical length parameters R may be deformed into an immersed (R, ε)-flat pair of pants. In other words, the (R, ε)-flat pairs of pants are deformations of existing real hyperbolic structures on topological pairs of pants. In this spirit, it appears immediately in the statements of the following propositions that representations in the lattice are found as deformations of representations in the Lie group. The following propositions avoid the recourse to mixing properties, only for some time. In the end, the Howe-Moore theorem (see 1.2.9) allows to prove that, for any simple connected real Lie group G with finite center, there are a lot of representations of the free group into G satisfying the conditions of the propositions.

Let π be a group with a finite presentation $\langle S|R\rangle$. A group representation ρ of π in a Lie group G is exactly determined by a family $(g_s)_{s\in S}$ of elements in G satisfying the relations in R. Deforming ρ consists in finding, for each s in S, an element h_s in G close to g_s, such that the family $(h_s)_{s\in S}$ yet satisfies the relations in R. Although the product $h_{s_m}\cdots h_{s_1}$, for each relation $s_m\cdots s_1$ in R, must be close to the identity element in G, provided that h_s is close enough to g_s for each s in S, it is difficult to guarantee in general that those products are actually trivial. Now, whenever the elements h_s are chosen in a lattice Λ in G, then the products $h_{s_m}\cdots h_{s_1}$ would actually be trivial, if they are sufficiently close to the identity element — since Λ is discrete, — hence giving rise to a representation of π into Λ associated to the family $(h_s)_{s\in S}$. This observation is not surprising at all but requires to be able to estimate the distance from $h_{s_m}\cdots h_{s_1}$ to $g_{s_m}\cdots g_{s_1}$ with respect to the distances from h_s to g_s. The following propositions provide a quantitative statement.

Let X denote the quotient manifold G/K of a real Lie group G by a maximal compact subgroup K. One may easily construct a Riemannian metric on G which is invariant under the action by multiplication on the left by G and on the right by K. However, neither the metric nor the maximal compact subgroup are canonical. For instance, the maximal compact subgroups of the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ are exactly the stabilizers of points in $\mathbb{H}^n_{\mathbb{R}}$, isomorphic to $\mathrm{O}(n)$. Hence, for each point $x\in\mathbb{H}^n_{\mathbb{R}}$, one may construct a metric m_x on G satisfying the latter invariance properties.

More generally, there is a whole family $(m_x)_{x\in X}$ of Riemannian metrics on G invariant under the action of G by multiplication on the left and satisfying

$$(R_g)^* m_x = m_{gx}$$

for all g in G and x in X. In particular, if g belongs to the stabilizer K_x of x, $(R_g)^* m_x = m_x$ which means that m_x is invariant under the action of K_x by multiplication on the right. The distance function corresponding to the metric m_x is denoted by d_x.

Let Λ be a lattice in G. The injectivity radius at a point τ in $\Lambda\backslash G$, with respect to the distance d_x induced on $\Lambda\backslash G$ from G, is defined as

$$\mathrm{inj}_x(\tau) = \frac{1}{2}\inf_{\lambda\in\Lambda-\{1\}} d_x(\lambda\tilde{\tau},\tilde{\tau})$$

where $\tilde{\tau} \in G$ is any lift of τ.

In the first place, let π denote the free group $\langle S \rangle$ generated by a finite alphabet S.

Proposition (section 1.2.2). *For any $\varepsilon \leq 1$, any representation $\rho : \pi \to G$ and any point x in X, if there exists a point τ in $\Lambda \backslash G$ satisfying*

$$\forall\, s \in S \quad d_x(\tau, \tau\rho(s)) < \varepsilon \operatorname{inj}_x(\tau)$$

then, given any lift $\tilde{\tau}$ of τ to G, there is a unique representation $\sigma : \pi \to \Lambda$ close to ρ in the sense that

$$\forall\, s \in S \quad d_x(\sigma(s)\tilde{\tau}, \tilde{\tau}\rho(s)) < \varepsilon \operatorname{inj}_x(\tau).$$

In the second place, for some integer m greater than 1, let π denote the group with presentation $\langle c_1, \ldots, c_m | c_m \cdots c_1 = 1 \rangle$.

Proposition (section 1.2.2). *For any $\varepsilon \leq 3^{1/m} - 1$, any representation $\rho : \pi \to G$ and any family of points $(x_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ in X such that $x_{j+1} = \rho(c_j)x_j$ for all j in $\mathbb{Z}/m\mathbb{Z}$, if there exists a point τ in $\Lambda \backslash G$ satisfying*

$$\forall\, j \in \mathbb{Z}/m\mathbb{Z} \quad d_{x_j}(\tau, \tau\rho(c_j)) < \varepsilon \operatorname{inj}_{x_j}(\tau)$$

then, given any lift $\tilde{\tau}$ of τ to G, there is a unique representation $\sigma : \pi \to \Lambda$ close to ρ in the sense that

$$\forall\, j \in \mathbb{Z}/m\mathbb{Z} \quad d_{x_j}(\sigma(c_j)\tilde{\tau}, \tilde{\tau}\rho(c_j)) < \varepsilon \operatorname{inj}_{x_j}(\tau).$$

Such a statement is also true for the finitely presented groups each of whose generators appears at most once in the relations altogether. Unfortunately all these groups are actually free and no analogue is known for arbitrary finitely presented groups. Nevertheless, since these statements deal with presentations — and not groups themselves — and that the images of the generators c_j by σ still satisfy the relation $c_m \cdots c_1 = 1$, this slight progress may let one hope that it is also possible with a larger class of finitely presented groups.

The propositions are true for any lattice in any real Lie group and do not resort to any specific geometric technique like decomposition into pairs of pants or Fenchel–Nielsen parameters. They deal with uniform and non uniform lattices simultaneously on the contrary to the theorems of Bowen. There is no limitation on the representation ρ. Besides there is a significant difference with the point of view presented by Bowen, since *passing to a finite-index subgroup* is not needed anymore.

The following statement is an application of the Howe–Moore theorem (see 1.2.9) to the former proposition.

Theorem (see section 1.2.3)**.** *Let G be a simple connected real Lie group with finite center, Λ a lattice in G and π a finitely generated free group. Let $S = \{s_1, s_2 \ldots\}$ be some free generating set of π, x be a point in X and $\varepsilon \leq 1$. Any representation $\rho : \pi \to G$, such that $\rho(s_1)$ leaves some large enough compact set K_1 and that $\rho(s_2)$ leaves some large enough compact set K_2 depending on $\rho(s_1)$ and so on, admits a small deformation conjugate to a representation $\sigma : \pi \to \Lambda$: more precisely, there exist τ in $\Lambda\backslash G$ and a lift $\tilde{\tau}$ in G such that*

$$\forall\, s \in S \quad d_x(\sigma(s)\tilde{\tau}, \tilde{\tau}\rho(s)) < \varepsilon\, \mathrm{inj}_x(\tau).$$

Representations of 3-manifolds

In three dimensions, spherical Cauchy–Riemann structures are also interesting. Those are the $(\mathbb{S}^3, \mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}}))$-structures, where $\mathbb{S}^3$ is seen as the boundary at infinity $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$ of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$. These structures are not part of Thurston's eight geometries. A spherical CR structure on a manifold M is called *uniformizable* if there exists an open subset Ω of $\mathbb{S}^3$ on which $\rho(\pi_1(M))$ acts freely and properly discontinuously, so that M is homeomorphic to quotient manifold $\rho(\pi_1(M))\backslash\Omega$. Like the complete (G, X)-structures, for any representation $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, there is at most one uniformizable spherical CR structure on the manifold M with holonomy ρ. Given a spherical CR structure with holonomy ρ or just a representation ρ, a candidate open subset is the discontinuity domain of $\rho(\pi_1(M))$, that is the largest open subset of $\mathbb{S}^3$ on which $\rho(\pi_1(M))$ acts properly discontinuously. In particular, whenever the discontinuity domain is empty, then the representation ρ cannot be the holonomy representation of a uniformizable spherical CR structure.

Only few examples of 3-dimensional hyperbolic manifolds carrying such structures and not many more representations of fundamental groups into $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$ are known. For instance, if M is the complement of the figure-eight knot, Falbel has shown that there are essentially two representations of $\pi_1(M)$ into $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, that the author denotes by ρ_1 and ρ_2, whose boundary representations $\pi_1(\partial M) \to \mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$ are unipotent [Fal08]. The representation ρ_1 is not the holonomy of a uniformizable structure since the domain of discontinuity of its image is empty. However it is shown that ρ_1 is the holonomy of a branched spherical CR structure on the figure-eight knot. Later, Falbel and Wang have shown that the complement of the figure-eight knot admits a branched spherical CR structure with holonomy ρ_2 [FW14] and Deraux and Falbel have shown it admits a uniformizable spherical CR structure with holonomy ρ_2 [DF15].

Chapter 2 introduces a method for constructing infinitely many non-conjugate representations of fundamental groups of closed hyperbolic 3-

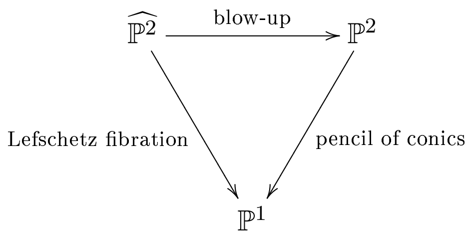
dimensional manifolds into a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$. The domain of discontinuity of those representations happens to be empty, so that they cannot arise as holonomies of uniformizable structures, unlike the example of Deraux-Falbel. Nevertheless, they still may be the holonomies of branched spherical CR structures.

Besides, since these representations take actually their values in a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, their existence may also be interpreted from the angle of the Kahn-Marković theorem.

The method relies on the careful examination of the properties of a complex hyperbolic surface, in section 2.1. It focuses on the particular example of Hirzebruch's surface Y_1, which was originally introduced as an example of a complex hyperbolic surface, that is the quotient of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ by a uniform lattice, isomorphic to $\pi_1(Y_1)$ [Hir83, YY84].

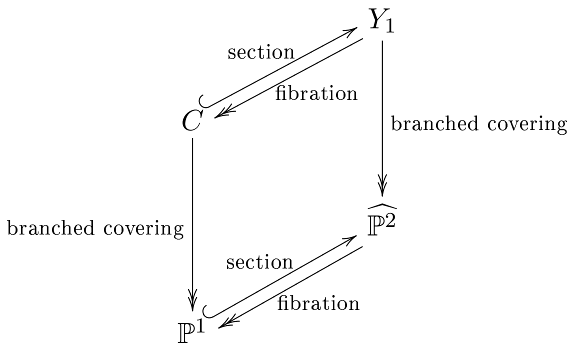
On the one hand, Y_1 is a branched covering space of degree 5^5 of a complex surface, denoted by $\widehat{\mathbb{P}^2}$, which is the blow-up of the complex projective plane $\mathbb{P}^2$ at 4 points (none three of which lie on the same line). The 6 lines in $\mathbb{P}^2$ passing through each pair among those 4 points form the *complete quadrilateral arrangement of lines* (see figure 2.1). Besides, the preimage by the blow-up $\widehat{\mathbb{P}^2} \to \mathbb{P}^2$ of each of the 4 points is isomorphic to the complex projective line $\mathbb{P}^1$. The branched covering map $Y_1 \to \widehat{\mathbb{P}^2}$ ramifies exactly over those $10 = 6 + 4$ lines in $\widehat{\mathbb{P}^2}$, with ramification index 5.

On the other hand, the conics in $\mathbb{P}^2$ passing through those 4 points give rise to a birational map $\mathbb{P}^2 \to \mathbb{P}^1$, called the *pencil of conics*. It lifts to a Lefschetz fibration $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ (see section 2.2).

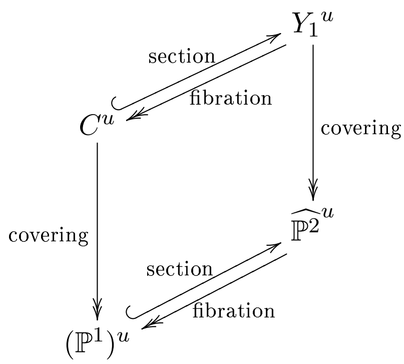


In addition, the Lefschetz fibration $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ admits sections $\widehat{\mathbb{P}^2} \hookleftarrow \mathbb{P}^1$. Furthermore, the union of the singular fibers under $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ consists of the proper transforms in $\widehat{\mathbb{P}^2}$ of the 6 lines of the complete quadrilateral arrangement in $\mathbb{P}^2$.

Finally, a Lefschetz fibration $Y_1 \to C$ over a complex curve C of genus 6 is derived as shown in the following commutative diagram (see proposition 2.4.1).



In particular, the branched covering map $Y_1 \to \widehat{\mathbb{P}^2}$ induces, by restriction, a branched covering map from each fiber under $Y_1 \to C$ into a fiber of $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$. Hence the properties of $Y_1 \to C$ may be read from those of $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$. The generic fibers of $Y_1 \to C$ are smooth curves of genus 76. There are also 4×5^2 singular fibers, each of which consists of 10 smooth curves intersecting normally at 5^2 points in total (see proposition 2.4.4). Denoting by ${Y_1}^u \to \widehat{\mathbb{P}^2}^u$ and $C^u \to (\mathbb{P}^1)^u$ the corresponding (unbranched) covering maps, one obtains the diagram



where there is neither ramification nor singular fibers anymore.

Section 2.3 is devoted to the careful study of the monodromy of the fibration $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$ (see corollary 2.2.7) and hence that of ${Y_1}^u \to C^u$ too. Since the fibers under $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$ are spheres with four punctures, the fibration induces a representation of $\pi_1((\mathbb{P}^1)^u)$ into the mapping class group $\mathrm{Mod}_{0,4}$ of a sphere, with 4 marked points. The monodromy representation proves to be an isomorphism and those groups are moreover isomorphic to the principal congruence subgroup $\Gamma(2)$ in $\mathrm{PSL}_2(\mathbb{Z})$ (of index 6). That fact has motivated the choice of the complex hyperbolic surface Y_1, so that the calculations and proofs are simpler than with more complicated mapping class groups. The elements in $\pi_1((\mathbb{P}^1)^u)$ whose images in $\mathrm{Mod}_{0,4}$ are *pseudo-Anosov* or reducible mapping classes are precisely determined: the classification corresponds to the classification of the elements of $\mathrm{PSL}_2(\mathbb{Z})$ as hyperbolic and parabolic elements ($\Gamma(2)$ contains no elliptic element).

Finally, let F_0 denote the generic fiber of $Y_1 \to C$. For any γ in $\pi_1(C^u)$, let M_γ denote the 3-dimensional manifold, obtained as the surface bundle over the circle with fiber F_0 and where the homeomorphism is the monodromy of the fibration ${Y_1}^u \to C^u$ along γ (see definition 2.3.7 and section 2.5). There is a natural mapping $M_\gamma \to Y_1$ and which induces a morphism

$$\rho_\gamma : \pi_1(M_\gamma) \to \pi_1(Y_1).$$

Since $\pi_1(Y_1)$ is isomorphic to a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, the morphism ρ_γ yields a representation into that lattice and in particular in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$.

It is remarkable that every mapping class in $\mathrm{Mod}_{0,4}$ can be realized as the monodromy along a curve in $(\mathbb{P}^1)^u$, of the fibration $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$. Since the generic fiber of $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ is a sphere with 4 marked points, all the possible surface bundles with the sphere as fiber and with monodromy preserving each of the 4 marked points are hence obtain in this way.
The same construction of surface bundles for the fibration $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$, instead of ${Y_1}^u \to C^u$ as above, produces representations of the fundamental groups of all those surface bundles. More precisely, the complex hyperbolic structure on Y_1 descends to a branched complex hyperbolic structure on $\widehat{\mathbb{P}^2}$ by the branched covering $Y_1 \to \widehat{\mathbb{P}^2}$. And the fibers of the latter surface bundles are seen as orbifolds with isotropy of order 5 at each of the four marked points. For γ in $\pi_1(C^u)$, the surface bundle M_γ is nothing but a branched covering of the orbifold surface bundle whose monodromy is the image of γ by $\pi_1(C^u) \to \pi_1((\mathbb{P}^1)^u)$.

Proposition. *For each element f of $\mathrm{Mod}_{0,4}$, consider the surface bundle M_f with monodromy f and with fiber the orbifold with the sphere as underlying space and with isotropy of order 5 at each of the four marked points. There is a representation of the orbifold fundamental group of M_f into a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$.*

Section 2.5 describes the manifold M_γ to a small extent, the group $\pi_1(M_\gamma)$ and properties of the representation ρ_γ with respect to the element γ in $\pi_1(C^u)$.

Proposition. *For any γ in $\pi_1(C^u)$, the limit set of the image of the representation $\rho_\gamma : \pi_1(M_\gamma) \to \pi_1(Y_1)$ is all of $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$.*

Proposition. *For any element γ in $\pi_1(C^u)$, if its image in $\pi_1(C)$ is not trivial, then*

1. *the kernel of ρ_γ is equal to the kernel of $\pi_1(F_0) \to \pi_1(Y_1)$,*

2. *the monodromy of the fibration ${Y_1}^u \to C^u$ along γ is pseudo-Anosov,*

3. *the kernel is not of finite type.*

Observe that, if the monodromy is pseudo-Anosov, then the surface bundle M_γ is a hyperbolic manifold, according to Thurston's hyperbolization theorem for surface bundles over the circle. In that case, the representation $\rho_\gamma : \pi_1(M_\gamma) \to \pi_1(Y_1)$ hence provides a representation of the fundamental group $\pi_1(M_\gamma)$ of the hyperbolic manifold M_γ, into a complex hyperbolic lattice.

Finally, the family of representations constructed in this way is the source of infinitely many conjugacy classes of representations of hyperbolic manifolds of three dimensions into a complex hyperbolic lattice.

Theorem. *For any two γ_1 and γ_2 in $\pi_1(C^u)$, if the image in $\pi_1(C)$ of γ_1 is not conjugate to that of γ_2 or its inverse, then either the groups $\pi_1(M_{\gamma_1})$ and $\pi_1(M_{\gamma_2})$ are not isomorphic or, if such an isomorphism $\Phi : \pi_1(M_{\gamma_1}) \to \pi_1(M_{\gamma_2})$ exists, then the representations ρ_{γ_1} and $\rho_{\gamma_2} \circ \Phi$ are not conjugate.*

Furthermore, the method seems reproducible with other complex hyperbolic lattices. Indeed, let Q_n be the quotient, in the sense of geometric invariant theory, of $(\mathbb{P}^1)^n$ by the diagonal action of $\mathrm{Aut}(\mathbb{P}^1)$. In other words, Q_n is the set of configurations of n marked points in the projective line. Let also Q_n^* denote the usual quotient, by the diagonal action of $\mathrm{Aut}(\mathbb{P}^1)$, of the subset of $(\mathbb{P}^1)^n$ formed by all the n-tuples of pairwise distinct points. The fibrations $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ and $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$ may actually be interpreted as the forgetful mappings $Q_5 \to Q_4$ and $Q_5^* \to Q_4^*$, respectively, which forget the last point of the configuration (see proposition 2.2.5). In passing, this observation explains morally the particular role of the fibration $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$.

It is remarkable that these spaces Q_n appear at the heart of the construction by Deligne–Mostow of complex hyperbolic lattices, as described below. The forgetful mappings $Q_n \to Q_p$ for $p < n$ (which forget, say, the last $n - p$ points of a configuration) provide natural fibrations for the Deligne–Mostow lattice quotients as well. Therefore, one might expect that the Deligne-Mostow lattices have the tendency to contain surface bundles, possibly with spherical CR structures.

There exist several constructions of complex hyperbolic lattices. The story has started with Picard at the end of the 19^{th} century and is still being written nowadays. Old and modern examples and construction methods cohabit. A glance at the survey of Parker [Par09] is sufficient to realize how rich this field is and the number of mathematicians it has attracted. Yet the relations between the variety of approaches are not clearly established. Some relatively recent points of view are worth one's attention [Tre16] and [McM13].

Originally, the Deligne–Mostow lattices were discovered by considering *hypergeometric functions*. Let $\mu = (\mu_1, \dots, \mu_n)$ be an n-tuple of real numbers in the interval $(0, 1)$ satisfying

$$\sum_{k=1}^{n} \mu_k = 2$$

and, for any distinct integers a and b of $\{1, \dots, n\}$, define

$$F_{ab}(z_1, \dots, z_n) = \int_{z_a}^{z_b} \prod_{k=1}^{n} (z - z_k)^{-\mu_k} \mathrm{d}z$$

where $z_1, \dots, z_n$ are elements in $\hat{\mathbb{C}}$ and the path of integration lies in $\hat{\mathbb{C}} - \{z_1, \dots, z_n\}$, apart from its end points. The functions F_{ab} are multi-valued functions, well defined if no two of the variables z_k coincide. Moreover, they span a vector space of dimension $n - 2$ and there exists a function h in the variables z_k such that

$$F_{ab}(\alpha(z_1), \dots, \alpha(z_n)) = h(z_1, \dots, z_n) F_{ab}(z_1, \dots, z_n)$$

for any $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ and for any distinct indices a and b. Therefore, one obtains a multi-valued mapping

$$F : \left\{ \begin{array}{ccc} Q_n^* & \longrightarrow & \mathbb{P}^{n-3} \\ Z = (z_1, \dots, z_n) & \longmapsto & [F_{a_1 b_1}(Z) : \dots : F_{a_{n-2} b_{n-2}}(Z)] \end{array} \right.$$

where $F_{a_1 b_1}, \dots, F_{a_{n-2} b_{n-2}}$ are linearily independent. Hence, F induces a *monodromy* representation from a fundamental group of Q_n^* onto a subgroup Γ_μ of $\mathrm{Aut}(\mathbb{P}^{n-3}) = \mathrm{PGL}_{n-2}(\mathbb{C})$. Furthermore, one may show that the monodromy preserves a Hermitian form of signature $(n-3, 1)$, so that Γ_μ is a subgroup of $\mathrm{PU}(n-3, 1) \simeq \mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^{n-3})$. Finally, if the n-tuple μ satisfies a integral condition, called ΣINT, then Deligne and Mostow show that the monodromy takes its values in a lattice [Par09, Theorem 3.2].

Note that the monodromy representation into $\mathrm{PU}(n-3, 1)$ exists, whether or not its image is a lattice. And the forgetful mappings provide fibrations. Therefore, even though the present construction focuses on the particular complex hyperbolic surface Y_1 and on the corresponding lattice in $\mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^{n-3})$, the method should generalize to a much larger class of surfaces bundles, so as to obtain representations of their fundamental groups into $\mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^2)$ and possibly spherical CR structures.

Chapter 1

Deforming representations

1.1 Real and complex hyperbolic spaces

A moral of the classification theorems of manifolds of dimensions two or three is that most of those manifolds are (real) hyperbolic.

1.1.1 Construction

Let $\mathbb{F}$ be the field of the real numbers $\mathbb{R}$, respectively the complex numbers $\mathbb{C}$, and consider a vector space V of dimension $n+1$ over $\mathbb{F}$ equipped with a non-degenerate bilinear, respectively sesquilinear, symmetric form $\phi : V \times V \to \mathbb{F}$ of signature $(1, n)$.

1.1.1 Example. The *Minkowski spacetime* $\mathbb{F}^{1,n}$ is the vector space $\mathbb{F}^{1+n}$, equipped with the form

$$\phi_{1,n}(w, z) = -\overline{w_0} z_0 + \overline{w_1} z_1 + \cdots + \overline{w_n} z_n.$$

Although the latter is the standard example from which the hyperbolic space is usually constructed, it is not canonical.

1.1.2 Notations. The group of orthogonal, respectively unitary, transformations of (V, ϕ) is denoted by $\mathrm{O}(V, \phi)$, respectively $\mathrm{U}(V, \phi)$. It consists of the linear automorphisms A of V preserving ϕ, in other terms, satisfying

$$\forall\, v, w \in V \quad \phi(Av, Aw) = \phi(v, w).$$

In particular, $\mathrm{O}(1, n)$, respectively $\mathrm{U}(1, n)$, denote the group of orthogonal, respectively unitary, transformations of $\mathbb{F}^{1,n}$. They all are real Lie groups.

The real hyperbolic space may be defined as follows. The level hypersurface

$$V_{-1} = \{v \in \mathbb{R}^{1+n} \mid \phi_{1,n}(v, v) = -1\}$$

is of a hyperboloid of two sheets, which are contained in the half spaces separated by the hyperplane of equation $z_0 = 0$. Moreover the scalar multiplication by -1 maps each component to the other. The upper component is the underlying space of what is known as the *hyperboloid model* of real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. Instead of choosing one of the components, since none is better than the other, it should in fact be defined as the quotient space $V_{-1}/\{\pm 1\}$.

In the complex case, the level hypersurface V_{-1} is a hyperboloid with one sheet. Furthermore, it is preserved by scalar multiplication by any complex number of modulus 1, so is it in the real case by scalar multiplication by ± 1. One may define complex hyperbolic space as the quotient space $V_{-1}/\mathbb{S}^1$, where $\mathbb{S}^1$ denotes the set of complex numbers of modulus 1.

However, the hypersurface V_{-1} of level -1 is not better than any other hypersurface of some negative level. Instead, real and complex hyperbolic spaces can be defined more canonically in the following way.

Consider the *light cone* V_0 and the *time cone* V_- defined as

$$V_0 = \{v \in V - \{0\} \mid \phi(v,v) = 0\} \quad \text{and} \quad V_- = \{v \in V \mid \phi(v,v) < 0\}.$$

These cones are stable by scalar multiplication. Moreover, they are preserved by the group of orthogonal, respectively unitary, transformations. Now consider the projection $\mathrm{P} : V - \{0\} \to \mathrm{P}(V)$ onto the projective space $\mathrm{P}(V) = (V - \{0\})/\mathbb{F}^*$.

1.1.3 Definition. The hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$ over $\mathbb{F}$ and of dimension n is $\mathrm{P}(V_-)$.

By definition, $\mathbb{H}^n_{\mathbb{F}}$ is an open subset of the projective space $\mathbb{P}^n_{\mathbb{F}}$ and is in particular a real, respectively complex, submanifold of dimension n.

1.1.4 Remark. If (V, ϕ) is chosen as the Minkowski spacetime $(\mathbb{F}^{1,n}, \phi_{1,n})$, then $P(V_-)$ is actually contained in the affine chart $U_0 = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n_{\mathbb{F}} \mid z_0 \neq 0\}$ where $[z_0 : \cdots : z_n]$ denotes the homogeneous coordinates in the projective space. The function

$$\begin{array}{ccc} U_0 & \longrightarrow & \mathbb{F}^n \\ [z_0 : \cdots : z_n] & \longmapsto & \left(\dfrac{z_1}{z_0}, \ldots, \dfrac{z_n}{z_0}\right) \end{array}$$

maps $\mathbb{H}^n_{\mathbb{F}}$ to the open unit ball centered at the origin, and gives rise to the *Klein ball model* of hyperbolic space.

1.1.5 Definition. The *boundary at infinity* of $\mathbb{H}^n_{\mathbb{F}}$, denoted by $\partial_\infty \mathbb{H}^n_{\mathbb{F}}$, is the topological boundary of $\mathbb{H}^n_{\mathbb{F}}$ in $\mathrm{P}(V)$. It is simply $\mathrm{P}(V_0)$.

1.1.6 Remark. In the Klein ball model, it corresponds to the unit sphere centered at the origin in the affine chart U_0.

The next step is to equip $\mathbb{H}^n_\mathbb{F}$ with a Riemannian, respectively Hermitian, metric. In the case of real hyperbolic space, viewed as the upper sheet of the level hypersurface V_{-1}, and in particular as a submanifold of V, the tangent space $\mathrm{T}_v\mathbb{H}^n_\mathbb{R}$, at a point v, is the kernel $\ker \phi_{1,n}(v,\cdot)$ of the linear form $w \mapsto \phi_{1,n}(v,w)$, which is simply the orthogonal $(\mathbb{R}v)^\perp$ of the line $\mathbb{R}v$. (Indeed, the differential at v of the mapping $v \mapsto \phi_{1,n}(v,v)$, whose hypersurface of level -1 is V_{-1}, is equal to $w \mapsto 2\phi_{1,n}(v,w)$.) Besides, the restriction of the bilinear symmetric form $\phi_{1,n}$, to the subspace $\mathrm{T}_v\mathbb{H}^n_\mathbb{R}$ of V, is positive definite: indeed, as $\phi_{1,n}$ is of signature $(1,n)$ and as it is negative definite on the line $\mathbb{R}v$ (for $\phi_{1,n}(v,v) = -1$), Sylvester's law of inertia guarantees that it is positive definite on the orthogonal $(\mathbb{R}v)^\perp$. Therefore, the restriction of the pseudo-Riemannian metric of V induces a Riemannian metric on $\mathbb{H}^n_\mathbb{R}$.

It remains to adapt the latter construction in the context of the previous definiton hyperbolic space $\mathbb{H}^n_\mathbb{F}$ on $\mathbb{F}$.
On the one hand, the trivial vector bundle $\mathrm{T}V_- = V_- \times V$ over V_- contains as a subbundle the bundle over V_- whose fiber over each point v is the orthogonal $(\mathbb{F}v)^\perp$ for the bilinear form ϕ. The pseudo-Riemannian, respectively pseudo-Hermitian, metric ϕ on the trivial bundle induces a Riemannian, respectively Hermitian, metric on the subbundle, for the same reason as before: since ϕ is negative definite on the line $\mathbb{F}v$, for all $v \in V_-$, its restriction to $(\mathbb{F}v)^\perp$ is positive definite. In addition, $(\mathbb{F}v)^\perp$ and hence the induced metric depend only on the class of v in $\mathrm{P}(V_-)$.
On the other hand, the differential $\mathrm{dP} : \mathrm{T}V_- \to \mathrm{T}\mathbb{H}^n_\mathbb{F}$ of the projection $\mathrm{P} : V_- \to P(V_-) = \mathbb{H}^n_\mathbb{F}$ induces, on each fiber of the subbundle, a surjective mapping $\mathrm{d}_v\mathrm{P} : (\mathbb{F}v)^\perp \to \mathrm{T}_v\mathbb{H}^n_\mathbb{F}$ which is an isomorphism for dimensional reasons. Consequently, pushing forward the Riemannian, respectively Hermitian, metric of the subbundle by the projection P yields a metric on $\mathbb{H}^n_\mathbb{F}$.

1.1.7 Definition. The *hyperbolic metric* on $\mathbb{H}^n_\mathbb{F}$ is the Riemannian, respectively Hermitian, metric construted as above. By a slight abuse of notations, it will still be denoted by ϕ.

Consider a vector subspace V' of V. More precisely, if $\mathbb{F}$ is $\mathbb{C}$, then V may been viewed not only as a vector space over $\mathbb{C}$, but also as a vector space over $\mathbb{R}$. Hence, for a unified treatment of the different cases, consider a subfield $\mathbb{F}'$ of $\mathbb{F}$ and a vector subspace V' of V, where V is viewed as a vector space over $\mathbb{F}'$. Assume moreover that the restriction of ϕ to V' is a bilinear form $\phi' : V' \times V' \to \mathbb{F}'$ that is non-degenerate and of signature $(1,n')$. Then the inclusion of (V',ϕ') into (V,ϕ) induces an isometric embedding of $\mathbb{H}^{n'}_{\mathbb{F}'}$ into $\mathbb{H}^n_\mathbb{F}$.

1.1.8 Examples.

1. The inclusion of $(\mathbb{F}'^{1,n'}, \phi_{1,n'})$ into $(\mathbb{F}^{1,n}, \phi_{1,n})$, mapping the standard basis of $\mathbb{F}'^{1,n'}$ to the first n' vectors of $\mathbb{F}^{1,n}$, induces an isometric embedding of $\mathbb{H}^{n'}_{\mathbb{F}'}$ into $\mathbb{H}^n_\mathbb{F}$.

2. For any two points x and y in $\mathbb{H}^n_{\mathbb{F}}$, represented by lines $\mathbb{F}v$ and $\mathbb{F}w$ in V, the restriction of ϕ to the vector subspace V' of V spanned by v and w is non-degenerate and of signature $(1,1)$. Therefore, there exists an isometric embedding of $\mathbb{H}^1_{\mathbb{F}}$ into $\mathbb{H}^n_{\mathbb{F}}$ passing through x and y.

3. Let x and y be any two points in $\mathbb{H}^1_{\mathbb{C}}$, represented by lines $\mathbb{C}v$ and $\mathbb{C}w$ in $\mathbb{C}^{1,1}$. Up to multiplying v or w by scalars, one may assume that $\phi_{1,1}(v,w)$ is a real number. Hence, the restriction of $\phi_{1,1}$ to the real vector subspace spanned by v and w is a real non-degenerate bilinear form of signature $(1,1)$. Therefore, there exists an isometric embedding of $\mathbb{H}^1_{\mathbb{R}}$ into $\mathbb{H}^1_{\mathbb{C}}$ passing through x and y.

4. According to the previous two examples, for any two points x and y in $\mathbb{H}^n_{\mathbb{F}}$, there exists an isometric embedding of $\mathbb{H}^1_{\mathbb{R}}$ into $\mathbb{H}^n_{\mathbb{F}}$ passing through x and y.

5. Note that $\mathbb{H}^1_{\mathbb{R}}$ is isometric to the Euclidean line $\mathbb{R}$ through

$$\gamma : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{H}^1_{\mathbb{R}} \\ t & \longmapsto & (\cosh t, \sinh t) \end{array} \right.$$

   where $\mathbb{H}^1_{\mathbb{R}}$ is identified with the upper sheet of the hypersurface V_{-1} of level -1 in $V = \mathbb{R}^{1,1}$. Indeed, the derivative $\gamma'(t) = (\sinh(t), -\cosh(t))$ satisfies $\phi_{1,1}(\gamma'(t), \gamma'(t)) = 1$.

1.1.9 Corollary. *The hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$ is geodesically complete.*

1.1.2 The group of isometries and the frame bundle

1.1.10 Proposition. *The projective orthogonal group*

$$\mathrm{PO}(V,\phi) = \mathrm{O}(V,\phi)/\{\pm 1\},$$

respectively the projective unitary group

$$\mathrm{PU}(V,\phi) = \mathrm{U}(V,\phi)/\mathbb{S}^1,$$

acts faithfully and isometrically on $\mathbb{H}^n_{\mathbb{F}}$.

Proof. The group $\mathrm{O}(V,\phi)$, respectively $U(V,\phi)$, acts on V_- equipped with the pseudo-Riemmanian, respectively pseudo-Hermitian, metric ϕ. For any element A in the group, as A is a linear transformation of V, its differential $\mathrm{d}_v A$ at a point v in V_- is A itself. Therefore, for any $w \in \mathrm{T}_v V_- = V$,

$$(A_*\phi)(w,w) = \phi((\mathrm{d}_v A)^{-1} w, (\mathrm{d}_v A)^{-1} w) = \phi(A^{-1}w, A^{-1}w) = \phi(w,w)$$

so that the action of the group preserves of metric on V_-.

Since the action on V_- maps lines to lines, the projection P transports it to an action on $\mathrm{P}(V_-) = \mathbb{H}^n_{\mathbb{F}}$. Moreover, since the metric on $\mathbb{H}^n_{\mathbb{F}}$ is the push-forward by P of the restriction of the metric ϕ to the subbundle and that the latter equation still holds with $w \in (\mathbb{F}v)^{\perp}$, it follows that the action of the group on $\mathbb{H}^n_{\mathbb{F}}$ is isometric as well.

Furthermore, an element A acts trivially on $\mathbb{H}^n_{\mathbb{F}}$ if, and only if, it preserves all the lines in V_- so that, for any $v \in V_-$, there exists a number λ_v such that $Av = \lambda_v v$. For any vectors v and w in V_-, up to replacing w by $-w$, one may assume that $\Re\phi(v, w) < 0$ so as to have

$$\phi(v + w, v + w) = \phi(v) + \phi(w) + 2\Re\phi(v, w) < 0$$

and $v + w \in V_-$. Now, since

$$\lambda_{v+w}(v + w) = A(v + w) = Av + Aw = \lambda_v v + \lambda_w w$$

it follows that $\lambda_v = \lambda_{v+w} = \lambda_w$. Thus A is a homothety (whose ratio is necessarily of modulus 1). Therefore, the projective orthogonal, respectively unitary, group acts faithfully on $\mathbb{H}^n_{\mathbb{F}}$. □

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian, respectively Hermitian, manifold of n dimensions.

1.1.11 Definition. The *frame bundle* $\mathcal{F}(M)$ over M, or more precisely the bundle of all orthonormal, respectively Hermitian, frames over M is the bundle over M whose fiber over any point x of M consists of the orthonormal, respectively Hermitian, bases of the tangent space $\mathrm{T}_x M$ at the point x.

The frame bundle $\mathcal{F}(M)$ is a fiber bundle over M: it is the pull-back of the identity matrix I_n by the submersion

$$\begin{array}{ccc} (\mathrm{T}M)^{\oplus n} & \longrightarrow & \mathrm{Sym}(n, \mathbb{F}) \\ v_1 \oplus \cdots \oplus v_n & \longmapsto & (\langle v_i, v_j \rangle)_{i,j=1..n} \end{array}$$

where $(\mathrm{T}M)^{\oplus n}$ denotes the n-times direct-sum vector bundle of the tangent bundle $\mathrm{T}M$ and $\mathrm{Sym}(n, \mathbb{F})$ the space of symmetric real, respectively anti-symmetric complex, n-by-n matrices.

1.1.12 Remark. It is of course possible to consider the bundle of all frames (not necessarily orthonormal or Hermitian) but only the bundle of orthonormal, respectively Hermitian, frames is relevant in the present context. For convenience, the adjectives orthonormal or Hermitian will be omitted.

1.1.13 Example. Each element of the frame bundle of $\mathbb{H}^n_{\mathbb{F}}$ corresponds to the data of a line $\mathbb{F}v_0$ in V_- and to n vectors $v_1, \ldots, v_n$ forming an orthonormal, respectively Hermitian, basis of $(\mathbb{F}v_0)^{\perp}$. Up to multiplying v_0 by a scalar, one may assume that $\phi(v_0, v_0) = -1$. It is remarquable that the family $(v_0, v_1, \ldots, v_n)$ is a basis of V such that the unique linear isomorphism $V \to \mathbb{F}^{1,n}$ mapping that basis to the standard basis of $\mathbb{F}^{1,n}$, is an isometry $(V, \phi) \to (\mathbb{F}^{1,n}, \phi_{1,n})$.

Any isometry f of $(M, \langle\cdot,\cdot\rangle)$ acts on $\mathcal{F}(M)$ by

$$f(v_1 \oplus \cdots \oplus v_n) = (\mathrm{d}_x f\, v_1) \oplus \cdots \oplus (\mathrm{d}_x f\, v_n)$$

mapping any frame $v_1 \oplus \cdots \oplus v_n$ over a point x of M to a frame over $f(x)$. In particular, for any orthogonal, respectively unitary, linear transformation A of V,

$$A(v_1 \oplus \cdots \oplus v_n) = (Av_1) \oplus \cdots \oplus (Av_n).$$

1.1.14 Proposition. *The group of isometries* $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ *of the hyperbolic space is isomorphic to the group* $\mathrm{PO}(V, \phi)$*, respectively* $\mathrm{PU}(V, \phi)$*, and acts freely and transitively on* $\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$ *and hence transitively on* $\mathbb{H}^n_{\mathbb{F}}$*.*

1.1.15 Remark. One should insists on the fact that, in the case of a Hermitian manifold, an isometry, strictly speaking, is required to preserve the Hermitian metric and not simply the induced Riemannian one. For example, the group of Riemannian isometries of $\mathbb{H}^n_{\mathbb{C}}$ contains, in addition to $\mathrm{PU}(V, \phi)$, anti-holomorphic isometries.

Proof. According to example 1.1.13, any frame of $\mathbb{H}^n_{\mathbb{F}}$ corresponds to a basis $(v_0, v_1, \ldots, v_n)$ of V such that the unique linear isomorphism $V \to \mathbb{F}^{1,n}$ mapping that basis to the standard basis of $\mathbb{F}^{1,n}$, is an isometry $(V, \phi) \to (\mathbb{F}^{1,n}, \phi_{1,n})$. Thus, for any two frames of $\mathbb{H}^n_{\mathbb{F}}$, corresponding to such bases $\mathcal{B}$ and $\mathcal{B}'$ of V, the unique linear isomorphism mapping $\mathcal{B}$ to $\mathcal{B}'$ is an orthogonal, respectively unitary, transformation. It follows that the projective orthogonal, respectively unitary, group acts transitively on $\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$.

Let f be an isometry of $\mathbb{H}^n_{\mathbb{F}}$. Up to composing f by a projective orthogonal, respectively unitary, transformation, one may assume that f preserves a frame over a point x in $\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$. Consequently, the differential $\mathrm{d}_x f$ of f at x is the identity mapping of the tangent space $\mathrm{T}_x\mathbb{H}^n_{\mathbb{F}}$. For any geodesic γ passing through x and directed by a tangent vector v at x, $f \circ \gamma$ is the geodesic passing through $f(x) = x$ and directed by $\mathrm{d}_x f(v) = v$. Therefore $f \circ \gamma = \gamma$, so that f is the identity mapping. Finally, any isometry of $\mathbb{H}^n_{\mathbb{F}}$ is a projective orthogonal, respectively unitary, transformation and $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ acts freely on the frame bundle. □

Given some frame τ over a point x of $\mathbb{H}^n_{\mathbb{F}}$, the action de $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ on $\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$ and on $\mathbb{H}^n_{\mathbb{F}}$ induces a bijection

$$\begin{array}{ccc} \mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}}) & \longrightarrow & \mathcal{F}(\mathbb{H}^n_{\mathbb{F}}) \\ g & \longmapsto & g \cdot \tau \end{array}$$

and a surjective mapping

$$\begin{array}{ccc} \mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}}) & \longrightarrow & \mathbb{H}^n_{\mathbb{F}} \\ g & \longmapsto & g \cdot x \end{array}$$

which however depend on the choice x and τ. The stabilizer of x, denoted by K_x, is isomorphic to the group $\mathrm{O}(n)$, respectively $\mathrm{U}(n)$, as it consists of the transformations mapping an orthonormal, respectively Hermitian, frame over x to another. In particular, K_x is compact. Consequently, the diagram

$$\begin{array}{ccc} \mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}}) & \xrightarrow{\sim} & \mathcal{F}(\mathbb{H}^n_{\mathbb{F}}) \\ \downarrow & & \downarrow \\ \mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})/K_x & \xrightarrow{\sim} & \mathbb{H}^n_{\mathbb{F}} \end{array}$$

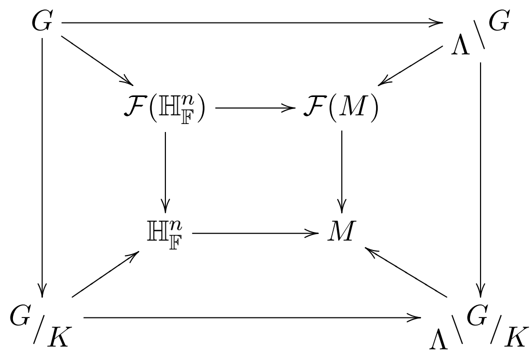
where the vertical arrows are the projection mappings and the horizontal ones are induced by the action on $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$, is commutative. Note that all these arrows are equivariant for the actions of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$

- on itself by translation on the left,
- on the frame bundle $\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$,
- on $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})/K_x$ by translation on the left,
- on the hyperbolic space $\mathbb{H}^n_{\mathbb{F}}$ by isometry.

Assume that the manifold M is connected. In the Riemannian case, if the manifold M is moreover orientable, then the frame bundle consists of two connected components: the frames of M may indeed be classified in positive and negative frames. In the Hermitian case, the frame bundle of M is connected.

In particular, since $\mathbb{H}^n_{\mathbb{R}}$ and $\mathbb{H}^n_{\mathbb{C}}$ are connected, the frame bundle $\mathcal{F}(\mathbb{H}^n_{\mathbb{R}})$ and the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ consist of two connected components, whereas the frame bundle $\mathcal{F}(\mathbb{H}^n_{\mathbb{C}})$ and the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$ are connected. The group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ is isomorphic to $\mathrm{PO}(1,n)$ and the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{C}})$ is isomorphic to $\mathrm{PU}(1,n)$. The identity component of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ is isomorphic to projective special orthogonal group $\mathrm{PSO}(1,n)$ and is denoted by $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ whereas the other component is denoted by $\mathrm{Isom}^-(\mathbb{H}^n_{\mathbb{R}})$.

Whenever the Riemannian, respectively Hermitian, manifold is isometric to a quotient of the form $\Lambda\backslash\mathbb{H}^n_{\mathbb{F}}$ where Λ is a lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$, then the fiber bundle of M is nothing but $\Lambda\backslash\mathcal{F}(\mathbb{H}^n_{\mathbb{F}})$. The following commutative diagram draws the complete picture of the spaces at stake and of the mappings between them.



The group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ is denoted by G for aesthetic reasons and K denotes the stabilizer K_x of a point x in $\mathbb{H}^n_{\mathbb{F}}$. In the diagram, the diagonal arrows are diffeomorphisms; the horizontal ones are covering maps, with automorphism group Λ; the vertical ones are principal K-bundles. Those mappings however depend on the choice of the point x in $\mathbb{H}^n_{\mathbb{F}}$ and of the frame τ over x, as well as on the isometry between M and $\Lambda\backslash\mathbb{H}^n_{\mathbb{F}}$ or rather on the covering map $\mathbb{H}^n_{\mathbb{F}} \to M$.

1.1.3 Invariant metrics, volume forms and measures

Let G be a real Lie group and K a compact subgroup of G. For instance, G may be $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ and K the stabilizer K_x of a point x in $\mathbb{H}^n_{\mathbb{F}}$.

The aim of the present paragraph is to show that G may be equipped with a Riemannian metric (and with the corresponding volume form and measure) satisfying remarkable invariance properties. As a consequence, the action of G on the quotient G/K, by multiplication on the left, proves to be isometric. The action of the group of isometries of the hyperbolic space is merely a particular case.

1.1.16 Proposition. *The group G may be equipped with a Riemannian metric, invariant on the left by G and on the right by K. The metric induced by that of G on the subgroup K is invariant on the left and on the right by K.*

Proof. Let ψ^G_e be some inner product on the tangent space T_eG to G at its identity element e. Its push-forward $(L_g)_*\psi^G_e$ by the left translation $L_g : h \mapsto gh$ by g is an inner product on T_gG, denoted by ψ^G_g. Hence, ψ^G is a Riemannian metric on G. And by construction, it is invariant on the left:

$$(L_g)_*\psi^G_h = (L_g)_*(L_h)_*\psi^G_e = (L_{gh})_*\psi^G_e = \psi^G_{gh} = \psi^G_{L_gh}$$

so that $(L_g)_*\psi^G = \psi^G$. Let dvol_{ψ^G} denote the volume form on G associated to ψ^G. The corresponding measure vol_{ψ^G} is a Radon measure (locally finite and inner regular).

The metric ψ^K induced on the subgroup K is also invariant on the left. And similarly, the measure corresponding to the volume form dvol_{ψ^K} is a Radon measure. In particular, its total measure is finite, since K is compact. Hence, one may define the mean ϕ^G of ψ^G, under the action of K by translation on the right, as

$$\phi^G_g = \frac{1}{\mathrm{vol}_{\psi^K}(K)} \int_K (R_k)_* \psi^G_{gk} \,\mathrm{dvol}_{\psi^K}(k)$$

where $R_k : h \mapsto hk^{-1}$ is the translation on the right. And by construction, the metric ϕ^G is invariant (under the action of K) on the right:

$$\begin{aligned}
\mathrm{vol}_{\psi^K}(K)\ (R_k)_*\phi^G_g &= (R_k)_* \left(\int_K (R_h)_* \psi^G_{gh} \,\mathrm{dvol}_{\psi^K}(h) \right) \\
&= \int_K (R_k)_*(R_h)_* \psi^G_{gh} \,\mathrm{dvol}_{\psi^K}(h) \\
&= \int_K (R_{L_k(h)})_* \psi^G_{R_k(g)L_k(h)} \,\mathrm{dvol}_{\psi^K}(h) \\
&= \int_K (R_h)_* \psi^G_{R_k(g)h} \underbrace{(L_k)_*\mathrm{dvol}_{\psi^K}(h)}_{\mathrm{dvol}_{\psi^K}(h)} = \mathrm{vol}_{\psi^K}(K)\ \phi^G_{R_k(g)}
\end{aligned}$$

so that $(R_k)_*\phi^G = \phi^G$. □

1.1.17 Proposition. *The metric ϕ^G on G induces a metric on G/K, denoted $\phi^{G/K}$, invariant on the left by G.*

In other words, the group G acts isometrically on the Riemannian manifold G/K.

Proof. Let $\pi : G \to G/K$ denote the projection mapping, which is a submersion: for any g in G, $\mathrm{d}_g\pi : \mathrm{T}_gG \to \mathrm{T}_{gK}(G/K)$ est surjective. As π is constant on the submanifold gK of G, the restriction of $\mathrm{d}_g\pi$ to $\mathrm{T}_g(gK)$ is zero. And for dimensional reasons, the restriction

$$\mathrm{d}_g\pi : \mathrm{T}_g(gK)^\perp \longrightarrow \mathrm{T}_{gK}(G/K)$$

is an isomorphism, where $\mathrm{T}_g(gK)^\perp$ denotes the orthogonal of $\mathrm{T}_g(gK)$ for the inner product ϕ^G_g on T_gG. Therefore, the push-forward by π of the restriction of ϕ^G to the vector subbundle of $\mathrm{T}G$, whose fiber over g is $\mathrm{T}_g(gK)^\perp$, is a metric on G/K. Indeed, it suffices to verify that $\pi_*\left(\phi^G_{g\,|\mathrm{T}_g(gK)^\perp}\right)$ depends only on the class of g in G/K: now, for any $k \in K$,

$$\pi_*\left(\phi^G_{gk|\mathrm{T}_{gk}(gK)^\perp}\right) = \pi_*(R_{k^{-1}})_*\left(\phi^G_{g\,|\mathrm{T}_g(gK)^\perp}\right) = (\underbrace{\pi \circ R_{k^{-1}}}_{\pi})_*\left(\phi^G_{g\,|\mathrm{T}_g(gK)^\perp}\right)$$

for ϕ^G_g is invariant on the right by K. □

1.1.18 Example. Recall the identification between G/K_x and $\mathbb{H}^n_{\mathbb{F}}$, so that the Riemannian metric ϕ^{G/K_x} may be transported to $\mathbb{H}^n_{\mathbb{F}}$. But the hyperbolic space is already equipped with its original Riemannian metric ϕ constructed above (if $\mathbb{F}$ is $\mathbb{C}$, take the real part of ϕ). In fact, the original metric ϕ and the one induced by that of G/K_x are proportional.

Indeed, as the transported metric ϕ^{G/K_x} is invariant under the action (on the left) of G and in particular of K_x, considering an orthonormal, respectively Hermitian, frame of the tangent space $\mathrm{T}_x\mathbb{H}^n_{\mathbb{F}}$ for the original metric ϕ_x, the number $\phi_x^{G/K_x}(v)$ of any vector v of the frame does not depend on v. Furthermore, for any two vectors v and w of the frame, as there exists an element of K_x mapping $v+w$ on $\sqrt{2}v$,

$$\phi_x^{G/K_x}(v+w) = 2\phi_x^{G/K_x}(v) = \phi_x^{G/K_x}(v) + \phi_x^{G/K_x}(w)$$

and thus $\phi_x^{G/K_x}(v,w) = 0$. Therefore the frame is also orthogonal for ϕ_x^{G/K_x}, up to a scalar factor. Consequently, the original metric ϕ and the transported one ϕ^{G/K_x} on $\mathrm{T}_x\mathbb{H}^n_{\mathbb{F}}$ are proportional. And since they are invariant on the left by the action de G, they are proportional on all of $\mathbb{H}^n_{\mathbb{F}}$, with the same scalar factor.

Up to dividing by that factor, one may assume that the metric on $\mathbb{H}^n_{\mathbb{F}}$ transported from G/K_x coincides with the original one.

1.1.4 Subgroups

Let $\overline{\mathbb{H}^n_{\mathbb{F}}}$ denote $\mathbb{H}^n_{\mathbb{F}} \cup \partial_\infty \mathbb{H}^n_{\mathbb{F}}$.

1.1.19 Lemma. *Let (γ_n) be a sequence in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$. If there exists a point x in $\mathbb{H}^n_{\mathbb{F}}$ such that*

$$\begin{cases} \lim_{n\to\infty} \gamma_n x = \lambda^+ \\ \lim_{n\to\infty} {\gamma_n}^{-1} x = \lambda^- \end{cases}$$

where λ^+ and λ^- are on the boundary at infinity $\partial_\infty \mathbb{H}^n_{\mathbb{F}}$, then

$$\begin{cases} \lim_{n\to\infty} \gamma_n y = \lambda^+ & \text{for any } y \in \overline{\mathbb{H}^n_{\mathbb{F}}} \setminus \{\lambda^-\}, \\ \lim_{n\to\infty} {\gamma_n}^{-1} y = \lambda^- & \text{for any } y \in \overline{\mathbb{H}^n_{\mathbb{F}}} \setminus \{\lambda^+\}. \end{cases}$$

Proof. By symmetry, it suffices to prove the first limit.

Let y and z be two distinct points in $\overline{\mathbb{H}^n_{\mathbb{F}}} \setminus \{\lambda^-\}$ and let x be a point on the geodesic $[y,z]$ joining y and z. Observe that

$$d({\gamma_n}^{-1}x, [y,z]) = d(x, [\gamma_n y, \gamma_n z]).$$

The left-hand side converges to $+\infty$ since ${\gamma_n}^{-1}x$ converges to λ^- and that y and z are distinct from λ^-. Assume that some subsequence $(\gamma_{\varphi(n)}y)$ converges to a limit a distinct from λ^+. Since the sequence $(\gamma_{\varphi(n)}x)$ converges to λ^+ and that the point $\gamma_{\varphi(n)}x$ belongs to the geodesic joining $\gamma_{\varphi(n)}y$ and $\gamma_{\varphi(n)}z$, $(\gamma_{\varphi(n)}z)$ must converge to λ^+. Moreover, $d(x,(\gamma_n y,\gamma_n z))$ converges to $d(x,(a,\lambda^+))$ which is finite except if $a=\lambda^+$. Therefore, $(\gamma_n y)$ converges to λ^+ as soon as y is distinct from λ^-. □

1.1.20 Definition. The *limit set* $\Lambda(\Gamma)$ of a subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ is the set $\overline{\Gamma x}\cap\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ of accumulation points in $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ of the orbit Γx of any point x in $\mathbb{H}^n_{\mathbb{F}}$.

1.1.21 Proposition. *If Γ does not preserve a point in $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ and that its limit set is not empty, then $\Lambda(\Gamma)$ is the smallest non-empty closed subset of $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ that is invariant by Γ.*

Proof. The limit set $\Lambda(\Gamma)$ is closed and invariant by Γ. Assume that $\Lambda(\Gamma)$ is not empty.

Let λ^+ be a point in $\Lambda(\Gamma)$. There exists a sequence (γ_n) in Γ such that λ^+ is the limit of $(\gamma_n x)$ for some point x in $\mathbb{H}^n_{\mathbb{F}}$. Up to passing to a subsequence, one may assume that $({\gamma_n}^{-1}x)$ converges to a point λ^- in $\overline{\mathbb{H}^n_{\mathbb{F}}}$. If λ^- were in $\mathbb{H}^n_{\mathbb{F}}$, then observing that

$$d({\gamma_n}^{-1}x,\lambda^-)=d(x,\gamma_n\lambda^-)$$

one would conclude that the sequence $(\gamma_n\lambda^-)$ converges to x, which contradicts the previous lemma. Therefore, λ^- belongs to $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$.

Let L a closed subset of $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ invariant by Γ. If Γ does not preserve a point at infinity, L does not consists of a single point. For any point ℓ in L, distinct from λ^-, $(\gamma_n\ell)$ converges to λ^+. Therefore, λ^+ belongs to L so that L contains $\Lambda(\Gamma)$. □

1.1.22 Examples.

1. The limit set $\Lambda(\Gamma)$ is empty if and only if the orbit Γx of some point x is contained in a compact subset of $\mathbb{H}^n_{\mathbb{F}}$. Now, since $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ acts properly on $\mathbb{H}^n_{\mathbb{F}}$, it implies that Γ is contained in a compact subgroup. Thus, $\Lambda(\Gamma)$ is empty if and only if Γ stabilizes some point x in $\mathbb{H}^n_{\mathbb{F}}$.

2. Let Γ_1 and Γ_2 be subgroups of $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$. If Γ_1 is a subgroup of Γ_2, then $\Lambda(\Gamma_1)\subset\Lambda(\Gamma_2)$.

3. The limit set of a uniform lattice Γ is all of $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$. Indeed, let x and y be any two points in $\mathbb{H}^n_{\mathbb{F}}$. Since the quotient $\Gamma\backslash\mathbb{H}^n_{\mathbb{F}}$ is compact, there exists an element γ in Γ such that $d(x,\gamma y)\leq D$, where D denotes the diameter of the quotient $\Gamma\backslash\mathbb{H}^n_{\mathbb{F}}$. Therefore, any point of $\mathbb{H}^n_{\mathbb{F}}$ is at a bounded distance from the orbit Γx of some point x and hence any point of $\partial_\infty\mathbb{H}^n_{\mathbb{F}}$ is arbitrarily close to the orbit Γx.

1.1.23 Proposition. *Let Γ_1 and Γ_2 be subgroups of* $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ *such that Γ_1 is a normal subgroup of Γ_2. If $\Lambda(\Gamma_1)$ is not empty and that Γ_2 does not preserve any point on the boundary at infinity, then $\Lambda(\Gamma_1)=\Lambda(\Gamma_2)$.*

Proof. Since Γ_1 is a subgroup of Γ_2, $\Lambda(\Gamma_1) \subset \Lambda(\Gamma_2)$. In particular, $\Lambda(\Gamma_2)$ is not empty either and, as Γ_2 does not preserve any point on the boundary at infinity, $\Lambda(\Gamma_2)$ is the smallest non-empty closed open subset invariant by Γ_2.

Let λ_1 be a point in $\Lambda(\Gamma_1)$ and (γ_n) a sequence in Γ_1 such that, for some point x in $\mathbb{H}^n_{\mathbb{F}}$, $(\gamma_n x)$ converges to λ_1. Then, for any γ in Γ_2, $(\gamma_n \gamma^{-1} x)$ converges to λ_1 as well and $(\gamma\gamma_n\gamma^{-1}x)$ converges to $\gamma\lambda_1$. Observe that the elements of the sequence $(\gamma\gamma_n\gamma^{-1})$ belong to Γ_1 since Γ_1 is a normal subgroup of Γ_2. Therefore, $\gamma\lambda_1$ belongs to $\Lambda(\Gamma_1)$, so that $\Lambda(\Gamma_1)$ is invariant by Γ_2. Finally, $\Lambda(\Gamma_1)$ is a non-empty closed subset invariant by Γ_2, which implies that $\Lambda(\Gamma_2) \subset \Lambda(\Gamma_1)$. $\square$

1.2 Deformation

1.2.1 Setting

Let π be a group with a finite presentation $\langle S|R\rangle$. A group representation ρ of π in a Lie group G is exactly determined by a family $(g_s)_{s\in S}$ of elements in G satisfying the relations in R. Deforming ρ consists in finding, for each s in S, an element h_s in G close to g_s, such that the family $(h_s)_{s\in S}$ yet satisfies the relations in R. Although the product $h_{s_m}\cdots h_{s_1}$, for each relation $s_m\cdots s_1$ in R, must be close to the identity element in G, provided that h_s is close enough to g_s for each s in S, it is difficult to guarantee in general that those products are actually trivial. Now, whenever the elements h_s are chosen in a lattice Λ in G, then the products $h_{s_m}\cdots h_{s_1}$ would actually be trivial, if they are sufficiently close to the identity element — since Λ is discrete, — hence giving rise to a representation of π into Λ associated to the family $(h_s)_{s\in S}$. This observation is not surprising at all but requires to be able to estimate the distance from $h_{s_m}\cdots h_{s_1}$ to $g_{s_m}\cdots g_{s_1}$ with respect to the distances from h_s to g_s. The following propositions provide a quantitative statement.

One may easily construct a Riemannian metric on a real Lie group G which is invariant under the action by multiplication on the left by G and on the right by a (maximal) compact subgroup K. However, neither the metric nor the compact subgroup are canonical. For instance, the maximal compact subgroups of the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{F}})$ are exactly the stabilizers of points in $\mathbb{H}^n_{\mathbb{F}}$. Hence, for each point $x \in \mathbb{H}^n_{\mathbb{F}}$, one may construct a metric m_x on G satisfying the latter invariance properties. In general, let X denote the quotient manifold G/K of the Lie group G by a compact subgroup K. The stabilizer of any point x in X of the form gK, under the action of G by multiplication on the left, is the compact subgroup gKg^{-1}, denoted by K_x. Given a point x in X, one may construct a Riemannian metric m_x on G invariant by multiplication on the left by G and on the right by K_x. For any other point, an analogous metric may be produced simply by pulling back m_x by the mapping $R_g : h \mapsto hg$. The pulled-back metric $(\mathrm{R}_g)^* m_x$

satisfies the same properties as the original one with the difference that it is now invariant by multiplication on the right by the group gK_xg^{-1} (instead of K_x) which is nothing but the stabilizer K_{gx} of the point gx.

In conclusion, G may be equipped with a family $(m_x)_{x\in X}$ of metrics, each of which is invariant by multiplication on the left by G and satisfies

$$(R_g)^* m_x = m_{gx}$$

for all g in G and x in X. In particular, if g belongs to K_x, $(R_g)^* m_x = m_x$. The distance function corresponding to the metric m_x is denoted by d_x.

Besides, for every point x, as the action of G on X on the left induces a canonical diffeomorphism between X and G/K_x and as the metric m_x is invariant under the action of K_x on the right, m_x induces a Riemannian metric on X which does not actually depend on the point x since $(R_g)^* m_x = m_{gx}$. On the other hand, as the metric m_x is invariant under the action of G by multiplication on the left, the action of G on X is isometric. For example, when G is the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$, it is not difficult to show that the metric induced on $\mathbb{H}^n_{\mathbb{R}}$ by the latter construction coincides with the hyperbolic metric, up to a scalar factor.

Although the metrics m_x are different in general, they are equivalent. Moreover, since a Lie group G containing a lattice is unimodular, the metrics m_x induce the same volume form on G. Furthermore, when G is the group $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$, for any points x and y in $\mathbb{H}^n_{\mathbb{R}}$, one may show that

$$e^{-d(x,y)} d_x \leq d_y \leq e^{d(x,y)} d_x$$

where d is the hyperbolic distance function on $\mathbb{H}^n_{\mathbb{R}}$, by considering for example the geodesic flow (on the unit tangent bundle) in the direction of the unit vector at x directed towards y. This illustrates the fact that the geodesic flow is contracting in some directions and dilating in others. Nevertheless, this inequality is not directly relating to the geodesic flow, but should probably be generalized in the context of real Lie groups.

Finally, the family $(d_x)_{x\in X}$ of distances is interesting because they allow to estimate quantitatively how distorted is a product when the multiplied elements are perturbed.

1.2.1 Proposition. *Let $g_1, \ldots, g_m$ and $h_1, \ldots, h_m$ be elements in G. And let x be a base point. Then*

$$d_x(g_m \cdots g_1, h_m \cdots h_1) \leq \sum_{i=1}^{m} d_{g_{i-1}\cdots g_1 x}(g_i, h_i).$$

Note that the distance function that comes into play to compute the distance between g_i and h_i is the one based at $g_{i-1} \cdots g_1 x$ and $g_{i-1} \cdots g_1$ is the suffix of $g_m \cdots g_1$ after g_i.

Proof. The triangle inequality implies that

$$d_x(g_m \cdots g_1, h_m \cdots h_1) \leq \sum_{i=1}^{m} d_x(h_m \cdots h_{i+1} g_i \cdots g_1, h_m \cdots h_i g_{i-1} \cdots g_1).$$

Now, since the distance functions are invariant by multiplication on the left,

$$d_x(h_m \cdots h_{i+1} g_i \cdots g_1, h_m \cdots h_i g_{i-1} \cdots g_1) = d_x(g_i \cdots g_1, h_i g_{i-1} \cdots g_1)$$

and by construction of the family of metrics

$$d_x(g_i \cdots g_1, h_i g_{i-1} \cdots g_1) = (\mathrm{R}_{g_{i-1} \cdots g_1}{}^* d_x)(g_i, h_i) = d_{g_{i-1} \cdots g_1 x}(g_i, h_i)$$

and the result follows. □

1.2.2 Corollary. *Let $g_1, \ldots, g_m$ and $h_1, \ldots, h_m$ be elements in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$. And let $x, x_1, \ldots, x_m$ be base points in $\mathbb{H}^n_{\mathbb{R}}$. Then*

$$d_x(g_m \cdots g_1, h_m \cdots h_1) \leq \sum_{i=1}^{m} e^{d(x_i, g_{i-1} \cdots g_1 x)} d_{x_i}(g_i, h_i)$$

and particularly when $x = x_1 = \cdots = x_n$,

$$d_x(g_m \cdots g_1, h_m \cdots h_1) \leq \sum_{i=1}^{m} e^{d(x, g_{i-1} \cdots g_1 x)} d_x(g_i, h_i)$$

Lattice and injectivity radius

Let Λ be a lattice in G, that is, a discrete subgroup such that the quotient $\Lambda \backslash G$ is of finite volume.

1.2.3 Remark. When G is $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ and that the quotient $\Lambda \backslash \mathbb{H}^n_{\mathbb{R}}$ is a manifold, its frame bundle may be identified with $\Lambda \backslash \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$, so does the frame bundle of $\mathbb{H}^n_{\mathbb{R}}$ with $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$. The geodesic flow on the frame bundle appears as a restriction of the action of $\mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ on $\Lambda \backslash \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ by multiplication on the right. Nevertheless, $\Lambda \backslash \mathbb{H}^n_{\mathbb{R}}$ need not be a manifold in order to consider the manifold $\Lambda \backslash \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$ and the action by multiplication on the right.

In general, each metric m_x, since it is invariant by multiplication on the left, induces a metric on the quotient $\Lambda \backslash G$ and, by a slight abuse of notation, d_x will indifferently denote the original distance function on G and the induced one on $\Lambda \backslash G$.

1.2.4 Definition. The *injectivity radius* at a point τ in $\Lambda \backslash G$, with respect to the distance d_x based at a point $x \in X$, is defined as

$$\mathrm{inj}_x(\tau) = \frac{1}{2} \inf_{\lambda \in \Lambda - \{1\}} d_x(\lambda \tilde{\tau}, \tilde{\tau})$$

where $\tilde{\tau} \in G$ is any lift of τ.

1.2.5 Properties. *The function* inj_x *is positive, Lipschitz continuous with constant* 1 *and with respect to the distance* d_x *and moreover satisfies*

$$\mathrm{inj}_x(\tau g) = (\mathrm{R}_g{}^* \mathrm{inj}_x)(\tau) = \mathrm{inj}_{gx}(\tau).$$

One of its interests lies in that an inequality of the form

$$d_x(\lambda\tilde{\tau}, \tilde{\tau}) < 2\,\mathrm{inj}_x(\tau)$$

guarantees that the element $\lambda \in \Lambda$ in question is trivial.

Proof. On the one hand, the function inj_x is positive because Λ is discrete. On the other hand, for all τ and τ' in $\Lambda\backslash G$, choosing any lifts $\tilde{\tau}$ and $\tilde{\tau}'$ such that $d_x(\tilde{\tau}', \tilde{\tau}) = d_x(\tau', \tau)$,

$$\mathrm{inj}_x(\tau) \leq \frac{1}{2} \inf_{\lambda\in\Lambda-\{1\}} \Big(d_x(\lambda\tilde{\tau}, \lambda\tilde{\tau}') + d_x(\lambda\tilde{\tau}', \tilde{\tau}') + d_x(\tilde{\tau}', \tilde{\tau}) \Big) \leq \mathrm{inj}_x(\tau') + d_x(\tau', \tau)$$

so that, by symmetry,

$$|\,\mathrm{inj}_x(\tau') - \mathrm{inj}_x(\tau)| \leq d_x(\tau', \tau).$$

Finally, $\mathrm{inj}_x(\tau g) = \frac{1}{2} \inf_{\lambda\in\Lambda-\{1\}} d_x(\lambda\tilde{\tau}g, \tilde{\tau}g) = \frac{1}{2} \inf_{\lambda\in\Lambda-\{1\}} d_{gx}(\lambda\tilde{\tau}, \tilde{\tau}) = \mathrm{inj}_{gx}(\tau)$. □

If the lattice Λ is uniform, that is, the quotient $\Lambda\backslash G$ is compact, then the function inj_x has a positive minimum. Moreover, since $\mathrm{R}_g{}^* \mathrm{inj}_x = \mathrm{inj}_{gx}$, this minimum does not depend on x and may simply be denoted by $\mathrm{inj}(\Lambda)$. In particular, an inequality of the form $d_x(\lambda\tilde{\tau}, \tilde{\tau}) < 2\,\mathrm{inj}(\Lambda)$ implies that λ is trivial. Therefore, the uniform case seems simpler. In general, the use of the functions inj_x ought to be necessary to deal with non uniformity.

1.2.2 Criteria to deform representations into a lattice

Of free groups

Let π denote the free group $\langle S\rangle$ generated by a finite *alphabet* S.

1.2.6 Proposition. *For any* $\varepsilon \leq 1$*, any representation* $\rho : \pi \to G$ *and any point* x *in* X*, if there exists a point* τ *in* $\Lambda\backslash G$ *satisfying*

$$\forall\, s \in S \quad d_x(\tau, \tau\rho(s)) < \varepsilon\,\mathrm{inj}_x(\tau)$$

then, given any lift $\tilde{\tau}$ *of* τ *to* G*, there is a unique representation* $\sigma : \pi \to \Lambda$ *close to* ρ *in the sense that*

$$\forall\, s \in S \quad d_x(\sigma(s)\tilde{\tau}, \tilde{\tau}\rho(s)) < \varepsilon\,\mathrm{inj}_x(\tau).$$

1.2.7 Remarks. 1. When $\tilde{\tau}$ is the identity element of G, the last inequality becomes $d_x(\sigma(s), \rho(s)) < \varepsilon \operatorname{inj}_x(\tau)$ which simply means that ρ and σ are close points in $\operatorname{Hom}(\pi, G)$ regarded as G^S. When $\tilde{\tau}$ is arbitrary, it means that the conjugacy classes of ρ and σ are close points in the topological quotient $\operatorname{Hom}(\pi, G)/G$.

2. It is not judicious to confine $\tilde{\tau}$ to be the identity element, since it would break the symmetry that exists between x, τ and ρ. Indeed, for any x, τ, ρ satisfying the condition of the proposition and for any g in G, the elements gx, τg^{-1}, $g\rho g^{-1}$ satisfy the condition as well.

3. This fact also answers the question about how the condition of the proposition depends on the point x. Changing the point x comes down to changing τ and ρ as described above. Thus, x may be chosen once and for all, so that the only unknown is τ. In particular, the representation ρ stays in the same conjugacy class.

Proof. For each generator s in S, since

$$d_x(\tau, \tau\rho(s)) = \inf_{\lambda \in \Lambda - \{1\}} d_x(\lambda\tilde{\tau}, \tilde{\tau}\rho(s))$$

there exists an element λ_s in Λ such that

$$d_x(\lambda_s\tilde{\tau}, \tilde{\tau}\rho(s)) = d_x(\tau, \tau\rho(s)) < \varepsilon \operatorname{inj}_x(\tau).$$

The element λ_s is unique. Indeed, if λ'_s were another element of Λ satisfying the latter inequality, then

$$d_x(\lambda_s^{-1}\lambda'_s\tilde{\tau}, \tilde{\tau}) = d_x(\lambda'_s\tilde{\tau}, \lambda_s\tilde{\tau}) < 2\varepsilon \operatorname{inj}_x(\tau) \leq \inf_{\lambda \in \Lambda - \{1\}} d_x(\lambda\tilde{\tau}, \tilde{\tau})$$

would imply that $\lambda_s^{-1}\lambda'_s = 1$, that is $\lambda_s = \lambda'_s$. Finally the morphism $\sigma : \pi \to \Lambda$, defined on the generators of S by $\sigma(s) = \lambda_s$, satisfies the conclusion of the proposition. □

Of a fundamental group of a pair of pants

Let π denote the presented group $\langle c_1, \ldots, c_m | c_m \cdots c_2 c_1 = 1 \rangle$ (with m greater than 1) which can be naturally viewed as a fundamental group of a surface diffeomorphic to the complement in a sphere of m open discs (whose closures are disjoint). When m is three, such a surface is often called a *pair of pants*. Although the group in question may be seen as a free group with $m - 1$ generators, one has to keep in mind that the group comes with a specific presentation.

1.2.8 Proposition. *For any $\varepsilon \le 3^{1/m} - 1$, any representation $\rho : \pi \to G$ and any family of points $(x_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ in X such that $x_{j+1} = \rho(c_j)x_j$ for all j in $\mathbb{Z}/m\mathbb{Z}$, if there exists a point τ in $\Lambda\backslash G$ satisfying*

$$\forall\, j \in \mathbb{Z}/m\mathbb{Z} \quad d_{x_j}(\tau, \tau\rho(c_j)) < \varepsilon\, \mathrm{inj}_{x_j}(\tau)$$

then, given any lift $\tilde{\tau}$ of τ to G, there is a unique representation $\sigma : \pi \to \Lambda$ close to ρ in the sense that

$$\forall\, j \in \mathbb{Z}/m\mathbb{Z} \quad d_{x_j}(\sigma(c_j)\tilde{\tau}, \tilde{\tau}\rho(c_j)) < \varepsilon\, \mathrm{inj}_{x_j}(\tau).$$

Proof. For each j in $\mathbb{Z}/m\mathbb{Z}$, since

$$d_{x_j}(\tau, \tau\rho(c_j)) = \inf_{\lambda \in \Lambda} d_{x_j}(\lambda\tilde{\tau}, \tilde{\tau}\rho(c_j))$$

there exists λ_j in Λ satisfying $d_{x_j}(\lambda_j\tilde{\tau}, \tilde{\tau}\rho(c_j)) = d_{x_j}(\tau, \tau\rho(c_j)) \le \varepsilon\, \mathrm{inj}_{x_j}(\tau)$. Each element λ_j is unique since $\varepsilon \le 3^{1/m} - 1 \le \sqrt{3} - 1 \le 1$.

In order to define the representation σ on the generators as $\sigma(c_j) = \lambda_j$, the elements λ_j must satisfy the relation $\lambda_m \cdots \lambda_2\lambda_1 = 1$. It suffices to show that $d_{x_1}(\lambda_m \cdots \lambda_1\tilde{\tau}, \tilde{\tau}) < 2\, \mathrm{inj}_{x_1}(\tau)$. Now, for all $1 \le p \le q \le m$,

$$\begin{aligned}
&d_{x_p}(\lambda_q \cdots \lambda_p\tilde{\tau}, \tilde{\tau}\rho(c_q \cdots c_p)) \\
&\le \sum_{j=p}^{q} d_{x_p}(\lambda_q \cdots \lambda_j\tilde{\tau}\rho(c_{j-1} \cdots c_p), \lambda_q \cdots \lambda_{j+1}\tilde{\tau}\rho(c_j \cdots c_p)) \\
&= \sum_{j=p}^{q} d_{\underbrace{\rho(c_{j-1} \cdots c_p)x_p}_{x_j}}(\lambda_j\tilde{\tau}, \tilde{\tau}\rho(c_j)) \\
&< \varepsilon \sum_{j=p}^{q} \mathrm{inj}_{x_j}(\tau)
\end{aligned}$$

Now $|\,\mathrm{inj}_{x_{j+1}}(\tau) - \mathrm{inj}_{x_j}(\tau)| = |\,\mathrm{inj}_{x_j}(\tau\rho(c_j)) - \mathrm{inj}_{x_j}(\tau)| \le d_{x_j}(\tau, \tau\rho(c_j))$ implies that $(1-\varepsilon)\,\mathrm{inj}_{x_j}(\tau) < \mathrm{inj}_{x_{j+1}}(\tau) < (1+\varepsilon)\,\mathrm{inj}_{x_j}(\tau)$ and thus

$$\begin{aligned}
d_{x_p}(\lambda_q \cdots \lambda_p\tilde{\tau}, \tilde{\tau}\rho(c_q \cdots c_p)) &< \varepsilon(1 + (1+\varepsilon) + \cdots + (1+\varepsilon)^{(q-p)})\,\mathrm{inj}_{x_p}(\tau) \\
&= ((1+\varepsilon)^{(q-p+1)} - 1)\,\mathrm{inj}_{x_p}(\tau).
\end{aligned}$$

In particular, for $p = 1$ and $q = m$, since $\rho(c_m \cdots c_1) = 1$,

$$d_{x_1}(\lambda_m \cdots \lambda_1\tilde{\tau}, \tilde{\tau}) < ((1+\varepsilon)^m - 1)\,\mathrm{inj}_{x_p}(\tau).$$

Finally it suffices to have $(1+\varepsilon)^m - 1 \le 2$ which happens exactly when $\varepsilon \le 3^{1/m} - 1$. □

1.2.3 Application of mixing

Since one is interested in Lie groups in some generality, a suitable statement about mixing is the following theorem of Roger Howe and Calvin Moore [Ben09, Theorem 3.2].

1.2.9 Theorem. *Let G be a semi-simple connected real Lie group with finite center and $R : G \to U(\mathcal{H})$ be a unitary representation of G in a Hilbert space $\mathcal{H}$ whose inner product is denoted by $\langle \cdot \mid \cdot \rangle$. If the subspace $\mathcal{H}^N$ of N-invariant vectors is $\{0\}$ for every non-trivial connected normal subgroup N of G, then for all v, w in $\mathcal{H}$,*

$$\lim_{g \to \infty} \langle R(g)\, v \mid w \rangle = 0.$$

A fundamental example of such a representation is that of the Hilbert space $L^2(\Lambda\backslash G)$ of square-integrable functions on the quotient $\Lambda\backslash G$. Indeed, the action of G on this space is defined, for all $v \in L^2(\Lambda\backslash G)$ and $\tau \in \Lambda\backslash G$, as

$$(g \cdot v)(\tau) = v(\tau g).$$

The action is unitary because G contains a lattice, so it is unimodular and the Haar measure is invariant on both sides. Since the constant functions are obviously G-invariant, the condition of the theorem is far from being satistied and one should instead consider the hyperplan $\mathcal{H}$ orthogonal to the constant functions, often denoted by $L^2_0(\Lambda\backslash G)$. Yet there may be non-zero N-invariant vectors if G has non-trivial connected normal subgroups N. This difficulty may be easily avoided by assuming that the connected real Lie group G is simple, that is, its Lie algebra is simple. In that case, the condition of theorem 1.2.9 boils down to $\mathcal{H}^G = \{0\}$ which is satisfied since the action of G on $\Lambda\backslash G$ is transitive so that any vector in $\mathcal{H}$ invariant by G must be constant and hence trivial.

1.2.10 Corollary. *Let Λ be a lattice in a simple connected real Lie group G with finite center. Then for all v, w in $L^2(\Lambda\backslash G)$, the quantity*

$$\int_{\Lambda\backslash G} v(\tau) w(\tau g)\, \mathrm{d}\tau \qquad \text{converges to} \qquad \frac{1}{\mathrm{vol}(\Lambda\backslash G)} \int_{\Lambda\backslash G} v \int_{\Lambda\backslash G} w$$

when g goes to infinity (leaves every compact set).

In order to invoke the previous statement, a little preparation is needed. First, let π denote the free group $\langle S \rangle$ generated by a finite alphabet S.

1.2.11 Lemma. *For any $\varepsilon \in (0,1]$, any representation $\rho : \pi \to G$ and any point x in X, if there exist a point τ in $\Lambda\backslash G$ and a family of points $(\tau_s)_{s \in S}$ in $\Lambda\backslash G$ satisfying*

$$\forall\, s \in S \quad d_x(\tau_s, \tau) < \frac{\varepsilon}{2+\varepsilon}\, \mathrm{inj}_x(\tau_s) \quad \text{and} \quad d_x(\tau_s, \tau\rho(s)) < \frac{\varepsilon}{2+\varepsilon}\, \mathrm{inj}_x(\tau_s)$$

then ρ, x and τ satisfy the condition of proposition 1.2.6.

Proof. For each generator s,

$$\mathrm{inj}_x(\tau_s) \leq \mathrm{inj}_x(\tau) + d_x(\tau, \tau_s) < \mathrm{inj}_x(\tau) + \frac{\varepsilon}{2+\varepsilon}\,\mathrm{inj}_x(\tau_s)$$

therefore

$$\frac{2}{2+\varepsilon}\,\mathrm{inj}_x(\tau_s) < \mathrm{inj}_x(\tau)$$

and $d_x(\tau, \tau\rho(s)) \leq d_x(\tau, \tau_s) + d_x(\tau_s, \tau\rho(s)) \leq \dfrac{2\varepsilon}{2+\varepsilon}\,\mathrm{inj}_x(\tau_s) \leq \varepsilon\,\mathrm{inj}_x(\tau)$. □

Although this lemma seems insignificant and its proof elementary, its interest lies in that τ and $\tau\rho(s)$ do not appear anymore as variables of the same function.

As for the presented group $\langle c_1, \ldots, c_m | c_m \cdots c_2 c_1 = 1\rangle$, a similar statement is true.

1.2.12 Lemma. *For any $\varepsilon \in (0, 3^{1/m} - 1]$, any representation $\rho : \pi \to G$ and any family of points $(x_j)_{j\in\mathbb{Z}/m\mathbb{Z}}$ in X such that $x_{j+1} = \rho(c_j)x_j$ for all j in $\mathbb{Z}/m\mathbb{Z}$, if there exist a point τ and a family of points $(\tau_j)_{j\in\mathbb{Z}/m\mathbb{Z}}$ in $\Lambda\backslash G$ satisfying*

$$d_{x_j}(\tau_j, \tau) \leq \frac{\varepsilon}{2+\varepsilon}\,\mathrm{inj}_{x_j}(\tau_j) \quad \text{and} \quad d_{x_j}(\tau_j, \tau\rho(c_j)) \leq \frac{\varepsilon}{2+\varepsilon}\,\mathrm{inj}_{x_j}(\tau_j)$$

for all j in $\mathbb{Z}/m\mathbb{Z}$, then ρ, $(x_j)_{j\in\mathbb{Z}/m\mathbb{Z}}$ and τ satisfy the condition of proposition 1.2.8.

The second lemma is not going to be used but is nevertheless stated in order to persuade that the obstacles preventing from dealing with arbitrary finitely presented groups are perhaps not related to this part of the reasoning. Besides the care taken so far to keep a tight rein on the quantificators is shattered by the non-quantitative statements about mixing. Hence there is no real interest anymore to distinguish between different presentations of the free group.

1.2.13 Theorem. *Let G be a simple connected real Lie group with finite center, Λ a lattice in G and π a finitely generated free group. Let $S = \{s_1, s_2 \ldots\}$ be some free generating set of π, x be a point in X and $\varepsilon \leq 1$. Any representation $\rho : \pi \to G$, such that $\rho(s_1)$ leaves some large enough compact set K_1 and that $\rho(s_2)$ leaves some large enough compact set K_2 depending on $\rho(s_1)$ and so on, admits a small deformation conjugate to a representation $\sigma : \pi \to \Lambda$: more precisely, there exist τ in $\Lambda\backslash G$ and a lift $\tilde{\tau}$ in G such that*

$$\forall\, s \in S \quad d_x(\sigma(s)\tilde{\tau}, \tilde{\tau}\rho(s)) < \varepsilon\,\mathrm{inj}_x(\tau).$$

Proof. Let $S = \{s_1, s_2 \dots\}$ be a free generating set of π, x be a point in X and ε be in $(0,1]$. Denote by $k_\varepsilon : (\Lambda\backslash G)^2 \to \mathbb{R}_+$ the function defined as $k_\varepsilon(\tau', \tau) = 1$ when

$$d_x(\tau', \tau) < \frac{\varepsilon}{2+\varepsilon} \operatorname{inj}_x(\tau')$$

and 0 otherwise. Then a representation $\rho : \pi \to G$ satisfies the condition of lemma 1.2.11 and hence that of proposition 1.2.6, whenever the integral

$$\int_{\Lambda\backslash G} \frac{\mathrm{d}\tau}{\operatorname{vol}(\Lambda\backslash G)} \int_{(\Lambda\backslash G)^S} \prod_{s\in S} k_\varepsilon(\tau_s, \tau) k_\varepsilon(\tau_s, \tau\rho(s)) \, \mathrm{d}\tau_s$$

is positive. Since the integrand is bounded between 0 and 1 and that $\Lambda\backslash G$ is of finite measure, Lebesgue's dominated convergence theorem and corollary 1.2.10 imply that

$$\lim_{\rho(s_1)\to\infty} \lim_{\rho(s_2)\to\infty} \cdots \int_{\Lambda\backslash G} \frac{\mathrm{d}\tau}{\operatorname{vol}(\Lambda\backslash G)} \int_{(\Lambda\backslash G)^S} \prod_{s\in S} k_\varepsilon(\tau_s, \tau) k_\varepsilon(\tau_s, \tau\rho(s)) \, \mathrm{d}\tau_s$$
$$= \int_{(\Lambda\backslash G)^S} \left(\left[\int_{\Lambda\backslash G} \prod_{s\in S} k_\varepsilon(\tau_s, \tau) \frac{\mathrm{d}\tau}{\operatorname{vol}(\Lambda\backslash G)} \right] \prod_{s\in S} \left[\int_{\Lambda\backslash G} k_\varepsilon(\tau_s, \tau) \frac{\mathrm{d}\tau}{\operatorname{vol}(\Lambda\backslash G)} \right] \mathrm{d}\tau_s \right)$$

where the successive limits $\lim_{\rho(s_1)\to\infty}$, $\lim_{\rho(s_2)\to\infty}$ and so on, are taken for every generator s in S, in the order of their indices s_1, s_2... Since the limit is positive, it means that if $\rho(s_1)$ leaves some compact set K_1 and that $\rho(s_2)$ leaves some compact set K_2 depending on $\rho(s_1)$ and so on, then the point x and the representation ρ satisfy the condition of lemma 1.2.11 and thus there exists a representation $\sigma : \pi \to \Lambda$ close to ρ, up to conjugacy, in the sense of proposition 1.2.6. $\square$

Chapter 2

Representations of 3-manifolds

Some notations, terminology and properties about the complex projective space, blow-ups in local charts and branched covering maps are given in appendices A.1 and A.2.

2.1 A complex hyperbolic surface

This section presents a particular construction of smooth complex algebraic surfaces, studied by Hirzebruch [Hir83]. Namba gave a more developed treatment [Nam87, section 1.4, example 6]. See also [Tre16] for generalizations. These algebraic varieties are obtained by resolving the singularities of some branched covering spaces of the complex projective plane. Under some conditions (see theorem 2.1.13), the surfaces happen to be quotients of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ by a lattice.

2.1.1 Construction of Hirzebruch

Consider an arrangement of a number k (greater than 2) of lines $D_1, \ldots, D_k$ in $\mathbb{P}^2$ whose equations are respectively $\ell_1 = 0, \ldots, \ell_k = 0$ where $\ell_1, \ldots, \ell_k$ are linear forms in the homogeneous coordinates z_1, z_2, z_3. Assume that not all lines of the arrangement pass through one point. And let n be an integer greater than 1.

2.1.1 Example. The *complete quadrilateral arrangement* in $\mathbb{P}^2$ is formed by the lines connecting each pair among four points in general position, that is to say, no three of them are colinear. There are three double intersection points and four triple ones which are the initial four points.

Any such four points are equivalent up to a projective transformation. Indeed, any three of the lines, not having a common triple point, give an affine coordinate system and, in suitable homogeneous coordinates $[z_1 : z_2 : z_3]$, the arrangement is given by the equation

$$z_1 z_2 z_3 (z_2 - z_1)(z_3 - z_2)(z_1 - z_3) = 0$$

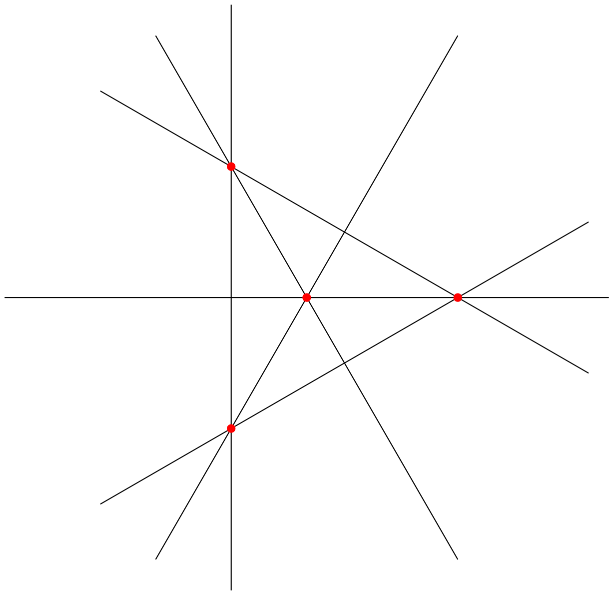


Figure 2.1: The complete quadrilateral arrangement.

and the four triple points by $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$. If one sets $z_4 = 0$, then the arrangement consists of the lines D_{ab} defined by the equation $z_a - z_b = 0$ where $\{a, b\}$ is any (unordered) subset of $\{1, 2, 3, 4\}$.

2.1.2 Remark. In [YY84], the authors set $z_0 = 0$ instead of $z_4 = 0$. This difference in the choice of indices, apparently insignificant, will prove helpful later.

2.1.3 Proposition (see [Hir83])**.** *The extension* $\mathbb{C}(\mathbb{P}^2)\left(\left(\frac{\ell_2}{\ell_1}\right)^{1/n}, \ldots, \left(\frac{\ell_k}{\ell_1}\right)^{1/n}\right)$ *of the function field* $\mathbb{C}(\mathbb{P}^2)$ *determines a normal algebraic surface* X *and an abelian branched covering map* $\chi : X \to \mathbb{P}^2$ *of degree* n^{k-1}*, ramified over the arrangement of lines with index* n*.*

The proposition is shown by defining and describing a complex surface X' and relating it to X in lemma 2.1.4, then by characterizing the singularities of X' in lemmas 2.1.6, 2.1.7 and 2.1.10 and finally by showing in corollary 2.1.8 that $X' = X$. Furthermore, lemma 2.1.7 states that the smooth complex surface Y obtained by resolving the singularities of X is an abelian branched covering space of some blow-up $\widehat{\mathbb{P}^2}$ of the projective plane $\mathbb{P}^2$. Local charts of Y are given in corollary 2.1.9. Lemma 2.1.11 describes the ramifications of the branched covering map $Y \to \mathbb{P}^2$.

See example A.2.1 for the definition of the branched covering map c_n.

2.1.4 Lemma ([Nam87, Lemma 1.4.6])**.** *Let* X' *be the fiber product with* *respect to the diagram*

$$\begin{array}{ccc} X' & \dashrightarrow & \mathbb{P}^{k-1} \\ {\scriptstyle \chi'} \downarrow & & \downarrow {\scriptstyle c_n} \\ \mathbb{P}^2 & \xrightarrow[\ell]{} & \mathbb{P}^{k-1} \end{array}$$

where $\ell : \mathbb{P}^2 \to \mathbb{P}^{k-1}$ *maps* $[z] = [z_1 : z_2 : z_3]$ *to* $[\ell_1(z) : \cdots : \ell_k(z)]$.

The morphism $\chi' : X' \to \mathbb{P}^2$ *is an abelian branched covering map of degree* n^{k-1} *and which ramifies over the arrangement of lines with index* n. *Moreover, the Galois group* $\mathrm{Aut}(\chi')$ *is naturally isomorphic to* $\mathrm{Aut}(c_n)$.

The normalization of X' *is isomorphic to* X.

Proof. As a set, the fiber product may be defined as

$$X' = \{(p, r) \in \mathbb{P}^2 \times \mathbb{P}^{k-1} \mid \ell(p) = c_n(r)\}$$

and the morphisms $X' \to \mathbb{P}^2$ and $X' \to \mathbb{P}^{k-1}$ as the restrictions to X' of the projections $\mathrm{pr}_1 : \mathbb{P}^2 \times \mathbb{P}^{k-1} \to \mathbb{P}^2$ and $\mathrm{pr}_2 : \mathbb{P}^2 \times \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$, respectively, on the first and on the second component. In particular, the fiber ${\chi'}^{-1}(p)$ of a point p lying on exactly m lines of the arrangement consists of n^{k-1-m} distinct points. Hence χ' is a branched covering map of degree n^{k-1}, ramified over the arrangement of lines with index n.

$\mathrm{Aut}(c_n)$ acts on $\mathbb{P}^2 \times \mathbb{P}^{k-1}$, trivially on the first component and naturally on the second. Hence this action restricts to an action on X' by automorphisms of $\mathrm{Aut}(\chi')$. Since the fibers under χ' and c_n are the same and that $\mathrm{Aut}(c_n)$ acts transitively on the fibers, $\mathrm{Aut}(c_n)$ and $\mathrm{Aut}(\chi')$ are naturally isomorphic.

Finally, let $N \to X'$ be the normalization of X'. The composition of $N \to X'$ and $X' \to \mathbb{P}^2$ is a mapping $N \to \mathbb{P}^2$ such that the induced morphism of function fields $\mathbb{C}(\mathbb{P}^2) \to \mathbb{C}(N)$ is exactly the extension

$$\mathbb{C}(\mathbb{P}^2) \longrightarrow \mathbb{C}(\mathbb{P}^2)\big((\frac{\ell_2}{\ell_1})^{1/n}, \ldots, (\frac{\ell_k}{\ell_1})^{1/n}\big)$$

Therefore N is isomorphic to X. □

2.1.5 Remark. The group $\mathrm{Aut}(\chi)$ is generated by the k automorphisms denoted by α_D, indexed by the lines D of the arrangement, satisfying for any lines D' and D'' of the arrangement

$$\left(\frac{\ell_{D'}}{\ell_{D''}}\right)^{1/n} \circ \alpha_D = e^{\frac{2\pi i}{n}(\delta_{D,D'} - \delta_{D,D''})} \left(\frac{\ell_{D'}}{\ell_{D''}}\right)^{1/n}$$

where δ is the Kronecker delta. The product $\prod_D \alpha_D$ is the identity.

For every line D of the arrangement, the automorphism α_D corresponds to a small loop turning around D counterclockwise.

2.1.6 Lemma. *A point q in X' is singular if and only if its image $\chi'(q)$ in $\mathbb{P}^2$ lies on more than two lines of the arrangement.*

Proof. Let q be a point in X' in the fiber of a point p in $\mathbb{P}^2$ lying on exactly m lines of the arrangement, say $D_1, \ldots, D_m$. The map c_n is defined in homogeneous coordinates as

$$c_n([u_1 : \cdots : u_n]) = [{u_1}^n : \cdots : {u_k}^n].$$

Choose affine coordinates

$$(v_1, \ldots, v_{k-1}) = \left(\frac{u_1}{u_k}, \ldots, \frac{u_{k-1}}{u_k}\right)$$

for $\mathbb{P}^{k-1}$, so that the defining equations of X' in the neighborhood of q are

$${v_s}^n = \frac{{u_s}^n}{{u_k}^n} = \frac{\ell_s}{\ell_k}$$

for s from 1 to $k-1$.

Choose also affine coordinates (w_1, w_2) for $\mathbb{P}^2$ centered at p. For s between 1 and $k-1$, $\frac{\ell_s}{\ell_k}$ may be written as $\alpha_s w_1 + \beta_s w_2 + \gamma_s$ where $\alpha_s, \beta_s, \gamma_s$ are complex numbers. Consider the map defined in these local coordinates by

$$(w_1, w_2, v_1, \ldots, v_{k-1}) \longmapsto ({v_s}^n - \alpha_s w_1 - \beta_s w_2 - \gamma_s)_{1 \le s \le k-1}$$

whose Jacobian matrix is the following.

$$\begin{bmatrix} -\alpha_1 & -\beta_1 & n{v_1}^{n-1} & & & \\ -\alpha_2 & -\beta_2 & & n{v_2}^{n-1} & & \\ \vdots & \vdots & & & \ddots & \\ -\alpha_{k-1} & -\beta_{k-1} & & & & n{v_{k-1}}^{n-1} \end{bmatrix}$$

When none of the coordinates $v_1, \ldots, v_{k-1}$ vanishes, which is generically true, the matrix is of rank $k-1$. Since the lines of the arrangement to which p belongs are exactly $D_1, \ldots, D_m$, the coordinates among $v_1, \ldots, v_{k-1}$ that vanish at q are exactly $v_1, \ldots, v_m$. Hence the $k-1$ by $k-1$ diagonal submatrix, formed by the last $k-1$ columns, is of rank $k-1-m$. And the submatrix formed by the first m lines of the first two columns is of rank $\min(2, m)$: indeed, it is equal to the rank of the family of the non-zero distinct linear forms $(w_1, w_2) \mapsto \alpha_s w_1 + \beta_s w_2$, for s from 1 to m (the coefficients γ_s are equal to zero). Therefore the rank of the Jacobian matrix at q is

$$\min(2, m) + k - 1 - m = \min(k-1, k-1+2-m).$$

If $m \le 2$, the Jacobian matrix is of rank $k-1$ everywhere in the neighborhood of q, so that q is a smooth point of X'.

If $m > 2$, one may assume, up to changing the local coordinates (w_1, w_2) of $\mathbb{P}^2$, that the equations in that chart of the lines D_1 and D_2 are respectively

$$\alpha_1 w_1 + \beta_1 w_2 = w_1 = 0 \quad \text{and} \quad \alpha_2 w_1 + \beta_2 w_2 = w_2 = 0.$$

Then $w_1 = {v_1}^n$ and $w_2 = {v_2}^n$ and the defining equations of X' in the local charts may be written, after eliminating w_1 and w_2, as

$$\alpha_s {v_1}^n + \beta_s {v_2}^n = {v_s}^n$$

for s from 3 to m (in the neighborhood of q, the coordinates w_1, w_2, v_{m+1}, …, v_{k-1} are holomorphic functions in the coordinates $v_1, \ldots, v_m$). Since those equations are homogeneous, X' has a singularity at q. □

2.1.7 Lemma ([Nam87, Proposition 1.4.9]). *The singularities of X' may be resolved by adequate blow-ups, so as to obtain a smooth algebraic surface Y and a morphism $\rho : Y \to X'$. Moreover, let $\tau : \widehat{\mathbb{P}^2} \to \mathbb{P}^2$ denote the blow-up of the projective plane at each of its points where more than two lines of the arrangement meet. There exists a morphism $\sigma : Y \to \widehat{\mathbb{P}^2}$ such that the following diagram is commutative.*

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\chi'} \\ \widehat{\mathbb{P}^2} & \xrightarrow[\tau]{} & \mathbb{P}^2 \end{array}$$

σ is a branched covering map of degree n^{k-1} and ramifies over the proper transforms in $\widehat{\mathbb{P}^2}$ of the lines of the arrangement and over the exceptional curves $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$. The ramification indices are equal to n.

Proof. Each singular point q of X' is resolved by specific blow-ups, which may be described locally. Assume that q is in the fiber of a point p in $\mathbb{P}^2$ lying on a number m (greater than 2) of lines of the arrangement, say $D_1, \ldots, D_m$.

As in the proof of the previous lemma, consider the affine coordinate system $(v_1, \ldots, v_{k-1})$ of $\mathbb{P}^{k-1}$. Blow up the $(v_1, \ldots, v_m)$-space by considering the m coordinate charts, indexed by an integer r between 1 and m,

$$(v_{1|r}, \ldots, v_{r-1|r}, v_r, v_{r+1|r}, \ldots, v_{m|r}, v_{m+1}, \ldots, v_{k-1})$$

defined by $v_{s|r} = v_s / v_r$ for s between 1 and m, different from r. Up to a permutation of the indices $1, \ldots, m$, one may assume for simplicity that $r = 1$.

Then choose an affine coordinate system (w_1, w_2) of $\mathbb{P}^2$, centered at p and where

$$\frac{\ell_1}{\ell_k} = w_1 \quad \text{and} \quad \frac{\ell_2}{\ell_k} = w_2.$$

The linear forms defining the arrangement of lines may be written as

$$\frac{\ell_s}{\ell_k} = \alpha_s w_1 + \beta_s w_2 + \gamma_s$$

for s between 1 and $k-1$ where $\alpha_s, \beta_s, \gamma_s$ are complex numbers such that $\gamma_s = 0$ for s between 1 and m. Blow up the (w_1, w_2)-space by considering the coordinate charts $(w_1, w_{2|1})$ and $(w_{1|2}, w_2)$ defined by $w_{2|1} = w_2/w_1$ and $w_{1|2} = w_1/w_2$.

In those local coordinates, the equations defining X' become

$$\begin{array}{lcll} {v_1}^n & = & w_1 & \\ {v_{s|1}}^n & = & \alpha_s + \beta_s w_{2|1} & \text{for } 2 \leq s \leq m \\ {v_s}^n & = & \alpha_s w_1 + \beta_s w_1 w_{2|1} + \gamma_s & \text{for } m < s < k. \end{array}$$

Similarly to the proof of the previous lemma, the Jacobian matrix of an adequate map associated to those equations is the following.

$$\begin{bmatrix} -1 & 0 & nv_1^{n-1} & & & & & \\ -\alpha_2 & -\beta_2 & & nv_{2|1}^{n-1} & & & & \\ \vdots & \vdots & & & \ddots & & & \\ -\alpha_m & -\beta_m & & & & nv_{m|1}^{n-1} & & \\ -\alpha_{m+1}-\beta_{m+1}w_{2|1} & -\beta_{m+1}w_1 & & & & & nv_{m+1}^{n-1} & \\ \vdots & \vdots & & & & & & \ddots \\ -\alpha_{k-1}-\beta_{k-1}w_{2|1} & -\beta_{k-1}w_1 & & & & & & & nv_{k-1}^{n-1} \end{bmatrix}$$

On the one hand, its rank is again generically $k-1$. On the other hand, the exceptional divisor, that is the curve obtained by resolving the singular point q, is defined in the new coordinates by the equation $v_1 = 0$. At any point in the exceptional divisor, the coordinates $v_{m+1}, \ldots, v_{k-1}$ never vanish and no two of the coordinates $v_{2|1}, \ldots, v_{m|1}$ may vanish simultaneously. Indeed, according to the equations, it would mean that such a point in the exceptional divisor belongs to the proper transforms of two lines of the arrangement passing through p, which is impossible because of the blow-up. A point on the exceptional divisor may belong at most to one of the proper transforms of the lines $D_1, \ldots, D_m$. In any case — for example when it belongs to D_2 — the Jacobian matrix is of rank $k-1$. Consequently, those equations define locally a smooth surface Y.

Besides, in the coordinate charts where Y has been locally defined and outside of the points where $v_{s|1} = 0$ (for some s between 3 and m) or $v_s = 0$ (for some s between $m+1$ and $k-1$), the coordinates $v_{3|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1}$ are locally holomorphic functions of the coordinates $v_1, v_{2|1}$. Therefore, $(v_1, v_{2|1})$ is a holomorphic coordinate system on a neighborhood in Y of the point satisfying $v_1 = v_{2|1} = 0$.

Finally, the morphisms $\chi' : X' \to \mathbb{P}^2$, $\tau : \widehat{\mathbb{P}^2} \to \mathbb{P}^2$ and the projection, denoted by $\rho : Y \to X'$, are defined in the coordinate charts by

$$\chi(w_1, w_2, v_1, \dots, v_{k-1}) = (w_1, w_2)$$

$$\tau(w_1, w_{2|1}) = (w_1, w_{2|1}w_1)$$

$$\begin{aligned} &\rho(w_1, w_{2|1}, v_1, v_{2|1}, \dots, v_{m|1}, v_{m+1}, \dots, v_k) \\ &= (w_1, w_{2|1}w_1, v_1, v_{2|1}v_1, \dots, v_{m|1}v_1, v_{m+1}, \dots, v_k) \end{aligned}$$

The morphism $\sigma : Y \to \widehat{\mathbb{P}^2}$ defined locally by

$$\sigma(w_1, w_{2|1}, v_1, v_{2|1}, \dots, v_{m|1}, v_{m+1}, \dots, v_k) = (w_1, w_{2|1})$$

is such that $\chi' \circ \rho = \tau \circ \sigma$.

Away from the exceptional divisor of ρ, the mapping σ behaves exactly like the branched covering map χ'. And in the neighborhood of the exceptional divisor, for instance in the $(v_1, v_{2|1})$ coordinate system, $\sigma(v_1, v_{2|1}) = (w_1, w_{2|1}) = ({v_1}^n, {v_{2|1}}^n)$. □

2.1.8 Corollary. *The variety X is normal and $X = X'$.*

Proof. According to lemma 2.1.7, the singularities of X' may be resolved by blow-ups. Hence the singularities are normal and the variety X' too. On the other hand, the normalisation of X' is X according to lemma 2.1.4. Therefore, $X = X'$. □

2.1.9 Corollary. *The variety Y is obtained by gluing together a family of affine algebraic varieties Z_{D_1,D_2,D_k}, indexed by any three lines D_1, D_2, D_k of the arrangement such that D_k does not pass through the intersection point of D_1 and D_2.*

More precisely, if $\mathbb{P}^2$ is endowed with the affine coordinate chart (w_1, w_2) where

$$\frac{\ell_1}{\ell_k} = w_1 \qquad \text{and} \qquad \frac{\ell_2}{\ell_k} = w_2$$

and, for s between 3 and $k-1$,

$$\frac{\ell_s}{\ell_k} = \alpha_s w_1 + \beta_s w_2 + \gamma_s$$

with $\gamma_s = 0$ if and only if s is not greater than some integer m, then Z_{D_1,D_2,D_k} is defined in the affine space, with coordinates

$$(w_1, w_{2|1}, v_1, v_{2|1}, \dots, v_{m|1}, v_{m+1}, \dots, v_{k-1}),$$

by the equations

$$
\begin{array}{lcll}
{v_1}^n & = & w_1 & \\
{v_{s|1}}^n & = & \alpha_s + \beta_s w_{2|1} & \text{for } 2 \leq s \leq m \\
{v_s}^n & = & \alpha_s w_1 + \beta_s w_1 w_{2|1} + \gamma_s & \text{for } m < s < k.
\end{array}
$$

The gluing of the affine varieties is given by the relations between the coordinates.

2.1.10 Lemma. *If a point p in $\mathbb{P}^2$ belongs to a number m, greater than 2, of lines of the arrangement, say $D_1, \ldots, D_m$, then each singular point q of X over p is resolved into a smooth curve C and the restriction $\sigma_{|C} : C \to \mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ is a branched covering map.*

$$
\begin{array}{ccccc}
C & \hookrightarrow & Y & \xrightarrow{\rho} & X \\
\downarrow{\scriptstyle \sigma_{|C}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \chi} \\
\mathbb{P}(\mathrm{T}_p\mathbb{P}^2) & \hookrightarrow & \widehat{\mathbb{P}^2} & \xrightarrow[\tau]{} & \mathbb{P}^2
\end{array}
$$

More precisely, $\sigma_{|C}$ is of degree n^{m-1}, ramified over the m points in $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ corresponding to the directions in $\mathrm{T}_p\mathbb{P}^2$ tangent to the lines of the arrangement passing through p. The Euler characteristic of C is $e(C) = n^{m-1}(2 - m) + m \cdot n^{m-2}$.

Proof. Let C denote $\rho^{-1}(q)$. In the typical coordinate chart

$$(w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$$

used in the proof of the previous lemma to resolve the singularities, C is locally defined by the equations

$$
\begin{array}{lcll}
v_1 & = & 0 & \\
w_1 & = & 0 & \\
{v_{s|1}}^n & = & \alpha_s + \beta_s w_{2|1} & \text{for } 2 \leq s \leq m \\
{v_s}^n & = & \gamma_s & \text{for } m < s < k
\end{array}
$$

and $v_{m+1}, \ldots, v_{k-1}$ are in fact uniquely determined by the choice of q in the fiber of p. The morphism $\sigma_{|C}$ is defined in this coordinate system as

$$\sigma_{|C}(0, w_{2|1}, 0, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1}) = (0, w_{2|1}) = (0, {v_{2|1}}^n).$$

Therefore, $\sigma_{|C}$ is a branched covering map of degree n^{m-1}, ramified over m points in $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ with index n, and the Euler characteristic of C is

$$e(C) = n^{m-1}(e(\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)) - m) + m\frac{n^{m-1}}{n} = n^{m-1}(2 - m) + m \cdot n^{m-2}. \quad \square$$

See remark 2.1.5 for the definition of the automorphisms α_D of χ.

2.1.11 Lemma. *Every automorphism α of χ extends as an automorphism of σ which coincides with α outside of the exceptional divisor of $\rho : Y \to X$.*

For each singular point q in X, lying over a point p in $\mathbb{P}^2$, $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q)$ is generated by the automorphisms α_D, for the lines D of the arrangement passing through p.

The automorphism of χ corresponding to a small loop turning around $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ counterclockwise is

$$\prod_{D \ni p} \alpha_D.$$

Finally, the Galois group $\mathrm{Aut}(\sigma_{|C})$ of $\sigma_{|C} : C \to \mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ is isomorphic to the quotient of $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q)$ by the cyclic subgroup generated by

$$\prod_{D \ni p} \alpha_D.$$

2.1.12 Notation. By a slight abuse of notation, α_D or the letter α will indifferently denote automophisms of $\mathbb{P}^{k-1}$, of X, of Y or even of C.

Proof. In order to show that the automorphisms of χ extend as automorphisms of σ, it suffices to prove it for the generators α_D. Furthermore it suffices to prove it locally (see 2.1.9).

Let p be a point in $\mathbb{P}^2$ which belongs to a number m, greater than 2, of lines of the arrangement, say $D_1, \ldots, D_m$, and let q be a singular point in X over p. Consider, without loss of generality, the $(w_1, w_2, v_1, \ldots, v_{k-1})$ coordinate system of X and the $(w_1, w_{2|1}, v_1, v_{2|1} \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$ coordinate system of Y. The point p has coordinates $(w_1, w_2) = (0, 0)$ and q has coordinates of the form $(0, \ldots, 0, v_{m+1}, \ldots, v_{k-1})$ where v_s is not zero for $m < s < k$.

In that coordinate system of X,

- if D is not the line at infinity D_k,

  $$\alpha_D(w_1, w_2, v_1, \ldots, v_{k-1}) = (w_1, w_2, v_1, \ldots, v_{s-1}, e^{\frac{2\pi i}{n}} v_s, v_{s+1}, \ldots, v_{k-1})$$

  for some s,

- if D is D_k,

  $$\alpha_D(w_1, w_2, v_1, \ldots, v_{k-1}) = (w_1, w_2, e^{-\frac{2\pi i}{n}} v_1, \ldots, e^{-\frac{2\pi i}{n}} v_{k-1}).$$

Therefore, in that coordinate system of Y,

1. if D is the line D_1 defined by the equation $w_1 = 0$,

   $$\begin{aligned}&\alpha_D(w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1}) \\ &= (w_1, w_{2|1}, e^{\frac{2\pi i}{n}} v_1, e^{-\frac{2\pi i}{n}} v_{2|1}, \ldots, e^{-\frac{2\pi i}{n}} v_{m|1}, v_{m+1}, \ldots, v_{k-1}),\end{aligned}$$

2. if D passes through p in $\mathbb{P}^2$ but is not D_1,

$$\alpha_D(w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$$
$$= (w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{s-1|1}, e^{\frac{2\pi i}{n}} v_{s|1}, v_{s+1|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$$

for some s,

3. if D is D_k,

$$\alpha_D(w_1, w_{2|1}, v_1, v_{2|1} \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$$
$$= (w_1, w_{2|1}, e^{-\frac{2\pi i}{n}} v_1, v_{2|1}, \ldots, v_{m|1}, e^{-\frac{2\pi i}{n}} v_{m+1}, \ldots, e^{-\frac{2\pi i}{n}} v_{k-1}),$$

4. if D does not pass through p in $\mathbb{P}^2$ and is not D_k,

$$\alpha_D(w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{k-1})$$
$$= (w_1, w_{2|1}, v_1, v_{2|1}, \ldots, v_{m|1}, v_{m+1}, \ldots, v_{s-1}, e^{\frac{2\pi i}{n}} v_s, v_{s+1}, \ldots, v_{k-1})$$

for some s.

In each case, α_D extends to the exceptional divisor of $\rho : Y \to X$.

Since q has coordinates of the form $(0, \ldots, 0, v_{m+1}, \ldots, v_{k-1})$ where v_s is not zero for $m < s < k$, it appears that $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{X})}(q)$ is the subgroup generated by the automorphisms $\alpha_{D_1}, \ldots, \alpha_{D_m}$.

Consider a loop in $\widehat{\mathbb{P}^2}$

$$\gamma : \begin{cases} [0, 2\pi] & \longrightarrow & \widehat{\mathbb{P}^2} \\ t & \longmapsto & (w_1(t), w_{2|1}(t)) = (\varepsilon e^{it}, w_{2|1}(0)) \end{cases}$$

turning around $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ and not meeting the proper transforms of the lines $D_1, \ldots, D_m$ (ε is arbitrarily small and $w_{2|1}$ is constant). Finding a lift $\tilde{\gamma} : [0, 2\pi] \to Y$ of γ amounts to finding continuous functions v_1, $v_{2|1}$, $\ldots$, $v_{m|1}$, v_{m+1}, $\ldots$, v_{k-1} satifying the equations

$$\begin{array}{lcll} v_1(t)^n & = & w_1(t) & \\ v_{s|1}(t)^n & = & \alpha_s + \beta_s w_{2|1}(t) & \text{for } 2 \leq s \leq m \\ v_s(t)^n & = & \alpha_s w_1(t) + \beta_s w_1(t) w_{2|1}(t) + \gamma_s & \text{for } m < s < k \end{array}$$

that is to say

$$\begin{array}{lcll} v_1(t)^n & = & \varepsilon e^{it} & \\ v_{s|1}(t)^n & = & v_{s|1}(0)^n & \text{for } 2 \leq s \leq m \\ v_s(t)^n & = & v_s(0)^n + \varepsilon(\alpha_s + \beta_s w_{2|1}(0))(e^{it} - 1) & \text{for } m < s < k. \end{array}$$

Thus

$$\tilde{\gamma}(2\pi) = (w_1(0), w_{2|1}(0), e^{\frac{2\pi i}{n}} v_1(0), v_{2|1}(0), \ldots, v_{m|1}(0), v_{m+1}(0), \ldots, v_{k-1}(0))$$
$$= \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_m(\tilde{\gamma}(0))$$

Since $\sigma_{|C}$ is the restriction of the Galois branched covering map σ, the morphism $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \to \mathrm{Aut}(\sigma_{|C})$ is surjective. The automorphism

$$\prod_{D \ni p} \alpha_D$$

fixes C so it is in the kernel of $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \to \mathrm{Aut}(\sigma_{|C})$. Finally, since $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q)$ has n^m elements and that $\mathrm{Aut}(\sigma_{|C})$ has as many element as the degree of $\sigma_{|C}$, that is n^{m-1}, the morphism $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \to \mathrm{Aut}(\sigma_{|C})$ is bijective, for cardinality reasons. □

2.1.13 Theorem (Miyaoka-Yau [Miy83]). *If the Chern classes of a compact complex surface Y of general type satisfy*

$$c_1(Y)^2 = 3c_2(Y)$$

then Y is the quotient of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ by a lattice.

In [Hir83], Hirzebruch finds three cases where, given an arrangement of lines and a exponent n, the corresponding surface Y is of general type and satisfies $c_1(Y)^2 = 3c_2(Y)$. Therefore those surfaces admit a complex hyperbolic structure. Hirzebruch denotes them by Y_1, Y_2 and Y_3.

2.1.14 Example. The surface Y_1 corresponds to the complete quadrilateral arrangement and to the exponent $n = 5$. Hence $\sigma : Y_1 \to \widehat{\mathbb{P}^2}$ is a branched covering map of degree 5^5 which ramifies over the six lines of the arrangement and the four exceptional curves, all with index 5.

The present thesis focuses on the surface Y_1.

2.1.2 Complex hyperbolic lattice

T. Yamazaki and M. Yoshida [YY84] have determined a lattice, that they denote by G_1, in the group of automorphisms of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ such that $\widehat{\mathbb{P}^2}$ appears as the quotient of $\mathbb{H}^2_{\mathbb{C}}$ by G_1 and that Hirzebruch's surface Y_1 is the quotient by the commutator subgroup $[G_1, G_1]$.

More precisely, $\widehat{\mathbb{P}^2}$ has the structure of a complex hyperbolic orbifold and Y_1 that of a complex hyperbolic manifold. Despite the orbifold structure, $\widehat{\mathbb{P}^2}$ is simpler than Y_1 and reflects also the complex hyperbolic structure.

Choose a base point a in the complement $\mathcal{D}$ of the branch locus of $\widehat{\mathbb{P}^2}$ and a loop $\rho(ij)$ based at a, for $i, j \in \{0, 1, 2, 3\}$ with $i < j$, turning around D_{ij}. A group presentation of the fundamental group $\pi_1(\mathcal{D}, a)$ is given by the generators $\rho(ij)$ and the relations

$$[\rho(ij)\rho(ik)\rho(jk), \rho(ij)] = 1,$$

$$[\rho(ij)\rho(ik)\rho(jk), \rho(ik)] = 1,$$
$$[\rho(ij)\rho(ik)\rho(jk), \rho(jk)] = 1$$

for $i < j < k$ and

$$\rho(01)\rho(02)\rho(12)\rho(03)\rho(13)\rho(23) = 1.$$

Let μ be $\exp(2\pi i\frac{3}{5})$. The group G_1 is the image of the representation $R : \pi_1(\mathcal{D}, a) \to \mathrm{PGL}_3(\mathbb{C})$ defined by $R(\rho(ij)) = R(ij)$ where

$$\begin{aligned}
R(12) &= I_3 + \begin{pmatrix} -\mu(1-\mu) & \mu(1-\mu) & 0 \\ 1-\mu & -(1-\mu) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
R(23) &= I_3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu(1-\mu) & \mu(1-\mu) \\ 0 & 1-\mu & -(1-\mu) \end{pmatrix} \\
R(13) &= I_3 + \begin{pmatrix} -\mu(1-\mu) & 0 & \mu(1-\mu) \\ (1-\mu)(1-\mu) & 0 & -(1-\mu)(1-\mu) \\ 1-\mu & 0 & -(1-\mu) \end{pmatrix} \\
R(01) &= I_3 + \begin{pmatrix} \mu^2-1 & 0 & 0 \\ \mu(1-\mu) & 0 & 0 \\ \mu(1-\mu) & 0 & 0 \end{pmatrix} \\
R(02) &= I_3 + \begin{pmatrix} 0 & -(1-\mu) & 0 \\ 0 & \mu^2-1 & 0 \\ 0 & -\mu(1-\mu) & 0 \end{pmatrix} \\
R(03) &= \mu I_3 + \mu \begin{pmatrix} 0 & 0 & -(1-\mu) \\ 0 & 0 & -(1-\mu) \\ 0 & 0 & \mu^2-1 \end{pmatrix}
\end{aligned}$$

In fact, G_1 is contained in the projective unitary group whose Hermitian form of signature $(+,+,-)$ is given by the Hermitian matrix

$$A_1 = \begin{pmatrix} \frac{-1}{\mu+\overline{\mu}} & \overline{\mu} & 1 \\ \mu & \frac{-1}{\mu+\overline{\mu}} & \overline{\mu} \\ 1 & \mu & \frac{-1}{\mu+\overline{\mu}} \end{pmatrix}.$$

2.2 The pencil of conics, an example of Lefschetz fibration

A Lefschetz fibration $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ is defined in this section and will allow to derive a similar one $Y_1 \to C$ in section 2.4.

A *conic* in the complex projective plane $\mathbb{P}^2$ is the zero-locus of a quadratic form in the variables z_1, z_2, z_3. The vector space $\mathrm{Sym}^2(\mathbb{C}^{3*})$ of all quadratic

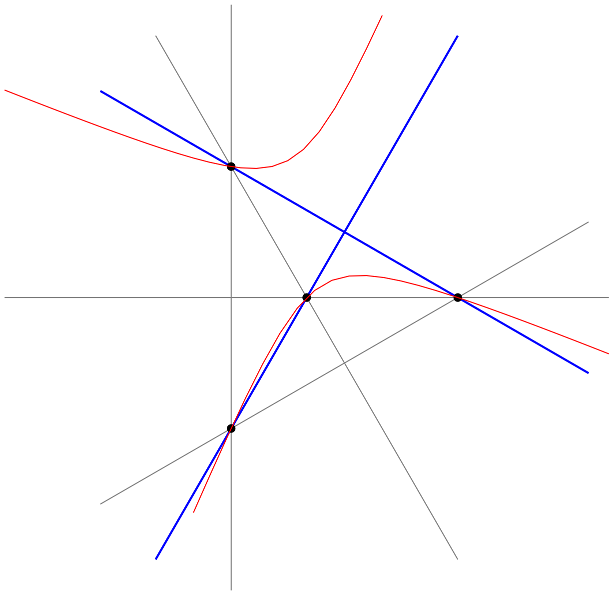


Figure 2.2: The pencil of conics. A generic fiber in red and one of the 3 singular fibers in blue.

forms on $\mathbb{C}^3$ is of dimension 6. Since the one and only way for two quadratic forms to define the same conic is to be proportional, the set of conics may be naturally identified with the projective space $\mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^{3*}))$.

The set of conics passing through four points given in $\mathbb{P}^2$, none three of which lie on the same line, say $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$, $p_4 = [1:1:1]$, corresponds to a line in $\mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^{3*}))$. For any fifth point (distinct from the first four), there is exactly one conic passing through the five points. And even when the fifth point happens to collide with any point p among the first four, prescribing in addition any line in the tangent plane $\mathrm{T}_p\mathbb{P}^2$, there is again exactly one conic passing through $p_1, \ldots, p_4$ and tangent to that line. Following the previous considerations, there is a natural mapping $f : \widehat{\mathbb{P}^2} \to \mathbb{P}^1$, where $\widehat{\mathbb{P}^2}$ denotes the projective plane blown up at the four points. Each exceptional curve in $\widehat{\mathbb{P}^2}$, obtained by blowing up a point p among the four, is naturally identified with $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$. The map f is a fibration whose fibers are the proper transforms in $\widehat{\mathbb{P}^2}$ of the conics passing through the four points. Moreover, for each point p among the four, f admits sections $\mathbb{P}^1 \to \mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ which maps a conic to its tangent line at p.

$$\begin{array}{ccc} \mathbb{P}(\mathrm{T}_p\mathbb{P}^2) & \hookrightarrow & \widehat{\mathbb{P}^2} \\ & \nwarrow\!\!\searrow & \downarrow f \\ & & \mathbb{P}^1 \end{array}$$

Among those conics, represented by points in $\mathbb{P}^1$, exactly three are singular. Each of them is the union of two lines, one passing through two among the four points and the second passing through the two others. Those six lines together form the complete quadrilateral arrangement. The points in $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ corresponding to the singular conics are the lines of the arrangement passing through p.

In coordinates, the pencil of conics may be defined as

$$[z_1 : z_2 : z_3] \mapsto [(z_1 - z_3)z_2 : z_1(z_2 - z_3)].$$

This is a rational mapping defined everywhere except at the points $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$, $p_4 = [1 : 1 : 1]$ where the polynomials $(z_1 - z_3)z_2$ and $z_1(z_2 - z_3)$ vanish simultaneously. Nevertheless, blowing up the projective plane at one of the four points, say p_3, one ends up with local coordinate charts $(w_1, w_{2|1})$ and $(w_{1|2}, w_2)$ defined as $w_s = z_s/z_3$ and $w_{r|s} = w_r/w_s$, for $r, s \in \{1, 2\}$, where the rational mapping extends in the neighborhood of the exceptional curve $\mathbb{P}(\mathrm{T}_{p_3}\mathbb{P}^2)$ as

$$(w_1, w_{2|1}) \mapsto [(w_1 - 1)w_{2|1} : w_1 w_{2|1} - 1]$$

and

$$(w_{1|2}, w_2) \mapsto [w_2 w_{1|2} - 1 : w_{1|2}(w_2 - 1)].$$

Note that the fiber over $[1 : 0]$ is the singular conic defined by $z_1(z_2 - z_3) = 0$, the one over $[0 : 1]$ is defined by $(z_1 - z_3)z_2 = 0$ and also the one over $[1 : 1]$ is defined by $(z_1 - z_2)z_3 = 0$. Those three are the only singular fibers.

The pencil of conics described above is a simple example of Lefschetz pencil or fibration.

2.2.1 Definition. A *Lefschetz pencil* or *Lefschetz fibration* f is respectively a rational mapping or morphism from a complex surface S to a complex curve C such that, for every point s in S (where f is defined),

1. either f is a submersion at s
2. or the differential $\mathrm{d}_s f$ of f at s is zero but the second symmetric differential $\mathrm{d}_s^2 f$ is a nondegenerate quadratic form.

2.2.2 Remarks.

1. The difference between a Lefschetz pencil or fibration is not worth spending to much time in the present context, where it seems sufficient to observe, in the example of the pencil of conics, that $\mathbb{P}^2 \to \mathbb{P}^1$ is a rational mapping undetermined at the four points, whereas $f : \widehat{\mathbb{P}^2} \to \mathbb{P}^1$ is well defined everywhere, after an adequate blow-up of $\mathbb{P}^2$.

2. If f happens to be a submersion everywhere (and also proper, which is guaranted when S is compact), then Ehresmann's fibration theorem yields that f is a differentiable fiber bundle. In general, except over a finite number of points in C, f is a fiber bundle whose fiber is called the *generic fiber* of the Lefschetz fibration.

3. Besides, the shape of the *singular fibers* are prescribed by the condition 2 in the previous definition. Indeed, at a point s of S where f is not submersive, the holomorphic analogue of Morse lemma holds that there exists local charts of S and C, centered at s and $f(s)$ respectively, where f is as simple as $(x, y) \mapsto xy$. Hence, in the neighborhood of s and up to a holomorphic transformation, the fiber passing through s is the union of two lines intersecting normally.

4. The singular fibers are sometimes required to have only one singular point, but this additional condition is not essential in the present account and is even unsatisfied in the sequel.

2.2.3 Examples.

1. In the local coordinate chart

$$(x, y) = \left(\frac{z_1 - z_3}{z_1}, \frac{z_2}{z_2 - z_3} \right)$$

centered at the point $[1 : 0 : 1]$, the rational mapping defining the pencil of conics is expressed as $f(x, y) = [xy : 1]$, so f may be easily expressed in the normal form without resorting to the Morse lemma.

2. In passing, many more examples of Lefschetz pencils arise in the way the pencil of conics is defined above with coordinates. Indeed, choose two homogeneous polynomials P and Q of a same nonzero degree d, in the variables z_1, z_2, z_3, with no common factor and consider the rational mapping

$$[z_1 : z_2 : z_3] \longmapsto [P : Q]$$

undetermined at the points where P and Q vanish simultaneously. The fiber over a point $[\lambda : \mu]$ is the curve defined by the equations $\mu P - \lambda Q = 0$ of degree d. In particular, the fiber over $[0 : 1]$ is $P = 0$, the fiber over $[1 : 0]$ is $Q = 0$ and those two intersect at isolated point. All of the fibers pass through the intersection points of $P = 0$ and $Q = 0$. For this reason, the rational map is called the pencil generated by P and Q and the set of points defined by $P = Q = 0$ is called the base of the pencil. Moreover, Bézout's theorem holds that the total number of intersection points of $P = 0$ and $Q = 0$, counted with their multiplicities, is equal to the product of the degrees of P and Q.

2.2.4 Theorem (Picard-Lefschetz formula)**.** *Let $f : S \to C$ be a Lefschetz fibration where S and C are compact. In local charts of S and C, centered respectively at a singular point s of a singular fiber and at $f(s)$, the monodromy of the generic fiber, corresponding to a loop in $C \setminus \{p\}$ turning counterclockwise around p, is a right-handed Dehn twist.*

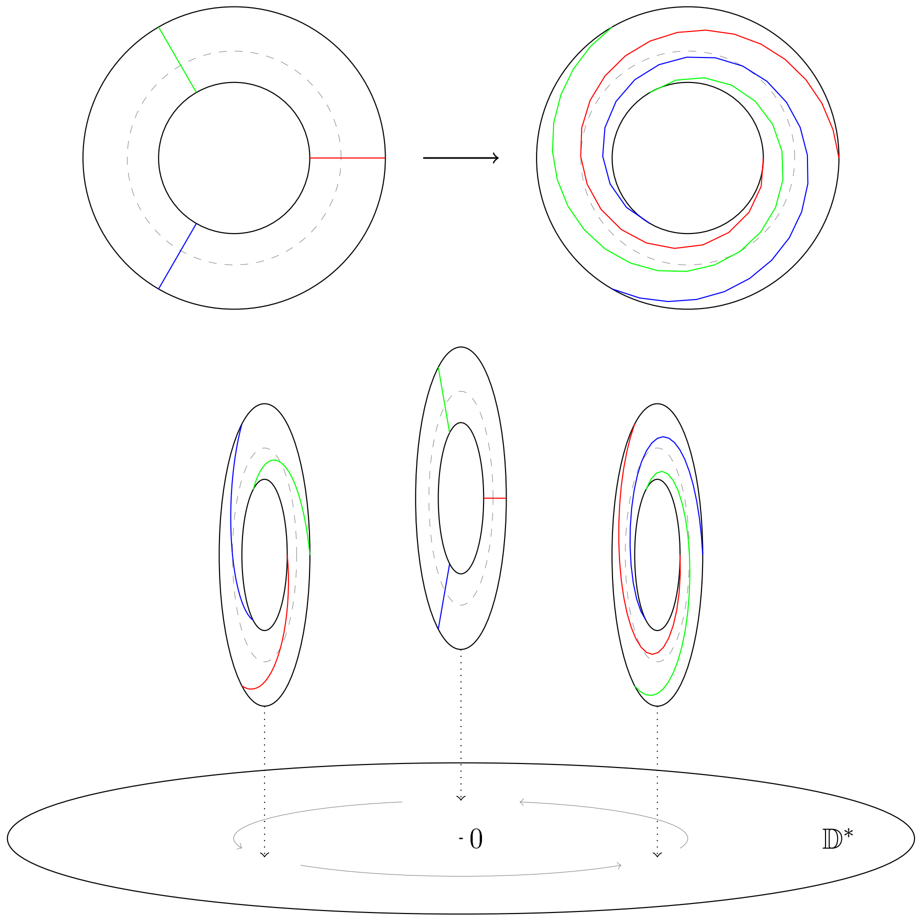


Figure 2.3: A right-handed Dehn twist and the monodromy along a loop turning about the 0, of the fibration $f : f^{-1}(\mathbb{D}) \cap \overline{\mathbb{D}}^2 \to \mathbb{D}$.

Proof. Consider a singular fiber over a critical value $c \in C$ and a local chart of C centered at c. For each singular point s of $f^{-1}(c)$, the holomorphic analogue of the Morse lemma holds that there exists a local coordinate chart of S, centered at s, where f is expressed as $f(x, y) = xy$. Up to homothetic transformations and restriction of the local charts, one may assume that f maps the closed bidisc $\overline{\mathbb{D}}^2$ to the closed disc $\overline{\mathbb{D}}$. After removing each open bidisc $\mathbb{D}^2$ centered at a singular point of the singular fiber, Ehresmann's fibration theorem applies over the whole open disc $\mathbb{D}$, even 0. Since $\mathbb{D}$ is contractible, the latter fibration is trivial. It remains to understand what happens in each bidisc.

The boundary left after removing each open bidisc is

$$\partial(\overline{\mathbb{D}}^2) = (\partial\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \partial\overline{\mathbb{D}}).$$

The restriction $f : f^{-1}(\mathbb{D}) \cap \partial(\overline{\mathbb{D}}^2) \to \mathbb{D}$ of the fibration to this boundary is trivial. Its fiber above 0 is

$$f^{-1}(0) \cap \partial(\overline{\mathbb{D}}^2) = (\partial\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \partial\overline{\mathbb{D}})$$

and one has the following trivialisation.

$$\begin{array}{ccc}
(x, z/x) & \longleftarrow\!\shortmid & (z, (x, 0)) \\
(z/y, y) & \longleftarrow\!\shortmid & (z, (0, y)) \\
f^{-1}(\mathbb{D}) \cap \partial(\overline{\mathbb{D}}^2) & \longleftarrow & \mathbb{D} \times ((\partial\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \partial\overline{\mathbb{D}})) \\
& {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\
& \mathbb{D} &
\end{array}$$

The fiber over a point z in $\mathbb{D}^*$ is

$$\begin{aligned} f^{-1}(z) \cap \overline{\mathbb{D}}^2 = \{(x, y) \in \overline{\mathbb{D}}^2 \mid xy = z\} &= \{(x, z/x) \mid |z| \leq |x| \leq 1\} \\ &= \{(z/y, y) \mid |z| \leq |y| \leq 1\}. \end{aligned}$$

It is biholomorphic to a closed annulus and one of the two coordinates, x or y, is enough to parametrize it. Its boundary has two connected components

$$\{(x, z/x) \mid |x| = 1\} = \{(z/y, y) \mid |z| = |y|\}$$

and

$$\{(x, z/x) \mid |x| = |z|\} = \{(z/y, y) \mid |y| = 1\}.$$

In order to understand the monodromy of the generic fiber when one turns around the origin in the punctured disc, that is to say, around the singular fiber, consider the parametrization $i\mathbb{H} \to \mathbb{D}^*, z \mapsto e^z$, where $i\mathbb{H}$ is the open half-plane of all complex numbers of negative real part. Let e^{z_0} be a base point in $\mathbb{D}^*$. It suffices to find a trivialization above $i\mathbb{H}$ interpolating the trivialization already given on the boundary.

$$
\begin{array}{ccccc}
f^{-1}(e^{z_0}) \cap (\overline{\mathbb{D}}^2) & & \overset{?}{\dashrightarrow} & & f^{-1}(e^{z}) \cap (\overline{\mathbb{D}}^2) \\
\uparrow & & & & \uparrow \\
f^{-1}(e^{z_0}) \cap \partial(\overline{\mathbb{D}}^2) & \longleftarrow & (\partial\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \partial\overline{\mathbb{D}}) & \longrightarrow & f^{-1}(e^{z}) \cap \partial(\overline{\mathbb{D}}^2) \\
(x, e^{z_0}/x) & \longleftarrow\!\!\dashv & (x,0) & \longmapsto & (x, e^{z}/x) \\
(e^{z_0}/y, y) & \longleftarrow\!\!\dashv & (0,y) & \longmapsto & (e^{z}/y, x) \\
(x,y) & & \xrightarrow{\text{if } |x|=1} & & (x, e^{z-z_0}y) \\
 & & \xrightarrow[\text{if } |y|=1]{} & & (e^{z-z_0}x, y)
\end{array}
$$

A mapping of the form

$$
\begin{array}{ccc}
(z,(x,y)) & \longmapsto & (e^{(z-z_0)\varphi(x,y)}x, e^{(z-z_0)\varphi(y,x)}y) \\
i\mathbb{H} \times (f^{-1}(e^{z_0}) \cap \overline{\mathbb{D}}^2) & \longrightarrow & f^{-1}(\mathbb{D}) \cap \overline{\mathbb{D}}^2 \\
\downarrow \mathrm{pr}_1 & & \downarrow f \\
i\mathbb{H} & \longrightarrow & \mathbb{D} \\
z & \longmapsto & e^{z}
\end{array}
$$

where $\varphi : f^{-1}(e^{z_0} \cap \overline{\mathbb{D}}^2) \longrightarrow [0,1]$ is a continuous function satisfying

$$\varphi(x,y) = \left\{ \begin{array}{l} 0 \text{ if } |x| = 1 \text{ and } |y| = |e^{z_0}| \\ 1 \text{ if } |y| = 1 \text{ and } |x| = |e^{z_0}| \end{array} \right. ,$$

is a trivialization above $i\mathbb{H}$. One may for instance define φ by

$$\varphi(x,y) = \frac{\ln|x|}{\ln|xy|}.$$

Finally, the parameter z corresponding to a counterclockwise loop around the origin, based at e^{z_0}, satisfies $z - z_0 = 2\pi i$. Therefore, the monodromy of the fibration along that loop, in $\overline{\mathbb{D}}^2$, is given by

$$(x,y) \longmapsto (e^{2\pi i\,\varphi(x,y)}x, e^{2\pi i\,\varphi(y,x)}y)$$

which is exactly a right-handed Dehn twist, about the loop $t \mapsto (x_0 e^{it}, y_0 e^{-it})$ or any homotopically equivalent one. □

The previous result allows to understand the behavior of the fibration in the neighborhood of each singular fiber, but not globally. In order to understand the global picture, there is actually another interpretation of the fibration $f : \widehat{\mathbb{P}^2} \to \mathbb{P}^1$.

2.2.5 Proposition. *For any integer greater than* 3*, let* Q_n *denote the quotient of* $(\mathbb{P}^1)^n$ *by the diagonal action of* $\operatorname{Aut}(\mathbb{P}^1)$*, in the sense of* geometric invariant theory*. Then* Q_4 *is isomorphic to* $\mathbb{P}^1$*,* Q_5 *to* $\widehat{\mathbb{P}^2}$ *and the diagram below is moreover commutative:*

$$\begin{array}{ccccc}
(v_1, v_2, v_3, v_4, v_5) & \longmapsto & \left[\frac{\det(v_1,v_4)}{\det(v_1,v_5)} : \frac{\det(v_2,v_4)}{\det(v_2,v_5)} : \frac{\det(v_3,v_4)}{\det(v_3,v_5)}\right] & & \\
\Big\downarrow & Q_5 & \longrightarrow & \widehat{\mathbb{P}^2} & \\
 & \Big\downarrow & & \Big\downarrow f & \\
 & Q_4 & \longrightarrow & \mathbb{P}^1 & \\
(v_1, v_2, v_3, v_4) & \longmapsto & \left[\frac{\det(v_1,v_3)}{\det(v_1,v_4)} : \frac{\det(v_2,v_3)}{\det(v_2,v_4)}\right] & &
\end{array}$$

($v_1, \ldots, v_5$ *denote nonzero vectors in* $\mathbb{C}^2$ *representing points in* $\mathbb{P}^1$*).*

2.2.6 Remarks.

1. The group $\operatorname{Aut}(\mathbb{P}^1)$ of the automorphisms of $\mathbb{P}^1$ is simply the group $\mathrm{PGL}_2(\mathbb{C})$ which is also $\mathrm{PSL}_2(\mathbb{C})$. The group acts transitively on triples of distinct points in $\mathbb{P}^1$. Hence the space Q_n becomes interesting only for n greater than 3.

2. The mapping $(v_1, v_2, v_3, v_4) \longmapsto \left[\dfrac{\det(v_1, v_3)}{\det(v_1, v_4)} : \dfrac{\det(v_2, v_3)}{\det(v_2, v_4)}\right]$ is nothing but the cross ratio of four points in $\mathbb{P}^1$, which is invariant by the diagonal action of $\operatorname{Aut}(\mathbb{P}^1)$. Note that the cross ratio is defined provided that none three of the four points are equal.

3. An element $(v_1, \ldots, v_n)$ in $(\mathbb{P}^1)^n$ is stable (respectively semi-stable) under the action of $\operatorname{Aut}(\mathbb{P}^1)$, in the sense of geometric invariant theory, if and only if the largest number of points among $v_1, \ldots, v_n$ that coincide is less (respectively not greater) than $n/2$.

   Let Q_n^* denote the quotient (in the usual sense), by the diagonal action of $\operatorname{Aut}(\mathbb{P}^1)$, of the subset of $(\mathbb{P}^1)^n$ formed by all the n-tuples of distinct points.

   For $n = 4$, Q_4^* is the subset of Q_4 of all stable points and the remainder consists of the classes of 4-tuples (z_1, z_2, z_3, z_4) two of whose components coincide [Dol03, Example 11.4].

   For $n = 5$, the difference between Q_5^* and Q_5 is the set of classes of 5-tuples $(z_1, z_2, z_3, z_4, z_5)$ such that $z_a = z_b$ for some distinct indices a and b. This set is hence the union of 10 lines of equation $z_a = z_b$ [Dol03, Example 11.5].

4. The ten lines of the form $z_a = z_b$ in Q_5 play symmetric roles, whereas the ten lines in $\widehat{\mathbb{P}^2}$ consists of the six lines of the arrangement and the four exceptional curves, apparently arising in a different way. This difference is related to the fact that the forgetful map $Q_5 \to Q_4$ does not treat equally the five components of 5-tuples.

Proof. The maps do not depend on the choice of the representatives $v_1, \dots, v_5$ and are well defined. Let $([z_1 : 1], [z_2 : 1], [z_3 : 1], [0 : 1], [1 : 0])$ be a representative of a point $(v_1, v_2, v_3, v_4, v_5)$ in Q_5^* (the proof is similar if the 5-tuple is not of that form). Then

$$\left[\frac{\det(v_1, v_4)}{\det(v_1, v_5)} : \frac{\det(v_2, v_4)}{\det(v_2, v_5)} : \frac{\det(v_3, v_4)}{\det(v_3, v_5)}\right] = [z_1 : z_2 : z_3]$$

and

$$\left[\frac{\det(v_1, v_3)}{\det(v_1, v_4)} : \frac{\det(v_2, v_3)}{\det(v_2, v_4)}\right] = \left[\frac{z_1 - z_3}{z_1} : \frac{z_2 - z_3}{z_2}\right] = f([z_1 : z_2 : z_3])$$

so that the diagram is commutative. □

2.2.7 Corollary. *The monodromy representation of the fibration $f : Q_5^* \to Q_4^*$ is a morphism $\pi_1(Q_4^*) \to \mathrm{Mod}_{0,4}$ such that the image of each generator of $\pi_1(Q_4^*)$ is a right-handed Dehn twist, as drawn in figure 2.4.*

2.2.8 Remark. The monodromy representation is a particular case of the *point pushing map* appearing in the *Birman exact sequence* (see [FM11, Theorem 4.6]).

Indeed, viewing Q_4^* as a sphere with 3 punctures, the monodromy along (the homotopy class) of a loop γ in Q_4^* is, according to corollary 2.2.7 and figure 2.4, the mapping class obtained by pushing the base point of Q_4^* along γ.

2.3 Mapping class groups

This section is devoted to a short presentation of mapping class groups, some examples and related notions, in order to better understand the monodromy of the pencil of conics.

Given two topological spaces X and Y, such that X is locally compact and Hausdorff, the compact-open topology on the set $\mathcal{C}(X, Y)$ of continuous functions from X to Y is the unique topology satisfying the following univeral property: for any topological space Z, the mapping

$$\begin{array}{ccc} \mathcal{C}(Z \times X, Y) & \longrightarrow & \mathcal{C}(Z, \mathcal{C}(X, Y)) \\ f & \longmapsto & \{z \mapsto f_z : x \mapsto f(z, x)\} \end{array}$$

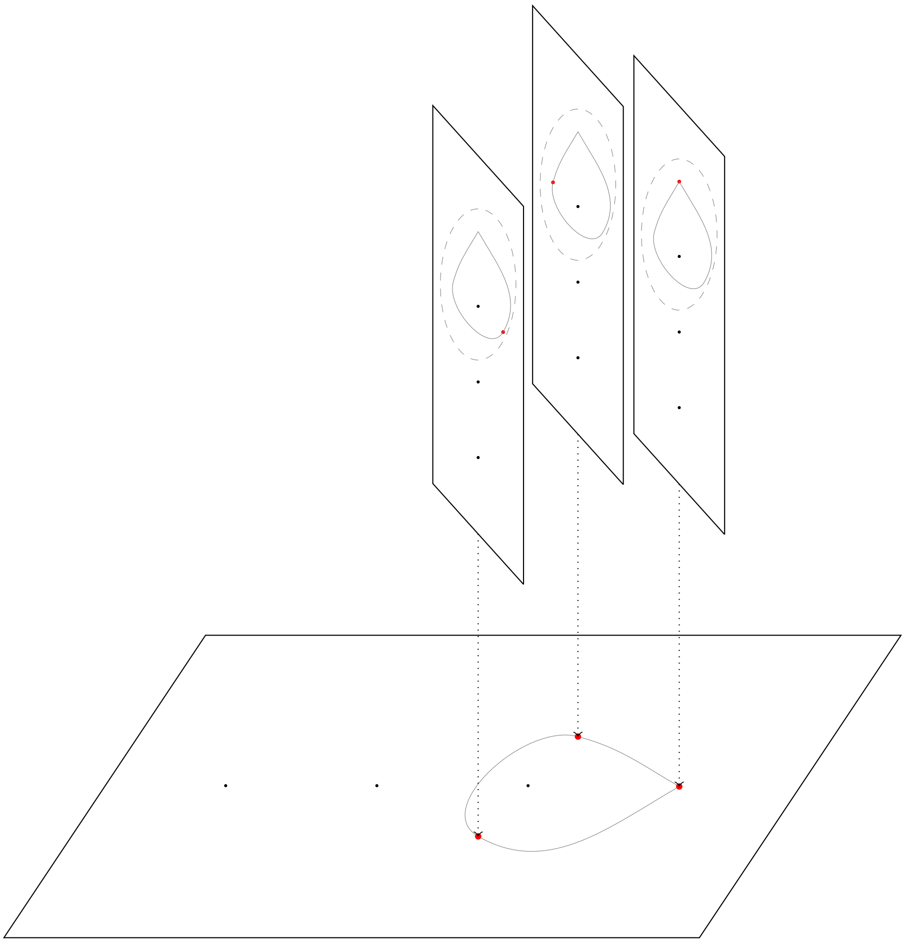


Figure 2.4: The monodromy of the forgetful mapping $Q_5^* \to Q_4^*$ along the grey loop is a right-handed Dehn twist along the dashed loop.

is bijective. In particular, a homotopy between two functions f_1 and f_2 in $\mathcal{C}(X, Y)$ is nothing but a continuous path in the topological space $\mathcal{C}(X, Y)$. The latter interpretation of homotopy allows to formulate a rather concise definition of mapping class groups.

The group $\mathrm{Homeo}(X)$ of all homeomorphisms of a locally compact and Hausdorff topological space X is a topological group, with the topology induced by the compact-open topology of $\mathcal{C}(X, X)$.

2.3.1 Definition. The *mapping class group* of X is 0-th homotopy group of the topological group $\mathrm{Homeo}(X)$ and is denoted by $\mathrm{Mod}(X)$.

When the space X admits an additional structure, one is rather interested in the subgroup of homeomorphisms preserving the structure in question. On the one hand, if X is an orientable manifold, the orientation-preserving homeomorphisms of X form a normal subgroup of index 2, denoted by $\mathrm{Homeo}^+(X)$, whose 0-th homotopy group is denoted by $\mathrm{Mod}^+(X)$. Nevertheless, only orientable surface are considered in the present context, so that the sign $+$ will be omitted and $\mathrm{Mod}(X)$ will always denote the 0-th homotopy group of $\mathrm{Homeo}^+(X)$, without any danger. On the other hand, if a particular subset A of X is marked (for example, a finite subset or the boundary of X if X is a manifold with boundary), the (orientation-preserving) homeomorphisms of X stabilizing each marked point in A form a subgroup, denoted by $\mathrm{Homeo}(X, A)$, whose 0-th homotopy group is denoted by $\mathrm{Mod}(X, A)$.

2.3.2 Notation. The mapping class group of a closed orientable surface of genus g and with n marked points is denoted by $\mathrm{Mod}_{g,n}$.

A natural approach to understand and describe the mapping class group $\mathrm{Mod}_{0,4}$ of the sphere with four marked points is to study the mapping class group $\mathrm{Mod}_{1,4}$ of the torus with four marked points. Indeed, the torus is a double branched covering space of the sphere, with ramification over 4 points. The automorphism group is generated by the *hyperelliptic involution*: identifying the torus with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, the hyperelliptic involution is induced by the linear transformation $(x, y) \mapsto (-x, -y)$ corresponding to the matrix $-\mathrm{I}_2$ (see figure 2.5). The hyperelliptic involution stabilizes four points of the torus.

2.3.3 Proposition. $\mathrm{Mod}_{1,1}$ *is naturally isomorphic to* $\mathrm{SL}_2(\mathbb{Z})$ *so that, for each mapping class, the corresponding matrix induces a transformation of the torus* $\mathbb{R}^2/\mathbb{Z}^2$*, which is a representative of that mapping class.*

Proof. Let O denote the image in the torus $\mathbb{R}^2/\mathbb{Z}^2$ of the origin $(0, 0)$ in the plane $\mathbb{R}^2$. Any homeomorphism $h : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ preserving O, induces an automorphism h_* of the fundamental group $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, O)$ depending only on the homotopy class of h. Since $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, O)$ is naturally isomorphic to $\mathbb{Z}^2$, h_* corresponds to an element of $\mathrm{GL}_2(\mathbb{Z})$ and that defines a group morphism $\mathrm{Mod}_{1,1} \to \mathrm{GL}_2(\mathbb{Z})$.

The action on the plane $\mathbb{R}^2$ of any matrix M in $\mathrm{GL}_2(\mathbb{Z})$ induces a homeomorphism h_M of the torus preserving O and such that the induced automorphism $(h_M)_*$ of the fundamental group $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, O)$ corresponds to the initial matrix M. Besides, the homeomorphism h_M preserves or reverses the orientation precisely when the determinant of M is respectively equal to $+1$ or -1. In other terms, h_M preserves the orientation if and only if M belongs to $\mathrm{SL}_2(\mathbb{Z})$. Therefore, the morphism $\mathrm{Mod}_{1,1} \to \mathrm{SL}_2(\mathbb{Z})$ is surjective.

Finally, let h be a homeomorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$ preserving O such that $h_* = \mathrm{id}_{\mathbb{Z}^2}$. h may be lifted to a homeomorphism $\tilde{h}$ of the plane $\mathbb{R}^2$ preserving $(0, 0)$ and such that

$$\forall\, t \in \mathbb{R}^2 \quad \forall\, \tau \in \mathbb{Z}^2 \quad \tilde{h}(t + \tau) = \tilde{h}(t) + h_*(\tau) = \tilde{h}(t) + \tau.$$

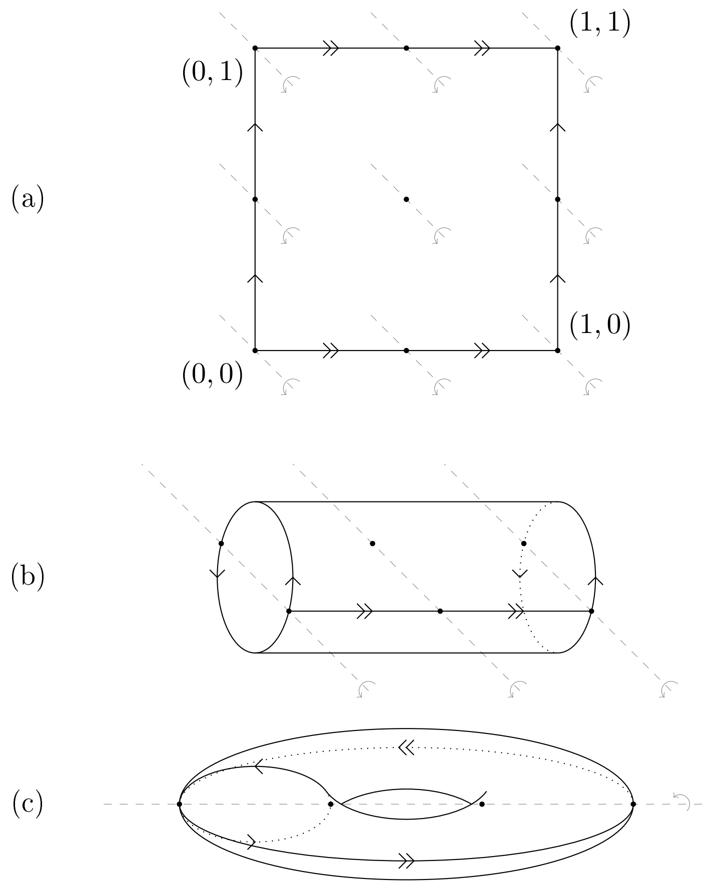


Figure 2.5: The hyperelliptic involution in three representations of the torus: (a) as the fundamental domain $[0,1]^2$ of the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translations, (b) as a fundamental domain of the action of $\mathbb{Z} \times \{0\}$ on the cylinder $\mathbb{R}^2/\{0\} \times \mathbb{Z}$, (c) as the usual embedding of the torus in the space.

Then a $\mathbb{Z}^2$-equivariant homotopy between $\tilde{h}$ and $\mathrm{id}_{\mathbb{R}^2}$ may be defined as

$$\begin{array}{ccc} [0,1] \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (s,t) & \longmapsto & (1-s)\,t + s\,\tilde{h}(t) \end{array}$$

and induces a homotopy between h and $\mathrm{id}_{\mathbb{R}^2/\mathbb{Z}^2}$. And since those two are homotopic, they are isotopic (see [FM11, Theorem 1.12]) so that the morphism $\mathrm{Mod}_{1,1} \to \mathrm{SL}_2(\mathbb{Z})$ is injective. □

2.3.4 Corollary. $\mathrm{Mod}_{0,4}$ *is isomorphic to the principal congruence subgroup of level* 2 *in* $\mathrm{PSL}_2(\mathbb{Z})$*, that is to say, the kernel of the morphism* $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ *induced by the reduction modulo* 2.

2.3.5 Notation. Let $\Gamma(2)$ denote the principal congruence subgroup of level 2 in $\mathrm{PSL}_2(\mathbb{Z})$, not to be confused with its counterpart in $\mathrm{SL}_2(\mathbb{Z})$.

Proof. The action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus commutes with the hyperelliptic involution. It induces therefore an action of $\mathrm{PSL}_2(\mathbb{Z})$ on the sphere. Moreover, as the former action preserves the set $(\frac{1}{2}\mathbb{Z}^2)/\mathbb{Z}^2$, the latter preserves the subset of the sphere consisting of the 4 ramification points. The subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ preserving each of those 4 points is exactly $\Gamma(2)$. Thus there exists a natural morphism $\Gamma(2) \to \mathrm{Mod}_{0,4}$. It follows from [FM11, Proposition 2.7] that this morphism is an isomorphism. □

Nielsen-Thurston classification. Any element of $\mathrm{Mod}_{g,n}$ admits a representative h which is either

1. *periodic*, that is to say, some power of h is the identity,
2. *reducible*, that is to say, h preserves some finite union of disjoint simple closed curves on the surface,
3. *pseudo-Anosov*, that is to say, there exists a pair of transverse measured foliations $(\mathcal{F}^s, \mu_s)$ and $(\mathcal{F}^u, \mu_u)$ on the surface and a number $\lambda > 1$ such that

$$h_*(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1}\mu_s) \quad \text{and} \quad h_*(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u)$$

(see Theorem 13.2 in [FM11] and also section 11.2 for a presentation of those objects).

2.3.6 Examples. When S is a sphere or a torus, the Nielsen-Thurston classification is quite elementary as it boils down to the study of 2-by-2 matrices. The group $\mathrm{Mod}_{1,1}$ is indeed isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ (see 2.3.3). Let A be a matrix in $\mathrm{SL}_2(\mathbb{Z})$ which is not the identity. Such a matrix is conjugate in $\mathrm{SL}_2(\mathbb{R})$ either to

1. a diagonal matrix whose entries are conjugate complex numbers of modulus 1, in which case $|\operatorname{tr}(A)| < 2$ and A acts on the plane as a finite-order rotation,
2. an upper triangular matrix whose diagonal entries are equal to 1, in which case $|\operatorname{tr}(A)| = 2$ and A acts on the plane as a transvection, hence preserving a line pointwise,
3. a diagonal matrix whose entries are real numbers, inverse of each other, in which case $|\operatorname{tr}(A)| > 2$ and the action of A on the plane has two privileged directions (or foliations), one that is contracted and one that is dilated.

The matrix A is respectively called *elliptic*, *parabolic* or *hyperbolic*. Consequently, the periodic, reducible or pseudo-Anosov nature of a mapping class is simply determined by the trace of the representative matrix.

Quite the same goes for $\mathrm{Mod}_{0,4}$ (see corollary 2.3.4). Let A be a matrix representing an element of $\Gamma(2)$. The action of A on the torus induces an action on the sphere, through the branched covering map mentionned above. Similarly, the periodic, reducible or pseudo-Anosov nature of a mapping class is determined by the absolute value of the trace of A. For example, since the absolute value of the trace of any matrix representing a non-trivial element of $\Gamma(2)$ is at least 2, $\mathrm{Mod}_{0,4}$ contains no periodic element.

On the contrary, determining the nature of a mapping class of a surface of higher genus is much more complex.

2.3.7 Definition. A *surface bundle over the circle* or a *mapping torus* is a quotient space of the form $(S \times \mathbb{R})/\mathbb{Z}$ where S is a closed surface and $\mathbb{Z}$ acts on $S \times \mathbb{R}$ by $n \cdot (x, t) = (h^n(x), t + n)$ where $h : S \to S$ is a homeomorphism. This space is denoted by M_h and the projection $\mathrm{pr}_2 : S \times \mathbb{R} \to \mathbb{R}$ induces a fibration $M_h \to \mathbb{R}/\mathbb{Z}$ over the circle, with fiber S.

2.3.8 Remarks. The previous construction depends, up to homeomorphism, only on the isotopy class of h, that is to say, on the class of h in $\mathrm{Mod}(S)$. Moreover, it only depends on the conjugacy class of the class of h in $\mathrm{Mod}(S)$. Furthermore, M_h and $M_{h^{-1}}$ are also homeomorphic. If A is a subset of S and h stabilizes each point in A, then M_h depends only on the conjugacy class of the class of h in $\mathrm{Mod}(S, A)$ and M_h contains the subset $A \times \mathbb{S}^1$.

Thurston has shown that if S is a closed surface of some genus $g \geq 2$ and if h is a homeomorphism of S, then the surface bundle M_h admits a hyperbolic structure if and only if h is pseudo-Anosov [Thu88, Ota96, FLP91].

The group $\Gamma(2)$ has multiple interests in the present context, which are not purely coincidental as shown in the following: it appears as a lattice in the group $\mathrm{Isom}^+(\mathbb{H}^2_{\mathbb{R}})$ and it is isomorphic to the mapping class group $\mathrm{Mod}_{0,4}$.

2.3.9 Proposition. *The* 3*-punctured sphere* $(\mathbb{P}^1)^u$ *is homeomorphic to the quotient* $\Gamma(2)\backslash\mathbb{H}^2_{\mathbb{R}}$*, which is a hyperbolic surface with* 3 *cusps. A presentation of* $\Gamma(2)$ *is* $\langle T_\infty, T_0, T_1 \mid T_\infty T_0 T_1 = 1\rangle$ *where*

$$T_\infty = (TS)^0 T^2 (TS)^{-0} = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$$

$$T_1 = (TS)^1 T^2 (TS)^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

$$T_0 = (TS)^2 T^2 (TS)^{-2} = \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}.$$

The group $\mathrm{PSL}_2(\mathbb{Z})$ is a lattice in $\mathrm{PSL}_2(\mathbb{R})$ generated by the elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and a fundamental domain is drawn in figure 2.6.

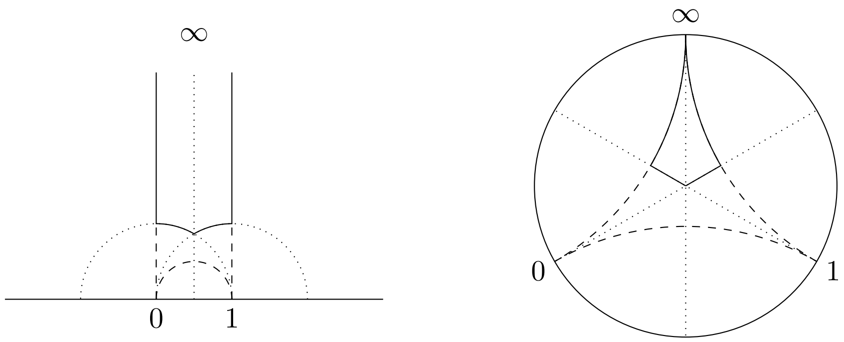


Figure 2.6: A fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ in the half-plane and disc models of hyperbolic plane. The skeleton of an ideal triangle drawn with dashes.

Alternatively, $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and TS. Those two are very particular elements of $\mathrm{PSL}_2(\mathbb{Z})$, since S is a hyperbolic rotation of angle π and TS is a hyperbolic rotation of angle $-\frac{2\pi}{3}$, whose centers are corners of the fundamental domain drawn in figure 2.6, respectively i and $e^{i\pi/3}$ in the half-plane model.

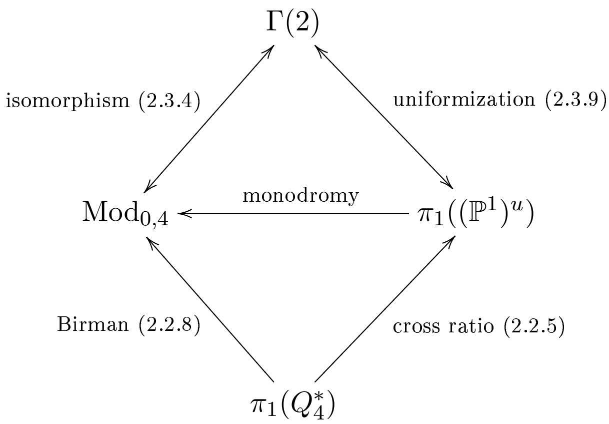
The group $\mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to the group $\mathfrak{S}_3$ of the permutations of 3 elements: it indeed consists of the 6 elements

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \begin{pmatrix} & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} \quad \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$$

and acts by permutations on the three points $\infty, 0, 1$ of the projective line over $\mathbb{Z}/2\mathbb{Z}$.

Proof of proposition 2.3.9. The orbit of the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ by the element TS of order 3 is the ideal triangle with vertices $\infty, 0, 1$. And the element S maps the ideal triangle on an adjacent copy of it. Hence a fundamental domain of the subgroup $\Gamma(2)$ is two adjacent copies of the ideal triangle. Therefore the quotient of $\mathbb{H}^2_{\mathbb{R}}$ by $\Gamma(2)$ is a sphere with three punctures. $\square$

This diagram sums up the facts presented above.



Finally, the monodromy morphism $\pi_1((\mathbb{P}^1)^u) \to \mathrm{Mod}_{0,4}$ is an isomorphism and it is quite elementary to determine whether the monodromy of a loop is pseudo-Anosov by calculating the trace of the corresponding element of $\Gamma(2)$. Such an element may be given

1. either in the form of a matrix, with the advantage of being able to compute its trace easily,

2. or as a product of the generators T_∞, T_0, T_1, which allows to read that element of $\Gamma(2)$ as a loop in the sphere with three punctures.

However, if an element of $\Gamma(2)$ is given in the latter form, rather than as a matrix, there is no direct method for calcutating either its entries or its trace other than computing the product.

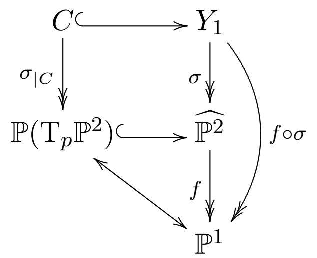
2.4 Fibration of Hirzebruch's surface

Composing $\sigma : Y_1 \to \widehat{\mathbb{P}^2}$ and $f : \widehat{\mathbb{P}^2} \to \mathbb{P}^1$ yields a fibration $f \circ \sigma : Y_1 \to \mathbb{P}^1$. Let p be one of the four triple intersection points of the arrangement of lines in $\mathbb{P}^2$, and q be one of the 5^2 points of X over p. Let C be the connected curve in Y_1 obtained by resolving the singular point q in X.

The restriction $\sigma_{|C} : C \to \mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ is a branched covering map of degree 5^2 which ramifies over the points in $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ corresponding to the lines of the arrangement passing through p. The exact same goes for $f \circ \sigma_{|C} : C \to \mathbb{P}^1$, since $f_{|\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)} : \mathbb{P}(\mathrm{T}_p\mathbb{P}^2) \to \mathbb{P}^1$ is an isomorphism. And since $n = 5$ and $m = 3$, the Euler characteristic of C is

$$e(C) = 5^{3-1}(2-3) + 3 \cdot 5^{3-2} = -10 = 2 - 2 \times 6$$

so that C is a smooth curve of genus 6.

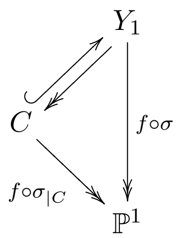


As well as the fibration $f : \widehat{\mathbb{P}^2} \to \mathbb{P}^1$ admits natural sections

$$\mathbb{P}^1 \to \mathbb{P}(\mathrm{T}_p\mathbb{P}^2) \subset \widehat{\mathbb{P}^2},$$

one may want to show that the inclusion $C \to Y_1$ is a section of a fibration $Y_1 \to C$.

2.4.1 Proposition. *There exists a fibration $Y_1 \to C$ with connected fibers, such that the inclusion $C \to Y_1$ is a section and that the following diagram is commutative.*



In other words, the composition of $Y_1 \to C$ and of the branched covering map $f \circ \sigma_{|C} : C \to \mathbb{P}^1$ is the Stein factorization of $f \circ \sigma : Y_1 \to \mathbb{P}^1$.

The curve C is of genus 6 and the generic fibers under $Y_1 \to C$ are smooth curves of genus 76. The singular fibers under $Y_1 \to C$ lie over the points of C over which the branched covering map $f \circ \sigma_{|C} : C \to \mathbb{P}^1$ is ramified, so that there are 3×5 such fibers.

The following proof resorts implicitly and repeatedly to proposition A.2.3.

Proof. For any point b in $\mathbb{P}^1$, $f^{-1}(b)$ is the proper transform in $\widehat{\mathbb{P}^2}$ of a conic in $\mathbb{P}^2$. $f^{-1}(b)$ and $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ meet at a single point, denoted by b_p.

If b is not one of the three points for which $f^{-1}(b)$ is singular, then $f^{-1}(b)$ does not intersect (the proper transforms of) the lines of the arrangement but intersects the four exceptional curves $\mathbb{P}(\mathrm{T}_{p'}\mathbb{P}^2)$. Since they intersect normally, $(f \circ \sigma)^{-1}(b)$ is smooth and the restriction $\sigma_{|(f\circ\sigma)^{-1}(b)} : (f \circ \sigma)^{-1}(b) \to f^{-1}(b)$ is a Galois branched covering map which ramifies exactly over the intersection of $f^{-1}(b)$ with the four exceptional curves.
Let Z be a connected component of $(f \circ \sigma)^{-1}(b)$. Consider the branched

covering map $\sigma_{|Z} : Z \to f^{-1}(b)$ and the corresponding unbranched one ${\sigma_{|Z}}^u : Z^u \to f^{-1}(b)^u$. Z^u (obtained from Z by removing the branch points) is still connected. Hence, given any base point $z \in Z^u$, the Galois group $\mathrm{Aut}(\sigma_{|Z})$ is naturally isomorphic to the image subgroup of $\alpha_z : \pi_1(f^{-1}(b)^u, \sigma(z)) \to \mathrm{Aut}(\sigma)$. Since $f^{-1}(b)^u$ is homeomorphic to a sphere with four punctures, the fundamental group $\pi_1(f^{-1}(b)^u, \sigma(z))$ is generated by the homotopy classes of four loops around the punctures (three are actually enough). The subgroup $\mathrm{Im}\,\alpha_z$ is hence generated (see 2.1.11) by (any three among) the four elements

$$\prod_{D \ni p'} \alpha_D.$$

Besides, $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q)$ is generated (see 2.1.11) by the automorphisms α_D, for the lines D passing through p. It appears, on the one hand, that $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \cap \mathrm{Aut}(\sigma_{|Z})$ is the cyclic subgroup generated by $\prod_{D \ni p} \alpha_D$ which acts trivially on C and, on the other hand, that $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q)\,\mathrm{Aut}(\sigma_{|Z}) = \mathrm{Aut}(\sigma)$.
As b_p belongs to $f^{-1}(b)$, $Z \cap \sigma^{-1}(b_p)$ is not empty. Let z be a point in the latter set and let α be an automorphism of σ such that $\alpha(z) \in C$. Since $\mathrm{Aut}(\sigma) = \mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) + \mathrm{Aut}(\sigma_{|Z})$, α may actually be chosen in $\mathrm{Aut}(\sigma_{|Z})$, so that $\alpha(z) \in Z \cap C$. And since $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \cap \mathrm{Aut}(\sigma_{|Z})$ acts trivially on C, $Z \cap C$ contains exactly one point.

If b is one of the three points for which $f^{-1}(b)$ is singular, $f^{-1}(b)$ is more precisely the union of (the proper tranforms of) two lines of the arrangement, say D_{12} and D_{34}, the former passing through triple intersection points denoted by p_1 and p_2 and the latter through p_3 and p_4. By a slight abuse of notations, the proper transforms, denoted by D_{12} and D_{34}, intersect at a point p_5 and each of them also intersects two of the exceptional curves, the former at p_1 and p_2, the latter at p_3 and p_4. Since the intersections are normal, $\sigma^{-1}(D_{12})$ is smooth and the restriction $\sigma : \sigma^{-1}(D_{12}) \to D_{12}$ is a Galois branched covering map of degree 5^4 ramified over p_1, p_2, p_5, with index 5. The exact same goes for $\sigma^{-1}(D_{34})$ over p_3, p_4, p_5.
Furthermore, if Z_{12} is a connected component of $\sigma^{-1}(D_{12})$, then $\mathrm{Aut}(\sigma_{|Z_{12}})$ is naturally isomorphic to the subgroup of $\mathrm{Aut}(\sigma)$ generated by

$$\alpha_{D_{34}} \qquad \prod_{D \ni p_1} \alpha_D \qquad \prod_{D \ni p_2} \alpha_D$$

and if Z_{34} is a connected component of $\sigma^{-1}(D_{34})$, then $\mathrm{Aut}(\sigma_{|Z_{34}})$ is naturally isomorphic to the subgroup of $\mathrm{Aut}(\sigma)$ generated by

$$\alpha_{D_{12}} \qquad \prod_{D \ni p_3} \alpha_D \qquad \prod_{D \ni p_4} \alpha_D.$$

Therefore, the subgroup of $\mathrm{Aut}(\sigma)$, denoted by H, preserving the connected components of $\sigma^{-1}(D_{12} \cup D_{34})$ is generated by $\alpha_{D_{12}}$, $\alpha_{D_{34}}$ and the four elements

$$\prod_{D \ni p'} \alpha_D$$

with $p' \in \{p_1, p_2, p_3, p_4\}$.
Let Z be a connected component of $\sigma^{-1}(D_{12} \cup D_{34})$ and let z be a point in Z such that $\sigma(z) = b_p$. Assuming that $p = p_1$, b_p is then the point in $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$ corresponding to the direction tangent to D_{12}. In particular, $\alpha_{D_{12}}(z) = z$ since $\sigma(z) = b_p$. Let α be an automorphism of σ such that $\alpha(z) \in C$. Since $\mathrm{Aut}(\sigma) = \mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) + H$, α may actually be chosen in H, so that $\alpha(z) \in Z \cap C$. And since $\mathrm{Stab}_{\mathrm{Aut}(\chi)}(q) \cap H$ acts trivially on z, $Z \cap C$ contains exactly one point.

In conclusion, each connected component of $(f \circ \sigma)^{-1}(b)$ meets C at exactly one point and one can define a fibration $Y_1 \to C$ by mapping any connected component of $(f \circ \sigma)^{-1}(b)$ to the only point in its intersection with C. This fibration is nothing but the Stein factorization of $f \circ \sigma$, since the fibers of $Y_1 \to C$ are exactly the connected components of those of $f \circ \sigma : Y_1 \to \mathbb{P}^1$.

As $f \circ \sigma_{|C} : C \to \mathbb{P}^1$ is a branched covering map of degree 5^2, a generic fiber $(f \circ \sigma)^{-1}(b)$ has then 5^2 connected components and total Euler characteristic

$$5^5(2-4) + 5^4 \times 4 = -6 \times 5^4$$

so that each connected component has Euler characteristic -6×5^2 and genus $1 + 3 \times 5^2 = 76$. □

2.4.2 Remark. The fibration $Y_1 \to C$ seems combinatorially complex since the base curve is of genus 6 with 15 ramification points and the generic fiber is of genus 76 with 4×5^2 marked points lying over the 4 marked points of the generic fiber of the pencil of conics. For instance, writing group presentations of fundamental groups of these spaces or of their corresponding mapping class groups is a laborious task.

However, recall that the much simpler manifold $\widehat{\mathbb{P}^2}$ bears an orbifold structure that is the quotient of the complex hyperbolic manifold Y_1. The fibration $Y_1 \to C$ itself arises from a fibration of $\widehat{\mathbb{P}^2}$. The base curve is a sphere with 3 punctures and the generic fiber is a conic with 4 marked points. The mapping class group of the generic fiber is much simpler than the mapping class group of a surface of higher genus, which makes the monodromy potentially simpler.

In the remainder of the present section, the base curve and the generic and singular fibers are studied in more detail.

2.4.3 Notation. In the following, fundamental groups of the spaces at play will be considered quite often. In order to avoid choosing base points each time, one should choose them once and for all. Let y_0 be a base point in ${Y_1}^u$ which will also serve as a base point of Y_1. Let c_0 denote the projection of y_0 to C so that c_0 will be the base point of both C and C^u. Besides, the fiber over c_0 of $Y_1 \to C$ will be denoted by F_0 and will be called the base fiber. The point y_0 belongs to F_0 and will be its base point. One may deduces base points similarly for $\widehat{\mathbb{P}^2}$, $\widehat{\mathbb{P}^2}^u$, $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$, $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)^u$, $\mathbb{P}^1$ and $(\mathbb{P}^1)^u$.

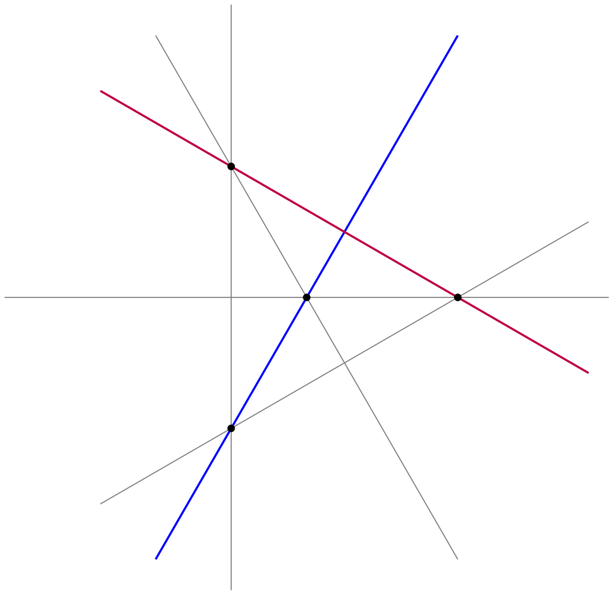


Figure 2.7: The two irreducible components of a singular conic.

2.4.4 Corollary. *The* 15 *singular fibers under* $Y_1 \to C$ *are isomorphic to*

$$(S_{12} \times I_{34}) \cup (I_{12} \times S_{34})$$

where S_{12} *and* S_{34} *are connected components of* $\sigma^{-1}(D_{12})$ *and* $\sigma^{-1}(D_{34})$ *respectively and* I_{12} *and* I_{34} *are the subsets of* S_{12} *and* S_{34} *respectively whose points lie over the intersection point of* D_{12} *and* D_{34}.

S_{12} *is a compact curve of genus* 6 *and a Galois branched covering space of* D_{12}, *of degree* 5^2, *ramified over three points and* I_{12} *consists of* 5 *points. The exact same goes for* S_{34} *and* I_{34}.

Proof. Let D_{12} and D_{34} denote the two irreducible components of a singular fiber of $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$. Then $\sigma^{-1}(D_{12})$ is a Galois branched covering space of D_{12} of degree $5^{5-1} = 5^4$ and ramified over 3 points with ramification index 5 (one is the point where D_{12} and D_{34} intersect and the other two are points

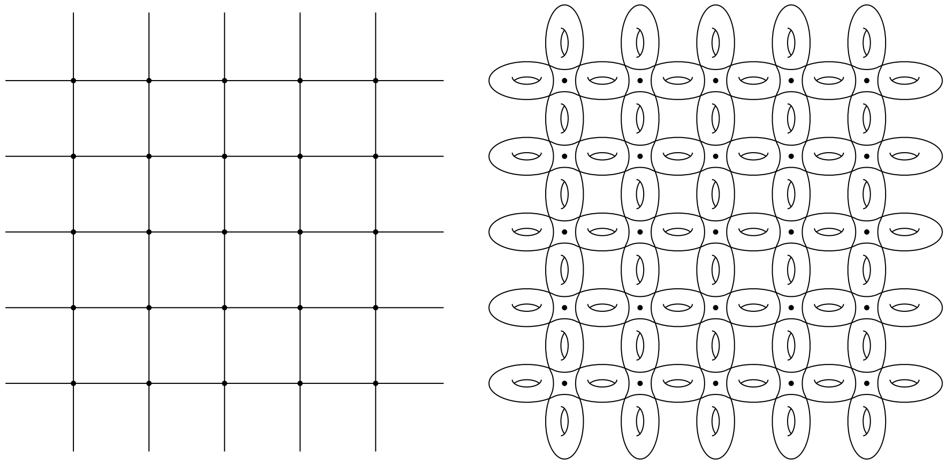


Figure 2.8: Two representations of the shape of the 15 singulars fibers: one on the left where the irreducible components are symbolically represented as line segments, one on the right where the irreducible components are more realistic whereas their intersection points are marked as thick dots.

where D_{12} intersects two of the four exceptional curves). Hence the Euler characteristic of $\sigma^{-1}(D_{12})$ is

$$5^4(2-3) + 3 \times 5^3 = -2 \times 5^3.$$

Let S_{12} be a connected component of $\sigma^{-1}(D_{12})$. The Galois group $\mathrm{Aut}(\sigma_{|S_{12}})$ is isomorphic to the quotient of the subgroup of $\mathrm{Aut}(\sigma)$ generated by the three elements (two are actually enough)

$$\alpha_{34}, \quad \alpha_{12}\alpha_{13}\alpha_{23}, \quad \alpha_{12}\alpha_{14}\alpha_{24}$$

by $\langle \alpha_{12} \rangle$. The fiber has then 5^2 connected components, so that each of them has Euler characteristic -2×5 and genus $1 + 5 = 6$. The same goes for the connected components of $\sigma^{-1}(D_{34})$. Let S_{34} be one of them and assume that it meets S_{12} at a point q. The Galois group $\mathrm{Aut}(\sigma_{|S_{34}})$ is isomorphic to the quotient of the subgroup of $\mathrm{Aut}(\sigma)$ generated by the elements

$$\alpha_{12}, \quad \alpha_{34}\alpha_{13}\alpha_{14}, \quad \alpha_{34}\alpha_{23}\alpha_{24}$$

by $\langle \alpha_{34} \rangle$. Since the intersection of the subgroups

$$\langle \alpha_{34}, \alpha_{12}\alpha_{13}\alpha_{23}, \alpha_{12}\alpha_{14}\alpha_{24} \rangle \quad \text{and} \quad \langle \alpha_{12}, \alpha_{34}\alpha_{13}\alpha_{14}, \alpha_{34}\alpha_{23}\alpha_{24} \rangle$$

is $\langle \alpha_{12}, \alpha_{34} \rangle$ which acts trivially on the point q, S_{12} and S_{34} meet at exactly one point.

The connected component of $\sigma^{-1}(D_{12} \cup D_{34})$ containing q is the union of the orbit of S_{12} under the action of $\langle \alpha_{34}, \alpha_{12}\alpha_{13}\alpha_{23}, \alpha_{12}\alpha_{14}\alpha_{24} \rangle$ and the orbit of S_{34} under the action of $\langle \alpha_{12}, \alpha_{34}\alpha_{13}\alpha_{14}, \alpha_{34}\alpha_{23}\alpha_{24} \rangle$. These orbits consists of five copies of S_{12} and S_{34} respectively (see figure 2.8). □

2.4.5 Remarks. The curves S_{12} and S_{34} are biholomorphic since they are covering spaces of lines of the arrangement which play symmetric roles.

The resolution of the 5^2 singularities of the singular fibers yields curves of genus $6 \times (5+5) + (5-1)^2 = 76$ (see figure 2.8), which is indeed equal to the genus of the generic fiber.

The following lemma aims at describing the kernel of a morphism from a free group to a finite abelian group. Consider the topological interpretation of a free group as a fundamental group of a wedge sum of circles. More precisely, the image in the torus $\mathbb{R}^m/\mathbb{Z}^m$ of the coordinate axes of $\mathbb{R}^m$ is a wedge sum of m circles, denoted by B_m, with a base point b. The fundamental group $\pi_1(B_m, b)$ is indeed a free group with m generators $c_1, \ldots, c_m$. The group morphism $\pi_1(B_m, b) \to \mathbb{Z}^m$ induced by the inclusion $B_m \to \mathbb{R}^m/\mathbb{Z}^m$ is nothing but the abelianization morphism, mapping the generator c_1 to the element $(1, 0, \ldots, 0)$ and so on.

2.4.6 Lemma. *If R is a subgroup of $\mathbb{Z}^m$ of index d then the torus $\mathbb{R}^m/R$ is naturally a covering space of $\mathbb{R}^m/\mathbb{Z}^m$, of degree d. Let $\hat{B}_m$ denote the covering space of B_m, obtained by pulling back B_m as follows.*

$$\begin{array}{ccc} (\hat{B}_m, \hat{b}) & \hookrightarrow & (\mathbb{R}^m/R, 0) \\ \downarrow & & \downarrow \\ (B_m, b) & \hookrightarrow & (\mathbb{R}^m/\mathbb{Z}^m, 0) \end{array}$$

Then the kernel of the morphism $\pi_1(B_m, b) \to \mathbb{Z}^m/R$ is isomorphic to the fundamental group $\pi_1(\hat{B}_m, \hat{b})$. Moreover, $\hat{B}_m$ has the homotopy type of a wedge sum of $d(m-1)+1$ circles.

If $R = k\mathbb{Z}^m$, then $d = k^m$ and the kernel of $\pi_1(B_m, b) \to (\mathbb{Z}/k\mathbb{Z})^m$ is generated by the elements ${c_1}^k, \ldots, {c_m}^k$ and the commutators $[{c_i}^p, {c_j}^q]$ for $1 \leq i, j \leq m$ and $1 \leq p, q \leq k$.

Proof. All the assertions are quite straightforward. The Euler characteristic of B_m is $e(B_m) = 1 - m$. Thus that of $\hat{B}_m$ is $e(\hat{B}_m) = d\, e(B_m) = d(1-m)$. Since $\hat{B}_m$ has the homotopy type of a wedge of circles, the number of those circles must be $d(m-1)+1$. □

2.4.7 Proposition. *As a covering space of $(\mathbb{P}^1)^u$, C^u admits a hyperbolic structure. More precisely, C^u is homeomorphic to the quotient of $\mathbb{H}^2_{\mathbb{R}}$ by the normal subgroup of $\Gamma(2)$ of index 5^2 formed by all the possible products of T_∞, T_0 and T_1 (and their inverses) where the numbers of occurrences of T_∞, T_0 and T_1 respectively (counted with their multiplicity, say, p for ${T_\infty}^p$) differ by multiples of 5. Besides, that group is generated by ${T_\infty}^5$, ${T_0}^5$, ${T_1}^5$ and the commutators of powers of T_∞, T_0, T_1.*

2.4.8 Remark. Since the generators T_∞, T_0, T_1 satisfy the relation

$$T_\infty T_0 T_1 = 1,$$

the previous properties may be written only in terms of two of the generators. As a fundamental group of a sphere with three punctures, $\pi_1(C^u)$ is indeed isomorphic to the free group with two generators, say T_∞ and T_0.

Any element of $\pi_1(C^u)$, written as a product of T_∞, T_0 and T_1, may be interpreted as (the homotopy class of) the lift to C^u of a loop in $(\mathbb{P}^1)^u$ obtained by turning around the puncture corresponding the factor T_∞, T_0 or T_1, each time one of them appears in the product. Representing a loop in $(\mathbb{P}^1)^u$ rather than in C^u is indeed easier since C^u is a Riemann surface of genus 6 with 15 punctures.

Proof. The unbranched covering map $\sigma : C^u \to (\mathbb{P}^1)^u$ induces a short exact sequence

$$1 \longrightarrow \pi_1(C^u, y) \xrightarrow{(\sigma_{|C^u})_*} \pi_1((\mathbb{P}^1)^u, \sigma(y)) \xrightarrow{\alpha_y} \mathrm{Aut}(\sigma_{|C^u}) \longrightarrow 1$$

where y denotes the base point of C^u. Identify $\pi_1((\mathbb{P}^1)^u, \sigma(y))$ with $\Gamma(2) = \langle T_\infty, T_0, T_1 \mid T_\infty T_0 T_1 = 1\rangle$ (see 2.3.9). With 2.1.11, the image of T_∞ by α_y is the automorphism α_D of σ where D is the line of the arrangement corresponding, in the identification of $\Gamma(2)\backslash\mathbb{H}^2_{\mathbb{R}}$ with $(\mathbb{P}^1)^u$ and $\mathbb{P}(\mathrm{T}_p\mathbb{P}^2)$, to the image of ∞. And similarly for 0 and 1.

Identify $\mathrm{Aut}(\sigma_{|C})$ with $(\mathbb{Z}/5\mathbb{Z})^2$ so that the morphism $\alpha_y : \Gamma(2) \to (\mathbb{Z}/5\mathbb{Z})^2$ maps T_∞ to $(1, 0)$, T_0 to $(0, 1)$ and T_1 to $(-1, -1)$. Since $\pi_1(C^u, y)$ is isomorphic to the kernel of α_y, it is also isomorphic to the subgroup of $\Gamma(2)$ formed by the products of T_∞, T_0 and T_1 (and their inverses) where the numbers of occurrences of T_∞, T_0 and T_1 respectively (counted with their multiplicity, say, p for ${T_\infty}^p$) differ by multiples of 5.

According to lemma 2.4.6, $\pi_1(C^u, y)$ is isomorphic to the subgroup of $\Gamma(2)$ generated by ${T_\infty}^5$, ${T_0}^5$, ${T_1}^5$ and the commutators of powers of T_∞, T_0, T_1. □

The Riemann surface C may also be uniformized. Instead of cusps and parabolic isometries as on $(\mathbb{P}^1)^u$ and C^u, consider the hyperbolic orbifold structure on $\mathbb{P}^1$ where the three points in $\mathbb{P}^1 \setminus (\mathbb{P}^1)^u$ have conic angle $2\pi/5$. Such a structure may be constructed by considering a (regular) hyperbolic triangle with angle $\pi/5$ at each vertex. Thus the quotient of $\mathbb{H}^2_{\mathbb{R}}$ by the triangle group $T(5, 5, 5)$ is an orbifold homeomorphic to $\mathbb{P}^1$: the triangle group $T(5, 5, 5)$ is the subgroup of index 2, formed by the orientation-preserving isometries, of the group generated by the reflections with respect to the sides of the hyperbolic triangle with angle $\pi/5$ at each vertex. It is generated by the rotations of angle $2\pi/5$ around the vertices of the triangle and any two adjacent translates of the triangle form a fundamental domain. Let

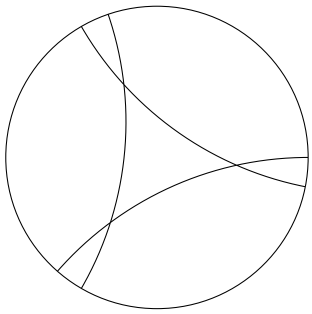


Figure 2.9: A hyperbolic triangle with angle $\pi/5$ at each vertex.

R_1, R_2, R_3 denote the rotations of angle $2\pi/5$ around the vertices of such a triangle, indexed such that they satisfy the relation $R_3R_2R_1 = 1$.

2.4.9 Proposition. *The Riemann surface C is homeomorphic to the quotient of $\mathbb{H}^2_{\mathbb{R}}$ by the normal subgroup of $T(5,5,5)$ of index 5^2 formed by all the possible products of R_1, R_2 and R_3 (and their inverses) where the numbers of occurrences of R_1, R_2 and R_3 respectively (counted with their multiplicity, say, p for ${R_1}^p$) differ by multiples of 5. Besides, that group is generated by the commutators of powers of R_1, R_2, R_3.*

Proof. Similar to the proof of proposition 2.4.7. □

2.4.10 Proposition. *The surjective morphism $\pi_1(C^u) \to \pi_1(C)$ induced by the inclusion $C^u \to C$ is the restriction (to the corresponding subgroups) of the morphism $\Gamma(2) \to T(5,5,5)$ mapping T_∞, T_0, T_1 to R_3, R_2, R_1 respectively. In particular, the kernel is the smallest normal subgroup of $\Gamma(2)$ generated by ${T_\infty}^5, {T_1}^5, {T_0}^5$. The kernel contains all the parabolic elements of $\pi_1(C^u)$.*

Proof. Following propositions 2.4.7 and 2.4.9, $\pi_1(C^u)$ and $\pi_1(C)$ are identified to subgroups of $\Gamma(2)$ and $T(5,5,5)$ respectively, in such a way that the diagram

$$\begin{array}{ccc} \pi_1(C^u) & \hookrightarrow & \Gamma(2) \simeq (\mathbb{P}^1)^u \\ \downarrow & & \downarrow \\ \pi_1(C) & \hookrightarrow & T(5,5,5) \end{array}$$

is commutative, where $T(5,5,5)$ is as the fundamental group of quotient orbifold and that the morphism $\Gamma(2) \to T(5,5,5)$ maps the generators T_∞, T_0, T_1 to R_3, R_2, R_1 respectively. Observe that the kernel of the latter morphism is the smallest normal subgroup of $\Gamma(2)$ generated by the three elements ${T_\infty}^5, {T_1}^5, {T_0}^5$. As these elements belong to $\pi_1(C^u)$, the kernel of the morphism $\pi_1(C^u) \to \pi_1(C)$ is also the smallest normal subgroup of $\pi_1(C^u)$ generated by ${T_\infty}^5, {T_1}^5, {T_0}^5$.

Any parabolic element of $\pi_1(C^u)$ is conjugate in $\Gamma(2)$ to a power of T_∞, T_0 or T_1, hence to a power of ${T_\infty}^5$, ${T_0}^5$ or ${T_1}^5$ according to 2.4.7. Therefore, any parabolic element is contained in the kernel. □

In section 2.1.2, a set of matrices generating the lattice G_1 is given. Recalling that the morphism $\pi_1(C) \to \pi_1(Y_1)$ is injective and identifying $\pi_1(Y_1)$ with the commutator subgroup $[G_1, G_1]$, the following proposition shows that the image of the morphism $\pi_1(C) \to G_1$ is a subgroup stabilizing a complex line in $\mathbb{H}^2_{\mathbb{C}}$.

2.4.11 Proposition. *The fundamental group of C is isomorphic to the commutator subgroup of the subgroup of $\mathrm{PGL}_3(\mathbb{C})$ generated by $R(ij)$, $R(jk)$, $R(ik)$ for some distinct indices i, j and k (two of them are actually sufficient).*

Choosing for example, $R(01)$, $R(02)$ and $R(12)$, it appears that the group in question preserves the line in $\mathbb{C}^3$ directed by $(0, 0, 1)$, which is positive. Therefore it preserves a complex plane in $\mathbb{C}^3$ with signature $(1, 1)$ and hence a complex line in $\mathbb{H}^2_{\mathbb{C}}$.

Proof. According to proposition 2.4.9, the fundamental group of C is isomorphic to the subgroup of the triangle group $T(5, 5, 5)$ generated the commutators of the elements R_1, R_2, R_3. These elements correspond to loops around the three lines of the arrangement passing a given triple point and hence to a triple of the form $R(ij)$, $R(jk)$, $R(ik)$ for some distinct indices i, j and k.

The remainder is straightforward computations. □

2.5 Representations

Recall the notations 2.4.3 about base points of fundamental groups. For any element γ in $\pi_1(C^u)$, let $M_\gamma \to \mathbb{R}/\mathbb{Z}$ be the surface bundle over the circle with fiber F_0 and where the homeomorphism is the monodromy of the fibration ${Y_1}^u \to C^u$ along γ. If a loop $\mathbb{R}/\mathbb{Z} \to C^u$ represents γ, then there is a natural mapping $M_\gamma \to Y_1$ such that the diagram

$$\begin{array}{ccc} M_\gamma & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \longrightarrow & C \end{array}$$

is commutative. For instance, if the loop $\mathbb{R}/\mathbb{Z} \to C^u$ happens to be an embedding or an immersion, then the same goes for $M_\gamma \to Y_1$.

The mapping $M_\gamma \to Y_1$ induces a morphism $\rho_\gamma : \pi_1(M_\gamma) \to \pi_1(Y_1)$ and hence a representation into a complex hyperbolic lattice. The manifold M_γ, the fibration $M_\gamma \to \mathbb{R}/\mathbb{Z}$ and of course the conjugacy class of the representation ρ_γ depend only on the conjugacy class of γ in $\pi_1(C^u)$. They do not moreover depend on the orientation of γ.

Since $\pi_1((\mathbb{P}^1)^u) \to \mathrm{Mod}_{0,4}$ is an isomorphism, every mapping class in $\mathrm{Mod}_{0,4}$ can be realized as the monodromy along a curve in $(\mathbb{P}^1)^u$, of the fibration $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$. The generic fiber of $\widehat{\mathbb{P}^2} \to \mathbb{P}^1$ is a sphere with 4 marked points. Therefore, all the possible surface bundles with the sphere as fiber and with monodromy preserving each of the 4 marked points are obtained in this way.
The same construction of surface bundles for the fibration $\widehat{\mathbb{P}^2}^u \to (\mathbb{P}^1)^u$, instead of ${Y_1}^u \to C^u$ as above, hence produces representations of the fundamental groups of all those surface bundles. More precisely, the complex hyperbolic structure on Y_1 descends to a branched complex hyperbolic structure on $\widehat{\mathbb{P}^2}$ by the branched covering $Y_1 \to \widehat{\mathbb{P}^2}$. And the fibers of the latter surface bundles are seen as orbifolds with isotropy of order 5 at each of the four marked points. For γ in $\pi_1(C^u)$, the surface bundle M_γ is nothing but a branched covering of the orbifold surface bundle whose monodromy is the image of γ by $\pi_1(C^u) \to \pi_1((\mathbb{P}^1)^u)$.

Proposition. *For each element f of $\mathrm{Mod}_{0,4}$, consider the surface bundle M_f with monodromy f and with fiber the orbifold with the sphere as underlying space and with isotropy of order 5 at each of the four marked points. There is a representation of the orbifold fundamental group of M_f into a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$.*

The group $\pi_1(M_\gamma)$ is isomorphic to the semi-direct product $\langle\gamma\rangle \ltimes \pi_1(F_0)$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(F_0) & \hookrightarrow & \pi_1(M_\gamma) & \rightleftarrows & \langle\gamma\rangle & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle \rho_\gamma} & & \downarrow & & \\
 & & & & \pi_1(Y_1) & \rightleftarrows & \pi_1(C) & &
\end{array}$$

2.5.1 Proposition. *For any γ in $\pi_1(C^u)$, the limit set of the image of the representation $\rho_\gamma : \pi_1(M_\gamma) \to \pi_1(Y_1)$ is all of $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$.*

The proposition shows that the representation ρ_γ is quite chaotic. If the limit set were not all $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$, then a natural question would have been to understand the quotient by the image of ρ_γ, of its domain of discontinuity, which would have given rise to a spherical Cauchy-Riemann structure. However, the domain of discontinuity will always be empty with this kind of construction which relies on a (singular) fibration of the complex hyperbolic manifold.

Proof. Since $\pi_1(Y_1)$ is (isomorphic to) a uniform lattice, its limit set is all of $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$ and $\pi_1(Y_1)$ does not preserve any point on the boundary. Besides, since the fundamental group of the fiber of ${Y_1}^u \to C^u$ is a normal subgroup of $\pi_1({Y_1}^u)$, its image by the surjective morphism $\pi_1({Y_1}^u) \to \pi_1(Y_1)$ is a normal subgroup N of $\pi_1(Y_1)$.

If the limit set of N were empty, then N would have been contained in a compact subgroup of $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$. As N is discrete, N would have been finite and $\pi_1(C)$ would have been of finite index in $\pi_1(Y_1)$ which is impossible.

Therefore, the limit set of N is equal to the limit set of $\pi_1(Y_1)$ (see proposition 1.1.23). Finally, since $\pi_1(M_\gamma)$ contains $\pi_1(F_0)$, the limit set of the image of $\pi_1(M_\gamma) \to \pi_1(Y_1)$ is all $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$. □

Furthermore, if the monodromy of the fibration along the loop γ is pseudo-Anosov, then the 3-manifold M_γ admits a real hyperbolic structure, according to Thurston's hyperbolization theorem of surface bundles over the circle. In that case, $\pi_1(M_\gamma)$ is isomorphic to a uniform lattice in $\mathrm{Isom}(\mathbb{H}^3_{\mathbb{R}})$ whose limit set is all of $\partial_\infty \mathbb{H}^3_{\mathbb{R}}$. However, determining that lattice or the manifold M_γ is a difficult problem and will not be addressed.

2.5.2 Proposition. *For any element γ in $\pi_1(C^u)$, if its image in $\pi_1(C)$ is not trivial, then*

1. *the kernel of ρ_γ is equal to the kernel of $\pi_1(F_0) \to \pi_1(Y_1)$,*
2. *the monodromy of the fibration ${Y_1}^u \to C^u$ along γ is pseudo-Anosov,*
3. *the kernel is not of finite type.*

2.5.3 Example. Consider the element

$$T_1 T_0 T_\infty = T_1 T_0 {T_1}^{-1} {T_0}^{-1} = [T_1, T_0] = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$

in $\Gamma(2)$ which corresponds to a element of $\pi_1(C^u)$, according to proposition 2.4.7. The trace of the matrix is 18 so that the monodromy along the corresponding loop is pseudo-Anosov. The corresponding element of $\mathrm{Mod}_{0,4}$ is the commutator of Dehn twists along intersecting loops.

Proof. As the morphism $\pi_1(C) \to \pi_1(Y_1)$ induced by the inclusion of C in Y_1 is injective, the image in $\pi_1(Y_1)$ of an element in $\pi_1(C^u)$ is trivial if and only if its image in $\pi_1(C)$ is trivial.

Any element of $\pi_1(M_\gamma)$ may be written as a product of the form $\gamma^m \omega$ with m in $\mathbb{Z}$ and ω in $\pi_1(F_0)$. The image of such an element by the composition $\pi_1(M_\gamma) \to \pi_1(Y_1) \to \pi_1(C)$ is the image of γ^m. Now, γ^m is in $\ker \rho_\gamma$ if and only if $m = 0$, hence $\ker \rho_\gamma$ is contained in $\pi_1(F_0)$.

Since γ is not in the kernel of the morphism $\pi_1(C^u) \to \pi_1(C)$, it is a hyperbolic element of $\pi_1(C^u)$, according to 2.4.10, so that the monodromy of the fibration along γ is pseudo-Anosov.

The kernel of $\pi_1(F_0) \to \pi_1(Y_1)$ is a subgroup invariant by the pseudo-Anosov monodromy of γ. According to [Ota96, Lemma 6.2.5], if such a subgroup is of finite type, then it is of finite index. However, since the limit set of the image of $\pi_1(F_0)$ in $\pi_1(Y_1)$ is all of $\partial_\infty \mathbb{H}^2_{\mathbb{C}}$, the image of $\pi_1(F_0) \to \pi_1(Y_1)$ cannot be finite and its kernel cannot be of finite index. Therefore, the kernel is not of finite type. □

2.5.4 Theorem. *For any two γ_1 and γ_2 in $\pi_1(C^u)$, if the image in $\pi_1(C)$ of γ_1 is not conjugate to that of γ_2 or its inverse, then either the groups $\pi_1(M_{\gamma_1})$ and $\pi_1(M_{\gamma_2})$ are not isomorphic or, if such an isomorphism $\Phi : \pi_1(M_{\gamma_1}) \to \pi_1(M_{\gamma_2})$ exists, then the representations ρ_{γ_1} and $\rho_{\gamma_2} \circ \Phi$ are not conjugate.*

Proof. Let γ_1 and γ_2 be two elements in $\pi_1(C^u)$. Assume that there exists an isomorphism $\Phi : \pi_1(M_{\gamma_1}) \to \pi_1(M_{\gamma_2})$ and that the representations $\rho_{\gamma_2} \circ \Phi$ and ρ_{γ_1} are conjugate. In other terms, there exists an element $\varphi_0\psi_0$ in $\pi_1({Y_1}^u)$, with φ_0 in the fundamental group of the fiber and ψ_0 in $\pi_1(C^u)$, such that the diagram

$$\begin{array}{ccc}
\pi_1(M_{\gamma_1}) & \xrightarrow{\Phi} & \pi_1(M_{\gamma_2}) \\
\downarrow{\scriptstyle \rho_{\gamma_1}} & & \downarrow{\scriptstyle \rho_{\gamma_2}} \\
\pi_1(Y_1) & \xrightarrow{\mathrm{Int}_{\rho(\varphi_0\psi_0)}} & \pi_1(Y_1)
\end{array}$$

is commutative, where $\mathrm{Int}_{\rho(\varphi_0\psi_0)}$ is the inner automorphisms of $\pi_1(Y_1)$ associated to $\rho(\varphi_0\psi_0)$. By replacing γ_1 by $\psi_0\gamma_1{\psi_0}^{-1}$, one may assume that $\psi_0 = 1$. Therefore the diagram

$$\begin{array}{ccc}
\pi_1(M_{\gamma_1}) & \xrightarrow{\Phi} & \pi_1(M_{\gamma_2}) \\
\downarrow{\scriptstyle \rho_{\gamma_1}} & & \downarrow{\scriptstyle \rho_{\gamma_2}} \\
\pi_1(Y_1) & \xrightarrow{\mathrm{Int}_{\rho(\varphi_0\psi_0)}} & \pi_1(Y_1) \\
& \searrow \quad \swarrow & \\
& \pi_1(C) &
\end{array}$$

is commutative. In particular, the images of $\pi_1(M_{\gamma_1})$ and $\pi_1(M_{\gamma_2})$ in $\pi_1(C)$ are equal. The image is generated indifferently by the image of γ_1 or γ_2 and is either trivial or an infinite cyclic subgroup. Hence the image γ_1 is equal to that of γ_2 or its inverse. □

Appendix A

Appendix

A.1 Notations and terminology

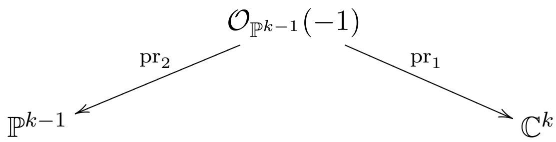
Let $\mathbb{P}^{k-1}$ denote the standard complex projective space of dimension $k-1$, defined as the quotient of $\mathbb{C}^k \setminus \{0\}$ by the action of $\mathbb{C}^*$ by homotheties and equipped with the homogeneous coordinates $[v_1 : \cdots : v_k]$. The field of meromorphic functions on $\mathbb{P}^{k-1}$ is the field $\mathbb{C}(\frac{v_2}{v_1}, \ldots, \frac{v_k}{v_1})$ of rational fractions, denoted by $\mathbb{C}(\mathbb{P}^{k-1})$.

More generally, for any complex vector space V of finite dimension, let $\mathbb{P}(V)$ denote the projectivization of V.

The tautological line bundle over $\mathbb{P}^{k-1}$ is defined as

$$\mathcal{O}_{\mathbb{P}^{k-1}}(-1) = \{(v, \ell) \in \mathbb{C}^k \times \mathbb{P}^{k-1} \mid v \in \ell\}$$

where each element ℓ in $\mathbb{P}^{k-1}$ is considered as a line in $\mathbb{C}^k$ passing through the origin.



The restriction to $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$ of the second projection $\mathrm{pr}_2 : \mathbb{C}^k \times \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ is the tautological line bundle. Indeed, the fiber of a point $\ell \in \mathbb{P}^{k-1}$ is the line $\ell \subset \mathbb{C}^k$.

The restriction to $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$ of the first projection $\mathrm{pr}_1 : \mathbb{C}^k \times \mathbb{P}^{k-1} \to \mathbb{C}^k$ is the blow-up of $\mathbb{C}^k$ at the origin. Indeed, it is bijective everywhere except over the origin of $\mathbb{C}^k$ whose fiber is $\mathbb{P}^{k-1}$.

Local charts and coordinates of $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$ may be given as follows. If U_r denotes the domain of the affine chart in $\mathbb{P}^{k-1}$ defined by $v_r \neq 0$ and with coordinates

$$v_{s|r} = \frac{v_s}{v_r} \quad \text{for } s \text{ between 1 and } k \text{ different form } r$$

then its inverse image in $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$ under $\mathrm{pr}_2 : \mathcal{O}_{\mathbb{P}^{k-1}}(-1) \to \mathbb{P}^{k-1}$ is the domain of the local chart with coordinates $(v_{1|r}, \ldots, v_{r-1|r}, v_r, v_{r+1|r}, \ldots, v_{k|r})$ corresponding to the point (v, ℓ) in $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$ with

$$v = (v_r v_{1|r}, \ldots, v_r v_{r-1|r}, v_r, v_r v_{r+1|r}, \ldots, v_r v_{k|r})$$

and

$$\ell = [v_{1|r} : \cdots : v_{r-1|r} : 1 : v_{r+1|r} : \cdots : v_{k|r}].$$

Finally, the blow-up of a complex manifold M at a point p may be realized by replacing a neighborhood of p, isomorphic to a neighborhood of the origin in $\mathbb{C}^k$, by the corresponding neighborhood in $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$. The *exceptional divisor* of such a blow-up is the preimage in the blow-up of the points which were blown-up. Moreover, if $(v_1, \ldots, v_k)$ are local coordinates for M, centered at p, local coordinates $(v_{1|r}, \ldots, v_{r-1|r}, v_r, v_{r+1|r}, \ldots, v_{k|r})$ for the blow-up of M at p may be defined for every $r \in \{1, \ldots, k\}$, similarly to $\mathcal{O}_{\mathbb{P}^{k-1}}(-1)$, as $v_{s|r} = v_s / v_r$ for s different from r.

A.2 Branched covering maps

A *branched covering map of finite degree* is a finite surjective morphism $\chi : Y \to X$ of varieties. Y is called a covering space of X. The isomorphisms $\alpha : Y \to Y$ such that $\chi \circ \alpha = \chi$ are called the automorphisms of χ and form a group denoted by $\mathrm{Aut}(\chi)$. If $\mathrm{Aut}(\chi)$ acts transitively on all fibers of $\chi : Y \to X$, then the covering map is called *Galois* or *regular* and $\mathrm{Aut}(\chi)$ is also referred to as the Galois group of the covering map. When, in addition, the Galois group is abelian, the covering map is called abelian.

A.2.1 Example. The morphism $c_n : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ defined by

$$c_n([u_1 : \cdots : u_k]) \to [{u_1}^n : \cdots : {u_k}^n]$$

is a branched covering map. The fiber ${c_n}^{-1}(p)$ over any point $p = [v_1 : \cdots : v_k]$ consists of n^{k-1-m} distinct points, where m is the number of homogeneous coordinates v_s of p that are equal to zero. m is the number of hyperplanes, of the following arrangement of hyperplanes, which contain p.

The arrangement in question is formed by the k hyperplanes D_s defined by the equations $v_s = 0$, which meet together in a rather simple way: for any distinct indices $s_1, \ldots, s_m$, the intersection $D_{s_1} \cap \cdots \cap D_{s_m}$ is merely the projective subspace of codimension m, defined by the equations $v_{s_1} = \cdots = v_{s_m} = 0$.

In particular, the fiber over any point in the complement of the arrangement of hyperplanes (this complement is an open and dense subset) consists of n^{k-1} points. In other words, c_n is a branched covering of degree n^{k-1} and which ramifies exactly over the previous arrangement of hyperplanes.

The morphisms $\alpha_s : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ defined for each index s by

$$\alpha_s([u_1 : \cdots : u_k]) \to [u_1 : \cdots : u_{s-1} : u_s e^{\frac{2\pi i}{n}} : u_{s+1} : \cdots : u_k]$$

are automorphisms of c_n. The subgroup of $\mathrm{Aut}(c_n)$ generated by the morphisms α_s acts transitively over every fiber. Therefore, c_n is an abelian covering map whose Galois group is generated by the automorphisms $\alpha_1, \ldots, \alpha_k$ satisfying ${\alpha_s}^k = \mathrm{id}$ and $\alpha_1 \circ \cdots \circ \alpha_k = \mathrm{id}$. The Galois group is obviously isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{k-1}$ but not canonically.

A.2.2 Notation. The Galois group $\mathrm{Aut}(c_n)$ may be identified with the additive group

$$\{(e_1, \ldots, e_k) \in (\mathbb{Z}/n\mathbb{Z})^k \mid \sum_{s=1}^{k} e_s \equiv 0[n]\}.$$

A branched covering map $\chi : Y \to X$ is said to be *ramified* along a hypersurface $f = 0$ in X, with ramification index p, if there exists local coordinates $(y_1, \ldots, y_n)$ of Y and $(x_1, \ldots, x_n)$ of X such that $x_n = f$ and that the image by χ of the point with coordinates $(y_1, \ldots, y_n)$ is the point with coordinates

$$(x_1, \ldots, x_n) = (y_1, \ldots, {y_n}^p).$$

The *branch locus* is the preimage in Y of the union of the hypersurfaces of X where χ is ramified. The *ramification locus* is the image in X of the branching locus. The unbranched covering associated to χ is the mapping $\chi^u : Y^u \to X^u$ where Y^u denotes the complement in Y of the branch locus and X^u the complement in X of the ramification locus. The mapping χ^u is a topological covering map.

Any branched covering map $\chi : Y \to X$ induces a finite field extension

$$\chi^* : \left\{ \begin{array}{ccc} \mathbb{C}(X) & \longrightarrow & \mathbb{C}(Y) \\ f & \longmapsto & f \circ \chi \end{array} \right.$$

between the field $\mathbb{C}(X)$ of meromorphic functions of X and that of Y. Conversely, given a normal variety X and a finite field extension $i : \mathbb{C}(X) \to L$, there is a branched covering map $\chi : Y \to X$ (unique up to isomorphism) such that $\chi^* = i$. The variety Y is the *normalization* of X in L.

The following proposition describes the relation between (unbranched) topological covering maps and fundamental groups.

A.2.3 Proposition. *Let X be a locally path-connected topological space. $\chi : Y \to X$ be a topological covering map and let x be a point in X.*

1. *There is a natural action (on the right) of $\pi_1(X, x)$ over $\chi^{-1}(x)$.*

2. *The image by χ of a path-connected component of Y is a path-connected component of X.*

3. *If X is path-connected, then the mapping $\chi^{-1}(x) \to \pi_0(Y)$, which maps any point y to the path-connected component of Y containing y, induces a bijection $\chi^{-1}(x)/\pi_1(X,x) \to \pi_0(Y)$.*

   In other words, the orbit of a point y in $\chi^{-1}(x)$ under the action of $\pi_1(X,x)$ is exactly the intersection of $\chi^{-1}(x)$ with the path-connected components of Y containing y.

4. *If χ is a Galois covering map, then, for any y in $\chi^{-1}(x)$, there exists a morphism $\alpha_y : \pi_1(X,x) \to \mathrm{Aut}(\chi)$ such that $yg = \alpha_y(g)y$ for any g in $\pi_1(X,x)$.*

5. *If χ is a Galois covering map and X is path-connected, then the restriction $\chi_{|Z} : Z \to X$ to a path-connected component Z of Y containing a point z is a Galois covering map whose Galois group $\mathrm{Aut}(\chi_{|Z})$ is naturally isomorphic to the subgroup $\mathrm{Im}\,\alpha_z$ of $\mathrm{Aut}(\chi)$.*

6. *If χ is a Galois covering map and Y is path-connected, then for any y in $\chi^{-1}(x)$,*

$$1 \longrightarrow \pi_1(Y,y) \xrightarrow{\chi_*} \pi_1(X,x) \xrightarrow{\alpha_y} \mathrm{Aut}(\chi) \longrightarrow 1$$

   is a short exact sequence.

Proof. 1. For any y in $\chi^{-1}(x)$ and any loop $\gamma : [0,1] \to X$ based at x, let $\tilde{\gamma} : [0,1] \to Y$ be the unique lift of γ satisfying $\tilde{\gamma}(0) = y$. Since its end point $\tilde{\gamma}(1)$ depends only on y and on the homotopy class $g \in \pi_1(X,x)$ of γ, it may be denoted by yg.

The map $(y,g) \mapsto yg$ so defined is moreover an action (on the right) of $\pi_1(X,x)$ over $\chi^{-1}(x)$. Indeed, for any y in $\chi^{-1}(x)$, if γ and γ' are loops in X based at x whose respective homotopy classes are g and g' in $\pi_1(X,x)$, then

- yg is the end point $\tilde{\gamma}(1)$ of the unique lift $\tilde{\gamma}$ of γ satisfying $\tilde{\gamma}(0) = y$,
- $(yg)g'$ is the end point $\tilde{\gamma}'(1)$ of the unique lift $\tilde{\gamma}'$ of γ' satisfying $\tilde{\gamma}'(0) = yg$,
- $y(gg')$ is the end point $\widetilde{\gamma\gamma'}(1)$ of the unique lift $\widetilde{\gamma\gamma'}$ of the concatenation $\gamma\gamma'$ satisfying $\widetilde{\gamma\gamma'}(0) = y$.

Now $\widetilde{\gamma\gamma'}$ is simply the concatenation $\tilde{\gamma}\tilde{\gamma}'$, so that their end points $y(gg')$ and $(yg)g'$ are the same.

2. Let Z be a path-connected component of Y, z be a point in Z and W be the path-connected component of X containing $\chi(z)$. For any point w in W, there exists a path $\gamma : [0,1] \to W$ from $\chi(z)$ to w. Let $\tilde{\gamma}$ be its unique lift satisfying $\tilde{\gamma}(0) = z$. Thus $\tilde{\gamma}$ connects z to a point whose image by χ is w. Finally $\chi(Z) = W$.

3. On the one hand, the mapping $\chi^{-1}(x) \to \pi_0(Y)$ is surjective. Indeed, for every path-connected component Z of Y, since X is path-connected, $\chi(Z) = X$ so that Z contains a point in the fiber of x. On the other, two points y and y' in the fiber over x belong to a same path-connected component of Y if and only if they are in a same orbit under the action of $\pi_1(X, x)$.

4. Let y be a point in $\chi^{-1}(x)$. Since $\mathrm{Aut}(\chi)$ acts freely and transitively over $\chi^{-1}(x)$, for any g in $\pi_1(X, x)$, there exists a unique automorphism $\alpha_y(g)$ satisfying $\alpha_y(g)y = yg$. That consideration yields a mapping $\alpha_y : \pi_1(X, x) \to \mathrm{Aut}(\chi)$. Moreover, for any g and g' in $\pi_1(X, x)$, let $\gamma : [0,1] \to X$ be a loop based at x representing g and $\tilde{\gamma}$ be its unique lift satisfying $\tilde{\gamma}(0) = y$ and hence $\tilde{\gamma}(1) = yg$. Then $\alpha_y(g') \circ \tilde{\gamma}$ is the unique lift of γ satisfying $\alpha_y(g') \circ \tilde{\gamma}(0) = \alpha_y(g')y$, so that

$$(\alpha_y(g')y)g = \alpha_y(g') \circ \tilde{\gamma}(1) = \alpha_y(g')(yg).$$

   Consequently,

$$\alpha_y(gg')y = y(gg') = (yg)g' = (\alpha_y(g)y)g' = \alpha_y(g)(yg') = \alpha_y(g)\alpha_y(g')y$$

   so that $\alpha_y(gg') = \alpha_y(g)\alpha_y(g')$.

5. Since Z is path-connected, $\pi_1(X, \chi(z))$ acts transitively over the intersection $\chi^{-1}(\chi(z)) \cap Z$. Hence, for any α in $\mathrm{Aut}(\chi_{|Z})$, there exists g in $\pi_1(X, \chi(z))$ such that $\alpha(z) = zg = \alpha_z(g)z$. Therefore each α in $\mathrm{Aut}(\chi_{|Z})$ coincides over Z with a unique element of $\mathrm{Im}\,\alpha_z$ so that $\mathrm{Im}\,\alpha_z$ and $\mathrm{Aut}(\chi_{|Z})$ are isomorphic.

6. First, χ_* is injective because, if the image $\chi \circ \gamma$ of a loop γ in Y is homotopically trivial, then γ itself is homotopically trivial.

   Since Y is path-connected, $\mathrm{Aut}(\chi) = \mathrm{Im}\,\alpha_y$ so that α_y is surjective.

   Finally, an element g in $\pi_1(X, x)$ is in the kernel of α_y if and only if $y = yg$ which means exactly that g is in the image $\chi_*\pi_1(Y, y)$. □

A.2.4 Example. Consider the branched covering map $c_n : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ defined in example A.2.1. Let X denote the open subset of $\mathbb{P}^{k-1}$ where none of the homogeneous coordinates $u_1, \ldots, u_k$ vanish and let x be the point $[1 : \cdots : 1]$. X is path-connected and the restriction $c_n : X \to X$ is a Galois unbranched covering map. The group $\pi_1(X,x)$ is generated by the homotopy classes g_s of the loops

$$\gamma_s : \begin{cases} [0,2\pi] & \longrightarrow & X \\ t & \longmapsto & [1:\cdots:1:e^{it}:1:\cdots:1] \end{cases}$$

for $1 \leq s \leq k$. Note that the loop γ_s consists in a turn around a hyperplane of the arrangement. The lift $\tilde{\gamma}_s$ of γ_s satisfying $\tilde{\gamma}_s(0) = x$ is

$$\tilde{\gamma}_s : \begin{cases} [0,2\pi] & \longrightarrow & X \\ t & \longmapsto & [1:\cdots:1:e^{i\frac{t}{n}}:1:\cdots:1] \end{cases}$$

so that the morphism $\alpha_x : \pi_1(X,x) \to \mathrm{Aut}(c_n)$ satisfies $\alpha_x(g_s) = \alpha_s$ for $1 \leq s \leq k$. Note that $\alpha_x(g_s)$ depends only on the hyperplane D_s around which the loop turns and the numer of turns, but not the choice of the loop.

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Représentations de groupes fondamentaux en géométrie hyperbolique

Résumé

Deux méthodes de construction de représentations de groupes sont présentées. La première propose une stratégie essayant de déterminer les représentations de groupes libres de type fini à valeurs dans tout réseau de groupes de Lie réel. La seconde, après avoir revu une construction d'une surface hyperbolique complexe, c'est-à-dire le quotient du plan hyperbolique complexe $\mathbb{H}^2_{\mathbb{C}}$ par un réseau de $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, et examiné soigneusement ses propriétés, produit une infinité de représentations non-conjuguées, à valeurs dans un réseau de $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, de groupes fondamentaux de variétés hyperboliques fermées de dimension 3, obtenues comme des fibrés en surfaces sur le cercle.

Mots-clés: représentations de groupes fondamentaux, réseaux de groupes de Lie, géométrie hyperbolique, structures CR sphériques.

Representations of fundamental groups in hyperbolic geometry

Abstract

Two construction methods of group representations are presented. The first one proposes a strategy to try to determine the representations of finitely generated free groups into any lattice in real Lie groups. The second, after reviewing a construction of a complex hyperbolic surface, that is the quotient of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ by a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, and examining its properties carefully, yields infinitely many non-conjugate representations into a lattice in $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$, of fundamental groups of closed hyperbolic 3-dimensional manifolds, obtained as surface bundles over the circle.

Keywords: representations of fundamental groups, lattices in Lie groups, hyperbolic geometry, spherical CR structures.