Geometric distance graphs, lattices and polytopes

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Geometric Distance Graphs, Lattices and Polytopes.

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Title. Geometric Distance Graphs, Lattices and Polytopes.

Abstract. A distance graph $G(X, D)$ is a graph whose set of vertices is the set of points X of a metric space (X, d), and whose edges connect the pairs $\{x, y\}$ such that $d(x, y) \in D$. In this thesis, we consider two problems that may be interpreted in terms of distance graphs in $\mathbb{R}^n$.

First, we study the famous sphere packing problem, in relation with the distance graph $G(\mathbb{R}^n, (0, 2r))$ for a given sphere radius r. Recently, Venkatesh improved the best known lower bound for lattice sphere packings by a factor $\log \log n$ for infinitely many dimensions n. We prove an effective version of this result, in the sense that we exhibit, for the same set of dimensions, finite families of lattices containing a lattice reaching this bound. Our construction uses codes over cyclotomic fields, lifted to lattices via Construction A. We also prove a similar result for families of symplectic lattices.

Second, we consider the unit distance graph G associated with a norm $\|\cdot\|$. The number $m_1(\mathbb{R}^n, \|\cdot\|)$ is defined as the supremum of the densities achieved by independent sets in G. If the unit ball corresponding with $\|\cdot\|$ tiles $\mathbb{R}^n$ by translation, then it is easy to see that $m_1(\mathbb{R}^n, \|\cdot\|) \geqslant \frac{1}{2^n}$. C. Bachoc and S. Robins conjectured that the equality always holds. We show that this conjecture is true for $n = 2$ and for several Voronoï cells of lattices in higher dimensions, by solving packing problems in discrete graphs.

Keywords. Distance graphs, Euclidean lattices, sphere packing, Minkowski-Hlawka bound, linear codes, parallelohedra, chromatic number.

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Titre. Graphes Métriques Géométriques, Réseaux et Polytopes.

Résumé. Un graphe métrique $G(X, D)$ est un graphe dont l'ensemble des sommets est l'ensemble X des points d'un espace métrique (X, d), et dont les arêtes relient les paires $\{x, y\}$ de sommets telles que $d(x, y) \in D$. Dans cette thèse, nous considérons deux problèmes qui peuvent être interprétés comme des problèmes de graphes métriques dans $\mathbb{R}^n$.

Premièrement, nous nous intéressons au célèbre problème d'empilements de sphères, relié au graphe métrique $G(\mathbb{R}^n,]0, 2r[)$ pour un rayon de sphère r donné. Récemment, Venkatesh a amélioré d'un facteur $\log \log n$ la meilleure borne inférieure connue pour un empilement de sphères donné par un réseau, pour une suite infinie de dimensions n. Ici nous prouvons une version effective de ce résultat, dans le sens où l'on exhibe, pour la même suite de dimensions, des familles finies de réseaux qui contiennent un réseaux dont la densité atteint la borne de Venkatesh. Notre construction met en jeu des codes construits sur des corps cyclotomiques, relevés en réseaux grâce à un analogue de la Construction A. Nous prouvons aussi un résultat similaire pour des familles de réseaux symplectiques.

Deuxièmement, nous considérons le graphe distance-unité G associé à une norme $\|\cdot\|$. Le nombre $m_1(\mathbb{R}^n, \|\cdot\|)$ est défini comme le supremum des densités réalisées par les stables de G. Si la boule unité associée à $\|\cdot\|$ pave $\mathbb{R}^n$ par translation, alors il est aisé de voir que $m_1(\mathbb{R}^n, \|\cdot\|) \geqslant \frac{1}{2^n}$. C. Bachoc et S. Robins ont conjecturé qu'il y a égalité. On montre que cette conjecture est vraie pour $n = 2$ ainsi que pour des régions de Voronoï de plusieurs types de réseaux en dimension supérieure, ceci en se ramenant à la résolution de problèmes d'empilement dans des graphes discrets.

Mots-clés. Graphes métriques, réseaux Euclidiens, empilement de sphères, borne de Minkowski-Hlawka, codes linéaires, polytopes pavant l'espace par translation, nombre chromatique.

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Contents

Introduction (en français)

Le *Problème d'Empilement de Sphères* est l'un des plus célèbres problèmes de géométrie. Reconnu de tous comme un problème difficile, il est extrêmement simple à énoncer : comment occuper la plus grande proportion d'espace dans $\mathbb{R}^n$ avec des sphères de même rayon et d'intérieurs disjoints ?

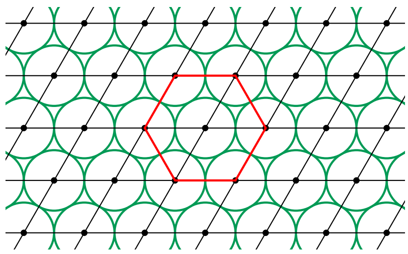


Figure 1. L'empilement optimal de cercles.

En dimension $n = 1$, la réponse est immédiate, puisque les intervalles pavent parfaitement la droite réelle. En dimension 2, le meilleur empilement de cercles est donné, sans surprise, par le réseau hexagonal (voir Figure 1). Ceci constitue néanmoins un premier résultat non trivial. Lagrange prouva d'abord en 1773 qu'il s'agissait du meilleur *empilement de sphère par réseau* en dimension 2, c'est-à-dire lorsque les centres des sphères forment un *réseau Euclidien*. Ensuite Thue [**Thu92**] fut le premier à fournir une preuve de son optimalité parmi tous les empilements, irréguliers compris. Une autre preuve historiquement importante est celle de Fejes Tóth [**FT50**]. En dimension 3, la célèbre *conjecture de Kepler* affirme que l'empilement optimal est donné par *le réseau cubique à faces centrées* (voir Figure 2). Gauss prouva en 1832 son optimalité parmi les réseaux, mais la conjecture de Kepler n'a été résolue que récemment par Hales ([**Hal06**], [**HAB**$^{+}$**17**]).

En dimensions 4, 5 (Korkine et Zolotareff, [**KZ77**], [**KZ73**]), 6, 7, 8 (Blichfeldt, [**Bli35**]), et 24 (Cohn et Kumar, [**CK09**]), les réseaux les plus denses sont connus, et on ne savait pas si ils fournissaient des empilements optimaux

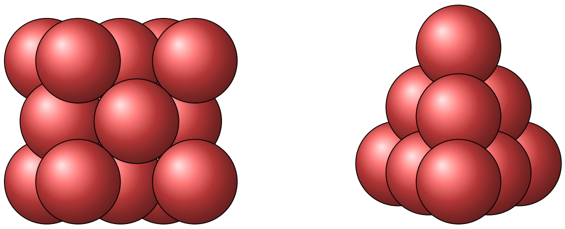


FIGURE 2. L'empilement optimal en dimension 3, donné par le réseau cubique à faces centrées. Il correspond à la manière usuelle d'empiler des objets sphériques dans l'espace.

en général, jusqu'au récent article de Viazovska ([Via17]). Elle a prouvé que le réseau E_8 donne le meilleur empilement de sphères en dimension 8. Quelques semaines plus tard, sa méthode fut adaptée pour prouver que l'empilement optimal en dimension 24 est donné par le réseau de Leech ([CKM+17]). Si il présente d'évidentes applications pratiques en dimensions 2 et 3, le problème d'empilements de sphères dans des dimensions plus grandes n'est pas seulement un beau problème mathématique. En effet, les empilements de sphères dans $\mathbb{R}^n$ peuvent être vus comme des analogues continus des *codes correcteurs d'erreurs* discrets, et possèdent des applications pour les réseaux de communications réels (voir [Zam14] pour plus de détails).

Un simple argument montre qu'un empilement optimal (sans imposer la structure de réseau) en dimension n doit avoir une densité supérieure à 2^{-n}. En effet, prenons un empilement de sphères de rayon r dans $\mathbb{R}^n$. Si il est optimal, il n'y a plus de place pour une sphère supplémentaire. Donc pour tout x de $\mathbb{R}^n$, il y a au moins un centre de sphère c tel que $d(x, c) < 2r$. Autrement dit, si on double le rayon, ce qui multiplie le volume recouvert par les sphères par 2^n, $\mathbb{R}^n$ est totalement recouvert. Donc la proportion d'espace occupée par les sphères était d'au moins 2^{-n}. Le premier sujet abordé dans cette thèse est l'étude de bornes inférieures asymptotiques pour le supremum Δ_n des densités atteintes par les réseaux en dimension n. Historiquement, le premier résultat marquant fut le *théorème de Minkowski-Hlawka* [Hla43], qui affirme que pour toute dimension n, $\Delta_n \geqslant \zeta(n)2^{-(n-1)}$, où $\zeta(n)$ désigne la fonction Zeta de Riemann. Plusieurs auteurs se sont consacrés à l'amélioration de cette borne ([Rog47], [DR47], [Bal92], [Van11]), jusqu'à la meilleure borne dûe à Venkatesh [Ven13], qui a récemment prouvé que, pour une suite infinie de dimensions, $\Delta_n \geqslant \dfrac{n \log \log n}{2^{n+1}}$. La plupart des preuves de ces résultats reposent sur un argument de moyenne appliqué à des familles infinies de réseaux bien choisies. On donnera une synthèse plus détaillée de ces résultats et de leurs preuves lors de l'introduction du Chapitre 2.

Toutefois, le but principal de ces théorèmes est d'assurer l'existence de réseaux denses : leurs preuves sont non constructives, et ne fournissent pas de stratégie pour construire explicitement un réseau dense pour une dimension donnée n. Si on peut difficilement s'attendre à obtenir un algorithme efficace fournissant des réseaux denses en grande dimension, réduire le creux entre résultats théoriques et pratiques demeure un challenge intéressant. Une idée naturelle dans ce sens consiste à réduire autant que possible la taille des familles contenant des réseaux denses. Jusqu'à maintenant, les travaux dans cette direction ([**Rus89**], [**GZ07**], voir l'introduction du Chapitre 2) ont fourni des familles de réseaux *finies*, bien que de tailles exponentielles, qui contiennent des réseaux atteignant la borne de Minkowski-Hlawka, ou l'une de ses versions améliorées.

Cependant, jusqu'à présent, aucune version effective du résultat de Venkatesh n'était connue. Il s'agit de la première contribution de cette thèse. On construit des familles finies de réseaux en relevant des codes linéaires sur des corps finis $\mathbb{F}_p$. En considérant une généralisation de la *Construction A* dans le cadre de la théorie des nombres, nos réseaux gardent la même structure algébrique que ceux de Venkatesh : ce sont des modules sur des corps cyclotomiques $\mathbb{Q}[\zeta_m]$, par conséquent invariants sous l'action du groupe des racines m-ièmes de l'unité. Outre la finitude de nos familles, notre preuve présente aussi des avantages techniques : par exemple l'argument de moyenne dans notre cas se ramène à un simple argument de comptage sur des ensembles finis. Aussi, en adaptant notre construction, on montre que l'on peut obtenir des réseaux symplectiques. Ces résultats sont prouvés en détails dans le Chapitre 2.

Le problème d'empilement de sphères peut être interprété comme un problème de théorie des graphes. Un empilement de boules unité dans $\mathbb{R}^n$ est complètement déterminé par l'ensemble Λ des centres de sphères. La distance entre deux éléments de Λ, par définition, ne peut être strictement inférieure à 2. Soit G le graphe dont les sommets sont les points de $\mathbb{R}^n$ et dont les arêtes relient les paires $x \neq y \in \mathbb{R}^n$ qui vérifient $d(x, y) < 2$. Alors l'ensemble Λ n'est autre qu'un *ensemble indépendant* dans G, et le problème d'empilement de sphères revient essentiellement à trouver un ensemble indépendant le plus grand possible dans G. Le graphe G est un exemple de *graphe métrique*. En général, un graphe métrique $G(X, D)$ est la donnée d'un espace métrique $X = (X, d)$ et d'un sous-ensemble $D \subset]0, \infty[$. Les sommets de $G(X, D)$ sont les éléments de X, et deux sommets $x, y \in X$ sont reliés dans $G(X, D)$ si et seulement si $d(x, y) \in D$. Pour le problème d'empilement de sphères, X est l'espace Euclidien usuel $\mathbb{R}^n$, et $D =]0, 2[$.

Le deuxième problème principal de cette thèse met aussi en jeu un graphe métrique. Le graphe en question est le *graphe distance-unité* $G(\mathbb{R}^n, \|\cdot\|)$: l'espace métrique est $\mathbb{R}^n$ muni d'une norme $\|\cdot\|$, et $D = \{1\}$. Historiquement, la question la plus étudiée concernant le graphe distance-unité est la détermination du nombre chromatique de $G(\mathbb{R}^n, \|\cdot\|_2)$, noté $\chi(\mathbb{R}^n, \|\cdot\|_2)$, lorsque la norme $\|\cdot\|_2$ est la norme Euclidienne. Il est évident que $\chi(\mathbb{R}, \|\cdot\|_2) = 2$. En dimension $n = 2$, la question est étonnamment difficile, et est connue comme le célèbre *problème de Hadwiger-Nelson* (on renvoie au livre de Soifer [**Soi08**] pour l'histoire de ce problème) : quel est le nombre minimal de couleurs nécessaires pour colorier le plan Euclidien sans que deux points à distance 1 l'un de l'autre ne reçoivent la même couleur ?

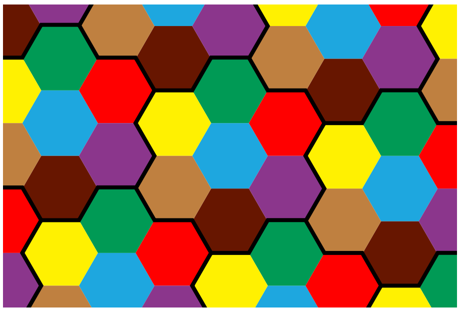


FIGURE 3. $\chi(\mathbb{R}^2, \|\cdot\|_2) \leqslant 7$.

D'une part, il y a une coloration naturelle du plan avec 7 couleurs, obtenue grâce à un pavage du plan par des hexagones réguliers de diamètre 1, comme illustré dans la Figure 3. D'autre part, le graphe de *Moser* (voir Figure 4) est contenu dans le plan et a pour nombre chromatique 4. On obtient donc aisément l'inégalité

$$4 \leqslant \chi(\mathbb{R}^2, \|\cdot\|_2) \leqslant 7.$$

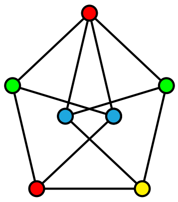


FIGURE 4. $4 \leqslant \chi(\mathbb{R}^2, \|\cdot\|_2)$.

Curieusement, ce sont les seules informations connues sur le nombre chromatique du plan Euclidien. Lorque que l'on impose aux classes de couleurs d'être mesurables, l'analogue du nombre chromatique est le *nombre chromatique mesurable* $\chi_m(\mathbb{R}^n, \|\cdot\|)$. Falconer [**Fal81**] a montré que $\chi_m(\mathbb{R}^2, \|\cdot\|_2) \geqslant 5$. Même si χ_m est plus facile à manipuler que χ, la détermination de $\chi_m(\mathbb{R}^n, \|\cdot\|_2)$ reste un problème ouvert.

Larman et Rogers [**LR72**] ont introduit un outil important pour obtenir des bornes inférieures sur $\chi_m(\mathbb{R}^n, \|\cdot\|)$: il s'agit du nombre $m_1(\mathbb{R}^n, \|\cdot\|)$, qui sera au centre des Chapitres 3 et 4. Un ensemble $A \subset (\mathbb{R}^n, \|\cdot\|)$ *évite la distance* 1 si pour tous $x, y \in A$, $\|x - y\| \neq 1$. Le nombre $m_1(\mathbb{R}^n, \|\cdot\|)$ est alors le supremum des densités qui peuvent être atteintes par un ensemble mesurable évitant la distance 1. Autrement dit, un ensemble évitant la distance 1 est un ensemble indépendant dans $G(\mathbb{R}^n, \|\cdot\|)$, et comme pour le problème d'empilement de sphères, calculer $m_1(\mathbb{R}^n, \|\cdot\|)$ consiste à trouver le plus grand (en termes de densité cette fois) ensemble indépendant dans le graphe distance-unité. La relation entre le nombre chromatique mesurable et le nombre $m_1(\mathbb{R}^n, \|\cdot\|)$ est donnée par l'inégalité :

$$\chi_m(\mathbb{R}^n) \geqslant \frac{1}{m_1(\mathbb{R}^n)}.$$

Cependant déterminer $m_1(\mathbb{R}^2, \|\cdot\|_2)$, et plus généralement $m_1(\mathbb{R}^n, \|\cdot\|_2)$, s'est avéré être encore un problème difficile. Une vue d'ensemble des résultats connus à propos de $m_1(\mathbb{R}^n, \|\cdot\|_2)$ sera donnée dans l'introduction du Chapitre 3.

Les difficultés rencontrées pour calculer χ_m et m_1 dans le cas Euclidien encouragent à considérer des variantes du problème initial. Par exemple, on peut remplacer le corps $\mathbb{R}$ par un corps plus général, et le produit scalaire Euclidien usuel par des formes quadratiques générales ([**BMK17**]). Récemment, DeCorte et Golubev ([**DG17**]) ont calculé des bornes inférieures sur le nombre mesurable chromatique du plan *hyperbolique*. Dans cette thèse, on considère une autre variante de ce problème. Comme cela sera décrit plus en détails dans l'introduction du Chapitre 3, le fait que les sphères Euclidiennes ne peuvent pas paver parfaitement $\mathbb{R}^n$ semble être un frein à ce qu'un ensemble évitant la distance 1 dans $\mathbb{R}^n$ puisse atteindre une densité de 2^{-n}. Inspiré par cette observation, on s'intéresse aux normes dont la boule unité est un polytope $\mathcal{P}$ qui pave $\mathbb{R}^n$ par translation. Dans cette situation, on trouve un exemple simple et naturel d'ensemble évitant la distance 1 de densité 2^{-n} (voir Figure 5), et cette construction semble optimale, comme l'ont conjecturé Bachoc et Robins.

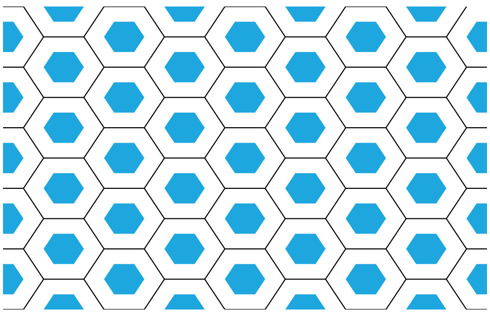


FIGURE 5. Lorsque la sphère unité pave l'espace par translation, il y a un ensemble évitant la distance 1 de densité $1/2^n$.

Le second objectif de cette thèse est de prouver cette conjecture pour plusieurs polytopes $\mathcal{P}$. Premièrement, nous allons montrer qu'elle est vraie en dimension 2 : pour toute norme $\|\cdot\|$ telle que la boule unité pave $\mathbb{R}^2$ par translation, $m_1(\mathbb{R}^2, \|\cdot\|) = 1/4$. Ceci implique que le nombre chromatique du plan dans ce cas est exactement 4. Nous étudions aussi les polytopes qui sont les cellules de Voronoï des célèbres réseaux A_n et D_n, pour toute dimension n. On prouve que la conjecture de Bachoc et Robins est vraie pour la première famille, tandis que pour la seconde on obtient un résultat légèrement plus faible. En dimension 3, il y a cinq types combinatoires de polytopes pavant $\mathbb{R}^3$ par translation (voir Chapitre 2, Section 3). On montre que la conjecture est vraie pour plusieurs cellules de Voronoï en dimension 3, qui couvrent quatre de ces cinq types combinatoires de polytopes (voir Chapitre 4).

Maintenant que l'on a parlé des résultats, il est temps de dire quelques mots à propos des méthodes utilisées. Bien que ce problème soit a priori énoncé dans un contexte continu, nous allons le transformer en un problème mettant en jeu des ensembles discrets. Si G est un sous-graphe de $G(\mathbb{R}^n, \|\cdot\|)$ induit par un ensemble discret $V \subset \mathbb{R}^n$, le nombre $m_1(\mathbb{R}^n, \|\cdot\|)$ est majoré par le *ratio d'indépendance* (voir Chapitre 2, Section 4) de G. Pour chaque polytope que l'on traitera, notre premier travail sera de construire un sous-graphe discret approprié, avant de prouver que son ratio d'indépendance ne peut dépasser $1/2^n$. Pour ce faire, nous attribuerons à V une structure de graphe auxiliaire, de façon à ce qu'un ensemble $A \subset V$ qui évite la distance polytope 1 puisse être décomposé de manière canonique comme une union disjointe de *blocs*. Dans le Chapitre 3, les graphes auxiliaires que l'on construit possèdent une propriété forte et utile : tout ensemble A évitant la distance 1 peut s'écrire comme union de cliques dont les voisinages dans le graphe sont disjoints. Par conséquent, la densité globale de A dans V est majorée par la *densité locale* d'une clique dans son voisinage fermé. Dans le Chapitre 4, on développe une stratégie plus générale dans le but de traiter des graphes plus compliqués, en introduisant notamment la notion de *fonction de distribution discrète*. Dès qu'un sous-ensemble $A \subset V$ est séparé en blocs disjoints, une fonction de distribution discrète répartit les points de V entre les blocs de A, en associant à chaque bloc un *voisinage*. Le challenge est de trouver une telle fonction qui assure que la densité de chaque bloc dans son voisinage est majorée par $1/2^n$. Autrement dit, on a transformé le problème initial en un problème d'empilement discret.

Contenu de la Thèse

Chapitre 1. On introduit les notions qui seront utilisées le long de la thèse. On fixe les notations et rappelle des notions basiques de topologie et de théorie des graphes, présente les réseaux Euclidiens sous plusieurs aspects, et donne un bref aperçu des résultats connus à propos des parallélohèdres.

Chapitre 2. Ce chapitre contient nos résultats sur le problème d'empilement de sphères. On utilise des codes sur des corps cyclotomiques afin de construire des familles finies de réseaux en grande dimension qui contiennent des réseaux denses. Ces résultats ont été publiés dans [**Mou17**].

Chapitre 3. Dans ce chapitre basé sur [**BBMP17**], qui est issu d'une collaboration avec Christine Bachoc, Thomas Bellitto et Arnaud Pêcher, on étudie la conjecture de Bachoc et Robins en dimension 2 ainsi que pour les cellules de Voronoï des réseaux A_n et D_n.

Chapitre 4. Ce chapitre contient d'autres résultat sur les ensembles évitant la distance 1. On y introduit le concept de fonction de distribution discrète, et on prouve la conjecture de Bachoc et Robins pour plusieurs autres polytopes, en particulier pour le *dodécahèdre allongé* en dimension 3.

Chapitre 5. Dans le dernier chapitre, on présente quelques perspectives pour de futurs travaux.

Introduction

The *Sphere Packing Problem* is one of the most famous problems in geometry. In spite of its acknowledged hardness, the question is surprisingly easy to state and to understand: what is the greatest proportion of $\mathbb{R}^n$ that can be filled by non overlapping spheres of same radius?

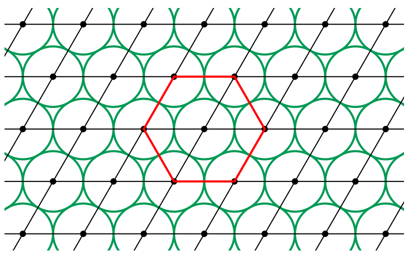


FIGURE 6. The optimal circle packing.

In dimension $n = 1$, the answer is immediate, since intervals fill perfectly the real line. In dimension 2, the best circle packing is given, as expected, by the hexagonal lattice (see Figure 6). However this is already a non-trivial result. Lagrange first proved in 1773 that it is the densest *lattice sphere packing* in dimension 2, namely when the centers of the spheres shape a *Euclidean lattice*. Then Thue [**Thu92**] was the first one to provide a proof that it is the best packing, even among the non regular packings. Another historically important proof is the one by Fejes Tóth [**FT50**]. In dimension 3, the famous *Kepler conjecture* asserts that the optimal packing is given by the so-called *face-centered cubic lattice* (see Figure 7). Gauss proved in 1832 that it is the best lattice packing, whereas the Kepler conjecture has been solved lately by Hales ([**Hal06**], [**HAB$^+$17**]).

In dimensions 4, 5 (Korkine and Zolotareff, [**KZ77**], [**KZ73**]), 6, 7, 8 (Blichfeldt, [**Bli35**]), and 24 (Cohn and Kumar, [**CK09**]), the densest lattice packings are known, and it was not known whether they were optimal packings, until the recent paper by Viazovska ([**Via17**]). She proved that the lattice E_8

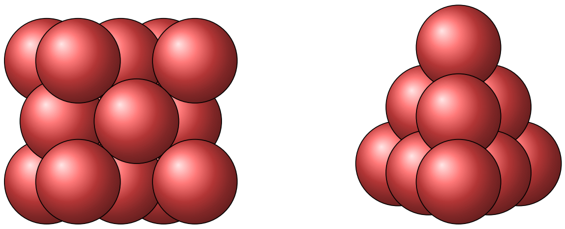


FIGURE 7. The optimal packing in dimension 3, given by the face-centered cubic lattice. It corresponds to the usual way to pack balls in the space.

provides the optimal sphere packing in dimension 8. A few weeks later, her method was adapted to prove that the densest packing in dimension 24 is given by the Leech lattice ([CKM+17]). If it presents an obvious practical interest in dimensions 2 and 3, the sphere packing problem in higher dimensions is not just a beautiful mathematical problem. Indeed, sphere packings in $\mathbb{R}^n$ may be seen as the continuous analogues of discrete *error correcting codes*, and admit some application in real communication channels (see [Zam14] for further details).

A natural argument shows that an optimal packing (non necessarily a lattice packing) in dimension n has density greater than 2^{-n}. Indeed, consider a packing of balls of radius r in $\mathbb{R}^n$. If it is optimal, there is no space for an additional sphere. So for every point x of $\mathbb{R}^n$, there is at least one center of sphere c such that $d(x, c) < 2r$. In other words, by doubling the size of the radius, which multiplies the volume of the union of the spheres by 2^n, $\mathbb{R}^n$ is completely covered. Thus the initial balls cover at least a 2^{-n} fraction of the space. The first topic studied in this thesis bears upon asymptotic lower bounds on the supreme density Δ_n achieved by a lattice packing in dimension n. Historically, the first important result is *Minkowski-Hlawka theorem* [Hla43], which asserts that for any dimension n, $\Delta_n \geqslant \zeta(n)2^{-(n-1)}$, where $\zeta(n)$ denotes the Riemann Zeta function. Some efforts have been done in order to enhance this lower bound ([Rog47],[DR47] ,[Bal92] ,[Van11]), and the best improvement is due to Venkatesh [Ven13], who recently proved that for an infinite sequence of dimensions, $\Delta_n \geqslant \dfrac{n \log \log n}{2^{n+1}}$. Usually, the proofs of these theorems consist in applying averaging arguments on well chosen infinite families of lattices. A more detailed review of these results and their proofs will be given in the introduction of Chapter 2.

However, the main goal of these theorems is to prove the existence of dense lattices: their proofs are non-constructive and do not provide a strategy to construct explicitly a dense lattice in a given dimension n. If one could hardly expect to obtain an efficient algorithm providing dense lattices in high dimension, it is an interesting challenge to cut down the gap between theoretical and practical results. A natural way to do so is to reduce as much as possible the size of the families containing dense lattices. Up to now, works in this direction ([Rus89], [GZ07], see the introduction of Chapter 2) have provided *finite*, although exponential-sized, families of lattices containing a lattice reaching Minkowski-Hlawka bound, or one of its improved version.

Nevertheless, there were no effective version of Venkatesh's result to date. This is the first contribution of this thesis. We construct finite families of lattices by lifting codes over finite fields $\mathbb{F}_p$. By considering a generalization of the well-known *Construction A* in the framework of number theory, our lattices afford the same strong algebraic structure as the ones introduced by Venkatesh: they are modules over the ring of integers of some cyclotomic field $\mathbb{Q}[\zeta_m]$, hence invariant under the action of the group of mth-roots of unity. In addition to the finiteness of our families, our proof also presents some technical advantages: for instance the averaging argument boils down to a straightforward counting argument of finite sets. Furthermore, we adapt our construction in order that our lattices become symplectic. These results are proved in details in Chapter 2.

The sphere packing problem may be interpreted in terms of graph theory. A packing of unit balls in $\mathbb{R}^n$ is completely determined by the set Λ of the centers of the spheres. Two elements in Λ cannot be at distance less than 2 from each other, by definition. Let G be the graph whose vertices are the points of $\mathbb{R}^n$ and whose edges are the pairs $x \neq y \in \mathbb{R}^n$ satisfying $d(x, y) < 2$. Then the set Λ is nothing but an *independent set* in G, and the sphere packing problem essentially amounts to finding the largest independent set in G. The graph G is an example of *distance graph*. A distance graph $G(X, D)$ in general is given by a metric space $X = (X, d)$ and a subset $D \subset (0, \infty)$. The vertices of $G(X, D)$ are the elements of X, and two vertices $x, y \in X$ are connected in $G(X, D)$ if an only if $d(x, y) \in D$. For the sphere packing problem, X is the usual Euclidean space $\mathbb{R}^n$, and $D = (0, 2)$.

The second main problem of this thesis also involves a distance graph. The graph at issue is the so-called *unit-distance graph* $G(\mathbb{R}^n, \|\cdot\|)$: the metric space is $\mathbb{R}^n$ equipped with a norm $\|\cdot\|$, and $D = \{1\}$. Historically, the most important question regarding the unit-distance graph is to determine the chromatic number of $G(\mathbb{R}^n, \|\cdot\|_2)$, denoted by $\chi(\mathbb{R}^n, \|\cdot\|_2)$, where the norm $\|\cdot\|_2$ is the Euclidean norm. It is obvious that $\chi(\mathbb{R}, \|\cdot\|_2) = 2$. In dimension $n = 2$, the question is amazingly hard, and is known as the celebrated *Hadwiger-Nelson problem* (see the book by Soifer [Soi08] for the history of this problem): what is the smallest number of colors required for coloring the plane in such a way that two points at Euclidean distance 1 from each other do not receive the same color?

There is a natural coloring of the plane using 7 colors, by tiling the plane with regular hexagons of diameter 1, see Figure 8. Moreover the *Moser* graph (see Figure 9) is contained in the plane and has chromatic number 4. So we easily obtain the inequality

$$4 \leqslant \chi(\mathbb{R}^2, \|\cdot\|_2) \leqslant 7.$$

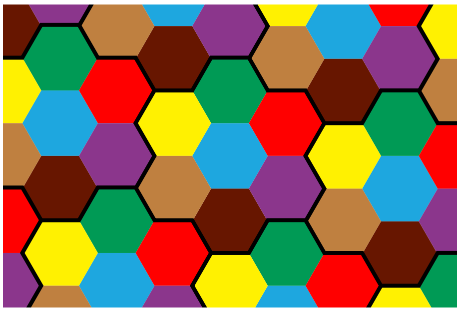


FIGURE 8. $\chi(\mathbb{R}^2, \|\cdot\|_2) \leqslant 7$.

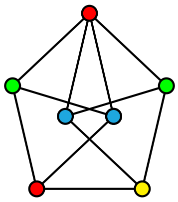


FIGURE 9. $4 \leqslant \chi(\mathbb{R}^2, \|\cdot\|_2)$.

Surprisingly, nothing more is known about the chromatic number of the Euclidean plane. The analogue of the chromatic number when we require the color classes to be measurable is the *measurable chromatic number* $\chi_m(\mathbb{R}^n, \|\cdot\|)$. Falconer [Fal81] proved that $\chi_m(\mathbb{R}^2, \|\cdot\|_2) \geqslant 5$. Even if χ_m is easier to handle than χ, determining $\chi_m(\mathbb{R}^n, \|\cdot\|_2)$ remains open.

An important tool that has been introduced by Larman and Rogers [LR72] in order to get lower bounds on $\chi_m(\mathbb{R}^n, \|\cdot\|)$ is the number $m_1(\mathbb{R}^n, \|\cdot\|)$, that will be the center of our attention in Chapters 3 and 4. A set $A \subset (\mathbb{R}^n, \|\cdot\|)$ *avoids distance* 1 if for any $x, y \in A$, $\|x - y\| \neq 1$. The number $m_1(\mathbb{R}^n, \|\cdot\|)$ is then the supreme density that can be achieved by a measurable set avoiding distance 1. In other words, a set avoiding distance 1 is an independent set in $G(\mathbb{R}^n, \|\cdot\|)$, and like for the sphere packing problem, computing $m_1(\mathbb{R}^n, \|\cdot\|)$ corresponds to finding the largest (in terms of density in this case) independent set in the unit distance graph. The relation between the measurable chromatic number and the number $m_1(\mathbb{R}^n, \|\cdot\|)$ is given by the inequality:

$$\chi_m(\mathbb{R}^n) \geqslant \frac{1}{m_1(\mathbb{R}^n)}.$$

However, the determination of $m_1(\mathbb{R}^2, \|\cdot\|_2)$, and more generally that of $m_1(\mathbb{R}^n, \|\cdot\|_2)$, turned out to be a difficult problem as well. An overview of the known results about $m_1(\mathbb{R}^n, \|\cdot\|_2)$ will be given in the introduction of Chapter 3.

The difficulties encountered in the computation of the numbers χ_m and m_1 in the Euclidean case encourage to consider variants of the original problem.

For instance, one could replace the field $\mathbb{R}$ by a more general field, and the usual Euclidean quadratic form by general quadratic forms ([BMK17]). Recently DeCorte and Golubev ([DG17]) have computed lower bounds on the measurable chromatic number of the *hyperbolic* plane. In this thesis, we consider another variant of this problem. As we will describe in the introduction of Chapter 3, the fact that Euclidean spheres cannot fill perfectly $\mathbb{R}^n$ is likely to prevent a set avoiding distance 1 in $\mathbb{R}^n$ to reach a density of 2^{-n}. In the light of this observation, we consider norms for which the unit ball is a polytope $\mathcal{P}$ tiling $\mathbb{R}^n$ by translation. In this situation, there is a simple example of a set avoiding distance 1 of density 2^{-n} (see Figure 10), and this construction seems to be optimal, as conjectured by Bachoc and Robins.

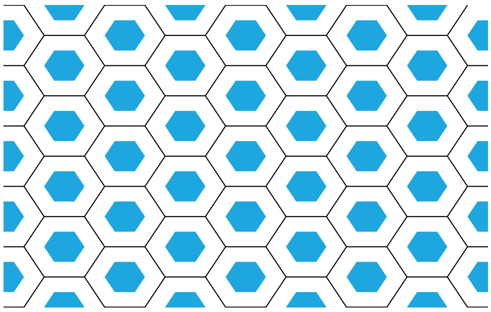


FIGURE 10. When the unit ball tiles space by translation, there is set avoiding distance 1 of density $1/2^n$.

The second goal of this thesis is to prove this conjecture for several polytopes $\mathcal{P}$. First, we will prove it in dimension 2: for any norm $\|\cdot\|$ such that the unit ball tiles $\mathbb{R}^2$ by translation, $m_1(\mathbb{R}^2, \|\cdot\|) = 1/4$. As a consequence the chromatic number of the plane in that case is exactly 4. We will also consider the polytopes that are the Voronoï cells of the famous lattices A_n and D_n, in any dimension n. We prove that Bachoc and Robins conjecture is true for the first family, and we obtain a slightly weaker result for the second one. In dimension 3, there are five combinatorial types of polytopes that tile $\mathbb{R}^3$ by translation (see Chapter 2, Section 3). We show that the conjecture is true for several Voronoï cells in dimension 3, that realize four combinatorial types of polytopes out of five (see Chapter 4).

Now that we have stated the results, it is worth to say a few words about the techniques employed. Although this problem is a priori stated in a continuous framework, we will turn it into a problem about discrete sets. If G is a subgraph of $G(\mathbb{R}^n, \|\cdot\|)$ induced by a discrete set $V \subset \mathbb{R}^n$, the number $m_1(\mathbb{R}^n, \|\cdot\|)$ is upper bounded by the *independence ratio* (see Chapter 2, Section 4) of G. For every polytope that we will consider, we will first need to construct an appropriate discrete graph, and then we will prove that its independence ratio cannot exceed $1/2^n$. To do so, we will give the set V an auxiliary graph structure, in such a way that a set $A \subset V$ avoiding polytope distance 1 may be decomposed in a canonical way as a disjoint union of *blocks*.

In Chapter 3, the auxiliary graphs that we construct afford a strong and useful property: any set A avoiding polytope distance 1 may be written as a union of cliques whose neighborhoods in the graph are disjoint. As a consequence the global density of A in V is upper bounded by the *local density* of a clique in its neighborhood. In Chapter 4, we develop a more general framework in order to handle more complicated graphs, by introducing the concept of *discrete distribution function*. Once that a set $A \subset V$ is split into disjoint blocks, a discrete distribution function breaks V up and assign to every block a *neighborhood*. The challenge is to find such a function which ensures that the density of each block in its neighborhood is upper bounded by $1/2^n$. In other words, the initial problem has turned into a... discrete packing problem.

Outline of the Thesis

Chapter 1. We introduce the material that will be used along the thesis. We fix some notation and recall very basic notions of topology and graph theory, present several aspects of Euclidean lattices, and give a short overview of known results about parallelohedra.

Chapter 2. This chapter contains our results on the sphere packing problem. We use codes over cyclotomic fields in order to construct finite families of lattices in high dimension that contain a dense lattice. These results have been published in [**Mou17**].

Chapter 3. In this chapter based on [**BBMP17**], which is joint work with Christine Bachoc, Thomas Bellitto and Arnaud Pêcher [**BBMP17**], we study Bachoc and Robins conjecture for parallelohedra in dimension 2 as well as for the Voronoï cells of the lattices A_n and D_n.

Chapter 4. This chapter contains further results on sets avoiding parallelohedron distance 1. We introduce the notion of discrete distribution function and we prove Bachoc and Robins conjecture for several new polytopes, especially for the so-called *elongated dodecahedron* in dimension 3.

Chapter 5. The last chapter is dedicated to concluding comments and perspectives for future work.

Chapter 1

Preliminaries

In this chapter, we introduce most of the material that we will need along this thesis. The main goal is to fix definitions and notations, and to recall, without proofs, well known results concerning the objects that we will handle.

1. Topological Preliminaries

1.1. Normed Spaces, Euclidean Spaces

Most of the time, our playground will be $E = \mathbb{R}^n$, the real vector space of dimension n, equipped with the *Lebesgue measure*. A *measurable* subset $A \subset \mathbb{R}^n$ will be understood as a set *measurable with respect to the Lebesgue measure*, and we will denote by $\mathrm{Vol}(A)$ its *volume*.

The space $E = \mathbb{R}^n$ will also be given a *norm*, that we will denote by $\|\cdot\|$. We keep the usual notations for the classical norms, for instance

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_i|, i = 1 \ldots n\},$$

where $|\cdot|$ is the usual absolute value in $\mathbb{R}$.

For $x \in E$, $r \leqslant 0$, we denote by $B(x, r)$ the *open ball* of radius r centred in x

$$B(x, r) = \{y \in E \mid ||x - y|| < r\},$$

and we write $B(r)$ instead of $B(0, r)$. Denote by V_n the volume of the unit ball in dimension n. By Stirling formula, we have

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \sim \frac{1}{\sqrt{n\pi}} \left(\sqrt{\frac{2\pi e}{n}}\right)^n$$

where $f \sim g$ means $\lim_{n\to\infty} f/g = 1$. Thus, if $\mathrm{Vol}(B(r)) = V$, we get that

$$r \sim \sqrt{\frac{n}{2\pi e}} V^{\frac{1}{n}}. \tag{1}$$

If A is a subset of E, we denote by $\mathring{A}$ its *interior*, by $\bar{A}$ its *closure* in E, and by ∂A its *boundary*. We also define the *diameter* of A:

$$\mathrm{Diam}(A) = \sup\{\|x-y\|, x, y \in A\}.$$

Finally, a *Euclidean space* is a vector space E isomorphic to $\mathbb{R}^n$, equipped with a *scalar product*, denoted by $\langle \cdot, \cdot \rangle$. This scalar product induces a norm on E, by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. The most classical Euclidean space is $\mathbb{R}^n$, together with the Euclidean norm $\| \cdot \|_2$, coming from the natural scalar product

$$\langle (x_1, \dots x_n), (y_1, \dots y_n) \rangle = \sum_{i=1}^{n} x_i y_i.$$

1.2. Density of a Set and Sphere Packings

We need to define the *density* of a measurable subset $A \subset \mathbb{R}^n$, in order to quantify the proportion of space covered by A in $\mathbb{R}^n$. One would like to take the limit in R of the quotient

$$\frac{\mathrm{Vol}(A \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)},$$

but this limit does not always exist. So we define the *upper density* of A:

$$\delta(A) = \limsup_{R\to\infty} \frac{\mathrm{Vol}(A \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)}.$$

From now on, we will forget the term *upper density*, and simply write *density.*

A *sphere packing* is a union of non overlapping balls having the same radius r. More precisely, it corresponds to a set $P \subset \mathbb{R}^n$ such that, for any $x \neq y \in P$,

$$B(x, r) \cap B(y, r) = \emptyset.$$

The *density of the packing* P is the density in $\mathbb{R}^n$ of the reunion of all the balls

$$\cup_{x\in P} B(x, r).$$

The famous *sphere packing problem* asks for the highest density that can be achieved by a sphere packing in dimension n.

1.3. Polytope Norms

Let $\mathcal{P}$ a convex symmetric polytope, centered at 0, and with a non empty interior. The *polytope norm* $\|\cdot\|_{\mathcal{P}}$ associated with $\mathcal{P}$ is defined by

$$\|x\|_{\mathcal{P}} = \inf\{\lambda \in \mathbb{R}_+ \mid x \in \lambda\mathcal{P}\}.$$

We also call *polytope distance* the distance induced by $\|\cdot\|_{\mathcal{P}}$.

If $B_{\mathcal{P}}(r) = \{x \in \mathbb{R}^n \mid \|x\|_{\mathcal{P}} < r\}$, we have by definition:

$$x \in B_{\mathcal{P}}(1) \Leftrightarrow x \in \mathring{\mathcal{P}} \text{ and } \|x\|_{\mathcal{P}} = 1 \Leftrightarrow x \in \partial\mathcal{P}.$$

Figure 1 presents an example of polytope distance, when the polytope is a regular hexagon in the plane.

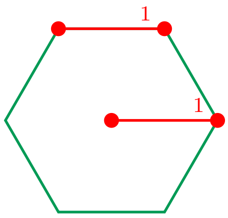


FIGURE 1. An example of polytope distance.

Several well-known norms are polytope norms, such as $\|\cdot\|_\infty$ and $\|\cdot\|_1$ for instance. Finally, suppose that a polytope $\mathcal{P}$ in $\mathbb{R}^n$ can be written as the direct product of two polytopes

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2,$$

with $\mathcal{P}_1 \subset \mathbb{R}^m$ and $\mathcal{P}_2 \subset \mathbb{R}^{n-m}$. Then the polytope norm on $\mathbb{R}^n$ corresponding to $\mathcal{P}$ is nothing but

$$\|(x_1, \ldots, x_n)\|_{\mathcal{P}} = \max\{\|(x_1, \ldots, x_m)\|_{\mathcal{P}_1}, \|(x_{m+1}, \ldots, x_n)\|_{\mathcal{P}_2}\}.$$

2. Euclidean Lattices

Euclidean lattices will assume a preponderant role in this thesis. There is a lot to say about this topic. In this short presentation, we only focus on the aspects that will be useful in the next chapters. Nice references on lattices are [CS87], [Mar03], or [Ebe13].

2.1. First Definitions

Let $E = \mathbb{R}^n$, equipped with the usual Euclidean scalar product. A *n-dimensional lattice* is a subset $\Lambda \subset \mathbb{R}^n$ with the property that there exists a basis $\mathcal{B} = \{e_1, \dots e_n\}$ of $\mathbb{R}^n$ such that

$$\Lambda = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n.$$

Such a basis is then called a *basis* of Λ, and we denote by $A_{\mathcal{B}}$ the matrix whose columns are the vectors $e_1, \dots, e_n$, written in the canonical basis of $\mathbb{R}^n$. For any $n \geqslant 2$, a lattice Λ admits infinitely many bases, but the following property holds: two bases $\mathcal{B}$ and $\mathcal{B}'$ of $\mathbb{R}^n$ span the same lattice Λ if and only if there exists a matrix M in

$$\{M \in \mathrm{GL}_n(\mathbb{Z}) \mid \det M = \pm 1\}$$

such that

$$A_{\mathcal{B}'} = M A_{\mathcal{B}}.$$

A *fundamental region* of Λ is a measurable set $\mathcal{R} \subset \mathbb{R}^n$ such that for any $\lambda \neq \lambda' \in \Lambda$, the measure of $(\lambda + \mathcal{R}) \cap (\lambda' + \mathcal{R})$ is 0, and

$$\mathbb{R}^n = \bigcup_{\lambda \in \Lambda} (\lambda + \mathcal{R}).$$

For instance, if $\mathcal{B}$ is a basis of Λ, the *fundamental parallelotope* of Λ associated with $\mathcal{B}$

$$\mathcal{P}_{\mathcal{B}} = \left\{ \sum_{i=1}^{n} x_i e_i \mid x_i \in [0,1] \right\}$$

is a fundamental region of Λ.

The *volume* of Λ is defined as the volume of a fundamental region of Λ. It does not depend on the choice of the fundamental region. For example, if $\mathcal{B}$ is a basis of Λ, we have

$$\mathrm{Vol}(\Lambda) = |\det(A_{\mathcal{B}})|.$$

The *Gram Matrix* of Λ with respect to one of its basis $\mathcal{B} = \{e_1, \dots e_n\}$ is the matrix whose coordinates are the scalar products between the elements of $\mathcal{B}$:

$$G_{\mathcal{B}} = (\langle e_i, e_j \rangle)_{1 \leqslant i,j \leqslant n} = A_{\mathcal{B}}^{tr} A_{\mathcal{B}},$$

where $A_{\mathcal{B}}^{tr}$ is the transpose of the matrix $A_{\mathcal{B}}$. If $\mathcal{B}'$ is another basis of Λ, we have $\det(G_{\mathcal{B}}) = \det(G_{\mathcal{B}'})$ and this value is the *determinant* of Λ. The determinant and the volume of Λ are related by the formula

$$\det(\Lambda) = \mathrm{Vol}(\Lambda)^2.$$

If Λ' is a lattice in $\mathbb{R}^n$ such that $\Lambda' \subset \Lambda$, then the index $|\Lambda/\Lambda'|$ is finite, and

$$\mathrm{Vol}(\Lambda') = |\Lambda/\Lambda'| \, \mathrm{Vol}(\Lambda).$$

The *symmetry group* (or *automorphism group*) of Λ is the group made by the *isometries* of $\mathbb{R}^n$ (*i.e.* the linear transformations preserving the scalar product) that send Λ onto itself.

Finally, we introduce the *dual lattice* $\Lambda^{\#}$ of Λ

$$\Lambda^{\#} = \{x \in \mathbb{R}^n, \forall\, y \in \Lambda, \langle x, y\rangle \in \mathbb{Z}\},$$

its volume is the inverse of the volume of Λ:

$$\mathrm{Vol}(\Lambda^{\#}) = \frac{1}{\mathrm{Vol}(\Lambda)}.$$

2.2. Voronoï Cell, Packing Radius, Covering Radius

Voronoï cell. The *Voronoï cell* of a lattice $\Lambda \subset \mathbb{R}^n$ is a very important region associated to Λ. It consists of the points of $\mathbb{R}^n$ that are closer to 0 than to any other vector of Λ

$$\mathcal{V} = \mathcal{V}_\Lambda = \{z \in \mathbb{R}^n \mid \forall\, x \in \Lambda, ||z - x|| \geqslant ||z||\},$$

and it is a fundamental region of Λ.

The *Voronoï vectors* of Λ are the vectors that define its Voronoï cell. More precisely, a vector $v \in \Lambda$ is a Voronoï vector of Λ if the intersection between $\mathcal{V}$ and the hyperplane

$$H_v = \left\{x \in \mathbb{R}^n \mid \langle x, v\rangle = \frac{1}{2}\langle v, v\rangle\right\}$$

is non empty. We say that v is *relevant* if this intersection is a *facet* of $\mathcal{V}$, that is a $(n-1)$-dimensional face of $\mathcal{V}$ (see Figure 2).

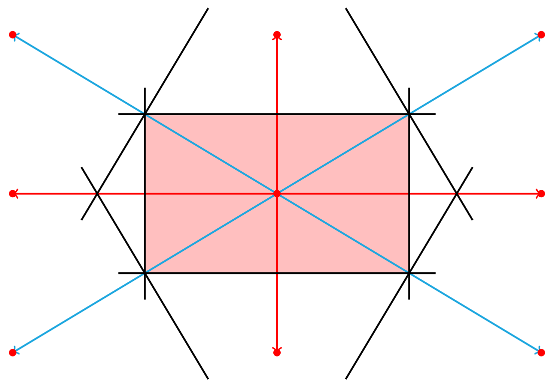


FIGURE 2. An example of Voronoï cell of a lattice with its Voronoï vectors. The red vectors are relevant, while the blue ones are not.

The relevant Voronoï vectors give a complete description of $\mathcal{V}$: a vector $x \in \mathbb{R}^n$ belongs to $\mathcal{V}$ if and only if, for any relevant Voronoï vector v,

$$|\langle x, v\rangle| \leqslant \frac{1}{2}\langle v, v\rangle.$$

We have the following nice characterization of the Voronoï vectors (see [CS92], Chapter 21, Section 3.G.):

PROPOSITION 1. *A vector $v \in \Lambda \setminus \{0\}$ is a Voronoï vector if and only if it is a shortest vector in the coset $v + 2\Lambda$.*

Moreover, v is relevant if and only if $\pm v$ are the only shortest vectors in that coset.

Packing Problem and Covering Problem. Previously we presented the sphere packing problem. For the *lattice sphere packing problem*, the centers of the spheres are required to shape a lattice. The *minimum* of a lattice Λ, denoted by μ, is the minimal norm among the non-zero vectors of Λ:

$$\mu = \mu_\Lambda = \min\{||x||, x \in \Lambda \setminus \{0\}\}.$$

The largest radius that one can take in order to get a sphere packing associated with Λ is obviously $\mu/2$: this radius is the *packing radius* of Λ. The density of the corresponding periodic packing is given by the ratio between the volume of a sphere and the volume of a fundamental region (see Figure 3).

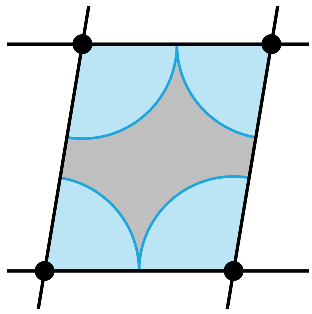


FIGURE 3. The density of a lattice packing is the ratio between the volume of a sphere and the volume of the lattice.

This density $\Delta(\Lambda)$ is completely determined by μ:

$$\Delta(\Lambda) = \frac{\mathrm{Vol}(B(\mu/2))}{\mathrm{Vol}(\Lambda)} = \left(\frac{\mu}{2}\right)^n \frac{V_n}{\mathrm{Vol}(\Lambda)}.$$

The lattice sphere packing problem asks for the supreme density Δ_n over all the lattices in dimension n. An equivalent formulation involves the so-called *Hermite invariant* of Λ:

$$\gamma(\Lambda) = \left(\frac{\mu}{\mathrm{Vol}(\Lambda)^{1/n}}\right)^2.$$

The analogue of Δ_n in this formulation is $\gamma_n = \sup_{\Lambda} \gamma(\Lambda)$, and both are related by the relation

$$\gamma_n = 4\left(\frac{\Delta_n}{V_n}\right)^{2/n}.$$

The *lattice covering problem* is often described as the dual problem of the lattice packing problem. The *covering radius* of Λ is the minimal radius r such that the union of balls of radius r, centered at the points of Λ, covers $\mathbb{R}^n$. Formally:

$$\tau = \tau_\Lambda = \sup_{z\in\mathbb{R}^n} \inf_{x\in\Lambda} ||z-x||.$$

The analogue of the density of the lattice is in this situation the *thickness* of Λ, measuring the average number of spheres in the covering that contain a point of $\mathbb{R}^n$:

$$\Theta(\Lambda) = \frac{\mathrm{Vol}(B(\tau))}{\mathrm{Vol}(\Lambda)} = \frac{V_n}{\mathrm{Vol}(\Lambda)}\tau^n.$$

A good lattice for the covering problem is a lattice whose thickness is small.

Voronoï cell, packing radius, and covering radius are related in the following way: the packing radius is the largest r such that $B(r) \subset \mathcal{V}$, and the covering radius is the smallest R such that $\mathcal{V} \subset B(R)$ (see Figure 4).

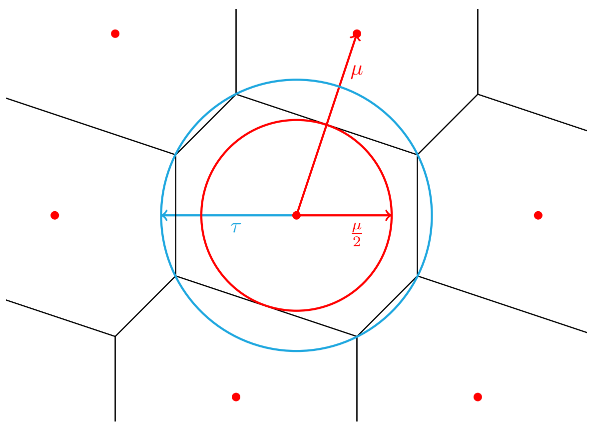


FIGURE 4. The packing radius and the covering radius of the lattice.

2.3. Fundamental Examples

The cubic lattice $\mathbb{Z}^n$. The most natural lattice is the *cubic lattice* $\mathbb{Z}^n$ made by the points in $\mathbb{R}^n$ having integer coordinates. A basis of $\mathbb{Z}^n$ is the canonical basis $\{e_1, \dots e_n\}$ of $\mathbb{R}^n$, and its Gram matrix in this basis is the identity matrix of size n. Moreover, we have:

- $\det(\mathbb{Z}^n) = \mathrm{Vol}(\mathbb{Z}^n) = 1$,
- $\mu(\mathbb{Z}^n) = 1$, so the packing radius of $\mathbb{Z}^n$ is $1/2$,

- $\tau(\mathbb{Z}^n) = \dfrac{\sqrt{n}}{2}$,
- The symmetry group of $\mathbb{Z}^n$ is the group generated by all permutations and sign changes of the coordinates.
- $\mathbb{Z}^n$ is *unimodular*: it is its own dual lattice,
- The Voronoï cell of $\mathbb{Z}^n$ is the hypercube whose vertices are the vectors $(\pm 1/2, \dots, \pm 1/2)$. The relevant Voronoï vectors are the $2n$ vectors $\pm e_i$.

The lattice A_n. For $n \geqslant 1$, let H be the hyperplane in dimension $n+1$ defined by

$$H = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 0\}.$$

In this n-dimensional vector space, the lattice A_n is defined by

$$A_n = H \cap \mathbb{Z}^{n+1}.$$

Let $\{e_0, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$. A basis of A_n is the family $\{e_0 - e_1, e_1 - e_2, \dots, e_{n-1} - e_n\}$. The Gram matrix of A_n in this basis is

$$G = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

and we have

- $\det(A_n) = n+1$ and $\mathrm{Vol}(A_n) = \sqrt{n+1}$,
- $\mu(A_n) = \sqrt{2}$, so the packing radius of A_n is $\sqrt{2}/2$,
- $\tau(A_n) = \dfrac{a(n+1-a)}{n+1}$, where a is the integer part of $(n+1)/2$.
- The symmetry group of A_n is the group generated by all permutations of the $n+1$ coordinates and the multiplication by -1.
- The relevant Voronoï vectors of A_n are all the vectors of the form $e_i - e_j$. The Voronoï cell of A_n will be described in more details in Chapter 3.
- Let $p_H : \mathbb{R}^{n+1} \to H$ denote the orthogonal projection on H. Then the dual lattice of A_n is $A_n^\# = p_H(\mathbb{Z}^{n+1})$.

Remark 1. *For $n = 2$, the lattice A_2 is the so-called* hexagonal lattice *(see Figure 5), which provides the densest packing in dimension 2, with $\Delta(A_2) = \dfrac{\pi}{\sqrt{12}} \approx 0.9069$.*

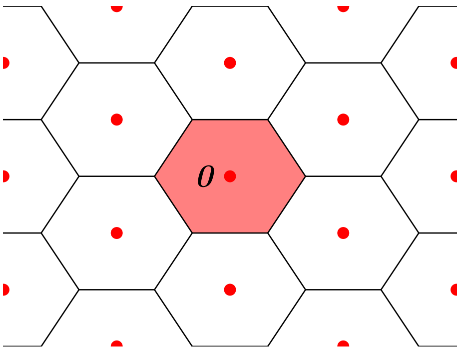


FIGURE 5. The hexagonal lattice A_2

The lattice D_n. For $n \geqslant 3$, D_n is the lattice made by the points in $\mathbb{Z}^n$ whose sum of coordinates is even:

$$D_n = \left\{ (x_1, \dots, x_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} x_i = 0 \bmod 2 \right\}.$$

Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. A basis of D_n is the family $\{e_1 + e_2, e_1 - e_2, e_2 - e_3 \dots, e_{n-1} - e_n\}$. The Gram matrix of D_n in this basis is

$$G = \begin{pmatrix} 2 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 2 & -1 & \ddots & 0 & 0 \\ 1 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

and we have

- $\det(D_n) = 4$ and $\mathrm{Vol}(D_n) = 2$,
- $\mu(D_n) = \sqrt{2}$, so the packing radius of D_n is $\sqrt{2}/2$,
- $\tau(D_3) = 1$, and for $n \geqslant 4$, $\tau = \dfrac{\sqrt{n}}{2}$.
- The symmetry group of D_n for $n \neq 4$ is the same as the symmetry group of $\mathbb{Z}^n$: the group generated by all permutations and sign changes of the coordinates. When $n = 4$, one has to add to the generators the Hadamard matrix $\dfrac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.
- The relevant Voronoï vectors of D_n are all the vectors of the form $e_i \pm e_j$. The Voronoï cell of D_n will also be described in Chapter 3.

- The dual lattice of D_n is

$$D_n^\# = D_n \cup \left(\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + D_n\right) \cup \left(\left(\frac{1}{2}, \ldots, -\frac{1}{2}\right) + D_n\right) \cup ((0, \ldots, 0, 1) + D_n).$$

REMARK 2. *For $n = 3$, we have $A_3 \simeq D_3$, and this lattice is the famous* face-centered cubic lattice, *which realizes the highest density $\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$ reached by a packing in dimension 3.*

2.4. Construction A

Construction A is a method to construct lattices from codes. A general reference about codes is [Rot06]. For relations between lattices and codes, one should look at [CS87] and [Ebe13]. A *$[n, k]_p$-linear code C* is a k-dimensional subspace of $\mathbb{F}_p^n$. The parameters n and k are respectively called the *length* and the *dimension* of C.

If C is a $[n, k]_p$-code, the *orthogonal code*, or *dual code $C^\perp$* of C is defined as

$$C^\perp = \left\{ (y_1, \ldots, y_n) \in \mathbb{F}_p^n \mid \forall (x_1, \ldots, x_n) \in C, \sum_{i=1}^{n} x_i y_i = 0 \right\}$$

and is a $[n, n-k]_p$-code.

Let us denote by π the canonical projection $\pi : \mathbb{Z}^n \to \mathbb{F}_p^n$. Let C be a $[n, k]_p$-code. The *lifted lattice* Λ_C obtained from C by Construction A is the preimage of C via π:

$$\Lambda_C = \pi^{-1}(C) = \{x \in \mathbb{Z}^n \mid \pi(x) \in C\}.$$

It satisfies the following properties:

- $p\mathbb{Z}^n \subset \Lambda_C \subset \mathbb{Z}^n$,
- $\mathrm{Vol}(\Lambda_C) = p^{n-k}$,
- Let $C^\perp$ be the orthogonal code of C. Then the dual lattice of Λ_C is

$$\Lambda_C^\# = \frac{1}{p} \Lambda_{C^\perp}.$$

EXAMPLE 1. *The lattice E_8, that has been recently proved to realize the best packing in dimension 8 (see [Via17]) can be constructed via Construction A: it is obtained by lifting the* extended Hamming code *$\mathcal{H}_8$, which is the $[8, 4]_2$-code generated by the lines of the matrix*

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Precisely:

$$E_8 = \frac{1}{\sqrt{2}} \Lambda_{\mathcal{H}_8}.$$

In Chapter 2, we will use a generalization of Construction A in the context of number fields.

2.5. Lattices from Algebraic Number Theory

Another way to construct Euclidean lattices involves ideals in algebraic number fields. For details about basic algebraic number theory, including all definitions and proofs that we ommit here, see [Sam67] or [Neu99].

Let $K/\mathbb{Q}$ be a number field of degree n. This degree may be broken down into $n = r_1 + 2r_2$, where r_1 (respectively $2r_2$) is the number of real (respectively complex) embeddings of K. More precisely, we label by $\sigma_1, \ldots, \sigma_{r_1}$ the real embeddings $K \to \mathbb{R}$, and by and $\sigma_{r_1+1}, \ldots, \sigma_{r_1+2r_2}$ the complex embeddings $K \to \mathbb{C}$, in such a way that for every $1 \leqslant j \leqslant r_2$, $\sigma_{r_1+r_2+j} = \bar{\sigma}_{r_1+j}$, where $\bar{\cdot}$ denotes the complex conjugation in $\mathbb{C}$. Thanks to these maps, there is a natural embedding ι of K into K_R, where $K_R = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^n \simeq K \otimes_{\mathbb{Q}} \mathbb{R}$:

$$\begin{array}{cccc} \iota & : & K & \to & K_{\mathbb{R}} \\ & & x & \mapsto & (\sigma_1(x), \ldots, \sigma_{r_1}(x), \sigma_{r_1+1}(x), \ldots, \sigma_{r_1+r_2}(x)) \end{array}.$$

The n-dimensional real vector space $K_{\mathbb{R}}$ is given a structure of Euclidean space by the *trace form* $\mathrm{tr} = \mathrm{tr}_{K/\mathbb{Q}}$ of K over $\mathbb{Q}$. Indeed, recall that, for any $x \in K$, $\mathrm{tr}(x) = \displaystyle\sum_{i=0}^{n} \sigma_i(x)$. Then the map

$$\begin{array}{ccccc} \beta & : & K \times K & \to & \mathbb{R} \\ & & (x, y) & \mapsto & \mathrm{tr}(x\bar{y}) \end{array}$$

is a positive-definite symmetric bilinear form, which induces a scalar product $\langle \cdot, \cdot \rangle$ on $K_{\mathbb{R}}$.

The *ring of integers* $\mathcal{O}_K$, and more generally every *fractional ideal* $\mathfrak{A}$ of K are free $\mathbb{Z}$-modules of rank n. So their images under ι are lattices in the Euclidean space $K_{\mathbb{R}}$. From now on, we will also denote the image $\iota(\mathfrak{A})$ by $\mathfrak{A}$.

The determinant of $\mathcal{O}_K$ is by definition the absolute value of the discriminant d_K of K. Thus $\mathrm{Vol}(\mathcal{O}_K) = \sqrt{|d_K|}$. Moreover, it is easy to see that the minimum of $\mathcal{O}_K$ is $\sqrt{n}$: indeed $||1|| = \sqrt{n}$ and the arithmetic geometric inequality gives $||x|| \geqslant \sqrt{n}$ for all $x \in \mathcal{O}_K$. Further, we will need bounds on the minimum and the covering radius of general fractional ideals:

Lemma 1 ([BF06], propositions 4.1 and 4.2.). *Let $\mathfrak{A}$ be a fractional ideal of K, where K is a number field of degree n over $\mathbb{Q}$. Then we have :*

(i) $\dfrac{\mu_{\mathfrak{A}}}{\mathrm{Vol}(\mathfrak{A})^{\frac{1}{n}}} \geqslant \dfrac{\sqrt{n}}{\sqrt{|d_K|}^{\frac{1}{n}}}$,

(ii) $\dfrac{\tau_{\mathfrak{A}}}{\mathrm{Vol}(\mathfrak{A})^{\frac{1}{n}}} \leqslant \dfrac{\sqrt{n}}{2}\sqrt{|d_K|}^{\frac{1}{n}}$.

Example 2. *In Chapter 2, we will focus on lattices coming from ideals in* cyclotomic fields. *Let K be the cyclotomic field $\mathbb{Q}[\zeta_m]$, where ζ_m is a primitive*

m-th root of unity. This is a totally imaginary field of degree $\phi(m)$ over $\mathbb{Q}$, where ϕ is the Euler's totient function, and its discriminant is (e.g [Was97])

$$|d_K| = \frac{m^{\phi(m)}}{\prod_{\substack{l \in \mathbb{P} \\ l | m}} l^{\phi(m)/(l-1)}} \tag{2}$$

where $\mathbb{P}$ is the set of prime numbers.

3. Parallelohedra

The polytopes that tile space by translation will be the center of Chapters 3 and 4. Here we present some characterisations of those polytopes, as well as a complete description of parallelohedra in dimensions 2 and 3.

3.1. Polytopes Tiling Space by Translation and Parallelohedra

We say that a convex body K *tiles* $\mathbb{R}^n$ *by translation* if there exists a family of vectors $T \subset \mathbb{R}^n$ such that:

- $\mathbb{R}^n = \cup_{t \in T}(K + t)$,
- For any $t \neq t' \in T$, $K + t$ and $K + t'$ have disjoint interiors.

A *parallelohedron* in dimension n is a polytope $\mathcal{P}$ that tiles $\mathbb{R}^n$ *face-to-face* by translation, *i.e.* there is a tiling such that the intersection between two translates of $\mathcal{P}$, if non empty, is a common face of both of them. Works by Minkowski [Min97], Venkov [Ven54], and McMullen [McM80] have led to a proof that the convex bodies tiling space by translation are exactly the parallelohedra. Moreover, they provide a characterization of such polytopes. These results can be summed up in the following theorem. Recall that a *facet* of a polytope $\mathcal{P}$ is a $(n-1)$-dimensional face of $\mathcal{P}$, and that a *belt* of $\mathcal{P}$ is a sequence of facets $F_1, \ldots, F_k$ such that for every i (defined modulo k), $F_i \cap F_{i+1}$ is a $(n-2)$-dimensional face of $\mathcal{P}$ which is a translate of $F_1 \cap F_2$ (see Figure 6).

Theorem 1 (Minkowski, Venkov, McMullen). *The convex bodies that tile $\mathbb{R}^n$ by translation are the parallelohedra. Moreover, a convex polytope $\mathcal{P}$ is a parallelohedron if and only if it satisfies the three following conditions:*

(1) It is centrally symmetric,
(2) Each of its facets is centrally symmetric,
(3) Each of its belts contains 4 or 6 facets.

Finally, we may assume that the set of translation vectors T is a lattice.

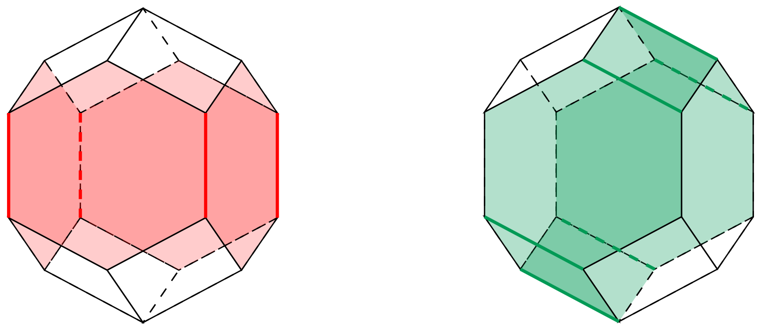


FIGURE 6. Two examples of belts of a same parallelohedron.

3.2. Voronoï's Conjecture

If we go back to the definition of a fundamental region of a lattice, it becomes clear that a fundamental region of a lattice, if convex, is a parallelohedron. In particular, the Voronoï cell of a lattice is a parallelohedron. Voronoï conjectured that the converse is also true, up to an affine transformation:

CONJECTURE 1 (Voronoi's Conjecture). *If $\mathcal{P}$ is a parallelohedron in $\mathbb{R}^n$, then there is an affine map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(\mathcal{P})$ is the Voronoï cell of a lattice $\Lambda \subset \mathbb{R}^n$.*

This conjecture has been solved for several families of parallelohedra. For instance, Voronoï himself [Vor08] proved it for *primitive parallelohedra*, that are the polytopes presenting a *facet-to-facet* tiling in dimension n such that in each vertex of a tile, exactly $n + 1$ tiles meet. Erdahl [Erd99] solved it for *zonotopal parallelohedra*, that are Minkowski sums of line segments. See [Gru07] of [Val03] for further details. Moreover, Delone [Del29] has shown that Voronoï's conjecture is true in dimensions up to $n = 4$. The most important result for us is the latter, since most of the parallelohedra that we will consider will be in dimension 2 and 3.

3.3. Parallelohedra in Dimensions 2 and 3

Following Delone's result, the understanding of parallelohedra in dimensions 2 and 3 boils down to the description of Voronoï cells of lattices in those dimensions. Here we give a short overview of the classification presented in [CS92].

Let $\Lambda \subset \mathbb{R}^n$ be a lattice. A *superbase* of Λ is a family of $n + 1$ vectors $v_0, v_1, \ldots, v_n$ such that

- $\{v_1, \ldots, v_n\}$ is a basis of Λ,
- $v_0 + v_1 + \ldots + v_n = 0$.

Let us define, for $i, j \in \{0, \ldots, n\}$, $i \neq j$, the Selling parameter

$$p_{i,j} = -\langle v_i, v_j \rangle.$$

A superbase is said to be *obtuse* if for all $i \neq j$, $p_{i,j} \geqslant 0$, and *strictly obtuse* if for all $i \neq j$, $p_{i,j} > 0$.

Even though every lattice obviously admits a superbase, it is not true that every lattice affords an obtuse superbase. Such a lattice is said to be *of Voronoï first kind*. For instance, A_n and $A_n^{\#}$ have an obtuse superbase, whereas D_n for $n \geqslant 4$ do not. Whenever a lattice has an obtuse superbase, there is a nice description of its Voronoï vectors:

THEOREM 2 ([CS92], Theorem 3). *Suppose Λ has an obtuse superbase $v_0, \ldots, v_n$. Then the $2^{n+1} - 2$ subsums*

$$v_S = \sum_{i \in S} v_i,$$

where S runs through all the $2^{n+1} - 2$ non trivial subsets of $\{0, \ldots, n\}$, are Voronoï vectors. For any such S, if $\bar{S}$ is the complementary set of S, $v_{\bar{S}} = -v_S$, and the v_S represent all the $2^n - 1$ non-zero cosets of $\Lambda \mod 2\Lambda$.

Furthermore, the v_S are all relevant if and only if $v_0, \ldots, v_n$ is strictly obtuse.

The remarkable fact is that in dimensions $n \leqslant 3$, all the lattices are of Voronoï first kind:

THEOREM 3 ([CS92], Theorem 8). *Let $n \leqslant 3$. Then every lattice Λ in $\mathbb{R}^n$ has an obtuse surperbase.*

Even more interesting, the combinatorial type of the Voronoï cell only depends on the Selling parameters whose value is 0. More precisely, the generic type of Voronoï cell is that of a lattice affording a strictly obtuse superbase. The other parallelohedra are degenerate cases, when $p_{i,j} = 0$ for some $i \neq j$, which geometrically corresponds to the shrinking of a family of parallel edges. We now illustrate this idea by enumerating the parallelohedra in dimensions 2 and 3.

Dimension 2. Let $\Lambda \subset \mathbb{R}^2$ be a lattice, with obtuse superbase v_0, v_1, v_2. When this superbase is strictly obtuse, the Voronoï cell of Λ is a hexagon. Since $v_0 = -v_1 - v_2$, at most one Selling parameter $p_{i,j}$ can be 0: in that case, the Voronoï cell of Λ is a rectangle. So these are the only two kinds of parallelohedra in dimension 2 (see Figure 7).

Dimension 3. Let $\Lambda \subset \mathbb{R}^3$ be a lattice, with obtuse superbase v_0, v_1, v_2, v_3. When this superbase is strictly obtuse, the Voronoï cell of Λ is a *truncated octahedron*, see Figure 8.

The truncated octahedron has 6 families of 6 parallel edges. Assume without loss of generality that $p_{0,1} = 0$. Then one of this families of edges is shrinked, and we get a *elongated dodecahedron*, see Figure 9.

There are two non equivalent ways of shrinking edges in the elongated dodecahedron:

- There are 4 families of 6 parallel edges. Shrinking one of these families corresponds to putting $p_{i,j} = 0$ with $i \in \{0, 1\}$ or $j \in \{0, 1\}$ (recall

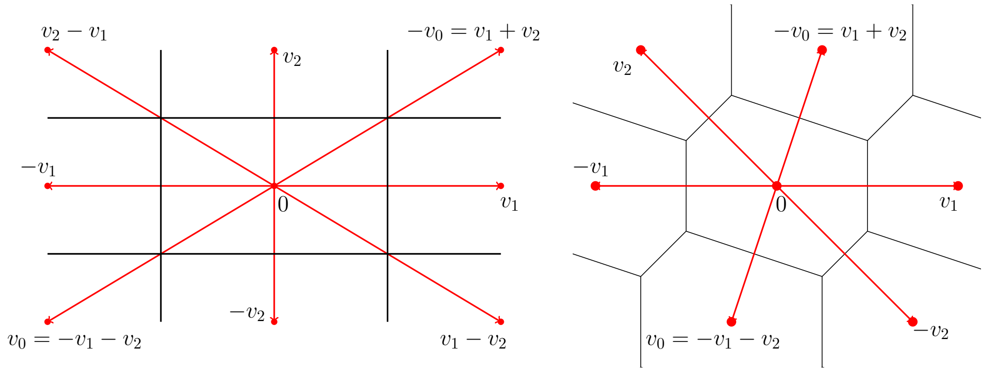


FIGURE 7. The two kinds of parallelohedra in dimension 2.

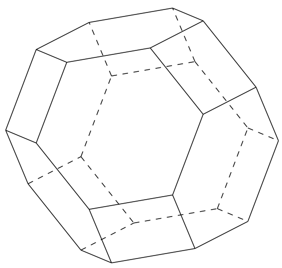


FIGURE 8. The truncated octahedron.

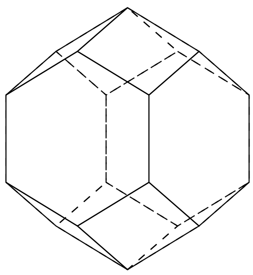


FIGURE 9. The elongated dodecahedron.

that $p_{0,1} = 0$). By this process, we obtain a *hexagonal prism*, see Figure 10.

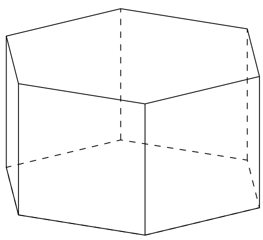


FIGURE 10. The hexagonal prism.

- There is one family of 4 parallel edges remaining. This family is shrinked when $p_{2,3} = 0$, and in this case we get a *rhombic dodecahedron*, see Figure 11.

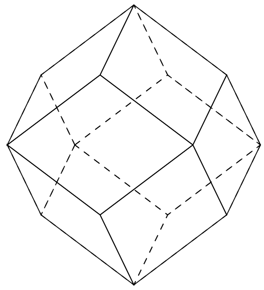


FIGURE 11. The rhombic dodecahedron.

Finally, from the two previous parallelohedra, there is only one way to shrink edges keeping a polytope in dimension 3. In both cases, one gets a *cuboïd*, see Figure 12. This the fifth and last combinatorial type of parallelohedron in dimension 3.

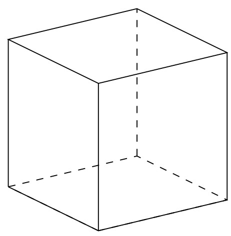


FIGURE 12. The cube.

REMARK 3. *In Figures 8, 9, 10, 11 and 12, the polytopes are presented in their more symmetrical shape. We point out that the truncated octahedron in Figure 8 is the Voronoï cell of the lattice* $A_3^\# \simeq D_3^\#$*, and that the rhombic dodecahedron in Figure 11 is the Voronoï cell of the lattice* $A_3 \simeq D_3$*. Of course, the cube in Figure 12 is the Voronoï cell of the* $\mathbb{Z}^3$.

4. Graphs

We conclude this preliminary chapter by some quick reminders on graph theory. If one is interested in graph theory in general, one could read [Bol98].

4.1. Definitions and Examples

A *graph* G is an ordered pair (V, E) such that the elements of E are pairs $\{x, y\}$, with $x \neq y \in V$. The set V is the set of *vertices* of G, and the elements of E are the *edges* of G. If $\{x, y\} \in E$, we say that the vertices x and y are *neighbors* in G. The *degree* of a vertex $v \in V$ is the number of its neighbors. A *path* between two vertices x and y is a chain $z_0 = x, z_1, \ldots, z_{t-1}, z_t = y$ such that for every $0 \leqslant i \leqslant t$, $z_i \in V$, and for every $0 \leqslant i \leqslant t-1$, $\{z_i, z_{i+1}\} \in E$. We say that t is the *length* of the path. Then the *graph distance* $d_G(x, y)$ between x and y in G is the minimal length of a path between x and y. If there is no path between x and y, we set $d_G(x, y) = +\infty$.

If A is a subset of V, and $x \in V$, the *graph distance* between x and A is naturally the minimal graph distance between x and an element of A:

$$d_G(x, A) = \inf_{a \in A} d_G(x, a).$$

Let $A \subset V$ be a subset of vertices. The *closed neighborhood* $N[A]$ of A in G is A augmented by its neighbors. In other words, it is the set of vertices at graph distance 0 or 1 from A:

$$N[A] = \{v \in V \mid d_G(v, A) \leqslant 1\}.$$

A *clique* of G is a subset $C \subset V$ such that, for every $x \neq y \in C$, $\{x, y\} \in E$. The *clique number* $\omega(G)$ is the maximal size of a clique in G. A *connected component* in G is a subset $A \subset V$ such that, for every $x \neq y \in A$, there is a path in G from x to y. The graph G is said to be *connected* if for every $x \neq y \in V$, there is a path in G from x to y, namely if V is a connected component in G.

The *complementary graph* $\bar{G} = (\bar{V}, \bar{E})$ of $G = (V, E)$ is the graph constructed as follows:

$$\begin{cases} \bar{V} = V \\ \{x, y\} \in \bar{E} \Leftrightarrow \{x, y\} \notin E \end{cases}.$$

Finally, if $G = (V, E)$ is a graph, for any subset $A \subset V$, we define the subgraph of G *induced* by A: its set of vertices is A, and two vertices $x, y \in A$ are connected in this new graph if and only if $\{x, y\} \in E$.

Example 3. *The first example that we present is the so-called* Moser graph*: it is a connected graph having* 7 *vertices. It is depicted in Figure* 13.

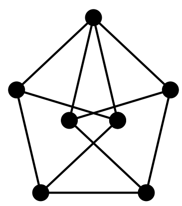


Figure 13. The Moser graph.

Now we present two precise kinds of graph, that will be of particular interest for us later.

Cayley Graphs. Let Γ be a group, and let S be a *generating set* of Γ. Suppose that S is *symmetric*, *i.e.* if $\gamma \in S$, then $\gamma^{-1} \in S$. We associate to Γ and S a graph, called *Cayley graph*, denoted by $G = G(\Gamma, S) = (V, E)$, defined as follows:

$$\begin{cases} V = \Gamma \\ \{x, y\} \in E \Leftrightarrow xy^{-1} \in S \end{cases}.$$

Since S is a generating set of Γ, the Cayley graph $G(\Gamma, S)$ is connected.

EXAMPLE 4. *There is a natural way to construct Cayley graphs on lattices. Let Λ be a lattice, and let $\mathcal{B}$ be a basis of Λ. Then we can associate to Λ the Cayley graph $G(\Lambda, S_{\mathcal{B}})$, where $S_{\mathcal{B}} = \{\pm\mathcal{E} \mid \epsilon \in \mathcal{E}\}$. However, there can be other interesting Cayley graphs associated with Λ. In Figure 14, we present, for the hexagonal lattice A_2, the Cayley graph structure previously defined, as well as the Cayley graph associated with the generating set made by the 6 shortest vectors of A_2.*

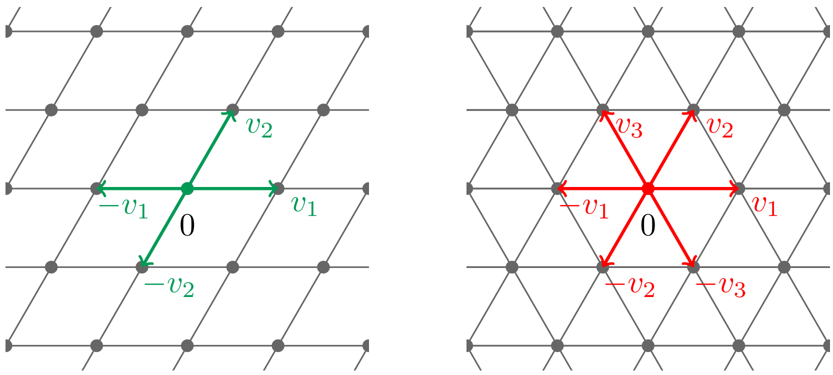


FIGURE 14. Two different Cayley graphs on A_2.

Distance Graphs. Let X be a metric space, and denote by d the associated distance $d : X \times X \to (0, +\infty)$. Let $D \subset (0, +\infty)$. Then we construct the *distance graph* $G = G(X, D) = (V, E)$ as follows:

$$\begin{cases} V = X \\ \{x, y\} \in E \Leftrightarrow d(x, y) \in D \end{cases}.$$

EXAMPLE 5. *Let $X = \mathbb{R}^n$, equipped with a norm $\|\cdot\|$. The* unit distance graph *$G(\mathbb{R}^n, \|\cdot\|) = (V, E)$ is the distance graph corresponding to $D = \{1\}$:*

$$\begin{cases} V = \mathbb{R}^n \\ \{x, y\} \in E \Leftrightarrow \|x - y\| = 1 \end{cases}.$$

4.2. Independent Sets and Chromatic Number of Finite Graphs

Here we recall the notion of independent set in a graph, and introduce the chromatic number of a graph. As a first step, before discussing the generalization of these notions to infinite graphs, we present them for finite graphs.

Independent Sets. An *independent set* in G is a subset $A \subset V$ such that for every $x, y \in A$, $\{x, y\} \notin E$. Note that an independent set in G is nothing but a clique in the complementary graph $\bar{G}$.

Let $G = (V, E)$ be a *finite* graph, *i.e.* the set of vertices V is finite. The *independence number* $\alpha(G)$ of G is the maximal size of an independent set of G:

$$\alpha(G) = \max\{|A| \mid A \subset V \text{ independent}\},$$

where $|A|$ denotes the cardinality of A. Note that we have, following the definition, $\alpha(G) = \omega(\bar{G})$. We also define the *independence ratio* of G, which is the quotient

$$\bar{\alpha}(G) = \frac{\alpha(G)}{|V|}.$$

This ratio measures the maximal density, in terms of finite sets, of an independent set in G.

EXAMPLE 6. *The independence number of the Moser graph is* 2*, see Figure* 15.

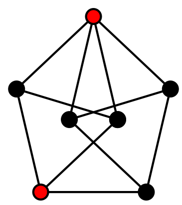


FIGURE 15. $\alpha(G) = 2$.

The Chromatic Number. The *chromatic number* $\chi(G)$ is the minimal number of colors required to color the vertices of G in such a way that if $x, y \in V$ and $\{x, y\} \in E$, x and y do not receive the same color. Obviously we have $\chi(G) \geqslant \omega(G)$.

EXAMPLE 7. *The chromatic number of the Moser graph is* 4*, see Figure* 16.

Now suppose G is finite, and fix a coloring of V with k colors, such that two neighbors do not get the same color. We can partition V into color classes. Every color class must be, following the definition, an independent set of G, so that its size is bounded from above by $\alpha(G)$. Since these k color classes are obviously disjoint, we get the inequality

$$|V| \leqslant k\alpha(G),$$

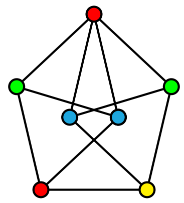


FIGURE 16. $\chi(G) = 4$.

which implies immediately the important inequality:

$$\chi(G) \geqslant \frac{1}{\bar{\alpha}(G)}. \tag{3}$$

In the sequel, we shall generalize this inequality to infinite graphs embedded in $\mathbb{R}^n$.

4.3. Generalization to Infinite Graphs in $\mathbb{R}^n$

In this paragraph, we suppose that the set of vertices V is a subset of $\mathbb{R}^n$.

Let $G = (V, E)$ be a *discrete* graph, *i.e.* the set V is discrete in $\mathbb{R}^n$. For any $R > 0$, and any subset $A \subset V$, we define A_R as the intersection

$$A_R = A \cap [-R, R]^n,$$

which is finite since V is discrete.

For $A \subset V$, we define the density of A in V:

$$\delta_V(A) = \limsup_{R \to \infty} \frac{|A \cap V_R|}{|V_R|} \tag{4}$$

This is a natural generalization of the density in finite sets, and the discrete analogue of the definition of density that we gave in $\mathbb{R}^n$. Based on this notion, we extend the definition of the independence ratio to discrete graphs:

$$\bar{\alpha}(G) = \sup\{\delta_V(A) \mid A \text{ independent set}\}.$$

With this definition, (3) still holds.

In Chapter 3, we will use the following equivalent formulation of $\bar{\alpha}(G)$:

LEMMA 2. *Let $G = (V, E)$ be a discrete graph with $V \subset \mathbb{R}^n$. If every $v \in V$ has finite degree, then*

$$\bar{\alpha}(G) = \limsup_{R \to \infty} \bar{\alpha}(G_R),$$

where G_R is the finite induced subgraph of G whose set of vertices is $V_R = V \cap [-R, R]^n$.

The proof of the lemma is rather technical, we omit it there. However one can find a proof in the Appendix in [BBMP17], along with a discussion on the importance of the hypothesis that all the vertices of the graph have finite degree.

In the case of the unit distance graph presented previously, the density of an independent set is the density of a set in $\mathbb{R}^n$. Since we only defined the density of measurable sets, we must introduce the *measurable chromatic number* χ_m, when the color classes are required to be measurable. Obviously, $\chi \leqslant \chi_m$. The equivalent of the independence ratio is in that case the number $m_1(\mathbb{R}^n, \|\cdot\|)$:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup\{\delta(A) \mid A \text{ independent measurable set }\},$$

and the equivalent of (3) is:

$$\chi_m(G(\mathbb{R}^n, \|\cdot\|)) \geqslant \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}.$$

The number $m_1(\mathbb{R}^n, \|\cdot\|)$ and its relations with the measurable chromatic number of the unit distance graph will be discussed at length in Chapter 3.

Chapter 2

On the density of cyclotomic lattices constructed from codes

This chapter is based on the publication [Mou17].

1. Introduction

The sphere packing problem in Euclidean spaces asks for the biggest proportion of space that can be filled by a collection of balls with disjoint interiors having the same radius. Here we focus on *lattice* sphere packings, where the centers of the balls are located at the points of a lattice, and we denote by Δ_n the supremum of the density that can be achieved by such a packing in dimension n. Let us recall that the exact value of Δ_n is known only for dimensions up to 8 [CS87] and for dimension 24 ([CK09]). For other dimensions, only lower and upper bounds are known. Moreover, asymptotically, the ratio between these bounds is exponential.

Here we focus on lower bounds. The first important result goes back to the celebrated Minkowski-Hlawka theorem [Hla43], stating the inequality $\Delta_n \geqslant \frac{\zeta(n)}{2^{n-1}}$ for all n, where $\zeta(n)$ denotes the Riemann zeta function. Later, Rogers [Rog47] improved this bound by a linear factor: he showed that $\Delta_n \geqslant \frac{cn}{2^n}$ for every $n \geqslant 1$, with $c \approx 0.73$. The constant c was successively improved by Davenport and Rogers [DR47] ($c = 1.68$), Ball [Bal92] ($c = 2$) and Vance [Van11] ($c = 2.2$ when n is divisible by 4). Recently Venkatesh has obtained a more dramatic improvement [Ven13], showing that for n big enough, $\Delta_n \geqslant \frac{65963n}{2^n}$. Most importantly, he proves that for infinitely many dimensions n, $\Delta_n \geqslant \frac{n \log \log n}{2^{n+1}}$, thus improving for the first time upon the linear growth of the numerator.

Unfortunately, all these results are of existential nature: their proofs are non constructive by essence, due to the fact that they generally use random arguments over infinite families of lattices. It is then natural to ask for effective versions of these results. It is worth to explain what we mean here by effectiveness. Indeed, designing a practical algorithm, *i.e.* running in polynomial time in the dimension, to construct dense lattices appears to be out of reach to date. More modestly, one aims at exhibiting finite and explicit sets of lattices, possibly of exponential size, in which one is guaranteed to find a dense lattice.

In this direction, the first to give an effective proof of Minkowski-Hlawka theorem was Rush [**Rus89**]. Later, Gaborit and Zémor [**GZ07**] provided an effective analogue of Roger's bound for the dimensions of the form $n = 2p$ with p a big enough prime number. In both constructions, the lattices are lifted from codes over a finite field, and run in sets of size of the form $\exp(kn^d \log n)$, with k a constant, $d = 1$ for [**GZ07**] and $d = 2$ for [**Rus89**].

Let us now explain with more details two ingredients that play a crucial role in the proofs of the results above. The first one is *Siegel's mean value theorem* [**Sie45**] which in particular states that, on average over the set $\mathcal{L}$ of n-dimensional lattices of volume 1,

$$\mathbb{E}_{\mathcal{L}}[|B(r) \cap (\Lambda \setminus \{0\})|] = \mathrm{Vol}(B(r)).$$

It follows that, if $\mathrm{Vol}(B(r)) < 1$, then there exists a lattice $\Lambda \in \mathcal{L}$ such that $B(r) \cap (\Lambda \setminus \{0\}) = \emptyset$, *i.e.* such that the minimum norm μ of its non zero vectors is greater than r. The density of the sphere packing associated to Λ then satisfies

$$\Delta(\Lambda) = \frac{\mathrm{Vol}(B(\mu))}{2^n} > \frac{1}{2^n}.$$

It is worth to point out that the same reasoning holds if $\mathrm{Vol}(B(r)) < 2$, because lattice vectors of given norm come by pairs $\{\pm x\}$. From this simple remark we get

$$\Delta_n > \frac{2}{2^n},$$

which is essentially Minkowski-Hlawka bound.

The second idea follows almost immediately from the previous observation: considering lattices affording a group of symmetries larger than the trivial $\{\pm \mathrm{Id}\}$ should allow to replace the factor 2 in the numerator by a greater value. To this end, one needs a family of lattices, invariant under the action of a group, for which an analogue of Siegel's mean value theorem holds. This idea is exploited in [**GZ07**], [**Van11**] and [**Ven13**]. In particular, this is how Venkatesh obtains the extra $\log \log n$ term, by considering cyclotomic lattices, *i.e.* lattices with an additional structure of $\mathbb{Z}[\zeta_m]$-modules. It turns out that, for a suitable choice of m, one can find such lattices in dimension $n = O(\frac{m}{\log \log m})$.

In this chapter, we consider cyclotomic lattices constructed from codes, in order to deal with finite families of lattices. We employ a generalization of the standard Construction A in the context of cyclotomic fields. To be more precise, the lattices that we take are the preimages through the standard surjection associated to a prime ideal $\mathfrak{P}$ of $\mathbb{Q}[\zeta_m]$

$$\mathbb{Z}[\zeta_m]^2 \to (\mathbb{Z}[\zeta_m]/\mathfrak{P})^2$$

of all one dimensional subspaces over the residue field $\mathbb{Z}[\zeta_m]/\mathfrak{P}$.

Our approach is simpler and more straightforward than the previous ones in several respects. On one hand, the analogue of Siegel's mean value theorem in our situation boils down to a simple counting argument on finite sets (see Lemma 5). On the other hand, the group action, which is, as in [GZ07], that of a cyclic group, is in our case easier to deal with, because it is a free action. As a consequence, we can cope with arbitrary orders m, while Gaborit and Zémor only consider prime orders.

Our main theorem is an effective version of Venkatesh's result:

Theorem 4. *For infinitely many dimensions n, a lattice Λ such that its density $\Delta(\Lambda)$ satisfies*

$$\Delta(\Lambda) \geqslant \frac{0.89n \log\log n}{2^n}$$

can be constructed with $\exp(1.5n \log n(1+o(1)))$ binary operations.

This result follows from a more general analysis of the density on average of the elements in the families of m-cyclotomic lattices described above, see Theorem 5 and Proposition 2 for precise statements.

A lattice Λ is said to be *symplectic* if there exists an isometry σ exchanging Λ and its dual lattice, and such that $\sigma^2 = -\mathrm{Id}$. Symplectic lattices are closely related to principally polarized Abelian varieties. In [Aut16], Autissier has adapted Venkatesh's approach to prove the existence of symplectic lattices with the same density. We show that, with some slight modifications, our construction leads to symplectic lattices, thus providing an effective version of Autissier's result (see Theorem 6 and Corollary 2).

The chapter is organized as follows: Section 2 introduces the construction of cyclotomic lattices from codes. In Section 3 we state and prove the main results discussed above. Section 4 is dedicated to the case of symplectic lattices.

2. Cyclotomic Lattices Constructed from Codes

A standard construction of lattices lifts codes over $\mathbb{F}_p$ to sublattices of $\mathbb{Z}^n$, this is the well known *Construction A* (see Chapter 2, subsection 2.4). Here we will deal with a slightly more general construction in the context of cyclotomic fields.

Let us consider $K = \mathbb{Q}[\zeta_m]$ and $K_{\mathbb{R}}$ the Euclidean space associated with K (see Chapter 2, subsection 2.5). Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$ lying over a prime number p which does not divide m. Then the quotient $F = \mathcal{O}_K/\mathfrak{P}$ is a finite field of cardinality $q = p^f$ for some f.

Let $E = K_{\mathbb{R}}^s$. We still denote by $\langle \cdot, \cdot \rangle$ the scalar product $\langle x, y \rangle = \sum_{i=1}^{s} \langle x_i, y_i \rangle$ induced on the $s\phi(m)$-dimensional $\mathbb{R}$-vector space E by that of $K_{\mathbb{R}}$. Let Λ_0 be a lattice in E which is a $\mathcal{O}_K$-submodule of E. We consider the canonical surjection

$$\pi : \Lambda_0 \to \Lambda_0/\mathfrak{P}\Lambda_0.$$

The norm $\|\cdot\|$ on E associated with $\langle \cdot, \cdot \rangle$ induces a weight on the quotient space $\Lambda_0/\mathfrak{P}\Lambda_0$: if $c \in \Lambda_0/\mathfrak{P}\Lambda_0$,

$$wt(c) = \min\{||z||, \pi(z) = c\}.$$

The quotient $\Lambda_0/\mathfrak{P}\Lambda_0$ is a vector space of dimension s over the finite field F. We will call a F-subspace C of $\Lambda_0/\mathfrak{P}\Lambda_0$ a *code*. We denote by k its dimension and by d its minimal weight, with respect to the weight defined above. Finally we denote by Λ_C the lattice obtained from C

$$\Lambda_C = \pi^{-1}(C)$$

and give in the following lemma a summary of its properties:

LEMMA 3. *Let C be a code of $\Lambda_0/\mathfrak{P}\Lambda_0$ of dimension k and minimal weight d. Then :*

(i) *The volume of Λ_C is*

$$\mathrm{Vol}(\Lambda_C) = q^{s-k}\,\mathrm{Vol}(\Lambda_0).$$

(ii) *The minimum of Λ_C is $\mu_{\Lambda_C} = \min\{d, \mu_{\mathfrak{P}\Lambda_0}\}$.*

(iii) *If $d \leqslant \mu_{\mathfrak{P}\Lambda_0}$, the packing density of Λ_C is:*

$$\Delta(\Lambda_C) = \frac{\mathrm{Vol}(B(d))}{2^n q^{s-k}\,\mathrm{Vol}(\Lambda_0)},$$

where $n = s\phi(m)$ is the dimension of E.

PROOF. (i) The lattice $\pi^{-1}(C)$ contains the lattice $\mathfrak{P}\Lambda_0$ and we have:

$$|\pi^{-1}(C)/\mathfrak{P}\Lambda_0| = |C| = q^k,$$

so

$$\mathrm{Vol}(\Lambda_C) = \frac{1}{q^k}\,\mathrm{Vol}(\mathfrak{P}\Lambda_0) = q^{s-k}\,\mathrm{Vol}(\Lambda_0).$$

(ii) and (iii) follow directly from the definitions.

□

To conclude this subsection, we state a lemma that relates the Euclidean ball and the discrete ball $\overline{B}(r) := \{c \in \Lambda_0/\mathfrak{P}\Lambda_0, wt(c) \leqslant r\}$.

LEMMA 4. *Assuming* $r < \dfrac{\mu_{\mathfrak{P}\Lambda_0}}{2}$, *we have:*

(i) $|\overline{B}(r)| = |\Lambda_0 \cap B(r)|$

(ii) $\mathrm{Vol}(B(r - \tau_{\Lambda_0})) \leqslant |\overline{B}(r)| \, \mathrm{Vol}(\Lambda_0) \leqslant \mathrm{Vol}(B(r + \tau_{\Lambda_0}))$.

(iii) *If* $\overline{B}(r) \cap (C \setminus \{0\}) = \emptyset$, *then*

$$\Delta(\Lambda_C) > \frac{\mathrm{Vol}(B(r))}{2^n q^{s-k} \, \mathrm{Vol}(\Lambda_0)}. \tag{5}$$

PROOF. (i) Let $c \in \Lambda_0 / \mathfrak{P}\Lambda_0$ such that $wt(c) \leqslant r$. We want to prove that c has exactly one representative $x \in \Lambda_0$ which satisfies $||x|| \leqslant r$. Indeed, if $y \in \Lambda_0$ with $y \neq x$ and $\pi(y) = \pi(x) = c$, we have $y = x + z$ with $z \in \mathfrak{P}\Lambda_0 \setminus \{0\}$. Then $||x - y|| = ||z|| \geqslant \mu_{\mathfrak{P}\Lambda_0} > 2r$, a contradiction.

(ii) Let us consider

$$A = \bigcup_{x \in \Lambda_0 \cap B(r)} (x + \mathcal{V}_{\Lambda_0})$$

where $\mathcal{V}_{\Lambda_0}$ is the Voronoï cell of Λ_0. The volume of A is

$$\mathrm{Vol}(A) = |\Lambda_0 \cap B(r)| \, \mathrm{Vol}(\Lambda_0) = |\overline{B}(r)| \, \mathrm{Vol}(\Lambda_0)$$

so the wanted inequalities will follow from the inclusions

$$B(r - \tau_{\Lambda_0}) \subset A \subset B(r + \tau_{\Lambda_0}).$$

Let us start with the second inclusion. If $z \in x + \mathcal{V}_{\Lambda_0}$, by definition of the covering radius, we have

$$||z - x|| \leqslant \tau_{\Lambda_0},$$

so if $||x|| \leqslant r$, $||z|| \leqslant r + \tau_{\Lambda_0}$. For the first inclusion, let y be such that $||y|| \leqslant r - \tau_{\Lambda_0}$. If x denotes the closest point to y in Λ_0, we have $y \in x + \mathcal{V}_{\Lambda_0}$ and $||x|| \leqslant ||y|| + ||x - y|| \leqslant r$, so that $y \in A$.

(iii) It follows directly from Lemma 3.

□

3. The Density of Cyclotomic Lattices Constructed from Codes

In this section, we introduce a certain family of lattices obtained from codes as described in the previous section, and show that for high dimensions, this family contains lattices having good density.

As before, $K = \mathbb{Q}[\zeta_m]$, $F = \mathcal{O}_K / \mathfrak{P} \simeq \mathbb{F}_q$. Let us set $s = 2$ and consider the Euclidean space $E = K_{\mathbb{R}}^2$, of dimension $2\phi(m)$, in which we fix $\Lambda_0 = \mathcal{O}_K^2$.

DEFINITION 1. *We denote by* $\mathcal{C}$ *the set of the* $(q+1)$ F*-lines of* $\Lambda_0 / \mathfrak{P}\Lambda_0 = F^2$, *and by* $\mathcal{L}_{\mathcal{C}}$ *the set of lattices of* E *constructed from the codes in* $\mathcal{C}$*:*

$$\mathcal{L}_{\mathcal{C}} = \{\Lambda_C, C \in \mathcal{C}\}.$$

The following lemma evaluates the average of the value of $|\overline{B}(r) \cap C \setminus \{0\}|$ over the family $\mathcal{C}$:

LEMMA 5. *We have:*

$$\mathbb{E}(|\overline{B}(r) \cap (C \setminus \{0\})|) < \frac{|\overline{B}(r)|}{q}.$$

PROOF. It is a straightforward computation:

$$\begin{aligned}\mathbb{E}(|\overline{B}(r) \cap (C \setminus \{0\})|) &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} |\overline{B}(r) \cap (C \setminus \{0\})| \\ &= \frac{1}{|\mathcal{C}|} \sum_{C \in \mathcal{C}} \sum_{\substack{c \in C \\ 0 < wt(c) \leqslant r}} 1 \\ &= \frac{1}{|\mathcal{C}|} \sum_{c \in \overline{B}(r) \setminus \{0\}} |\{C \in \mathcal{C} \ , \ c \in C\}|.\end{aligned}$$

There is exactly one line passing through every non zero vector in F^2. So

$$\mathbb{E}(|\overline{B}(r) \cap (C \setminus \{0\})|) = \frac{|\overline{B}(r) \setminus \{0\}|}{|\mathcal{C}|} < \frac{|\overline{B}(r)|}{q}.$$

$\square$

From now on, q will vary with m, so we adopt the notation q_m instead of q. We show that the family $\mathcal{L}_\mathcal{C}$ of lattices contains, when m is big enough and when q_m grows in a suitable way with m, lattices having high density.

THEOREM 5. *For every $1 > \varepsilon > 0$, if $\phi(m)^2 m = o(q_m{}^{\frac{1}{\phi(m)}})$, then for m big enough, the family of lattices $\mathcal{L}_\mathcal{C}$ contains a lattice $\Lambda \subset \mathbb{R}^{2\phi(m)}$ satisfying*

$$\Delta(\Lambda) > \frac{(1-\varepsilon)m}{2^{2\phi(m)}}.$$

We start with a technical lemma.

LEMMA 6. *Let $\rho_m = \sqrt{\frac{\phi(m)}{\pi e}} (q_m \operatorname{Vol}(\Lambda_0))^{\frac{1}{2\phi(m)}}$. If $\phi(m)^2 m = o(q_m{}^{\frac{1}{\phi(m)}})$, then*

(i) $\displaystyle\lim_{m \to \infty} \frac{\phi(m)\tau_{\Lambda_0}}{\rho_m} = 0$,

(ii) *For m big enough, $\rho_m < \frac{\mu_{\mathfrak{P}\Lambda_0}}{2}$.*

PROOF. (i) We have:

$$\frac{\phi(m)\tau_{\Lambda_0}}{\rho_m} = \frac{\sqrt{\pi e \phi(m)}\tau_{\Lambda_0}}{(q_m \operatorname{Vol}(\Lambda_0))^{\frac{1}{2\phi(m)}}}.$$

Since $\Lambda_0 = \mathcal{O}_K \times \mathcal{O}_K$, we have $\tau_{\Lambda_0} = \sqrt{2}\tau_{\mathcal{O}_K}$ and $\mathrm{Vol}(\Lambda_0) = \mathrm{Vol}(\mathcal{O}_K)^2$. Then, by (ii) of Lemma 1,

$$\frac{\tau_{\mathcal{O}_K}}{\mathrm{Vol}(\mathcal{O}_K)^{\frac{1}{\phi(m)}}} \leqslant \frac{\sqrt{\phi(m)}}{2}|d_K|^{\frac{1}{2\phi(m)}}.$$

Applying $|d_K| \leqslant m^{\phi(m)}$ (following (2)), we obtain

$$\frac{\tau_{\mathcal{O}_K}}{\mathrm{Vol}(\mathcal{O}_K)^{\frac{1}{\phi(m)}}} \leqslant \frac{\sqrt{m\phi(m)}}{2}.$$

So

$$\frac{\phi(m)\tau_{\Lambda_0}}{\rho_m} \leqslant \sqrt{\frac{\pi e}{2}}\phi(m)\sqrt{m}\, q_m^{-\frac{1}{2\phi(m)}}$$

which tends to 0 when m goes to infinity, by hypothesis.

(ii) We have:

$$\begin{aligned}\rho_m = \sqrt{\frac{\phi(m)}{\pi e}}\left(q_m \mathrm{Vol}(\Lambda_0)\right)^{\frac{1}{2\phi(m)}} &\leqslant \frac{1}{2}\sqrt{\phi(m)}q_m^{\frac{1}{2\phi(m)}}|d_K|^{\frac{1}{2\phi(m)}} \\ &\leqslant \frac{1}{2}\sqrt{\phi(m)}q_m^{\frac{1}{2\phi(m)}}\sqrt{m}\end{aligned}.$$

Because $\mathfrak{P}\Lambda_0 = \mathfrak{P} \times \mathfrak{P}$, $\mu_{\mathfrak{P}\Lambda_0} = \mu_{\mathfrak{P}}$. Then, by (i) of Lemma 1, since $\mathrm{Vol}(\mathfrak{P}) = q_m\sqrt{|d_K|}$,

$$\mu_{\mathfrak{P}} \geqslant q_m^{\frac{1}{\phi(m)}}\sqrt{\phi(m)}.$$

The hypothesis on q_m ensures in particular that for m big enough, we have $m < q_m^{\frac{1}{\phi(m)}}$, and thus

$$\rho_m < \frac{1}{2}\sqrt{\phi(m)}q_m^{\frac{1}{\phi(m)}} \leqslant \frac{\mu_{\mathfrak{P}\Lambda_0}}{2}.$$

$\square$

Now we can prove Theorem 5.

Proof of Theorem 5. Let us fix $1 > \varepsilon > 0$. Let $r_m > 0$ be the radius such that $\mathrm{Vol}(B_{r_m}) = (1-\varepsilon)mq_m \mathrm{Vol}(\Lambda_0)$. By (1), $r_m \sim \rho_m$, where ρ_m is the radius defined in Lemma 6. Applying Lemma 5, we get

$$\mathbb{E}(|\overline{B}(r_m) \cap (C \setminus \{0\})|) < \frac{|\overline{B}(r_m)|}{q_m}.$$

Because $r_m \sim \rho_m$, by (ii) of Lemma 6, $r_m < \frac{\mu_{\mathfrak{P}\Lambda_0}}{2}$, so we can apply (ii) of Lemma 4, so that

$$\begin{aligned}\mathbb{E}(|\overline{B}(r_m) \cap (C \setminus \{0\})|) < \frac{\mathrm{Vol}(B(r_m + \tau_{\Lambda_0}))}{q_m \mathrm{Vol}(\Lambda_0)} &= \frac{\mathrm{Vol}(B(r_m))}{q_m \mathrm{Vol}(\Lambda_0)}\left(1 + \frac{\tau_{\Lambda_0}}{r_m}\right)^{2\phi(m)} \\ &= (1-\varepsilon)m\left(1 + \frac{\tau_{\Lambda_0}}{r_m}\right)^{2\phi(m)}.\end{aligned}$$

Now applying (i) of Lemma 6, we have $\lim_{m\to\infty}\left(1+\frac{\tau_{\Lambda_0}}{r_m}\right)^{2\phi(m)} = 1$, and so, for m big enough,

(6) $$\mathbb{E}(|\overline{B}(r_m)\cap(C\setminus\{0\})|) < m.$$

Now comes the crucial argument involving the action of the m-roots of unity. From (6), there is at least one code C in $\mathcal{C}$ which satisfies $|\overline{B}(r_m)\cap(C\setminus\{0\})| < m$. Because the codes that we consider are stable under the action of the m-roots of unity, which preserves the weight of the codewords, and because the length of every non zero orbit under this action is m, we can conclude that $\overline{B}(r_m)\cap(C\setminus\{0\}) = \emptyset$, and so by (iii) of Lemma 4 that,

$$\Delta(\Lambda_C) > \frac{\mathrm{Vol}(B(r_m))}{2^{2\phi(m)}q_m\,\mathrm{Vol}(\Lambda_0)} = \frac{(1-\varepsilon)m}{2^{2\phi(m)}}.$$

□

Theorem 5 shows that for every big enough dimension of the form $n = 2\phi(m)$ our construction provides lattices having density approaching $\frac{m}{2^n}$, thus larger than $\frac{cn}{2^n}$ with $c = 1/2$. A particular sequence of dimensions leads to a better lower bound:

Corollary 1. *For infinitely many dimensions, the family $\mathcal{L}_\mathcal{C}$ contains a lattice $\Lambda\subset\mathbb{R}^n$ satisfying*

$$\Delta(\Lambda) \geqslant \frac{0.89n\log\log n}{2^n}.$$

Proof. To get the optimal gain between m and $2\phi(m)$, we take $m = \prod_{\substack{l\in\mathbb{P}\\ l\leqslant X}} l$, where X is a positive real number, which tends to infinity. Thanks to Mertens' theorem [Har27], we can evaluate:

(7) $$\frac{m}{\phi(m)} \sim e^\gamma\log\log m.$$

where γ is the Euler-Mascheroni constant which satisfies $\gamma > 0.577$.

So we get

(8) $$m \sim \phi(m)e^\gamma\log\log m \sim \frac{e^\gamma}{2}n\log\log n.$$

Let us set $\delta := 2e^{-\gamma}0.89$. Because $\frac{e^\gamma}{2} > 0.89$, $\delta < 1$. Then by Theorem 5, we get a lattice $\Lambda\subset\mathbb{R}^n$ such that

$$\Delta(\Lambda) > \frac{\delta m}{2^n}.$$

So by (8), for m big enough,

$$\Delta(\Lambda) \geqslant \frac{0.89n\log\log n}{2^n}.$$

□

Finally we evaluate the complexity of constructing a lattice Λ with the desired density:

PROPOSITION 2. *Let* $n = 2\phi(m)$. *For every* $1 > \varepsilon > 0$, *the construction of a lattice* $\Lambda \subset \mathbb{R}^n$ *satisfying*

$$\Delta(\Lambda) > \frac{(1-\varepsilon)m}{2^{2\phi(m)}}$$

requires $\exp(1.5n \log n(1 + o(1)))$ *binary operations.*

We need to find a prime ideal $\mathfrak{P}$ such that $q_m = |\mathcal{O}_K/\mathfrak{P}|$ satisfies the condition required in Theorem 5. Let us recall that $q_m = p_m^{f_m}$ where p_m is the prime number lying under $\mathfrak{P}$, and f_m is the order of p_m in the group $(\mathbb{Z}/m\mathbb{Z})^*$ (see [Was97]). We will restrict our attention to the case $f_m = 1$, *i.e.* when $p_m = 1 \mod m$. In that case, p_m decomposes totally in $\mathbb{Q}[\zeta_m]$, and $q_m = p_m$. We use Siegel-Walfisz theorem in order to give an upper bound for the smallest such prime number:

LEMMA 7. *For* m *big enough, there is a prime number* p_m *congruent to* $1 \mod m$ *such that:*

$$\frac{1}{2}(m^3 \log m)^{\phi(m)} \leqslant p_m \leqslant (m^3 \log m)^{\phi(m)}.$$

PROOF. Let us denote by $\pi(x, m, a)$ the number of primes $p < x$ such that $p = a \mod m$. Siegel-Walfisz theorem (see [IH04]) gives for any $A > 0$:

$$\pi(x, m, a) = \frac{Li(x)}{\phi(m)} + \mathcal{O}\left(\frac{x}{(\log x)^A}\right),$$

where the implied constant depends only on A, and $Li(x) = \displaystyle\int_2^x \frac{dt}{\log t}$. Applying this theorem to $x = (m^3 \log m)^{\phi(m)}$, $a = 1$, and $A = 2$ we get

$$\pi(x, m, 1) - \pi(x/2, m, 1) = \frac{1}{\phi(m)} \int_{x/2}^x \frac{dt}{\log t} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right).$$

We have $\displaystyle\frac{1}{\phi(m)} \int_{x/2}^x \frac{dt}{\log t} > \frac{x}{2\phi(m)\log x}$, which grows faster than the error term since $\log x \sim 3\phi(m)\log(m)$, and thus ensures the existence of a prime p_m between $x/2$ and x. □

PROOF OF PROPOSITION 2. Applying Lemma 7, the complexity of finding q_m satisfying the condition of Theorem 5 is

$$\mathcal{O}(m^3 \log m)^{\phi(m)} = e^{3\phi(m)\log(m)(1+o(1))} = e^{1.5n\log(n)(1+o(1))}.$$

The corresponding family of lattices $\mathcal{L}_\mathcal{C}$ has $q_m + 1$ elements. By construction, each of these lattices is generated by vectors with coefficients which are polynomial in n. So, the cost of computing their density, which can be done with $2^{O(n)}$

operations, following [HPS11], is negligible compared with the enumeration of the family. □

4. Symplectic Cyclotomic Lattices

For a survey about symplectic lattices, we refer to [Ber97]. Here we briefly introduce this notion.

Let E be a Euclidean space, and Λ a lattice in E. Then an *isoduality* is an isometry σ of E such that $\sigma(\Lambda) = \Lambda^{\#}$. If Λ affords an isoduality, then it is called *isodual*. If moreover σ satisfies $\sigma^2 = -\mathrm{Id}$, then Λ is called *symplectic*.

Now we explain how to change the lattice Λ_0 in such a way that our construction provides symplectic lattices.

Let

$$\Lambda_0 = \alpha^{-1}\mathcal{O}_K \times \alpha\,\mathfrak{P}^{-1}\mathcal{O}_K^{\#},$$

where $\alpha = (q|d_K|)^{\frac{1}{2\phi(m)}}$. The volume of Λ_0 is now

$$\mathrm{Vol}(\Lambda_0) = \mathrm{Vol}(\mathcal{O}_K)\,\mathrm{Vol}(\mathfrak{P}^{-1}\mathcal{O}_K^{\#}) = \frac{\mathrm{Vol}(\mathcal{O}_K)\,\mathrm{Vol}(\mathcal{O}_K^{\#})}{q} = \frac{1}{q}. \tag{9}$$

Let us define the map

$$\begin{array}{cccc} \sigma & : & K_{\mathbb{R}}^2 & \to & K_{\mathbb{R}}^2 \\ & & (x_1, x_2) & \mapsto & (-\overline{x_2}, \overline{x_1}) \end{array}.$$

It is clear that σ is an isometry, and that $\sigma^2 = -\mathrm{Id}$.

In the following lemma, we show that the lattices we defined in Definition 1 are now symplectic:

LEMMA 8. *If C is a F-line of $\Lambda_0/\mathfrak{P}\Lambda_0$, then the lattice Λ_C is symplectic.*

PROOF. Let us prove that $\sigma(\Lambda_C) \subset \Lambda_C^{\#}$. Let us take $(x_1, x_2) \in \Lambda_C$. We have to show that for every $(y_1, y_2) \in \Lambda_C$, $\langle \sigma(x_1, x_2), (y_1, y_2)\rangle \in \mathbb{Z}$, that is

$$\mathrm{tr}(-x_2 y_1) + \mathrm{tr}(x_1 y_2) \in \mathbb{Z}. \tag{10}$$

According to the definition of C, we have $C = F(u_1, u_2)$ with $u_1 \in \alpha^{-1}\mathcal{O}_K$ and $u_2 \in \alpha\,\mathfrak{P}^{-1}\mathcal{O}_K^{\#}$. So there exists $\lambda, \mu \in \mathcal{O}_K$ such that

$$\begin{cases} x_1 = \lambda u_1 & \mod \alpha^{-1}\mathfrak{P} \\ x_2 = \lambda u_2 & \mod \alpha\,\mathcal{O}_K^{\#} \end{cases} \quad \text{and} \quad \begin{cases} y_1 = \mu u_1 & \mod \alpha^{-1}\mathfrak{P} \\ y_2 = \mu u_2 & \mod \alpha\,\mathcal{O}_K^{\#} \end{cases}.$$

This implies that

$$\mathrm{tr}(x_1 y_2) = \mathrm{tr}(\lambda\mu u_1 u_2) \mod \mathbb{Z}$$

and

$$\mathrm{tr}(x_2 y_1) = \mathrm{tr}(\lambda\mu u_1 u_2) \mod \mathbb{Z},$$

so that (10) is satisfied.

To conclude the proof it is enough to notice that $\mathrm{Vol}(\Lambda_C) = q\,\mathrm{Vol}(\Lambda_0) = 1$, which implies $\sigma(\Lambda_C) = \Lambda_C^{\#}$. $\square$

We again consider the set $\mathcal{C}$ of lines of $\Lambda_0/\mathfrak{P}\Lambda_0$. It is clear that the result of Lemma 5 remains valid for this new family of codes. The general strategy underlying the proof of Theorem 5 applies to the family of lattices associated to these codes, so that we get the following analogues in this context :

THEOREM 6. *For every $1 > \varepsilon > 0$, if $\phi(m)^2 m = o(q_m^{\frac{1}{\phi(m)}})$, then for m big enough, the family of symplectic lattices $\mathcal{L}_{\mathcal{C}}$ contains a lattice $\Lambda \subset \mathbb{R}^{2\phi(m)}$ satisfying*

$$\Delta(\Lambda) > \frac{(1-\varepsilon)m}{2^{2\phi(m)}}.$$

COROLLARY 2. *For infinitely many dimensions, the family $\mathcal{L}_{\mathcal{C}}$ contains a symplectic lattice $\Lambda \subset \mathbb{R}^n$ satisfying*

$$\Delta(\Lambda) \geqslant \frac{0.89 n \log\log n}{2^n}.$$

The proofs of Theorem 6 and Corollary 2 are similar to those of Theorem 5 and Corollary 1. However, we need to prove that Lemma 6 still holds, even if we changed Λ_0:

LEMMA 9. *Let* $\rho_m = \sqrt{\dfrac{\phi(m)}{\pi e}}(q_m \mathrm{Vol}(\Lambda_0))^{\frac{1}{2\phi(m)}} = \sqrt{\dfrac{\phi(m)}{\pi e}}$.
If $\phi(m)^2 m = o(q_m^{\frac{1}{\phi(m)}})$, then

(i) $\lim\limits_{m\to\infty} \dfrac{\phi(m)\tau_{\Lambda_0}}{\rho_m} = 0$,

(ii) *For m big enough,* $\rho_m < \dfrac{\mu_{\mathfrak{P}\Lambda_0}}{2}$.

PROOF. (i) We have:

$$\frac{\phi(m)\tau_{\Lambda_0}}{\rho_m} = \sqrt{\pi e \phi(m)}\tau_{\Lambda_0}.$$

Let us set $\mathfrak{A}_1 = \alpha^{-1}\mathcal{O}_K$ and $\mathfrak{A}_2 = \alpha\mathfrak{P}^{-1}\mathcal{O}_K^{\#}$. Then $\Lambda_0 = \mathfrak{A}_1 \times \mathfrak{A}_2$ and the covering radius of Λ_0 is $\tau_{\Lambda_0} = \sqrt{\tau_{\mathfrak{A}_1}^2 + \tau_{\mathfrak{A}_2}^2}$. So we have to bound both covering radii $\tau_{\mathfrak{A}_1}$ and $\tau_{\mathfrak{A}_2}$. Applying (ii) of Lemma 1, and because $\mathrm{Vol}(\mathfrak{A}_1) = \mathrm{Vol}(\mathfrak{A}_2) = \dfrac{1}{\sqrt{q}}$, we have, for $i \in \{1,2\}$,

$$\tau_{\mathfrak{A}_i} \leqslant \frac{\sqrt{\phi(m)}}{2}|d_K|^{\frac{1}{2\phi(m)}} q^{-\frac{1}{2\phi(m)}} \leqslant \frac{\sqrt{m\phi(m)}\, q^{-\frac{1}{2\phi(m)}}}{2}.$$

So

$$\tau_{\Lambda_0} \leqslant \sqrt{2}\max\{\tau_{\mathfrak{A}_1}, \tau_{\mathfrak{A}_2}\} \leqslant \sqrt{m\phi(m)} q^{-\frac{1}{2\phi(m)}}$$

and finally

$$\frac{\phi(m)\tau_{\Lambda_0}}{\rho_m} \leqslant \sqrt{\pi e}\,\phi(m)\sqrt{m}\,q_m^{-\frac{1}{2\phi(m)}}$$

which tends to 0 when m goes to infinity, by hypothesis.

(ii) Let us set $\mathfrak{B}_1 = \alpha^{-1}\mathfrak{P}$ and $\mathfrak{B}_2 = \alpha\,\mathcal{O}_K^{\#}$. Then $\mathfrak{P}\Lambda_0 = \mathfrak{B}_1 \times \mathfrak{B}_2$, and clearly $\mu_{\mathfrak{P}\Lambda_0} = \min\{\mu_{\mathfrak{B}_1}, \mu_{\mathfrak{B}_2}\}$. Then, applying (i) of Lemma 1, since $\mathrm{Vol}(\mathfrak{B}_1) = \mathrm{Vol}(\mathfrak{B}_2) = \sqrt{q}$, we have, for $i \in \{1, 2\}$,

$$\mu_{\mathfrak{B}_i} \geqslant \frac{\sqrt{\phi(m)}\,q^{\frac{1}{2\phi(m)}}}{|d_K|^{\frac{1}{2\phi(m)}}} \geqslant \frac{\sqrt{\phi(m)}\,q^{\frac{1}{2\phi(m)}}}{\sqrt{m}}.$$

So

$$\mu_{\mathfrak{P}\Lambda_0} \geqslant \frac{\sqrt{\phi(m)}\,q^{\frac{1}{2\phi(m)}}}{\sqrt{m}}.$$

The hypothesis on q_m ensures in particular that for m big enough, m satisfies $\sqrt{m} < q_m^{\frac{1}{2\phi(m)}}$, and thus

$$\rho_m = \sqrt{\frac{\phi(m)}{\pi e}} < \frac{1}{2}\,\frac{\sqrt{\phi(m)}\,q^{\frac{1}{2\phi(m)}}}{m} \leqslant \frac{\mu_{\mathfrak{P}\Lambda_0}}{2}.$$

$\square$

Since the condition on the growth of q_m does not change, the estimation for the complexity of construction in this context is the same:

Proposition 3. *Let $n = 2\phi(m)$. For every $1 > \varepsilon > 0$, the construction of a symplectic lattice $\Lambda \subset \mathbb{R}^n$ satisfying*

$$\Delta(\Lambda) > \frac{(1-\varepsilon)m}{2^{2\phi(m)}}$$

requires $\exp(1.5n \log n(1 + o(1)))$ *binary operations.*

Chapter 3

On the density of sets avoiding parallelohedron distance 1

This chapter is based on [BBMP17], which is joint work with Christine Bachoc, Thomas Bellitto and Arnaud Pêcher.

1. Introduction

A set *avoiding distance* 1 is a set A in a normed vector space $(\mathbb{R}^n, \|\cdot\|)$ such that $\|x-y\| \neq 1$ for every $x, y \in A$. The number $m_1(\mathbb{R}^n, \|\cdot\|)$ measures the highest proportion of space that can be filled by a set avoiding distance 1. More precisely, $m_1(\mathbb{R}^n, \|\cdot\|)$ is the supremum of the *densities* (see Chapter 2, subsection 1.2 for a precise definition) of Lebesgue measurable sets $A \subset \mathbb{R}^n$ avoiding distance 1.

The problem of determining $m_1(\mathbb{R}^n, \|\cdot\|)$ has been mostly studied in the Euclidean case. The number $m_1(\mathbb{R}^n) = m_1(\mathbb{R}^n, \|\cdot\|_2)$ was introduced by Larman and Rogers in [LR72] as a tool to study the *measurable chromatic number* $\chi_m(\mathbb{R}^n)$ of $\mathbb{R}^n$, which is the minimal number of colors required to color $\mathbb{R}^n$ in such a way that two points at Euclidean distance 1 have distinct colors, and that the color classes are measurable. Determining $\chi_m(\mathbb{R}^n)$ has turned out to be a very difficult problem, that has only been solved in dimension 1, and that is wide open in any other dimension, including the familiar dimension 2, where it is only known that $5 \leqslant \chi_m(\mathbb{R}^2) \leqslant 7$ (see [Fal81], [Szé02], and [Soi08, Chapter 3] for a detailed historical account).

The connection between $m_1(\mathbb{R}^n)$ and $\chi_m(\mathbb{R}^n)$ lies in the following inequality:

$$\chi_m(\mathbb{R}^n) \geqslant \frac{1}{m_1(\mathbb{R}^n)},$$

so, from an upper bound for $m_1(\mathbb{R}^n)$, one obtains a lower bound for $\chi_m(\mathbb{R}^n)$.

A natural approach to build a set avoiding distance 1, that works for any norm, starts from a packing of unit balls. Let Λ be a set such that if $x, y \in \Lambda$, then the unit open balls $B(x, 1)$ and $B(y, 1)$ do not overlap. Then the set $A = \cup_{\lambda\in\Lambda} B(\lambda, 1/2)$ of disjoint balls of radius $1/2$ is a set avoiding 1 and its density is $\frac{\delta}{2^n}$ where n is the dimension of the space and δ is the density of the packing. This construction is illustrated in Figure 1.

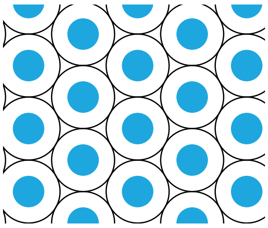


FIGURE 1. A set avoiding distance 1 built from a sphere packing.

In the Euclidean plane, the density of an optimal packing of discs of radius 1 is 0.9069 and this approach therefore provides a lower bound of $0.9069/4 = 0.2267$ for $m_1(\mathbb{R}^2, \|\cdot\|_2)$. The best known construction is not much better than that: by refining this idea, Croft manages to build in [Cro67] a set of density 0.2293, which is an arrangement of balls cut out by hexagons, see Figure 2.

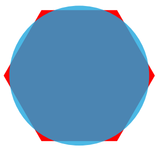


FIGURE 2. The block of Croft's set: the intersection of a hexagon and a ball, depicted in dark blue.

Regarding upper bounds, Erdős conjectured (see [Szé02]) that

$$m_1(\mathbb{R}^2) < \frac{1}{4}.$$

The best upper bound up to now is due to Keleti, Matolcsi, de Oliveira Filho and Ruzsa [KMdOFR16], who have shown $m_1(\mathbb{R}^2) \leqslant 0.258795$. Moser, Larman and Rogers (see [LR72]) generalized Erdős' conjecture to higher dimensions: for every $n \geqslant 2$,

$$m_1(\mathbb{R}^n) < \frac{1}{2^n}.$$

A weaker result has been proved in [KMdOFR16]: a set avoiding distance 1 necessarily has a density strictly smaller than $\frac{1}{2^n}$ if it has a *block structure*, *i.e.* if it may be decomposed as a disjoint union $A = \cup A_i$ such that if x and y are in the same block A_i then $\|x - y\| < 1$ and if they are not, $\|x - y\| > 1$. However, without this assumption, the known upper bounds are pretty far from 2^{-n}, even asymptotically: the best asymtotic bound is $m_1(\mathbb{R}^n) \leqslant (1+o(1))(1.2)^{-n}$ (see [**LR72**], [**BPT15**]).

Going back to the general case of an arbitrary norm, the method described previously to build a set avoiding distance 1 from a packing is still valid. We make the remark that if the unit ball tiles $\mathbb{R}^n$ by translation, it provides a set of density exactly $1/2^n$, as illustrated in Figure 3. Moreover, it is likely that this construction of a set avoiding distance 1 is optimal, as conjectured by Bachoc and Robins:

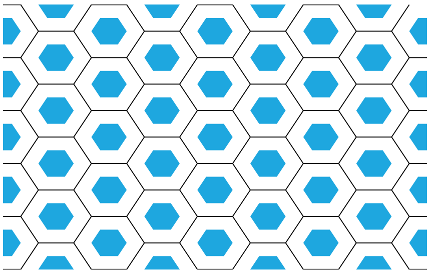


FIGURE 3. The natural construction of density $1/2^n$.

CONJECTURE 2 (Bachoc, Robins). *If $\|\cdot\|$ is a norm such that the unit ball tiles $\mathbb{R}^n$ by translation, then*

$$m_1(\mathbb{R}^n, \|\cdot\|) = \frac{1}{2^n}.$$

In this chapter, we prove Conjecture 1 in dimension 2:

THEOREM 7. *If $\|\cdot\|$ is a norm such that the unit ball tiles $\mathbb{R}^2$ by translation, then*

$$m_1(\mathbb{R}^2, \|\cdot\|) = \frac{1}{4}.$$

Recall that the only convex bodies that tile space by translation are the *parallelohedra*, *i.e.* the polytopes that admit a face-to-face tiling by translation (see Chapter 2, subsection 3.1). For a given parallelohedron $\mathcal{P}$, we denote by $\|\cdot\|_{\mathcal{P}}$ the norm whose unit ball is $\mathcal{P}$.

The Voronoï cell of a lattice is a parallelohedron. Conversely, Voronoï conjectured that all parallelohedra are, up to affine transformations, the Voronoï cells of lattices (see Chapter 2, subsection 3.2). On the other hand, $m_1(\mathbb{R}^n, \|\cdot\|)$ is clearly left unchanged under the action of a linear transformation applied to the norm. So, in the light of Voronoï's conjecture, it is natural to consider the polytopes that are Voronoï cells of lattices.

The most obvious family of lattices is the family of cubic lattices $\mathbb{Z}^n$, whose Voronoï cells are hypercubes. We will see that in this case, Conjecture 1 holds trivially. The next families of lattices to consider are arguably the root lattices A_n and D_n, where

$$A_n = \{x \in \mathbb{Z}^{n+1} \ : \ \sum_{i=1}^{n+1} x_i = 0\} \quad (n \geqslant 2).$$

and

$$D_n = \{x \in \mathbb{Z}^n \ : \ \sum_{i=1}^{n} x_i \equiv 0 \bmod 2\} \quad (n \geqslant 4).$$

We will prove Conjecture 1 for the Voronoï cells of the lattices A_n in every dimensions $n \geqslant 2$. For the lattices D_n, we can only show the inequality

$$m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leqslant \frac{1}{(3/4)2^n + n - 1}$$

which is however asymptotically of the order $O\left(\frac{1}{2^n}\right)$.

Let us now give an idea of the method that we use to prove these results. The strategy is to transfer the study of sets avoiding distance 1 to a discrete setting, in which such sets can be decomposed as the disjoint union of small pieces (in other words they afford a kind of block structure). Computing the optimal density of a set avoiding distance 1 in the discrete setting amounts then to understanding how these blocks fit together locally.

To be more precise, we consider discrete subsets V of $\mathbb{R}^n$, seen as induced subgraphs of the *unit distance graph* $G(\mathbb{R}^n, \|\cdot\|)$. This is the graph whose vertices are the points of $\mathbb{R}^n$ and whose edges connect the vertices x and y if and only if $\|x - y\| = 1$.

If $G = (V, E)$ is a finite induced subgraph of $G(\mathbb{R}^n, \|\cdot\|)$, then it is well known that (see [LR72])

$$m_1(\mathbb{R}^n, \|\cdot\|) \leqslant \frac{\alpha(G)}{|V|},$$

where as usual $\alpha(G)$ denotes the *independence number* of G and $|V|$ is the number of its vertices. We use a generalization of this inequality to discrete graphs (see Subsection 2.2). Of course, the most difficult task is to design an appropriate discrete subset V, *i.e.* one that provides a good upper bound of $m_1(\mathbb{R}^n, \|\cdot\|)$ and at the same time is easy to analyse.

For the regular hexagon in the plane, we follow an idea due to Dmitry Shiryaev [Shi] who proposed an auxiliary graph satisfying the following remarkable property: if two points x and y are at graph distance 2, then they are at polytope distance 1. This implies that a set avoiding polytope distance 1 is a union of cliques whose closed neighborhoods are disjoint. The density of such a set is bounded by the supremum of the *local densities* of the cliques in their closed neighborhood. In the case of a general hexagonal Voronoï cell in the plane, this approach doesn't work straightforwardly and we need to introduce a different graph with a slightly weaker property. The construction of such an auxiliary graph is also a key ingredient of our proofs of the bounds for the Voronoï cells of A_n and D_n.

The chapter is organized as follows: in Section 2, we set our problem formally and make some preliminary work. In Section 3, we prove Theorem 7. Section 4 is dedicated to the families of lattices A_n (Theorem 10) and D_n (Theorem 11). In Section 5, we discuss the chromatic number of the unit distance graph $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$.

2. Preliminaries

2.1. The Density of a Set Avoiding Polytope Distance 1

Let $\mathbb{R}^n$ be equipped with a norm $\|\cdot\|$. A set $S \subset \mathbb{R}^n$ is said to *avoid 1* if for every $x, y \in S$, $d(x, y) = \|x - y\| \neq 1$. Recall that the *density* of a measurable set $A \subset \mathbb{R}^n$ with respect to Lebesgue measure is

$$\delta(A) = \limsup_{R\to\infty} \frac{\mathrm{Vol}(A \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)}.$$

We denote by $m_1(\mathbb{R}^n, \|\cdot\|)$ the supremum of the densities achieved by measurable sets avoiding distance 1:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup_{\substack{S\subset\mathbb{R}^n \text{measurable} \\ S \text{ avoiding } 1}} \delta(S).$$

Let $\mathcal{P}$ be a polytope tiling $\mathbb{R}^n$ by translations, and let $\Lambda \subset \mathbb{R}^n$ such that $\bigcup_{\lambda\in\Lambda} (\lambda + \mathcal{P}) = \mathbb{R}^n$ and for every $\lambda \neq \lambda'$, $(\lambda + \mathring{\mathcal{P}}) \cap (\lambda' + \mathring{\mathcal{P}}) = \emptyset$. Then, the set

$$A = \bigcup_{\lambda\in\Lambda} (\lambda + \frac{1}{2}\mathring{\mathcal{P}})$$

avoids 1, and has density $\frac{1}{2^n}$. This set gives a lower bound for m_1:

Proposition 4. *If $\mathcal{P}$ is a polytope tiling $\mathbb{R}^n$ by translation, and $\|\cdot\|_{\mathcal{P}}$ the norm associated with $\mathcal{P}$, then*

$$m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geqslant \frac{1}{2^n}.$$

2.2. Discretization of the Problem

A set avoiding distance 1 in $\mathbb{R}^n$ is exactly an independent set in $G(\mathbb{R}^n, \|\cdot\|)$, *i.e.* a subset S of vertices such that, for all $x, y \in S$, $\|x - y\| \neq 1$. Therefore $m_1(\mathbb{R}^n, ||.||)$ is the supremum of the densities achieved by independent sets. It is the analogue of the *independence ratio* $\bar{\alpha}(G) = \dfrac{\alpha(G)}{|V|}$ of a finite graph G, see Chapter 2, Section 4.

Discrete induced subgraphs of $G(\mathbb{R}^n, \|\cdot\|)$ provide upper bounds of $m_1(\mathbb{R}^n, \|\cdot\|)$ thanks to the following lemma:

LEMMA 10. *Let $G = (V, E)$ be a discrete induced subgraph of $G(\mathbb{R}^n, \|\cdot\|)$. Then*

$$m_1(\mathbb{R}^n, \|\cdot\|) \leqslant \bar{\alpha}(G).$$

PROOF. By Lemma 2, we may assume without loss of generality that G is finite. In this case the result is well known: the proof below is for the sake of completeness.

Let $R > 0$ be a real number, and let $X \in [-R, R]^n$ chosen uniformly at random. For $S \subset \mathbb{R}^n$, the probability that X is in S is

$$\mathbb{P}(X \in S) = \frac{\mathrm{Vol}(S \cap [-R, R]^n)}{\mathrm{Vol}([-R, R]^n)}.$$

Notice that $\limsup_{R\to\infty} \mathbb{P}(X \in S) = \delta(S)$.

Let $S \subset \mathbb{R}^n$ be a set avoiding 1. We define the random variable

$$N = |(X + V) \cap S|.$$

On one hand, we have:

$$\begin{aligned}\mathbb{E}\left[\frac{N}{|V|}\right] &= \frac{1}{|V|}\mathbb{E}\left[\sum_{v\in V} \mathbb{1}_{\{X+v\in S\}}\right] \\ &= \frac{1}{|V|}\sum_{v\in V} \mathbb{P}(X \in S - v).\end{aligned}$$

For every v, we have $\limsup_{R\to\infty} \mathbb{P}(X \in S - v) = \delta(S - v) = \delta(S)$.

On the other hand, since for $v_1, v_2 \in V$, $\|(X + v_1) - (X + v_2)\| = \|v_1 - v_2\|$, and $(X + V) \cap S \subset S$, the set

$$\{v \in V \mid X + v \in S\}$$

is an independent set in G, so that, for any $R > 0$,

$$\frac{N}{|V|} \leqslant \bar{\alpha}(G).$$

Thus we get,

$$\delta(S) \leqslant \bar{\alpha}(G).$$

□

In order to give a first example, we consider the most natural lattice: the cubic lattice. The associated tiling and norm are respectively the cubic tiling and the well known sup norm $\|x\|_\infty = \sup_{1\leqslant i\leqslant n} |x_i|$. More precisely, if $L = 2\mathbb{Z}^n$, the Voronoï cell of L is the cube whose vertices are the points of coordinates $(\pm 1, \pm 1, \dots, \pm 1)$.

PROPOSITION 5. *For every $n \geqslant 1$, we have:*

$$m_1(\mathbb{R}^n, \|\cdot\|_\infty) = \frac{1}{2^n}$$

PROOF. Let $V = \{0,1\}^n \subset \mathbb{R}^n$ and let G be the subgraph of $G(\mathbb{R}^n, \|\cdot\|)$ induced by V. Following the definition of V, for every $v, v' \in V$ with $v \neq v'$, we have $\|v - v'\|_\infty = 1$. So G is a complete graph, thus its independence number is 1. Since it has 2^n vertices, applying Lemma 10, we get

$$m_1(\mathbb{R}^n, \|\cdot\|_\infty) \leqslant \frac{\alpha(G)}{|V|} = \frac{1}{2^n}.$$

□

3. Parallelohedron Norms in the Plane

In this section, we prove Theorem 7. It is well known that the parallelohedra in dimension 2, are, up to an affine transformation, the Voronoï cells of a lattice, and that their combinatorial type is either that of a square or of a hexagon (see Figure 4).

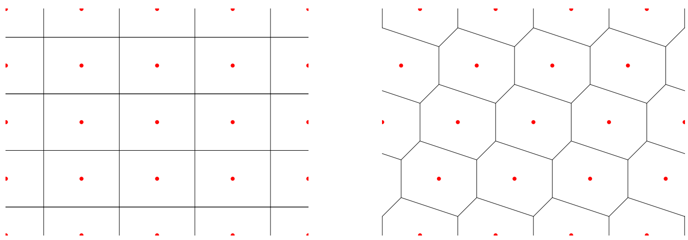


FIGURE 4. The two kinds of Voronoï cells of lattices in the plane.

We have already seen that $m_1(\mathbb{R}^2, \|\cdot\|_\infty) = \dfrac{1}{4}$, so it remains to deal with hexagons. Even though it is not true that every hexagonal Voronoï cell is linearly equivalent to the regular hexagon, we will first consider the regular hexagon in order to present in this basic case, the ideas that will be used in the general case.

3.1. The Regular Hexagon

The following result is due to Dmitry Shiryaev [Shi]:

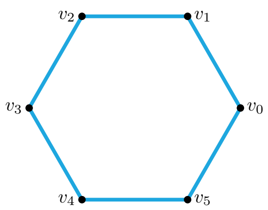
Theorem 8. *If $\mathcal{P}$ is the regular hexagon in the plane, then*

$$m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = \frac{1}{4}.$$

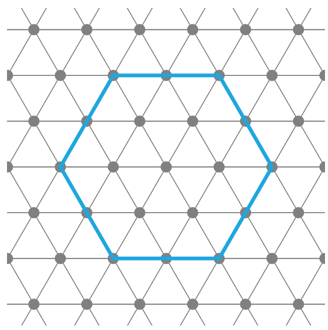
Let $\mathcal{P}$ be the regular hexagon in $\mathbb{R}^2$. We denote by S its set of vertices and by $\partial\mathcal{P}$ its boundary. Thus, $\|x\|_{\mathcal{P}} = 1$ if and only if $x \in \partial\mathcal{P}$. We label the vertices of $\mathcal{P}$ modulo 6 as described in Figure 5a.

The set $\frac{1}{2}S$ spans a lattice V. Let us consider $G_{\mathcal{P}}$, the subgraph of $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$ induced by V. We shall prove that $\bar{\alpha}(G_{\mathcal{P}}) \leqslant 1/4$. To do so, we introduce an auxiliary graph $\tilde{G} = (\tilde{V}, \tilde{E})$, which is the Cayley graph with the same set of vertices $\tilde{V} = V$ corresponding to the generating set $\frac{1}{2}S$. In other words, for $x, y \in V$, $(x, y) \in \tilde{E}$ if and only if $x - y \in \frac{1}{2}S$. This graph is drawn in Figure 5b.

We denote by $\tilde{d}(x, y)$ the distance between two vertices x and y in the graph $\tilde{G}$, *i.e.* the minimal length of a path in $\tilde{G}$ between x and y. We define the distance $\tilde{d}(A, B)$ in $\tilde{G}$ between two subsets of vertices A and B as the minimal distance between a vertex of A and a vertex of B. The following lemma will be crucial for the proof of Theorem 8:



(A) The regular hexagon.



(B) The Cayley graph $\tilde{G}$.

Lemma 11. *Let u_1 and u_2 be two vertices of $\tilde{G}$. Then:*

(Property D) $$\tilde{d}(u_1, u_2) = 2 \Rightarrow \|u_1 - u_2\|_{\mathcal{P}} = 1.$$

Proof. Since $\tilde{G}$ is vertex-transitive, we may assume without loss of generality that $u_1 = 0$. The vertices u at graph distance 2 from 0 must be of the form $\frac{v_i}{2} + \frac{v_j}{2}$. It is not hard to check that if $u_2 = \frac{v_i + v_j}{2}$ is neither 0 nor another $\frac{v_k}{2}$ (in which case $\tilde{d}(0, u_2) < 2$), then it is a point of $\partial\mathcal{P}$ (see also Figure 5b). □

Remark 4. *It can be noted, although it will not be useful, that the equivalence $\tilde{d}(u_1, u_2) = 2 \Leftrightarrow \|u_1 - u_2\|_{\mathcal{P}} = 1$ holds here.*

For a set $A \subset \tilde{V}$, we define its closed neighborhood

$$N[A] = \{v \in \tilde{V} \text{ such that } \tilde{d}(v, A) \leqslant 1\} = A + \left(\{0\} \cup \frac{1}{2}S\right).$$

Now we consider the *cliques* of $\tilde{G}$, that is the sets $C \subset \tilde{V}$ such that for every $u \neq v \in C$, $\tilde{d}(u, v) = 1$. We will use the following lemma several times: it shows that for any graph $\tilde{G}$ satisfying (Property D), if $A \subset \tilde{V}$ avoids polytope distance 1, then A is a union of cliques whose closed neighborhoods are disjoint:

LEMMA 12. *Let $\|\cdot\|_{\mathcal{P}}$ be a polytope norm in $\mathbb{R}^n$, and $G_{\mathcal{P}}$ an induced subgraph of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$. Assume there exists an auxiliary graph $\tilde{G}$ with the same vertices V as $G_{\mathcal{P}}$ satisfying (Property D). Let $A \subset V$ avoiding polytope distance 1. Then A may be written as a union of cliques of $\tilde{G}$*

$$A = \bigcup_{C \in \mathcal{C}} C$$

such that if $C, C' \in \mathcal{C}$ with $C \neq C'$, then

$$N[C] \cap N[C'] = \emptyset.$$

PROOF. Let us consider the decomposition of A in connected components with respect to $\tilde{G}$. Following Lemma 11, since A avoids polytope distance 1, a connected component C cannot contain two vertices at graph distance 2 from each other. So C must be a clique.

Assume that two different cliques C and C' of A share a common neighbor. Thus $\tilde{d}(C, C') \leqslant 2$. Since C and C' are two disjoint connected components, $\tilde{d}(C, C') > 1$. So $\tilde{d}(C, C') = 2$, which is impossible, since A avoids polytope distance 1. □

Now we define the local density of a clique C of $\tilde{G}$: $\delta^0(C) = \dfrac{|C|}{|N[C]|}$. In the next lemma, we analyse the different possible cliques of the graph $\tilde{G}$ that we constructed for the regular hexagon, and determine their local density:

LEMMA 13. *For every clique $C \subset \tilde{G}$,*

$$\delta^0(C) \leqslant \frac{1}{4}$$

PROOF. Let C be a clique of $\tilde{G}$. Since $\tilde{G}$ is vertex transitive, we can assume without loss of generality that $0 \in C$. Up to the action of the dihedral group $\mathcal{D}_3$ on V, there are only three possible cliques in $\tilde{G}$ containing 0, and one can easily determine their neighborhoods (see Figure 6):

- $C = \{0\}$: its neighborhood is $\{0\} \cup \frac{1}{2}S$. Thus $\delta^0(C) = \frac{1}{7}$.
- $C = \left\{0, \frac{v_0}{2}\right\}$, and $\delta^0(C) = \frac{2}{10} = \frac{1}{5}$.
- $C = \left\{0, \frac{v_0}{2}, \frac{v_1}{2}\right\}$, and $\delta^0(C) = \frac{3}{12} = \frac{1}{4}$.

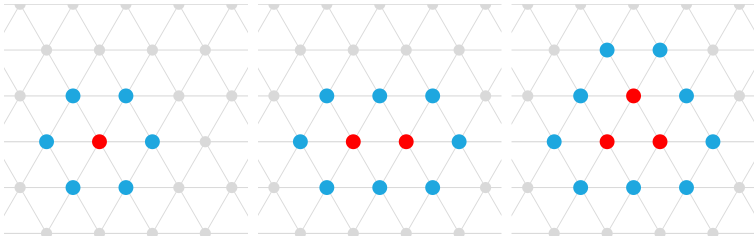


FIGURE 6. The possible cliques and their neighborhoods.

□

We have all the ingredients to prove that the density of a set avoiding 1 for the regular hexagon cannot exceed 1/4:

PROOF OF THEOREM 8. Following Lemma 10, it is sufficient to prove that $\bar{\alpha}(G_{\mathcal{P}}) \leqslant \frac{1}{4}$. If $A \subset V$ is a set avoiding 1, it may be written as the union of cliques in $\tilde{G}$, whose neighborhoods are disjoint (Lemma 12). So the density of A is bounded from above by the maximum local density of a clique in $\tilde{G}$. So, from Lemma 13, $\bar{\alpha}(G_{\mathcal{P}}) \leqslant \frac{1}{4}$. □

3.2. General Hexagonal Voronoï Cells

In this subsection, we deal with a general hexagonal Voronoï cell $\mathcal{P}$ of the plane, and prove:

THEOREM 9. *If $\mathcal{P}$ is an hexagonal Voronoï cell in the plane, then*

$$m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = \frac{1}{4}.$$

Let $\mathcal{P}$ be the hexagonal Voronoï cell of a lattice $L \subset \mathbb{R}^2$. Let $\{\beta_0, \beta_1\}$ be a basis of L such that the vectors β_0 , β_1, $\beta_2 = \beta_1 - \beta_0$, and their opposites define the faces of $\mathcal{P}$. We label the vertices v_i, for $0 \leqslant i \leqslant 5$, of $\mathcal{P}$ in such a way that $\beta_i = v_i + v_{i+1}$, where i is defined modulo 6. This situation is depicted in Figure 7.

In order to prove Theorem 9, just like in the case of the regular hexagon, we shall construct a graph $G_{\mathcal{P}}$ induced by $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$, and prove that $\bar{\alpha}(G_{\mathcal{P}}) \leqslant 1/4$. Unfortunately, in general, the vertices of $\mathcal{P}$ do not span a lattice. We will use a different point of view in order to build $G_{\mathcal{P}}$, together with an auxiliary graph $\tilde{G}$ that will satisfy a weaker version of (Property D).

For the set V of vertices of $G_{\mathcal{P}}$, we take the lattice $\frac{1}{2}L$, together with the translates of the vertices $V_{\mathcal{P}}$ of $\mathcal{P}$ by $\frac{1}{2}L$. We set $A = \frac{1}{2}L$ and $B = V_{\mathcal{P}} + \frac{1}{2}L$ so that $V = A \cup B$; this construction is represented in Figure 8 where the vertices of A are depicted in red, and those of B in green.

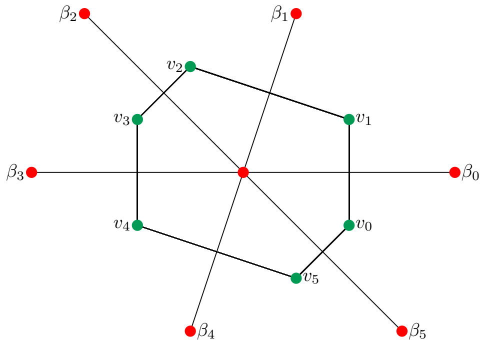
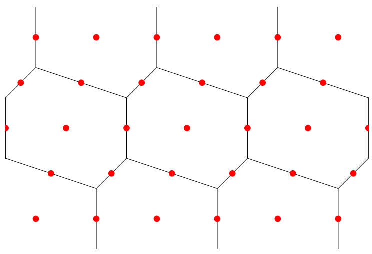
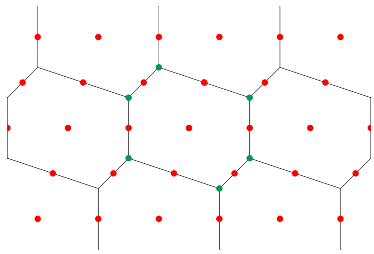


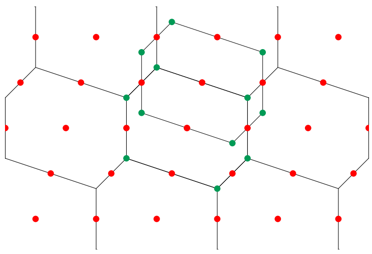
FIGURE 7. The vectors β_i and the vertices of the hexagon.



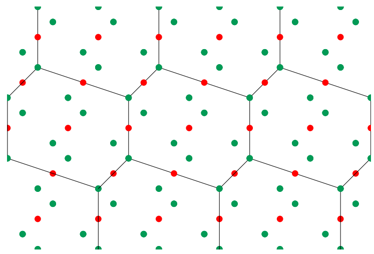
(A) We start with $A = \dfrac{1}{2}L$,



(B) We add the vertices of $\mathcal{P}$...



(C) ... and their translates by A.



(D) The set of vertices V.

FIGURE 8. Constructing the vertices of $G_{\mathcal{P}}$.

Let us note that for every i mod 6, $v_{i+2} = v_i \mod L$. Indeed,

$$v_{i+2} - v_i = v_{i+2} + v_{i+1} - (v_i + v_{i+1}) = \beta_{i+1} - \beta_i = \beta_{i+2}.$$

As a consequence, we may write V as the disjoint union of three sets:

$$V = \frac{1}{2}L \cup (\frac{1}{2}L + v_0) \cup (\frac{1}{2}L + v_1),$$

and this implies that the density of B in V is twice that of A.

Now, let us construct the auxiliary graph $\tilde{G} = (\tilde{V}, \tilde{E})$. It has the same vertices as $G_{\mathcal{P}}$, i.e. $\tilde{V} = V$. Let us describe the edges of $\tilde{G}$. By construction, there are exactly 7 vertices of V in the interior of $\mathcal{P}$: the center $0 \in A$, and six points of B denoted $s_0, \dots, s_5$, with

$$s_i = \frac{v_{i-1} + v_{i+1}}{2}.$$

For every point of $a \in A$, we define the edges $(a, a+s_i)$ and $(a+s_i, a+s_{i+1})$ for i from 0 to 5. This is illustrated in Figure 9.

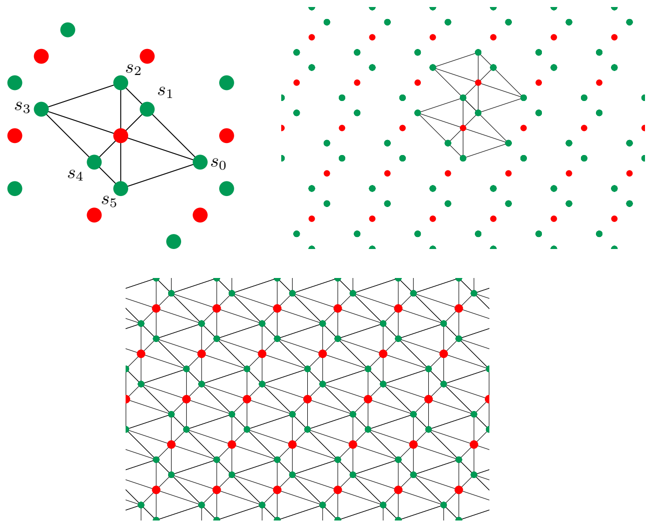


FIGURE 9. Constructing the edges of $\tilde{G}$.

REMARK 5. *In the case of the regular hexagon, this construction leads to the same graph $\tilde{G}$ that we considered in Subsection 3.1.*

Let us describe the neighborhood (with respect to $\tilde{G}$) of each type of point. By construction, a point in A has 6 neighbors, and they all belong to B. A vertex $a + s_i$ of B also has six neighbors. Three of them are elements of A, namely a, $a + \dfrac{\beta_i}{2}$ and $a + \dfrac{\beta_{i-1}}{2}$ and the other three are elements of B, namely, $a + s_{i-1}$, $a + s_{i+1}$ and $a + v_i$. Figure 10 illustrates the neighborhoods of the vertices of $\tilde{G}$.

It should be noted that (Property D) is not in general fulfilled by $\tilde{G}$: indeed, the vertices s_0 and s_3 are at graph distance 2 in $\tilde{G}$ but not (in general) at polytope distance 1. However, this property continues to hold *for points that share a common neighbor in B*. We prove this in the next lemma, which will play the role of Lemma 11 for this new graph $\tilde{G}$:

LEMMA 14. *If two vertices $x, y \in V$ are at distance 2 from each other in $\tilde{G}$ and have a common neighbor $z \in B$, then $\|x - y\|_{\mathcal{P}} = 1$.*

PROOF. First suppose that at least one of the two vertices is in A. In this case we may assume $x = 0$. Then z is one of the s_i, and following the analysis

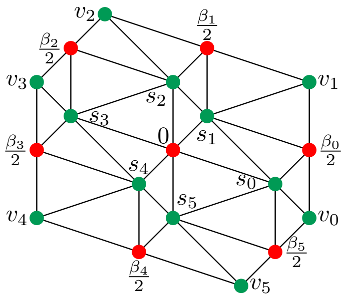


FIGURE 10. The basic pattern in $\tilde{G}$.

of the neighbors of s_i, y must be in the set $\{0, s_{i-1}, s_{i+1}, \frac{\beta_i}{2}, \frac{\beta_{i-1}}{2}, v_i\}$. The first three are obviously not at graph distance 2 from 0, so y is one of the last three vertices, and they all are in $\partial\mathcal{P}$. Thus, $\|x-y\|_{\mathcal{P}} = 1$.

Now suppose $x, y, z \in B$. Then we may assume without loss of generality $x = s_{i-1}$, and $z = s_i$. Since z has only three neighbors in B, y can be either s_{i+1} or v_i. We have:

$$s_{i+1} - s_{i-1} = \frac{v_i + v_{i+2}}{2} - \frac{v_i + v_{i-2}}{2} = \frac{v_{i+2} - v_{i-2}}{2} = \frac{v_{i+2} + v_{i+1}}{2} = \frac{\beta_{i+1}}{2}$$

and

$$v_i - s_{i-1} = v_i - \frac{v_i + v_{i-2}}{2} = \frac{v_i - v_{i-2}}{2} = \frac{v_i + v_{i+1}}{2} = \frac{\beta_i}{2}.$$

In both cases $\|x-y\|_{\mathcal{P}} = 1$. □

Let $U \subset V$ be a set of vertices avoiding polytope distance 1, let C be a connected component of U and let $N[C]$ be its closed neighborhood. We define:

$$N_B[C] = N[C] \cap B$$

and

$$\delta_B^0(C) = \frac{|C|}{|N_B[C]|}.$$

The following lemma is the analogue of Lemma 12 in this situation: we show that if C and C' are two different connected components, then $N_B[C]$ and $N_B[C']$ must be disjoint:

LEMMA 15. *Let $U \subset V$ be a set avoiding polytope distance 1. If $C \neq C'$ are two connected components of U, then*

$$N_B[C] \cap N_B[C'] = \emptyset.$$

PROOF. If a vertex $z \in B$ is in both $N_B[C]$ and $N_B[C']$, then there is $x \in C$, $y \in C'$ such that $\tilde{d}(x,z) = \tilde{d}(z,y) = 1$. Since C and C' are connected components of U, we have $\tilde{d}(x,y) > 1$. Thus $\tilde{d}(x,y) = 2$ and by Lemma 14, $\|x-y\|_{\mathcal{P}} = 1$, which is impossible, since U avoids 1. $\square$

Now we study the different possible connected components:

LEMMA 16. *Let $U \subset V$ be a set avoiding polytope distance 1. If C is a connected component of U, then*

$$\delta_B^0 \leqslant \frac{3}{8}.$$

PROOF. We enumerate the possible connected components. Let us start with the isolated points. Up to translations by $\frac{1}{2}L$, we have:

- $C = \{0\} \subset A$. Its neighborhood is made of six vectors from B. So $\delta_B^0(C) = 1/6$.
- $C = \{s_i\} \subset B$. We know that such a vertex has three neighbors in B, thus $\delta_B^0(C) = 1/4$.

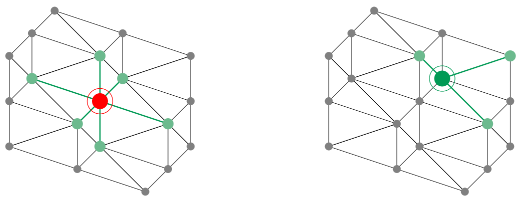


FIGURE 11. The two possible types of connected component with one element. The circled vertices denote the elements of C and the figure represents all their neighbors in B.

We now focus on the connected components of size 2. Since a vertex in A has all its neighbors in B, such a connected component cannot contain two elements of A. Thus, up to translation, we only have:

- $C = \{0, s_i\}$, and the only neighbor in B that is not a neighbor of 0 is v_i. Thus $\delta_B^0 = 2/7$.
- $C = \{s_i, s_{i+1}\}$ and the neighbors in B are $s_{i-1}, v_i, s_{i+2}, v_{i+1}$. Thus $\delta_B^0 = 2/6 = 1/3$.

There are up to translations two kinds of connected components of size three:

- $C = \{0, s_i, s_{i+1}\}$. The only neighbor of s_{i+1} in B that is not a neighbor of $\{0, s_i\}$ is v_{i+1}. Thus $\delta_B^0 = 3/8$.
- $C = \{0, s_i, -s_i\}$. The only neighbor of $-s_i$ in B that is not a neighbor of $\{0, s_i\}$ is $-v_i$. Thus $\delta_B^0 = 3/8$.

It is easy to check, applying Lemma 14, that we have enumerated all kinds of connected components of U. $\square$

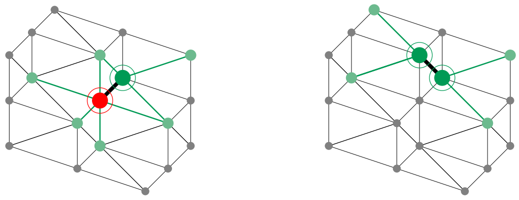


FIGURE 12. The two possible types of connected component with two elements.

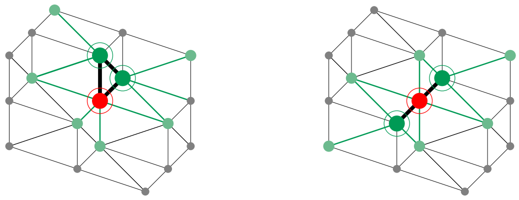


FIGURE 13. The two possible types of connected component with three elements.

Finally we can put everything together and complete the proof of Theorem 9:

PROOF OF THEOREM 9. Let $U \subset V$ avoiding polytope distance 1. We define

$$\delta_B(U) = \limsup_{R \to \infty} \frac{|U \cap V_R|}{|B \cap V_R|}$$

where as usual $V_R = V \cap [-R, R]^n$. We have:

$$\delta_{G_\mathcal{P}}(U) \leqslant \delta_B(U) \times \delta_{G_\mathcal{P}}(B),$$

and since $V = A \cup B$ and B is twice as dense as A in $G_\mathcal{P}$,

$$\delta_{G_\mathcal{P}}(U) \leqslant \frac{2}{3}\delta_B(U).$$

From Lemma 15, we have $\delta_B(U) \leqslant \sup_{C \subset U} \delta_B^0(C)$ where C runs over the connected components of U. Then Lemma 16 shows that

$$\delta_B(U) \leqslant \frac{3}{8}$$

and we get

$$\delta_{G_\mathcal{P}}(U) \leqslant \frac{2}{3} \times \frac{3}{8} = \frac{1}{4}.$$

$\square$

4. The Norms Associated with the Voronoï Cells of the Lattices A_n and D_n

4.1. The Lattice A_n

Here we consider for any $n \geqslant 2$, the lattice

$$A_n = \mathbb{Z}^{n+1} \cap H,$$

where H is the hyperplane $H = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1},\ \sum_{i=1}^{n+1} x_i = 0\}$. Let $\mathcal{P}$ be the Voronoï cell of A_n. We shall prove:

Theorem 10. *For every dimension $n \geqslant 2$, if $\mathcal{P}$ is the Voronoï cell of the lattice A_n, then*

$$m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2^n}.$$

In fact, for $n = 2$, the Voronoï cell of A_2 is nothing but the regular hexagon. We are going to generalize to all dimensions $n \geqslant 2$ the strategy that we used in subsection 3.1.

Let us recall the description of the Voronoï cell $\mathcal{P}$ of A_n given in [CS87], Chapter 21, section 3.

The orthogonal projection on H is denoted by p_H. Let, for $1 \leqslant i \leqslant n$ and $j := (n+1) - i$,

$$\begin{aligned} v_i &= p_H((\underbrace{0, \ldots, 0}_{i \text{ times}}, \underbrace{1, \ldots, 1}_{j \text{ times}})) \\ &= (0, \ldots, 0, 1, \ldots, 1) - \frac{j}{n+1}(1, \ldots, 1) \\ &= (\underbrace{\frac{-j}{n+1}, \ldots, \frac{-j}{n+1}}_{i \text{ times}}, \underbrace{\frac{i}{n+1}, \ldots, \frac{i}{n+1}}_{j \text{ times}}). \end{aligned}$$

Let S be the simplex whose vertices are 0 and the vectors v_i. Then the vertices of $\mathcal{P}$ are the images of the non zero vertices of S under the permutation group $\mathfrak{S}_{n+1}$. In other words, the set of vertices of $\mathcal{P}$ is

$$V_{\mathcal{P}} = \{p_H(u) \mid u \in V_0\}, \text{ where } V_0 = \{0,1\}^{n+1} \setminus \{(0, \ldots, 0), (1, \ldots, 1)\}.$$

We also analyze the boundary of $\mathcal{P}$, in order to understand the norm associated with $\mathcal{P}$. The non zero vertices of S are supported by the hyperplane $H_{0,n}$ of H defined by $H_{0,n} = \{x = (x_0, \ldots x_n) \in H \mid x_n - x_0 = 1\}$. Applying $\mathfrak{S}_{n+1}$, we find that the faces of $\mathcal{P}$ are supported by all the hyperplanes

$H_{i,j} = \{x = (x_0, \dots x_n) \in H \mid x_j - x_i = 1\}$, for $i \neq j$. So

$$\begin{cases} x \in \mathcal{P} \text{ if and only if for all } i \neq j,\ x_j - x_i \leqslant 1 \\ x \in \partial\mathcal{P} \text{ if and only if } \max\limits_{i \neq j}(x_j - x_i) = \max\limits_j x_j - \min\limits_i x_i = 1, \end{cases}$$

and more generally the norm $\|x\|_{\mathcal{P}}$ of a vector $x \in H$ is given by

$$\|x\|_{\mathcal{P}} = \max_j x_j - \min_i x_i.$$

Note that if $x = p_H(u)$, because $H^{\perp} = \mathbb{R}(1, \dots, 1)$, we have

$$\max_j x_j - \min_i x_i = \max_j u_j - \min_i u_i.$$

The vertices of $\mathcal{P}$ generate a lattice, which is the dual lattice of A_n:

LEMMA 17. *The vertices of $\mathcal{P}$ span over $\mathbb{Z}$ the lattice $A_n^{\#} = p_H(\mathbb{Z}^{n+1})$.*

PROOF. Let $v \in V_{\mathcal{P}}$. There is $u \in V_0$ such that $v = p_H(u)$. Because $(u - p_H(u)) \in H^{\perp}$, we have, for every $x \in A_n = \mathbb{Z}^{n+1} \cap H$,

$$\langle x, v \rangle = \langle x, p_H(u) \rangle = \langle x, u \rangle = \sum_{i, u_i = 1} x_i \in \mathbb{Z},$$

so $\mathrm{span}_{\mathbb{Z}}(V_{\mathcal{P}}) \subset A_n^{\#}$.

Now let us take $x \in A_n^{\#} = p_H(\mathbb{Z}^{n+1})$, so $x = p_H(z_0, \dots, z_n)$, with $z_i \in \mathbb{Z}$. Then

$$x = p_H(z_0, \dots, z_n) = \sum_{i=0}^{n} z_i p_H(0, \dots, 0, \underbrace{1}_{i}, 0 \dots, 0) \in \mathrm{span}_{\mathbb{Z}}(V_{\mathcal{P}}).$$

Thus $\mathrm{span}_{\mathbb{Z}}(V_{\mathcal{P}}) = A_n^{\#}$. □

We consider the subgraph $G_{\mathcal{P}}$ of $G(\mathbb{R}^n, \| \cdot \|_{\mathcal{P}})$ induced by the set of vertices $\frac{1}{2}A_n^{\#}$, and the auxiliary graph $\tilde{G}$ which is the Cayley graph on $\frac{1}{2}A_n^{\#}$ associated with the generating set $\frac{1}{2}V_{\mathcal{P}}$. These graphs are the generalizations of the graphs that we considered in subsection 3.1. Here we show that $\tilde{G}$ satisfies the same remarkable property:

LEMMA 18. *The graph $\tilde{G}$ satisfies* (Property D).

PROOF. We follow the proof of Lemma 11. We may assume $x = 0$, and we need to show that, for $v, v' \in V_{\mathcal{P}}$, if $\frac{v + v'}{2} \neq 0$ then it is either some $\frac{v''}{2} \in \frac{1}{2}V_{\mathcal{P}}$, or an element of $\partial\mathcal{P}$. Equivalently, we study $v + v'$ and show that one of the three following situations occurs:

$$\begin{cases} v + v' = 0, \\ v + v' = v'' \in V_{\mathcal{P}}, \\ \|v + v'\|_{\mathcal{P}} = 2. \end{cases}$$

Let u and u' be elements of $V_0 = \{0,1\}^{n+1} \setminus \{(0,\dots,0),(1,\dots,1)\}$ such that $v = p_H(u)$ and $v' = p_H(u')$. The coordinates of the vector $u+u'$ belong to $\{0,1,2\}$, but cannot be all 0 nor all 2. We explore the possible cases:

- If $u+u' = (1,\dots,1)$, then $p_H(u+u') = (0,\dots,0)$, and $v+v' = 0$.
- If the coordinates of $u+u'$ are only 0's and 1's, then $u+u' \in V_0$, and thus $v+v' \in V_{\mathcal{P}}$.
- If the coordinates of $u+u'$ are only 1's and 2's, we may decompose $u+u'$ as $u+u' = (1,\dots,1)+w$, and w must be an element of V_0. This implies that $v+v' = p_H(w) \in V_{\mathcal{P}}$.
- The last remaining case is when both 0's and 2's appear in the coordinates of $U = u+u'$. Then, $\max_j U_j - \min_i U_i = 2$, that is $\|v+v'\|_{\mathcal{P}} = 2$.

□

Because $\tilde{G}$ satisfies (Property D), Lemma 12 is satisfied by $\tilde{G}$. So we can proceed to analyze the cliques of $\tilde{G}$, and for each of them, determine its local density. Since $\tilde{G}$ is vertex transitive, we only describe the cliques containing 0. For $u \in V_0$, we define its support $I = \{i \in \{1,\dots,n+1\},\ u_i = 1\}$.

Lemma 19. *The cliques of $\tilde{G}$ containing 0 are the sets of the form*

$$\left\{0, \frac{p_H(u_1)}{2}, \dots, \frac{p_H(u_s)}{2}\right\}$$

such that if I_i is the support of u_i, then

$$I_1 \subset I_2 \subset \ldots \subset I_s.$$

In particular, since $s \leqslant n$, a clique can not contain more than $n+1$ vertices.

Proof. Let C be a clique of $\tilde{G}$, and assume $0 \in C$. Then the other elements of C must belong to $\frac{1}{2}V_{\mathcal{P}}$ and since C is a clique, they must be adjacent in the graph. In other words, if $\frac{v}{2}, \frac{v'}{2} \in C$, then $\frac{v-v'}{2} \in \frac{1}{2}V_{\mathcal{P}}$. Let $v \neq v' \in V_{\mathcal{P}}$, and $u, u' \in V_0$ such that $v = p_H(u)$ and $v' = p_H(u')$. We denote by I and I' the respective supports of u and u'. For $i \in \{1,\dots,n+1\}$, the ith coordinate of $u-u'$ is:

$$\begin{cases} 1 & \text{if } i \in I \setminus I', \\ -1 & \text{if } i \in I' \setminus I, \\ 0 & \text{otherwise.} \end{cases}$$

If both 1 and -1 appear in the coordinates of $u-u'$, then $\|v-v'\|_{\mathcal{P}} = 2$, and $v-v' \notin V_{\mathcal{P}}$. By definition of V_0 and since $v \neq v'$, the coordinates of $u-u'$ must take two different values. Two cases remain: if $u-u'$ contains only 0's and 1's, $u-u' \in V_0$ and $v-v' \in V_{\mathcal{P}}$; and if it contains only 0's and -1's, then we can write $u-u' = w-(1,\dots,1)$, with $w \in V_0$, so that $v-v' \in V_{\mathcal{P}}$ as well.

To conclude, we find that $v-v' \in V_{\mathcal{P}}$ if and only if $I \subset I'$ or $I' \subset I$.

□

LEMMA 20. *For every clique C of $\tilde{G}$,*

$$\delta^0(C) \leqslant \frac{1}{2^n}.$$

PROOF. Let $\left\{0, \frac{p_H(u_1)}{2}, \ldots, \frac{p_H(u_s)}{2}\right\}$ be a clique. By symmetry, we may assume that

$$u_i = (\underbrace{1, \ldots, 1}_{w_i}, 0, \ldots, 0),$$

where $w_i = |I_i|$. We want to count the vertices in

$$N[C] = \frac{1}{2}\left(\{0, p_H(u_1), \ldots, p_H(u_s)\} + V_{\mathcal{P}}\right).$$

Since $0 \in C$, the set $(\{0, p_H(u_1), \ldots, p_H(u_s)\} + V_{\mathcal{P}})$ must contain all the images of $V_0 \cup \{0\}$ by p_H: there are $2^{n+1} - 1$ such vertices. We count, for each $i = 1, \ldots, s$, how many new neighbors are provided by $p_H(u_i) + V_{\mathcal{P}}$. We find that

- The vector
$$u_1 = (\underbrace{1, \ldots, 1}_{w_1}, 0, \ldots, 0),$$
provides $(2^{w_1} - 1)(2^{n+1-w_1} - 1)$ new neighbors.
- The vector
$$u_2 = (\underbrace{1, \ldots, 1}_{w_1}, \underbrace{1, \ldots, 1}_{w_2 - w_1}, 0, \ldots, 0),$$
provides $2^{w_1}(2^{w_2-w_1} - 1)(2^{n+1-w_2} - 1)$ new neighbors.
- For any $2 \leqslant i \leqslant s$, u_i provides $2^{w_{i-1}}(2^{w_i-w_{i-1}} - 1)(2^{n+1-w_i} - 1)$ new neighbors.

By summing all the values, if we set $w_0 = 0$, we get:

$$\begin{aligned}|N[C]| &= 2^{n+1} - 1 + \sum_{i=1}^{s} 2^{w_{i-1}}(2^{w_i-w_{i-1}} - 1)(2^{n+1-w_i} - 1) \\ &= (s+1)2^{n+1} - (2^{w_s} + \sum_{i=1}^{s} 2^{n+1-(w_i-w_{i-1})}).\end{aligned}$$

Since $w_s \leqslant n$ and for every i, $(w_i - w_{i-1}) \geqslant 1$, we have

$$2^{w_s} + \sum_{i=1}^{s} 2^{n+1-(w_i-w_{i-1})} \leqslant (s+1)2^n,$$

and this implies

$$|N[C]| \geqslant (s+1)2^{n+1} - (s+1)2^n = (s+1)2^n.$$

Finally, the local density of C satisfies:

$$\delta^0(C) = \frac{|C|}{|N[C]|} = \frac{s+1}{|N[C]|} \leqslant \frac{1}{2^n},$$

and we may note that this bound is sharp if and only if $w_s = n$ and for every i, $w_i - w_{i-1} = 1$, that is when C is a maximal clique of the form

$$\{0, (1,0,\dots,0), (1,1,0,\dots,0), \dots, (1,\dots,1,0,0), (1,\dots,1,0)\}.$$

□

Now we can conclude the proof of Theorem 10:

Proof of Theorem 10. Following Lemma 20 and Lemma 12, $\bar{\alpha}(G_{\mathcal{P}}) \leqslant \frac{1}{2^n}$, which leads to the theorem, following Lemma 10. □

4.2. The Lattice D_n, $n \geqslant 4$

We apply the same method as for A_n to another classical family of lattices. For $n \geqslant 4$, the lattice D_n is defined by

$$D_n = \{x = (x_1, \dots, x_n) \in \mathbb{Z}^n, \sum_{i=0}^{n} x_i = 0 \mod 2\}.$$

The same construction provides again a graph that satisfies (Property D). Unfortunately, the analysis of the neighborhoods of the cliques does not lead to the wanted $\frac{1}{2^n}$ upper bound. Nevertheless, we can prove:

Theorem 11. *For every dimension $n \geqslant 4$, if $\mathcal{P}$ is the Voronoï cell of the lattice D_n, then*

$$m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leqslant \frac{1}{(3/4)2^n + n - 1}.$$

Let us describe the Voronoï cell of D_n. Again we refer to [CS87] for further details. Let S be the simplex whose vertices are 0, $(0,\dots,0,1)$, $\left(\frac{1}{2},\dots,\frac{1}{2}\right)$, and the vectors $\left(\underbrace{0,\dots,0}_{i},\frac{1}{2},\dots,\frac{1}{2}\right)$, for $2 \leqslant i \leqslant n-2$. Then the Voronoï cell $\mathcal{P}$ of D_n is the union of the images of S by the group generated by all permutations of the coordinates and sign changes of evenly many coordinates. Note that some of the vertices of S are not extreme points of $\mathcal{P}$ anymore. Actually, there are two types of vertices of $\mathcal{P}$:

$$\begin{cases} 2n \text{ vectors of the form } (\pm 1, 0, \dots, 0) & \text{(type 1)}, \\ 2^n \text{ vectors of the form } \left(\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right) & \text{(type 2)}. \end{cases}$$

The non zero vectors of S are contained in the hyperplane of $\mathbb{R}^n$ defined by the equation $x_{n-1} + x_n = 1$. The faces of $\mathcal{P}$ are supported by the images of this hyperplane under the action of the group *i.e.* the hyperplanes defined by the equations of the form $\pm x_i \pm x_j = 1$, with $i \neq j$. Thus,

$$\begin{cases} x \in \mathcal{P} \text{ if and only if for all } i \neq j,\ |x_i| + |x_j| \leqslant 1 \\ x \in \partial\mathcal{P} \text{ if and only if } \max_{i \neq j}(|x_i| + |x_j|) = 1, \end{cases}$$

and the norm $\|x\|_{\mathcal{P}}$ of a vector $x \in \mathbb{R}^n$ is

$$\|x\|_{\mathcal{P}} = \max_{i \neq j}(|x_i| + |x_j|).$$

As in the case of A_n, the vertices of $\mathcal{P}$ span the dual lattice of D_n:

LEMMA 21. *The vertices of $\mathcal{P}$ span over $\mathbb{Z}$ the dual lattice $D_n^{\#}$.*

PROOF. It is immediate to check that for every $x \in D_n$ and $v \in V_{\mathcal{P}}$, $\langle x, v \rangle \in \mathbb{Z}$, so $\operatorname{span}_{\mathbb{Z}}(V_{\mathcal{P}}) \subset D_n^{\#}$. The converse follows directly from the following decomposition of $D_n^{\#}$:

$$D_n^{\#} = D_n \cup \left(\left(\frac{1}{2}, \dots, \frac{1}{2} \right) + D_n \right) \cup \left(\left(\frac{1}{2}, \dots, -\frac{1}{2} \right) + D_n \right) \cup ((0, \dots, 0, 1) + D_n).$$

□

Once again, let $G_{\mathcal{P}}$ be the subgraph of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ induced by $V = \frac{1}{2} D_n^{\#}$, and let $\tilde{G}$ be the auxiliary graph which is the Cayley graph on V associated with the generating set $\frac{1}{2} V_{\mathcal{P}}$. It also satisfies (Property D):

LEMMA 22. *The graph $\tilde{G}$ satisfies* (Property D).

PROOF. We follow the proof of Lemma 18. Let $v, v' \in V_{\mathcal{P}}$. We distinguish three cases depending on the type of v and v':

- If both v and v' are of type 1, $v + v'$ is either 0, or, up to permutation of the coordinates, of the form $(\pm 2, 0, \dots, 0)$ or $(\pm 1, \pm 1, 0, \dots, 0)$, and $\|v + v'\|_{\mathcal{P}} = 2$.
- If both v and v' are of type 2, the non zero coordinates of $v + v'$ are 1 or -1. If $v + v' \neq 0$, then either it is a vertex of $V_{\mathcal{P}}$ of type 1, or it has at least two coordinates whose absolute values are equal to 1, and so $\|v + v'\|_{\mathcal{P}} = 2$.
- If v is of type 1 and v' is of type 2, then $v + v'$ is either a vertex of $V_{\mathcal{P}}$ of type 2, or, up to a permutation of coordinates, of the form $\left(\pm\frac{3}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2} \right)$, and $\|v + v'\|_{\mathcal{P}} = 2$.

□

It remains to analyze the neighborhoods of the cliques of $\tilde{G}$. We first determine the possible cliques of $\tilde{G}$. We may assume that they contain 0.

LEMMA 23. *Up to symmetry, a clique of* $\tilde{G}$ *containing* 0 *must be a subset of the maximal clique*

$$C_{\max} = \left\{0, \frac{v_1}{2}, \frac{v_2}{2}, \frac{v_3}{2}\right\} \text{ where } \begin{cases} v_1 = (0, \dots, 0, 1) \\ v_2 = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \\ v_3 = \left(-\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) \end{cases}.$$

PROOF. Let $v, v' \in V_{\mathcal{P}}$ such that $\dfrac{v - v'}{2} \in \dfrac{1}{2}V_{\mathcal{P}}$. The conclusion follows from the following facts:

- Both v and v' can not be of type 1, because the difference of two such vectors, is either 0 or has polytope norm 2.
- If v and v' are both of type 2, then v and v' must differ by only one coordinate, otherwise $\|v - v'\|_{\mathcal{P}} = 2$.
- If v is of type 1, say $v = (0, \cdots, 0, \underbrace{\pm 1}_{i}, 0 \cdots, 0)$, if v' is of type 2 and $\dfrac{v - v'}{2} \in \dfrac{1}{2}V_{\mathcal{P}}$, then the ith coordinate of v' must have the same sign as the ith coordinate of v.

□

Then, we analyze the local density of the cliques:

LEMMA 24. *For every clique of* $\tilde{G}$,

$$\delta^0(C) \leqslant \frac{1}{(3/4)2^n + n - 1}.$$

PROOF. By enumerating the neighbors of every element in $C_{\max}$ and by couting the intersections of the different neighborhoods, we find that:

- If $C = \{0\}$, $\delta^0(C) = \dfrac{1}{1 + 2^n + n}$.
- If $C = \left\{0, \dfrac{v_1}{2}\right\}$,

$$\delta^0(C) = \frac{2}{2^n + 2^{n-1} + 4n} = \frac{1}{(3/4)2^n + 2n}.$$

  Note that for $n \geqslant 6$, this density is already greater than $\dfrac{1}{2^n}$.
- If C is one of the two symmetric cliques $\left\{0, \dfrac{v_2}{2}\right\}$ and $\left\{0, \dfrac{v_3}{2}\right\}$,

$$\delta^0(C) = \frac{2}{2 \times 2^n + 2n} = \frac{1}{2^n + n}.$$

- By symmetry, the cliques of the form $\left\{0, \frac{v_i}{2}, \frac{v_j}{2}\right\}$ have the same number of neighbors. If C is one of them,
$$\delta^0(C) = \frac{3}{2 \times 2^n + 2^{n-1} + 3n - 1} = \frac{1}{(5/6)2^n + n - 1/3},$$
which is also greater than $\frac{1}{2^n}$.
- Finally,
$$\delta^0(C_{\max}) = \frac{4}{3 \times 2^n + 4n - 4} = \frac{1}{(3/4)2^n + n - 1},$$
which is the highest possible value of $\delta^0(C)$.

$\square$

5. The Chromatic Number of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$

In this section, we discuss the chromatic number $\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ of the unit distance graph associated with a parallelohedron. We start with the construction of a natural coloring of $\mathbb{R}^n$ with 2^n colors, leading to:

Proposition 6. *Let $\mathcal{P}$ be a parallelohedron in $\mathbb{R}^n$. Then*
$$\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leqslant 2^n.$$

Proof. By assumption, there is a lattice Λ such that $\mathbb{R}^n$ is the disjoint union $\cup_{\lambda\in\Lambda}(\lambda + \mathcal{P})$. We may also write $\mathbb{R}^n$ as the disjoint union
$$\mathbb{R}^n = \bigcup_{\lambda\in\frac{1}{2}\Lambda} \left(\lambda + \frac{1}{2}\mathcal{P}\right) = \bigcup_{\lambda\in\frac{1}{2}\Lambda} B_{\mathcal{P}}\left(\lambda, \frac{1}{2}\right).$$
If H is a coset of $\frac{1}{2}\Lambda\Big/\Lambda$, then
$$A_H = \bigcup_{\lambda\in H} B_{\mathcal{P}}\left(\lambda, \frac{1}{2}\right)$$
is a set avoiding distance 1. So the points in A_H can receive the same color. This concludes the proof, since $\mathbb{R}^n$ is the disjoint union of all A_H where H runs through the 2^n cosets. $\square$

In order to bound $\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ from below, we can take advantage of the induced subgraphs that we have constructed in previous sections. In particular, whenever we have a discrete induced subgraph $G_{\mathcal{P}}$ of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ satisfying $\bar{\alpha}(G_{\mathcal{P}}) = \frac{1}{2^n}$, we obtain as an immediate consequence that
$$\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geqslant \chi(G_{\mathcal{P}}) \geqslant \frac{1}{\bar{\alpha}(G_{\mathcal{P}})} = 2^n.$$

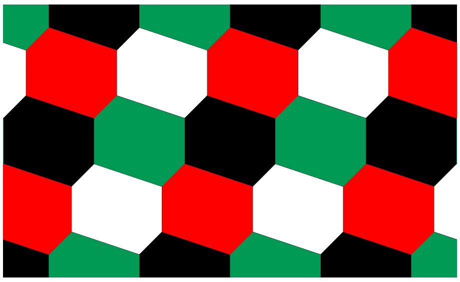


FIGURE 14. $\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \leqslant 2^n$.

Thus we have proved:

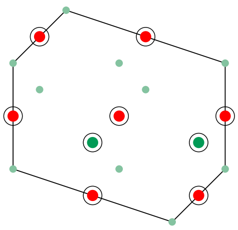
COROLLARY 3. *Let* $\mathcal{P}$ *be a parallelohedron in* $\mathbb{R}^2$*. Then*

$$\chi(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = 4.$$

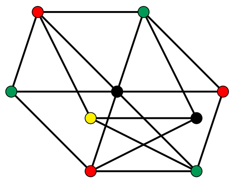
COROLLARY 4. *Let* $\mathcal{P}$ *be the Voronoï cell of the lattice* A_n *in* $\mathbb{R}^n$*. Then*

$$\chi(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) = 2^n.$$

REMARK 6. *We want to point out the fact that in dimension* 2*, one can find a* finite *induced subgraph of* $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ *with chromatic number* 4*. This is illustrated in Figure* 15*: consider the induced subgraph of* $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ *whose vertices are highlighted in Figure* 15a *and whose edges are drawn in Figure* 15b*. Then one can easily see that the chromatic number of this graph is* 4*.*



(A) Consider the subgraph induced by the highlighted vertices.



(B) Its chromatic number is 4.

FIGURE 15. $\chi(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) \geqslant 4$.

Chapter 4

Discrete distribution functions and application to polytope distance graphs

1. Introduction

In this chapter, we generalize the method that we used in the previous chapter in order to prove that the bound $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geqslant \dfrac{1}{2^n}$ is sharp for other polytopes $\mathcal{P}$.

All the bounds that we have obtained so far derive from the same strategy: we consider a discrete induced subgraph G of the unit distance graph $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ for which we want to prove that $\bar{\alpha}(G) \leqslant 1/2^n$. For this, we give to this set of vertices an *auxiliary graph* structure $\tilde{G}$, such that the sets avoiding 1 in G can be written as the union of cliques in $\tilde{G}$ whose closed neighborhoods are disjoints. This is a consequence of (Property D). Finally, we obtain our upper bound on the *global* density of a set avoiding 1 in G by computing the highest *local* density of a clique of $\tilde{G}$ in its closed neighborhood.

However, given a discrete induced subgraph G in $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$, finding an auxiliary graph having (Property D) and providing a good upper bound, is not always possible. We shall generalize this method in several respects.

First, we keep the idea of writing a set avoiding polytope distance 1 as a disjoint union of small pieces. Indeed, such sets were seen to be easier to handle in the Euclidean plane. In [**KMdOFR16**], the authors prove that the density of a set affording *block structure* is strictly lower than $1/2^n$, where a set A in $(\mathbb{R}^2, \|\cdot\|_2)$ is said to have block structure when it can be written as a disjoint union

$$A = \cup_{i \in I} A_i$$

such that for any $i \in I$, $\mathrm{Diam}(A_i) < 1$, and for any $i \neq j$, the distance between A_i and A_j is greater than 1. Here we adapt this idea in a discrete framework, by forcing our independent sets to have a kind of block structure. As we did in the previous chapter, we add an *auxiliary graph* structure $\tilde{G}$ to a discrete subset $V \subset \mathbb{R}^n$. We want $\tilde{G}$ to satisfy the following property: in the decomposition

$$A = \cup_{C \in \mathcal{C}} C$$

of a set A avoiding polytope distance 1 into connected components in $\tilde{G}$, every connected component C satisfies $\mathrm{Diam}(C) < 1$. However, we do not require a priori the condition $d_{\mathcal{P}}(C, C') > 1$.

Second, once a set $A \subset V$ avoiding polytope distance 1 is split into blocks, we can obtain an upper bound on the density of A by bounding from above the *local density* of each block. Nevertheless, there are several ways to define the local density of a block. In the previous chapter, the strong (Property D) naturally implied a definition: the local density of block was its density in its closed neighborhood in the auxiliary graph $\tilde{G}$. In a more general context, the challenge is to find a good division of the points of V between the blocks of A; in other words, each block receives a certain amount of points of V, that we will refer to as its *neighborhood*. Then the local density of a block is its density in that neighborhood. This idea is inspired by the methods usually employed to solve packing problems. For instance, the well known Voronoï region of a sphere in a packing in $\mathbb{R}^n$ is the set made by the points in $\mathbb{R}^n$ that are closer to that sphere than to any other sphere in the packing. The highest density of a sphere in its Voronoï region gives an upper bound for the density of the whole packing. In dimension 2, the smallest, in terms of volume, local Voronoï region is a regular hexagon, and it is the Voronoï cell of the optimal hexagonal lattice packing. In dimension 3, the local upper bound given by Voronoï regions is not sharp. Indeed, following the *Kepler conjecture* proved by Hales ([**Hal06**], [**HAB+17**]), the best packing in dimension 3 is realized by the *face-centered cubic* lattice and has density about 0.74, whereas the *dodecahedral conjecture*, claimed by Fejes Tóth [**Fej43**], and proved almost sixty years later by Hales and McLaughlin [**HM10**], asserts that the minimum volume of a Voronoï cell is a regular dodecahedron of inradius 1. The density of a unit ball in this region is approximately 0.755. The proof of Kepler conjecture requires also a partition of the space, but more sophisticated than the one involving Voronoï regions.

In this chapter, we will define the discrete analogue of the Voronoï distribution, as well as more general *discrete distribution functions*. We shall see that with good choices of functions, we can extend the scope of polytopes for which we can prove that the bound $1/2^n$ is sharp: by using a somehow simpler graph, we provide an alternative proof for the hexagonal Voronoï cells in dimension 2. More dramatically, we employ this method to handle the regular *elongated dodecahedron*. This is the main result of the chapter, see Theorem 13. Finally, we will consider polytopes that may be written as a product $\mathcal{P} = \mathcal{P}_0 \times [-1, 1]^m$, where $\mathcal{P}_0$ is a parallelohedron in dimension n. Precisely, we will show that $m_1(\mathbb{R}^{n+m}, \|\cdot\|_{\mathcal{P}}) = 1/2^m m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}_0})$. We will first give an intrinsic argument, before replacing this result in the context of this chapter, by using discrete distribution functions. This is motivated by one of the five types of polytope in dimension 3: the hexagonal prisms.

The chapter is organized as follows: in Section 2, we introduce the notion of discrete distribution function of discrete sets. Then, as a first example, in Section 3, we use discrete distribution functions to prove in another way Theorem 7 of Chapter 3 . Section 4 is dedicated to the elongated dodecahedron, it contains the main theorem of the chapter. Finally, in Section 5, we deal with hexagonal prisms, and their generalizations in higher dimension.

2. Discrete Distribution Functions

2.1. Definitions

Let $V \subset \mathbb{R}^n$ be a discrete set, and let A be a subset of V. Recall (see Chapter 2, Section 4) that the density of A in V is defined as the limit sup:

$$\delta_V(A) = \limsup_{R\to\infty} \frac{|A \cap V_R|}{|V_R|}.$$

We want to bound $\delta_V(A)$ from above. To do so, we associate to each point $a \in A$ a region, a kind of *neighborhood*, in such a way that a has a low density in its neighborhood. In other words, the points of V are allocated to the elements of A. A single point v in V may eventually be shared between several points of A. More precisely, we introduce the notion of discrete distribution function:

Definition 2. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$. A discrete distribution function of V with respect to A is a map*

$$\begin{array}{cccc} f & : & A \times V & \to & [0,1] \\ & & (a,x) & \mapsto & f(a,x) \end{array}$$

satisfying the following properties:

$$\forall\, a \in A, \quad f(a,a) = 1, \tag{11}$$

$$\forall\, x \in V, \quad \sum_{a\in A} f(a,x) \leqslant 1. \tag{12}$$

Moreover, for technical reasons, we also require

$$\exists\, \rho > 0 \mid \forall\, (a,x) \in A \times V, \|a-x\|_\infty > \rho \Rightarrow f(a,x) = 0. \tag{13}$$

Once we have a discrete distribution function f, we can define in a natural way the neighborhood of an element, its volume, and its local density corresponding to f. The *f-neighborhood* of $a \in A$ is made by the points in V that are, eventually partially, allocated to a. Its f*-volume* is the sum of the contributions $f(a,x)$, and the f*-density* of a with respect to f is defined locally as the inverse of the f-volume of its f-neighborhood:

Definition 3. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$, and let $f : A \times V \to [0,1]$ be a discrete distribution function of V with respect to A. We define:*

- *The* f-neighborhood *of $a \in A$ as the set of points x in V such that $f(a,x) > 0$:*

$$N_f(a) = \{x \in V \mid f(a,x) > 0\}.$$

- *The* f-volume *of the f-neighborhood of a:*

$$\mathrm{Vol}_f(N_f(a)) = \sum_{x \in N_f(a)} f(a,x) = \sum_{x \in V} f(a,x)$$

- *The* f-density *of an element $a \in A$ in its f-neighborhood:*

$$\delta_f(a) = \frac{1}{\sum_{x \in V} f(a,x)} = \frac{1}{\mathrm{Vol}_f(N_f(a))}.$$

Remark 7. *Note that condition (13) implies that the f-volume of $N_f(a)$ is finite, because V is discrete. Moreover, the condition (11) ensures that this volume cannot be 0 and that $\delta_f(a) \leqslant 1$.*

The density of A in V is bounded by the supremum of the local densities δ_f achieved by the elements of a, as proved in the following lemma:

Lemma 25. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$, and let $f : A \times V \to [0,1]$ be a discrete distribution function of V with respect to A. We set:*

$$\delta_f^* = \sup_{a \in A} \delta_f(a).$$

Suppose that V satisfies, for any constant $r > 0$,

$$\lim_{R \to \infty} \frac{|V_R \setminus V_{R-r}|}{|V_R|} = 0. \tag{14}$$

Then

$$\delta_V(A) \leqslant \delta_f^*.$$

Remark 8. *The condition (14) is a technical requirement that allows to take limits when we deal with infinite graphs. We will apply our method to periodic sets, therefore fulfilling the condition.*

Proof. Let $\rho > 0$ be such that $f(a,x) = 0$ whenever $\|a - x\|_\infty > \rho$, and let us fix $R > \rho$. We have:

$$\frac{|A_R|}{|V_R|} = \frac{|A_{R-\rho}|}{|V_R|} + \frac{|A_R \setminus A_{R-\rho}|}{|V_R|}.$$

Following the assumption on ρ, we have, for every $a \in A_{R-\rho}$,

$$\sum_{x \in V} f(a,x) = \sum_{x \in V_R} f(a,x).$$

Consequently,

$$\begin{aligned} \frac{|A_{R-\rho}|}{|V_R|} = \frac{\sum_{a \in A_{R-\rho}} 1}{|V_R|} &= \frac{\sum_{a \in A_{R-\rho}} \delta_f(a) \sum_{x \in V_R} f(a,x)}{|V_R|} \\ &\leqslant \delta_f^* \frac{\sum_{x \in V_R} (\sum_{a \in A_{R-\rho}} f(a,x))}{|V_R|} \\ &\leqslant \delta_f^* \frac{|V_R|}{|V_R|} = \delta_f^*. \end{aligned}$$

Thus

$$\frac{|A_R|}{|V_R|} \leqslant \delta_f^* + \frac{|A_R \setminus A_{R-\rho}|}{|V_R|}. \tag{15}$$

The hypothesis on V allows to conclude the proof, because $A_R \setminus A_{R-\rho}$ is contained in $V_R \setminus V_{R-\rho}$. □

2.2. Discrete distribution functions associated with partitions

Here we introduce a particular type of discrete distribution function, associated with a given partition of the set A:

Definition 4. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$. Suppose that A may be written as a disjoint union of finite subsets $C \in \mathcal{C}$. A discrete distribution function of V associated with the partition $\mathcal{C}$ is a map*

$$\begin{array}{cccc} f & : & \mathcal{C} \times V & \to & [0,1] \\ & & (C,x) & \mapsto & f(C,x) \end{array}$$

satisfying the following properties:

$$\forall\, C \in \mathcal{C},\ \forall\, c \in C, \quad f(C,c) = 1, \tag{16}$$

$$\forall\, x \in V, \quad \sum_{C \in \mathcal{C}} f(C,x) \leqslant 1, \tag{17}$$

and

$$\exists\, \rho > 0 \mid \forall\, (C,x) \in \mathcal{C} \times V, d_\infty(x,C) > \rho \Rightarrow f(C,x) = 0, \tag{18}$$

where $d_\infty(x,C) = \min_{c \in C} \|x - c\|_\infty$.

This definition induces the following analogue of Definition 3. In particular, we get a local density function on the elements of the partition $\mathcal{C}$:

Definition 5. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$, and let f be a discrete distribution function of V associated with a partition $\mathcal{C}$ of A. We define:*

- *The* f-neighborhood *of $C \in \mathcal{C}$:*
$$N_f(C) = \{x \in V \mid f(C,x) > 0\}.$$
- *The* f-volume *of the f-neighborhood of C:*
$$\mathrm{Vol}_f(N_f(C)) = \sum_{x \in N_f(C)} f(C,x) = \sum_{x \in V} f(C,x)$$
- *The* f-density *of C in its f-neighborhood:*
$$\delta_f(C) = \frac{|C|}{\sum_{x \in V} f(C,x)} = \frac{|C|}{\mathrm{Vol}_f(N_f(C))}.$$

A discrete distribution function $f_{\mathcal{C}}$ associated with a partition $\mathcal{C}$ naturally provides a discrete distribution function in the sense of Definition 2. Indeed, if $A = \bigcup_{C \in \mathcal{C}} C$, for every $a \in A$, there is a unique $C_a \in \mathcal{C}$ such that $a \in C_a$. Then it is immediate to check that the map

$$\begin{array}{rccl} f & : & A \times V & \to & [0,1] \\ & & (a,x) & \mapsto & \begin{cases} 1 & \text{if } x \in A \\ \dfrac{1}{|C_a|} f_{\mathcal{C}}(C_a,x) & \text{else} \end{cases} \end{array}$$

satisfies conditions (12) and (13). Not surprisingly, we get the following consequence of Lemma 25:

Lemma 26. *Let $A \subset V$ be two discrete subsets of $\mathbb{R}^n$, and let f be a discrete distribution function of V associated with a partition $\mathcal{C}$ of A. We set:*
$$\delta_f^* = \sup_{C \in \mathcal{C}} \delta_f(C).$$
Suppose that V satisfies, for any constant $r > 0$,
$$\lim_{R \to \infty} \frac{|V_R \setminus V_{R-r}|}{|V_R|} = 0. \tag{19}$$
Then
$$\delta_V(A) \leqslant \delta_f^*.$$

2.3. Scope of Application and Examples

Although we defined discrete distribution functions in the general framework of discrete sets, we will use this notion in a more specific context. In order to get a natural partition of a subset A of a discrete set V in $\mathbb{R}^n$, we will give V an auxiliary graph structure. Then the elements of $\mathcal{C}$ will be the connected components of A in V. However, the graph structure must be chosen in such a way that the connected components of a set avoiding polytope distance 1 are always finite. Moreover, we want the possible connected components to be easy to enumerate. A possible way to do so is to construct a graph such that for every pair $x, y \in V$ such that $d_{\mathcal{P}}(x, y) > 1$, any path from x to y contains a point z at polytope distance 1 from x. As a consequence, a connected component of a set avoiding polytope distance 1 must have a diameter strictly lower than 1. Such a graph will be constructed in order to treat the elongated dodecahedron (see Section 4).

Once we fixed a graph structure on V, the decomposition of A into connected components is completely determined. Then we can still define several different distribution functions associated with this decomposition. Let us now present two elementary examples of discrete distribution functions.

Suppose that the auxiliary graph $\tilde{G}$ satisfies (Property D). By definition, this means that if the graph distance in $\tilde{G}$ between two vertices x and y in V is 2, then $\|x - y\|_{\mathcal{P}} = 1$. As a consequence, the connected components in $\tilde{G}$ of a set avoiding polytope distance 1 must be cliques. Incidentally, the strategy that we employed in the previous chapter may be interpreted in terms of discrete distribution functions, as presented in the following first example:

Example 8. *Suppose that the auxiliary graph $\tilde{G}$ satisfies* (Property D). *Let $A = \bigcup_{C \in \mathcal{C}} C$ be the decomposition of a set avoiding 1 A into connected components in $\tilde{G}$. Then it is easy to see that the function*

$$\begin{array}{rcl} f \; : \; \mathcal{C} \times V & \to & [0,1] \\ (C, x) & \mapsto & \begin{cases} 1 & \text{if } x \in N[C] \\ 0 & \text{else} \end{cases} \end{array}$$

is a discrete distribution function. In the previous chapter, we computed implicitly the number δ_f^* for several graphs $\tilde{G}$.

Our second example is a discrete analogue of the well-known Voronoï partition of space associated with a packing. Note that here, no auxiliary graph is a priori required. However, we will always use it for decompositions induced by graph structures.

Example 9. *Let $A \subset V$ a set avoiding 1, and let $\mathcal{C}$ be a partition of A in finite subsets. For $x \in V$, we define*

$$d_x = d_{\mathcal{P}}(x, A)$$

and

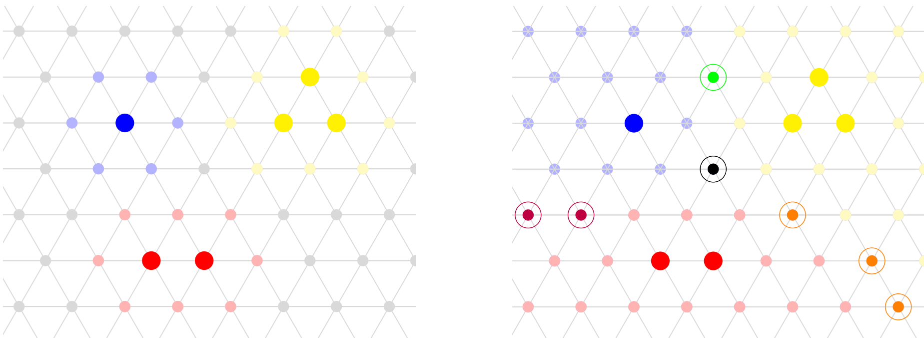
$$n_x = |\{C \in \mathcal{C} \mid d_{\mathcal{P}}(x, C) = d_x\}|,$$

that are both finite, since V is discrete. Let us consider the function

$$\begin{array}{rcl} f \; : \; \mathcal{C} \times V & \to & [0,1] \\ (C, x) & \mapsto & \begin{cases} \dfrac{1}{n_x} & \text{if } d_{\mathcal{P}}(x, C) = d_x \\ 0 & \text{else} \end{cases} . \end{array}$$

The condition (17) *is obviously satisfied. Moreover, if we assume that A is saturated, then the condition* (18) *is also satisfied.*

In Figure 1 we consider the graph that we introduced in order to treat the regular hexagon. We have seen that this graph has (Property D). In Figure 1a, one can see the points that contribute to every connected component of A. In Figure 1b, we consider the same subset, but we depict the f-neighborhoods of the connected components with respect to the discrete Voronoï distribution function. This figure aims at showing that for a graph satisfying (Property D), even if the function described in Example 8 does not lead to the expected local bound $1/2^n$, one may hope to reach better bounds by considering the Voronoï distribution associated with this graph, or more sophisticated distribution functions.



(A) The neighborhoods of the connected components for the function of Example 8, when the auxiliary graph has (Property D). They only contain the neighbors of the connected component in the auxiliary graph $\tilde{G}$.

(B) The neighborhoods corresponding to the discrete Voronoï distribution of Example 9. The circled points are the ones that are in several neighborhoods. For instance the black point is shared among the three cliques.

FIGURE 1. Several neighborhoods for the same decomposition.

3. An alternative proof in dimension 2

In this section, we illustrate the computation of local bounds by using several discrete distribution functions on the simple example of a set A avoiding hexagonal Voronoï distance 1 in the plane, contained in a specific discrete set V. We will see that the two functions that we displayed in the previous section do not lead to the expected bound $1/4$. Nevertheless we will introduce a function f for which computing $\delta_f^* = 1/4$ is not hard. This will provide an alternative proof of the fact that $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = 1/4$.

3.1. Definition and description of the graph

The graph that we are going to study is nothing but a subgraph of the graph that we used for general hexagonal Voronoï cells. Recall that the set of vertices of the latter was

$$\frac{1}{2}L \cup (\frac{1}{2}L + v_0) \cup (\frac{1}{2}L + v_1),$$

where L was the translation lattice of the polytope $\mathcal{P}$, and v_0 and v_1 were two consecutive vertices of $\mathcal{P}$. Here we deal with a slightly simpler set of vertices,

$$V = \frac{1}{2}L \cup (\frac{1}{2}L + v_0).$$

Remark that we only keep two cosets modulo $\frac{1}{2}L$ out of three. The edges of the auxiliary graph $\tilde{G}$ are the ones induced by the edges of our previous auxiliary graph. Note that now green points cannot be connected anymore. The edges that remain simply correspond to the semi-edges of the hexagon $\mathcal{P}$. Both graphs are pictured in Figure 2.

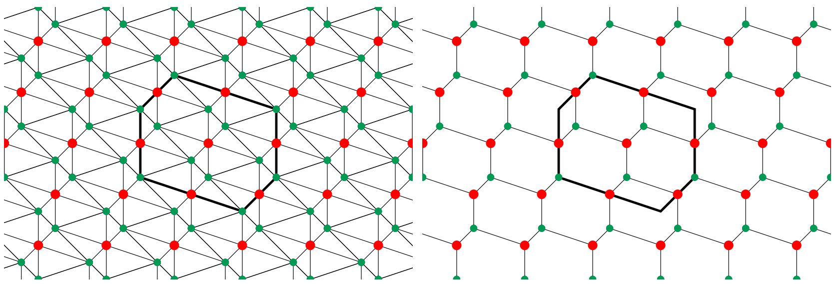


FIGURE 2. Comparison between the previous and the new graph.

It is almost immediate to check that this new graph $\tilde{G}$ verifies (Property D). The distribution functions will always be associated with the decomposition of a set A avoiding polytope distance 1 into connected components, that are cliques in this particular case, in the graph $\tilde{G}$.

The size of these connected components cannot exceed 2. Moreover, one can note that red vertices and green vertices play symmetrical roles, as depicted in Figure 3. As a consequence, there are, up to symmetries, only two kinds of connected components in $\tilde{G}$, that are represented in Figure 4.

3.2. The bound given by the previous method

Since the graph $\tilde{G}$ is a regular graph of degree three, the local densities, with respect to the function defined in Example 8, of the connected components of size 1 and 2 are respectively 1/4 and 1/3 (see Figure 5). So our previous method leads to the bound 1/3 for that graph.

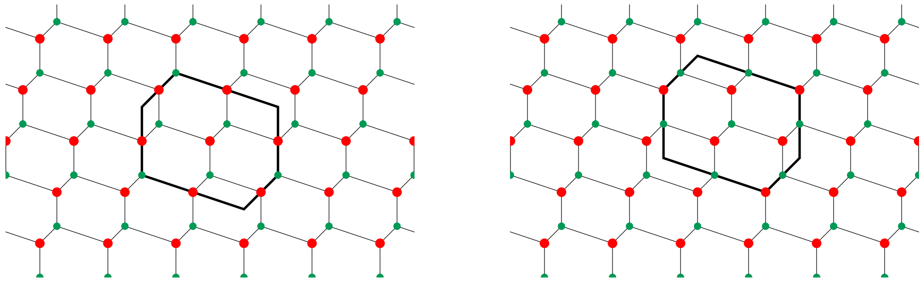


FIGURE 3. Red vertices and green vertices play symmetrical roles.

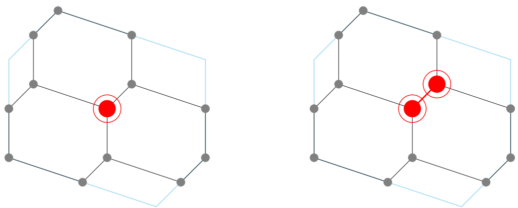


FIGURE 4. The two different kinds of cliques in $\tilde{G}$.

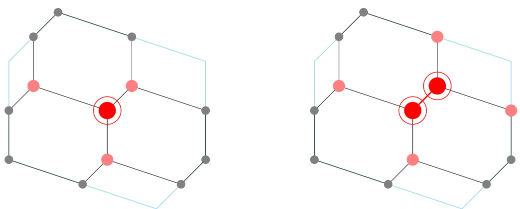


FIGURE 5. The neighborhoods of the cliques in $\tilde{G}$.

3.3. The bound given by the Voronoï distribution

Now we compute the number δ_f^* when f is the Voronoï distribution function described in Example 9.

Because $\tilde{G}$ satisfies (Property D), the Voronoï neighborhood of a connected component must contain its neighbors in $\tilde{G}$. Hence the bound obtained with the Voronoï distribution cannot be worse than the previous one.

Since the isolated vertices already have a local density bounded from above by $1/4$, we only care about the connected components of size 2.

Let us fix such a connected component C_0, and suppose that a saturated set A avoiding polytope distance 1 contains C_0. We already know that for

$x \in V$, if $d_{\mathcal{P}}(x, C_0) < 1$, then x is entirely attributed to C, namely

$$\begin{cases} f(C_0, x) = 1 \\ f(C, x) = 0 \text{ for any } C \neq C_0. \end{cases}$$

Moreover, if $d_{\mathcal{P}}(x, C_0) > 1$, there must be $C \neq C_0$ such that $d_{\mathcal{P}}(x, C_0) \leqslant 1$, otherwise A would not be saturated. So $f(C_0, x) = 0$. In other words, we need to understand the minimal contribution of the points at polytope distance 1 from C_0 in its Voronoï cell.

It is possible that such a point x does not contribute to the Voronoï cell of C_0, precisely when there is another connected component at polytope distance strictly less than 1 from x. If this does not happen, then x is in the f-neighborhood of C_0, and is shared between the connected components at polytope distance 1 from it. We want to know the highest possible local density of C_0 in its neighborhood, or, equivalently, its smallest f-neighborhood.

Let Ω be the set made by all the cliques in $\widetilde{G}$. We say that two elements C and C' of Ω are *compatible* if there is no $c \in C$, $c' \in C'$ such that $d_{\mathcal{P}}(c, c') = 1$. In other words, C and C' can coexist in a set avoiding polytope distance 1. Here two elements C and C' of Ω are compatible if and only if $d_{\mathcal{P}}(C, C') > 1$. In more complicated sets, such as the one we will consider for the elongated dodecahedron, this equivalence does not hold. Let us define

$$\mathcal{S} = \{C \in \Omega \text{ compatible with } C_0 \mid d_{\mathcal{P}}(C, C_0) \leqslant 2\}.$$

The elements of $\mathcal{S}$ are precisely the cliques that can contribute to the f-volume of the f-neighborhood of C_0. Indeed, for x at polytope distance 1 from C_0, by the triangular inequality, a connected component such that $f(C, x) > 0$ must be at polytope distance at most 2 from C_0.

A subset $S \subset \mathcal{S}$ is said to be *admissible*, if any two $C, C' \in S$ are compatible. In order to compute the highest possible f-density of C_0, we look through all the configurations associated with the maximal admissible subset of $\mathcal{S}$, and we compute the f-volume of the Voronoï neighborhood of C_0 in that configuration. Note that finding these maximal subsets amounts to compute the maximal cliques in the graph whose vertices are the elements of S, in which two elements C and C' are connected if and only if they are compatible. With the help of a computer, one finds that the maximal f-density of a clique is

$$\delta_f^* = \frac{2}{6 + 5/3} = \frac{6}{23} \approx 0.261.$$

An example of such a configuration is given in Figure 6: among the ten points at distance 1 from C_0, five do not contribute to its Voronoï cell, and five are shared between three different connected components. This bound is not exactly 1/4 but is clearly better than the one that we obtained with the previous method.

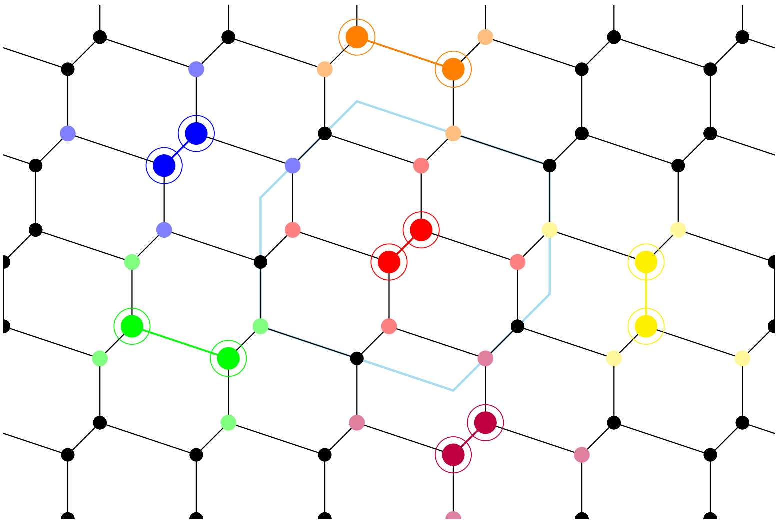


FIGURE 6. An example of a configuration with local Voronoï density 0.261. The clique C_0 is the one depicted in red. The five black points at polytope distance 1 from C_0 are the ones that are in its Voronoï neighborhood. They all are at distance 1 from exactly three cliques.

3.4. Achieving $1/4$ with another distribution function

Finally we introduce a more specific distribution function and show that it leads to the expected bound $1/4$:

THEOREM 12. *Let us consider the graph introduced in subsection 3.1. Let A be a subset of V avoiding polytope distance 1, and let $\mathcal{C}$ be its partition into connected components of $\tilde{G}$. Consider the function $f : \mathcal{C} \times V \to [0,1]$ defined as follows: for any $x \in V$, $C \in \mathcal{C}$,*

$$f(C,x) = \begin{cases} 1 & \text{if } d_{\mathcal{P}}(x,C) = 0, \\ 2/3 & \text{if } 0 < d_{\mathcal{P}}(x,C) < 1, \\ 1/3 & \text{if } d_{\mathcal{P}}(x,C) = 1, \\ 0 & \text{if } d_{\mathcal{P}}(x,C) > 1. \end{cases}$$

Then f is a discrete distribution function of V with respect to $\mathcal{C}$. Moreover,

$$\delta_f^* = \frac{1}{4}.$$

PROOF. First we need to prove that f is a discrete distribution function. Although it trivially verifies condition (18), we need to check that is satisfies condition (17) as well.

- First let us suppose $x \in A$. Then there is a unique $C_x \in \mathcal{C}$ such that $x \in C_x$. Obviously, there cannot be another clique C such that $d_{\mathcal{P}}(x, C) \leqslant 1$. Hence $\sum_{C \in \mathcal{C}} f(C, x) = f(C_x, x) = 1$.
- Now assume $d_{\mathcal{P}}(x, A) < 1$. Because $\tilde{G}$ has (Property D), there is a unique $C \in \mathcal{C}$ such that $d_{\mathcal{P}}(x, C) = 1$. In other words, x is a neighbor of an element $a \in A$. We want to know how many connected components can be at polytope distance exactly 1 from x. To do so, we look at the points at polytope distance 1 from x that are not at distance 1 from a. We also forget about the neighbors of a in $\tilde{G}$: in the worst-case scenario, such a point would be in C. We find that there are only three "free" points, and we see immediately (Figure 7) that they cannot intersect more than one connected component. So in that case, there is at most one clique at polytope distance 1 from x, and $\sum_{C \in \mathcal{C}} f(C, x) \leqslant 2/3 + 1/3 = 1$.

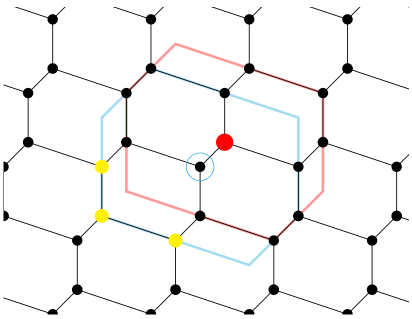


FIGURE 7. A neighbor of an element of A can be at polytope distance 1 from at most one clique: suppose that the red point is in A. The three yellow points are the only points at polytope distance 1 from the circled point that are compatible with the red one. These three points cannot contain more than one clique in A.

- Finally, when $d_{\mathcal{P}}(x, A) = 1$, we only need to check that x can not be shared between more than three cliques. This fact can be observed by looking at the points at polytope distance 1 from x (see Figure 8).

Hence f satisfies condition (17). It remains to see that $\delta_f^* = 1/4$. First assume C is a single point. The definition of f implies that each one of the three neighbors of C in $\tilde{G}$ (respectively the nine points at polytope distance 1 from C) provides a contribution greater than $2/3$ (respectively $1/3$) for the f-volume of $N_f[C]$ (see Figure 9). As a consequence,

$$\sum_{x \in V} f(C, x) \geqslant 1 + 3.\frac{2}{3} + 9.\frac{1}{3} = 6,$$

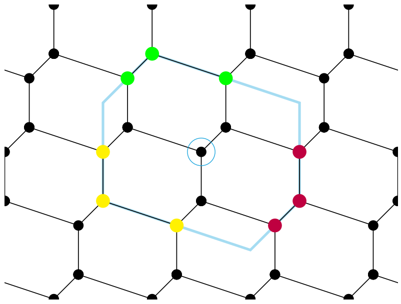


FIGURE 8. A vertex in V cannot be at polytope distance 1 from more than three connected components in A: each group of three points depicted with the same color cannot intersect more than one connected component in $\tilde{G}$.

so that $\delta_f(C) \leqslant 1/6$.

Now assume C is a clique with two elements. Then C has four neighbors in $\tilde{G}$ and ten points at polytope distance 1 (see Figure 9), so that, as before

$$\sum_{x \in V} f(C,x) \geqslant 2 + 4.\frac{2}{3} + 10.\frac{1}{3} = 8,$$

and hence $\delta_f(C) \leqslant 2/8 = 1/4$.

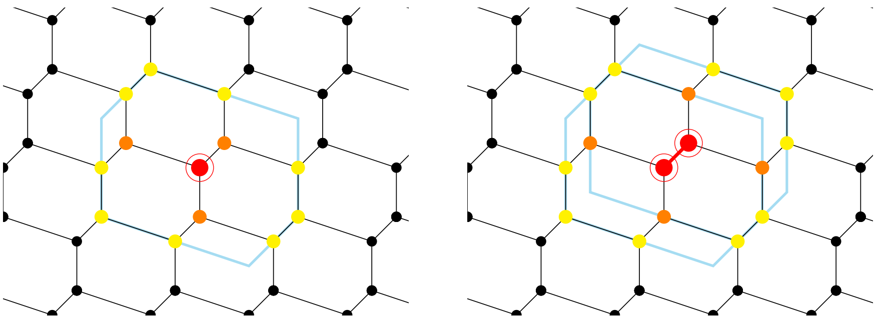


FIGURE 9. The points in the f-neighborhood of the cliques: the contribution of the red points is 1, that of the orange ones is at least $2/3$, and that of the yellow points is at least $1/3$.

□

4. The Elongated Dodecahedron

This section is dedicated to the computation of $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ when $\mathcal{P}$ is an elongated dodecahedron.

4.1. Description of the Polytope

Following the description of lattices in dimension three (see Chapter 2, subsection 3.3), the elongated dodecahedra are the Voronoï cells of the lattices having exactly one Selling parameter which is 0. Such a polytope is in general composed by four hexagonal faces, corresponding to two pairs of orthogonal Voronoï vectors, and by four other pairs of faces, all being parallelograms. Here we consider the most symmetrical case, where the hexagonal faces are identical, and, as a consequence, the height parallelograms are rhombi, identical as well. A representation of such a polytope (see Figure 10) is the Voronoï cell $\mathcal{P}$ of the lattice $\Lambda \subset \mathbb{R}^3$ spanned by the basis $\mathcal{B} = \{(2,0,0),(0,2,0),(-1,-1,2)\}$.

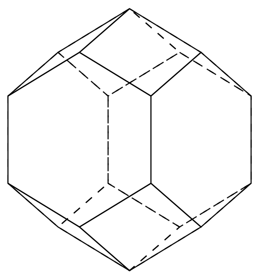


FIGURE 10. The Elongated Dodecahedron.

The aim of this section is the determination of the number $m_1(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}})$:

THEOREM 13. *Let $\mathcal{P}$ be the Voronoï cell of the lattice Λ spanned by $\mathcal{B}$. Then*

$$m_1(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}}) = 1/8.$$

Let us portray precisely the polytope $\mathcal{P}$. Its symmetry group $\mathfrak{S}$ is the group generated by the permutation of the two first coordinates, and sign changes of any coordinate.

The Voronoï vectors of Λ are the images under $\mathfrak{S}$ of the two vectors $(2,0,0)$ and $(-1,-1,2)$. As a consequence, the polytope norm of an element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with respect to $\mathcal{P}$ is given by:

$$\|x\|_{\mathcal{P}} = \max\left\{|x_1|, |x_2|, \frac{|x_1| + |x_2| + 2|x_3|}{3}\right\}.$$

In the sequel, the vertices of $\mathcal{P}$, the middle of its edges, and the centers of its facets will be of particular interest. Table 1 gives an overview of all the orbits of these points under the action of $\mathfrak{S}$. In order to locate each point, we precise the type of faces containing it. As a complement of this description, Figure 11 depicts a representative of each kind of face of $\mathcal{P}$.

Type of point	Representative	Number	Corresponding facets
Vertex	$(0,0,3/2)$	2	4 rhombi.
	$(1,0,1)$	8	2 rhombi, 1 hexagon.
	$(1,1,1/2)$	8	1 rhombus, 2 hexagons.
Middle of edge	$(1,1,0)$	4	2 hexagons.
	$(1,1/2,3/4)$	16	1 rhombus, 1 hexagons.
	$(0,1/2,5/4)$	8	2 rhombi.
Center of facet	$(1,0,0)$	4	1 hexagon.
	$(1/2,1/2,1)$	8	1 rhombus.

TABLE 1. The vertices, middle of edges, and centers of facets of the Elongated Dodecahedron.

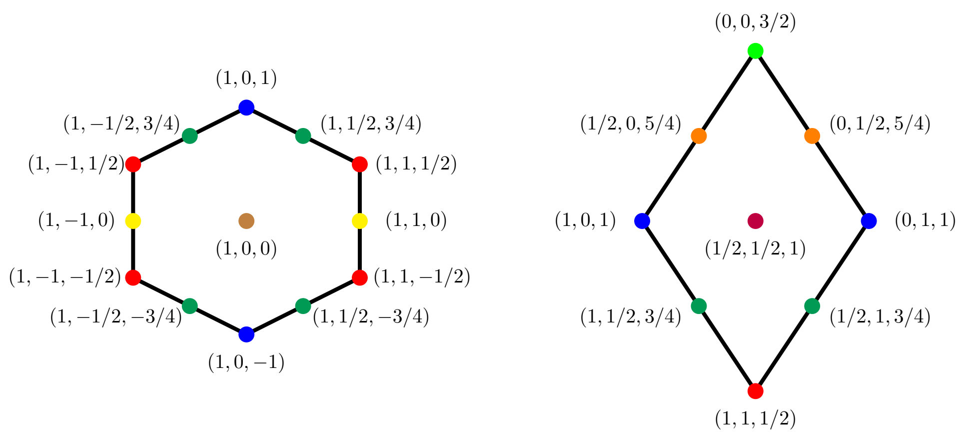


FIGURE 11. The faces of the Elongated Dodecahdron.

4.2. Definition and Description of the Induced Subgraph and its Auxiliary Graph

Using the union $\mathcal{A}$ of the vertices, middles of edges, and centers of facets of $\mathcal{P}$, we construct the induced subgraph G of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ that will be proved to satisfy $\bar{\alpha}(G) = 1/8$. Precisely, the set of vertices V of G is defined as follows:

$$V = \frac{1}{2}\Lambda + \mathcal{A}.$$

This set turns out to be a lattice, and moreover it is the dual lattice of the translation lattice Λ:

LEMMA 27. *Let $\Lambda_1 \subset \mathbb{R}^3$ be the lattice spanned by the basis*

$$\mathcal{B}_1 = \{(1/2, 0, 1/4), (-1/2, 0, 1/4), (0, 1/2, -1/4)\}.$$

We have:

$$V = \Lambda_1 = \Lambda^{\#}.$$

PROOF. It is immediate to check that $\frac{1}{2}\Lambda$, which is generated by the centers of the facets of $\mathcal{P}$, is contained in Λ_1. The Gram matrices of the bases $\frac{1}{2}\mathcal{B}$ and $\mathcal{B}_1$ of these lattices are respectively

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ -1/2 & -1/2 & 3/2 \end{pmatrix} \text{ and } \frac{1}{16}\begin{pmatrix} 5 & -3 & -1 \\ -3 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix},$$

so that $\operatorname{Vol}\left(\frac{1}{2}\Lambda\right) = 1$ and $\operatorname{Vol}(\Lambda_1) = 1/8$. Hence, $|\Lambda_1 \big/ (\frac{1}{2}\Lambda)| = 8$. Putting together the points of $\mathcal{A}$ that agree modulo $\frac{1}{2}\Lambda$, we may write:

$$V = \frac{1}{2}\Lambda + \mathcal{R},$$

where

$$\mathcal{R} = \{0, (1/2, 0, \pm 1/4), (0, 1/2, \pm 1/4), (0, 0, \pm 1/2), (1/2, 1/2, 0)\}.$$

The eight points of $\mathcal{R}$ are nothing but representatives of the height cosets of $\Lambda_1 \big/ (\frac{1}{2}\Lambda)$. Hence $V = \Lambda_1$.

Finally, for every $u \in \mathcal{B}, v \in \mathcal{B}_1$, $\langle u, v\rangle \in \mathbb{Z}$, thus $\Lambda_1 \subset \Lambda^{\#}$. To conclude, it is sufficient to note that

$$\operatorname{Vol}(\Lambda_1) = 1/8 = 1/\operatorname{Vol}(\Lambda) = \operatorname{Vol}(\Lambda^{\#}).$$

$\square$

Before defining the edges of the auxiliary graph $\tilde{G}$ associated with G, we enumerate in Table 2, up to symmetry, all the 33 points of V that are contained in the interior of $\mathcal{P}$. We give for each orbit its size and the polytope norm of its elements.

Representative	Length of the Orbit	Polytope Norm
$(0,0,0)$	1	0
$(0,0,1/2)$	2	$1/3$
$(1/2,0,1/4)$	8	$1/2$
$(1/2,1/2,0)$	4	$1/2$
$(0,0,1)$	2	$2/3$
$(1/2,1/2,1/2)$	8	$2/3$
$(0,1/2,3/4)$	8	$2/3$

TABLE 2. The points of V inside $\mathcal{P}$.

Now we construct our auxiliary graph $\tilde{G} = (\tilde{V}, \tilde{E})$, where as usual, $\tilde{V} = V$. The edges of $\tilde{G}$ are defined as follows: for $x, y \in V$,

$$(x, y) \in \tilde{E} \Leftrightarrow 0 < \|x - y\|_{\mathcal{P}} \leqslant 1/2.$$

In particular, $\tilde{G}$ is a Cayley graph, since the set of points x satisfying $0 < \|x\|_{\mathcal{P}} \leqslant 1/2$ generates V.

We shall study the possible connected components in $\tilde{G}$ of a set $A \subset V$ avoiding polytope distance 1. First we prove that the diameter of these components must be strictly less than 1:

LEMMA 28. *Let $A \subset V$ be a set avoiding polytope distance 1, and let $\mathcal{C}$ be the partition of A into connected components in $\tilde{G}$. Then, for any $C \in \mathcal{C}$,*

$$\mathrm{Diam}(C) < 1.$$

PROOF. To prove this lemma, it is enough to check that every path in $\tilde{G}$ that goes from 0 to a point outside $\mathcal{P}$ must hit the boundary of $\mathcal{P}$. Following the definition of $\tilde{G}$, the neighbors of 0 are the vectors of polytope norm 1/3 and 1/2. Following table 2, the neighbors of these points having a greater polytope norm are all the points having polytope norm 2/3, and some points on the boundary of $\mathcal{P}$. Finally, all the further neighbors of the points at polytope distance 2/3 from 0 have polytope norm 1, which is sufficient to conclude the proof. □

As a consequence, there are, up to translation, only finitely many possible connected components appearing in the decomposition of A. It is not hard to compute, up to symmetry, all the connected components containing 0, and to see that their size is upper bounded by 8. In Table 3, we give the number, up to symmetry, of types of possible connected component containing 0 of each size.

Size of the connected component	1	2	3	4	5	6	7	8
Number of orbits	1	3	6	14	16	13	5	2

TABLE 3. The numbers of types connected components containing 0 of each size.

4.3. Computation of $\bar{\alpha}(G)$

In order to prove Theorem 13, we introduce a discrete distribution function associated with the partition of a set $A \subset V$ avoiding 1 into connected components in $\tilde{G}$ and prove that the local bound provided by this function, and as a consequence $\bar{\alpha}(G)$, are precisely 1/8:

THEOREM 14. *Let $A \subset V$ be a set avoiding 1, and let $\mathcal{C}$ be its partition into connected components in $\tilde{G}$. Let x be an element of V. Define*

$$d_x = d_{\mathcal{P}}(x, A).$$

Depending on the value of d_x, we also define

$$n_x = \begin{cases} |\{C \in \mathcal{C} \mid d_{\mathcal{P}}(x, C) = d_x\}| & \text{if } d_x \leqslant 1/2, \\ |\{C \in \mathcal{C} \mid d_{\mathcal{P}}(x, C) = d_x \text{ and } |C| \geqslant 4\}| & \text{otherwise.} \end{cases}$$

Finally, we introduce the following discrete distribution f of V associated with $\mathcal{C}$:

$$\begin{array}{rcccl} f & : & \mathcal{C} \times V & \to & [0,1] \\ & & (C, x) & \mapsto & \begin{cases} \dfrac{1}{n_x} & \text{if } d_{\mathcal{P}}(x, C) = d_x \text{ and } n_x \geqslant 1 \\ 0 & \text{otherwise} \end{cases} \end{array}.$$

Then

$$\delta_f^* = \frac{1}{8}.$$

Following Lemma 26, Theorem 13 is an immediate consequence of Theorem 14. Before entering into the details of the proof of the latter, it is worth explaining the underlying idea of the definition of the distribution function f: for a small connected component C, the contribution of the points close to C suffices to have $\delta_f(C) \leqslant 1/8$, while a bigger component will need the contribution of further points. In order to optimize this contribution, such points are only shared between big connected components.

Rather than presenting directly the proof of Theorem 14, we split it into several lemmas. Their proofs require some computations that have been made by computer. In order to keep the writing as light as possible, we only give here the most important results, and postpone the results of intermediate computations and checks to Appendix A.

As a first step, we deal with the connected components of small size:

LEMMA 29. *Let $C \in \mathcal{C}$, with $|C| \leqslant 3$. Then*

$$\delta_f(C) \leqslant 1/8.$$

PROOF. Following the definition of f, the elements of V that contribute to the f-volume of $N_f[C]$ are the points at polytope distance 1/3 and 1/2 from C. If such a point is shared with another connected component C', then C' must be at distance 2/3 from C. By looking at all the possible admissible configurations of such connected components around C, and computing the local density of C in this configuration, we check that this density cannot exceed 1/8. In Table 1, we recap, for each type of connected component, up to symmetry, the maximal local density that it can reach. □

From now on, we focus on big connected components. First of all, we show that they cannot be too close from each other:

LEMMA 30. *Let $C, C' \in \mathcal{C}$, with $|C| \geqslant 4$ and $|C'| \geqslant 4$. Then*

$$d_{\mathcal{P}}(C, C') > 1.$$

PROOF. Obviously, the distance between C and C' cannot be 1, and must exceed $1/2$, because C and C' are disjoint connected components. So we only have to check that it cannot be $2/3$. For all the connected components C such that $|C| \geqslant 5$, one can check that every point at polytope distance $2/3$ from C is at polytope distance 1 from another element of C, and as a consequence cannot be in C'. This property is also satisfied by all the types of connected component of size 4 but one. For this last remaining case, we check that the two points at distance $2/3$ from C that are allowed by C do not have any admissible neighbor, so that $|C'| = 1$. $\square$

Using Lemma 30, we get an upper bound on the number n_x when $d_{\mathcal{P}}(x, A) = 2/3$:

LEMMA 31. *Let $x \in V$ such that $d_{\mathcal{P}}(x, A) = 2/3$. Then*

$$n_x = \{C \in \mathcal{C} \mid d_{\mathcal{P}}(x, C) = 2/3 \text{ and } |C| \geqslant 4\} \leqslant 2.$$

PROOF. Without loss of generality, we may assume that $x = 0$. Following Lemma 30, it is sufficient to understand how many points can be picked among the ones at distance $2/3$ from x, in such a way that the distance between any two of them is strictly bigger than 1. By looking at these 18 points (see Table 2), one can easily check that if the third coordinates of u and v have same sign, then $\|u - v\|_{\mathcal{P}} \leqslant 1$. So x will be at most shared among two big connected components. $\square$

Finally, we can show that the local density of the big connected components is also upper bounded by $1/8$:

LEMMA 32. *Let $C \in \mathcal{C}$, with $|C| \geqslant 4$. Then*

$$\delta_f(C) \leqslant 1/8.$$

PROOF. We have seen in the proof of Lemma 30 that whenever $|C| \geqslant 5$, no point of A can be located at polytope distance $2/3$ from C. As an immediate consequence, for all x in the closed neighborhood of C in $\tilde{G}$, $f(C, x) = 1$ and $f(x, C') = 0$ for $C \neq C'$. Unfortunately, this closed neighborhood is not in general large enough to conclude.

Fix a connected component $C \in \mathcal{C}$, with $|C| \geqslant 4$. After the closed neighborhood of C in $\tilde{G}$, the next points that can bring a contribution to the f-volume of $N_f[C]$ are the ones at polytope distance $2/3$ from C. Some of these points have no neighbor in $\tilde{G}$ that are not at polytope distance 1 from an element of C. In other words, they must be in the f-neighborhood of C. Moreover, following Lemma 31, this contribution is at least $1/2$. We denote by $M(C)$ the set made by those points. Consequently the contribution of $M(C)$ in the

f-volume of $N_f(C)$ is at least $\frac{1}{2}|M(C)|$. Finally, we just have to check that if we add the minimal contributions brought by the closed neighborhood of C and by $M(C)$, the local density of C with respect to f cannot exceed $1/8$. In tables 2, 3, 4, 5 and 6, we give all the bounds for every type of connected components. We also may note that our results show, in particular, that a set of density $1/8$ is uniquely made by connected components of size 8. □

So we have proved Theorem 14:

Proof of Theorem 14. It is an immediate consequence of Lemma 29 and Lemma 32. □

5. Hexagonal Prisms

In this section, we use our results concerning hexagonal Voronoï cells in the plane in order to deduce easily the number $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ for a kind of polytope in dimension 3 very similar to the hexagons: hexagonal prisms.

A lattice whose Voronoï cell is a hexagonal prism is a lattice of the form (see Chapter 2, subsection 3.3)

$$\Lambda = \Lambda_0 \times t\mathbb{Z} \subset \mathbb{R}^2 \times \mathbb{R},$$

where $\Lambda_0 \subset \mathbb{R}^2$ is a planar lattice whose Voronoï cell is a hexagon $\mathcal{H}$. We may assume for instance that $t = 2$: then the Voronoï cell of Λ is

$$\mathcal{P} = \mathcal{H} \times [-1, 1],$$

and the polytope norm $\|\cdot\|_{\mathcal{P}}$ of an element $v = (X, z) \in \mathbb{R}^2 \times \mathbb{R}$ is given by

$$\|v\|_{\mathcal{P}} = \max\{\|X\|_{\mathcal{H}}, |z|\}.$$

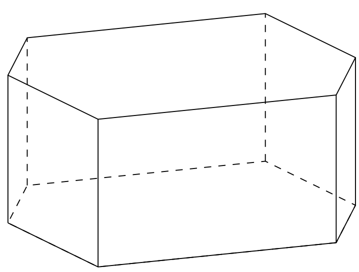


Figure 12. A hexagonal prism.

We are going to prove:

Theorem 15. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a hexagonal prism. Then*

$$m_1(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}}) = \frac{1}{8}.$$

5.1. A first natural proof

For $i \in \{0,1\}$, let us denote by H_i the hyperplane

$$H_i = \{(X,z) \in \mathbb{R}^2 \times \mathbb{R} \mid z = i\},$$

and by $[-R,R]_i^2$ the 2-dimensional square $[-R,R]^2 \times \{i\}$.

Let us consider the subgraph $G_{\{0,1\}}$ of $G(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}})$ induced by $H_0 \cup H_1$. If A is a subset of $H_0 \cup H_1$, we denote by A_i the intersection $A \cap H_i$, for $i \in \{0,1\}$. Then, the *density* of A in $H_0 \cup H_1$ is

$$\delta_{\{0,1\}}(A) = \frac{1}{2}(\delta_{H_0}(A_0) + \delta_{H_1}(A_1)),$$

where for $i \in \{0,1\}$, δ_{H_i} is the 2-dimensional density

$$\delta_{H_i} = \limsup_{R \to \infty} \frac{\mathrm{Vol}(A_i \cap [-R,R]_i^2)}{\mathrm{Vol}([-R,R]_i^2)}.$$

If we introduce the number

$$m_1(G_{\{0,1\}}) = \sup\{\delta_{\{0,1\}}(A) \mid A \subset H_0 \cup H_1,\ A \text{ avoiding } \mathcal{P} - \text{distance } 1\},$$

the following analogue of Lemma 10 holds:

$$m_1(G(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}})) \leqslant m_1(G_{\{0,1\}}).$$

The graph $G_{\{0,1\}}$ is the union of two copies of the 2-dimensional unit distance graph $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}})$, at $\mathcal{P}$-distance 1 one from the other. We are going to show that the density of an independent set in $G_{\{0,1\}}$ cannot exceed one half of the supreme density of a set avoiding $\mathcal{H}$-distance 1 in $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}})$, namely $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. To do so, to any $A \subset H_0 \cup H_1$ avoiding $\mathcal{P}$-distance 1, we associate another subset, having the same density as A in $H_0 \cup H_1$ (see Figure 13):

LEMMA 33. *Let us consider the natural projection:*

$$\begin{array}{cccc} \pi & : & H_0 \cup H_1 & \to & H_0 \cup H_1 \\ & & (X,z) & \mapsto & (X,0) \end{array}.$$

Let $A \subset H_0 \cup H_1$ be a set avoiding polytope distance 1. Then the restriction $\pi_{|A}$ of π to A is injective, $\pi(A)$ is again a set avoiding 1, and A and $\pi(A)$ have the same density in $H_0 \cup H_1$.

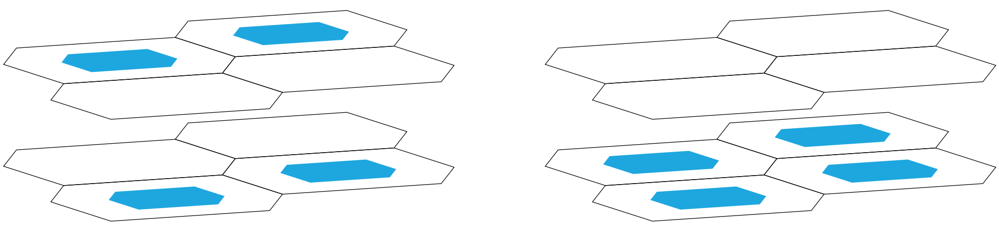


FIGURE 13. A set A and its projection $\pi(A)$.

PROOF. Suppose $\pi(X_1, z_1) = \pi(X_2, z_2)$, then $X_1 = X_2$, and if $z_1 \neq z_2$, then $|z_1 - z_2| = 1$, so that $\|(X_1, z_1) - (X_2, z_2)\|_{\mathcal{P}} = 1$. A contradiction.

In the same manner, if $\|\pi(X_1, z_1) - \pi(X_2, z_2)\|_{\mathcal{P}} = 1$, then $\|X_1 - X_2\|_{\mathcal{H}} = 1$, which would imply $\|(X_1, z_1) - (X_2, z_2)\|_{\mathcal{P}} = 1$, since $|z_1 - z_2| \leqslant 1$. So $\pi(A)$ still avoids polytope distance 1 with respect to $\mathcal{P}$.

Finally, because $\pi_{|A}$ is injective, A and $\pi(A)$ have the same density in $H_0 \cup H_1$. $\square$

As a consequence, we only have to bound the density of $\pi(A)$ in $H_0 \cup H_1$. This leads us to the bound $m_1(G_{\{0,1\}}) \leqslant 1/8$, which immediately implies Theorem 15:

THEOREM 16. *The number $m_1(G_{\{0,1\}})$ satisfies:*

$$m_1(G_{\{0,1\}}) \leqslant \frac{1}{8}.$$

PROOF. Let $A \subset H_0 \cup H_1$ be a set avoiding $\mathcal{P}$-distance 1. Following Lemma 33, we may replace A with $\pi(A)$, and $\pi(A)$ is nothing but a set avoiding $\mathcal{H}$-distance 1 in H_0. So its density in H_0 is upper bounded by 1/4, following Theorem 12. Since the density of H_0 in $H_0 \cup H_1$ is obviously 1/2, the density of $\pi(A)$ in $H_0 \cup H_1$, and consequently that of A, is upper bounded by 1/8. $\square$

5.2. Reformulation in terms of discrete distribution function

We can interpret the previous result in terms of discrete distribution functions, by using the graphs that we employed in order to prove Theorem 12.

Let $G_{\mathcal{H}}$ and $\tilde{G_{\mathcal{H}}}$ be respectively the induced subgraph of $G(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}})$ that we have introduced in Section 3, and its associated auxiliary graph. Let G be the subgraph of $G(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ induced by

$$V = V_{\mathcal{H}} \times \{0, 1\},$$

where $V_{\mathcal{H}}$ is the set of vertices of $G_{\mathcal{H}}$ and $\tilde{G_{\mathcal{H}}}$. We denote by V_i for $i \in \{0, 1\}$ the subset $\{v = (X, z) \in V \mid z = i\}$, so that $V = V_0 \cup V_1$.

The discrete analogue of Lemma 33, whose proof is exactly the same, is the following lemma:

LEMMA 34. *Let us consider the natural projection:*

$$\begin{array}{cccc} \pi & : & V & \to & V \\ & & (X, z) & \mapsto & (X, 0) \end{array}.$$

Let $A \subset V$ be a set avoiding polytope distance 1. Then the restriction $\pi_{|A}$ of π to A is injective, $\pi(A)$ is again a set avoiding 1, and A and $\pi(A)$ have the same density in V.

Then we can construct a discrete distribution of V with respect to $\pi(A)$, which provides another proof of Theorem 15:

THEOREM 17. *Let $A \subset V$ be a set avoiding polytope distance 1, and let $A_0 = \pi(A)$. By identifying V_0 and $V_{\mathcal{H}}$, we can put on V_0 the auxiliary graph structure of $\tilde{G}_H$. Let $\mathcal{C}$ be the decomposition of A_0 into connected components of this graph, and let $f_0 : \mathcal{C} \times V_0 \to [0,1]$ the associated discrete distribution function introduced in Theorem 12. Then the function*

$$\begin{array}{rcrcl} f & : & \mathcal{C} \times V & \to & [0,1] \\ & & (C,v) & \mapsto & f_0(C,\pi(v)) \end{array}$$

is a discrete distribution function of V with respect to $\mathcal{C}$, and satisfies

$$\delta_f^* = \frac{1}{8}.$$

PROOF. First of all, let us explain the underlying idea in the definition of f: we want the f-volume of the f-neighborhood in G of a connected component to be twice the f_0-volume of its f_0-neighborhood in $G_{\mathcal{H}}$ (see Figure 14, and compare with Figure 9).

It is clear that f is a discrete distribution function, since for every $v \in V$, $\sum_{C\in\mathcal{C}} f(C,v) = \sum_{C\in\mathcal{C}} f_0(C,\pi(v)) \leqslant 1$. It is also easy to see, following the definition of f, that for any $C \in \mathcal{C}$,

$$\sum_{v\in V} f(C,v) = 2\sum_{v\in V_0} f_0(C,v) \leqslant 2\frac{|C|}{\delta_{f_0}^*},$$

so that, following Theorem 12,

$$\delta_f^* = \frac{1}{2}\delta_{f_0}^* = \frac{1}{8}.$$

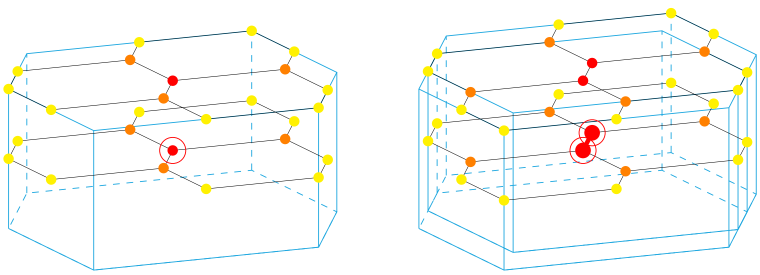


FIGURE 14. For one point in $G_{\mathcal{H}}$ contributing to a clique, there are two points giving the same contribution in G, so that the local density of a clique is reduced by one half. This is the analogue in that situation of Figure 9.

□

5.3. Generalizations

The method that we have just presented with hexagonal prisms in dimension 3 admits an immediate generalization. Indeed, if we replace the hexagon $\mathcal{H}$ with any polytope $\mathcal{P}_0$ in $\mathbb{R}^n$, then the product $\mathcal{P} = \mathcal{P}_0 \times [-1,1]$ satisfies $m_1(\mathbb{R}^{n+1}, \|\cdot\|_{\mathcal{P}}) = \dfrac{1}{2} m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}_0})$.

THEOREM 18. *Let $\mathcal{P}_0$ be a parallelohedron in $\mathbb{R}^n$, and let*

$$\mathcal{P} = \mathcal{P}_0 \times [-1,1] \subset \mathbb{R}^{n+1}.$$

Then

$$m_1(\mathbb{R}^{n+1}, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2} m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}_0}).$$

With a trivial induction, $[-1,1]$ may be replaced with any hypercube:

THEOREM 19. *Let $\mathcal{P}_0$ be a parallelohedron in $\mathbb{R}^n$, and let*

$$\mathcal{P} = \mathcal{P}_0 \times [-1,1]^m \subset \mathbb{R}^{n+m}.$$

Then

$$m_1(\mathbb{R}^{n+m}, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2^m} m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}_0}).$$

Chapter 5

Concluding Comments and Open Questions

This last chapter provides a quick overview of the results contained in this thesis, and presents some related open questions. It is divided in two parts, corresponding to the two main problems tackled.

1. Lattice Sphere Packings

In Chapter 2, we have constructed finite families of lattices containing a lattice whose sphere packing density reaches the highest asymptotic lower bound Δ_n. Besides the most natural challenge of improving Δ_n, there are several other open questions related to this topic, that seem to be more approachable.

The lattices that we have constructed are invariant under the action of a cyclic group. It is an interesting question whether larger symmetry groups can also lead to dense lattices. Such constructions may be explored with the goal of building smaller families of lattices with a density exponentially decreasing in the dimension. More precisely, given a finite group $G \subset O(\mathbb{R}^n)$, we would like to construct finite families of lattices in $\mathbb{R}^n$ invariant under the action of G and containing a lattice Λ whose density $\Delta(\Lambda)$ satisfies

$$\Delta(\Lambda) \geqslant \alpha^{-n},$$

for some α. In the context of coding theory a result of this flavor has been obtained by Bazzi in [**BM06**].

Averaging arguments over families of lattices obtained via Construction A have been used in the literature for problems different from the sphere packing problem, such as the covering problem, or problems related with communications (see for example [ELZ05]). It is also an interesting challenge, for these different questions, to try to reduce the size of the families, by adding some algebraic structure to the lattices.

2. Sets Avoiding Distance 1

In this thesis, we have proved Bachoc and Robins conjecture in dimension 2. Namely, for every parallelohedron $\mathcal{P} \subset \mathbb{R}^2$, $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = 1/4$. In dimension 3, we have proved this conjecture for several polytopes, covering four combinatorial types of 3-dimensional parallolehedra out of five. However, we only considered the elongated dodecahedron and the rhombic dodecahedron in their most symmetrical form. In order to settle the dimension 3, the number $m_1(\mathbb{R}^3, \|\cdot\|_{\mathcal{P}})$ should be studied for these polytopes in their general form, as well as for the last remaining type of polytope: the truncated octahedron. For the latter, we have a natural discrete induced subgraph G that is likely to satisfy $\bar{\alpha}(G) = 1/8$: the graph induced by the set of vertices obtained by translating all the centers of faces of the polytope by $\frac{1}{2}L$, where L is the translation lattice. However, this construction provides, in comparison with the previous polytopes, a larger amount of points inside the polytope, which makes harder the analysis of the graph. It seems to be worth to construct a smaller induced subgraph, easier to analyze, with independence ratio 1/8.

In Chapter 3, we have found an upper bound on $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ when $\mathcal{P}$ is the Voronoï region of the lattice D_n. In particular, for $n = 4$, this bound is 1/15, whereas the expected value of $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}})$ is 1/16. Even if the bound 1/15 is locally optimal for the Voronoï distribution in the graph that we considered, we did not find any set avoiding polytope distance 1 in the graph with a global density exceeding 1/16. It is still possible that with a good choice of distribution function, one could prove that the independence ratio of that graph is 1/16.

More generally, the would be to find a standard construction which, given a polytope $\mathcal{P}$ in dimension n, provides a discrete graph G, together with a discrete distribution function enabling to prove that $\bar{\alpha}(G) = 1/2^n$. Even though we may think that the construction involving the centers of faces of the polytope leads to a good set of vertices, we do not have a general good distribution function in order to analyze it.

After the norms whose unit ball is a parallelohedron, the natural step towards Euclidean norm is to consider norms whose unit ball is a polytope in general. For instance, in dimension 2, what is the value of $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}})$

for a given polytope $\mathcal{P}$? What is the chromatic number of the plane in that case? Does it bring any information on the chromatic number of the Euclidean plane?

Finally, a more general problem is to find upper bounds for $m_1(\mathbb{R}^n, \|\cdot\|)$ for any norm. We know that for the Euclidean norm, $m_1(\mathbb{R}^n, \|\cdot\|_2)$ decreases exponentially in n. Is that true in general? A possible way to attack this question is to use Fourier analysis and linear programming, as it was done in [**dOFV10**] and [**BPT15**] in the Euclidean case.

Chapter A

Intermediate computational results for the Elongated Dodecahedron

Here we give the results of the computations of the maximal local densities achieved by the possible connected components in the auxiliary graph that we have constructed for the elongated dodecahedron. The Sage code used to obtain these results is available on GitHub [1].

The following table concerns the small connected components (Lemma 29). Recall that the maximal density of a connected component C is computed, in that case, by checking all the configurations of connected components around C that have an influence on the contribution on the neighbors of C.

Size	Representative	Maximal Local Density
1	$\{(0,0,0)\}$	0.0938
2	$\{(0,0,0),(0,1/2,1/4)\}$	0.1112
2	$\{(0,0,0),(0,0,1/2)\}$	0.0834
2	$\{(0,0,0),(1/2,1/2,0)\}$	0.0953
3	$\{(0,0,0),(1/2,0,1/4),(0,1/2,3/4)\}$	0.0968
3	$\{(0,0,0),(1/2,0,1/4),(1/2,1/2,1/2)\}$	0.12
3	$\{(0,0,0),(0,1/2,-1/4),(0,1/2,1/4)\}$	0.1072
3	$\{(0,0,0),(1/2,0,-1/4),(1/2,1/2,0)\}$	0.1225
3	$\{(0,0,0),(0,0,1/2),(0,0,1)\}$	0.0909
3	$\{(0,0,0),(1/2,1/2,0),(1/2,1/2,1/2)\}$	0.0938

Table 1. The maximal local densities for the small connected components.

[1] https://gist.githubusercontent.com/PMoustrou

Table 2 gives the maximal local densities of the connected components of size 4. For such a component C, like for the smaller ones, we can compute the minimal contribution of the neighborhood of C to its local density. However, for one particular kind, it is not enough. The remaining contribution then comes from the points in $M(C)$ (see Chapter 4, Lemma 32).

Representative	$\lvert N[C]\rvert$	Maximal of $N(C)$ Contribution of $N(C)$	$\lvert M(C)\rvert$	Maximal Local Density
$\{(0,0,0),(0,0,1/2),$ $(0,1/2,-1/4),(0,1/2,3/4)\}$	38	38	26	0.0785
$\{(0,0,0),(0,0,1/2),$ $(0,1/2,-1/4),(0,1/2,3/4)\}$	35	35	30	0.08
$\{(0,0,0),(1/2,-1/2,0),$ $(1/2,-1/2,1/2),(1/2,0,3/4)\}$	39	39	30	0.0741
$\{(0,0,0),(0,0,1/2),$ $(0,1/2,1/4),(0,1/2,3/4)\}$	34	34	24	0.0870
$\{(0,0,0),(1/2,0,-3/4),$ $(1/2,0,-1/4),(1/2,1/2,0)\}$	37	37	30	0.0770
$\{(0,0,0),(1/2,-1/2,-1/2),$ $(1/2,-1/2,0),(1/2,0,1/4)\}$	37	37	30	0.0785
$\{(0,0,0),(0,0,1/2),$ $(0,0,1),(0,1/2,3/4)\}$	37	37	24	0.0817
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,1/4),(1/2,1/2,1/2)\}$	36	36	32	0.0770
$\{(0,0,0),(1/2,0,1/4),$ $(1/2,1/2,0),(1/2,1/2,1/2)\}$	34	34	30	0.0817
$\{(0,0,0),(0,1/2,1/4),$ $(1/2,0,1/4),(1/2,1/2,1/2)\}$	32	28	18	0.1082
$\{(0,0,0),(1/2,-1/2,0),$ $(1/2,0,-1/4),(1/2,0,1/4)\}$	32	32	30	0.0852
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,-1/4),(1/2,1/2,0)\}$	32	32	32	0.0834
$\{(0,0,0),(1/2,1/2,-1/2),$ $(1/2,1/2,0),(1/2,1/2,1/2)\}$	40	40	28	0.0741
$\{(0,0,0),(0,0,1/2),$ $(1/2,1/2,0),(1/2,1/2,1/2)\}$	38	38	28	0.0770

TABLE 2. The maximal local densities of the connected components of size 4. Note that for $C = \{(0,0,0),(0,1/2,1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$, 8 points in $N[C]$ can be shared, between at most two connected components. So the minimal contribution of $N[C]$ is in that case 28 instead of 32. There is no other similar case.

Tables 3, 4, 5 and 6 give the maximal local densities of the connected components of size bigger than 5. Recall that for all these components C the whole neighborhood of C fully contributes to the local density of C. The remaining contribution still comes from the points in $M(C)$.

Representative	$\lvert N[C]\rvert$	$\lvert M(C)\rvert$	Maximal Local Density
$\{(0,0,0),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,1/2,-1/2),(1/2,1/2,1/2)\}$	42	32	0.0863
$\{(0,0,0),(1/2,-1/2,0),$ $(1/2,-1/2,1/2),(1/2,0,-1/4),(1/2,0,3/4)\}$	43	31	0.0855
$\{(0,0,0),(0,0,1/2),$ $(0,1/2,-1/4),(0,1/2,1/4),(0,1/2,3/4)\}$	40	24	0.0962
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$	41	32	0.0878
$\{(0,0,0),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	38	30	0.0944
$\{(0,0,0),(0,0,1/2),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$	40	30	0.0910
$\{(0,0,0),(0,1/2,1/4),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$	38	32	0.0926
$\{(0,0,0),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,0,3/4),(1/2,1/2,1/2)\}$	41	30	0.0893
$\{(0,0,0),(0,0,1/2),$ $(1/2,-1/2,0),(1/2,-1/2,1/2),(1/2,0,3/4)\}$	42	30	0.0878
$\{(0,0,0),(0,1/2,-3/4),$ $(1/2,0,-3/4),(1/2,0,-1/4),(1/2,1/2,0)\}$	45	32	0.0820
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,-3/4),(1/2,0,-1/4),(1/2,1/2,0)\}$	40	32	0.0893
$\{(0,0,0),(1/2,0,-3/4),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,0)\}$	41	30	0.0893
$\{(0,0,0),(1/2,-1/2,-1/2),$ $(1/2,-1/2,0),(1/2,-1/2,1/2),(1/2,0,1/4)\}$	42	30	0.0878
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,1/4),(1/2,1/2,-1/2),(1/2,1/2,1/2)\}$	43	33	0.0841
$\{(0,0,0),(0,1/2,1/4),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$	39	30	0.0926
$\{(0,0,0),(0,1/2,1/4),$ $(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	36	32	0.0962

TABLE 3. The maximal local densities of the connected components of size 5.

Representative	$\lvert N[C]\rvert$	$\lvert M(C)\rvert$	Maximal Local Density
$\{(0,0,0),(0,1/2,-1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,-1/2),(1/2,1/2,1/2)\}$	45	33	0.0976
$\{(0,0,0),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,-1/2),(1/2,1/2,0),(1/2,1/2,1/2)\}$	44	30	0.1017
$\{(0,0,0),(1/2,-1/2,0),(1/2,-1/2,1/2),(1/2,0,-1/4),(1/2,0,1/4),(1/2,0,3/4)\}$	44	30	0.1017
$\{(0,0,0),(0,0,1/2),(1/2,-1/2,0),(1/2,-1/2,1/2),(1/2,0,-1/4),(1/2,0,3/4)\}$	46	31	0.0976
$\{(0,0,0),(0,1/2,-1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	42	32	0.1035
$\{(0,0,0),(0,0,1/2),(0,1/2,-1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)\}$	46	32	0.0968
$\{(0,0,0),(0,1/2,-1/4),(0,1/2,1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,1/2)]\}$	43	32	0.1017
$\{(0,0,0),(0,0,1/2),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	43	30	0.1035
$\{(0,0,0),(0,1/2,1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	40	32	0.1072
$\{(0,0,0),(0,1/2,1/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,0,3/4),(1/2,1/2,1/2)\}$	44	32	0.1
$\{(0,0,0),(0,1/2,-3/4),(0,1/2,-1/4),(1/2,0,-3/4),(1/2,0,-1/4),(1/2,1/2,0)\}$	46	32	0.0968
$\{(0,0,0),(0,1/2,-1/4),(1/2,0,-3/4),(1/2,0,-1/4),(1/2,0,1/4),(1/2,1/2,0)\}$	44	32	0.1
$\{(0,0,0),(0,0,1/2),(0,1/2,1/4),(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	40	32	0.1072

TABLE 4. The maximal local densities of the connected components of size 6.

Representative	$\|N[C]\|$	$\|M(C)\|$	Maximal Local Density
$\{(0,0,0),(0,1/2,-1/4),$ $(1/2,0,-1/4),(1/2,0,1/4),$ $(1/2,1/2,-1/2),(1/2,1/2,0),(1/2,1/2,1/2)\}$	46	32	0.1130
$\{(0,0,0),(0,1/2,-1/4),$ $(0,1/2,1/4),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,1/2,-1/2),(1/2,1/2,1/2)\}$	47	33	0.1103
$\{(0,0,0),(0,0,1/2),$ $(1/2,-1/2,0),(1/2,-1/2,1/2),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,0,3/4)\}$	47	30	0.1130
$\{(0,0,0),(0,0,1/2),$ $(1/2,-1/2,0),(1/2,-1/2,1/2),$ $(1/2,0,-1/4),(1/2,0,1/4),(1/2,0,3/4)\}$	47	32	0.1112
$\{(0,0,0),(0,1/2,-1/4),$ $(0,1/2,1/4),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	44	32	0.1167

TABLE 5. The maximal local densities of the connected components of size 7.

Representative	$\|N[C]\|$	$\|M(C)\|$	Maximal Local Density
$\{(0,0,0),(0,1/2,-1/4),$ $(0,1/2,1/4),(1/2,0,-1/4),(1/2,0,1/4),$ $(1/2,1/2,-1/2),(1/2,1/2,0),(1/2,1/2,1/2)\}$	48	32	0.125
$\{(0,0,0),(0,0,1/2),$ $(0,1/2,-1/4),(0,1/2,1/4),(1/2,0,-1/4),$ $(1/2,0,1/4),(1/2,1/2,0),(1/2,1/2,1/2)\}$	48	32	0.125

TABLE 6. The maximal local densities of the connected components of size 8.

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