Comportement en temps long d'équations de type Vlasov : études mathématiques et numériques

Romain Horsin

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UFR de Mathématiques

Comportement en temps long d'équations de type Vlasov: Etudes mathématiques et numériques.

Thèse soutenue à Rennes le 01/12/2017

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Comportement en Temps Long d'Equations de Type Vlasov

Etudes Mathématiques et Numériques

HORSIN Romain

Thèse de doctorat préparée à l'INRIA Rennes-Bretagne Atlantique et à l'Université de Rennes 1, sous la direction des professeurs Erwan FAOU et Frédéric ROUSSET

Résumé

Cette thèse porte sur le comportement en temps long de solutions d'équations de type Vlasov, principalement le modèle Vlasov-HMF. On s'intéressera en particulier au phénomène d'amortissement Landau, prouvé mathématiquement dans divers cadres, pour plusieurs équations de type Vlasov, comme l'équation de Vlasov-Poisson ou le modèle Vlasov-HMF, et présentant certaines analogies avec le phénomène d'amortissement non visqueux pour l'équation d'Euler 2D. Ce document a pour but de présenter et de prouver les principaux résultats obtenus par l'auteur et ses directeurs, à savoir:

- Un théorème d'amortissement Landau pour des solutions numériques du modèle Vlasov-HMF, obtenues par discrétisation en temps de ce dernier *via* des méthodes de splitting, ainsi que des estimations de convergence pour ces schémas.

- Un théorème d'amortissement Landau pour des solutions du modèle Vlasov-HMF linéarisé autour d'états stationnaires non-homogènes.

- L'étude numérique du théorème précédent au niveau non-linéaire par de nombreuses simulations.

- La convergence d'un schéma discrétisant en temps l'équation d'Euler 2D au moyen d'un intégrateur de Crouch-Grossman symplectique.

Mots-clés: Equations de type Vlasov, équation d'Euler, équations de transport, amortissement Landau, dispersion, état stationnaire, méthodes de splitting, méthodes semi-Lagrangiennes, intégrateur symplectique, intégrateur de Crouch-Grossman, analyse d'erreur rétrograde, systèmes hamiltoniens, coordonnées action-angle.

On Long Time Behavior of Certain Vlasov Equations

Mathematics and Numerics.

Abstract

This thesis concerns the long time behavior of certain Vlasov equations, mainly the Vlasov-HMF model. We shall be in particular interested in the celebrated phenomenon of Landau damping, proved mathematically in various frameworks, for several Vlasov equations, such as the Vlasov-Poisson equation or the Vlasov-HMF model, and exhibiting certain analogies with the inviscid damping phenomenon for the 2D Euler equation.This document is meant to describe and prove rigorously the main results obtained by the author and his advisors, which are the following :

- A Landau damping theorem for numerical solutions of the Vlasov-HMF model, constructed by means of time-discretizations by splitting methods, as well as convergence estimates for these numerical schemes.

- A Landau damping theorem for solutions of the Vlasov-HMF model linearized around inhomogeneous stationary states.

- The numerical confrontation of the previous result at the nonlinear level, by a large amount of simulations.

- The convergence of a scheme that discretizes in time the 2D Euler equation by means of a symplectic Crouch-Grossmann integrator.

Key-words: Vlasov equations, Euler equation, transport equations, Landau damping, scattering, stationary state, splitting methods, semi-Lagrangian methods, symplectic integrator, Crouch-Grossman integrator, backward error analysis, hamiltonian systems, angle-action variables.

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Organisation/Organization

Pour les lecteurs francophones: Le chapitre 1, rédigé en français, contient une introduction au sujet de recherche et une brève présentation des résultats obtenus durant cette thèse. Le reste du document est écrit en anglais, et est organisé comme suit:

- Le chapitre 2 est une présentation relativement détaillée des résultats obtenus par l'auteur et ses directeurs. Il contient les énoncés complets de ces résultats, ainsi que des résumés de preuves retenant les arguments essentiels.

- Les chapitres 3, 4 et 6 exposent les preuves complètes des résultats présentés dans le chapitre 2, et peuvent être chacun lus indépendamment du reste du manuscrit.

- Le chapitre 5 contient de nombreuses simulations numériques, qui ont pour but d'étudier numériquement les résultats du chapitre 4 au niveau non-linéaire.

For english-speaking readers:

- Chapter 1, written in french, is an introduction to the scientific context surrounding this thesis, and to our topics of interest. It is concluded by a short presentation of the new results obtained by the author and his advisors. The rest of the document is written in english.

- Chapter 2 introduces separately each one of these results. It contains complete statements, and sketches of proofs that retain the essential arguments.

- Chapter 3, 4 and 6 contain the complete proofs of the results presented in chapter 2. Each one of these chapters may be read independently from the rest of the document.

- Chapter 5 contains a large number of numerical simulations, and is designed to study numerically the main result of chapter 4 at the nonlinear level.

Contents

Chapter 1. Introduction **15**
1.1 Physique des plasmas & équations de Vlasov 16
1.1.1 Physique des plasmas 16
1.1.2 Théorie cinétique: les équations de Vlasov 17
1.1.3 Le modèle Vlasov-HMF 19
1.2 Amortissement Landau: panorama 20
1.2.1 Définition de l'amortissement Landau 20
1.2.2 Un exemple simple: le transport libre 22
1.2.3 Amortissement Landau pour Vlasov-HMF 22
1.2.4 Résultats pour l'équation de Vlasov-Poisson 26
1.2.5 A propos des états non-homogènes 27
1.3 Nouveaux résultats et quelques perspectives 28

Chapter 2. Main results of the thesis **33**
2.1 On numerical Landau damping for the Vlasov-HMF model 34
2.1.1 Time discretization of the Vlasov-HMF model 34
2.1.2 Main result 35
2.1.3 Sketch of proof for the estimate (2.1.6) 36
2.1.4 Sketch of proof for the estimates (2.1.7) and (2.1.8) 38
2.2 On linear Landau damping around inhomogeneous stationary states of the Vlasov-HMF model 40
2.2.1 Notations and settings 40
2.2.2 Main result 42
2.2.3 Sketch of proof for Theorem 2.2.2 45
2.3 On time-discretization of the 2D Euler equation by a symplectic Crouch-Grossman integrator 47
2.3.1 Notations and settings 48
2.3.2 Time-discretization and main result 48
2.3.3 Sketch of proof for Theorem 2.3.2 50

Chapter 3. On Numerical Landau Damping for Splitting Methods Applied to the Vlasov-HMF Model **53**
3.1 Introduction 54
3.2 Landau damping for the Vlasov-HMF model, main result 56

3.3 Backward error analysis . . . 60
3.4 Estimates . . . 62
3.4.1 Estimate of $M_{T,s-1}(\mathsf{z})$ . . . 62
3.4.2 Estimate of $N_{T,s,\nu}(\mathsf{g})$ . . . 66
3.4.3 Estimate of $\|\mathsf{g}\|_{\mathcal{H}^{s-4}_{\nu}}$ . . . 67
3.4.4 Proof of Theorem 3.3.3 and estimate (3.2.5) . . . 68
3.5 Proof of the convergence estimate (3.2.6) . . . 69
3.5.1 Estimate of $M_{T,s-3}(d)$ . . . 70
3.5.2 Estimate of $\|\delta\|_{\mathcal{H}^{s-6}_{\nu-1}}$ . . . 74
3.5.3 Estimate of $N_{T,s-2,\nu-1}(\delta)$ . . . 75
3.5.4 Proof of Proposition 3.5.1 . . . 76
3.6 Proof of the convergence estimate (3.2.7) for Strang splitting . . . 76
3.6.1 Estimate of $M_{T,s-4}(d)$ . . . 77
3.6.2 Estimate of $\|\delta\|_{\mathcal{H}^{s-7}_{\nu-2}}$ . . . 79
3.6.3 Estimate of $N_{T,s-3,\nu-2}(\delta)$ . . . 81
3.6.4 Proof of proposition 3.6.1 . . . 83
3.7 Appendix . . . 83
3.7.1 Proof of Lemma 3.4.3 . . . 83
3.7.2 Proof of Lemma 3.4.5 . . . 86

Chapter 4. On linear Landau damping around inhomogeneous stationary states of the Vlasov-HMF model **87**
4.1 Introduction . . . 88
4.2 Main results . . . 91
4.2.1 Action angle variables and applications . . . 91
4.2.2 Linear Landau damping . . . 95
4.2.3 About the Penrose criterion . . . 97
4.2.4 Organization . . . 98
4.3 Proof of Theorem 4.2.4 . . . 98
4.3.1 Study of the kernels $K_{\mathcal{C}}$ and $K_{\mathcal{S}}$ . . . 99
4.3.2 Study of the magnetization . . . 100
4.4 Proof of Corollary 4.2.5 . . . 105
4.5 Proofs of Theorem 4.2.6 and Proposition 4.2.8 . . . 106
4.6 Examples of stable stationary states . . . 111
4.6.1 Sufficient conditions of existence and stability . . . 111
4.6.2 Example of stable stationary states: Maxwell-Boltzmann distributions 113
4.7 Action-angle variables . . . 116
4.7.1 Elliptic integrals, elliptic functions, and elliptic trigonometry . . . 116
4.7.2 Action-angle variables on U_+ or U_- . . . 118
4.7.3 Action-angle in $U_\circ$ . . . 130

Chapter 5. Numerics **143**
5.1 Introduction & notations . . . 144

5.2 Numerical experiments . . . 148
5.2.1 Semi-Lagrangian scheme . . . 148
5.2.2 Tests to study the damping rates . . . 149
5.2.3 Some remarks on the numerical scheme . . . 150
5.3 Confrontation of Theorem 4.2.4 . . . 152
5.3.1 Inhomogeneous Maxwell-Boltzmann stationary states . . . 152
5.3.2 First numerical test . . . 153
5.3.3 Second numerical test . . . 155
5.4 Evolution of the distribution function - The case of Maxwell-Boltzmann Stationary States . . . 157
5.4.1 The homogeneous case . . . 157
5.4.2 Inhomogeneous case - first example . . . 159
5.4.3 Inhomogeneous case - second example . . . 161
5.4.4 Inhomogeneous case: the case of a small magnetization . . . 163
5.5 Evolution of the distribution function - The case of Lynden-Bell distributions 166
5.5.1 Lynden-Bell distributions - first example . . . 166
5.5.2 Lynden-Bell distributions - second example . . . 169
5.6 Evolution of the distribution function - The case Compactly Supported Stationary States . . . 172
5.6.1 The case where $\mathrm{Supp}(\eta) \subset U_\circ$ . . . 172
5.6.2 The case where $\mathrm{Supp}(\eta) = U_\circ$ . . . 175
5.6.3 The case where $U_\circ \subset \mathrm{Supp}(\eta)$ . . . 178
5.6.4 Smooth stationary state with compact support, first example . . . 181
5.6.5 Smooth stationary state with compact support, second example . . . 184
5.7 Examples of instability . . . 187
5.7.1 Bump-on-tail instability . . . 187
5.7.2 Violation of the Penrose criterion . . . 190
5.7.3 "Big" perturbations . . . 192

Chapter 6. On time-discretization of the 2D Euler equation by a symplectic Crouch-Grossman integrator 195
6.1 Introduction . . . 196
6.2 Main result . . . 197
6.2.1 Notations . . . 197
6.2.2 Functional framework . . . 199
6.2.3 Statement of the main result . . . 200
6.3 Stability estimates for the exact flows . . . 202
6.3.1 Notations . . . 202
6.3.2 A stability Lemma for some transport equations . . . 202
6.3.3 Stability estimates . . . 205
6.4 Numerical stability . . . 207
6.4.1 Properties of the midpoint flow . . . 207
6.4.2 Stability estimates . . . 212

6.5 Convergence estimates 218
6.5.1 Local errors 218
6.5.2 An *a priori* global error estimate 221
6.5.3 Convergence of the semi-discrete scheme 223
6.6 Appendix A: Solving Poisson's equation 223

Bibliography **225**

Chapter 1

Introduction

On s'intéresse dans cette thèse au comportement en temps long d'équations de type Vlasov, et plus particulièrement au célèbre phénomène d'amortissement Landau. Ce résumé se veut être une introduction générale à ce sujet, ainsi qu'une présentation du reste du manuscrit.
Dans la première partie, nous décrivons le contexte physique dans lequel sont établies les équations de type Vlasov, à savoir la théorie cinétique de la physique des plasmas. La seconde partie est une introduction à l'amortissement Landau. Nous y en donnons une définition, et décrivons précisément le phénomène dans les cas de l'équation de transport libre et de l'équation de Vlasov-HMF. Nous évoquerons également, dans de moindres détails, les divers résultats d'amortissement Landau de la littérature actuelle, et discuterons de questions ouvertes.
Nous conclurons par une présentation succincte des nouveaux résultats obtenus par l'auteur et ses directeurs, et discuterons de futures perspectives de recherche.

1.1 Physique des plasmas & équations de Vlasov

1.1.1 Physique des plasmas

Lorsqu'un gaz est chauffé à température suffisamment haute, les collisions entre atomes sont telles que les électrons se séparent des noyaux, donnant naissance à un ensemble hétéroclite de particules (ions, électrons, particules neutres, etc...), tout de même globalement neutre. Le mouvement des particules y est contraint par plusieurs facteurs: des champs électromagnétiques créés par les particules chargées distantes, les possibles collisions avec les particules voisines, etc ... Par analogie avec le plasma sanguin, un tel gaz ionisé est appelé un "**plasma**", nom qui est dû au physicien I. Langmuir. De par leur nature, et sans doute leur rareté sur terre, les plasmas sont souvent considérés comme étant un quatrième état de la matière, après les liquides, les gaz, et les solides.
Les plasmas sont néanmoins très répandus dans le reste de l'univers. Ils sont par exemple les constituants principaux des étoiles ou des vents solaires. Sur terre ils sont utilisés pour la fusion thermonucléaire, *ie* la production d'énergie par la fusion de deux particules en une particule plus lourde, un processus nécessitant en effet de hautes températures pour détacher les électrons des autres particules. Si les plasmas sont par exemple naturellement confinés par la masse d'une étoile, la question de leur confinement en temps long sur terre est toujours ouverte. Dans le cadre de la fusion thermonucléaire par exemple, le confinement est opéré dans une chambre appelée *tokamak*, au moyen d'un champ magnétique extérieur. Ces questions de confinement, et plus largement la fusion thermonucléaire, sont des sujets actuels de recherche. Dans cette optique, nous renvoyons le lecteur intéressé au site du célèbre projet ITER [1].

[1] https://www.iter.org

1.1.2 Théorie cinétique: les équations de Vlasov

Un plasma contient *a priori* des particules de type $\alpha \in \{1, 2, ..., M\}$, comme les électrons, les protons ou les particules neutres. La **théorie cinétique** signifie la description du plasma par la fonction de distribution $f_\alpha(t, x, v)$ de chaque type de particule α, où $t, x, v \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ sont les variables de temps, position, et vitesse, respectivement. La fonction $f_\alpha(t, x, v)$ représente le nombre de particules de type α se trouvant à un temps t dans un volume élémentaire $\mathrm{d}x\mathrm{d}v$ centré au point (x, v).
Nous pouvons citer l'exemple typique suivant

$$f_\alpha(v) = \exp\left(-\frac{\mathbf{m}_\alpha |v|^2}{2kT}\right), \tag{1.1.1}$$

où $\mathbf{m}_\alpha$ est la masse de la particule α, k la constante de Boltzmann, T la température du plasma, et $|\cdot|$ la norme Euclidienne sur $\mathbb{R}^3$. Il s'agit de la distribution de **Maxwell-Boltzmann** des vitesses, dans un plasma à l'équilibre thermodynamique.
Si les collisions dans le plasma sont assez fréquentes, les distributions de particules de type Maxwell-Boltzmann (1.1.1) sont maintenues sur des temps longs. Dans ce cas, la **théorie fluide** des plasmas, qui considère essentiellement des moyennes en vitesse de la fonction de distribution, est suffisante. Les quantités physiquement pertinentes, comme la densité ou la vitesse du fluide, sont alors fonctions de x et t. La théorie cinétique est indispensable dans le cas, par exemple, de plasmas à très hautes températures, où les collisions peuvent être suffisamment rares pour que la fonction de distribution de telle ou telle particule reste éloignée de la distribution de Maxwell-Boltzmann pendant des temps long. Dans ce cas, la vitesse v est la variable pertinente. Nous renvoyons le lecteur intéressé à [43] pour plus de détails.

Considérons le modèle N-corps suivant: la dynamique des particules $(x_n, v_n)_{1 \le n \le N}$ d'un certain type α est donnée par les lois

$$\frac{\mathrm{d}x_n(t)}{\mathrm{d}t} = v_n(t) \quad \text{et} \quad \frac{\mathrm{d}v_n(t)}{\mathrm{d}t} = \frac{\mathbf{F}(t, x_n(t), v_n(t))}{\mathbf{m}_\alpha}, \tag{1.1.2}$$

où $\mathbf{F}$ est la force extérieure, et $\mathbf{m}_\alpha$ la masse de la particule α. En l'absence de collisions, ou lorsque les effets collectifs dominent les collisions entre particules, la force extérieure est

$$\mathbf{F}(t, x, v) = \mathbf{q}_\alpha(\mathbf{E}(t, x) + v \times \mathbf{B}(t, v)), \tag{1.1.3}$$

où $\mathbf{E}$ est le champ électrique, $\mathbf{B}$ le champ magnétique, et $\mathbf{q}_\alpha$ la charge de la particule α.
La physique statistique permet d'obtenir, lorsque N tend vers l'infini, un modèle cinétique vérifié par la fonction de distribution $f_\alpha(t, x, v)$ de la α-ième particule, à savoir **l'équation de Vlasov**

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha + \frac{\mathbf{q}_\alpha}{\mathbf{m}_\alpha}(\mathbf{E} + v \times \mathbf{B}) \cdot \nabla_v f_\alpha = 0. \tag{1.1.4}$$

La partie libre de l'équation est l'équation de transport libre

$$\partial_t + v \cdot \nabla_x$$

donnant le mouvement des particules en l'absence de forces extérieures. La partie potentielle

$$\frac{\mathbf{q}_\alpha}{\mathbf{m}_\alpha}(\mathbf{E} + v \times \mathbf{B}) \cdot \nabla_v$$

décrit l'action des forces extérieures.
Le champ électromagnétique $(\mathbf{E}, \mathbf{B})$ est produit par les particules selon les **équations de Maxwell**:

$$\begin{cases} \nabla_x \cdot (\varepsilon_0 \mathbf{E}(t,x)) = \sigma(t,x), \quad \nabla_x \cdot \mathbf{B}(t,x) = 0, \\ \nabla_x \times \mathbf{B}(t,x) = \mu_0 j(t,x) + \dfrac{1}{c^2}\partial_t \mathbf{E}(t,x), \quad \nabla_x \times \mathbf{E}(t,x) = -\partial_t \mathbf{B}(t,x), \end{cases} \tag{1.1.5}$$

où ε_0 et μ_0 désignent respectivement la permittivité et perméabilité du vide, et où c est la vitesse de la lumière dans le vide. Les symboles ∇, $\nabla\cdot$ et $\nabla\times$ désignent respectivement les opérateurs gradient, divergence, et rotationnel, en dimension 3. La densité de charge σ et la densité de courant j sont quant à elles données par

$$\sigma(t,x) = \sum_{\alpha=1}^{M} \mathbf{q}_\alpha \int_{\mathbb{R}^3} f_\alpha(t,x,v)\mathrm{d}v \quad \text{et} \quad j(t,x) = \sum_{\alpha=1}^{M} \mathbf{q}_\alpha \int_{\mathbb{R}^3} v f_\alpha(t,x,v)\mathrm{d}v. \tag{1.1.6}$$

Les équations (1.1.4), (1.1.5) and (1.1.6) constituent le **système de Vlasov-Maxwell**.

De l'équation de Vlasov-Maxwell, on peut obtenir celle de Vlasov-Poisson pour la fonction de distribution $f(t,x,v)$ des électrons dans un plasma, lorsque l'on suppose que le champ électromagnétique vérifie les équations de Maxwell stationnaires:

$$\begin{cases} \nabla_x \cdot (\varepsilon_0 \mathbf{E}(t,x)) = \sigma(t,x), \quad \nabla_x \times \mathbf{E}(t,x) = 0, \\ \nabla_x \times \mathbf{B}(t,x) = \mu_0 j(t,x), \quad \nabla_x \cdot \mathbf{B}(t,x) = 0. \end{cases} \tag{1.1.7}$$

Les champs électriques et magnétiques sont découplés, et si l'on suppose de plus que la force de Lorentz $v \times \mathbf{B}(t,x)$ est négligeable, alors la seule force extérieure pertinente provient du champ électrique $\mathbf{E}(t,x)$. Dans ce cas, on ne considére que les premières et quatrièmes équations de Maxwell

$$\nabla_x \cdot (\varepsilon_0 \mathbf{E}(t,x)) = \sigma(t,x) \quad \text{et} \quad \nabla_x \times \mathbf{E}(t,x) = 0.$$

Le rotationnel de $\mathbf{E}$ étant nul, il existe un potentiel ϕ tel que $\mathbf{E}(t,x) = -\nabla_x \phi(t,x)$, et la première équation de Maxwell donne l'équation de **Poisson**

$$-\Delta_x \phi(t,x) = \varepsilon_0^{-1} \sigma(t,x),$$

où Δ est le Laplacien (la divergence du gradient). En supposant que seuls les électrons font osciller significativement la densité de charge, leur fonction de distribution $f(t,x,v)$ est

solution du système de **Vlasov-Poisson**

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \dfrac{\mathbf{e}}{m} \mathbf{E} \cdot \nabla_v f = 0 \\ \mathbf{E}(t,x) = -\nabla_x \phi(t,x) \\ \Delta_x \phi(t,x) = \dfrac{\mathbf{e}}{\varepsilon_0} \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \end{cases} \tag{1.1.8}$$

où m est la masse d'un électron, et $\mathbf{e}$, la charge élémentaire.

Pour conclure ce paragraphe, précisons que l'équation de Vlasov-Poisson fait aussi sens dans le contexte de la dynamique des galaxies (voir [18], [57], [58]), où l'interaction moyenne est donnée par la force d'attraction gravitationnelle de Newton, au lieu de la répulsion Coulombienne. Alors, l'équation the Vlasov-Poisson modélise la distribution d'étoiles de masses identiques dans une galaxie. $f(t,x,v)$ est ainsi la densité d'étoiles se situant près d'un point (x,v) au temps t, et vérifie

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathbf{E} \cdot \nabla_v f = 0 \\ \mathbf{E}(t,x) = -\nabla_x \phi(t,x) \\ \Delta_x \phi(t,x) = m\mathcal{G} \displaystyle\int_{\mathbb{R}^3} f(t,x,v) \mathrm{d}v, \end{cases} \tag{1.1.9}$$

où m est la masse d'une étoile, et $\mathcal{G}$ la constante de gravitation.

1.1.3 Le modèle Vlasov-HMF

Le modèle Vlasov-HMF (hamiltonian mean-field) est un modèle-jouet unidimensionnel qui conserve plusieurs caractéristiques des équations de Vlasov décrites ci-dessus. Il s'agit d'une simplification du modèle de Vlasov-Poisson particulièrement agréable pour effectuer des simulations numériques, et permettant d'effectuer un grand nombre de calcul analytiques. Notons qu'à cet effet ses états stationnaires sont des fonctions du Hamiltonien du pendule, comme nous aurons l'occasion de le voir ci-dessous. De ce point de vue, ce modèle apparait comme très pertinent dans la littérature physique (voir [4], [5], [6], [7], [21], [22]). Il présente aussi de nombreuses similitudes avec le modèle de Kuramoto pour les oscillateurs couplés, dans sa limite continue (voir [15], [32], [41]).
Le modèle de Vlasov-HMF est

$$\begin{cases} \partial_t f(t,x,v) + \{f, H[f]\}(t,x,v) = 0, \\ H[f(t)](x,v) = \dfrac{v^2}{2} - \Lambda \displaystyle\int_{\mathbb{T}\times\mathbb{R}} f(t,y,u) \cos(x-y) \mathrm{d}y \mathrm{d}u, \end{cases} \tag{1.1.10}$$

où $(t,x,v) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$, où $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, et où le crochet de Poisson $\{\cdot,\cdot\}$ est donné par

$$\{f,g\} = \partial_x f \partial_v g - \partial_x g \partial_v f.$$

Nous avons introduit un paramètre $\Lambda = \pm 1$ pour couvrir les cas de potentiels répulsifs et attractifs, par analogie avec les équations de Vlasov-Poisson qui modélisent respectivement les plasmas sans collisions et la dynamique des galaxies.
L'équation (1.1.10) est une équation de transport unidimensionnelle associée à un champ de vecteur hamiltonien séparable. D'un point de vue numérique, il s'agit ainsi d'un candidat bien adapté à une discrétisation en temps par des méthodes de splitting (voir [39] ou le chapitre 3).

1.2 Amortissement Landau: panorama

1.2.1 Définition de l'amortissement Landau

Considérons l'équation de Vlasov-Poisson adimensionnée, en dimension $d \geq 1$ d'espace, avec conditions de bord périodiques:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - E[f](t,x) \cdot \nabla_v f = 0, \\ E[f](t,x) = \Lambda \nabla_x \Delta_x^{-1} \left[\displaystyle\int_{\mathbb{R}^d} f(t,x,v) \mathrm{d}v - \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t,x,v) \mathrm{d}x \mathrm{d}v \right], \\ f(0,x,v) = f^0(x,v), \\ \Lambda = \pm 1, \end{cases} \tag{1.2.1}$$

avec $(t,x,v) \in \mathbb{R} \times \mathbb{T}^d \times \in \mathbb{R}^d$, où $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. Comme précédemment un paramètre $\Lambda = \pm 1$ permet de couvrir les deux applications à la physique des plasmas et à la dynamique des galaxies.
Dans le cadre de la régularité analytique (ou Gevrey) par exemple, on peut démontrer l'existence locale de solutions *via* un Théorème de Cauchy-Kovalevskaya (voir [62], [63]). L'existence globale peut être ensuite obtenue en propageant la régularité de plusieurs manières possibles. Parmi l'abondante littérature sur ce sujet, nous renvoyons par exemple à [14], [42] et [56], et également à [10] et [61] pour des discussions sur le sujet. Notre principal sujet d'intérêt est le comportement en temps long des solutions.
Les solutions de (1.2.1) qui nous intéressent sont à vrai dire celles de la forme

$$f(t,x,v) = \eta(v) + r(t,x,v), \tag{1.2.2}$$

où $\eta(v)$ est une solution stationnaire de (1.2.1) spatialement homogène, et où la perturbation de moyenne nulle, r, est petite dans un cadre fonctionnel convenable.
L'**amortissement Landau** désigne l'homogénéisation en temps long de telles solutions. Ce phénomène fut prédit en 1946 par L. Landau, qui, résolvant l'équation linéarisée autour de η au moyen de transformées de Fourier et Laplace, et, étudiant les singularités dans le domaine fréquentiel, conclut que le champ électrique $E[f](t,x)$ décroissait vers zero, à vitesse exponentielle (voir [51]).
Cela signifie physiquement que les oscillations de la charge sont amorties, et que le plasma retourne à la neutralité électrique en temps long. Notons que l'équation (1.2.1) satisfait les lois de conservations suivantes :

- Conservation de la masse:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)\mathrm{d}x\mathrm{d}v = 0.$$

- Conservation des normes L^p : pour tout $1 \leq p \leq \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d} |f(t,x,v)|^p\mathrm{d}x\mathrm{d}v = 0.$$

- Conservation de l'energie:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{T}^d\times\mathbb{R}^d} \frac{v^2}{2} f(t,x,v)\mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\mathbb{T}^d} |E[f](t,x)|^2\mathrm{d}x\right] = 0.$$

La conservation de l'énergie nous dit en particulier que l'amortissement du champ électrique est couplé à un gain d'énergie cinétique. Il est également notable que l'amortissement Landau prédise un phénomène irréversible, pour une équation temporellement réversible.
La décroissance vers zero du champ électrique peut être mathématiquement exprimée en terme de décroissance des modes de Fourier de

$$\rho(t,x) = \int_{\mathbb{R}^d} r(t,x,v)\mathrm{d}v, \tag{1.2.3}$$

la densité de la perturbation r. Plus précisément, définissons la transformée de Fourier d'une fonction f sur $\mathbb{T}^d \times \mathbb{R}^d$ par

$$\hat{f}_k(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(x,v)e^{-ik\cdot x}e^{-i\xi\cdot v}\mathrm{d}x\mathrm{d}v, \quad k \in \mathbb{Z}^d, \xi \in \mathbb{R}^d.$$

Notons alors que

$$\hat{\rho}_k(t) = \hat{r}_k(t,0),$$

et que

$$E[r(t)](x) = \Lambda\sum_{k\in\mathbb{Z}^d}(ik)|k|^{-2}\hat{\rho}_k(t)e^{ik\cdot x}.$$

Ainsi, nous prendrons pour définition de l'amortissement Landau la convergence vers zero des modes $\hat{\rho}_k(t)$:

$$\forall k \in \mathbb{Z}^d, \quad \lim_{t\to+\infty}|\hat{\rho}_k(t)| = 0. \tag{1.2.4}$$

Notons qu'avec une telle définition la vitesse de décroissance est *a priori* dépendante de la valeur de la fréquence spatiale k.
Ce type de résultat fut démontré rigoureusement au niveau non-linéaire par C. Mouhot et C. Villani dans [61], bien que des résultats antérieurs furent obtenus dans [20] et [46]. Les deux prochains paragraphes ont pour but d'introduire de manière concise les mathématiques se trouvant derrière l'obtention de résultats de type (1.2.4).

1.2.2 Un exemple simple: le transport libre

La perturbation (de moyenne nulle) r vérifie l'équation

$$\begin{cases} \partial_t r + v \cdot \nabla_x r + E[r](t,x) \cdot \nabla_v r + E[r](t,x) \cdot \nabla_v \eta = 0, \\ E[r](t,x) = \Lambda \nabla_x \Delta_x^{-1} \displaystyle\int_{\mathbb{R}^d} r(t,x,v) \mathrm{d}v, \\ r(0,x,v) = r^0(x,v). \end{cases} \tag{1.2.5}$$

Nous choisissons ici de présenter plus en détail l'amortissement Landau dans le cadre simple de la partie libre de (1.2.5), à savoir l'équation de transport libre

$$\partial_t r + v \cdot \nabla_x r = 0, \tag{1.2.6}$$

dont la solution est $r(t,x,v) = r^0(x - vt, v)$. Dans le cas présent où la variable d'espace appartient au tore, l'homogénéisation en temps long est la conséquence de l'effet de **mélange des phases** (the **phase-mixing effect**): la vitesse angulaire des particules évolue exactement comme la vitesse plane. La perturbation spirale dans l'espace des phases et s'homogénéise en temps long.
En variables de Fourier, l'équation (1.2.6) devient

$$\partial_t \hat{r}_k(\xi) - k \cdot \nabla_\xi \hat{r}_k(\xi) = 0, \tag{1.2.7}$$

qui se trouve être à nouveau une équation de transport. Dans ces variables, l'effet de mélange des phases décrit la transmission d'énergie depuis les basses fréquences spatiales k vers les hautes fréquences cinétiques à une vitesse proportionnelle à k, l'équation (1.2.7) ayant pour solution $\hat{r}_k(\xi) = \hat{r}_k^0(\xi + kt)$.
Mathématiquement il s'agit d'un échange de régularité en vitesse contre de la décroissance en Fourier. En effet, bien que le mode $k = 0$ soit préservé, pour une fréquence fixée ξ, tous les autres modes $\hat{r}_k(t,\xi)$ $(k \neq 0)$ décroissent vers zéro lorsque t tend vers l'infini, à une vitesse qui dépend de k et de la régularité de r^0 (selon v) (il suffit essentiellement d'utiliser le Lemme de Riemann-Lebesgue). Cela implique en particulier la décroissance vers zero des modes $\hat{\rho}_k(t)$, à une vitesse qui dépend de k et de la régularité de r^0.

1.2.3 Amortissement Landau pour Vlasov-HMF

Nous venons de présenter l'amortissement Landau pour la partie libre de l'équation (1.2.5), et le but de ce paragraphe est de présenter brièvement le cas de l'équation non-linéaire. Pour plus de cohérence avec le reste du document, où l'on s'intéresse principalement au modèle de Vlasov-HMF, nous choisissons d'évoquer ici brièvement le traitement de l'amortissement Landau non-linéaire par E. Faou et F. Rousset, dans le cadre du modèle Vlasov-HMF (1.1.10) (voir [40]). Nous renvoyons à [10, 61] pour le cas de l'équation de Vlasov-Poisson.
Considérons les espaces de Sobolev à poids, munis de la norme

$$\|f\|_{\mathcal{H}^s_\nu} = \left(\sum_{|p|+|q| \leq s} \int_{\mathbb{R}} (1 + |v|^2)^\nu \left| \partial_x^p \partial_v^q f(x,v) \right|^2 \mathrm{d}x \mathrm{d}v \right)^{1/2}, \tag{1.2.8}$$

avec $\nu > 1/2$ et $s \geq 0$. Mentionnons qu'il est assez aisé de prouver que le modèle Vlasov-HMF est globalement bien posé dans ces espaces, *via* des techniques classiques pour les équations de transports associées à un champ de vecteur de divergence nulle (voir par exemple [2]).
E. Faou et F. Rousset ont considéré des solutions du type

$$f(t,x,v) = \eta(v) + \varepsilon r(t,x,v), \quad f(0,x,v) = f^0(x,v) = \eta(v) + \varepsilon r^0(x,v), \tag{1.2.9}$$

avec η un état stationnaire spatialement homogène de (1.1.10), et $\varepsilon \in (0,1)$. L'analyse en temps long menée dans [40] est plus à rapprocher de celle de [10] que des travaux de C. Mouhot et C. Villani, au sens où la première étape est de filtrer l'effet du transport libre en considérant la fonction

$$g(t,x,v) = r(t,x+tv,v). \tag{1.2.10}$$

La fonction g vérifie alors

$$\begin{cases} \partial_t g = \{\phi(t,g), \eta + \varepsilon g\} \\ \phi(t,g)(x,v) = -\Lambda \displaystyle\int_{\mathbb{R}\times\mathbb{T}} (\cos(x-y+t(v-u)))\, g(t,y,u)\mathrm{d}y\mathrm{d}u, \quad \Lambda = \pm 1. \end{cases} \tag{1.2.11}$$

Le potentiel $\phi(t,g)$ satisfait la relation

$$\phi(t,g)(x,v) = -\frac{\Lambda}{2} \sum_{k=\pm 1} \zeta_k(t) e^{ikx} e^{iktv}, \quad \zeta_k(t) = \hat{g}_k(t,kt) = \hat{r}_k(t,0), \tag{1.2.12}$$

de sorte que sa décroissance en temps est liée à celle des modes ζ_k, comme évoqué ci-dessus pour le système de Vlasov-Poisson. Notons que dans le cas de Vlasov-HMF, seuls les modes $\zeta_{-1}(t)$ et $\zeta_1(t)$ sont non nuls.
Les coefficients $\zeta_k(t)$ sont les modes de Fourier de la densité de la perturbation et jouent pour Vlasov-HMF le même rôle crucial dans la dynamique en temps long que dans le cadre de l'équation de Vlasov-Poisson (voir [10, 61]). Ils vérifient en effet une équation intégrale de Volterra

$$\begin{aligned} \zeta_k(t) = \hat{g}_k(0,kt) &+ \int_0^t K(k,t-\sigma)\zeta_k(\sigma)\mathrm{d}\sigma \\ &+ \varepsilon\Lambda \sum_{\ell=\pm 1} \frac{1}{2} \int_0^t \zeta_\ell(\sigma)\hat{g}_{k-\ell}(\sigma, kt-\ell\sigma)k\ell(t-\sigma)\mathrm{d}\sigma, \end{aligned} \tag{1.2.13}$$

où le noyau K est défini par

$$K(k,t) = \Lambda k \frac{1}{2}\, kt\, \hat{\eta}_0(kt) \mathbb{1}_{\{t\geq 0\}}, \quad k = \pm 1. \tag{1.2.14}$$

Sous de bonnes hypothèses sur $K(k,t)$, l'équation de Volterra vérifiée par les modes ζ_k permet de contrôler leur décroissance en temps par celle des termes non-linéaires. Plus

précisément, introduisons la notion de stabilité suivante: η est dit linéairement stable s'il vérifie l'hypothèse suivante:

$$\textbf{(H)} \quad \eta(v) \in \mathcal{H}^5_3 \quad \text{et} \quad \exists\, \kappa > 0, \quad \inf_{\operatorname{Im}\tau \le 0} \left| 1 - \int_{\mathbb{R}} e^{-i\tau t} K(k,t) \mathrm{d}t \right| \ge \kappa, \quad k = \pm 1.$$

Nous appellerons "**critère de Penrose**" l'hypothèse **(H)**, ainsi que toute assertion de ce type, par analogie avec les énoncés similaires utilisés pour le système de Vlasov-Poisson dans [10, 61]. Dans le cas d'un potentiel répulsif ($\Lambda = -1$), elle est en particulier vérifiée par les états $\eta(v) = G(|v|)$ avec G décroissante, qui sont stables au sens de Lyapounov pour l'équation non-linéaire (voir [59]). Cette hypothèse de stabilité **(H)** permet en outre le contrôle de la partie linéaire de (1.2.13) par les termes non-linéaires, en résolvant cette équation en variable de Fourier temporelle, essentiellement.
Le point de vue d'E. Faou et F. Rousset dans [40] (voir aussi [10] pour Vlasov-Poisson) est de traiter les équations (1.2.11) et (1.2.13) comme un système fermé, controlé uniformément en temps par une norme tenant compte de la régularité Sobolev de g et de la décroissance des modes ζ_k.
Une des principales difficultés est la présence de résonances dans l'équation (1.2.13): lorsque $kt \sim \ell\sigma$, $\zeta_\ell(\sigma)$ a un effet important sur $\zeta_k(t)$ au sens où la décroissance du terme non-linéaire correspondant est faible comparée à celle des autres termes. Le contrôle de ces résonances, appelées aussi "**plasma echoes**", est crucial pour prouver l'amortissement Landau non-linéaire, comme expliqué dans [10, 61].
Posons

$$N_{T,s,\nu}(g) = \sup_{t\in[0,T]} \frac{\|g(t)\|_{\mathcal{H}^s_\nu}}{\langle t\rangle^3}, \quad M_{T,\gamma}(\zeta) = \sup_{t\in[0,T]} \sup_{k\in\{\pm1\}} \langle t\rangle^\gamma |\zeta_k(t)| \tag{1.2.15}$$

et

$$Q_{T,s,\nu}(g) = N_{T,s,\nu}(g) + M_{T,s-1}(\zeta) + \sup_{[0,T]} \|g(t)\|_{\mathcal{H}^{s-4}_\nu}, \tag{1.2.16}$$

où $\langle x\rangle = (1+|x|^2)^{1/2}$ pour $x \in \mathbb{R}$.
Dans [40], le résultat suivant est démontré

Théorème 1.2.1 (E. Faou & F. Rousset, [40]). *Fixons $s \ge 7$, $\nu > 1/2$ et $R_0 > 0$ tel que $Q_{0,s,\nu}(g) \le R_0$, et supposons que $\eta \in \mathcal{H}^{s+4}_\nu$ vérifie l'hypothèse* **(H)**. *Alors il existe $R > 0$ et $\varepsilon_0 > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0]$ et pour tout $T \ge 0$, la solution de (1.2.11) vérifie l'estimation*

$$Q_{T,s,\nu}(g) \le R.$$

La preuve du Théorème 1.2.1 utilise un argument de bootstrap: d'une borne *a priori* sur $Q_{T,s,\nu}(g)$ les auteurs en obtiennent une nouvelle qui dépend de manière adéquate de ε, et la propagent en temps en utilisant la petitesse de ε. Des estimations d'énergie sont menées sur l'équation (1.2.11) en bénéficiant de la structure de transport *via* un commutateur, permettant d'estimer les normes de Sobolev. La contribution $M_{T,s-1}(\zeta)$ est traitée *via* l'équation de Volterra (1.2.13), pour laquelle le résultat suivant est prouvé:

Lemme 1.2.1 (E. Faou & F. Rousset, [40]). *Soit $\gamma \geq 0$, et supposons que $\eta \in \mathcal{H}^{\gamma+3}$ vérifie* **(H)**. *Il existe $C > 0$ tel que pour tout $T > 0$,*

$$M_{T,\gamma}(\zeta) \leq C M_{T,\gamma}(F),$$

où

$$F_n(t) = \hat{g}_k(0, kt) + \varepsilon\Lambda \sum_{\ell=\pm 1} \frac{1}{2} \int_0^t \zeta_\ell(\sigma) \hat{g}_{k-\ell}(\sigma, kt - \ell\sigma) k\ell(t-\sigma) \mathrm{d}\sigma.$$

La décroissance en temps de $\hat{g}_k(0, kt)$ est une conséquence de l'effet de mélange des phases, tandis que la seconde contribution non-linéaire dans $M_{T,\gamma}(F)$ est contrôlée par des estimations intégrales tenant précisément compte des résonances localisées.
Puisque pour tout $t \geq 0$,

$$g(t, x, v) = g(0, x, v) + \int_0^t \{\phi(\sigma, g(\sigma)), \eta + \varepsilon g(\sigma)\}(x, v) \mathrm{d}\sigma, \tag{1.2.17}$$

le Théorème 1.2.1 permet en fait de montrer que le comportement en temps long de $g(t)$ est donné par une intégrale en temps convergente. Posant

$$g^\infty(x, v) = g(0, x, v) + \int_0^\infty \{\phi(\sigma, g(\sigma)), \eta + \varepsilon g(\sigma)\}(x, v) \mathrm{d}\sigma, \tag{1.2.18}$$

E. Faou et F. Rousset démontrèrent en effet l'estimation de scattering suivante

Théorème 1.2.2 (E. Faou & F. Rousset, [40]). *Fixons $s \geq 7$, $\nu > 1/2$ et supposons que $\eta \in \mathcal{H}^{s+4}_\nu$ vérifie l'hypothèse* **(H)**. *Supposons que $g(0, x, v)$ soit dans $\mathcal{H}^s_\nu$. Alors il existe $\varepsilon_0 > 0$ et une constante $C > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0]$, $g^\infty(x, v) \in \mathcal{H}^{s-4}_\nu$ et pour tout $r \leq s - 4$ et $r \geq 1$,*

$$\forall\, t \geq 0, \quad \|g(t, x, v) - g^\infty(x, v)\|_{\mathcal{H}^r_\nu} \leq \frac{C}{\langle t \rangle^{s-r-3}}. \tag{1.2.19}$$

Comme conséquence de ces résultats il est possible de montrer la décroissance polynomiale des modes de Fourier de $f(t, x, v) = \eta(v) + \varepsilon g(t, x - vt, v)$: pour tout $\alpha + \beta = s - 4$, $k \neq 0$ et $\xi \in \mathbb{R}$,

$$\left|\hat{f}_k(t, \xi)\right| = \varepsilon \left|\hat{g}_k(t, \xi + kt)\right| \lesssim \frac{\varepsilon}{\langle k \rangle^\alpha \langle \xi + kt \rangle^\beta}.$$

Cela implique la convergence faible

$$f(t, x, v) \rightharpoonup_{t \to +\infty} \eta(v) + \frac{\varepsilon}{2\pi} \int_{\mathbb{T}} g^\infty(x, v) \mathrm{d}x,$$

à vitesse polynomiale.
Il est à noter que la preuve d'E. Faou et F. Rousset ne traite en fait que le cas d'un potentiel répulsif $\Lambda = -1$. Il est cependant assez aisé de voir que leur analyse persiste dans le cas

d'un potentiel attractif $\Lambda = 1$. L'article [40] d'E. Faou et F. Rousset contient aussi une généralisation au cas d'un noyau d'interaction qui est un polynôme trigonométrique de degré fini, *ie* à support compact en variables de Fourier.
Enfin, mentionnons que si l'on ne s'intéresse qu'à la dynamique linéarisée, c'est à dire à l'équation

$$\zeta_k(t) = \hat{g}_k(0, kt) + \int_0^t K(k, t-\sigma)\zeta_k(\sigma)\mathrm{d}\sigma,$$

alors les résonances ne jouent aucun rôle, et la décroissance en temps des modes $\zeta_k(t)$ est uniquement la conséquence du critère de Penrose et de l'expression explicite de la solution du transport libre. Il suffit en effet d'appliquer le Lemme 1.2.1 pour contrôler la décroissance en temps de $\zeta_k(t)$ par celle de $\hat{g}_k(0, kt)$, laquelle est une conséquence de l'effet de mélange des phases, comme expliqué dans le paragraphe 1.2.2.

1.2.4 Résultats pour l'équation de Vlasov-Poisson

Un résultat de scattering similaire au Théorème 1.2.2 fut prouvé par J. Bedrossian, N. Masmoudi et C. Mouhot dans le cadre de la régularité Gevrey (voir [10]), impliquant également un résultat d'amortissement Landau non-linéaire. Leur preuve raccourcit celle du Théorème originel de C. Mouhot et C. Villani dans [61].
Parmi les différences qui peuvent être notées avec l'analyse en temps long de E. Faou et F. Rousset, la plus remarquable semble être le traitement des résonances (plasma echoes). Leur contrôle dans le cas de l'équation Vlasov-HMF, en utilisant la régularité Sobolev, est essentiellement possible car le potentiel est à support compact en Fourier. Dans le cas de Vlasov-Poisson, où il n'y a pas cette structure particulière, l'accumulation potentielle des résonances peut compliquer leur contrôle. Les auteurs de [10] ou [61] travaillent ainsi en régularité analytique ou Gevrey, plutôt qu'en régularité finie.
On peut ainsi se demander si le Théorème de C. Mouhot et C. Villani dans [61] peut être étendu dans le cadre de solutions à régularité finie, comme la régularité Sobolev. II fut prouvé dans [55] (en dimension $d = 1$) qu'il existait des ondes stationnaires à densité non nulle, appelées "ondes BGK", arbitrairement proches de la distribution de Maxwell-Boltzmann

$$\eta(v) = e^{-\frac{v^2}{2}}$$

dans $H^s(\mathbb{T} \times \mathbb{R})$ avec $s < 3/2$. Néanmoins, il fut aussi prouvé dans [55] que cet obstacle disparaissait pour des régularités Sobolev suffisamment hautes.
Des travaux récents de J. Bedrossian et I. Tristani (voir [8] et [73]) ont montré que les résonances pouvaient être supprimées dans le cadre d'une régime collisionnel faible, c'est à dire en ajoutant à l'équation de Vlasov-Poisson un petit paramètre $\delta > 0$ devant un opérateur de collision. Dans ce cadre J. Bedrossian et I. Tristani ont obtenu, indépendamment, des résultats d'amortissement Landau en régularité Sobolev, uniformes vis à vis du paramètre de collision δ.
Mentionnons également les récents travaux de J. Bedrossian, N. Masmoudi et C. Mouhot (voir [12]) où un résultat d'amortissement Landau est prouvé en régularité Sobolev dans le

cadre de l'équation de Vlasov-Poisson sur l'espace entier $\mathbb{R}^3 \times \mathbb{R}^3$, avec un champ électrique intégrable.
Pour terminer, nous faisons brièvement mention de similarités avec l'équation d'Euler bidimensionnelle, pour une perturbation de l'écoulement de Couette (Couette flow). En formulation vorticité, il s'agit de

$$\begin{cases} \partial_t \omega + y\nabla_x \omega + U \cdot \nabla \omega = 0, \\ U = \nabla^\perp \Delta^{-1} \omega, \quad \omega(t=0,x,y) = \omega_0(x,y), \end{cases} \tag{1.2.20}$$

où $(x,y) \in \mathbb{T} \times \mathbb{R}$, $\nabla^\perp = (-\partial_y, \partial_x)$, et où ω_0 est de moyenne nulle. Il est prouvé dans [9] que, si la donnée initiale ω_0 est assez petite dans une norme de Gevrey convenable, alors il existe $u_\infty(y)$ telle que le champ de vecteur vitesse exhibe le phénomène *d'amortissement non visqueux* suivant

$$U(t,x,y) \to (u_\infty(y), 0).$$

Ceci est à rapprocher de l'amortissement du champ électrique dans le cadre de l'amortissement Landau. Quelques différences sont notables, car dans ce nouveau cas la convergence est forte et à vitesse *algébrique*. Nous renvoyons le lecteur intéressé à [9] pour plus de détails.

1.2.5 A propos des états non-homogènes

Fut laissée ouverte par E. Faou et F. Rousset la question de la possibilité d'amortissement Landau autour des états stationnaires *non-homogènes*: on peut se demander s'il est possible de prouver une estimation de scattering dans le style de (1.2.19) en partant d'une donnée initiale du type

$$f^0(x,v) = \eta(x,v) + \varepsilon r^0(x,v), \tag{1.2.21}$$

où $\eta(x,v)$ est un état stationnaire non-homogène de (1.1.10). Les travaux susmentionnés sur l'amortissement Landau pour l'équation de Vlasov-Poisson ([8], [10], [12], [61], [73]) ne considèrent aussi que des états spatialement homogènes.
Nous considérons des solutions stationnaires non-homogènes de (1.1.10) du type

$$\eta(x,v) = G\left(h_0(x,v)\right), \quad h_0(x,v) = \frac{v^2}{2} - M_0 \cos(x). \tag{1.2.22}$$

pour une fonction G donnée, et où nous devons noter que h_0 est le Hamiltonien associé au système du pendule. M_0 est un réel positif, connu dans la littérature physique comme étant la *magnétisation* de η. Il est assez aisé de voir que η est un état stationnaire de (1.1.10) dès que M_0 vérifie l'équation

$$M_0 = \int_{\mathbb{T}\times\mathbb{R}} G\left(\frac{v^2}{2} - M_0 \cos(x)\right) \cos(x) \mathrm{d}x\mathrm{d}v. \tag{1.2.23}$$

Dans le chapitre 4, nous donnerons des exemples de fonctions G décroissantes pour lesquelles une solution M_0 strictement positive existe, dans le cas d'un potentiel attractif $\Lambda = 1$. Pour le moment, citons les exemples suivants, abondants dans la littérature physique:

- Distributions de Maxwell-Boltzmann (voir [5], [7], [28], [29]):

$$\eta(x,v) = \alpha e^{-\beta h_0(x,v)} \tag{1.2.24}$$

- Distributions de Lynden-Bell (voir [27]):

$$\eta(x,v) = \frac{\alpha}{1+e^{\beta h_0(x,v)}} \tag{1.2.25}$$

- Distributions de Tsallis à support compact (voir [24] et [27]):

$$\eta(x,v) = \alpha(E - h_0(x,v))^{\frac{1}{q-1}} \mathbb{1}_{\{E \geq h_0(x,v)\}}(x,v), \quad \text{with} \quad q > 1. \tag{1.2.26}$$

- Distributions Water-bag (voir [7]):

$$\eta(x,v) = \alpha \mathbb{1}_{\{E \geq h_0(x,v)\}}(x,v). \tag{1.2.27}$$

Notons que des solutions non-nulles de (1.2.23) n'existent que si les divers paramètres α, β, q, E vérifient certaines conditions.
Il existe dans les littératures physique et mathématique des critères de stabilité non-linéaire (voir [53] ou [4]), qui sont utilisés dans [53] pour prouver la stabilité orbitale de ces états stationnaires non-homogènes.
La possibilité d'un amortissement Landau autour de ces états est néanmoins une question ouverte, bien que récemment étudiée dans la littérature physique (voir [5] et [7]), dans le cas de l'équation linéarisée autour de η. L'approche est essentiellement basée sur le fait que h_0 est le Hamiltonien associé au système du pendule, et est donc en tant que tel intégrable. On peut ainsi considérer les **coordonnées angle-action** correspondantes (voir [45] ou le chapitre 4 pour une description précise de ces coordonnées). Le point important est que, dans ces nouvelles variables, on peut intégrer le flot de h_0 comme dans le cas du transport libre, et les auteurs de [5] et [7] peuvent alors résoudre l'équation linéarisée dans ces variables au moyen de transformées de Fourier et Laplace, dans le style de L. Landau ([51]). A cause de singularités dans le changement de variable, il est prédit par les auteurs que la vitesse de l'amortissement du champ électrique est algébrique, et pas meilleure que $1/t^2$ (ou $1/t^3$, dans le cas de certaines symétries).

1.3 Nouveaux résultats et quelques perspectives

Le point de départ de cette thèse fut la lecture des travaux d'E. Faou et F. Rousset dans [40] concernant l'amortissement Landau pour le modèle Vlasov-HMF, décrits dans la partie précédente. Par sa simplicité, le modèle Vlasov-HMF est particulièrement adapté à la réalisation de simulations numériques.

De ce point de vue, notre premier sujet d'intérêt fut la discrétisation en temps du modèle Vlasov-HMF (1.1.10) par des **méthodes de splittings** entre les parties linéaire et non-linéaire de l'équation (voir par exemple [45] pour des généralités sur ces méthodes). Nous avons principalement démontré que la conclusion du Théorème 1.2.2 de E. Faou et F. Rousset restait vraie après ces discrétisations temporelles, de sorte que les solutions numériques présentaient un *effet d'amortissement Landau numérique*. Nous avons aussi prouvé que l'état final numérique était proche de l'état final continu, ainsi que des estimations de convergence pour le schéma en temps.
Bien que la littérature mathématique abonde de résultats de convergence pour ces méthodes de splitting appliquées aux équations de type Vlasov (citons par exemple [17], [25], [34]), nous attirons l'attention que sur le fait que les nôtres décrivent en fait le comportement en temps long des solutions numériques, à savoir l'*amortissement Landau numérique*.
Ces travaux ont fait l'objet de la publication [39]. Ils sont présentés *via* une formulation complète des résultats et des idées de preuves dans la première partie du chapitre 2, et les preuves complètes sont exposées dans le chapitre 3.
Nous laissons ouverte la possibilité d'étendre nos résultats à une discrétisation complète, n'ayant traité que la discrétisation en temps. Le chapitre 3 contiendra une courte discussion sur ce sujet.

Dans un second temps nous nous sommes intéressés à la question de l'amortissement Landau autour d'états stationnaires non-homogènes du modèle Vlasov-HMF, décrite dans la partie précédente. Nous avons considéré des solutions de l'équation Vlasov-HMF (1.1.10) avec un potentiel attractif ($\Lambda = 1$) partant d'une donnée initiale donnée par (1.2.21) et (1.2.22), et avons prouvé un résultat d'amortissement Landau *linéaire* pour les solutions de l'équation linéarisée associée, avec une vitesse d'amortissement *algébrique*.
Dans le cas de solutions stationnaires homogènes ($M_0 = 0$), nous avons vu dans les paragraphes 1.2.2 et 1.2.3 que le phénomène d'amortissement Landau l'équation linéarisée s'analyse par l'usage de la transformée de Fourier et de l'expression explicite de la solution du transport libre. L'amortissement Landau est alors principalement la conséquence de l'effet de mélange des phases et du critère de Penrose.
Dans le cas présent de solutions stationnaires non-homogènes ($M_0 > 0$), nous nous ramenons à une situation similaire, en considérant les coordonnées action-angle associées au Hamiltonien h_0, dans lesquelles le flot de ce Hamiltonien s'intègre. Nous pouvons dans ces nouvelles coordonnées retrouver l'effet de mélange des phases. La contrepartie est que les singularités du changement de variables forcent la vitesse d'amortissement à être algébrique.
L'utilisation des coordonnées action-angle rapproche notre analyse de récents travaux de la littérature physique ([5] et [7]), discutés dans le paragraphe précédent. Néanmoins, notre approche diffère dans le sens où nous ne résolvons pas explicitement l'équation linéarisée au moyen de la transformée de Fourier, mais démontrons une estimation de stabilité pour des équations intégrales de Volterra, dans l'esprit du Lemme 1.2.1. Nous formulons un critère de stabilité linéaire de type Penrose, analogue à **(H)**, qui se trouve être relié aux critères de stabilités utilisés pour les travaux sur la *stabilité Orbitale* (voir [53]).

Ces travaux sont présentés *via* une formulation complète des résultats et des idées de preuves dans la second partie du chapitre 2, et les preuves complètes sont données dans le chapitre 4. Dans le chapitre 5, nous proposons des simulations permettant d'observer numériquement les vitesses d'amortissement algébriques, dans le cas de l'équation non-linéaire.
Un perspective de recherche serait l'extension de nos résultats au cas non-linéaire. A ce sujet, nous nous attendons, d'un point de vue purement technique, à ce que les singularités du changement en variables angle-action dans des régions précises de l'espace des phases soient un sérieux obstacle à un résultat d'amortissement Landau non-linéaire dans le style du Théorème 1.2.2. En effet, l'analyse du phénomène non-linéaire nécessite *a priori* de manipuler des solutions suffisamment régulières, comme discuté dans le paragraphe 1.2.3., pour pouvoir contrôler les résonances (plasma echoes). Nous craignons une perte de régularité qui serait uniquement due aux singularités locales du changement de variable. Cela suggère de s'intéresser éventuellement à des solutions à support compact, pour éviter les régions de l'espace des phases posant problème.
Dans cette optique, le chapitre 5 fut conçu pour nous aider, et aider le lecteur intéressé, à comprendre le cas non-linéaire et à mettre en lumière ces nouveaux phénomènes. Il contient de nombreuses simulations numériques qui nous montrent le comportement en temps long de plusieurs solutions numériques de l'équation de Vlasov-HMF non-linéaire. Cela illustrera en particulier l'ébauche de discussion ci-dessus concernant les singularités dans l'espace des phases et les possibles avantages à utiliser des solutions à support compact.

Enfin, nous nous sommes également intéressés à la discrétisation en temps de l'équation d'Euler bi-dimensionnelle (1.2.20), avec des conditions de bord périodiques. Plus précisément, nous construisons une approximation de la solution exacte ω par un **intégrateur de Crouch-Grossman** (voir [30]): sur un pas de temps, le champ de vecteur vitesse U est d'abord figé au temps initial pour le défaire de sa dépendance en la fonction ω. L'équation de transport "élémentaire" qui en résulte est alors discrétisée en temps *via* un intégrateur **symplectique**, à savoir le **point-milieu implicite**. Nous prouvons une estimation de convergence pour de tels schémas en temps, en norme de Sobolev.
Notre approche utilise essentiellement des estimations de stabilité classiques pour les équations de transport (voir par exemple [72], [69], [70] et [71]), qui nous permettent à la fois de contrôler la régularité des solutions exactes et numériques, et d'estimer les erreurs de consistance locales dues au figement du champ de vecteur vitesse et à la méthode du point-milieu. La préservation de la structure symplectique par l'intégrateur du point-milieu est également un ingrédient indispensable de notre preuve.
Ces travaux sont présentés *via* une formulation complète des résultats et des idées de preuves dans la troisième partie du chapitre 2, et les preuves complètes sont données dans le chapitre 6.
Nous laissons ouverte la possibilité d'une extension de notre résultat à des schémas complètement discrets, n'ayant traité que la discrétisation en temps. Travaillant dans le cadre de la régularité Sobolev avec conditions de bord périodiques, l'approche qui parait naturelle serait d'obtenir une discrétisation complète *via* de l'interpolation par les polynômes

trigonométriques, en utilisant les séries de Fourier discrètes. L'erreur *d'aliasing* inhérente à cette méthode d'interpolation nous empêche pour le moment de prouver la stabilité du schéma complètement discret.

Chapter 2

Main results of the thesis

This chapter is meant to be a concise presentation of the results obtained by the author and his advisors. In section 1, 2, 3, we describe the main results of chapters 3, 4, 6, respectively. For each of them we shall provide the reader with complete statements, and sketches of proofs that retain the essential arguments.

2.1 On numerical Landau damping for the Vlasov-HMF model

In this section we introduce the reader to the main result of chapter 3, which says essentially that the conclusion of Theorem 1.2.2 persists through time-discretization of the Vlasov-HMF model (1.1.10) by splitting methods. This work was the subject of the journal paper [39]. This section has connections with subsections 1.1.3 and 1.2.3 of the previous chapter, and we shall in particular use notations already defined there.

2.1.1 Time discretization of the Vlasov-HMF model

We consider time-discretizations of the Vlasov-HMF equation (1.1.10) by splitting methods between the free part

$$\begin{cases} \partial_t f(t,x,v) + v\partial_x f(t,x,v) = 0, \\ f(0,x,v) = f^0(x,v), \end{cases} \tag{2.1.1}$$

whose solution is explicitly given by $\varphi_T^t(f^0)(x,v) = f^0(x-vt,v)$, and the potential part

$$\begin{cases} \partial_t f(t,x,v) + \Lambda\partial_x \left(\displaystyle\int_{\mathbb{R}\times\mathbb{T}} \cos(x-y) f(t,y,u)\mathrm{d}y\mathrm{d}u \right) \partial_x f(t,x,v) = 0, \\ f(0,x,v) = f^0(x,v), \end{cases} \tag{2.1.2}$$

whose solution is explicitly given by $\varphi_P^t(f^0)(x,v) = f^0(x, v - tE[f^0,x])$, where

$$E[f,x] = \Lambda\partial_x \left(\int_{\mathbb{R}\times\mathbb{T}} \cos(x-y) f(y,u)\mathrm{d}y\mathrm{d}u \right)$$

is indeed kept constant during the evolution of (2.1.2).
We consider the following Lie splittings

$$f^{n+1} = \varphi_P^h \circ \varphi_T^h(f^n) \quad \text{or} \quad f^{n+1} = \varphi_T^h \circ \varphi_P^h(f^n), \tag{2.1.3}$$

where $h > 0$ is the time-step. It can be shown (see for instance [25] for the Vlasov-Poisson case) that these schemes yield an order one approximation $f^n(x,v)$ of $f(t,x,v)$ at time $t = nh$, where $f(t,x,v)$ is the solution of the Vlasov-HMF equation (1.1.10), and for a fixed-time horizon.
One may also consider the following Strang splitting method

$$f^{n+1} = \varphi_T^{h/2} \circ \varphi_P^h \circ \varphi_T^{h/2}(f^n), \tag{2.1.4}$$

which gives an order two approximation $f^n(x,v)$ of $f(t,x,v)$ at time $t = nh$.
One may construct from this schemes approximations $g^n(x,v)$ of the function $g(t,x,v)$ defined by (1.2.9) and (1.2.10) at time $t = nh$ by setting

$$f^n(x,v) = \eta(v) + r^n(x,v) \quad \text{and} \quad g^n(x,v) = r^n(x + nhv, v), \tag{2.1.5}$$

where η is a spatially homogeneous stationary state of (1.1.10).
Such time-discretizations by splitting methods are common for transport equations associated with separable hamiltonian vector fields, and we refer for instance the reader to [37] or [45] for more detail on splitting methods.
Full-discretization involves moreover an interpolation procedure in space (cubic splines for instance, see [67]). These **Semi-Lagrangian methods** are widely used to discretize Vlasov equations, and we refer the reader to [17], [25], [26], [33] or [34] for previous works on this topic, where higher order splitting methods may be considered. Our work deals with time-discretizations only.

2.1.2 Main result

In the framework of weighted Sobolev spaces (see (1.2.8)), we shall prove in chapter 3 the following semi-discrete version of Theorem 1.2.2:

Theorem 2.1.1. *Let us fix $s \geq 7$, $\nu > 1/2$ and assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the stability assumption* (**H**). *Assume that $g(0,x,v)$ is in $\mathcal{H}^s_\nu$. For a time step h, let $g^n(x,v)$, $n \geq 0$, be the sequence of functions defined by the formula (2.1.5) from iterations of the splitting methods (2.1.3) (Lie), or (2.1.4) (Strang), with $g^0(x,v) = g(0,x,v)$. Then there exists $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and every $h \in (0, h_0]$, there exists $g^\infty_h(x,v) \in \mathcal{H}^{s-4}_\nu$ such that for all $r \leq s-4$ and $r \geq 1$,*

$$\forall\, n \geq 0, \quad \|g^n(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^r_\nu} \leq \frac{C}{\langle nh \rangle^{s-r-3}}. \tag{2.1.6}$$

If moreover $\nu > 3/2$ and $s \geq 8$, we have for the Lie splitting methods (2.1.3) the estimate

$$\|g^n(x,v) - g(nh,x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq Ch \quad \forall n \in \mathbb{N}, \tag{2.1.7}$$

where $g(t,x,v)$ is the solution defined by (1.2.9) and (1.2.10) associated with the continuous equation with the same initial value.
In the case of the Strang splitting method (2.1.4), we have if $\nu > 5/2$ and $s \geq 9$ the second order estimate

$$\|g^n(x,v) - g(nh,x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq Ch^2 \quad \forall n \in \mathbb{N}. \tag{2.1.8}$$

The scattering estimate (2.1.6) shows that the nonlinear Landau damping Theorem 1.2.2 persists through time-discretization. The question of whether it still holds through space-discretization is left open.

The estimates (2.1.7) and (2.1.8) exhibit convergence rates in time of order 1 and 2 respectively for the numerical solutions, and hold uniformly in time. That last point fails for the functions $f^n(x,v)$ given by the splitting methods (2.1.3) and (2.1.4). It is indeed easy to check, using the formula $f^n(x,v) = \eta(v) + \varepsilon g^n(x - nhv, v)$ that we have an estimate of the form

$$\forall\, n \geq 0, \quad \|f^n(x,v) - f(nh,x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq C\varepsilon\langle nh\rangle^{s-6}h.$$

for the Lie splitting methods (2.1.3). In the case of the Strang splitting (2.1.4), we have from the same arguments:

$$\forall\, n \geq 0, \quad \|f^n(x,v) - f(nh,x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq C\varepsilon\langle nh\rangle^{s-7}h^2.$$

Hence we obtain convergence results that hold for a fixed time horizon $nh \leq T$, and the error in $\mathcal{H}^\sigma$ grows like T^σ.
Let us point that many convergence results for splitting methods applied to Vlasov equations may be found in the literature. We refer for instance to [34] for the case of compactly supported data, and to [17], [25] or [26] for the Vlasov-Poisson case. The convergence estimates (2.1.7) and (2.1.8) (and their aforementioned consequences) are specific in the sense that they describe in fact the long time behavior of the numerical solutions (the *numerical Landau damping*), with constants that are uniform in time.
Let us also mention as an easy consequence of (1.2.19), (2.1.6), (2.1.7) and (2.1.8), that the following estimates hold for the limit state of the equation: for the Lie splitting

$$\|g^\infty(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq Ch,$$

and for the Strang splitting

$$\|g^\infty(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq Ch^2.$$

Finally, let us say that the proof of Theorem (2.1.1) in [39] or in chapter 3 only deals with the case of a repulsive potential ($\Lambda = -1$), though it can be adapted with any major difficulty to the attractive case $\Lambda = 1$.

2.1.3 Sketch of proof for the estimate (2.1.6)

Our proof may be compared with the classical *backward error analysis* methods of Geometric Numerical Integration (see for instance [45, 52]): we express the numerical solution as the exact solution of a continuous Vlasov equation with time dependent kernel (with a poor regularity in time). We saw (see formulae (1.2.17) and (1.2.18)) that the long time behavior of the exact solution is essentially controlled by a time convergent integral, which ensures the existence of the continuous limit function $g^\infty(x,v)$.
The effect of the splitting approximation is essentially to discretize this convergent integral. As the integrand converges algebraically when the time goes to infinity, the numerical solution

also yields a convergent time integral, even if the time appears in a discontinuous way in the evolution equation.
We show in fact that the solution $g^n(x,v)$ given by the splitting method (2.1.3) (Lie) or (2.1.4) (Strang), and formula (2.1.5) coincides at times $t = nh$ with the solution $\mathsf{g}(t,x,v)$ of the equation

$$\partial_t \mathsf{g} = \{\Phi_h(t,\mathsf{g}(t)),\eta\} + \varepsilon\{\Phi_h(t,\mathsf{g}(t)),\mathsf{g}(t)\}, \tag{2.1.9}$$

where $\Phi_h(t,\mathsf{g}(t)) = \phi(\mathsf{s}_h(t),\mathsf{g}(t))$ with the definition of ϕ given in (1.2.12).
In the case of Lie splittings (2.1.3) , $\mathsf{s}_h(t)$ are given by the formulae

$$\mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h + h, \quad \text{and} \quad \mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h, \tag{2.1.10}$$

respectively. In the case of the Strang splitting method (2.1.4), $\mathsf{s}_h(t)$ is given by the formula

$$\mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h + \frac{h}{2}. \tag{2.1.11}$$

Note that

$$\Phi_h(t,\mathsf{g}(t)) = -\frac{\Lambda}{2} \sum_{k\in\{\pm1\}} e^{ikx} e^{ik\mathsf{s}_h(t)v} \mathsf{z}_k(t), \quad \text{with} \quad \mathsf{z}_k(t) = \hat{\mathsf{g}}_k(t, k\mathsf{s}_h(t)). \tag{2.1.12}$$

The modes $\mathsf{z}_k(t)$ are in fact approximations of the terms $\zeta_k(t)$ defined in (1.2.12), and play the same crucial part in the dynamics for the semi-discrete equation (2.1.9). Indeed, we prove that they satisfy a semi-discrete version of the Volterra equation (1.2.13), namely

$$\mathsf{z}_k(t) = \int_0^t K(k,t-\sigma)\mathsf{z}_k(\sigma)\,\mathrm{d}\sigma + F_k(t) + G_k(t) \quad k = \pm 1, \tag{2.1.13}$$

where the kernel $K(k,t)$ has been introduced in (1.2.14), and where

$$F_k(t) := \hat{\mathsf{g}}_k(0,k\mathsf{s}_h(t)) + \varepsilon\frac{\Lambda}{2} \sum_{\ell\in\{\pm1\}} \int_0^t \mathsf{z}_\ell(\sigma)\hat{g}_{k-\ell}(\sigma, k\mathsf{s}_h(t) - \ell\mathsf{s}_h(\sigma))k\ell(\mathsf{s}_h(t) - \mathsf{s}_h(\sigma))\mathrm{d}\sigma \tag{2.1.14}$$

and

$$G_k(t) = \int_0^t \Big(K(k,\mathsf{s}_h(t) - \mathsf{s}_h(\sigma)) - K(k,t-\sigma)\Big)\mathsf{z}_k(\sigma)\,\mathrm{d}\sigma. \tag{2.1.15}$$

$F_k(t)$ contains the nonlinearities, where one must in particular handle the plasma echoes (see chapter 1), *ie* the terms where $k\mathsf{s}_h(t) \sim \ell\mathsf{s}_h(\sigma)$. $G_k(t)$ is the error coming from the discretization in time of the kernel.
We consider, as in the continuous case (see [40] or chapter 1), the equations (2.1.9) and (2.1.13) as a coupled system, which is controlled by the norm $Q_{T,s,\nu}(\mathsf{g})$ (defined by (1.2.15) and (1.2.16) in chapter 1). Note that few differences exist with the continuous case, as one must also handle additional terms that reflect the discretization error (the terms G_k, essentially).
In chapter 3, we prove the following result:

Theorem 2.1.2. *Let us fix $s \geq 7$, $\nu > 1/2$ and $R_0 > 0$ such that $Q_{0,s,\nu}(\mathsf{g}) \leq R_0$, and assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* (**H**). *Then there exists $R > 0$, $h_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h_0]$ and for every $T \geq 0$, the solution of* (2.1.9) *satisfies the estimate*

$$Q_{T,s,\nu}(\mathsf{g}) \leq R.$$

Once this result is obtained, estimate (2.1.6) can be proved. Indeed, we may write

$$\mathsf{g}(t,x,v) = \mathsf{g}(0,x,v) + \int_0^t \left\{\Phi_h(\sigma, \mathsf{g}(\sigma)), \eta + \varepsilon \mathsf{g}(\sigma)\right\}(x,v)\mathrm{d}\sigma. \tag{2.1.16}$$

We can prove (see subsection 3.3.4 of chapter 3), using the bound given by Theorem 2.1.2, that the above integral converges in appropriate Sobolev norms, and then obtain the estimate (2.1.6) by setting

$$g_h^\infty(x,v) = \mathsf{g}(0,x,v) + \int_0^\infty \left\{\Phi_h(\sigma, \mathsf{g}(\sigma)), \eta + \varepsilon \mathsf{g}(\sigma)\right\}(x,v)\mathrm{d}\sigma. \tag{2.1.17}$$

About the proof of Theorem 2.1.2

The proof of Theorem 2.1.2 is very similar to the one of Theorem 1.2.1 from [40], that has also been discussed in chapter 1.
It uses indeed a bootstrap argument: assuming an *a priori* bound on $Q_{T,s,\nu}(\mathsf{g})$, we infer an estimate on that same quantity that scales properly in ε, and conclude that the bound holds uniformly in time by using the smallness of ε. The proof is organized as follows.
Using the *a priori* bound on $Q_{T,s,\nu}(\mathsf{g})$, we first control in Proposition 3.4.1 (of chapter 3) the factor $M_{T,s-1}(\mathsf{z})$. Essentially we use Lemma 1.2.1 as in the continuous case, to control the linear part of (2.1.13) by the nonlinear terms. The term $F_k(t)$ is controlled by integral estimates that account carefully for the localized resonances (the plasma echoes), while G_k is estimated by a Taylor expansion, and scales properly with respect to h.
By using this new control on $M_{T,s-1}(\mathsf{z})$ and the *a priori* bound on $Q_{T,s,\nu}(\mathsf{g})$, the first and third factor in the definition of $Q_{T,s,\nu}(\mathsf{g})$ are then respectively estimated in Propositions 3.4.4 and 3.4.6 (of chapter 3), by means of energy estimates with a commutator trick.

2.1.4 Sketch of proof for the estimates (2.1.7) and (2.1.8)

The proofs of these estimates use essentially the same arguments as above, coupled with the fact that the electric field is discretized in time by the rectangle method for Lie splittings (2.1.3), and by the midpoint rule for Strang splittings (2.1.4). This will yield the order one and order two convergence estimates.
In view of Theorems 2.1.2 and 1.2.1, we can assume that for all $\alpha \in \{0,1,2\}$, $Q_{T,s,\nu-\alpha}(\mathsf{g})$ and $Q_{T,s,\nu-\alpha}(g)$ are bounded by the same constant $R > 0$, provided that $\nu > 5/2$. Those bounds on the exact and numerical solutions are essential to prove the convergence estimates, and we think of it as the *stability* of the schemes.

Though we aim to estimate $g^n(x,v) - g(nh,x,v)$, we take in fact advantage of the continuous Vlasov equations (1.2.11) and (2.1.9) that are satisfied respectively by g and $\mathtt{g}$, and we shall rather consider the continuous function $\delta(t) = g(t) - \mathtt{g}(t)$. At the beginning of section 3.5 of chapter 3, we derive the following equation, satisfied by the function δ :

$$\partial_t \delta(t) = \{\phi(t,\delta(t)),\eta\} + \varepsilon\{\phi(t,\delta(t)),g(t)\} + \varepsilon\{\phi(t,g(t)),\delta(t)\} - \varepsilon\{\phi(t,\delta(t)),\delta(t)\} + \mathcal{R}(t) \tag{2.1.18}$$

where the remainder $\mathcal{R}(t)$ is defined by formula (3.5.2) in chapter 3. Hence the function δ satisfies an equation that is similar to the ones satisfied by g and $\mathtt{g}$, and thus our strategy to prove (2.1.7) and (2.1.8) is to control the factor $Q_{T,s,\nu}(\delta)$ (with a minor modification) uniformly in time, and to show that it scales in h or h^2.
Let us point that the remainder $\mathcal{R}$ reflects precisely the error attributable to the discretization in time. The control of $\mathcal{R}$ brings technical complications in the proof (especially for Strang splittings), in comparison with the proof of (2.1.6), where the errors attributable to the discretization in time do not have a major influence. It also requires to lose regularity in physical and Fourier variables.

Order one estimates

Let T be a positive real number, and let us consider, for $s \geq 8$ and $\nu > 3/2$, the quantity

$$Q_{T,s-2,\nu-1}(\delta) = M_{T,s-3}(d) + N_{T,s-2,\nu-1}(\delta) + \sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}$$

with $N_{T,s,\gamma}$ and $M_{T,\gamma}$ defined in (1.2.15), and where we set

$$d_k(t) = \hat{g}_k(t,kt) - \hat{\mathtt{g}}_k(t,kt).$$

In chapter 3, we prove the following result, which implies (2.1.7):

Proposition 2.1.3. *For $s \geq 8$, $\nu > 3/2$, assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**. *Then there exists $R_1 > 0$, $h_0 > 0$ and $\epsilon_0 > 0$ such that for every $h \in (0,h_0]$, every $\varepsilon \in (0,\varepsilon_0]$ and every $T \geq 0$, the solution of (2.1.18) satisfies the estimate*

$$Q_{T,s-2,\nu-1}(\delta) \leq R_1 h.$$

The proof of the above Proposition is made of a bootstrap argument similar to the one presented for the control of $Q_{T,s,\nu}(\mathtt{g})$ in the previous subsection, though it is more technical. It is organized as follows: assuming an *a priori* bound on $Q_{T,s-2,\nu-1}(\delta)$, the factors $M_{T,s-3}(d)$, $\sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}$ and $N_{T,s-2,\nu-1}(\delta)$ are successively controlled, in Propositions 3.5.3, 3.5.4 and 3.5.5.

Order two estimates

The estimate (2.1.8) is proved by considering as previously the quantity $Q_{T,s-3,v-2}(\delta)$, for which we prove the following result

Proposition 2.1.4. *Let us fix $s \geq 9$ and $\nu > 5/2$. Assuming that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**, *there exists $R_2 > 0$, $h_0 > 0$ and $\varepsilon_0 > 0$ such that for every $h \in (0, h_0]$, every $\varepsilon \in (0, \varepsilon_0]$ and every $T > 0$, the solution of (2.1.18) satisfies the estimate*

$$Q_{T,s-3,\nu-2}(\delta) \leq R_2 h^2.$$

The proof of this result is similar the proof of Proposition (2.1.3), though it is more complicated and technical. It is organized as follows: assuming an *a priori* bound on $Q_{T,s-3,\nu-2}(\delta)$, the factors $M_{T,s-4}(d)$, $\sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-7}_{\nu-2}}$ and $N_{T,s-3,\nu-2}(\delta)$ are successively controlled, in Propositions 3.5.3, 3.5.4 and 3.5.5. The crucial point, and also the main source of technical complications, is the cancellation provided by the midpoint rule, that must be constantly captured throughout the estimates. We can summarize it by the following easy Lemma:

Lemma 2.1.5. *For $t \in \mathbb{R}$ and a fixed $h > 0$, let $\mathsf{s}_h(t)$ be given by formula (2.1.11). Then, for all $n \in \mathbb{N}$,*

$$\int_{nh}^{(n+1)h} (\sigma - \mathsf{s}_h(\sigma)) \mathrm{d}\sigma = 0.$$

Taking advantage of this Lemma allows us to obtain an estimate on the remainder $\mathcal{R}$ that scales in h^2, by means of second order Taylor-expansions. Let us point that this requires an appropriate splitting of the remainder $\mathcal{R}$ into

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2,$$

where $\mathcal{R}_1$ will essentially fall under the scope of Lemma 2.1.5, and $\mathcal{R}_2$ is a Taylor-remainder that scales properly in h.

2.2 On linear Landau damping around inhomogeneous stationary states of the Vlasov-HMF model

In this section we introduce the reader to the main result of chapter 4, where we prove a linear Landau damping Theorem for perturbations of inhomogeneous stationary solutions of the Vlasov-HMF model (1.1.10).

2.2.1 Notations and settings

We consider the Vlasov-HMF model with an attractive potential, that reads

$$\begin{cases} \partial_t f(t,x,v) + \{f, H[f]\}(t,x,v) = 0, \\ H[f](t,x,v) = \dfrac{v^2}{2} - \phi[f](t,x) \\ \phi[f](t,x) = \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \cos(x-y) f(t,y,v) \mathrm{d}y \mathrm{d}v \end{cases} \tag{2.2.1}$$

with $(t,x,v) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}$. The potential may be expressed as the following Fourier series

$$\phi[f](t,x) = \mathcal{C}[f]\cos(x) + \mathcal{S}[f]\sin(x), \tag{2.2.2}$$

with

$$\mathcal{C}[f] = \int_{\mathbb{T}\times\mathbb{R}} \cos(y) f(y,v)\mathrm{d}y\mathrm{d}v \quad \text{and} \quad \mathcal{S}[f] = \int_{\mathbb{T}\times\mathbb{R}} \sin(y) f(y,v)\mathrm{d}y\mathrm{d}v.$$

We consider stationary solutions $\eta(x,v)$ of (2.2.1) defined by

$$\eta(x,v) = G\left(H[\eta](x,v)\right), \tag{2.2.3}$$

where the function $G : [-e, +\infty[\to \mathbb{R}$, $e > 0$, has sufficient regularity. It is easy to see that such states are stationary solutions of (2.2.1), but not clear that one may define a function G such that a solution η of (2.2.3) exists. To that extent, up to a translation $x \mapsto x + x_0$ one can always assume that $\mathcal{S}[\eta] = 0$, and write that

$$H[\eta] = \frac{v^2}{2} - \mathcal{C}[\eta]\cos(x).$$

The notation $M_0 = \mathcal{C}[\eta]$ will usually be used. M_0 is called the magnetization of η, and one quite easily sees that η solves equation (2.2.3) if and only if M_0 satisfies the equation

$$M_0 = \mathcal{C}\left[G\left(\frac{v^2}{2} - M_0\cos(x)\right)\right].$$

Note that this equation is satisfied by $M_0 = 0$, whatever the choice of G is, and in chapter 4 we shall exhibit examples of functions G for which a solution $M_0 > 0$ exists.
We fix a triplet (η, G, M_0) such that

$$\eta(x,v) = G\left(h_0(x,v)\right), \quad h_0(x,v) = \frac{v^2}{2} - M_0\cos(x), \quad M_0 > 0. \tag{2.2.4}$$

We consider solutions of equation (2.2.1) that are perturbations of such a stationary state, namely

$$\begin{cases} f(t,x,v) = \eta(x,v) + r(t,x,v) \\ r(0,x,v) = r^0(x,v), \end{cases} \tag{2.2.5}$$

where r^0 is the initial perturbation. We will retain only the linearized equation around η, that reads

$$\partial_t r(t,x,v) - \{\eta, \phi[r]\}(t,x,v) + \{r, h_0\}(t,x,v) = 0. \tag{2.2.6}$$

Most of our work will be to study the evolution of the coefficients $\mathcal{C}(t) = \mathcal{C}[r(t)]$ and $\mathcal{S}(t) = \mathcal{S}[r(t)]$, for which we derive a closed system of Volterra equations, that will play the same crucial part in the evolution as the closed equations for the Fourier coefficients of the density $\int_v f\mathrm{d}v$ that are obtained in the study of the stability of homogeneous stationary states for

the Vlasov-HMF model or the Vlasov-Poisson system (see chapter 1).
Let $\psi_t(x,v)$ be the flow of the Hamiltonian h_0, that is the flow associated with the ODE

$$\begin{cases} \dot{x} &= \partial_v h_0(x,v) \\ \dot{v} &= -\partial_x h_0(x,v), \end{cases}$$

which are the equations of motion for the celebrated pendulum system. As in the homogeneous case (see [40], or chapter 1), we pull-back the perturbation by the flow ψ_t, by setting

$$g(t,x,v) = r(t,\psi_t(x,v)) = r \circ \psi_t(x,v), \tag{2.2.7}$$

where r is defined by (2.2.5) and satisfies (2.2.6).
Defining the projection $\mathrm{X} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}$ by $\mathrm{X}(x,v) = x$, we prove (see Proposition 4.1.1 in chapter 4) that the function g satisfies the equation

$$\partial_t g = \mathcal{C}(t)\{\eta, \cos(\mathrm{X}\circ\psi_t)\} + \mathcal{S}(t)\{\eta, \sin(\mathrm{X}\circ\psi_t)\}. \tag{2.2.8}$$

and that the coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$ satisfy the following Volterra equations

$$\mathcal{C}(t) = F_{\mathcal{C}}(t) + \int_0^t \mathcal{C}(s)K_{\mathcal{C}}(t-s)\mathrm{d}s \quad \text{and} \quad \mathcal{S}(t) = F_{\mathcal{S}}(t) + \int_0^t \mathcal{S}(s)K_{\mathcal{S}}(t-s)\mathrm{d}s, \tag{2.2.9}$$

with

$$F_{\mathcal{C}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w, \quad F_{\mathcal{S}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w,$$

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}} \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\}\cos(\mathrm{X}\circ\psi_t), \quad K_{\mathcal{S}}(t) = -\mathbb{1}_{\{t\geq 0\}} \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \sin(\mathrm{X})\}\sin(\mathrm{X}\circ\psi_t).$$

2.2.2 Main result

The fact that $\mathcal{S}(t)$ and $\mathcal{C}(t)$ satisfy the Volterra equations (2.2.9) will allow us to obtain decay in time for these two coefficients, provided that the source terms $F_{\mathcal{C}}(t)$ and $F_{\mathcal{S}}(t)$ decay also in time. For that we shall prove a stability result for Volterra integral equations (see Lemma 4.3.2 and Corollary 4.3.3 in chapter 6). The remaining part of the work is then to obtain decay in time for the source terms. Now in the homogeneous case, this is in practice given by the Lebesgue-Riemann Lemma, and the decay is the consequence of the phase-mixing effect associated with the free transport flow, as explained in chapter 1. In the present case, we capture the phase-mixing effect by finding an appropriate system of coordinates in which the flow ψ_t is integrable. To that extent, we consider the angle-action variables associated with h_0, and for that we define the following charts $U_+, U_-, U_\circ$:

$$\begin{aligned} U_+ &= \{(x,v)\in\mathbb{T}\times\mathbb{R} \,|\, v>0 \quad \text{and} \quad h_0(x,v) > M_0\}, \\ U_- &= \{(x,v)\in\mathbb{T}\times\mathbb{R} \,|\, v<0 \quad \text{and} \quad h_0(x,v) > M_0\}, \quad \text{and} \\ U_\circ &= \{(x,v)\in\mathbb{T}\times\mathbb{R} \,|\, h_0(x,v) < M_0\}. \end{aligned} \tag{2.2.10}$$

We have the following result:

Theorem 2.2.1. *For $* \in \{\pm, \circ\}$, there exists a symplectic change of variables $(x, v) \mapsto (\theta, a)$ from U_* to*

$$W_* = \{(\theta, a) \in \mathbb{R}^2 \,|\, a \in J_*,\, \theta \in (-\pi, \pi)\},$$

with $J_\pm = \left(\frac{4}{\pi}\sqrt{M_0}, +\infty\right)$ and $J_\circ = \left(0, \frac{8}{\pi}\sqrt{M_0}\right)$, such that the flow of h_0 in the variables (θ, a) in U_* is $a(t) = a(0)$ and $\theta(t) = t\omega_*(a(0)) + \theta(0)$, and where $\omega_*(a) = \partial_a h_0(a)$.

For a given function $f(x, v)$, we write f^* for the restriction of f to the chart U_*, and we define the Fourier coefficients in variable θ of f by

$$f^*_\ell(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x(\theta, a), v(\theta, a))e^{-i\ell\theta}\mathrm{d}\theta$$

for $a \in J_*$, where $x(\theta, a)$ and $v(\theta, a)$ are given by the change of variable on U_*. The notations $C^*_\ell(a)$ and $S^*_\ell(a)$ will be employed for the Fourier coefficients of the functions

$$\theta \mapsto \cos(x(\theta, a)) \quad \text{and} \quad \theta \mapsto \sin(x(\theta, a)),$$

respectively, and both restricted to U_*.
We shall also use the following notations: for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, and for functions $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, we will write

$$|\alpha| = \alpha_1 + \alpha_2 \quad \text{and} \quad \partial^\alpha_{x,v} f = \partial^{\alpha_1}_x \partial^{\alpha_2}_v f.$$

Our main result is then the following:

Theorem 2.2.2. *Let $\eta(x, v) = G(h_0(x, v))$ with G a decreasing function that satisfies the assumption*

$$\max_{n\leq 10} \left\| \langle y\rangle^\mu G^{(n)}(y)\right\|_{L^\infty(\mathbb{R})} \leq C_\mu,$$

with $\mu > 2$, and assume that there exists $\kappa > 0$ such that

$$\inf_{\mathrm{Im}(\xi)\leq 0} |1 - \hat{K}_\mathcal{C}(\xi)| \geq \kappa \quad \text{and} \quad \inf_{\mathrm{Im}(\xi)\leq 0} |1 - \hat{K}_\mathcal{S}(\xi)| \geq \kappa, \tag{2.2.11}$$

where $\hat{K}(\xi)$ stands for the complex-variable Fourier transform of some function K.
Let us assume that the initial perturbation r^0 satisfies

$$\max_{|\alpha|\leq m} \left\| \langle v\rangle^\nu \partial^\alpha_{x,v} r^0(x, v)\right\|_{L^\infty} \leq C_{m,\nu},$$

for some $\nu > 2$, and where

$$m \geq 5 + \frac{3p}{2},$$

with

$$p = \max\left\{k \geq 1, \quad \partial^\alpha_{x,v} r^0(0, 0) = 0, \quad \forall 1 \leq |\alpha| \leq k\right\}.$$

Then, if r^0 satisfies the orthogonality condition

$$\sum_{*\in\{\pm,\circ\}} \int_{J_*} C_0^*(a)(r^0)_0^*(a)\mathrm{d}a = 0, \tag{2.2.12}$$

there exists $C > 0$ such that for all $t \geq 0$

$$|\mathcal{C}(t)| \leq \frac{C}{(1+t)^\alpha} \quad \textit{and} \quad |\mathcal{S}(t)| \leq \frac{C}{(1+t)^2},$$

with

- $\alpha = 5/2$ *if* $p \geq 1$ *or* $\int_{\mathbb{R}\times\mathbb{T}} r^0(x,v)\mathrm{d}x\mathrm{d}v = 0$,
- $\alpha = 3$ *if both conditions are satisfied*
- $\alpha = 2$ *if none of them is.*

Let us point that, contrary to the homogeneous case, the damping rates for the coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$ are algebraic, and do not depend on the regularity of the initial perturbation r^0. The explanation is the following: the decay in time of the source terms $F_\mathcal{C}$ and $F_\mathcal{S}$ is driven by the phase-mixing effect *in action-angle variables*, where ψ_t is integrable. However, the change of variables has strong singularities near the origin $h_0 \sim -M_0$, and these singularities interfere with the phase-mixing effect, and prevent a better decay in time.
The condition (2.2.11) will be called the Penrose criterion, by analogy with the similar statements in [10, 40, 61] that are stability hypothesis usually used to ensure Landau damping. We shall relate it to a more standard stability criterion (see [53, 4]) which was used to prove the nonlinear orbital stability of the states of type (2.2.4).
In chapter 5 we shall study numerically the damping rates predicted by Theorem 2.2.2 for the linearized dynamics, but at the nonlinear level.
As a consequence of Theorem 2.2.2 we obtain the following scattering result for the function $g(t) = r \circ \psi_t$.

Corollary 2.2.3. *Under the assumptions of Theorem 2.2.2 with $p = 0$, we obtain that:*

- *There exists $g_\infty(x,v)$ such that when $t \to +\infty$, we have*

$$\|g(t) - g_\infty\|_{L^1_{x,v}} \lesssim \frac{1}{\langle t\rangle}. \tag{2.2.13}$$

- *There exists $r_\infty(x,v)$ that depends only on h, that is to say $r_\infty(x,v) = R_\infty(h(x,v))$ such that for every test function ϕ, we have that*

$$\int_{\mathbb{T}\times\mathbb{R}} r(t,x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v \to_{t\to+\infty} \int_{\mathbb{T}\times\mathbb{R}} r_\infty(x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v.$$

Contrary to Landau damping around homogeneous steady states (see Theorem 1.2.2), the scattering estimate (2.2.3) do not hold in some regular norm, but only in L^1, and only holds in the present case of the linearized dynamics. We leave open the question of nonlinear Landau damping in a regular norm, as Sobolev's or Gevrey's. However, we shall show in chapter 5 a quite large amount of numerical simulations performed for the nonlinear Vlasov-HMF model, where r^0 will moreover be chosen small in an appropriate norm, and which should help to understand the nonlinear case.

2.2.3 Sketch of proof for Theorem 2.2.2

Let us now explain how we prove our main Theorem 2.2.2. As in most of the papers of the Landau damping literature (see [8, 10, 12, 40, 41, 61, 73]), we shall use the Penrose criterion (2.2.11) to derive stability estimates on the Volterra equations (2.2.9), in the style of Lemma 1.2.1, to control the decay of $\mathcal{C}(t)$ or $\mathcal{S}(t)$ by the one of $F_{\mathcal{C}}(t)$ or $F_{\mathcal{S}}(t)$. The latter will be captured by estimating certain oscillatory integrals in action-angle variables. More precisely, we prove in chapter 4 the following result:

Theorem 2.2.4. *Consider $f(x,v)$ and $\varphi(x,v)$ two functions such that*

$$\max_{|\alpha|\leq m} \left\| \langle v\rangle^{\mu} \partial^{\alpha}_{x,v} f(x,v) \right\|_{L^\infty} \leq C_{m,\mu} \quad \text{and} \quad \max_{|\alpha|\leq M} \left\| \partial^{\alpha}_{x,v} \varphi(x,v) \right\|_{L^\infty} \leq C_M,$$

for some $\mu > 2$, where $\langle v\rangle = (1+v^2)^{1/2}$.
Let p and q be defined by

$$p = \max\{n \geq 1,\ \partial^{\alpha}_{x,v} f(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\},\ q = \max\{n \geq 1,\ \partial^{\alpha}_{x,v} \varphi(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\}.$$

Then, if

$$m \geq 5 + p + \frac{p+q}{2} \quad \text{and} \quad M \geq \max\left(7 + q + \frac{p+q}{2}, m+2\right),$$

there exists $C > 0$ such that for all $t \geq 1$, we have

$$\left| \int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} f_0^*(a)\overline{\varphi_0^*(a)}\mathrm{d}a \right| \leq \frac{C}{(1+t)^{\frac{p+q}{2}+2}}.$$

We prove in section 4.3 of chapter 4 that the Theorem yields the following decay estimates

$$K_{\mathcal{C}}(t) \lesssim \frac{1}{(1+t)^3}, \quad K_{\mathcal{S}}(t) \lesssim \frac{1}{(1+t)^2}, \quad F_{\mathcal{C}}(t) \lesssim \frac{1}{(1+t)^{\alpha}}, \quad F_{\mathcal{S}}(t) \lesssim \frac{1}{(1+t)^2}, \tag{2.2.14}$$

where α depends on the initial perturbation r^0 in a way explained in Theorem 2.2.2. To obtain the decay estimates claimed in Theorem 2.2.2 for the coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$, we use then a general result on Volterra equation, in the style of Lemma 1.2.1.
More precisely, consider the following Volterra equation

$$y(t) = K * y(t) + F(t), \quad t \in \mathbb{R} \tag{2.2.15}$$

where K, y, F vanish for $t < 0$. We prove in section 4.3 of chapter 4 the following Lemma:

Lemma 2.2.5. *Assume that $K \in L^1(\mathbb{R})$ is such that*

$$\min_{\mathrm{Im}(\xi)\leq 0} |1 - \hat{K}(\xi)| \geq \kappa.$$

Then the following holds:

i) There exists $C > 0$ such that

$$\|y\|_{L^\infty} \leq C\|F\|_{L^\infty}.$$

ii) If $\langle t\rangle^2 K \in L^\infty$ and $\langle t\rangle^2 F \in L^\infty$, then there exists $C > 0$ such that

$$\|\langle t\rangle^2 y\|_{L^\infty} \leq C\|\langle t\rangle^2 F\|_{L^\infty}.$$

iii) If $\langle t\rangle^3 K \in L^\infty$ and $\langle t\rangle^\alpha F \in L^\infty$ for $\alpha \in [2,3]$, then there exists $C > 0$ such that

$$\|\langle t\rangle^\alpha y\|_{L^\infty} \leq C\|\langle t\rangle^\alpha F\|_{L^\infty}.$$

This Lemma is proved by using a Paley-Wiener Theorem, which under the assumption on $\hat{K}$ shows the existence of a resolvent Kernel for the equation (2.2.15), that belongs in $L^1(\mathbb{R}_+)$. The estimates of the Lemma are then obtained by using an explicit formulation for the solution $y(t)$ of (2.2.15), given by the resolvent Kernel. With the decay estimates (2.2.14), and the Penrose assumption (2.2.11), we apply the Lemma to the Volterra equations (2.2.9), and prove at the end of section 4.3 of chapter 4 the Theorem 2.2.2.

About the proof of Theorem 2.2.4

The starting point of the proof of this Theorem is the following: Fourier expanding, we have in fact the identity

$$\int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}v = \sum_{*\in\{\pm,\circ\}}\sum_{\ell\in\mathbb{Z}}\int_{J_*} f_\ell^*(a)\varphi_{-\ell}^*(a)e^{it\ell\omega_*(a)}\mathrm{d}a.$$

From this formula with integrate with respect to a to gain a decay with respect to t. This stationary phase argument, that typically depends on the possible cancellation of $\partial_a\omega_*(a)$, is in our case possible as $\partial_a\omega_*(a)$ never approaches zero. However, the decay in time depends also on the cancellations at the boundary points, and this is precisely where problems may come, due to the singularities of the action-angle variables.
We can distinguish two zones: first, near the *separatix* $h_0 = M_0$, where the action angle variables induce logarithmic singularities for the Fourier coefficients f_ℓ^*. However, near this point, $\omega_*(a)$ also exhibits a logarithmic singularity, and it can be shown that $\partial_a\omega_*(a)$ goes to infinity fast enough to ensure a decay in time which is essentially driven by the regularity of f and φ.
In the vicinity $h_0 \sim -M_0$ (or, equivalently, $a \sim 0$, or $(x,v) \sim (0,0)$), the situation is more delicate: in this zone, the Hamiltonian h_0 is essentially a perturbation of the Harmonic

oscillator, for which no Landau damping is expected (ω being constant). However, we can prove that $\partial_a \omega_*(a)$ does not vanish near this point. But this is not enough: indeed the action angle variable of the harmonic oscillator involves an algebraic singularity of order $\sqrt{a}$. This explains why the rate of decay of the integral with respect to the time is mainly driven by the behavior of f and φ near $(0,0)$, which corresponds to a local behavior of $f_\ell^\circ(a)\varphi_\ell^\circ(a)$ in $a^{\frac{p+q}{2}}$, and which yields the main contribution for the decay in the previous Theorem.

About the Penrose criterion

Written in this form, the Penrose criterion (2.2.11) is difficult to check, but we can relate it to a more tractable statement. We prove first the following result:

Theorem 2.2.6. *Let η be a state defined by (2.2.4), and assume that G satisfies the hypothesis of Theorem 2.2.2. Assume moreover that $G' < 0$ and*

$$1 - \hat{K}_{\mathcal{C}}(0) > 0 \quad \textit{and} \quad 1 - \hat{K}_{\mathcal{S}}(0) > 0. \tag{2.2.16}$$

Then the Penrose criterion (2.2.11) holds true.

We should point that, as long as G is assumed to be decreasing, the proof of this Theorem relies only on the *geometry*, in the sense that it depends almost exclusively on properties of the change of variable. Indeed, in section 4.5 of chapter 4 we derive an explicit expression of the imaginary part of $\hat{K}_{\mathcal{C}}(\xi)$ (or $\hat{K}_{\mathcal{S}}(\xi)$), and we prove that, for $\xi \neq 0$, it can only vanish if all the coefficients $C_\ell^*(a)$ (or $S_\ell^*(a)$) vanish as well. Explicit expressions for these coefficients, which are proved in section 4.7 of chapter 4, show that this cannot happen. So neither $\hat{K}_{\mathcal{C}}(\xi)$ nor $\hat{K}_{\mathcal{S}}(\xi)$ can achieve the value 1 when $\xi \neq 0$. Hence one must only check the case $\xi = 0$, which is ruled out by (2.2.16).
Therefore all the work now reduces to the verification of (2.2.16). The assumption is automatically satisfied in the case of $K_{\mathcal{S}}$, as long as $G' < 0$ (see section 4.5 of chapter 4). In the case of $K_{\mathcal{C}}$, we prove in section 4.5 of chapter 4 that, by elementary computations, the assumption is equivalent to the following notion of stability (see also [4, 53]):

Definition 2.2.7. *A state $\eta(x,v)$ defined by (2.2.4) is said to be linearly stable if*

$$1 + \int_{\mathbb{R}\times\mathbb{T}} \frac{\partial_v \eta(x,v)}{v} \cos^2(x) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} C_0^*(a)^2 \mathrm{d}a > 0.$$

In chapter 4, we shall exhibit examples of stationary states $\eta(x,v)$ that satisfy 2.2.7, and thus the Penrose criterion (2.2.11).

2.3 On time-discretization of the 2D Euler equation by a symplectic Crouch-Grossman integrator

In this section we present the main result of chapter 6. It concerns time-discretization of the Euler equation (1.2.20) mentioned in the previous chapter, with periodic boundary

conditions, by means of a symplectic Crouch-Grossman integrator. Our main result is a convergence estimate for the aforementioned time scheme, in the framework of periodic Sobolev spaces.

2.3.1 Notations and settings

We consider the two-dimensional Euler equation with periodic boundary conditions

$$\begin{cases} \partial_t \omega - U(\omega) \cdot \nabla \omega = 0, \\ \omega(0, z) = \omega_0(z), \end{cases} \tag{2.3.1}$$

where $\omega(t, z) \in \mathbb{R}$, with $t \in \mathbb{R}_+$, and $z \in \mathbb{T}^2$. The divergence-free velocity vector field U is given by the formula

$$U(\omega) = J \nabla \Delta^{-1} \omega.$$

using the canonical symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Δ^{-1} stands for the inverse of the Laplace operator on functions with average 0 on $\mathbb{T}^2$, and ∇ is the two-dimensional gradient operator.
We shall work on the classical periodic Sobolev space $H^s(\mathbb{T}^2)$, $s \in \mathbb{R}$, equipped with the norm

$$\|v\|_{H^s(\mathbb{T}^2)} = \left(\sum_{k \in \mathbb{Z}^2} |\hat{v}_k|^2 \langle k \rangle^{2s} \right)^{1/2}, \tag{2.3.2}$$

where the Fourier coefficients of some function $v : \mathbb{T}^2 \to \mathbb{R}$ are defined by

$$\hat{v}_k = \int_{\mathbb{T}^2} v(z) e^{-ik \cdot z} \mathrm{d}z, \quad k \in \mathbb{Z}^2.$$

Theorem 2.3.1 ([2]). *Let $s > 1$ and $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. There exists an unique solution $\omega(t, z) \in C^0(\mathbb{R}_+, H^s(\mathbb{T}^2)) \cap C^1(\mathbb{R}_+, H^{s-1}(\mathbb{T}^2))$ of equation (2.3.1) with initial data ω_0.*

2.3.2 Time-discretization and main result

We shall consider the following time-discretization of (2.3.1) by a Crouch-Grossman integrator (see [30]), that proceeds in two steps. First, freezing the velocity vector field at time $t = 0$, which gives the hamiltonian transport equation

$$\partial_t f - J \nabla \Delta^{-1} \omega_0 \cdot \nabla f = 0 \tag{2.3.3}$$

with initial data ω_0.
The second step is to discretize the flow of the vector field $J\nabla\Delta^{-1}\omega_0$ by the implicit midpoint method. More precisely, we define $\Phi_t(z)$ as the unique solution of the implicit equation

$$\Phi_t(z) = z + tJ\nabla\Delta^{-1}\omega_0\left(\frac{z+\Phi_t(z)}{2}\right),$$

and if t is small enough, $\omega_0 \circ \Phi_t(z)$ should be an approximation of the solution $\omega(t,z)$ of (2.3.3).
Thus the semi-discrete operator $\mathcal{S}_t$ is defined by

$$\begin{cases} \mathcal{S}_t(\omega_0(z)) = \omega_0\left(\Phi_t(z)\right), \\ \Phi_t(z) = z + tJ\nabla\Delta^{-1}\omega_0\left(\dfrac{z+\Phi_t(z)}{2}\right). \end{cases} \tag{2.3.4}$$

If $\tau \in]0,1[$ is the time-step, let $(\omega_n)_{n\in\mathbb{N}}$ by given by

$$\begin{cases} \omega_n(z) = \mathcal{S}_\tau(\omega_{n-1}) = \mathcal{S}_\tau^n(\omega_0) \\ \mathcal{S}_\tau^0(\omega_0) = \omega_0. \end{cases} \tag{2.3.5}$$

Our main result is the following:

Theorem 2.3.2. *Let $s \geq 6$ and $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. Let $\omega(t,z) \in C^0\left(\mathbb{R}_+, H^s(\mathbb{T}^2)\right)$ be the unique solution of equation (2.3.1) given by Theorem 2.3.1, with initial data ω_0. For a time step $\tau \in]0,1[$, let $(\omega_n)_{n\in\mathbb{N}}$ be the sequence of functions starting from ω_0 and defined by formula (2.3.5) from iterations of the semi-discrete operator (2.3.4). For a fixed time horizon $T > 0$, let $B = B(T)$ be such that*

$$\sup_{t\in[0,T]} \|\omega(t)\|_{H^s(\mathbb{T}^2)} \leq B.$$

There exists two positive constants R_0 and R_1, and an increasing continuous function $R : \mathbb{R}_+ \to \mathbb{R}_+$, such that, if τ satisfies the hypothesis

$$\tau < \max\left(\frac{1}{R_0 B}, \frac{B}{TR(B)e^{R_1T(1+B)}}\right),$$

the semi-discrete scheme enjoys the following convergence estimate: for all $n \in \mathbb{N}$ such that $t_n = n\tau \leq T$,

$$\|\omega_n - \omega(t_n)\|_{H^{s-4}(\mathbb{T}^2)} \leq \tau t_n R(B) e^{R_1T(1+B)}.$$

Moreover

$$R(B) \leq R_1\left(B + B^3\right).$$

The global error scales in τ, though the reader familiar with the midpoint method, which is known to produce a global error of order two, may wonder why it does not scale in τ^2.

This is due to the freezing effect, *ie* the fact that the error in Sobolev space between the solution f of (2.3.3) and ω at time t only scales in t.
This asks the question of the interest of using the implicit midpoint method. In the problem, it is essential due to its symplecticity: through area preservation, it ensures that for all n, ω_n has average zero, so that we may define $\psi_n = \Delta^{-1}\omega_n$ at each step, and then the midpoint integrator associated with the vector field $J\nabla\psi_n$. Although the proof uses as well extensively the special structure of the implicit midpoint integrator, in the sense that it is the composition of Euler's forward and backward integrators with half time steps, it is thus possible that our result may be extended to a larger class of symplectic integrators.
The result holds for time-discretization only. Fully discrete schemes should involve moreover an interpolation procedure at each step. Our main idea for spatial discretization is to use trigonometric polynomial interpolation, *ie* discrete Fourier series, whose error in periodic Sobolev spaces is polynomial with respect to the space step size. This work is still in progress, as aliasing errors keeps us from proving the stability of the fully discrete scheme, for the time being.

2.3.3 Sketch of proof for Theorem 2.3.2

The prove uses an expansion of the error $\omega_n - \omega(t_n)$ that reads inductively

$$\omega_{n+1} - \omega(t_{n+1}) = \underbrace{\mathcal{S}_\tau(\omega_n) - \mathcal{S}_\tau(\omega(t_n))}_{(1)} + \underbrace{\mathcal{S}_\tau(\omega(t_n)) - \varphi_{F,\tau}(\omega(t_n))}_{(2)} + \underbrace{\varphi_{F,\tau}(\omega(t_n)) - \omega(t_{n+1})}_{(3)}, \tag{2.3.6}$$

where $\varphi_{F,\tau}(\omega(t_n))$ is the value at time $t = \tau$ of the function $f(t,z)$ that solves the frozen equation

$$\partial_t f - J\nabla\Delta^{-1}\omega(t_n)\cdot\nabla f = 0,$$

with initial data $\omega(t_n)$.
In equation (2.3.6), we expect (1) to be controlled in H^{s-4} norm by the H^{s-4} norm of $\omega_n - \omega(t_n)$, essentially, provided that the *stability* of the operator $\mathcal{S}_\tau$ is proved. The terms (2) and (3) represent respectively the local *consistency* errors attributable to the midpoint integrator and to the freezing effect, and are expected to scale respectively in τ^3 and τ^2.
The proof may once more be related to classical *backward analysis methods*, as we prove that for any function h, $\mathcal{S}_\tau(h)$ coincides at time $t = \tau$ with a transport equation on $[0,\tau]$, whose initial data is h. From that point, any of the terms of (2.3.6) may be estimated in H^{s-4} by applying properly the following stability Lemma for transport equations:

Lemma 2.3.3. *For any $s \geq 0$, there exists a constant $C > 0$ such that, for vector fields X and functions h, if g solves the equation*

$$\partial_t g - X\cdot\nabla g = h.$$

then for all $t \in \mathbb{R}$ and $\sigma \in \{0,1,2\}$, g enjoys the estimates

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{H^s(\mathbb{T}^2)} \leq & C\left[\|X(t)\|_{H^{s+\sigma}(\mathbb{T}^2)}\left(\|g(t)\|_{H^s(\mathbb{T}^2)} + \|g(t)\|_{H^{3-\sigma}(\mathbb{T}^2)}\right)\right]\|g(t)\|_{H^s(\mathbb{T}^2)} \\ & + \|\nabla\cdot X(t)\|_{L^\infty(\mathbb{T}^2)}\|g(t)\|^2_{H^s(\mathbb{T}^2)} + 2\|h(t)\|_{H^s(\mathbb{T}^2)}\|g(t)\|_{H^s(\mathbb{T}^2)},\end{aligned} \tag{2.3.7}$$

where $\nabla\cdot$ is the divergence operator.

That being said the application of Lemma 2.3.3 requires bounds on the regularity of $\omega(t)$ and of the sequence $(\omega_n)_{n\in\mathbb{N}}$.

Step 1. Stability

The stability on one time step of the Euler flow and of the flow of the frozen equation are easily deduced from Lemma 2.3.3. Concerning the semi-discrete flow, the stability on one time step is expressed by the following result

Proposition 2.3.4. *Let $s \geq 3$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0, and $\tau \in]0,1[$. There exists two positive constants R_0, R_1, independent of ω_0, such that, if $\tau R_0 \|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$, then for all $t \in [0,\tau]$,*

$$\|\mathcal{S}_t(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq e^{R_1 t\left(1+t\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}} \|\omega_0\|_{H^s(\mathbb{T}^2)},$$

where the operator $\mathcal{S}_t$ is defined by formula (2.3.4).

The proof of Proposition 2.3.4 uses the fact that the midpoint integrator is the composition of Euler's backward and forward integrators with half time steps, namely

$$\Phi_t(z) = \mathcal{E}_t \circ \mathcal{E}_t^*(z), \quad \text{with} \quad \mathcal{E}_t(z) = z + \frac{t}{2} J\nabla\Delta^{-1}\omega_0(z) \quad \text{and} \quad \mathcal{E}_t^* = \mathcal{E}_{-t}^{-1}.$$

Essentially, one may express, for any function $g \in H^s(\mathbb{T}^2)$, the functions $g\circ\mathcal{E}_t$ and $g\circ\mathcal{E}_t^*$ as the values at time t of two solutions of two transport equations with initial data g. Then Proposition 2.3.4 is obtained by applying Lemma 2.3.3 to these two functions (see Lemmas 6.4.3 and 6.4.5 in chapter 6).

Step 2. Consistency

The second step is to prove an *a priori* consistency estimate, namely:

Proposition 2.3.5. *Let $s \geq 5$, and $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. Let $\omega(t) \in C^0\left(\mathbb{R}_+, H^s(\mathbb{T}^2)\right)$ be the unique solution of equation (2.3.1) with initial data ω_0, given by Theorem 2.3.1. For a time step $\tau \in]0,1[$, let $(\omega_n)_{n\in\mathbb{N}}$ be the sequence of functions starting from ω_0 and defined by formula (2.3.5) from iterations of the semi-discrete operator (2.3.4). Assume that there exists a time $T_0 > 0$ and a constant $L > 0$ such that*

$$\sup_{t\in[0,T_0]} \|\omega(t)\|_{H^s(\mathbb{T}^2)} \leq B \quad \text{and} \quad \sup_{t_n\leq T_0} \|\omega_n\|_{H^2(\mathbb{T}^2)} \leq 2B.$$

Then there exists two positive constants R_0, R_1 such that, if $\tau R_0 B < 1$,

$$\|\omega_n - \omega(t_n)\|_{H^{s-4}(\mathbb{T}^2)} \leq \tau R(B) t_n e^{R_1 T_0(1+B)},$$

for all $t_n \leq T_0 + \tau$, where

$$R(B) \leq R_1\left(B + B^3\right).$$

This Proposition is proved by induction using the relation (2.3.6), and applying properly Lemma 2.3.3. It requires also the local stability in the form of Proposition 2.3.4. One of the main tool is also the local error due to the midpoint integrator, which we capture by proving (in Proposition 6.4.1 of chapter 6) that for any function h, we have for all $t \in [0, \tau]$ a relation of the form

$$\partial_t \mathcal{S}_t(h) = J\nabla\Delta^{-1}h \cdot \nabla \mathcal{S}_t(h) + \mathcal{O}(\tau^2),$$

that holds essentially in H^{s-4} norm.
By using this, Lemma 2.3.3, and Proposition 2.3.4, we prove in Corollary 6.4.7 and Proposition 6.5.3 (of chapter 6) that the terms (1) and (2) of (2.3.6) may be respectively controlled in H^{s-4} norm by

$$(1 + \mathcal{O}(\tau)) \, \|\omega_n - \omega(t_n)\|_{H^{s-4}(\mathbb{T}^2)} \quad \text{and} \quad R(B)\tau^3.$$

By using Lemma 2.3.3, we prove in Proposition 6.5.1 that the term (3) scales in τ^2, which with the above estimates implies Proposition 2.3.5 by induction.

Step 3. Bootstrapping the numerical solution, and conclusion

Proposition 2.3.5 will essentially imply Theorem 2.3.2, provided that one proves that the numerical solution may be bootstrapped up to the time horizon T. For that one must, in order to apply Proposition 2.3.4, essentially, control the H^2 norm of ω_n uniformly with respect to n. We obtain this control in section 6.5.3 of chapter 6, using the bound on the H^s norm of the exact solution and the global *a priori* consistency estimate from Proposition 2.3.5. It requires also the smallness hypothesis on τ from the Theorem.

Chapter 3

On Numerical Landau Damping for Splitting Methods Applied to the Vlasov-HMF Model

The topic of this chapter was introduced and summarized in section 2.1 of chapter 2, and was also the subject of the journal paper [39].
We consider time discretizations of the Vlasov-HMF equation based on splitting methods between the linear and nonlinear parts. We consider solutions starting in a small Sobolev neighborhood of a spatially homogeneous state satisfying a linearized stability criterion. We prove that the numerical solutions exhibit a scattering behavior to a modified state, which implies a nonlinear Landau damping effect with polynomial rate of damping. Moreover, we prove that the modified state is close to the continuous one and provide error estimates with respect to the time step size.
The present chapter is self-consistent an may be read independently from the others.

3.1 Introduction

In this chapter we consider time discretizations of the Vlasov-HMF model. A long time analysis of the Vlasov-HMF model around homogenous stationary states has been performed in [40] where a Landau damping result is proved in Sobolev regularity. The purpose of the present chapter is in essence to show that this result persists through time discretization by splitting methods.
The Vlasov-HMF model with a repulsive potential reads

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \partial_x\Big(\int_{\mathbb{R}\times\mathbb{T}} P(x-y)f(t,y,u)\mathrm{d}u\mathrm{d}y\Big)\partial_v f(t,x,v), \tag{3.1.1}$$

where $(x,v) \in \mathbb{T}\times\mathbb{R}$ and the kernel $P(x)$ is given by $P(x) = \cos(x)$. We consider initial data under the form $f_0(x,v) = \eta(v) + \varepsilon r_0(x,v)$ where ε is a small parameter and r_0 is of size one (in a suitable functional space). This means that we study small perturbations of a stationary solution $\eta(v)$. Writing the exact solution as

$$f(t,x,v) = \eta(v) + \varepsilon r(t,x,v),$$

and setting

$$g(t,x,v) = r(t,x+tv,v), \tag{3.1.2}$$

the main result given in [40] is that if ε is small enough, $g(t,x,v)$ converges towards some $g^\infty(x,v)$ when t goes to ∞ in Sobolev regularity. This results implies a Landau damping phenomenon for the solution.

In this chapter, we consider the time discretization of (3.1.1) by splitting methods based on the decomposition of the equation between the free part

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = 0, \quad f(0,x,v) = f^0(x,v), \tag{3.1.3}$$

whose solution is given explicitely by $\varphi_T^t(f^0)(x,v) := f^0(x-tv,v)$, and the potential part

$$\partial_t f(t,x,v) = \partial_x\Big(\int_{\mathbb{R}\times\mathbb{T}} P(x-y)f(t,y,u)\mathrm{d}u\mathrm{d}y\Big)\partial_v f(t,x,v), \quad f(0,x,v) = f^0(x,v), \tag{3.1.4}$$

whose solution is explicitely given by

$$\varphi_P^t(f^0) = f^0(x, v + tE(f^0, x)),$$

where $E(f,x) = \partial_x\Big(\int_{\mathbb{R}\times\mathbb{T}} P(x-y)f(y,u)\mathrm{d}u\mathrm{d}y\Big)$ is indeed kept constant during the evolution of (3.1.4).

The Lie splittings we consider are given by the formulas

$$f^{n+1} = \varphi_P^h \circ \varphi_T^h(f^n), \quad \text{or} \quad f^{n+1} = \varphi_T^h \circ \varphi_P^h(f^n), \tag{3.1.5}$$

where $h > 0$ is the time step. The functions $f^n(x,v)$ defined above are *a priori* order one approximations of $f(t,x,v)$ at time $t = nh$.

We also consider the Strang splitting

$$f^{n+1} = \varphi_T^{h/2} \circ \varphi_P^h \circ \varphi_T^{h/2}(f^n) \tag{3.1.6}$$

that should provide an order two approximation $f^n(x,v)$ of $f(t,x,v)$ at time $t = nh$ (the same being expected for the symmetric splitting where the roles of T and P are swapped).

We can then define the sequence of function $r^n(x,v)$ by the formula

$$f^n(x,v) = \eta(v) + \varepsilon r^n(x,v), \tag{3.1.7}$$

and the functions

$$g^n(x,v) = r^n(x + nhv, v) \tag{3.1.8}$$

which have to be thought as approximations of $g(t,x,v)$ at time $t = nh$.

The main result of our chapter is that if ε and h are small enough, $g^n(x,v)$ converges towards a limit function $g_h^\infty(x,v)$ when the n goes to ∞. Moreover, this solution is close to the exact limit function $g^\infty(x,v)$ with an error estimate that scales in h for the Lie splitting, and in h^2 for the Strang splitting. Note that our results also imply convergence results in time which are uniform for positive times for $g^n(x,v)$ and give explicit convergence bounds for $f^n(x,v)$ in $\mathcal{H}^s$ (Sobolev space, see (3.2.1) below) that depend on the final time T in a polynomial way.

The main idea of our proof can be compared with the classical *backward error analysis* methods widely used in Geometric Numerical Integration, see for instance [45], [52]: we express the numerical solution as the exact solution of a continuous Vlasov type equation with time dependent kernel (with a poor regularity in time). Usually for Hamiltonian systems, the analysis has to be refined to make this equation independent of the time, implying the existence of a modified energy that is preserved by the numerical scheme. This "time averaging" introduces in general a remaining error term which is exponentially small (with respect to the time step) for finite dimensional systems (see [16], [45], [52], [66]) or requires the use of a CFL (Courant-Friedrichs-Lewy) condition for semilinear Hamiltonian equations to be controlled (see [38],[37]).

Here the situation is completely different. The long time behavior of the solution is essentially controlled by a time convergent integral, which is a consequence of the dispersive effect of the free flow and ensures the existence of the continuous limit function $g^\infty(x,v)$ (The *Landau damping* effect, see [61], [10], [40]). As we will observe in the next section, the effect of the splitting approximation is essentially to discretize this convergent integral. As the integrand converges algebraically when the time goes to infinity, the numerical solution also yields a convergent time integral, even if the time appears in a discontinuous way in the evolution equation.

The proof of the uniform convergence estimates is based on a similar argument, but requires slightly more regularity for the functions than for the continuous case.

3.2 Landau damping for the Vlasov-HMF model, main result

Before stating our main result, we first recall the scattering result derived in [40] (see also [61], [10] for similar result with analytic or Gevrey regularity that are valid for much more singular interaction potentials).

We work in the following weighted Sobolev spaces: for a given $\nu > 1/2$, we set

$$\|f\|^2_{\mathcal{H}^s_\nu} = \sum_{|p|+|q|\leqslant s} \int_{\mathbb{T}\times\mathbb{R}} (1+|v|^2)^\nu |\partial_x^p \partial_v^q f|^2 \mathrm{d}x\mathrm{d}v, \tag{3.2.1}$$

and we shall denote by $\mathcal{H}^s_\nu$ the corresponding function space. We shall denote by $\hat{\cdot}$ or $\mathcal{F}$ the Fourier transform on $\mathbb{T}\times\mathbb{R}$ given by

$$\hat{f}_k(\xi) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} f(x,v) e^{-ikx-i\xi v}\mathrm{d}x\mathrm{d}v. \tag{3.2.2}$$

We shall need a stability property of the reference state η in order to control the linear part of the Vlasov equation (3.3.1). Let us denote by $\eta = \eta(v)$ the spatially homogeneous stationary state and let us define the functions

$$K(n,t) = -np_n\, nt\, \hat{\eta}_0(nt), \quad K_1(n,t) = -np_n\, nt\, \hat{\eta}_0(nt)\mathbb{1}_{t\geq 0}, \quad t\in\mathbb{R}, \quad n\in\mathbb{Z}, \tag{3.2.3}$$

where $(p_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of the kernel $P(x)$. We shall denote by $\hat{K}_1(n,\tau) = \int_{\mathbb{R}} e^{-i\tau t} K_1(n,t)\,\mathrm{d}t$ the Fourier transform of $K_1(n,\cdot)$. We shall assume that η satisfies the following condition

$$(\mathbf{H}) \quad \eta(v)\in\mathcal{H}^5_3 \quad \text{and} \quad \exists\,\kappa>0, \quad \inf_{\mathrm{Im}\,\tau\leq 0} |1-\hat{K}_1(n,\tau)| \geq \kappa, \quad n=\pm 1.$$

Note that thanks to the localization property of η in the first part of the assumption, the Fourier transform of K can be indeed continued in the half plane $\mathrm{Im}\,\tau \leq 0$. Here, the assumption is particularly simple due to the fact that for our kernel, there are only two

non-zero Fourier modes. This assumption is very similar to the one used in [61], [10] and can be related to the standard statement of the Penrose criterion. In particular it is verified for the states $\eta(v) = \rho(|v|)$ with ρ non-increasing which are also known to be Lyapounov stable for the nonlinear equation (see [59]).

We also use the notation $\langle x \rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}$. In [40], the following result is proved:

Theorem 3.2.1. *Let us fix $s \geq 7$, $\nu > 1/2$ and assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* (**H**). *Assume that $g(0,x,v)$ is in $\mathcal{H}^s_\nu$. Then there exists $\varepsilon_0 > 0$ and a constant $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists $g^\infty(x,v) \in \mathcal{H}^{s-4}_\nu$ such that for all $r \leq s-4$ and $r \geq 1$,*

$$\forall\, t \geq 0, \quad \|g(t,x,v) - g^\infty(x,v)\|_{\mathcal{H}^r_\nu} \leq \frac{C}{\langle t \rangle^{s-r-3}}. \tag{3.2.4}$$

In this chapter, we prove the following semi-discrete version of the previous result:

Theorem 3.2.2. *Let us fix $s \geq 7$, $\nu > 1/2$ and assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* (**H**). *Assume that $g(0,x,v)$ is in $\mathcal{H}^s_\nu$. For a time step h, let $g^n(x,v)$, $n \geq 0$, be the sequence of functions defined by the formula (3.1.8) from iterations of the splitting methods (3.1.5) (Lie), or (3.1.6) (Strang), with $g^0(x,v) = g(0,x,v)$. Then there exists $\varepsilon_0 > 0$, $h_0 > 0$ and a constant $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and every $h \in (0, h_0]$, there exists $g^\infty_h(x,v) \in \mathcal{H}^{s-4}_\nu$ such that for all $r \leq s-4$ and $r \geq 1$,*

$$\forall\, n \geq 0, \quad \|g^n(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^r_\nu} \leq \frac{C}{\langle nh \rangle^{s-r-3}}. \tag{3.2.5}$$

If moreover $\nu > 3/2$ and $s \geq 8$, we have for the Lie splitting methods (3.1.5) the estimate

$$\|g^n(x,v) - g(nh,x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq Ch \quad \forall n \in \mathbb{N}, \tag{3.2.6}$$

where $g(t,x,v)$ is the solution (3.1.2) associated with the continuous equation with the same initial value.
In the case of the Strang splitting method (3.1.6), we have if $\nu > 5/2$ and $s \geq 9$ the second order estimate

$$\|g^n(x,v) - g(nh,x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq Ch^2 \quad \forall n \in \mathbb{N}. \tag{3.2.7}$$

Let us make the following comments:

a) The estimates (3.2.6) and (3.2.7) exhibit a convergence rates in time of order 1 and 2 respectively for the numerical solutions. These estimates hold *uniformly in time*. Note however that these results do not imply convergence results uniform in time for the functions $f^n(x,v)$ to $f(nh,x,v)$ given by the splitting methods (3.1.5) and (3.1.6). It is easy to check, using the formula $f^n(x,v) = \eta(v) + \varepsilon g^n(x - nhv, v)$ that we have an estimate of the form

$$\forall\, n \geq 0, \quad \|f^n(x,v) - f(nh,x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq C\varepsilon \langle nh \rangle^{s-6} h.$$

for the Lie splitting methods (3.1.5). In the case of the Strang splitting (3.1.6), we have from the same arguments:

$$\forall\, n \geq 0, \quad \|f^n(x,v) - f(nh,x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq C\varepsilon\langle nh\rangle^{s-7}h^2.$$

Hence we obtain convergence results which are global in time only if we measure the error in L^2. If we measure the error in H^σ, $\sigma > 0$, then for a fixed time horizon $nh \leq T$, the error grows like T^σ. This is however better than the rough e^T, estimate that is usually obtained through Gronwall type arguments (Note that convergence results can be found in [34] for the case of compactly supported data, and in [25] for the Vlasov-Poisson case).
Let us also mention as an easy consequence of (3.2.4), (3.2.5), (3.2.6) and (3.2.7), that the following estimates hold for the limit state of the equation: For the Lie splitting

$$\|g^\infty(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq Ch,$$

and for the Strang splitting

$$\|g^\infty(x,v) - g^\infty_h(x,v)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq Ch^2.$$

b) The long time behavior of the exact solution (3.1.2) is essentially controlled by a time convergent integral (see [40]). We shall see (Proposition 3.3.1 below) that the splitting method provides a discretization of this integral, but essentially without changing the decay in time of the integrand. Thus the numerical solution also yields a convergent time integral, even if the time appears in a discontinous way, giving us the long time behavior.
Moreover, this discretization is performed by rectangle methods in the case of Lie splittings, and by the midpoint rule in the case of Strang splitting. Estimates (3.2.6) and (3.2.7) reflect the respective accuracies of these two methods. The second order estimate requires a more refined analysis than the first order, for it is obtained by tracking the cancellations provided by the midpoint rules. We mention that this result remains true for Strang splitting of the form (3.1.6) where the role of T and P are exchanged but the complete proof is given for (3.1.6) only (the time integration rule being the trapezoidal rule and the arguments identical for both cases).
Finally, let us mention that the proofs of the convergence results (for Lie or Strang) widely use the long time behavior of both the exact and discrete solutions, in particular uniform bounds on their regularity. This can be understood as stability results for the numerical schemes. The convergence results are essentially the combination of these stability results, and the accuracy of the discretization of the integral.

c) Our results hold only for time discretization of the equation. Fully discrete scheme including for example an interpolation procedure at each step (semi-Lagrangian methods) traditionally exhibit *recurrence* phenomena due the discretization in the v variable. Indeed, the Landau damping effect reflects essentially the fact that the solution of the free Vlasov equation is a superposition of travelling wave in the Fourier variable ξ. At the discrete level, the ξ variable is only discretized by a finite number of points which causes numerical

interactions of these travelling waves preventing the mixing effect to occur for very long time. Typically, the previous result is hence valid *a priori* for a time of order $\mathcal{O}(1/\delta v)$ only, if δv is the size of the mesh variable in v. Solutions exist to remedy these difficulties, for examples by putting absorbers in the Fourier spaces, see for instance [35]. The analysis of these space discretization effects will be the subject of further studies.

d) Our result should also hold without any major modification in the case of the Vlasov-HMF equation with an attractive potential, namely

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = -\partial_x\Big(\int_{\mathbb{R}\times\mathbb{T}} P(x-y)f(t,y,u)\mathrm{d}u\mathrm{d}y\Big)\partial_v f(t,x,v).$$

The main difference is that the Penrose criterion **(H)** is less easier to satisfy in that case: it is not sufficient for the stationary state to be a decreasing function of $|v|$ anymore.

Let us finally explain how the previous scattering results imply Landau damping effects for the solution $f(t,x,v)$. Let us recall the following elementary Lemma:

Lemma 3.2.3. *For every α, β, γ, $s \in \mathbb{N}$ with $\alpha+\beta = s$, and $\gamma < \nu - \frac{1}{2}$. we have the following inequality:*

$$\forall\, k \in \mathbb{Z}, \quad \forall\, \xi \in \mathbb{R}, \quad |\partial_\xi^\gamma \hat{f}_k(\xi)| \leqslant 2^{s/2} C(\nu)\langle k\rangle^{-\alpha}\langle \xi\rangle^{-\beta}\|f\|_{\mathcal{H}^s_\nu}, \tag{3.2.8}$$

where $C(\nu)$ depends only on $\nu > 1/2$.

Proof. We have by using the Cauchy-Schwarz inequality that

$$\begin{aligned}\left|k^\alpha \xi^\beta \partial_\xi^\gamma \hat{f}_k(\xi)\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}\times\mathbb{R}} \partial_x^\alpha \partial_v^\beta (v^\gamma f(x,v)) e^{-ikx}e^{-iv\xi}\mathrm{d}x\mathrm{d}v\right| \\ &\leqslant C\|f\|_{\mathcal{H}^s_\nu}\Big(\int_{\mathbb{R}}(1+|v|^2)^{\gamma-\nu}\mathrm{d}v\Big)^{1/2}.\end{aligned}$$

The previous inequality with $\alpha = \beta = 0$ yields the result when $k = 0$ or $|\xi| \leqslant 1$ and we conclude by using $\langle x\rangle \leqslant 2^{\alpha/2}|x|^2$ for $|x| > 1$ and the fact that $\nu - \gamma > 1/2$. □

As a consequence of Theorem 3.2.2, we have the nonlinear Landau damping effect for the semi-discrete solution: The functions $g^n(x,v)$ being bounded in $\mathcal{H}^{s-4}_\nu$, $f^n(x,v) = \eta(v) + \varepsilon r^n(x,v) = \eta(v) + \varepsilon g^n(x-nhv,v)$ satisfy

$$\forall\, k \in \mathbb{Z}^*, \quad \forall \xi \in \mathbb{R}, \quad \forall\, \alpha+\beta = s-4, \quad |\hat{f}^n_k(\xi)| = \varepsilon|\hat{g}^n_k(\xi+knh)| \leq \frac{C\varepsilon}{\langle \xi+knh\rangle^\alpha\langle k\rangle^\beta},$$

the last estimate being a consequence of the embedding Lemma 3.2.3. This yields that for every $k \neq 0$, $\hat{f}^n_k(\xi)$ tends to zero when $nh \to \infty$ with a polynomial rate, but with a speed depending on k.

Moreover, by setting

$$\eta^\infty_h(v) := \eta(v) + \frac{\varepsilon}{2\pi}\int_{\mathbb{T}} g^\infty_h(x,v)\mathrm{d}x,$$

we have by the previous Lemma 3.2.3 that for $r \leq s-4$,

$$\forall\, \xi \in \mathbb{R}, \quad |\hat{f}_0^n(\xi) - \hat{\eta}_h^\infty(\xi)| \leq \frac{C}{\langle \xi \rangle^r \langle nh \rangle^{s-r-3}}.$$

In other words, $f^n(x,v)$ converges weakly towards $\eta_h^\infty(v)$. Moreover, this weak limit $\eta_h^\infty(v)$ is $\mathcal{O}(h)$ for Lie splittings (or $\mathcal{O}(h^2)$ for Strang splitting) close to the exact limit

$$\eta^\infty(v) := \eta(v) + \frac{\varepsilon}{2\pi} \int_{\mathbb{T}} g^\infty(x,v) \mathrm{d}x,$$

which exists by Theorem 3.2.1.

3.3 Backward error analysis

The unknown $g(t,x,v)$ defined in (3.1.2) is solution of the equation

$$\partial_t g = \{\phi(t, g(t)), \eta\} + \varepsilon \{\phi(t, g(t)), g\}. \tag{3.3.1}$$

where

$$\phi(t,g)(x,v) = \int_{\mathbb{R}\times\mathbb{T}} (\cos(x - y + t(v-u))) g(y,u) \mathrm{d}u \mathrm{d}y, \tag{3.3.2}$$

and $\{f,g\} = \partial_x f \partial_v g - \partial_v f \partial_x g$ is the usual microcanonical Poisson bracket. In Fourier space, we have the expression:

$$\phi(t,g) = \frac{1}{2} \sum_{k \in \{\pm 1\}} e^{ikx} e^{iktv} \hat{g}_k(kt).$$

In the evolution of the solution $g(t,x,v)$ of (3.3.1), an important role is played by the quantity

$$\zeta_k(t) = \hat{g}_k(t, kt), \quad k \in \{\pm 1\}, \tag{3.3.3}$$

such that

$$\phi(t, g(t)) = \frac{1}{2} \sum_{k \in \{\pm 1\}} e^{ikx} e^{iktv} \zeta_k(t).$$

Note that for $k \neq 0$, $\zeta_k(t)$ is the Fourier coefficient in x of the density $\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\, dv$.

The following result shows that semi-discrete solution $g^n(x,v)$ also satisfies an equation of the form (3.3.1), but with a discontinuous dependence with respect to the time:

Proposition 3.3.1. *For $h > 0$, the solution $g^n(x,v)$ given by the splitting method (3.1.5) (Lie) or (3.1.6) (Strang), and formula (3.1.8) coincides at times $t = nh$ with the solution $\mathsf{g}(t,x,v)$ of the equation*

$$\partial_t \mathsf{g} = \{\Phi_h(t, \mathsf{g}(t)), \eta\} + \varepsilon \{\Phi_h(t, \mathsf{g}(t)), \mathsf{g}(t)\}, \tag{3.3.4}$$

where $\Phi_h(t, \mathsf{g}(t)) = \phi(\mathsf{s}_h(t), \mathsf{g}(t))$ *with the definition of* ϕ *given in* (3.3.2).
In the case of Lie splittings (3.1.5) , $\mathsf{s}_h(t)$ *are given by the formulas*

$$\mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h + h, \quad \textit{and} \quad \mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h, \tag{3.3.5}$$

respectively. In the case of the Strang splitting method (3.1.6), $\mathsf{s}_h(t)$ *is given by the formula*

$$\mathsf{s}_h(t) = \left\lfloor \frac{t}{h} \right\rfloor h + \frac{h}{2}. \tag{3.3.6}$$

Proof. We prove the result in the case of Strang splitting (3.1.6)-(3.1.8), the proof being analogous for Lie splittings.
By definition, the function $f^n(x, v)$ satisfies the recurrence relation (3.1.6). Hence, we have (using the linearity of φ_T^t and the fact that $\varphi_T^t(\eta) = \eta$ for all $t \in \mathbb{R}$)

$$\begin{aligned}
\eta + \varepsilon g^n &= \varphi_T^{-nh}(f^n) \\
&= \varphi_T^{-nh} \circ \varphi_T^{h/2} \circ \varphi_P^h \circ \varphi_T^{h/2}(f^{n-1}) \\
&= \varphi_T^{-nh+h/2} \circ \varphi_P^h \circ \varphi_T^{h/2+(n-1)h} \varphi_T^{-(n-1)h}(f^{n-1}) \\
&= \varphi_T^{-nh+h/2} \circ \varphi_P^h \circ \varphi_T^{nh-h/2}(\eta + \varepsilon g^{n-1}).
\end{aligned}$$

Now we verify that for $t \in [0, h]$, the application $t \mapsto \varphi_T^{-nh+h/2} \circ \varphi_P^t \circ \varphi_T^{nh-h/2}(\eta + \varepsilon g^{n-1})$ is the solution of the equation

$$\partial_t \tilde{\mathsf{g}} = \{\phi(\mathsf{s}_h(t), \tilde{\mathsf{g}}), \tilde{\mathsf{g}}\}.$$

with inital data $\tilde{\mathsf{g}}(0) = \eta + \varepsilon g^{n-1}$. Using the fact that η is a stationary state of the equation, we easily get the result. □

For notational convenience, we will often write in the following $\mathsf{s}(t)$ instead of $\mathsf{s}_h(t)$. As in (3.3.3), we define

$$\mathsf{z}_k(t) = \hat{\mathsf{g}}_k(t, k\mathsf{s}(t)), \quad k \in \{\pm 1\}, \tag{3.3.7}$$

such that

$$\Phi_h(t, \mathsf{g}(t)) = \phi(\mathsf{s}(t), \mathsf{g}(t)) = \frac{1}{2} \sum_{k \in \{\pm 1\}} e^{ikx} e^{ik\mathsf{s}(t)v} \mathsf{z}_k(t).$$

Of course, we expect the $\mathsf{z}_k(t)$ to be approximations of the terms $\zeta_k(t)$ defined in (3.3.3).

Lemma 3.3.2. *Let* $h_0 > 0$ *be given. There exist two constants* c *and* $C > 0$ *such that for all* $h \in (0, h_0]$ *and all* $t > 0$,

$$c\langle t \rangle \leq \langle \mathsf{s}_h(t) \rangle \leq C\langle t \rangle$$

and for all t *and* σ,

$$c\langle t \pm \sigma \rangle \leq \langle \mathsf{s}_h(t) \pm \mathsf{s}_h(\sigma) \rangle \leq C\langle t \pm \sigma \rangle.$$

Proof. For $t \in \mathbb{R}$, we can write $t = nh + \mu$ with $\mu \in [0, h)$. In the case of Strang splitting (3.3.6), we thus have $\mathsf{s}(t) = nh + h/2 = t + h/2 - \mu$. Hence we have $t - \mathsf{s}(t) \in [-h/2, h/2)$ which clearly implies the first inequality. The second is proved using the fact that with similar calculations $t \pm \sigma = \mathsf{s}(t) \pm \mathsf{s}(\sigma) + \mathcal{O}(h)$. The proof is analogous for Lie splittings (3.3.5). □

As in [40] we introduce the weighted norms:

$$N_{T,s,\nu}(\mathsf{g}) = \sup_{t\in[0,T]} \frac{\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu}}{\langle t\rangle^3}, \quad M_{T,\gamma}(\mathsf{z}) = \sup_{t\in[0,T]} \sup_{k\in\{\pm 1\}} \langle t\rangle^\gamma |\mathsf{z}_k(t)| \tag{3.3.8}$$

and

$$Q_{T,s,\nu}(\mathsf{g}) = N_{T,s,\nu}(\mathsf{g}) + M_{T,s-1}(\mathsf{z}) + \sup_{[0,T]} \|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu}. \tag{3.3.9}$$

We shall prove the following result:

Theorem 3.3.3. *Let us fix $s \geq 7$, $\nu > 1/2$ and $R_0 > 0$ such that $Q_{0,s,\nu}(\mathsf{g}) \leq R_0$, and assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* (**H**)*. Then there exist $R > 0$, $h_0 > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, $h \in (0, h_0]$ and for every $T \geq 0$, the solution of (3.3.4) satisfies the estimate*

$$Q_{T,s,\nu}(\mathsf{g}) \leq R.$$

This result is a semi-discrete version of the main Theorem in [40] where the same norms are used to control the solution $g(t)$ of the equation (3.3.1). Let us also mention that it holds for any of the three formulas (3.3.6)-(3.3.5) defining $\mathsf{s}(t)$, the only property being used is the fact that $\mathsf{s}(t)$ satisfies Lemma 3.3.2.

3.4 Estimates

We fix now $s \geq 7$ and R_0 as in the previous Theorem. In the following a priori estimates, C stands for a number which may change from line to line and which is independent of R_0, R, h, ε and T.

3.4.1 Estimate of $M_{T,s-1}(\mathsf{z})$

Towards the proof of Theorem 3.3.3, we shall first estimate $\mathsf{z}_k(t)$, $k = \pm 1$.

Proposition 3.4.1. *Assuming that $\eta \in \mathcal{H}^{s+2}_\nu$ verifies the assumption* (**H**)*, then there exist $C > 0$ and $h_0 > 0$ such that for every $T > 0$ and $h \in (0, h_0]$, every solution of (3.3.4) such that $Q_{T,s,\nu}(\mathsf{g}) \leq R$ enjoys the estimate*

$$M_{T,s-1}(\mathsf{z}) \leq C\big(R_0 + \varepsilon R^2\big). \tag{3.4.1}$$

Proof. The main ingredient of the proof of the previous result is to write the equation (3.3.4) in Fourier:

$$\hat{\mathsf{g}}_n(t,\xi) = \hat{\mathsf{g}}_n(0,\xi) + \int_0^t p_n \mathsf{z}_n(\sigma)\hat{\eta}_0(\xi - n\mathsf{s}(\sigma))(n^2\mathsf{s}(\sigma) - n\xi)\mathrm{d}\sigma \\ + \varepsilon \sum_{k\in\mathbb{Z}} p_k \int_0^t \mathsf{z}_k(\sigma)\hat{\mathsf{g}}_{n-k}(\sigma, \xi - k\mathsf{s}(\sigma))(nk\mathsf{s}(\sigma) - k\xi)\mathrm{d}\sigma, \quad (3.4.2)$$

for all $(n,\xi) \in \mathbb{Z}\times\mathbb{R}$, with $p_k = \frac{1}{2}$ for $k \in \{\pm 1\}$ and $p_k = 0$ for $k \neq \pm 1$, and where the $\mathsf{z}_k(t)$ are defined by (3.3.7). Setting $\xi = n\mathsf{s}(t)$ in (3.4.2), the equation satisfied by $(\mathsf{z}_n(t))_{n=\pm 1}$ can be written under the almost closed form

$$\mathsf{z}_n(t) = \hat{\mathsf{g}}_n(0, n\mathsf{s}(t)) - \int_0^t p_n \mathsf{z}_n(\sigma)\hat{\eta}_0(n(\mathsf{s}(t) - \mathsf{s}(\sigma)))n^2(\mathsf{s}(t) - \mathsf{s}(\sigma))\mathrm{d}\sigma \\ - \varepsilon \sum_{k\in\{\pm 1\}} p_k \int_0^t \mathsf{z}_k(\sigma)\hat{\mathsf{g}}_{n-k}(\sigma, n\mathsf{s}(t) - k\mathsf{s}(\sigma))kn(\mathsf{s}(t) - \mathsf{s}(\sigma))\mathrm{d}\sigma. \quad (3.4.3)$$

Remark 3.4.2. *Note that, for every $m \in \mathbb{N}$, we have for $t \in [mh, (m+1)h]$ the formula*

$$\mathsf{z}_n(t) - \hat{\mathsf{g}}_n(mh, n\mathsf{s}(t)) = -\int_{mh}^t p_n \mathsf{z}_n(\sigma)\hat{\eta}_0(n(\mathsf{s}(t) - \mathsf{s}(\sigma)))n^2(\mathsf{s}(t) - \mathsf{s}(\sigma))\mathrm{d}\sigma \\ - \varepsilon \sum_{k\in\{\pm 1\}} p_k \int_{mh}^t \mathsf{z}_k(\sigma)\hat{\mathsf{g}}_{n-k}(\sigma, n\mathsf{s}(t) - k\mathsf{s}(\sigma))kn(\mathsf{s}(t) - \mathsf{s}(\sigma))\mathrm{d}\sigma.$$

As $\mathsf{s}(t) - \mathsf{s}(\sigma) = 0$ for almost every $\sigma \in [mh, t]$, we notice that the function $t \mapsto \hat{\mathsf{g}}_n(t, n\mathsf{s}(t)) = \mathsf{z}_n(t)$ is constant on the small intervalls $[mh, (m+1)h]$. This is due to the fact that the electric field is constant during the evolution of (3.1.4).

To study the equation (3.4.3), we shall first consider the corresponding linear equation, that is to say that we shall first see

$$F_n(t) := \hat{\mathsf{g}}_n(0, n\mathsf{s}(t)) - \varepsilon \sum_{k\in\{\pm 1\}} p_k \int_0^t \mathsf{z}_k(\sigma)\hat{g}_{n-k}(\sigma, n\mathsf{s}(t) - k\mathsf{s}(\sigma))kn(\mathsf{s}(t) - \mathsf{s}(\sigma))\mathrm{d}\sigma \quad (3.4.4)$$

as a given source term and we shall study the linear integral equation

$$\mathsf{z}_n(t) = \int_0^t K(n, \mathsf{s}(t) - \mathsf{s}(\sigma))\mathsf{z}_n(\sigma)\,\mathrm{d}\sigma + F_n(t) \quad n = \pm 1, \quad (3.4.5)$$

where the kernel $K(n,t)$ has been introduced in (3.2.3). We rewrite this equation as

$$\mathsf{z}_n(t) = \int_0^t K(n, t-\sigma)\mathsf{z}_n(\sigma)\,\mathrm{d}\sigma + F_n(t) + G_n(t) \quad n = \pm 1, \quad (3.4.6)$$

where

$$G_n(t) = \int_0^t \Big(K(n, \mathsf{s}(t) - \mathsf{s}(\sigma)) - K(n, t-\sigma)\Big)\mathsf{z}_n(\sigma)\,\mathrm{d}\sigma \tag{3.4.7}$$

is an error term due to the time discretization. To study the linear equation (3.4.5)-(3.4.6), we use the result given by the following Lemma. The proof of this Lemma can be found in [40]. For completeness, we recall it in Appendix of the present chapter.

Lemma 3.4.3. *Let $\gamma \geq 0$, and assume that $\eta \in \mathcal{H}^{\gamma+3}_\nu$ satisfies* (**H**). *Then, there exists $C > 0$ such for every $T \geq 0$, we have*

$$M_{T,\gamma}(\mathsf{z}) \leq C M_{T,\gamma}(F + G).$$

From this Lemma and (3.2.8), we first get that

$$M_{T,s-1}(\mathsf{z}) \leq C\big(\|\mathsf{g}(0)\|_{\mathcal{H}^s_\nu} + M_{T,s-1}(G) + \varepsilon M_{T,s-1}(F^1) + \varepsilon M_{T,s-1}(F^2)\big) \tag{3.4.8}$$

where

$$F^1_n(t) = n^2 p_{-n}\int_0^t \mathsf{z}_{-n}(\sigma)\hat{\mathsf{g}}_{2n}(\sigma, n(\mathsf{s}(t)+\mathsf{s}(\sigma)))(\mathsf{s}(t)-\mathsf{s}(\sigma))\,\mathrm{d}\sigma, \quad n = \pm 1,$$
$$F^2_n(t) = -n^2 p_n\int_0^t \mathsf{z}_{n}(\sigma)\hat{\mathsf{g}}_{0}(\sigma, n(\mathsf{s}(t)-\mathsf{s}(\sigma)))(\mathsf{s}(t)-\mathsf{s}(\sigma))\,\mathrm{d}\sigma, \quad n = \pm 1.$$

Let us estimate F^1_n. By using again (3.2.8) and the definition (3.3.8) of $N_{\sigma,s,\nu}$, we get using Proposition 3.3.1 that

$$|F^1_n(t)| \leq C\int_0^t \frac{(\mathsf{s}(t)-\mathsf{s}(\sigma))\langle\sigma\rangle^3 M_{\sigma,s-1}(\mathsf{z})N_{\sigma,s,\nu}(\mathsf{g})}{\langle\sigma\rangle^{s-1}\langle\mathsf{s}(t)+\mathsf{s}(\sigma)\rangle^s}\,\mathrm{d}\sigma \leq C\frac{R^2}{\langle t\rangle^{s-1}}\int_0^{+\infty}\frac{1}{\langle\sigma\rangle^{s-4}}\,\mathrm{d}\sigma \leq C\frac{R^2}{\langle t\rangle^{s-1}}$$

provided $s \geq 6$. This yields that for all $T \geq 0$

$$M_{T,s-1}(F^1) \leq CR^2.$$

To estimate F^2_n, we split the integral into two parts: we write

$$F^2_n(t) = I^1_n(t) + I^2_n(t)$$

with

$$I^1_n(t) = -n^2 p_n\int_0^{\frac{t}{2}} \mathsf{z}_{n}(\sigma)\hat{\mathsf{g}}_{0}(\sigma, n(\mathsf{s}(t)-\mathsf{s}(\sigma)))(\mathsf{s}(t)-\mathsf{s}(\sigma))\,\mathrm{d}\sigma, \quad n = \pm 1,$$
$$I^2_n(t) = -n^2 p_n\int_{\frac{t}{2}}^t \mathsf{z}_{n}(\sigma)\hat{\mathsf{g}}_{0}(\sigma, n(\mathsf{s}(t)-\mathsf{s}(\sigma)))(\mathsf{s}(t)-\mathsf{s}(\sigma))\,\mathrm{d}\sigma, \quad n = \pm 1.$$

For I^1_n, we proceed as previously,

$$|I^1_n(t)| \leq CR^2\int_0^{\frac{t}{2}} \frac{\langle\sigma\rangle^3(\mathsf{s}(t)-\mathsf{s}(\sigma))}{\langle\sigma\rangle^{s-1}\langle\mathsf{s}(t)-\mathsf{s}(\sigma)\rangle^s}\,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}}\int_0^{+\infty}\frac{1}{\langle\sigma\rangle^{s-4}}\,\mathrm{d}\sigma$$

and hence since $s \geq 6$, we have

$$M_{T,s-1}(I^1) \leq CR^2.$$

To estimate I_n^2, we shall rather use the last factor in the definition of $Q_{T,s,\nu}$ in (3.3.9). By using again (3.2.8), we write

$$|I_n^2(t)| \leq C \int_{\frac{t}{2}}^t \frac{M_{\sigma,s-1}(\mathsf{z})}{\langle\sigma\rangle^{s-1}} \frac{\|\mathsf{g}(\sigma)\|_{\mathcal{H}_\nu^{s-4}}}{\langle \mathsf{s}(t)-\mathsf{s}(\sigma)\rangle^{s-5}} \,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}} \int_0^{+\infty} \frac{1}{\langle\sigma\rangle^{s-5}} \,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}}$$

and hence since $s \geq 7$, we find again

$$M_{T,s-1}(I^2) \leq CR^2.$$

Finally, we have

$$\begin{aligned} \langle t\rangle^{s-1}|G_n(t)| &\leq M_{T,s-1}(\mathsf{z}) \int_0^t \frac{\langle t\rangle^{s-1}}{\langle\sigma\rangle^{s-1}} \left| \int_{t-\sigma}^{\mathsf{s}(t)-\mathsf{s}(\sigma)} |\partial_t K(n,\theta)| \mathrm{d}\theta \right| \mathrm{d}\sigma \\ &\leq CM_{T,s-1}(\mathsf{z}) \int_0^t \left| \int_{t-\sigma}^{\mathsf{s}(t)-\mathsf{s}(\sigma)} \frac{\langle t\rangle^{s-1}}{\langle\theta\rangle^{s+2}\langle\sigma\rangle^{s-1}} \mathrm{d}\theta \right| \mathrm{d}\sigma, \end{aligned}$$

where we used the fact that $\eta \in \mathcal{H}_\nu^{s+2}$ and Lemma 3.2.3. Now, since

$$\left| \int_{t-\sigma}^{\mathsf{s}(t)-\mathsf{s}(\sigma)} \frac{\mathrm{d}\theta}{\langle\theta\rangle^{s+2}} \right| \leq \frac{Ch}{\langle t-\sigma\rangle^{s+2}},$$

we get

$$M_{T,s-1}(G) \leq ChM_{T,s-1}(\mathsf{z}) \int_0^t \frac{\langle t\rangle^{s-1}}{\langle t-\sigma\rangle^{s+2}\langle\sigma\rangle^{s-1}} \mathrm{d}\sigma.$$

As the integral

$$\int_0^t \frac{\langle t\rangle^{s-1}}{\langle t-\sigma\rangle^{s+2}\langle\sigma\rangle^{s-1}} \mathrm{d}\sigma \leq C \int_0^{t/2} \frac{1}{\langle t\rangle^3\langle\sigma\rangle^{s-1}} \mathrm{d}\sigma + C \int_{t/2}^t \frac{1}{\langle t-\sigma\rangle^{s+2}} \mathrm{d}\sigma.$$

is uniformly bounded in time, we conclude that

$$M_{T,s-1}(G) \leq ChM_{T,s-1}(\mathsf{z}). \tag{3.4.9}$$

By combining the last estimates and (3.4.8), we thus obtain (3.4.1).

$$M_{T,s-1}(\mathsf{z}) \leq C\big(R_0 + hM_{T,s-1}(\mathsf{z}) + \varepsilon R^2\big).$$

By taking $h \leq h_0$ small enough, this ends the proof of Proposition 3.4.1. □

3.4.2 Estimate of $N_{T,s,\nu}(\mathsf{g})$

Proposition 3.4.4. *Assuming that $\eta \in \mathcal{H}^{s+2}_\nu$ verifies the assumption* **(H)**, *then there exists $C > 0$ and $h_0 > 0$ such that for every $T > 0$ and $h \in (0, h_0]$, every solution of (3.3.4) such that $Q_{T,s,\nu}(\mathsf{g}) \leq R$ enjoys the estimate*

$$N_{T,s,\nu}(\mathsf{g}) \leq C(R_0 + \varepsilon R^2)(1 + \varepsilon R)e^{C\varepsilon R}.$$

Proof. To prove Proposition 3.4.4, we shall use energy estimates. We set $\mathcal{L}_t[\mathsf{g}]$ the operator

$$\mathcal{L}_t[\mathsf{g}]f = \{\phi(t, \mathsf{g}), f\}$$

such that g solves the equation

$$\partial_t \mathsf{g} = \mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)](\eta + \varepsilon \mathsf{g}).$$

For any linear operator D, we thus have by standard manipulations that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D\mathsf{g}(t)\|^2_{L^2} &= 2\varepsilon\langle D\mathsf{g}(t), D(\mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)]\mathsf{g}(t))\rangle_{L^2} + 2\langle D\mathsf{g}(t), D(\mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)](\eta))\rangle_{L^2} \\ &= 2\varepsilon\langle D\mathsf{g}(t), \mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)]D\mathsf{g}(t)\rangle_{L^2} + 2\varepsilon\langle D\mathsf{g}(t), [D, \mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)]]\mathsf{g}(t)\rangle_{L^2} \\ &\quad +2\langle D\mathsf{g}(t), D(\mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)](\eta))\rangle_{L^2},\end{aligned}$$

where $[D, \mathcal{L}_{\mathsf{s}(t)}]$ denotes the commutator between the two operators D and $\mathcal{L}_{\mathsf{s}(t)}$. The first term in the previous equality vanishes since $\mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}]$ is the transport operator associated with a divergence free Hamiltonian vector field. Consequently, we get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D\mathsf{g}(t)\|^2_{L^2} \leqslant 2\varepsilon\ \|D\mathsf{g}(t)\|_{L^2}\|[D, \mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(t)]]\mathsf{g}(t)\|_{L^2} \\ + 2\|D\mathsf{g}(t)\|_{L^2}\|D(\mathcal{L}_{\mathsf{s}(t)}[\mathsf{g}(\sigma)](\eta))\|_{L^2}. \quad (3.4.10)\end{aligned}$$

To get the estimates of Proposition 3.4.4, we shall use the previous estimates with the operator $D = D^{m,p,q}$ defined as the Fourier multiplier by $k^p\xi^q\partial^m_\xi$ for $(m, p, q) \in \mathbb{N}^{3d}$ such that $p + q \leqslant s$, $m \leqslant \nu$ and the definition (3.2.1) of the $\mathcal{H}^s_\nu$ norm. To evaluate the right hand-side of (3.4.10), we shall use the following Lemma, whose proof is given in Appendix (see also [40]).

Lemma 3.4.5. *For $p + q \leqslant \gamma$ and $r \leqslant \nu$, and functions $h(x, v)$ and $g(x, v)$, we have the estimates*

$$\|\big[D^{r,p,q}, \mathcal{L}_\sigma[g]\big]h\|_{L^2} \leq C\big(m_{\sigma,\gamma+1}(\zeta)\|h\|_{\mathcal{H}^1_\nu} + m_{\sigma,2}(\zeta)\|h\|_{\mathcal{H}^\gamma_\nu}\big), \quad (3.4.11)$$

$$\|D^{r,p,q}\big(\mathcal{L}_\sigma[g]\big)h\|_{L^2} \leq C\big(m_{\sigma,\gamma+1}(\zeta)\|h\|_{\mathcal{H}^1_\nu} + m_{\sigma,2}(\zeta)\|h\|_{\mathcal{H}^{\gamma+1}_\nu}\big), \quad (3.4.12)$$

for all σ, where the sequence ζ_k is defined by $\zeta_k = \hat{g}_k(k\sigma)$, $k \in \{\pm 1\}$, and where

$$m_{\sigma,\gamma}(\zeta) = \langle\sigma\rangle^\gamma\Big(\sup_{k\in\{\pm1\}}|\zeta_k|\Big),$$

with a constant C depending only on γ, and in particular, not depending on σ.

Let us finish the proof of Proposition 3.4.4. By using the previous lemma with $\gamma = s$, $\sigma = \mathsf{s}(t)$, $g = \mathsf{g}(t)$ and $h = \mathsf{g}(t)$ or $h = \eta$, we obtain from (3.4.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu}^2 \le C\langle t\rangle^2 m_{t,s-1}(\mathsf{z}(t))\big(\|\eta\|_{\mathcal{H}^1_\nu} + \varepsilon\|\mathsf{g}(t)\|_{\mathcal{H}^1_\nu}\big)\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu} + \frac{C}{\langle t\rangle^{s-3}} m_{t,s-1}(\mathsf{z}(t))\|\eta\|_{\mathcal{H}^{s+1}_\nu}\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu} + \frac{C\varepsilon}{\langle t\rangle^{s-3}} m_{t,s-1}(\mathsf{z}(t))\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu}^2.$$

This yields, using the fact that $M_{t,\gamma}(\mathsf{z}) = \sup_{\sigma\in[0,t]} m_{\sigma,\gamma}(\mathsf{z}(\sigma))$,

$$\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu} \le \|\mathsf{g}(0)\|_{\mathcal{H}^s_\nu} + C\langle t\rangle^3 M_{t,s-1}(\mathsf{z})\big(\|\eta\|_{\mathcal{H}^{s+1}_\nu} + \varepsilon R\big) + C\varepsilon R\int_0^t \frac{1}{\langle\sigma\rangle^{s-3}}\|\mathsf{g}(\sigma)\|_{\mathcal{H}^s_\nu}\,\mathrm{d}\sigma$$

for $t \in [0,T]$. From the Gronwall inequality, we thus obtain

$$\|\mathsf{g}(t)\|_{\mathcal{H}^s_\nu} \le \Big(\|\mathsf{g}(0)\|_{\mathcal{H}^s_\nu} + C\langle t\rangle^3 M_{t,s-1}(\mathsf{z})\big(\|\eta\|_{\mathcal{H}^{s+1}_\nu} + \varepsilon R\big)\Big)e^{C\varepsilon R\int_0^{+\infty}\frac{\mathrm{d}\sigma}{\langle\sigma\rangle^{s-3}}}.$$

By using Proposition 3.4.1, this yields

$$N_{T,s,\nu}(\mathsf{g}) \le \Big(R_0 + C(R_0+\varepsilon R^2)(1+\varepsilon R)\Big)e^{C\varepsilon R}.$$

This ends the proof of Proposition 3.4.4. □

3.4.3 Estimate of $\|\mathsf{g}\|_{\mathcal{H}^{s-4}_\nu}$

To close the argument, it only remains to estimate $\|\mathsf{g}\|_{\mathcal{H}^{s-4}_\nu}$.

Proposition 3.4.6. *Assuming that $\eta \in \mathcal{H}^{s+2}_\nu$ verifies the assumption* **(H)**, *then there exists $C > 0$ and $h_0 > 0$ such that for every $T > 0$ and $h \in (0, h_0]$, every solution of (3.3.4) such that $Q_{T,s,\nu}(\mathsf{g}) \le R$ enjoys the estimate*

$$\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu} \le C\big(R_0 + \varepsilon R^2\big)e^{C\varepsilon R}, \quad \forall t \in [0,T].$$

Proof. We use again (3.4.10) with $D = D^{m,p,q}$ but now with $p+q \le s-4$. By using Lemma 3.4.5, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu}^2 \le C m_{t,s-3}(\mathsf{z}(t))\big(\|\eta\|_{\mathcal{H}^{s-3}_\nu}\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu} + \varepsilon\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu}^2\big). \tag{3.4.13}$$

This yields

$$\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu} \le \|\mathsf{g}(0)\|_{\mathcal{H}^{s-4}_\nu} + C\|\eta\|_{\mathcal{H}^{s-3}_\nu} M_{t,s-1}(\mathsf{z})\int_0^t \frac{1}{\langle\sigma\rangle^2}\,\mathrm{d}\sigma + C\varepsilon M_{t,s-1}(\mathsf{z})\int_0^t \frac{1}{\langle\sigma\rangle^2}\|\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-4}_\nu}\,\mathrm{d}\sigma.$$

By using Proposition 3.4.1, we thus get

$$\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu} \leq C\big(R_0+\varepsilon R^2\big)+C\varepsilon R\int_0^t \frac{1}{\langle\sigma\rangle^2}\|\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-4}_\nu}\,\mathrm{d}\sigma.$$

From the Gronwall inequality, we finally find

$$\|\mathsf{g}(t)\|_{\mathcal{H}^{s-4}_\nu} \leq C\big(R_0+\varepsilon R^2\big)e^{C\varepsilon R}.$$

This ends the proof of Proposition 3.4.6.

□

3.4.4 Proof of Theorem 3.3.3 and estimate (3.2.5)

The proof of Theorem 3.3.3 follows from the a priori estimates in Propositions 3.4.1, 3.4.4 and 3.4.6 and a continuation argument. Indeed, by combining the estimates of these three propositions, we get that

$$Q_{T,s,\nu}(\mathsf{g}) \leq C(R_0+\varepsilon R^2)(1+\varepsilon R)e^{C\varepsilon R}$$

assuming that $Q_{T,s,\nu}(\mathsf{g}) \leq R$. Consequently, let us choose R such that $R > CR_0$, then for ε sufficiently small we have $R > C(R_0+\varepsilon R^2)(1+\varepsilon R)e^{C\varepsilon R}$ and hence by usual continuation argument, we obtain that the estimate $Q_{T,s,\nu}(\mathsf{g}) \leq R$ is valid for all times.

To prove (3.2.5), let us define $g_h^\infty(x,v)$ by

$$g_h^\infty(x,v) = \mathsf{g}(0,x,v)+\int_0^{+\infty}\mathcal{L}_{\mathsf{s}(\sigma)}[\mathsf{g}(\sigma)](\eta+\epsilon\mathsf{g}(\sigma))\mathrm{d}\sigma,$$

To justify the convergence of the integral, we note that thanks to (3.4.12), we have for all σ

$$\big\|\mathcal{L}_{\mathsf{s}(\sigma)}[\mathsf{g}(\sigma)](\eta+\epsilon\mathsf{g}(\sigma))\big\|_{\mathcal{H}^{s-4}_\nu} \leq C\Big(m_{\sigma,s-3}(\mathsf{z}(\sigma))\,\|\eta+\epsilon\mathsf{g}(\sigma)\|_{\mathcal{H}^1_\nu}+m_{\sigma,2}(\mathsf{z}(\sigma))\,\|\eta+\epsilon\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-3}_\nu}\Big),$$

where $\mathsf{z}_k(t) = \hat{\mathsf{g}}_k(t,k\mathsf{s}(t))$, $k=\pm1$, and

$$m_{\sigma,\gamma}(\mathsf{z}(\sigma)) = \langle\sigma\rangle^\gamma\left(\sup_{k=\pm1}|\mathsf{z}_k(\sigma)|\right).$$

By interpolation, we have

$$\|\eta+\epsilon\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-3}_\nu} \leq C\|\eta+\epsilon\mathsf{g}(\sigma)\|^{\frac{3}{4}}_{\mathcal{H}^{s-4}_\nu}\|\eta+\epsilon\mathsf{g}(\sigma)\|^{\frac{1}{4}}_{\mathcal{H}^{s}_\nu},$$

and thus, using the bound provided by Theorem 3.3.3, we have

$$\|\eta+\epsilon\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-3}_\nu} \leq C(R)\langle\sigma\rangle^{\frac{3}{4}}.$$

Using again Theorem 3.3.3, we thus find that

$$\left\|\mathcal{L}_{\mathsf{s}(\sigma)}[\mathsf{g}(\sigma)](\eta+\epsilon\mathsf{g}(\sigma))\right\|_{\mathcal{H}^{s-4}_{\nu}} \leq C(R)\left(\frac{1}{\langle\sigma\rangle^2}+\frac{1}{\langle\sigma\rangle^{s-3-\frac{3}{4}}}\right), \tag{3.4.14}$$

and $g_h^\infty(x,v)$ is then well defined, and belongs to $\mathcal{H}^{s-4}_\nu$.
Since we have for all t

$$\mathsf{g}(t)-g_h^\infty(x,v)=\int_t^{+\infty}\mathcal{L}_{\mathsf{s}(\sigma)}[\mathsf{g}(\sigma)](\eta+\epsilon\mathsf{g}(\sigma))\mathrm{d}\sigma,$$

we find by using again (3.4.14) that

$$\|\mathsf{g}(t)-g_h^\infty\|_{\mathcal{H}^{s-4}} \leq C(R)\Big(\int_t^{+\infty}\frac{1}{\langle\sigma\rangle^2}+\frac{1}{\langle\sigma\rangle^{s-3-\frac{3}{4}}}\,\mathrm{d}\sigma\Big)\leq\frac{C(R)}{\langle t\rangle}.$$

In a similar way, by using again (3.4.12), we have for $r\leq s-4$ and $r\geq 1$,

$$\|\mathsf{g}(t)-g_h^\infty\|_{\mathcal{H}^r_\nu} \leq C(R)\Big(\int_t^{+\infty}\frac{1}{\langle\sigma\rangle^{s-r-2}}+\frac{1}{\langle\sigma\rangle^{s-3}}\,\mathrm{d}\sigma\Big)\leq C(R)\big(\frac{1}{\langle t\rangle^{s-r-3}}+\frac{1}{\langle t\rangle^{s-4}}\big)\leq\frac{C(R)}{\langle t\rangle^{s-r-3}},$$

which gives the result using the fact that $\mathsf{g}(nh)=g^n$ the solution given by the numerical scheme.

3.5 Proof of the convergence estimate (3.2.6)

We shall now prove the convergence estimate (3.2.6). Note that in view of the previous result and of the analysis in [40], the functions $\mathsf{g}(t,x,v)$ and $g(t,x,v)$ satisfy the same estimates. In particular, we can assume that $Q_{T,s,\nu}(\mathsf{g})$, $Q_{T,s,\nu-1}(\mathsf{g})$, $Q_{T,s,\nu}(g)$ and $Q_{T,s,\nu-1}(g)$ are both uniformly bounded by the same constant R, provided that $\nu>3/2$.

By using the equations (3.3.1) and (3.3.4), we get that $\delta=g-\mathsf{g}$ solves the equation

$$\partial_t\delta(t)=\{\phi(t,\delta(t)),\eta\}+\varepsilon\{\phi(t,\delta(t)),g(t)\}+\varepsilon\{\phi(t,g(t)),\delta(t)\}-\varepsilon\{\phi(t,\delta(t)),\delta(t)\}+\mathcal{R} \tag{3.5.1}$$

with

$$\mathcal{R}(t,x,v)=\{\phi(t,\mathsf{g}(t))-\phi(\mathsf{s}(t),\mathsf{g}(t)),\eta\}+\varepsilon\{\phi(t,\mathsf{g}(t))-\phi(\mathsf{s}(t),\mathsf{g}(t)),\mathsf{g}(t)\} \tag{3.5.2}$$

and with zero initial data. It will be useful to use the expression of $\mathcal{R}$ in Fourier which reads

$$\begin{aligned}\hat{\mathcal{R}}_n(t,\xi)&=np_n\Big(\hat{\mathsf{g}}_n(t,nt)\hat{\eta}_0(\xi-nt)(nt-\xi)-\hat{\mathsf{g}}_n(t,n\mathsf{s}(t))\hat{\eta}_0(\xi-n\mathsf{s}(t))(n\mathsf{s}(t)-\xi)\Big)\\&+\varepsilon\sum_{k=\pm1}kp_k\Big(\hat{\mathsf{g}}_k(t,kt)\hat{\mathsf{g}}_{n-k}(t,\xi-kt)(nt-\xi)-\hat{\mathsf{g}}_k(t,k\mathsf{s}(t))\hat{\mathsf{g}}_{n-k}(t,\xi-k\mathsf{s}(t))(n\mathsf{s}(t)-\xi)\Big).\end{aligned} \tag{3.5.3}$$

Let T be a positive real number. By using the weighted norms defined in (3.3.8), we now consider for $s \geq 8$ and $\nu > 3/2$, the quantity

$$Q_{T,s-2,\nu-1}(\delta) = M_{T,s-3}(d) + N_{T,s-2,\nu-1}(\delta) + \sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}$$

with $N_{T,s,\gamma}$ and $M_{T,\gamma}$ defined in (3.3.8), and where we set

$$d_k(t) = \hat{g}_k(t, kt) - \hat{\mathsf{g}}_k(t, kt).$$

We shall prove the following result:

Proposition 3.5.1. *For $s \geq 8$, $\nu > 3/2$, assume that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**. *Then there exists $R_1 > 0$, $h_0 > 0$ and $\epsilon_0 > 0$ such that for every $h \in (0, h_0]$, every $\varepsilon \in (0, \varepsilon_0]$ and every $T \geq 0$, the solution of (3.5.1) satisfies the estimate*

$$Q_{T,s-2,\nu-1}(\delta) \leq R_1 h.$$

Proposition 3.5.1 clearly implies the convergence estimate (3.2.6). It actually proves that the interpolation $\mathsf{g}(t)$ of the sequence of functions $g^n(x, v)$ given by the splitting methods (Strang or Lie) is always at least an approximation of order one of the exact solution $g(t)$ at all times. The proof will use the same steps as in the proof of Theorem 3.3.3.

Remark 3.5.2. *We mentioned that $\mathsf{z}_k(t)$ is expected to be an approximation of $\zeta_k(t)$. By Taylor expanding in ξ, it is indeed clear that Proposition 3.5.1, together with the uniform bound on $Q_{T,s,\nu}(\mathsf{g})$ and $Q_{T,s,\nu}(g)$, implies that*

$$\sup_{\substack{t\in\mathbb{R}\\ k=\pm1}} |\zeta_k(t) - \mathsf{z}_k(t)| = \mathcal{O}(h).$$

In fact, we could have proved without any major difference Proposition 3.5.1 using the quantity $\tilde{d}_k(t) = \zeta_k(t) - \mathsf{z}_k(t)$ instead of $d_k(t)$. The use of $d_k(t)$ will however be crucial to prove the second order estimate, since it is clear that the quantity $\tilde{d}_k(0)$ does not scale in h^2.

Let us now begin the proof of Proposition 3.5.1.

3.5.1 Estimate of $M_{T,s-3}(d)$

We shall first estimate $d_k(t)$, $k = \pm 1$.

Proposition 3.5.3. *Assuming that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**, *there exists $C > 0$, $\varepsilon_0 > 0$ and $h_0 > 0$ such that for every $h \in (0, h_0]$, every $\varepsilon \in (0, \varepsilon_0]$ and every $T > 0$, every solution of (3.5.1) such that $Q_{T,s-2,\nu-1}(\delta) \leq R_1 h$ enjoys the estimate*

$$M_{T,s-3}(d) \leq C(R)h(1 + (\varepsilon + \varepsilon h)R_1^2),$$

where $C(R)$ is a number that depends only on R (one can take $C(R) = (1 + R + R^2)C$).

Proof. From (3.5.1), we obtain by taking the Fourier transform, integrating in time and setting $\xi = nt$ that

$$d_n(t) = \int_0^t K(n, t-\sigma) d_n(\sigma)\,\mathrm{d}\sigma + G_n(t) + H_n(t) \tag{3.5.4}$$

where

$$H_n(t) = \int_0^t \hat{\mathcal{R}}_n(\sigma, nt)\,\mathrm{d}\sigma$$

and

$$G_n(t) = \varepsilon \sum_{k=\pm 1} \int_0^t k p_k \Big(d_k(\sigma)\hat{g}_{n-k}(\sigma, nt-k\sigma) + \zeta_k(\sigma)\hat{\delta}_{n-k}(\sigma, nt-k\sigma) \\ - d_k(\sigma)\hat{\delta}_{n-k}(\sigma, nt-k\sigma)\Big) n(\sigma - t)\,\mathrm{d}\sigma.$$

The kernel $K(k,t)$ is still defined by (3.2.3).
By using Lemma 3.4.3, we find the estimate

$$M_{T,s-3}(d) \leq C\left(M_{T,s-3}(G) + M_{T,s-3}(H)\right). \tag{3.5.5}$$

To estimate $M_{T,s-3}(G)$, we proceed as in the proof of Theorem 3.3.3. We first split $G = G^1 + G^2$ where G^1 corresponds to the term $k = -n$ in the sum and G^2 corresponds to $k = n$. For G^1, we obtain

$$|G_n^1| \leq C\varepsilon \int_0^t (t-\sigma)\Big(\frac{1}{\langle\sigma\rangle^{s-3}} \frac{\langle\sigma\rangle^3}{\langle t+\sigma\rangle^s} M_{\sigma,s-3}(d) N_{\sigma,s,\nu}(g) \\ + \frac{1}{\langle\sigma\rangle^{s-1}} \frac{\langle\sigma\rangle^3}{\langle t+\sigma\rangle^{s-2}} M_{\sigma,s-1}(\zeta) N_{\sigma,s-2,\nu-1}(\delta) \\ + \frac{1}{\langle\sigma\rangle^{s-3}} \frac{\langle\sigma\rangle^3}{\langle t+\sigma\rangle^{s-2}} M_{\sigma,s-3}(d) N_{\sigma,s-2,\nu-1}(\delta)\Big)\mathrm{d}\sigma,$$

and hence that

$$|G_n^1| \leq C\varepsilon(RR_1 h + h^2 R_1^2) \frac{1}{\langle t\rangle^{s-2}}$$

since $s \geq 8$. To estimate G_n^2, we can again write

$$G_n^2 = J_n^1 + J_n^2$$

with

$$J_n^1 = \int_0^{\frac{t}{2}} j_n\,\mathrm{d}\sigma, \quad J_n^2 = \int_{\frac{t}{2}}^t j_n\,\mathrm{d}\sigma$$

with

$$j_n = \varepsilon n p_n \Big(d_n(\sigma)\hat{g}_0(\sigma, nt-n\sigma) + \zeta_n(\sigma)\hat{\delta}_0(\sigma, nt-n\sigma) - d_n(\sigma)\hat{\delta}_0(\sigma, nt-n\sigma)\Big) n(\sigma - t).$$

As in the proof of Proposition 3.4.1, we can prove by using the same estimates as above that

$$|J_n^1| \leq C\varepsilon(RR_1h + R_1^2h^2)\frac{1}{\langle t\rangle^{s-2}}.$$

To estimate J_n^2, we also proceed as in the proof of Proposition 3.4.1 and write

$$|J_n^2| \leq C\varepsilon \int_{\frac{t}{2}}^t (t-\sigma)\Big(\frac{1}{\langle\sigma\rangle^{s-3}}\frac{1}{\langle t-\sigma\rangle^{s-4}}M_{\sigma,s-3}(d)\|g\|_{\mathcal{H}_\nu^{s-4}} + \frac{1}{\langle\sigma\rangle^{s-1}}\frac{1}{\langle t-\sigma\rangle^{s-6}}M_{\sigma,s-1}(\zeta)\|\delta\|_{\mathcal{H}_{\nu-1}^{s-6}} + \frac{1}{\langle\sigma\rangle^{s-3}}\frac{1}{\langle t-\sigma\rangle^{s-6}}M_{\sigma,s-3}(d)\|\delta\|_{\mathcal{H}_{\nu-1}^{s-6}}\Big)\,\mathrm{d}\sigma.$$

This yields, since $s \geq 8$,

$$|J_n^2| \leq C\varepsilon(RR_1h + R_1^2h^2)\frac{1}{\langle t\rangle^{s-3}}.$$

We have thus proven that

$$M_{T,s-3}(G) \leq C\varepsilon(RR_1h + R_1^2h^2).$$

It remains to estimate $M_{T,s-3}(H)$. We shall prove that

$$M_{T,s-3}(H) \leq C(R)h.$$

At first, we can write

$$H_n = H_n^L + \varepsilon H_n^P \tag{3.5.6}$$

with

$$H_n^L = \int_0^t np_n\Big(\hat{\mathsf{g}}_n(\sigma,n\sigma)\hat{\eta}_0(nt-n\sigma)(n\sigma-nt) - \hat{\mathsf{g}}_n(\sigma,n\mathsf{s}(\sigma))\hat{\eta}_0(nt-n\mathsf{s}(\sigma))(n\mathsf{s}(\sigma)-nt)\Big)\,\mathrm{d}\sigma$$

and

$$H_n^P = \int_0^t \sum_{k=\pm1} kp_k\Big(\hat{\mathsf{g}}_k(\sigma,k\sigma)\hat{\mathsf{g}}_{n-k}(\sigma,nt-k\sigma)(n\sigma-nt) - \hat{\mathsf{g}}_k(\sigma,k\mathsf{s}(\sigma))\hat{\mathsf{g}}_{n-k}(\sigma,nt-k\mathsf{s}(\sigma))(n\mathsf{s}(\sigma)-nt)\Big)\mathrm{d}\sigma.$$

We shall focus on the estimate of H_n^P, the estimate of H_n^L is easier to obtain since η can be assumed as smooth as we need (here $\eta \in \mathcal{H}_\nu^{s+4}$ suffices).
By Taylor expanding up to the first order, we have

$$\big|H_n^P\big| \leq h\sum_{k=\pm1} p_k \int_0^t \sup_{\alpha+\beta\leq1}\sup_{\xi\in[k\mathsf{s}(\sigma);k\sigma]}\Big|\partial_\xi^\alpha\hat{\mathsf{g}}_k(\sigma,\xi)\partial_\xi^\beta\hat{\mathsf{g}}_{n-k}(\sigma,nt-\xi)\Big|\,|n\sigma-nt|\,\mathrm{d}\sigma.$$

As previously, we distinguish the cases $k = n$ and $k = -n$.
When $k = -n$, we have by Lemma 3.2.3

$$\int_0^t \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[-n\mathsf{s}(\sigma);-n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{-n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{2n}(\sigma,nt-\xi)\right| |\sigma-t|\,\mathrm{d}\sigma \\ \leq C\int_0^t \frac{\langle\sigma\rangle^6 N_{T,s,\nu}(\mathsf{g})N_{T,s,\nu}(\mathsf{g})}{\langle\sigma\rangle^s\langle t+s\rangle^{s-1}}\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}},$$

since $s \geq 8$.
When $k = n$, we split as previously the integral into

$$\int_0^t \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[n\mathsf{s}(\sigma);n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{0}(\sigma,nt-\xi)\right| |n\sigma-nt|\,\mathrm{d}\sigma \\ = \int_0^{t/2} \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[n\mathsf{s}(\sigma);n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{0}(\sigma,nt-\xi)\right| |n\sigma-nt|\,\mathrm{d}\sigma \\ + \int_{t/2}^t \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[n\mathsf{s}(\sigma);n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{0}(\sigma,nt-\xi)\right| |n\sigma-nt|\,\mathrm{d}\sigma.$$

Using the same estimates as above, we have for the first term

$$\left|\int_0^{t/2} \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[n\mathsf{s}(\sigma);n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{0}(\sigma,nt-\xi)\right| |n\sigma-nt|\,\mathrm{d}\sigma\right| \leq \frac{CR^2}{\langle t\rangle^{s-1}}.$$

For the second term, we rather use the quantity $\|\mathsf{g}\|_{\mathcal{H}_\nu^{s-4}}$ instead of $N_{T,s,\nu}(\mathsf{g})$. We obtain the estimate

$$\left|\int_{t/2}^t \sup_{\alpha+\beta\leq 1} \sup_{\xi\in[n\mathsf{s}(\sigma);n\sigma]} \left|\partial_\xi^\alpha \hat{\mathsf{g}}_{n}(\sigma,\xi)\partial_\xi^\beta \hat{\mathsf{g}}_{0}(\sigma,nt-\xi)\right| |n\sigma-nt|\,\mathrm{d}\sigma\right| \\ \leq C\int_{t/2}^t \frac{\langle\sigma\rangle^3 N_{T,s,\nu}(\mathsf{g})\,\|\mathsf{g}(\sigma)\|_{\mathcal{H}_\nu^{s-4}}}{\langle\sigma\rangle^s\langle t-s\rangle^{s-5}}\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-3}},$$

since $s \geq 8$.
Hence

$$M_{T,s-3}(H^P) \leq ChR^2,$$

and by the same arguments

$$M_{T,s-3}(H^L) \leq ChR^2.$$

Therefore

$$M_{T,s-3}(H) \leq ChR^2,$$

which concludes the proof of proposition 3.5.3. □

3.5.2 Estimate of $\|\delta\|_{\mathcal{H}^{s-6}_{\nu-1}}$

Proposition 3.5.4. *Assuming that $\eta \in \mathcal{H}^{s+4}$ satisfies the assumption* **(H)**, *there exists $C>0$, $\varepsilon_0>0$ and $h_0>0$ such that for every $h\in(0,h_0]$, every $\varepsilon\in(0,\varepsilon_0]$ and every $T>0$, every solution of (3.5.1) such that $Q_{T,s-2,\nu-1}(\delta)\leq R_1 h$ enjoys the estimate*

$$\sup_{t\in[0,T]}\|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}\leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1)$$

where $C(R)$ is a number that depends only on R.

Proof. By using the notation $\mathcal{L}_t[g]h=\{\phi(t,g),h\}$, we can rewrite (3.5.1) as

$$\partial_t\delta(t)=\mathcal{L}_t[\delta(t)](\eta-\varepsilon\delta)+\varepsilon\mathcal{L}_t[\delta(t)]g+\varepsilon\mathcal{L}_t[g(t)]\delta+\mathcal{R}. \tag{3.5.7}$$

Let D be the linear operator defined as the Fourier multiplier by $k^p\xi^q\partial_\xi^m$ for $(m,p,q)\in\mathbb{N}^{3d}$ such that $p+q\leq s-6$, and $m\leq\nu-1$. From an energy estimate as in proof of Proposition 3.4.4, we obtain that

$$\begin{aligned}\|D\delta(t)\|_{L^2}^2=\int_0^t\Big(&-\varepsilon\langle[D,\mathcal{L}_\sigma[\delta(\sigma)]]\delta(\sigma),D\delta(\sigma)\rangle_{L^2}+\varepsilon\langle[D,\mathcal{L}_\sigma[g(\sigma)]]\delta(\sigma),D\delta(\sigma)\rangle_{L^2}\\&+\langle D\left(\mathcal{L}_\sigma[\delta(\sigma)](\eta+\varepsilon g(\sigma))\right),D\delta(\sigma)\rangle_{L^2}+\varepsilon\langle D\mathcal{R},D\delta(\sigma)\rangle_{L^2}\Big)\mathrm{d}\sigma.\end{aligned} \tag{3.5.8}$$

By using Lemma 3.4.5, we thus obtain that

$$\begin{aligned}\sup_{[0,T]}\|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}\leq C\int_0^T\Big(&\varepsilon(m_{\sigma,s-5}(d(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-1}}+m_{\sigma,2}(d(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}})\\&+\varepsilon(m_{\sigma,s-5}(\zeta(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-1}}+m_{\sigma,2}(\zeta(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}})\\&+m_{\sigma,s-5}(d(\sigma))\|\eta+\varepsilon g(\sigma)\|_{\mathcal{H}^1_{\nu-1}}+m_{\sigma,2}(d(\sigma))\|\eta+\varepsilon g(\sigma)\|_{\mathcal{H}^{s-5}_{\nu-1}}+\|D\mathcal{R}\|_{L^2}\Big)\,\mathrm{d}\sigma.\end{aligned}$$

Next, we can use the fact that $Q_{t,s,\nu}(g)\leq R$ and Proposition 3.5.3 to obtain that

$$\begin{aligned}\sup_{t\in[0,T]}\|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}}&\leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1)\int_0^T\frac{1}{\langle\sigma\rangle^2}\,\mathrm{d}\sigma+C\int_0^T\|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}}\,\mathrm{d}\sigma\\&\leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1)+C\int_0^T\|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}}\,\mathrm{d}\sigma.\end{aligned}$$

It remains to estimate the last integral to conclude. By using the expression (3.5.3), we get as in the proof of Lemma 3.4.5 that

$$\|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}}\leq Ch\Big(m^{(1)}_{\sigma,s-5}(\mathrm{g}(\sigma))(\|\eta\|_{\mathcal{H}^1_\nu}+\|\mathrm{g}(\sigma)\|_{\mathcal{H}^1_\nu})+m^{(1)}_{\sigma,2}(\mathrm{g}(\sigma))(\|\eta\|_{\mathcal{H}^{s-5}_\nu}+\|\mathrm{g}(\sigma)\|_{\mathcal{H}^{s-5}_\nu})\Big)$$

where we have set

$$m^{(1)}_{\sigma,\gamma}(h) = \langle\sigma\rangle^{\gamma} \sup_{n=\pm 1} \sup_{\xi\in[n\sigma, n\mathsf{s}(\sigma)]} (|\hat{h}_n(\xi)| + |\partial_\xi \hat{h}_n(\xi)|).$$

Note that thanks to lemma 3.2.3, we have that

$$m^{(1)}_{\sigma,s-5}(\mathsf{g}(\sigma)) \leq C\frac{\langle\sigma\rangle^{s-5+3}}{\langle\sigma\rangle^{s}}\|\mathsf{g}(\sigma)\|_{\mathcal{H}^s_\nu} \leq \frac{C}{\langle\sigma\rangle^2}R \tag{3.5.9}$$

since $Q_{t,s,\nu}(\mathsf{g}) \leq R$. This yields

$$\int_0^t \|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq CRh\int_0^t \frac{1}{\langle\sigma\rangle^2}\,\mathrm{d}\sigma \leq CRh. \tag{3.5.10}$$

Consequently, we get that

$$\sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-6}_{\nu-1}} \leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1).$$

This ends the proof of Proposition 3.5.4. □

3.5.3 Estimate of $N_{T,s-2,\nu-1}(\delta)$

Proposition 3.5.5. *Assuming that $\eta \in \mathcal{H}^{s+4}$ satisfies the assumption* **(H)**, *there exists $C>0$, $\varepsilon_0>0$ and $h_0>0$ such that for every $h\in(0,h_0]$, every $\varepsilon\in(0,\varepsilon_0]$ and every $T>0$, every solution of (3.5.1) such that $Q_{T,s-2,\nu-1}(\delta) \leq hR_1$ enjoys the estimate*

$$N_{T,s-2,\nu-1}(\delta) \leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1)$$

where $C(R)$ is a number that depends only on R.

Proof. We proceed as in the previous proof. Let D be the linear operator defined as the Fourier multiplier by $k^p\xi^q\partial_\xi^m$ for $(m,p,q)\in\mathbb{N}^{3d}$ such that $p+q\leq s-2$, and $m\leq\nu-1$. Using (3.5.8) with this operator D, and Lemma 3.4.5, we obtain that

$$\begin{aligned}\|\delta(t)\|^2_{\mathcal{H}^{s-2}_{\nu-1}} \leq C\int_0^t \Big(&\varepsilon(m_{\sigma,s-1}(d(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-1}}\|\delta(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}} + m_{\sigma,2}(d(\sigma))\|\delta(\sigma)\|^2_{\mathcal{H}^{s-2}_{\nu-1}})\\ &+\varepsilon(m_{\sigma,s-1}(\zeta(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-1}}\|\delta(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}} + m_{\sigma,2}(\zeta(\sigma))\|\delta(\sigma)\|^2_{\mathcal{H}^{s-2}_{\nu-1}})\\ &+m_{\sigma,s-1}(d(\sigma))\|\eta+\varepsilon g(\sigma)\|_{\mathcal{H}^1_{\nu-1}}\|\delta(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}}\\ &+m_{\sigma,2}(d(\sigma))\|\eta+\varepsilon g(\sigma)\|_{\mathcal{H}^{s-1}_{\nu-1}}\|\delta(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}} + \|D\mathcal{R}\|_{L^2}\|\delta(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}}\Big)\,\mathrm{d}\sigma.\end{aligned}$$

Using now $Q_{t,s,\nu}(g)\leq R$ and Proposition 3.5.3, we obtain that

$$\begin{aligned}\|\delta(t)\|^2_{\mathcal{H}^{s-2}_{\nu-1}} \leq C\langle t\rangle^6 h^2 C(R)^2(1+(\varepsilon+\varepsilon h)R_1^2)^2(1+hR_1)^2+\\ \langle t\rangle^3 hC(R)(1+(\varepsilon+\varepsilon h)R_1^2)\int_0^t \|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}}\,\mathrm{d}\sigma.\end{aligned}$$

It remains to estimate the last integral to conclude. By using the expression (3.5.3), we get as in the proof of Lemma 3.4.5 that

$$\|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}} \leq Ch\Big(m^{(1)}_{\sigma,s-1}(\mathsf{g}(\sigma))(\|\eta\|_{\mathcal{H}^1_\nu}+\|\mathsf{g}(\sigma)\|_{\mathcal{H}^1_\nu})+m^{(1)}_{\sigma,2}(\mathsf{g}(\sigma))(\|\eta\|_{\mathcal{H}^{s-1}_\nu}+\|\mathsf{g}(\sigma)\|_{\mathcal{H}^{s-1}_\nu})\Big)$$

with

$$m^{(1)}_{\sigma,\gamma}(h) = \langle\sigma\rangle^\gamma \sup_{n=\pm1}\sup_{\xi\in[n\sigma,n\mathsf{s}(\sigma)]}(|\hat{h}_n(\xi)|+|\partial_\xi\hat{h}_n(\xi)|).$$

As in the previous proof we have that

$$m^{(1)}_{\sigma,s-1}(\mathsf{g}(\sigma)) \leq C\frac{\langle\sigma\rangle^{s-1+3}}{\langle\sigma\rangle^s}\|\mathsf{g}(\sigma)\|_{\mathcal{H}^s_\nu} \leq C\langle\sigma\rangle^2 R \tag{3.5.11}$$

since $Q_{t,s,\nu}(\mathsf{g}) \leq R$. This yields

$$\int_0^t \|\mathcal{R}(\sigma)\|_{\mathcal{H}^{s-2}_{\nu-1}} \leq C(R)h\langle t\rangle^3. \tag{3.5.12}$$

Collecting all the above estimates, we obtain

$$\sup_{t\in[0,T]}\langle t\rangle^{-3}\|\delta(t)\|_{\mathcal{H}^{s-2}_{\nu-1}} \leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1).$$

This ends the proof of Proposition 3.5.5. □

3.5.4 Proof of Proposition 3.5.1

From Propositions 3.5.3, 3.5.4 and 3.5.5, we have the estimate

$$Q_{T,s-2,\nu-1}(\delta) \leq C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1),$$

under the assumption $Q_{T,s-2,\nu-1}(\delta) \leq hR_1$. Choosing $R_1 > C(R)$, we have

$$C(R)h(1+(\varepsilon+\varepsilon h)R_1^2)(1+hR_1) < R_1 h,$$

if ε and h are small enough. Hence, by an usual continuation argument, the estimate $Q_{T,s-2,\nu-1}(\delta) \leq R_1 h$ is valid for all $T > 0$, thus proving Proposition 3.5.1 and with it the convergence estimate 3.2.6.

3.6 Proof of the convergence estimate (3.2.7) for Strang splitting

From now on, we only consider the case of Strang splitting (3.1.6), and thus $\mathsf{s}(t)$ is given by formula (3.3.6). Proposition 3.5.1 implies that, up to the loss of two derivatives, $\mathsf{g}(t,x,v)$ is an approximation of order one (with respect to h) of the exact solution of the Vlasov-HMF

equation $g(t,x,v)$.
Getting the rate of order 2 brings technical complications in order to take advantage of the cancellations provided by the midpoint rule.
In view of theorem 3.3.3 and the main result of [40], we can assume that for all $\alpha \in \{0,1,2\}$, $Q_{T,s,\nu-\alpha}(\mathsf{g})$ and $Q_{T,s,\nu-\alpha}(g)$ are bounded by the same constant $R>0$, provided that $\nu > 5/2$.

To prove the result, we proceed as before and start from the equation (3.5.1) on the error term $\delta(t)$. From now on, using the weighted norms defined in (3.3.8), we consider the quantity:

$$Q_{T,s-3,\nu-2}(\delta) = M_{T,s-4}(d) + N_{T,s-3,\nu-2}(\delta) + \sup_{t\in[0,T]} \|\delta(t)\|_{\mathcal{H}^{s-7}_{\nu-2}}.$$

The convergence result (3.2.7) will be a consequence of the following proposition:

Proposition 3.6.1. *Let us fix $s \geq 9$ and $\nu > 5/2$. Assuming that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**, *there exists $R_2 > 0$, $h_0 > 0$ and $\varepsilon_0 > 0$ such that for every $h \in (0,h_0]$, every $\varepsilon \in (0,\varepsilon_0]$ and every $T>0$, the solution of (3.5.1) satisfies the estimate*

$$Q_{T,s-3,\nu-2}(\delta) \leq R_2 h^2.$$

The crucial point of the proof of proposition 3.6.1 is the cancellation provided by the midpoint rule. We can summarize it by the following easy lemma:

Lemma 3.6.2. *For $t \in \mathbb{R}$ and a fixed $h>0$, let $\mathsf{s}(t) = \mathsf{s}_h(t)$ be given by formula (3.3.6). Then, for all $n \in \mathbb{N}$,*

$$\int_{nh}^{(n+1)h} (\sigma - \mathsf{s}(\sigma))\mathrm{d}\sigma = 0.$$

3.6.1 Estimate of $M_{T,s-4}(d)$

Proposition 3.6.3. *Assuming that $\eta \in \mathcal{H}^{s+4}_\nu$ satisfies the assumption* **(H)**, *there exists $C>0$, $\varepsilon_0>0$ and h_0 such that for every $h \in (0,h_0]$, every $\varepsilon \in (0,\varepsilon_0]$ and every $T>0$, every solution of (3.5.1) such that $Q_{T,s-3,\nu-2}(\delta) \leq R_2 h^2$ enjoys the estimate*

$$M_{T,s-4}(d) \leq C(R)h^2(1+(\varepsilon+\varepsilon h^2)R_2^2)$$

Proof. From Equation (3.5.4) and Lemma 3.4.3, we still get that

$$M_{T,s-4}(d) \leq C\left(M_{T,s-4}(G) + M_{T,s-3}(H)\right).$$

By using the same arguments as in the proof of Proposition 3.5.3, we can easily obtain that

$$M_{T,s-4}(G) \leq C\varepsilon(RR_2h^2 + h^4R_2^2).$$

It thus remains the estimate of $M_{T,s-4}(H)$ that requires a refined analysis of the cancellations in the integral. By using again the decomposition (3.5.6), we shall focus on the term H^P

that is more difficult. By Taylor expanding up to second order, we find that

$$H_n^P = \int_0^t \sum_{k=\pm 1} k^2 p_k(\sigma - \mathsf{s}(\sigma))\Big(\partial_\xi \hat{\mathsf{g}}_k(\mathsf{s}(\sigma), k\mathsf{s}(\sigma))\hat{\mathsf{g}}_{n-k}(\mathsf{s}(\sigma), nt - k\mathsf{s}(\sigma)) \\ + \hat{\mathsf{g}}_k(\mathsf{s}(\sigma), k\mathsf{s}(\sigma))\partial_\xi \hat{\mathsf{g}}_{n-k}(\mathsf{s}(\sigma), nt - k\mathsf{s}(\sigma))\Big)(n\mathsf{s}(\sigma) - nt)\,\mathrm{d}\sigma + K_n$$

where

$$|K_n| \leq h^2 \sum_{k=\pm 1} p_k \int_0^t \sup_{\substack{\alpha+\beta_1 \leq 2,\, \alpha \neq 2 \\ \alpha+\beta_2 \leq 2,\, \alpha \neq 2}} \sup_{\tau, \xi \in [k\sigma, k\mathsf{s}(\sigma)]} |\partial_t^\alpha \partial_\xi^{\beta_1} \hat{\mathsf{g}}_k(\tau, \xi)||\partial_t^\alpha \partial_\xi^{\beta_2} \hat{\mathsf{g}}_{n-k}(\tau, nt - \xi)||n\sigma - nt|\,\mathrm{d}\sigma.$$

To estimate the remainder, K_n, we can again distinguish the cases $k = n$ and $k = -n$. Since $Q_{T,s,\nu}(\mathsf{g}) \leq R$, by using the equation (3.3.4) and Lemma 3.4.5, we also have that

$$\|\partial_t \mathsf{g}(t)\|_{\mathcal{H}_\nu^{s-1}} \leq C(R)\langle t \rangle^2.$$

Since $\nu > 5/2$, we can use Lemma 3.2.3 to obtain, by similar arguments to the ones used in the proof of Proposition 3.5.3, that

$$|K_n| \leq h^2 C(R)\Big(\int_0^t \frac{1}{\langle \sigma \rangle^{s-4}} \frac{1}{\langle t + \sigma \rangle^{s-3}}(t - \sigma)\,d\sigma + \int_{t/2}^t \frac{1}{\langle \sigma \rangle^{s-4}} \frac{1}{\langle t - \sigma \rangle^{s-3}}(t - \sigma)\,\mathrm{d}\sigma\Big)$$

and hence that

$$|K_n| \leq \frac{1}{\langle t \rangle^{s-4}} C(R)h^2.$$

To estimate the main term in H_n^P, assuming that $Nh \leq T \leq (N+1)h$ for some N, we can split the time integral into

$$\int_0^t = \sum_{j=0}^{N-1} \int_{jh}^{(j+1)h} + \int_{Nh}^t$$

and we observe that all the integrals $\int_{jh}^{(j+1)h}$ vanish due to the symmetry of $\sigma - \mathsf{s}(\sigma)$ (see Lemma 3.6.2). We thus obtain that

$$|H_n^P| \leq \frac{1}{\langle t \rangle^{s-4}} C(R) \int_{Nh}^t |t - \mathsf{s}(\sigma)|\,d\sigma + \frac{1}{\langle t \rangle^{s-4}} C(R)h^2 \leq \frac{1}{\langle t \rangle^{s-4}} C(R)h^2.$$

From the same argument (slightly easier since η does not depend on time and is smoother), we can also prove that

$$|H_n^L| \leq \frac{1}{\langle t \rangle^{s-4}} C(R)h^2.$$

Consequently, by collecting the previous estimates, we find that

$$M_{T,s-4}(G) \leq C(R)h^2(1 + (\varepsilon + \varepsilon h^2)R_2^2).$$

This ends the proof. □

3.6.2 Estimate of $\|\delta\|_{\mathcal{H}^{s-7}_{\nu-2}}$

Proposition 3.6.4. *Assuming $\eta \in \mathcal{H}^{s+4}_{\nu}$ satisfies the assumption* **(H)**, *there exists $C > 0$, ε_0 and h_0 such that for every $h \in (0, h_0]$, every $\varepsilon \in (0, \varepsilon_0]$ and every $T > 0$, every solution of (3.5.1) such that $Q_{T,s-3,\nu-2}(\delta) \leq h^2 R_2$ enjoys the estimate*

$$\sup_{[0,T]} \|\delta\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq C(R)h^2(1 + (\varepsilon + \varepsilon h^2)R_2^2)(1 + h^2 R_2).$$

Proof. We can start as in the proof of Proposition 3.5.4. By using the energy identity (3.5.8) with D the Fourier multiplier by $k^p \xi^q \partial_\xi^m$ for $(m, p, q) \in \mathbb{N}^{3d}$ such that $p + q \leq s - 7$, and $m \leq \nu - 2$ and Lemma 3.4.5, we first obtain that

$$\begin{aligned}
\|D\delta(t)\|_{L^2}^2 \leq C \int_0^t \|D\delta(t)\|_{L^2} \Big(& \varepsilon(m_{\sigma,s-6}(d(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(d(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}}) \\
& + \varepsilon(m_{\sigma,s-6}(\zeta(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(\zeta(\sigma))\|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}}) \\
& + m_{\sigma,s-6}(d(\sigma))\|\eta + \varepsilon g(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(d(\sigma))\|\eta + \varepsilon g(\sigma)\|_{\mathcal{H}^{s-6}_{\nu-2}} \Big) \, \mathrm{d}\sigma \\
& + \left| \int_0^t \langle D\mathcal{R}, D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right|.
\end{aligned}$$

This yields

$$\begin{aligned}
\|D\delta(t)\|_{L^2}^2 &\leq C(R)h^2(1 + (\varepsilon + \varepsilon h^2)R_2^2)(1 + h^2 R_2) \int_0^t \frac{\|D\delta(\sigma)\|_{L^2}}{\langle \sigma \rangle^2} \, \mathrm{d}\sigma + \left| \int_0^t \langle D\mathcal{R}, D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right| \\
&\leq C(R)h^2(1 + (\varepsilon + \varepsilon h^2)R_2^2)(1 + h^2 R_2) \sup_{\sigma \in [0,t]} \|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}} + \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right|.
\end{aligned} \tag{3.6.1}$$

The main difficulty is now to use the cancellation in the midpoint quadrature rule in order to estimate the last integral. Let us define

$$I(t) = \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right|.$$

By Taylor expanding, integrating by parts once and using the estimate (3.5.10), we first write that

$$\begin{aligned}
I(t) &\leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma)) \rangle_{L^2} \, \mathrm{d}\sigma \right| + h \sup_{\sigma \in [0,T]} \|\partial_t \delta(\sigma)\|_{\mathcal{H}^{s-8}_{\nu-2}} \sup \int_0^t \|\mathcal{R}\|_{\mathcal{H}^{s-6}_{\nu-2}} \, \mathrm{d}\sigma \\
&\leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma)) \rangle_{L^2} \, \mathrm{d}\sigma \right| + h^2 C(R) \sup_{\sigma \in [0,T]} \|\partial_t \delta(\sigma)\|_{\mathcal{H}^{s-8}_{\nu-2}}.
\end{aligned}$$

By using the equation (3.5.7) and Lemma 3.4.5, we have that

$$\|\partial_t \delta(t)\|_{\mathcal{H}^{s-8}_{\nu-2}} \leq C m_{s-7}(d(t))\|\eta\|_{\mathcal{H}^{s-7}_{\nu-2}} + \varepsilon\big(m_{s-7}(d(t))(\|\delta\|_{\mathcal{H}^{s-7}_{\nu-2}} + \|g\|_{\mathcal{H}^{s-7}_{\nu-2}}) + m_{s-7}(\zeta)\|\delta\|_{\mathcal{H}^{s-7}_{\nu-2}}\big)$$

and thus by using Proposition 3.6.3 we find

$$\|\partial_t \delta(t)\|_{\mathcal{H}^{s-8}_{\nu-2}} \leq C(R)h^2(1+(\varepsilon+\varepsilon h^2)R_2^2)+\varepsilon h^4 R_2^2.$$

This yields

$$I(t) \leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2}\, \mathrm{d}\sigma \right| + C(R)h^4(1+(\varepsilon+\varepsilon h^2)R_2^2).$$

Next, by using the definition of $\mathcal{R}$ provided in (3.5.3), we observe that by Taylor expanding in time and in the ξ variable, we can write

$$\mathcal{R}(t,x,v) = \mathcal{R}^1(t,x,v) + \mathcal{R}^2(t,x,v) \tag{3.6.2}$$

where $\mathcal{R}^1$ is given in Fourier by

$$\hat{\mathcal{R}}^1_n(t,\xi) = (t-\mathsf{s}(t))\Big(np_n\big(\partial_\xi \hat{\mathsf{g}}_n(\mathsf{s}(t),n\mathsf{s}(t))\hat{\eta}(\xi-n\mathsf{s}(t))+\hat{\mathsf{g}}_n(\mathsf{s}(t),n\mathsf{s}(t))\partial_\xi\hat{\eta}(\xi-n\mathsf{s}(t))\big)(n\mathsf{s}(t)-\xi)$$
$$+\varepsilon\sum_{k=\pm 1} kp_k(\partial_\xi\hat{\mathsf{g}}_k(\mathsf{s}(t),n\mathsf{s}(t))\hat{\mathsf{g}}_{n-k}(t,\xi-k\mathsf{s}(t))+\hat{\mathsf{g}}_k(\mathsf{s}(t),n\mathsf{s}(t))\partial_\xi\hat{\mathsf{g}}_{n-k}(\mathsf{s}(t),\xi-k\mathsf{s}(t)))(n\mathsf{s}(t)-\xi)\Big)$$

and the remainder $\mathcal{R}^2$ is such that

$$\|\mathcal{R}^2(t)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq Ch^2\Big(m^{(2)}_{t,s-6}(\mathsf{g})\big(\|\eta\|_{\mathcal{H}^1_\nu}+\varepsilon\|\mathsf{g}\|_{\mathcal{H}^1_\nu}\big)+m^{(2)}_{t,2}(\mathsf{g})\big(\|\eta\|_{\mathcal{H}^{s-6}_\nu}+\varepsilon\|\mathsf{g}\|_{\mathcal{H}^{s-6}_\nu}\big)\Big)$$

with

$$m^{(2)}_{t,\gamma}(h) = \langle t\rangle^\gamma \sup_{n=\pm 1}\sup_{\xi\in[nt,n\mathsf{s}(t)]}\sum_{\alpha+\beta\leq 2,\,\alpha\neq 2}|\partial_t^\alpha\partial_\xi^\beta \hat{h}_n(\xi)|.$$

Since $\nu > 5/2$, we have that

$$m^{(2)}_{t,s-6}(\mathsf{g}) \leq C\frac{1}{\langle t\rangle^2}\|\mathsf{g}\|_{\mathcal{H}^{s-4}_\nu}+\frac{1}{\langle t\rangle^2}\|\partial_t\mathsf{g}\|_{\mathcal{H}^{s-4}_\nu}$$

and hence thanks to the equation (3.3.4) and Theorem 3.3.3, we obtain that

$$m^{(2)}_{t,s-6}(\mathsf{g}) \leq \frac{C(R)}{\langle t\rangle^2}.$$

This yields

$$\|\mathcal{R}^2(t)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq \frac{C(R)}{\langle t\rangle^2},$$

and hence

$$I(t) \leq \left| \int_0^t \langle D\mathcal{R}^1, D\delta(\mathsf{s}(\sigma))\rangle_{L^2}\, \mathrm{d}\sigma \right| + C(R)h^4(1+(\varepsilon+\varepsilon h^2)R_2^2) + C(R)h^2 \sup_{\sigma\in[0,t]}\|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}}. \tag{3.6.3}$$

Thanks to the definition of $\mathcal{R}^1$, assuming that $Nh \leq t \leq (N+1)h$, we obtain that

$$\left| \int_0^t \langle D\mathcal{R}^1(\sigma), D\delta(\mathsf{s}(\sigma)) \rangle_{L^2} \, d\sigma \right| = \left| \int_{Nh}^t \langle D\mathcal{R}^1(\sigma), D\delta(\mathsf{s}(\sigma)) \rangle_{L^2} \, d\sigma \right| \leq C(R) h^2 \sup_{\sigma \in [0,t]} \|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}}.$$

Here we have also used the cancellation argument of Lemma 3.6.2. Consequently, from (3.6.1), (3.6.3) and the above estimate, we obtain that

$$\begin{aligned} \sup_{\sigma \in [0,T]} \|\delta(\sigma)\|^2_{\mathcal{H}^{s-7}_{\nu-2}} \leq C(R) h^4 (1 + (\varepsilon + \varepsilon h^2) R_2^2)(1 + h^2 R_2) \\ + C(R) h^2 (1 + (\varepsilon + \varepsilon h^2) R_2^2)(1 + h^2 R_2) \sup_{\sigma \in [0,T]} \|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}}. \end{aligned}$$

By using the Young inequality this yields

$$\sup_{\sigma \in [0,T]} \|\delta(\sigma)\|_{\mathcal{H}^{s-7}_{\nu-2}} \leq C(R) h^2 (1 + (\varepsilon + \varepsilon h^2) R_2^2)(1 + h^2 R_2).$$

This ends the proof.

□

3.6.3 Estimate of $N_{T,s-3,\nu-2}(\delta)$

Proposition 3.6.5. *Assuming $\eta \in \mathcal{H}^{s+4}$ satisfies the assumption* **(H)**, *there exists $C > 0$, $\varepsilon_0 > 0$, and h_0 such that for every $h \in (0, h_0]$, every $\varepsilon \in (0, \varepsilon_0]$ and every $T > 0$, every solution of (3.5.1) such that $Q_{T,s-3,\nu-2}(\delta) \leq h^2 R_2$ enjoys the estimate*

$$N_{T,s-3,\nu-2}(\delta) \leq C(R) h^2 (1 + (\varepsilon + \varepsilon h^2) R_2^2)(1 + h^2 R_2).$$

Proof. We can start once more as in the proof of Proposition 3.5.4. By using the energy identity (3.5.8) with D the Fourier multiplier by $k^p \xi^q \partial_\xi^m$ for $(m,p,q) \in \mathbb{N}^{3d}$ such that $p+q \leq s-3$, and $m \leq \nu - 2$ and Lemma 3.4.5, we first obtain that

$$\begin{aligned} \|D\delta(t)\|^2_{L^2} \leq C \int_0^t \|D\delta(t)\|_{L^2} \Big(& \varepsilon (m_{\sigma,s-2}(d(\sigma)) \|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(d(\sigma)) \|\delta(\sigma)\|_{\mathcal{H}^{s-3}_{\nu-2}}) \\ & + \varepsilon (m_{\sigma,s-2}(\zeta(\sigma)) \|\delta(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(\zeta(\sigma)) \|\delta(\sigma)\|_{\mathcal{H}^{s-3}_{\nu-2}}) \\ & + m_{\sigma,s-2}(d(\sigma)) \|\eta + \varepsilon g(\sigma)\|_{\mathcal{H}^1_{\nu-2}} + m_{\sigma,2}(d(\sigma)) \|\eta + \varepsilon g(\sigma)\|_{\mathcal{H}^{s-3}_{\nu-2}} \Big) \, \mathrm{d}\sigma \\ & + \left| \int_0^t \langle D\mathcal{R}, D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right|. \end{aligned}$$

This yields

$$\begin{aligned} \|D\delta(t)\|^2_{L^2} &\leq C(R)^2 h^2 (1 + (\varepsilon + \varepsilon h^2) R_2^2)(1 + h^2 R_2) \int_0^t \langle \sigma \rangle^2 \|D\delta(\sigma)\|_{L^2} \, \mathrm{d}\sigma + \left| \int_0^t \langle D\mathcal{R}, D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right| \\ &\leq C(R)^2 \langle t \rangle^6 h^4 (1 + (\varepsilon + \varepsilon h^2) R_2^2)^2 (1 + h^2 R_2)^2 + \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\sigma) \rangle_{L^2} \, \mathrm{d}\sigma \right|. \end{aligned} \tag{3.6.4}$$

As in the previous proof we consider

$$I(t) = \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\sigma)\rangle_{L^2} \, \mathrm{d}\sigma \right|,$$

and obtain as previously

$$\begin{aligned} I(t) &\leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2} \, \mathrm{d}\sigma \right| + h \sup_{\sigma\in[0,T]} \|\partial_t \delta(\sigma)\|_{\mathcal{H}^{s-4}_{\nu-2}} \sup \int_0^t \|\mathcal{R}\|_{\mathcal{H}^{s-2}_{\nu-2}} \, \mathrm{d}\sigma \\ &\leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2} \, \mathrm{d}\sigma \right| + h^2 C(R)\langle t\rangle^3 \sup_{\sigma\in[0,T]} \|\partial_t \delta(\sigma)\|_{\mathcal{H}^{s-4}_{\nu-2}}. \end{aligned}$$

Here we have used estimate (3.5.12). By using the equation (3.5.7) and Lemma 3.4.5, we have that

$$\|\partial_t \delta(t)\|_{\mathcal{H}^{s-4}_{\nu-2}} \leq C m_{s-3}(d(t))\|\eta\|_{\mathcal{H}^{s-3}_{\nu-2}} + \varepsilon\big(m_{s-3}(d(t))(\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}} + \|g\|_{\mathcal{H}^{s-3}_{\nu-2}}) + m_{s-3}(\zeta)\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}}\big)$$

and thus by using Proposition 3.6.3 we find

$$\|\partial_t \delta(t)\|_{\mathcal{H}^{s-4}_{\nu-2}} \leq C(R)\langle t\rangle^3 h^2(1 + (\varepsilon + \varepsilon h^2)R_2^2) + \langle t\rangle^3 \varepsilon h^4 R_2^2.$$

This yields

$$I(t) \leq \left| \int_0^t \langle D\mathcal{R}(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2} \, \mathrm{d}\sigma \right| + \langle t\rangle^6 C(R)^2 h^4 (1 + (\varepsilon + \varepsilon h^2)R_2^2)^2.$$

As in the previous proof, we write

$$\mathcal{R}(t,x,v) = \mathcal{R}^1(t,x,v) + \mathcal{R}^2(t,x,v)$$

where $\mathcal{R}^1$ is given in Fourier by

$$\begin{aligned} \hat{R}^1_n(t,\xi) = (t-\mathsf{s}(t))\Big(& np_n\big(\partial_\xi \hat{\mathsf{g}}_n(\mathsf{s}(t), n\mathsf{s}(t))\hat{\eta}(\xi - n\mathsf{s}(t)) + \hat{\mathsf{g}}_n(\mathsf{s}(t), n\mathsf{s}(t))\partial_\xi\hat{\eta}(\xi - n\mathsf{s}(t))\big)(n\mathsf{s}(t) - \xi) \\ & + \varepsilon \sum_k kp_k(\partial_\xi \hat{\mathsf{g}}_k(\mathsf{s}(t), n\mathsf{s}(t))\hat{\mathsf{g}}_{n-k}(t, \xi - k\mathsf{s}(t)) + \hat{\mathsf{g}}_k(\mathsf{s}(t), n\mathsf{s}(t))\partial_\xi \hat{\mathsf{g}}_{n-k}(\mathsf{s}(t), \xi - k\mathsf{s}(t)))(n\mathsf{s}(t) - \xi)\Big) \end{aligned}$$

and the remainder $\mathcal{R}^2$ is such that

$$\|\mathcal{R}^2(t)\|_{\mathcal{H}^{s-3}_{\nu-2}} \leq Ch^2\Big(m^{(2)}_{t,s-2}(\mathsf{g})\big(\|\eta\|_{\mathcal{H}^1_\nu} + \varepsilon\|\mathsf{g}\|_{\mathcal{H}^1_\nu}\big) + m^{(2)}_{t,2}(\mathsf{g})\big(\|\eta\|_{\mathcal{H}^{s-2}_\nu} + \varepsilon\|\mathsf{g}\|_{\mathcal{H}^{s-2}_\nu}\big)\Big)$$

with

$$m^{(2)}_{t,\gamma}(h) = \langle t\rangle^\gamma \sup_{n=\pm 1} \sup_{\xi\in[nt, n\mathsf{s}(t)]} \sum_{\alpha+\beta\leq 2,\, \alpha\neq 2} |\partial_t^\alpha \partial_\xi^\beta \hat{h}_n(\xi)|.$$

Since $\nu > 5/2$, we have that

$$m^{(2)}_{t,s-2}(\mathsf{g}) \leq C\langle t\rangle^2 \|\mathsf{g}\|_{\mathcal{H}^{s-4}_\nu} + \langle t\rangle^2 \|\partial_t \mathsf{g}\|_{\mathcal{H}^{s-4}_\nu}.$$

and hence thanks to the equation (3.3.4) and Theorem 3.3.3, we obtain that

$$m^{(2)}_{t,s-2}(\mathsf{g})C(R)\langle t\rangle^2$$

This yields

$$\|\mathcal{R}^2(t)\|_{\mathcal{H}^{s-3}_{\nu-2}} \leq C(R)\langle t\rangle^2$$

and hence

$$I(t) \leq \left|\int_0^t \langle D\mathcal{R}^1, D\delta(\mathsf{s}(\sigma))\rangle_{L^2}\,\mathrm{d}\sigma\right| + C(R)^2\langle t\rangle^6 h^4(1+(\varepsilon+\varepsilon h^2)R_2^2)^2 + C(R)\langle t\rangle^3 h^2 \sup_{[0,t]}\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}}. \tag{3.6.5}$$

Thanks to the definition of $\mathcal{R}^1$, assuming that $Nh \leq t \leq (N+1)h$, we obtain that

$$\left|\int_0^t \langle D\mathcal{R}^1(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2}\,\mathrm{d}\sigma\right| = \left|\int_{Nh}^t \langle D\mathcal{R}^1(\sigma), D\delta(\mathsf{s}(\sigma))\rangle_{L^2}\,\mathrm{d}\sigma\right| \leq C(R)\langle t\rangle^3 h^2 \sup_{[0,t]}\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}}.$$

Once more we have used the cancellation argument (see lemma 3.6.2). Consequently, from (3.6.4), (3.6.5) and the above estimate, we obtain that

$$\sup_{[0,T]}\|\delta\|^2_{\mathcal{H}^{s-3}_{\nu-2}} \leq C(R)^2\langle t\rangle^6 h^4(1+(\varepsilon+\varepsilon h^2)R_2^2)^2 + C(R)\langle t\rangle^3 h^2(1+(\varepsilon+\varepsilon h^2)R_2^2)(1+h^2R_2)\sup_{[0,T]}\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}}.$$

By using the Young inequality this yields

$$\sup_{t\in[0,T]}\langle t\rangle^{-3}\|\delta\|_{\mathcal{H}^{s-3}_{\nu-2}} \leq C(R)h^2(1+(\varepsilon+\varepsilon h^2)R_2^2)(1+h^2R_2).$$

This ends the proof. □

3.6.4 Proof of proposition 3.6.1

Using Propositions 3.6.3, 3.6.4 and 3.6.5, we have proven that

$$Q_{T,s-3,\nu-2}(\delta) \leq C(R)h^2(1+(\varepsilon+\varepsilon h^2)R_2^2)(1+h^2R_2)$$

if $Q_{T,s-3,\nu-2}(\delta) \leq h^2R_2$. We can thus obtain Proposition 3.6.1 with the same bootstrap argument as before by choosing $R_2 > C(R)$ and then by taking ε and h sufficiently small.

3.7 Appendix

3.7.1 Proof of Lemma 3.4.3

Let us first note that the equation (3.4.6) only involves $K(n,t)$ for positive t, hence $K(n,t)$ can be replaced here by $K_1(n,t)$ (see (3.2.3)). Let us take $T > 0$, and let us set for the

purpose of the proof $K(t) = K_1(n,t)$, $F(t) = (F_n(t) + G_n(t))\mathbb{1}_{0\le t\le T}$. Since we only consider the cases $n = \pm 1$, we do not write down anymore explicitly the dependence in n. We consider the equation

$$y(t) = K * y(t) + F(t), \quad t \in \mathbb{R} \tag{3.7.1}$$

setting $y(t) = 0$ for $t \le 0$. Note that the solution of this equation coincides with $\zeta_n(t)$ on $[0,T]$ since the modification of the source term for $t \ge T$ does not affect the past. By taking the Fourier transform in t (that we still denote by $\hat{\cdot}$), we obtain

$$\hat{y}(\tau) = \hat{K}(\tau)\hat{y}(\tau) + \hat{F}(\tau), \quad \tau \in \mathbb{R}, \tag{3.7.2}$$

with $\hat{K}(\tau) = \hat{K}(n,\tau)$. Under the assumption (**H**), the solution of (3.7.2) is given explicitely by the formula

$$\hat{y}(\tau) = \frac{\hat{F}(\tau)}{1 - \hat{K}(\tau)}. \tag{3.7.3}$$

Let us observe that since $(1+v^2)\eta_0 \in \mathcal{H}^5$, we have by (3.2.8) that for $\alpha \le 2$ and for $t > 0$

$$|\partial_t^\alpha K(t)| \le \frac{C}{\langle t\rangle^4} \in L^1(\mathbb{R}_+). \tag{3.7.4}$$

Note that by definition of $K(t)$, the function $K(t)$ is continuous in $t = 0$, but not C^1. Using an integration by parts on the definition of the Fourier transform, we then get that

$$|\partial_\tau^\alpha \hat{K}(\tau)| \le \frac{C}{\langle \tau\rangle^2}, \quad \alpha \le 2. \tag{3.7.5}$$

To get this, we have used that the function $t\,\hat{\eta}_0(t)$ vanishes at zero.

By using this estimate on $\hat{K}$, **(H)** and that $\hat{F}(\tau) \in H^1_\tau$ (the Sobolev space in τ) since F is compactly supported in time, we easily get that y defined via its Fourier transform by (3.7.3) belongs to H^1_τ. This implies that $\langle t\rangle y \in L^2$ and thus that $y \in L^1_t$. These remarks justify the use of the Fourier transform and that the function y defined through its Fourier transform via (3.7.3) is a solution of (3.7.1). Moreover, thanks to (3.7.3) and **(H)**, we get that $\hat{y}$ can be continued as an holomorphic function in $\mathrm{Im}\,\tau \le 0$ and thanks to a Paley Wiener type argument, that y vanishes for $t \le 0$. We have thus obtained an L^1 solution of (3.7.1) that vanishes for $t \le 0$. By a Gronwall type argument, we easily get that there is a unique solution in this class of (3.7.1) and thus we have obtained the expression of the unique solution.

We can thus now focus on the proof of the estimate stated in Lemma 3.4.3. Note that a L^2-based version of this estimate would be very easily obtained. The difficulty here is to get the uniform L^∞ in time estimate we want to prove.

We shall first prove the estimate for $\gamma = 0$. Let us take $\chi(\tau) \in [0,1]$ a smooth compactly supported function that vanishes for $|\tau| \ge 1$ and which is equal to one for $|\tau| \le 1/2$. We define $\chi_R(\tau) = \chi(\tau/R)$ and $\chi_R(\partial_t)$ the corresponding operator in t variable corresponding

to the convolution with the inverse Fourier transform of $\chi_R(\tau)$. Thanks to (3.7.5), we have that for R large

$$\langle t\rangle^2|(1-\chi_R(\partial_t))K(t)| \le C\sum_{\alpha\le 2}\|\partial_\tau^\alpha((1-\chi_R(\tau))\hat{K}(\tau))\|_{L^1(\mathbb{R})} \le C\int_{|\tau|\ge R/2}\frac{1}{\langle\tau\rangle^2} \le \frac{C}{R}$$

and hence

$$\|(1-\chi_R(\partial_t))K(t)\|_{L^1(\mathbb{R})} \le \frac{C}{R} \le \frac{1}{2} \tag{3.7.6}$$

for R sufficiently large. This choice fixes R.

To estimate the solution y of (3.7.1), we shall write that

$$y = \chi_{2R}(\partial_t)y + (1-\chi_{2R}(\partial_t))y =: y^l + y^h.$$

By applying $(1-\chi_{2R}(\partial_t))$ to (3.7.1), we get that

$$y^h = K * y^h + (1-\chi_{2R}(\partial_t))F = \big((1-\chi_R(\partial_t)K\big) * y^h + (1-\chi_{2R}(\partial_t))F$$

since $(1-\chi_R) = 1$ on the support of $1-\chi_{2R}$. Therefore, we obtain thanks to (3.7.6) and the fact that $\chi_{2R}(\partial_t)$ is a convolution operator with a L^1 function, that

$$\|y^h\|_{L^\infty} \le \frac{1}{2}\|y^h\|_{L^\infty} + C\|F\|_{L^\infty}$$

and hence

$$\|y^h\|_{L^\infty} \le 2C\|F\|_{L^\infty}.$$

For the low frequencies, we can use directly the form (3.7.1) of the equation: We can write that

$$\hat{y}^l(\tau) = \frac{\chi_{2R}(\tau)}{1-\hat{K}(\tau)}\chi_R(\tau)\hat{F}(\tau).$$

Since the denominator does not vanish thanks to (**H**), we obtain again that y^l can be written as the convolution of an L^1 function - which is the inverse Fourier transform of $\chi_{2R}(\tau)/(1-\hat{K}(\tau))$ - by the function $\chi_R(\partial_t)F$ which is a convolution of F by a smooth function. Thus we obtain by using again the Young inequality that

$$\|y^l\|_{L^\infty} \le C\|F\|_{L^\infty}.$$

Since $\|y\|_{L^\infty} \le \|y^l\|_{L^\infty} + \|y^h\|_{L^\infty}$, we get the desired estimate for $\gamma = 0$. To get the estimate for arbitrary γ, we can proceed by induction. We observe that

$$ty(t) = K * (ty) + F^1$$

with $F^1 = (tK) * y + tF$. Using the result $\gamma = 0$, we obtain that $\|ty\|_{L^\infty} \le C\|F^1\|_{L^\infty}$. Now since $\eta_0 \in \mathcal{H}^{\gamma+3}$, for $\gamma = 1$, we obtain that $tK \in L^1$ and thus

$$\|F^1\|_{L^\infty} \le C\big(\|tF\|_{L^\infty} + \|y\|_{L^\infty}\big) \le C\|(1+t)F\|_{L^\infty}.$$

The higher order estimates follow easily in the same way.

3.7.2 Proof of Lemma 3.4.5

We give the proof of (3.4.11), the proof of the second estimate being slightly easier. In the Fourier side, we have for $\mathcal{L}_\sigma[\mathsf{g}](h)$ the expression

$$(\mathcal{F}\mathcal{L}_\sigma[\mathsf{g}]h)_n(\xi) = \sum_{k\in\{\pm 1\}} kp_k\mathsf{z}_k(\sigma)\hat{h}_{n-k}(\sigma,\xi - k\mathsf{s}(\sigma))(n\mathsf{s}(\sigma)-\xi).$$

Consequently, we obtain that

$$\begin{aligned}\big(\mathcal{F}([D^{r,p,q},\mathcal{L}_\sigma[\mathsf{g}]]h)\big)_n(\xi) = \\ \sum_{k\in\{\pm 1\}} kp_k\mathsf{z}_k(\sigma)\Big(n^p\xi^q\partial_\xi^r\big(\hat{h}_{n-k}(\sigma,\xi-k\mathsf{s}(\sigma))(n\mathsf{s}(\sigma)-\xi)\big) - \\ \big((n-k)^p(\xi-\mathsf{s}(\sigma))^q\partial_\xi^r\hat{h}_{n-k}(\sigma,\xi-k\mathsf{s}(\sigma))(n\mathsf{s}(\sigma)-\xi)\big)\Big).\end{aligned}$$

For $k=\pm 1$, we can thus expand the above expression into a finite sum of terms under the form

$$I_n^k(\sigma,\xi) = kp_k\mathsf{z}_k(\mathsf{s}(\sigma))k^{p_1}(n-k)^{p-p_1+\alpha}\big(k\mathsf{s}(\sigma)\big)^{q_1+\alpha}(\xi-k\mathsf{s}(\sigma))^{q-q_1+\beta}\partial_\xi^{r_1}\hat{h}_{n-k}(\sigma,\xi-k\mathsf{s}(\sigma))$$

where

$$0\le p_1\le p,\ 0\le q_1\le q,\quad m-1\le r_1\le m,\quad \alpha+\beta = r_1-m+1,\ \alpha,\ \beta\ge 0.$$

Moreover, if $r_1 = r$, then we have $p_1+q_1>0$.

We have to estimate $\sum_n\int_\xi|\sum_{k\in\pm1}I_n^k(\sigma,\xi)|^2\,d\xi$ by isometry of the Fourier transform.

We note that for a fixed $k\in\{\pm1\}$ then for $|n-k|+|\xi-k\mathsf{s}(\sigma)|\le|k|\mathsf{s}(\sigma)$, we have

$$|I_n^k(\sigma,\xi)|\le C\mathsf{s}(\sigma)^{p+q+1}|\mathsf{z}_k(\mathsf{s}(\sigma))||n-k||\partial_\xi^{r_1}\hat{h}_{n-k}(\sigma,\xi-k\mathsf{s}(\sigma))|$$

whereas for $|n-k|+|\xi-k\mathsf{s}(\sigma)|\ge|k|\mathsf{s}(\sigma)$, we have

$$|I_n^k(\sigma,\xi)|\le C\langle\sigma\rangle^2|\mathsf{z}_k(\mathsf{s}(\sigma))|(|n-k|+|\xi-k\mathsf{s}(\sigma)|)^\gamma|\partial_\xi^{r_1}\hat{h}_{n-k}(\sigma,\xi-k\mathsf{s}(\sigma))|.$$

Consequently by taking the L^2 norm, we find that

$$\|\sum_{k\in\pm1}I_n^k(\sigma,\xi)\|_{L^2}\le C\big(m_{\sigma,\gamma+1}(\mathsf{z})\|h(\sigma)\|_{\mathcal{H}^1}+m_{\sigma,2}(\mathsf{z})\|h(\sigma)\|_{\mathcal{H}^m}\big).$$

This ends the proof of the Lemma.

Chapter 4

On linear Landau damping around inhomogeneous stationary states of the Vlasov-HMF model

The topic of this chapter was introduced in chapter 2, section 2.2.
We consider solutions of the Vlasov-HMF equation with an attractive potential starting from a perturbation of an inhomogeneous stationary state which satisfies a linearized stability criterion (Penrose criterion). We prove that the solutions of the linearized equation around the stationary state exhibit a scattering behavior to the average in the proper variables of their initial values, which implies a linear Landau damping effect. We prove moreover that the damping rate is algebraic, and essentially independent of the regularity of the initial data.
The present chapter can be read independently from the others.

4.1 Introduction

In this chapter are interested in Landau damping for the Vlasov-HMF model. As discussed in chapter 1, a long time analysis of small perturbations of a spatially homogeneous stationary state has been recently performed in [40], where a nonlinear Landau damping Theorem has been proved for these states, in Sobolev regularity. Our purpose here is to study the case of inhomogeneous stationary states, a problem that was also subject to recent interests in the Physics literature (see [5, 7]).
We consider the Vlasov-HMF model with an attractive potential

$$\begin{cases} \partial_t f(t,x,v) + \{f, H[f]\}(t,x,v) = 0, \\ H[f](t,x,v) = \dfrac{v^2}{2} - \phi[f](t,x) \\ \phi[f](t,x) = \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \cos(x-y) f(t,y,v) \mathrm{d}y\mathrm{d}v \end{cases} \tag{4.1.1}$$

with $(t,x,v) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and where

$$\{f,g\} = \partial_x f \partial_v g - \partial_v f \partial_x g \tag{4.1.2}$$

is the Poisson bracket. The potential will usually be expressed as the following Fourier series

$$\phi[f](t,x) = \mathcal{C}[f]\cos(x) + \mathcal{S}[f]\sin(x), \tag{4.1.3}$$

with

$$\mathcal{C}[f] = \int_{\mathbb{T}\times\mathbb{R}} \cos(y) f(y,v)\mathrm{d}y\mathrm{d}v, \quad \text{and} \quad \mathcal{S}[f] = \int_{\mathbb{T}\times\mathbb{R}} \sin(y) f(y,v)\mathrm{d}y\mathrm{d}v,$$

where we use in the integration the normalized Lebesgue measure on the torus. Note that when f is spatially even, *ie* $f(-x,v) = f(x,v)$, we have $\mathcal{S}[f] = 0$.

Our interest lies in stationary solutions of (4.1.1) that depend on both x and v. More precisely, fix a function $G : [-e, +\infty[\to \mathbb{R}$, $e > 0$, with sufficient regularity, and consider a function $\eta(x,v)$ satisfying the equation

$$\eta(x,v) = G\left(H[\eta](x,v)\right). \tag{4.1.4}$$

Such functions (4.1.4) are stationary states of (4.1.1), since

$$\{\eta, H[\eta]\} = \{G\left(H[\eta]\right), H[\eta]\} = 0.$$

Note that the existence of a solution η of (4.1.4) is *a priori* not clear. To that extent, note that for any integrable function η, we may write

$$H[\eta] = \frac{v^2}{2} - \mathcal{C}[\eta]\cos(x) - \mathcal{S}[\eta]\sin(x).$$

Up to a translation $x \mapsto x + x_0$, one can always assume that $\mathcal{S}[\eta] = 0$, such that

$$H[\eta] = \frac{v^2}{2} - \mathcal{C}[\eta]\cos(x).$$

Let us indeed specify the argument: we can always define M_0 and $x_0 \in [-\pi, \pi]$ such that

$$\phi[\eta](x) = M_0 \cos(x - x_0).$$

For that it suffices to set $M_0 = \sqrt{\mathcal{C}[\eta]^2 + \mathcal{S}[\eta]^2}$, and to take x_0 as the solution of

$$M_0 \cos(x_0) = \mathcal{C}[\eta] \quad \text{and} \quad M_0 \sin(x_0) = \mathcal{S}[\eta].$$

We have then that

$$\phi[\eta](x) = M_0 \cos(x_0)\cos(x) + M_0 \sin(x_0)\sin(x) = M_0 \cos(x - x_0).$$

By the rotational invariance of the Vlasov-HMF model (4.1.1), we may then assume that, up to the translation $x \to x + x_0$, we have

$$\phi[\eta](x) = M_0 \cos(x), \quad \text{such that} \quad \mathcal{C}[\eta] = M_0 \quad \text{and} \quad \mathcal{S}[\eta] = 0.$$

M_0 is usually called the magnetization of η, and one quite easily sees that η solves equation (4.1.4) if and only if M_0 satisfies the equation

$$M_0 = \mathcal{C}\left[G\left(\frac{v^2}{2} - M_0\cos(x)\right)\right].$$

Note that this equation is satisfied by $M_0 = 0$, whatever the choice of G is, and in this case $\partial_x \eta(x, v) = 0$. Landau damping being well-understood for these spatially homogeneous states, we aim to study the case where a solution $M_0 > 0$ exists. In section 4.6, we shall exhibit sufficient conditions on G to ensure the existence of a strictly positive magnetization. A typical example that we shall consider is when G is an exponential function (see also [29]). From now on, we fix a triplet (η, G, M_0) such that

$$\eta(x, v) = G\left(h_0(x, v)\right), \quad h_0(x, v) = \frac{v^2}{2} - M_0\cos(x), \quad M_0 > 0. \tag{4.1.5}$$

We are interested in solutions of equation (4.1.1) that are perturbations of such a stationary state, namely

$$\begin{cases} f(t,x,v) = \eta(x,v) + r(t,x,v) \\ r(0,x,v) = r^0(x,v). \end{cases} \tag{4.1.6}$$

where r^0 is the initial perturbation. Note that

$$H[f] = H[\eta] - \phi[r],$$

such that $r(t,x,v)$ satisfies the equation

$$\partial_t r(t,x,v) - \{\eta, \phi[r]\}(t,x,v) + \{r, H[\eta]\}(t,x,v) - \{r, \phi[r]\}(t,x,v) = 0. \tag{4.1.7}$$

In this paper, we will retain the linear part of this equation, namely the linearized equation around η, that reads

$$\partial_t r(t,x,v) - \{\eta, \phi[r]\}(t,x,v) + \{r, h_0\}(t,x,v) = 0. \tag{4.1.8}$$

Let us set $\mathcal{C}(t) = \mathcal{C}[r(t)]$ and $\mathcal{S}(t) = \mathcal{S}[r(t)]$. Solving equation (4.1.8) using characteristics, one can obtain a closed system of Volterra integral equations for $\mathcal{C}(t)$ and $\mathcal{S}(t)$, that will play the same crucial part in the evolution as the closed equations for the Fourier coefficients of the density $\int_v f dv$ that are obtained in the study of the stability of homogeneous stationary states for the HMF model or the Vlasov-Poisson system (see [10, 40, 61], or chapter 1).
Let $\psi_t(x,v)$ the flow of the Hamiltonian h_0, that is the flow associated with the ODE

$$\begin{cases} \dot{x} &= \partial_v h_0(x,v) \\ \dot{v} &= -\partial_x h_0(x,v). \end{cases}$$

The flow ψ_t is globally well defined and defines a symplectic application. In particular it preserves the Poisson bracket

$$\{f,g\} \circ \psi_t(x,v) = \{f \circ \psi_t, g \circ \psi_t\}(x,v).$$

Note that for a given function $f(x,v)$, the function $(t,x,v) \mapsto f(t,x,v) = f(\psi_t(x,v))$ solves the equation

$$\partial_t f(t,x,v) = \{f, h_0\}.$$

As in the homogeneous case (see [40]), we shall pull-back the perturbation by the flow ψ_t, and thus consider the function

$$g(t,x,v) = r(t, \psi_t(x,v)) = r \circ \psi_t(x,v) \tag{4.1.9}$$

where r is defined by (4.1.6) and satisfies (4.1.8). Defining the projection $\mathrm{X} : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ by $\mathrm{X}(x,v) = x$, we have the following result:

Proposition 4.1.1. *The function $g(t,x,v)$ satisfies the equation*

$$\partial_t g = \mathcal{C}(t)\{\eta, \cos(\mathrm{X}\circ\psi_t)\} + \mathcal{S}(t)\{\eta, \sin(\mathrm{X}\circ\psi_t)\}. \tag{4.1.10}$$

where

$$\mathcal{C}(t) = \mathcal{C}[r(t)] = \mathcal{C}[g\circ\psi_{-t}] = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}(y,w))g(t,\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w \tag{4.1.11}$$

and

$$\mathcal{S}(t) = \mathcal{S}[r(t)] = \mathcal{S}[g\circ\psi_{-t}] = \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}(y,w))g(t,\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w. \tag{4.1.12}$$

Moreover, the coefficients $\mathcal{C}(t) = \mathcal{C}[r(t)]$ and $\mathcal{S}(t) = \mathcal{S}[r(t)]$ satisfy the following Volterra integral equations

$$\mathcal{C}(t) = F_{\mathcal{C}}(t) + \int_0^t \mathcal{C}(s)K_{\mathcal{C}}(t-s)\mathrm{d}s \quad \textit{and} \quad \mathcal{S}(t) = F_{\mathcal{S}}(t) + \int_0^t \mathcal{S}(s)K_{\mathcal{S}}(t-s)\mathrm{d}s, \tag{4.1.13}$$

with

$$F_{\mathcal{C}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w, \quad F_{\mathcal{S}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w,$$

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}}\int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\}\cos(\mathrm{X}\circ\psi_t), \quad K_{\mathcal{S}}(t) = -\mathbb{1}_{\{t\geq 0\}}\int_{\mathbb{T}\times\mathbb{R}} \{\eta, \sin(\mathrm{X})\}\sin(\mathrm{X}\circ\psi_t).$$

This Proposition will be proved at the beginning of subsection 4.3.2. Note that the coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$ exists globally in time provided that $r^0(x,v)$ is smooth enough, by standard well-posedness results for (4.1.1).
Our main result is that, if η and r^0 have sufficient regularity, the coefficients $\mathcal{S}(t)$ and $\mathcal{C}(t)$ decay in time, with algebraic rates of damping. As a consequence it will be shown that $r = g\circ\psi_{-t}$ converges weakly towards a final state R_∞ that is a function of h_0.

4.2 Main results

4.2.1 Action angle variables and applications

As a one-dimensional Hamiltonian system, the system associated with the Hamiltonian $h_0(x,v)$ is integrable (it is the classical dynamical system for the motion of a pendulum). We will need relatively precise informations about the corresponding action-angle change of variable. Let us define the three following zones U_+, U_- and $U_\circ$ as follows:

$$\begin{aligned} U_+ &= \{(x,v)\in\mathbb{T}\times\mathbb{R}\,|\,v>0 \quad \text{and} \quad h_0(x,v) > M_0\},\\ U_- &= \{(x,v)\in\mathbb{T}\times\mathbb{R}\,|\,v<0 \quad \text{and} \quad h_0(x,v) > M_0\}, \quad \text{and}\\ U_\circ &= \{(x,v)\in\mathbb{T}\times\mathbb{R}\,|\,h_0(x,v) < M_0\}. \end{aligned} \tag{4.2.1}$$

Recall that $h_0(x,v) \geq -M_0$ and the center of the "eye" $U_\circ$ corresponds to the point $(x,v) = (0,0)$ which minimizes h_0. The set

$$\{(x,v) \in \mathbb{T} \times \mathbb{R} \quad | \quad h_0(x,v) = M_0\}$$

will usually be called "the separatix". Let us first recall the following Theorem:

Theorem 4.2.1. *Setting $h(x,v) = \frac{v^2}{2} - M_0 \cos(x)$, then for $* \in \{\pm, \circ\}$, there exists symplectic change of variable $(x,v) \mapsto (\psi, h)$ from U_* to the set*

$$V_* := \{(\psi, h) \in \mathbb{R}^2, | h \in I_*,\, \psi \in (-r_*(h), r_*(h))\},$$

where $r_*(h)$ is a positive function, $I_\pm = (M_0, +\infty)$ and $I_\circ = (-M_0, M_0)$ such that the flow of the pendulum in the variable (ψ, h) is $h(t) = h(0)$ and $\psi(t) = t + \psi(0)$.
Moreover, there exists a symplectic change of variables $(\psi, h) \mapsto (\theta, a)$ from V_* to

$$W_* = \{(\theta, a) \in \mathbb{R}^2 \,|\, a \in J_*,\, \theta \in (-\pi, \pi)\},$$

with $J_\pm = (\frac{4}{\pi}\sqrt{M_0}, +\infty)$ and $J_\circ = (0, \frac{8}{\pi}\sqrt{M_0})$ such that

$$\theta(\psi, h) = \omega_*(h)\psi, \quad \text{and} \quad \partial_h a(h) = \frac{1}{\omega_*(h)} = \frac{\pi}{r_*(h)},$$

so that the flow of the pendulum in the variables (θ, a) in U_* is $a(t) = a(0)$ and $\theta(t) = t\omega_*(a(0)) + \theta(0)$.

This Theorem is explicit in the sense that formulae exist for the construction of the different changes of variables, by involving Jacobi elliptic functions. Fourier series in variable θ may be in particular considered, and for a given function $f(x,v)$, we can define the restriction f^* of f to the set U_*, and the Fourier coefficients

$$f^*_\ell(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(x(\theta, a), v(\theta, a)) e^{-i\ell\theta} \mathrm{d}\theta$$

for $a \in J_*$, where $x(\theta, a)$ and $v(\theta, a)$ are given by the change of variable on U_*. Note that for given functions f and φ, we have the decomposition

$$\int_{\mathbb{R}\times\mathbb{T}} f(x,v)\varphi(x,v)\mathrm{d}x\mathrm{d}v = \sum_{*\in\{\pm,\circ\}} \sum_{\ell\in\mathbb{Z}} \int_{J_*} f^*_\ell(a)\varphi^*_{-\ell}(a)\mathrm{d}a. \tag{4.2.2}$$

Finally, let us notice that the jacobian of the change of variable $h \mapsto a(h)$ is $\partial_h a(h) = \frac{1}{\omega^*(h)}$, and we have in particular

$$\int_{J_*} f^*_\ell(a)\varphi^*_\ell(a)\mathrm{d}a = \int_{I_*} f^*_\ell(a(h))\varphi^*_{-\ell}(a(h))\frac{1}{\omega_*(h)}\mathrm{d}h. \tag{4.2.3}$$

We will usually write $f^*_\ell(h)$ for the quantity $f^*_\ell(a(h))$, as natural variables to compute the expressions and the singularities of the relevant functions in action-angle variables shall be

indeed expressed more easily in the variables (θ, h), which are not symplectic, but on which integrals can be easily computed as well as the flow of the system. Moreover, in this case,

$$f_0^*(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x(\theta,h), v(\theta,h))\mathrm{d}\theta$$

can be seen as an average of f on the isocurve $\{(x,v)\,|\,h_0(x,v)=h\}$, while ψ is the arclength on this curve, the jacobian $\frac{1}{\omega_*(h)}$ appearing in the standard co-area formula, which is another way to see (4.2.2)-(4.2.3).
The notations $C_\ell^*(a)$ and $S_\ell^*(a)$ will be used for the Fourier coefficients of the functions

$$\theta \mapsto \cos(x(\theta,a)) \quad \text{and} \quad \theta \mapsto \sin(x(\theta,a)),$$

respectively, and both restricted to U_*. These coefficients can be calculated explicitly using elliptic functions (see Propositions 4.7.4 and 4.7.10), and we shall write

$$\cos(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} C_\ell^*(a)e^{i\ell\theta} \quad \text{and} \quad \sin(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} S_\ell^*(a)e^{i\ell\theta}, \tag{4.2.4}$$

for $(\theta, a) \in J_* \times (-\pi, \pi)$.

Before stating our first important result, let us fix some notations. For a two-dimensional integer $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we shall set

$$|\alpha| = \alpha_1 + \alpha_2.$$

We shall also write $\partial_{x,v}^\alpha$ for the operator acting on functions $f : \mathbb{T}\times\mathbb{R} \to \mathbb{C}$ by the formula

$$\partial_{x,v}^\alpha f(x,v) = \partial_x^{\alpha_1}\partial_v^{\alpha_2} f(x,v).$$

In section 4.7 we prove the following result:

Theorem 4.2.2. *Consider $f(x,v)$ and $\varphi(x,v)$ two functions such that*

$$\max_{|\alpha|\le m} \left\|\langle v\rangle^\mu \partial_{x,v}^\alpha f(x,v)\right\|_{L^\infty} \le C_{m,\mu} \quad \text{and} \quad \max_{|\alpha|\le M} \left\|\partial_{x,v}^\alpha \varphi(x,v)\right\|_{L^\infty} \le C_M,$$

for some $\mu > 2$, where $\langle v\rangle = (1+v^2)^{1/2}$.
Let p and q be defined by

$$p = \max\{n\ge 1,\ \partial_{x,v}^\alpha f(0,0) = 0,\ \forall\alpha,\ 1\le|\alpha|\le n\},\ q = \max\{n\ge 1,\ \partial_{x,v}^\alpha \varphi(0,0) = 0,\ \forall\alpha,\ 1\le|\alpha|\le n\}.$$

Let $\psi_t(x,v)$ denote the flow of the Hamiltonian $h_0(x,v) = \frac{v^2}{2} - M_0\cos(x)$. Then, if

$$m \ge 5 + p + \frac{p+q}{2} \quad \text{and} \quad M \ge \max\left(7 + q + \frac{p+q}{2}, m+2\right),$$

there exists $C > 0$ such that for all $t \ge 1$, we have

$$\left|\int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}}\int_{J_*} f_0^*(a)\varphi_0^*(a)\mathrm{d}a\right| \le \frac{C}{(1+t)^{\frac{p+q}{2}+2}}.$$

Let us explain this Theorem as follows: the starting point of the proof is the Fourier expansion (4.2.2), which yields

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}v &= \sum_{*\in\{\pm,\circ\}}\sum_{\ell\in\mathbb{Z}}\int_{J_*} f_\ell^*(a)\varphi_{-\ell}^*(a)e^{it\ell\omega_*(a)}\mathrm{d}a \\ &= \sum_{*\in\{\pm,\circ\}}\sum_{\ell\in\mathbb{Z}}\int_{I_*} f_\ell^*(h)\overline{\varphi_\ell^*(h)}e^{it\ell\omega_*(h)}\frac{1}{\omega_*(h)}\mathrm{d}h.\end{aligned}$$

Now we can use a stationary phase argument by integrating with respect to h to gain a decay with respect to t. Typically, this kind of analysis depends on the possible cancellation of $\partial_h\omega_*(h)$. In our case, the situation seems to be very favourable, as $\partial_h\omega_*(h)$ never approaches zero, as shown in Section 4.7. However, the problems come from the singularities of the action angle variables. We can distinguish two zones: near the separatix $h = M_0$. In this case, the action angle variables induce logarithmic singularities. Essentially it means that the Fourier coefficients f_ℓ^* involve logarithmic singularities near $h = M_0$. However, near this point, $\omega_*(h)$ also exhibits a logarithmic singularity, and it can be shown that $\partial_h\omega_*(h)$ goes to infinity fast enough to ensure a decay in time which is essentially driven by the regularity of f and φ. So the problems are not at the separatix.
Near the point $h = -M_0$, the situation is more delicate: in this zone, the pendulum Hamiltonian is essentially a perturbation of the Harmonic oscillator, for which no Landau damping is expected (ω being constant). However, we can prove that $\partial_h\omega_*(h)$ does not vanish near this point. But this is not enough: indeed the action angle variable of the harmonic oscillator involves algebraic singularity of order $\sqrt{h+M_0}$. This explains why the rate of decay of the integral with respect to the time is mainly driven by the behavior of f and φ near $(0,0)$ which corresponds of a local behavior of $f_\ell^\circ(h)\varphi_\ell^\circ(h)$ in $(h+M_0)^{\frac{p+q}{2}}$ which yields the main contribution for the decay in the previous Theorem.
Theorem 4.2.2 will be a straightforward consequence of Propositions 4.7.6 and 4.7.12, proven in section 4.7.

Remark 4.2.3. *In view of the previous analysis, the result of the Theorem may be improved if special cancellations occur. More precisely, if, for all $\ell \in \mathbb{Z}$,*

$$f_\ell^\circ(h)\varphi_{-\ell}^\circ(h) = 0,$$

then there is no contribution coming from the eye, and we have

$$\int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}v = \sum_{*\in\{\pm\}}\sum_{\ell\in\mathbb{Z}}\int_{J_*} f_\ell^*(a)\varphi_{-\ell}^*(a)e^{it\ell\omega_*(a)}\mathrm{d}a.$$

Therefore the singularity near the point $h \sim -M_0$ does not interfere, and the decay in time is essentially driven by the regularity of f and g. More precisely, if r is such that $M \geq 3+r$ and $m \geq 1+r$, then the conclusion of the Theorem becomes

$$\left|\int_{\mathbb{T}\times\mathbb{R}} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm\}}\int_{J_*} f_0^*(a)\varphi_0^*(a)\mathrm{d}a\right| \leq \frac{C}{(1+t)^{r+1}},$$

by applying Proposition 4.7.6 of section 4.7.
This is typically the situation if one takes $f(x,v) = \sin(x)$ *and* $\varphi(x,v) = \cos(x)$ *(see Proposition 4.7.10).*

4.2.2 Linear Landau damping

For a function $F(t)$, we define

$$\hat{F}(\xi) = \int_{\mathbb{R}} F(t) e^{-it\xi} \mathrm{d}t$$

its Fourier transform.

Theorem 4.2.4. *Let* $\eta(x,v) = G(h_0(x,v))$ *with* G *a decreasing function that satisfies the assumption*

$$\max_{n \leq 10} \left\| \langle y \rangle^{\mu} G^{(n)}(y) \right\|_{L^\infty(\mathbb{R})} \leq C_\mu,$$

with $\mu > 2$*, and assume that there exists* $\kappa > 0$ *such that*

$$\min_{\mathrm{Im}\xi \leq 0} |1 - \hat{K}_{\mathcal{C}}(\xi)| \geq \kappa \quad \textit{and} \quad \min_{\mathrm{Im}\xi \leq 0} |1 - \hat{K}_{\mathcal{S}}(\xi)| \geq \kappa. \tag{4.2.5}$$

Let us assume that the initial perturbation r^0 *satisfies*

$$\max_{|\alpha| \leq m} \left\| \langle v \rangle^{\nu} \partial^{\alpha}_{x,v} r^0(x,v) \right\|_{L^\infty} \leq C_{m,\nu},$$

for some $\nu > 2$*, and where*

$$m \geq 5 + \frac{3p}{2},$$

with

$$p = \max \left\{ k \geq 1, \quad \partial^{\alpha}_{x,v} r^0(0,0) = 0, \quad \forall 1 \leq |\alpha| \leq k \right\}.$$

Then, if r^0 *satisfies the orthogonality condition*

$$\sum_{* \in \{\pm, \circ\}} \int_{J_*} C_0^*(a) (r^0)_0^*(a) \mathrm{d}a = 0, \tag{4.2.6}$$

there exists $C > 0$ *such that for all* $t \geq 0$

$$|\mathcal{C}(t)| \leq \frac{C}{(1+t)^{\alpha}} \quad \textit{and} \quad |\mathcal{S}(t)| \leq \frac{C}{(1+t)^2},$$

with

- $\alpha = 5/2$ *if* $p \geq 1$ *or* $\int_{\mathbb{R}\times\mathbb{T}} r^0(x,v) \mathrm{d}x \mathrm{d}v = 0$,
- $\alpha = 3$ *if both conditions are satisfied*
- $\alpha = 2$ *if none of them is.*

Let us make the following comments:

a) The damping rates of the two coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$ are algebraic, and essentially independent of the assumptions on the regularity of r^0. This may be explained as follows: through the assumptions on $\hat{K}_{\mathcal{C}}$ and $\hat{K}_{\mathcal{S}}$, the quantities $\mathcal{C}(t)$ and $\mathcal{S}(t)$ inherit the decay in time of the source terms $F_{\mathcal{C}}(t)$ and $F_{\mathcal{S}}(t)$ of the two Volterra equations that they satisfy (see (4.1.13)), by a Paley-Wiener Theorem, essentially. Now these sources terms fall under the scope of Theorem 4.2.2, and thus the damping rates are mainly driven by the behavior of the functions $\cos(X)r^0$ and $\sin(X)r^0$ in the vicinity $(x,v) \sim (0,0)$, and both of then have a zero of finite order there, independently from the regularity of r^0.
Note that the result may be slightly improved when special cancellations occur. Indeed, if the perturbation r^0 is such that $(r^0)^{\circ}_{\ell}(h) = 0$ for even ℓ, then by Remark 4.2.3 one may take $\alpha = 3$, even if $p = 0$. An example of this situation is if one considers $r^0(x,v) = \sin(x)\eta(x,v)$, for which the Theorem predicts that $\alpha = 5/2$, and where we may in fact obtain $\alpha = 3$ by using Remark 4.2.3. This will be observed numerically in chapter 5, for the nonlinear equation.
This is of course a major difference with the existing results on Landau damping around spatially homogeneous stationary states (see [10, 40, 61]). In this case the role of $\mathcal{C}(t)$ and $\mathcal{S}(t)$ in the dynamic is played by the Fourier modes of the density $\int_v f(t,x,v)\mathrm{d}v$, which satisfy as well Volterra integral equations, for which the decay in time of the source terms is completely driven by the regularity of r^0, according the Riemann-Lebesgue Lemma, essentially.

b) The orthogonality condition (4.2.6) seems natural since this condition is propagated by the flow of the linear equation (4.1.10). Indeed by using the action-angle variables given by Theorem 4.2.1, we can set $\mathrm{g}^*(t,\theta,a) = g(t,x(\theta,a),v(\theta,a))$, $* \in \{\pm,\circ\}$, since the change of variable is symplectic, we get for g^* the equation

$$\partial_t \mathrm{g}^* = \mathcal{C}(t)\{G(h),\cos(x(\cdot + t\omega,\cdot))\} + \mathcal{S}(t)\{G(h),\sin(x(\cdot + t\omega,\cdot))\}$$

where the Poisson bracket is now computed in the variable (θ,a). Since $G(h)$ depends only on a, we get from the above equation that

$$\partial_t \mathrm{g}^*_0 = \partial_t \int_\theta \mathrm{g}^*(\theta,a)\,\mathrm{d}\theta = 0.$$

c) We shall call assumption (4.2.5) the Penrose criterion, by analogy with the stability conditions of the same name that are usually used in the Landau damping literature. This is a typical assumption which allows one to solve the Volterra equations on $\mathcal{C}(t)$ and $\mathcal{S}(t)$ in Fourier, by a Paley-Wiener Theorem. We refer to [8, 10, 12, 40, 41, 61, 73] for similar statements.

As a corollary of Theorem 4.2.4, we get a scattering result for the solution g of (4.1.10) and the weak convergence of $r(t,x,v) = g(t,\psi_{-t}(x,v))$ the solution of (4.1.8) towards an asymptotic state $r_\infty(x,v)$ that depends only on $h_0(x,v)$.

Corollary 4.2.5. *Under the assumptions of Theorem 4.2.4 with $p = 0$, we obtain that:*

- *There exists $g_\infty(x,v)$ such that when $t \to +\infty$, we have*

$$\|g(t) - g_\infty\|_{L^1_{x,v}} \lesssim \frac{1}{\langle t \rangle}. \tag{4.2.7}$$

- *There exists $r_\infty(x,v)$ that depends only on h, that is to say $r_\infty(x,v) = R_\infty(h(x,v))$ such that for every test function ϕ, we have that*

$$\int_{\mathbb{T}\times\mathbb{R}} r(t,x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v \to_{t\to+\infty} \int_{\mathbb{T}\times\mathbb{R}} r_\infty(x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v.$$

Let us make the following additional comments:

a) In this special case of the linearized dynamics (4.1.8), we can express r_∞ in terms of the initial data r^0, and obtain that

$$R_\infty(h) = \int_{-\pi}^{\pi} r^0(x^*(\theta,h), v^*(\theta,h))d\theta, \quad h \in I_*, * \in \{\pm, \circ\}.$$

It also shows that

$$f(\theta,h) = \eta(h) + r(t, x^*(\theta,h), v^*(\theta,h)), \quad h \in I_*, * \in \{\pm,\circ\},$$

converges weakly to its initial average with respect to θ.
This is a behavior that we should relate to linear Landau damping around spatially homogeneous stationary states, since in this case the function

$$f(t,x,v) = \eta(v) + r(t,x,v)$$

converges to its initial average with respect to x (see [61] for instance for a complete discussion on linear Landau damping).

b) The first estimate of Corollary 4.2.5 shows that g exhibits a scattering behavior, which is also the case in both linear and nonlinear Landau damping around spatially homogeneous stationary states. The difference is that the present scattering estimate (4.2.7) holds in L^1 norm, whereas in the homogeneous case scattering estimates may be proved in regular norm, such as Sobolev's ([40]) or Gevrey's ([10]). The question of linear (and nonlinear) Landau damping around inhomogeneous stationary states in regular norms is left open, though we shall provide the reader with numerical simulations to investigate this matter, in chapter 5.

4.2.3 About the Penrose criterion

Written in this form, the Penrose criterion (4.2.5) is difficult to check, but we can relate it to a more classical condition that was found in [53] or [4] to ensure orbital stability of inhomogeneous stationary states in the nonlinear equation.
First we shall prove that the verification of Penrose criterion (4.2.5) a the frequency $\xi = 0$ is sufficient, by proving the following Theorem.

Theorem 4.2.6. *Let η be a state defined by (4.1.5), and assume that G satisfies the hypothesis of Theorem 4.2.4. Assume moreover that $G' < 0$ and*

$$1 - \hat{K}_{\mathcal{C}}(0) > 0 \quad \text{and} \quad 1 - \hat{K}_{\mathcal{S}}(0) > 0. \tag{4.2.8}$$

Then the Penrose criterion (4.2.5) holds true.

Let us now define the following notion of stability (see also [4, 53]).

Definition 4.2.7. *A state $\eta(x, v)$ defined by (4.1.5) is said to be linearly stable if*

$$1 + \int_{\mathbb{R}\times\mathbb{T}} \frac{\partial_v \eta(x,v)}{v} \cos^2(x) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} C_0^*(a)^2 \mathrm{d}a > 0.$$

Now this condition is easier to verify than the Penrose criterion (4.2.5), and in section 4.6 we shall exhibit examples of states η that satisfies it, typically Maxwell-Boltzmann stationary states. They will also satisfy the Penrose criterion (4.2.5), because of Theorem 4.2.6, and of the following

Proposition 4.2.8. *Let η be a state defined by (4.1.5). Assume that G satisfies the hypothesis of Theorem 4.2.4, and that $G' < 0$. Then (4.2.8) holds true if and only if η is stable in the sense of Definition 4.2.7.*

4.2.4 Organization

For the sake of clarity, let us explain how the rest of the chapter is organized. In section 3, we collect and prove some results concerning Volterra integral equations, and use them to prove Theorem 4.2.4. In section 4, we prove corollary 4.2.5. Section 5 is dedicated to the Penrose criterion, and we prove there Theorem 4.2.6 and Proposition 4.2.8. In section 4.6 we exhibit examples of inhomogeneous stationary states which are stable in the sense of definition 4.2.7. Finally, section 7 contains all the technical results that we shall need concerning angle-action variables, and we prove there Theorem 4.2.2.

4.3 Proof of Theorem 4.2.4

The proof proceeds as follows: we prove that the kernels $K_{\mathcal{C}}$ and $K_{\mathcal{S}}$ have sufficient decay in time, which, with the Penrose criterion, will allow us to apply a Paley-Wiener Theorem to the Volterra equations (4.1.13). This Theorem will yield a control of the decay in time of $\mathcal{C}(t)$ and $\mathcal{S}(t)$ by the one of the source terms $F_{\mathcal{C}}(t)$ and $F_{\mathcal{S}}(t)$, and the latter will be guaranteed by Theorem 4.2.2.

4.3.1 Study of the kernels $K_{\mathcal{C}}$ and $K_{\mathcal{S}}$

As a preliminary, we shall first use Theorem 4.2.2 in order to get the decay rates of the kernels. We shall prove the following result.

Proposition 4.3.1. *Let $\eta(x,v) = G(h_0(x,v))$ with G a decreasing function that satisfies the assumption*

$$\max_{n\leq 10} \left\| \langle y\rangle^{\mu} G^{(n)}(y) \right\|_{L^\infty(\mathbb{R})} \leq C_\mu,$$

with $\mu > 2$, Then there exist a constant C such that

$$|K_{\mathcal{C}}(t)| \leq \frac{C}{(1+t)^3} \quad \textit{and} \quad |K_{\mathcal{S}}(t)| \leq \frac{C}{(1+t)^2}.$$

Proof. Recall that

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}} \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\} \cos(\mathrm{X}\circ\psi_t).$$

We notice that, as

$$\int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\}\, \mathrm{d}x\mathrm{d}v = -\int_{\mathbb{T}\times\mathbb{R}} v\sin(\mathrm{X}) G'(h_0(x,v))\mathrm{d}x\mathrm{d}v = 0,$$

by oddness, we actually may write that

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}} \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\}\, (1-\cos(\mathrm{X}\circ\psi_t)).$$

We apply Theorem 4.2.2 with the functions $\{\eta, \cos(\mathrm{X})\}(x,v)$ and $1-\cos(\mathrm{X}(x,v))$. We have $p=1$ and $q=1$, and we have that for all $*\in\{\circ,\pm\}$,

$$f_0^*(h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x(h,\theta),v(h,\theta))\mathrm{d}\theta = \frac{\omega_*(h)G'(h)}{2\pi}\int_{-\pi}^{\pi} \partial_\theta(\cos(x(h,\theta)))\mathrm{d}\theta = 0.$$

Applying Theorem 4.2.2 yields then

$$|K_{\mathcal{C}}(t)| \leq \frac{1}{(1+t)^3}.$$

Concerning $K_{\mathcal{S}}(t)$, it suffices to apply Theorem 4.2.2 with the functions $\{\eta, \sin(\mathrm{X})\}(x,v)$ and $\sin(\mathrm{X}(x,v))$. We have this time $p=q=0$, and $S_0^*(h)=0$ for all $*\in\{\circ,\pm\}$ (see (4.7.22) and (4.7.35)). Hence the application of Theorem 4.2.2 yields

$$|K_{\mathcal{S}}(t)| \leq \frac{1}{(1+t)^2}.$$

□

4.3.2 Study of the magnetization

We shall now start the proof of Theorem 4.2.4. First we shall prove Proposition 4.1.13.
Proof of [Proposition 4.1.1]. Let us first prove (4.1.10). If r solves (4.1.8), the function

$$g(t,x,v) = r(t,\psi_t(x,v)) = r\circ\psi_t(x,v)$$

satisfies

$$\begin{aligned}\partial_t g(t,x,v) &= \{r,h_0\}(t,\psi_t(x,v)) + \{\eta,\phi[r]\}\,(t,\psi_t(x,v)) - \{r,h_0\}\,(t,\psi_t(x,v))\\ &= \{\eta,\phi[g\circ\psi_{-t}]\}(t,\psi_t(x,v)).\end{aligned}$$

Hence as η is invariant by the flow ψ_t, g solves

$$\partial_t g(t,x,v) = \{\eta,\phi[g\circ\psi_{-t}]\circ\psi_t\}(t,x,v).$$

Since ψ_t preserves the volume,

$$\begin{aligned}\phi[g\circ\psi_{-t}]\circ\psi_t(x,v) &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(x,v)-y)g(t,\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w\\ &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(x,v)-\mathrm{X}(y,w))g(t,\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w\\ &= \cos(\mathrm{X}\circ\psi_t(x,v))\mathcal{C}(t) + \sin(\mathrm{X}\circ\psi_t(x,v))\mathcal{S}(t),\end{aligned}$$

with $\mathcal{C}(t) = \mathcal{C}[g\circ\psi_{-t}] = \mathcal{C}[r(t)]$ and $\mathcal{S}(t) = \mathcal{S}[g\circ\psi_{-t}] = \mathcal{S}[r(t)]$.
Thus we can write

$$\partial_t g = \mathcal{C}(t)\{\eta,\cos\left(\mathrm{X}\circ\psi_t\right)\} + \mathcal{S}(t)\{\eta,\sin\left(\mathrm{X}\circ\psi_t\right)\},$$

which proves (4.1.10).
We deduce that

$$g(t,x,v) = r^0(x,v) + \int_0^t \mathcal{C}(s)\{\eta,\cos\left(\mathrm{X}\circ\psi_s\right)\} + \mathcal{S}(s)\{\eta,\sin\left(\mathrm{X}\circ\psi_s\right)\}\mathrm{d}s.$$

Using this formula and the fact that ψ_t preserves the Poisson bracket, we calculate that

$$\begin{aligned}\mathcal{C}(t) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}(y,w))g(t,\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}(y,w))r^0(\psi_{-t}(y,w))\mathrm{d}y\mathrm{d}w\\ &+ \int_0^t \mathcal{C}(s)\int_{\mathbb{T}\times\mathbb{R}} \cos(X)\{\eta,\cos(\mathrm{X}\circ\psi_{s-t})\}\mathrm{d}s\\ &+ \int_0^t \mathcal{S}(s)\int_{\mathbb{T}\times\mathbb{R}} \cos(X)\{\eta,\sin(X\circ\psi_{s-t})\}\mathrm{d}s.\end{aligned}$$

Note that the flow ψ_t is reversible with respect to the transformation $\nu(x,v) = (x,-v)$, that is we have $\psi_t\circ\nu = -\nu\circ\psi_{-t}$. But as the Hamiltonian is even in x, the flow is also reversible with respect to $(x,v)\mapsto(-x,v)$. Hence the transformation $\mu(x,v) := (-x,-v)$ satisfies $\psi_t\circ\mu = \mu\circ\psi_t$, and this transformation preserves the Poisson bracket and is an isometry.

Let us apply this to the last term in the previous equation. We thus have for any $\sigma \in \mathbb{R}$

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \sin(\mathrm{X}\circ\psi_\sigma)\} &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\mu)\{\eta, \sin(\mathrm{X}\circ\psi_\sigma)\}\circ\mu \\ &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \sin(\mathrm{X}\circ\mu\circ\psi_\sigma)\} \\ &= -\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \sin(\mathrm{X}\circ\psi_\sigma)\} = 0,\end{aligned}$$

as $\mathrm{X}\circ\mu = -\mathrm{X}$. For the same reason, we have

$$\int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X})\{\eta, \cos(\mathrm{X}\circ\psi_\sigma)\} = 0.$$

Now using the identities $\eta\circ\nu = \eta$ and $\mathrm{X}\circ\nu = \mathrm{X}$, and the evenness of the cosine function, we have

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \cos(\mathrm{X}\circ\psi_{s-t})\} &= -\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\nu)\{\eta, \cos(\mathrm{X}\circ\psi_{s-t}\circ\nu)\} \\ &= -\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \cos(\mathrm{X}\circ(-\nu)\circ\psi_{t-s})\} \\ &= -\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \cos(\mathrm{X}\circ\psi_{t-s})\}.\end{aligned}$$

Integrating by parts that last integral yields then

$$\int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X})\{\eta, \cos(\mathrm{X}\circ\psi_{s-t})\} = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_{t-s})\{\eta, \cos(\mathrm{X})\}.$$

Using the oddness of the sine function, we have by similar manipulations

$$\int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X})\{\eta, \sin(\mathrm{X}\circ\psi_{s-t})\} = -\int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}\circ\psi_{t-s})\{\eta, \sin(\mathrm{X})\}.$$

This ends the proof. ∎

To study the coefficients $\mathcal{C}(t)$ and $\mathcal{S}(t)$, we shall use general results on Volterra integral equations written under the form

$$y(t) = K * y(t) + F(t), \quad t \in \mathbb{R} \tag{4.3.1}$$

where K, y, F vanish for $t < 0$. Let us first recall the following Paley-Wiener result on Volterra integral equations (Theorem 4.1 of [44], see also [32, 65]).

Lemma 4.3.2 (Paley-Wiener)**.** *Assume that $K \in L^1(\mathbb{R})$ is such that*

$$\min_{\mathrm{Im}(\xi)\le 0} |1 - \hat{K}(\xi)| \ge \kappa.$$

Then there exists a unique resolvent kernel $R \in L^1(\mathbb{R}_+)$ which vanishes for $t \le 0$ such that

$$R(t) = -K(t) + K * R(t). \tag{4.3.2}$$

Note that using R, the solution of (4.3.1) can be written as

$$y(t) = F(t) - R * F(t). \tag{4.3.3}$$

We shall then use the following corollary.

Corollary 4.3.3. *Under the assumptions of Lemma 4.3.2, the following holds:*

i) There exists $C > 0$ such that

$$\|y\|_{L^\infty} \leq C\|F\|_{L^\infty}. \tag{4.3.4}$$

ii) If $\langle t\rangle^2 K \in L^\infty$ and $\langle t\rangle^2 F \in L^\infty$, then there exists $C > 0$ such that

$$|\langle t\rangle^2 y\|_{L^\infty} \leq C\|\langle t\rangle^2 F\|_{L^\infty}.$$

iii) If $\langle t\rangle^3 K \in L^\infty$ and $\langle t\rangle^\alpha F \in L^\infty$ for $\alpha \in [2,3]$, then

$$\|\langle t\rangle^\alpha y\|_{L^\infty} \leq C\|\langle t\rangle^\alpha F\|_{L^\infty}.$$

Proof. To get i) it suffices to use (4.3.3) and the Young inequality.

Let us prove ii). We first observe that

$$t^{\frac{1}{2}} y(t) = K * (t^{\frac{1}{2}} y) + \int_0^t (t^{\frac{1}{2}} - s^{\frac{1}{2}}) K(t-s) y(s)\, \mathrm{d}s + t^{\frac{1}{2}} F.$$

By using i), we obtain that

$$\|t^{\frac{1}{2}} y\|_{L^\infty} \lesssim \|y\|_{L^\infty} \sup_t \int_0^t (t^{\frac{1}{2}} - s^{\frac{1}{2}}) |K(t-s)|\, \mathrm{d}s + \|\langle t\rangle^{\frac{1}{2}} F\|_{L^\infty} \lesssim \|\langle t\rangle^{\frac{1}{2}} F\|_{L^\infty}. \tag{4.3.5}$$

Next, we can write that

$$ty(t) = K * (ty) + \int_0^t (t^{\frac{1}{2}} - s^{\frac{1}{2}}) K(t-s)\, s^{\frac{1}{2}} y(s)\, \mathrm{d}s + t^{\frac{1}{2}} \int_0^t (t^{\frac{1}{2}} - s^{\frac{1}{2}}) K(t-s) y(s)\, \mathrm{d}s + tF.$$

Consequently, by using i) and the assumptions on K, we obtain that

$$\|ty\|_{L^\infty} \lesssim \sup_t \left(\int_0^t \frac{(t-s)^{\frac{1}{2}}}{\langle t-s\rangle^2}\, \mathrm{d}s \right) \|\langle t\rangle^{\frac{1}{2}} y\|_{L^\infty} + \sup_t \left(t^{\frac{1}{2}} \int_0^t \frac{(t-s)^{\frac{1}{2}}}{\langle t-s\rangle^2} \frac{1}{\langle s\rangle^{\frac{1}{2}}} \mathrm{d}s \right) \|t^{\frac{1}{2}} y\|_{L^\infty} + \|\langle t\rangle F\|_{L^\infty}$$

and by (4.3.5),

$$\|ty\|_{L^\infty} \lesssim \|\langle t\rangle F\|_{L^\infty}. \tag{4.3.6}$$

Note that we have used that

$$t^{\frac{1}{2}} \int_0^t \frac{(t-s)^{\frac{1}{2}}}{\langle t-s\rangle^2} \frac{1}{\langle s\rangle^{\frac{1}{2}}} \mathrm{d}s \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}}} \int_{\frac{t}{2}}^t \frac{1}{\langle s\rangle^{\frac{1}{2}}}\, \mathrm{d}s + \int_0^{\frac{t}{2}} \frac{1}{\langle t-s\rangle^{\frac{3}{2}}}\, \mathrm{d}s \lesssim 1.$$

We then estimate $t^2 y$, and for that we write

$$t^2 y = K * t^2 y + F_2$$

where by similar manipulations as above, the source term F_2 may be estimated as follows

$$|F_2| \lesssim t^{\frac{1}{2}} \left(\left(\langle \cdot \rangle^{\frac{1}{2}} |K| \right) * (\langle \cdot \rangle |y|) \right) + \left(\langle \cdot \rangle^{\frac{1}{2}} |K| \right) * \left(\langle \cdot \rangle^{\frac{3}{2}} |y| \right) + t \left(\left(\langle \cdot \rangle^{\frac{1}{2}} |K| \right) * \left(\langle \cdot \rangle^{\frac{1}{2}} |y| \right) \right) + t^{\frac{3}{2}} \left(\langle \cdot \rangle^{\frac{1}{2}} |K| \right) * (|y|) + t^2 |F|.$$

By using again i) and (4.3.5), and similar arguments as above, we obtain that

$$\|t^2 y\|_{L^\infty} \lesssim \|\langle t \rangle^{\frac{3}{2}} y\|_{L^\infty} + \|\langle t \rangle^2 F\|_{L^\infty}.$$

To conclude, we can use first the interpolation inequality

$$\|\langle t \rangle^{\frac{3}{2}} y\|_{L^\infty} \lesssim \|\langle t \rangle y\|_{L^\infty}^{\frac{1}{2}} \|\langle t \rangle^2 y\|_{L^\infty}^{\frac{1}{2}}.$$

Then we apply the Young inequality: for any $\delta > 0$,

$$\|\langle t \rangle y\|_{L^\infty}^{\frac{1}{2}} \|\langle t \rangle^2 y\|_{L^\infty}^{\frac{1}{2}} \leq \frac{\|\langle t \rangle y\|_{L^\infty}}{2\delta} + \frac{\delta \|\langle t \rangle^2 y\|_{L^\infty}}{2}.$$

Choosing δ small enough, we conclude that

$$\|t^2 y\|_{L^\infty} \lesssim \|\langle t \rangle y\|_{L^\infty} + \|\langle t \rangle^2 F\|_{L^\infty}$$

and the result follows by using (4.3.6).

To prove iii), we can use the same arguments. We first write

$$ty = K * (ty) + F_1$$

with

$$F_1(t) = tF + (tK) * y.$$

Since $tK \in L^1$, we get by using (4.3.4) that

$$\|ty\|_{L^\infty} \lesssim \|F_1\|_{L^\infty} \lesssim \|\langle t \rangle F\|_{L^\infty}.$$

Next, we write

$$t^2 y = K * t^2 y + F_2, \quad F_2 = (tK) * ty + tF^1$$

and by Young's inequality

$$\|F_2\|_{L^\infty} \lesssim \|tK\|_{L^1} \|ty\|_{L^\infty} + \|t^2 F\|_{L^\infty} + \|t((tK) * y)\|_{L^\infty} \lesssim \|tF\|_{L^\infty} + \|t^2 F\|_{L^\infty} + \|t((tK) * y)\|_{L^\infty}.$$

It remains to see that

$$|(tK) * y| \lesssim \int_0^t \frac{1}{\langle t - s \rangle^2} \frac{1}{\langle s \rangle} \|\langle t \rangle y\|_{L^\infty} \mathrm{d}s \lesssim \frac{1}{\langle t \rangle} \|ty\|_{L^\infty},$$

such that

$$\|F_2\|_{L^\infty} \lesssim \|\langle t\rangle^2 F\|_{L^\infty}.$$

We conclude by using again (4.3.4) that

$$\|t^2 y\|_{L^\infty} \lesssim \|\langle t\rangle^2 F\|_{L^\infty}.$$

$t^3 y$ is estimated in the same way as above. □

We shall then apply the Corollary to the two Volterra equations (4.1.13) to prove Theorem 4.2.4, starting with the one satisfied by $\mathcal{C}(t)$. Note that by using Proposition 4.3.1, and the Penrose criterion (4.2.5), we get that the kernel $K_{\mathcal{C}}$ matches the assumptions of Corollary 4.3.3 iii). We have moreover that

$$F_{\mathcal{C}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w$$

can be estimated by

$$|F_{\mathcal{C}}(t)| \lesssim \frac{1}{\langle t\rangle^\alpha}.$$

with $\alpha \in \{2, 5/2, 3\}$. Indeed, we apply Theorem 4.2.2 (using the orthogonality condition (4.2.6)) with the functions $\cos(\mathrm{X}(x,v))$ and $r^0(x,v)$, for which we have $p \geq 0$ and $q = 0$. Now without further assumptions this implies that $\alpha = 2$. If we assume that $p \geq 1$, then we obtain $\alpha = 5/2$. If we assume that r^0 has average zero, then

$$F_{\mathcal{C}}(t) = \int_{\mathbb{T}\times\mathbb{R}} (\cos(\mathrm{X}\circ\psi_t(y,w)) - 1)r^0(y,w)\mathrm{d}y\mathrm{d}w,$$

and we apply Theorem 4.2.2 with the functions $\cos(\mathrm{X}(x,v)) - 1$ and $r^0(x,v)$, for which we have now $p \geq 0$ and $q = 1$, which implies that $\alpha = 5/2$, and that $\alpha = 3$ if we assume moreover that $p \geq 1$.
Therefore the application of Corollary 4.3.3 to the first Volterra equation of (4.1.13) yields the estimate on $\mathcal{C}(t)$ claimed in Theorem 4.2.4.

In the case of the second Volterra equation of (4.1.13), satisfied by $\mathcal{S}(t)$, we have

$$F_{\mathcal{S}}(t) = \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w.$$

The application of Theorem 4.2.2 with the functions $\sin(\mathrm{X}(x,v))$ and $r^0(x,v)$, for which we have $p \geq 0$ and $q = 0$, yields the estimate

$$|F_{\mathcal{S}}(t)| \lesssim \frac{1}{\langle t\rangle^2}.$$

As the kernel $K_{\mathcal{S}}$ falls under the scope of Corollary 4.3.3 ii), the estimate on $\mathcal{S}(t)$ claimed in Theorem 4.2.4 follows.

4.4 Proof of Corollary 4.2.5

Let us first study the asymptotic behavior of g, and define $g_\infty(x,v)$ by

$$g_\infty(x,v) = r^0(x,v) + \int_0^{+\infty} \Big(\mathcal{C}(s)\,\{\eta, \cos(\mathrm{X})\} \circ \psi_s(x,v) + \mathcal{S}(s)\,\{\eta, \sin(\mathrm{X})\} \circ \psi_s(x,v) \Big)\, \mathrm{d}s. \quad (4.4.1)$$

Note that the above integral is convergent in $L^1_{x,v}$. Indeed, by using that ψ_s is measure preserving and Theorem 4.2.4, we get that

$$\|\mathcal{C}(s)\,\{\eta, \cos(\mathrm{X})\} \circ \psi_s(x,v) + \mathcal{S}(s)\,\{\eta, \sin(\mathrm{X})\} \circ \psi_s(x,v)\|_{L^1_{x,v}} \lesssim \frac{1}{\langle s\rangle^2}.$$

As

$$g(t,x,v) = r^0(x,v) + \int_0^{t} \Big(\mathcal{C}(s)\,\{\eta, \cos(\mathrm{X})\} \circ \psi_s(x,v) + \mathcal{S}(s)\,\{\eta, \sin(\mathrm{X})\} \circ \psi_s(x,v) \Big)\, \mathrm{d}s,$$

this also yields that

$$\|g(t) - g_\infty\|_{L^1_{x,v}} \lesssim \int_t^{+\infty} \frac{1}{\langle s\rangle^2}\, \mathrm{d}s \lesssim \frac{1}{\langle t\rangle}. \quad (4.4.2)$$

Now, let us study the weak convergence of $r(t,x,v)$. Let us observe that for every test function $\phi(x,v)$, we have by volume preservation that

$$\int_{\mathbb{T}\times\mathbb{R}} r(t,x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}\times\mathbb{R}} g(t,x,v)\phi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v = \int_{\mathbb{T}\times\mathbb{R}} g_\infty(x,v)\phi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v + \mathcal{O}\left(\frac{1}{\langle t\rangle}\right).$$

Therefore, we only need to study the limit when $t \to +\infty$ of

$$I(t) = \int_{\mathbb{T}\times\mathbb{R}} g_\infty(x,v)\phi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v.$$

Let us define

$$r_\infty(x,v) = R_\infty(h(x,v))$$

with

$$R_\infty(h) = \int_{(-\pi,\pi)} r^0(x(\theta,h), v(\theta,h))\mathrm{d}\theta, \quad h \in I_*,\ * \in \{\pm, \circ\}.$$

By using the expression (4.4.1), we obtain that

$$I(t) = \int_{\mathbb{T}\times\mathbb{R}} r^0(x,v)\phi(\psi_t(x,v))\,\mathrm{d}x\mathrm{d}v + \int_0^{+\infty} \left(\mathcal{C}(s) \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \cos(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v + \mathcal{S}(s) \int_{\mathbb{T}\times\mathbb{R}} \{\eta, \sin(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v \right) \mathrm{d}s.$$

Now, thanks to Theorem 4.2.2, we obtain that

$$\int_{\mathbb{T}\times\mathbb{R}} r^0(x,v)\phi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v \to_{t\to+\infty} \sum_{*\in\{\pm,\circ\}} \int_{J_*} (r^0)^*_0(a)\phi^*_0(a)\mathrm{d}a = \int_{\mathbb{T}\times\mathbb{R}} r_\infty(x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v.$$

Next, we observe that $\{\eta,\cos(\mathrm{X})\}^*_0 = \{\eta,\sin(\mathrm{X})\}^*_0 = 0$. Consequently, by using again Theorem 4.2.2, we obtain that

$$\left|\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\cos(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v\right| + \left|\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\sin(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v\right| \lesssim \frac{1}{\langle t-s\rangle^2}.$$

Consequently, we find that

$$\left|\int_0^{+\infty}\left(\mathcal{C}(s)\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\cos(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v + \mathcal{S}(s)\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\sin(\mathrm{X})\}\phi(\psi_{t-s}(x,v))\,\mathrm{d}x\mathrm{d}v\right)\mathrm{d}s\right| \lesssim \int_0^{+\infty}\frac{1}{\langle s\rangle^2}\frac{1}{\langle t-s\rangle^2}\,\mathrm{d}s \lesssim \frac{1}{\langle t\rangle^2}.$$

We have thus proven that

$$\int_{\mathbb{T}\times\mathbb{R}} r(t,x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v \to_{t\to+\infty} \int_{\mathbb{T}\times\mathbb{R}} r_\infty(x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v,$$

which concludes the proof of corollary 4.2.5.

4.5 Proofs of Theorem 4.2.6 and Proposition 4.2.8

Proof of Theorem 4.2.6.

Case of $K_{\mathcal{C}}$

Let us start with the study of $K_{\mathcal{C}}$, which is given by

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}}\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\cos(\mathrm{X})\}\cos(\mathrm{X}\circ\psi_t).$$

From Proposition 4.3.1, we have that $K_{\mathcal{C}} \in L^1(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$. Therefore $\hat{K}_{\mathcal{C}}(\xi)$ defines an analytic function on $\{\mathrm{Im}(\xi) < 0\}$ which is continuous on $\{\mathrm{Im}(\xi)\leq 0\}$.
Moreover, the Riemann-Lebesgue Lemma shows that $\lim_{|\xi|\to+\infty}\hat{K}_{\mathcal{C}}(\xi) = 0$ with $\xi = \gamma + i\tau$, $\tau\leq 0$. The criterion (4.2.5) is thus automatically satisfied for large ξ and it suffices to study if $1-\hat{K}_{\mathcal{C}}(\xi)$ can vanish for ξ in a compact set (by uniform continuity if it does not vanish for every ξ in this compact set then it will not vanish uniformly).
That being said we shall first calculate a rather convenient expression of $K_{\mathcal{C}}(t)$ in terms of angle-action variables, and then take its complex-variable Fourier transform to check the Penrose criterion (4.2.5) on a compact set.

Step 1. Computing $\hat{K}_{\mathcal{C}}$

Using the preservation of the Poisson bracket we have that

$$K_{\mathcal{C}}(t) = -(2\pi)^{-1}\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\circ,\pm\}} \int_{J_*\times(-\pi,\pi)} \cos(\mathrm{X}\circ\psi_t(\theta,a))\partial_a\eta(a)\partial_\theta \cos(x(\theta,a))\mathrm{d}a\mathrm{d}\theta.$$

By using for $(\theta,a)\in J_*\times(-\pi,\pi)$ the identity $\psi_t(\theta,a)=\theta+t\omega_*(a)$ and the Fourier expansion (4.2.4) for the cosine function, we infer that

$$\begin{aligned} K_{\mathcal{C}}(t) &= -(2\pi)^{-1}\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell,\ell'\in\mathbb{Z}} \int_{J_*\times(-\pi,\pi)} \partial_a\eta(a) i\ell' C^*_{\ell'}(a) C^*_{\ell}(a) e^{it\ell\omega_*(a)} e^{i(\ell+\ell')\theta}\mathrm{d}a\mathrm{d}\theta \\ &= -\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{J_*} \partial_a\eta(a)(-i\ell) C^*_{-\ell}(a) C^*_{\ell}(a) e^{it\ell\omega_*(a)}\mathrm{d}a \\ &= \mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{J_*} \partial_a\eta(a) i\ell |C^*_{\ell}(a)|^2 e^{it\ell\omega_*(a)}\mathrm{d}a, \end{aligned}$$

where we have also used the identity $C^*_{-\ell}(a)=\overline{C^*_{\ell}(a)}$. Note that as $\eta(a)=G(h_0(a))$ and $\omega_*(a)=\partial_a h_0(a)$, we can write that

$$K_{\mathcal{C}}(t) = -2\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a))\ell\omega_*(a)|C^*_{\ell}(a)|^2 \sin(t\ell\omega_*(a))\mathrm{d}a.$$

Therefore

$$K_{\mathcal{C}}(t) = \mathbb{1}_{\{t\geq 0\}} \frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathcal{C}}(t),$$

with

$$Q_{\mathcal{C}}(t) = 2 \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a))|C^*_{\ell}(a)|^2 \cos(t\ell\omega_*(a))\mathrm{d}a.$$

Taking the Fourier transform we may write that,

$$\hat{K}_{\mathcal{C}}(\xi) = \int_0^{+\infty} e^{-it\xi} \frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathcal{C}}(t)\mathrm{d}t = -Q_{\mathcal{C}}(0) + i\xi \int_0^{+\infty} e^{-it\xi} Q_{\mathcal{C}}(t)\mathrm{d}t,$$

where we have integrated by parts (this is at least legitimate for $\mathrm{Im}(\xi)<0$).
Note that $Q_{\mathcal{C}}(0)<0$ if G is decreasing and that the assumption (4.2.8) can actually be restated as

$$1-\hat{K}_{\mathcal{C}}(0) = 1+Q_{\mathcal{C}}(0) > 0.$$

Now we calculate that

$$\int_0^{+\infty} e^{-it\xi} Q_{\mathcal{C}}(t)\mathrm{d}t = 2 \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a))|C^*_{\ell}(a)|^2 \int_0^{\infty} e^{-it\xi} \cos(t\ell\omega_*(a))\mathrm{d}a.$$

However,

$$\begin{aligned}\int_0^\infty e^{-it\xi}\cos(t\ell\omega_*(a))\mathrm{d}t &= \frac{1}{2}\int_0^\infty e^{-it(\xi-\ell\omega_*(a)}\mathrm{d}t + \frac{1}{2}\int_0^\infty e^{-it(\xi+\ell\omega_*(a))}\mathrm{d}t \\ &= \frac{1}{2i}\left(\frac{1}{\xi-\ell\omega_*(a)}+\frac{1}{\xi+\ell\omega_*(a)}\right).\end{aligned}$$

Hence we can write

$$\begin{aligned}\hat{K}_{\mathcal{C}}(\xi) &= -Q_{\mathcal{C}}(0)+\xi\sum_{*\in\{\pm,\circ\}}\sum_{\ell>0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\frac{1}{2i}\left(\frac{1}{\xi-\ell\omega_*(a)}+\frac{1}{\xi+\ell\omega_*(a)}\right)\mathrm{d}a \\ &= -Q_{\mathcal{C}}(0)+\sum_{*\in\{\pm,\circ\}}\sum_{\ell\neq 0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\frac{\xi}{\xi-\ell\omega_*(a)}\mathrm{d}a \\ &= \sum_{*\in\{\pm,\circ\}}\sum_{\ell\neq 0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\left(\frac{\xi}{\xi-\ell\omega_*(a)}-1\right)\mathrm{d}a \\ &= \sum_{*\in\{\pm,\circ\}}\sum_{\ell\neq 0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\left(\frac{\ell\omega_*(a)}{\xi-\ell\omega_*(a)}\right)\mathrm{d}a.\end{aligned}$$

Step 2. Verification of the Penrose criterion

We shall now check assumption (4.2.5) on a compact set, by showing that the imaginary part of $\hat{K}_{\mathcal{C}}$ does not vanish. We use the notation $\xi=\gamma+i\tau$, with $\tau\leq 0$. Note that from the above computations we have

$$\begin{aligned}\mathrm{Im}\,\hat{K}_{\mathcal{C}}(\xi) &= -\sum_{*\in\{\pm,\circ\}}\sum_{\ell\neq 0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\tau\ell\omega_*(a)\frac{1}{(\gamma-\ell\omega_*(a))^2+\tau^2} \\ &= -\sum_{*\in\{\pm,\circ\}}\sum_{\ell>0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\tau\ell\omega_*(a)\left(\frac{1}{(\gamma-\ell\omega_*(a))^2+\tau^2}-\frac{1}{(\gamma+\ell\omega_*(a))^2+\tau^2}\right) \\ &= -\sum_{*\in\{\pm,\circ\}}\sum_{\ell>0}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\tau\ell\omega_*(a)\left(\frac{(\gamma+\ell\omega_*(a))^2-(\gamma-\ell\omega_*(a))^2}{((\gamma-\ell\omega_*(a))^2+\tau^2)((\gamma+\ell\omega_*(a))^2+\tau^2)}\right) \\ &= -4\gamma\tau\sum_{\substack{*\in\{\pm,\circ\}\\ \ell>0}}\int_{J_*}G'(h_0(a))|C_\ell^*(a)|^2\ell^2\omega_*(a)^2\left(\frac{1}{((\gamma-\ell\omega_*(a))^2+\tau^2)((\gamma+\ell\omega_*(a))^2+\tau^2)}\right).\end{aligned}$$

Hence we see that the sign of $\mathrm{Im}\,\hat{K}_{\mathcal{C}}(\xi)$ is the same as the one of γ. Moreover, as long as $\gamma\neq 0$, $\tau\neq 0$, $(-\tau\gamma)^{-1}\mathrm{Im}\,\hat{K}_{\mathcal{C}}(\xi)$ is a sum of strictly positive terms, as the coefficients $C_\ell^*(a)$ explicitly given in section 4.7 are non-zero everywhere (see Propositions 4.7.4 and 4.7.10). Therefore $\hat{K}_{\mathcal{C}}(\xi)$ does not achieve the value 1 when ξ is neither real nor purely imaginary. It remains to see if this still holds when ξ crosses the axis, and for that we shall calculate the limit $\lim_{\tau\to 0}\hat{K}_{\mathcal{C}}(\gamma+i\tau)$ for $\gamma\neq 0$.

We shall use the fact that $\partial_a\omega_*$ does not vanish on each chart U^* (see section 4.7). As a matter of fact $\partial_a\omega_*$ is strictly negative on $U_\circ$ and strictly positive on U_+ and U_-.Thus we

can make the change of variable $u = -\ell\omega(a)$ to get that

$$\hat{K}_{\mathcal{C}}(\gamma + i\tau) = \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{\mathbb{R}} \phi_{*,\ell}(u) \frac{\gamma + u - i\tau}{(\gamma+u)^2 + \tau^2} \, \mathrm{d}u$$

with

$$\phi_{*,\ell}(u) = -u \frac{G'(h_0(a(u/\ell)))|C^*_\ell(a(u/\ell))|^2}{|\ell\partial_a\omega_*(a(u/\ell))|} \mathbb{1}_{\ell\omega_*(J_*)}(u)$$

with the notation $a(u) = (\omega_*)^{-1}(-u)$. Defining then the Poisson kernel P by

$$P : u \to P(u) = \frac{\tau}{\pi(u^2+\tau^2)}.$$

we have in fact obtained that the imaginary part of $\hat{K}_{\mathcal{C}}(\gamma + i\tau)$ is given by

$$-\pi \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} (P * \phi_{*,\ell})(-\gamma).$$

There we can use the well-known approximation result

$$\lim_{\tau\to 0} (P * \phi_{*,\ell})(-\gamma) = \phi(-\gamma).$$

Consequently, we find that

$$\lim_{\tau\to 0} \mathrm{Im}\, \hat{K}_{\mathcal{C}}(\xi) = -\sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \gamma \frac{G'(h_0(a(-\gamma/\ell)))|C^*_\ell(a(-\gamma/\ell))|^2}{|\ell\partial_a\omega_*(a(-\gamma/\ell))|}.$$

Again since G' does not vanish and since the coefficients $C^*_\ell(a)$ are non-zero everywhere, the above sum can vanish only if $\gamma = 0$, such that $\hat{K}_{\mathcal{C}}(\gamma + i\tau)$ cannot achieve the value 1 when $\gamma \neq 0$ and $\tau = 0$. Consequently, it only remains to check that $\hat{K}_{\mathcal{C}}(\gamma + i\tau)$ does not achieve the value 1 when $\gamma = 0$ and $\tau = 0$, and this is ruled out by the assumptions of the Theorem.

Case of $K_{\mathcal{S}}$

For $K_{\mathcal{S}}$ we start the computations as previously, using the preservation of the Poisson bracket, which shows that

$$K_{\mathcal{S}}(t) = (2\pi)^{-1} \mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\circ,\pm\}} \int_{J_*\times(-\pi,\pi)} \sin(\mathrm{X}\circ\psi_t(\theta,a))\partial_a\eta(a)\partial_\theta \sin(x(\theta,a))\mathrm{d}a\mathrm{d}\theta.$$

By using for $(\theta, a) \in J_* \times (-\pi, \pi)$ the identity $\psi_t(\theta, a) = \theta + t\omega_*(a)$ and the Fourier expansion (4.2.4) for the sine function, we infer that

$$\begin{aligned} K_{\mathcal{S}}(t) &= (2\pi)^{-1} \mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell,\ell'\in\mathbb{Z}} \int_{J_*\times(-\pi,\pi)} \partial_a \eta(a) i\ell' S^*_{\ell'}(a) S^*_{\ell}(a) e^{it\ell\omega_*(a)} e^{i(\ell+\ell')\theta} \mathrm{d}a\mathrm{d}\theta \\ &= \mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{J_*} \partial_a \eta(a)(-i\ell) S^*_{-\ell}(a) S^*_{\ell}(a) e^{it\ell\omega_*(a)} \mathrm{d}a \\ &= -\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{J_*} \partial_a \eta(a) i\ell |S^*_{\ell}(a)|^2 e^{it\ell\omega_*(a)} \mathrm{d}a, \end{aligned}$$

where we have used as previously the identity $S^*_{-\ell}(a) = \overline{S^*_{\ell}(a)}$. We conclude that

$$K_{\mathcal{S}}(t) = 2\mathbb{1}_{\{t\geq 0\}} \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a)) \ell\omega_*(a) |S^*_{\ell}(a)|^2 \sin(t\ell\omega_*(a)) \mathrm{d}a,$$

such that

$$K_{\mathcal{S}}(t) = \mathbb{1}_{\{t\geq 0\}} \frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathcal{S}}(t),$$

with

$$Q_{\mathcal{S}}(t) = -2 \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a)) |S^*_{\ell}(a)|^2 \cos(t\ell\omega_*(a)) \mathrm{d}a.$$

As in the previous case the assumption $1 - K_{\mathcal{S}}(0) > 0$ is the same as $1 + Q_{\mathcal{S}}(0) > 0$. From then the argument is identical as for the case of $K_{\mathcal{C}}$, since the coefficients $S^*_{\ell}(a)$ are non-zero everywhere (see Propositions 4.7.4 and 4.7.10).

■

Proof of Proposition 4.2.8. In the case of $K_{\mathcal{C}}$, we saw in the previous proof that

$$1 - \hat{K}_{\mathcal{C}}(0) > 0 \Leftrightarrow 1 + Q_{\mathcal{C}}(0) > 0,$$

with

$$Q_{\mathcal{C}}(t) = 2 \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a)) |C^*_{\ell}(a)|^2 \cos(t\ell\omega_*(a)) \mathrm{d}a.$$

Now we can use Parseval's identity

$$\sum_{\ell\in\mathbb{Z}} |C^*_{\ell}(a)|^2 = \frac{1}{2\pi} \int_{(-\pi,\pi)} \cos^2(x(\theta, a)) \mathrm{d}\theta$$

to write that

$$\begin{aligned}
Q_{\mathcal{C}}(0) &= \sum_{*\in\{\pm,\circ\}} \sum_{\ell\neq 0} \int_{J_*} G'(h_0(a))|C_\ell^*(a)|^2 \mathrm{d}a \\
&= \frac{1}{2\pi} \sum_{*\in\{\pm,\circ\}} \int_{J^*\times(-\pi,\pi)} G'(h_0(a)) \cos^2(x(\theta,a)) \mathrm{d}\theta \mathrm{d}a - \sum_{*\in\{\pm,\circ\}} \int_{J_*} G'(h_0(a))|C_0^*(a)|^2 \mathrm{d}a \\
&= \int_{\mathbb{T}\times\mathbb{R}} G'(h_0(x,v)) \cos^2(x) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} G'(h_0(a))|C_0^*(a)|^2 \mathrm{d}a \\
&= \int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v \eta(x,v)}{v} \cos^2(x) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} |C_0^*(a)|^2 \mathrm{d}a,
\end{aligned}$$

where we have also used area preservation. Hence we have

$$1 + Q_{\mathcal{C}}(0) > 0 \Leftrightarrow 1 + \int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v \eta(x,v)}{v} \cos^2(x) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} |C_0^*(a)|^2 \mathrm{d}a > 0,$$

which proves the result in the case of $K_{\mathcal{C}}$.
Now in the case of $K_{\mathcal{S}}$, we saw in the previous proof that

$$1 - \hat{K}_{\mathcal{S}}(0) > 0 \Leftrightarrow 1 + Q_{\mathcal{S}}(0) > 0,$$

with

$$Q_{\mathcal{S}}(t) = -2 \sum_{*\in\{\pm,\circ\}} \sum_{\ell>0} \int_{J_*} G'(h_0(a))|S_\ell^*(a)|^2 \cos(t\ell\omega_*(a)) \mathrm{d}a.$$

Using Parseval's formula as previously, and as $S_0^*(a) = 0$ on each chart, we obtain this time that

$$1 + Q_{\mathcal{S}}(0) > 0 \Leftrightarrow 1 - \int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v \eta(x,v)}{v} \sin^2(x) \mathrm{d}x\mathrm{d}v > 0,$$

which is guaranteed by the assumption $G' < 0$. ∎

4.6 Examples of stable stationary states

In this section we study the existence of stationary states of the kind (4.1.5), and exhibit examples of such states that satisfy the stability hypothesis (4.2.7).

4.6.1 Sufficient conditions of existence and stability

The following Proposition provides a sufficient condition on the function G such that an inhomogeneous state of the kind (4.1.5) exists.

Proposition 4.6.1. *Let $G : [-e, +\infty[\to \mathbb{R}_+$ be a C^1 function such that $G, G' \in L^1([-e, +\infty[)$. Assume that there exists $\zeta > 0$ such that*

$$\int_{\mathbb{T}\times\mathbb{R}} G\left(\frac{v^2}{2} - \zeta\cos(x)\right)\cos(x)\mathrm{d}x\mathrm{d}v \geq \zeta, \tag{4.6.1}$$

and that

$$1 + \int_{\mathbb{T}\times\mathbb{R}} G'\left(\frac{v^2}{2}\right)\cos^2(x)\mathrm{d}x\mathrm{d}v > 0. \tag{4.6.2}$$

Then there exists a solution $M_0 > 0$ to the equation

$$M_0 = \int_{\mathbb{T}\times\mathbb{R}} G\left(\frac{v^2}{2} - M_0\cos(x)\right)\cos(x)\mathrm{d}x\mathrm{d}v.$$

In particular, $\eta(x, v) = G\left(\frac{v^2}{2} - M_0\cos(x)\right)$ is an inhomogeneous stationary solution of (4.1.1).

Proof. The proof is very easy: consider the function

$$F(z) = \int_{\mathbb{T}\times\mathbb{R}} G\left(\frac{v^2}{2} - z\cos(x)\right)\cos(x)\mathrm{d}x\mathrm{d}v - z.$$

We have $F(0) = 0$ (as the cosine function has average 0 on $(-\pi, \pi)$), and the hypothesis imply that $F(\zeta) \geq 0$ and $F'(0) < 0$. Hence either $F(\zeta) = 0$ and the proof is done, or $F(\zeta) > 0$, and the intermediary value Theorem shows that there exists $M_0 \in (0, \zeta)$ such that $F(M_0) = 0$. □

The next Proposition gives a sufficient condition to fulfill the stability assumption of definition 4.2.7, which is moreover independent of the angle-action variables.

Proposition 4.6.2. *Let $G : [-e, +\infty[\to \mathbb{R}_+$ be a C^1 function such that $G, G' \in L^1([-e, +\infty[)$, and η be defined by (4.1.5) with $M_0 > 0$. Assume that $G' < 0$, and that η satisfies*

$$1 + \int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v\eta(x,v)}{v}\cos^2(x)\mathrm{d}x\mathrm{d}v - \frac{\left(\int_{\mathbb{T}\times\mathbb{R}} \cos(x)\frac{\partial_v\eta(x,v)}{v}\mathrm{d}x\mathrm{d}v\right)^2}{\int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v\eta(x,v)}{v}\mathrm{d}x\mathrm{d}v} > 0.$$

Then η is stable in the sense of definition 4.2.7.

Proof. By area preservation, one has that

$$\frac{\left(\int_{\mathbb{T}\times\mathbb{R}} \cos(x)\frac{\partial_v\eta(x,v)}{v}\mathrm{d}x\mathrm{d}v\right)^2}{\int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v\eta(x,v)}{v}\mathrm{d}x\mathrm{d}v} = \frac{\left(\sum_{*\in\{\circ,\pm\}}\int_{J_*} C_0^*(a)\frac{\partial_a\eta(a)}{\omega_*(a)}\mathrm{d}a\right)^2}{\sum_{*\in\{\circ,\pm\}}\int_{J_*}\frac{\partial_a\eta(a)}{\omega_*(a)}\mathrm{d}a},$$

where we should point that the denominators do not vanish since $G' < 0$. Hence it suffices to check that

$$\sum_{*\in\{\circ,\pm\}} \int_{J_*} C_0^*(a)^2 \frac{\partial_a \eta(a)}{\omega_*(a)} \mathrm{d}a \leq \left(\sum_{*\in\{\circ,\pm\}} \int_{J_*} C_0^*(a) \frac{\partial_a \eta(a)}{\omega_*(a)} \mathrm{d}a \right)^2 \left(\sum_{*\in\{\circ,\pm\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} \mathrm{d}a \right)^{-1}. \tag{4.6.3}$$

Now for $* \in \{\circ, \pm\}$, we define on J_* a function

$$F(a) = \left(\sum_{*\in\{\circ,\pm\}} \int_{J_*} \frac{\partial_a \eta(\alpha)}{\omega_*(\alpha)} \mathrm{d}\alpha \right)^{-1} \frac{\partial_a \eta(a)}{\omega_*(a)},$$

which is positive, since $G' < 0$. The Cauchy-Schwarz inequality implies then that for all $* \in \{\circ, \pm\}$

$$\sum_{*\in\{\circ,\pm\}} \int_{J_*} C_0^*(a)^2 F(a) \mathrm{d}a \geq \left(\sum_{*\in\{\circ,\pm\}} \int_{J_*} C_0^*(a) F(a) \mathrm{d}a \right)^2,$$

where we have also used the preservation of the area. Multiplying both sides of the inequality by the real number

$$\sum_{*\in\{\circ,\pm\}} \int_{J_*} \frac{\partial_a \eta(a)}{\omega_*(a)} \mathrm{d}a,$$

which is negative (since $G' < 0$), we obtain that (4.6.3) is true, and the proof is done. □

4.6.2 Example of stable stationary states: Maxwell-Boltzmann distributions

Here we study the case where the function G is an exponential. As we consider averages of G against cosine functions, we introduce the modified Bessel functions of the first kind:

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z\cos(x)} \cos(nx) \mathrm{d}x = \int_{\mathbb{T}} e^{z\cos(x)} \cos(nx) \mathrm{d}x.$$

We shall use the following assymptotics (see formulae 9.6.10 and 9.7.1 of [1]):

$$I_n(z) = \left(\frac{z}{2}\right)^n \left[\frac{1}{n!} + \frac{z^2}{4(n+1)!} + \mathcal{O}(z^4) \right] \quad \text{when} \quad z \to 0, \tag{4.6.4}$$

$$I_n(z) = \left(\frac{e^z}{\sqrt{2\pi z}} \right) \left[1 - \frac{4n^2 - 1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right] \quad \text{when} \quad z \to +\infty. \tag{4.6.5}$$

We shall also use the following result

Proposition 4.6.3 ([50]). *For all $n \in \mathbb{N}$ and $z \in \mathbb{R}$, we have*

$$z\frac{I_n'(z)}{I_n(z)} < \sqrt{z^2+n^2} \quad \textit{and} \quad \frac{I_{n+1}(z)}{I_n(z)} > \frac{\sqrt{(n+1)^2+z^2}-(n+1)}{z}.$$

We shall prove the following easy result, which shows there exists inhomogeneous Maxwell-Boltzmann distributions

$$\eta(x,v) = \alpha e^{-\beta h_0(x,v)}$$

that are stationary solutions of (4.1.1) of the kind (4.1.5) (see also [29]).

Proposition 4.6.4. *Let $\alpha, \beta \in \mathbb{R}_+^*$, and $G(s) = \alpha e^{-\beta s}$. Then if $\alpha\sqrt{\beta} < \frac{2}{\sqrt{2\pi}}$, G satisfies (4.6.1) and (4.6.2).*

Proof. We have for any $z > 0$

$$\int_{\mathbb{T}\times\mathbb{R}} G\left(\frac{v^2}{2} - z\cos(x)\right)\cos(x)\mathrm{d}x\mathrm{d}v = \alpha\int_{\mathbb{T}\times\mathbb{R}} e^{-\beta\frac{v^2}{2}}e^{\beta z\cos(x)}\cos(x)\mathrm{d}x\mathrm{d} = \alpha\sqrt{\frac{2\pi}{\beta}}I_1(\beta z),$$

and (4.6.1) is clearly guaranteed by (4.6.5) for z sufficiently large.
Using the first formula of (4.7.9)

$$1 + \int_{\mathbb{T}\times\mathbb{R}} G'\left(\frac{v^2}{2}\right)\cos^2(x)\mathrm{d}x\mathrm{d}v = 1 - \alpha\sqrt{\beta}\sqrt{2\pi}\left[\frac{I_0(0)+I_2(0)}{2}\right] = 1 - \alpha\sqrt{\beta}\frac{\sqrt{2\pi}}{2}.$$

That last quantity is positive when $\alpha\sqrt{\beta} < \frac{2}{\sqrt{2\pi}}$, and this concludes the proof. □

Now we shall prove that the inhomogeneous states given by Proposition 4.6.4 are stable in the sense of definition 4.2.7

Proposition 4.6.5. *Let η be a stationary solution of (4.1.1) given by*

$$\eta(x,v) = \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)},$$

with $\alpha, \beta, M_0 \in \mathbb{R}_+^*$, $\alpha\sqrt{\beta} < \frac{2}{\sqrt{2\pi}}$, and M_0 given by Proposition 4.6.1, and satisfying

$$M_0 = \alpha\int_{\mathbb{T}\times\mathbb{R}} e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)}\cos(x)\mathrm{d}x\mathrm{d}v.$$

Then η is stable in the sense of definition 4.2.7.

Proof. We shall prove that the assumptions of Proposition 4.6.2 are fulfilled, which will imply the result. First, we have

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}} \frac{\partial_v\eta(x,v)}{v}\cos^2(x)\mathrm{d}x\mathrm{d}v &= -\alpha\beta\int_{\mathbb{T}\times\mathbb{R}} e^{-\beta\frac{v^2}{2}}e^{\beta M_0\cos(x)}\cos^2(x)\mathrm{d}x\mathrm{d}v \\ &= -\alpha\sqrt{\beta}\frac{(2\pi)^{1/2}}{2}\left[I_0(\beta M_0) + I_2(\beta M_0)\right],\end{aligned}$$

using the first formula of (4.7.9). Then have also

$$\int_{\mathbb{T}\times\mathbb{R}} \cos(x)\frac{\partial_v \eta(x,v)}{v}\mathrm{d}v\mathrm{d}x = -\alpha\beta \int_{\mathbb{T}\times\mathbb{R}} \cos(x) e^{-\frac{v^2}{2}} e^{\beta M_0 \cos(x)}\mathrm{d}x\mathrm{d}v = -\alpha\sqrt{\beta}(2\pi)^{1/2} I_1(\beta M_0)$$

and

$$\int_{\mathbb{T}\times\mathbb{R}} \frac{\partial_v \eta(x,v)}{v}\mathrm{d}v\mathrm{d}x = -\alpha\sqrt{\beta}(2\pi)^{1/2} I_0(\beta M_0).$$

Hence, by Proposition 4.6.2, it is sufficient to verify that

$$1 - \frac{\alpha\sqrt{\beta}}{2}(2\pi)^{1/2}\left[I_0(\beta M_0) + I_2(\beta M_0)\right] + \alpha\sqrt{\beta}(2\pi)^{1/2}\frac{I_1(\beta M_0)^2}{I_0(\beta M_0)} > 0.$$

Note that

$$\frac{\left[I_0(\beta M_0) + I_2(\beta M_0)\right]}{2} = I_1'(\beta M_0).$$

Moreover, we know that

$$M_0 = \alpha \int_{\mathbb{T}\times\mathbb{R}} e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)}\cos(x)\mathrm{d}x\mathrm{d}v,$$

which is equivalent to

$$M_0 = \alpha\sqrt{\frac{2\pi}{\beta}} I_1(\beta M_0).$$

In other words,

$$\alpha\sqrt{\beta} = \frac{\beta M_0}{(2\pi)^{1/2} I_1(\beta M_0)}.$$

Hence it is sufficient to show that

$$1 - \frac{\beta M_0 I_1'(\beta M_0)}{I_1(\beta M_0)} + \frac{\beta M_0 I_1(\beta M_0)}{I_0(\beta M_0)} > 0.$$

But Proposition 4.6.3 implies that

$$\frac{\beta M_0 I_1(\beta M_0)}{I_0(\beta M_0)} > \sqrt{1 + (\beta M_0)^2} - 1$$

and

$$-\frac{\beta M_0 I_1'(\beta M_0)}{I_1(\beta M_0)} > -\sqrt{1 + (\beta M_0)^2}.$$

Hence

$$1 - \frac{\beta M_0 I_1'(\beta M_0)}{I_1(\beta M_0)} + \frac{\beta M_0 I_1(\beta M_0)}{I_0(\beta M_0)} > 0,$$

and the proof is done. □

4.7 Action-angle variables

In this section we shall recall how angle-action variables are constructed on each chart U_*. It will involve elliptic integrals and Jacobi's elliptic functions, whose definitions and main properties are summarized in the following subsection.

4.7.1 Elliptic integrals, elliptic functions, and elliptic trigonometry

For $k \in (0,1)$ and $\phi \in (-\pi/2, \pi/2)$, we define the incomplete elliptic integrals by

$$E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin(y)}\mathrm{d}y \quad \text{and} \quad F(\phi,k) = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin(y)}}\mathrm{d}y$$

and the complete elliptic integrals by

$$\boldsymbol{E}(k) = E\left(\frac{\pi}{2},k\right) \quad \text{and} \quad \boldsymbol{K}(k) = F\left(\frac{\pi}{2},k\right).$$

We will use the following standard notations: The complementarity modulus $k' = \sqrt{1-k^2}$, $\boldsymbol{K}'(k) = \boldsymbol{K}(k')$ and Jacobi's nome

$$q(k) = \exp(-\pi \boldsymbol{K}'(k)/\boldsymbol{K}(k)).$$

We collect below some useful results for these functions.

Proposition 4.7.1. *The functions $\boldsymbol{E}(z)$, $\boldsymbol{K}(z)$ and $q(z)$ extend as analytic functions of z^2 for $|z|<1$, satisfying $\boldsymbol{E}(0) = \boldsymbol{K}(0) = \frac{\pi}{2}$ and $q(0) = 0$, and we have*

$$\begin{aligned} \boldsymbol{E}(z) &\sim \frac{\pi}{2}\left(1+\frac{1}{4}z^2\right) \quad \textit{when} \quad z \to 0, \\ \boldsymbol{K}(z) &\sim \frac{\pi}{2}\left(1-\frac{1}{4}z^2\right) \quad \textit{when} \quad z \to 0, \\ q(z) &\sim \frac{z^2}{16} \quad \textit{when} \quad z \to 0. \end{aligned} \tag{4.7.1}$$

Moreover, these functions have logarithmic singularities in $z=1$:

$$\begin{aligned} \boldsymbol{E}(z) &\sim 1 - \frac{1}{2}(1-z)\log(1-z) \quad \textit{when} \quad z \to 1, \\ \boldsymbol{K}(z) &\sim -\frac{1}{2}\log(1-z) \quad \textit{when} \quad z \to 1, \\ q(z) &\sim 1 + \frac{\pi^2}{\log(1-z)} \quad \textit{when} \quad z \to 1. \end{aligned} \tag{4.7.2}$$

More precisely, for all $n \geq 1$ there exists constants C_n such that

$$\begin{aligned}
&\|(1-z)^n \partial_z^{n+1} \boldsymbol{E}(z)\|_{L^\infty(\frac{1}{2},1)} \leq C_n, \\
&\|(1-z)^n \partial_z^{n} \boldsymbol{K}(z)\|_{L^\infty(\frac{1}{2},1)} \leq C_n, \\
&\left\|\log(1-z)^2 (1-z)^n \partial_z^n \Big(\frac{1}{\boldsymbol{K}(z)}\Big)\right\|_{L^\infty(\frac{1}{2},1)} \leq C_n, \\
&\left\|\log(1-z)^2 (1-z)^n \partial_z^n q(z)\right\|_{L^\infty(\frac{1}{2},1)} \leq C_n, \\
&\left\|(1-z)^n \partial_z^n \Big(\frac{1}{1-q(z)}\Big)\right\|_{L^\infty(\frac{1}{2},1)} \leq C_n.
\end{aligned} \tag{4.7.3}$$

Proof. The statements of (4.7.1), (4.7.2) and (4.7.3) concerning $\boldsymbol{E}(z)$ and $\boldsymbol{K}(z)$ are consequences of the power series expansions (900.00) and (900.05) of [19] for the function $\boldsymbol{K}(z)$, and (900.07) and (900.10) for the function $\boldsymbol{E}(z)$. In particular, near $z = 1$, we have

$$\boldsymbol{K}(z) = \log(4/z')K_1(z') + K_2(z')$$

where K_1 and K_2 are smooth functions of $(z')^2 = 1 - z^2$ and $K_1(0) = 1$. In other words, we have for $z \in (1/2, 1)$,

$$\boldsymbol{K}(z) = \log(1-z)A(z) + B(z) > 0$$

with A and B smooth functions of z^2 and $A(1) = -\frac{1}{2}$. The estimates on $1/\boldsymbol{K}(z)$ follow from this formula.
The first statement (4.7.1) concerning the function $q(z)$ is a consequence of formula (900.05) of [19]. The second (4.7.2) of the expansion

$$q(z) = \exp(-\pi \boldsymbol{K}(\sqrt{1-z^2})/\boldsymbol{K}(z)) = \sum_{n\geq 0} \frac{(-1)^n}{n!} \left(\frac{\pi \boldsymbol{K}(\sqrt{1-z^2})}{\boldsymbol{K}(z)}\right)^n,$$

that holds near $z = 1$. Note that as $\boldsymbol{K}(z)$ is an analytic function z^2, $\boldsymbol{K}(\sqrt{1-z^2})$ is an analytic function of z for $|z| < 1$ which is bounded as well as its derivatives in the vicinity of $z = 1$. This completes the proof of (4.7.3). □

The Jacobi elliptic functions are then defined as follows: first, we define the amplitude $\mathrm{am}(u, k)$ by the formula

$$F(\mathrm{am}(u,k), k) = u.$$

The first Jacobi elliptic function is then

$$\mathrm{sn}(u,k) = \sin(\mathrm{am}(u,k)). \tag{4.7.4}$$

The second and third Jacobi elliptic functions are defined by the formulae

$$\mathrm{cn}(u,k) = \sqrt{1-\mathrm{sn}^2(u,k)} \quad \text{and} \quad \mathrm{dn}(u,k) = \sqrt{1-k^2\mathrm{sn}^2(u,k)}.$$

We have the following Fourier series for these functions (see formulae (908.00)–(908.03) of [19]):

$$\begin{aligned}
\mathrm{am}(u,k) &= \frac{\pi u}{2\boldsymbol{K}(k)} + 2\sum_{m=0}^{\infty} \frac{q(k)^{m+1}}{(m+1)(1+q(k)^{2(m+1)})} \sin\left((m+1)\frac{\pi u}{\boldsymbol{K}(k)}\right),\\
\mathrm{sn}(u,k) &= \frac{2\pi}{k\boldsymbol{K}(k)} \sum_{m=1}^{\infty} \frac{q(k)^{m-\frac{1}{2}}}{1-q(k)^{2m-1}} \sin\left((2m-1)\frac{\pi u}{2\boldsymbol{K}(k)}\right),\\
\mathrm{cn}(u,k) &= \frac{2\pi}{k\boldsymbol{K}(k)} \sum_{m=1}^{\infty} \frac{q(k)^{m-\frac{1}{2}}}{1+q(k)^{2m-1}} \cos\left((2m-1)\frac{\pi u}{2\boldsymbol{K}(k)}\right),\\
\mathrm{dn}(u,k) &= \frac{\pi}{2\boldsymbol{K}(k)} + \frac{2\pi}{\boldsymbol{K}(k)} \sum_{m=1}^{\infty} \frac{q(k)^{m}}{1+q(k)^{2m}} \cos\left(m\frac{\pi u}{\boldsymbol{K}(k)}\right).
\end{aligned} \tag{4.7.5}$$

The following formulae will also be useful (see (2.14), (2.24) in [60]):

$$\begin{aligned}
\mathrm{sn}^2(u,k) &= \frac{\boldsymbol{K}(k)-\boldsymbol{E}(k)}{k^2\boldsymbol{K}(k)} - \frac{2\pi^2}{k^2\boldsymbol{K}(k)^2} \sum_{m=1}^{\infty} \frac{mq(k)^m}{1-q(k)^{2m}} \cos\left(m\frac{\pi u}{\boldsymbol{K}(k)}\right)\\
\mathrm{sn}(u,k)\mathrm{cn}(u,k) &= \frac{2\pi^2}{k^2\boldsymbol{K}(k)^2} \sum_{m=1}^{\infty} \frac{mq(k)^m}{1+q(k)^{2m}} \sin\left(m\frac{\pi u}{\boldsymbol{K}(k)}\right)\\
\mathrm{sn}(u,k)\mathrm{dn}(u,k) &= \frac{\pi^2}{k\boldsymbol{K}(k)^2} \sum_{m=1}^{\infty} \frac{(2m-1)q(k)^{m-\frac{1}{2}}}{1+q(k)^{2m-1}} \sin\left((2m-1)\frac{\pi u}{2\boldsymbol{K}(k)}\right).
\end{aligned} \tag{4.7.6}$$

We shall also need the following elliptic trigonometry identities (see formulae 120.02, 122.00, 122.03 in [19])

$$\begin{aligned}
&\mathrm{sn}(-u,k) = -\mathrm{sn}(u,k), \quad \mathrm{cn}(-u,k) = \mathrm{cn}(u,k), \quad \mathrm{dn}(-u,k) = \mathrm{dn}(u,k),\\
&\mathrm{sn}(u+\boldsymbol{K}(k),k) = \mathrm{sn}(\boldsymbol{K}(k)-u,k), \quad \mathrm{cn}(u+\boldsymbol{K}(k),k) = -k'\frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)},
\end{aligned} \tag{4.7.7}$$

of which two straightforward consequences are the following equalities

$$-\mathrm{sn}(u-\boldsymbol{K}(k),k) = \mathrm{sn}(u+\boldsymbol{K}(k),k) \quad \text{and} \quad -\mathrm{cn}(u-\boldsymbol{K}(k),k) = \mathrm{cn}(u+\boldsymbol{K}(k),k). \tag{4.7.8}$$

Finally, we recall for completion some classical trigonometry identities which we often use: for a real number z,

$$\begin{aligned}
&2\cos^2(z) = 1+\cos(2z), \quad 2\sin^2(z) = 1-\cos(2z), \quad \sin(2z) = 2\sin(z)\cos(z),\\
&\arcsin(\cos(z)) = \sqrt{1-z^2}.
\end{aligned} \tag{4.7.9}$$

4.7.2 Action-angle variables on U_+ or U_-.

We will use the following notations: $\epsilon_+ = 1$, and $\epsilon_- = -1$. The action-angle coordinates are constructed on $U_\pm$ as follows.

Proposition 4.7.2. *For $* \in \{\pm\}$, there exists a symplectic change of variable $(x, v) \mapsto (\psi, h)$ from U_* to the set*

$$V_* := \{(\psi, h) \in \mathbb{R}^2, |h \in (M_0, +\infty),\, \psi \in (-r_*(h), r_*(h))\},$$

with

$$r_*(h) = \frac{1}{k(h)\sqrt{M_0}} \boldsymbol{K}\left(\frac{1}{k(h)}\right), \quad \textit{where} \quad k(h) = \sqrt{\frac{h + M_0}{2M_0}},$$

such that the flow of the pendulum in the variables (ψ, h) is $h(t) = h(0)$ and $\psi(t) = t + \psi(0)$. There exists then a second symplectic change of variables $(\psi, h) \mapsto (\theta, a)$ from V_* to

$$W_* = \left\{(\theta, a) \in \mathbb{R}^2, |a \in J_* = \left(\frac{4}{\pi}\sqrt{M_0}, +\infty\right), \theta \in (-\pi, \pi)\right\},$$

such that

$$\begin{cases} a(h) &= \dfrac{4}{\pi} k(h)\sqrt{M_0}\boldsymbol{E}\left(\dfrac{1}{k(h)}\right) \\ \theta(\psi, h) &= \omega_*(h)\psi \end{cases} \qquad \textit{with} \quad \omega_*(h) = \frac{\pi k(h)\sqrt{M_0}}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)},$$

and so that the flow of the pendulum in the variables (θ, a) is $a(t) = a(0)$ and $\theta(t) = t\omega_*(a) + \theta(0)$.
Moreover, we can easily express x and v as functions of the variables (θ, h) with the formulae

$$x(\theta, h) = \epsilon_* 2 \operatorname{am}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right), \tag{4.7.10}$$

$$v(\theta, h) = \epsilon_* 2k(h)\sqrt{M_0}\operatorname{dn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right). \tag{4.7.11}$$

Proof. Setting $h(x, v) = \frac{v^2}{2} - M_0\cos(x)$, we have on U_*

$$v(x, h) = \epsilon_*\sqrt{2(h + M_0\cos(x))}.$$

Note that $v(x, h) = \partial_x S(x, h)$, where

$$\begin{aligned} S(x, h) &= \epsilon_* \int_0^x \sqrt{2(h + M_0\cos(y))}\mathrm{d}y \\ &= \epsilon_* 2\sqrt{2(h + M_0)}\int_0^{x/2}\sqrt{1 - \frac{2M_0}{h + M_0}\sin^2(y)}\mathrm{d}y \\ &= \epsilon_* 4\sqrt{M_0}k(h)\int_0^{x/2}\sqrt{1 - \frac{\sin^2(y)}{k(h)^2}}\mathrm{d}y \\ &= \epsilon_* 4k(h)\sqrt{M_0}E\left(\frac{x}{2}, \frac{1}{k(h)}\right). \end{aligned}$$

We define then the variable $\psi(x,h)$ by

$$\psi(x,h) = \partial_h S(x,h) = \epsilon_* \int_0^x \frac{1}{\sqrt{2(h+M_0\cos(y))}} \mathrm{d}y = \epsilon_* \frac{1}{k(h)\sqrt{M_0}} F\left(\frac{x}{2}, \frac{1}{k(h)}\right).$$

By construction, the variables (ψ, h) are symplectic, and S is the mixed-variable generatrix function (see formula (5.5) p197 of [45]). We have by the above formulae

$$\dot{\psi}(t) = \dot{x}(t)\partial_x \psi(x(t), h(x(t), v(t))) = v(t) \frac{\epsilon_*}{\sqrt{2(h(x(t),v(t)) + M_0\cos(x(t)))}} = 1,$$

and the preservation of the hamiltonian reads $\dot{h} = 0$, such that the flow associated with h_0 is in these new variables $\dot{\psi}(t) = 1, \dot{h}(t) = 0$.
Setting now

$$r_*(h) = \int_0^\pi \frac{1}{\sqrt{2(h+M_0\cos(y))}} \mathrm{d}y = \frac{1}{k(h)\sqrt{M_0}} \boldsymbol{K}\left(\frac{1}{k(h)}\right),$$

we have $\psi \in [-r_*(h), r_*(h)]$, and the orbits of the flow are periodic with period $2r_*(h)$, which is unsatisfying for us. Thus the second step is to perform another transformation which shall force all trajectories to evolve in a torus. If we define

$$g_*(h) = \frac{1}{\pi} r_*(h) = \frac{1}{\pi} \int_0^\pi \frac{1}{\sqrt{2(h+M_0\cos(y))}} \mathrm{d}y = \frac{\partial}{\partial h} a_*(h) > 0,$$

with

$$a_*(h) = \frac{1}{\pi} \int_0^\pi \sqrt{2(h+M_0\cos(y))} \mathrm{d}y = \frac{4}{\pi} k(h)\sqrt{M_0} \boldsymbol{E}\left(\frac{1}{k(h)}\right),$$

and set $\theta(x,h) = \frac{1}{g(h)}\psi(x,h)$, then for each $* \in \{\pm\}$, the variables (θ, a) belong to $J_* \times (-\pi, \pi)$ and are symplectic. Moreover the flow reduces to

$$\dot{\theta}(t) = \frac{\dot{\psi}(t)}{g_*(h)} = \omega_*(a) \quad \text{and} \quad \dot{a}(t) = 0,$$

where $\omega_*(a_*(h)) = 1/g_*(h) = 1/\partial_h a_*(h)$. Note that we have

$$g_*(h) = \frac{1}{\pi k(h)\sqrt{M_0}} \boldsymbol{K}\left(\frac{1}{k(h)}\right),$$

which gives the formula for $\omega_*(h)$.
We can express x in terms of θ and h using the formula

$$\theta = \epsilon_* \frac{\pi}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)} F\left(\frac{x}{2}, \frac{1}{k(h)}\right).$$

Using the definition

$$\mathrm{sn}(u,k) = \sin(\mathrm{am}(u,k)) \quad \text{for} \quad F(\mathrm{am}(u,k),k) = u,$$

we obtain

$$\sin(x(\theta,h)/2) \;=\; \epsilon_* \mathrm{sn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right)$$

and hence

$$x(\theta,h) = \epsilon_* 2\,\mathrm{am}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right).$$

Moreover, we have

$$\begin{aligned} v(\theta,h) &= \epsilon_*\sqrt{2(h+M_0\cos(x))} \\ &= \epsilon_*\sqrt{2(h+M_0)}\sqrt{1-\frac{1}{k(h)^2}\sin^2(x(\theta,h)/2)} \\ &= \epsilon_* 2k(h)\sqrt{M_0}\,\mathrm{dn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right), \end{aligned}$$

and this ends the proof. □

We consider now the asymptotics of these functions. Note that the variables (θ,h) are not symplectic, but we will use them to examine these asymptotics. Note moreover that the change $a_*(h)$ defined above allows to compute easily integrals in (θ,h) by using $\mathrm{d}a = g_*(h)\mathrm{d}h = \frac{1}{\omega_*(h)}\mathrm{d}h$.

Proposition 4.7.3. *For $* \in \{\pm\}$, the functions $\omega_*(h)$, $x(\theta,h)$ and $v(\theta,h)$ are analytic for $\theta \in (-\pi,\pi)$ and $h \in (M_0,+\infty)$.*
The function ω_* exhibits the following asymptotic behavior

$$\begin{aligned} &\omega_*(h) \sim \sqrt{2h} \quad \textit{when} \quad h \to +\infty, \\ &\omega_*(h) \sim \frac{2\pi\sqrt{M_0}}{\log\left(\frac{1}{(h-M_0)}\right)} \quad \textit{when} \quad h \to M_0^+, \end{aligned} \tag{4.7.12}$$

and there exists constants, C_r, $\omega_r \neq 0$ and $\alpha_r \neq 0$ such that for all $r \geq 1$,

$$\begin{aligned} &\left\| \sqrt{h}\left[h^{-\frac{1}{2}+r}\partial_h^r\omega_*(h) - \omega_r\right]\right\|_{L^\infty(2M_0,+\infty)} \leq C_r, \\ &\left\|\log(h-M_0)^2\left[\log(h-M_0)^2(h-M_0)^r\partial_h^r\omega_*(h) - \alpha_r\right]\right\|_{L^\infty(M_0,2M_0)} \leq C_r. \end{aligned} \tag{4.7.13}$$

The change of variable $(h,\theta) \mapsto (x,v)$ satisfies the following estimates: for large h it converges towards the "identity" in the sense that

$$\begin{aligned} &x(\theta,h) \sim \epsilon_*\theta \quad \textit{when} \quad h \to +\infty, \\ &v(\theta,h) \sim \epsilon_*\sqrt{2h} \quad \textit{when} \quad h \to +\infty. \end{aligned} \tag{4.7.14}$$

More precisely, for $r, s \geq 0$ there exist constants $C_{r,s}$ such that

$$\begin{aligned} &\left\| h^{r+1} \partial_h^r \partial_\theta^s (x(\theta,h) - \epsilon_* \theta) \right\|_{L^\infty((-\pi,\pi)\times(2M_0,+\infty))} \leq C_{r,s}, \\ &\left\| \sqrt{h} \left[h^{-\frac{1}{2}+r} \partial_h^r v(\theta,h) - \epsilon_* \omega_r \right] \right\|_{L^\infty((-\pi,\pi)\times(2M_0,+\infty))} \leq C_r, \quad \textit{and} \\ \textit{for} \quad s \geq 1, \quad &\left\| \sqrt{h} \left[h^{\frac{1}{2}+r} \partial_h^r \partial_\theta^s v(\theta,h) \right] \right\|_{L^\infty((-\pi,\pi)\times(2M_0,+\infty))} \leq C_{r,s}. \end{aligned} \tag{4.7.15}$$

Finally, we have for $r, s \geq 1$,

$$\begin{aligned} &\| |(h-M_0)^r| \log(h-M_0)|^{-s+2} \partial_h^r \partial_\theta^s (x(\theta,h) - \epsilon_* \theta) \|_{L^\infty((-\pi,\pi)\times(M_0,2M_0))} \leq C_{r,s} \quad \textit{and} \\ &\| |(h-M_0)^r| \log(h-M_0)|^{-s+3} \partial_h^r \partial_\theta^s (v(\theta,h)) \|_{L^\infty((-\pi,\pi)\times(M_0,2M_0))} \leq C_{r,s} \end{aligned} \tag{4.7.16}$$

for some constants $C_{r,s}$, and

$$\| |\log(h-M_0)|^{-s} \partial_\theta^s (x(\theta,h) - \epsilon_* \theta) \|_{L^\infty((-\pi,\pi)\times(M_0,2M_0))} \leq C_s$$

for $s \geq 1$ and some constant C_s.

Proof. We begin with the study of the function ω_*, and prove (4.7.12) and (4.7.13).

When $h \to +\infty$, $k(h)$ goes to $+\infty$, and $1/k(h)$ goes to 0. Hence as $\boldsymbol{K}(z)$ extends near $z \sim 0$ as a smooth function in z^2 with $\boldsymbol{K}(0) = \frac{\pi}{2}$, we have that

$$\begin{aligned} \omega_*(h) &= \frac{\pi k(h) \sqrt{M_0}}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)} \\ &= \sqrt{2(h+M_0)} \left(1 + \Omega\left(\frac{1}{h+M_0} \right) \right), \end{aligned}$$

where Ω is analytic in a neighborhood of 0. This shows that on $(2M_0, +\infty)$, we have $\omega_*(h) = \sqrt{2h}(1 + \tilde{\Omega}(1/h))$ for some analytic function $\tilde{\Omega}$ on $(0, \frac{1}{2M_0})$. This gives the first asymptotic of (4.7.12), and also the first estimate of (4.7.13).

When $h \to M_0^+$, $1/k(h)$ goes to 1 and is smooth in a neighborhood of M_0. Moreover $1 - \frac{1}{k(h)} \sim \frac{1}{2M_0}(h - M_0)$ when $h \to M_0^+$. Asymptotics (4.7.2) show that

$$\omega_*(h) = \frac{\pi k(h) \sqrt{M_0}}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)} \sim \pi \sqrt{M_0} \frac{2}{(-\log(1 - \frac{1}{k(h)}))} \quad \text{when} \quad h \to M_0^+,$$

from which we infer the asymptotics of ω_*. The second estimate of (4.7.13) is easily deduced using the estimate on $1/\boldsymbol{K}$ from (4.7.3), and also the fact that $1/k(h)$ is smooth on $(M_0, 2M_0)$.

Let us now study the functions $x(\theta,h)$ and $v(\theta,h)$. Using the expansions of (4.7.5), and the expressions of $x(\theta,h)$ and $v(\theta,h)$ from Proposition 4.7.2, we write

$$x(\theta,h) = \epsilon_* \theta + \epsilon_* 4 \sum_{m=0}^{\infty} \frac{q(1/k(h))^{m+1}}{(m+1)(1 + q(1/k(h))^{2(m+1)})} \sin((m+1)\theta)$$

and

$$\begin{aligned} v(\theta,h) &= \epsilon_*\omega_*(h)\left(1+4\sum_{m=0}^{\infty}\frac{q(1/k(h))^{m+1}}{1+q(1/k(h))^{2(m+1)}}\cos((m+1)\theta)\right) \\ &= \omega_*(h)\partial_\theta x(\theta,h). \end{aligned}$$

Note that by construction, $x(\theta,h)$ is bounded for $(\theta,h)\in(-\pi,\pi)\times(M_0,+\infty)$. It is then clear that we have to consider and study the auxiliary function

$$R(\theta,q)=4\sum_{m=0}^{\infty}\frac{q^{m+1}}{1+q^{2(m+1)}}e^{i(m+1)\theta}.$$

This function is well defined for $|q|<1$ and when $q\to 0$, we have

$$R(\theta,q)\sim 4qe^{i\theta}. \tag{4.7.17}$$

Moreover, we have the estimate

$$\|\partial_q^r\partial_\theta^s R(\theta,q)\|_{L^\infty(0,\frac{1}{2})}\leq C_{r,s} \tag{4.7.18}$$

for all $r,s\geq 0$ and some constant $C_{r,s}$.To prove this, note that as $(1+x^2)^{-1}$ and its derivatives are bounded near $x=0$, we can write that

$$\partial_q^r\partial_\theta^s R(q,\theta)=\sum_{m\geq r-1}^{\infty}(m+1)^{r+s}q^{m+1-r}e^{i(m+1)\theta}R_m^{r,s}(q)$$

where $R_m^{r,s}(q)\leq C_{r,s}$ for all m and $q\in(0,\frac{1}{2})$. Estimate (4.7.18) follows easily. In addition, when $q\to 1$, we have

$$\|R(\theta,q)\|_{L^\infty(\frac{1}{2},1)}\leq 4\sum_{m=0}^{\infty}q^{m+1}\leq 4\left(\frac{q}{1-q}\right). \tag{4.7.19}$$

We also want to estimate the derivatives of this function. We can proceed as previously, and use the fact that $x\mapsto(1+x^2)^{-1}$ is bounded as well as all its derivatives near $x=1$, so that we can write for $r\geq 1$ and $s\geq 0$;

$$\partial_q^r\partial_\theta^s R(\theta,q)=\sum_{m\geq r-1}^{\infty}(m+1)^{r+s}q^{m+1-r}e^{i(m+1)\theta}R_m^{r,s}(q)$$

where $R_m^{r,s}(q)\leq C_{r,s}$ for all m and $q\in(\frac{1}{2},1)$. We deduce that

$$\|\partial_q^r\partial_\theta^s R(\theta,q)\|_{L^\infty(\frac{1}{2},1)}\leq\frac{C_{r,s}}{(1-q)^{r+s+1}}, \tag{4.7.20}$$

for some constant $C_{r,s}$.

We can now study the asymptotic behaviors of $x(\theta,h)$ and $v(\theta,h)$, starting with the case when $h \to +\infty$. In this case $1/k(h)$ goes to 0, and $q(1/k(h))$ is a smooth function of $1/k(h)^2 = 2M_0/(h+M_0) \to 0$. As we have

$$x(\theta,a) = \varepsilon_* \left[\theta + \mathrm{Im}\left(R\left(\theta, q\left(\frac{1}{k(h)}\right)\right)\right)\right],$$

we deduce from then (4.7.17), (4.7.18) and the Faà di Bruno formula that

$$\partial_h^r \partial_\theta^s (x(\theta,h) - \epsilon_* \theta) = \mathcal{O}(h^{-r-1})$$

uniformly in θ. The results for v are obtained from the previous result and the fact that $v(h,\theta) = \omega_*(h)\partial_\theta x(\theta,h) \sim \epsilon_* \omega_*(h) \sim \epsilon_* \sqrt{2h}$ when $h \to 0$. In other words asymptotics (4.7.14) and estimates (4.7.15) are proved.

It remains to study the behaviors of $x(\theta,h)$ and $v(\theta,h)$ when $h \to M_0^+$. Using the properties of $1/(1-q(z))$ (see (4.7.2) and (4.7.3)) near $z=1$, we obtain from (4.7.20)

$$\|\partial_q^r \partial_\theta^s R(q(1/k(h)),\theta)\|_{L^\infty(M_0,2M_0)} \leq C \log\left(\frac{1}{h-M_0}\right)^{r+s+1},$$

and moreover, from (4.7.3)

$$\partial_h^r q(1/k(h)) = \mathcal{O}\left(\left(\frac{1}{h-M_0}\right)^r \frac{1}{\log(h-M_0)^2}\right).$$

This shows that for $s \geq 1$,

$$\partial_\theta^s (x(\theta,h) - \epsilon_* \theta) = \mathcal{O}\left(\log\left(\frac{1}{h-M_0}\right)^{s+1}\right),$$

and using the Faà di Bruno formula, we see that for $s \geq 1$ and $r \geq 1$,

$$\partial_h^r \partial_\theta^s (x(\theta,h) - \epsilon_* \theta) = \mathcal{O}\left(\left(\frac{1}{h-M_0}\right)^r \log\left(\frac{1}{h-M_0}\right)^{s-1}\right).$$

As $v(\theta,h) = \omega_*(h)\partial_\theta x(\theta,h)$, we deduce from the estimates on $\omega_*(h)$ that

$$\partial_h^r \partial_\theta^s v(\theta,h) = \mathcal{O}\left(\left(\frac{1}{h-M_0}\right)^r \log\left(\frac{1}{h-M_0}\right)^{s-3}\right),$$

and this proves estimates (4.7.16). To conclude the proof, let us say that the analyticity of $\omega_*(h)$, $x(\theta,h)$ and $v(\theta,h)$ stated in the Proposition follows from the analyticity properties of the special functions stated at the beginning of section 4.7. □

Now let us consider a function $f(x,v)$ that is continuous and its restriction f^* to U_*. We are interested in the behavior of the Fourier coefficients

$$f^*_\ell(a) = \frac{1}{2\pi}\int_0^{2\pi} f^*(x(\theta,a),v(\theta,a))e^{-i\ell\theta}\mathrm{d}\theta.$$

Note that by a slight abuse of notation, we will also denote by

$$f^*_\ell(h) = \frac{1}{2\pi}\int_0^{2\pi} f^*(x(\theta,h),v(\theta,h))e^{-i\ell\theta}\mathrm{d}\theta$$

these coefficients in the variable h.
In the special cases where f is either the cosine or the sine function, we have the following explicit expressions:

Proposition 4.7.4. *For $* \in \{\pm\}$, and $(\theta,a) \in J_* \times (-\pi,\pi)$,*

$$\cos(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} C^*_\ell(a)e^{i\ell\theta} \quad \textit{and} \quad \sin(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} S^*_\ell(a)e^{i\ell\theta}.$$

with, in terms of the variable h,

$$\begin{aligned} C^*_0(h) &= 1 - 2k(h)^2 + 2k(h)^2\frac{\boldsymbol{E}\left(\frac{1}{k(h)}\right)}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)} \\ C^*_\ell(h) = C^*_{-\ell}(h) &= \frac{2\pi^2 k(h)^2}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)^2}\left(\frac{|\ell| q\left(\frac{1}{k(h)}\right)^{|\ell|}}{1-q\left(\frac{1}{k(h)}\right)^{2|\ell|}}\right), \quad \ell > 0 \end{aligned} \tag{4.7.21}$$

and

$$\begin{aligned} S^*_0(h) &= 0 \\ S^*_\ell(h) = -S^*_{-\ell}(h) &= \epsilon_*(-i)\frac{2\pi^2 k(h)^2}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)^2}\left(\frac{|\ell| q\left(\frac{1}{k(h)}\right)^{|\ell|}}{1+q\left(\frac{1}{k(h)}\right)^{2|\ell|}}\right), \quad \ell > 0. \end{aligned} \tag{4.7.22}$$

Proof. Recall that we have

$$\sin(x(\theta,h)/2) = \epsilon_* \mathrm{sn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta, \frac{1}{k(h)}\right).$$

Hence using the expansion of sn^2 in (4.7.6) and the second formula of (4.7.9), we obtain

$$\begin{aligned}\cos(x(\theta,h)) &= 1-2\mathrm{sn}^2\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta,\frac{1}{k(h)}\right)\\ &= 1-2k(h)^2\frac{\boldsymbol{K}\left(\frac{1}{k(h)}\right)-\boldsymbol{E}\left(\frac{1}{k(h)}\right)}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)}\\ &+ \frac{4\pi^2k(h)^2}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)^2}\sum_{m=1}^{\infty}\frac{mq\left(\frac{1}{k(h)}\right)^m}{1-q\left(\frac{1}{k(h)}\right)^{2m}}\cos(m\theta).\end{aligned}$$

Formulae (4.7.21) follow easily. Moreover, we have

$$x(\theta,h)=\epsilon_*2\,\mathrm{am}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta,\frac{1}{k(h)}\right).$$

Hence, using the third formula of (4.7.9), the definitions of the functions $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$, and the expansion of $\mathrm{sn}(u,k)\mathrm{cn}(u,k)$ from (4.7.6), we infer

$$\begin{aligned}\sin(x(\theta,h)) &= 2\epsilon_*\mathrm{sn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta,\frac{1}{k(h)}\right)\mathrm{cn}\left(\frac{1}{\pi}\boldsymbol{K}\left(\frac{1}{k(h)}\right)\theta,\frac{1}{k(h)}\right)\\ &= 2\epsilon_*\frac{2k(h)^2\pi^2}{\boldsymbol{K}(1/k(h))^2}\sum_{m=1}^{\infty}\frac{mq(1/k(h))^m}{1+q(1/k(h))^{2m}}\sin(m\theta).\end{aligned}$$

Formulae (4.7.22) follow easily. □

For some smooth function f, we can estimate the generalized Fourier coefficients $f_\ell^*(a)$ in the following way.

Proposition 4.7.5. *Assume that f is a function satisfying*

$$\max_{|\alpha|\leq m}\left\|\langle v\rangle^\mu\partial_{x,v}^\alpha f\right\|_{L^\infty(U_*)}\leq C_{p,m}$$

for some $m\geq 1$ and $\mu\geq 0$. Then, we have

$$\begin{aligned}\partial_h^r f_\ell^*(h) &= \mathcal{O}\left(\frac{1}{|\ell|^s}\left(\frac{1}{h-M_0}\right)^r\log\left(\frac{1}{h-M_0}\right)^s\right)\quad \textit{when}\quad h\to M_0^+\\ \partial_h^r f_\ell^*(h) &= \mathcal{O}\left(\frac{1}{|\ell|^s}\frac{1}{h^{\mu/2}}\right)\quad \textit{when}\quad h\to+\infty\end{aligned}\tag{4.7.23}$$

for $r+s\leq m$.

Proof. We have

$$\partial_h^r f_\ell^*(h) = \frac{1}{\ell^s}(-i)^s \frac{1}{2\pi}\int_{(-\pi,\pi)} \partial_h^r \partial_\theta^s (f^*(x(\theta,h),v(\theta,h)))e^{-i\ell\theta}\mathrm{d}\theta,$$

and using Faà di Bruno formula, the hypothesis on f, and (4.7.16), we infer that when $h \to M_0^+$,

$$\partial_h^r f_\ell^*(h) = \mathcal{O}\left(\frac{1}{|\ell|^s}\Big(\frac{1}{h-M_0}\Big)^r \log\Big(\frac{1}{h-M_0}\Big)^s\right).$$

Moreover, by the same arguments, and using also (4.7.15), we obtain

$$\partial_h^r f_\ell^*(h) = \mathcal{O}\Big(\frac{1}{|\ell|^s}\frac{1}{h^{p/2}}\Big),$$

when $h \to +\infty$. □

From this result, we obtain the following:

Proposition 4.7.6. *Assume that f an φ are real functions satisfying*

$$\max_{|\alpha|\le m} \|\langle v\rangle^\mu \partial_{x,v}^\alpha f\|_{L^\infty(U_*)} \le C_{\mu,m} \quad \text{and} \quad \max_{|\alpha|\le M} \|\partial_{x,v}^\alpha \varphi\|_{L^\infty(U_*)} \le C_M.$$

for some $m \ge 1+r$, $\mu > 2$ and $M \ge 3+r$. Then we have for all $t \ge 1$,

$$\left|\int_{U_*} \varphi(\psi_t(x,v))f(x,v)\mathrm{d}v - \int_{M_0}^{+\infty} f_0^*(h)\varphi_0^*(h)\frac{1}{\omega_*(h)}\mathrm{d}h\right| \le \frac{C}{(1+t)^{r+1}}.$$

In this statement, φ has to be thought as a test function so that we can think M to be as big as we need. Note that φ is not assumed to be localized in v since we shall mainly use this result for test functions that do not depend on v like $\cos(x)$ or $\sin(x)$.

Proof. Recall that for $* \in \{\pm\}$, we have by formulae (4.2.2) and (4.2.3), and the identity $\psi_t(\theta,h) = \theta + t\omega_*(h)$,

$$\int_{U_*} \varphi(\psi_t(x,v))f(x,v)\mathrm{d}v\mathrm{d}x = \sum_{\ell\in\mathbb{Z}} \int_{(M_0,+\infty)} f_\ell^*(h)\varphi_{-\ell}^*(h)e^{it\ell\omega_*(h)}\frac{1}{\omega_*(h)}\mathrm{d}h.$$

We shall then integrate by parts with respect to h, using in particular the fact that $\partial_h\omega_*$ does not vanish on $U_\pm$. For $\ell \neq 0$, we may thus define two operators $\mathrm{D}_{\ell,h}$ and $\mathrm{D}_{\ell,h}^\top$ acting on function F of h by

$$\mathrm{D}_{\ell,h}F = \frac{1}{i\ell\partial_h\omega_*}\partial_h F \quad \text{and} \quad \mathrm{D}_{\ell,h}^\top F = -\partial_h\left(\frac{F}{i\ell\partial_h\omega_*}\right).$$

We shall in particular consider iterations of the operator $\mathrm{D}_{\ell,h}^\top$, and use the special notation

$$(\mathrm{D}_{\ell,h}^\top)^0 F = -\frac{F}{i\ell\partial_h\omega_*}.$$

We have the following useful Lemma (whose proof is postponed to the end of the current one):

Lemma 4.7.7. *For all* $0 \leq \beta \leq r+1$,

$$(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \mathcal{O}(|\ell|^{-\beta}h^{-\mu/2}) \quad \textit{when} \quad h \to +\infty,$$

and

$$(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \mathcal{O}\left(|\ell|^{-\beta}(h-M_0)\log\left(\frac{1}{h-M_0}\right)^{m(\beta)}\right) \quad \textit{when} \quad h \to M_0^{+},$$

for some integer $m(\beta) > 0$.

This Lemma says essentially that the singularities coming from f_{ℓ}^{*} and $\varphi_{-\ell}^{*}$ at the separatrix are cancelled by the one of $\partial_h\omega_*$. This shows in particular that for all $0 < \beta \leq r+1$,

$$\lim_{h\to M_0^{+}}(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\frac{1}{\partial_h\omega_*(h)} = 0,$$

$$\lim_{h\to+\infty}(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\frac{1}{\partial_h\omega_*(h)} = 0,$$

and that

$$\left\|(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\right\|_{L^1((M_0,+\infty))} \leq \frac{C_\beta}{|\ell|^{\beta}},$$

using also (4.7.13) and the hypothesis $\mu > 2$.
Integrating by parts $r+1$ times while using Lemma 4.7.7, we have for $\ell \neq 0$

$$\begin{aligned}
\int_{(M_0,+\infty)} f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)e^{i\ell t\omega_*(h)}\frac{\mathrm{d}h}{\omega_*(h)} &= \frac{1}{t}\int_{(M_0,+\infty)} f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\mathrm{D}_{\ell,h}(e^{i\ell t\omega_*(h)})\frac{\mathrm{d}h}{\omega_*(h)}\\
&= \frac{1}{t}\left[-(\mathrm{D}_{\ell,h}^{\top})^{0}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)e^{i\ell t\omega_*(h)}\right]_{M_0}^{+\infty}\\
&+ \frac{1}{t}\int_{(M_0,+\infty)} e^{i\ell t\omega_*(h)}(\mathrm{D}_{\ell,h}^{\top})\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\mathrm{d}h,\\
&= \frac{1}{t}\int_{(M_0,+\infty)} \mathrm{D}_{\ell,h}(e^{i\ell t\omega_*(h)})(\mathrm{D}_{\ell,h}^{\top})\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\mathrm{d}h,\\
&= \frac{1}{t^{r+1}}\int_{(M_0,+\infty)} e^{i\ell t\omega_*(h)}(\mathrm{D}_{\ell,h}^{\top})^{r+1}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\mathrm{d}h,
\end{aligned}$$

such that

$$\left|\int_{(M_0,+\infty)} f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)e^{i\ell t\omega_*(h)}\frac{\mathrm{d}h}{\omega_*(h)}\right| \leq \frac{1}{t^{r+1}}\left\|(\mathrm{D}_{\ell,h}^{\top})^{r+1}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\right\|_{L^1((M_0,+\infty))} \leq \frac{C_{r+1}}{t^{r+1}|\ell|^{r+1}},$$

and summing in ℓ gives the result. □

Proof of [Lemma 4.7.7]. Let us first prove the estimate when $h \sim M_0$. We shall rather prove by induction on $\beta \leq r+1$ that

$$(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \mathcal{O}\left(|\ell|^{-\beta}(h-M_0)\log\left(\frac{1}{h-M_0}\right)^{m(\beta)}\right), \tag{4.7.24}$$

and, if $\beta \leq r$,

$$\partial_h(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \mathcal{O}\left(|\ell|^{-\beta}\log\left(\frac{1}{h-M_0}\right)^{n(\beta)}\right), \tag{4.7.25}$$

for some integers $m(\beta), n(\beta)$.
When $\beta = 0$, we have by Proposition 4.7.5, (4.7.12) and (4.7.13)

$$-(\mathrm{D}_{\ell,h}^{\top})^{0}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \frac{f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)}{\omega_{*}(h)\partial_h\omega_{*}(h)} = \mathcal{O}\left((h-M_0)\log\left(\frac{1}{(h-M_0)}\right)^{3}\right),$$

and

$$\begin{aligned}\partial_h\left(\frac{f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)}{\omega_{*}(h)\partial_h\omega_{*}(h)}\right) &= \frac{\partial_h\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\right)}{\omega_{*}(h)\partial_h\omega_{*}(h)} + \left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\right)\partial_h\left(\frac{1}{\omega_{*}(h)\partial_h\omega_{*}(h)}\right)\\ &= \mathcal{O}\left(\log\left(\frac{1}{(h-M_0)}\right)^{3}\right).\end{aligned}$$

For $\beta \geq 1$, if (4.7.24) and (4.7.25) hold at rank $\beta-1$, then by using the formula

$$(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = -\partial_h\left(\frac{(\mathrm{D}_{\ell,h}^{\top})^{\beta-1}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)}{i\ell\partial_h\omega_{*}(h)}\right),$$

and the estimate (4.7.13) for $\partial_h\omega_{*}(h)^{-1}$, one easily proves (4.7.24) at rank β. As long as $\beta \leq r$, one deduces then (4.7.25) at rank β by writing that

$$\log\left(\frac{1}{h-M_0}\right)^{-m(\beta)}(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right) = \mathcal{O}\left(|\ell|^{-\beta}(h-M_0)\right),$$

which shows that

$$\partial_h\left(\log\left(\frac{1}{h-M_0}\right)^{-m(\beta)}(\mathrm{D}_{\ell,h}^{\top})^{\beta}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\omega_{*}(h)^{-1}\right)\right) = \mathcal{O}(|\ell|^{-\beta}),$$

and gives the result.
The asymptotics when $h \to +\infty$ are easier to obtain, as for any $s \leq \beta \leq r+1$, Proposition 4.7.5 implies that

$$\partial_h^{s}\left(f_{\ell}^{*}(h)\varphi_{-\ell}^{*}(h)\right) = \mathcal{O}(h^{-\mu/2}),$$

while (4.7.13) shows that

$$\partial_h^{s}\left(\frac{1}{\omega_{*}(h)\partial_h\omega_{*}(h)}\right) = \mathcal{O}(1),$$

such that Leibniz's formula yields the result. Note that the contribution $|\ell|^{-\beta}$ comes obvi-

ously from iterations of the operator $\mathrm{D}^{\top}_{\ell,h}$. ∎

4.7.3 Action-angle in $U_\circ$

In this subsection we provide a rather complete description of the change of variable in $U_\circ$.

Proposition 4.7.8. *There exists a symplectic change of variable $(x,v)\mapsto(\psi,h)$ from $U_\circ$ to the set*

$$V_\circ := \{(\psi,h)\in\mathbb{R}^2, |h\in(-M_0,M_0),\ \psi\in(-r_\circ(h),r_\circ(h))\},$$

with

$$r_\circ(h)=\frac{2k(h)}{\sqrt{M_0}}\boldsymbol{K}(k(h)),\quad \textit{where}\quad k(h)=\sqrt{\frac{h+M_0}{2M_0}}.$$

such that the flow of the pendulum in the variable (ψ,h) is $h(t)=h(0)$ and $\psi(t)=t+\psi(0)$. There exists then a second symplectic change of variables $(\psi,h)\mapsto(\theta,a)$ from $V_\circ$ to

$$\{(\theta,a)\in\mathbb{R}^2, |a\in J_\circ=\left(0,\frac{8}{\pi}\sqrt{M_0}\right),\ \theta\in(-\pi,\pi)\},$$

such that

$$\begin{cases} a(h) &= \dfrac{8\sqrt{M_0}}{\pi}(\boldsymbol{E}(k(h))-(1-k(h)^2)\boldsymbol{K}(k(h))) \\ \theta(\psi,h) &= \omega_\circ(h)\psi \end{cases} \qquad \textit{with}\quad \omega_\circ(h)=\frac{\pi\sqrt{M_0}}{2\boldsymbol{K}(k(h))}$$

and so that the flow of the pendulum in the variables (θ,a) is $a(t)=a(0)$ and $\theta(t)=t\omega_\circ(a(0))+\psi(0)$.
Moreover, we can easily express (x,v) as functions of the variables (θ,h) with the formulae

$$x(\theta,h) = 2\arcsin\left(k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right)\right), \tag{4.7.26}$$

$$v(\theta,h) = 2k(h)\sqrt{M_0}\,\mathrm{cn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right). \tag{4.7.27}$$

Proof. In this case, we have $h\in(-M_0,M_0)$ and we can write

$$v(x,h)=\epsilon_*\sqrt{2(h+M_0\cos(x))}$$

defined for $h+M_0\cos(x)\geq 0$, where $\epsilon_*=1$ if $v\geq 0$ and $\epsilon_*=-1$ if $v\leq 0$. Using this representation, both sets $U_{\circ,+}=U_\circ\cap\{v\geq 0\}$ and $U_{\circ,-}=U_\circ\cap\{v\leq 0\}$ can be parametrized as

$$\begin{aligned} U_{\circ,*} &= \{(x,h)\in\mathbb{T}\times(-M_0,M_0)\,|\,h\geq -M_0\cos(x)\} \\ &= \{(x,h)\,|\,h\times(-M_0,M_0), x\in(-x_0(h),x_0(h))\}, \end{aligned}$$

where $x_0(h)$ is the solution in $[0,\pi]$ of the equation $h + M_0\cos(x_0(h)) = 0$. Note that have

$$\sin^2(x_0(h)/2) = k(h)^2.$$

For $x \in (-x_0(h), x_0(h))$, let us define $\Theta(x,h) \in (-\pi/2, \pi/2)$ as the unique solution of

$$k(h)\sin(\Theta(x,h)) = \sin(x/2).$$

This solution is well defined when $x \in (-x_0(h), x_0(h))$ as $\frac{1}{k(h)}\sin(\frac{x}{2}) \in (0,1)$ in this interval. Note that $\Theta(0,h) = 0$, $\Theta(-x_0(h),h) = -\frac{\pi}{2}$ and $\Theta(x_0(h),h) = \frac{\pi}{2}$. Moreover, by taking the derivative with respect to x, we have

$$\begin{aligned} k(h)\cos(\Theta(x,h))\partial_x\Theta(x,h) &= \frac{1}{2}\cos(x/2) \\ &= \frac{1}{2}\sqrt{1-k(h)^2\sin(\Theta(x,h))^2}. \end{aligned}$$

In particular, we have

$$\sqrt{1-\sin^2(\Theta(x,h))} = \frac{1}{2k(h)\partial_x\Theta(x,h)}\sqrt{1-k(h)^2\sin(\Theta(x,h))^2}.$$

Then we have

$$U_{\circ,*} = \{(x,h) | h \in (-M_0, M_0), \Theta(x,h) \in (-\pi/2, \pi/2)\},$$

and we can define the generatrix function $S(x,h)$ on $U_{\circ,*}$ by the formula

$$\begin{aligned} S(x,h) &= \epsilon_* \int_{x_0(h)}^{x} \sqrt{2(h+M_0\cos(y))}\mathrm{d}y \\ &= \epsilon_* \int_{x_0(h)}^{x} \sqrt{2(h+M_0) - 4M_0\sin^2(y/2)}\mathrm{d}y \\ &= \epsilon_* 2k(h)\sqrt{M_0}\int_{x_0(h)}^{x} \sqrt{1-\frac{\sin^2(y/2)}{k(h)^2}}\mathrm{d}y \\ &= \epsilon_* 2k(h)\sqrt{M_0}\int_{x_0(h)}^{x} \sqrt{1-\sin^2(\Theta(y,h))}\mathrm{d}y \\ &= \epsilon_* 2k(h)\sqrt{M_0}\int_{x_0(h)}^{x} \sqrt{1-\sin^2(\Theta(y,h))}\frac{1}{\partial_x\Theta(y,h)}\partial_x\Theta(y,h)\mathrm{d}y \\ &= \epsilon_* 4k(h)^2\sqrt{M_0}\int_{\frac{\pi}{2}}^{\Theta(x,h)} \frac{1-\sin^2(\phi)}{\sqrt{1-k(h)^2\sin^2(\phi)}}\mathrm{d}\phi \end{aligned}$$

Whence

$$\begin{aligned} S(x,h) = \epsilon_* 4\sqrt{M_0}(E(\Theta(x,h),k(h)) - \boldsymbol{E}(k(h))) \\ - \epsilon_* 4\sqrt{M_0}(1-k(h)^2)(F(\Theta(x,h),k(h)) - \boldsymbol{K}(k(h))). \end{aligned}$$

Note that this function is equal to zero on the axis $\{v=0, x\in[0,\pi]\}$ and has a discontinuity in the axis $\{v=0, x\in[-\pi,0]\}$. We can then define

$$\begin{aligned}\psi(x,h) &= \frac{\partial}{\partial h}S(x,h)\\ &= \epsilon_*\int_{x_0(h)}^{x}\frac{1}{\sqrt{2(h+M_0\cos(y))}}\mathrm{d}y\\ &= \epsilon_*\frac{1}{\sqrt{M_0}}\int_{x_0(h)}^{x}\frac{1}{\sqrt{1-k(h)^2\sin^2(\Theta(y,h))}}\partial_x\Theta(x,h)\mathrm{d}y\\ &= \epsilon_*\frac{1}{\sqrt{M_0}}(F(\Theta(x,k(h)),k(h))-\boldsymbol{K}(k(h))),\end{aligned}$$

where we used the fact that $h+M_0\cos(x_0(h))=0$.
On a period, we thus see that $\psi(x,h)\in\left(-2\frac{\boldsymbol{K}(k(h))}{\sqrt{M_0}},2\frac{\boldsymbol{K}(k(h))}{\sqrt{M_0}}\right)$. Hence the function

$$\theta=\epsilon_*\frac{\pi\sqrt{M_0}}{2\boldsymbol{K}(k(h))}\psi=\epsilon_*\frac{\pi}{2}\frac{F(\Theta(x,k(h)),k(h))}{\boldsymbol{K}(k(h))}-\epsilon_*\frac{\pi}{2}$$

belongs to $(-\pi,\pi)$, and is such that the point $x_0(h)$ correspond to the angle $\theta=0$. The frequency is

$$\omega_\circ(h)=\frac{\pi\sqrt{M_0}}{2\boldsymbol{K}(k(h))}=1/g_\circ(h).$$

The action is given by

$$a_\circ(h)=\frac{8\sqrt{M_0}}{\pi}(\boldsymbol{E}(k(h))-(1-k(h)^2)\boldsymbol{K}(k(h))$$

as

$$\begin{aligned}\partial_h a_\circ(h) &= \frac{8\sqrt{M_0}}{\pi}\frac{1}{4M_0k(h)}(\partial_k\boldsymbol{E}(k(h))-(1-k(h)^2)\partial_k\boldsymbol{K}(k(h)+2k(h)\boldsymbol{K}(k(h))\\ &= \frac{8\sqrt{M_0}}{\pi}\frac{1}{4M_0k(h)}(k(h)\boldsymbol{K}(k(h))=g_\circ(h).\end{aligned}$$

Using the properties of the elliptic functions, we have

$$\Theta(x(\theta,h),h))=\epsilon_*\mathrm{am}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\epsilon_*\frac{\pi}{2}\right),k(h)\right)$$

hence

$$\sin(x(\theta,h)/2)=\epsilon_*k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\epsilon_*\frac{\pi}{2}\right),k(h)\right). \tag{4.7.28}$$

Now using the first formula of (4.7.8) we see that the expression (4.7.28) does actually not depend on the value of $\epsilon^*=\pm1$, and thus

$$\sin(x(\theta,h)/2)=k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right),$$

which yields

$$x(\theta,h) = 2\arcsin\left(k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right)\right).$$

Moreover, we have

$$\begin{aligned} v(\theta,h) &= \epsilon_*\sqrt{2(h+M_0\cos(x(\theta,h))} \\ &= \epsilon_*\sqrt{2(h+M_0)}\sqrt{1-\frac{1}{k(h)^2}\sin^2(x(\theta,h)/2)} \\ &= \epsilon_*\sqrt{2(h+M_0)}\sqrt{1-\sin^2(\Theta(x(\theta,h),h)} \\ &= \epsilon_* 2k(h)\sqrt{M_0}\,\mathrm{cn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\epsilon_*\frac{\pi}{2}\right),k(h)\right), \end{aligned}$$

and using the second formula of (4.7.8), it yields

$$v(\theta,h) = 2k(h)\sqrt{M_0}\,\mathrm{cn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right),$$

and concludes the proof. □

Proposition 4.7.9. *The function $\omega_\circ(h)$, $x(\theta,h)$ and $v(\theta,h)$ are analytic for $\theta\in(-\pi,\pi)$ and $h\in(-M_0,M_0)$.*
The function $\omega_\circ$ exhibits the following asymptotic behavior

$$\begin{aligned} &\omega_\circ(h) \sim \frac{\pi\sqrt{M_0}}{(-\log(M_0-h))} \quad \textit{when} \quad h\to M_0^-, \\ &\omega_\circ(h) = \sqrt{M_0} - \frac{1}{4\sqrt{2}}(h+M_0) \quad \textit{when} \quad h\to -M_0, \end{aligned} \tag{4.7.29}$$

and there exists constants, C_r, ω_r with $\omega_1=\frac{1}{4\sqrt{2}}$ and $\alpha_r\neq 0$ such that for all $r\geq 1$,

$$\begin{aligned} &\left\|(h+M_0)^{-1}\left[\partial_h^r\omega_\circ(h)-\omega_r\right]\right\|_{L^\infty(-M_0,0)} \leq C_r, \\ &\left\|\log(h-M_0)^2\left[\log(h-M_0)^2(h-M_0)^r\partial_h^r\omega_\circ(h)-\alpha_r\right]\right\|_{L^\infty(0,M_0)} \leq C_r. \end{aligned} \tag{4.7.30}$$

The change of variable $(h,\theta)\mapsto(x,v)$ satisfies the following estimates: When $h\to -M_0$ it converges towards the action-angle variable of the harmonic oscillator $\frac{1}{2}(v^2+M_0x^2)$ in the sense that

$$\begin{aligned} &x(\theta,h) \sim 2\sqrt{\frac{h+M_0}{2M_0}}\cos(\theta) \quad \textit{when} \quad h\to -M_0, \\ &v(\theta,h) \sim -2\sqrt{M_0}\sqrt{\frac{h+M_0}{2M_0}}\sin(\theta) \quad \textit{when} \quad h\to -M_0. \end{aligned} \tag{4.7.31}$$

More precisely, for $r, s \geq 0$ there exist constants $C_{r,s}$ such that

$$\begin{aligned}
&\left\| (h+M_0)^{r-\frac{1}{2}} \partial_h^r \partial_\theta^s \Big[x(\theta,h) - 2\sqrt{\frac{h+M_0}{2M_0}} \cos(\theta) \Big] \right\|_{L^\infty((-\pi,\pi)\times(-M_0,0))} \leq C_{r,s}, \\
&\left\| (h+M_0)^{r-\frac{1}{2}} \partial_h^r \partial_\theta^s \Big[v(\theta,h) + 2\sqrt{M_0}\sqrt{\frac{h+M_0}{2M_0}} \sin(\theta) \Big] \right\|_{L^\infty((-\pi,\pi)\times(-M_0,0))} \leq C_{r,s}.
\end{aligned} \tag{4.7.32}$$

Finally, we have for $r, s \geq 1$,

$$\begin{aligned}
&\| |(M_0-h)^r| \log(M_0-h)|^{-s+2} \partial_h^r \partial_\theta^s (x(\theta,h)) \|_{L^\infty((-\pi,\pi)\times(0,M_0))} \leq C_{r,s} \quad \text{and} \\
&\| |(M_0-h)^r| \log(M_0-h)|^{-s+3} \partial_h^r \partial_\theta^s (v(\theta,h)) \|_{L^\infty((-\pi,\pi)\times(0,M_0))} \leq C_{r,s}
\end{aligned} \tag{4.7.33}$$

for some constants $C_{r,s}$, and

$$\| |\log(M_0-h)|^{-s} \partial_\theta^s (x(\theta,h)) \|_{L^\infty((-\pi,\pi)\times(0,M_0))} \leq C_s$$

for $s \geq 1$ and some constant C_s.

Proof. Let us first prove (4.7.29) and (4.7.30), starting with the study of $\omega_\circ$ when $h \to M_0^-$. We have, by using (4.7.2), that

$$\omega_\circ(h) = \frac{\pi\sqrt{M_0}}{2\boldsymbol{K}(k(h))} \sim \frac{\pi\sqrt{M_0}}{(-\log(1-k(h)))}$$

and we obtain the result using

$$1 - k(h) = 1 - \sqrt{1 + \frac{h-M_0}{2M_0}} \sim \frac{M_0 - h}{4M_0}.$$

This proves the first part of (4.7.29). Note that the second estimate of (4.7.30) follows from the estimate on the function $1/\boldsymbol{K}$ of (4.7.3), and the smoothness of $k(h)$ in the vicinity $h \sim M_0$.
When $h \to -M_0$, $\omega_\circ(h)$ is an analytic function of $k(h)^2 = \frac{h+M_0}{2M_0}$, and we have using (4.7.1)

$$\begin{aligned}
\omega_\circ(h) = \frac{\pi\sqrt{M_0}}{2\boldsymbol{K}(k(h))} &= \sqrt{M_0}\left(1 - \frac{1}{4}k(h)^2\right) + \mathcal{O}((h+M_0)^2) \\
&= \sqrt{M_0} - \frac{1}{4\sqrt{2}}(h+M_0) + \mathcal{O}((h+M_0)^2).
\end{aligned}$$

The first estimate of (4.7.30) follows easily.

Let us now study the functions $x(\theta,h)$ and $v(\theta,h)$. Using (4.7.5) and the expression (4.7.26) and (4.7.27) of $x(\theta,h)$ and $v(\theta,h)$, we obtain the expansions

$$v(\theta,h) = \sqrt{M_0}\frac{4\pi}{\boldsymbol{K}(k(h))} \sum_{m=0}^{\infty} \frac{q(k(h))^{m+\frac{1}{2}}}{1+q(k(h))^{2m+1}} \cos\left((2m+1)\left(\theta + \frac{\pi}{2}\right)\right)$$

and

$$\sin(x(\theta,h)/2) = k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right)$$
$$= \frac{2\pi}{\boldsymbol{K}(k(h))}\sum_{m=0}^{\infty}\frac{q(k(h))^{m+\frac{1}{2}}}{1-q(k(h))^{2m+1}}\sin\left((2m+1)\left(\theta+\frac{\pi}{2}\right)\right).$$

This, together with the fact that $\arcsin(z)\sim z$ is analytic in the vicinity of $z=0$ and the expansions (4.7.1), shows that $v(\theta,h)$ and $x(\theta,h)$ are analytic functions of $\sqrt{h+M_0}$ when $h\to -M_0$, and that

$$v(\theta,h)\sim 8\sqrt{q(k(h))}\sqrt{M_0}\cos\left(\theta+\frac{\pi}{2}\right)\sim -2k(h)\sqrt{M_0}\sin(\theta)$$

and

$$x(\theta,h)\sim 8\sqrt{q(k(h))}\sin\left(\theta+\frac{\pi}{2}\right)\sim 2k(h)\cos(\theta)$$

which yields asymptotics (4.7.31), and estimates (4.7.32) follow easily.
It remains to prove (4.7.33). The analysis is similar to what we did for U_+ and U_-, as we have

$$v(\theta,h) = \sqrt{M_0}\frac{4\pi}{\boldsymbol{K}(k(h))}R(\theta,q(k(h)))$$

with

$$R(\theta,q) = \sum_{m=0}^{\infty}\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}}\cos\left((2m+1)\left(\theta+\frac{\pi}{2}\right)\right).$$

By doing an analysis similar to the one performed for U_+ and U_-, we have

$$\left\|\partial_q^r\partial_\theta^s R(\theta,q(k(h)))\right\|_{L^\infty(0,M_0)}\leq C\log\left(\frac{1}{M_0-h}\right)^{r+s+1},$$

Moreover (4.7.3) shows that

$$\partial_h^r q(k(h)) = \mathcal{O}\left(\left(\frac{1}{M_0-h}\right)^r\frac{1}{\log(M_0-h)^2}\right),$$

and

$$\partial_h^r\frac{1}{\boldsymbol{K}(k(h))} = \mathcal{O}\left(\left(\frac{1}{M_0-h}\right)^r\frac{1}{\log(M_0-h)^2}\right).$$

We deduce from these estimates that we have the same asymptotics as in the case of U_+ and U_-:

$$\partial_h^r\partial_\theta^s v(\theta,h) = \mathcal{O}\left(\left(\frac{1}{h-M_0}\right)^r\log\left(\frac{1}{h-M_0}\right)^{s-3}\right).$$

Now we can perform a similar analysis for

$$\sin(x(\theta,h)/2) = k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right),k(h)\right)$$
$$= \frac{2\pi}{\boldsymbol{K}(k(h))}\sum_{m=0}^{\infty}\frac{q(k(h))^{m+\frac{1}{2}}}{1-q(k(h))^{2m+1}}\sin\left((2m+1)\left(\theta+\frac{\pi}{2}\right)\right)$$

after noticing that

$$\frac{1}{1-q^{2m+1}} = \frac{1}{1-q}\left(\frac{1-q}{1-q^{2m+1}}\right) \leq \frac{C}{2m+1}\left(\frac{1}{1-q}\right)$$

when $q \in (\frac{1}{2},1)$. To obtain the conclusion for $x(\theta,h)$, we just have to be careful as arcsin has singularities in ± 1: recall that we have the expansion (see (4.4.41) in [1])

$$\arcsin(x) = -\pi/2 + \sqrt{2(1+x)}\left(\sum_{n=0}^{+\infty}\frac{(2n)!}{8^n(2n+1)(n!)^2}(1+x)^n\right)$$

and

$$\arcsin(x) = \pi/2 - \sqrt{2(1-x)}\left(\sum_{n=0}^{+\infty}\frac{(2n)!}{8^n(2n+1)(n!)^2}(1-x)^n\right).$$

In our context, it will happen at the point $\pm x_0(h)$ and the singularity will be of order $\sqrt{M_0-h}$. However, this singularity is weaker than the other one coming from functions $q(z)$ and $\boldsymbol{K}(z)$ in the vicinity $z \sim 1$ (see (4.7.2)). This finishes the proof of (4.7.33). □

Now let us consider a function $f(x,v)$ that is smooth and its restriction f° to $U_\circ$. We are as previously interested in the behavior of the Fourier coefficients

$$f^\circ_\ell(a) = \frac{1}{2\pi}\int_{(-\pi,\pi)} f^\circ(x(\theta,a),v(\theta,a))e^{-i\ell\theta}\mathrm{d}\theta.$$

With the usual abuse of notation, we will also denote by

$$f^\circ_\ell(h) = \frac{1}{2\pi}\int_{(-\pi,\pi)} f^\circ(x(\theta,h),v(\theta,h))e^{-i\ell\theta}\mathrm{d}\theta$$

these coefficients in the variable h. In the case where f is the sine or cosine function we have as previously explicit expressions of these coefficients:

Proposition 4.7.10. *For* $(\theta,a) \in (-\pi,\pi)\times J_\circ$,

$$\cos(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} C^\circ_\ell(a)e^{i\ell\theta} \quad \textit{and} \quad \sin(x(\theta,a)) = \sum_{\ell\in\mathbb{Z}} S^\circ_\ell(a)e^{i\ell\theta}$$

with, in terms of the variable h,

$$\begin{aligned} C_0^\circ(h) &= -1 + 2\frac{\boldsymbol{E}(k(h))}{\boldsymbol{K}(k(h))} \\ C_{2\ell}^\circ(h) &= (-1)^{|\ell|}\frac{2\pi^2}{\boldsymbol{K}(k(h))^2}\left(\frac{|\ell| q(k(h))^{|\ell|}}{1-q(k(h))^{2|\ell|}}\right), \quad \ell \neq 0 \\ C_{2\ell+1}^\circ &= 0, \end{aligned} \tag{4.7.34}$$

and

$$\begin{aligned} S_{2\ell}^\circ(h) &= 0 \\ S_{2\ell-1}^\circ(h) = S_{-(2\ell-1)}^\circ(h) &= \frac{(-1)^{\ell-1}\pi^2}{2\boldsymbol{K}(k(h))^2}\left(\frac{(2\ell-1)q(k(h))^{\ell-\frac{1}{2}}}{1-q(k(h))^{2\ell-1}}\right), \quad \ell \geq 1. \end{aligned} \tag{4.7.35}$$

Proof. Recall that we have

$$\sin(x(\theta,h)/2) = k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right).$$

By using the third formula of (4.7.9), the definition of the function $\mathrm{dn}(u,k)$ and the expansion of $\mathrm{sn}(u,k)\mathrm{dn}(u,k)$ in (4.7.6), we have

$$\begin{aligned} \sin(x(\theta,h)) &= 2\sin(x(\theta,h)/2)\sqrt{1-\sin^2(x(\theta,h)/2)} \\ &= 2k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right)\mathrm{dn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right) \\ &= \frac{\pi^2}{\boldsymbol{K}(k(h))^2}\sum_{m=1}^{\infty}\frac{(2m-1)q(k(h))^{m-\frac{1}{2}}}{1+q(k(h))^{2m-1}}\sin\left((2m-1)\left(\theta+\frac{\pi}{2}\right)\right), \\ &= \frac{\pi^2}{\boldsymbol{K}(k(h))^2}\sum_{m=1}^{\infty}\frac{(2m-1)q(k(h))^{m-\frac{1}{2}}}{1+q(k(h))^{2m-1}}\left[\frac{e^{(2m-1)i\theta}e^{(2m-1)\frac{i\pi}{2}} - e^{-(2m-1)i\theta}e^{-(2m-1)\frac{i\pi}{2}}}{2i}\right], \end{aligned}$$

which yields (4.7.35).
Using now the first and fourth formulae of (4.7.9), and the expansion of $\mathrm{sn}^2(u,k)$ from (4.7.6), we obtain

$$\begin{aligned} \cos(x(\theta,h)) &= \cos\left(2\arcsin\left(k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right)\right)\right) \\ &= 2\cos^2\left(\arcsin\left(k(h)\mathrm{sn}\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right)\right)\right) - 1 \\ &= 1 - 2k(h)^2\mathrm{sn}^2\left(\frac{2}{\pi}\boldsymbol{K}(k(h))\left(\theta+\frac{\pi}{2}\right), k(h)\right) \\ &= 1 - 2\frac{\boldsymbol{K}(k(h)) - \boldsymbol{E}(k(h))}{\boldsymbol{K}(k(h))} \\ &\quad + \frac{4\pi^2}{\boldsymbol{K}(k(h))^2}\sum_{m=1}^{\infty}\frac{mq(k(h))^m}{1-q(k(h))^{2m}}\cos(2m\theta + m\pi), \end{aligned}$$

which yields (4.7.34). □

As in the case of $U_\pm$ we can establish estimates for general Fourier coefficients f° as follows:

Proposition 4.7.11. *Assume that f is a function satisfying*

$$\max_{|\alpha|\leq m} \|\partial^\alpha_{x,v} f\|_{L^\infty(U_\circ)} \leq C_m$$

for some $m \geq p+2$, with p defined by

$$p = \max\{n \geq 1,\ \partial^\alpha_{x,v} f(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\}$$

(and with the convention that $p = 0$ if this set is empty). Then, as long as $r + s \leq m$ and $s + p + 2 \leq m$, we have that for $\ell \neq 0$,

$$\partial^r_h f^\circ_\ell(h) = \mathcal{O}\left(\frac{1}{|\ell|^s}\Big(\frac{1}{h-M_0}\Big)^r \log\Big(\frac{1}{h-M_0}\Big)^s\right) \quad \textit{when} \quad h \to M_0^-, \tag{4.7.36}$$

and that

$$f^\circ_\ell(h) = \frac{1}{|\ell|^s}\left(c_\ell\,(h+M_0)^{\frac{p+1}{2}} + (h+M_0)^{\frac{p+2}{2}} r_\ell(\sqrt{h+M_0})\right), \tag{4.7.37}$$

where c_ℓ is a number uniformly bounded in ℓ and $r_\ell \in W^{m-(p+2)-s,\infty}$ uniformly in ℓ.

Proof. The estimates near the separatix are exactly the same as in the case of U_+ and U_-. Let us then focus on the asymptotic near $h = -M_0$. Taylor-expanding, we can always write that

$$f(x,v) = f(0,0) + F^1(x,v)\cdot(x,v) = f(0,0) + F^1(0,0)\cdot(x,v) + F^2(x,v)\cdot(x,v)^{(2)}$$

where $F^1(x,v)$ is linear and $F^2(x,v)$ is bilinear. We may write for $\ell \neq 0$ that

$$f^\circ_\ell(a) = \frac{1}{2\pi}\int_{(-\pi,\pi)} \left[F^1(0,0)\cdot(x,v) + F^2(x(\theta,a),v(\theta,a))\cdot(x(\theta,a),v(\theta,a))^{(2)}\right] e^{-i\ell\theta}\,\mathrm{d}\theta. \tag{4.7.38}$$

We obtain then from Proposition 4.7.9 that $U(\theta,a) = (x(\theta,a),v(\theta,a))$ can be expanded when h is near $-M_0$ as

$$U(\theta,a) = \sum_{n\geq 1} a_n(\theta)(h+M_0)^{\frac{n}{2}},$$

where the functions a_n are smooth since $(x(\theta,a),v(\theta,a))$ is an analytic function of $\sqrt{h+M_0}$ uniformly in θ. By plugging this expansion in (4.7.38) and by integrating by parts s times we get (4.7.37) for $p = 0$. If $p > 0$, we just notice that by further Taylor expansion, we have

$$f(x,v) = f(0,0) + F^{p+1}(0,0)\cdot(x,v)^{(p)} + F^{p+2}(x,v)(x,v)^{(p+1)},$$

where $F^{p+1}(x,v)$ is $p+1$-linear and F^{p+2} $p+2$ linear. It suffices then to proceed as above. □

From this result, we obtain the following:

Proposition 4.7.12. *Assume that f an φ are real functions satisfying*

$$\max_{|\alpha|\leq m} \|\partial^\alpha_{x,v} f\|_{L^\infty(U_\circ)} \leq C_m \quad \text{and} \quad \max_{|\alpha|\leq M} \|\partial^\alpha_{x,v} \varphi\|_{L^\infty(U_\circ)} \leq C_{r,s}$$

for some $m \geq 0$. Let p and q defined as

$$p = \max\{n \geq 1,\ \partial^\alpha_{x,v} f(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\},\ q = \max\{n \geq 1,\ \partial^\alpha_{x,v} \varphi(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\}.$$

Then, for $m \geq 5 + p + \frac{p+q}{2}$, $M \geq 7 + q + \frac{p+q}{2}$, $M \geq m + 2$, we have for $t \geq 1$

$$\left| \int_{U_\circ} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}v - \int_{-M_0}^{M_0} f_0^\circ(h)\varphi_0^\circ(h)\frac{1}{\omega_\circ(h)}\mathrm{d}h \right| \leq \frac{C}{(1+t)^{\frac{p+q}{2}+2}}.$$

Proof. We begin as in Proposition 4.7.6 and write that

$$\int_{U_\circ} f(x,v)\varphi(\psi_t(x,v))\mathrm{d}x\mathrm{d}v = \sum_{\ell\in\mathbb{Z}} \int_{-M_0}^{M_0} f_\ell^\circ(h)\varphi_{-\ell}^\circ(h) e^{it\ell\omega_\circ(h)} \frac{1}{\omega_\circ(h)}\mathrm{d}h.$$

By taking a smooth nonnegative function $\chi(s)$ such that $\chi = 1$ for $s \leq \delta$ and $\chi = 0$ for $s \geq 2\delta$ with δ small enough, we can split the integral into

$$\begin{aligned}\int_{-M_0}^{M_0} f_\ell^\circ(h)\varphi_{-\ell}^\circ(h) e^{it\ell\omega_\circ(h)} \frac{1}{\omega_\circ(h)}dh &= \int_{-M_0}^{M_0} \chi\left(\frac{h+M_0}{M_0}\right) f_\ell^\circ(h)\varphi_{-\ell}^\circ(h) e^{it\ell\omega_\circ(h)} \frac{1}{\omega_\circ(h)}\mathrm{d}h \\ &\quad + \int_{-M_0}^{M_0} \left(1 - \chi\left(\frac{h+M_0}{M_0}\right)\right) f_\ell^\circ(h)\varphi_{-\ell}^\circ(h) e^{it\ell\omega_\circ(h)} \frac{1}{\omega_\circ(h)}\mathrm{d}h \\ &= I_\ell^1 + I_\ell^2.\end{aligned}$$

As in the proof of Proposition 4.7.6, the idea is again to integrate by parts as long as we can, *ie* as long as the contributions from the boundary points $h \sim \pm M_0$ vanish. The term I_ℓ^2 can be handled as before for U_+ and U_- : as $\partial_h\omega_\circ$ does not vanish, only the contribution at the separatix $h \sim M_0$ matters, and this yields a decay by $(1+t)^{-r}$ assuming enough regularity. As a matter of fact, we just need to take m large enough in order to choose $r \geq \frac{p+q}{2} + 2$. We shall now focus on I_ℓ^1 which contains the contribution from the center $h \sim -M_0$. By using Proposition 4.7.11, we can expand I_ℓ^1 under the form

$$\begin{aligned}I_\ell^1 = \frac{1}{|\ell|^s}\tilde{c}_\ell \int_{-M_0}^{M_0} (h+M_0)^{1+\frac{p+q}{2}} \chi\left(\frac{h+M_0}{M_0}\right) e^{it\ell\omega_\circ(h)} \frac{1}{\omega_\circ(h)}\mathrm{d}h \\ + \frac{1}{|\ell|^s}\tilde{c}_\ell \int_{-M_0}^{M_0} (h+M_0)^{1+\frac{1+p+q}{2}} \chi\left(\frac{h+M_0}{M_0}\right) \tilde{r}_\ell(\sqrt{h+M_0}) \frac{1}{\omega_\circ(h)} e^{it\ell\omega_\circ(h)}\mathrm{d}h,\end{aligned}$$

where $\tilde{r}_\ell$ is uniformly in ℓ in W^{m-p-2} (since we always assume that M is much bigger than m, $M \geq m + s$), and where the constant c_ℓ is uniformly bounded in ℓ. It is important to

notice that $\omega_\circ(h)$ and all its derivatives are non-zero smooth functions in $]-M_0, c]$ for any $c \in (0, M_0)$, so that during the integration by parts, $\omega_\circ$ will not play any major part.
Let us first consider the case where $p+q$ is even, and write $p+q=2k$. Then the polynomial contributions in $(h+M_0)$ are $(h+M_0)^{k+1}$ in the first integral and $(h+M_0)^{k+1+\frac{1}{2}}$ in the second integral above. We can thus integrate by parts $k+2$ times in each of the two integrals (as in the proof of Proposition 4.7.6), in order to obtain that

$$|I_\ell^1| \lesssim \frac{1}{|\ell|^s} \frac{1}{(1+t)^{k+2}}.$$

Taking $s=2$ and summing with respect to ℓ, we get the result.
In the case $p+q=2k+1$, the polynomial contributions in $(h+M_0)$ are $(h+M_0)^{k+\frac{3}{2}}$ in the first integral and $(h+M_0)^{k+2}$ in the second integral. For the latter we can thus integrate by parts $k+3$ times as previously to get a decay like $1/(1+t)^{k+3}$, which is $1/(1+t)^{\frac{p+q}{2}+\frac{5}{2}}$, and is already faster than the expected decay. For the first integral, we can integrate by parts $k+3$ times, except for the most singular term where we can integrate by parts only $k+2$ times without boundary terms to obtain integrals under the form

$$\tilde{I}_\ell^1 = \frac{1}{(1+t)^{k+2}|\ell|^s} \tilde{c}_\ell \int_{-M_0}^{M_0} (h+M_0)^{-\frac{1}{2}} \chi\left(\frac{h+M_0}{M_0}\right) \tilde{\chi}(h+M_0) e^{it\ell\omega_\circ(h)} \mathrm{d}h.$$

where $\tilde{\chi}$ is a smooth function. Next, since $\partial_h \omega_\circ$ does not vanish, we can make the change of variable $u = \omega_\circ(h) - \omega_\circ(-M_0)$. By observing that this allows to write $h+M_0 = uA(u)$ where A is smooth and does not vanish (so in particular, we have that $A(0) \neq 0$), we can thus write

$$\tilde{I}_\ell^1 = \frac{1}{(1+t)^{k+2}|\ell|^s} \tilde{c}_\ell \int_0^X \frac{1}{u^{\frac{1}{2}}} \Psi(u) e^{it\ell u} \, \mathrm{d}u$$

where Ψ is smooth and compactly supported in $[0, X)$. Taylor-expanding the function Ψ, we obtain that

$$\tilde{I}_\ell^1 = \frac{1}{(1+t)^{k+2}|\ell|^s} \tilde{c}_\ell \int_0^X \frac{1}{u^{\frac{1}{2}}} e^{it\ell u} \, \mathrm{d}u + \frac{1}{(1+t)^{k+2}|\ell|^s} \tilde{c}_\ell \int_0^X u^{\frac{1}{2}} \Psi^1(u) e^{it\ell u} \, \mathrm{d}u.$$

For the second integral above, we can integrate by parts once to obtain an estimate by $\frac{1}{(1+t)^{k+3}|\ell|^s}$. To handle the first integral we use Lemma 4.7.13 below, which yields the decay

$$1/(1+t)^{k+2+\frac{1}{2}}.$$

By noticing that $k+\frac{1}{2}+2 = \frac{p+q}{2}+2$, we finally get the result. □

Lemma 4.7.13. *Consider the integral*

$$I(t) = \int_0^X \frac{1}{u^{\frac{1}{2}}} e^{itu} \, \mathrm{d}u.$$

Then we have that for $t \geq 1$,

$$|I(t)| \lesssim \frac{1}{t^{\frac{1}{2}}}.$$

Proof. Let us set $v = tu$ in the integral, we obtain that

$$I(t) = \frac{1}{t^{\frac{1}{2}}} \int_0^{tX} \frac{1}{v^{\frac{1}{2}}} e^{iv} \, \mathrm{d}v = \frac{1}{t^{\frac{1}{2}}} \int_0^1 \frac{1}{v^{\frac{1}{2}}} e^{iv} \, \mathrm{d}v + \frac{1}{t^{\frac{1}{2}}} \int_1^{tX} \frac{1}{v^{\frac{1}{2}}} e^{iv} \, \mathrm{d}v$$

(assuming that t is sufficiently large so that $tX \geq 1$). The first integral in the RHS above is clearly uniformly bounded by $1/t^{\frac{1}{2}}$. For the second integral in the above right hand side, we can integrate by parts once to get that

$$\left| \frac{1}{t^{\frac{1}{2}}} \int_1^{tX} \frac{1}{v^{\frac{1}{2}}} e^{iv} \, \mathrm{d}v \right| \lesssim \frac{1}{t^{\frac{1}{2}}} \left(1 + \int_1^{+\infty} \frac{1}{v^{\frac{3}{2}}} \, \mathrm{d}v \right).$$

□

Chapter 5

Numerics

In this chapter we study the Vlasov-HMF model from the numerical point of vue. It contains a large amount of numerical simulations, and was designed first to study at the nonlinear level the damping rates predicted by Theorem 4.2.4 for the linearized dynamics. Our aim is also to understand whether further extensions of Theorem 4.2.4 for the nonlinear dynamics are possible.
This chapter is of course deeply related to chapter 4, and should not be read independently from the latter. We shall in particular use notations introduced there.
This is a joint work with Pierre Navaro (University of Rennes 1).

5.1 Introduction & notations

Most of the notations of this chapter were used in chapter 4. For the reader's convenience, we shall however specify once again some of them. We consider the Vlasov-HMF equation with an attractive potential

$$\begin{cases} \partial_t f(t,x,v) + \{f, H[f]\}(t,x,v) = 0 \\ H[f](x,v) = \dfrac{v^2}{2} - \phi[f](x), \end{cases} \tag{5.1.1}$$

where the potential ϕ is expressed as the following Fourier series

$$\phi[f](x) = \mathcal{C}[f]\cos(x) + \mathcal{S}[f]\sin(x),$$

with

$$\mathcal{C}[f] = \int_{\mathbb{T}\times\mathbb{R}} \cos(y) f(y,u)\mathrm{d}y\mathrm{d}u \quad \text{and} \quad \mathcal{S}[f] = \int_{\mathbb{T}\times\mathbb{R}} \sin(y) f(y,u)\mathrm{d}y\mathrm{d}u. \tag{5.1.2}$$

We consider an initial data of the kind

$$f^0(x,v) = \eta(x,v) + \varepsilon r^0(x,v), \tag{5.1.3}$$

with $0 < \varepsilon << 1$, and where η is a stationary solution of (5.1.1). More precisely, we fix a function $G : \mathbb{R} \to \mathbb{R}$ and consider a state $\eta(x,v)$ satisfying the equation

$$\eta(x,v) = G\left(h_0(x,v)\right), \tag{5.1.4}$$

with

$$h_0(x,v) = \frac{v^2}{2} - M_0\cos(x), \tag{5.1.5}$$

where $M_0 = \mathcal{C}[\eta] > 0$ is the magnetization of η (see chapter 4). We will moreover impose that,

$$\int_{\mathbb{T}\times\mathbb{R}} \eta(x,v)\mathrm{d}x\mathrm{d}v = 1. \tag{5.1.6}$$

The existence of such states was discussed in the section 6 of chapter 4. In this chapter, we shall consider the following examples

- Maxwell-Boltzmann distributions (see [5], [7], [29] and [28]):

$$\eta(x,v) = \alpha e^{-\beta h_0(x,v)} \tag{5.1.7}$$

- Lynden-Bell distributions (see [27]):

$$\eta(x,v) = \frac{\alpha}{1+e^{\beta h_0(x,v)}} \tag{5.1.8}$$

- Tsallis distributions with compact support (see [24] and [27]):

$$\eta(x,v) = \alpha(E - h_0(x,v))^{\frac{1}{q-1}} \mathbb{1}_{\{h_0(x,v)<E\}}, \quad \text{with} \quad q > 1. \tag{5.1.9}$$

- Smooth distributions with compact support:

$$\eta(x,v) = G\left(h_0(x,v)\right), \quad G \in C_c^\infty(]-M_0, +\infty[). \tag{5.1.10}$$

Conditions should exist on the parameters α, β, E, or q, for the self-consistent equation (5.1.4) to be satisfied (see for instance section 6 of chapter 4 in the case of Maxwell-Boltzmann distributions).

The solution $f(t,x,v)$ of (5.1.1) associated with the initial data (5.1.3) is written

$$f(t,x,v) = \eta(x,v) + \varepsilon r(t,x,v), \tag{5.1.11}$$

where r satisfies the equation

$$\partial_t r + \{r, h_0\} - \{\eta, \phi[r]\} - \varepsilon\{r, \phi[r]\} = 0.$$

The linearized equation around η reads

$$\partial_t r + \{r, h_0\} - \{\eta, \phi[r]\} = 0. \tag{5.1.12}$$

In chapter 4, we studied the long time behavior of the coefficients

$$\mathcal{C}[f](t) = \mathcal{C}[f(t)] \quad \text{and} \quad \mathcal{S}[f](t) = \mathcal{S}[f(t)]$$

for the linearized dynamics, and under suitable assumptions on r^0 and η, Theorem 4.2.4 states that

$$|\mathcal{C}[f](t) - M_0| \lesssim \frac{1}{(1+t)^\alpha} \quad \text{and} \quad |\mathcal{S}[f](t)| \lesssim \frac{1}{(1+t)^2}, \tag{5.1.13}$$

where α takes his values in $\left\{2, \frac{5}{2}, 3\right\}$ (we refer to Theorem 4.2.4 for a classification of the values of α).
We consider two examples of initial perturbations that were also subject of recent interests in

the Physics literature (see [5] for instance), and for which numerical experiments similar to those that we will show in this chapter were conducted. The first case is a cosine perturbation:

$$r^0(x,v) = \cos(x)\eta(x,v).$$

Now we should point that this perturbation does not satisfy the orthogonality condition of Theorem 4.2.4, but we actually expect to see the coefficient $\mathcal{C}[f](t)$ being damped to its long-time average, with a rate $1/t^3$ (and note that the coefficient $\mathcal{S}[f](t)$ vanishes as the initial data is spatially even). Let us explain our theory.
We may in fact write the initial data as

$$f^0(x,v) = \eta(x,v)(1+\varepsilon\cos(x)) = \eta(x,v)(1+\varepsilon) + \varepsilon(\cos(x)-1)\eta(x,v).$$

Our theory is then that, if ε is small enough, there exists a magnetization $M_0(\varepsilon) = M_0 + \mathcal{O}(\varepsilon)$ such that $\eta(x,v)(1+\varepsilon)$ is a stationary state of the Vlasov-HMF equation whose magnetization is $M_0(\varepsilon)$. If we assume this to be true, then we are actually considering a small perturbation of this new steady state that is $\varepsilon(\cos(x)-1)\eta(x,v)$, and which satisfies the orthogonality condition of Theorem 4.2.4. Then the application of the latter predicts, for the linearized dynamics, that

$$|\mathcal{C}[f](t) - M_0 - \mathcal{O}(\varepsilon)| \lesssim \frac{1}{(1+t)^3}. \tag{5.1.14}$$

The second type of initial perturbation that we consider is a sine perturbation:

$$r^0(x,v) = \sin(x)\eta(x,v).$$

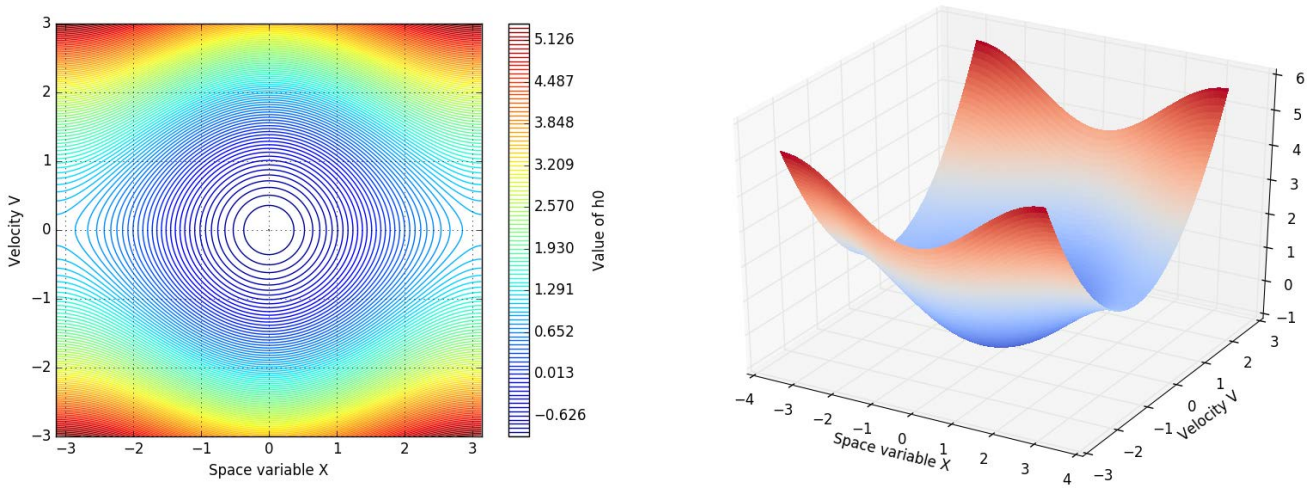
The application of Theorem 4.2.4 predicts the estimates (5.1.13) with $\alpha = 3$ (see also the discussion **a)** below the Theorem 4.2.4 about special cancellations for sine perturbations.)

The first goal of this chapter is to verify numerically the damping rates of (5.1.13) for a sine perturbation, as well as the estimate (5.1.14) for a cosine perturbation, for the nonlinear equation. Our interest is indeed to understand if Theorem 4.2.4 can be extended to the nonlinear level. The smallness of ε, an usual hypothesis for nonlinear Landau damping (see [10, 61, 40]) that was not needed in linearized analysis of chapter 4, should be necessary in the nonlinear case. The numerical tests that we shall employ to confront Theorem 4.2.4 will be specified in section 2, and the study of the numerical damping rates will be done in section 3.

Theorem 4.2.4 concerns the decay in time of the Fourier modes of the potential, which we proved for the linearized equation. For the nonlinear dynamics we strongly expect this decay in time to be linked with the regularity of the function $g(t) = r(t) \circ \psi_t$, where $\psi_t(x,v)$ is the flow of the Hamiltonian h_0, an usual feature of nonlinear Landau damping (see [8, 10, 40, 61]). Now recall that the linear analysis of chapter 4 uses the expression of the flow ψ_t in angle-action variables, where we are able to capture a phase-mixing effect that is essential for Landau damping. It was moreover shown in chapter 4 that the aforementioned change of variables requires the division of the space into the three following charts

$$\begin{aligned} U_\circ &= \{(h,v) \in \mathbb{T}\times\mathbb{R} \quad | \quad h_0(x,v) \in]-M_0, M_0[\} \\ U_+ &= \{(h,v) \in \mathbb{T}\times\mathbb{R} \quad | \quad h_0(x,v) \in]M_0, +\infty[, v>0\} \\ U_- &= \{(h,v) \in \mathbb{T}\times\mathbb{R} \quad | \quad h_0(x,v) \in]M_0, +\infty[, v<0\}\,. \end{aligned} \tag{5.1.15}$$

The figure below shows the division of the box $[-\pi, \pi] \times [-3, 3]$ into the three areas. The separatix is $\{(h, v) \in \mathbb{T} \times \mathbb{R} \quad | \quad h_0(x, v) = M_0\}$.



(a) *Contour lines* (b) *3D plot*

Figure 5.1: *Plot of $h_0(x, v)$ with $M_0 = 0.946$. The eye $U_\circ$ corresponds to the blue ellipse-shaped trajectories in the phase space (LHS figure), and U_+, U_- correspond respectively to the above and below trajectories.*

Corollary 4.2.5 shows in the case of the linearized dynamics the existence of a function g^∞ such that

$$\|g(t) - g^\infty\|_{L^1(\mathbb{T}\times\mathbb{R})} \lesssim \frac{1}{(1+t)}, \tag{5.1.16}$$

implying the existence of a final state $\eta^\infty \in L^1(\mathbb{T} \times \mathbb{R})$ such that

$$f(t) \rightharpoonup \eta^\infty \quad \text{and} \quad \|\eta^\infty - \eta\|_{L^1(\mathbb{T}\times\mathbb{R})} \lesssim \varepsilon. \tag{5.1.17}$$

Note moreover that η^∞ is a function of h_0. The chapter will contain simulations describing the evolution of $f(t)$, to observe whether the weak convergence result (5.1.17) may hold at the nonlinear level.
Nevertheless, justifying (5.1.16) for the nonlinear dynamics could be very difficult mathematically. Based on the existing Landau damping literature, in which trades between regularity and decay are crucial, we expect in fact any extension of Theorem 4.2.4 at the nonlinear level to require uniform-in-time bounds on g in a regular norm, as Gevrey's or Sobolev's. If true, this should imply that the scattering estimate (5.1.16) holds in a regular norm as well, and, at the end, that η^∞ is close to η (with respect to ε) in a regular norm. However, the flow ψ_t expressed in angle-action variables has a singularity near the separatix (the singularity of the frequency ω_*, see chapter 4, section 7), and we fear that this singularity may interfere with other mechanisms of Landau damping , such that η^∞ is far from η in regular norm.
A possible solution to prevent this would be to start from an initial data f^0 that is compactly

supported in $U_\circ$, to keep the poor behavior of $\psi_t(x,v)$ at the separatix from interfering, as long as the support stays included in $U_\circ$ in the long time. If this is the case, we might expect η^∞ to be close to η in a regular norm.
Besides the numerical nonlinear confrontation of Theorem 4.2.4, the aim of the chapter will be as well to show, in section 4, 5 and 6, numerical simulations plotting the evolution of the distribution function $f(t)$, with various initial data, and discuss the possibility of nonlinear Landau damping for each example. We shall analyse the possible obstacles to nonlinear Landau damping, and illustrate in particular the above discussion on singularities near the separatix.

5.2 Numerical experiments

In this section we provide the reader with a rather complete description of the numerical scheme that we shall use, as well as a detailed formulation of the tests that we shall perform to confront the theory, such that our numerical experiments should be reproducible.

5.2.1 Semi-Lagrangian scheme

We employ a time discretization of (5.1.1) by a Lie splitting method (see also chapter 3), based on the decomposition of the equation between the free part

$$\begin{cases} \partial_t f(t,x,v) + v\partial_x f(t,x,v) = 0 \\ f(0,x,v) = f^0(x,v), \end{cases} \tag{5.2.1}$$

whose solution is explicitly given by $\varphi_t^t(f^0)(x,v) = f^0(x-vt,v)$, and the potential part

$$\begin{cases} \partial_t f(t,x,v) + \partial_x \phi[f(t)](x)\partial_v f(t,x,v) = 0 \\ f(0,x,v) = f^0(x,v), \end{cases} \tag{5.2.2}$$

whose solution is explicitly given by $\varphi_P^t(f^0)(x,v) = f^0(x, v - tE[f^0](x))$, with $E[f^0](x) = \partial_x\phi[f^0](x)$
Starting from an initial data f^0, the Lie splitting that we consider is given by the iterative relation

$$f^{n+1} = \varphi_P^{\delta_t} \circ \varphi_t^{\delta_t}(f^n), \quad n \in \mathbb{N}, \tag{5.2.3}$$

where δ_t is the time step.
The functions $f^n(x,v)$ defined above are approximations of $f(t,x,v)$ at time $t = n\delta_t$ (see [39] or chapter 3).
The fully-discrete scheme involves in addition an interpolation procedure at each step, using cubic splines. More precisely, we confuse the space $\mathbb{T}\times\mathbb{R}$ with $[-\pi,\pi]\times[-3,3]$. This domain is then discretized into $N\times N$ uniform grid, which we shall write $(x_i,v_j)_{0\le i,j\le N-1}$. Let us set

$$\delta_x = \frac{2\pi}{N} \quad \text{and} \quad \delta_v = \frac{6}{N}.$$

We choose $N = 2^m$, with m large, in order to use the fast Fourier transform (FFT), and the reverse fast Fourier transform (IFFT).
Indeed, the computation of the electric field $E[f]$ is done by a spectral method. If, for a function on $\mathbb{T} \times \mathbb{R}$, we set

$$\rho_f(x) = \int_{\mathbb{R}} f(x,v)\mathrm{d}v,$$

then for $k = \pm 1$,

$$\hat{E}[f]_k = \frac{ik}{2}\hat{\rho}_{f_k}, \tag{5.2.4}$$

where $(\hat{g}_k)_{k\in\mathbb{Z}}$ is the Fourier transform of a function $g(x)$ on $\mathbb{T}$.
The practical implementation of the fully-discrete scheme associated with the Lie splitting (5.2.3) reads:

- Assume that f_N^n is given as a two-dimensional distribution on the grid. Interpolate $x \mapsto f_N^n(x,\cdot)$ at the points $(x_i - \delta_t v_i, \cdot)_{0\le i\le N-1}$, and let $f_N^{n+1/2}$ be the corresponding two-dimensional distribution.

- Compute

$$\rho_{f_N^{n+1/2}}(x_i) = \delta_v \sum_{j=0}^{N-1} f_N^{n+1/2}(x_i, v_j),$$

  and then, using (5.2.4),

$$\left(E[f_N^{n+1/2}](x_i)\right)_{0\le i\le N-1} = IFFT\left(\frac{ik_N}{2} FFT\left((\rho_{f_N^{n+1/2}}(x_i))_{0\le i\le N-1}\right)\right),$$

  where k_N represents the sequence of discrete Fourier modes.

- Interpolate then $v \mapsto f_N^{n+1/2}(\cdot, v)$ at the points $\left(\cdot, v_i - \delta_t E[f_N^{n+1/2}](x_i)\right)_{0\le i\le N-1}$, and let f_N^{n+1} be the corresponding two-dimensional distribution.

This defines a sequence $(f_N^n)_{n\in\mathbb{N}}$ of two-dimensional distributions, such that $f_N^{n+1}(x_i, v_j)$ is *a priori* an approximation of $f(n\delta_t, x_i, v_j)$, when $\delta_t \to 0$, and $N \to +\infty$.

5.2.2 Tests to study the damping rates

The result of Theorem 4.2.4 may be tested numerically by computing at each step of the time-loop the coefficients $\mathcal{C}[f_N^n]$ and $\mathcal{S}[f_N^n]$. More precisely, once f_N^n is constructed, then compute

$$\mathcal{C}[f_N^n] = \delta_x\delta_v \sum_{0\le i,j\le N-1} \cos(x_i) f_N^n(x_i, v_j)$$

and

$$\mathcal{S}[f_N^n] = \delta_x\delta_v \sum_{0\le i,j\le N-1} \sin(x_i) f_N^n(x_i, v_j).$$

Saving the results at each step, and denoting by n_f the number of iterations, we have then at our disposal the two sequences $(\mathcal{C}[f_N^n])_{0\leq n\leq n_f}$ and $(\mathcal{S}[f_N^n])_{0\leq n\leq n_f}$.
Once the stationary state $\eta(x,v)$ is chosen, we will have the two following examples of initial perturbations:

$$r^0(x,v) = \eta(x,v)\cos(x) \tag{5.2.5}$$

or

$$r^0(x,v) = \eta(x,v)\sin(x). \tag{5.2.6}$$

In the case of a cosine perturbation 5.2.5, we perform the following tests

- The sequence
$$\frac{|\mathcal{C}[f_N^n] - M_0|}{\varepsilon}.$$
should converge towards a non-zero constant as $n \to n_f$. This should illustrate the above discussion around estimate (5.1.14).

- The sequence
$$\left|\mathcal{C}[f_N^n] - \mathcal{C}[f_N^{n_f}]\right|$$
should decay towards zero at a rate $1/(n\delta_t)^3$, when n increases.

In the case of a sine perturbation 5.2.5, the tests are the following

- The sequence
$$|\mathcal{C}[f_N^n] - M_0|.$$
should converge towards zero as $n \to n_f$.

- The sequence
$$\left|\mathcal{C}[f_N^n] - \mathcal{C}[f_N^{n_f}]\right|$$
should decay towards zero at a rate $1/(n\delta_t)^3$, when n increases.

- The sequence
$$\left|\mathcal{S}[f_N^n] - \mathcal{S}[f_N^{n_f}]\right|$$
should decay towards zero at a rate $1/(n\delta_t)^2$, when n increases.

5.2.3 Some remarks on the numerical scheme

Doing some numerical simulations with the method described above, we quickly realized that the quantities $\mathcal{C}[f](t)$ and $\mathcal{S}[f](t)$ (minus their long time averages) are actually decaying as oscillatory damped waves, and that there exists a transitory regime before the expected decay in time may be observed. This means that, to confront with good accuracy the predicted rates $1/t^2$ and $1/t^3$, we must reach the long time regime, so that the effect of the oscillations becomes sufficiently weak. Moreover, and it will be illustrated below, the accuracy of the numerical damping rates depends on the resolution that we choose. In other words, the

larger we choose N, the more the numerical damping rates will be close to the predicted ones, $1/t^2$ and $1/t^3$.
As each step of the time-loop requires two interpolation procedures in space, choosing large values for both N and n_f results in very heavy computations to reach the long-time regime. To give the reader an idea of the cost: the scheme was first implemented using the *Python* language, with $N = 1024$, $\delta_t = 0.1$, and $n_f = 10^4$. Even after an optimization of the code, it took us about 90 minutes to reach an acceptable long-time regime.
To remedy these computational difficulties a practical solution would be to use parallelization methods, and take advantage of the available computer architecture. In particular we draw the reader's attention to the semi-Lagrangian library *SeLalib* [1], which we consulted, and where such methods can be found. The interpolation and advection procedures were already implemented, so our contribution was to add an HMF-Poisson solver, *ie* the discrete version of (5.2.4), and an initial data f^0 with a proper value for M_0.
Another solution would be to use Particle-In-Cell methods (PIC), and take advantage of its efficiency and simplicity. The idea is to "throw" the particles $(x_i, v_j)_{0\leq i,j\leq N-1}$ on the grid according the initial distribution function f^0, and follow them in time by solving the canonical equations of motions. More precisely:

- Assign to each particle (x_i, v_j) the initial weight $\omega_{i,j} = Cf^0(x_i, v_j)$, where C is determined by the normalization condition

$$\sum_{0\leq i,j\leq N-1} \omega_{i,j} = 1.$$

- The particles move then on the grid according to the canonical equations of motion for the HMF model:

$$\begin{cases} \dot{x_i}(t) = v_i(t) \\ \dot{v_i}(t) = M_y(t)\cos(x_i(t)) - M_x(t)\sin(x_i(t)), \end{cases} \tag{5.2.7}$$

  with

$$M_x(t) = \sum_{0\leq i,j\leq N-1} \omega_{i,j}\cos(x_i(t)) \quad \text{and} \quad M_y(t) = \sum_{0\leq i,j\leq N-1} \omega_{i,j}\sin(x_i(t)).$$

- Discretize in time the canonical equations of motion (5.2.7), using a symplectic integrator, such as the semi-implicit Euler method or the implicit midpoint method (see [45]).

- Here M_x and M_y play the role of $\mathcal{C}[f]$ and $\mathcal{S}[f]$, and it suffices then to follow the evolution of those quantities during the time-loop, to confront the theoretical predictions.

[1] http://selalib.gforge.inria.fr

- The distribution function $f(t,x,v)$ is approximated (at times $t = n\delta_t$) by a sum of Dirac masses:

$$f(t,x,v) \sim f_N(t,x,v) = \sum_{0\leq i,j\leq N-1} \omega_{i,j} \mathbb{1}_{x=x_i(t)}(x) \mathbb{1}_{v=v_j(t)}(v). \tag{5.2.8}$$

PIC methods allow one to follow the canonical evolution of the particles, instead of the evolution of the distribution function, but this is sufficient to compute M_x and M_y.
Amongst the main advantages, it is much more faster than the semi-Lagrangian scheme, since no interpolation procedure is needed. The numerical experiments seem however to show that, to confront Theorem 4.2.4 with comparable accuracies, PIC methods requires a higher resolution than the semi-Lagrangian scheme. One could argue that PIC methods are highly parallelizable, such that choosing high resolutions is possible, but then again, parallelized semi-Lagrangian methods are available as well (see [1] as previously).
In addition, as foretold, PIC methods do not give us the evolution of the (numerical) distribution function, contrary to the semi-Lagrangian scheme, although one could construct another approximation of the exact solution using (5.2.8).
Our first goal is of course to confront Theorem 4.2.4, but we also are interested in the long-time behavior of the distribution function, as an illustration of the discussion of the introduction on a possible nonlinear theory. For that reason, it seemed us preferable to chose the semi-Lagrangian scheme over PIC methods. Nonetheless, let us mention the paper [5], where the damping rates are verified numerically with a PIC method.

5.3 Confrontation of Theorem 4.2.4

In this section we confront numerically Theorem 4.2.4 using small perturbations of Maxwell-Boltzmann stationary states.

5.3.1 Inhomogeneous Maxwell-Boltzmann stationary states

As foretold the initial data will read (5.1.3). In this section, η will be a Maxwell-Boltzmann distribution

$$\eta(x,v) = \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)}.$$

Recall that M_0 must solve the self-consistent equation

$$M_0 = \int_{\mathbb{T}\times\mathbb{R}} \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)} \cos(x) \mathrm{d}x\mathrm{d}v. \tag{5.3.1}$$

In theory (see [29] or section 6 of chapter 4), it can be shown that this equation, with the normalization condition (5.1.6), has a positive solution M_0 if and only if $\beta > 2$. Otherwise $M_0 = 0$ is the only solution of the equation. We solve numerically (5.3.1) using dichotomy. The following figure shows the evolution of M_0 as a function of β :

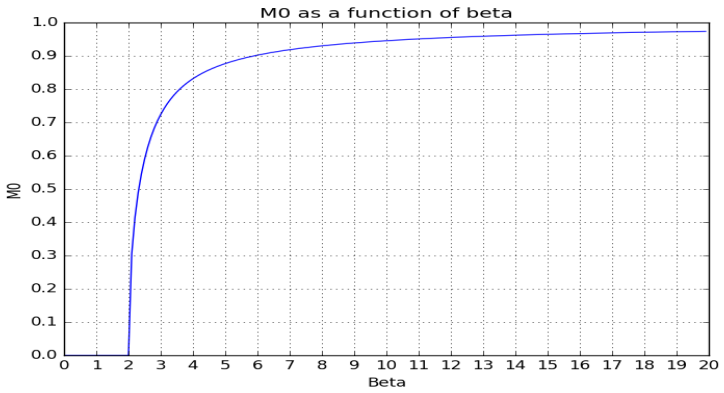


Figure 5.2: *Evolution of M_0 as a function of β. $\beta = 2$ is the critical value, below which M_0 is zero.*

We shall choose the same initial data as the authors of [5], to compare the present numerical experiments, obtained with the semi-Lagrangian scheme, with theirs, obtained with a PIC method. Our first choice of initial data is

$$f^0(x,v) = \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)}(1 + \varepsilon\cos(x)), \tag{5.3.2}$$

with $\varepsilon = 0.1$, $\beta = 10$, and M_0 as the numerical positive solution of (5.3.1) (we find $M_0 \sim 0.946$). α is a normalization factor to satisfy (5.1.6).
Next we will consider

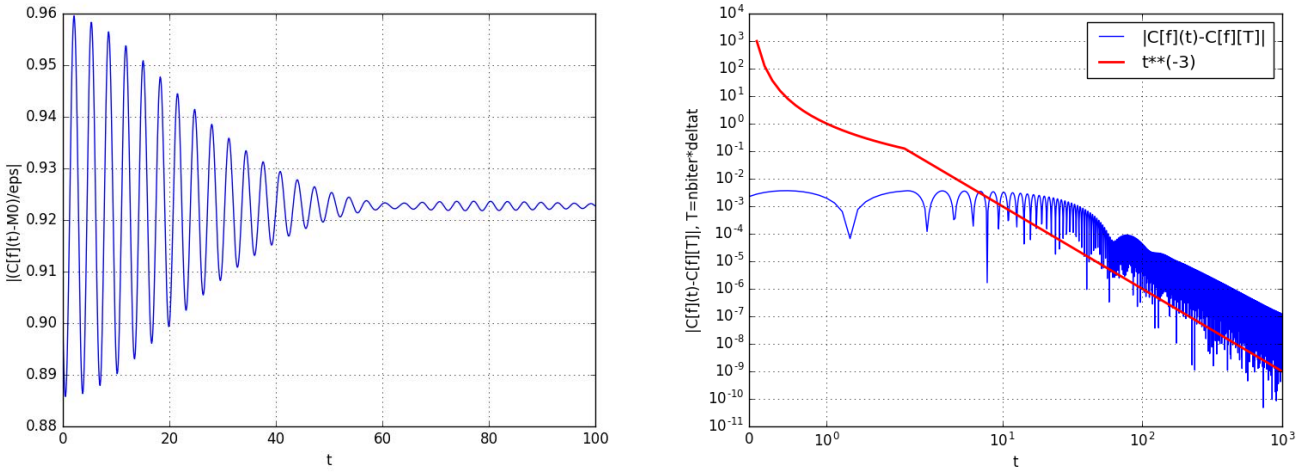
$$f^0(x,v) = \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)}(1 + \varepsilon\sin(x)), \tag{5.3.3}$$

with $\varepsilon = 0.1$, $\beta = 10$, and $M_0 \sim 0.946$ as previously, and α is still a normalization factor.

5.3.2 First numerical test

f^0 is given by (5.3.2), hence the symmetry in variable x shows that $\mathcal{S}[f](t) = 0$ for all t, and thus we only have to study the long time behavior of $\mathcal{C}[f](t)$. We choose $\delta_t = 0.1$, $n_f = 10000$, and $N = 2048$.

Plotted below are the evolution in time of $(\mathcal{C}[f](t) - M_0)/\varepsilon$ and $\mathcal{C}[f](t)$ minus its long time average:

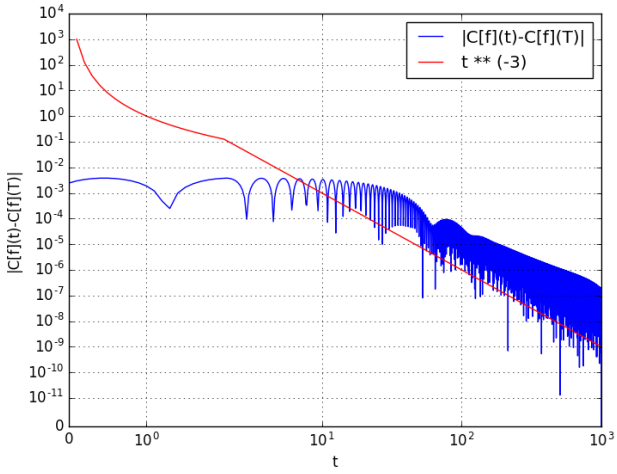


(a) *Evolution of* $(\mathcal{C}[f](t) - M_0)/\varepsilon$.

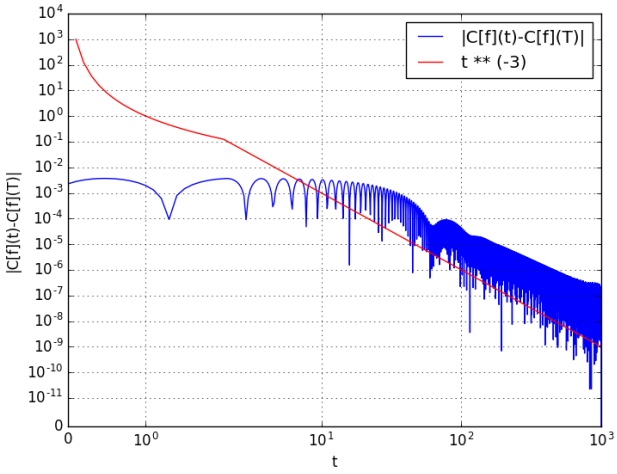
(b) *Evolution of* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$.

Figure 5.3: *Evolution of* $(\mathcal{C}[f](t) - M_0)/\varepsilon$ *and* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$, $T = n_f \times \delta_t$. *Comparison with expected decay (red curve).*

The above figure on the LHS shows that $\mathcal{C}[f](t)$ converges to $M_0 + \mathcal{O}(\varepsilon)$, as expected. To obtain the rate of convergence, we subtract from $\mathcal{C}[f](t)$ its long time average, and plot the corresponding evolution in the above RHS figure, with the expected decay t^{-3}.
The above plots seem to be in perfectly good accordance with our expectations, and also with the numerical experiment conducted in [5].
We draw the reader's attention to the choice of N. Numerically, we observed that the decay of $\mathcal{C}[f](t)$ towards its long time average seems to be related to N : the larger N is, the closer the decay is to the theoretical one, that is t^{-3}. For instance, the two figures below show the evolution of $\mathcal{C}[f](t) - \mathcal{C}[f](T)$ with $N = 512$ and $N = 1024$, with still $\delta_t = 0.1$:



(a) *Evolution of* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$, $N = 512$.



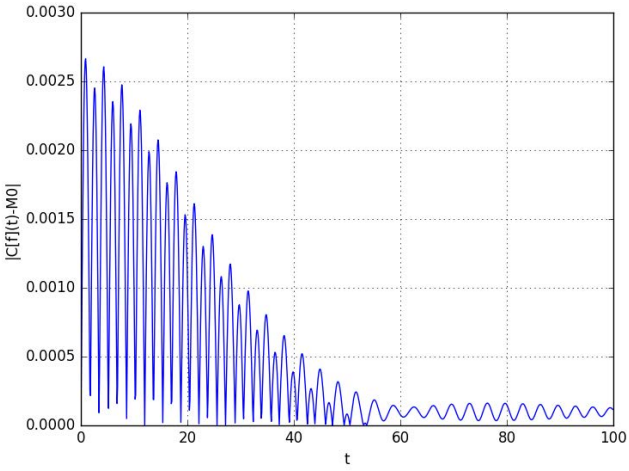
(b) *Evolution of* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$, $N = 1024$.

Figure 5.4: *Evolution of* $(\mathcal{C}[f](t) - M_0)/\varepsilon$ *and* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$, $T = n_f \times \delta_t$. *Comparison with expected decay (red curve).*

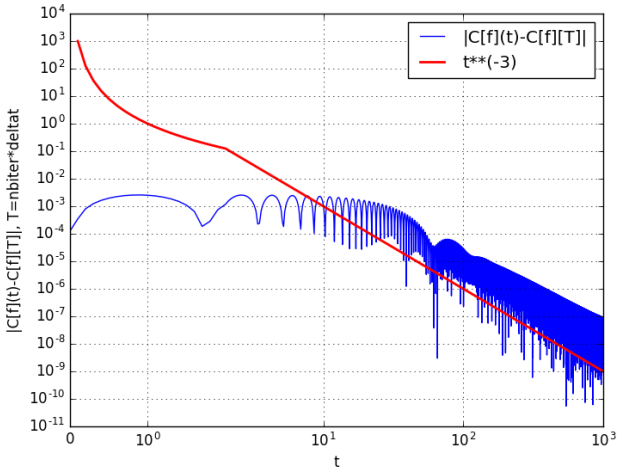
We also observed that choosing smaller values for δ_t does not improve significantly the accuracy of the present simulations, nor does changing the Lie splitting for higher order splitting methods (see [25]). One must moreover take into consideration the time and cost of the computations when choosing N, keeping in mind that we need to reach the long-time regime. As regards all these constraints, $N = 2048$ seemed to be an acceptable choice.

5.3.3 Second numerical test

f^0 is now given by (5.3.3). As previously we have $\delta_t = 0.1$, $n_f = 10000$, and $N = 2048$. To confront the theory, we first plot the evolution of $\mathcal{C}[f](t) - M_0$, and as previously, we subtract from $\mathcal{C}[f](t)$ its long time average, and plot the corresponding evolution:



(a) *Evolution of* $\mathcal{C}[f](t) - M_0$.



(b) *Evolution of* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$.

Figure 5.5: *Evolution of* $\mathcal{C}[f](t) - M_0$ *and* $\mathcal{C}[f](t) - \mathcal{C}[f](T)$, $T = n_f \times \delta_t$. *Comparison with expected decay.*

Next, we subtract also from $\mathcal{S}[f](t)$ its long time average, and plot the corresponding curve:

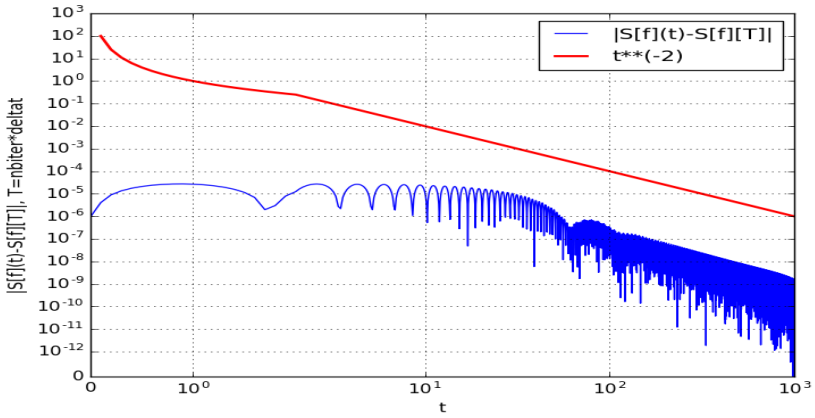


Figure 5.6: *Evolution of* $\mathcal{S}[f](t) - \mathcal{S}[f](T)$, $T = n_f \times \delta_t$. *The red curve is the expected decay* $1/t^2$.

As previously the simulations are in perfectly good accordance with our expectations (and also with the the conclusions of [5]).

5.4 Evolution of the distribution function - The case of Maxwell-Boltzmann Stationary States

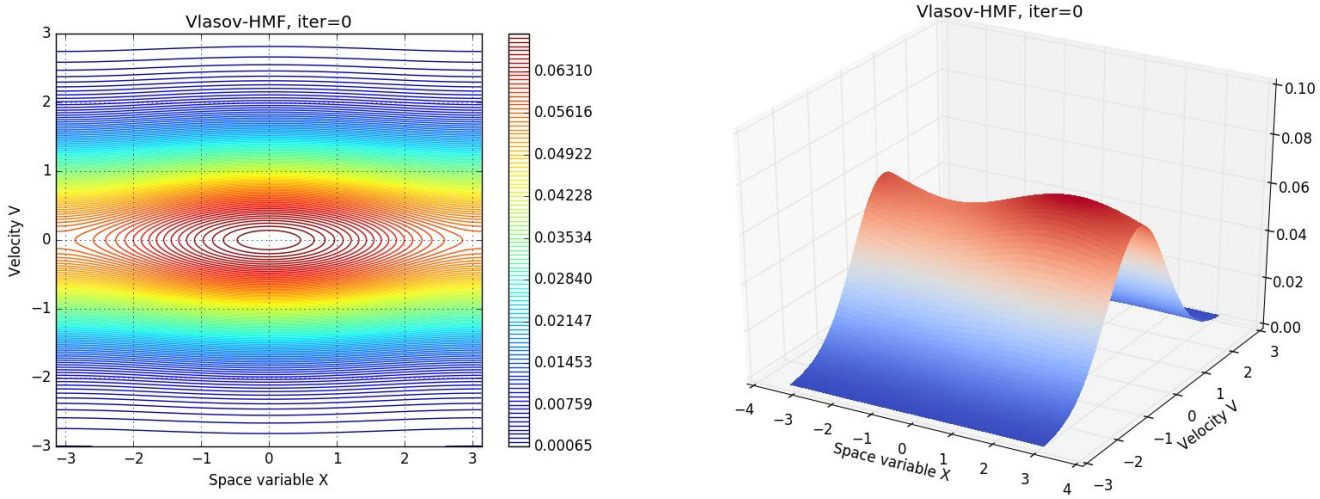
The aim of this section is to present numerical illustrations of the evolution in time of the distribution fonction f, starting from a small perturbation of a Maxwell-Boltzmann stationary state. In this entire section the values of the scheme parameters are $\delta_t = 0.1$ and $N = 1024$. We reduced the value of N for the purpose of time and cost of computations. Indeed, our goal is now to observe the long-time behavior of the distribution function, not to confront with high accuracy of the damping rates as previously, and in order to do so, the choice $N = 1024$ appeared perfectly acceptable. We perform 10000 iterations in time.

5.4.1 The homogeneous case

As a first example, and for the purpose of comparison with the inhomogeneous examples to come, we give an illustration of the situation in the homogeneous case. That is, the initial data is:

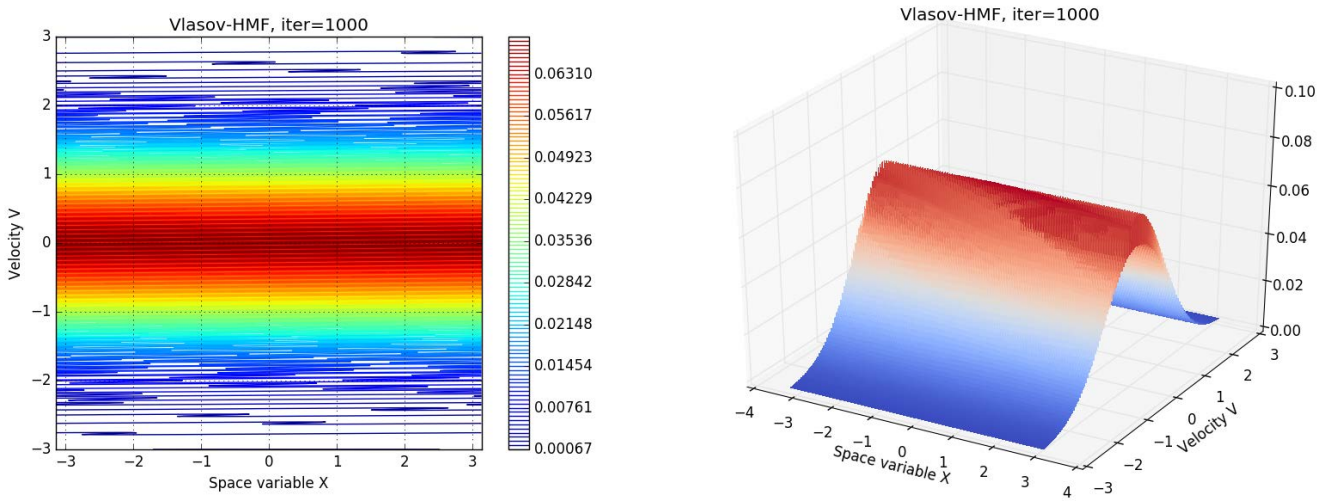
$$f^0(x, v) = \alpha e^{-\beta \frac{v^2}{2}} (1 + \varepsilon \cos(x)),$$

with $\varepsilon = 0.1$ and $\beta = 1$. With this choice of parameters the stability criterion is satisfied. Below we plot the evolution of the distribution function in contour lines and in $3D$ plot.



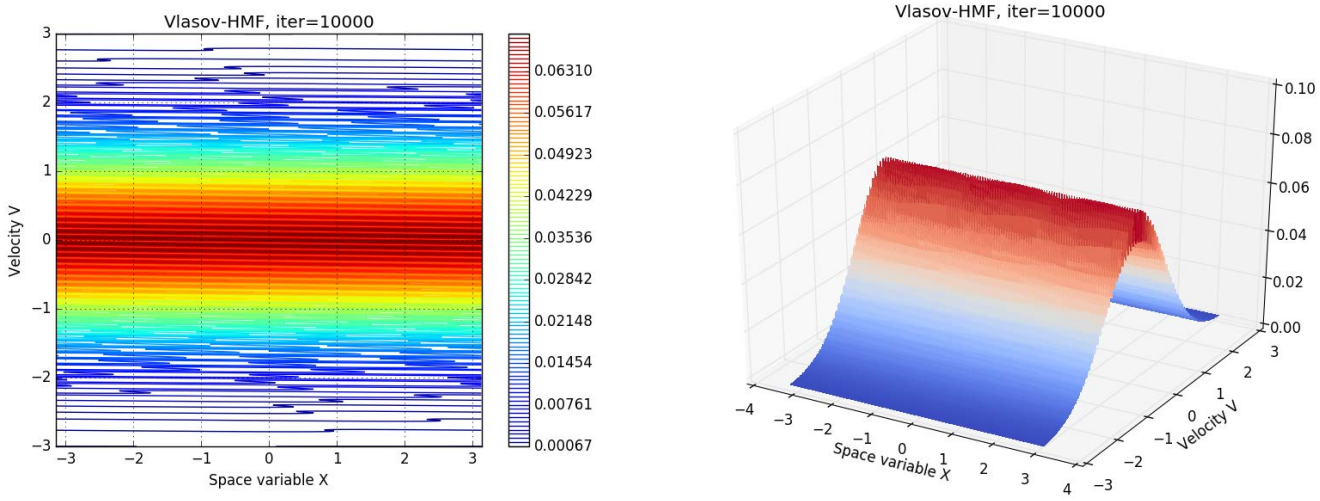
(a) *Contour lines.* (b) *3D plot*

Figure 5.7: *Initial distribution*



(a) *Contour lines.* (b) *3D plot*

Figure 5.8: *Distribution function after* 1000 *iterations in time.*



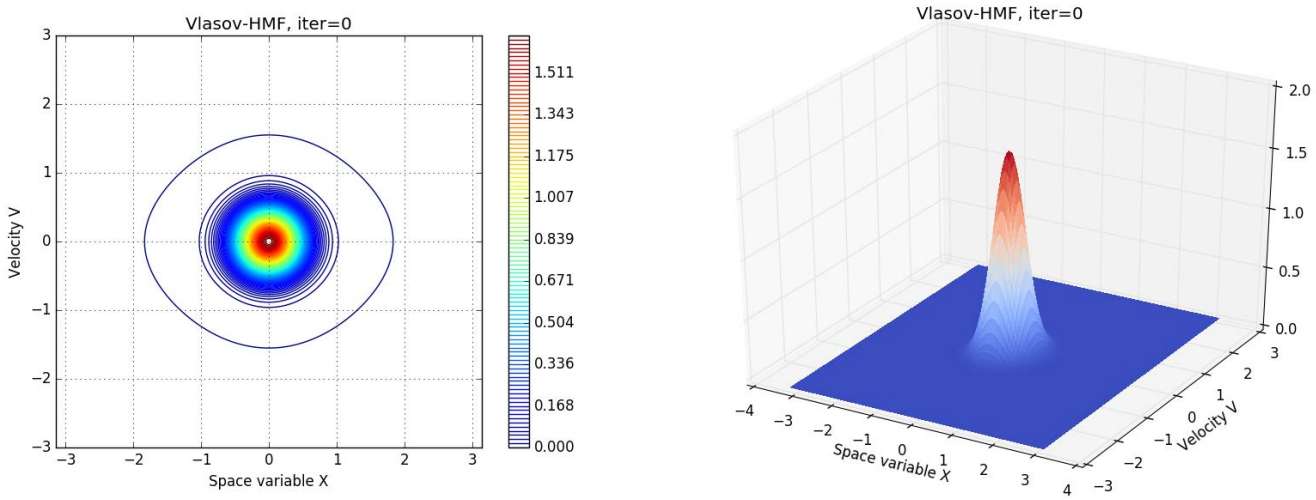
(a) *Contour lines.* (b) *3D plot*

Figure 5.9: *Distribution function after* 10000 *iterations in time.*

We observe indeed the Landau damping effect as proved in [40] : the distribution function exhibits a scattering behavior to a spatially homogeneous modified state, close to the stationary state in a regular norm. Moreover, since we started from a smooth initial data f^0, the weak convergence occurs at a high rate.

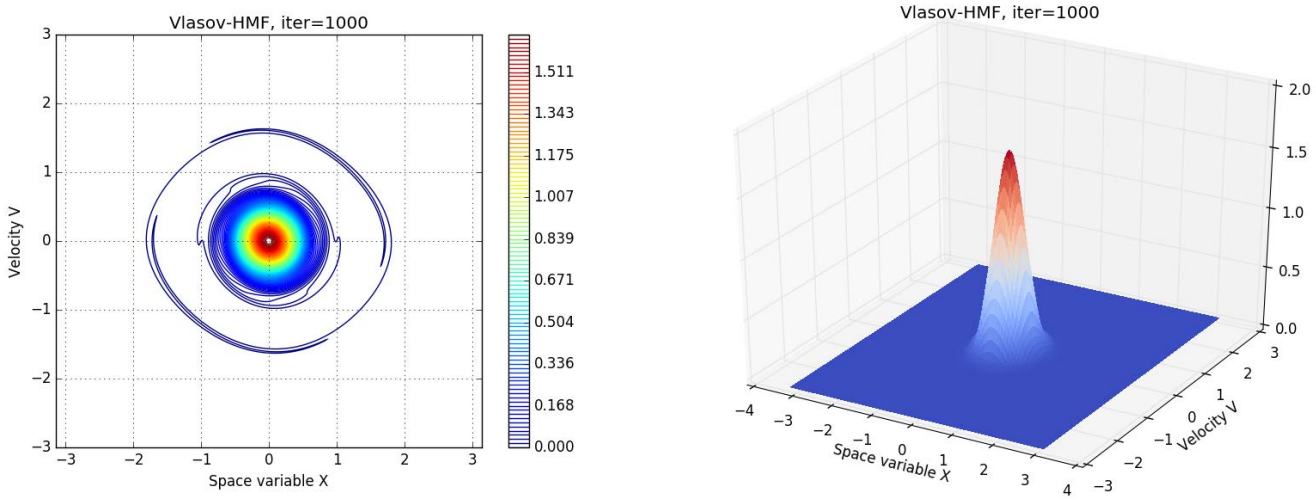
5.4.2 Inhomogeneous case - first example

Here we are interested in the long-time behavior of the distribution function associated with the initial data (5.3.2), for which we already observed the decay of the electric field. We draw the reader's attention to the fact that the mass of the initial data is highly concentrated in the area $U_\circ$, almost as if f^0 was compactly supported in it.
The evolution of the distribution function is plotted bellow:



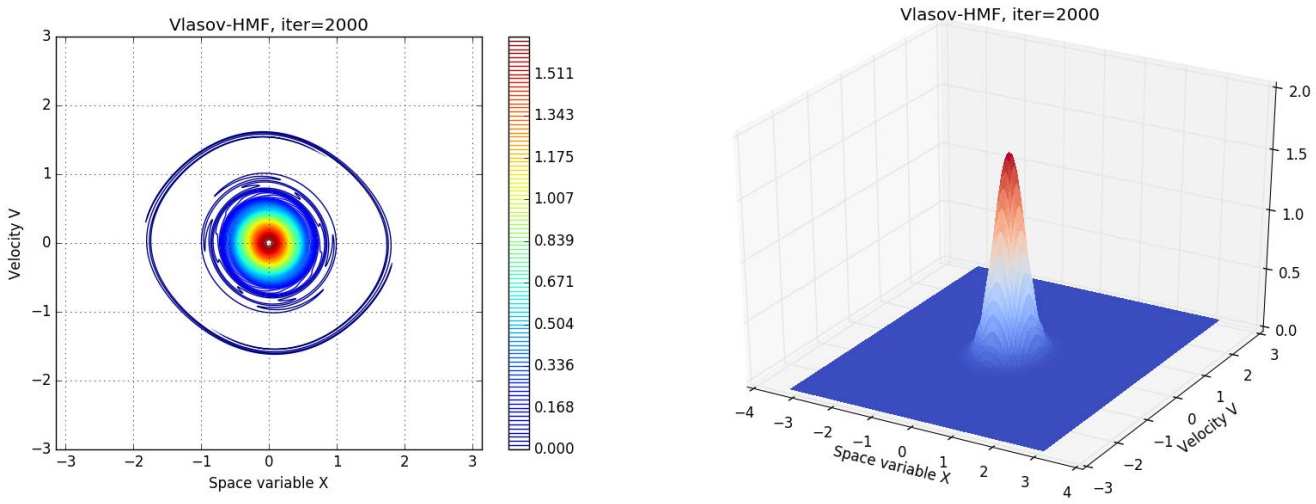
(a) *Contour lines.* (b) *3D plot*

Figure 5.10: *Initial distribution.*



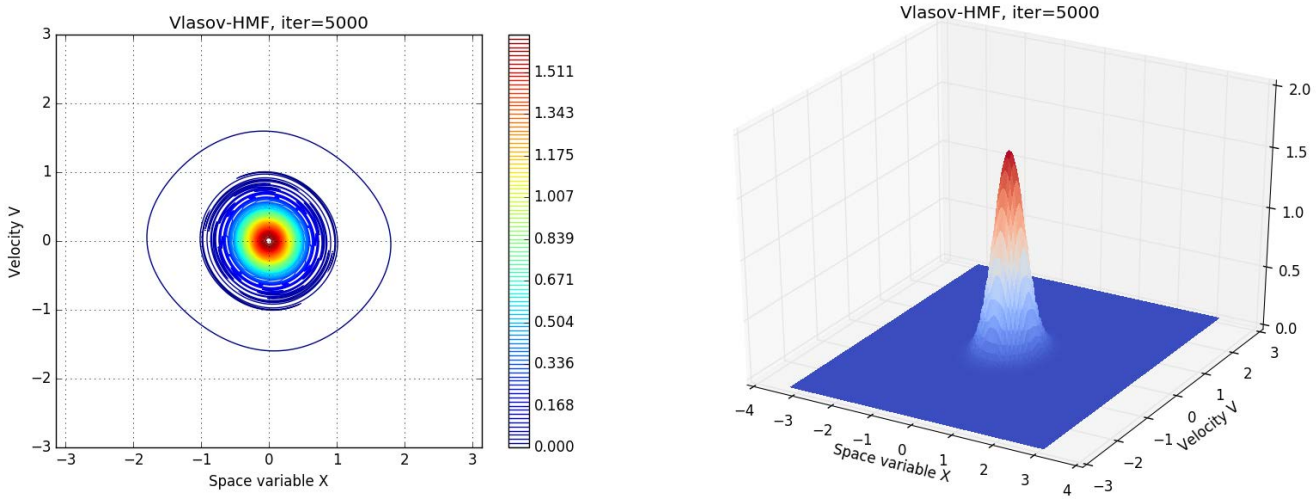
(a) *Contour lines.* (b) *3D plot*

Figure 5.11: *Distribution function after* 1000 *iterations in time.*



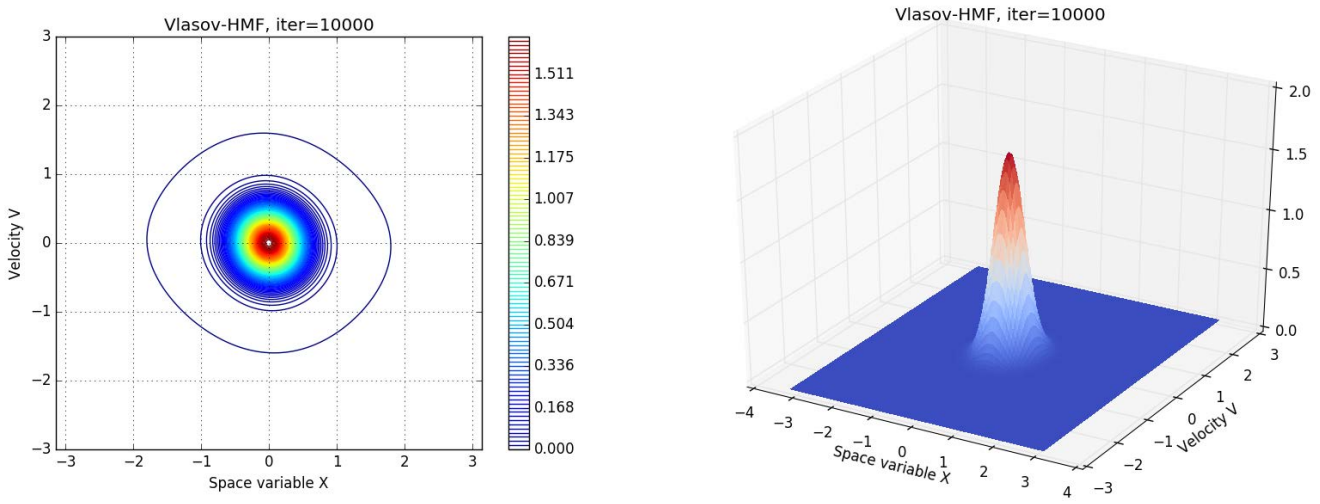
(a) *Contour lines.* (b) *3D plot*

Figure 5.12: *Distribution function after* 2000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.13: *Distribution function after* 5000 *iterations in time.*



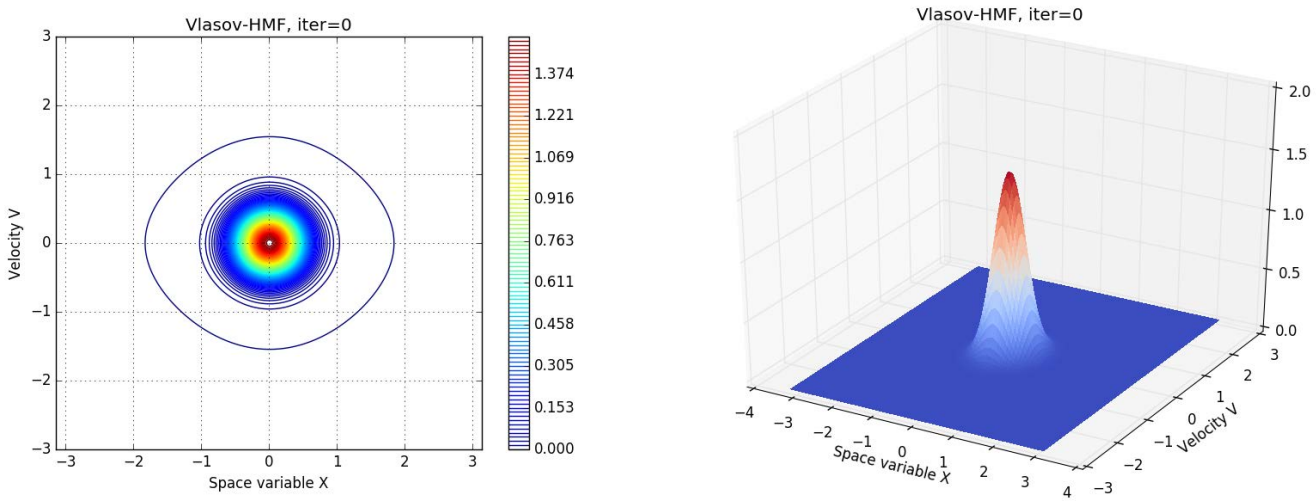
(a) *Contour lines.* (b) *3D plot*

Figure 5.14: *Distribution function after* 10000 *iterations in time.*

The first comment that we can make is that the present simulations seem to be in perfectly good accordance with (5.1.16) and (5.1.17).
Moreover it seems reasonable to think that the final state is bounded and close to the stationary state in some regular norm, so (5.1.16) and (5.1.17) may in this case hold in a regular norm.
This may be a confirmation of our analysis on the concentration of the distribution function in $U_\circ$. Of course, neither f nor f^0 are compactly supported in the eye, but perhaps their masses are sufficiently concentrated in it such that they behave as compactly supported functions, at least numerically. It is also possible that the extremely good regularity properties of the distribution function (which is a Maxwellian, essentially) overcome the singularities at the separatix.
Anyhow, it seems that the singularities of $\psi_t(x, v)$ near the separatix do not have a major influence on the behavior of f, and we observe a scattering behavior towards some small perturbation of $\eta(x, v)$, close in regular norm.

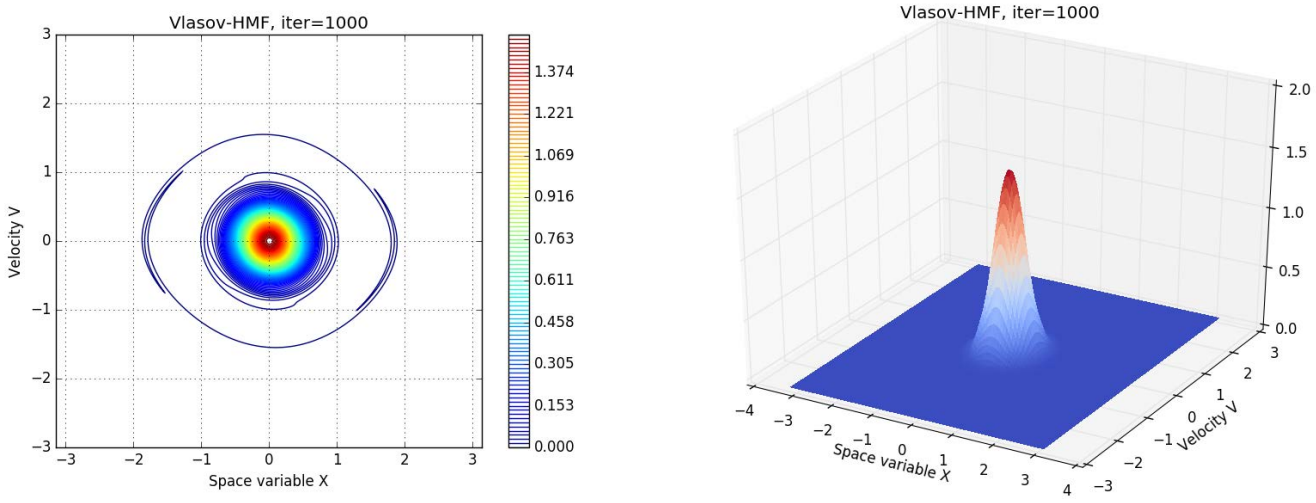
5.4.3 Inhomogeneous case - second example

We provide here the reader with illustrations of the long-time behavior of the distribution function associated with the initial data (5.3.3), plotted on the following figures:



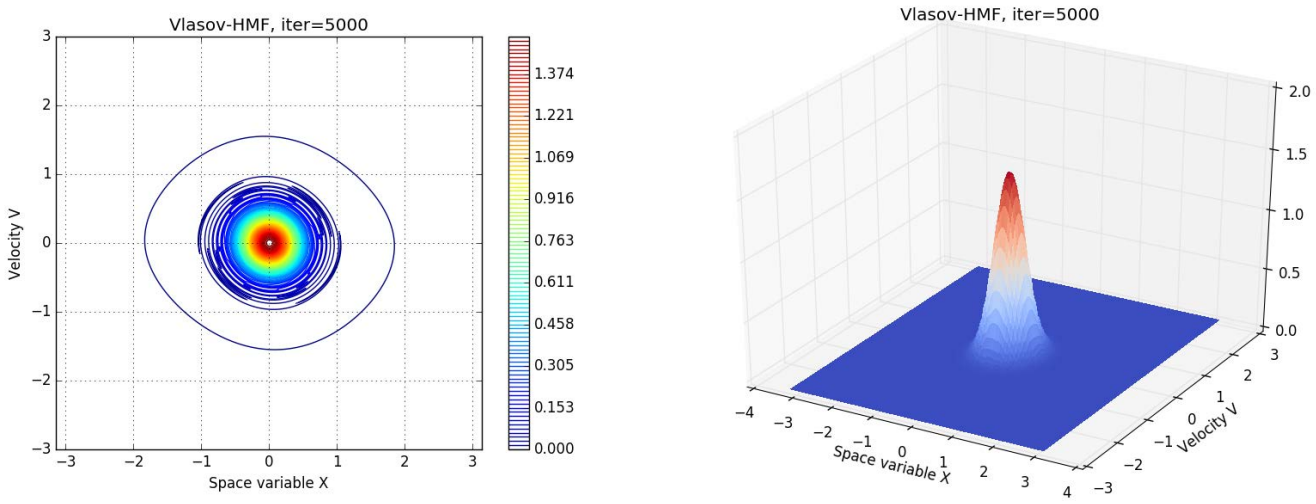
(a) *Contour lines.* (b) *3D plot*

Figure 5.15: *Initial distribution.*



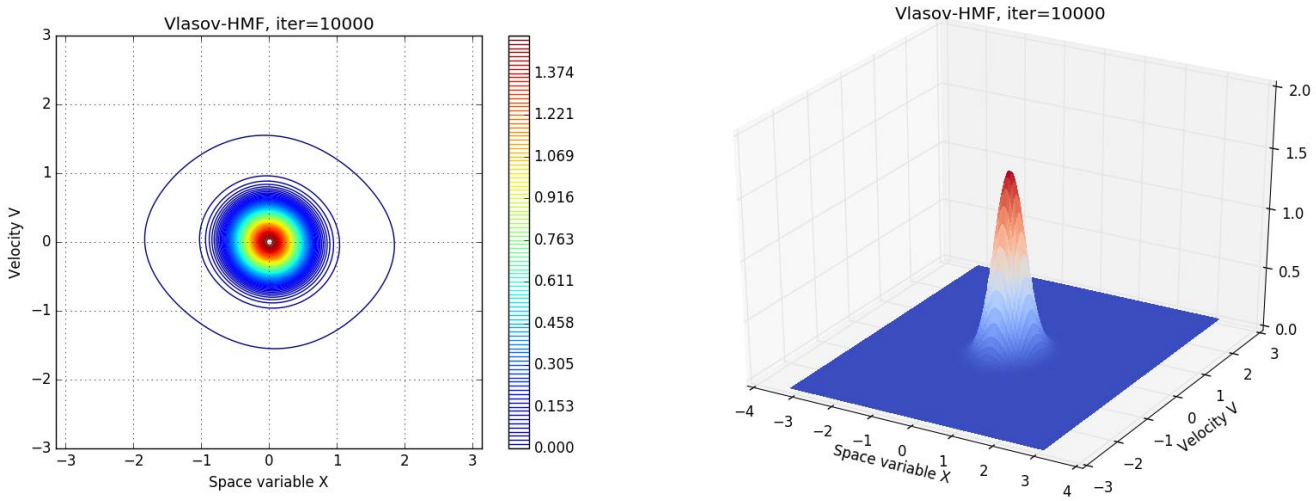
(a) *Contour lines.* (b) *3D plot*

Figure 5.16: *Distribution function after* 1000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.17: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.18: *Distribution function after* 10000 *iterations in time.*

Thus the situation is analogous to the previous example, and seems to confirm that in the case of Gaussian functions, the poor behavior of $\psi_t(x, v)$ near the separatix from does not have a major influence.

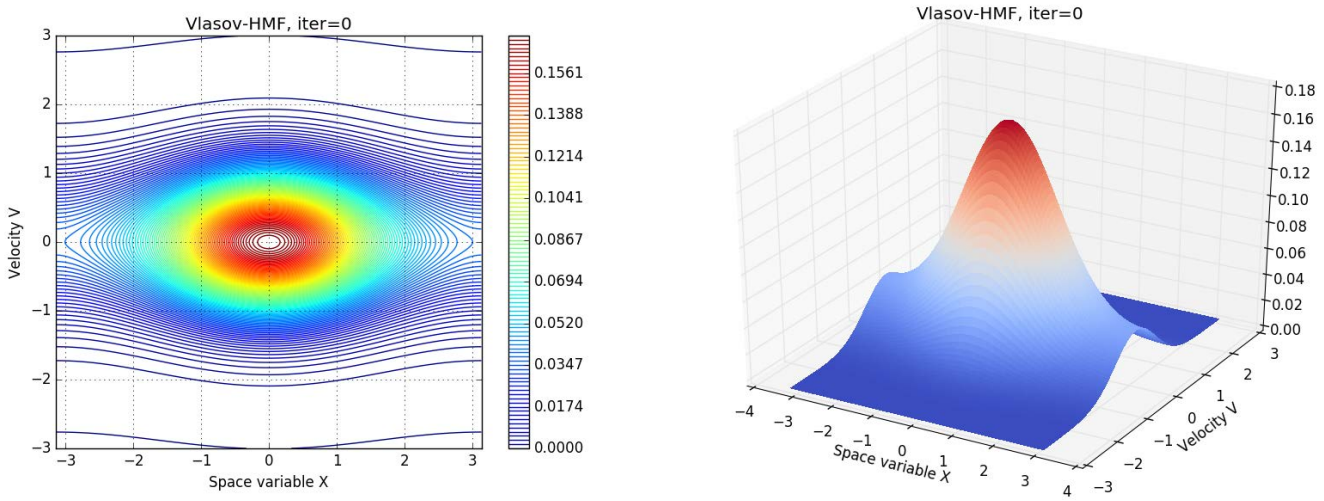
5.4.4 Inhomogeneous case: the case of a small magnetization

The two previous examples were similar in the sense that the initial distribution, because of the choice of the parameters, is highly concentrated in the eye $U_\circ$. As regards as our

above analysis on the effect that this might have on a potential Landau damping result, we wish to study now a more "spreaded" Maxwell-Boltzmann stationary state η, which has a non-negligible part of its mass on the neighborhood of the separatix. In order to do so, β must be chosen as small as possible, keeping in mind that it must be strictly larger than 2, so that M_0 can be chosen strictly positive. Here the initial data is then

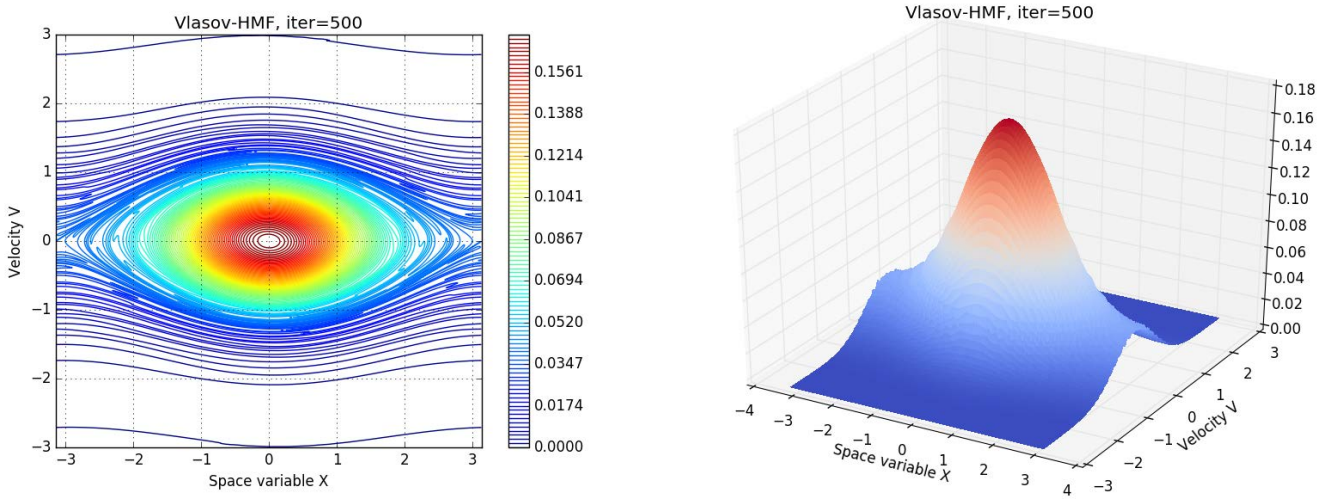
$$f^0(x,v) = \alpha e^{-\beta h_0(x,v)}(1+\varepsilon\cos(x)),$$

with $\beta = 2.1$, $\varepsilon = 0.1$, and $M_0 \sim 0.3$. α is still a normalization coefficient.



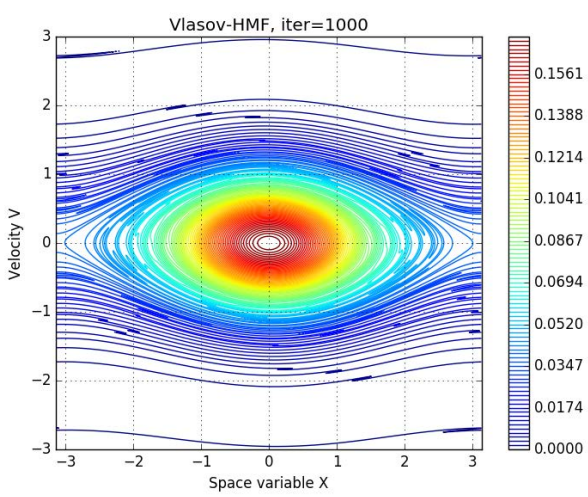
(a) *Contour lines.* (b) *3D plot*

Figure 5.19: *Initial data.*

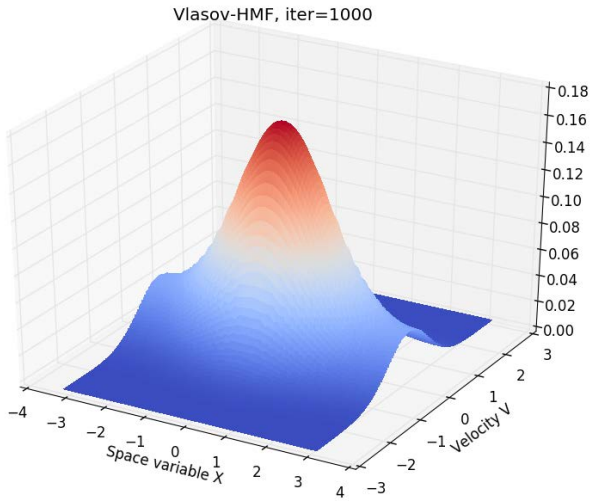


(a) *Contour lines.* (b) *3D plot*

Figure 5.20: *Distribution function after* 500 *iterations in time.*

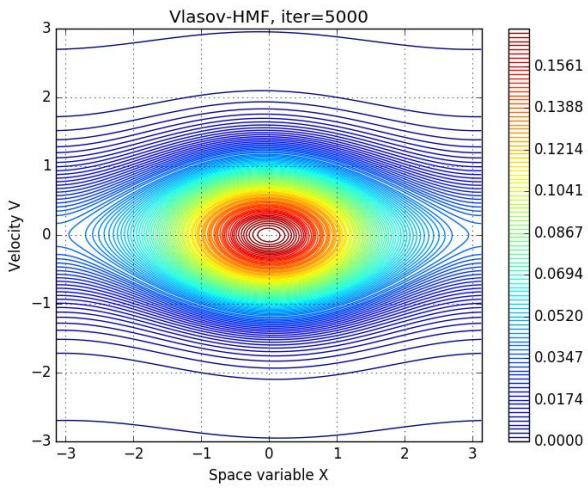


(a) *Contour lines.*

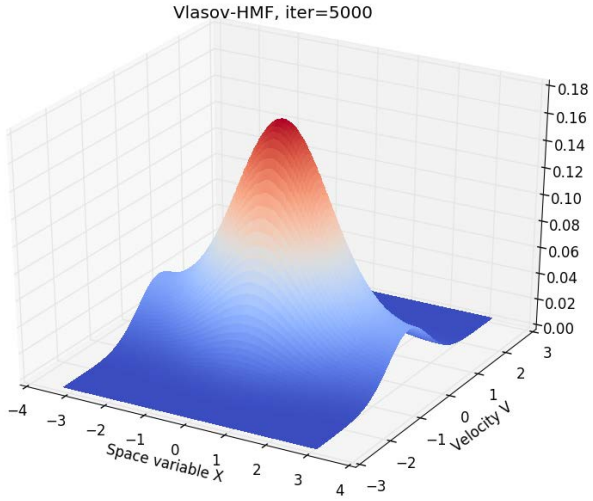


(b) *3D plot*

Figure 5.21: *Distribution function after* 1000 *iterations in time.*

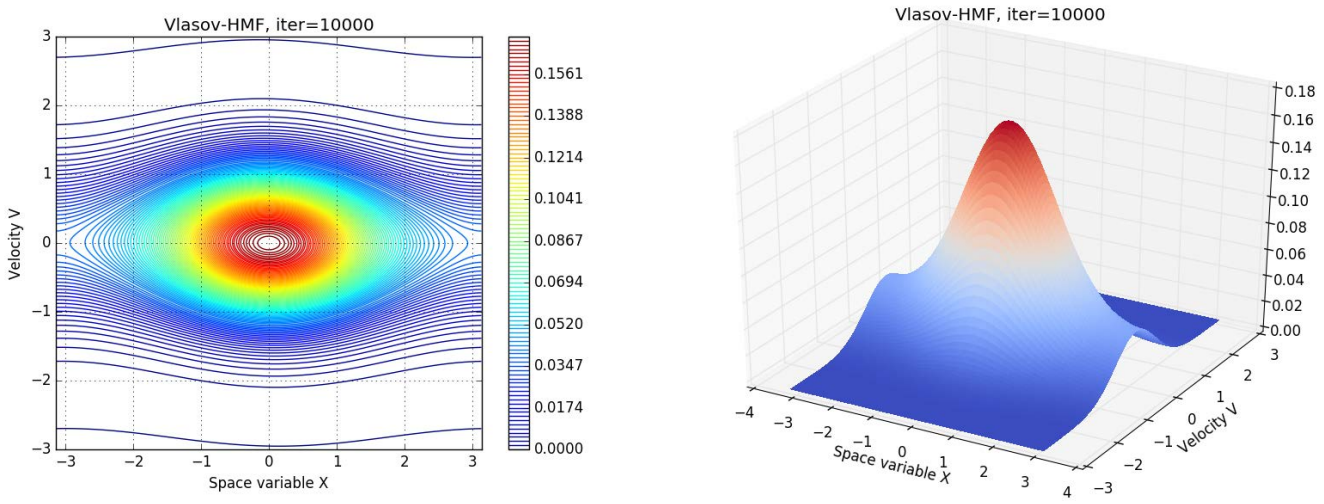


(a) *Contour lines.*



(b) *3D plot*

Figure 5.22: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.23: *Distribution function after* 10000 *iterations in time.*

We observe that the situation is similar to the previous examples: the distribution function seems to scatter to a modified state which appears to be close in a regular norm. Although the distribution function clearly interferes with the separatix, it is possible that its extremely good regularity properties of the distribution function overcome the singularities of the flow $\psi_t(x, v)$. Anyhow, the examples of this section seems to show a Gaussian stationary state is an ideal candidate to a nonlinear Landau damping Theorem.

5.5 Evolution of the distribution function - The case of Lynden-Bell distributions

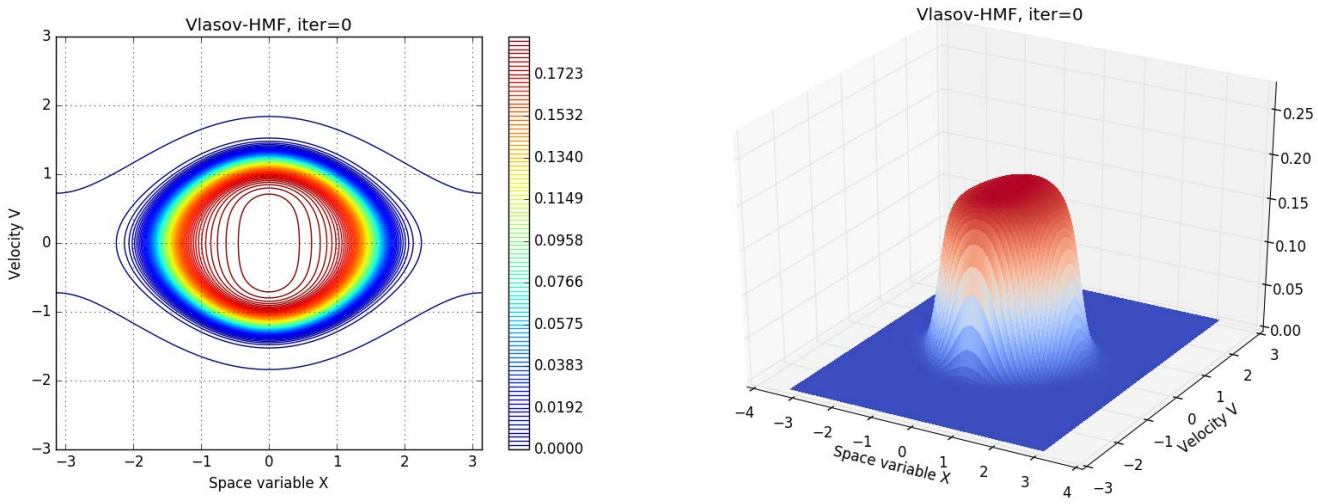
In this section we present illustrations of the situation where the distribution function starts from an initial data of the kind (5.1.3), with η a Lynden-Bell distribution. The parameters are still $\delta_t = 0.1$ and $N = 1024$, and we perform 10000 iterations in time.

5.5.1 Lynden-Bell distributions - first example

We pick here

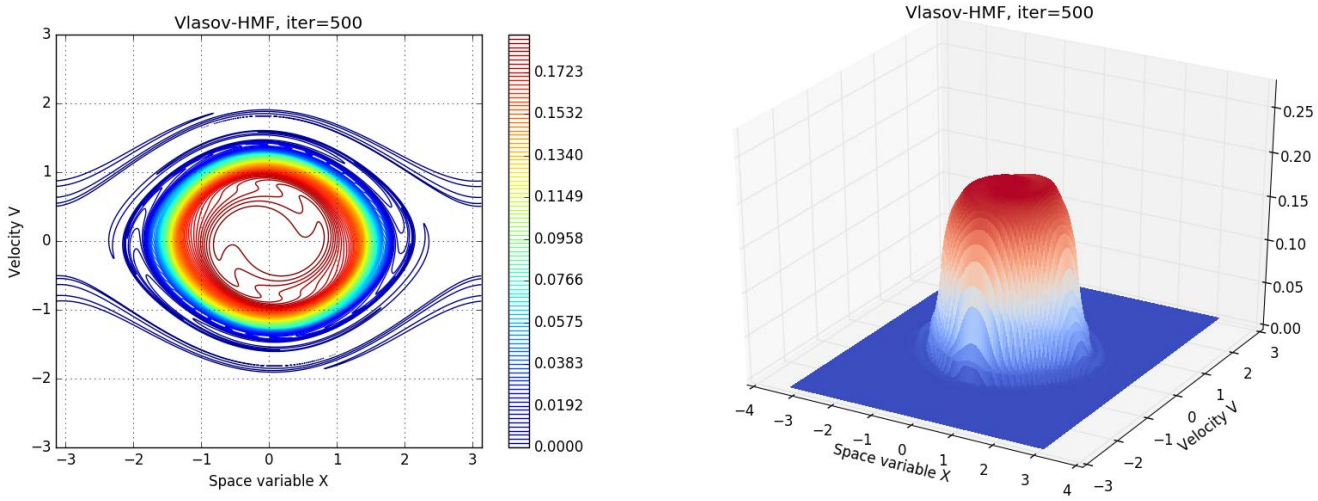
$$f^0(x, v) = \frac{\alpha}{1 + e^{\beta h_0(x,v)}}(1 + \varepsilon \cos(x)),$$

with $\varepsilon = 0.1$, $\beta = 10$, $M_0 \sim 0.7$, and where α is obtained by normalization.



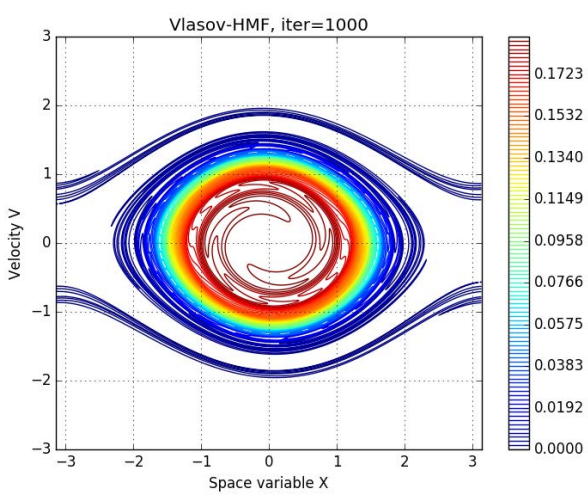
(a) *Contour lines.* (b) *3D plot*

Figure 5.24: *Initial data.*

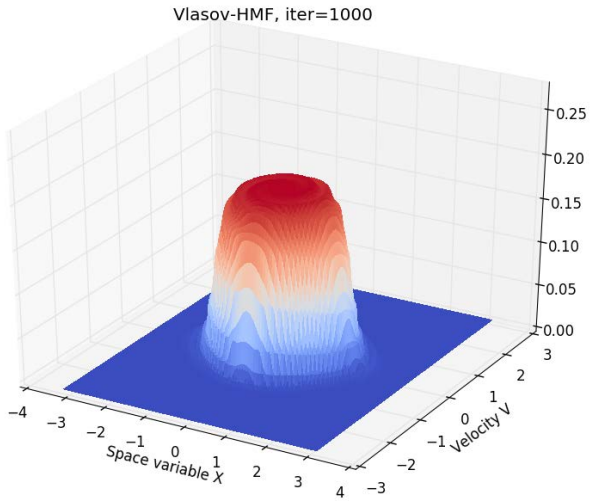


(a) *Contour lines.* (b) *3D plot*

Figure 5.25: *Distribution function after* 500 *iterations in time.*

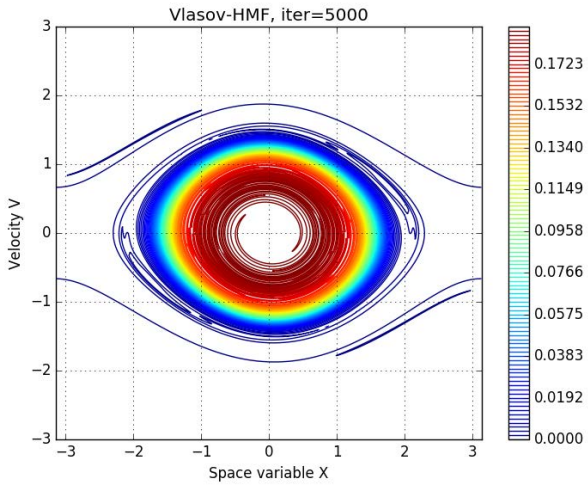


(a) *Contour lines.*

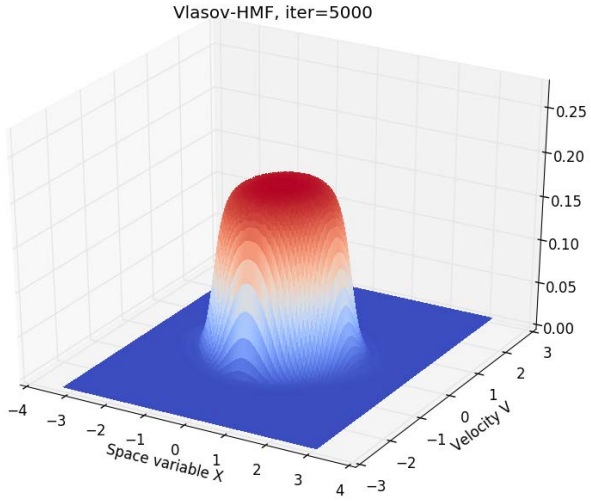


(b) *3D plot*

Figure 5.26: *Distribution function after* 1000 *iterations in time.*

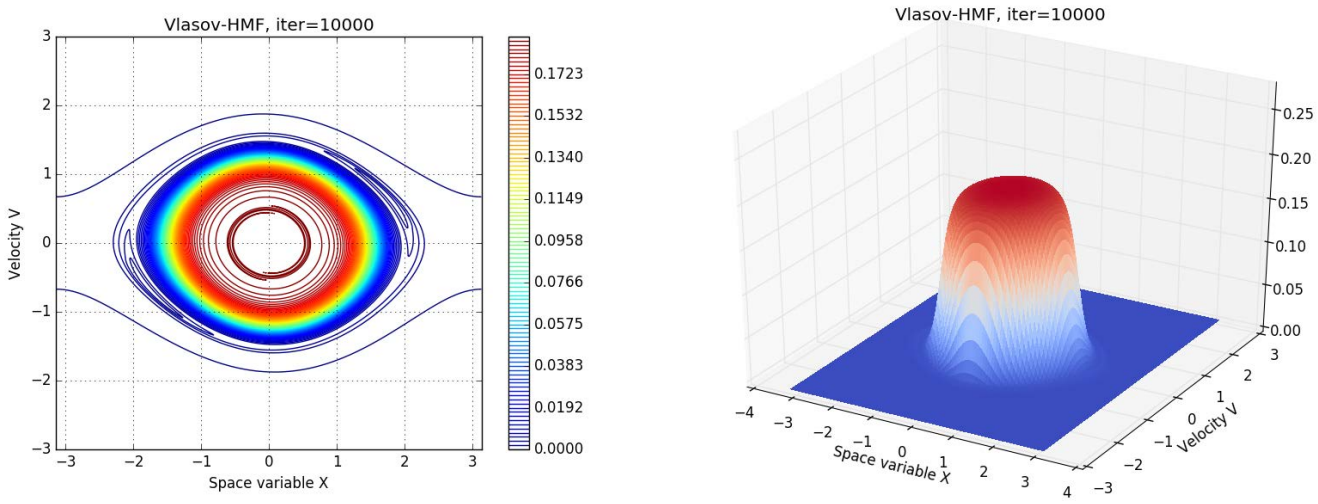


(a) *Contour lines.*



(b) *3D plot*

Figure 5.27: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.28: *Distribution function after* 10000 *iterations in time.*

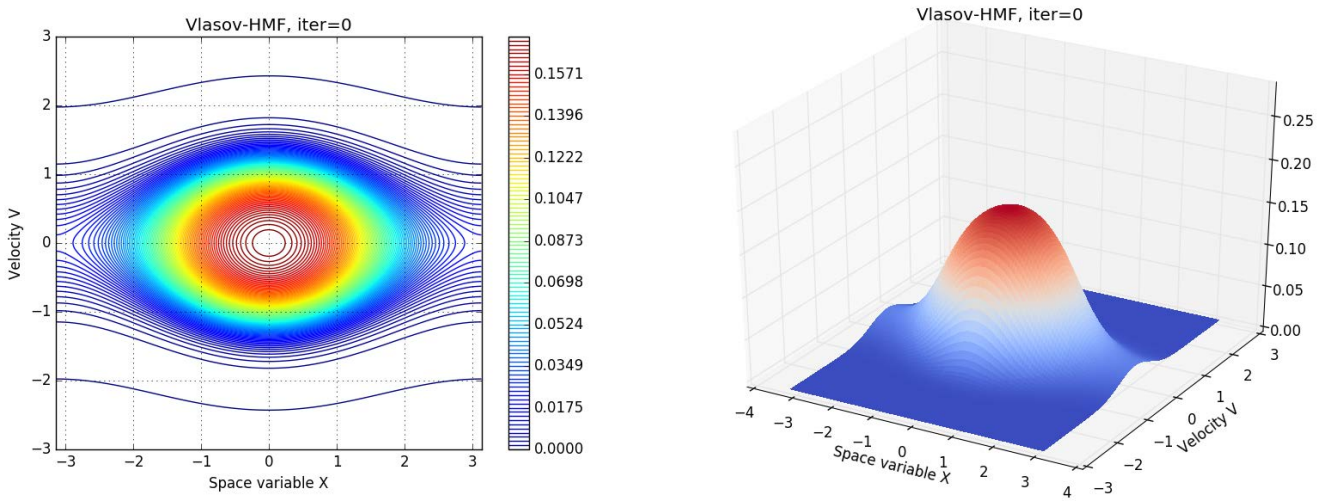
This example shows the possible obstacle to nonlinear Landau damping that we did expect: singularities at the separatix seem to show that the final distribution is not close in regular norm, though it is in L^1 norm.

5.5.2 Lynden-Bell distributions - second example

We pick here

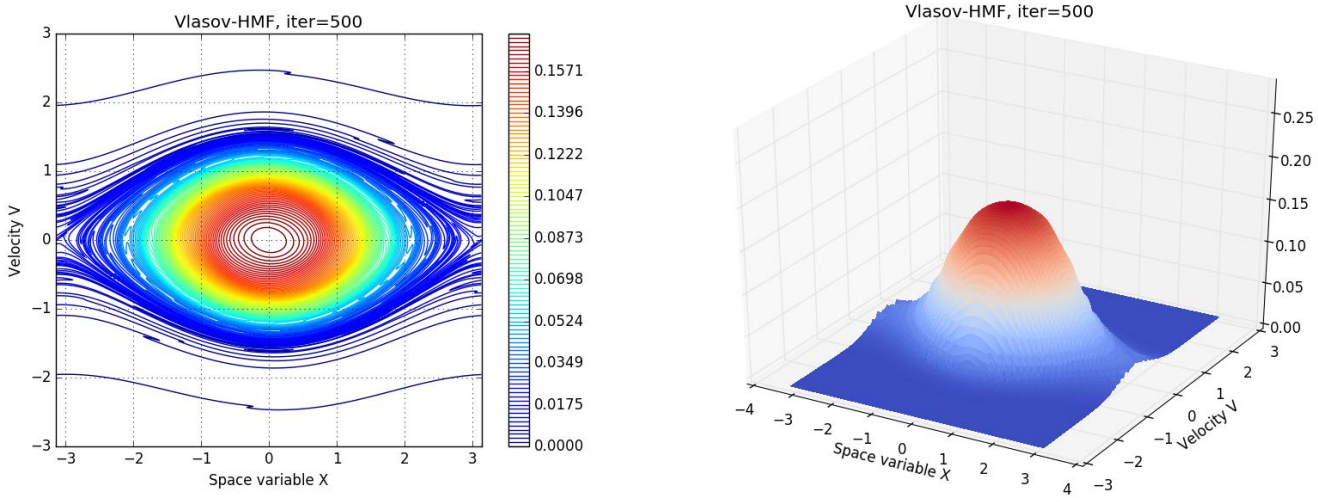
$$f^0(x,v) = \frac{\alpha}{1+e^{\beta h_0(x,v)}}(1+\varepsilon\cos(x)),$$

with $\varepsilon = 0.1$, $\beta = 4$, $M_0 \sim 0.47$, and where α is obtained by normalization.



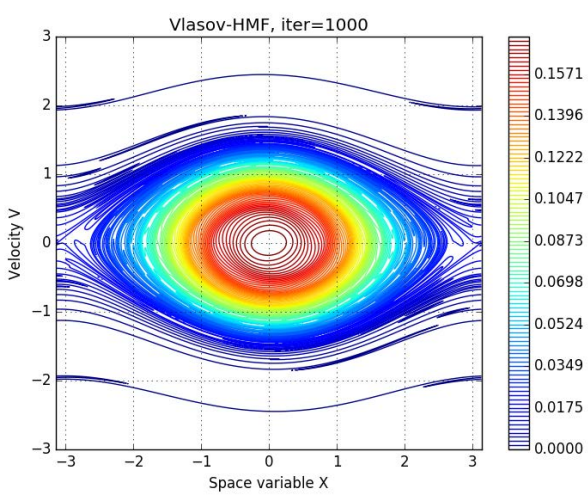
(a) *Contour lines.* (b) *3D plot*

Figure 5.29: *Initial data.*

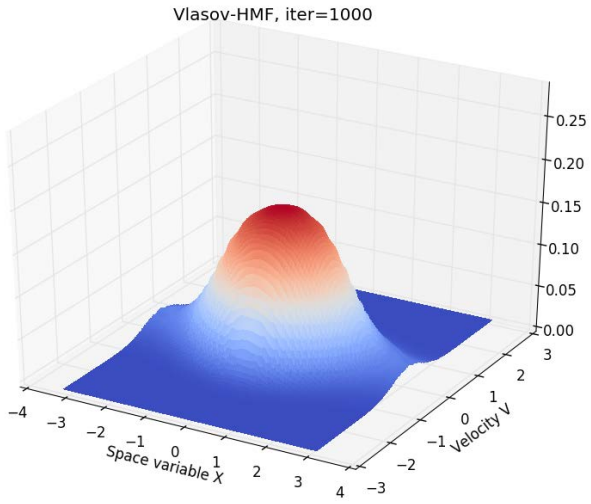


(a) *Contour lines.* (b) *3D plot*

Figure 5.30: *Distribution function after* 500 *iterations in time.*

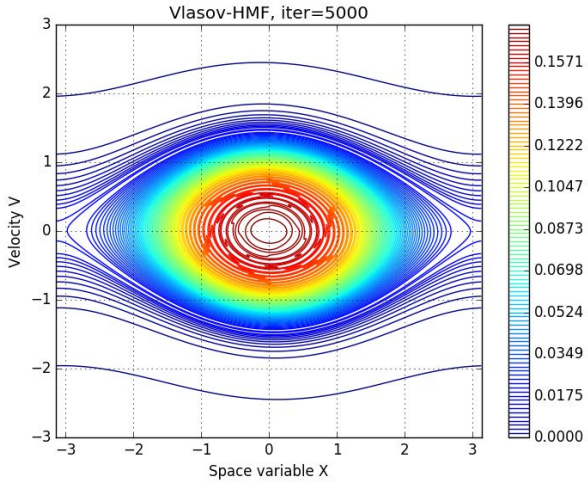


(a) *Contour lines.*

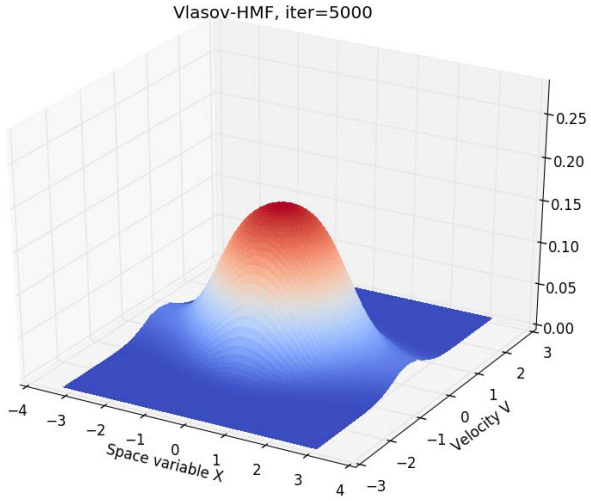


(b) *3D plot*

Figure 5.31: *Distribution function after* 1000 *iterations in time.*

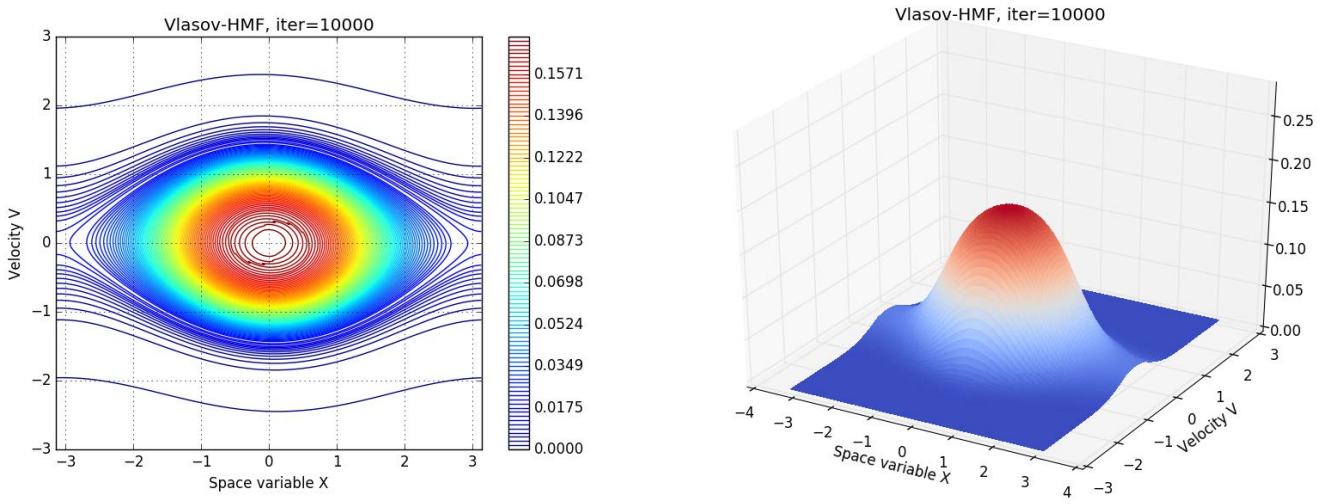


(a) *Contour lines.*



(b) *3D plot*

Figure 5.32: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.33: *Distribution function after* 10000 *iterations in time.*

The situation seems more comfortable here, and the behavior of $f(t)$ is comparable to the case of Maxwell-Boltzmann distributions, and it is thus possible that we have nonlinear Landau damping.

5.6 Evolution of the distribution function - The case Compactly Supported Stationary States

This section is dedicated to the confrontation of the discussion of the introduction on compactly supported solutions.
We will first consider an initial data of the kind (5.1.3), with η a compactly supported Tsallis distribution (see (5.1.9)), and we will choose the support properly. These states have of course singularities (when $h_0(x,v) \sim E$), so nonlinear Landau damping should not be expected. However, they are interesting in the sense that to have any hope of proving nonlinear Landau damping for compactly supported distributions, as discussed in the introduction, we should observe, somehow, a preservation property of the support. That is our expectation for these Tsallis distributions, which are moreover orbitally stable (see [53]).
In a second time we shall consider initial data of the kind (5.1.3), with η a compactly supported in $U_\circ$ and smooth, and we expect to observe nonlinear Landau damping.
In this section we have as previously $\delta_t = 0.1$ and $N = 1024$.

5.6.1 The case where Supp$(\eta) \subset U_\circ$

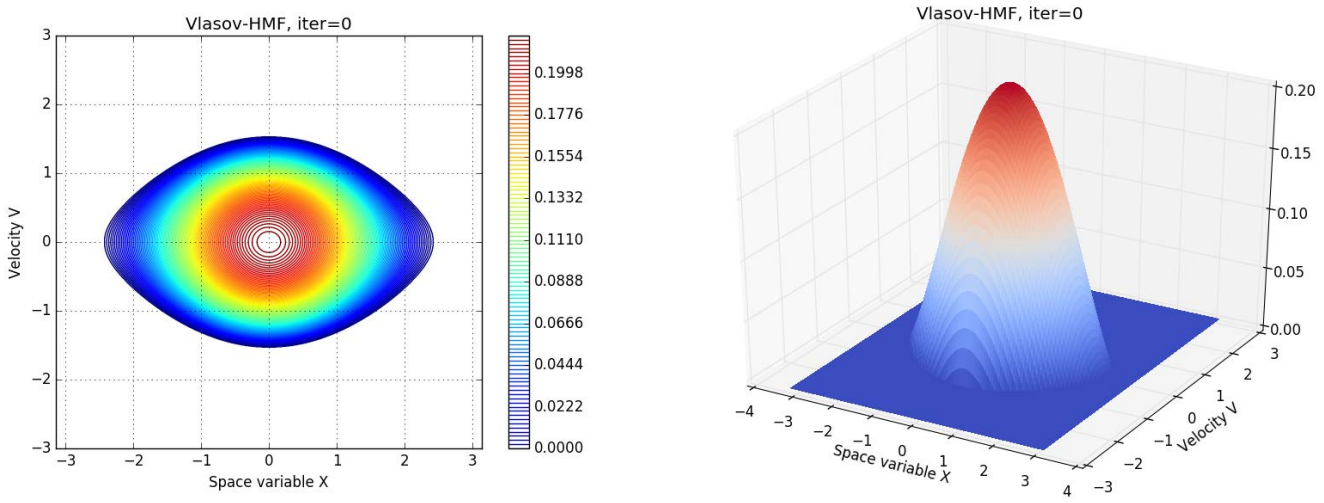
We begin with the case where the initial data is

$$f^0(x,v) = \eta(x,v)(1 + \varepsilon \cos(x)),$$

where

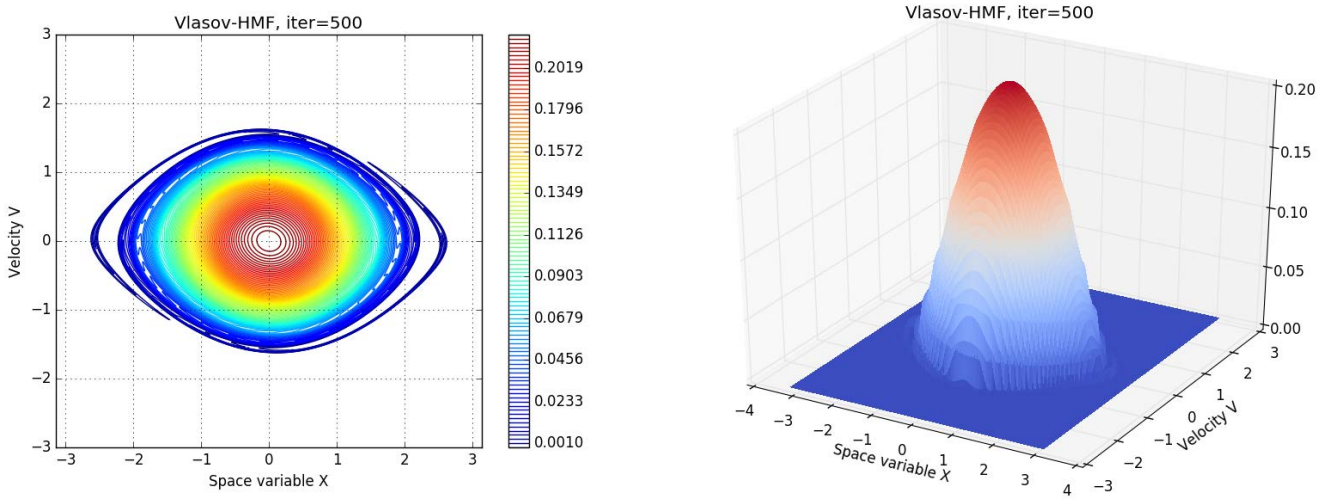
$$\eta(x,v) = \alpha\left(E - h_0(x,v)\right)_+ ,$$

with $\varepsilon = 0.1$, $E = 0.5$ and $M_0 \sim 0.6643$. Therefore, η and f^0 have compact support in $U_\circ$. Hence the support of the distribution does not intersect the separatix, and our goal is to see if this continues to be true in the long time.



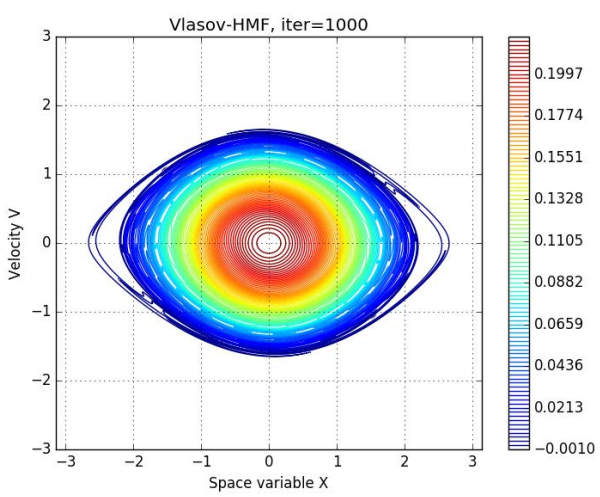
(a) *Contour lines.* (b) *3D plot*

Figure 5.34: *Initial data.*

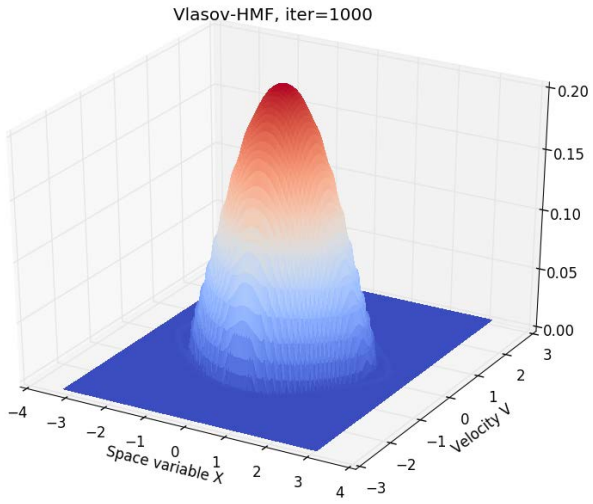


(a) *Contour lines.* (b) *3D plot*

Figure 5.35: *Distribution function after* 500 *iterations in time.*

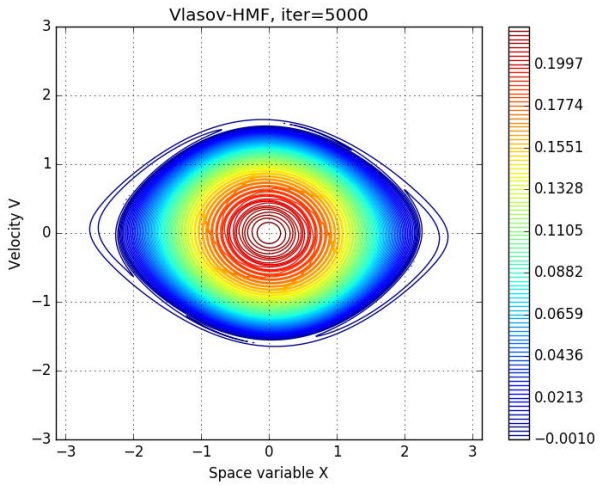


(a) *Contour lines.*

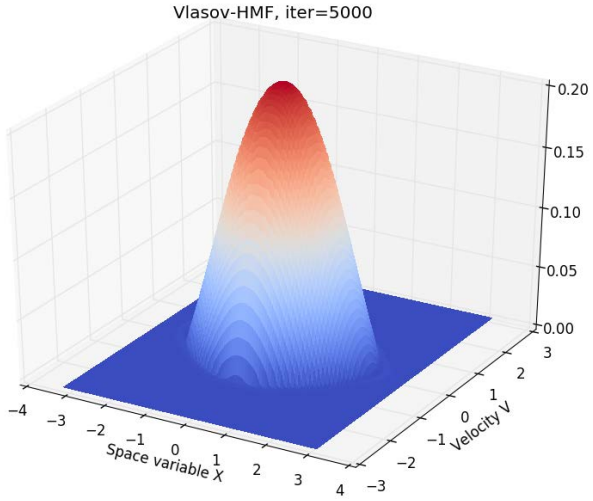


(b) *3D plot*

Figure 5.36: *Distribution function after* 1000 *iterations in time.*

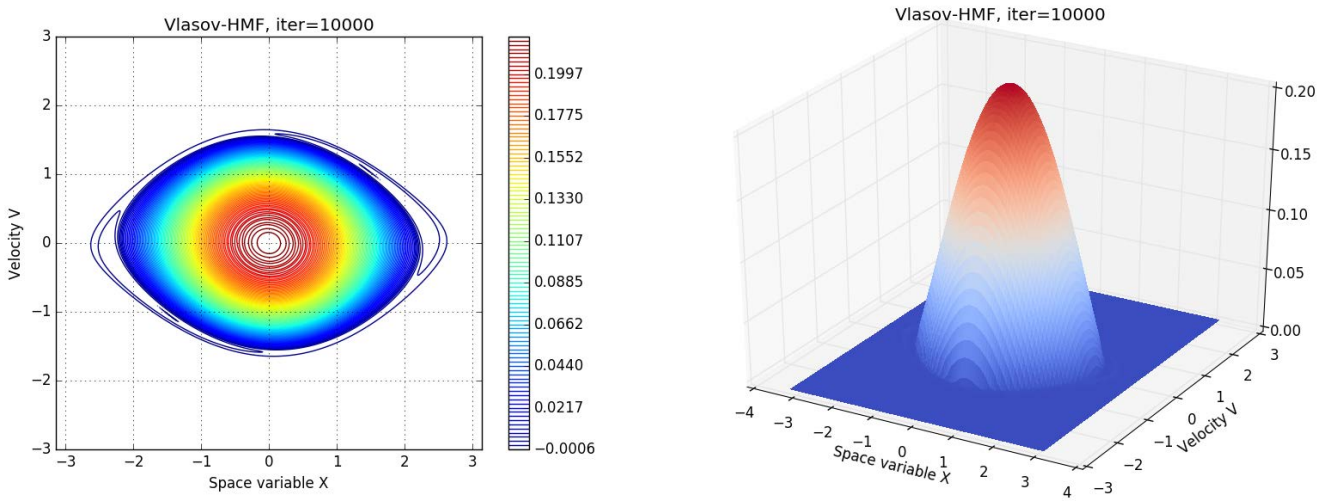


(a) *Contour lines.*



(b) *3D plot*

Figure 5.37: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.38: *Distribution function after* 10000 *iterations in time.*

We observe that the support of f seems to stay included in $U_\circ$ at all times. The final state is close to the stationary state in L^1 norm, and as expected we do not have nonlinear Landau damping due to singularities at the boundary of the support.

5.6.2 The case where Supp$(\eta) = U_\circ$

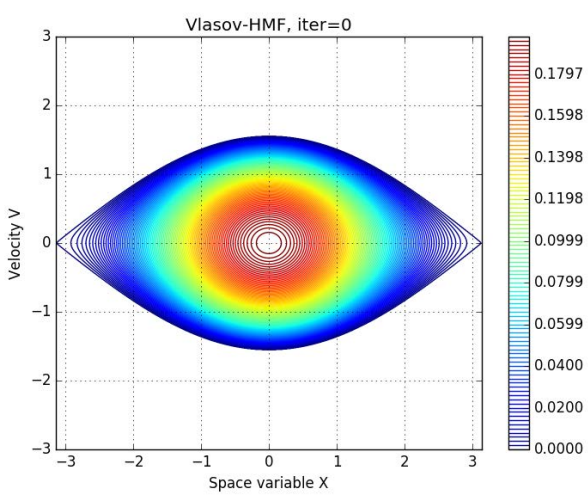
The second example is the initial data is

$$f^0(x,v) = \eta(x,v)(1+\varepsilon\cos(x)),$$

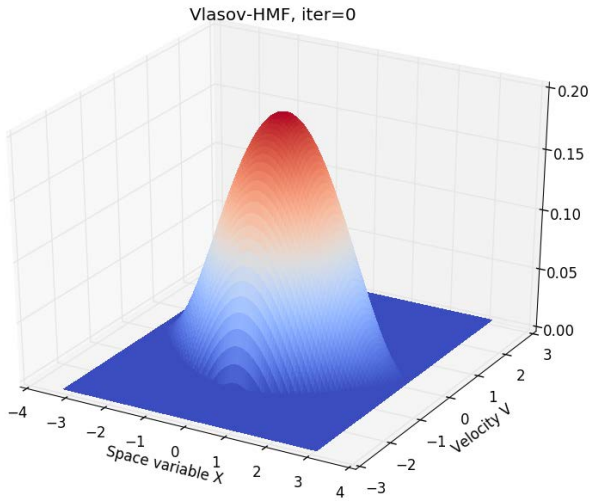
where

$$\eta(x,v) = \alpha\left(E - h_0(x,v)\right)_+,$$

with $\varepsilon = 0.1$, $E = 0.6$ and $M_0 \sim 0.6$. This time the support of the initial data coincides perfectly with $U_\circ$.

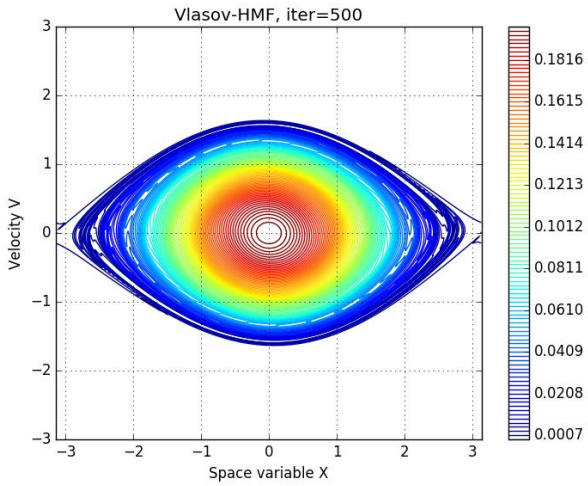


(a) *Contour lines.*

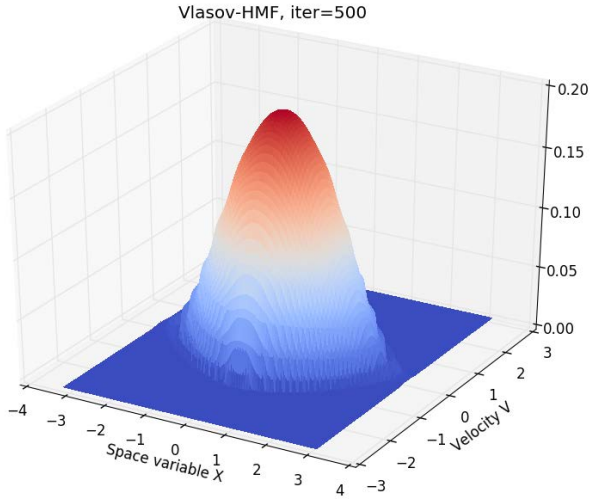


(b) *3D plot*

Figure 5.39: *Initial data.*

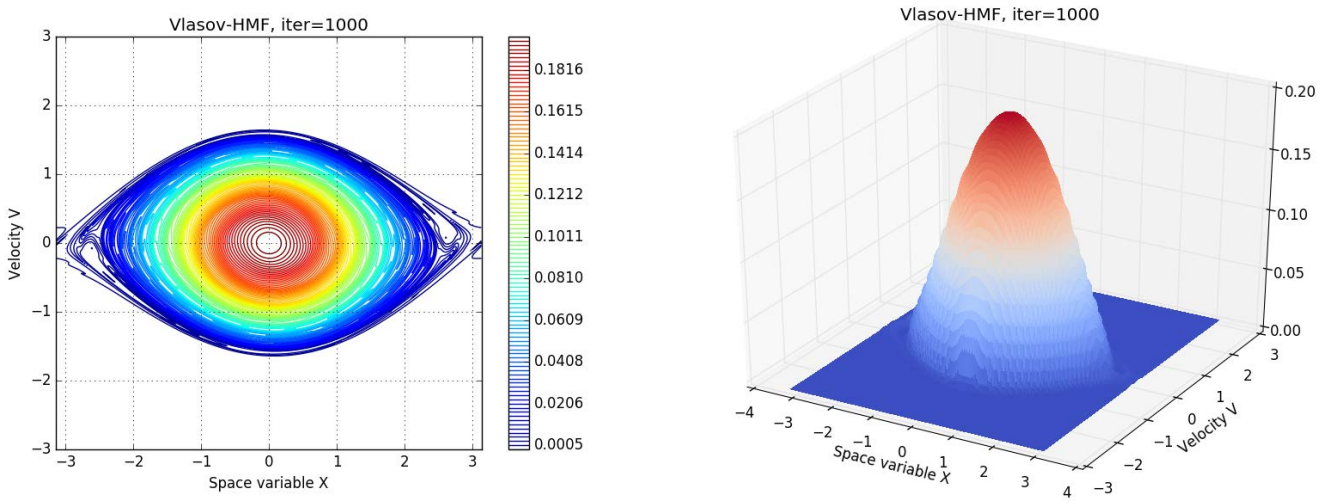


(a) *Contour lines.*



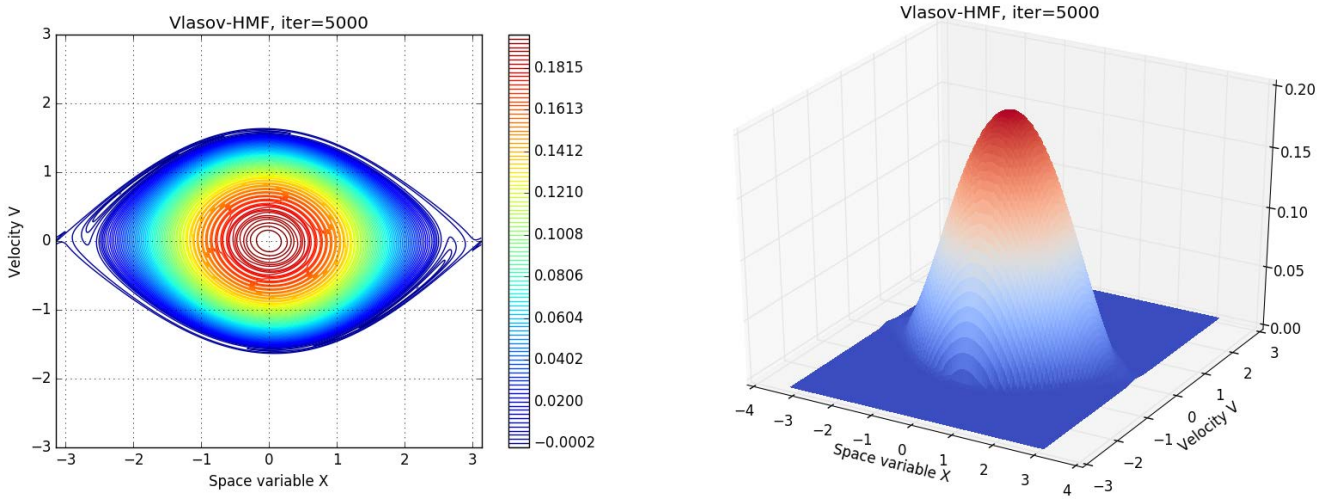
(b) *3D plot*

Figure 5.40: *Distribution function after* 500 *iterations in time.*



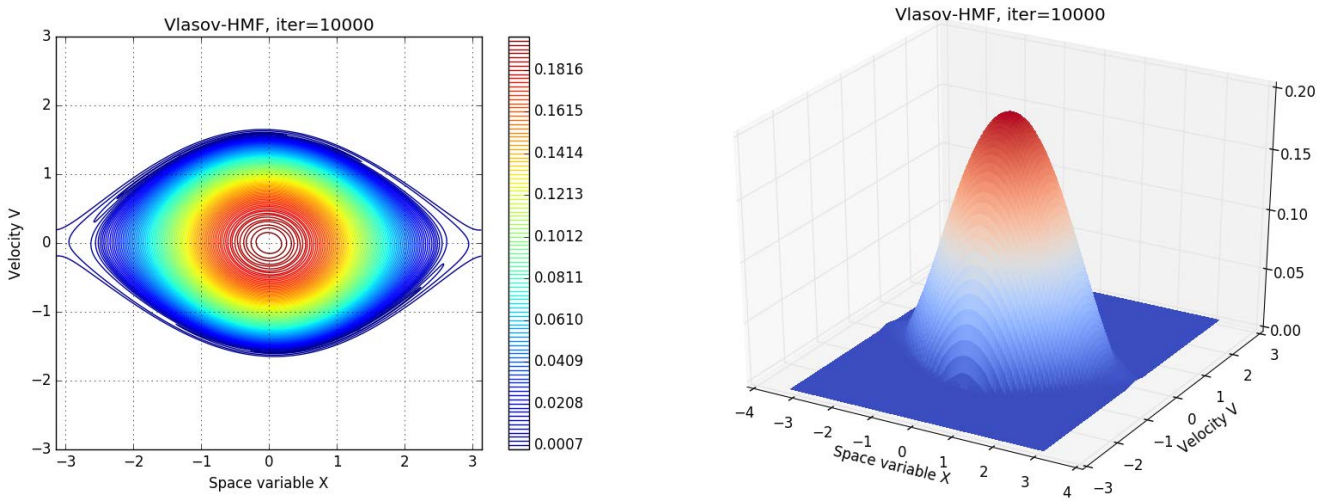
(a) *Contour lines.* (b) *3D plot*

Figure 5.41: *Distribution function after* 1000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.42: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.43: *Distribution function after* 10000 *iterations in time.*

The situation remains the same as in the previous example: its seems that the support is preserved over long times, and the final state is close to η in L^1 norm only. As previously, singularities at the boundary of the support are an obstacle to nonlinear Landau damping.

5.6.3 The case where $U_\circ \subset$ Supp(η)

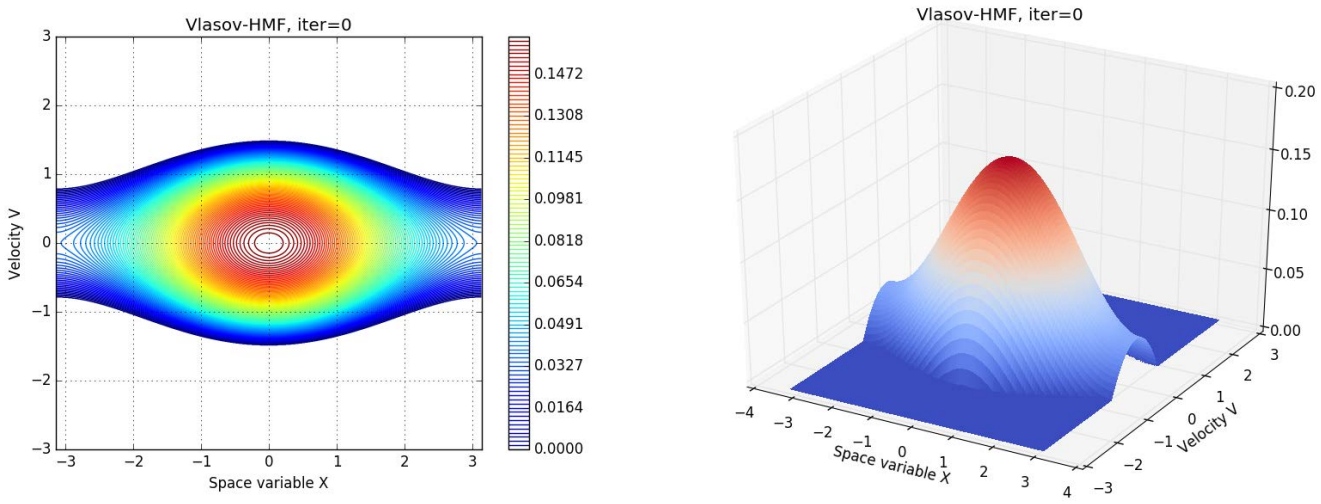
The last compactly supported initial data is

$$f^0(x,v) = \eta(x,v)(1+\varepsilon\cos(x)),$$

where

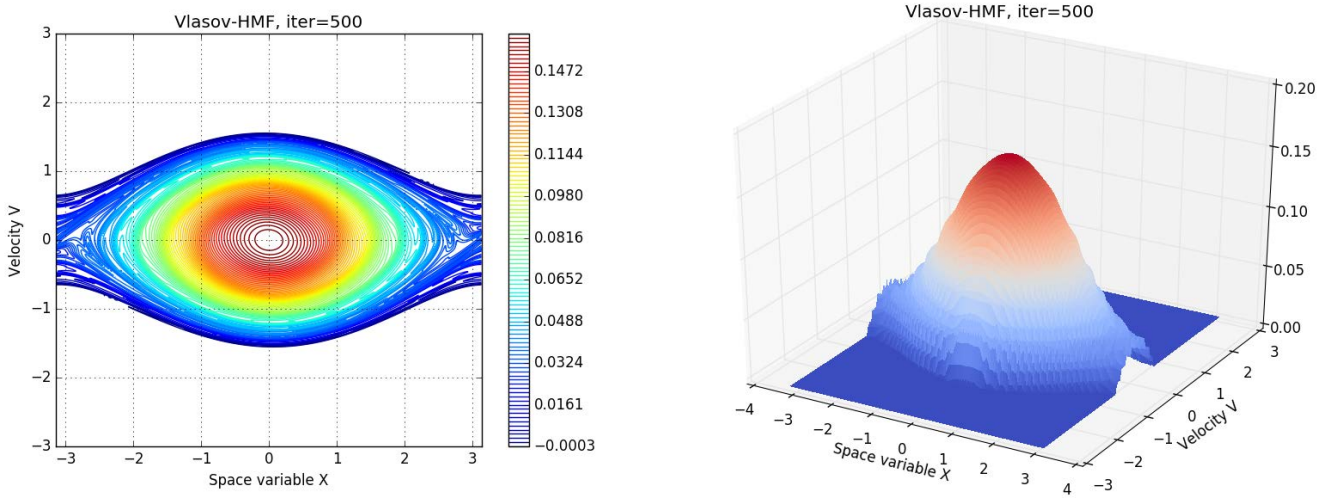
$$\eta(x,v) = \alpha\left(E - h_0(x,v)\right)_+,$$

with $\varepsilon = 0.1$, $E = 0.7$ and $M_0 \sim 0.3937$. The support of f^0 intersects all the zones $U_+, U_-, U_\circ$, and we expect to observe the effect of the singularities at the separatix.



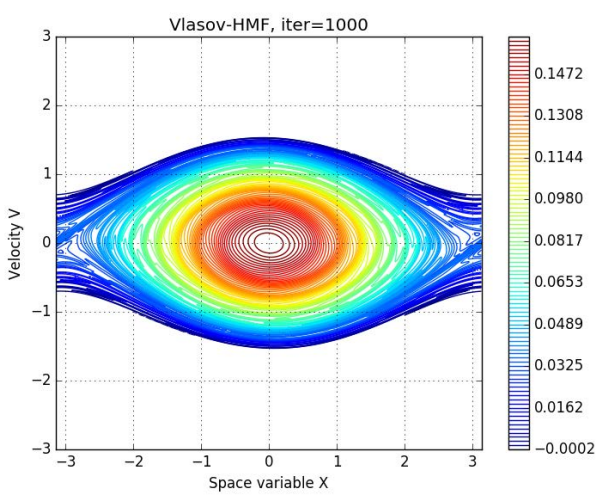
(a) *Contour lines.* (b) *3D plot*

Figure 5.44: *Initial data.*

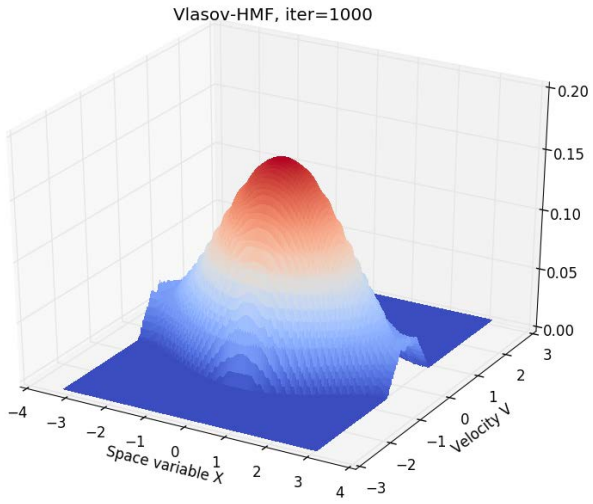


(a) *Contour lines.* (b) *3D plot*

Figure 5.45: *Distribution function after* 500 *iterations in time.*

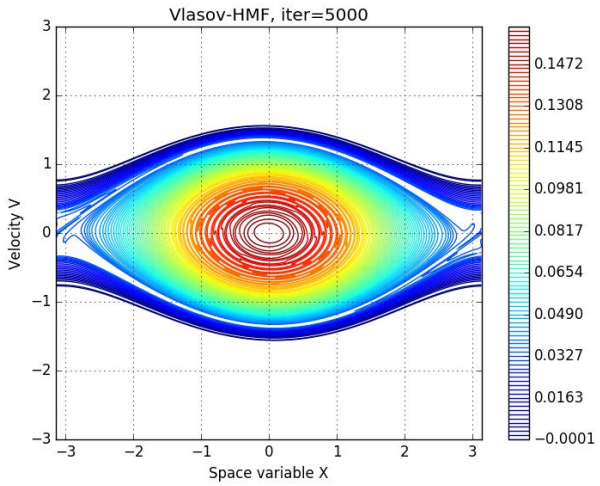


(a) *Contour lines.*

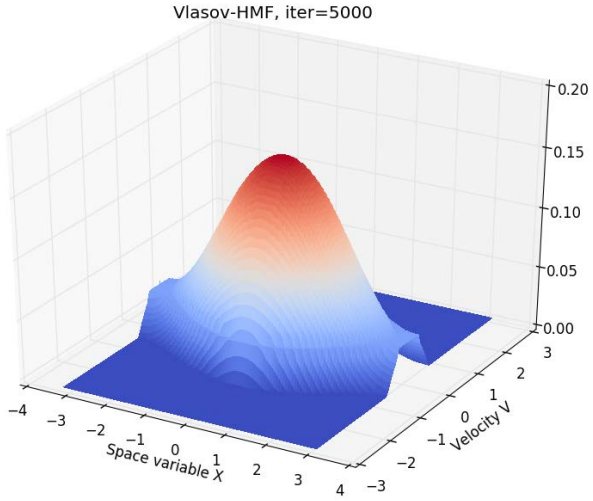


(b) *3D plot*

Figure 5.46: *Distribution function after* 1000 *iterations in time.*

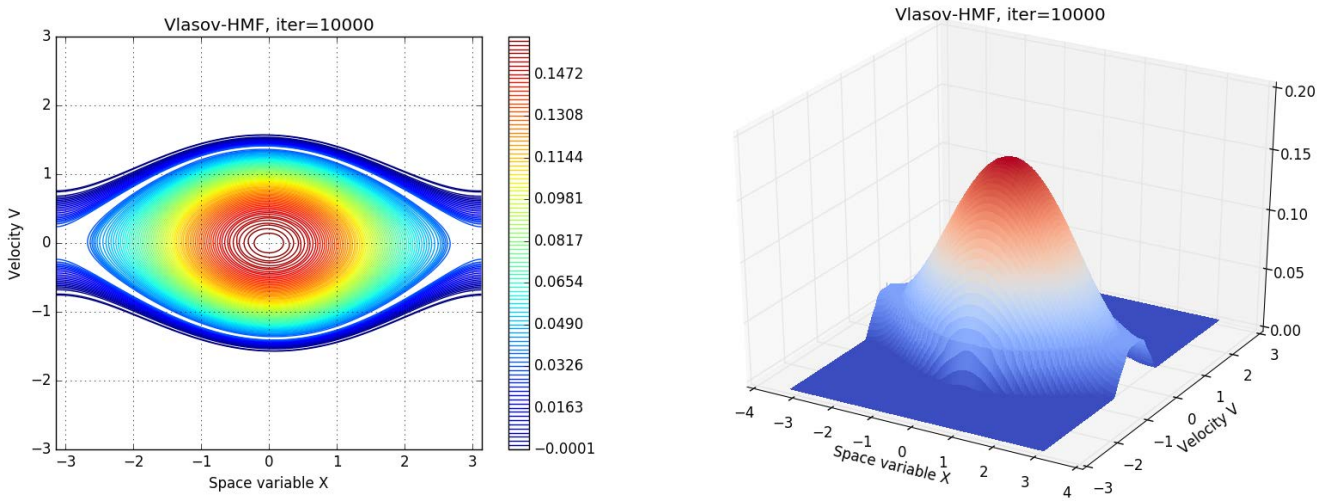


(a) *Contour lines.*



(b) *3D plot*

Figure 5.47: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.48: *Distribution function after* 10000 *iterations in time.*

As previously the support seems to be preserved over long times. Moreover, this exemple clearly exhibits the singularities at the sepratix (see in the above picture how the distribution function flattens near this region), and the final state is clearly far from η in regular norm, though it is as previously close in L^1 norm.

5.6.4 Smooth stationary state with compact support, first example

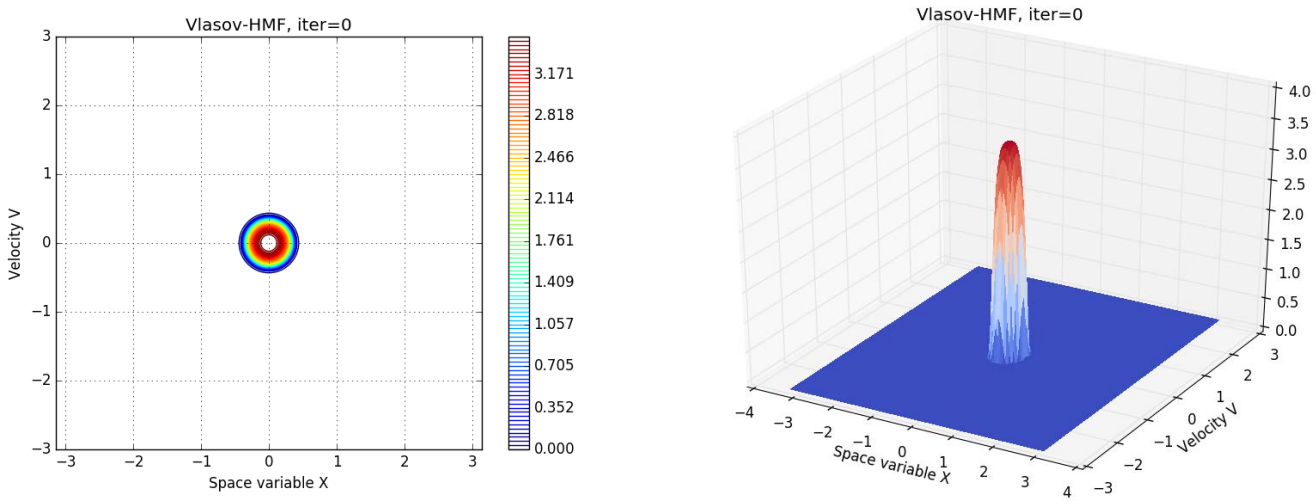
We consider the function $g_{\beta,R} : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\begin{cases} g_{\beta,R}(z) = \beta h(R-z) \times h(z+R) \\ h(z) = e^{-1/z} \mathbb{1}_{\{z>0\}}, \end{cases} \tag{5.6.1}$$

and initial data of the kind

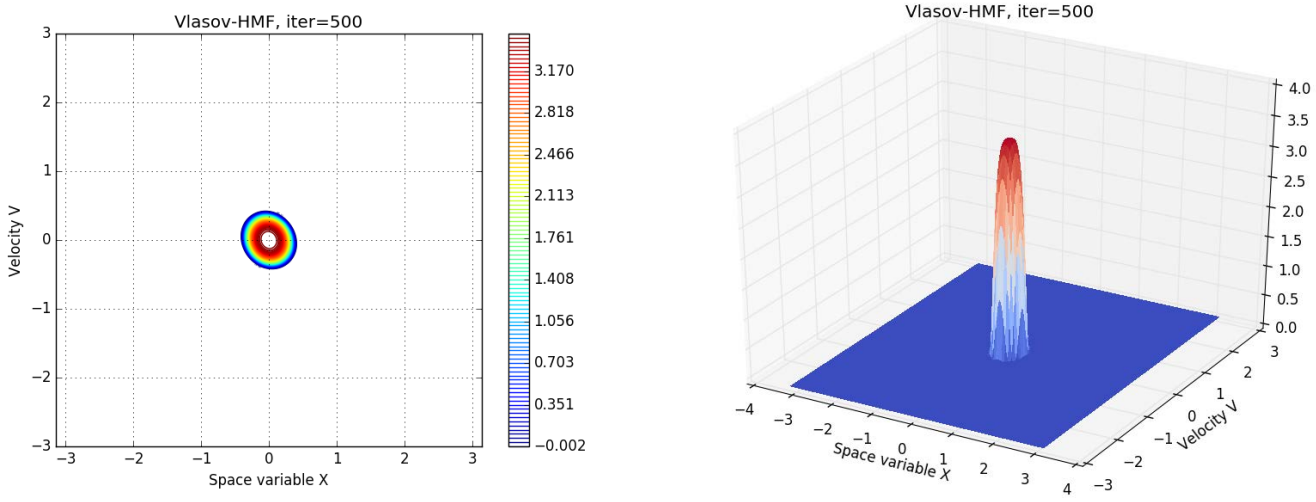
$$f^0(x,v) = a g_{\beta,R}\left(h_0(x,v) + M_0\right)\left(1 + \varepsilon \cos(x)\right), \tag{5.6.2}$$

with $\varepsilon = 0.1$, $R = 1$, $\beta = 10$, $M_0 = 0.986$, and α a normalization coefficient as usual. This initial data is smooth and compactly supported in $U_\circ$, and as regards as the previous discussions, we expect to observe nonlinear Landau damping.



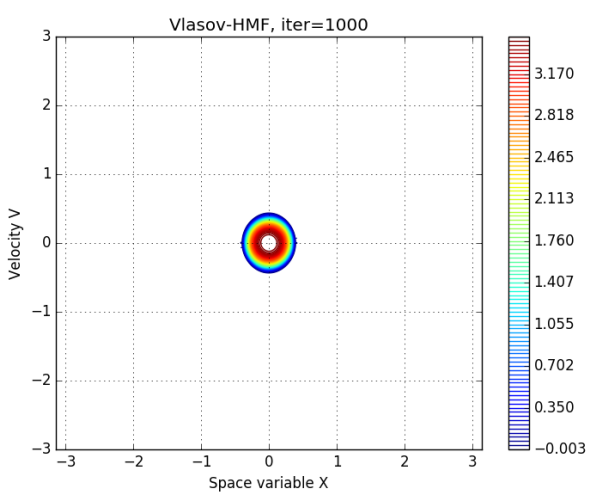
(a) *Contour lines.* (b) *3D plot*

Figure 5.49: *Initial data.*

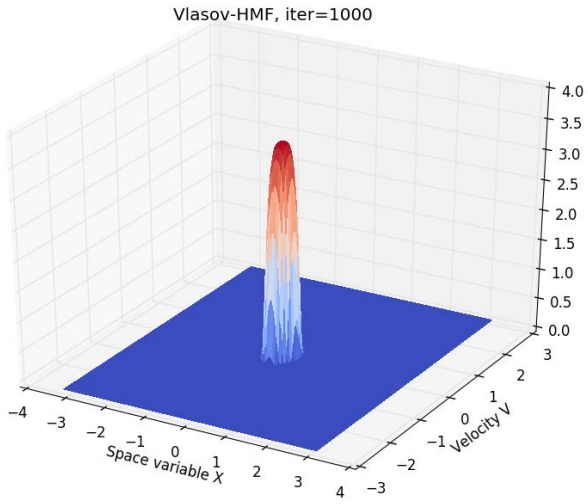


(a) *Contour lines.* (b) *3D plot*

Figure 5.50: *Distribution function after* 500 *iterations in time.*

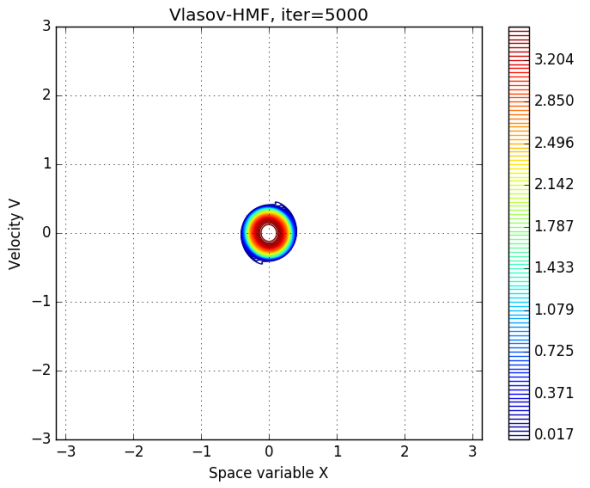


(a) *Contour lines.*

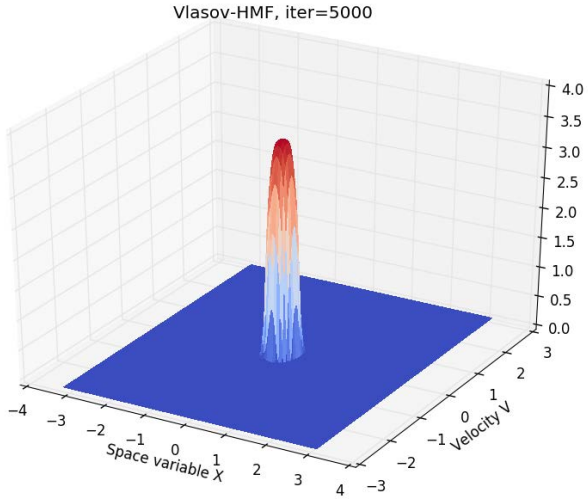


(b) *3D plot*

Figure 5.51: *Distribution function after* 1000 *iterations in time.*

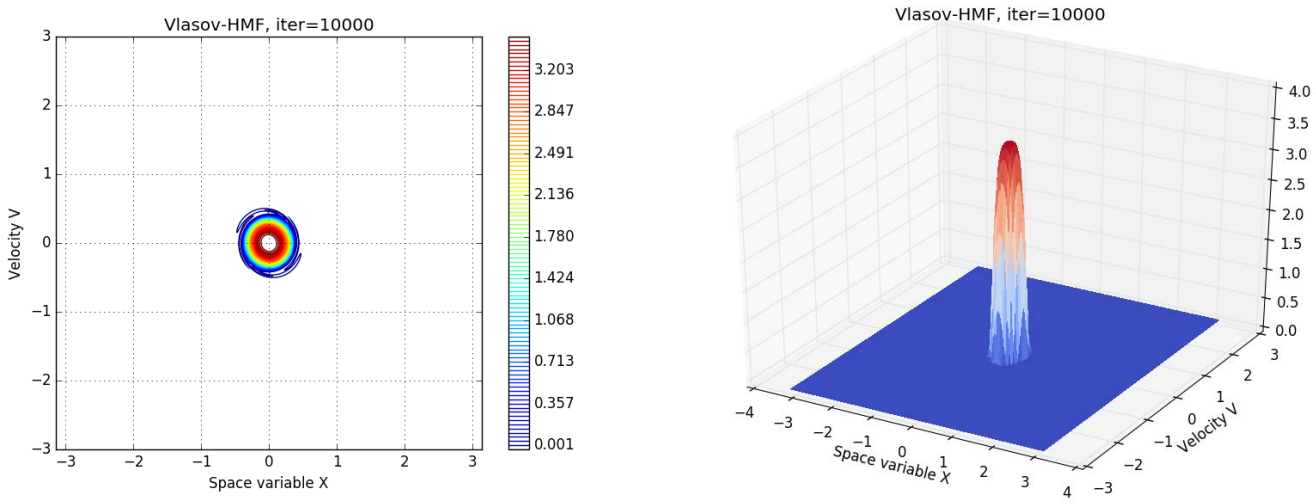


(a) *Contour lines.*



(b) *3D plot*

Figure 5.52: *Distribution function after* 5000 *iterations in time.*



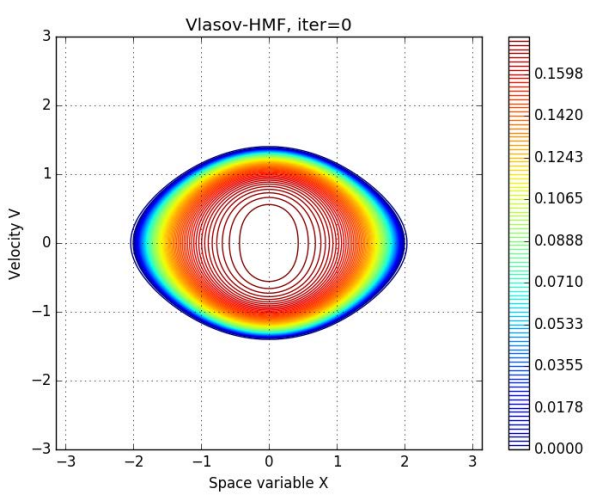
(a) *Contour lines.* (b) *3D plot*

Figure 5.53: *Distribution function after* 10000 *iterations in time.*

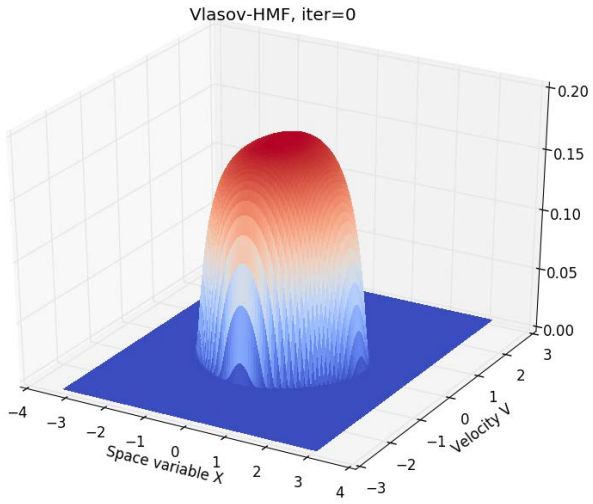
As previously the support seems to be preserved, and the final state seems to be close to the stationary state in regular norm, though is is not completely clear form the above LHS figure (possibly due to numerical complications). Thus it seems that, as in the case of Maxwell-Boltzmann distributions, we might expect nonlinear Landau damping for these smooth compactly supported states.

5.6.5 Smooth stationary state with compact support, second example

As the final example of this section, we consider an initial data defined by formulas (5.6.1) and (5.6.2), with $\varepsilon = 0.1$, $R = 5$, $\beta = 5$, $M_0 = 0.678$, and α a normalization coefficient as previously.

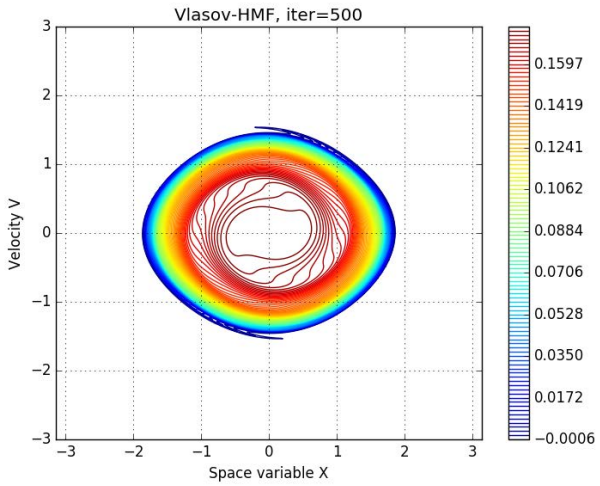


(a) *Contour lines.*

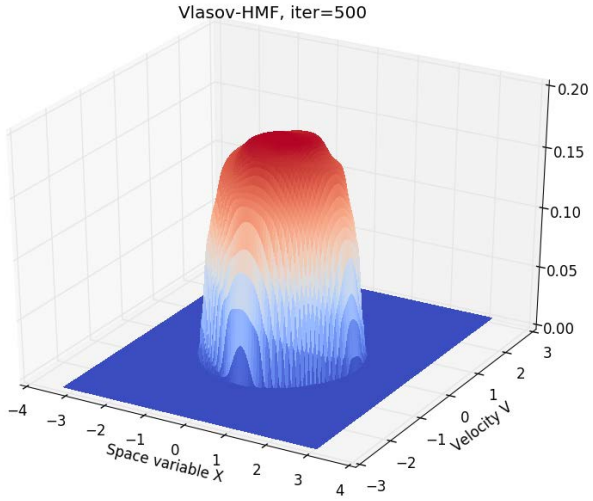


(b) *3D plot*

Figure 5.54: *Initial data.*

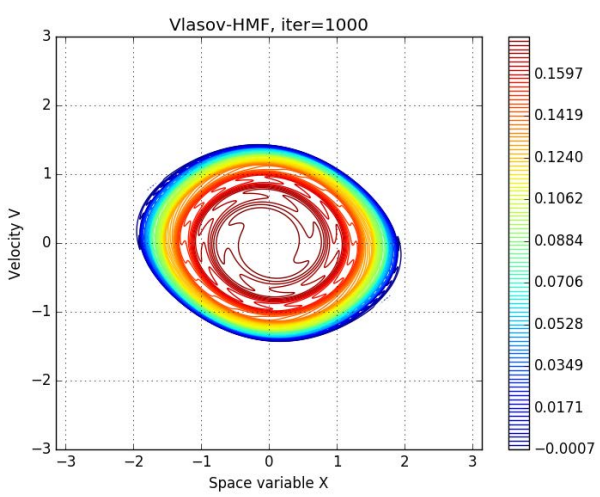


(a) *Contour lines.*

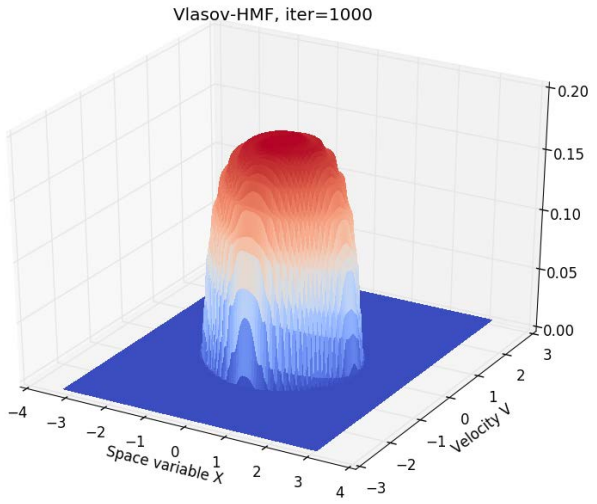


(b) *3D plot*

Figure 5.55: *Distribution function after* 500 *iterations in time.*

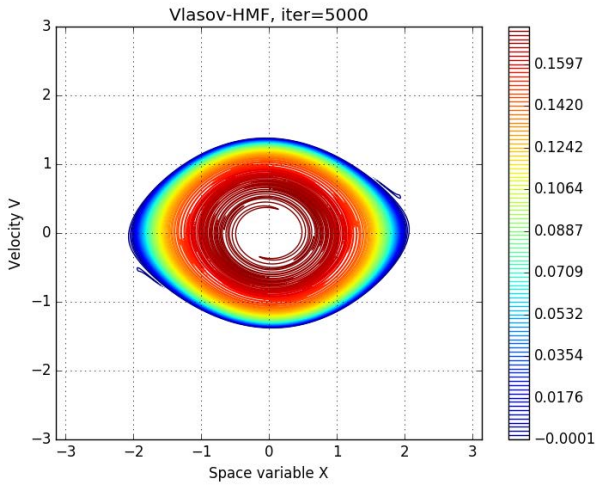


(a) *Contour lines.*

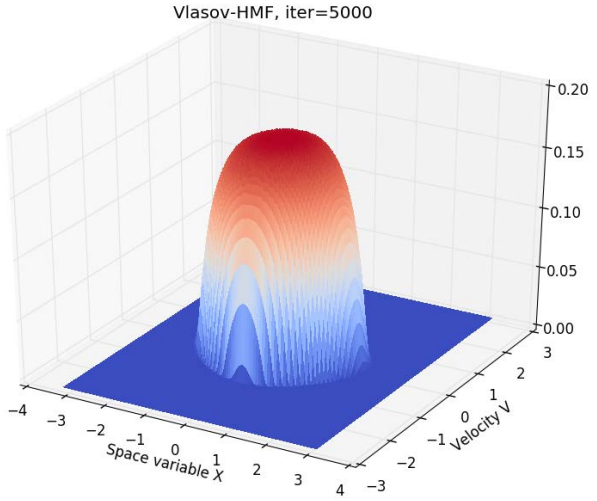


(b) *3D plot*

Figure 5.56: *Distribution function after* 1000 *iterations in time.*

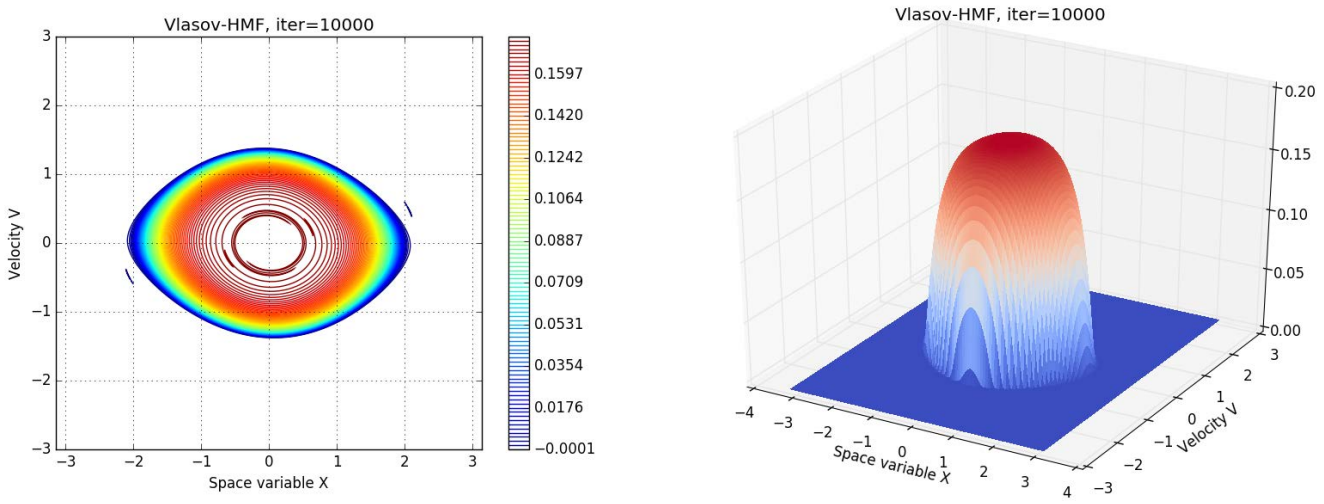


(a) *Contour lines.*



(b) *3D plot*

Figure 5.57: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.58: *Distribution function after* 10000 *iterations in time.*

It seems that as previously we have indeed a nonlinear Landau damping phenomenon. As a conclusion of this section we may say that a nonlinear Landau damping Theorem may be proved for smooth compactly supported states, as discussed in the introduction. This appears to be confirmed by the example of subsection 5.6.4 and of the present subsection. Moreover, the figures from subsections 5.6.1 to 5.6.3 seem to show that the Vlasov-HMF model preserves quite well the support of the states over long times, which is encouraging for further investigations on nonlinear Landau damping for smooth compactly supported states.

5.7 Examples of instability

In the previous sections we exhibited solutions of the Vlasov-HMF equation for which a nonlinear Landau damping behavior may perhaps be proved, and also solutions for which it definitely cannot, due to the singularities of the flow $\psi_t(x,v)$ near the separatix. In this last subsection, we shall show other examples of instability, caused by other obstacles to nonlinear Landau damping. The parameters are still $\delta_t = 0.1$ and $N = 1024$.

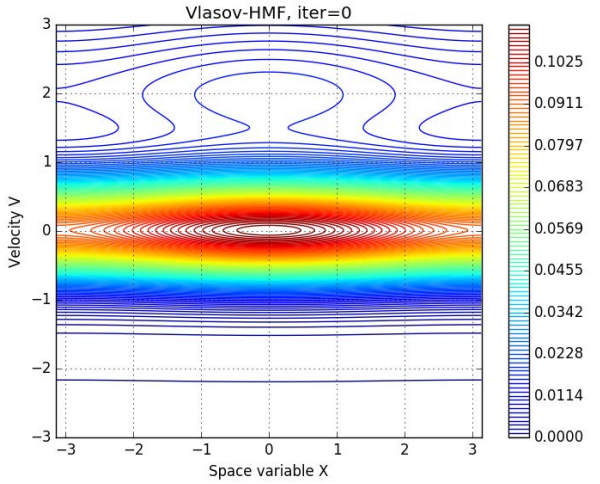
5.7.1 Bump-on-tail instability

The first example is the well-known bump-on-tail instability. The initial data is

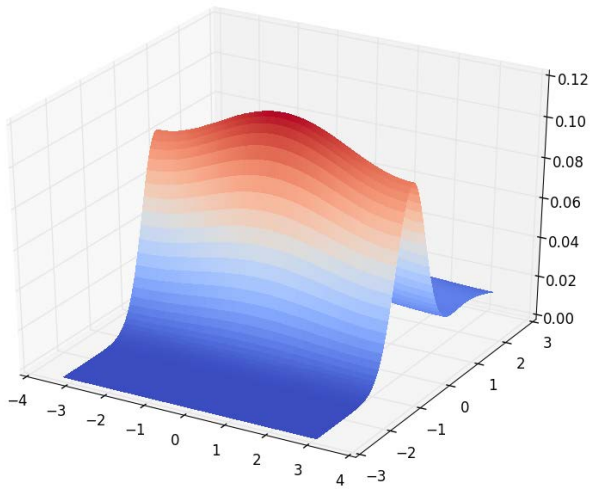
$$f^0(x,v) = \alpha \left(e^{-2v^2} + 0.15 e^{-\frac{(v-2)^2}{2}} \right) (1 + \varepsilon \cos(x)),$$

with $\varepsilon = 0.1$ and α a normalization coefficient.
We are in the homogeneous case ($M_0 = 0$, $\eta(x,v) = \eta(v)$). The Penrose criterion **(H)** (see

chapter 1) reflects then essentially the fact that the state η has one bump, when $v = 0$. Instability occurs when η has a secondary bump, which implies that the stability criterion is violated. Below we present numerical illustrations of this so-called "bump-on-tail" instability.

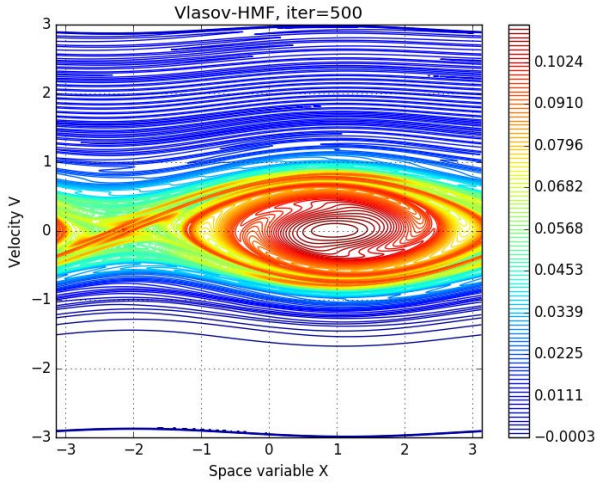


(a) *Contour lines.*

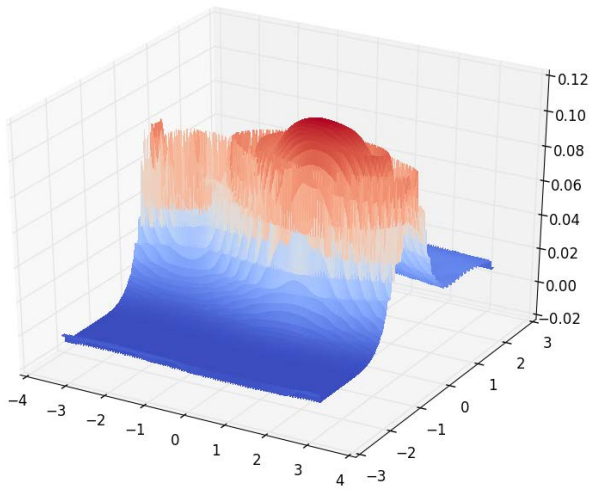


(b) *3D plot*

Figure 5.59: *Initial data.*

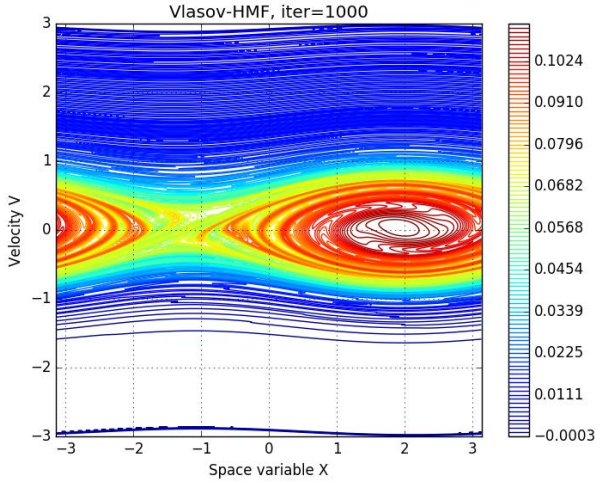


(a) *Contour lines.*

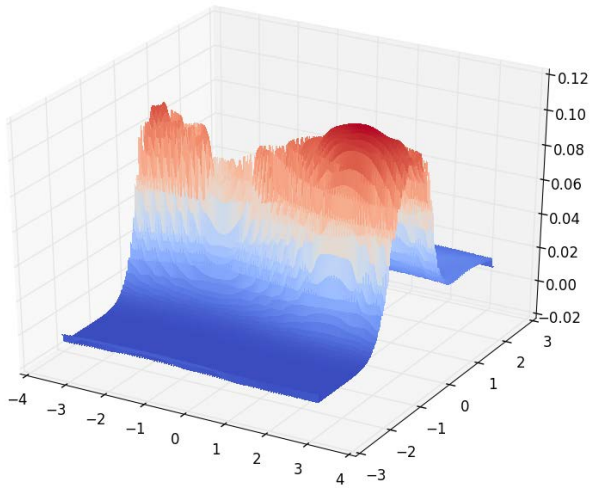


(b) *3D plot*

Figure 5.60: *Distribution function after* 500 *iterations in time.*

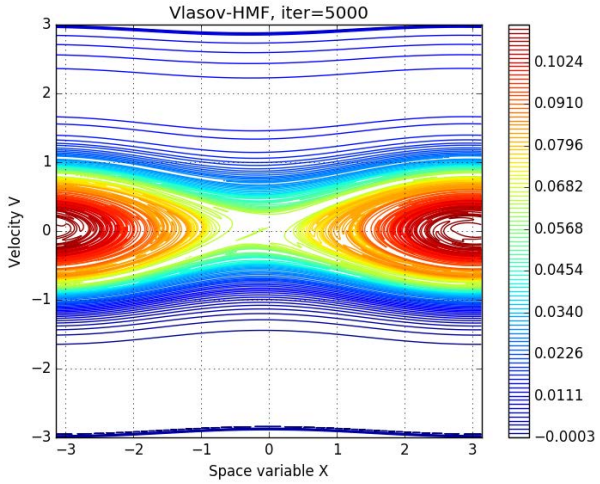


(a) *Contour lines.*

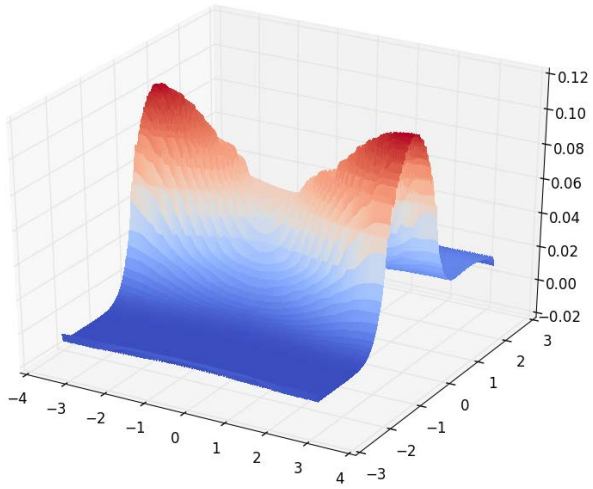


(b) *3D plot*

Figure 5.61: *Distribution function after* 1000 *iterations in time.*

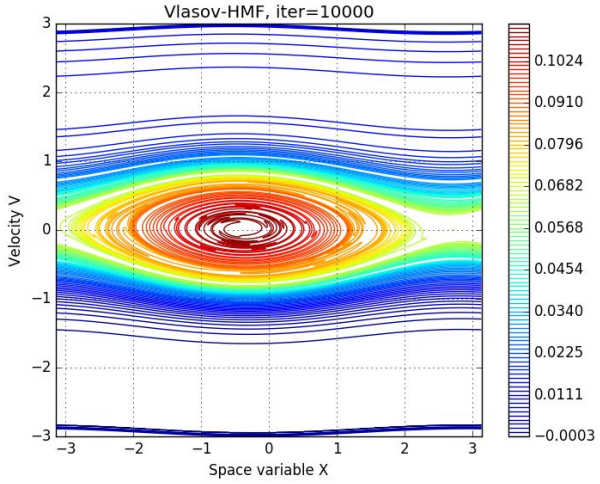


(a) *Contour lines.*

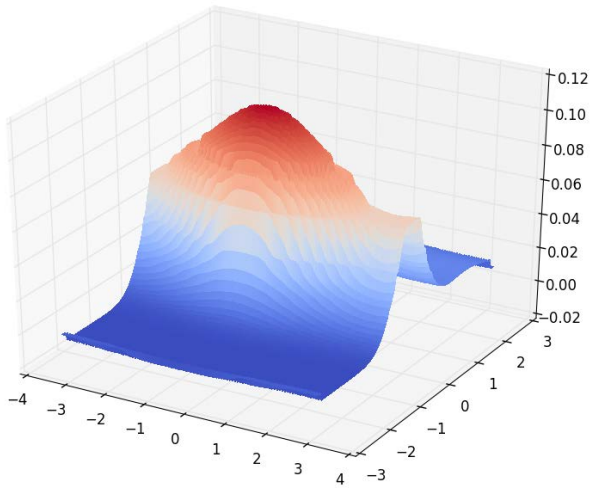


(b) *3D plot*

Figure 5.62: *Distribution function after* 5000 *iterations in time.*



(a) *Contour lines.*



(b) *3D plot*

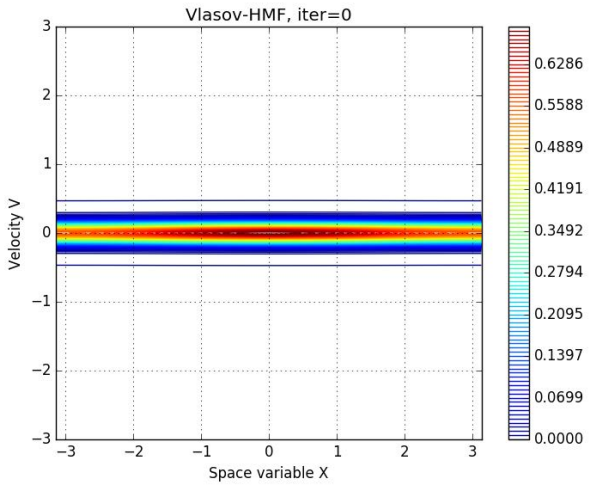
Figure 5.63: *Distribution function after* 10000 *iterations in time.*

5.7.2 Violation of the Penrose criterion

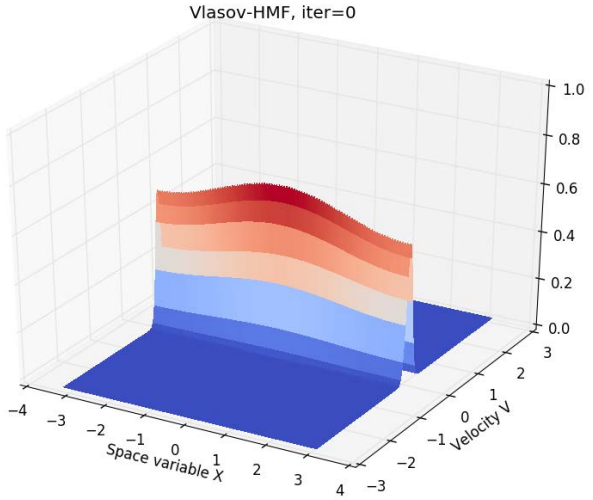
The initial data is here

$$f^0(x,v) = \alpha e^{-\beta\frac{v^2}{2}}(1+\varepsilon\cos(x)),$$

with $\varepsilon = 0.1$, $\beta = 100$, and α a normalization coefficient. With this choice of β the Penrose criterion **(H)** (see chapter 1, subsection 1.2.3) is violated, and we have instability, as it is shown in the simulations below.

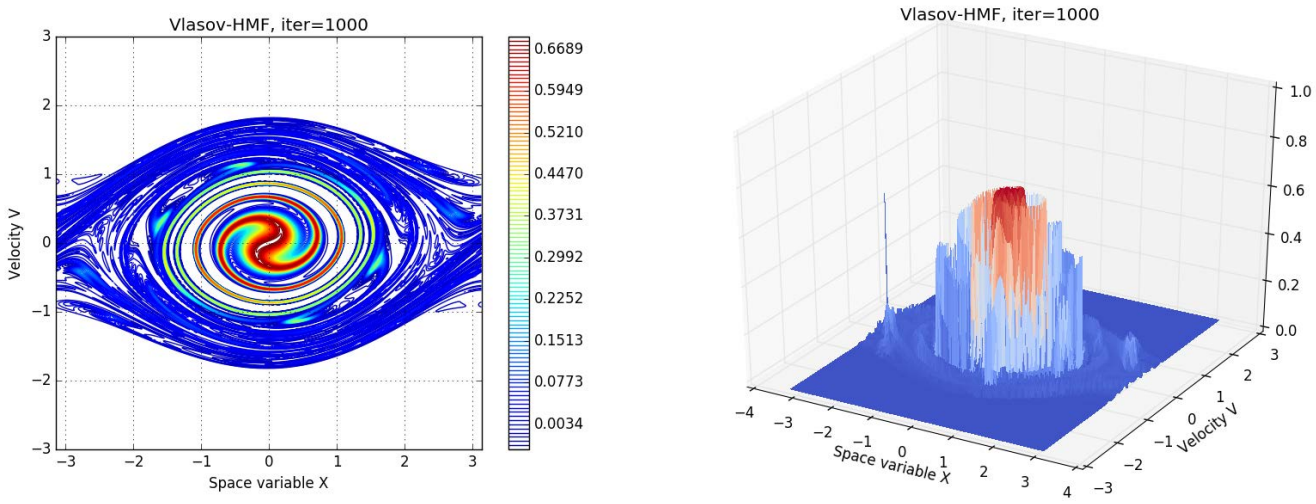


(a) *Contour lines.*



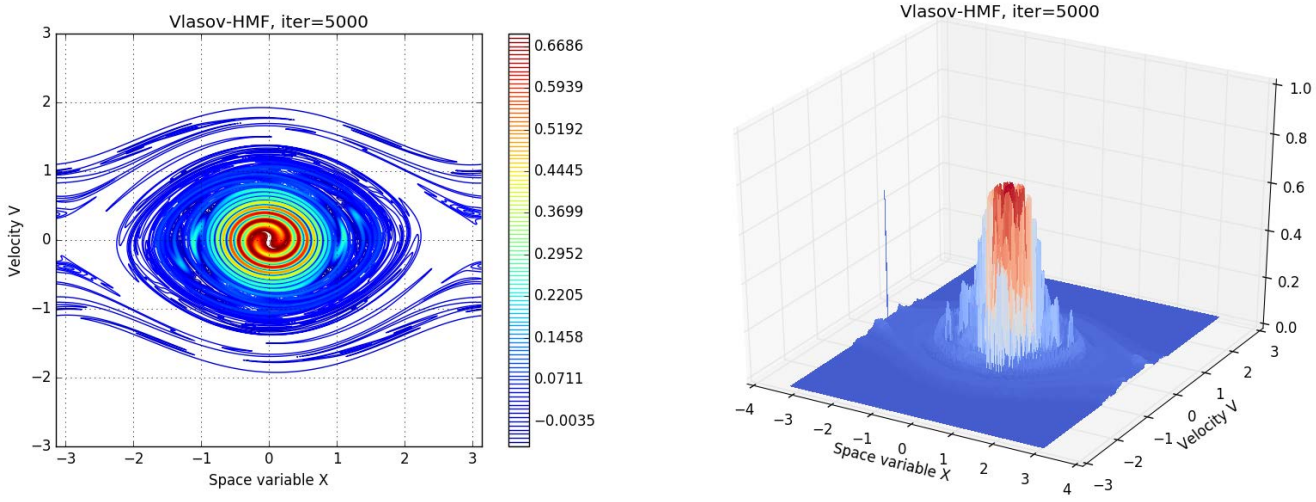
(b) *3D plot*

Figure 5.64: *Initial distribution*



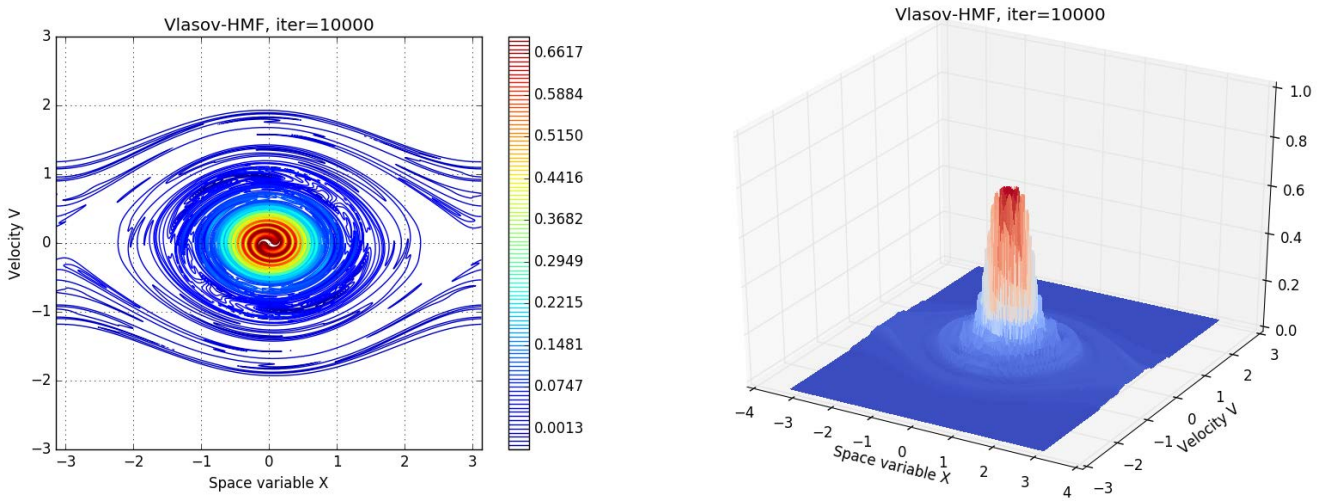
(a) *Contour lines.* (b) *3D plot*

Figure 5.65: *Distribution function after* 1000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

Figure 5.66: *Distribution function after* 1000 *iterations in time.*



(a) *Contour lines.* (b) *3D plot*

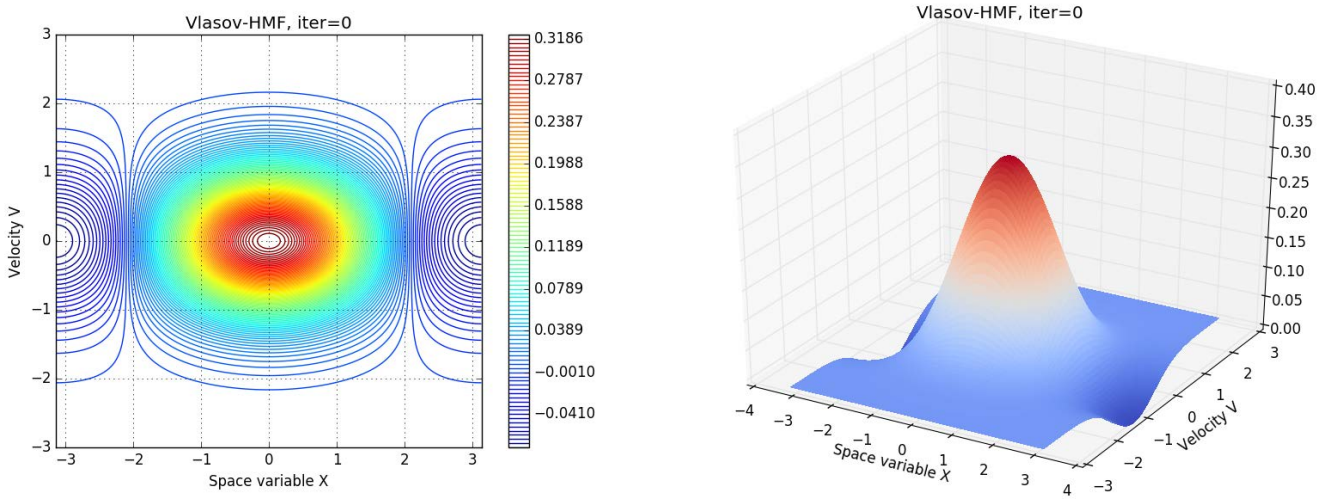
Figure 5.67: *Distribution function after* 10000 *iterations in time.*

5.7.3 "Big" perturbations

The initial data is

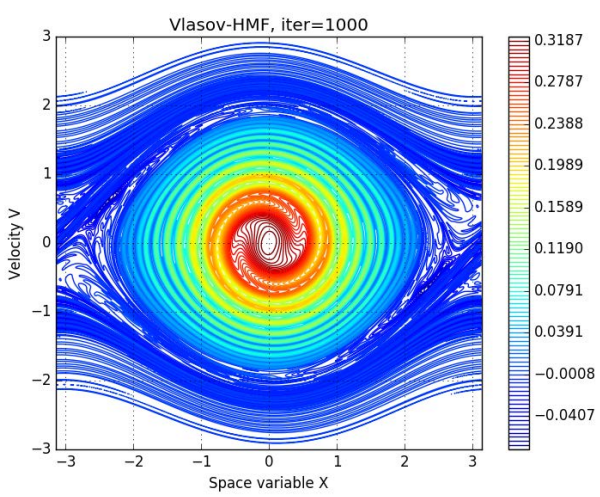
$$f^0(x,v) = \alpha e^{-\beta h^0(x,v)}(1+\varepsilon\cos(x)),$$

with $\beta = 2.1$, $M_0 \sim 0.3$, and $\varepsilon = 2$. Here the value of ε is too large, and we dot not observe Landau damping as the initial data is not a small perturbation of a stationary state anymore.

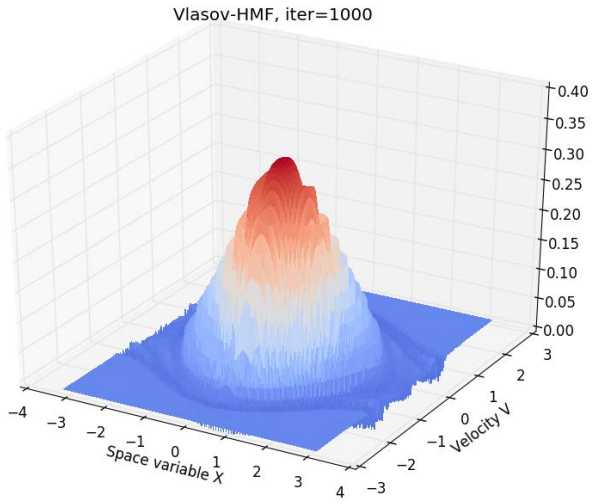


(a) *Contour lines.* (b) *3D plot*

Figure 5.68: *Initial distribution*

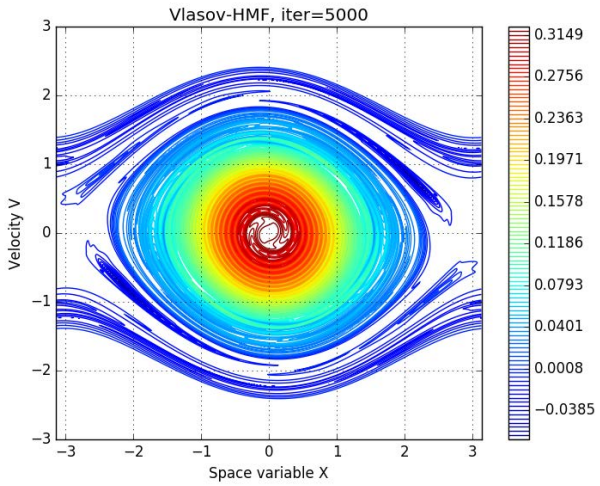


(a) *Contour lines.*

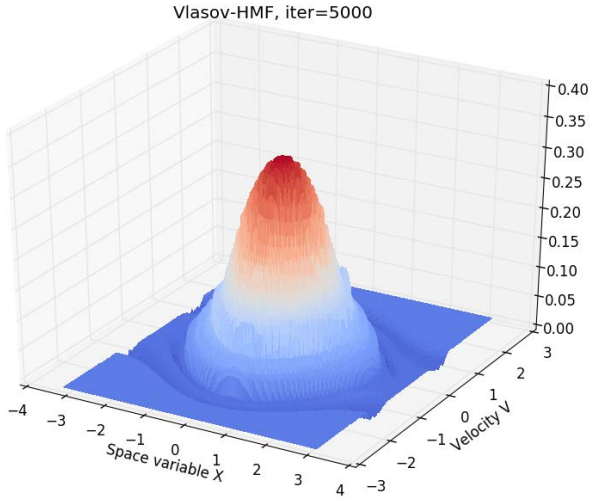


(b) *3D plot*

Figure 5.69: *Distribution function after* 1000 *iterations in time.*

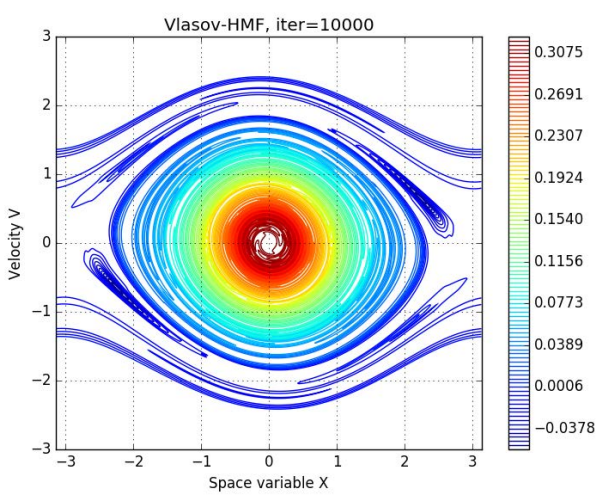


(a) *Contour lines.*

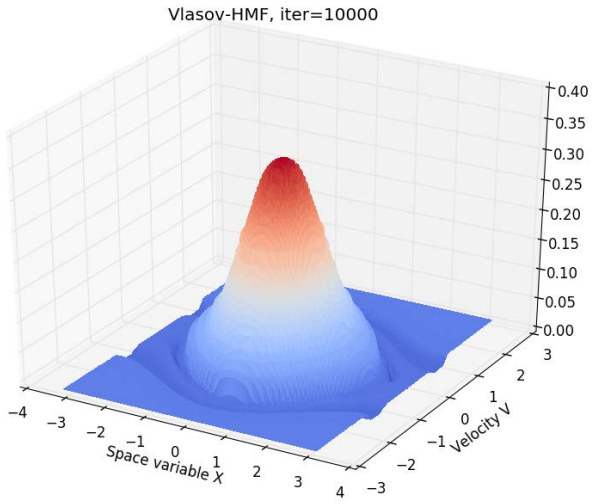


(b) *3D plot*

Figure 5.70: *Distribution function after* 1000 *iterations in time.*



(a) *Contour lines.*



(b) *3D plot*

Figure 5.71: *Distribution function after* 10000 *iterations in time.*

Chapter 6

On time-discretization of the 2D Euler equation by a symplectic Crouch-Grossman integrator

The topic of this chapter was introduced in the third section of chapter 2.
We consider a discretization in time of the two-dimensional Euler equation, with periodic boundary conditions, by means of a symplectic Crouch-Grossman integrator. Our purpose is to prove the convergence of this time scheme in periodic Sobolev norms, with a convergence rate of order one with respect to the time step size.
This chapter may be read independently from the others.

6.1 Introduction

In this chapter we consider time-discretizations of the two-dimensional Euler equation for perfect incompressible fluids, written in vorticity form, and with periodic boundary conditions. It is a transport equation associated with an hamiltonian vector field, whose flow hence preserves the symplectic structure. Our purpose is to construct a time-discretization which preserves as well the symplectic structure, and to prove the convergence of the semi-discrete scheme.
The time-discretization will employ a *Crouch-Grossman integrator* (see [30]). By this denomination we mean that the time scheme requires two steps. As we consider a transport equation with a time-dependent velocity vector field, the first step is indeed to freeze this vector field at a fixed-time, and thus obtain an "elementary" transport equation associated with an autonomous hamiltonian vector field. The second step is to discretize in time the flow of this hamiltonian by means of a symplectic integrator, namely the celebrated implicit midpoint method (see [37], [45]).
The two-dimensional Euler equation in a periodic box, written in vorticity form, reads

$$\begin{cases} \partial_t \omega - U(\omega) \cdot \nabla \omega = 0, \\ \omega(0,z) = \omega_0(z), \end{cases} \tag{6.1.1}$$

where $\omega(t,z) \in \mathbb{R}$, with $t \in \mathbb{R}_+$, and $z \in \mathbb{T}^2$, the two-dimensional torus defined by

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2.$$

The divergence-free velocity vector field U is given by the formula

$$U(\omega) = J\nabla\Delta^{-1}\omega.$$

using the canonical symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Δ^{-1} stands for the inverse of the Laplace operator on function with average 0 on $\mathbb{T}^2$ (see appendix A), and ∇ is the two-dimensional gradient operator.
Note that the vorticity form (6.1.1) is formally equivalent to standard formulations of the Euler equation that may be found in the literature (see [2] for instance). Note moreover that

a more usual, but equivalent, convention may be found throughout the literature, where J is replaced by $-J$, and where the sign in front of $U(\omega)$ in (6.1.1) has to be changed.

In this chapter we consider the time-discretization of (6.1.1) by a Crouch-Grossman integrator (see [30]), that proceeds in two steps. First, freezing the velocity vector field at time $t = 0$, which gives the hamiltonian transport equation

$$\begin{cases} \partial_t f - J\nabla\psi \cdot \nabla f = 0 \\ \Delta\psi = \omega_0 \end{cases} \tag{6.1.2}$$

with initial data ω_0.
The second step is to discretize the flow of the vector field $J\nabla\psi$ by the implicit midpoint method. More precisely, we define $\Phi_t(z)$ as the unique solution of the implicit equation

$$\Phi_t(z) = z + tJ\nabla\psi\left(\frac{z + \Phi_t(z)}{2}\right),$$

and if t is small enough, $\omega_0 \circ \Phi_t(z)$ should be an approximation of the solution $\omega(t, z)$ of (6.1.2).
Therefore we define the semi-discrete operator $\mathcal{S}_t$ by

$$\begin{cases} \mathcal{S}_t(\omega_0(z)) = \omega_0\left(\Phi_t(z)\right), \\ \Phi_t(z) = z + tJ\nabla\psi\left(\dfrac{z + \Phi_t(z)}{2}\right), \\ \Delta\psi = \omega_0. \end{cases} \tag{6.1.3}$$

If $\tau \in]0, 1[$ is the time-step, we define a sequence $(\omega_n)_{n\in\mathbb{N}}$ by

$$\begin{cases} \omega_n(z) = \mathcal{S}_\tau(\omega_{n-1}) = \mathcal{S}_\tau^n(\omega_0) \\ \mathcal{S}_\tau^0(\omega_0) = \omega_0. \end{cases} \tag{6.1.4}$$

The main result of the chapter is that, if τ is small enough, $\omega_n(z) = \mathcal{S}_t^n(\omega_0)(z)$ is an order one approximation in H^s (Sobolev space, see definition 6.2.1) of $\omega(t, z)$ at time $t_n = n\tau$, where ω solves (6.1.1).

6.2 Main result

6.2.1 Notations

We shall consider functions defined on the two-dimensional torus, also seen as periodic functions on $\mathbb{R}^2$, with period 2π in each variables. Therefore a function f on $\mathbb{T}^2$ will be usually written as

$$f(z) = f(x, y), \quad \text{with} \quad z = (x, y) \in [0, 2\pi]^2.$$

If λ is the Lebesgue measure on $\mathbb{R}^2$, then $(2\pi)^{-2}\lambda$ induces a measure $\tilde{\lambda}$ on $\mathbb{T}^2$ such that,

$$\int_{\mathbb{T}^2} f(z)\mathrm{d}\tilde{\lambda}(z) = \frac{1}{(2\pi)^2}\int_{[0,2\pi]^2} f(z)\mathrm{d}\lambda(z).$$

We shall rather use the notation

$$\int_{\mathbb{T}^2} f(z)\mathrm{d}z = \int_{\mathbb{T}^2} f(z)\mathrm{d}\tilde{\lambda}(z).$$

The notation $|\cdot|$ will refer to any norm on $\mathbb{T}^2$.
If $\alpha = (\alpha_1, \alpha_2)$ belongs to $\mathbb{N}^2$, we will write

$$|\alpha| = \alpha_1 + \alpha_2 \quad \text{and} \quad \partial_z^\alpha f(z) = \partial_x^{\alpha_1}\partial_y^{\alpha_2} f(z).$$

The operators ∇, Δ, and $\Delta\cdot$ are defined by

$$\nabla f(z) = (\partial_x f(z), \partial_y f(z))^\top, \quad \Delta f(z) = \partial_x^2 f(z) + \partial_y^2 f(z), \quad \nabla \cdot X = \partial_x X_1 + \partial_y X_2,$$

where X is two-dimensional vector field $X = (X_1, X_2) : \mathbb{T}^2 \to \mathbb{T}^2$. In particular, $\nabla^2 f$ will be the Hessian matrix of f.
We shall also write

$$\partial_z^\alpha X = \left(\partial_z^\alpha X_1, \partial_z^\alpha X_2\right)^\top.$$

The differential of X shall be denoted by $\partial_z X$, as follows

$$\partial_z X = \left(\partial_z^{(1,0)} X, \partial_z^{(0,1)} X\right).$$

For an element $k = (k_1, k_2) \in \mathbb{Z}^2$, we define the quantities $|k|$ and $\langle k\rangle$ by

$$|k| = \sqrt{k_1^2 + k_2^2}, \quad \langle k\rangle = \left(1 + k_1^2 + k_2^2\right)^{1/2}.$$

The scalar product between $k = (k_1, k_2) \in \mathbb{Z}^2$ and $z = (x, y) \in \mathbb{T}^2$ is defined by

$$k \cdot z = k_1 x + k_2 y.$$

In preparation of computations involving Fourier coefficients, it is rather useful, for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, to define the following notations:

$$(ik)^\alpha = (ik_1)^{\alpha_1}(ik_2)^{\alpha_2} \quad \text{and} \quad |k|^\alpha = |k_1|^{\alpha_1}|k_2|^{\alpha_2}.$$

Finally, the notation C will always refer to a positive constant of which the exact value has no interest in our eyes.

6.2.2 Functional framework

For $p \in [1, +\infty[$, $\ell^p(\mathbb{Z}^2)$ and $L^p(\mathbb{T}^2)$ are the classical Lebesgue spaces on $\mathbb{Z}^2$ and $\mathbb{T}^2$, respectively equipped with the norms

$$\|(u_k)_{k\in\mathbb{Z}^2}\|_{\ell^p(\mathbb{Z}^2)} = \left(\sum_{k\in\mathbb{Z}^2} |u_k|^p\right)^{1/p} \quad \text{and} \quad \|u\|_{L^p(\mathbb{T}^2)} = \left(\int_{\mathbb{T}^2} |u(z)|^p \mathrm{d}z\right)^{1/p}.$$

We shall use the notation

$$\langle u, v\rangle_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} f(z)\overline{g(z)}\mathrm{d}z$$

for the usual inner product associated with the norm $\|\cdot\|_{L^2(\mathbb{T}^2)}$.
We will also consider the Lebesgue spaces $\ell^\infty(\mathbb{Z}^2)$ and $L^\infty(\mathbb{T}^2)$, respectively equipped with the norms

$$\|(u_k)_{k\in\mathbb{Z}^2}\|_{\ell^\infty(\mathbb{Z}^2)} = \sup_{k\in\mathbb{Z}^2} |u_k|$$
$$\|u\|_{L^\infty(\mathbb{T}^2)} = \sup\left\{M \in \mathbb{R} \;\middle|\; \tilde{\lambda}\left(\{z \;\middle|\; |u(z)| > M\}\right) = 0\right\}.$$

The two-dimensional Fourier series are a natural tool for functions on $\mathbb{T}^2$. Let u be such a function. The Fourier coefficients of u are given by

$$\hat{u}_k = \int_{\mathbb{T}^2} u(z)e^{-ik\cdot z}\mathrm{d}z, \quad k \in \mathbb{Z}^2.$$

The Fourier series of u is defined by

$$\sum_{k\in\mathbb{Z}^2} \hat{u}_k e^{ikz}.$$

We refer to [47] for basic convergence properties of Fourier series.
For $s \in \mathbb{R}$, $H^s(\mathbb{T}^2)$ is the classical periodic Sobolev space, equipped with the norm

$$\|u\|_{H^s(\mathbb{T}^2)} = \left(\sum_{k\in\mathbb{Z}^2} |\hat{u}_k|^2\langle k\rangle^{2s}\right)^{1/2} \sim \left(\sum_{0\le|\alpha|\le s} \|\partial_z^\alpha u\|^2_{L^2(\mathbb{T}^2)}\right)^{1/2}. \tag{6.2.1}$$

We refer to [47] for basic properties of the Sobolev spaces. We shall essentially use the Sobolev embeddings

$$H^1(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2), \tag{6.2.2}$$

and

$$H^s(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2), \quad \text{for all } s > 1. \tag{6.2.3}$$

Note that, using the Bochner integral, the L^p and Sobolev norms are naturally extended for vector-valued functions on $\mathbb{T}^2$, for which the above Sobolev embeddings still hold. We shall also use in addition the following Lemma (see [72] for instance):

Lemma 6.2.1. *Assume that $F : \mathcal{U} \to M_2(\mathbb{R})$ is a C^∞ map satisfying $F(0) = 0$, where $\mathcal{U}$ is an open subset of $M_2(\mathbb{R})$ containing 0. For any $s > 1$ and $A \in H^s(\mathbb{T}^2)$, such that $A \in \mathcal{U}$ almost everywhere,*

$$\|F(A)\|_{H^s(\mathbb{T}^2)} \leq C(F)C_s\left(\|A\|_{L^\infty(\mathbb{T}^2)}\right)\left(1 + \|A\|_{H^s(\mathbb{T}^2)}\right),$$

where $C_s : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuous function.

The two-dimensional Euler equation is globally well-posed in these Sobolev spaces. More precisely, we have the following result (see for instance [2] for a complete proof):

Theorem 6.2.2. *Let $s > 1$ and $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. There exists an unique solution $\omega(t, z) \in C^0(\mathbb{R}_+, H^s(\mathbb{T}^2)) \cap C^1(\mathbb{R}_+, H^{s-1}(\mathbb{T}^2))$ of equation (6.1.1) with initial data ω_0.*

6.2.3 Statement of the main result

Our goal is to prove the following convergence Theorem:

Theorem 6.2.3. *Let $s \geq 6$ and $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. Let $\omega(t, z) \in C^0\left(\mathbb{R}_+, H^s(\mathbb{T}^2)\right)$ be the unique solution of equation (6.1.1) given by Theorem 6.2.2, with initial data ω_0. For a time step $\tau \in]0, 1[$, let $(\omega_n)_{n\in\mathbb{N}}$ be the sequence of functions starting from ω_0 and defined by formula (6.1.4) from iterations of the semi-discrete operator (6.1.3). For a fixed time horizon $T > 0$, let $B = B(T)$ be such that*

$$\sup_{t\in[0,T]} \|\omega(t)\|_{H^s(\mathbb{T}^2)} \leq B.$$

There exists two positive constants R_0 and R_1, and an increasing continuous function $R : \mathbb{R}_+ \to \mathbb{R}_+$, such that, if τ satisfies the hypothesis

$$\tau < \max\left(\frac{1}{R_0 B}, \frac{B}{TR(B)e^{R_1 T(1+B)}}\right),$$

the semi-discrete scheme enjoys the following convergence estimate: for all $n \in \mathbb{N}$ such that $t_n = n\tau \leq T$,

$$\|\omega_n - \omega(t_n)\|_{H^{s-4}(\mathbb{T}^2)} \leq \tau t_n R(B)e^{R_1 T(1+B)}.$$

Moreover

$$R(B) \leq R_1\left(B + B^3\right).$$

Let us make the following comments:

a) The convergence estimate depends on $B = B(T)$, the bound for the H^s norm of the exact solution on $[0, T]$. The best known upper bound for $B(T)$ is a double exponential in time, namely

$$\ln(B(T)) \lesssim \left(1 + \ln^+\left(\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\right)e^{CT} - 1,$$

where $\ln^+(x) = \ln(x)\mathbb{1}_{\{x>1\}}$.
We refer to [31] for more details on this type of estimate.

b) Although we used the implicit midpoint integrator, which is known to produce in general a global error that scales in τ^2, the global error scales in τ. This is due to the freezing effect, *ie* the fact that the error in Sobolev space between the solution f of (6.1.2) and ω at time t only scales in t. However the use of a symplectic integrator is essential in our problem: through area preservation, it ensures that for all n, ω_n has average zero, so that we may define $\psi_n = \Delta^{-1}\omega_n$ at each step, and then the midpoint integrator associated with the vector field $J\nabla\psi_n$.
The restriction on the regularity $s \geq 6$ comes from the fact that we shall prove that the local error attributable to the midpoint integrator scales in τ^3. Smaller values of s should be admissible if we are willing to let the local midpoint error to scale in τ^2 only, which is the case for the freezing error anyway. In view of (6.1.3), one should require the numerical velocity vector field $U(\omega_n)$ to be at least Lipschitz, in order to solve the implicit midpoint equation. This should impose at least the restriction $s \geq 4$.

c) Our proof may be compared with the classical *backward error analysis* methods widely used in geometric numerical integration (see [37], [45]): the semi-discrete operator $\mathcal{S}_\tau(\omega_0)$ coincides, at time $t = \tau$, with $\mathcal{S}_t(\omega_0)$. and it turns out that $\mathcal{S}_t(\omega_0)$ satisfies a transport equation. The local consistency errors are then obtain by means of standard energy estimates for transport equations, essentially. From that perspective our proof may be related to the paper [25], where convergence estimates are proved for time-discretizations of the Vlasov-Poisson equation by splitting methods, by means of stability estimates for the associated transport operator.

d) The main difficulty of the proof is to obtain stability estimates for the operator $\mathcal{S}_t(\omega_0) = \omega_0 \circ \Phi_t$, essentially because Φ_t does not satisfy an ODE of the type

$$\partial_t \Phi_t(z) = F(\Phi_t(z)).$$

The trick is to use the fact that the implicit midpoint integrator is the composition of the forward and backward Euler integrators, with half-time step. More precisely, set

$$\mathcal{E}_t(z) = z + \frac{t}{2} J\nabla\psi(z) \quad \text{and} \quad \mathcal{E}_t^* = \mathcal{E}_{-t}^{-1}(z),$$

then

$$\Phi_t = \mathcal{E}_t \circ \mathcal{E}_t^*.$$

One may then easily derive ODE's like

$$\partial_t \mathcal{E}_t(z) = F_1(t, z) \quad \text{and} \quad \partial_t \mathcal{E}_t^*(z) = F_2(t, \mathcal{E}_t^*(z)),$$

and obtain stability estimates for $g \circ \mathcal{E}_t$ and $g \circ \mathcal{E}_t^*$ in H^s, for any function $g \in H^s$, by means of standard energy estimates on transport equations.

e) The proof proceeds in three steps. In section 3 we derive general stability estimates for certain transport equations, which will be the main tool of the chapter. In section 4 we prove stability estimates for the semi-discrete operator using the ideas described in **d)**. In section 5, we analyse the local errors attributable to the freezing of the velocity vector field and to the midpoint discretization, by means described in **c)**, and prove the main result.

6.3 Stability estimates for the exact flows

6.3.1 Notations

Let $s \geq 2$ and $\omega_0 \in H^s(\mathbb{T}^2)$. The solution $\omega(t,z)$ of the Euler equation with initial data ω_0 given by Theorem 6.2.2, namely

$$\begin{cases} \partial_t \omega - J\nabla\Delta^{-1}\omega \cdot \nabla\omega = 0 \\ \omega(t=0,z) = \omega_0(z), \end{cases} \tag{6.3.1}$$

will be from now on written

$$\varphi_{E,t}(\omega_0)(z) = \omega(t,z). \tag{6.3.2}$$

Let ψ be the solution of the Poisson equation $\Delta\psi = \omega_0$. Proposition 6.6.2 and the Sobolev embedding (6.2.3) imply that

$$\sup_{0\leq|\alpha|\leq s} \|\partial_z^\alpha \psi\|_{L^\infty} \leq C \|\psi\|_{H^{s+2}(\mathbb{T}^2)} \leq C \|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

The Cauchy-Lipschitz Theorem ensures then that the flow $\Psi^t(z)$ associated with the vector field $J\nabla\psi$ is well-defined and exists globally in time, and the function $f(t,z) = \omega_0(\Psi^t(z))$ solves globally in time the frozen equation

$$\begin{cases} \partial_t f - J\nabla\psi \cdot \nabla f = 0 \\ \Delta\psi = \omega_0 \\ f(t=0,z) = \omega_0(z) \end{cases} \tag{6.3.3}$$

with initial data ω_0. We shall write as previously

$$\varphi_{F,t}(\omega_0)(z) = f(t,z), \tag{6.3.4}$$

6.3.2 A stability Lemma for some transport equations

We first prove, in the Lemma below, estimates for transport operators, that will in particular apply to Euler's equation and to the frozen equation.
For a two-dimensional vector field $X : \mathbb{T}^2 \to \mathbb{T}^2$, let $L(X)$ be the operator defined for functions g by

$$L(X)g = X \cdot \nabla g, \tag{6.3.5}$$

and let us define for $\alpha \in \mathbb{N}^2$ the commutator

$$[\partial_z^\alpha, L(X)] = \partial_z^\alpha L(X) - L(X)\partial_z^\alpha. \tag{6.3.6}$$

Lemma 6.3.1. *For any $s \geq 0$, there exists a constant $C > 0$ such that, for indices $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq s$, vector fields X and functions g, we have*

$$\|[\partial_z^\alpha, L(X)]\, g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{s+1}(\mathbb{T}^2)} \left[\|g\|_{H^s(\mathbb{T}^2)} + \|g\|_{H^2(\mathbb{T}^2)} \right], \tag{6.3.7}$$

$$\|[\partial_z^\alpha, L(X)]\, g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{s+2}(\mathbb{T}^2)} \left[\|g\|_{H^s(\mathbb{T}^2)} + \|g\|_{H^1(\mathbb{T}^2)} \right], \tag{6.3.8}$$

$$\|[\partial_z^\alpha, L(X)]\, g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^s(\mathbb{T}^2)} \left[\|g\|_{H^s(\mathbb{T}^2)} + \|g\|_{H^3(\mathbb{T}^2)} \right], \tag{6.3.9}$$

and

$$\|\partial_z^\alpha L(X) g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{s+1}(\mathbb{T}^2)} \left[\|g\|_{H^{s+1}(\mathbb{T}^2)} + \|g\|_{H^2(\mathbb{T}^2)} \right]. \tag{6.3.10}$$

Proof. We will prove estimates (6.3.7), (6.3.8) and (6.3.9). The proof of estimate (6.3.10) is almost identical to the proof of estimate (6.3.7), and easier, so we will not detail it.
All of this is obvious when $|\alpha| = 0$, and for $|\alpha| \geq 1$, the starting point is to use Leibniz's formula as follows

$$[\partial_z^\alpha, L(X)]\, g = \partial_z^\alpha (X \cdot \nabla g) - X \cdot \partial_z^\alpha \nabla g = \sum_{\substack{\beta+\gamma=\alpha \\ \gamma \neq \alpha}} \binom{\alpha}{\beta} \partial_z^\beta X \cdot \partial_z^\gamma \nabla g.$$

Any term of the sum may be estimated using the Sobolev embeddings (6.2.2) or (6.2.3) in several ways, which will produce the three estimates (6.3.7), (6.3.8) and (6.3.9).

Proof of estimate (6.3.7)

When $|\gamma| \geq 1$, we have by the Sobolev embedding (6.2.3)

$$\begin{aligned} \left\| \partial_z^\beta X \cdot \partial_z^\gamma \nabla g \right\|_{L^2(\mathbb{T}^2)} &\leq \left\| \partial_z^\beta X \right\|_{L^\infty(\mathbb{T}^2)} \|\partial_z^\gamma \nabla g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|+2}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|-|\gamma|+2}(\mathbb{T}^2)} \|g\|_{H^s(\mathbb{T}^2)} \leq C \|X\|_{H^{s+1}(\mathbb{T}^2)} \|g\|_{H^s(\mathbb{T}^2)}. \end{aligned}$$

When $|\gamma| = 0$, we have by Hölder's inequality and the Sobolev embedding (6.2.2)

$$\begin{aligned} \left\| \partial_z^\beta X \cdot \nabla g \right\|_{L^2(\mathbb{T}^2)} &\leq \left\| \partial_z^\beta X \right\|_{L^4(\mathbb{T}^2)} \|\nabla g\|_{L^4(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|+1}(\mathbb{T}^2)} \|g\|_{H^2(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|+1}(\mathbb{T}^2)} \|g\|_{H^2(\mathbb{T}^2)} \leq C \|X\|_{H^{s+1}(\mathbb{T}^2)} \|g\|_{H^2(\mathbb{T}^2)}. \end{aligned}$$

This proves estimate (6.3.7).

Proof of estimate (6.3.8)

It suffices to use Sobolev's embedding (6.2.3) for any $\gamma \neq \alpha$, as follows

$$\begin{aligned} \left\| \partial_z^\beta X \cdot \partial_z^\gamma \nabla g \right\|_{L^2(\mathbb{T}^2)} &\leq \left\| \partial_z^\beta X \right\|_{L^\infty(\mathbb{T}^2)} \|\partial_z^\gamma \nabla g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|+2}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|-|\gamma|+2}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \leq C \|X\|_{H^{s+2-|\gamma|}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)}. \end{aligned}$$

For any $|\gamma| > 0$, the RHS is

$$C \|X\|_{H^{s+2-|\gamma|}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \leq C \|X\|_{H^{s+2}(\mathbb{T}^2)} \|g\|_{H^s(\mathbb{T}^2)},$$

and for $|\gamma| = 0$, it is

$$C \|X\|_{H^{s+2}(\mathbb{T}^2)} \|g\|_{H^1(\mathbb{T}^2)}.$$

This proves (6.3.8).

Proof of estimate (6.3.9)

For any $|\gamma| \geq 2$, we may use the Sobolev embedding (6.2.3) as previously

$$\begin{aligned}\left\|\partial_z^\beta X \cdot \partial_z^\gamma \nabla g\right\|_{L^2(\mathbb{T}^2)} &\leq \left\|\partial_z^\beta X\right\|_{L^\infty(\mathbb{T}^2)} \|\partial_z^\gamma \nabla g\|_{L^2(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|+2}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|-|\gamma|+2}(\mathbb{T}^2)} \|g\|_{H^{|\gamma|+1}(\mathbb{T}^2)} \leq C \|X\|_{H^s(\mathbb{T}^2)} \|g\|_{H^s(\mathbb{T}^2)}.\end{aligned}$$

When $|\gamma| = 1$, we may use the Sobolev embedding (6.2.2) and Hölder's inequality

$$\begin{aligned}\left\|\partial_z^\beta X \cdot \partial_z^\gamma \nabla g\right\|_{L^2(\mathbb{T}^2)} &\leq \left\|\partial_z^\beta X\right\|_{L^4(\mathbb{T}^2)} \|\partial_z^\gamma \nabla g\|_{L^4(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|+1}(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|-|\gamma|+1}(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)} \leq C \|X\|_{H^s(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)}.\end{aligned}$$

Finally, when $|\gamma| = 0$, Sobolev's embedding (6.2.3) gives us the estimate

$$\begin{aligned}\left\|\partial_z^\beta X \cdot \nabla g\right\|_{L^2(\mathbb{T}^2)} &\leq \left\|\partial_z^\beta X\right\|_{L^2(\mathbb{T}^2)} \|\nabla g\|_{L^\infty(\mathbb{T}^2)} \leq C \|X\|_{H^{|\beta|}(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)} \\ &\leq C \|X\|_{H^{|\alpha|}(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)} \leq C \|X\|_{H^s(\mathbb{T}^2)} \|g\|_{H^3(\mathbb{T}^2)}.\end{aligned}$$

Collecting the three previous estimates yields (6.3.9). □

The previous Lemma allows us to obtain the following stability result, which will be the main technical tool of the chapter.

Lemma 6.3.2. *For any $s \geq 0$, there exists a constant $C > 0$ such that, for vector fields X and functions h, if g solves the equation*

$$\partial_t g - X \cdot \nabla g = h.$$

then for all $t \in \mathbb{R}$, g enjoys the estimates

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 \leq & C \left[\|X(t)\|_{H^{s+1}(\mathbb{T}^2)} \left(\|g(t)\|_{H^s(\mathbb{T}^2)} + \|g(t)\|_{H^2(\mathbb{T}^2)}\right)\right] \|g(t)\|_{H^s(\mathbb{T}^2)} \\ &+ \|\nabla \cdot X(t)\|_{L^\infty(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 + 2 \|h(t)\|_{H^s(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)},\end{aligned} \tag{6.3.11}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 \leq & C \left[\|X(t)\|_{H^{s+2}(\mathbb{T}^2)} \left(\|g(t)\|_{H^s(\mathbb{T}^2)} + \|g(t)\|_{H^1(\mathbb{T}^2)}\right)\right] \|g(t)\|_{H^s(\mathbb{T}^2)} \\ &+ \|\nabla \cdot X(t)\|_{L^\infty(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 + 2 \|h(t)\|_{H^s(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)},\end{aligned} \tag{6.3.12}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 \leq & C \left[\|X(t)\|_{H^s(\mathbb{T}^2)} \left(\|g(t)\|_{H^s(\mathbb{T}^2)} + \|g(t)\|_{H^3(\mathbb{T}^2)}\right)\right] \|g(t)\|_{H^s(\mathbb{T}^2)} \\ &+ \|\nabla \cdot X(t)\|_{L^\infty(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)}^2 + 2 \|h(t)\|_{H^s(\mathbb{T}^2)} \|g(t)\|_{H^s(\mathbb{T}^2)}.\end{aligned} \tag{6.3.13}$$

Proof. We prove the Lemma by an energy estimate. With the notations introduced in (6.3.5) and (6.3.6), we have for all $|\alpha| \leq s$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_z^\alpha g(t)\right\|_{L^2(\mathbb{T}^2)}^2 &= \langle \partial_z^\alpha L(X)g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)} + \langle \partial_z^\alpha h, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)} \\ &= \langle [\partial_z^\alpha, L(X)]\, g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)} + \langle L(X)\partial_z^\alpha g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)} + \langle \partial_z^\alpha h, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)}.\end{aligned}$$

The last term is obviously controlled as follows

$$\left|\langle \partial_z^\alpha h, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)}\right| \leq \left\|h(t)\right\|_{H^s(\mathbb{T}^2)} \left\|g(t)\right\|_{H^s(\mathbb{T}^2)}.$$

Also,

$$\langle L(X)\partial_z^\alpha g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} X(t,z)\cdot\nabla\partial_z^\alpha g(t,z)\overline{\partial_z^\alpha g(t,z)}\mathrm{d}z = -\frac{1}{2}\int_{\mathbb{T}^2} \nabla\cdot X(t,z)|\partial_z^\alpha g(t,z)|^2\mathrm{d}z,$$

such that

$$\left|\langle L(X)\partial_z^\alpha g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)}\right| \leq \frac{1}{2}\left\|\nabla\cdot X(t)\right\|_{L^\infty(\mathbb{T}^2)} \left\|g(t)\right\|_{H^s(\mathbb{T}^2)}^2.$$

Finally,

$$\left|\langle [\partial_z^\alpha, L(X)]\, g, \partial_z^\alpha g\rangle_{L^2(\mathbb{T}^2)}\right| \leq \left\|[\partial_z^\alpha, L(X)]\, g\right\|_{L^2(\mathbb{T}^2)} \left\|g(t)\right\|_{H^s(\mathbb{T}^2)},$$

such that estimates (6.3.7), (6.3.8) and (6.3.9) from Lemma 6.3.1 yields respectively the estimates (6.3.11), (6.3.12) and (6.3.13). □

6.3.3 Stability estimates

We will prove in the Proposition below the stability in H^s of the flows $\varphi_{E,t}$ and $\varphi_{F,t}$, defined respectively by (6.3.1) & (6.3.2), and (6.3.3) & (6.3.4).
Throughout this subsection we shall use the following property: for any $s \geq 0$ and function $g \in H^s(\mathbb{T}^2)$, the vector field $J\nabla\Delta^{-1}g$ enjoys the estimate

$$\left\|J\nabla\Delta^{-1}g\right\|_{H^{s+1}(\mathbb{T}^2)} \leq C\left\|g\right\|_{H^s(\mathbb{T}^2)}. \tag{6.3.14}$$

This is an easy consequence of Proposition 6.6.2.

Proposition 6.3.3. *Let $s \geq 2$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0, and $B > 0$ such that $\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)} \leq B$. There exists two positive constants L_0 and L_1, both independent of ω_0, such that, if*

$$T_0 < \frac{1}{L_0 B},$$

then for all $t \in [0, T_0]$,

$$\left\|\varphi_{E,t}(\omega_0)\right\|_{H^s(\mathbb{T}^2)} \leq \min\left(2, e^{BL_1 t}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}, \tag{6.3.15}$$

and

$$\left\|\varphi_{F,t}(\omega_0)\right\|_{H^s(\mathbb{T}^2)} \leq \min\left(2, e^{BL_1 t}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}. \tag{6.3.16}$$

Proof. We begin with the stability estimate (6.3.15). $\varphi_{E,t}(\omega_0)$ satisfies the equation

$$\partial_t \varphi_{E,t}(\omega_0) - J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0)\cdot\nabla\varphi_{E,t}(\omega_0) = 0.$$

Applying Lemma 6.3.2 with the divergence-free vector field $X(t) = J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0)$, estimate (6.3.11) (with $s \geq 2$) implies that for all $t \geq 0$,

$$\|\varphi_{E,t}(\omega_0)\|^2_{H^s(\mathbb{T}^2)} \leq \|\omega_0\|^2_{H^s(\mathbb{T}^2)} + C\int_0^t \|\varphi_{E,\sigma}(\omega_0)\|_{H^s(\mathbb{T}^2)}\|\varphi_{E,\sigma}(\omega_0)\|^2_{H^s(\mathbb{T}^2)}\,\mathrm{d}\sigma.$$

Here we have also used the inequality (6.3.14) with the function $g = \varphi_{E,t}(\omega_0)$.
If $T_0 > 0$ is such that the following estimate holds,

$$\sup_{t\in[0,T_0]}\|\varphi_{E,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq 2\|\omega_0\|_{H^s(\mathbb{T}^2)},$$

the previous inequality implies then that

$$(1 - 2BCT_0)\sup_{t\in[0,T_0]}\|\varphi_{E,t}(\omega_0)\|^2_{H^s(\mathbb{T}^2)} \leq \|\omega_0\|^2_{H^s(\mathbb{T}^2)}.$$

Hence if T_0 is chosen such that $2BCT_0 < 3/4$, we obtain the estimate

$$\sup_{t\in[0,T_0]}\|\varphi_{E,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} < 2\|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

By a bootstrap argument this implies that a time $T_0 > 0$ can be chosen such that, if $2BCT_0 < 3/4$, $\varphi_{E,t}(\omega_0)$ enjoys the estimate

$$\sup_{t\in[0,T_0]}\|\varphi_{E,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq 2\|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

Using Gronwall's Lemma we infer easily that

$$\|\varphi_{E,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq e^{BL_1 t}\|\omega_0\|_{H^s(\mathbb{T}^2)},$$

which proves estimate (6.3.15), with $L_0 = 8C/3$, and $L_1 = 2C$.
To prove estimate (6.3.16), we use the fact that $\varphi_{F,t}(\omega_0)$ satisfies the equation

$$\partial_t \varphi_{F,t}(\omega_0) - J\nabla\Delta^{-1}\omega_0\cdot\nabla\varphi_{F,t}(\omega_0) = 0,$$

with initial data ω_0. Applying once more Lemma 6.3.2 with the divergence-free vector field $X(t) = J\nabla\Delta^{-1}\omega_0$, estimate (6.3.11) yields as previously

$$\|\varphi_{F,t}(\omega_0)\|^2_{H^s(\mathbb{T}^2)} \leq \|\omega_0\|^2_{H^s(\mathbb{T}^2)} + C\int_0^t \|\omega_0\|_{H^s(\mathbb{T}^2)}\|\varphi_{F,\sigma}(\omega_0)\|^2_{H^s(\mathbb{T}^2)}\,\mathrm{d}\sigma,$$

where we have also applied the inequality (6.3.14) with the function $g = \omega_0$.
Hence if $2BCT_0 < 3/4$, it implies that for all $t \in [0, T_0]$,

$$\|\varphi_{F,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq 2\|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

On the other hand, Gronwall's Lemma implies that

$$\|\varphi_{F,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq e^{BL_1 t} \|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

Therefore,

$$\|\varphi_{F,t}(\omega_0)\|_{H^s(\mathbb{T}^2)} \leq \min\left(2, e^{BL_1 t}\right) \|\omega_0\|_{H^s(\mathbb{T}^2)},$$

and estimate (6.3.16) is proven. □

6.4 Numerical stability

6.4.1 Properties of the midpoint flow

Proposition 6.4.1. *Let $s \geq 3$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0. Let ψ be the solution of the Poisson equation $\Delta\psi = \omega_0$, and let $\tau \in]0,1[$. There exists a positive constant R_0, independent of ω_0, such that, if $\tau \|\omega_0\|_{H^2(\mathbb{T}^2)} R_0 < 1$, the following properties hold:*

i) *For all $t \in [0,\tau]$ and $z \in \mathbb{T}^2$, there exists an unique solution $\Phi_t(z)$ of the implicit equation*

$$\Phi_t(z) = z + tJ\nabla\psi\left(\frac{z + \Phi_t(z)}{2}\right). \tag{6.4.1}$$

ii) *The function $\Phi_t(z) : [0,\tau] \times \mathbb{T}^2 \to \mathbb{T}^2$ is C^{s-1}, and for all $t \in [0,\tau]$, $z \mapsto \Phi_t(z)$ is a symplectic global diffeomorphism on $\mathbb{T}^2$. Moreover, $\Phi_t^{-1} = \Phi_{-t}$.*

iii) *For all $t \in [0,\tau]$, the mappings*

$$\mathcal{E}_t(z) = z + \frac{t}{2}J\nabla\psi(z) \quad \text{and} \quad \mathcal{E}_t^*(z) = \mathcal{E}_{-t}^{-1}(z)$$

are as well global diffeomorphisms on $\mathbb{T}^2$, and

$$\Phi_t = \mathcal{E}_t \circ \mathcal{E}_t^*.$$

iv) *Let $V(t,z)$ be the vector field defined by*

$$V(t,z) = \partial_t\Phi_{-t} \circ \Phi_t(z).$$

If $s \geq 4$, there exists a C^{s-4} vector field $(t,z) \mapsto \mathcal{R}(t,z)$ such that

$$V(t,z) = -J\nabla\psi(z) + t^2\mathcal{R}(t,z).$$

Proof.

Proof of assertion **i)**

Let us first note that, using Proposition 6.6.2 and the Sobolev embedding (6.2.3), we have for any $\sigma \geq 0$,

$$\sup_{0\leq|\alpha|\leq\sigma} \|\partial_z^\alpha \psi\|_{L^\infty(\mathbb{T}^2)} \leq C \|\psi\|_{H^{\sigma+2}(\mathbb{T}^2)} \leq C \|\omega_0\|_{H^\sigma(\mathbb{T}^2)}. \tag{6.4.2}$$

The existence (and uniqueness) of $\Phi_t(z)$ follows then easily: for $t \in [0,\tau]$, and $z \in \mathbb{T}^2$, we define a function $F_{t,z} : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$F_{t,z}(\zeta) = z + tJ\nabla\psi\left(\frac{z+\zeta}{2}\right).$$

For any $\zeta_1, \zeta_2 \in \mathbb{T}^2$, we have by the mean-value Theorem

$$|F_{t,z}(\zeta_1) - F_{t,z}(\zeta_2)| = t\left|J\nabla\psi\left(\frac{z+\zeta_1}{2}\right) - J\nabla\psi\left(\frac{z+\zeta_2}{2}\right)\right| \leq \tau C \sup_{|\alpha|=2} \|\partial_z^\alpha\psi\|_{L^\infty(\mathbb{T}^2)} |\zeta_1-\zeta_2| < |\zeta_1-\zeta_2|,$$

provided that, using (6.4.2), $\tau R_0 \|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$, for some appropriate constant $R_0 = R_0(C)$. In that case $F_{t,z}$ is a contraction mapping on $\mathbb{T}^2$, such that Banach's fixed point Theorem gives us an unique solution $\Phi_t(z)$ to the equation

$$F_{t,z}\left(\Phi_t(z)\right) = \Phi_t(z).$$

This proves the assertion **i)**.

Proof of assertion **ii)**

Let us now consider, for $\varepsilon \in]0,1[$, the function $G :]-\varepsilon, \tau+\varepsilon[\times\mathbb{T}^2\times\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$G(t,\zeta,z) = \zeta - z - tJ\nabla\psi\left(\frac{z+\zeta}{2}\right). \tag{6.4.3}$$

Then by (6.4.2), $\nabla\psi$ belongs to H^{s+1}, which is continuously embedded in C^{s-1}, such that G has class C^{s-1} and, in addition, for all $(t,z) \in [0,\tau]\times\mathbb{T}^2$,

$$G(t,\Phi_t(z),z) = 0.$$

Moreover,

$$\partial_\zeta G(t,\Phi_t(z),z) = A_t\left(JY^t(z)\right),$$

with

$$Y^t(z) = \nabla^2\psi\left(\frac{\Phi_t(z)+z}{2}\right), \tag{6.4.4}$$

and where, if Y is a 2×2 square matrix,

$$A_t(Y) = I_2 - \frac{t}{2}Y. \tag{6.4.5}$$

Thanks to (6.4.2),

$$\sup_{|\alpha|=2} \|\partial_z^\alpha \psi\|_{L^\infty(\mathbb{T}^2)} \leq C \|\omega_0\|_{H^2(\mathbb{T}^2)},$$

and thus it is well-known that, if $(\tau+\varepsilon)C\|\omega_0\|_{H^2(\mathbb{T}^2)} < 2$, $A_t\left(JY^t(z)\right)$ is invertible for all $(t,z) \in [0,\tau]\times\mathbb{T}^2$ and

$$A_t\left(JY^t(z)\right)^{-1} = \sum_{n=0}^{+\infty} \frac{t^n}{2^n}\left(JY^t(z)\right)^n. \tag{6.4.6}$$

We may assume that this is true, given the assumption on τ, and choosing ε small enough. As $s-1>1$, we can apply the implicit function Theorem, which shows then that the function $(t,z)\mapsto\Phi_t(z)$ has C^{s-1} regularity on $]-\varepsilon,\tau+\varepsilon[\times\mathbb{T}^2$.
In addition we are allowed to differentiate the equation

$$G(t,\Phi_t(z),z) = 0$$

with respect to z, and it implies that

$$A_t(JY^t(z))\partial_z\Phi_t(z) = A_{-t}(JY^t(z)), \tag{6.4.7}$$

Since $A_t(JY^t(z))$ and $A_{-t}(JY^t(z))$ are invertible, then so is $\partial_z\Phi_t(z)$. By the local inverse Theorem and the open mapping Theorem, $\Phi_t(\cdot)$ is therefore a local diffeomorphism on $\mathbb{T}^2$, and an open mapping. In particular $\Phi_t\left(\mathbb{T}^2\right)$ is open, and, by continuity, also compact, thus closed. By connectedness, we conclude that $\Phi_t(\cdot)$ is onto. It is also one-to-one, since if $\Phi_t(z_1)=\Phi_t(z_2)$, then by the mean-value Theorem

$$|z_1-z_2| = \left|tJ\nabla\psi\left(\frac{z_1+\Phi_t(z_1)}{2}\right) - tJ\nabla\psi\left(\frac{z_2+\Phi_t(z_1)}{2}\right)\right| \leq \frac{tC\|\omega_0\|_{H^2(\mathbb{T}^2)}}{2}|z_1-z_2|,$$

which implies that $z_1=z_2$ if $t\|\omega_0\|_{H^2(\mathbb{T}^2)}C<2$. Thus $\Phi_t(\cdot)$ is a global diffeomorphism mapping $\mathbb{T}^2$ to itself.
It remains to show that it is symplectic, *ie* that

$$\partial_z\Phi_t(z)^\top J\partial_z\Phi_t(z) = J.$$

Using (6.4.7), the identity $J^\top = J^{-1} = -J$ and the symmetry of the matrix $Y^t(z)$, we have

$$\begin{aligned}
& \partial_z\Phi_t(z)^\top J\partial_z\Phi_t(z) = J \\
\Leftrightarrow\quad & A_{-t}(JY^t(z))^\top\left(A_t(JY^t(z))^\top\right)^{-1} J\left(A_t(JY^t(z))\right)^{-1}A_{-t}(JY^t(z)) = J \\
\Leftrightarrow\quad & \left(A_t(JY^t(z))^\top\right)^{-1} J\left(A_t(JY^t(z))\right)^{-1} = \left(A_{-t}(JY^t(z))^\top\right)^{-1} J\left(A_{-t}(JY^t(z))\right)^{-1} \\
\Leftrightarrow\quad & A_t(JY^t(z))JA_t(JY^t(z))^\top = A_{-t}(JY^t(z))JA_{-t}(JY^t(z))^\top \\
\Leftrightarrow\quad & A_t(JY^t(z))JA_{-t}(Y^t(z)J) = A_{-t}(JY^t(z))JA_t(Y^t(z)J) \\
\Leftrightarrow\quad & J = J,
\end{aligned}$$

the last line being easily obtained by expanding each sides of the penultimate equality. Finally, as we have for all $z \in \mathbb{T}^2$,

$$\Phi_t(z) = z + tJ\nabla\psi\left(\frac{z+\Phi_t(z)}{2}\right),$$

one infers that for all $z \in \mathbb{T}^2$,

$$z = \Phi_t^{-1}(z) + tJ\nabla\psi\left(\frac{z+\Phi_t^{-1}(z)}{2}\right),$$

which shows that $\Phi_t^{-1} = \Phi_{-t}$.

Proof of assertion **iii)**

The mappings $\mathcal{E}_t$ and $\mathcal{E}_t^*$ are defined by

$$\mathcal{E}_t(z) = z + \frac{t}{2}J\nabla\psi(z) \quad \text{and} \quad \mathcal{E}_t^*(z) = z + \frac{t}{2}J\nabla\psi(\mathcal{E}_t^*(z)).$$

Thus, using the above notation (6.4.5),

$$\partial_z\mathcal{E}_t(z) = A_{-t}(J\nabla^2\psi(z)) \quad \text{and} \quad A_t(J\nabla^2\psi(\mathcal{E}_t^*(z)))\partial_z\mathcal{E}_t^*(z) = I_2,$$

such that we may repeat the previous arguments (local inverse Theorem, open mapping Theorem) to conclude that $\mathcal{E}$ and $\mathcal{E}^*$ are global diffeomorphisms on $\mathbb{T}^2$.
Moreover, $\zeta = \mathcal{E}_t^*(z)$ is by definition the unique solution of the equation

$$\zeta = z + \frac{t}{2}J\nabla\psi(\zeta),$$

which is also solved by $\zeta = \frac{z+\Phi_t(z)}{2}$. Hence $\mathcal{E}_t^*(z) = \frac{z+\Phi_t(z)}{2}$, and

$$\Phi_t = \mathcal{E}_t \circ \mathcal{E}_t^*.$$

Proof of assertion **iv)**

In that part of the proof we shall need the following derivatives of the function G defined by (6.4.3):

$$\begin{cases} \partial_t G(t,\zeta,z) = -J\nabla\psi\left(\dfrac{z+\zeta}{2}\right) \\ \partial_\zeta G(t,\zeta,z) = I_2 - \dfrac{t}{2}J\nabla^2\psi\left(\dfrac{z+\zeta}{2}\right) \\ \partial_z G(t,\zeta,z) = -I_2 - \dfrac{t}{2}J\nabla^2\psi\left(\dfrac{z+\zeta}{2}\right) \\ \partial_z\partial_t G(t,\zeta,z) = \partial_\zeta\partial_t G(t,\zeta,z) = -\dfrac{1}{2}J\nabla^2\psi\left(\dfrac{z+\zeta}{2}\right) \\ \partial_z\partial_\zeta G(t,\zeta,z) = -\dfrac{t}{4}J\nabla^3\psi\left(\dfrac{z+\zeta}{2}\right) \\ \partial_t^2 G(t,\zeta,z) = (0,0)^\top. \end{cases} \tag{6.4.8}$$

We will write the second order Taylor-expansion in time of $\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)$, and for that we will need the expressions of

$$\Phi_t(z), \quad \partial_t \Phi_t(z), \quad \partial_t(\Phi_{-t}(z)) \circ \Phi_t(z), \quad \frac{\mathrm{d}}{\mathrm{d}t}[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)]$$

at time $t = 0$.
First of all, using (6.4.3) and (6.4.8) and evaluating the identities

$$G(t, \Phi_t(z), z) = 0 \quad \text{and} \quad \partial_t G(t, \Phi_t(z), z) + \partial_\zeta G(t, \Phi_t(z), z)\partial_t \Phi_t(z) = 0$$

at $t = 0$ gives us

$$\Phi^0(z) = z \quad \text{and} \quad \partial_t \Phi_t(z)_{|t=0} = J\nabla\psi(z). \tag{6.4.9}$$

In addition, we know that

$$G(-t, \Phi_{-t}(z), z) = 0. \tag{6.4.10}$$

Differentiating (6.4.10) with respect to the time, we obtain

$$-\partial_t G(-t, \Phi_{-t}(z), z) + \partial_\zeta G(-t, \Phi_{-t}(z), z)\partial_t(\Phi_{-t}(z)) = 0.$$

It holds for all $z \in \mathbb{T}^2$, and thus, pulling-back by the map $z \mapsto \Phi_t(z)$, we infer that

$$-\partial_t G(-t, z, \Phi_t(z)) + \partial_\zeta G(-t, z, \Phi_t(z))\,[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)] = 0. \tag{6.4.11}$$

Evaluating (6.4.11) at $t = 0$ and using (6.4.8), we obtain

$$[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)]_{|t=0} = -J\nabla\psi(z). \tag{6.4.12}$$

Differentiating (6.4.11) with respect to t we have,

$$\begin{aligned} -\partial_z\partial_t G(-t, z, \Phi_t(z))\partial_t\Phi_t(z) - \partial_t\partial_\zeta G(-t, z, \Phi_t(z))\,[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)] \\ + \partial_z\partial_\zeta G(-t, z, \Phi_t(z))\partial_t\Phi_t(z) \cdot [\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)] \\ + \partial_\zeta G(t, \Phi_t(z), z)\frac{\mathrm{d}}{\mathrm{d}t}\,[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)] = 0. \end{aligned}$$

Evaluating this expression at $t = 0$ with the help of (6.4.8), (6.4.9) and (6.4.12) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)]_{|t=0} = 0.$$

Using this and (6.4.12), we conclude by a Taylor expansion that for all $t \in [0, \tau]$,

$$\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z) = -J\nabla\psi(z) + \frac{t^2}{2}\mathcal{R}(t, z).$$

Moreover the Taylor remainder has the regularity of

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,[\partial_t(\Phi_{-t}(z)) \circ \Phi_t(z)]\,,$$

which is C^{s-4}, as $(t, z) \mapsto \partial_t(\Phi_{-t}(z))$ is C^{s-2} and $(t, z) \mapsto \Phi_t(z)$ is C^{s-1}. □

Remark 6.4.2. *In particular, if a function g has average 0, then the function $g \circ \Phi_t$ has also average 0, as Φ_t preserves the volume.*
This justifies our choice of a symplectic integrator, as it implies that at each step of the scheme (6.1.4), $\omega_n = \mathcal{S}_\tau^n(\omega_0)$ has average 0, and we may define the divergence-free vector field $J\nabla\Delta^{-1}\omega_n$, and thus compute ω_{n+1}, and so on.

6.4.2 Stability estimates

Our analysis of the stability of the semi-discrete operator defined by (6.1.3) is based on the fact that the implicit midpoint rule is the composition of Euler's backward and forward methods, with half time-steps, as it was shown in the third point of Proposition 6.4.1.
Therefore, to control the regularity (in space) of some function $g \circ \Phi_t$, we shall first analyse the effect of $\mathcal{E}_t$ (Lemma 6.4.3 below), and then the effect of $\mathcal{E}_t^*$ (Lemma 6.4.5 below).

Lemma 6.4.3. *Let $s \geq 3$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0, and $\tau \in]0,1[$. Let ψ be the solution of the Poisson equation $\Delta\psi = \omega_0$, and let $\mathcal{E}_t$ be the half time-step forward Euler integrator defined for $t \in [0,\tau]$ by the formula*

$$\mathcal{E}^t(z) = z + \frac{t}{2} J\nabla\psi(z).$$

There exists two positive constants R_0 and R_1, independent of ω_0, such that, if $\tau \left\|\omega_0\right\|_{H^2(\mathbb{T}^2)} R_0 < 1$, then for all $g \in H^s(\mathbb{T}^2)$ and all $t \in [0,\tau]$,

$$\left\|g \circ \mathcal{E}_t\right\|_{H^s(\mathbb{T}^2)} \leq e^{R_1 t\left(1+t\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}} \left\|g\right\|_{H^s(\mathbb{T}^2)}.$$

Proof. The idea of our proof is to derive a transport equation whose initial data is g and whose final data is $g \circ \mathcal{E}^t$, and to obtain the conclusion by Lemma 6.3.2.
Let us consider the transport equation

$$\begin{cases} \partial_t r(t,z) - X(t,z) \cdot \nabla r(t,z) = 0 \\ r(0,z) = g(z), \end{cases} \tag{6.4.13}$$

with

$$X(t,z) = \frac{1}{2}\left(I_2 + \frac{t}{2} J\nabla^2\psi(z)\right)^{-1} J\nabla\psi(z). \tag{6.4.14}$$

Note that the inversion of the above matrix has already been justified in the proof of Proposition 6.4.1, under the hypothesis $\tau \left\|\omega_0\right\|_{H^2(\mathbb{T}^2)} R_0 < 1$. As

$$\mathcal{E}_{-t}^*(z) = z - \frac{t}{2} J\nabla\psi(\mathcal{E}_{-t}^*(z)),$$

we have

$$\partial_t(\mathcal{E}_{-t}^*(z)) = -X(t, \mathcal{E}_{-t}^*(z)),$$

such that for all $(t,z) \in [0,\tau] \times \mathbb{T}^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} r(t, \mathcal{E}^*_{-t}(z)) = 0,$$

and thus for all $(t,z) \in [0,\tau] \times \mathbb{T}^2$,

$$r(t, \mathcal{E}^*_{-t}(z)) = g(z).$$

In other words, as $\mathcal{E}^*_{-t}(z) = \mathcal{E}_t^{-1}(z)$, we have

$$r(t,z) = g \circ \mathcal{E}_t(z).$$

That being said, using estimate (6.3.13) from Lemma 6.3.2 (with the inequality $s \geq 3$), we may write that for all $t \in [0,\tau]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|r(t,\cdot)\|^2_{H^s(\mathbb{T}^2)} \leq C \|X(t,\cdot)\|_{H^s(\mathbb{T}^2)} \|r(t,\cdot)\|^2_{H^s(\mathbb{T}^2)} + \|\nabla \cdot X(t,\cdot)\|_{L^\infty(\mathbb{T}^2)} \|r(t,\cdot)\|^2_{H^s(\mathbb{T}^2)}.$$

Using Lemma 6.4.4 below and Gronwall's Lemma, we infer that

$$\|r(t,\cdot)\|_{H^s(\mathbb{T}^2)} \leq \|r(0,\cdot)\|_{H^s(\mathbb{T}^2)} e^{R_1 t\left(1+t\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}},$$

which gives the desired conclusion, as $r(0) = g$ and $r(t) = g \circ \mathcal{E}_t$. □

Lemma 6.4.4. *Let $s \geq 3$. For $t \in [-\tau,\tau]$, with τ satisfying the hypothesis $\tau R_0 \|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$ of Proposition 6.4.1 and Lemma 6.4.3, and $z \in \mathbb{T}^2$, let us consider the vector field $X(t,z)$ defined by (6.4.14). There exists a constant $C > 0$ such that*

$$\|X(t,\cdot)\|_{H^s(\mathbb{T}^2)} \leq C\left(1 + |t| \|\omega_0\|_{H^s(\mathbb{T}^2)}\right) \|\omega_0\|_{H^s(\mathbb{T}^2)} \tag{6.4.15}$$

and

$$\|\nabla \cdot X(t,\cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C\left(1 + |t| \|\omega_0\|_{H^s(\mathbb{T}^2)}\right) \|\omega_0\|_{H^s(\mathbb{T}^2)}. \tag{6.4.16}$$

Proof. From definition (6.4.14), we may write that

$$2X(t,z) = J\nabla\psi(z) + F(tJ\nabla^2\psi(z))J\nabla\psi(z),$$

where $F : \mathcal{U} \subset M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a smooth function defined on a sufficiently small neighborhood $\mathcal{U}$ of 0_2, the zero element of the vector space $M_2(\mathbb{R})$, by the formula

$$F(A) = \left(I_2 + \frac{1}{2}A\right)^{-1} - I_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} A^n.$$

Since (see (6.4.2))

$$|t| \left\|J\nabla^2\psi\right\|_{L^\infty(\mathbb{T}^2)} \leq C\tau \|\omega_0\|_{H^2(\mathbb{T}^2)},$$

we way assume, up to a proper modification of R_0, that $tJ\nabla^2\psi$ belongs to $\mathcal{U}$ almost everywhere. Applying Lemma 6.2.1, we infer that

$$\|X(t,\cdot)\|_{H^s(\mathbb{T}^2)} \leq \|J\nabla\psi\|_{H^s(\mathbb{T}^2)} + C_s\left(\left\|tJ\nabla^2\psi\right\|_{L^\infty(\mathbb{T}^2)}\right)\left(1+\left\|tJ\nabla^2\psi\right\|_{H^s(\mathbb{T}^2)}\right)\|J\nabla\psi\|_{H^s(\mathbb{T}^2)},$$

where $C_s : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuous function. In view of the hypothesis $\tau R_0 \|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$ and of estimate (6.4.2), we may assume that for all $|t| \leq \tau$,

$$C_s\left(\left\|tJ\nabla^2\psi\right\|_{L^\infty(\mathbb{T}^2)}\right) \leq C,$$

for some appropriate constant C. Thus we obtain the estimate

$$\|X(t,\cdot)\|_{H^s(\mathbb{T}^2)} \leq C\left(1+|t|\,\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)},$$

which is precisely estimate (6.4.15).
Estimate (6.4.16) follows quickly using the Sobolev embedding (6.2.3) as follows

$$\|\nabla\cdot X(t,\cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C\,\|X(t,\cdot)\|_{H^3(\mathbb{T}^2)} \leq \|X(t,\cdot)\|_{H^s(\mathbb{T}^2)} \leq C\left(1+|t|\,\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)},$$

since $s \geq 3$. ∎

Lemma 6.4.5. *Let $s \geq 3$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0, and $\tau \in]0,1[$. Let ψ be the solution of the Poisson equation $\Delta\psi = \omega_0$, and let $\mathcal{E}_t^*$ be the half time-step backward Euler integrator defined for $t \in [0,\tau]$ by the formula*

$$\mathcal{E}_t^*(z) = z + \frac{t}{2}J\nabla\psi(\mathcal{E}_t^*(z)).$$

There exists two positive constants R_0, R_1, independent of ω_0, such that, if $\tau R_0\|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$, then for all $g \in H^s(\mathbb{T}^2)$ and $t \in [0,\tau]$,

$$\|g\circ\mathcal{E}_t^*\|_{H^s(\mathbb{T}^2)} \leq e^{R_1 t\left(1+t\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}}\|g\|_{H^s(\mathbb{T}^2)}.$$

Proof. Our proof ressembles the proof of Lemma 6.4.3, as we take advantage of the fact that one travels from $g\circ\mathcal{E}_t^*$ to g along the flow $\mathcal{E}_{-t} = \mathcal{E}_t^{*-1}$. Thus, instead of deriving a new equation that will transport us from g to $g\circ\mathcal{E}_t^*$, we shall use once more equation (6.4.13) (with the time reversed, essentially), with this time $g\circ\mathcal{E}_t^*$ for initial data, and g for final data, and we shall conclude by Lemma 6.3.2.
Let us consider the transport equation

$$\begin{cases} \partial_\sigma r(\sigma,t,z) + X(-\sigma,z)\cdot\nabla r(\sigma,t,z) = 0 \\ r(0,t,z) = g\circ\mathcal{E}_t^*(z), \end{cases} \tag{6.4.17}$$

where the auxiliary variable σ belongs to $[0,t]$, and where X was defined by (6.4.14). Note that, as

$$\mathcal{E}_\sigma^*(z) = z + \frac{\sigma}{2}J\nabla\psi(\mathcal{E}_\sigma^*(z)),$$

we have the identity

$$\partial_\sigma \mathcal{E}^*_\sigma(z) = X(-\sigma, \mathcal{E}^*_\sigma(z)).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} r(\sigma, t, \mathcal{E}^*_\sigma(z)) = 0,$$

such that for all $\sigma \in [0,t]$ and $z \in \mathbb{T}^2$,

$$r(\sigma, t, \mathcal{E}^*_\sigma(z)) = r(0,t,z) = g \circ \mathcal{E}^*_t(z),$$

and thus

$$r(t,t,z) = g \circ \mathcal{E}^*_t \circ \mathcal{E}_{-t}(z) = g(z).$$

Therefore the transport equation (6.4.17) has the expected initial and final data. However, it will be more convenient to deal with the function

$$u(\sigma, t, z) = r(-\sigma, t, z),$$

with $\sigma \in [-t, 0]$. (Essentially, we do this to apply Gronwall's Lemma in its usual statement at the end of the proof.)
u satisfies on $[-t,0]$ the transport equation

$$\begin{cases} \partial_\sigma u(\sigma,t,z) - X(\sigma,z)\cdot\nabla u(\sigma,t,z) = 0 \\ u(0,t,z) = r(0,t,z) = g\circ\mathcal{E}^*_t(z), \\ u(-t,t,z) = r(t,t,z) = g(z). \end{cases} \tag{6.4.18}$$

That being said, estimate (6.3.13) from Lemma 6.3.2 shows that for any $\sigma \in [-t,0]$,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left\|u(\sigma,t,\cdot)\right\|^2_{H^s(\mathbb{T}^2)} \leq \left\|X(\sigma,\cdot)\right\|_{H^s(\mathbb{T}^2)}\left\|u(\sigma,t,\cdot)\right\|^2_{H^s(\mathbb{T}^2)} + \left\|\nabla\cdot X(\sigma,\cdot)\right\|_{L^\infty(\mathbb{T}^2)}\left\|u(\sigma,t,\cdot)\right\|^2_{H^s(\mathbb{T}^2)}.$$

Gronwall's Lemma and Lemma 6.4.4 above imply then that

$$\left\|u(\sigma,t,\cdot)\right\|^2_{H^s(\mathbb{T}^2)} \leq \left\|u(-t,t,\cdot)\right\|^2_{H^s(\mathbb{T}^2)} \exp\left(\int_{-t}^{\sigma} 2C\left(1+|\theta|\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\mathrm{d}\theta\right).$$

Taking $\sigma = 0$ gives us the estimate

$$\left\|u(0,t,\cdot)\right\|_{H^s(\mathbb{T}^2)} \leq \left\|u(-t,t,\cdot)\right\|_{H^s(\mathbb{T}^2)} e^{R_1 t\left(1+t\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}},$$

which gives the desired conclusion, using the second and third lines of (6.4.18). □

Proposition 6.4.6. *Let $s \geq 3$, $\omega_0 \in H^s(\mathbb{T}^2)$ with average 0, and $\tau \in]0,1[$. There exists two positive constants R_0, R_1, independent of ω_0, such that, if $\tau R_0 \left\|\omega_0\right\|_{H^2(\mathbb{T}^2)} < 1$, then for all $t \in [0,\tau]$,*

$$\left\|\mathcal{S}_t(\omega_0)\right\|_{H^s(\mathbb{T}^2)} \leq e^{R_1 t\left(1+t\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\right)\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}}\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)},$$

where the operator $\mathcal{S}_t$ is defined by formula (6.1.3).

Proof. As already seen, we may write

$$\mathcal{S}_t(\omega_0) = \omega_0 \circ \Phi_t = \omega_0 \circ \mathcal{E}_t \circ \mathcal{E}_t^*,$$

with $\mathcal{E}_t^*$ and $\mathcal{E}_t$ defined in Proposition 6.4.1. We shall apply Lemma 6.4.5 and 6.4.3. However, note that in these Lemmas, we derived a bound for $g \circ \mathcal{E}_t^*$ (or $g \circ \mathcal{E}_t$) where g is a function that depends only on z. Hence, to apply these Lemmas, we shall consider the function

$$f(\sigma, t, z) = \omega_0 \circ \mathcal{E}_\sigma \circ \mathcal{E}_t^*(z),$$

with $t, \sigma, \in [0, \tau]$.
In view of the hypothesis $\tau R_0 \|\omega_0\|_{H^2(\mathbb{T}^2)} < 1$, we may apply Lemma 6.4.5, which shows that for all $t, \sigma \in [0, \tau]$,

$$\|f(\sigma, t, \cdot)\|_{H^s(\mathbb{T}^2)} \leq e^{Ct\left(1+t\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}} \|\omega_0 \circ \mathcal{E}_\sigma\|_{H^s(\mathbb{T}^2)},$$

for some constant $C > 0$.
Applying now Lemma 6.4.3, we may also write that for all $\sigma \in [0, \tau]$,

$$\|\omega_0 \circ \mathcal{E}_\sigma\|_{H^s(\mathbb{T}^2)} \leq e^{R_1 t\left(1+\sigma\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}} \|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

Hence, for all $t, \sigma \in [0, \tau]$,

$$\|f(\sigma, t, \cdot)\|_{H^s(\mathbb{T}^2)} \leq e^{Ct\left(2+(t+\sigma)\|\omega_0\|_{H^s(\mathbb{T}^2)}\right)\|\omega_0\|_{H^s(\mathbb{T}^2)}} \|\omega_0\|_{H^s(\mathbb{T}^2)}.$$

This gives the result by taking $\sigma = t$, and $R_1 = 2C$. □

Corollary 6.4.7. *Let $s \geq 5$, and $u, v \in H^s(\mathbb{T}^2)$ with average 0. Let $\tau \in]0, 1[$. There exists two positive constants R_0, R_1, independent of u and v, such that, if*

$$\tau R_0 \max\left(\|u\|_{H^2(\mathbb{T}^2)}, \|v\|_{H^{s-3}(\mathbb{T}^2)}\right) < 1,$$

for all $t \in [0, \tau]$,

$$\|\mathcal{S}_t(u) - \mathcal{S}_t(v)\|_{H^{s-4}(\mathbb{T}^2)} \leq \|u - v\|_{H^{s-4}(\mathbb{T}^2)} e^{\tau R_1\left(1+\|u\|_{H^{s-3}(\mathbb{T}^2)}\right)} + R_1\tau^3 \|u\|_{H^{s-3}(\mathbb{T}^2)}.$$

where the operator $\mathcal{S}_t$ is defined by formula (6.1.3).

Proof. We may apply Proposition 6.4.1 and define on $[0, \tau]$ the midpoint integrator associated with u, namely

$$\Phi_t(z) = z + tJ\nabla\Delta^{-1}u\left(\frac{z + \Phi^t(z)}{2}\right).$$

As

$$\mathcal{S}_t(u) = u \circ \Phi_t,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{S}_t(u)\circ\Phi_{-t}\right]=0,$$

such that $\mathcal{S}_t(u)$ satisfies the transport equation

$$\begin{cases}\partial_t\mathcal{S}_t(u)+V_u(t)\cdot\nabla\mathcal{S}_t(u)=0,\\ V_u(t,z)=\partial_t\Phi_{-t}\circ\Phi_t(z).\end{cases}$$

Moreover, assertions **ii)** and **iv)** from Proposition 6.4.1 show that V_u has C^{s-2} regularity, and that there exists a C^{s-4} vector field $R_u(t,z)$ such that for all $(t,z)\in[0,\tau]\times\mathbb{T}^2$,

$$V_u(t,z)=-J\nabla\Delta^{-1}u+t^2R_u(t,z).$$

With the same arguments, there exists a C^{s-2} vector field $V_v(t,z)$ such that

$$\begin{cases}\partial_t\mathcal{S}_t(v)+V_v(t)\cdot\nabla\mathcal{S}_t(v)=0,\\ V_v(t,z)=-J\nabla\Delta^{-1}v+t^2R_v(t,z).\end{cases}$$

Moreover $R_v(t,z)$ has C^{s-4} regularity. Therefore $\mathcal{S}_t(u)-\mathcal{S}_t(v)$ solves the equation

$$\partial_t(\mathcal{S}_t(u)-\mathcal{S}_t(v))+V_u(t)\cdot\nabla\left(\mathcal{S}_t(u)-\mathcal{S}_t(v)\right)=(V_v(t)-V_u(t))\cdot\nabla\mathcal{S}_t(v).$$

Applying estimate (6.3.12) from Lemma 6.3.2, we infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\right\|^2_{H^{s-4}(\mathbb{T}^2)}&\leq C\left\|V_u(t)\right\|_{H^{s-2}(\mathbb{T}^2)}\left\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\right\|^2_{H^{s-4}(\mathbb{T}^2)}\\&\quad+\left\|\nabla\cdot V_u(t)\right\|_{L^\infty(\mathbb{T}^2)}\left\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\right\|^2_{H^{s-4}(\mathbb{T}^2)}\\&\quad+2\left\|(V_v(t)-V_u(t))\cdot\nabla\mathcal{S}_t(v)\right\|_{H^{s-4}(\mathbb{T}^2)}\left\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\right\|_{H^{s-4}(\mathbb{T}^2)}.\end{aligned}$$

Since $V_u(t)$ has class C^{s-2}, with $s\geq 5$, we may write that for all $t\in[0,\tau]$,

$$\left\|V_u(t)\right\|_{H^{s-2}(\mathbb{T}^2)}\leq C\quad\text{and}\quad\left\|\nabla\cdot V_u(t)\right\|_{L^\infty(\mathbb{T}^2)}\leq C.$$

Also, we may find a C^{s-4} vector field $R(t,z)$ such that

$$V_v(t)-V_u(t)=J\nabla\Delta^{-1}u-J\nabla\Delta^{-1}v+t^2R(t,z).$$

Using estimate (6.3.10) from Lemma 6.3.1, this implies that

$$\begin{aligned}\left\|(V_v(t)-V_u(t))\cdot\nabla\mathcal{S}_t(v)\right\|_{H^{s-4}(\mathbb{T}^2)}&\leq C\left(\left\|J\nabla\Delta^{-1}u-J\nabla\Delta^{-1}v\right\|_{H^{s-3}(\mathbb{T}^2)}+t^2\right)\left\|\mathcal{S}_t(v)\right\|_{H^{s-3}(\mathbb{T}^2)}\\&\leq C\left(\left\|u-v\right\|_{H^{s-4}(\mathbb{T}^2)}+t^2\right)\left\|\mathcal{S}_t(v)\right\|_{H^{s-3}(\mathbb{T}^2)}.\end{aligned}$$

Using Proposition 6.4.6, we may write, under the hypothesis $\tau R_0\left\|v\right\|_{H^{s-3}(\mathbb{T}^2)}<1$,

$$\left\|\mathcal{S}_t(v)\right\|_{H^{s-3}(\mathbb{T}^2)}\leq C\left\|v\right\|_{H^{s-3}(\mathbb{T}^2)}.$$

Collecting the previous estimates, and applying Lemma 6.5.2, we infer that

$$\begin{aligned}\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\|_{H^{s-4}(\mathbb{T}^2)} \leq \|u-v\|_{H^{s-4}(\mathbb{T}^2)} + \int_0^t C\,\|\mathcal{S}_\sigma(u)-\mathcal{S}_\sigma(v)\|_{H^{s-4}(\mathbb{T}^2)}\,\mathrm{d}\sigma \\ + C\tau\,\|u\|_{H^{s-3}(\mathbb{T}^2)}\left(\|u-v\|_{H^{s-4}(\mathbb{T}^2)}+\tau^2\right).\end{aligned}$$

Applying Gronwall's Lemma we conclude that

$$\|\mathcal{S}_t(u)-\mathcal{S}_t(v)\|_{H^{s-4}(\mathbb{T}^2)} \leq \|u-v\|_{H^{s-4}(\mathbb{T}^2)}\,(1+C\tau\,\|u\|_{H^{s-3}(\mathbb{T}^2)})e^{\tau C} + C\tau^3\,\|u\|_{H^{s-3}(\mathbb{T}^2)}\,e^{C\tau}.$$

One obtains then the desired conclusion with the inequality

$$1+x\leq e^x$$

that holds for any $x\geq 0$, and with an appropriate choice of the constant $R_1 = R_1(C)$. □

6.5 Convergence estimates

6.5.1 Local errors

Proposition 6.5.1. *Let $s\geq 2$ and $\omega_0\in H^s(\mathbb{T}^2)$ with average 0. There exists two positive constants R_0, R_1 independent of ω_0, such that, if $\tau R_0\,\|\omega_0\|_{H^s(\mathbb{T}^2)}<1$, then for all $t\in[0,\tau]$,*

$$\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\|_{H^{s-1}(\mathbb{T}^2)} \leq R_1\tau^2\,\|\omega_0\|^3_{H^s(\mathbb{T}^2)}.$$

Proof. $\varphi_{E,t}(\omega_0)$ and $\varphi_{F,t}(\omega_0)$ satisfy respectively the transport equations

$$\partial_t\varphi_{E,t}(\omega_0) - J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0)\cdot\nabla\varphi_{E,t}(\omega_0)=0$$

and

$$\partial_t\varphi_{F,t}(\omega_0) - J\nabla\Delta^{-1}\omega_0\cdot\nabla\varphi_{F,t}(\omega_0)=0,$$

with initial data ω_0.
Hence

$$\begin{aligned}\partial_t(\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)) - J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0)\cdot\nabla\left(\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right) \\ = J\nabla\left(\Delta^{-1}\omega_0-\Delta^{-1}\varphi_{E,t}(\omega_0)\right)\cdot\nabla\varphi_{F,t}(\omega_0).\end{aligned}$$

Therefore, using estimate (6.3.12) from Lemma 6.3.2, applied with the divergence-free vector field

$$X(t) = J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0),$$

which satisfies (using Proposition 6.6.2)

$$\left\|J\nabla\Delta^{-1}\varphi_{E,t}(\omega_0)\right\|_{H^{s+1}(\mathbb{T}^2)} \leq C\,\|\varphi_{E,t}(\omega_0)\|_{H^s(\mathbb{T}^2)}.$$

we obtain the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|^2_{H^{s-1}(\mathbb{T}^2)} \le C\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|^2_{H^{s-1}(\mathbb{T}^2)}\left\|\varphi_{E,t}(\omega_0)\right\|_{H^s(\mathbb{T}^2)} + 2\left\|J\nabla\left(\Delta^{-1}\omega_0-\Delta^{-1}\varphi_{E,t}(\omega_0)\right)\cdot\nabla\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}.$$

In view of the hypothesis $\tau R_0\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}<1$ we may apply Proposition 6.3.3, which shows that

$$\left\|\varphi_{E,t}(\omega_0)\right\|_{H^s(\mathbb{T}^2)} \le C\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}.$$

Also, using estimate (6.3.10) form Lemma 6.3.1, and Propositions 6.3.3 and 6.6.2, we have

$$\begin{aligned}\left\|J\nabla\left(\Delta^{-1}\omega_0-\Delta^{-1}\varphi_{E,t}(\omega_0)\right)\cdot\nabla\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)} &\le C\left\|\Delta^{-1}(\omega_0-\varphi_{E,t}(\omega_0))\right\|_{H^{s+1}(\mathbb{T}^2)}\left\|\varphi_{F,t}(\omega_0)\right\|_{H^s(\mathbb{T}^2)}\\ &\le C\left\|\omega_0-\varphi_{E,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\end{aligned}$$

However, using once more Proposition 6.3.3 and estimate (6.3.10) from Lemma 6.3.1,

$$\begin{aligned}\left\|\varphi_{E,t}(\omega_0)-\omega_0\right\|_{H^{s-1}(\mathbb{T}^2)} &\le \int_0^t\left\|J\nabla\Delta^{-1}\varphi_{E,\sigma}(\omega_0)\cdot\nabla\varphi_{E,\sigma}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}\mathrm{d}\sigma\\ &\le \int_0^t\left\|\varphi_{E,\sigma}(\omega_0)\right\|^2_{H^s(\mathbb{T}^2)} \le Ct\left\|\omega_0\right\|^2_{H^s(\mathbb{T}^2)}.\end{aligned}$$

Collecting the previous estimates, we infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|^2_{H^{s-1}(\mathbb{T}^2)} &\le C\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|^2_{H^{s-1}(\mathbb{T}^2)}\\ &\quad + C\left\|\omega_0\right\|^3_{H^s(\mathbb{T}^2)}t\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}.\end{aligned}$$

Using Lemma 6.5.2 below, we conclude that

$$\begin{aligned}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)} &\le C\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}\int_0^t\left\|\varphi_{E,\sigma}(\omega_0)-\varphi_{F,\sigma}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)}\mathrm{d}\sigma\\ &\quad + C\left\|\omega_0\right\|^3_{H^s(\mathbb{T}^2)}\int_0^t\sigma\mathrm{d}\sigma.\end{aligned}$$

Thus, if $\tau C\left\|\omega_0\right\|_{H^s(\mathbb{T}^2)}<1$, which we may assume, choosing if necessary $R_0>C$, we have

$$\sup_{t\in[0,\tau]}\left\|\varphi_{E,t}(\omega_0)-\varphi_{F,t}(\omega_0)\right\|_{H^{s-1}(\mathbb{T}^2)} \le R_1\left\|\omega_0\right\|^3_{H^s(\mathbb{T}^2)}\tau^2,$$

for some appropriate constant $R_1(C)$. □

The previous proof uses the following result, inspired by Lemma 2.9 of [36].

Lemma 6.5.2. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function, and $y : \mathbb{R} \to \mathbb{R}_+$ be a differentiable function satisfying the inequality*

$$\forall t \in \mathbb{R}, \quad \frac{\mathrm{d}}{\mathrm{d}t} y(t) \leq 2C_1 y(t) + 2C_2 \sqrt{y(t)} f(t),$$

where C_1 and C_2 are two positive constants. Then

$$\forall t \in \mathbb{R}, \quad \sqrt{y(t)} \leq \sqrt{y(0)} + C_1 \int_0^t \sqrt{y(\sigma)} \mathrm{d}\sigma + C_2 \int_0^t f(\sigma) \mathrm{d}\sigma.$$

Proof. For $\varepsilon > 0$, we define $y_\varepsilon = y + \varepsilon$. We have then

$$\frac{\mathrm{d}}{\mathrm{d}t} \sqrt{y_\varepsilon(t)} = \frac{1}{2\sqrt{y_\varepsilon(t)}} \frac{\mathrm{d}}{\mathrm{d}t} y(t) \leq \frac{C_1 y(t)}{\sqrt{y_\varepsilon(t)}} + \frac{C_2 \sqrt{y(t)}}{\sqrt{y_\varepsilon(t)}} f(t).$$

Therefore

$$\sqrt{y_\varepsilon(t)} \leq \sqrt{y_\varepsilon(0)} + C_1 \int_0^t \frac{y(\sigma)}{\sqrt{y_\varepsilon(\sigma)}} \mathrm{d}\sigma + C_2 \int_0^t \frac{\sqrt{y(\sigma)}}{\sqrt{y_\varepsilon(\sigma)}} f(\sigma) \mathrm{d}\sigma.$$

Taking then the limit $\varepsilon \to 0$ proves the Lemma. □

Proposition 6.5.3. *Let $s \geq 5$ and $\omega_0 \in H^s(\mathbb{T}^2)$, with average 0. There exists two positive constants R_0 and R_1, independent of ω_0, such that, if $\tau \|\omega_0\|_{H^{s-3}(\mathbb{T}^2)} R_0 < 1$, then for all $t \in [0, \tau]$,*

$$\|\varphi_{F,t}(\omega_0) - \mathcal{S}_t(\omega_0)\|_{H^{s-4}(\mathbb{T}^2)} \leq R_1 \tau^3 \|\omega_0\|_{H^{s-3}(\mathbb{T}^2)}.$$

Proof. In view of the hypothesis $\tau \|\omega_0\|_{H^{s-3}(\mathbb{T}^2)} R_0 < 1$ and $s \geq 5$, we may assume that we are in the frame of Propositions 6.4.1 and 6.4.6, and thus that their respective conclusions hold.
That being said, it was shown in the proof of Corollary 6.4.7 that there exists a C^{s-2} vector field $V(t, z)$ such that $\mathcal{S}_t(\omega_0)$ solves the equation

$$\partial_t \mathcal{S}_t(\omega_0) + V(t, \cdot) \cdot \nabla \mathcal{S}_t(\omega_0) = 0,$$

on $[0, \tau]$, with initial data ω_0. Moreover, it was also shown with the help of Proposition 6.4.1, that there exists a C^{s-4} vector field $\mathcal{R} : [0, \tau] \times \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$V(t, z) + J\nabla\Delta^{-1}\omega_0(z) = t^2 \mathcal{R}(t, z). \tag{6.5.1}$$

Meanwhile, $\varphi_{F,t}(\omega_0)$ satisfies the equation

$$\partial_t \varphi_{F,t}(\omega_0) - J\nabla\Delta^{-1}\omega_0 \cdot \nabla \varphi_{F,t}(\omega_0) = 0.$$

Hence we have

$$\partial_t \left(\varphi_{F,t}(\omega_0) - \mathcal{S}_t(\omega_0)\right) - J\nabla\Delta^{-1}\omega_0 \cdot \nabla \left(\varphi_{F,t}(\omega_0) - \mathcal{S}_t(\omega_0)\right) = \left(V + J\nabla\Delta^{-1}\omega_0\right) \cdot \nabla \mathcal{S}_t(\omega_0).$$

Therefore, using estimate (6.3.12) from Lemma 6.3.2, applied to the vector field

$$X = J\nabla\Delta^{-1}\omega_0,$$

that satisfies (using Proposition 6.6.2)

$$\left\|J\nabla\Delta^{-1}\omega_0\right\|_{H^{s-2}(\mathbb{T}^2)} \leq C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)},$$

we obtain the estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|^2_{H^{s-4}(\mathbb{T}^2)} &\leq C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|^2_{H^{s-4}(\mathbb{T}^2)}\\ &\quad+2\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)}\left\|\left(V+J\nabla\Delta^{-1}\omega_0\right)\cdot\nabla\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)}.\end{aligned}$$

Moreover, using identity (6.5.1) and the C^{s-4} regularity of $\mathcal{R}$, we have

$$\left\|\left(V+J\nabla\Delta^{-1}\omega_0\right)\cdot\nabla\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq Ct^2\left\|\mathcal{S}_t(\omega_0)\right\|_{H^{s-3}(\mathbb{T}^2)}.$$

By Proposition 6.4.6 and the hypothesis $\tau R_0\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}<1$, we may write that

$$\left\|\mathcal{S}_t(\omega_0)\right\|_{H^{s-3}(\mathbb{T}^2)} \leq C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}.$$

Collecting the previous estimates, we infer that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|^2_{H^{s-4}(\mathbb{T}^2)} &\leq C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|^2_{H^{s-4}(\mathbb{T}^2)}\\ &\quad+C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}t^2\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)}.\end{aligned}$$

Lemma 6.5.2 gives us then the estimate

$$\begin{aligned}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} &\leq C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}\int_0^t\left\|\varphi_{F,\sigma}(\omega_0)-\mathcal{S}_\sigma(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)}\mathrm{d}\sigma\\ &\quad+C\tau^3\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}.\end{aligned}$$

Therefore, if $\tau C\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)}<1$, which we may assume, choosing if necessary $R_0>C$, we conclude that

$$\sup_{t\in[0,\tau]}\left\|\varphi_{F,t}(\omega_0)-\mathcal{S}_t(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq R_1\tau^3\left\|\omega_0\right\|_{H^{s-3}(\mathbb{T}^2)},$$

for some appropriate constant $R_1(C)$. □

6.5.2 An *a priori* global error estimate

Proposition 6.5.4. *Let $s\geq 5$, and $\omega_0\in H^s(\mathbb{T}^2)$ with average 0. Let $\omega(t)\in C^0\left(\mathbb{R}_+,H^s(\mathbb{T}^2)\right)$ be the unique solution of equation (6.1.1) with initial data ω_0, given by Theorem 6.2.2. For a time step $\tau\in]0,1[$, let $(\omega_n)_{n\in\mathbb{N}}$ be the sequence of functions starting from ω_0 and defined*

by formula (6.1.4) *from iterations of the semi-discrete operator* (6.1.3). *Assume that there exists a time* $T_0 > 0$ *and a constant* $B > 0$ *such that*

$$\sup_{t\in[0,T_0]} \|\omega(t)\|_{H^s(\mathbb{T}^2)} \leq B \quad \textit{and} \quad \sup_{t_n\leq T_0} \|\omega_n\|_{H^2(\mathbb{T}^2)} \leq 2B.$$

Then there exists two positive constants R_0, R_1 *such that, if* $\tau R_0 B < 1$,

$$\|\omega_n - \omega(t_n)\|_{H^{s-4}(\mathbb{T}^2)} \leq \tau R(B) t_n e^{R_1 T_0(1+B)},$$

for all $t_n \leq T_0 + \tau$, *where* $R : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing continuous function that satisfies*

$$R(B) \leq R_1 \left(B + B^3\right).$$

Proof. We shall use the notations introduced previously

$$\omega_n = \mathcal{S}_\tau^n(\omega_0) \quad \text{and} \quad \omega(t_n) = \varphi_{E,t_n}(\omega_0).$$

We may choose R_0 such that, if $\tau R_0 B < 1$, Proposition 6.4.6 holds when applied to any term of the sequence $\mathcal{S}_\tau^n(\omega_0)$, for $t_n \leq T_0$. This implies that $\mathcal{S}_\tau^n(\omega_0)$ belongs to H^s for all $t_n \leq T_0 + \tau$.
That being said, the semi-discrete error is inductively expanded as follows

$$\begin{aligned}\left\|\mathcal{S}_\tau^{n+1}(\omega_0) - \varphi_{E,t_{n+1}}(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} &\leq \left\|\mathcal{S}_\tau\left(\mathcal{S}_\tau^n(\omega_0)\right) - \mathcal{S}_\tau\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)} \\ &+ \left\|\mathcal{S}_\tau\left(\varphi_{E,t_n}(\omega_0)\right) - \varphi_{F,\tau}\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)} \\ &+ \left\|\varphi_{F,\tau}\left(\varphi_{E,t_n}(\omega_0)\right) - \varphi_{E,\tau}\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)},\end{aligned}$$

for all $t_{n+1} \leq T_0 + \tau$.
Applying Corollary 6.4.7, and using the hypothesis $\tau R_0 B < 1$, we may find a positive constant R_1 such that

$$\left\|\mathcal{S}_\tau\left(\mathcal{S}_\tau^n(\omega_0)\right) - \mathcal{S}_\tau\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq e^{\tau R_1(1+B)} \left\|\mathcal{S}_\tau^n(\omega_0) - \varphi_{E,t_n}(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} + R_1 B \tau^3.$$

Applying now Propositions 6.5.3 and 6.5.1, we may moreover write that

$$\left\|\mathcal{S}_\tau\left(\varphi_{E,t_n}(\omega_0)\right) - \varphi_{F,\tau}\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq R_1 B \tau^3$$

and

$$\left\|\varphi_{F,\tau}\left(\varphi_{E,t_n}(\omega_0)\right) - \varphi_{E,\tau}\left(\varphi_{E,t_n}(\omega_0)\right)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq R_1 B^3 \tau^2.$$

Therefore,

$$\left\|\mathcal{S}_\tau^{n+1}(\omega_0) - \varphi_{E,t_{n+1}}(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq e^{\tau R_1(1+B)} \left\|\mathcal{S}_\tau^n(\omega_0) - \varphi_{E,t_n}(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} + R(B)\tau^2,$$

with

$$R(B) \leq R_1 \left(B + B^3\right).$$

which implies by induction that for all $t_n \leq T_0 + \tau$,

$$\left\|\mathcal{S}_\tau^n(\omega_0) - \varphi_{E,t_n}(\omega_0)\right\|_{H^{s-4}(\mathbb{T}^2)} \leq R(B)\tau^2 \sum_{i=0}^{n-1} e^{t_i R_1(1+B)} \leq \tau R(B) t_n e^{R_1 T_0(1+B)},$$

which concludes the proof. □

6.5.3 Convergence of the semi-discrete scheme

Here we shall prove our main result, namely Theorem 6.2.3. It will be a consequence of Proposition 6.5.4, and the only remaining task is to bootstrap controls of the same order on the H^s norm of the exact solution and on the H^2 norm of the numerical solution up to a fixed time horizon.

Proof of Theorem 6.2.3.
Let T and $B = B(T)$ be such that

$$\sup_{t\in[0,T]} \|\omega(t)\|_{H^s(\mathbb{T}^2)} \leq B.$$

We shall first prove by induction on $n \in \mathbb{N}$, with $t_n \leq T$, that

$$\sup_{t_k\leq t_n} \|\omega_k\|_{H^2(\mathbb{T}^2)} \leq 2B.$$

This clearly holds for $n = 0$. Let now $n \geq 1$, with $t_n \leq T$, and assume that the following induction hypothesis holds:

$$\sup_{t_k\leq t_n} \|\omega_k\|_{H^2(\mathbb{T}^2)} \leq 2B.$$

By applying Proposition 6.5.4 with $s = 6$ and $T_0 = t_n$, we may find two positive constants R_0, R_1 such that, if $\tau R_0 B < 1$, then for any $t_k \leq t_{n+1}$,

$$\|\omega_k\|_{H^2(\mathbb{T}^2)} \leq \|\omega(t_k)\|_{H^2(\mathbb{T}^2)} + \|\omega_k - \omega(t_k)\|_{H^2(\mathbb{T}^2)} \leq B + \tau t_k e^{R_1 T_0(1+B)} R(B),$$

with

$$R(B) \leq R_1 \left(B + B^3\right).$$

If τ satisfies

$$\tau < \frac{B}{TR(B)e^{TR_1(1+B)}},$$

this yields

$$\|\omega_k\|_{H^2(\mathbb{T}^2)} \leq 2B,$$

for any $t_k \leq t_{n+1}$. This concludes the induction and shows that

$$\sup_{t_n\leq T} \|\omega_n\|_{H^2(\mathbb{T}^2)} \leq 2B.$$

One obtains then the conclusion of Theorem 6.2.3 by applying Proposition 6.5.4 with $T_0 = T - \tau$.

■

6.6 Appendix A: Solving Poisson's equation

Proposition 6.6.1. *Let $f \in L^2(\mathbb{T}^2)$. Assume that*

$$\int_{\mathbb{T}^2} f(z)\mathrm{d}z = 0.$$

Then there exists an unique $u \in H^2(\mathbb{T}^2)$ such that

$$\begin{cases} \Delta u = f \\ \displaystyle\int_{\mathbb{T}^2} u(z)\mathrm{d}z = 0. \end{cases}$$

Proof. Assume first that $u \in H(\mathbb{T}^2)$ satisfies the equation. Since the average of f on the torus is 0, we can write

$$f(z) = \sum_{k \in \mathbb{Z}^{2*}} \hat{f}_k e^{ik\cdot z}.$$

Setting also

$$u(z) = \sum_{k \in \mathbb{Z}^{2*}} \hat{u}_k e^{ik\cdot z},$$

the Poisson equation simply reads

$$|k|^2 \hat{u}_k = \hat{f}_k, \quad k \in \mathbb{Z}^2.$$

Reciprocally, if the coefficients $\hat{u}_k$ are defined by the above formula (for $k \in \mathbb{Z}^{2*}$), and if we set

$$u(z) = \sum_{k \in \mathbb{Z}^{2*}} \hat{u}_k e^{ik\cdot z},$$

then $u \in H(\mathbb{T}^2)$ (this will be precisely proven in the next Proposition), has average 0, and satisfies $\Delta u = f$. □

Proposition 6.6.2. *Assume that u and f satisfy*

$$\Delta u = f.$$

Then for all $s \geq 2$,

$$\sup_{0 \leq |\alpha| \leq s} \left\| \partial_z^\alpha u \right\|_{L^2(\mathbb{T}^2)} \leq C \left\| f \right\|_{H^{s-2}(\mathbb{T}^2)}. \tag{6.6.1}$$

Proof. For any $0 \leq |\alpha| \leq s$, we have

$$\begin{aligned} \left\| \partial_z^\alpha u \right\|_{L^2(\mathbb{T}^2)}^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2\alpha} |\hat{u}_k|^2 \leq C \sum_{k \in \mathbb{Z}^2} |k|^{2\alpha - 4} |k|^2 |\hat{u}_k|^2 \\ \leq C \sum_{k \in \mathbb{Z}^2} \langle k \rangle^{2s-4} \left| \hat{f}_k \right|^2 = C \left\| f \right\|_{H^{s-2}(\mathbb{T}^2)}^2. \end{aligned}$$

□

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Résumé

Cette thèse porte sur le comportement en temps long de solutions d'équations de type Vlasov, principalement le modèle Vlasov-HMF. On s'intéresse en particulier au phénomène d'amortissement Landau, prouvé mathématiquement dans divers cadres, pour plusieurs équations de type Vlasov, comme l'équation de Vlasov-Poisson ou le modèle Vlasov-HMF, et présentant certaines analogies avec le phénomène d'amortissement non visqueux pour l'équation d'Euler 2D. Les résultats qui y sont décrits sont les suivants.
Le premier est un théorème d'amortissement Landau pour des solutions numériques du modèle Vlasov-HMF, obtenues par discrétisation en temps de ce dernier *via* des méthodes de splitting. Nous prouvons en outre la convergence des schémas numériques. Le second est un théorème d'amortissment Landau pour des solutions du modèle Vlasov-HMF linéarisé autour d'états stationnaires inhomogènes. Ce théorème est accompagné de nombreuses simulations numériques destinées à étudier numériquement le cas non-linéaire, et semblant mettre en lumière de nouveaux phénomènes. Enfin, le dernier résultat porte sur la discrétisation en temps de l'équation d'Euler 2D par un intégrateur de Crouch-Grossman symplectique. Nous prouvons la convergence du schéma.

Abstract

This thesis concerns the long time behavior of certain Vlasov equations, mainly the Vlasov-HMF model. We are in particular interested in the celebrated phenomenon of Landau damping, proved mathematically in various frameworks, for several Vlasov equations, such as the Vlasov-Poisson equation or the Vlasov-HMF model, and exhibiting certain analogies with the inviscid damping phenomenon for the 2D Euler equation. The results described in the document are the following.
The first one is a Landau damping theorem for numerical solutions of the Vlasov-HMF model, constructed by means of time-discretizations by splitting methods. We prove moreover the convergence of the schemes. The second result is a Landau damping theorem for solutions of the Vlasov-HMF model linearized around inhomogeneous stationary states. We provide moreover a quite large amount of numerical simulations, which are designed to study numerically the nonlinear case, and which seem to show new phenomenons. The last result is the convergence of a scheme that discretizes in time the 2D Euler equation by means of a symplectic Crouch-Grossmann integrator.