Equations de Hamilton-Jacobi sur des réseaux et applications à la modélisation du trafic routier

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Résumé : Cette thèse porte sur l'analyse et l'homogénéisation d'équations aux dérivées partielles (EDP) posées sur des réseaux avec des applications en trafic routier. Deux types de travaux ont été réalisés : le premier axe de travail consiste à considérer des modèles microscopiques de trafic routier et d'établir une connexion entre ces modèles et des modèles macroscopiques du genre de ceux introduit par Imbert et Monneau [IM13]. Une telle connexion va permettre de justifier rigoureusement les modèles macroscopiques du trafic routier. En effet, les modèles microscopiques décrivent la dynamique de chaque véhicule individuellement et sont donc plus faciles à justifier du point de vue modélisation. Par contre, ces modèles ne sont pas utilisables pour décrire le trafic à grande échelle (des villes par exemple). Les modèles macroscopiques font le jeu inverse : ils sont fort pour décrire le trafic à grande échelle mais du point de vue modélisation, ils sont compliqués à mettre en oeuvre pour prédire toutes les situations du trafic (par exemple trafic libre ou congestionné). Le passage du microscopique au macroscopique est fait en s'appuyant sur la théorie des solutions de viscosité et en particulier les techniques d'homogénéisation. Le second axe consiste à considérer une équation d'Hamilton-Jacobi avec une jonction qui bouge en temps. Cette équation peut décrire la circulation des voitures sur une route avec la présence d'un véhicule particulier (plus lent que les voitures par exemple). On prouve l'existence et l'unicité (par un principe de comparaison) d'une solution de viscosité pour cette EDP.

Abstract : This thesis deals with the analysis and homogenization of partial differential equations (PDE) posed on networks with application to traffic. Two types of work are done : the first line of work consists to consider microscopic traffic models in order to establish a connection between these models and macroscopic models like the one introduced by Imbert and Monneau [IM13]. Such connection allows to justify rigorously the macroscopic models of traffic. In fact, microscopic models describe the dynamic of each vehicle individually and so they are easy to justify from the modelization point of view. On the other hand, these models are complicated to implement in order to describe the traffic at large scales (cities for example). Macroscopic models do the opposite : they are effective for describing the traffic at large scales but from the modelization point of view, they are incapable to predict all traffic situations (for example free or congested flow). The passage from microscopic to macroscopic is done using the viscosity solutions theory and in particular homogenization technics. The second line of work consists to consider a Hamilton-Jacobi equation coupled by a junction condition which moves in time. This equation can describe the circulation of cars on a road with the presence of a particular vehicle (slower than the cars for example). We prove existence and uniqueness (by a comparison principle) of viscosity solution of this PDE.

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Chapitre 1

Introduction

Cette thèse porte sur l'analyse et l'homogénéisation d'équations aux dérivées partielles (EDP) posées sur des réseaux avec des applications en trafic routier. Cette introduction est divisée en deux parties : dans la première partie, on explique de manière détaillée les modèles de trafic microscopiques et macroscopiques et puis on présente une bibliographie sur la théorie des solutions de viscosité. La deuxième partie de cette introduction sera consacrée à présenter les résultats de cette thèse.

Nous commençons cette introduction en décrivant les deux grandes familles de modèles qui existent pour décrire le trafic routier : les modèles microscopiques et les modèles macroscopiques.

1.1 Les modèles microscopiques

Les modèles microscopiques étudient la dynamique de chaque voiture individuellement. Pour modéliser les trajectoires des véhicules, on distingue trois classes de modèles :

- les modèles de type poursuite ("car following model") qui décrivent comment les véhicules adaptent leur position, leur vitesse ou leur accélération en fonction des véhicules environnants. Dans ce type de modèle, le comportement du conducteur dépend de la situation devant lui : s'il n'est pas précédé par un autre véhicule, il circule librement ("free flow"). Sinon, il doit adapter son comportement de conduite en fonction de la distance avec le véhicule "leader". De nombreux modèles de poursuite ont été proposés jusqu'à présent, citons par exemple [BHN+95, Pip53, CHM58, GHR61, BM99, New61, THH00, NS92].
- les modèles avec changement de voie qui décrivent la manière dont les véhicules changent de voie de circulation. Les comportements de changement de voie sont les plus difficiles à comprendre, à mesurer et à modéliser. Il existe deux types de changement de voie :
  i) les changements de voie de confort qui visent à améliorer le confort du conducteur et éviter une voie à vitesse limitée par exemple. Un véhicule effectue généralement un changement de voie de confort afin de circuler à la vitesse qu'il désire.

ii) les changements de voie obligatoires qui visent à rejoindre une voie pour rejoindre une destination spécifique. Lorsqu'un véhicule approche d'une bifurcation, son choix d'itinéraire peut le contraindre à rejoindre une voie spécifique. Il effectue alors un changement de voie dit "obligatoire".

Plusieurs modèles de changement de voie sont proposés dans la littérature, citons par exemple [ABAKM96, LD06, LL08, Hid02].

– les modèles d'insertion qui régissent la manière dont les véhicules s'insèrent dans une voie. Lorsqu'un véhicule s'approche d'un " cédez-le-passage " ou d'un "stop", il s'engage uniquement si l'intersection est libre et que les conditions élémentaires de sécurité sont respectées. C'est sur ce principe que repose les modèles d'insertion. La réponse est binaire : le véhicule s'engage ou le véhicule ne s'engage pas.
Les modèles d'acceptation de créneaux [YK96, Lee06, MF89] sont à la base de nombreux modèles d'insertion : ils considèrent que le véhicule non prioritaire s'engage si et seulement si aucun véhicule prioritaire ne se situe dans une zone de priorité. Le modèle considère donc deux zones de priorité dont les longueurs ne sont pas nécessairement identiques. Si un véhicule prioritaire se situe dans l'une de ces deux zones, alors le véhicule non-prioritaire attendra que le véhicule prioritaire ait quitté la zone de priorité avant de s'engager.

Les modèles microscopiques combinent généralement ces trois types de modèles. L'avantage de l'approche microscopique est sa facilité du point de vue modélisation car les indicateurs de trafic des modèles microscopiques comme la vitesse ou la distance intervéhiculaire sont individuels. Par contre, ces modèles ne sont pas utilisables à grande échelles et les temps de calcul générés lors de la simulation seraient beaucoup trop longs surtout s'il existe une congestion ou si la modélisation intègre de gros réseaux maillés.

Dans cette thèse, on s'interesse seulement à des modèles microscopique du type "car following model" sur une seule route, voilà pourquoi on présente quelques modèles de ce type. Afin d'unifier les notations dans les exemples qui suivent, on utilisera les notations suivantes :

– U_i représente la position du véhicule i et $U_{i+1} > U_i$.
– $U_i^{'}$ représente la vitesse du véhicule i.
– $U_i^{''}$ représente l'accélération du véhicule i.
– T représente le temps de réaction du conducteur i pour prendre en compte les variations du comportement de son véhicule leader U_{i+1}.

Modèle de Gazis, Herman et Rothery (GHR). Le modèle de Gazis, Herman et Rothery (GHR) [GHR61] est l'un des premiers travaux sur la modélisation du conducteur. Ce modèle est également connu sous le nom modèle General Motors (GM). Son expression est donnée par :

$$U_i^{''}(t+T) = \beta \frac{\left(U_i^{'}(t+T)\right)^m}{(U_{i+1}(t) - U_i(t))^l} \left(U_{i+1}^{'}(t) - U_i^{'}(t)\right)$$

où β, m et l sont des constantes de calibration pour chaque état du véhicule et du trafic. L'inconvénient de ce modèle est que son utilisation demande la calibration des paramètres

β, l et m pour chaque situation du véhicule et du trafic. D'où la nécessité d'avoir des données expérimentales qui sont difficiles à obtenir. De plus, d'après [BM99] , les travaux réalisés sur ce modèle présentent des contradictions au niveau des paramètres β, l et m.

Modèle de Kometani : Collision Avoidance model Le modèle Collision Avoidance a été présenté dans [KS59]. Son but est de déterminer l'interdistance de sécurité permettant au conducteur d'éviter les collisions lorsque son véhicule meneur freine de façon imprévisible. Cette interdistance minimale est donnée par l'expression

$$U_{i+1}(t) - U_i(t) = \alpha \left(U'_{i+1}(t)\right)^2 + \beta \left(U'_i(t+T)^2\right) + \gamma U'_i(t+T) + \delta$$

où α et β représentent, respectivement, l'inverse de la capacité maximale de décélération des véhicules $i+1$ et i. Les constantes de calibration γ et δ représentent, respectivement, l'inverse du temps de réaction et une distance. Le temps T nécessaire à la prise en compte d'une modification de la vitesse du conducteur peut être interprété comme un temps de réaction.

L'avantage de ce modèle est la focalisation sur la distance de sécurité qui est un critère très important dans l'étude de la sécurité du trafic. Par contre, son inconvénient réside dans l'absence de la prise en compte du critère de l'anticipation spatiale.

Les modèles à vitesse optimale : Optimal velocity Model (OVM) Ces modèles considèrent que la vitesse d'un véhicule dépend plutôt d'une vitesse qui est fonction de la distance entre celui-ci et son leader. Le premier modèle de ce genre a été proposé par Newell [New61] et est de la forme suivante :

$$U'_i(t+T) = V\left(U_{i+1}(t) - U_i(t)\right) \tag{1.1.1}$$

où V est la vitesse optimale. Généralement, V est une fonction croissante, borné et positive. Par un développement de taylor de l'équation (1.1.1), on obtient le modèle suivant

$$U''_i(t) = a\left(V\left(U_{i+1}(t) - U_i(t)\right) - U'_i(t)\right) \tag{1.1.2}$$

où $a = \dfrac{1}{T}$ est la sensibilité du conducteur i. Les modèles qui en découlent sont maintenant connus sous l'appellation "Optimal Velocity Model" (OVM). Il a été démontré que ce type de modèle est apte à décrire différentes situations de trafic. En 1995, Bando et al [BHN$^+$95] a proposé une fonction de vitesse optimale donnée par

$$V(p) = \frac{V_{max}}{2}\left(\tanh(p - S_c) + \tanh(S_c)\right)$$

où S_c représente une interdistance critique. Dans cette thèse, on considèrera les modèles (1.1.1) et (1.1.2) perturbés en négligeant le temps de réaction. Dans ces modèles, on multiplie le terme $V(U_{i+1} - U_i)$ par une fonction qui simule une perturbation locale à l'origine. Cette fonction peut dépendre du temps (cas d'un feux tricolore) ou non (cas d'un péage par exemple).

1.2 Les modèles macroscopiques

Les modèles macroscopiques sont issus d'une analogie hydrodynamique de l'écoulement. L'objet de ces modèles est de pouvoir caractériser le comportement global du trafic, à une échelle d'étude relativement importante. Ils sont donc tout particulièrement utilisés dans le cadre de la modélisation de grands réseaux. L'application des modèles macroscopiques couvre la simulation du trafic pour la gestion des infrastructures et de la dynamique du trafic ainsi que l'évaluation des politiques de ces gestions à posteriori. Le modèle le plus connu (et présent dans la thèse) est le modèle LWR (Lighthill-Whitham-Richars) qui a été introduit dans [Ric56, LW55]. Ce modèle fait intervenir trois variables : le flux, la densité et la vitesse moyenne. Il est donné par l'équation suivante :

$$\rho_t + (f(\rho))_x = 0 \tag{1.2.1}$$

où $\rho(t,x)$ représente la densité des voitures en un point x de l'espace et à l'instant t, f est une fonction qui décrit le flux et définie par $f(\rho) = \rho V(\rho)$, où V représente la vitesse moyenne des véhicules. La fonction f est aussi connue par le diagramme fondamental. Le diagramme fondamental essaie de reproduire des situations d'équilibre d'un site donné. Il représente les caractéristiques du réseau sur lequel roule les véhicules. Plusieurs modèles de fonction flux ont été proposés et en général ils doivent répondre aux observations suivantes :

- Lorsque la densité des véhicules est suffisamment faible, ces véhicules peuvent circuler à la vitesse désirée, appelée vitesse libre.
- En augmentant le nombre de véhicules dans la section, les interactions deviennent plus importantes et les vitesses pratiquées diminuent.
- Quand la section est saturée et donc la densité est maximale, la vitesse et le flux sont nuls.

On cite quelques formes proposées dans la littérature pour le diagramme fondamental :

Modèle de Greenshields. Greenshields [GCM+35] a supposé que la vitesse est une fonction décroissante de la densité. La vitesse dans ce modèle est donnée par l'expression suivante :

$$V(\rho) = V_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

où V_{max} et ρ_{max} représente respectivement la vitesse moyenne maximale et la densité maximale. On obtient alors un diagramme fondamental concave qui fait apparaître deux régimes : Le régime fluide pour lequel le flux augmente si la densité augmente et un régime congestionné pour lequel le flux diminue si la densité augmente. La limite entre les deux parties coincide avec un état critique caractérisé par un flux de trafic maximal qui est atteint pour une densité critique ρ_c. L'avantage de ce modèle vient de sa simplicité car le modèle est linéaire. Par contre, la densité critique de ce modèle n'appartient pas à l'intervalle proposé dans les études expérimentales de Bourrel [Bou03] ($[0.2\rho_{max}, 0.3\rho_{max}]$).

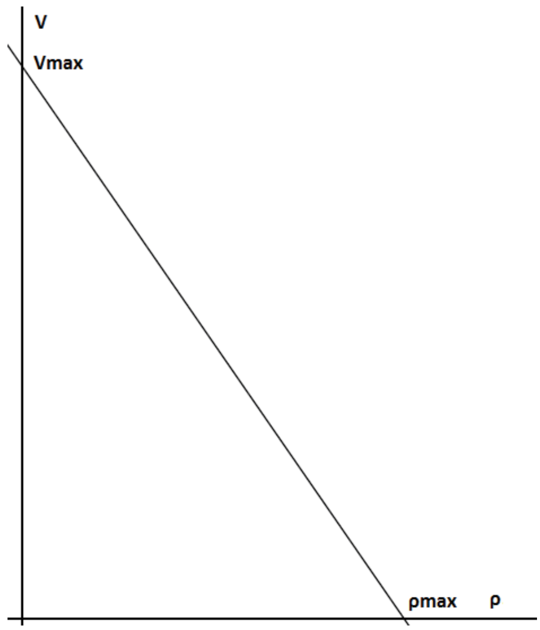

FIGURE 1.1 – Représentation schématique de la vitesse de Greenshields.

Modèle de Greenberg. Greenberg [Gre59] a proposé une relation logaritmique entre la vitesse et la densité donnée par

$$V(\rho) = V_{max} \log\left(\frac{\rho_{max}}{\rho}\right).$$

Le point fort de ce modèle est que la densité critique est inférieure à celle introduite par Greenshields [GCM+35] ce qui constitue une amélioration de son modèle. Toutefois elle reste supérieure à la valeur $0,3\rho_{max}$ donnée dans [Bou03].

Le principal inconvénient de ce modèle est que la vitesse tend vers l'infini lorsque la densité tend vers zéro d'après l'équation. Cela montre son incapacité à prédire la vitesse pour de faibles densités.

Modèle de Underwood. Underwood [Und08] a développé un modèle exponentiel pour pallier les inconvénients des modèles de Greenberg et de Greenshields dans le cas où le trafic est libre (faible densité et vitesse élevée). Ce modèle est donnée par

$$V(\rho) = V_{max} \exp\left(-\frac{\rho}{\rho_c}\right).$$

où ρ_c est la densité optimale, c'est-à-dire la densité qui correspond au flux maximal. Le principal inconvénient de ce modèle est que la vitesse devient nulle seulement lorsque la densité tend vers l'infini. Par conséquent, ce modèle ne peut pas être utilisé pour prédire la vitesse à haute densité.

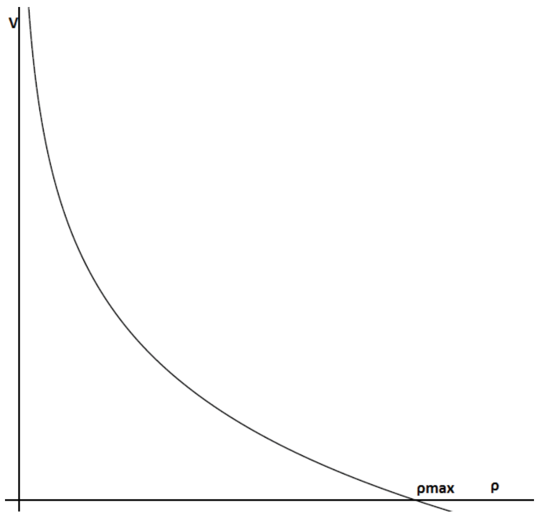

FIGURE 1.2 – Représentation schématique de la vitesse de Greenberg.

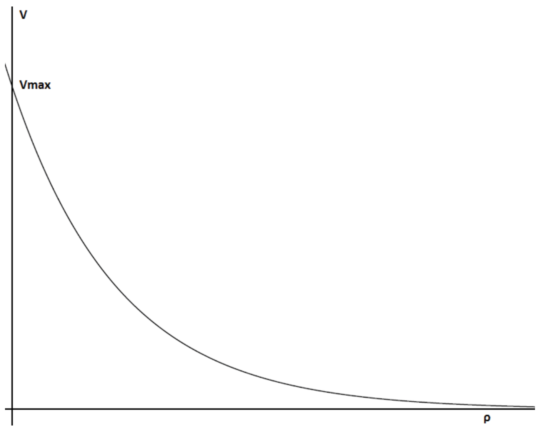

FIGURE 1.3 – Représentation schématique de la vitesse d'Underwood.

Dans cette thèse, on retrouve seulement le modèle LWR, voilà pourquoi on ne parlera pas d'autres modèles macroscopiques. Le lecteur souhaitant découvrir d'autres modèles peut se référer par exemple à [AR00, Zha02, HB01].

Remarquons qu'en introduisant la fonction de Moskowitz [MN63], on peut voir que le modèle LWR (1.2.1) peut être vue comme une équation d'Hamilton Jacobi. La fonction de Moskowitz M donne le nombre total de véhicules s'étant écoulés au point x entre un temps initial et le temps $t > 0$. Elle est définie par

$$M\left(t,x\right) = g\left(t\right) - \int_{0}^{x} \rho\left(t,y\right) dy,$$

où

$$g(t) = \int_0^t f(\rho(s,0))\, ds.$$

On obtient donc la formulation Hamilton-Jacobi :

$$M_t + H(M_x) = 0,$$

où

$$H(p) = -f(-p).$$

Dans cette thèse, le problème macroscopique considéré dans les trois premiers chapitres est un modèle de type LWR intégré complété par une condition au point 0. Dans ce paragraphe, on présente le modèle pour un domaine plus général (la jonction) et la formulation Hamilton-Jacobi de ce modèle est faite dans la sous section 1.4.

La jonction. Considérons $n \geq 1$ vecteurs unitaires $e_i \in \mathbb{R}^2$. On définit les n branches de la jonction comme étant les demi-droites générées par ces vecteurs unitaires. Pour $i = 1, ..., m$, on définit les branches entrantes à la jonction

$$J_i = (-\infty, 0] \cdot e_i \quad \text{et} \quad J_i^* = J_i \backslash \{0\}.$$

Pour $i = m+1, ..., n$, on définit les branches sortantes de la jonction

$$J_i = (0, +\infty] \cdot e_i \quad \text{et} \quad J_i^* = J_i \backslash \{0\}.$$

La jonction J est définie par

$$J = \cup_{i=1}^n J_i.$$

Le point $x = 0$ est appelé point de la jonction. La définition de la jonction étant complète,

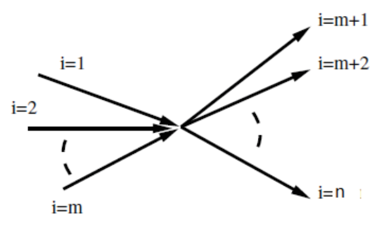

Figure 1.4 – Une jonction.

on peut énoncer le modèle macroscopique de trafic sur la jonction. On présente au début les variables utilisées dans ce modèle :

- La fonction $\rho^i(t,x)$ qui représente la densité au temps t et à la position x sur la branche i.
- Pour tout $i = 1,...,n$, $f^i : \mathbb{R} \to \mathbb{R}$ est une fonction qui décrit le trafic sur la branche i de la jonction. De plus, il existe une valeur critique ρ_c^i tel que f^i est croissante sur $(-\infty, \rho_c^i]$ et f^i est décroissante sur $[\rho_c^i, +\infty)$. On suppose aussi que $\lim_{p\to\pm\infty} f^i(p) = -\infty$, (un exemple de cette fonction f est le diagramme fondamental de Greenshields).
- On suppose qu'il existe une suite $\left(\gamma^i\right)_i$ tel que,
  - pour $i = 1,...,m$, γ_i représente la proportion des voitures qui entrent dans la jonction en venant de la branche i.
  - pour $i = m+1,...,n$, γ_i représente la proportion des voitures qui sortent de la jonction vers la branche i.

Le problème macroscopique est donné par, pour tout $t > 0$,

$$\begin{cases} \rho_t^i + \left(f^i\left(\rho^i\right)\right)_x = 0 \quad \text{si } x \in J_i^* \text{ pour tout } i = 1,...,n \\ \sum\limits_{i=1}^{m} f^i\left(\rho^i(t,0)\right) = \min\left(\min\limits_{i=1,...,m} \dfrac{1}{\gamma^i} f_D^i\left(\rho^i\left(t,0^-\right)\right), \min\limits_{i=m+1,...,n} \dfrac{1}{\gamma^i} f_S^i\left(\rho^i\left(t,0^+\right)\right)\right) \end{cases} \tag{1.2.2}$$

où f_D^i est la fonction demande définie par

$$f_D^i(p) = \begin{cases} f^i(p) & \text{si } p \leq \rho_c^i, \\ f^i\left(\rho_c^i\right) & \text{si } p > \rho_c^i, \end{cases}$$

et f_S^i est la fonction d'offre définie par

$$f_S^i(p) = \begin{cases} f^i\left(\rho_c^i\right) & \text{si } p \leq \rho_c^i, \\ f^i(p) & \text{si } p > \rho_c^i. \end{cases}$$

La deuxième condition dans (1.2.2) signifie que le flux qui passe par le point 0 est égal au minimum entre le flux qui arrive (qui correspond à la fonction demande) et celui qui peut sortir (qui correspond à la fonction offre). Cette condition est présentée dans [Leb05].

1.3 Connexion micro-macro

Généralement, les modèles microscopiques sont considérés plus précis car la dynamique de chaque véhicule est décrite individuellement d'une manière assez précise. Au contraire, les modèles macroscopiques sont plus compliqués à justifier du point de vue modélisation et sont basés sur des hypothèses difficilement vérifiables, d'où la nécéssité d'établir une connexion entre les modèles microscopiques et les modèles macroscopiques dans le but de justifier ces derniers en se basant sur la modélisation des premiers.

Plusieurs techniques ont été proposées pour les connexions micro-macro et la dérivation d'un modèle macroscopique à partir d'un modèle microscopique a été étudiée par plusieurs auteurs. En 1970, Payne [Pay71] a utilisé la méthode d'expansion du gradient pour dériver un modèle LWR à partir d'un modèle de Newel [New61]. Les auteurs de [HHST02] ont établi une connexion micro-macro en supposant que les variables macroscopiques en un point x de l'espace peuvent être définies par les variables microscopiques si un véhicule est présent en x. Sinon, les variables macroscopiques en x sont définies par interpolation linéaire. Dans [AKRM02], les auteurs ont dérivé un modèle Aw-Rascale à partir d'un modèle du second ordre du type "follow the leader". Le lecteur peut se référer également à [CS15, Ros14, DFR15] pour approfondir ce sujet.

Dans cette thèse, on utilise la technique de l'homogénéisation [LPV86, FIM09b, GIM15] pour établir la connexion micro-macro. Dans tous les travaux de cette thèse, on prouve après un changement d'échelle convenable que la primitive de la mesure empirique de la position des véhicules converge vers une solution d'un problème d'Hamilton-Jacobi défini sur une jonction. La solution de ce dernier représente la primitive de la densité des véhicules du problème macroscopique.

1.4 Les solutions de viscosité

Dans cette thèse, les équations étudiées sont de type Hamilton-Jacobi. On utilise donc la notion de solution de viscosité pour définir des solutions à ces équations. On va faire une introduction de cette notion et on donne quelques références pour approfondir la connaissance de cette théorie. La notion de solutions de viscosité a été introduite par M.G Crandall et P.L Lions [CL83, Cra81] pour traiter les équations d'Hamilton-Jacobi du premier ordre. Le mot "viscosité" vient de la motivation de définir ces solutions par la méthode de viscosité évanescente. Cette notion de solution fournit des résultats d'existence, d'unicité et de stabilité pour une grande variété d'EDP non linéaires. Un bon exemple qui a été utilisé par les auteurs [Bar97, Car04, Bar94, Koi] pour illustrer la puissance de cette notion est la fameuse équation eikonal sur l'intervalle $[-1, 1]$,

$$\begin{cases} |Du| = 1 \\ u(-1) = u(1) = 0. \end{cases} \tag{1.4.1}$$

On remarque donc qu'il y a un problème d'unicité et de stabilité des solutions généralisées (solutions presque partout). La théorie des solutions de viscosité résoud ce problème. En effet, pour cette nouvelle définition de solution, la seule solution de (1.4.1) est $u = 1 - |x|$, donc le problème d'unicité est résolu. Pour une présentation générale et complète de cette théorie, on renvoie à [CIL92, Bar13, Car04, Bar94, Bar97, DI02, Koi]. Dans la suite, on décrit plus précisement quelques aspects de cette théorie qui seront utilisés dans cette thèse :

Homogénéisation. La théorie de l'homogénéisation des équations d'Hamilton-Jacobi a été dévéloppée depuis les travaux de Lions, Papanicolaou et Varadhan [LPV86] et il est difficile de donner une liste complète de références. Citons par exemple [AB03, BS01, BS00a, BS00b, CSW05, Eva89, Eva92, Ish99, LS05, Mül87, Roq98, FIM09a, IMR08, FIM09b, IM08].

La plupart des travaux d'homogénéisation utilisent la méthode de la fonction test perturbée introduite par Evans [Eva89]. La construction des correcteurs est liée au comportement en temps long des solutions des équations d'Hamilton-Jacobi. Ce type de problème a été étudié par plusieurs auteurs, le lecteur est renvoyé à [BS00a, Roq98, BS00a, Fat98, Roq98, BR03]. Citons enfin les articles [FIM09b, FIM09a, FIM12b] pour des résultats d'homogénéisation pour les opérateurs non locaux.

Dans notre travail, on a emprunté des idées de ces travaux comme les résultats sur les solutions de viscosité pour des opérateurs non locaux, la fonction de distribution cumulative, et l'idée d'injecter le système d'EDOs dans une seule EDP et puis d'homogénéiser cette EDP.

Dans cette thèse, on utilise également la technique de l'homogénéisation spécifiée. Au contraire de l'homogénéisation classique qui va moyenniser les phénomènes et donc oublier les petits défauts, le but de l'homogénéisation spécifiée est de décrire ces défauts de manière précise. On prouve après un changement d'échelle que la solution converge vers la solution de l'Hamiltonien effectif attendu dont le flux à l'origine est limité au sens de Imbert et Monneau [IM13]. En d'autres termes, l'homogénéisation dans notre travail fournit une condition de jonction au point de discontinuité de l'Hamiltonien effectif. L'idée principale pour compléter ce travail est de construire un correcteur qui converge vers la fonction test particulière introduite dans [IM13]. La technique de construction de ce correcteur repose sur l'étude du problème de la cellule sur un domaine tronqué avec des bonnes conditions aux bords et ensuite de passer à la limite sur la taille du domaine comme dans [GIM15, AT15] (voir Section 1.5 pour plus de détails).

Solutions de viscosités pour des équations avec un terme intégro-différentiel. [BI08, BBP97, AT96, Son86, Awa91a, Awa91b, JK06]. Les auteurs ont étendu la notion de solutions de viscosité pour ce type d'équations. Par exemple, Alvarez et Tourin [AT96] ont obtenu l'éxistence des solutions par la méthode de Perron introduite par Ishii [Ish87]. Barles et Imbert [BI08] ont prouvé un principe de comparaison en utilisant une nouvelle définition des solutions de viscosité qui combine fonctions test et sous-différentiels. Dans cette thèse, on utilise la notion de solution de viscosité des équations non-locales introduite par Slepcev [Sle03, FIM09b].

Equations d'Hamilton-Jacobi où l'Hamiltonien est discontinu. Il est naturel de tomber sur des équations d'Hamilton-Jacobi pour lesquelles le Hamiltonien est discontinu par rapport à la variable d'espace car ce type d'équations intervient dans des domaines très variés comme la théorie du contrôle, l'économie ou le trafic routier. . Notons qu'il existe plusieurs travaux sur les équations d'Hamilton-Jacobi avec un hamiltonien discontinu [BBC13, BBC14, RSZ14, RZ13, CR07, Dup92, DE04, GS06, IM13, GH13]. Du point de vue solution de viscosité, l'existence des solutions ne présente pas une grande difficulté car la méthode de Perron s'applique facilement. De même, dans la plupart des cas, la stabilité des solutions ne présente pas un problème. Par contre, la difficulté principale qu'on rencontre si l'hamiltonien est discontinu est de pouvoir prouver un principe de comparaison car la technique de dédoublement des variables [Cra81, Lio82, Bar94] ne s'applique pas.

Par exemple, les auteurs [IM13] ont démontré un principe de comparaison pour une équation d'Hamilton-Jacobi en construisant une fonction test, "vertex test function". Plus récemment, une autre méthode pour obtenir le principe de comparaison a été proposé dans [BBCI16].

Ajoutons que les auteurs dans [LS16] ont proposé une autre preuve mais dans le cas stationnaire.

Equation d'Hamilton-Jacobi sur les réseaux. L'étude du trafic routier sur les réseaux est un sujet important et a été étudiée par plusieurs auteurs [BK08, EFNS08, GP06] essentiellement d'un point de vue lois de conservation scalaires. La théorie des équations d'Hamilton-Jacobi sur les réseaux est beaucoup plus récente et a connu une grande évolution. Les premiers travaux ont été obtenus dans [Sch06] pour l'équation eikonale. Quelques années plus tard, des travaux sur la même équation sont apparus dans [CSM12, SC13]. Les auteurs dans [IMZ13, ACCT13] ont étendu les résultats au cas où l'Hamiltonien est convexe par rapport à la variable gradient et leur approche repose surtout sur la théorie du contrôle optimal. Dans [ACCT13], un problème de contrôle optimal a été étudié dans $\mathbb{R}^2$ en supposant que les trajectoires du système contrôlé restent dans le réseau. Dans [CMS13], les auteurs ont étudié la méthode de viscosité évanescente pour les équations d'Hamilton-Jacobi sur des jonctions (le réseau le plus simple, voir en dessous).

Dans [BBC13, BBC14] , les auteurs ont étudié un problème de contrôle régional, c'est-à-dire avec des dynamiques et des coûts réguliers de chaque côté de l'hyperplan mais sans imposer une hypothèse de compatibilité ou de continuité le long de l'hyperplan. Dans cette thèse, on utilisera les résultats obtenus par Imbert et Monneau dans [IM13, IM14] où les auteurs ont considéré une équation d'Hamilton-Jacobi discontinue et ont introduit la notion de "*flux-limited solutions*" (voir aussi [LS16]). Le lien entre la théorie développée dans [BBC13, BBC14] et les "*flux-limited solutions*" dans [IM13, IM14] est exploré dans [BBCI16]. En particulier, [BBCI16] contient une preuve plus simple du principe de comparaison que dans [IM13].

Dans le reste de ce paragraphe, on montre, comment on obtient une équation d'Hamilton-Jacobi à partir de (1.2.2) (voir [IMZ13]) et puis on présente les résultats obtenus pour cette équation dans [IM13]. On définit

$$\begin{cases} U^i(t,x) = g(t) - \dfrac{1}{\gamma^i}\displaystyle\int_x^0 \rho^i(t,y)\,dy & \text{si } x \in J_i^*, i = 1, ..., m \\ U^i(t,x) = g(t) + \dfrac{1}{\gamma^i}\displaystyle\int_0^x \rho^i(t,y)\,dy & \text{si } x \in J_i^*, i = m+1, ..., n \end{cases}$$

où

$$g(t) = -\int_0^t \sum_{j=1}^m f^j\left(\rho^j\left(s, 0^-\right)\right) ds.$$

Remarque 1.4.1 (Interprétation trafic routier de U^i). *Soit x_1 et x_2 deux point de J_i. On remarque que*

$$U^i(t,x_2) - U^i(t,x_1) = \int_{x_1}^{x_2} \rho^i(t,y)\,dy$$

et donc la fonction U^i peut être interprétée comme une fonction qui labélise les voitures sur la branche i.

Pour $i = 1...m$, et en utilisant l'équation (1.2.2) pour $x \neq 0$, on a

$$\begin{aligned} U_t^i + \frac{1}{\gamma^i} f^i\left(\gamma^i U_x^i\right) &= g'(t) - \frac{1}{\gamma^i}\int_x^0 \rho_t^i(t,y)\,dy + \frac{1}{\gamma^i} f^i\left(\rho^i(t,x)\right) \\ &= g'(t) + \frac{1}{\gamma^i}\int_x^0 \left(f^i\left(\rho^i(t,y)\right)\right)_y dy + \frac{1}{\gamma^i} f^i\left(\rho^i(t,x)\right) \\ &= g'(t) + \frac{1}{\gamma^i} f^i\left(\rho^i\left(t,0^-\right)\right). \end{aligned} \tag{1.4.2}$$

De même, pour $i = m+1, ..., n$, on a

$$U_t^i + \frac{1}{\gamma^i} f^i\left(\gamma^i U_x^i\right) = g'(t) + \frac{1}{\gamma^i} f^i\left(\rho^i\left(t,0^+\right)\right). \tag{1.4.3}$$

Le but est de montrer que les égalités (1.4.2) et (1.4.3) sont nulles. Rappelons d'abord que, pour $i = 1, ..., m$, on a

$$\gamma^i = \frac{f^i\left(\rho^i\left(t,0^-\right)\right)}{\sum\limits_{j=1}^{m} f^j\left(\rho^j\left(t,0^-\right)\right)} \tag{1.4.4}$$

et pour $i = m+1, ..., n$

$$\gamma^i = \frac{f^i\left(\rho^i\left(t,0^+\right)\right)}{\sum\limits_{j=m+1}^{n} f^j\left(\rho^j\left(t,0^+\right)\right)}. \tag{1.4.5}$$

D'autre part, en utilisant le fait que le flux de voitures entrant est égal au flux de voitures sortant, on obtient que

$$\sum_{j=1}^{m} f^j\left(\rho^j\left(t,0^-\right)\right) = \sum_{j=m+1}^{n} f^j\left(\rho^j\left(t,0^+\right)\right). \tag{1.4.6}$$

Les égalités (1.4.4),(1.4.5) et (1.4.6) impliquent directement que

$$U_t^i + \frac{1}{\gamma^i} f^i\left(\gamma^i U_x^i\right) = 0 \quad \text{pour tout } i = 1, ..., n.$$

Pour $x \in J$, on définit

$$u(t,x) = -U^i(t,x) \quad \text{si } x \in J_i.$$

Par un simple calcul, on obtient que

$$u_t + H_i(u_x) = 0 \quad \text{si } x \in J_i^*$$

où $H_i(p) = -\frac{1}{\gamma^i} f^i\left(-\gamma^i p\right)$.

D'autre part, la deuxième équation de (1.2.2) et la définition de g impliquent que

$$u_t(t,0) = \min\left(\min_{i=1,\dots,m} \frac{1}{\gamma^i} f_D^i\left(-\gamma^i \partial^i u(t,0)\right), \min_{i=m+1,\dots,n} \frac{1}{\gamma^i} f_S^i\left(-\gamma^i \partial^i u(t,0)\right)\right)$$

où $\partial^i u(t,0) = \lim_{x\to 0, x\in J_i} u_x(t,x)$. On déduit donc la formulation suivante

$$u_t(t,0) + \max\left(\max_{i=1,\dots,m} H_i^+\left(\partial^i u(t,0)\right), \max_{i=m+1,\dots,n} H_i^-\left(\partial^i u(t,0)\right)\right) = 0.$$

Une extension de l'équation (1.2.2) peut être obtenue en ajoutant dans la deuxième équation une constante $-A$ dans le minimum. Cette constante peut être vue comme un limiteur du flux (voir [IM13]). Du point de vue trafic routier, cela signifie que le flux des voitures au point de jonction ne peut pas dépasser $-A$. En particulier, pour que la constante $-A$ limite effectivement le flux des voitures, il faut que

$$-A < \min_{i=1,\dots,n}\left(\max_{p\in\mathbb{R}} \frac{1}{\gamma^i} f^i(p)\right). \tag{1.4.7}$$

En effet, si (1.4.7) n'est pas vraie, alors

$$\begin{aligned}\min\left(-A, \min_{i=1,\dots,m} \frac{1}{\gamma^i} f_D^i\left(\rho^i\left(t,0^-\right)\right), \min_{i=m+1,\dots,n} \frac{1}{\gamma^i} f_S^i\left(\rho^i\left(t,0^+\right)\right)\right) = \\ \min\left(\min_{i=1,\dots,m} \frac{1}{\gamma^i} f_D^i\left(\rho^i\left(t,0^-\right)\right), \min_{i=m+1,\dots,n} \frac{1}{\gamma^i} f_S^i\left(\rho^i\left(t,0^+\right)\right)\right).\end{aligned}$$

Plus précisément, on obtient l'équation d'Hamilton-Jacobi suivante :

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{si } x \in J_i^* \\ u_t + F_A(u_x) = 0 & \text{si } x = 0 \end{cases} \tag{1.4.8}$$

où $u_x(t,0) = \left(\partial^1 u(t,0), \dots\dots, \partial^n u(t,0)\right)$ et pour $p \in \mathbb{R}^n$,

$$F_A(p) = \max\left(A, \max_{i=1,\dots,m} H_i^+(p_i), \max_{i=m+1,\dots,n} H_i^-(p_i)\right).$$

On associe à (1.4.8) une condition initiale :

$$u(0,x) = u_0(x). \tag{1.4.9}$$

Pour définir la notion de solutions de viscosité pour l'équation (1.4.8), Imbert et Monneau [IM13] ont introduit la classe des fonctions test suivante : pour $T > 0$, on définit $J_T = (0,T) \times J$ et la classe des fonctions test sur J_T par

$$C^1\left(J_T\right) = \left\{\varphi \in C\left(J_T\right),\ \text{la réstriction de } \varphi \text{ sur } (0,T) \times J_i \text{ est } C^1 \text{ pour tout } i = 1,...,n\ \right\}.$$

Il faut de plus rappeler la définition de l'enveloppe semi-continue supérieure et inférieure u^* et u_* d'une fonction u localement bornée sur $(0,T) \times J$,

$$u^*\left(t,x\right) = \limsup_{(s,y)\to(t,x)} u\left(s,y\right) \quad \text{et} \quad u_*\left(t,x\right) = \liminf_{(s,y)\to(t,x)} u\left(s,y\right).$$

Définition 1.4.1. Soit $u : [0,T) \times \mathbb{R} \to \mathbb{R}$.

- On dit que u est une sous-solution (resp. sur-solution) de (1.4.8) si pour tout $\varphi \in C^1\left(J_T\right)$ tel que $u^* - \varphi$ admet un maximum local (resp. $u_* - \varphi$ admet un minimum local) en (t_0, x_0), on a

$$\begin{cases} \varphi_t + H_i\left(\varphi_x\right) \leq 0 & (\text{resp.} \geq 0) \ \text{en } (t_0,x_0) \ \text{si } x_0 \in J_i^* \\ \varphi_t + F_A\left(\varphi_x\right) \leq 0 & (\text{resp.} \geq 0) \ \text{en } (t_0,x_0) \ \text{si } x_0 = 0. \end{cases}$$

- On dit que u est une sous-solution (resp. sur-solution) de (1.4.8),(1.4.9) sur $[0,T)\times J$ si de plus, on a

$$u^*\left(0,x\right) \leq u_0\left(x\right) \quad (\text{resp. } u_*\left(0,x\right) \geq u_0\left(x\right)\ ) \ \text{ pour tout } x \in J.$$

- On dit u est une solution de viscosité si u est une sous et sur-solution.

Un des résultats majeur de [IM13] (qui repose sur un principe de comparaison) est le suivant :

Théorème 1.4.1. *Supposons que u_0 est uniformément continue. Alors il existe une unique solution de viscosité pour l'équation* (1.4.8),(1.4.9) *tel que pour tout $T > 0$, il existe une constante $C_T > 0$ tel que*

$$\left|u\left(t,x\right) - u_0\left(x\right)\right| \leq C_T \quad \textit{pour tout } (t,x) \in [0,T) \times J.$$

1.5 Résultats de cette thèse

On présente dans cette sous-section les résultats de la thèse en laissant les détails pour les chapitres suivants.

1.5.1 Homogénéisation d'un modèle du premier ordre

Le but de ce travail [FSZ17b] est de dériver un modèle de trafic macroscopique à partir d'un modèle microscopique. L'idée est de redimensionner le problème microscopique, qui décrit les dynamiques de chaque véhicule individuellement dans le dessein d'obtenir un modèle macroscopique qui décrit la dynamique de la densité des véhicules. La motivation pour dériver des modèles macroscopiques à partir des modèles microscopiques est que les modèles macroscopiques sont plus adaptés à simuler le trafic à grandes échelle. De plus, les modèles microscopiques sont fondés sur des hypothèses facilement vérifiables et donc la dérivation d'un modèle macroscopique permet de le justifier rigoureusement.

Dans ce travail, on suppose qu'il existe une perturbation locale qui ralentit les véhicules et on veut comprendre l'influence de cette perturbation sur la dynamique macroscopique. Cette perturbation peut être constante en temps et dans ce cas, elle représente par exemple un péage ou un accident de voiture près de la route. Elle peut dépendre du temps (périodiquement) et représenter, par exemple, un feu de circulation. Nous

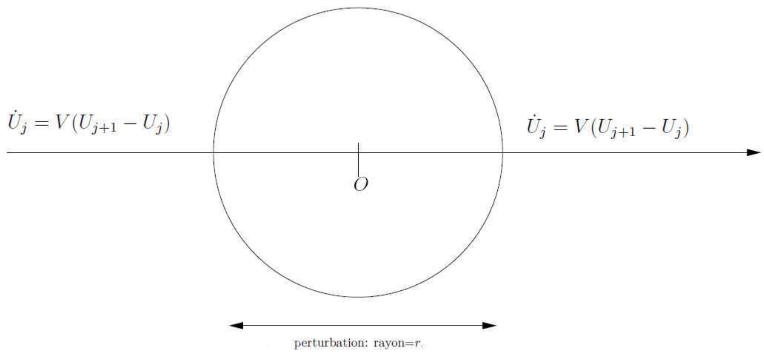

Figure 1.5 – Représentation schématique du modèle microscopique.

désignons par $U_j(t)$ la position du véhicule j et on suppose que la vitesse de chaque véhicule est donnée par une fonction V. Pour obtenir le résultat d'homogénéisation, on procède comme dans [FIM09b, FIM09a, FIM12b, FS15, FSZ17a] et on remet à l'échelle le modèle microscopique qui décrit les dynamiques de chaque véhicule pour obtenir un problème macroscopique qui décrit la densité des véhicules. Si la perturbation locale se situe au voisinage de zéro, c'est naturel d'obtenir à l'échelle macroscopique une équation d'Hamilton-Jacobi avec une condition au point de jonction $x = 0$, (voir la figure 1.7, $-u_x^0$ est la densité des véhicules et l'Hamiltonien effectif $\overline{H}$ est défini plus loin), car le rayon de la perturbation converge à zéro quand on change l'échelle.

Dans ce travail [FSZ17b], on utilise les idées développées dans [FIM09b] pour passer du modèle microscopique au modèle macroscopique. La difficulté principale est que le cadre de notre travail n'est pas périodique et donc la construction des correcteurs est plus compliquée. En particulier, on utilise les idées développées par Achdou et Tchou dans [AT15] et par Galise, Imbert et Monneau dans [GIM15] et dans les cours de Lions au collège de France [Lio14] et qui consiste à construire des correcteurs sur des domaines tronqués.

Le modèle microscopique. On considère le modèle microscopique du premier ordre suivant,

$$\dot{U}_j(t) = V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(t, U_j(t)\right), \tag{1.5.1}$$

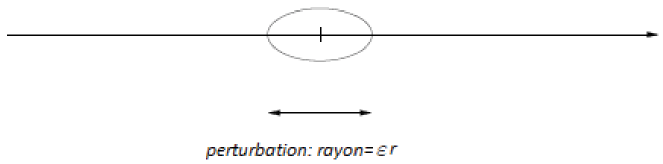

FIGURE 1.6 – La perturbation après le changement d'échelle.

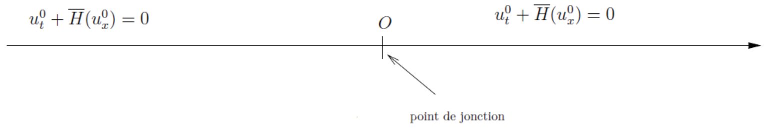

FIGURE 1.7 – Représentation schématique du modèle macroscopique.

où $U_j : [0,+\infty) \to \mathbb{R}$ est la position du véhicule j et $\dot{U}_j$ sa vitesse. La fonction $\phi : \mathbb{R} \times \mathbb{R} \to [0,1]$ représente une perturbation locale au voisinage du point zero. V est la vitesse optimale de ce modèle (voir [BT10]). On suppose que V et ϕ vérifient les hypothèses suivantes (voir Figures 1.8 et 1.9) :

Hypothèse (A)

(A1) $V : \mathbb{R} \to \mathbb{R}^+$ est lipschitzienne, positive.
(A2) V est croissante sur $\mathbb{R}$.
(A3) Il existe $h_0 \in (0,+\infty)$ tel que pour tout $h \leq h_0$, $V(h) = 0$.
(A4) Il existe $h_{max} \in (h_0,+\infty)$ tel que $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
(A5) Il existe $p_0 \in [-1/h_0, 0)$ tel que la fonction $p \mapsto pV(-1/p)$ est décroissante sur $[-1/h_0, p_0)$ et croissante sur $[p_0, 0)$.
(A6) La fonction $\phi : \mathbb{R} \times \mathbb{R} \to [0,1]$ est lipschitzienne et il existe $r > 0$ tel que $\phi(t,x) = 1$ si $|x| \geq r$. On suppose aussi que ϕ est $\mathbb{Z}$-périodique en temps.

Remarque 1.5.1. *Les hypothèses (A1)-(A2)-(A3)-(A5) sont satisfaites par plusieurs fonctions de vitesse optimale. Le fait que V donne la vitesse d'un véhicule justifie l'hypothèse que V est régulière, continue et positive, car le véhicule avance toujours dans notre modèle. De plus, un véhicule roule plus vite s'il y a plus d'espace devant lui ce qui explique l'hypothèse (A2). L'hypothèse (A3) découle du fait que l'on suppose qu'aucune collision aura lieu et pour cela on a introduit une distance de sécurité h_0 à notre modèle : si la distance entre le véhicule i et le véhicule $i+1$ est inférieure à h_0, le conducteur i doit s'arrêter. L'hypothèse (A5) est utilisée pour obtenir enfin un Hamiltonien effectif qui satisfait les propriétés dans [IM13] pour utiliser les résultats de cet article. Finalement,* *on a ajouté (A4) car on a besoin de travailler avec une fonction V' à support compact. Un exemple de vitesse optimale est celle du modèle de Newell [New61] (voir aussi [Edi61]) est donné par :*

$$V(h) = \begin{cases} 0 & \text{si } h \leq h_0, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h-h_0}{b}\right)^n\right)\right) & \text{si } h_0 < h \leq h_{max}, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h_{max}-h_0}{b}\right)^n\right)\right) & \text{si } h > h_{max}, \end{cases}$$

avec $n \in \mathbb{N}\backslash\{0\}$ et $b \in [0, +\infty)$.

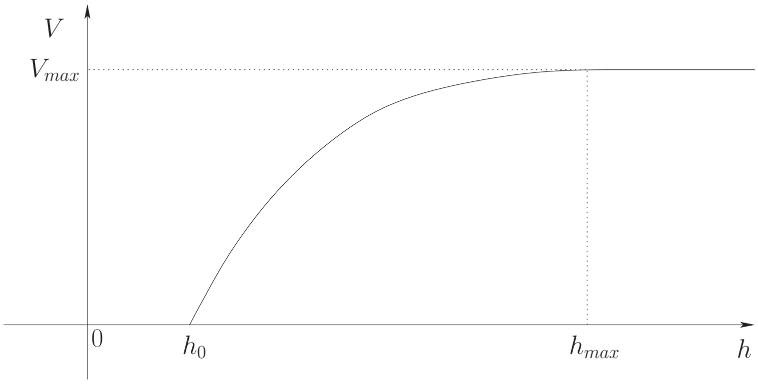

FIGURE 1.8 – Représentation schématique de la vitesse optimale V.

La fonction ϕ peut être vue comme la modélisation d'un feu tricolore situé au point zéro, et un exemple de fonction peut être donné pour $t \in [0, 1]$, par

$$\phi(t,x) = \begin{cases} 1 & \text{si } |x| > r \\ \dfrac{(\phi_0(t) - 1)}{r}x + \phi_0(t) & \text{si } x \in [-r, 0] \\ \dfrac{(1 - \phi_0(t))}{r}x + \phi_0(t) & \text{si } x \in (0, r] \end{cases}$$

avec ϕ_0 une fonction $\mathbb{Z}$-périodique définie de la manière suivante

$$\phi_0(t) = \begin{cases} 4t & \text{si } 0 < t < \frac{1}{4}, \ \textit{la fin du feux rouge} \\ 1 & \text{si } \frac{1}{4} < t < \frac{1}{2}, \ \textit{feux vert} \\ -4t + 3 & \text{si } \frac{1}{2} < t < \frac{3}{4}, \ \textit{feux orange} \\ 0 & \text{si } \frac{3}{4} < t < 1, \ \textit{feux rouge.} \end{cases}$$

Le modèle macroscopique. On présente maintenant le problème macroscopique qui est une équation d'Hamilton-Jacobi avec une condition au point de jonction $x = 0$. Cette équation a été introduite par Imbert et Monneau dans [IM13]. On introduit au début l'Hamiltonien effectif : soit $k_0 = 1/h_0$, on définit

$$\overline{H}(p) = \begin{cases} -p - k_0 & \text{si } p < -k_0, \\ -V\left(\dfrac{-1}{p}\right)|p| & \text{si } -k_0 \le p \le 0, \\ p & \text{si } p > 0. \end{cases} \tag{1.5.2}$$

L'Hamiltonien $\overline{H}$ est continu, coercitif $\left(\lim_{|p|\to+\infty} \overline{H}(p) = +\infty\right)$ et d'après (A5), il existe un unique point $p_0 \in [-k_0, 0]$ tel que

$$\begin{cases} \overline{H} \text{ est décroissant sur } (-\infty, p_0), \\ \overline{H} \text{ est croissant sur } (p_0, +\infty). \end{cases} \tag{1.5.3}$$

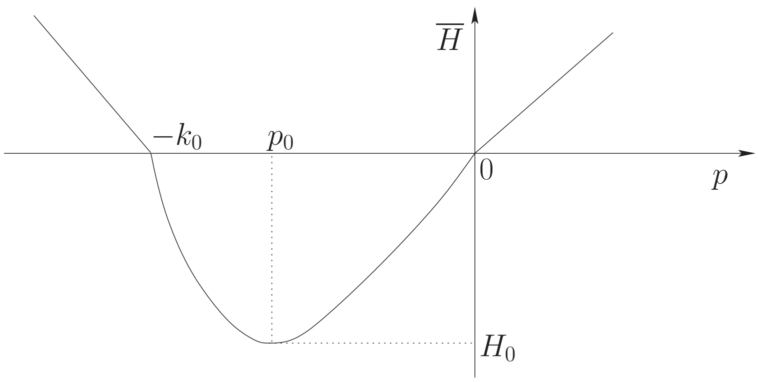

FIGURE 1.9 – Représentation schématique de $\bar{H}$.

Le problème limite est

$$\begin{cases} u_t^0 + \overline{H}(u_x^0) = 0 & \text{si } (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t^0 + \overline{H}(u_x^0) = 0 & \text{si } (t,x) \in (0,+\infty) \times (0,+\infty) \\ u_t^0 + F_{\overline{A}}\left(u_x^0(t,0^-), u_x^0(t,0^+)\right) = 0 & \text{si } (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{si } x \in \mathbb{R}, \end{cases} \tag{1.5.4}$$

où $\overline{A}$ est une constante à déterminer et $F_{\overline{A}}$ est definie par

$$F_{\overline{A}}(p_1, p_2) = \max\left(\overline{A}, \overline{H}^+(p_1), \overline{H}^-(p_2)\right)$$

avec

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{si } p \le p_0 \\ \overline{H}(p_0) & \text{si } p \ge p_0 \end{cases} \quad \text{et} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{si } p \le p_0, \\ \overline{H}(p) & \text{si } p \ge p_0. \end{cases}$$

On suppose que la condition initiale vérifie la propriété suivante :

$$\text{(A0)} \quad -k_0 \le (u_0)_x \le 0$$

qui signifie que la densité initiale des voitures est positive et inférieure à k_0. Dans (1.5.4), $-u^0_x$ est la densité des voitures (voir sous-section 1.4 pour les détails). D'après le Théorème 1.4.1, il existe une unique solution de viscosité u^0 de (1.5.4).

On rappelle maintenant un théorème qui permet de réduire la classe des fonctions test au point de jonction $x = 0$ à une seule fonction test (voir [IM13, Théorème 2.5]). Ce résultat sera utilisé en particulier dans la preuve d'homogénéisation au point de jonction. Il s'agit d'une définition équivalente à la définition de sous et sur-solution au point de jonction. Considérons le problème suivant :

$$u_t + \overline{H}(u_x) = 0 \quad \text{pour } t \in (0,T) \text{ et } x \in \mathbb{R}\backslash\{0\}. \tag{1.5.5}$$

Théorème 1.5.1 (Réduction de la classe des fonctions test)**.** *Soit $A \in [H_0, +\infty)$ avec $H_0 = \min_{\mathbb{R}} \overline{H}$. Soit $p^A_\pm \in \mathbb{R}$ la solution de*

$$\overline{H}\left(p^A_+\right) = \overline{H}^+\left(p^A_+\right) = A = \overline{H}^-\left(p^A_-\right) = \overline{H}\left(p^A_-\right) \tag{1.5.6}$$

et fixons une fonction test ϕ^0 tel que la réstriction de ϕ^0 sur $\mathbb{R}^+$ et sur $\mathbb{R}^-$ soit C^1 et qui satisfait

$$\phi^0_x(0^\pm) = p^A_\pm.$$

Soit $u : (0,T) \times \mathbb{R} \to \mathbb{R}$,

i) si u est une sous-solution semi-continue supérieurement de (1.5.5), alors u est une sous-solution de

$$u_t\,(t,0) + F_{H_0}\left(u_x(t,0^-), u_x(t,0^+)\right) = 0.$$

ii) Soit $A > H_0$ et $t_0 \in (0,T)$, si u est une sous-solution semi-continue supérieure de (1.5.5) et si pour toute fonction test φ touchant u par au-dessus en $(t_0, 0)$ avec

$$\varphi(t,x) = \psi(t) + \phi^0(x), \tag{1.5.7}$$

pour une certaine fonction $\psi \in C^1(0,+\infty)$, on a

$$\varphi_t + F_A\left(\varphi_x(t_0,0^-), \varphi_x(t_0,0^+)\right) \le 0 \quad \textit{en } (t_0,0),$$

alors u est une sous-solution de

$$u_t + F_A\left(u_x(t_0,0^-), u_x(t_0,0^+)\right) = 0 \quad \textit{en } (t_0,0).$$

iii) Soit $t_0 \in (0,T)$, si u est une sur-solution semi-continue inférieurement de (1.5.5) *et si pour toute fonction test φ satisfaisant* (1.5.7) *touchant u par en-dessous en* $(t_0, 0)$ *on a*

$$\varphi_t + F_A\left(\varphi_x(t_0,0^-), \varphi_x(t_0,0^+)\right) \ge 0 \quad \textit{en } (t_0,0),$$

alors u est sur-solution de

$$u_t + F_A\left(u_x(t_0,0^-), u_x(t_0,0^+)\right) = 0 \quad \textit{en } (t_0,0),$$

Résultat principal. On introduit la fonction de distribution cumulative (voir [FIM09b]),

$$\rho(t,y) = -\left(\sum_{i\geq 0} F\left(y - U_i(t)\right) + \sum_{i<0}\left(-1 + F\left(y - U_i(t)\right)\right)\right),$$

avec

$$F(x) = \begin{cases} 1 & \text{si } x \geq 0 \\ 0 & \text{si } x < 0. \end{cases}$$

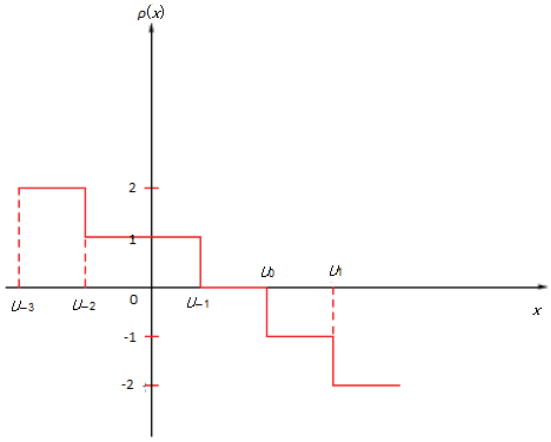

FIGURE 1.10 – Représentation de la fonction ρ.

On remarque que $\rho\left(t, U_j(t)\right) = -\left(j+1\right)$ et donc la fonction ρ est la primitive de la mesure empirique de la position des voitures. Par un changement d'échelle, on définit ρ^ε tel que $\rho^\varepsilon\left(t,y\right) = \varepsilon\rho\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right)$, voir les figures 1.5.1 et 1.5.1. Le résultat principal de ce travail est

Théorème 1.5.2. *Supposons qu' à l'instant initial, on a pour tout $i \in \mathbb{Z}$,*

$$U_i(0) \leq U_{i+1}(0) - h_0.$$

On suppose de plus qu'il existe $R > 0$ tel que pour tout $i \in \mathbb{Z}$, si $|U_i(0)| \geq R$, on a

$$U_{i+1}(0) - U_i(0) = h,$$

avec $h \geq h_0$. On définit la fonction u_0 par $u_0(x) = -x/h$ pour tout $x \in \mathbb{R}$. Alors il existe $\overline{A} \in [H_0, 0]$ tel que la fonction ρ^ε converge vers l'unique solution u^0 de (1.5.4).

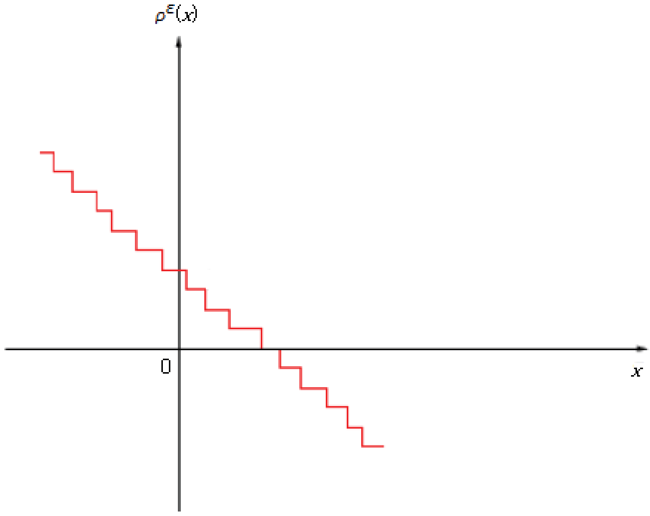

FIGURE 1.11 – Représentation de la fonction ρ^{ε}.

Remarque 1.5.2. *Comme dans [FSZ17a], on peut démontrer qu'il existe une constante* $C > 0$ *tel que*

$$0 \leq U_{i+1}(t) - U_i(t) \leq C.$$

On déduit donc que

$$0 \leq \varepsilon U_{i+1}(t) - \varepsilon U_i(t) \leq \varepsilon C. \tag{1.5.8}$$

De plus, on remarque que la fonction ρ^{ε} *donne à chaque position* $\varepsilon U_i(t)$ *au temps* εt *un label* ε. *L' équation* (1.5.8) *implique que la distance entre deux véhicules consécutifs tend vers zéro quand* ε *tend vers zéro, ce qui peut être vu comme un passage du micro au macro. Il est naturel donc que* ρ^{ε} *converge la fonction qui labélise les véhicules à l'échelle macroscopique, qui est* u^0, *la solution de* (1.5.4).

De plus, nous avons obtenu les deux résultats naturels suivants : le premier concerne le limiteur de flux $\overline{A}$ et qui affirme que la limitation de flux du trafic au point de jonction $x = 0$ est plus importante si à l'échelle microscopique, l'impact de la perturbation locale sur la vitesse est plus important.

Proposition 1.5.1 (Monotonicité du limiteur de flux)**.** *Supposons que l'hypothèse (A) est satisfaite et soit* $\phi_1, \phi_2 : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$ *deux fonctions satisfaisant (A6). Soit* $\overline{A}_1$ *et* $\overline{A}_2$ *leur limiteur de flux respectifs donnés par le Théorème 1.5.2. Si pour tout* $(t,x) \in \mathbb{R} \times \mathbb{R}$, *on a*

$$\phi_1(t,x) \leq \phi_2(t,x),$$

alors

$$\overline{A}_1 \geq \overline{A}_2.$$

Remarque 1.5.3. *Nous rappelons (voir sous-section 1.4) que l'équation* (1.4.8) *est dérivée par du modèle macroscopique de trafic suivant*

$$f\left(\rho\left(t,0^-\right)\right)=\min\left(-A,f_D\left(\rho\left(t,0^-\right)\right),f_S\left(\rho\left(t,0^+\right)\right)\right)$$

où ρ est la densité des voitures , f est la fonction qui décrit le flux et f_D, f_S sont respectivement les fonctions offre et demande de f. On remarque donc que si $\phi_1 \leq \phi_2$, le flux est plus limité pour ϕ_1.

Le deuxième résultat concerne les bornes de la densité des véhicules. En effet, la solution u^0 de l'équation (1.5.4) est la primitive de la densité et donc le résultat suivant implique que la densité des voitures est bornée.

Proposition 1.5.2. *Supposons que les hypothèses (A0)-(A) sont satisfaites. Soit u^0 la solution de l'équation* (1.5.4)*, alors pour tout $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$,*

$$-k_0 \leq u_x^0 \leq 0.$$

Ce résultat implique aussi qu'on utilise la définition de l'Hamiltonien effectif $\overline{H}$ (1.5.2) seulement pour $p \in [-k_0, 0]$. Toutefois, on a définit $\overline{H}$ de manière à ce qu'il soit coercitif pour pouvoir utiliser les résultats de [IM13] et pour la construction des correcteurs. Notons donc que la définition de $\overline{H}$ pour $p \notin [-k_0, 0]$ n'est pas unique.

Le chemin pour obtenir le résultat de convergence. On explique maintenant en quelques étapes, comment obtenir le résultat.

i) **Injection du système d'EDO dans une EDP :**
comme dans [FIM12b, FIM09b], on injecte le système d'équations différentielles ordinaires dans une équation aux dérivées partielles. On cherche donc à construire un opérateur non-local M qui vérifie

$$M\left[\rho\left(t,\cdot\right)\right]\left(U_j\left(t\right)\right)=-V\left(U_{j+1}-U_j\right). \tag{1.5.9}$$

En effet, si l'on construit M qui vérifie (1.5.9), on obtient que ρ est une solution de viscosité discontinue de

$$u_t + M\left[u(t,\cdot)\right](x)\cdot\phi\left(t,x\right)\cdot|u_x| = 0 \quad \text{sur } (0,+\infty)\times\mathbb{R}.$$

L'opérateur non-local convenable qui vérifie (1.5.9) est donné par

$$M[U](x)=\int_{-\infty}^{+\infty} J(z)E\left(U(x+z)-U(x)\right)dz-\frac{3}{2}V_{max}$$

avec

$$E(z)=\begin{cases} 0 & \text{si } z\geq 0\\ 1/2 & \text{si } -1\leq z<0\\ 3/2 & \text{si } z<-1\end{cases} \qquad \text{et} \quad J=V' \text{ sur } \mathbb{R}.$$

On déduit donc que ρ^ε est une solution de viscosité de

$$u_t^\varepsilon + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)\cdot\phi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\cdot|u_x^\varepsilon| = 0 \quad \text{sur } (0,+\infty)\times\mathbb{R}, \tag{1.5.10}$$

avec

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+\varepsilon z) - U(x)\right)dz - \frac{3}{2}V_{max}.$$

Remarque 1.5.4 (Formulation Lagrangienne)**.** *Une autre méthode pour injecter le système d'EDO dans une EDP est de considérer la formulation lagrangienne comme dans [FS15], en considérant la fonction*

$$v : [0,T]\times\mathbb{R}\to\mathbb{R}, \quad v(t,y) = U_{\lfloor y\rfloor}(t).$$

Cette fonction satisfait pour tout $(t,y)\in[0,T]\times\mathbb{R}$

$$\begin{cases} v_t(t,y) = V\left(v(t,y+1) - v(t,y)\right)\cdot\phi\left(t, v(t,y)\right), \\ v(0,y) = v_0(y). \end{cases} \tag{1.5.11}$$

La difficulté de cette formulation est que la fonction ϕ *est évaluée en* $v(t,y)$ *et non en un point physique de la route. La notion de jonction dans ce cas n'est pas bien définie et c'est pour cela que nous avons décidé d'utiliser une autre formulation* (1.5.10) *(dans laquelle la perturbation sera évaluée sur un point physique de la route) en utilisant la même technique que dans le papier de Forcadel, Imbert et Monneau [FIM09b], c'est-à-dire en utilisant la primitive de la mesure empirique des positions des véhicules. De cette manière, on peut utiliser les résultats de Imbert et Monneau [IM13] pour les hamiltoniens quasi-convexes avec des conditions de jonction*

ii) **Convergence de la solution continue :** la seconde étape consiste à montrer que l'unique solution u^ε de l'équation

$$\begin{cases} u_t^\varepsilon + \cdot M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)\cdot\phi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\cdot|u_x^\varepsilon| = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.5.12}$$

converge localement uniformément vers la solution de (1.5.4). L'existence de la solution du problème (1.5.12) peut être obtenue facilement par la méthode de Perron [IMR08, Preuve du Theórème 6], [AT96] ou [Imb05] après la construction des barrières. La preuve du principe de comparaison est similaire à la preuve du principe de comparaison dans [FIM09b].

Théorème 1.5.3. *Supposons que les hypothèses (A0) et (A) sont satisfaites. Il existe une unique solution* u^ε *de* (1.5.12) *qui satisfait (pour une certaine constante* K_1*)*

$$u_0(x) \le u^\varepsilon(t,x) \le u_0(x) + K_1 t.$$

La preuve de convergence repose sur la construction de bons correcteurs [LPV86, FIM12b, IMR08, IM08] qui seront utilisé dans la méthode de la fonction test perturbée introduite par Evans dans [Eva89]. La forme particulière de l'équation limite (1.5.4) nécessite la construction de deux correcteurs, un pour $x \neq 0$ et l'autre pour $x = 0$.

– Si $x \neq 0$: on utilise l'Ansatz classique

$$u^{\varepsilon}(t,x) = u^{0}(t,x) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{1.5.13}$$

En effet, en utilisant le fait que u^{ε} est solution de (1.5.12) et en notant $\lambda = -u_t^0(t,x)$ et $p = u_x^0(t,x)$, l'expression (1.5.13) implique que pour $x \neq 0$, un correcteur v doit vérifier : pour tout $p \in [-k_0, 0]$, il existe un unique $\lambda \in \mathbb{R}$, tel qu' il existe une solution bornée v de

$$\begin{cases} M_p[v](x) \cdot |v_x + p| = \lambda, \quad x \in \mathbb{R}, \\ v \quad \text{est } \mathbb{Z}\text{−periodique}, \end{cases}$$

avec

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+z) - U(x) + p \cdot z\right) dz - \frac{3}{2} V_{max}.$$

On remarque facilement que $\lambda = \overline{H}(p)$ et que l'on peut choisir $v = 0$.

– Si $x = 0$: dans ce cas, formellement, en utilisant le Théorème 1.5.1, on suppose qu'au voisinage de zéro,

$$u^0(t,x) = u^0(t,0) + \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}$$

où $\overline{p}_+$ et $\overline{p}-$ sont deux constantes qui vérifient

$$\begin{cases} \overline{H}\left(\overline{p}_+\right) = \overline{H}^+\left(\overline{p}_+\right) = \overline{A} \\ \overline{H}\left(\overline{p}_-\right) = \overline{H}^-\left(\overline{p}_-\right) = \overline{A}. \end{cases}$$

Sur ce voisinage, on utilise l'Ansatz suivant

$$u^{\varepsilon}(t,x) = u^{0}(t,0) + \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{1.5.14}$$

Formellement, en utilisant (1.5.14), on a

$$\begin{aligned} w_t\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) &= u_t^{\varepsilon}(t,x) - u_t^0(t,0) \\ &= -M^{\varepsilon}\left[\frac{u^{\varepsilon}(t,\cdot)}{\varepsilon}\right](x) |u_x^{\varepsilon}(t,x)| \phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + F_{\overline{A}}\left(\overline{p}_-, \overline{p}_+\right) \\ &= -M^{\varepsilon}\left[\frac{u^{\varepsilon}(t,\cdot)}{\varepsilon}\right](x) |u_x^{\varepsilon}(t,x)| \phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + \overline{A}. \end{aligned} \tag{1.5.15}$$

De plus, on a que

$$M^{\varepsilon}\left[\frac{u^{\varepsilon}(t,\cdot)}{\varepsilon}\right](x) = \int_{-\infty}^{+\infty} J(z)E\left(\frac{u^{\varepsilon}(t,x+\varepsilon z)-u^{\varepsilon}(t,x)}{\varepsilon}\right)dz - \frac{3}{2}V_{max}$$
$$= \int_{-\infty}^{+\infty} J(z)E\left(w\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+z\right)-w\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\right)dz - \frac{3}{2}V_{max}$$
$$= M\left[w\left(\frac{t}{\varepsilon},\cdot\right)\right]\left(\frac{x}{\varepsilon}\right). \tag{1.5.16}$$

D'où, en utilisant les notations $y = \dfrac{x}{\varepsilon}$ et $s = \dfrac{t}{\varepsilon}$ et en combinant (1.5.15)-(1.5.16), on obtient que

$$w_s(s,y) + M[w(s,\cdot)](y)\cdot\phi(s,y)\cdot|w_y(s,y)| = \overline{A}.$$

Théorème 1.5.4. *Il existe une unique constante $\lambda = \overline{A}$ tel qu' il existe une solution w de*

$$w_t + M[w(t,\cdot)](x)\cdot\phi(t,x)\cdot|w_x| = \lambda \tag{1.5.17}$$

et tel que $w^{\varepsilon}(t,x) = \varepsilon\, w\left(\dfrac{t}{\varepsilon},\dfrac{x}{\varepsilon}\right)$ satisfait localement uniformément :

$$w^{\varepsilon}(t,x) \underset{\varepsilon\to 0}{\longrightarrow} W(x) = \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}.$$

Pour construire le couple $(\overline{A}, w)$, on utilise l'idée introduite par Achdou, Tchou [AT15] et Galise, Imbert et Monneau [GIM15], qui consiste à considérer un problème dans la cellule sur un domaine tronqué puis de passer à la limite quand la taille du domaine tend vers l'infini. Pour prédire les bonnes pentes à l'infini, il faut donc imposer de bonnes conditions aux bords. Dans notre cas, l'équation (1.5.17) étant non-locale, il est difficile d'imposer ces conditions aux bords. C'est pour cette raison que nous introduisons un nouvel Hamiltonien qui sera égal à l'opérateur non-local dans l'intervalle $[-R,R]$ et qui sera remplacé par un hamiltonien local en dehors de $[-R-1,R+1]$ afin de pouvoir imposer des conditions aux limites. Plus précisément, pour $l \in (r,+\infty), r << l$ et $r \leq R << l$, on considère le problème suivant : trouver $\lambda_{l,R}$ tel qu'il existe une solution $w^{l,R}$ de

$$\begin{cases} w_t^{l,R} + G_R\left(t,x,[w^{l,R}(t,\cdot)],w_x^{l,R}\right) = \lambda_{l,R} & \text{si } (t,x)\in\mathbb{R}\times(-l,l) \\ w_t^{l,R} + \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{si } (t,x)\in\mathbb{R}\times\{-l\} \\ w_t^{l,R} + \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{si } (t,x)\in\mathbb{R}\times\{l\} \\ w^{l,R} \text{ est 1-periodique en } t & \end{cases}$$

avec

$$G_R(t,x,[U],q) = \psi_R(x)\cdot\phi(t,x)\cdot M[U](x)\cdot|q| + (1-\psi_R(x))\cdot\overline{H}(q),$$

et $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \to [0,1]$, avec

$$\psi_R \equiv \begin{cases} 1 & \text{sur } [-R, R] \\ 0 & \text{à l'extérieur de } [-R-1, R+1], \end{cases}$$

et

$$\psi_R(x) < 1 \ \forall x \notin [-R, R].$$

Le limiteur du flux effectif $\overline{A}$ est alors construit comme la limite quand l tend vers $+\infty$ puis R tend vers $+\infty$ de $\lambda_{l,R}$. La fonction $w^{l,R}$ permet de construire la fonction w du Théorème 1.5.4 avec les bonnes pentes à l'infini. On a également le résultat suivant qui caractérise le limiteur du flux
Théorème 1.5.5 (Limiteur du flux effectif)**.** *Supposons que l'hypothèse (A) est satisfaite. On définit l'ensemble de fonctions suivant*

$$\mathcal{S} = \left\{ w \textit{ tel que } \exists m \in Lip\,(\mathbb{R}) \textit{ et } C \geq 0 \textit{ tel que } ||w - m||_{L^\infty(\mathbb{R})} \leq C \right\}.$$

Alors on a

$$\overline{A} = \inf \left\{ \lambda \in \mathbb{R} : \exists\ w \in \mathcal{S} \textit{ solution de (1.5.17)} \right\}.$$

La construction de ces correcteurs nous permet de prouver le résultat de convergence en utilisant la méthode de la fonction test perturbé et le Théorème 1.5.1,
Théorème 1.5.6. *Supposons que les hypothèses (A0)-(A) sont satisfaites. Pour $\varepsilon > 0$, soit u^ε la solution de viscosité de* (1.5.12). *Il existe $\overline{A} \in [H_0, 0]$ donnée par le Théorème 1.5.4 tel que u^ε converge localement uniformément vers l'unique solution de viscosité u^0 de l'équation* (1.5.4).

iii) **Lien entre le système d'EDO et l'EDP :** enfin, le résultat principal de ce travail (Théorème 1.5.2) est obtenu par le principe de comparaison pour l'équation (1.5.12). En effet, posons $u_0(x) = -x/h$ avec $h \geq h_0$. On sait d'après l'étape i) que ρ^ε est solution de (1.5.10). De plus, on peut démontrer que

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon)$$

avec $f(\varepsilon) \to 0$ quand $\varepsilon \to 0$. D'où, en utilisant le principe de comparaison de l'équation (1.5.12), on obtient que

$$u^\varepsilon(t, x) - f(\varepsilon) \leq \rho^\varepsilon(t, x) \leq u^\varepsilon(t, x) + f(\varepsilon) \tag{1.5.18}$$

où u^ε est l'unique solution de viscosité de l'équation (1.5.12) pour $u_0(x) = -x/h$. En passant à la limite quand $\varepsilon \to 0$ dans (1.5.18) et en utilisant l'étape ii), on obtient notre résultat.

On présente maintenant très rapidement un travail très similaire au travail précédent.

1.5.2 Homogénéisation d'un modèle d'ordre un avec deux vitesses différentes

On présente une première généralisation au cas où les vitesses sont différentes avant et après la jonction. On considère le modèle microscopique d'ordre un suivant

$$\dot{U}_j(t) = V_1(U_{j+1}(t) - U_j(t))\varphi\left(U_j(t)\right) + V_2(U_{j+1}(t) - U_j(t))\left(1 - \varphi\left(U_j(t)\right)\right), \quad (1.5.19)$$

où V_1 et V_2 sont les deux vitesses optimales du modèle et φ est une fonction de transition lipschitzienne,

$$\varphi(x) = \begin{cases} 1 & \text{si } x \leq -r \\ 0 & \text{si } x > r\ . \end{cases}$$

Comme dans le travail précédent, on montre que la fonction de distribution cumulative converge vers la solution d'une équation d'Hamilton-Jacobi avec une condition au point de jonction. La différence ici est que l'on obtient deux Hamiltoniens effectifs différents car à l'échelle microscopique, on a deux vitesses optimales différentes. Plus précisément, l'équation limite est

$$\begin{cases} u_t^0 + \overline{H}_1(u_x^0) = 0 & \text{si} \quad (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t^0 + \overline{H}_2(u_x^0) = 0 & \text{si} \quad (t,x) \in (0,+\infty) \times (0,+\infty) \\ u_t^0 + F_{\overline{A}}(u_x^0(t,0^-), u_x^0(t,0^+)) = 0 & \text{si} \quad (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{si} \quad x \in \mathbb{R} \end{cases}$$

avec

$$F_A(p_1,p_2) = \max\left(\overline{A}, \overline{H}_1^+(p_1), \overline{H}_2^-(p_2)\right);$$

où $\overline{H}_1^+$ et $\overline{H}_2^-$ représentent respectivement la partie croissante de $\overline{H}_1$ et décroissante de $\overline{H}_2$. La différence principale avec [FSZ17b] est qu'on doit définir deux opérateurs non-locaux et donc la construction des correcteurs sera un peu plus compliquée.

1.5.3 Homogénéisation d'un modèle du second ordre

On présente maintenant les résultats de [FSZ17a]. On considère un modèle microscopique d'ordre deux et on prouve comme dans le travail précédent que la primitive de la mesure empirique converge vers la solution de l'équation (1.5.4). L'intérêt de travailler avec un modèle du second ordre est qu'il est plus réaliste car il considère l'accélération du véhicule. Cependant, le fait de travailler avec un modèle du second ordre ajoute beaucoup de difficultés techniques. En effet, on a une combinaison entre les difficultés provenant du fait d'avoir un modèle contenant une perturbation locale, mais aussi du fait qu'il s'agit d'un modèle du second ordre. La différence principale avec [FSZ17b] est l'injection du système d'EDO dans un système d'EDP. En effet, on utilise l'idée de définir une nouvelle variable qui dépend de la position et de la vitesse des véhicules, pour obtenir un système de deux systèmes d'EDO, et puis d'injecter ce système dans un système de deux EDPs (voir [FS15]). On donne dans ce paragraphe les principales différence de ce travail avec le précédent [FSZ17b]. On considère un modèle microscopique du second ordre qui simule la présence d'une perturbation locale. Le modèle considéré est une version modifiée du modèle de Bando *et al* dans [BHN$^+$95]. Plus précisément, on considère un modèle de poursuite "follow-the-leader" de la forme suivante

$$\ddot{U}_j(t) = a\left(V(U_{j+1}(t) - U_j(t)) \cdot \phi\left(U_j(t)\right) - \dot{U}_j(t)\right). \tag{1.5.20}$$

La plupart des notations de ce modèle sont les mêmes que dans (1.5.1). La différence ici est que l'on considère un problème du second ordre et donc on introduit $\ddot{U}_j\left(t\right)$ qui représente l'accélération du véhicule j. De plus, la constante a représente la sensibilité des conducteurs. On suppose que l'hypothèse (A) (voir sous-section 1.5.1) est satisfaite pour ce modèle. Pour simplifier la présentation, on supposera que ϕ ne dépend pas du temps. On ajoute l'hypothèse (A'7) à (A) et on note (A')=(A)+(A'7).

(A'7)(Monotonicité) $\quad a \geq 4\left\|V'\right\|_\infty \left\|\phi\right\|_\infty + 4\left\|\phi'\right\|_\infty \left\|V\right\|_\infty.$

Remarque 1.5.5 (Remarque sur (A'7))**.** *L'hypothèse (A'7) implique que pour tout* $(b, x) \in \mathbb{R}^2$*, la fonction*

$$f : z \mapsto \frac{a}{2}z - 2V\left(b + z\right)\phi\left(x - z\right)$$

est croissante. Ce résultat est important car il implique que le système que l'on considèrera plus tard dans ce travail est monotone au sens de Ishii et Koike [IK91], ce qui implique l'unicité de la solution que l'on considère. Dans certains travaux, cette hypothèse a un sens physique comme dans le cas d'un modèle de Frenkel-Kontorova [FIM12b]. Dans notre cas, d'un point de vue modélisation, cette hypothèse signifie que les conducteurs sont capables d'adapter leur vitesse assez rapidement pour éviter les collisions. Dans le cas où il n'y a pas de perturbation ($\phi \equiv 1$), cette hypothèse est en particulier utilisée dans certains travaux pour obtenir la stabilité du système d'EDO.

Le théorème suivant est le résultat principal de ce travail.

Théorème 1.5.7. *Supposons que l'hypothèse (A) est satisfaite. Alors, il existe une unique constante* $\overline{A} \in [H_0, 0]$ *tel que* ρ^ε *converge localement uniformément vers l'unique solution de* (1.5.4)

Puisque le modèle macroscopique est le même que dans le premier travail (1.5.4), on explique directement le chemin de convergence. Pour chaque point, on donne les différences avec le premier travail.

Le chemin pour obtenir le résultat de convergence.

i) **Injection du système d'EDO dans une EDP** : on utilise l'idée de [FIM09a, FIM12b, FS15] et on considère pour tout $j \in \mathbb{Z}$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha}\dot{U}_j(t) \quad \text{avec} \quad \alpha = \frac{a}{2}.$$

En utilisant cette nouvelle fonction, on obtient le système d'EDP suivant qui est équivalent à (1.5.20) pour tout $j \in \mathbb{Z}$, pour tout $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \dot{\Xi}_j(t) = \alpha\left(U_j(t) - \Xi_j(t)\right) + 2V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right) \end{cases}$$

et puis on introduit la fonction de distribution cumulative de $(\Xi_j)_j$, définie par

$$\sigma^\varepsilon(t, y) = -\varepsilon\left(\sum_{j\geq 0} F\left(y - \varepsilon\Xi_j\left(t/\varepsilon\right)\right) + \sum_{j<0}\left(-1 + F\left(y - \varepsilon\Xi_j\left(t/\varepsilon\right)\right)\right)\right).$$

On construit donc deux opérateurs non-locaux M et L tel que

$$\begin{cases} M\left(\rho(t, U_j\left(t\right)), \left[\sigma\left(t, \cdot\right)\right]\right)\left(U_j\right) = -U_j'\left(t\right) \\ L\left(\Xi_j, \sigma\left(t, \Xi_j\left(t\right)\right), \left[\rho\left(t, \cdot\right)\right]\right)\left(\Xi_j\left(t\right)\right) = -\Xi_j'\left(t\right) \end{cases}$$

pour déduire que $(\rho^\varepsilon, \sigma^\varepsilon)$ est une solution de viscosité discontinue pour $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + M^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}(t, x), \left[\frac{\xi^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\xi^\varepsilon}{\varepsilon}(t, x), \left[\frac{u^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |\xi_x^\varepsilon| = 0. \end{cases}$$

La définition des opérateurs non-locaux est un peu compliquée donc on préfère de ne pas entrer dans ces détails qui n'ont pas un impact direct sur l'idée générale de la preuve.

ii) **Convergence de la solution continue.**

(A0) (Borne du gradient.) Soit u_0 et ξ_0^ε deux fonctions lipschitziennes tel que

$$-k_0 \leq (u_0)_x \leq 0$$

$$-k_0 \leq (\xi_0^\varepsilon)_x \leq 0,$$

et

$$0 \leq \xi_0^\varepsilon(x) - u_0(x) \leq \varepsilon.$$

On considère l'équation suivante

$$\begin{cases} u_t^\varepsilon + M^\varepsilon\left(\dfrac{u^\varepsilon}{\varepsilon}(t,x),\left[\dfrac{\xi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + L^\varepsilon\left(\dfrac{x}{\varepsilon},\dfrac{\xi^\varepsilon}{\varepsilon}(t,x),\left[\dfrac{u^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\xi_x^\varepsilon| = 0 \\ u^\varepsilon(0,x) = u_0(x) \\ \xi^\varepsilon(0,x) = \xi_0^\varepsilon(x) \end{cases} \tag{1.5.21}$$

et on démontre le théorème suivant

Théorème 1.5.8. *Supposons que l'hypothèse (A') est satisfaite. Pour $\varepsilon > 0$, soit $(u^\varepsilon, \xi^\varepsilon)$ la solution de (1.5.21). Alors il existe $\bar{A} \in [H_0, 0]$ tel que u^ε et ξ^ε convergent localement uniformément vers l'unique solution de viscosité u^0 de (1.5.4).*

Remarque 1.5.6. *Classiquement, l'existence et l'unicité de la solution de l'équation (1.5.21) sont obtenues par Perron et par le principe de comparaison qui est obtenu en utilisant la monotonicité du système (voir (A'7)).*

Pour la construction des correcteurs, on fait comme avant, c'est-à-dire, on construit deux correcteurs.

– Si $x \neq 0$: on utilise l'Ansatz

$$\begin{cases} u^\varepsilon(t,x) = u^0(t,x) + \varepsilon w\left(\dfrac{x}{\varepsilon}\right) \\ \xi^\varepsilon(t,x) = u^0(t,x) + \varepsilon \chi\left(\dfrac{x}{\varepsilon}\right). \end{cases}$$

On remarque que les correcteurs w et χ ne dépendent pas du temps et cela revient à la forme de ϕ qui ne dépend pas du temps. En procédant comme en sous-section 1.5.1, on remarque que le couple (w,χ) doit vérifier : pour tout $p \in [-k_0, 0]$, il existe un unique $\lambda \in \mathbb{R}$, tel qu'il existe une unique solution bornée (w,χ) d'un système d'équations non-locales. On obtient que pour $p \in [-k_0, 0]$, on a $\lambda = \overline{H}(p) = -V\left(\dfrac{-1}{p}\right)|p|$ et on peut choisir $(w,\chi) = \left(0, -\dfrac{p}{\alpha}V\left(\dfrac{-1}{p}\right)\right)$.

– si $x = 0$: comme dans la sous-section 1.5.1, on utilise l'Ansatz suivant

$$\begin{cases} u^\varepsilon(t,x) = u^0(t,0) + \varepsilon w\left(\dfrac{t}{\varepsilon},\dfrac{x}{\varepsilon}\right) \\ \xi^\varepsilon(t,x) = u^0(t,0) + \varepsilon \chi\left(\dfrac{t}{\varepsilon},\dfrac{x}{\varepsilon}\right) \end{cases} \tag{1.5.22}$$

En faisant le même calcul formel de l'étape ii) dans la sous-section 1.5.1, on obtient qu'un correcteur (w,χ) doit vérifier le théorème suivant :

Théorème 1.5.9. *Il existe une unique constante $\lambda = \overline{A}$ tel qu'il existe une solution (w,χ) de*

$$\begin{cases} M(w(x),[\chi(\cdot)])\cdot|w_x| = \lambda \\ L(x,\chi(x),[w(\cdot)])\cdot|\chi_x| = \lambda \end{cases}$$

et tel que localement uniformément, on a

$$(w^{\varepsilon},\chi^{\varepsilon})\underset{\varepsilon\to 0}{\longrightarrow} W(x)$$

où W est la même fonction donnée dans le Théorème 1.5.4.

La construction du correcteur (w,χ) est similaire à la construction du correcteur dans le résultat précédent : on construit le correcteur sur un domaine tronqué puis on prend l vers $+\infty$ et R vers $+\infty$: il existe $\lambda_{l,R}\in\mathbb{R}$ tel qu'il existe une solution $(w^{l,R},\chi^{l,R})$ de

$$\begin{cases}\begin{cases}G_R^1(x,w^{l,R}(x),[\chi^{l,R}],w_x^{l,R})=\lambda_{l,R}\\ G_R^2(x,\chi^{l,R}(x),[w^{l,R}],\chi_x^{l,R})=\lambda_{l,R}\end{cases} & \text{si } x\in(-l,l)\\ \begin{cases}\overline{H}^+(w_x^{l,R})=\lambda_{l,R}\\ \overline{H}^+(\chi_x^{l,R})=\lambda_{l,R}\end{cases} & \text{si } x=l\\ \begin{cases}\overline{H}^-(w_x^{l,R})=\lambda_{l,R}\\ \overline{H}^-(\chi_x^{l,R})=\lambda_{l,R}\end{cases} & \text{si } x=-l\end{cases}$$

avec

$$G_R^1(x,w(x),[\chi],q)=\psi_R(x)M(w(x),[\chi])(x)|q|+(1-\psi_R(x))\overline{H}(q),$$

$$G_R^2(x,\chi(x),[w],q)=\quad\psi_R(x)L\left(x,\chi(x),[w]\right)(x)|q|+(1-\psi_R(x))\overline{H}(q).$$

De plus, $\psi_R\in C^{\infty}$, $\psi_R:\mathbb{R}\to[0,1]$, avec

$$\psi_R\equiv\begin{cases}1 & \text{sur } [-R,R]\\ 0 & \text{sur } (-\infty,-R-1]\cup[R+1,+\infty),\end{cases}$$

et

$$\psi_R(x)<1\ \forall x\notin[-R,R].$$

Remarque 1.5.7. *Quand on fait la preuve de convergence, on aura besoin de démontrer que le gradient des fonctions u^{ε} et ξ^{ε} est borné. En comparant les deux travaux (ce travail et celui d'ordre un), on remarque que la preuve dans ce cas est plus compliquée car c'est un système couplé. Ajoutons aussi que la preuve du principe de comparaison pour le système* (1.5.21) *est aussi un peu plus compliquée que celle de l'équation* (1.5.12).

iii) **Lien entre le système d'EDO et l' EDP :** le théorème suivant implique directement notre résultat principal. Ce résultat est obtenu en procédant comme dans le point iii) dans la sous-section 1.5.1 en utilisant le principe de comparaison pour (1.5.21).

Théorème 1.5.10. *Supposons que (A') est satisfaite et que le couple* $(U_i(0), \Xi_i(0))_i$ *satisfait*

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0,$$

et

$$U_{i+1}(0) - U_i(0) \leq h_{max}.$$

On suppose de plus qu'il existe $R > 0$ *tel que, pour tout* $i \in \mathbb{Z}$*, si* $|U_i(0)| \geq R$

$$U_{i+1}(0) - U_i(0) = h$$

et si $|\Xi_i(0)| \geq R$

$$\Xi_{i+1}(0) - \Xi_i(0) = h,$$

avec $h \in [h_0, h_{max}]$*. On définit deux fonctions* u_0 *et* ξ_0^ε *par* $u_0(x) = \xi_0^\varepsilon(x) = -x/h$ *pour tout* $x \in \mathbb{R}$*. Alors il existe un unique* $\overline{A} \in [H_0, 0]$ *tel que les fonctions* ρ^ε *et* σ^ε *convergent localement uniformément vers l'unique solution* u^0 *de* (1.5.4).

On présente maintenant les résultats du dernier travail de cette thèse.

1.5.4 Un principe de comparaison pour une équation d'Hamilton-Jacobi sur un domaine qui varie en temps

Le but de ce travail est de démontrer un principe de comparaison pour une équation d'Hamilton-Jacobi sur un domaine qui varie en temps. Les bornes du domaine, ou les "points de jonction" du domaine sont des fonctions de classe C^1. La motivation pour considérer cette équation est son application au sujet du trafic routier. En effet, on montre que l'on peut dériver cette équation à partir d'un modèle de trafic du premier ordre qui étudie l'interaction entre un bus et les autres véhicules sur une route. L'idée consiste à supposer que la grandeur de ce bus, ainsi que sa vitesse, réduisent la densité des véhicules qui passent à côté du bus. Un court travail d'homogénéisation est aussi fait pour renforcer l'interprétation trafic routier de cette équation. Les points de jonction sont au temps t sont les points $b_i(t) \in \mathbb{R}$ et on définit pour $i \in \{1, ..., N+1\}$,

$$B_i = \{(t, x) \in (0, T) \times \mathbb{R}, \text{tel que } b_{i-1}(t) < x < b_i(t)\}.$$

On introduit la "fonction de limitation du flux" (voir [IM13]). Pour $i \in \{1, ..., N\}$, $t \in \mathbb{R}^+$ et $p = (p_1, p_2) \in \mathbb{R}^2$

$$F_{A_i}(t, p_1, p_2) = \max\Big(A_i(t), H_{i,i}^+(t, p_1), H_{i+1,i}^-(t, p_2)\Big)$$

où A_i est une fonction localement lipschitzienne et $H_{i,i}^+$ (resp. $H_{i+1,i}^-$) est la partie croissante (resp. décroissante) de l'Hamiltonien $H_{i,i}$ (resp. $H_{i+1,i}$) dont la définition est donné plus tard. L'équation considérée est

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{si } (t,x) \in B_i, i = 1,..,N+1 \\ \dfrac{d}{dt}\left(u(t,b_i(t))\right) + F_{A_i}\left(t, u_x^{i,-}(t,x), u_x^{i,+}(t,x)\right) = 0 & \text{si } x = b_i(t), i = 1,...,N \\ u(0,x) = u_0(x) & \text{pour } x \in \mathbb{R}, \end{cases} \tag{1.5.23}$$

où $u_t = \dfrac{\partial u}{\partial t}$ et $u_x = \dfrac{\partial u}{\partial x}$ sont respectivement la dérivée en temps et en espace de u. De plus, on a

$$u_x^{i,+}(t,b_i(t)) = \lim_{\substack{(t,x)\to(t,b_i(t)) \\ x>b_i(t)}} u_x(t,x)$$
$$u_x^{i,-}(t,b_i(t)) = \lim_{\substack{(t,x)\to(t,b_i(t)) \\ x<b_i(t)}} u_x(t,x).$$

Hypothèses et notations (B).

- (B1) Les fonctions $b_1,...,b_N$ sont des fonctions dérivables tel que $b_{i+1} > b_i$. On désigne par $b_0 = -\infty$ et $b_{N+1} = +\infty$. De plus, on suppose que pour tout $j \in \{1,...,N\}$, b'_j est une fonction localement lipschitzienne.
- (B2) Les Hamiltoniens $H_1,...,H_{N+1} : \mathbb{R} \to \mathbb{R}$ satisfaient les propriétés suivantes : pour tout $i \in \{1,...,N+1\}$,

$$\begin{cases} H_i \text{ est continu,} \\ H_i \text{ est sur-linéair, c'est-à-dire, } \lim\limits_{|p|\to+\infty} \dfrac{H_i(p)}{|p|} = +\infty. \end{cases}$$

- (B3) Pour $i \in \{1,...,N+1\}$ et pour $k \in \{i,i+1\}$, $H_{k,i}(t,p) = H_k(p) - b'_i(t)p$. De plus, on suppose que pour tout $i \in \{1,...,N\}$, $k \in \{i,i+1\}$ et pour tout $t \in \mathbb{R}^+$, l'Hamiltonien $H_{k,i}(t,\cdot)$ est quasi-convexe. On désigne par $H^+_{k,i}(t,\cdot)$ et $H^-_{k,i}(t,\cdot)$ respectivement la partie croissante et la partie décroissante de $H_{k,i}(t,\cdot)$.
- (B4) Pour tout $i \in \{1,...,N\}$, le limiteur de flux $A_i : [0,T] \to \mathbb{R}$ est une fonction localement lipschitzienne.

Remarque 1.5.8. *On peut démontrer (voir chapitre 5) dans le cas d'une seule fonction b et d'un seul Hamiltonien H que l'équation* (1.5.23) *peut-être dériver d'un modèle interaction véhicule-bus sur une route simple [Leb05] suivant*

$$\begin{cases} \rho_t + (f(\rho))_x = 0 & si\ x \neq b(t) \\ \tilde{f}(t,\rho(t,x^-)) = \min\left(B(t), \tilde{f}_D(t,\rho(t,x^-)), \tilde{f}_S(t,\rho(t,x^+))\right) & si\ x = b(t) \end{cases}$$

où b est une fonction linéaire qui décrit la trajectoire du bus, ρ est la densité des véhicules au temps t et en position x, f est une fonction strictement concave (comme le modèle *de Greenshield [GCM+35]), qui atteint un maximum strict en une densité critique ρ_c, qui décrit le flux des véhicules et $\tilde{f}(t,p) = f(p) - b'(t) \cdot p$. La fonction B est le limiteur du flux passant à côté du bus au temps t. La définition de $\tilde{f}$ implique que pour tout t, la fonction $\tilde{f}(t,\cdot)$ atteint un unique maximum en un point $\tilde{\rho}_c(t)$. Les fonctions $\tilde{f}_D$ et $\tilde{f}_S$ sont respectivement les fonctions de demande et d'offre définies par*

$$\tilde{f}_D(t,p) = \begin{cases} \tilde{f}(t,\tilde{\rho}_c(t)) & si\ p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t,p) & si\ p < \tilde{\rho}_c(t) \end{cases}$$

et

$$\tilde{f}_S(t,p) = \begin{cases} \tilde{f}(t,p) & si\ p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t,\tilde{\rho}_c(t)) & si\ p < \tilde{\rho}_c(t). \end{cases}$$

En particulier, on a démontré que $H(p) = -f(-p)$, et on remarque directement que H et b vérifient les hypothèses (B2) et (B3) en rappelant que la vitesse du bus b' est constante.

L'équation (1.5.23) est similaire à l'équation introduite par Imbert et Monneau dans [IM13]. La différence ici est que l'on considère une jonction qui bouge dans le temps. La stabilité, l'existence et même la réduction de la classe des fonctions test pour (1.5.23) sont facilement obtenues en adaptant les preuves de ces résultats dans [IM13]. Dans ce travail, on prouve un principe de comparaison pour l'équation (1.5.23). On emprunte l'idée introduite par les auteurs dans [BBCI16] et on utilise une procédure de localisation pour insérer la bonne fonction test dans l'étape de dédoublement des variables.

Solution de viscosité de l'équation (1.5.23)**.** Pour définir les solutions de viscosité du problème (1.5.23), on commence par définir la classe des fonctions test. Pour $T > 0$, soit $D = (0,T) \times \mathbb{R}$.

Fonctions test. On note $\mathcal{C}^1(D)$ la classe des fonctions test. Si $\varphi \in \mathcal{C}^1(D)$, alors
- φ est continue.
- La réstriction de φ sur J_i est C^1.
- Pour tout $i = 1, ..., N$, la fonction $\varphi(t, b_i(t))$ est C^1 en temps. De plus,

$$\begin{aligned}\frac{d}{dt}\varphi(t,b_i(t)) &= \varphi_t^+(t,b_i(t)) + b_i'(t)\,\varphi_x^+(t,b_i(t)) \\ &= \varphi_t^-(t,b_i(t)) + b_i'(t)\,\varphi_x^-(t,b_i(t)).\end{aligned}$$

Définition 1.5.1. Soit $u : [0,T] \times \mathbb{R} \to \mathbb{R}$.

i) On dit que u est une sous-solution (resp. sur-solution) de (1.5.23) sur $[0,T] \times \mathbb{R}$ si $u^*(0,x) \leq u_0(x)$ (resp. $u_*(0,x) \geq u_0(x)$) et si pour toute fonction test $\varphi \in \mathcal{C}^1(D)$ touchant u^* par au-dessus (resp. touchant u_* par au-dessous) en $(t_0,x_0) \in D$, on a

$$\varphi_t + H_i(u_x) \leq 0 \ \ (\text{resp.} \geq 0) \ \ \text{en } (t_0,x_0) \ \ \text{si } (t_0,x_0) \in J_i$$

$$\frac{d}{dt}\varphi(t_0,b_{i_0}(t_0)) + F_{A_i}\left(t_0,u_x^{i,-},u_x^{i,+}\right) \leq 0 \ \ (\text{resp.} \geq 0) \ \ \text{si } x_0 = b_i(t_0).$$

i) On dit que u est une solution de viscosité de (1.5.23) si u est une sous-solution et une sur-solution de (1.5.23).

Comme dans [IM13, IM14], on arrive à réduire la classe des fonctions test aux points de jonction.

Théorème 1.5.11 (Réduction de la classe des fonctions test)**.** *Supposons que l'hypothèse (B) est satisfaite. On fixe $i \in \{1, ..., N\}$ et on suppose que*

$$A_i(t) \geq A_i^0(t) = \max\left(\min_{\mathbb{R}} H_{i,i}(t, \cdot), \min_{\mathbb{R}} H_{i+1,i}(t, \cdot)\right).$$

Soit $t_0 \in (0, T)$ et soit $p_{i,i}^{A_i(t_0)}$ et $q_{i+1,i}^{A_i(t_0)}$ deux constantes satisfaisantes

$$\begin{cases} H_{i,i}\left(t_0, p_{i,i}^{A_i(t_0)}\right) = H_{i,i}^{-}\left(t_0, p_{i,i}^{A_i(t_0)}\right) = A_i(t_0) \\ H_{i+1,i}\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = H_{i+1,i}^{+}\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = A_i(t_0). \end{cases}$$

On considère l'équation d'Hamilton-Jacobi suivante

$$u_t + H_k(u_x) = 0 \quad \text{si } (t, x) \in B_k, k = i, i+1. \tag{1.5.24}$$

– *Soit $u : (0, T) \times \mathbb{R} \to \mathbb{R}$ une sous-solution semi-continue supérieure de* (1.5.24) *et satisfaisant*

$$u(t, b_i(t)) = \limsup_{(s,y)\to(t,b_i(t)),y>b_i(s)} u(s, y) = \limsup_{(s,y)\to(t,b_i(t)),y<b_i(s)} u(s, y). \tag{1.5.25}$$

Si pour toute fonction test φ touchant u par au-dessus en $(t_0, b_i(t_0))$ avec

$$\varphi(t, x) = g(t) + q_{i+1,i}^{A_i(t_0)}(x - b_i(t)) 1_{\{x-b_i(t)>0\}} + p_{i,i}^{A_i(t_0)}(x - b_i(t)) 1_{\{x-b_i(t)<0\}} \tag{1.5.26}$$

pour un certain $g \in C^1(0, +\infty)$, on a

$$\frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}\left(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))\right) \leq 0$$

alors u est une sous-solution de

$$\frac{d}{dt}u(t, b_i(t)) + F_{A_i}\left(t, u_x^{i,-}, u_x^{i,+}\right) = 0 \quad \text{en } (t_0, b_i(t_0)).$$

– *Soit $u : (0, T) \times \mathbb{R} \to \mathbb{R}$ une sur-solution semi-continue inférieure de* (1.5.24)*. Si pour toute fonction test φ touchant u par au-dessous en $(t_0, b_i(t_0))$ avec φ définie comme dans* (1.5.26)*, on a*

$$\frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}\left(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))\right) \geq 0$$

alors u est une sur-solution de

$$\frac{d}{dt}u(t, b_i(t)) + F_{A_i}\left(t, u_x^{i,-}, u_x^{i,+}\right) = 0 \quad \text{en } (t_0, b_i(t_0)).$$

Théorème 1.5.12. *Supposons que l'hypothèse (B) est satisfaite et que la condition initiale u_0 est Lipschitzienne. Alors il existe une unique solution u de* (1.5.23) *dans* $[0,T)\times\mathbb{R}$ *et une constante $C_T > 0$ tel que*

$$|u(t,x) - u_0(x)| \leq C_T.$$

Le résultat principale de ce travail est le théorème suivant. On donne après une idée de la preuve.

Théorème 1.5.13 (Principe de Comparaison)**.** *Soit $T > 0$. Supposons que u_0 est Lipschitzienne. Soit u une sous-solution semi-continue supérieure et v une sur-solution semi-continue inférieure de* (1.5.23)*, et qu'il existe une constante $K > 0$, tel que pour tout $t \in [0,T]$, on a $u(t,x) \leq u_0(x) + Kt$ et $v(t,x) \geq u_0(x) - Kt$, alors on a*

$$u(t,x) \leq v(t,x) \quad \textit{pour tout } (t,x) \in [0,T] \times \mathbb{R} .$$

Pour prouver le principe de comparaison, nous avons emprunté l'idée de Barles, Briani, Chasseigne et Imbert [BBCI16] qui consiste à conserver le terme classique $\dfrac{x-y}{\varepsilon}$ dans la preuve et d'étudier son signe. Comme toujours, on sait que dans le cas d'équations d'Hamilton-Jacobi discontinues, le problème quand on fait la preuve vient des points de jonction. En effet, après une procédure de localisation, la preuve ne présente aucun problème si le point de maximum n'est pas un point de jonction. Sinon, on doit introduire la bonne fonction test. Plus précisément, on a procédé de la manière suivante : on a introduit d'abord

$$M = \sup_{(t,x)\in[0,T)\times\mathbb{R}} \{u(t,x) - v(t,x)\} .$$

Le but est de prouver que $M \leq 0$. On suppose par absurde que $M > 0$. Ensuite, pour L et R deux constantes tel que $L < \min\limits_{[0,T]} b_1$ et $R > \max\limits_{[0,T]} b_N$ et pour $\eta > 0$, nous introduisons

$$M_\eta = \sup_{\substack{t\in[0,T] \\ L\leq x\leq R}} \left\{ u(t,x) - v(t,x) - \frac{\eta}{T-t} \right\} . \tag{1.5.27}$$

Si $M_\eta \leq 0$ ou $M_\eta > 0$ et atteint en (t_η, x_η) avec $x_\eta \neq b_i(t_\eta)$ pour tout i, on continue la preuve de manière classique. Sinon, c'est-à-dire, si $M_\eta > 0$ et il existe i_0 tel que $x_\eta = b_{i_0}(t_\eta)$ alors dans ce cas, nous avons introduit un nouveau supremum

$$M_{\nu,\alpha} = \sup_{\substack{t,s\in[0,T] \\ L\leq x\leq R}} \left\{ \begin{array}{l} u(t,x) - v(s, x + b_{i_0}(s) - b_{i_0}(t)) - \dfrac{\eta}{T-t} - \dfrac{(t-s)^2}{2\nu} \\ -\alpha\left(x - b_{i_0}(t)\right)^2 - (t - t_\eta)^2 \end{array} \right\}$$

Ce nouveau supremum est atteint en (t_ν, s_ν, x_ν) et on a que

$$x_\nu \neq b_i(t_\nu) \quad \text{pour tout } i \neq i_0.$$

L'étape suivante consiste à dédoubler les variables en espace en utilisant la fonction test $\frac{(x+b_{i_0}(s)-b_{i_0}(t)-y)^2}{2\varepsilon}$ et puis on continue comme en [BBCI16] en étudiant le signe de $\frac{x+b_{i_0}(s)-b_{i_0}(t)-y}{\varepsilon}$. Notre preuve nécessite donc d'introduire au début un maximum en (t,x). Le choix de M_η (1.5.27) est justifié à la fin de la preuve du théorème lors du passage aux limites. Plus précisément, si l'on considère le supremum M_β défini par

$$M_\beta = \sup_{\substack{t\in[0,T]\\ x\in\mathbb{R}}} \left\{ u(t,x) - v(t,x) - \frac{\eta}{T-t} - \beta x^2 \right\}.$$

alors le terme $2\beta x$ empêche d'obtenir la contradiction à la fin de la preuve.

Le second résultat de ce travail est un résultat d'homogénéisation. On énonce le résultat sans donner l'idée de la preuve car elle est très similaire aux travaux précédents (même plus simple). On considère un modèle qui modélise une restriction de la densité des véhicules (comme un bus ou plus généralement une congestion qui bouge en temps), pour $(t,x)\in\mathbb{R}^+\times\mathbb{R}$,

$$\rho_t + \left(f(\rho)\,\phi\,(x-b(t))\right)_x + \left(g(\rho)\,(1-\phi\,(x-b(t)))\right)_x = 0$$

où ρ est la densité des véhicules, b représente la position de la congestion, f est la fonction qui décrit le flux à l'extérieur de la zone d'influence de la congestion, g la fonction qui décrit le flux dans la zone d'influence de la congestion et ϕ est une fonction de transition. On impose les hypothèses suivantes :

Hypothèses (C).

- (C1) La fonction f est le diagramme fondamental de Greenshield [GCM$^+$35] et est définie par

$$f(\rho) = \rho V_{max}\left(1-\frac{\rho}{\rho_{max}}\right)$$

  où V_{max} représente la vitesse moyenne maximale et ρ_{max} est la densité des véhicules maximale loin du bus.
- (C2) La fonction flux près du bus g est donnée par

$$g(\rho) = \rho V_{max}\left(1-\frac{\rho}{\sigma_{max}}\right)$$

  où σ_{max} est la densité maximale des véhicules près du bus. De plus, on suppose que $\sigma_{max} < \rho_{max}$.
- (C3) b est la fonction trajectoire du bus et est donnée par

$$b(t) = V_b t \quad \text{et on suppose que } 0 < V_b < V_{max}.$$

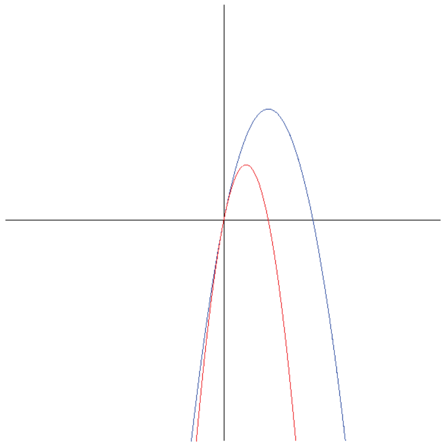

FIGURE 1.12 – Représentation schématique de f (bleu) et de g (rouge) .

– (C4) La fonction ϕ est une fonction de transition de classe C^1 et est définie par

$$\phi(x) = \begin{cases} 0 & \text{si } x \in [-r, r] \\ 1 & \text{si } x < -r-1 \text{ ou } x > r+1. \end{cases}$$

On suppose que la densité initiale satisfait

$$0 \leq \rho(0, x) \leq \begin{cases} \rho_{max} & \text{si } |x| > r+1 \\ \sigma_{max} & \text{si } |x| \leq r+1. \end{cases}$$

En posant

$$\begin{cases} H(p) = -f(-p) \\ F(p) = -g(-p) \\ \tilde{H}(p) = H(p) - b'(t)\, p = H(p) - V_b p \\ \tilde{F}(p) = F(p) - b'(t)\, p = F(p) - V_b p \\ \tilde{F}_0 = \min_{\mathbb{R}} \tilde{F}, \end{cases}$$

on a le résultat suivant : soit u^ε l'unique solution de

$$\begin{cases} u_t^\varepsilon + H(u_x^\varepsilon)\, \phi\left(\dfrac{x - V_b t}{\varepsilon}\right) + F(u_x^\varepsilon)\left(1 - \phi\left(\dfrac{x - V_b t}{\varepsilon}\right)\right) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ u^\varepsilon(0, x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{1.5.28}$$

On suppose que u_0 est Lipschitzienne et satisfait

$$(C0) \begin{cases} -\rho_{max} \leq (u_0)_x \leq 0 & \text{si } |x| > r+1 \\ -\sigma_{max} \leq (u_0)_x \leq 0. & \text{si } |x| \leq r+1. \end{cases} \tag{1.5.29}$$

Théorème 1.5.14. *Supposons que (C) et (C0) sont satisfaites. Pour* $\varepsilon > 0$*, soit* u^{ε} *l'unique solution de* (1.5.28)*. Alors il existe* $A \in \left[\tilde{F}_0, 0\right]$ *tel que* u^{ε} *converge localement uniformément vers l'unique solution de viscosité* u^0 *de l'équation suivante*

$$\begin{cases} u_t + H\left(u_x\right) = 0 & si\ x \neq b\left(t\right) \\ \dfrac{d}{dt} u\left(t, b\left(t\right)\right) + \max\left(A, \tilde{H}^{+}\left(u_x^{-}\right), \tilde{H}^{-}\left(u_x^{+}\right)\right) = 0 & si\ x = b\left(t\right) \\ u\left(0, x\right) = u_0\left(x\right). \end{cases}$$

De plus, on a

$$-\rho_{max} \leq u_x^0 \leq 0.$$

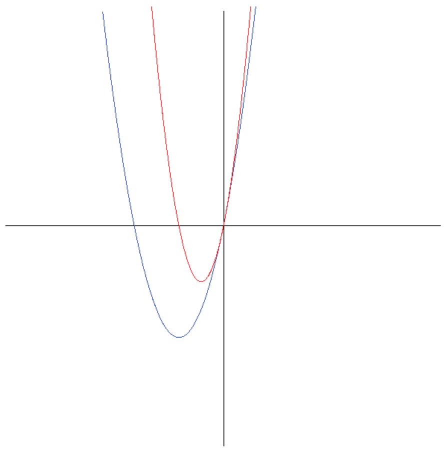

FIGURE 1.13 – Représentation schématique de H (bleu) et de F (rouge) .

Chapitre 2

Specified homogenization of a discrete traffic model leading to an effective junction condition

N. Forcadel [1], W. Salazar [1], M. Zaydan [1]

Abstract

In this chapter, we focus on deriving traffic flow macroscopic models from microscopic models containing a local perturbation such as a traffic light. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider a local perturbation located at the origin that slows down the vehicles. At the macroscopic scale, we obtain an explicit Hamilton-Jacobi equation left and right of the origin and a junction condition at the origin (in the sense of [IM13]) which keeps the memory of the local perturbation. As it turns out, the macroscopic model is equivalent to a LWR model, with a flux limiting condition at the junction. Finally, we also present qualitative properties concerning the flux limiter at the junction.

AMS Classification : 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords : specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

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2.1 Introduction

The goal of this chapter is to derive a macroscopic model for traffic flow problems from a microscopic model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles. The main motivation for deriving macroscopic models from microscopic models comes from the fact that macroscopic models are more adapted to simulate traffic at large scales. Moreover, microscopic models are based on assumptions that are easier to verify and therefore to derive a macroscopic model allows to rigorously verify it.

Several techniques were proposed for the micro-macro connections and the derivation of a macroscopic model from a microscopic model was studied by several authors. In 1970, Payne [Pay71] used the method of expansion of the gradient to derive a LWR model from a Newel model [New61]. The authors of [HHST02] established a micro-macro connection supposing that the macroscopic variables in a point x of the space can be defined by the microscopic variables if a vehicle is present at x. Otherwise, the macroscopic variables x are defined by linear interpolation. In [AKRM02], the authors derived an Aw-Rascale model from a second order model of the type "follow the leader". The reader can also refer to [CS15, Ros14, DFR15].

The originality of our work is that we assume that there is a local perturbation that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. This local perturbation can be constant in time and represent a slowdown near a school or due to a car crash near the road. It can also depend (periodically) in time and represent for example a traffic light. The schematic representation of the microscopic model is given in Figure 2.1.

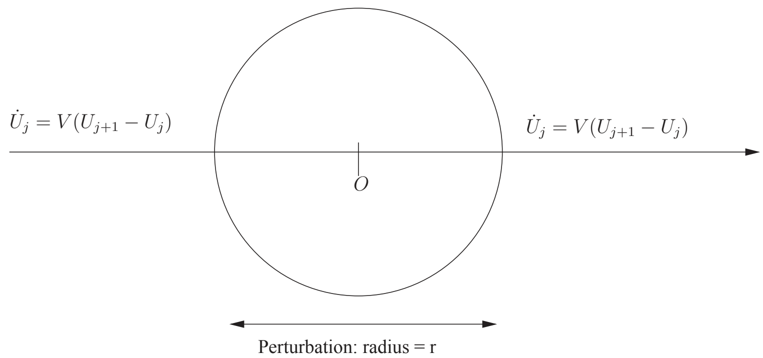

FIGURE 2.1 – Schematic representation of the microscopic model.

We denote by $U_j(t)$, $j \in \mathbb{Z}$, the position of the j th vehicle and we assume that the velocity of each vehicle is given by the function V. In order to obtain our homogenization result, we proceed as in [FIM09a, FIM09b, FIM12b, FS15] and rescale the microscopic model which describes the dynamics of each vehicle, to obtain a macroscopic model that describes the density of vehicles. If the local perturbation is located around zero, at the macroscopic scale it is natural to get an Hamilton-Jacobi equation with a junction condition at the origin (see Figure 2.2, $-u_x^0$ is the density of vehicles and the effective Hamiltonian $\overline{H}$ is defined later in the chapter), since the size of the perturbation goes to zero when we do the rescaling. This junction condition keeps the memory of the presence of the local perturbation.

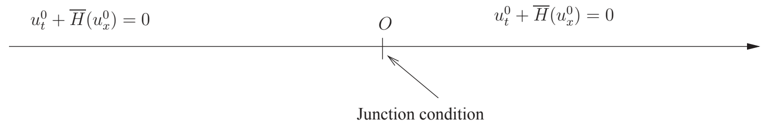

FIGURE 2.2 – Schematic representation of the macroscopic model.

Recently, the theory of Hamilton-Jacobi equations with junction or more generally on networks has known important developments in particular since the works of Achdou, Camilli, Cutri, and Tchou [ACCT13] and the work of Achdou and Imbert, Monneau, and Zidani [IMZ13].

In [ACCT13], an optimal control problem was studied in $\mathbb{R}^2$ supposing that the trajectories of the controlled system stay in a network. In [CMS13], the authors studied the vanishing viscosity method for Hamilton-Jacobi equations on junctions. In [BBC13, BBC14, AOT16], the authors studied a regional control problem with regular dynamics and costs on each side of the hyperplane and a condition on the interfaces separating the regions where the dynamic and running cost are different.

In this chapter, we will use the results obtained by Imbert and Monneau in [IM13, IM14] where the authors considered a discontinuous Hamilton-Jacobi equation and introduced the notion of "flux-limited solutions" (see also [LS16]). The link between the theory developed in [BBC13, BBC14] and "flux-limited solutions" in [IM13, IM14] is explored in [BBCI16]. In particular, [BBCI16] contains a simpler proof of the comparison principle than in [IM13]. Finally, let us mention in this context the lectures of Lions at the "College de France" [Lio16] and the recent work of Lions and Souganidis [LS16].

In this chapter, we will use the ideas developed in [FIM09b] in order to pass from microscopic models to macroscopic ones. In particular, we will show that this problem can be seen as an homogenization result. The difficulty here is that, due to the local perturbation, we are not in a periodic setting and so the construction of suitable correctors is more complicated. In particular, we will use the idea developpped by Achdou and Tchou in [AT15], by Galise, Imbert, and Monneau in [GIM15] , and in the lectures of Lions at the "College de France" [Lio14], which consists in constructing correctors on truncated domains.

Finally, we would like to mention the work [Sal16] where the second author provided a numerical scheme for the computation of an approximation of the flux limiter $\overline{A}$.

2.2 Main results

2.2.1 The microscopic model

In this chapter, we are interested in a first order microscopic model of the form

$$\dot{U}_j(t) = V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(t, U_j(t)\right), \tag{2.2.1}$$

where $U_j : [0, +\infty) \to \mathbb{R}$ denotes the position of the $j-$th vehicle and $\dot{U}_j$ is its velocity. The function $\phi : \mathbb{R} \times \mathbb{R} \to [0, 1]$ simulates the presence of a local perturbation around the origin. We denote by r the radius of influence of the perturbation.

The function V is called the optimal velocity function and we make the following assumptions on V and ϕ :

Assumption (A)

(A1) $V : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative.
(A2) V is non-decreasing on $\mathbb{R}$.
(A3) There exists a $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
(A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
(A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.
(A6) The function $\phi : \mathbb{R} \times \mathbb{R} \to [0, 1]$ is Lipschitz continuous and there exists $r > 0$ such that $\phi(t, x) = 1$ for $|x| \geq r$. We assume also that ϕ is $\mathbb{Z}$-periodic in time.

Remark 2.2.1. *Assumptions (A1)-(A2)-(A3)-(A5) are satisfied by several classical optimal velocity functions. To be more precise, since V gives the velocity of a vehicle it is normal to assume that the function should be regular, continuous and non-negative (the vehicles only go forward) which explains assumption (A1). Moreover, a vehicle should go faster if he has more space in front of it, which explains assumption (A2). Assumption (A3) comes from the fact that we want to avoid any collisions and we added a safety distance h_0 to our model : if a vehicles has less than h_0 in front of it the vehicles should not advance. Assumption (A5) plays a crucial role at the macroscopic scale. In fact, the reader will see later that the macroscopic model is the Hamilton-Jacobi equation with junction condition at the point zero introduced by Imbert and Monneau in [IM13]. Assumption (A5) provides an essential assumption used by the authors in [IM13] which is the quasi-convexity of the Hamiltonian. We have added assumption (A4) to work with V' with a bounded support. But by modifying slightly the classical optimal velocity functions, we obtain a function that satisfies all the assumptions. For instance, in the case of the Greenshields based models [GCM$^+$35](see also [BT10]) :*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max}\left(1 - \left(\dfrac{h_0}{h}\right)^n\right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max}\left(1 - \left(\dfrac{h_0}{h_{max}}\right)^n\right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N}\backslash\{0\}$. *Another optimal velocity function, based on the Newell model [New61](see also [Edi61]), is given by :*

$$V(h) = \begin{cases} 0 & \text{for } h \leq h_0, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h-h_0}{b}\right)^n\right)\right) & \text{for } h_0 < h \leq h_{max}, \\ V_{max}\left(1 - \exp\left(-\left(\dfrac{h_{max}-h_0}{b}\right)^n\right)\right) & \text{for } h > h_{max}, \end{cases}$$

with $n \in \mathbb{N}\backslash\{0\}$ *and* $b \in [0, +\infty)$. *See Figure 2.3 for a schematic representation of an optimal velocity function satisfying assumption (A).*

Remark 2.2.2. *We will give an example of the function* ϕ. *We will define* ϕ *on the interval* $[0,1]$ *since it's a* $\mathbb{Z}$*-periodic function. For* $t \in [0,1]$,

$$\phi(t,x) = \begin{cases} 1 & \text{if } |x| > r \\ \dfrac{(\phi_0(t) - 1)}{r}x + \phi_0(t) & \text{if } x \in [-r, 0] \\ \dfrac{(1 - \phi_0(t))}{r}x + \phi_0(t) & \text{if } x \in (0, r]. \end{cases}$$

where ϕ_0 *is defined in the following form*

$$\phi_0(t) = \begin{cases} 4t & \text{if } 0 < t < \frac{1}{4}, \text{ The end of the red light time} \\ 1 & \text{if } \frac{1}{4} < t < \frac{1}{2}, \text{ Green light time} \\ -4t + 3 & \text{if } \frac{1}{2} < t < \frac{3}{4}, \text{ Orange light time} \\ 0 & \text{if } \frac{3}{4} < t < 1, \text{ Red light time.} \end{cases}$$

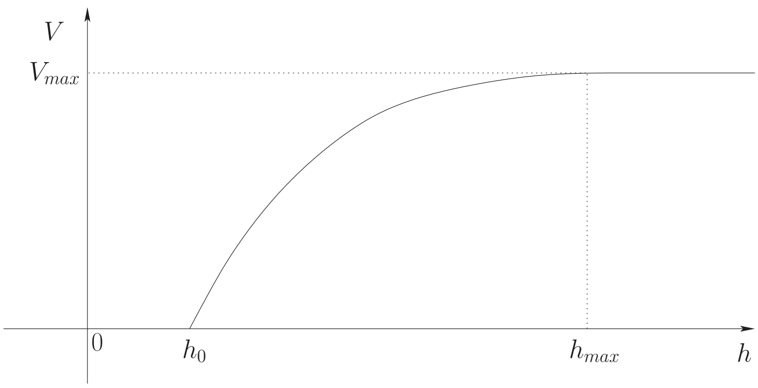

FIGURE 2.3 – Schematic representation of the optimal velocity function V.

2.2.2 The macroscopic model

We recall that $k_0 = 1/h_0$ and we define $\overline{H} : \mathbb{R} \to \mathbb{R}$, by

$$\overline{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\dfrac{-1}{p}\right)|p| & \text{for } -k_0 \le p \le 0, \\ p & \text{for } p > 0. \end{cases} \tag{2.2.2}$$

Note that such a $\overline{H}$ is continuous, coercive $\left(\lim\limits_{|p|\to+\infty} \overline{H}(p) = +\infty\right)$ and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \overline{H} \text{ is decreasing on } (-\infty, p_0), \\ \overline{H} \text{ is increasing on } (p_0, +\infty). \end{cases} \tag{2.2.3}$$

We denote by

$$H_0 = \min_{p\in\mathbb{R}} \overline{H}(p) = \overline{H}(p_0) \tag{2.2.4}$$

and we refer to Figure 2.4 for a schematic representation of $\overline{H}$.

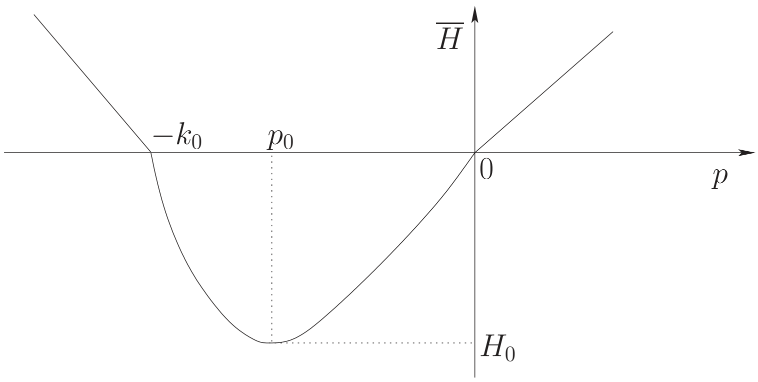

Figure 2.4 – Schematic representation of $\bar{H}$.

The macroscopic model of this chapter is the Hamilton-Jacobi equation with flux limiting condition at the junction point introduced by Imbert and Monneau in [IM13] and is given by

$$\begin{cases} u_t^0 + \overline{H}(u_x^0) = 0 & \text{for } (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t^0 + \overline{H}(u_x^0) = 0 & \text{for } (t,x) \in (0,+\infty) \times (0,+\infty) \\ u_t^0 + F_{\overline{A}}\left(u_x^0(t,0^-), u_x^0(t,0^+)\right) = 0 & \text{for } (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.2.5}$$

where $\overline{A}$ has to be determined and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_1, p_2) = \max\left(\overline{A}, \overline{H}^+(p_1), \overline{H}^-(p_2)\right), \tag{2.2.6}$$

with

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \le p_0, \\ \overline{H}(p_0) & \text{if } p \ge p_0, \end{cases} \quad \text{and} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{if } p \le p_0, \\ \overline{H}(p) & \text{if } p \ge p_0. \end{cases} \tag{2.2.7}$$

Remark 2.2.3. *We notice that in the case of traffic flow,* (2.2.5) *is equivalent (deriving in space) to a LWR model (see [LW55, Ric56, IMZ13, LK99]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and $-u_x^0$ corresponds to the density of vehicles.*

2.2.3 Main result : transition from micro to macro

In this chapter, we will study the traffic flow when the number of vehicles per unit length tends to infinity by introducing the rescaled "cumulative distribution function" of vehicles, ρ^ε, defined by

$$\rho^\varepsilon(t,y) = \varepsilon\rho\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right) \tag{2.2.8}$$

where

$$\rho(t,y) = -\left(\sum_{i\ge 0} H(y - U_i(t)) + \sum_{i<0} (-1 + H(y - U_i(t)))\right), \tag{2.2.9}$$

with

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2.2.10}$$

The following theorem is the main result of this chapter

Theorem 2.2.4 (Junction condition by homogenization : application to traffic flow)**.** *Assume (A) and that we have for all $i \in \mathbb{Z}$,*

$$h_0 \le U_{i+1}(0) - U_i(0). \tag{2.2.11}$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \ge R$

$$U_{i+1}(0) - U_i(0) = h, \tag{2.2.12}$$

with $h \ge h_0$. For the particular initial condition $u_0(x) = -x/h$ for all $x \in \mathbb{R}$, there exists $\overline{A} \in [H_0, 0]$ such that the function ρ^ε defined by (2.2.9) *converges towards the unique solution u^0 of* (2.2.5)*.*

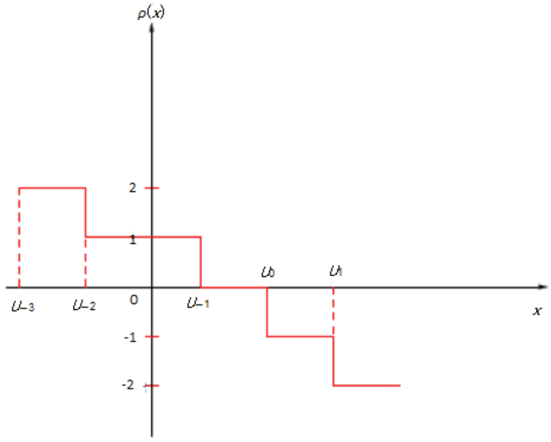

FIGURE 2.5 – Schematic representation of the function ρ.

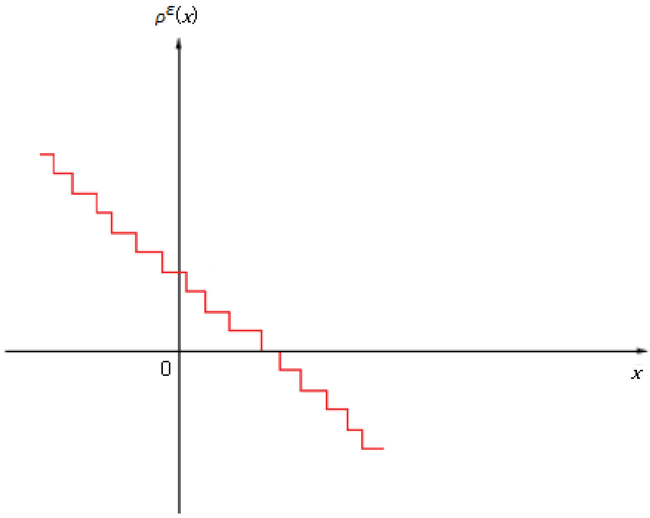

FIGURE 2.6 – Schematic representation of the function ρ^ε.

Remark 2.2.5 (Comparaison with the work [FSZ17a])**.** *We would like first to mention that this work was done before [FSZ17a]. The idea remains the same, to pass from microscopic model to a macroscopic model, this chapter presents the main idea clearer than in [FSZ17a] since we consider a first order microscopic model and so we have to construct only one non-local operator. At the contrary, in [FSZ17a], we have to define two non-local operators whose definition is complicated (authors have dedicated one section for the definition of the non-local operators). This fact will implies that results as comparison principle, stability and control of oscillations in [FSZ17a] are much more complicated and it takes long time for the reader to understand the details of each proof.*

Let us mention also that in this chapter, the perturbation depends on the time variable which represents an improvement espacially from the traffic flow point of view since we can now imagine a traffic light situation which was unrealizable in [FSZ17a].

2.2.4 Strategy of the proof of the main result

We will show now in three steps the path to obtain the convergence result.

1) **Injecting the system of ODEs into a single PDE :** as in [FIM12b, FIM09b], we inject the system of ordinary differential equations in a partial differential equation. We look to construct a non-local operator M which satisfies

$$M\left[\rho\left(t,\cdot\right)\right]\left(U_j\left(t\right)\right) = -V\left(U_{j+1}-U_j\right). \tag{2.2.13}$$

In fact, if we construct M satisfying (2.2.13), we obtain that ρ is a discontinuous viscosity solution of

$$u_t + M\left[u(t,\cdot)\right](x)\cdot\phi\left(t,x\right)\cdot|u_x| = 0 \quad \text{on } (0,+\infty)\times\mathbb{R}.$$

The "suitable" non-local operator which satisfies (2.2.13) is given by

$$M[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+z)-U(x)\right)dz - \frac{3}{2}V_{max}$$

with

$$E(z) = \begin{cases} 0 & \text{if } z \geq 0 \\ 1/2 & \text{if } -1 \leq z < 0 \\ 3/2 & \text{if } z < -1, \end{cases} \qquad \text{and} \quad J = V' \text{ on } \mathbb{R}. \tag{2.2.14}$$

We deduce that ρ^ε is a discontinuous viscosity solution of

$$u_t^\varepsilon + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)\cdot\phi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\cdot|u_x^\varepsilon| = 0 \quad \text{on } (0,+\infty)\times\mathbb{R}, \tag{2.2.15}$$

with

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+\varepsilon z)-U(x)\right)dz - \frac{3}{2}V_{max}$$

Theorem 2.2.6. *The cumulative distribution function ρ defined by* (2.2.9) *is a discontinuous viscosity solution of*

$$\rho_t + M[\rho(t,\cdot)](x)\cdot\phi(t,x)\cdot|\rho_x| = 0 \quad \textit{for } (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{2.2.16}$$

Conversely, if u is a bounded and continuous viscosity solution of (2.2.16) satisfying for some time $T>0$, and for all $t\in(0,T)$

$$u(t,x) \textit{ is decreasing in } x,$$

then the points $U_j(t)$, defined by $u(t,U_j(t)) = -(j+1)$ for $j\in\mathbb{Z}$, satisfy the system (2.2.1) on $(0,T)$.

2) **Convergence of the continuous solution :** we couple equation (2.2.15) with the following initial condition

$$u^\varepsilon(0,x) = u_0(x) \quad \text{on } \mathbb{R}. \tag{2.2.17}$$

We also assume that the initial condition satisfies the following assumption :

(A0) (Gradient bound) The function u_0 is Lipschitz continuous and satisfies

$$-k_0 := -1/h_0 \leq (u_0)_x \leq 0 \quad \text{for all } x \in \mathbb{R}. \tag{2.2.18}$$

Remark 2.2.7. *This condition ensures that initially the vehicles have a security distance between them and since we are working with a first order model, this security distance will be preserved. In fact, h_0 (from assumption (A3)) is called the safety distance. However, since we work with Eulerian coordinates, we use k_0 which is the inverse of the safety distance. We choose u_0 a regular function such that for all ε,*

$$|\rho^\varepsilon(0,x) - u_0(x)| \leq f(\varepsilon),$$

with $f(\varepsilon) \to 0$ as ε goes to 0. This is explained in (2.2.12).

The second step is to prove that the unique solution u^ε of the equation

$$\begin{cases} u_t^\varepsilon + \cdot M^\varepsilon \left[\dfrac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x) \cdot \phi\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right) \cdot |u_x^\varepsilon| = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{2.2.19}$$

converges locally uniformly towards the unique solution of (2.2.5).

Theorem 2.2.8 (Junction condition by homogenization)**.** *Assume (A) and (A0). For $\varepsilon > 0$, let u^ε be the viscosity solution of (2.2.19). Then there exists $\overline{A} \in [H_0, 0]$ such that u^ε converges locally uniformly to the unique viscosity solution u^0 of (2.2.5) (in the sense of Definition 2.3.3).*

Existence and uniqueness of solution of (2.2.19) is done Section 2.3. The proof of convergence relies on the construction of good correctors [FIM12b, IMR08] in order to use them in the perturbed test function method introduced by evans [Eva89]. The particular form of the limit equation (2.2.5) requires to construct two correctors, one for $x \neq 0$ and one for $x = 0$.

– If $x \neq 0$: we use the classical Ansatz

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{2.2.20}$$

In fact, using the fact that u^ε is a solution of (2.2.19) and denoting by $\lambda = -u_t^0(t,x)$ and $p = u_x^0(t,x)$, the expression (2.2.20) implies that for $x \neq 0$, a corrector v must verify : for all $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of

$$\begin{cases} v_t + M_p[v](x) \cdot |v_x + p| = \lambda, \quad x \in \mathbb{R}, \\ v \quad \text{is } \mathbb{Z}\text{−periodique}, \end{cases} \tag{2.2.21}$$

with

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z) E\left(U(x+z) - U(x) + p \cdot z\right) dz - \frac{3}{2} V_{max}.$$

Proposition 2.2.9 (Homogenization left and right of the perturbation)**.** *Assume (A). Then for every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution v of* (2.2.21)*. In particular, we have that $\lambda = \overline{H}(p)$.*

– If $x = 0$: in this case, formally, using Theorem 2.3.10 we assume that near zero,

$$u^0(t,x) = u^0(t,0) + \overline{p}_+ 1_{\{x>0\}} + \overline{p}_- 1_{\{x<0\}}$$

where $\overline{p}_+$ and $\overline{p}-$ are the two constants satisfying

$$\begin{cases} \overline{H}\left(\overline{p}_+\right) = \overline{H}^+\left(\overline{p}_+\right) = \overline{A} \\ \overline{H}\left(\overline{p}_-\right) = \overline{H}^-\left(\overline{p}_-\right) = \overline{A}. \end{cases} \tag{2.2.22}$$

We then use the following Ansatz

$$u^\varepsilon(t,x) = u^0(t,0) + \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{2.2.23}$$

Formaly, using (2.2.23), we have

$$\begin{aligned} w_t\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) &= u_t^\varepsilon(t,x) - u_t^0(t,0) \\ &= -M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)|u_x^\varepsilon(t,x)|\phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + F_{\overline{A}}\left(\overline{p}_-, \overline{p}_+\right) \\ &= -M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x)|u_x^\varepsilon(t,x)|\phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) + \overline{A}. \end{aligned} \tag{2.2.24}$$

Moreover, we have

$$\begin{aligned} M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right](x) &= \int_{-\infty}^{+\infty} J(z) E\left(\frac{u^\varepsilon(t,x+\varepsilon z) - u^\varepsilon(t,x)}{\varepsilon}\right) dz - \frac{3}{2} V_{max} \\ &= \int_{-\infty}^{+\infty} J(z) E\left(w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} + z\right) - w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)\right) dz - \frac{3}{2} V_{max} \\ &= M\left[w\left(\frac{t}{\varepsilon}, \cdot\right)\right]\left(\frac{x}{\varepsilon}\right). \end{aligned} \tag{2.2.25}$$

Therefore, using the notations $y = \dfrac{x}{\varepsilon}$ and $s = \dfrac{t}{\varepsilon}$ and combaining (1.5.15)-(1.5.16), we obtain that

$$w_s(s,y) + M\left[w(s,\cdot)\right](y) \cdot \phi(s,y) \cdot |w_y(s,y)| = \overline{A}.$$

Theorem 2.2.10 (Existence of a global corrector for the junction)**.** *Assume (A). We consider for $\lambda \in \mathbb{R}$, the following problem*

$$\begin{cases} w_t + M[w(t,\cdot)](x) \cdot \phi(t,x) \cdot |w_x| = \lambda & \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}. \\ w \text{ is } 1 - \text{periodic in time} \end{cases} \tag{2.2.26}$$

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution w of (2.2.26) with $\lambda = \overline{A}$ and such that there exists a constant C and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$ anf $t >$,

$$|w(t,x) - m(x)| \le C. \tag{2.2.27}$$

ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists a γ_0 such that for every $\gamma \in (0, \gamma_0)$, we have

$$\begin{cases} w(t,x+h) - w(t,x) \ge (\overline{p}_+ - \gamma)h - C & \text{for } x \ge r \text{ and } h \ge 0, \\ w(t,x-h) - w(t,x) \ge (-\overline{p}_- - \gamma)h - C & \text{for } x \le -r \text{ and } h \ge 0. \end{cases} \tag{2.2.28}$$

iii) (Rescaling w) For $\varepsilon > 0$, we set

$$w^\varepsilon(t,x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

then (along a subsequence $\varepsilon_n \to 0$) we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \le C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}(W_x) = \overline{A} & \text{for all } x \in \mathbb{R}\backslash\{0\}, \end{cases} \tag{2.2.29}$$

In particular, we have (with $W(0) = 0$)

$$W(x) = \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}. \tag{2.2.30}$$

iv)(Uniqueness of the flux limiter $\overline{A}$) We define the following set of functions

$$\mathcal{S} = \left\{ w \text{ s.t. } \exists m \in Lip(\mathbb{R}) \text{ and } C \ge 0 \text{ s.t. } ||w(t,\cdot) - m||_{L^\infty(\mathbb{R})} \le C \text{ for all } t > 0 \right\}.$$

Then we have

$$\overline{A} = \inf \{\lambda \in \mathbb{R} : \exists\ w \in \mathcal{S} \text{ solution of (2.2.26) } \}.$$

v)(Monotonicity of the flux-limiter $\overline{A}$) Let $\phi_1, \phi_2 : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$ be two functions satisfying (A6). Let $\overline{A}_1$ and $\overline{A}_2$ be their respective flux limiters given by Theorem 2.2.8. If, for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, we have

$$\phi_1(t,x) \le \phi_2(t,x),$$

then

$$\overline{A}_1 \ge \overline{A}_2.$$

3) **Link between the system of ODEs and the PDE :** finally, the main result of this work (Theorem 2.2.4) is obtained by the comparison principle for equation (2.2.19). In fact, if $u_0(x) = -x/h$ with $h \geq h_0$, we know from step 1 that ρ^ε is a solution of (2.2.15). Moreover, we can prove that

$$|\rho^\varepsilon(0,x) - u_0(x)| \leq f(\varepsilon)$$

with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, using the comparison principle for equation (2.2.19), we obtain that

$$u^\varepsilon(t,x) - f(\varepsilon) \leq \rho^\varepsilon(t,x) \leq u^\varepsilon(t,x) + f(\varepsilon) \tag{2.2.31}$$

where u^ε is the unique viscosity solution of (2.2.19) with $u_0(x) = -x/h$. Passing to the limit as $\varepsilon \to 0$ in (2.2.31) and using step ii), we obtain our result.

2.2.5 Organization of the chapter

Section 2.3 contains the definition of the viscosity solutions for the problems we consider in the entire chapter and it also contains some results for those problems. Section 2.4 contains the proof of the point 1) of Subsection 2.2.4. Point 2 of Subsection 2.2.4 is divided into two sections : in Section 2.5 , we construct correctors and in Section 2.6 we prove the convergence result. Finally, Section 2.7 contains the proof of the main Theorem 2.2.4 i.e. point 3) of Subsection 2.2.4.

2.3 Viscosity solutions for (2.2.19) and (2.2.5)

2.3.1 Definition of the non-local operators

In this subsection, we recall the definition of the non-local operators. For $p \in \mathbb{R}$,

$$M_p[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+z) - U(x) + p\cdot z\right)dz - \frac{3}{2}V_{max}. \tag{2.3.1}$$

In particular, for $p = 0$, we obtain the non-local operator M,

$$M[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+z) - U(x)\right)dz - \frac{3}{2}V_{max}. \tag{2.3.2}$$

Finally, for $\varepsilon > 0$, we recall the definition of the non-local operator M^ε,

$$M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z)E\left(U(x+\varepsilon z) - U(x)\right)dz - \frac{3}{2}V_{max}. \tag{2.3.3}$$

To each operator M_p (resp. M, M^ε), we associate the operator $\tilde{M}_p$ (resp. $\tilde{M}, \tilde{M}^\varepsilon$) which is defined in the same way except that the function E is replaced by the function $\tilde{E}$, defined by

$$\tilde{E}(z) = \begin{cases} 0 & \text{if } z > 0 \\ 1/2 & \text{if } -1 < z \leq 0 \\ 3/2 & \text{if } z \leq -1. \end{cases} \tag{2.3.4}$$

Remark 2.3.1. *Using the fact that E and V are bounded, we get that for every function U and every $x \in \mathbb{R}$, we have for $p \in \mathbb{R}$,*

$$-M_0 = -\frac{3}{2} V_{max} \leq M_p[U](x) \leq 0. \tag{2.3.5}$$

2.3.2 Definitions of viscosity solutions

We will use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u :

$$u^*(t,x) = \limsup_{s \to t, y \to x} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{s \to t, y \to x} u(s,y).$$

In order to give a general definition for all the non-local problems we consider, we will give the definition for the following equation, with $p \in \mathbb{R}$, for all $(t,x) \in (0,+\infty) \times \mathbb{R}$,

$$\begin{cases} u_t + \psi(x) \cdot M_p[u(t,\cdot)](x) \cdot \phi(t,x) \cdot |p + u_x| + (1 - \psi(x)) \cdot \overline{H}(u_x) = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{2.3.6}$$

with $\psi : \mathbb{R} \to [0,1]$ a Lipschitz continuous function.

Definition 2.3.1 (Viscosity solutions for (2.3.6))**.** Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.3.6) on $[0,T] \times \mathbb{R}$, if $u(0,x) \leq u_0(x)$ (resp. $u(0,x) \geq u_0(x)$) and for all $(t,x) \in (0,T) \times \mathbb{R}$ and for all $\varphi \in C^2([0,T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point (t,x), we have

$$\varphi_t + \psi(x) \cdot \phi(t,x) \cdot M_p[u(t,\cdot)](x) \cdot |p + \varphi_x| + (1 - \psi(x))\overline{H}(\varphi_x) \leq 0$$

$$\left(\text{resp.} \quad \varphi_t + \psi(x) \cdot \phi(t,x) \cdot \tilde{M}_p[u(t,\cdot)](x) \cdot |p + \varphi_x| + (1 - \psi(x))\overline{H}(\varphi_x) \geq 0\right).$$

We say that a function u is a viscosity solution of (2.3.6) if u^* and u_* are respectively a sub-solution and a super-solution of (2.3.6).

Remark 2.3.2. *We use this definition in order to have a stability result for the non-local term. We refer to [Sle03] for such kind of definition and to [FIM09b, Proposition 4.2]. For the readers convenience we give the stability result and its proof.*

Proposition 2.3.3 (Stability of the solutions of (2.3.6))**.** *Let $(u_n)_n$ be a sequence of uniformly bounded upper semi-continuous functions (resp. lower semi-continuous functions) and let $\overline{u}$ denote* $\limsup^* u_n$ *(resp.* $\underline{u} = \liminf_* u_n$*). Let $(t_n, x_n, p_n) \to (t_0, x_0, p)$ in $\mathbb{R}^3$ be such that $u_n(t_n, x_n) \to \overline{u}(t_0, x_0)$ (resp. $u_n(t_n, x_n) \to \underline{u}(t_0, x_0)$). Then*

$$\limsup_{n\to+\infty}{}^* M_{p_n}[u_n(t_n,\cdot)](x_n) \geq M_p[\overline{u}(t_0,\cdot)](x_0) \tag{2.3.7}$$

$$\left(\textit{resp.} \liminf_{n\to+\infty}{}_* \tilde{M}_{p_n}[u_n(t_n,\cdot)](x_n) \leq \tilde{M}_p[\underline{u}(t_0,\cdot)](x_0) \right). \tag{2.3.8}$$

In order to prove Proposition 2.3.3, we use the following lemma which proof can be found in [Sle03].

Lemma 2.3.4. *1 Let $(f_n)_n$ be a sequence of measurable functions on $\mathbb{R}$, and consider*

$$\overline{f} = \limsup{}^* f_n$$

and

$$\underline{f} = \liminf{}_* f_n. \tag{2.3.9}$$

Let $(a_n)_n$ be a sequence of $\mathbb{R}$ converging to zero. Then

$$\mathcal{L}(\{f_n \geq a_n\} \backslash \{\overline{f} \geq 0\}) \to 0 \quad \textit{as} \quad n \to +\infty$$

and

$$\mathcal{L}(\{\underline{f} > 0\} \backslash \{f_n > a_n\}) \to 0 \quad \textit{as} \quad n \to +\infty,$$

where $\mathcal{L}(A)$ denotes the Lesbegue measure of measurable set A.

Proof of Proposition 2.3.3. We just do the proof for $\overline{u}$. Let $\varepsilon > 0$. Using (2.2.14), we have that

$$E(\beta) = \frac{1}{2} 1_{\{\beta\in[-1,0)\}} + \frac{3}{2} 1_{\{\beta<-1\}} = \frac{1}{2} 1_{\{\beta<0\}} + 1_{\{\beta<-1\}}.$$

We get that

$$\begin{aligned}
&\int_{\mathbb{R}} J(z) E(u_n(t_n, x_n+z) - u_n(t_n,x_n) + p_n.z) dz \\
&\qquad - \int_{\mathbb{R}} J(z) E(\overline{u}(t_0, x_0+z) - \overline{u}(t_0,x_0) + p.z) dz \\
&= \int_{\mathbb{R}} J(z) \left\{ 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z<-1\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z<-1\}} \right\} dz \qquad (2.3.10)\\
&\qquad + \int_{\mathbb{R}} \frac{1}{2} J(z) \left\{ 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z<0\}} - 1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z<0\}} \right\} dz
\end{aligned}$$

Using Lemma 2.3.4, we have for n big enough,

$$\begin{aligned}
&\int_{\mathbb{R}} J(z)\left\{1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq -1\}} - 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z\geq -1\}}\right\} dz \\
&\quad \geq -\int_{\mathbb{R}} J(z) 1_{\{A_n(z)\setminus A(z)\}} \geq -\frac{\varepsilon}{2}, \\
&\int_{\mathbb{R}} \frac{1}{2} J(z)\left\{1_{\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq 0\}} - 1_{\{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z\geq 0\}}\right\} dz \\
&\quad \geq -\frac{1}{2}\int_{\mathbb{R}} J(z) 1_{\{B_n(z)\setminus B(z)\}} \geq -\frac{\varepsilon}{2},
\end{aligned} \tag{2.3.11}$$

with

$$\begin{cases} A_n(z) = & \{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z\geq -1\} \\ & \cup\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq -1\} \\ A(z) = & \{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq -1\} \end{cases}$$

and

$$\begin{cases} B_n(z) = & \{u_n(t_n,x_n+z)-u_n(t_n,x_n)+p_n.z\geq 0\} \\ & \cup\{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq 0\} \\ B(z) = & \{\overline{u}(t_0,x_0+z)-\overline{u}(t_0,x_0)+p.z\geq 0\}. \end{cases}$$

Using (2.3.10) and (2.3.11), we deduce that

$$M_{p_n}[u_n(t_n,\cdot)](x_n) \geq M_p[\overline{u}(t_0,\cdot)](x_0) - \varepsilon, \tag{2.3.12}$$

for n big enough. This implies (2.3.7).

□

Definition 2.3.2 (Class of test functions for (2.2.5))**.** We denote by $J_\infty := (0,+\infty)\times\mathbb{R}$, $J_\infty^+ := (0,+\infty)\times[0,+\infty)$ and $J_\infty^- := (0,\infty)\times(-\infty,0]$. We define a class of test functions on J_∞ by

$$\mathcal{C}^1(J_\infty) = \left\{\varphi\in C(J_\infty),\ \text{the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ is } C^1\right\}.$$

Definition 2.3.3 (Viscosity solutions for (2.2.5))**.** Let $\overline{H}$ be given by (2.2.2) and $\overline{A}\in\mathbb{R}$. An upper semi-continuous (resp. lower semi-continuous) function $u:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (2.2.5) if $u(0,x)\leq u_0(x)$ (resp. $u(0,x)\geq u_0(x)$) and for all $(t,x)\in J_\infty$ and for all $\varphi\in\mathcal{C}^1(J_\infty)$ such that

$$u\leq\varphi \text{ (resp. } u\geq\varphi) \quad \text{in a neighbourhood of } (t,x)\in J_\infty \quad \text{and} \quad u(t,x)=\varphi(t,x),$$

we have

$$\begin{array}{ll} \varphi_t(t,x)+\overline{H}(\varphi_x(t,x))\leq 0 \quad (\text{resp. } \geq 0) & \text{if } x\neq 0, \\ \varphi_t(t,x)+F_{\overline{A}}(\varphi_x(t,0^-),\varphi_x(t,0^+))\leq 0 \quad (\text{resp. } \geq 0) & \text{if } x=0. \end{array}$$

We say that a function u is a viscosity solution of (2.2.5) if u^* and u_* are respectively a sub-solution and a super-solution of (2.2.5). We refer to this solution as an $\overline{A}-$flux limited solution.

2.3.3 Results for viscosity solutions of (2.3.6)

Proposition 2.3.5 (Comparison principle for (2.3.6))**.** *Assume (A0) and (A). Let u be a sub-solution of (2.3.6) and v be a super-solution of (2.3.6). Let us also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$u(t, x) \leq u_0(x) + Kt \quad \text{and} \quad -v(t, x) \leq -u_0(x) + Kt. \tag{2.3.13}$$

Then we have $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

Proof. The only difficulty in proving the comparison principle comes from the non-local term, but in our case the proof is similar to the proof of [FIM09b, Theorem 4.4]. For the readers convenience we give the details of the proof of Proposition 2.3.5.

Let us introduce

$$M = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \{u(t, x) - v(t, x)\}.$$

We want to prove that $M \leq 0$, we argue by contradiction and assume that $M > 0$.

Step 1 : the test function. Let us introduce the following test function,

$$\varphi(t, x, y) = u(t, x) - v(t, y) + p(x - y) - \frac{\eta}{T - t} - e^{Bt}\frac{(x - y)^2}{2\varepsilon} - \alpha x^2,$$

where η, ε, and α are small, strictly positive parameters, and B is a constant to be chosen later. We can notice that we have

$$\lim_{|x|,|y|\to+\infty} \varphi(t, x, y) = -\infty.$$

In fact, using (2.3.13) and (A0) we have

$$\begin{aligned}\varphi(t, x, y) \leq & u_0(x) - u_0(y) + p(x - y) + 2KT - \alpha x^2 - e^{Bt}\frac{(x - y)^2}{2\varepsilon}\\ \leq & (k_0 + |p|)|x - y| + 2KT - \alpha x^2 - e^{Bt}\frac{(x - y)^2}{2\varepsilon}.\end{aligned}$$

Using the fact that our test function is upper semi-continuous we can see that it reaches a maximum at a finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. We also have the following result.

Lemma 2.3.6. *For α and η small enough, we have*
- $M_{\eta,\varepsilon,\alpha} := \varphi(\bar{t}, \bar{x}, \bar{y}) \geq M/2 > 0$.
- $|\bar{x} - \bar{y}| \to 0$ *as* $\varepsilon \to 0$.
- $\alpha|\bar{x}| \to 0$ *as* $\alpha \to 0$.

Proof of Lemma 2.3.6. By definition of M, for all $\theta > 0$, $\exists (t_\theta, x_\theta) \in [0,T] \times \mathbb{R}$ such that

$$\begin{aligned}
& M - \theta \leq u(t_\theta, x_\theta) - v(t_\theta, x_\theta) \\
\Rightarrow \quad & M - \theta - \frac{\eta}{T - t_\theta} - \alpha x_\theta^2 \leq u(t_\theta, x_\theta) - v(t_\theta, x_\theta) - \frac{\eta}{T - t_\theta} - \alpha x_\theta^2,
\end{aligned}$$

choosing η and α small enough we have $0 < M/2 \leq M_{\eta,\varepsilon,\alpha}$. Using this result we can see that we have

$$\alpha \bar{x}^2 + e^{Bt} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} \leq u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) + p(\bar{x} - \bar{y}) \leq (k_0 + |p|)|\bar{x} - \bar{y}| + 2KT,$$

where we have used (2.3.13) and (A0), this inequality allows us to directly deduce the rest of the lemma.

□

Step 2 : case $\bar{t} = 0$. In this particular case, using Lemma 2.3.6 we have

$$\begin{aligned}
& 0 < \varphi(0, \bar{x}, \bar{y}) = u_0(\bar{x}) - u_0(\bar{y}) + p(\bar{x} - \bar{y}) - \frac{\eta}{T} - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon} - \alpha \bar{x}^2, \\
\Rightarrow \quad & \frac{\eta}{T} \leq (k_0 + |p|)|\bar{x} - \bar{y}|,
\end{aligned}$$

using the fact that $|\bar{x} - \bar{y}| \to 0$ as $\varepsilon \to 0$, we get a contradiction.

Step 3 : case $\bar{t} > 0$. By duplication of the time variable and passing to the limit, we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + Be^{B\bar{t}} \frac{(\bar{x} - \bar{y})^2}{2\varepsilon}$$

and

$$\begin{aligned}
a &\leq -M_p[u(\bar{t}, \cdot)](\bar{x}) \cdot \phi(\bar{t}, \bar{x}) \cdot |e^{B\bar{t}} p_\varepsilon + 2\alpha \bar{x}| - \psi(\bar{x}) \overline{H}(e^{B\bar{t}} p_\varepsilon + 2\alpha \bar{x}) \\
b &\geq -\tilde{M}_p[v(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot |e^{B\bar{t}} p_\varepsilon| - \psi(\bar{y}) \overline{H}(e^{B\bar{t}} p_\varepsilon),
\end{aligned}$$

with $p_\varepsilon = (x - \bar{y})/\varepsilon$. Combining these inequalities we obtain

$$
\begin{aligned}
Be^{B\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \frac{\eta}{(T-\bar{t})^2} \leq & \tilde{M}_p[v(\bar{t},\cdot)](\bar{y}) \cdot \phi(\bar{t},\bar{y}) \cdot |e^{B\bar{t}}p_\varepsilon| \\
& - M_p[u(\bar{t},\cdot)](\bar{x}) \cdot \phi(\bar{t},x) \cdot |e^{B\bar{t}}p_\varepsilon + 2\alpha\bar{x}| \\
& + \psi(\bar{y})\overline{H}(e^{B\bar{t}}p_\varepsilon) - \psi(\bar{x})\overline{H}(e^{B\bar{t}}p_\varepsilon + 2\alpha\bar{x}) \\
\leq & M_0|p_\varepsilon| \cdot ||\phi'||_\infty \cdot |\bar{x}-\bar{y}|e^{B\bar{t}} \\
& + \tilde{M}_p[v(\bar{t},\cdot)](\bar{y}) \cdot \phi(\bar{t},\bar{x}) \cdot |e^{B\bar{t}}p_\varepsilon| + o_\alpha(1) \\
& - M_p[u(\bar{t},\cdot)](\bar{x}) \cdot \phi(\bar{t},\bar{x}) \cdot |e^{B\bar{t}}p_\varepsilon| \\
& + \psi(\bar{x})\left(\overline{H}(e^{B\bar{t}}p_\varepsilon) - \overline{H}(e^{B\bar{t}}p_\varepsilon + 2\alpha\bar{x})\right) \\
& + ||\psi'||_\infty|\bar{x}-\bar{y}|\overline{H}(e^{B\bar{t}}p_\varepsilon) \\
\leq & M_0|p_\varepsilon| \cdot ||\phi'||_\infty \cdot |\bar{x}-\bar{y}|e^{B\bar{t}} \\
& + \tilde{M}_p[v(\bar{t},\cdot)](\bar{y}) \cdot \phi(\bar{t},\bar{x}) \cdot |e^{B\bar{t}}p_\varepsilon| + o_\alpha(1) \\
& - M_p[u(\bar{t},\cdot)](\bar{x}) \cdot \phi(\bar{t},\bar{x}) \cdot e^{B\bar{t}}|p_\varepsilon| \\
& + ||\psi'||_\infty|\bar{x}-\bar{y}|\overline{H}(e^{B\bar{t}}p_\varepsilon),
\end{aligned}
\tag{2.3.14}
$$

where we have used (2.3.5), the fact that ϕ is Lipschitz continuous, Lemma 2.3.6, and the fact that ψ is Lipschitz continuous for the second inequality. Moreover, we have used Lemma 2.3.6 and the fact that $\overline{H}$ is Lipschitz continuous for the last inequality.

As in [FIM09b], we define

$$
\mathcal{A} := \left\{ z : \ \tilde{E}\left(v(\bar{t},z) - v(\bar{t},\bar{y}) + p(z-\bar{y})\right) \leq E\left(u(\bar{t},z) - u(\bar{t},\bar{x}) + p(z-\bar{x})\right) \right\}.
$$

However, we know that $M_{\eta,\varepsilon,\alpha}$ is reached at $(\bar{t},\bar{x},\bar{y})$, we have that for all $z \in \mathbb{R}$, $\varphi(\bar{t},z,z) \leq \varphi(\bar{t},\bar{x},\bar{y})$ meaning that we have

$$
\begin{aligned}
u(\bar{t},z) - u(\bar{t},\bar{x}) + p(z-\bar{x}) \leq \quad & v(\bar{t},z) - v(\bar{t},\bar{y}) + p(z-\bar{y}) \\
& + \left(\alpha z^2 - e^{B\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} - \alpha\bar{x}^2\right).
\end{aligned}
\tag{2.3.15}
$$

This implies that

$$
\mathcal{A}^c \subset \{|z| \geq R_{\varepsilon,\alpha}\}, \quad \text{with } R^2_{\varepsilon,\alpha} = \frac{1}{\alpha}\left(e^{B\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \alpha\bar{x}^2\right).
$$

We distinguish two cases.

Case 1 : there exists a constant $C_\varepsilon > 0$, such that for any α small enough we have

$$
\frac{|\bar{x}-\bar{y}|}{2\varepsilon} \geq C_\varepsilon. \tag{2.3.16}
$$

In this case, we have

$$\{|z-\bar{y}| \geq R_{\varepsilon,\alpha}\} \subset \left\{|z| \geq \overline{R}_{\varepsilon,\alpha}\right\},$$

with $\overline{R}_{\varepsilon,\alpha} = -|\bar{y}| + R_{\varepsilon,\alpha} \to +\infty$ as $\alpha \to 0$ (see Da Lio *et al.* Lemma 2.5 in [DLFM08]). This implies that

$$\begin{aligned}\tilde{M}_p\left[v(\bar{t},\cdot)\right](\bar{y}) &= \int_{\mathbb{R}} J(z-\bar{y})\tilde{E}\left(v(\bar{t},z) - v(\bar{t},\bar{y}) + p(z-\bar{y})\right)dz - \frac{3}{2}V_{max} \\ &\leq \int_{\mathbb{R}} J(z-\bar{y})E\left(u(\bar{t},z) - v(\bar{t},\bar{x}) + p(z-\bar{x})\right)dz - \frac{3}{2}V_{max} + o_\alpha(1).\end{aligned}$$

Using this and the fact that by definition $\forall p \in \mathbb{R}\ |\overline{H}(p)| \leq V_{max}|p|$, (2.3.14) becomes

$$\begin{aligned}\frac{\eta}{T^2} + Be^{B\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} &\leq \left(2M_0||\phi'||_\infty + 2V_{max}||\psi'||_\infty\right)\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}e^{B\bar{t}} + o_\alpha(1) \\ &\quad + e^{B\bar{t}}|p_\varepsilon|\phi(\bar{t},\bar{x}).\int_{\mathbb{R}} J(z-\bar{y}).E\left(u(\bar{t},z) - u(\bar{t},\bar{x}) + p(z-x)\right)dz \\ &\quad - e^{B\bar{t}}|p_\varepsilon|\phi(\bar{t},x)\int_{\mathbb{R}} J(z-\bar{x}).E\left(u(\bar{t},z) - u(\bar{t},\bar{x}) + p(z-x)\right)dz \\ &\leq \left(2M_0||D\phi||_\infty + 3||\phi||_\infty.||DJ||_{L^1(\mathbb{R})}\right)\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}e^{B\bar{t}} + o_\alpha(1) \\ &\quad + 2V_{max}||\psi'||_\infty\frac{(\bar{x}-\bar{y})^2}{2\varepsilon}e^{B\bar{t}}\end{aligned}$$

Choosing

$$B = 2M_0||\phi'||_\infty + 3||\phi||_\infty.||DJ||_{L^1(\mathbb{R})} + 2V_{max}||\psi'||_\infty,$$

we get a contradiction for α small enough.

Case 2 : there exists a subsequence α_n, such that

$$\frac{|\bar{x}-\bar{y}|}{2\varepsilon} \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.3.17}$$

In this case, (2.3.14), gives us a contradiction, choosing $B = 2M_0||\phi'||_\infty + 2V_{max}||\psi'||_\infty$ and passing to the limit as $n \to +\infty$.

□

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and for $\overline{r}, \overline{R} > 0$, the set

$$\mathcal{Q}_{\overline{r},\overline{R}}(t_0,x_0) = (t_0 - \overline{r}, t_0 + \overline{r}) \times (x_0 - \overline{R}, x_0 + \overline{R}). \tag{2.3.18}$$

Theorem 2.3.7 (Comparison principle on bounded sets for (2.3.6))**.** *Assume (A). Let u be a sub-solution of* (2.3.6) *and let v be a super-solution of* (2.3.6) *on the open set $\mathcal{Q}_{\overline{r},\overline{R}} \subset (0,T)\times\mathbb{R}$. We assume that u (resp. v) is upper semi-continuous (resp. lower semi-continuous) on $\overline{\mathcal{Q}}_{\overline{r},\overline{R}}$. Also assume that*

$$u \le v \quad \textit{outside } \mathcal{Q}_{\overline{r},\overline{R}},$$

then

$$u \le v \quad \textit{on } \mathcal{Q}_{\overline{r},\overline{R}}.$$

Démonstration. The proof of this theorem is similar to the one of Proposition 2.3.5, so we skip it.

□

Lemma 2.3.8 (Existence of barriers for (2.3.6))**.** *Assume (A0) and (A). There exists a constant $K_1 > 0$ such that*

$$u^+(t,x) = K_1 t + u_0(x) \quad \textit{and} \quad u^-(t,x) = u_0(x),$$

are respectively super and sub-solutions of (2.3.6).

Démonstration. We define $K_1 = M_0 \cdot (|p| + k_0) + |H_0|$. Let us prove that u^+ is a super-solution of (2.3.6). Using assumption (A0) and the form of the non-local operator and of $\overline{H}$, we have

$$\begin{aligned}\phi(t,x)\psi(x)M_p[u_0](x)\cdot|p+(u_0)_x| + (1-\psi(x))\cdot\overline{H}((u_0)_x) \quad &\ge -M_0\cdot|p+(u_0)_x| + H_0\\ &\ge -M_0(|p|+k_0) - |H_0|\\ &= -K_1,\end{aligned}$$

where we used (2.3.5) and (2.2.4). The proof for u^- is simpler, it uses (2.3.5) and (2.2.4),

$$\phi(t,x)\psi(x)M_p[u_0](x)\cdot|p+(u_0)_x| + (1-\psi(x))\cdot\overline{H}((u_0)_x) \le 0.$$

□

Applying Perron's method (see [IMR08, Proof of Theorem 6], [AT96] or [Imb05] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 2.3.9 (Existence and uniqueness of viscosity solutions for (2.3.6))**.** *Assume (A0) and (A). Then, there exists a unique continuous solution u of (2.3.6) which satisfies (for some constant K_1)*

$$u_0(x) \le u(t,x) \le u_0(x) + K_1 t$$

2.3.4 Results for viscosity solutions of (2.2.5)

Now we recall a result concerning the reduction of the class of test functions at the junction point for equation (2.2.5). We will also consider the following problem,

$$u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0,T) \text{ and } x \in \mathbb{R}\backslash\{0\}. \tag{2.3.19}$$

Theorem 2.3.10 (Reduction of the class of test functions)**.** *(see [IM13, Theorem 2.7]). Let $\overline{H}, \overline{H}^+$ and $\overline{H}^-$ given by* (2.2.2) *and* (2.2.7) *and consider $A \in [H_0, +\infty)$ with H_0 defined in* (2.2.4)*. Given arbitrary solutions $p_\pm^A \in \mathbb{R}$ of*

$$\overline{H}\left(p_+^A\right) = \overline{H}^+\left(p_+^A\right) = A = \overline{H}^-\left(p_-^A\right) = \overline{H}\left(p_-^A\right), \tag{2.3.20}$$

let us fix any time independent test function $\phi^0(x)$ satisfying

$$\phi_x^0(0^\pm) = p_\pm^A.$$

Given a function $u : (0,T) \times \mathbb{R} \to \mathbb{R}$, the following properties hold true.

i) If u is an upper semi-continuous sub-solution of (2.3.19) and satisfies

$$u\,(t,0) = \limsup_{(s,y)\to(t,0), y\in\mathbb{R}_+^*} u\,(s,y) = \limsup_{(s,y)\to(t,0), y\in\mathbb{R}_-^*} u\,(s,y)\,, \tag{2.3.21}$$

then u is a H_0-flux limited sub-solution.

ii) Given $A > H_0$ and $t_0 \in (0,T)$, if u is an upper semi-continous sub-solution of (2.3.19) and satisfies (2.3.21) *and if for any test function φ touching u from above at $(t_0, 0)$ with*

$$\varphi(t,x) = \psi(t) + \phi^0(x), \tag{2.3.22}$$

for some $\psi \in C^1(0,+\infty)$, we have

$$\varphi_t + F_A\left(\varphi_x(t_0,0^-), \varphi_x(t_0,0^+)\right) \leq 0 \quad \text{at } (t_0,0),$$

then u is an A-flux limited sub-solution at $(t_0,0)$.

iii) Given $t_0 \in (0,T)$, if u is a lower semi-continuous super-solution of (2.3.19) *and if for any test function φ satisfying* (2.3.22) *touching u from above at $(t_0,0)$ we have*

$$\varphi_t + F_A\left(\varphi_x(t_0,0^-), \varphi_x(t_0,0^+)\right) \geq 0 \quad \text{at } (t_0,0),$$

then u is an A-flux limited super-solution at $(t_0,0)$.

Démonstration. The proof of Theorem 2.3.10 can be founded in [IM13, Theorem 2.5]. □

We recall now the comparison principle for equation (2.2.5). To simplify, we will give the result under assumptions we need in our chapter. Note that authors in [IM13] proved this result under weaker assumptions.

Theorem 2.3.11 (Comparison principle for (2.2.5))**.** *Assume (A0). Let u and v be respectively sub-solution and super-solution of* (2.2.5)*. Let us also assume that there exists a constant $K > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$,*

$$u(t,x) \leq u_0(x) + Kt \quad \textit{and} \quad -v(t,x) \leq -u_0(x) + Kt.$$

Then we have $u(t,x) \leq v(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

2.3.5 Control of the oscillations for (2.2.19)

Theorem 2.3.12 (Control of the oscillations)**.** *Let $T > 0$. Assume (A0)-(A) and let u be a solution of* (2.2.19)*, with $\varepsilon = 1$. Then for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t \in [0,T]$, we have*

$$-k_0(x-y) - 1 \leq u(t,x) - u(t,y) \leq 0, \tag{2.3.23}$$

with k_0 defined in (2.2.18)*.*

Démonstration. In this proof we used the barriers given by Lemma 2.3.8 (with $p = 0$ and $\psi \equiv 1$), which means that the solution u of (2.2.19) with $\varepsilon = 1$ satisfies for all $(t,x) \in [0,+\infty) \times \mathbb{R}$,

$$0 \leq u(t,x) - u_0(x) \leq M_0 k_0 t. \tag{2.3.24}$$

In the rest of the proof we will use the following notation :

$$\Omega = \left\{(t,x,y) \in [0,T) \times \mathbb{R}^2 \text{ s.t. } x \geq y\right\}.$$

Proof of the upper inequality for the control of the space oscillations. We introduce,

$$N = \sup_{(t,x,y)\in\Omega} \{u(t,x) - u(t,y)\}.$$

We want to prove that $N \leq 0$. We argue by contradiction and assume that $N > 0$.

Step 1 : the test function. For $\eta, \alpha > 0$, small parameters, we define

$$\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

Using (2.3.24), we have that

$$\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2M_0k_0T - \alpha(x^2+y^2) \leq -\alpha(x^2+y^2) + 2M_0k_0T,$$

where we used assumption (A0) for the second inequality. Therefore we have

$$\lim_{|x|,|y|\to+\infty} \varphi(t,x,y) = -\infty.$$

Since φ is upper-semi continuous, it reaches a maximum at a point that we denote by $(\bar{t},\bar{x},\bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \dfrac{N}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \to 0 \text{ as } \alpha \to 0. \end{cases}$$

Step 2 : $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

where we used that u_0 is non-increasing, and we get a contradiction. The fact that $\bar{x} > \bar{y}$, comes directly from the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 3 : utilisation of the equation. We split the variables by introducing,

$$M_{\nu,\alpha} = \sup_{\substack{t,s\in[0,T] \\ x\geq y}} \left\{ u(t,x) - u(s,y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2 - \frac{(t-s)^2}{2\nu} - h\left(t,x,y\right) \right\}$$

where

$$h\left(t,x,y\right) = \left(t-\bar{t}\right)^2 + \left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2.$$

Easily, $M_{\nu,\alpha}$ is reached at a point (t_0, s_0, x_0, y_0). Moreover, we have that

$$\begin{aligned} 0 < M_{\nu,\alpha} &\leq u_0\left(x_0\right) - u_0\left(y_0\right) + 2M_0k_0T - \frac{\left(t_0-s_0\right)^2}{2\nu} \\ &\leq 2M_0k_0T - \frac{\left(t_0-s_0\right)^2}{2\nu} \end{aligned} \tag{2.3.25}$$

where we have used in the second inequality the fact that u_0 is non-increasing. Inequality (2.3.25) implies that

$$\frac{\left(t_0-s_0\right)^2}{2\nu} \leq 2M_0k_0T$$

and therefore

$$\lim_{\nu\to 0} |t_0 - s_0| = 0.$$

We denote by

$$\begin{cases} \hat{t} = \lim\limits_{\nu\to 0} t_0 = \lim\limits_{\nu\to 0} s_0 \\ \hat{x} = \lim\limits_{\nu\to 0} x_0 \\ \hat{y} = \lim\limits_{\nu\to 0} y_0. \end{cases}$$

We have that

$$u(t_0,x_0) - u(s_0,y_0) - \frac{\eta}{T-t_0} - \alpha x_0^2 - \alpha y_0^2 - h\left(t_0,x_0,y_0\right) \geq M_{\nu,\alpha} \geq \varphi\left(\bar{t},\bar{x},\bar{y}\right). \tag{2.3.26}$$

Passing to the limite as ν goes to zero in (2.3.26), we obtain that

$$\varphi\left(\bar{t},\bar{x},\bar{y}\right) - h\left(\hat{t},\hat{x},\hat{y}\right) \geq \varphi\left(\hat{t},\hat{x},\hat{y}\right) - h\left(\hat{t},\hat{x},\hat{y}\right) \geq \varphi\left(\bar{t},\bar{x},\bar{y}\right). \tag{2.3.27}$$

Using the definition of the function h and inequality (2.3.27), we deduce that

$$\begin{cases} \hat{t} = \bar{t} \\ \hat{x} = \bar{x} \\ \hat{y} = \bar{y}. \end{cases}$$

We deduce that for ν small enough, $t_0, s_0 \neq 0$ and $x_0 > y_0$. Writting the viscosity inequalities and passing to the limit as $\nu \to 0$, we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} &\leq \tilde{M}[u(\bar{t},\cdot)](\bar{y}) \cdot |2\alpha\bar{y}| \cdot \phi(\bar{t},\bar{y}) - M[u(\bar{t},\cdot)](\bar{x}) \cdot \phi(\bar{t},\bar{x}) \cdot |2\alpha\bar{x}| \\ &\leq 2M_0 \cdot \alpha(|\bar{x}| + |\bar{y}|), \end{aligned}$$

passing to the limit as α goes to 0, we obtain a contradiction.

Proof of the lower inequality for the control of the space oscillations Let us introduce,

$$N = \sup_{(t,x,y)\in\Omega} \left\{u(t,y) - u(t,x) - 1 - k_0(x-y)\right\}.$$

We want to prove that $N \leq 0$. We argue by contradiction and assume that $N > 0$.

Step 1 : the test function. For $\alpha, \eta > 0$, small parameters we consider the function

$$\varphi(t,x,y) = u(t,y) - u(t,x) - 1 - k_0(x-y) - \alpha(x^2+y^2) - \frac{\eta}{T-t}.$$

We have that

$$\begin{aligned} \varphi(t,x,y) \leq& \quad u_0(y) - u_0(x) - \alpha(x^2+y^2) + 2M_0k_0T - k_0(x-y) - 1 \\ \leq& \quad -\alpha(x^2+y^2) + 2M_0k_0T. \end{aligned}$$

Therefore, we have

$$\lim_{|x|,|y|\to+\infty} \varphi(t,x,y) = -\infty.$$

Using the fact that φ is upper-semi continuous we deduce that φ reaches a maximum at a finite point that we denote $(\bar{t},\bar{x},\bar{y}) \in \Omega$. Classically we have for η and α small enough,

$$\begin{cases} 0 < \dfrac{N}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \alpha|\bar{x}|, \alpha|\bar{y}| \to 0 \text{ as } \alpha \to 0. \end{cases}$$

Step 2 : $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume that $\bar{t} = 0$. Using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$ and (A0), we have

$$\frac{\eta}{T} < u(0, \bar{y}) - u(0, \bar{x}) - k_0(\bar{x} - \bar{y}) - 1 \leq -1,$$

which is a contradiction. Hence $\bar{t} > 0$. Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we also deduce that $\bar{x} > \bar{y}$.

Step 3 : Utilisation of the equation By duplicating the time variable and passing to the limit, we get that

$$\frac{\eta}{(T - \bar{t})^2} \leq \tilde{M}[u(\bar{t}, \cdot)](\bar{x}) \cdot |-k_0 - 2\alpha\bar{x}| \cdot \phi(\bar{t}, \bar{x}) - M[u(\bar{t}, \cdot)](\bar{y}) \cdot \phi(\bar{t}, \bar{y}) \cdot |-k_0 + 2\alpha\bar{y}|.$$

Using the fact that $\tilde{M} \leq 0$, if we prove that

$$M[u(\bar{t}, \cdot)](\bar{y}) = \int_{\mathbb{R}} J(z) E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - \frac{3}{2} V_{max} = 0. \tag{2.3.28}$$

then we obtain a contradiction. Let us prove (2.3.28). Let $z \in (h_0, h_{max}]$. If $\bar{y} + z \geq \bar{x}$, using that u is non-increasing in space, we get

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) \leq -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, for α small enough, we obtain

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq -k_0 z + \alpha\left(2z\bar{y} + z^2\right) \leq -k_0 z + \alpha\left(2h_{max}\bar{y} + h_{max}^2\right) < -1.$$

This implies that we have for all $z \in (h_0, h_{max}]$,

$$E(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) = \frac{3}{2}.$$

Injecting this in the non-local term, we get our result.

$\square$

2.4 Proof of Theorem 2.2.6

Proof of Theorem 2.2.6. Theorem 2.2.6 is a consequence of the following lemma.

Lemma 2.4.1 (Link between the velocities)**.** *Assume (A). Let $((U_j)_j)$ be the solution of (2.2.1) with*

$$U_{j+1}(0) - U_j(0) > h_0. \tag{2.4.1}$$

Then we have

$$\dot{U}_j(t) = -M[u(t, \cdot)](U_j(t)) \cdot \phi(t, U_j(t)), \tag{2.4.2}$$

where E and J are defined in (2.2.14) *and $u(t,x)$ is a continuous function such that*

$$\begin{cases} u(t,x) = \rho_*(t,x) = \rho(t,x) \text{ for } x = U_j(t),\ j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \tag{2.4.3}$$

with ρ defined in (2.2.9) *(with $\varepsilon = 1$).*

Démonstration. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. Using the fact that $u(U_j) = -(j+1)$ and (2.4.3), we have for all $z \in [0, +\infty)$,

$$\begin{cases} 0 \geq u(U_j + z) - u(U_j) > u(U_{j+1}) - u(U_j) = -1 & \text{if } z \in [0, U_{j+1} - U_j) \\ -1 \geq u(U_j + z) - u(U_j) & \text{if } z \in [U_{j+1} - U_j, +\infty). \end{cases}$$

Given that u is continuous, this implies that

$$M[u](U_j) = \int_0^{U_{j+1}-U_j} \frac{1}{2} J(z)dz + \int_{U_{j+1}-U_j}^{+\infty} \frac{3}{2} J(z)dz - \frac{3}{2} V_{max} = -V\left(U_{j+1} - U_j\right).$$

Combining this result with (2.2.1), we obtain (2.4.2).

□

Noticing that because of (2.4.3), we have for $x = U_j(t),\ j \in \mathbb{Z}$,

$$\tilde{M}[\rho_*(t,\cdot)](x) = \tilde{M}[u(t,\cdot)](x) = M[u(t,\cdot)](x),$$

and using Lemma 2.4.1, and Definition 2.3.1, we can see that ρ_* is a discontinuous viscosity super-solution of (2.2.16). We obtain a similar result for ρ^*, therefore, ρ is a discontinuous viscosity solution of (2.2.16).

We prove the converse. For the readers convenience we recall Proposition 4.8 from [FIM09b] that we will use later. The proof of this proposition remains almost the same in our case the only difference being the definition of the functions E and $\tilde{E}$.

Lemma 2.4.2. *Let $\theta : \mathbb{R} \to \mathbb{R}$ a non-decreasing and upper semi-continuous (resp. lower semi-continuous) function. Assume also that*

$$\theta(v) - v \text{ is } 1\text{-periodic in } v.$$

Assume that $\varepsilon = 1$ in (2.2.15)*. Consider also a sub-solution (resp. a super-solution) u of* (2.2.15)*. Then $\theta(u)$ is also a sub-solution (resp. a super-solution) of* (2.2.15)*.*

Using Lemma 2.4.2 we can conclude that $\rho_* = \lceil u \rceil$ (resp. $\rho^* = \lfloor u \rfloor$) is a viscosity super-solution (resp. sub-solution) of

$$\partial_t \rho - \tilde{c}(t,x)\partial_x \rho = 0 \quad \text{with } \tilde{c}(t,x) = M[u(t,\cdot)](x) \cdot \phi(t,x) = \tilde{M}[u(t,\cdot)](x) \cdot \phi(t,x).$$

Using the fact that u is decreasing in space, we define

$$U_i(t) = \inf\{x,\ u(t,x) \leq -(i+1)\} = (u(t,\cdot))^{-1}(-i-1)$$

and we consider the functions $t \mapsto U_i(t)$. They are continuous because u is decreasing in x and is continuous in (t,x).

We now prove that the functions U_i are viscosity solutions of (2.2.1). Let φ be a test function such that $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t,x) = -(i+1) + \varphi(t) - x$. It satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

and

$$\hat{\varphi}(t,x) \leq \rho_*(t,x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).$$

This implies that

$$\begin{aligned} &\varphi_t(t_0) + \tilde{c}(t_0, U_i(t_0)) \geq 0 \\ \Leftrightarrow\ &\varphi_t(t_0) \geq -\tilde{c}(t_0, U_i(t_0)) = -\bar{c}_i(t_0) = V(U_{i+1}(t_0) - U_i(t_0)).\phi(t, U_i(t_0)). \end{aligned}$$

This proves that U_i are viscosity super-solutions of (2.2.1). The proof for sub-solutions is similar and we skip it. Moreover, since $\bar{c}_i$ is continuous, we deduce that $U_i \in C^1$ and it is therefore a classical solution of (2.2.1).

□

2.5 Construction of correctors (proofs of Proposition 2.2.9 and Theorem 2.2.10)

In this section, we will construct the correctors far and near the junction point.

2.5.1 Proof of Proposition 2.2.9

Proof of Proposition 2.2.9. Let us prove that $v = 0$ is an obvious solution of (2.2.21) with $\lambda = \overline{H}(p)$, for $p \in [-k_0, 0]$. First, let us notice that if $p = 0$ the result is obvious since by definition of $\overline{H}$, we have $\overline{H}(0) = 0$ and $M_0[0](x)$ is finite (for all $x \in \mathbb{R}$). Let us now consider $p < 0$, we have for all $x \in \mathbb{R}$,

$$\begin{aligned} M_p[0](x) &= \int_{-\infty}^{+\infty} J(z)E(pz)dz - \frac{3}{2}V_{max} \\ &= \int_{0}^{+\infty} J(z)E(pz)dz - \frac{3}{2}V_{max} \\ &= \int_{0}^{-1/p} \frac{1}{2}J(z)dz + \int_{-1/p}^{+\infty} \frac{3}{2}J(z)dz - \frac{3}{2}V_{max} \\ &= \frac{1}{2}\left(V\left(\frac{-1}{p}\right) - V(0)\right) + \frac{3}{2}\left(\left(\lim_{h\to+\infty} V(h)\right) - V\left(\frac{-1}{p}\right)\right) - \frac{3}{2}V_{max} \\ &= -V\left(\frac{-1}{p}\right), \end{aligned}$$

where we have used assumption (A3) for the second line, the definition of E and J (see (2.2.14)) for the third and fourth lines. Finally, using this result and the definition of $\overline{H}$, we notice that $\overline{H}(p) = M_p[0](x)|p| = \lambda$. The uniqueness of λ is classical (see for instance [FIM09a, Proof of Proposition 4.6]) so we skip it.

□

2.5.2 Proof of Theorem 2.2.10

This subsection contains the proof of Theorem 2.2.10. To do this, we will construct correctors on truncated domains and then pass to the limit as the size of the domain goes to infinity. This idea comes from [AT15] and [GIM15]. The difficulty in our non-local case is that it is non-standard to well define boundary conditions. In order to overcome this difficulty, we will replace the non-local operator by a local one near the boundary. More precisely, for $l \in (r, +\infty)$, $r << l$ and $r \le R << l$, we want to find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} w_t^{l,R} + G_R\left(t, x, [w^{l,R}(t,\cdot)], w_x^{l,R}\right) = \lambda_{l,R} & \text{if } (t,x) \in \mathbb{R} \times (-l, l) \\ w_t^{l,R} + \overline{H}^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } (t,x) \in \mathbb{R} \times \{-l\} \\ w_t^{l,R} + \overline{H}^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } (t,x) \in \mathbb{R} \times \{l\} \\ w^{l,R} \text{ is 1-periodic in } t. \end{cases} \tag{2.5.1}$$

with

$$G_R(t, x, [U], q) = \psi_R(x)\phi(t,x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \overline{H}(q), \tag{2.5.2}$$

and $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \to [0,1]$, with

$$\psi_R \equiv \begin{cases} 1 & \text{on } [-R, R] \\ 0 & \text{outside } [-R-1, R+1], \end{cases} \quad \text{and} \quad \psi_R(x) < 1 \ \forall x \notin [-R, R]. \tag{2.5.3}$$

To G_R, we associate $\tilde{G}_R$ which is defined in the same way but the operator M is replaced by $\tilde{M}$.

Remark 2.5.1. *The operator G_R is used to have a local operator near the boundary and then to well define the boundary conditions.*

Comparison principle for a truncated problem

Proposition 2.5.2 (Comparison principle on truncated domains)**.** *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with and $l_2 >> R$.*

$$\begin{cases} v_t + \tilde{G}_R(t, x, [v(t,\cdot)], v_x) \ge \lambda & \textit{for } (t,x) \in \mathbb{R} \times (l_1, l_2) \\ v_t + \overline{H}^+(v_x) \ge \lambda & \textit{for } (t,x) \in \mathbb{R} \times \{l_2\} \\ v(t,x) \ge U_0(t) & \textit{for } (t,x) \in \mathbb{R} \times \{l_1\} \\ v \textit{ is 1-periodic in } t, \end{cases} \tag{2.5.4}$$

where U_0 is continuous, and for $\varepsilon_0 > 0$

$$\begin{cases} u_t + G_R(t, x, [u\,(t, \cdot)], u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t, x) \in \mathbb{R} \times (l_1, l_2) \\ u_t + \overline{H}^+(u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t, x) \in \mathbb{R} \times \{l_2\} \\ u(t, x) \leq U_0\,(t) & \textit{for } (t, x) \in \mathbb{R} \times \{l_1\} \\ u \textit{ is 1-periodic in } t, \end{cases} \tag{2.5.5}$$

Then we have $u \leq v$ in $\mathbb{R} \times [l_1, l_2]$.

Démonstration. The only difficulty in proving this result is the comparison at the boundary $\{l_2\}$. However, for x close to l_2, the function G_R is actually the effective Hamiltonian $\overline{H}$. Therefore, we can proceed as in the proof of [GIM15, Proposition 4.1] and so we skip the proof.

□

Remark 2.5.3. *We have a similar result for $l_1 < l_2 < -r$ and if for all $x \in [l_2, l_2 + h_{max}]$, $u(t, x) \leq v(t, x)$ and the following conditions are imposed at $x = l_1$:*

$$\begin{cases} v_t + \overline{H}^-(v_x) \geq \lambda & \textit{for } x = l_1, \\ u_t + \overline{H}^-(u_x) \leq \lambda - \varepsilon_0 & \textit{for } x = l_1. \end{cases}$$

Existence of correctors on a truncated domain

Proposition 2.5.4 (Existence of correctors on truncated domains)**.** *There exists a unique $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $w^{l,R}$ of (2.5.1). Moreover, there exists a constant C (depending only on k_0, V_{max} and $|H_0|$), and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \textit{for } x, y \in [-l, l], \\ |w^{l,R}(t, x) - m^{l,R}(x)| \leq C & \textit{for } (t, x) \in \mathbb{R} \times [-l, l], \end{cases} \tag{2.5.6}$$

with $H_0 = \min \overline{H}$.

Démonstration. In order to construct a corrector on the truncated domain, we will classically consider the approximated problem

$$\begin{cases} \delta v^\delta + v^\delta_t + G_R(t, x, [v^\delta\,(t, \cdot)], v^\delta_x) = 0 & \text{for } (t, x) \in \mathbb{R} \times (-l, l) \\ \delta v^\delta + v^\delta_t + \overline{H}^-(v^\delta_x) = 0 & \text{for } (t, x) \in \mathbb{R} \times \{-l\} \\ \delta v^\delta + v^\delta_t + \overline{H}^+(v^\delta_x) = 0 & \text{for } (t, x) \in \mathbb{R} \times \{l\} \\ v^\delta \text{ is 1-periodic in } t \end{cases} \tag{2.5.7}$$

Step 1 : construction of barriers. Using that 0 and $\delta^{-1}C_0$ are respectively sub and super-solution of (2.5.7) with $C_0 = |H_0|$, the comparison principle and Perron's method for 1-periodic solutions, we deduce that there exists a continuous viscosity solution, v^δ of (2.5.7) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta}. \tag{2.5.8}$$

Step 2 : control of the space oscillations of v^δ.

Lemma 2.5.5. *The function v^δ satisfies for all $t \in \mathbb{R}$ and for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x-y) - 1 \leq v^\delta(t,x) - v^\delta(t,y) \leq 0,$$

with k_0 defined in (A0).

Proof of Lemma 2.5.5. In the rest of the proof we will use the following notation,

$$\Omega = \left\{(t,x,y) \in \mathbb{R} \times [-l,l]^2 \text{ such that } x \geq y\right\}.$$

Step 2.1 : proof of the upper inequality. Let $\varepsilon > 0$. We want to prove that

$$N = \sup_{(t,x,y)\in\Omega} \left\{v^\delta(t,x) - v^\delta(t,y)\right\} \leq 0.$$

We argue by contradiction and assume that $N > 0$. We then consider

$$N_\nu = \sup_{t,s\in\mathbb{R}, x\geq y} \left\{v^\delta(t,x) - v^\delta(s,y) - \frac{(t-s)^2}{2\nu}\right\}.$$

Since $N > 0$, we deduce that $N_\nu > 0$. Remark also that we consider the supremum of a continuous, 1-periodic in t and s function, so we deduce that N_ν is reached at a point $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Given that $N_\nu > 0$, we deduce that $\bar{x} \neq \bar{y}$ if ν is small enough (classicaly we have that $|t-s| \to 0$ as $\nu \to 0$). Therefore, we can use the viscosity inequalities for (2.5.7).

-If $(\bar{x}, \bar{y}) \in (-l,l)^2$, we have

$$\delta v^\delta(\bar{t},\bar{x}) + \frac{\bar{t}-\bar{s}}{\nu} + G_R(\bar{x}, [v^\delta(\bar{t},\cdot)], 0) \leq 0$$
$$\delta v^\delta(\bar{s},\bar{y}) + \frac{\bar{t}-\bar{s}}{\nu} + G_R(\bar{y}, [v^\delta(\bar{s},\cdot)], 0) \geq 0,$$

combining these two inequalities with the fact that $G_R(x, [U], 0) = 0$, we obtain

$$\delta\left(v^\delta(\bar{t},x) - v^\delta(\bar{s},\bar{y})\right) \leq 0.$$

-If $x = l$ and $\bar{y} \in (-l,l)$, similarly we obtain

$$\delta\left(v^\delta(\bar{t},x) - v^\delta(\bar{s},\bar{y})\right) \leq 0,$$

where we have used the fact that $\overline{H}^+(0) = 0$.

-If $x \in (-l,l)$ and $\bar{y} = -l$, we obtain

$$\delta\left(v^\delta(\bar{t},x) - v^\delta(\bar{s},\bar{y})\right) \leq H_0 \leq 0,$$

where we used the fact that $\overline{H}^-(0) = H_0$.

-If $x = l$ and $\bar{y} = -l$, we obtain

$$\delta\left(v^\delta(\bar{t},x) - v^\delta(\bar{s},\bar{y})\right) \leq H_0 \leq 0.$$

For every value of x and $\bar{y}$ we obtain a contradiction, therefore we have $N \leq 0$.

Step 2.2 : proof of the lower inequality. We want to prove that

$$N = \sup_{(t,x,y)\in\Omega} \Big\{ v^\delta(t,y) - v^\delta(t,x) - k_0(x-y) - 1 \Big\} \le 0.$$

We argue by contradiction and assume that $N > 0$. We then consider

$$N_\nu = \sup_{t,s\in\mathbb{R},x\ge y} \left\{ v^\delta(t,y) - v^\delta(s,x) - k_0\,(x-y) - 1 - \frac{(t-s)^2}{2\nu} \right\}.$$

Since $N > 0$, we get $N_\nu > 0$. Remark also that we consider the supremum of a continuous, 1-periodic in t and s function, so we deduce that N_ν is reached at a point $(\bar t, \bar s, \bar x, \bar y)$. Given that $N_\nu > 0$, we deduce that $\bar x \ne \bar y$ if ν is small enough (classicaly we have that $|\bar t - \bar s| \to 0$ as $\nu \to 0$). Therefore, we can use the viscosity inequalities for (2.5.7).

Case 1 : $\bar y \in (-l, l)$**.** If $\bar y \in (-l, l)$, we have

$$\delta v^\delta(\bar t, \bar y) + \frac{\bar t - \bar s}{\nu} + \psi_R(\bar y) M[v^\delta\,(\bar t, \cdot)](\bar y)\cdot\phi(\bar t,\bar y)\cdot|-k_0| + (1-\psi_R(\bar y))\overline{H}(-k_0) \le 0 \quad (2.5.9)$$

We claim that $M[v^\delta\,(\bar t, \cdot)](\bar y) = 0$.

Indeed, for all $z > h_0$, if $x > \bar y + z$ using the fact that the maximum is reached for $(\bar t, \bar s, x, \bar y)$, we deduce that

$$v^\delta(\bar t, \bar y + z) - v^\delta(\bar t, \bar y) \le -k_0 z < -1.$$

On the contrary, if $x \le \bar y + z$, using the fact that v^δ is continuous, non-increasing in space, and the fact that $v^\delta(\bar s, x) - v^\delta(\bar t, \bar y) < -1$, we deduce that

$$v^\delta(\bar t, \bar y + z) - v^\delta(\bar t, \bar y) \le v^\delta(\bar t, x) - v^\delta(\bar t, \bar y) < -1.$$

We can therefore, conclude that for all $z \in (h_0, +\infty)$, $E(v^\delta(\bar t, \bar y + z) - v^\delta(\bar t, \bar y)) = -\frac{3}{2}$ and so we get $M[v^\delta\,(\bar t, \cdot)](\bar y) = 0$. Using also that $\overline{H}(-k_0) = 0$, equation (2.5.9) becomes

$$\delta v^\delta(\bar t, \bar y) + \frac{\bar t - \bar s}{\nu} \le 0.$$

Moreover, whether $\bar x \in (-l, l)$ or $\bar x = l$, since the non-local operator is negative and $H^+\,(-k_0) < 0$, we have that

$$-\delta v^\delta(\bar s, x) - \frac{\bar t - \bar s}{\nu} \le 0.$$

We deduce that

$$\delta\left(v^\delta(\bar t, \bar y) - v^\delta(\bar s, \bar x)\right) \le 0,$$

which is a contradiction.

Case 2 : $\bar y = -l$**.** In this situation, the viscosity inequality becomes

$$\delta v^\delta(\bar t, \bar y) + \frac{\bar t - \bar s}{\nu} + \overline{H}^-(-k_0) \le 0.$$

Using the fact that $\overline{H}^-(-k_0) = \overline{H}(-k_0) = 0$, and as in the previous case, we obtain a contradiction. This ends the proof of the lemma.

□

Step 3 : control of the time oscillations of v^δ.

Lemma 2.5.6. *The function v^δ satisfies for all $x \in [-l, l]$ and for all $t, s \in \mathbb{R}$,*

$$\left| v^\delta(t, x) - v^\delta(s, x) \right| \le C_1$$

with $C_1 = \frac{3}{2} V_{max} k_0 + |H_0| + 1$.

Démonstration. Since v^δ is 1-periodic in t, it is sufficient to show that for all $x \in [-l, l]$ and for all $t, s \in \mathbb{R}$ such that $t \ge s$, we have that

$$v^\delta(t, x) - v^\delta(s, x) \le C_2(t - s) + 1. \tag{2.5.10}$$

with $C_2 = C_1 - 1$. In order to prove that, we will fix $x_0 \in (-l, l)$ and $s_0 \in \mathbb{R}$, and we will prove that if $t \ge s_0$, then

$$v^\delta(t, x_0) \le v^\delta(s_0, x_0) + C_2(t - s_0) + 1. \tag{2.5.11}$$

We define

$$w^\delta(t, x) = v^\delta(s_0, x_0) + C_2(t - s_0) + k_0 |x - x_0| + 1.$$

Using the space oscillation of v^δ, we have that $v^\delta(s_0, x) \le w^\delta(s_0, x)$. On the other hand, we can check that w^δ is a super solution of (2.5.7) on $(s_0, +\infty) \times [-l, l]$ using that

$$\begin{cases} w^\delta(t, x) \ge 0 \\ |H_0| \ge -\overline{H}, -\overline{H}^+, -\overline{H}^- \\ \frac{3}{2} V_{max} \ge -M[U](x) \\ 1 \ge \phi. \end{cases}$$

Finally, using the comparaison principle on $[s_0, +\infty) \times [-l, l]$, we deduce that

$$v^\delta(t, x) \le w^\delta(t, x).$$

In particular, for $x = x_0$, we obtain (2.5.11). We deduce that (2.5.10) is true even if $x = \pm l$ because v^δ is continuous. The proof is now complete. ☐

Step 4 : Lipschitz estimate.

Lemma 2.5.7. *There exists a Lipschitz continuous function m^δ, such that there exists a constant C, (independent of l, R and δ) such that*

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \le C|x - y| & \text{for all } x, y \in [-l, l], \\ |v^\delta(t, x) - m^\delta(x)| \le C & \text{for all } (t, x) \in \mathbb{R} \times [-l, l]. \end{cases} \tag{2.5.12}$$

Proof of Lemma 2.5.7. Let us define m^δ as an affine function in each interval of the form $[ih_0,(i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(0, ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta(0,(i+1)h_0).$$

Since $m^\delta, v^\delta(0,\cdot)$ are non-increasing and $|v^\delta(0,(i+1)h_0) - v^\delta(0, ih_0)| \le k_0h_0 + 1 = 2$, we deduce that $\forall x \in [ih_0,(i+1)h_0]$,

$$-2 \le v^\delta(0,(i+1)h_0) - m^\delta(ih_0) \le v^\delta(0,x) - m^\delta(x) \le v^\delta(0, ih_0) - m^\delta((i+1)h_0) \le 2,$$

and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \le 2k_0|x - y|.$$

Using the time oscillations of v^δ, we deduce that

$$|v^\delta(t,x) - m^\delta(x)| \le C \quad \text{for all } (t,x) \in \mathbb{R} \times [-l, l]$$

with $C = \frac{3}{2}V_{max}k_0 + |H_0| + 3$. □

Step 4 : passing to the limit as δ goes to 0. Using (2.5.8) and (2.5.12), we deduce that there exists $\delta_n \to 0$ such that

$$\begin{array}{ll} \delta_n v^{\delta_n}(0,0) \to -\lambda_{l,R} & \text{as } n \to +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) \to m^{l,R} & \text{as } n \to +\infty, \end{array}$$

the second convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R}(t,x) = \limsup_{\delta_n \to 0}{}^*(v^{\delta_n} - v^{\delta_n}(0,0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \to 0}{}_*(v^{\delta_n} - v^{\delta_n}(0,0)).$$

Therefore, we have that $\lambda_{l,R}, m^{l,R}, \overline{w}^{l,R}$ and $\underline{w}^{l,R}$ satisfy

$$\begin{array}{r} H_0 \le \lambda_{l,R} \le 0, \\ |\overline{w}^{l,R} - m^{l,R}| \le C, \\ |\underline{w}^{l,R} - m^{l,R}| \le C, \\ |m^{l,R}_x| \le C. \end{array} \tag{2.5.13}$$

By stability of the solutions we have that $\overline{w}^{l,R} - 2C$ and $\underline{w}^{l,R}$ are respectively a sub-solution and a super-solution of (2.5.1) and

$$\overline{w}^{l,R} - 2C \le \underline{w}^{l,R}.$$

By Perron's method we can construct a solution $w^{l,R}$ of (2.5.1) and thanks to (2.5.8) and (2.5.13), $m^{l,R}$, $\lambda_{l,R}$ and $w^{l,R}$ satisfy (2.5.6).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 2.5.4.

□

Proposition 2.5.8 (First definition of the flux limiter)**.** *The following limits exist (up to a subsequence)*

$$\begin{cases} \overline{A}_R = \lim\limits_{l\to+\infty} \lambda_{l,R} \\ \\ \overline{A} = \lim\limits_{R\to+\infty} \overline{A}_R. \end{cases} \tag{2.5.14}$$

Moreover, we have

$$H_0 \leq \overline{A}_R, \overline{A} \leq 0.$$

Démonstration. This results comes from the fact that we have the following bound on $\lambda_{l,R}$ which is independent of l and R (see Proposition 2.5.4),

$$H_0 \leq \lambda_{l,R} \leq 0.$$

□

Remark 2.5.9. *This proposition does not ensure the uniqueness of the flux limiter $\overline{A}$. However, since we know that such a limit exists, we can obtain the convergence result. The uniqueness of $\overline{A}$ is given in Theorem 2.2.10.*

Proposition 2.5.10 (Control of the slopes on a truncated domain)**.** *Assume that l and R are big enough. Let $w^{l,R}$ be the solution of (2.5.1) given by Proposition 2.5.4. We also assume that up to a sub-sequence $\overline{A} = \lim\limits_{R\to+\infty} \lim\limits_{l\to+\infty} \lambda_{l,R} > H_0$. Then there exits $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, there exists a constant C (independent of l and R) such that for all $x \geq r$ and $h \geq 0$*

$$w^{l,R}(t, x+h) - w^{l,R}(t, x) \geq (\overline{p}_+ - \gamma)h - C. \tag{2.5.15}$$

Similarly, for all $x \leq -r$ and $h \geq 0$,

$$w^{l,R}(t, x-h) - w^{l,R}(t, x) \geq (-\overline{p}_- - \gamma)h - C. \tag{2.5.16}$$

Démonstration. We only prove (2.5.15) since the proof for (2.5.16) is similar. For $\mu > 0$ small enough, we denote by p_+^μ the real number such that

$$\overline{H}(p_+^\mu) = \overline{H}^+(p_+^\mu) = \lambda_{l,R} - \mu.$$

Using that

$$H_0 < \lambda_{l,R} \leq 0,$$

we deduce that p_+^μ exists, is unique and satisfies $-k_0 \le p_+^\mu \le 0$ for μ small enough.

Let us now consider the function $w^+ = p_+^\mu x$ that satisfies

$$\overline{H}(w_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}.$$

We also have

$$\begin{aligned} M[w^+](x) &= \int_{\mathbb{R}} J(z) E(p_+^\mu (x+z) - p_+^\mu x) dz - \frac{3}{2} V_{max} \\ &= \int_0^{\frac{-1}{p_+^\mu}} \frac{1}{2} J(z) dz + \int_{\frac{-1}{p_+^\mu}}^{+\infty} \frac{3}{2} J(z) dz - \frac{3}{2} V_{max} \\ &= - V\left(\frac{-1}{p_+^\mu}\right). \end{aligned}$$

For all $x \in (r, l)$, using that $\phi(t, x) = 1$, we deduce that

$$M[w^+](x) \cdot \phi(t,x) \cdot |w_x^+| = -V\left(\frac{-1}{p_+^\mu}\right) \cdot |p_+^\mu| = \overline{H}(p_+^\mu) = \lambda_{l,R} - \mu,$$

and so the restriction of w^+ to $(r, l]$ satisfies

$$\begin{cases} w_t^+ + G_R(t, x, [w^+], w_x^+) = \lambda_{l,R} - \mu & \text{for } (t,x) \in \mathbb{R} \times (r, l) \\ w_t^+ + \overline{H}^+(w_x^+) = \lambda_{l,R} - \mu & \text{for } (t,x) \in \mathbb{R} \times \{l\}. \end{cases}$$

Let us denote by $g(t,x) = w^{l,R}(t,x) - w^{l,R}(0, x_0)$ and $u(x) = w^+(x) - w^+(x_0) - C$, for some $x_0 \in (r, l)$ and C defined as in Proposition 2.5.4. Then we have

$$g(t, x_0) \ge -C = u(x_0).$$

Using that g is a solution of (2.5.4) and u is a solution of (2.5.5) (with $\varepsilon_0 = \mu$) joint to the comparison principle (Proposition 2.5.2) we get that

$$w^{l,R}(t,x) - w^{l,R}(t, x_0) = g(t,x) \ge u(x) = p_+^\mu (x - x_0) - C.$$

This implies that for all $h \ge 0$ and for all $x \in (r, l)$,

$$w^{l,R}(t, x+h) - w^{l,R}(t,x) \ge p_+^\mu h - C.$$

Finally, if we choose $\gamma_0 < |p_0 - \bar{p}_+|$ (with p_0 defined in (2.2.4)), then

$$\overline{H}(\bar{p}_+ - \gamma) = \overline{H}^+(\bar{p}_+ - \gamma),$$

and we can choose $\mu > 0$ such that

$$p_+^\mu = \overline{p}_+ - \gamma.$$

This implies inequality (2.5.15).

□

Proof of Theorem 2.2.10. The proof is performed in two steps.

Step 1 : proof of i) and ii). The goal is to pass to the limit as $l \to +\infty$ and then as $R \to +\infty$. Using Proposition 2.5.4, there exists $l_n \to +\infty$, such that

$$m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty,$$

the convergence being locally uniform. We also define

$$\overline{w}^R(t,x) = \limsup_{l_n \to +\infty}{}^* \left(w^{l_n,R} - w^{l_n,R}(0,0)\right),$$
$$\underline{w}^R(t,x) = \liminf_{l_n \to +\infty}{}_* \left(w^{l_n,R} - w^{l_n,R}(0,0)\right).$$

Thanks to (2.5.6), we know that $\overline{w}^R$ and $\underline{w}^R$ are finite and satisfy

$$m^R - C \le \underline{w}^R \le \overline{w}^R \le m^R + C.$$

By stability of viscosity solutions, $\overline{w}^R - 2C$ and $\underline{w}^R$ are respectively a sub and a super-solution of

$$w_t^R + G_R(x, [w^R(t,\cdot)], w_x^R) = \overline{A}_R \quad \text{for } (t,x) \in \mathbb{R} \times \mathbb{R} \tag{2.5.17}$$

Therefore, using Perron's method, we can construct a solution w^R of (2.5.17) with $m^R, \overline{A}^R$ and w^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \le C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(t,x) - m^R(x)| \le C & \text{for } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ H_0 \le \overline{A}_R \le 0. \end{cases} \tag{2.5.18}$$

Using Proposition 2.5.10, if $\overline{A} > H_0$, we know that there exists a γ_0 and a constant C, such that for all $\gamma \in (0, \gamma_0)$,

$$\begin{cases} w^R(t,x+h) - w^R(t,x) \ge (\overline{p}_+ - \gamma)h - C & \text{for all } x \ge r,\ h \ge 0, \\ w^R(t,x-h) - w^R(t,x) \ge (-\overline{p}_- - \gamma)h - C & \text{for all } x \le -r,\ h \ge 0. \end{cases} \tag{2.5.19}$$

We now pass to the limit as $R \to +\infty$. We consider (up to some subsequence)

$$\begin{cases} \overline{w}(t,x) = \limsup\limits_{R \to +\infty}{}^* \left(w^R - w^R(0,0)\right), \\ \\ \underline{w}(t,x) = \liminf\limits_{R \to +\infty}{}_* \left(w^R - w^R(0,0)\right), \\ \\ \overline{A} = \lim\limits_{R \to +\infty} \overline{A}_R, \\ \\ m = \lim\limits_{R \to +\infty} (m^R - m^R(0)). \end{cases}$$

The last convergence being locally uniform. Thanks to (2.5.18), we know that $\overline{w}$ and $\underline{w}$ are finite and satisfy

$$m - C \le \underline{w} \le \overline{w} \le m + C.$$

By stability of viscosity solutions, $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub and a super-solution of (2.2.26) with $\lambda = \overline{A}$. Using Perron's method, we can then construct a solution w of (2.2.26) with $\lambda = \overline{A}$ that satisfies (2.2.27) and (2.2.28).

Step 2 : proof of iii). We are now interested in the rescaled function $w^\varepsilon(t,x) = \varepsilon w\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right)$. Using (2.2.28), we have that

$$w^\varepsilon(t,x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Therefore, we can find a sequence $\varepsilon_n \to 0$, such that

$$w^{\varepsilon_n} \to W \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$. Like in [IM13], arguing as in the proof of convergence away from the junction point, we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.$$

For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

where we have used (2.2.28). Therefore we get

$$W_x = \overline{p}_+ \quad \text{for } x > 0,$$

this result remains valid even if $\overline{A} = H_0$ (in this particular case $W_x = p_0$). Similarly, we get

$$W_x = \overline{p}_- \quad \text{for } x < 0.$$

which implies (2.2.29) and (2.2.30). This ends the proof of Theorem 2.2.10.

□

Step 3 : proof of iv). Up to a sub-sequence, we assume that $\overline{A} = \lim\limits_{R\to+\infty} \lim\limits_{l\to+\infty} \lambda_{l,R}$. We want to prove that $\overline{A} = \inf E$, where

$$E = \{\lambda \in [h_0, 0] : \exists\, w \in \mathcal{S} \text{ solution of } (2.2.26)\},$$

with

$$\mathcal{S} = \{w \text{ s.t. } \exists m \in Lip\,(\mathbb{R}) \text{ and a } C > 0 \text{ s.t. } |w(t,x) - m(x)| \leq C \text{ for all } t > 0\}.$$

We argue by contradiction and assume that there exists a $\lambda < \overline{A}$ and a function $w^\lambda \in \mathcal{S}$ solution of (2.2.26). We assume that $w^\lambda(0,0) = 0$ (if we are not in this situation, we do a translation since we have $w^\lambda - w^\lambda(0,0) \in \mathcal{S}$. Arguing as in the proof of step 2, we deduce that the function

$$w^\varepsilon_\lambda(t,x) = \varepsilon w^\lambda\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

has a limit W^λ (with $W^\lambda(0)=0$) which satisfies

$$\overline{H}(W^\lambda_x)=\lambda \quad \text{for } x>0,$$

which means that for all $x>0$,

$$W^\lambda_x \le p^\lambda_+ < \overline{p}_+ \quad \text{with } \overline{H}(p^\lambda_+)=\overline{H}^+(p^\lambda_+)=\lambda. \tag{2.5.20}$$

Similarly we have for all $x<0$,

$$W^\lambda_x \ge p^\lambda_- > \overline{p}_- \quad \text{with } \overline{H}(p^\lambda_-)=\overline{H}^-(p^\lambda_-)=\lambda. \tag{2.5.21}$$

These inequalities imply that for all $\gamma>0$, there exists a constant $\tilde{C}_\gamma>0$ such that

$$w^\lambda(t,x) \le \begin{cases} (p^\lambda_+ + \gamma)x + \tilde{C}_\gamma & \text{for } x>0, \\ (p^\lambda_- - \gamma)x + \tilde{C}_\gamma & \text{for } x<0, \end{cases} \tag{2.5.22}$$

In fact, if w^λ does not satisfies (2.5.22), we cannot have (2.5.20) and (2.5.21). Using point ii) of Theorem 2.2.10, we get

$$w^\lambda < w \quad \text{for } |x| \ge \tilde{R}$$

if γ is small enough and $\tilde{R}$ big enough. This implies that there exists a constant $C_{\tilde{R}}>0$ such that for all $x\in\mathbb{R}$, we have

$$w^\lambda(t,x) < w(t,x) + C_{\tilde{R}}.$$

Let us now introduce, $u(t,x)=w(t,x)+C_{\tilde{R}}-\overline{A}t$ and $u_\lambda(t,x)=w^\lambda(t,x)-\lambda t$ both solutions of (2.2.15) with $\varepsilon=1$ and $u_\lambda(0,x)\le u(0,x)$. Therefore, the comparison principle implies

$$w^\lambda(t,x)-\lambda t \le w(t,x)+C_{\tilde{R}}-\overline{A}t$$

Dividing by t and passing to the limit as t goes to infinity, we get

$$\overline{A}\le\lambda,$$

which is a contradiction.

Step 4 : proof of v). In order to establish the monotonicity, we have to consider the approximated truncated cell problem (2.5.7). Let us consider v^δ_1 and v^δ_2 viscosity solutions of (2.5.7), respectively for ϕ_1 and ϕ_2, with $0\le\phi_1\le\phi_2$. First, using the fact that the non-local operator is negative, we have

$$G^2_R(x,[U],q) \le G^1_R(x,[U],q),$$

with

$$G^i_R(x,[U],q)=\phi_i(t,x)\cdot M[U](x)\cdot\psi_R(x)\cdot|q| + (1-\psi_R(x))\overline{H}(q), \quad \text{for } i=1,2.$$

Therefore, we have

$$0 = \delta v_1^\delta + \left(v_1^\delta\right)_t + G_R^1(x, [v_1^\delta(t,\cdot)], (v_1^\delta)_x) \geq \delta v_1^\delta + \left(v_1^\delta\right)_t + G_R^2(x, [v_1^\delta(t,\cdot)], (v_1^\delta)_x),$$

meaning that v_1^δ is a sub-solution of (2.5.7) with ϕ_2. The comparison principle and (2.5.8) imply that

$$0 \leq \delta v_1^\delta \leq \delta v_2^\delta \leq |H_0|.$$

Passing to the limit as $\delta \to 0$, we obtain

$$0 \geq \lambda_{l,R}^1 \geq \lambda_{l,R}^2 \geq H_0.$$

Passing to the limit as $l, R \to +\infty$, we get the result.

2.6 Convergence (proof of Theorem 2.2.8)

This section contains the proof of the homogenization result (Theorem 2.2.8). This proof relies on the existences of correctors (Proposition 2.2.9 and Theorem 2.2.10).

We begin with two useful lemmas for the proof of Theorem 2.2.8. The first result is a direct consequence of Perron's method and Lemma 2.3.8.

Lemma 2.6.1 (Barriers uniform in ε). *Assume (A0) and (A). There exists a constant $C > 0$ (depending only on M_0 and k_0) such that for all $t > 0$ and $x \in \mathbb{R}$,*

$$|u^\varepsilon(t,x) - u_0(x)| \leq Ct.$$

The following lemma is a direct result of Theorem 2.3.12 and the fact that

$$u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Lemma 2.6.2 (Uniform gradient bound). *Assume (A0) and (A). Then the solution u^ε of (2.2.19) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$-k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0. \tag{2.6.1}$$

Before passing to the proof of Theorem 2.2.8, let us show how it allows us to obtain the following result

Corollary 2.6.3. *Assume (A0)-(A). Let u^0 be the unique solution of (2.2.5), then we have for all $(t,x) \in [0,T] \times \mathbb{R}$,*

$$-k_0 \leq u_x^0 \leq 0,$$

with k_0 defined in (A0).

Remark 2.6.4 (Extension of the effective Hamiltonian)**.** *This result implies in particular that in the case of traffic flow, the effective Hamiltonian only needs to be computed for* $p \in [-k_0, 0]$*. However, for the construction of the correctors it is necessary to work with a coercive Hamiltonian in* $\mathbb{R}$ *that is why we extend the function* $\overline{H}$ *in* (2.2.2)*.*

Proof of Corollary 2.6.3. We want to prove that for all $t \in [0, +\infty)$ and for all $x, y \in \mathbb{R}$, $x \geq y$,

$$-k_0(x-y) \leq u^0(t,x) - u^0(t,y) \leq 0. \tag{2.6.2}$$

Using Lemma 2.6.2, we have that the solution u^ε of (2.2.19), satisfies for all $(t,x,y) \in [0,+\infty) \times \mathbb{R} \times \mathbb{R}$, with $x \geq y$,

$$-k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0.$$

Now using Theorem 2.2.8, passing to the limit as $\varepsilon \to 0$, we obtain the result.

□

We now turn to the proof of Theorem 2.2.8.

Proof of Theorem 2.2.8. We introduce

$$\overline{u}(t,x) = \limsup_{\varepsilon \to 0}{}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\varepsilon \to 0}{}_* u^\varepsilon. \tag{2.6.3}$$

Thanks to Lemma 2.6.1, we know that these functions are well defined. We want to prove that $\overline{u}$ and $\underline{u}$ are respectively a sub-solution and a super-solution of (2.2.5). In this case, the comparison principle (Theorem 2.3.11) will imply that $\overline{u} \leq \underline{u}$. But, by construction, we have $\underline{u} \leq \overline{u}$, hence we will get $\underline{u} = \overline{u} = u^0$, the unique solution of (2.2.5).

Let us prove that $\overline{u}$ is a sub-solution of (2.2.5) (the proof for $\underline{u}$ is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi \in \mathcal{C}^1(J_\infty)$ (in the sense of Definition 2.3.2), and a point $(\bar{t}, x) \in (0, +\infty) \times \mathbb{R}$ such that

$$\begin{cases} \overline{u}(\bar{t},x) = \varphi(\bar{t},x) & & \\ \overline{u} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x) & \text{with } \bar{r} > 0 \\ \overline{u} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x) & \text{with } \eta > 0 \\ \varphi_t(\bar{t},x) + \overline{H}(x, \varphi_x(\bar{t},x)) = \theta & \text{with } \theta > 0, & \end{cases} \tag{2.6.4}$$

where $\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x)$ is defined in (2.3.18) and

$$\overline{H}(x, \varphi_x(\bar{t},x)) := \begin{cases} \overline{H}\,(\varphi_x(\bar{t},x)) & \text{if } x \neq 0, \\ F_{\overline{A}}\,(\varphi_x(\bar{t},0^-), \varphi_x(\bar{t},0^+)) & \text{if } x = 0. \end{cases}$$

Using the previous Lemmas 2.6.2-2.6.1, we can assume (up to changing φ at infinity) that for ε small enough, we have

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x).$$

Using the previous lemmas we get that the function $\overline{u}$ satisfies for all $t > 0$ and $x, y \in \mathbb{R}$, $x \geq y$,

$$\begin{array}{c} |\overline{u}(t,x) - u_0(x)| \leq Ct, \\ -k_0(x-y) \leq \overline{u}(t,x) - \overline{u}(t,y) \leq 0. \end{array} \tag{2.6.5}$$

First case : $x \neq 0$. We only consider $x > 0$, since the other case ($x < 0$) is treated in the same way. We define $p = \varphi_x(\bar{t}, x)$ which according to (2.6.5) satisfies

$$-k_0 \leq p \leq 0.$$

We choose $\bar{r}$ small enough so that $x - 2\bar{r} > 0$. Let us prove that the test function φ satisfies in the viscosity sense, the inequality

$$\varphi_t + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x) \cdot \phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot |\varphi_x| \geq \frac{\theta}{2} \quad \text{for } (t,x) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, x). \tag{2.6.6}$$

Recalling the definition of the function ϕ (assumption (A6)), we notice that for ε small enough we have

$$\phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) = 1 \quad \text{for all } (t,x) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, x).$$

In fact, the last equality is true since for all $(t,x) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, x)$, we have that $r < \dfrac{x}{\varepsilon}$ since

$$0 < \bar{x} - 2\bar{r} < \bar{x} - \bar{r} \leq x.$$

For all $(t,x) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, x)$, we have for $\bar{r}$ small enough

$$\begin{aligned} \varphi_t(t,x) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x) \cdot |\varphi_x(t,x)| &= \varphi_t(\bar{t},x) + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x) \cdot |\varphi_x(\bar{t},\bar{x})| \\ &= \theta - \overline{H}(p) + o_{\bar{r}}(1) + \tilde{M}^\varepsilon \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right](x) \cdot |p| \\ &=: \Delta, \end{aligned} \tag{2.6.7}$$

where we have used (2.6.4). We recall from Proposition 2.2.9 that for $-k_0 \leq p \leq 0$,

$$\overline{H}(p) = M_p[0](0)|p|.$$

Moreover, for all $z \in [h_0, h_{max}]$, and for ε and $\bar{r}$ small enough we have

$$\begin{aligned} \frac{\varphi(t, x+\varepsilon z) - \varphi(t,x)}{\varepsilon} &= z\varphi_x(t,y) + \varepsilon z^2 \varphi_{xx}(t, \xi(x, x+\varepsilon z)) \\ &\leq pz + o_{\bar{r}}(1) + c\varepsilon, \end{aligned}$$

where we have used the fact that $\varphi \in \mathcal{C}^2$ and that $z \in [h_0, h_{max}]$. Now using the fact that $\tilde{E}$ is decreasing we have

$$\tilde{E}(pz + c\varepsilon + o_{\bar{r}}(1)) \leq \tilde{E}\left(\frac{\varphi(t, x+\varepsilon z) - \varphi(t,x)}{\varepsilon}\right).$$

Using this result and replacing the non-local operators in (2.6.7) by their definition (see (2.3.1)), we obtain

$$\Delta \geq \theta + o_{\bar r}(1) + |p| \int_{h_0}^{h_{max}} J(z)\tilde{E}(pz + c\varepsilon + o_{\bar r}(1))dz$$
$$- |p| \int_{h_0}^{h_{max}} J(z)\tilde{E}(pz)dz. \tag{2.6.8}$$

We can see that if we have $p = 0$, we obtain directly our result. However, if $-k_0 \leq p < 0$,

$$\int_{\mathbb{R}} J(z)\tilde{E}(pz + c\varepsilon + o_{\bar r}(1))dz = - V\left(\frac{-1 - c\varepsilon + o_{\bar r}(1)}{p}\right) - \frac{1}{2}V\left(-\frac{c\varepsilon + o_{\bar r}(1)}{p}\right) + \frac{3}{2}V_{max},$$
$$\int_{\mathbb{R}} J(z)\tilde{E}(pz)dz = - V\left(\frac{-1}{p}\right) + \frac{3}{2}V_{max}. \tag{2.6.9}$$

Injecting (2.6.9) in (2.6.8) and choosing ε and $\bar r$ small enough, we obtain

$$\begin{aligned}\Delta \geq\ & \theta + o_{\bar r}(1) + |p| \cdot \left[-V\left(\frac{-1 - c\varepsilon + o_{\bar r}(1)}{p}\right) + V\left(\frac{-1}{p}\right)\right] \\ \geq\ & \theta + o_{\bar r}(1) - ||V'||_\infty \cdot (c\varepsilon + o_{\bar r}(1)) \\ \geq\ & \frac{\theta}{2},\end{aligned}$$

where we have used assumption (A1) for the second line.

Getting a contradiction. By definition, we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar r, \bar r}(\bar t, x).$$

Using the comparison principle on bounded subsets for (2.2.15) (Theorem 2.3.7), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar r, \bar r}(\bar t, x).$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar r, \bar r}(\bar t, x)$ and this contradicts the fact that $\overline{u}(\bar t, x) = \varphi(\bar t, x)$.

Second case : $x = 0$. Using Theorem 2.3.10, we may assume that the test-function in (2.6.4) has the following form

$$\varphi(t, x) = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{2\bar r, 2\bar r}(\bar t, 0), \tag{2.6.10}$$

where g is a C^1 function defined in $(0, +\infty)$ and $\overline{p}_-, \overline{p}_-$ the two constants satisfying (2.2.22). The last line in condition (2.6.4) becomes

$$g'(t) + F_{\overline{A}}(\overline{p}_-, \overline{p}_+) = g'(t) + \overline{A} = \theta \quad \text{at } (\bar t, 0). \tag{2.6.11}$$

Let us consider the solution w of (2.2.26) provided by Theorem 2.2.10, and let us denote by

$$\varphi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(t,x) & \text{on } \mathcal{Q}_{2\bar r,2\bar r}(\bar t,0), \\ \varphi(t,x) & \text{outside } \mathcal{Q}_{2\bar r,2\bar r}(\bar t,0). \end{cases} \tag{2.6.12}$$

We would like to prove that this function satisfies in the viscosity sense, for $\bar r$ and ε small enough,

$$\varphi^\varepsilon_t(t,x) + \tilde M^\varepsilon \left[\frac{\varphi^\varepsilon}{\varepsilon}(t,\cdot)\right](x)\cdot \phi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\cdot |\varphi^\varepsilon_x| \geq \frac{\theta}{2} \quad \text{on } \mathcal{Q}_{\bar r,\bar r}(\bar t,0).$$

Let h be a test function touching φ^ε from below at $(t_1,x_1)\in \mathcal{Q}_{\bar r,\bar r}(\bar t,0)$, so we have

$$w\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(h(t_1,x_1) - g(t_1)\right),$$

and

$$w(s,y) \geq \frac{1}{\varepsilon}\left(h(\varepsilon s,\varepsilon y) - g(\varepsilon s)\right),$$

for (s,y) in a neighbourhood of $\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right)$. Therefore, using the fact that w is a solution of (2.2.26) for $\lambda = \overline{A}$ (Theorem 2.2.10) , we have

$$h_t(t_1,x_1) - g'(t_1) + \tilde M\left[w\left(\frac{t_1}{\varepsilon},\cdot\right)\right]\left(\frac{x_1}{\varepsilon}\right)\cdot\phi\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right)\cdot |h_x(t_1,x_1)| \geq \overline{A}.$$

This implies that (using (2.6.11) and taking $\bar r$ small enough)

$$h_t(t_1,x_1) + \tilde M\left[w\left(\frac{t_1}{\varepsilon},\cdot\right)\right]\left(\frac{x_1}{\varepsilon}\right)\cdot\phi\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right)\cdot |h_x(t_1,x_1)| \geq \overline{A} + g'(t_1) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon h_{max} \leq \bar r$, we deduce from the previous inequality and using the fact that $\tilde M$ is a non-local operator with a bounded support, that we have

$$h_t(t_1,x_1) + \tilde M^\varepsilon\left[\frac{\varphi^\varepsilon(t_1,\cdot)}{\varepsilon}\right](x_1)\cdot\phi\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right)\cdot |h_x(t_1,x_1)| \geq \frac{\theta}{2}.$$

Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{2\bar r,2\bar r}(\bar t,0)\backslash \mathcal{Q}_{\bar r,\bar r}(\bar t,0).$$

Using the fact that $w^\varepsilon \to W$, and using (2.2.30), we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } \mathcal{Q}_{2\bar r,2\bar r}(\bar t,0)\backslash \mathcal{Q}_{\bar r,\bar r}(\bar t,0).$$

Combining this with (2.6.12), we get that

$$u^\varepsilon + \frac{\eta}{2} \le \varphi^\varepsilon \quad \text{outside } \mathcal{Q}_{\bar r,\bar r}(\bar t, 0),$$

By the comparison principle on bounded subsets (Theorem 2.3.7) the previous inequality holds in $\mathcal{Q}_{\bar r,\bar r}(\bar t, 0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar t, 0)$, we obtain

$$\overline{u}(\bar t, 0) + \frac{\eta}{2} \le \varphi(\bar t, 0) = \overline{u}(\bar t, 0),$$

which is a contradiction.

$\square$

2.7 Proof of Theorem 2.2.4

This section is devoted to the proof of Theorem 2.2.4, which is a direct application of our convergence result, Theorem 2.2.8.

Proof of Theorem 2.2.4. We recall that in Theorem 2.2.4, we have $u_0(x) = -x/h$, with $h \ge h_0$. First, we would like to prove that for all $\varepsilon > 0$, we have

$$|\rho^\varepsilon(0, x) - u_0(x)| \le f(\varepsilon) \quad \text{for all } x \in \mathbb{R}, \tag{2.7.1}$$

with $f(\varepsilon) \to 0$ as ε goes to 0. To do this, we define a piecewise affine function v satisfying

$$\rho^1(0, x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.$$

Given that for all $i \in \mathbb{Z}$, $U_{i+1}(0) - U_i(0) \ge h_0$, we notice that v is k_0-Lipschitz continuous and by definition of $\rho^1(0, x)$, we have

$$\left|\rho^1(0, x) - v(x)\right| \le 1 \quad \text{for all } x \in \mathbb{R}.$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by

$$i_0 = \sup \{i \in \mathbb{Z},\ \text{s.t. } \ U_i(0) \le -R\}.$$

Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \le -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \le U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf \{i \in \mathbb{Z},\ \text{s.t. } \ U_i(0) \ge R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \geq R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \geq U_{i_1}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} - i_1 - 1.$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0,x) = \varepsilon\rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0,x) - u_0(x)| \leq & \left|\rho^\varepsilon(0,x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)\right| + \left|\varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x)\right| \\ \leq & \varepsilon + \varepsilon \max\left(\left|\frac{U_{i_1}(0)}{h} - i_1 - 1\right|, \left|\frac{U_{i_0}(0)}{h} - i_0 - 1\right|\right). \end{aligned} \tag{2.7.2}$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0,x) - u_0(x)| \leq & \left|\rho^\varepsilon(0,x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)\right| + \left|\varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right)\right| \\ \leq & \varepsilon + \varepsilon \max_{y \in \left[U_{i_0}(0), U_{i_1}(0)\right]} \left(|v(y) - u_0(y)|\right), \end{aligned} \tag{2.7.3}$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (2.7.2) and (2.7.3) and choosing

$$\begin{aligned} f(\varepsilon) = \varepsilon + \varepsilon \max\Bigg(&\left|\frac{U_{i_0}(0)}{h} - i_0 - 1\right|, \\ &\max_{y \in \left[U_{i_0}(0), U_{i_1}(0)\right]} \left(|v(y) - u_0(y)|\right), \left|\frac{U_{i_1}(0)}{h} - i_1 - 1\right|\Bigg) \end{aligned}$$

we deduce (2.7.1). Notice also that thanks to (2.7.1), we have

$$|(\rho^\varepsilon)^*(0,x) - u_0(x)| \leq f(\varepsilon) + \varepsilon. \tag{2.7.4}$$

Therefore, we have

$$u_0(x) - f(\varepsilon) \leq \rho^\varepsilon(0,x) \leq (\rho^\varepsilon)^*(0,x) \leq u_0(x) + f(\varepsilon) + \varepsilon.$$

Using the fact that ρ^ε is a viscosity solution of (2.2.15) and the comparison principle (Proposition 2.3.5) we deduce that (with u^ε the continuous solution of (2.2.15) associated to the initial condition $u_0(x) = -x/h$)

$$u^\varepsilon(t,x) - f(\varepsilon) \leq \rho^\varepsilon(t,x) \leq (\rho^\varepsilon)^*(t,x) \leq u^\varepsilon(t,x) + f(\varepsilon) + \varepsilon,$$

where we have used the fact that (2.2.15) is invariant by addition of constants to the solutions. Passing to the limit as $\varepsilon \to 0$ and using Theorem 2.2.8 we get that $\rho^\varepsilon \to u^0$, which ends the proof of Theorem 2.2.4.

□

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Chapitre 3

Derivation of a macroscopic LWR model from a microscopic *follow-the-leader* model by homogenization

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Abstract

The goal of this chapter is to derive a traffic flow macroscopic model from a microscopic model with a transition function. At the microscopic scale, we consider a first order model of the form "follow the leader" i.e. the velocity of each vehicle depends on the distance to the vehicle in front of it. We consider two different velocities and a transition zone. The transition zone represents a local perturbation operated by a Lipschitz function. After rescaling, we prove that the ”cumulative distribution function” of the vehicles converges towards the solution of a macroscopic homogenized Hamilton-Jacobi equation with a flux limiting condition at junction which can be seen as a LWR model.

3.1 Introduction

The goal of this chapter is to present a rigorous derivation of a traffic flow macroscopic model by homogenization of a *follow-the-leader* model, see [GHR61, PH71]. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles. Several studies have been done about the connection between microscopic

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and macroscopic traffic flow model. This type of connection is important since it allows us to deduce macroscopic models rigorously and without using strong assumptions. We refer for example to [AKRM02, CR15, DFR15] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR (Lighthill-Whitham-Richards) model). More recently, another kind of macroscopic models appears. These models rely on the Moskowitz function and make appear an Hamilton-Jacobi equation. This is the setting of our work which is a generalization of [FSZ17b]. Indeed, authors in [FSZ17b] considered a single road and one velocity throughout this road with a local perturbation at the origin while we consider two different velocities and a transition zone which can be seen as a local perturbation thats slows down the vehicles. At the macroscopic scale, we get an Hamilton-Jacobi equation with a junction condition at zero and an effective flux limiter. In order to have our homogenization result, we will construct the correctors. The main new technical difficulties comes from the construction of correctors and in particular the gradient estimates are more complicated from that in [FSZ17b] because the gradient on the left and on the right may differ.

3.2 The microscopic model

In this chapter, we consider a "follow the leader" model of the following form

$$\dot{U}_j(t) = V_1(U_{j+1}(t) - U_j(t))\varphi\left(U_j(t)\right) + V_2(U_{j+1}(t) - U_j(t))\left(1 - \varphi\left(U_j(t)\right)\right),$$

where U_j denotes the position of the j-th vehicle and $\dot{U}_j$ its velocity. The function φ simulates the presence of a local perturbation around the origin which allows us to pass from the optimal velocity function V_1 (on the left of the origin) to V_2 (on the right). We make the following assumptions on V_1, V_2 and φ.

Assumption (A)

- (A1) $V_1, V_2 : \mathbb{R} \to \mathbb{R}^+$ are Lipschitz continuous, non-negative and non-decreasing.
- (A2) For $i = 1, 2$, there exists a $h_0^i \in (0, +\infty)$ such that
$$V_i(h) = 0 \text{ for all } h \leq h_0^i.$$
- (A3) For $i = 1, 2$, there exists a $h_{max}^i \in (0, +\infty)$ such that
$$V_i(h) = V_{imax} \text{ for all } h \geq h_{max}^i.$$
- (A4) For $i = 1, 2$, there exists a real $p_0^i \in [-1/h_0^i, 0)$ such that the function $p \mapsto pV_i(-1/p)$ is decreasing on $[-1/h_0^i, p_0^i)$ and increasing on $[p_0^i, 0)$.
- (A5) The function $\varphi : \mathbb{R} \to [0, 1]$ is Lipschitz continuous and
$$\varphi(x) = \begin{cases} 1 & \text{if } x \leq -r \\ 0 & \text{if } x > r\ . \end{cases}$$

3.3 The homogenization result

We introduce the "cumulative distribution function" of the vehicles :

$$\rho(t,y) = -\left(\sum_{i\geq 0} H\left(y - U_i(t)\right) + \sum_{i<0}\left(-1 + H\left(y - U_i(t)\right)\right)\right)$$

and we make the following rescaling

$$\rho^{\varepsilon}(t,y) = \varepsilon\rho\left(t/\varepsilon, y/\varepsilon\right).$$

ρ^{ε} is a discontinuous solution of the following equation : for $(t,x) \in (0,+\infty)\times\mathbb{R}$,

$$\begin{cases} u_t^{\varepsilon} + \left(M_1^{\varepsilon}\left[\dfrac{u^{\varepsilon}(t,\cdot)}{\varepsilon}\right](x)\varphi\left(\dfrac{x}{\varepsilon}\right) + M_2^{\varepsilon}\left[\dfrac{u^{\varepsilon}(t,\cdot)}{\varepsilon}\right](x)\left(1-\varphi\left(\dfrac{x}{\varepsilon}\right)\right)\right)\cdot |u_x^{\varepsilon}| = 0 \\ \\ u^{\varepsilon}(0,x) = u_0(x) \end{cases} \tag{3.3.1}$$

where the non-local operators M_i^{ε}, $(i = 1,2)$ are defined by

$$M_i^{\varepsilon}[U](x) = \int_{-\infty}^{+\infty} J_i(z)E\left(U(x+\varepsilon z) - U(x)\right)dz - \frac{3}{2}V_{imax} \tag{3.3.2}$$

with

$$E(z) = \begin{cases} 0 \quad \text{if } z \geq 0, \\ 1/2 \quad \text{if } -1 \leq z < 0 \\ 3/2 \quad \text{if } z < -1. \end{cases}, \quad J_1 = V_1' \text{ and } J_2 = V_2' \text{ on } \mathbb{R}. \tag{3.3.3}$$

We also assume that the initial condition satisfies the following assumption.

(A0) (Gradient bound). Let $k_0 = \max\left(k_0^1, k_0^2\right)$ with $k_0^i = 1/h_0^i$. The function u_0 is Lipschitz continuous and satisfies

$$-k_0 \leq (u_0)_x \leq 0.$$

We have the following theorem (see [FSZ17b]).

Theorem 3.3.1. *Assume (A0) and (A). Then, there exists a unique viscosity solution u^{ε} of (3.3.1). Moreover, the function u^{ε} is continuous and there exists a constant K such that*

$$u_0(x) \leq u^{\varepsilon}(t,x) \leq u_0(x) + Kt.$$

We will introduce now the macroscopic model which is a Hamilton-Jacobi equation on a junction. The Hamiltonians $\overline{H}_1$ and $\overline{H}_2$ are called effective Hamiltonians (see Proposition 2.9 in [FSZ17b]) and are defined as follows : for $i = 1, 2$

$$\overline{H}_i(p) = \begin{cases} -p - k_0^i & \text{for } p < -k_0^i, \\ -V_i\left(\dfrac{-1}{p}\right) \cdot |p| & \text{for } -k_0^i \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases} \tag{3.3.4}$$

with

$$H_0^i = \min_{p \in \mathbb{R}} \overline{H}_i(p) \quad \text{and } H_0 = \max\left(H_0^1, H_0^2\right). \tag{3.3.5}$$

Now we can define the limit problem. We refer to [IM13] for more details about existence and uniqueness of solution for this type of equation.

$$\begin{cases} u_t^0 + \overline{H}_1(u_x^0) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (-\infty, 0) \\ u_t^0 + \overline{H}_2(u_x^0) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times (0,+\infty) \\ u_t^0 + F_{\overline{A}}(u_x^0(t,0^-), u_x^0(t,0^+)) = 0 & \text{for} \quad (t,x) \in (0,+\infty) \times \{0\} \\ u^0(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{3.3.6}$$

where $\overline{A}$ has to be determined and $F_{\overline{A}}$ is defined by

$$F_A(p_1, p_2) = \max\left(\overline{A}, \overline{H}_1^+(p_1), \overline{H}_2^-(p_2)\right);$$

$\overline{H}_1^+$ and $\overline{H}_2^-$ represent respectively the increasing and the decreasing part of $\overline{H}_1$ and $\overline{H}_2$. The following theorem is our main result in this chapter.

Theorem 3.3.2. *There exists $\overline{A} \in [H_0, 0]$ such that the function u^ε defined by Theorem 3.3.1 converge locally uniformly towards the unique solution u^0 of* (3.3.6)*.*

Remark 3.3.3. *Formally, if we derive (3.3.6), we will obtain a scalar conservation law with discontinuous flux whose literature is very rich, see for example [Die95]. However, the passage from microscopic to macroscopic models are more difficult in this setting and in particular on networks. On the contrary, the approach proposed in this chapter can be extended to models posed on networks (see [FS]).*

3.4 Correctors for the junction

The key ingredient to prove the convergence result is to construct correctors for the junction. Given $\overline{A} \geq H_0$, we introduce two real numbers $\overline{p}_1, \overline{p}_2 \in \mathbb{R}$, such that

$$\overline{H}_2(\overline{p}_2) = \overline{H}_2^+(\overline{p}_2) = \overline{H}_1(\overline{p}_1) = \overline{H}_1^-(\overline{p}_1) = \overline{A}. \tag{3.4.1}$$

Due to the form of $\overline{H}_1$ and $\overline{H}_2$ this two real numbers exist and are unique. We consider now the following problem : find $\lambda \in \mathbb{R}$ such that there exists a solution w of the following global-in-time Hamilton-Jacobi equation

$$(M_1[w](x) \cdot \varphi(x) + M_2[w](x) \cdot (1 - \varphi(x))) \cdot |w_x| = \lambda \quad \text{for } x \in \mathbb{R} \tag{3.4.2}$$

with

$$M_i[U](x) = \int_{-\infty}^{+\infty} J_i(z) E\left(U(x+z) - U(x)\right) dz - \frac{3}{2} V_{imax}, i = 1, 2. \tag{3.4.3}$$

Theorem 3.4.1 (Existence of a global corrector for the junction)**.** *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution w of (3.4.2) with $\lambda = \overline{A}$ and such that there exists a constant $C > 0$ and a globally Lipschitz continuous function m such that for all $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C. \tag{3.4.4}$$

ii) (Bound from below at infinity) If $\bar{A} > H_0$, then there exists γ_0 such that for every $\gamma \in (0, \gamma_0)$, we have

$$\begin{cases} w(x-h) - w(x) \geq (-\overline{p}_1 - \gamma)h - C & \text{for } x \leq -r \text{ and } h \geq 0, \\ w(x+h) - w(x) \geq (\overline{p}_2 - \gamma)h - C & \text{for } x \geq r \text{ and } h \geq 0. \end{cases} \tag{3.4.5}$$

iii) (Rescaling w) For $\varepsilon > 0$, we set

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right),$$

then (along a subsequence $\varepsilon_n \to 0$) we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}_1(W_x) = \overline{A} & \text{for all } x < 0, \\ \overline{H}_2(W_x) = \overline{A} & \text{for all } x > 0. \end{cases} \tag{3.4.6}$$

In particular, we have (with $W(0) = 0$)

$$W(x) = \overline{p}_1 x 1_{\{x<0\}} + \overline{p}_2 x 1_{\{x>0\}}. \tag{3.4.7}$$

3.5 Proof of Theorem 3.4.1

This section contains the proof of Theorem 3.4.1. To do this, we will construct correctors on truncated domains and then pass to the limit as the size of the domain goes to infinity. For $l \in (r, +\infty)$, $r << l$ and $r \leq R << l$, we want to find $\lambda_{l,R}$, such that there exists a solution $w^{l,R}$ of

$$\begin{cases} Q_R\left(x, [w^{l,R}], w_x^{l,R}\right) = \lambda_{l,R} & \text{if } x \in (-l, l) \\ \overline{H}_1^-(w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in \{-l\} \\ \overline{H}_2^+(w_x^{l,R}) = \lambda_{l,R} & \text{if } x \in \{l\}, \end{cases} \tag{3.5.1}$$

with

$$Q_R(x,[U],q) = \psi_R(x)\cdot M_2[U](x)\cdot(1-\varphi(x))\cdot|q| + (1-\psi_R(x))\cdot\overline{H}_2(q) \tag{3.5.2}$$

$$+\,\Phi_R(x)\cdot M_1[U](x)\cdot\varphi(x)\cdot|q| + (1-\Phi_R(x))\cdot\overline{H}_1(q) \tag{3.5.3}$$

and $\psi_R, \Phi_R \in C^\infty$, $\psi_R, \Phi_R : \mathbb{R} \to [0,1]$, with

$$\psi_R \equiv \begin{cases} 1 & x \le R \\ 0 & x > R+1 \end{cases} \quad \text{and} \quad \Phi_R \equiv \begin{cases} 1 & x \ge -R \\ 0 & x < -R-1. \end{cases} \tag{3.5.4}$$

Proposition 3.5.1 (Existence of correctors on truncated domains)**.** *There exists a unique $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solutions $w^{l,R}$ of (3.5.1). Moreover, there exists a constant C (depending only on k_0), and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0^1 \le \lambda_{l,R} \le 0, \\ |m^{l,R}(x) - m^{l,R}(y)| \le C|x-y| & \text{for } x,y \in [-l,l], \\ |w^{l,R}(x) - m^{l,R}(x)| \le C & \text{for } x \in \mathbb{R}\times[-l,l]. \end{cases} \tag{3.5.5}$$

Proof. We only give the main steps of the proof. Classically, we will consider the approximated problem depending on the parameter δ and then take δ to 0.

$$\begin{cases} \delta v^\delta + Q_R(x,[v^\delta], v_x^\delta) = 0 & \text{for } x \in (-l,l) \\ \delta v^\delta + \overline{H}_1^-(v_x^\delta) = 0 & \text{for } x \in \{-l\} \\ \delta v^\delta + \overline{H}_2^+(v_x^\delta) = 0 & \text{for } x \in \{l\} \end{cases} \tag{3.5.6}$$

Step 1 : construction of barriers . We remark first that the functions

$$v_1^+(x) = -k_0^1(x-l) + \frac{|H_0^2|}{\delta}$$

and

$$v_2^+(x) = -k_0^2(x-l) + \frac{|H_0^1|}{\delta}$$

are super-solutions of (3.5.6). Using stability of super-solutions, we deduce that the function

$$v^+(x) = \min\left(v_1^+(x), v_2^+(x)\right)$$

is a super-solution of (3.5.6). Using Perron's method and 0 and v^+ as barriers, we deduce that there exists a continuous viscosity solution v^δ of (3.5.6) which satisfies

$$0 \le v^\delta \le v^+. \tag{3.5.7}$$

Step 2 : control of the space oscillations of v^δ. The function v^δ satisfies for all $x, y \in [-l,l]$, $x \ge y$,

$$-k_0(x-y) - 1 \le v^\delta(x) - v^\delta(y) \le 0,$$

with $k_0 = \max(k_0^1, k_0^2)$ (see [FSZ17b, Lemma 6.5]).
Step 3 : construction of a Lipschitz estimate. As in [FSZ17b, Lemma 6.6] we can construct a Lipschitz continuous function m^δ, such that there exists a constant C, (independent of l, R and δ) such that

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x-y| & \text{for all } x, y \in [-l, l], \\ |v^\delta(x) - m^\delta(x)| \leq C & \text{for all } x \in [-l, l]. \end{cases} \tag{3.5.8}$$

Step 4 : passing to the limit as δ goes to 0. Classicaly, taking δ to zero, we get $\lambda_{l.R}, w^{l,R}$ and $m^{l,R}$ satisfiying (3.5.5). The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 3.5.1. □

Proposition 3.5.2. *The following limits exist (up to a subsequence)*

$$\overline{A}_R = \lim_{l \to +\infty} \lambda_{l,R}, \quad \text{and} \quad \overline{A} = \lim_{R \to +\infty} \overline{A}_R.$$

Moreover, we have

$$H_0 \leq \overline{A}_R, \overline{A} \leq 0.$$

Proposition 3.5.3 (Control of the slopes on a truncated domain)**.** *Assume that l and R are big enough. Let $w^{l,R}$ be the solution of (3.5.1) given by Proposition 3.5.1. We also assume that up to a sub-sequence $\overline{A} = \lim_{R\to+\infty} \lim_{l\to+\infty} \lambda_{l,R} > H_0$. Then there exits a $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, there exists a constant C (independent of l and R) such that for all $x \leq -r$ and $h \geq 0$*

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\overline{p}_1 - \gamma)h - C. \tag{3.5.9}$$

Similarly, for all $x \geq r$ and $h \geq 0$,

$$w^{l,R}(x+h) - w^{l,R}(x) \geq (\overline{p}_2 - \gamma)h - C. \tag{3.5.10}$$

Proof. We only prove (3.5.9) since the proof for (3.5.10) is similar. For $\sigma > 0$ small enough, we denote by p_-^σ the real number such that

$$\overline{H}_1(p_-^\sigma) = \overline{H}_1^-(p_-^\sigma) = \lambda_{l,R} - \sigma.$$

Let us now consider the function $w^- = p_-^\sigma x$ that satisfies

$$\overline{H}_1(w_x^-) = \lambda_{l,R} - \sigma \quad \text{for } x \in \mathbb{R}.$$

We also have

$$M_1[w^-](x) = -V_1\left(\frac{-1}{p_-^\sigma}\right).$$

For all $x \in (-l,-r)$, using that $\varphi(x) = 1$ and $\psi_R(x) = 1$, we deduce that w^- satisfies

$$\begin{cases} Q_R\left(x, [w^-], w_x^-\right) = \lambda_{l,R} - \mu & \text{for } x \in (-l,-r) \\ \overline{H}_1^-\left(w_x^+\right) = \lambda_{l,R} - \mu & \text{for } x \in \{-l\}. \end{cases}$$

Using the comparaison principle, we deduce that for all $h \geq 0$, for all $x \in (-l,-r)$, we have that

$$w^{l,R}(x-h) - w^{l,R}(x) \geq -p_-^\sigma h - 2C.$$

Finally, for γ_0 and σ small enough, we can set $p_-^\sigma = \overline{p}_1 + \gamma$. $\square$

Proof of Theorem 3.4.1. The proof is performed in two steps.
Step 1 : proof of i) and ii). The goal is to pass to the limit as $l \to +\infty$ and then as $R \to +\infty$. There exists $l_n \to +\infty$, such that

$$m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty,$$

the convergence being locally uniform. We also define

$$\begin{aligned} \overline{w}^R(x) &= \limsup_{l_n \to +\infty}{}^* \left(w^{l_n,R} - w^{l_n,R}(0)\right), \\ \underline{w}^R(x) &= \liminf_{l_n \to +\infty}{}_* \left(w^{l_n,R} - w^{l_n,R}(0)\right). \end{aligned}$$

Thanks to (3.5.5), we know that $\overline{w}^R$ and $\underline{w}^R$ are finite and satisfy

$$m^R - C \leq \underline{w}^R \leq \overline{w}^R \leq m^R + C.$$

By stability of viscosity solutions, $\overline{w}^R - 2C$ and $\underline{w}^R$ are respectively a sub and a super-solution of

$$Q_R(x, [w^R], w_x^R) = \overline{A}_R \quad \text{for } x \in \mathbb{R} \tag{3.5.11}$$

Therefore, using Perron's method, we can construct a solution w^R of (3.5.11) with $m^R, \overline{A}^R$ and w^R satisfying

$$\begin{cases} |m^R(x) - m^R(y)| \leq C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \leq C & \text{for } x \in \mathbb{R} \times \mathbb{R}, \\ H_0 \leq \overline{A}_R \leq 0. \end{cases} \tag{3.5.12}$$

Using Proposition 3.5.3, if $\overline{A} > H_0$, we know that there exists γ_0 and $C > 0$, such that for all $\gamma \in (0, \gamma_0)$,

$$\begin{cases} w^R(x-h) - w^R(x) \geq (-\overline{p}_1 - \gamma)h - C & \text{for all } x \leq -r,\ h \geq 0, \\ w^R(x+h) - w^R(x) \geq (\overline{p}_2 - \gamma)h - C & \text{for all } x \geq r,\ h \geq 0. \end{cases} \tag{3.5.13}$$

Passing to the limit as $R \to +\infty$ and proceeding as above, the proof is complete.

Step 2 : proof of iii). Using (3.4.5), we have that

$$w^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Therefore, we can find a sequence $\varepsilon_n \to 0$, such that

$$w^{\varepsilon_n} \to W \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$. Like in [GIM15](Appendix A.1), we have that

$$\overline{H}_1(W_x) = \overline{A} \quad \text{for } x < 0 \quad \text{and} \quad \overline{H}_2(W_x) = \overline{A} \quad \text{for } x > 0.$$

For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

$$W_x \geq \overline{p}_2 - \gamma,$$

where we have used (3.4.5). Therefore we get

$$W_x = \overline{p}_2 \quad \text{for } x > 0,$$

Similarly, we get $W_x = \overline{p}_1$ for $x < 0$. This ends the proof of Theorem 3.4.1.

□

3.6 Proof of convergence

In this section, we will prove our homogenization result. Classicaly, the proof relies on the existence of correctors. We will just prove the convergence result at the junction point since at any other point, the proof is classical using that $v = 0$ is a corrector, see [FSZ17b].

Proof of Theorem 3.3.2. We introduce

$$\overline{u}(t,x) = \limsup_{\varepsilon\to 0}{}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon. \tag{3.6.1}$$

Let us prove that $\overline{u}$ is a sub-solution of (3.3.6) at the point 0, (the proof for $\underline{u}$ is similar and we skip it). The definition of viscosity solution for Hamilton-Jacobi equation is presented in Section 2 in [IM13]. We argue by contradiction and assume that there exist a test function $\Psi \in \mathcal{C}^1(J_\infty)$ such that

$$\begin{cases} \overline{u}(\bar t, 0) = \Psi(\bar t, 0) & \\ \overline{u} \leq \Psi & \text{on } \mathcal{Q}_{\bar r,\bar r}(\bar t, 0) \quad \text{with } \bar r > 0 \\ \overline{u} \leq \Psi - 2\eta & \text{outside } \mathcal{Q}_{\bar r,\bar r}(\bar t, 0) \quad \text{with } \eta > 0 \\ \Psi_t(\bar t, 0) + F_{\overline{A}}(\Psi_x(\bar t, 0^-), \Psi_x(\bar t, 0^+)) = \theta & \text{with } \theta > 0. \end{cases} \tag{3.6.2}$$

According to [IM13], we may assume that the test function has the following form

$$\Psi(t,x) = g(t) + \overline{p}_1 x 1_{\{x<0\}} + \overline{p}_2 x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{\bar r, 2\bar r}(\bar t, 0), \tag{3.6.3}$$

The last line in condition (3.6.2) becomes

$$g'(\bar{t}) + F_{\overline{A}}(\overline{p}_1, \overline{p}_2) = g'(\bar{t}) + \overline{A} = \theta. \tag{3.6.4}$$

Let us consider w the solution of (3.4.2) provided by Theorem 3.4.1, and let us denote

$$\Psi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0), \\ \Psi(t,x) & \text{outside } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0). \end{cases} \tag{3.6.5}$$

We claim that Ψ^ε is a viscosity solution on $Q_{\bar{r},\bar{r}}(\bar{t},0)$ of the following problem,

$$\Psi_t^\varepsilon + \left(\tilde{M}_1^\varepsilon\left[\frac{\Psi^\varepsilon}{\varepsilon}(t,\cdot)\right](x)\varphi\left(\frac{x}{\varepsilon}\right) + \tilde{M}_2^\varepsilon\left[\frac{\Psi^\varepsilon}{\varepsilon}(t,\cdot)\right](x)\left(1-\varphi\left(\frac{x}{\varepsilon}\right)\right)\right)\cdot|\Psi_x^\varepsilon| \geq \frac{\theta}{2}.$$

Indeed, let h be a test function touching φ^ε from below at $(t_1,x_1) \in \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0)$, so we have that the function $\chi(y) = \frac{1}{\varepsilon}(h(t_1,\varepsilon y) - g(t_1))$ touches w from below at $\frac{x_1}{\varepsilon}$ which implies that

$$\left(\tilde{M}_1[w]\left(\frac{x_1}{\varepsilon}\right)\varphi\left(\frac{x_1}{\varepsilon}\right) + \tilde{M}_2[w]\left(\frac{x_1}{\varepsilon}\right)\left(1-\varphi\left(\frac{x_1}{\varepsilon}\right)\right)\right)\cdot|h_x(t_1,x_1)| \geq \overline{A}. \tag{3.6.6}$$

Using (3.6.4) and the fact that $h_t(t_1,x_1) = g'(t_1)$ and computing (3.6.6), we get the desired result.

Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \Psi = g(t) + \overline{p}_1 x 1_{\{x<0\}} + \overline{p}_2 x 1_{\{x>0\}} \quad \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0)\backslash\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0).$$

Using the fact that $w^\varepsilon \to W$, and using (3.4.7), we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \Psi^\varepsilon \quad \text{on } \mathcal{Q}_{\bar{r},2\bar{r}}(\bar{t},0)\backslash\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0).$$

Combining this with (3.6.5), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \Psi^\varepsilon \quad \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0).$$

By the comparison principle on bounded subsets the previous inequality holds in $\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality at $(\bar{t},0)$, we obtain the following contradiction

$$\overline{u}(\bar{t},0) + \frac{\eta}{2} \leq \Psi(\bar{t},0) = \overline{u}(\bar{t},0).$$

□

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Chapitre 4

Homogenization of second order discrete model with local perturbation and application to traffic flow

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Abstract

The goal of this chapter is to derive a traffic flow macroscopic model from a second order microscopic model with a local perturbation. At the microscopic scale, we consider a Bando model of the type following the leader, i.e. the acceleration of each vehicle depends on the distance of the vehicle in front of it. We consider also a local perturbation like an accident at the roadside that slows down the vehicles. After rescaling, we prove that the "cumulative distribution functions" of the vehicles converges towards the solution of a macroscopic homogenized Hamilton-Jacobi equation with a flux limiting condition at junction which is equivalent to LWR (Lighthill-Whitham-Richards) model.

AMS Classification : 35D40, 90B20, 35B27, 35F20, 45K05.

Keywords : specified homogenization, Hamilton-Jacobi equations, integro-differential operators, Slepčev formulation, viscosity solutions, traffic flow, microscopic models, macroscopic models.

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4.1 Introduction

The modelling and simulation of traffic flow is a challenging task in particular in order to design infrastructure. Indeed, there are some examples in which the construction of a new infrastructure did not improve the traffic. For example, in Stuttgart, Germany, after investments into the road network in 1969, the traffic situation did not improve until a section of newly build road was closed for traffic again (see [Knö69]). This is known as the Braess' paradox. In the past years, a lot of work has been done concerning the modelling and simulation of traffic flows problems.

Traffic flow can be modelled at different scales depending on the level of details one wants to observe : the microscopic scale (describes the dynamics of each of the vehicles), the macroscopic scale (describes the dynamics of the density of vehicles) and the mesoscopic scale (describes the dynamics of the density of vehicles but the car-to-car interactions are not lost).

Microscopic models are considered more justifiable because the behaviour of every single vehicle can be described with high precision whereas macroscopic models are based on assumptions which are less verifiable. Another way to justify macroscopic models is to derive them from microscopic models by rescaling arguments.

The problem of deriving macroscopic models from microscopic ones has already been studied for models of the type following the leader (i.e. the velocity or the acceleration of each vehicle depends only on the distance to the vehicle in front of it). We refer for example to [AKRM02, DFR15, Hel98, HP10] where the authors rescaled the empirical measure and obtained a scalar conservation law (LWR model). In particular, passing from microscopic to macroscopic model for second-order models was instead investigated in [AKRM02, Gre01], where the Aw-Rascle model is derived as the limit of a second order follow-the-leader model.

In this chapter we establish a connection between a car-following model and a fluid-dynamic model. This result is a generalization of the results of [FSZ17b] to a second order microscopic model. We consider a second order microscopic model of *follow-the-leader* type with a local perturbation. In such model, the whole traffic flow is determined by the dynamics of the very first vehicle (the *leader*). We will establish a connection between this second order discrete model and a macroscopic model equivalent to a LWR model. The idea is to rescale the microscopic model, which describes the dynamics of each vehicle individually, in order to get a macroscopic model which describes the dynamics of density of vehicles.

The model we study here is similar to the one considered in [FS15], but in our work, as in [FSZ17b], we assume that there is a local perturbation (located at the origin for example) that slows down the vehicles and we want to understand how this local perturbation influences the macroscopic dynamics. Due to this perturbation, it is natural to get an Hamilton-Jacobi equation with a junction condition at the origin and an effective flux limiter. Further, our result is stronger than the one in [FSZ17b] because our microscopic model is a second order model which is more realistic than the first order model considered in the last chapter. From a mathematical point of view the fact of considering a second order model presents many technical difficulties. First, we need to consider a system of two non-local PDEs instead of a single equation [FIM12b, FS15]. Moreover, the two functions that we consider have to satisfy certain properties that derive from the physical characteristics of the microscopic model and those properties need to be proven for the system of non-local PDEs which is more complicated in the case of a second order model than in the case of a first order model.

Paper organization. The chapter is organized as follows. In Section 4.2, we present the microscopic model for which we will present an homogenization result. In Section 4.3, we inject the system of ODEs into a system of PDEs and we present our main results. Section 4.3.3 is dedicated to the definition of the non-local operators which appear in the PDEs given in Section 4.3. In Section 4.4, we introduce the notion of viscosity solutions for the considered problems and give stability, existence and uniqueness results. In Sections 4.5 and 4.6 we present the correctors necessaries for the proof of convergence which is located in Section 4.7. Section 4.8 contains the proof of existence of correctors for the junction, where we use the idea developed in [AT15, GIM15] and in the lectures of Lions at the "College de France" [Lio14], which consists in constructing correctors on truncated domains. In Section 4.9 we show the link between the system of ODEs and the system of PDEs which proof is in Appendix 4.11. Finally in Appendix 4.10 we analyse the properties of the microscopic model.

4.2 A first main result

In this chapter, we are interested in a second order microscopic model that can simulate the presence of a local perturbation. In order to do that, we considered a modified version of the model introduced by Bando *et al* in [BHN+95]. More precisely, we consider a "follow-the-leader" model of the following form

$$\ddot{U}_j(t) = a\left(V(U_{j+1}(t) - U_j(t)) \cdot \phi\left(U_j(t)\right) - \dot{U}_j(t)\right), \tag{4.2.1}$$

where U_j denotes the position of the j-th vehicle, $\dot{U}_j$ its velocity and $\ddot{U}_j$ its acceleration. The function ϕ simulates the presence of a local perturbation located at the origin and we denote by r its radius of influence. In this model, a and V represent respectively the drivers sensibility and the optimal velocity function. We make the following assumptions on V, ϕ and on the coefficient a.

Assumption (A)

- (A1) $V : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative.
- (A2) V is non-decreasing on $\mathbb{R}$.
- (A3) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V(h) = 0$.
- (A4) There exists $h_{max} \in (h_0, +\infty)$ such that for all $h \geq h_{max}$, $V(h) = V(h_{max}) =: V_{max}$.
- (A5) There exists a real $p_0 \in [-1/h_0, 0)$ such that the function $p \mapsto pV(-1/p)$ is decreasing on $[-1/h_0, p_0)$ and increasing on $[p_0, 0)$.

- (A6) The function $\phi : \mathbb{R} \to (0,1]$ is Lipschitz continuous and $\phi(x) = 1$ for $|x| \geq r$. We denote by $\phi_0 = \min\limits_{x\in[-r,r]} \phi(x) > 0$.
- (A7)(Monotonicity). $a \geq 4\,||V'||_\infty\,||\phi||_\infty + 4\,||\phi'||_\infty\,||V||_\infty$.

Remark 4.2.1 (Remark on (A6))**.** *In the case $\phi = 0$ on an open interval (therefore $\phi_0 = 0$) all the vehicles left of the perturbation would come to a full stop. This case lacks any interest and therefore we can assume that $\phi_0 > 0$.*

Remark 4.2.2 (Remark on (A7))**.** *Assumption (A7) yields that for all $(b,x) \in \mathbb{R}^2$, the function*

$$f : z \mapsto \frac{a}{2}z - 2V\left(b+z\right)\phi\left(x-z\right)$$

is non-decreasing. This result is particularly important later in the chapter because it implies that the systems we consider later in this work are monotone in the sense of Ishii and Koike [IK91], which will imply the uniqueness of the solution we consider.

As we said in the introduction, in order to obtain an homogenization result for (4.3.1), we will inject the system of ODEs into a system of PDEs. To do so, we proceed as in [FIM09b, FSZ17b] by introducing the rescaled "cumulative distribution function", which is the primitive of the rescaled empirical measure, defined by,

$$\rho^\varepsilon(t,y) = -\varepsilon\left(\sum_{i\geq 0} F\left(y - \varepsilon U_i\left(t/\varepsilon\right)\right) + \sum_{i<0}\left(-1 + F\left(y - \varepsilon U_i\left(t/\varepsilon\right)\right)\right)\right) \tag{4.2.2}$$

with

$$F(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{4.2.3}$$

The macroscopic model

We define $\overline{H} : \mathbb{R} \to \mathbb{R}$, by

$$\overline{H}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V\left(\dfrac{-1}{p}\right)\cdot|p| & \text{for } -k_0 \leq p < 0, \\ p & \text{for } p \geq 0. \end{cases} \tag{4.2.4}$$

Note that such $\overline{H}$ is continuous, coercive and because of (A5), there exists a unique point $p_0 \in [-k_0, 0]$ such that

$$\begin{cases} \overline{H} & \text{is decreasing on } (-\infty, p_0) \\ \overline{H} & \text{is increasing on } (p_0, +\infty), \end{cases} \tag{4.2.5}$$

and we denote by

$$H_0 := \overline{H}(p_0) = \min_{p\in\mathbb{R}} \overline{H}(p) < 0. \tag{4.2.6}$$

We want to show that the rescaled "cumulative distribution function" converges to the solution of the following macroscopic model.

$$\begin{cases} u^0_t + \overline{H}(u^0_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(-\infty,0) \\ u^0_t + \overline{H}(u^0_x) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times(0,+\infty) \\ u^0_t + F_{\overline{A}}(u^0_x(t,0^-), u^0_x(t,0^+)) = 0 & \text{for} \quad (t,x) \in (0,+\infty)\times\{0\} \\ u^0(0,x) = u_0(x) & \text{for} \quad x\in\mathbb{R}. \end{cases} \tag{4.2.7}$$

where $\overline{A}$ has to be determined and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_1,p_2) = \max\left(\overline{A}, \overline{H}^+(p_1), \overline{H}^-(p_2)\right), \tag{4.2.8}$$

with

$$\overline{H}^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \le p_0 \\ \overline{H}(p_0) & \text{if } p \ge p_0 \end{cases} \quad \text{and} \quad \overline{H}^+(p) = \begin{cases} \overline{H}(p_0) & \text{if } p \le p_0 \\ \overline{H}(p) & \text{if } p \ge p_0. \end{cases} \tag{4.2.9}$$

The initial condition u_0 is a function that satisfies

$$-k_0 \le (u_0)_x \le 0 \quad \text{and for all } \varepsilon > 0 \quad \rho^\varepsilon(0,x) = \left\lfloor \frac{u_0(x)}{\varepsilon} \right\rfloor. \tag{4.2.10}$$

According to [IM13], for all $\overline{A}\in\mathbb{R}$, there exists a unique solution u^0 of (4.2.7).

Remark 4.2.3. *We notice that in the case of traffic flow, (4.2.7) is equivalent (deriving in space) to a LWR model (see [LW55, Ric56]) with a flux limiting condition at the origin. In fact, the fundamental diagram of the model is $pV(1/p)$ and $-u^0_x$ corresponds to the density of vehicles.*

Passage from a microscopic to a macroscopic model

The main result of this chapter is the following convergence result.

Theorem 4.2.4 (Passage from a microscopic to a macroscopic model)**.** *Assume (A). There exists a unique $\overline{A} \in [H_0, 0]$ such that the function ρ^ε defined by* (4.2.2) *converges locally uniformly towards the unique solution of* (4.2.7)*.*

4.3 Main results

4.3.1 Injecting the system of ODEs into a system of PDEs

In the rest of the chapter, we will work with an equivalent formulation of (4.2.1). We borrow the idea from [FIM09a, FIM12b, FS15] and consider for all $j \in \mathbb{Z}$,

$$\Xi_j(t) = U_j(t) + \frac{1}{\alpha}\dot{U}_j(t) \quad \text{with} \quad \alpha = \frac{a}{2}.$$

Using this new function, we obtain the following system of ODEs equivalent to (4.2.1) for all $j \in \mathbb{Z}$, for all $t \in (0, +\infty)$,

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \\ \dot{\Xi}_j(t) = \alpha\left(U_j(t) - \Xi_j(t)\right) + 2V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right). \end{cases} \tag{4.3.1}$$

In Appendix 4.10, we give some properties of system 4.3.1, such as maximal velocities of the vehicles and minimal and maximal distance between two consecutive vehicles.

We now introduce the "cumulative distribution function" for $(\Xi_j)_j$, defined by

$$\sigma^\varepsilon(t, y) = -\varepsilon\left(\sum_{i\geq 0} F\left(y - \varepsilon\Xi_i\left(t/\varepsilon\right)\right) + \sum_{i<0}\left(-1 + F\left(y - \varepsilon\Xi_i\left(t/\varepsilon\right)\right)\right)\right). \tag{4.3.2}$$

Under assumption (A), $(\rho^\varepsilon, \sigma^\varepsilon)$ is a discontinuous viscosity solution (see Theorem 4.3.4) of the following non-local equation, for all $(t, x) \in (0, +\infty) \times \mathbb{R}$,

$$\begin{cases} u_t^\varepsilon + M^\varepsilon\left(\frac{u^\varepsilon}{\varepsilon}(t, x), \left[\frac{\xi^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |u_x^\varepsilon| = 0 \\ \xi_t^\varepsilon + L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\xi^\varepsilon}{\varepsilon}(t, x), \left[\frac{u^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) \cdot |\xi_x^\varepsilon| = 0. \end{cases} \tag{4.3.3}$$

The definition of M^ε and L^ε is postponed to the next section. We submit equation (4.3.3) to the following initial condition. For all $x \in \mathbb{R}$,

$$\begin{cases} u^\varepsilon(0, x) = u_0(x) \\ \xi^\varepsilon(0, x) = \xi_0^\varepsilon(x). \end{cases} \tag{4.3.4}$$

We also assume that the initial condition satisfies the following assumption.

(A0) (Gradient bound) Let $k_0 = 1/h_0$. The functions u_0 and ξ_0^ε are Lipschitz continuous functions, such that

$$-k_0 \leq (u_0)_x \leq 0 \tag{4.3.5}$$

$$-k_0 \leq (\xi_0^\varepsilon)_x \leq 0, \tag{4.3.6}$$

and

$$0 \leq \xi_0^\varepsilon(x) - u_0(x) \leq \varepsilon. \tag{4.3.7}$$

Remark 4.3.1. *The initial conditions u_0 and ξ_0^ε are "regular" functions such that for all $\varepsilon > 0$ we have*

$$|\rho^\varepsilon(0,x) - u_0(x)| \leq f(\varepsilon) \quad \text{and} \quad |\sigma^\varepsilon(0,x) - \xi_0^\varepsilon(x)| \leq g(\varepsilon), \tag{4.3.8}$$

with $f(\varepsilon), g(\varepsilon) \to 0$ as ε goes to 0. For $\varepsilon = 1$, the conditions on the gradients translate the fact that at the initial time there is at least h_0 meters between two consecutive vehicles. In the rest of the paper we are interested in the behaviour of ρ^ε and σ^ε as ε goes to 0. This in fact translates to studying the behaviour of the traffic as the number of vehicles per unit length goes to infinity. For $\varepsilon = 1$ condition (4.3.7) translate the fact that at initial time the velocity of the vehicles must be bounded so the ordering of the vehicles is kept. The fact that ξ_0^ε depends on ε comes from the rescaling. In fact, given that σ^ε is the "cumulative distribution function" of $(\Xi_j)_j$ which are defined using the velocity of the vehicles, an ε appears multiplying the velocity when rescaling (see [FS15, Remark 1.2]). Therefore, ξ_0^ε tends to u_0 as ε goes to zero. Finally, to simplify the notations, we denote by $\xi_0 = \xi_0^\varepsilon$ for $\varepsilon = 1$.

4.3.2 Convergence result

Theorem 4.2.4 is a consequence of the following theorems. The proof of Theorem 4.3.2 is postponed until Section 4.7 and the proof of Theorem 4.3.4 is postponed until Section 4.9.

Theorem 4.3.2 (Junction condition by homogenization)**.** *Assume (A) and (A0). For $\varepsilon > 0$, let $(u^\varepsilon, \xi^\varepsilon)$ be the solution of (4.3.3)-(4.3.4). Then there exists $\bar{A} \in [H_0, 0]$ such that u^ε and ξ^ε converge locally uniformly to the unique viscosity solution u^0 of (4.2.7).*

Remark 4.3.3. *This result is formally obtained by functions U_j and Ξ_j. In fact, we can prove that*

$$0 \leq U_i' \leq V_{max}.$$

Therefore recalling the definition of Ξ_j, we obtain that

$$0 \leq \Xi_j^\varepsilon(t) - U_j^\varepsilon(t) \leq \varepsilon \frac{V_{max}}{\alpha}$$

where

$$\begin{cases} \Xi_j^\varepsilon(t) = \varepsilon \Xi_j\left(\dfrac{t}{\varepsilon}\right) \\ U_j^\varepsilon(t) = \varepsilon U_j\left(\dfrac{t}{\varepsilon}\right). \end{cases}$$

Taking ε to zero, we can see that Ξ_j^ε and U_j^ε converge to the same limit.

Theorem 4.3.4 (Junction condition by homogenization : application to traffic flow)**.** *Assume (A) and that at initial time $(U_i(0), \Xi_i(0))_i$ satisfies*

$$0 \leq \Xi_i(0) - U_i(0) \leq \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \geq h_0,$$

and

$$U_{i+1}(0) - U_i(0) \le h_{max}.$$

We also assume that there exists a constant $R > 0$ such that, for all $i \in \mathbb{Z}$, if $|U_i(0)| \ge R$

$$U_{i+1}(0) - U_i(0) = h \tag{4.3.9}$$

and if $|\Xi_i(0)| \ge R$

$$\Xi_{i+1}(0) - \Xi_i(0) = h, \tag{4.3.10}$$

with $h \in [h_0, h_{max}]$. We define two function u_0 and ξ_0^ε (satisfying (A0)) by $u_0(x) = \xi_0^\varepsilon(x) = -x/h$ for all $x \in \mathbb{R}$. Then there exists a unique $\overline{A} \in [H_0, 0]$ such that the functions ρ^ε and σ^ε defined by (4.2.2) *and* (4.3.2) *converge locally uniformly towards the unique solution u^0 of* (4.2.7)*.*

The following theorem ensures that when we use (4.2.7) we only evaluate the function $\overline{H}$ in the interval $[-k_0, 0]$. The proof of Theorem 4.3.5 is postponed until Section 4.7.

Theorem 4.3.5 (Gradient bound)**.** *Assume (A0)-(A). Let u^0 be the unique solution of* (4.2.7)*, then we have for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$-k_0 \le u_x^0 \le 0,$$

with k_0 defined in (A0).

4.3.3 Definition of the non-local operators

In this section, we clarify equation (4.3.3). We will give the definition of M and L, and then the definition of M^ε and L^ε. To do this, we first introduce the following functions.

$$E(z) = \begin{cases} -\alpha & \text{if } z \ge 0 \\ 0 & \text{if } z < 0, \end{cases} \quad F(z) = \begin{cases} 1 & \text{if } z < 0 \\ 0 & \text{if } z \ge 0, \end{cases} \quad I(z) = \begin{cases} 1 & \text{if } z \ge -1 \\ 0 & \text{if } z < -1, \end{cases}$$

$$\tilde{E}(z) = \begin{cases} -\alpha & \text{if } z > 0 \\ 0 & \text{if } z \le 0, \end{cases} \quad \tilde{F}(z) = \begin{cases} 1 & \text{if } z \le 0 \\ 0 & \text{if } z > 0, \end{cases} \quad \text{and} \quad \tilde{I}(z) = \begin{cases} 1 & \text{if } z > -1 \\ 0 & \text{if } z \le -1. \end{cases}$$

For $x, p \in \mathbb{R}$, we then define the following non-local operators

$$\begin{aligned} M_p\left(U(x), [\Sigma]\right)(x) &= \int_0^D E(\Sigma(x+z) - U(x) + pz)dz, \\ K_p\left(\Sigma(x), [U]\right)(x) &= \int_0^D F(U(x-z) - \Sigma(x) - pz)dz, \\ N_p\left(\Sigma(x), [U]\right)(x) &= \int_0^D I(U(x+z) - \Sigma(x) + pz)dz, \end{aligned}$$

with $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$ (see Appendixes 4.10 and 4.11 for more details on where the constant D comes from). We can now define L_p. For $x, y \in \mathbb{R}$,

$$\begin{aligned} L_p\left(y, \Sigma(x), [U]\right)(x) &= \alpha K_p\left(\Sigma(x), [U]\right)(x) \\ &- 2V\Big(N_p\left(\Sigma(x), [U]\right)(x) + K_p\left(\Sigma(x), [U]\right)(x)\Big)\phi\left(y - K_p\left(\Sigma(x), [U]\right)(x)\right). \end{aligned} \tag{4.3.11}$$

In the same way, we define $\tilde{M}_p, \tilde{K}_p$ and $\tilde{N}_p$ by replacing E, F and I respectively by $\tilde{E}, \tilde{F}$ and $\tilde{I}$. Similary,

$$\begin{aligned} \tilde{L}_p\left(y, \Sigma(x), [U]\right)(x) &= \alpha \tilde{K}_p\left(\Sigma(x), [U]\right)(x) \\ &- 2V\Big(\tilde{N}_p\left(\Sigma(x), [U]\right)(x) + \tilde{K}_p\left(\Sigma(x), [U]\right)(x)\Big) \cdot \phi\Big(y - \tilde{K}_p\left(\Sigma(x), [U]\right)(x)\Big). \end{aligned} \tag{4.3.12}$$

For $p = 0$, we define

$$M(U(x), [\Sigma])(x) := M_0(U(x), [\Sigma]) = \int_0^D E(\Sigma(x+z) - U(x))dz, \tag{4.3.13}$$

$$K\left(\Sigma(x), [U]\right)(x) := K_0\left(\Sigma(x), [U]\right)(x) = \int_0^D F(U(x-z) - \Sigma(x))dz, \tag{4.3.14}$$

$$N\left(\Sigma(x), [U]\right)(x) := N_0\left(\Sigma(x), [U]\right)(x) = \int_0^D I(U(x+z) - \Sigma(x))dz, \tag{4.3.15}$$

and

$$\begin{aligned} L\left(y, \Sigma(x), [U]\right)(x) &= \alpha K\left(\Sigma(x), [U]\right)(x) \\ &- 2V\Big(N\left(\Sigma(x), [U]\right)(x) + K\left(\Sigma(x), [U]\right)(x)\Big) \cdot \phi\left(y - K\left(\Sigma(x), [U]\right)(x)\right). \end{aligned} \tag{4.3.16}$$

Remark 4.3.6 (Remarks on the non-local operators)**.** *First let us notice that the domain of integration in the non-local operators is bounded by a constant $D := h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$, this comes from the fact that the velocities of the vehicles as well as the distance between two consecutive vehicles from model 4.3.1 are bounded (see Appendix 4.10). In particular, there exists a constant $M_0 > 0$ (independent of p), such that we have the following bounds on the non-local operators,*

$$\begin{aligned} -M_0 \leq -\alpha D \leq M_p(U(x), [\Sigma])(x) &\leq 0, \\ M_0 \geq D \geq K_p(\Sigma(x), [U])(x) &\geq 0, \\ M_0 \geq D \geq N_p(\Sigma(x), [U])(x) &\geq 0, \\ M_0 \geq \alpha D \geq L_p(y, \Sigma(x), [U])(x) &\geq -2V_{max} \geq -M_0, \end{aligned}$$

with $M_0 = \max(2V_{max}, \alpha D, D)$.

Finally, we would like to point out that given the fact that the function V is non-decreasing (assumption (A2)) and that the function $F \geq 0$ and therefore $K(\Sigma, [U])(x) \geq 0$, we have

$$L(y, \Sigma(x), [U])(x) \geq -2V\Big(N\left(\Sigma(x), [U]\right)(x)\Big). \tag{4.3.17}$$

Finally, we introduce for $\varepsilon > 0$,

$$M^\varepsilon\left(U(x),[\Sigma]\right)(x) = \int_0^D E(\Sigma(x+\varepsilon z) - U(x))dz, \tag{4.3.18}$$

$$K^\varepsilon\left(\Sigma(x),[U]\right)(x) = \int_0^D F(U(x-\varepsilon z) - \Sigma(x))dz, \tag{4.3.19}$$

$$N^\varepsilon\left(\Sigma(x),[U]\right)(x) = \int_0^D I(U(x+\varepsilon z) - \Sigma(x))dz, \tag{4.3.20}$$

$$\tag{4.3.21}$$

and

$$\begin{aligned} L^\varepsilon\left(y,\Sigma(x),[U]\right)(x) &= \alpha K^\varepsilon\left(\Sigma(x),[U]\right)(x) \\ &- 2V\Big(N^\varepsilon\left((\Sigma(x),[U]\right)(x) + K^\varepsilon\left(\Sigma(x),[U]\right)(x)\Big)\cdot\phi\left(y - K^\varepsilon\left(\Sigma(x),[U]\right)(x)\right). \end{aligned} \tag{4.3.22}$$

The bounds provided by Remark 4.3.6 remain valid for the non-local operators depending on $\varepsilon > 0$.

Remark 4.3.7 (Lagrangian formulation)**.** *Another way to treat this problem is to consider a Lagrangian formulation, like in [FS15], considering the functions,*

$$u(t,y) = U_{\lfloor y\rfloor}(t) \quad \textit{and} \quad \xi(t,y) = \Sigma_{\lfloor y\rfloor}(t).$$

The couple (u,ξ) *satisfies for all* $(t,y)\in[0,T]\times\mathbb{R}$

$$\begin{cases} u_t(t,y) = \alpha\left(\xi(t,y) - u(t,y)\right) \\ \xi_t(t,y) = \alpha\left(u(t,y) - \xi(t,y)\right) + 2V\left(u(t,y+1) - u(t,y)\right)\cdot\phi(u(t,y)) \\ u(0,y) = u_0(y) \\ \xi(0,y) = \xi_0^\varepsilon(y). \end{cases}$$

We note that the system we obtain is much more simple. Nevertheless, the difficulty with this formulation is that the function ϕ *is evaluated at* $u(t,y)$ *and not at a physical point of the road. At the macroscopic scale, we then expect to get a junction condition located at* $u = 0$*. The notion of junction in this case is not well defined and this is why we use the formulation (4.3.3)(where the perturbation function is evaluated at a point of the road). This will allow us to use the results of Imbert and Monneau [IM13] concerning quasi-convex Hamiltonians with a junction condition.*

4.4 Viscosity Solutions

This section is devoted to the definition and useful results for viscosity solutions of the problems considered in this chapter. The reader is referred to the user's guide of Crandall, Ishii, Lions [CIL92] and the book of Barles [Bar94] for an introduction to viscosity solutions. In order to give a general definition, we will give the definition of viscosity solutions for the following equation, with $p \in \mathbb{R}$, and for all $(t,x) \in (0,+\infty)\times\mathbb{R}$,

$$\begin{cases} u_t + \psi(x)\cdot M_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+u_x| + (1-\psi(x))\cdot\overline{H}\left(u_x\right) = 0 \\ \xi_t + \psi(x)\cdot L_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\xi_x| + (1-\psi(x))\cdot\overline{H}\left(\xi_x\right) = 0 \\ u(0,x) = u_0(x) \\ \xi(0,x) = \xi_0(x), \end{cases} \tag{4.4.1}$$

with $\psi : \mathbb{R} \to [0,1]$ a Lipschitz continuous function. We also use the following notations for the upper and lower semi-continuous envelopes of a locally bounded function u :

$$u^*(t,x) = \limsup_{s\to t, y\to x} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{s\to t, y\to x} u(s,y).$$

4.4.1 Definitions

Definition 4.4.1 (Viscosity solutions for (4.4.1))**.** Let $T>0$. Let $u : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ and $\xi : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) functions. We say that (u,ξ) is a viscosity sub-solution (resp. super-solution) of (4.4.1) on $[0,T]\times\mathbb{R}$ if $u(0,x)\le u_0(x)$ and $\xi(0,x)\le\xi_0(x)$ (resp. $u(0,x)\ge u_0(x)$ and $\xi(0,x)\ge\xi_0(x)$) and for all $(t,x)\in(0,T)\times\mathbb{R}$, and for any test function $\varphi\in C^1((0,T)\times\mathbb{R})$ such that $u-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have

$$\varphi_t + \psi(x)\cdot M_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+\varphi_x| + (1-\psi(x))\cdot\overline{H}(\varphi_x) \le 0,$$
$$(resp.\quad \varphi_t + \psi(x)\cdot \tilde{M}_p\left(u(t,x),[\xi(t,\cdot)]\right)(x)\cdot|p+\varphi_x| + (1-\psi(x))\cdot\overline{H}(\varphi_x) \ge 0),$$

and for all $(t,x)\in(0,T)\times\mathbb{R}$ and any test function $\varphi\in C^1((0,T)\times\mathbb{R})$ such that $\xi-\varphi$ attains a local maximum (resp. a local minimum) at the point (t,x), we have

$$\varphi_t + \psi(x)\cdot L_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\varphi_x| + (1-\psi(x))\cdot\overline{H}(\varphi_x) \le 0,$$

$$(\text{resp.}\quad \varphi_t + \psi(x)\cdot \tilde{L}_p\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|p+\varphi_x| + (1-\psi(x))\cdot\overline{H}(\varphi_x) \ge 0).$$

We say that (u,ξ) is a viscosity solution of (4.4.1) if (u^*,ξ^*) and (u_*,ξ_*) are respectively a sub-solution and a super-solution of (4.4.1).

Proposition 4.4.1 (Stability result for (4.4.1))**.** *Let (u_n,ξ_n) be a sequence of uniformly bounded upper semi-continuous functions (resp. lower semi-continuous) and let $(p_n)_n$ be such that $p_n\to p$.*

We assume that (u_n,ξ_n) is a sub-solution (resp. a super-solution) of (4.4.1) *with $p=p_n$.*

Let $(\overline{u},\overline{\xi}) = (\limsup^* u_n, \limsup^* \xi_n)$ (resp. $(\underline{u},\underline{\xi}) = (\liminf_* u_n, \liminf_* \xi_n)$).

Then $(\overline{u},\overline{\xi})$ (resp. $(\underline{u},\underline{\xi})$) is a sub-solution (resp. a super-solution) of (4.4.1).

Proof. The proof is classical and we refer to [FIM09b]. The only point to note is that both Hamiltonians in (4.4.1) are monotone with respect to the non-local operators (this is a consequence of assumption (A7) for the non-local operator K_p).

□

4.4.2 Viscosity solutions for (4.2.7)

The theory of viscosity solutions for Hamilton-Jacobi equations on networks was recently treated in several papers. We give here some results for viscosity solutions of (4.2.7) that will be used in the rest of chapter and we refer to [IM13] for the general theory and for the proofs.

Definition 4.4.2 (Class of test function for (4.2.7))**.** We denote by $J_\infty := (0,+\infty)\times\mathbb{R}$, $J_\infty^+ := (0,+\infty)\times(0,+\infty)$ and $J_\infty^- := (0,+\infty)\times(-\infty,0)$, we define a class of test function on J_∞ by

$$C^1(J_\infty) = \{\varphi \in C(J_\infty), \text{the restriction of } \varphi \text{ to } J_\infty^+ \text{ and to } J_\infty^- \text{ are } C^1\}.$$

Definition 4.4.3 (Viscosity solution for (4.2.7))**.** An upper semi-continuous (resp. lower semi-continuous) function $u : [0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (4.2.7) if $u(0,x) \le u_0(x)$ (resp. $u(0,x) \ge u_0(x)$) and for all $(t,x)\in J_\infty$ and for all $\varphi \in C^1(J_\infty)$ such that

$$u \le \varphi \quad (\text{resp. } u \ge \varphi) \text{ in a neighbourhood of } (t,x)\in J_\infty$$

and

$$u(t,x) = \varphi(t,x),$$

we have

$$\begin{aligned} \varphi_t(t,x) + \overline{H}(\varphi_x(t,x)) &\le 0 \quad (\text{resp. } \ge 0) \qquad \text{if } x \neq 0 \\ \varphi_t(t,x) + F_A(\varphi_x(t,0^-),\varphi_x(t,0^+)) &\le 0 \quad (\text{resp. } \ge 0) \qquad \text{if } x = 0. \end{aligned}$$

We say that a function u is a viscosity solution of (4.2.7) if u^* and u_* are respectively a sub-solution and a super-solution of (4.2.7).We refer to this solution as A-flux-limited solution.

Now we recall an equivalent definition (Theorem 2.5 in [IM13]) for sub and super solution at the junction. We will also consider the following problem,

$$u_t + \overline{H}(u_x) = 0 \quad \text{for } t \in (0,T) \text{ and } x \in \mathbb{R}\backslash\{0\}. \tag{4.4.2}$$

Theorem 4.4.2 (Reduction of the class of test functions)**.** *Let $\overline{H}$ given by (4.2.4) and consider $A \in [H_0,+\infty)$ with H_0 defined in (4.2.6). Given arbitrary solutions $p_\pm^A \in \mathbb{R}$ of*

$$\overline{H}\left(p_+^A\right) = \overline{H}^+\left(p_+^A\right) = A = \overline{H}^-\left(p_-^A\right) = \overline{H}\left(p_-^A\right), \tag{4.4.3}$$

let us fix any time independent test function $\phi^0(x)$ satisfying

$$\phi_x^0(0^\pm) = p_\pm^A.$$

Given a function $u : (0,T)\times\mathbb{R}\to\mathbb{R}$, the following properties hold true.

1. *If u is an upper semi-continuous sub-solution of (4.4.2) satisfying*

$$u(t,0) = \limsup_{(t,y)\to(t,0),y\in J_i^*} u(s,y), \tag{4.4.4}$$

then u is a H_0-flux limited sub-solution.

2. *Given $A > H_0$ and $t_0 \in (0,T)$, if u is an upper semi-continous sub-solution of (4.4.2) satisfying* (4.4.4) *and if for any test function φ touching u from above at* $(t_0, 0)$ *with*

$$\varphi(t,x) = \psi(t) + \phi^0(x), \tag{4.4.5}$$

for some $\psi \in C^1(0,+\infty)$, we have

$$\varphi_t + F_A\left(\varphi_x\left(t_0,0^-\right), \varphi_x\left(t_0,0^+\right)\right) \le 0 \quad \text{at } (t_0,0),$$

then u is a A-flux limited sub-solution at $(t_0, 0)$*.*

3. *Given $t_0 \in (0,T)$, if u is a lower semi-continuous super-solution of (4.4.2) and if for any test function φ satisfying (4.4.5) touching u from above at* $(t_0, 0)$ *we have*

$$\varphi_t + F_A\left(\varphi_x\left(t_0,0^-\right), \varphi_x\left(t_0,0^+\right)\right) \ge 0 \quad \text{at } (t_0,0),$$

then u is a A-flux limited super-solution at $(t_0, 0)$*.*

4.4.3 Existence and uniqueness of viscosity solution for (4.4.1) with $p = 0$

We recall that for $p = 0$, our equation is

$$\begin{cases} u_t + M(u(t,x),[\xi(t,\cdot)])(x)\cdot|u_x| = 0 & \text{for} \quad (t,x)\in(0,+\infty)\times\mathbb{R}, \\ \xi_t + L\left(x,\xi(t,x),[u(t,\cdot)]\right)(x)\cdot|\xi_x| = 0 & \text{for} \quad (t,x)\in(0,+\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{for} \quad x\in\mathbb{R}, \\ \xi(0,x) = \xi_0(x) & \text{for} \quad x\in\mathbb{R}. \end{cases} \tag{4.4.6}$$

Lemma 4.4.3 (Existence of barriers for (4.4.6))**.** *Assume (A) and (A0). There exists a constant $K_1 > 0$ such that*

$$(u^-,\xi^-) = (u_0 - K_1 t, \xi_0 - K_1 t) \quad \text{and} \quad (u^+,\xi^+) = (u_0 + K_1 t, \xi_0 + K_1 t) \tag{4.4.7}$$

are respectively sub-solution and super-solution of (4.4.6).

Proof. We define $K_1 = M_0 k_0$. Let us prove that (u^+, ξ^+) is a super-solution of (4.4.6). In fact, we have that

$$u_t^+ + \tilde{M}(u^+(t,x),[\xi^+(t,\cdot)])(x)|u_x^+| \ge K_1 - M_0 k_0 = 0,$$

where we have used Remark 4.3.6 for the second inequality. Similarly, using that $\tilde{K} \geq 0$ and $K_1 \geq 2||V||_\infty k_0$, we have that

$$\xi_t^+ + \tilde{L}(x, \xi^+(t,x), [u^+(t,\cdot)])(x)|\xi_t^+| \geq 0.$$

The proof that (u^-, ξ^-) is a sub-solution is similar and we skip it.

□

Proposition 4.4.4 (Comparaison principle)**.** *Let $T > 0$. Assume (A)-(A0). Let (u, ξ) and (v, ζ) be respectively a sub-solution and a super-solution of (4.4.6). We also assume that there exists a constant $C > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$, we have*

$$u_0(x) - Ct \leq u(t,x) \leq u_0(x) + Ct, \qquad \xi_0(x) - Ct \leq \xi(t,x) \leq \xi_0(x) + Ct \tag{4.4.8}$$

and

$$u_0(x) - Ct \leq v(t,x) \leq u_0(x) + Ct, \qquad \xi_0(x) - Ct \leq \zeta(t,x) \leq \xi_0(x) + Ct \tag{4.4.9}$$

If

$$u(0,x) \leq v(0,x) \quad \textit{and} \quad \xi(0,x) \leq \zeta(0,x) \quad \textit{for all } x \in \mathbb{R},$$

then

$$u(t,x) \leq v(t,x) \quad \textit{and} \quad \xi(t,x) \leq \zeta(t,x) \quad \textit{for all } x \in \mathbb{R}, t \in [0,T].$$

Proof. Let us introduce

$$\overline{M} = \sup_{t \in [0,T], x \in \mathbb{R}} \max\left(u(t,x) - v(t,x), \xi(t,x) - \zeta(t,x)\right).$$

We want to prove that $\overline{M} \leq 0$. We argue by contradiction by assuming that $M > 0$.

Step 1 : test functions. We introduce the following test functions

$$\varphi(t,x,y) = u(t,x) - v(t,y) - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right)$$

and

$$\psi(t,x,y) = \xi(t,x) - \zeta(t,y) - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right),$$

with η, γ small parameters, and A a constant to be chosen later. We denote by $\Phi(t,x,y) = \max\left(\varphi(t,x,y), \psi(t,x,y)\right)$. Using (4.4.8) and (4.4.9) we have that

$$\begin{aligned}\varphi(t,x,y) &\leq u_0(x) - u_0(y) + 2CT - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right)\\ &\leq 2CT + k_0|x-y| - \frac{\eta}{T-t} - e^{At}\left(\frac{(x-y)^2}{2\varepsilon} + \gamma\frac{x^2}{2}\right).\end{aligned}$$

We have a similar result for ψ which yields that

$$\lim_{|x|,|y|\to+\infty} \Phi = -\infty.$$

Using the fact that our test functions are upper semi continuous, we can see that Φ reaches a maximum at some finite point that we denote by $(\bar{t}, \bar{x}, \bar{y}) \in [0, T) \times \mathbb{R} \times \mathbb{R}$. Classicaly we have for η and γ small enough,

$$\begin{cases} M_{\eta,\varepsilon,\gamma} = \Phi(\bar{t}, \bar{x}, \bar{y}) \geq \dfrac{\overline{M}}{2} > 0, \\ |\bar{x} - \bar{y}| \to 0 \quad \text{as} \quad \varepsilon \to 0, \\ \gamma|\bar{x}| \to 0 \quad \text{as} \quad \gamma \to 0. \end{cases}$$

Step 2 : $\bar{t} > 0$ for ε small enough. By contradiction, let us assume that Φ reaches its maximum for $\bar{t} = 0$. Let us for instance assume that $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. In this case, we have

$$0 < \frac{\overline{M}}{2} \leq u(0, \bar{x}) - v(0, \bar{y}) - \frac{\eta}{T - t} \leq k_0|\bar{x} - \bar{y}| - \frac{\eta}{T - t}.$$

Therefore, $\dfrac{\eta}{T} < k_0|\bar{x} - \bar{y}|$ and for ε small enough we get a contradiction. In the same way, we get a contradiction if we assume that $\phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$.

Step 3 : utilisation of the equation in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) \tag{4.4.10}$$

$$a + M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x})|e^{A\bar{t}} \left(p_\varepsilon + \gamma\bar{x}\right)| \leq 0 \tag{4.4.11}$$

$$b + \tilde{M}(v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)])(\bar{y})|e^{A\bar{t}} p_\varepsilon| \geq 0 \tag{4.4.12}$$

with $p_\varepsilon = \dfrac{\bar{x} - \bar{y}}{\varepsilon}$. Combining (4.4.10), (4.4.11) and (4.4.12), we obtain

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} + Ae^{A\bar{t}} \left(\frac{(\bar{x} - \bar{y})^2}{2\varepsilon} + \gamma \frac{\bar{x}^2}{2} \right) &\leq e^{A\bar{t}} p_\varepsilon \tilde{M}(v(\bar{t}, \bar{y}), [\zeta(\bar{t}, \cdot)])(\bar{y}) \\ &\quad - e^{A\bar{t}} p_\varepsilon M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) + o_\gamma, \end{aligned} \tag{4.4.13}$$

where we have used the fact that $M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x})$ is finite according to Remark 4.3.6.

We distinguish two cases.

Case 1 : there exists a subsequence γ_n such that

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \to 0 \quad \text{as } n \to +\infty.$$

In this case, taking γ going to zero in (4.4.13) yields a contradiction.

Case 2 : there exists a constant $C_\varepsilon > 0$ such that for any γ small enough we have,

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \geq C_\varepsilon.$$

Changing variables in (4.4.13) we can write

$$\begin{aligned}
\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} &\leq |e^{A\bar{t}}p_\varepsilon| \int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz \\
&\quad - |e^{A\bar{t}}p_\varepsilon| \int_{\bar{x}}^{D+x} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz + o_\gamma(1) \\
&\leq |e^{A\bar{t}}p_\varepsilon| \int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y})) - E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz \\
&\quad + |e^{A\bar{t}}p_\varepsilon| \left| \int_{\bar{y}}^{\bar{x}} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz \right| \\
&\quad + |e^{A\bar{t}}p_\varepsilon| \left| \int_{D+\bar{x}}^{D+\bar{y}} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz \right| + o_\gamma(1).
\end{aligned} \tag{4.4.14}$$

We define

$$\mathcal{A} = \Big\{ z \in \mathbb{R} : \tilde{E}\left(\zeta(\bar{t},z)-v(\bar{t},\bar{y})\right) \leq E\left(\xi(\bar{t},z)-u(\bar{t},\bar{x})\right) \Big\}.$$

The inequality $\varphi(\bar{t},\bar{x},\bar{y}) \geq \psi(\bar{t},z,z)$ yields

$$\zeta(\bar{t},z)-v(\bar{t},\bar{y}) \geq \xi(\bar{t},z)-u(\bar{t},\bar{x}) + e^{A\bar{t}}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \gamma\frac{\bar{x}^2}{2} - \gamma\frac{z^2}{2}\right).$$

This implies that

$$\mathcal{A}^c \subset \{|z| \geq R_{\varepsilon,\gamma}\} \quad \text{with } R^2_{\varepsilon,\gamma} = \frac{2}{\gamma}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \gamma\frac{\bar{x}^2}{2}\right).$$

Moreover $\tilde{R}_{\varepsilon,\gamma} = R_{\varepsilon,\gamma} - |\bar{y}| \to +\infty$ as $\gamma \to 0$ (see Da Lio *et al.* in [DLFM08, Lemma 2.5]). This implies that

$$\begin{aligned}
\int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz &= \int_{[\bar{y},D+\bar{y})\cap\mathcal{A}} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz \\
&\quad + \int_{[\bar{y},D+\bar{y})\cap\mathcal{A}^c} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz.
\end{aligned}$$

However, from Remark 4.3.6, we have that for γ small enough

$$\begin{aligned}
0 \leq \int_{[\bar{y},D+\bar{y})\cap\mathcal{A}^c} -\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}])dz &= \int_{[\bar{y},D+\bar{y}]\cap\{|z|\geq R_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz \\
&= \int_{[0,D]\cap\{|z+\bar{y}|\geq R_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t},z+\bar{y})-v(\bar{t},\bar{y}))dz \\
&\leq \int_{[0,D]\cap\{|z|\geq \tilde{R}_{\varepsilon,\gamma}\}} -\tilde{E}(\zeta(\bar{t},z+\bar{y})-v(\bar{t},\bar{y}))dz \\
&= 0.
\end{aligned}$$

We deduce that for γ small enough,

$$\int_{\bar{y}}^{D+\bar{y}} \tilde{E}(\zeta(\bar{t},z)-v(\bar{t},\bar{y}))dz = \int_{\bar{y}}^{D+\bar{y}} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz.$$

Then for γ small enough (4.4.14) implies

$$\begin{aligned}
C_{A,\eta,\varepsilon} &\leq \left|e^{A\bar{t}}p_\varepsilon\right| \left|\int_{\bar{y}}^{\bar{x}} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz + \int_{D+\bar{x}}^{D+\bar{y}} E(\xi(\bar{t},z)-u(\bar{t},\bar{x}))dz\right| + o(\gamma) \\
&\leq 2\alpha e^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{\varepsilon} + o_\gamma.
\end{aligned}$$

with $C_{A,\eta,\varepsilon} = \dfrac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\dfrac{(\bar{x}-\bar{y})^2}{2\varepsilon}$. Choosing $A = 4\alpha$, we get a contradiction.

Step 4 : utilisation of equation in the case $\Phi(\bar{t},\bar{x},\bar{y}) = \psi(\bar{t},\bar{x},\bar{y})$. By duplication of the time variable and passing to the limit, we have that there exist two real numbers $a, b \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\left(\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} + \gamma\frac{\bar{x}^2}{2}\right) \tag{4.4.15}$$

$$a + L(\bar{x},\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x}) \cdot |e^{A\bar{t}}(p_\varepsilon + \gamma\bar{x})| \leq 0 \tag{4.4.16}$$

$$b + \tilde{L}(\bar{y},\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) \cdot |e^{A\bar{t}}p_\varepsilon| \geq 0 \tag{4.4.17}$$

with $p_\varepsilon = \frac{\bar{x}-\bar{y}}{\varepsilon}$. Combining (4.4.15), (4.4.16) and (4.4.17), we obtain that

$$C_{A,\eta,\varepsilon} \leq |e^{A\bar{t}}p_\varepsilon| \left(\tilde{L}(\bar{y},\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) - L(\bar{x},\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})\right) + o_\gamma. \tag{4.4.18}$$

We recall that we defined L and $\tilde{L}$ using K and V (see (4.3.11) and (4.3.12)). Therefore, we can see that the right part of inequality (4.4.18) is finite (using Remark 4.3.6). We distinguish two cases.

Case 1 : there exists a subsequence γ_n such that

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \to 0 \quad \text{as } n \to +\infty.$$

In this case, taking γ to zero in (4.4.18) yields a contradiction.

Case 2 : there exists a constant $C_\varepsilon > 0$, such that for any γ small enough we have

$$\frac{|\bar{x}-\bar{y}|}{\varepsilon} \geq C_\varepsilon.$$

To simplify, we introduce

$$\begin{array}{ll} L = L(\bar{x}, \xi(\bar{t},\bar{x}), u[(\bar{t},\cdot)])(\bar{x}) & \tilde{L} = \tilde{L}(\bar{y}, \zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}), \\ K = K(\xi(\bar{t},\bar{x}), [u(\bar{t},\cdot)])(\bar{x}) & \tilde{K} = \tilde{K}(\zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}), \\ N = N(\xi(\bar{t},\bar{x}), [u(\bar{t},\cdot)])(\bar{x}) & \tilde{N} = \tilde{N}(\zeta(\bar{t},\bar{y}), [v(\bar{t},\cdot)])(\bar{y}). \end{array}$$

As above, we can prove

$$\tilde{K} - K \leq |\bar{x}-\bar{y}| \quad \text{and} \quad N - \tilde{N} \leq |\bar{x}-\bar{y}|.$$

We have that

$$\begin{aligned} \tilde{L} - L &= \alpha \tilde{K} - 2V\left(\tilde{N} + \tilde{K}\right)\phi\left(\bar{y} - \tilde{K}\right) - L \\ &\leq \alpha\left(K + |\bar{x}-\bar{y}|\right) - 2V\left(\tilde{N} + K + |\bar{x}-\bar{y}|\right)\phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right) - L \\ &\leq \alpha\left(K + |\bar{x}-\bar{y}|\right) - 2V\left(N + K\right)\phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right) - L \\ &\leq \alpha|\bar{x}-\bar{y}| + 2V\left(N+K\right)\left(\phi\left(\bar{x}-K\right) - \phi\left(\bar{y} - K - |\bar{x}-\bar{y}|\right)\right) \\ &\leq \alpha|\bar{x}-\bar{y}| + 2\,||V||_\infty\,||\phi'||_\infty|\bar{x}-\bar{y}|, \end{aligned} \tag{4.4.19}$$

where we have used for the first inequality the monotonicity (see Remark 4.2.2). The monotonicity of V yields the second inequality. The third and the final inequalities come from the definition of L and the fact that ϕ and V are Lipschitz functions. Finally, combining (4.4.19) with (4.4.18), we obtain

$$\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \leq e^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{\varepsilon}\left(\alpha + 2\,||V||_\infty\,||\phi'||_\infty\right) + o_\gamma(1). \tag{4.4.20}$$

Taking $A = 2\left(\alpha + 2\,||V||_\infty\,||\phi'||_\infty\right)$, we get a contradiction in (4.4.20). The proof of Proposition 4.4.4 is now complete.

□

We now give a comparison principle on bounded sets, to do this, we define for a given point $(t_0, x_0) \in (0,T) \times \mathbb{R}$ and for $\overline{r}, \overline{R} > 0$, the set

$$\mathcal{Q}_{\overline{r},\overline{R}}(t_0,x_0) = (t_0 - \overline{r}, t_0 + \overline{r}) \times (x_0 - \overline{R}, x_0 + \overline{R}).$$

Proposition 4.4.5 (Comparison principle on bounded sets for (4.4.6))**.** *Assume (A). Let* (u, ξ) *be a sub-solution of (4.4.6) and let* (v, ζ) *be a super-solution of (4.4.6) on the open set* $\mathcal{Q}_{\overline{r},\overline{R}} \subset (0,T) \times \mathbb{R}$*. Also assume that*

$$u \leq v \quad \textit{and} \quad \xi \leq \zeta \quad \textit{outside } \mathcal{Q}_{\overline{r},\overline{R}},$$

then

$$u \leq v \quad \textit{and} \quad \xi \leq \zeta \quad \textit{on } \mathcal{Q}_{\overline{r},\overline{R}}.$$

Proof. The proof of similar to the proof of Proposition 4.4.4 so we skip it. □

Applying Perron's method (see [IMR08, Proof of Theorem 6], [AT96] or [Imb05] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 4.4.6 (Existence and uniqueness of viscosity solutions for (4.4.6))**.** *Assume (A0) and (A). Then, there exists a unique solution* (u, ξ) *of (4.4.6). Moreover, the functions* u *and* ξ *are continuous and there exists a constant* $K_1 > 0$ *such that*

$$u_0(x) - K_1 t \leq u(t,x) \leq u_0(x) + K_1 t \quad \textit{and} \quad \xi_0(x) - K_1 t \leq \xi(t,x) \leq \xi_0(x) + K_1 t \tag{4.4.21}$$

4.4.4 Control of the oscillations for (4.4.6)

We now present a theorem that provides a control on the oscillations in space of the solution of (4.4.6). This is a very important theorem, first because it will allow us to prove Theorem 4.3.5 and also because it presents some of the arguments we use later to build the correctors at the junction.

Theorem 4.4.7 (Control of the space oscillations)**.** *Let* $T > 0$*. Assume (A0)-(A) and let* (u, ξ) *be the solution of (4.4.6) provided by Theorem 4.4.6. Then for all* $x, y \in \mathbb{R}, x \geq y$ *and for all* $t \in [0,T]$*, we have*

$$-k_0(x-y) - 1 \leq u(t,x) - u(t,y) \leq 0 \tag{4.4.22}$$

and

$$-k_0(x-y) - 1 \leq \xi(t,x) - \xi(t,y) \leq 0, \tag{4.4.23}$$

with k_0 *defined in (A0).*

Proof. We use the following notation,

$$\Omega = \{(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R} \quad \text{s.t. } x \geq y\}.$$

Proof of the upper bound. We introduce

$$\overline{M} = \sup_{(t,x,y)\in\Omega} \max\left(u(t,x) - u(t,y), \xi(t,x) - \xi(t,y)\right).$$

We want to prove that $\overline{M} \leq 0$. We argue by contradiction and assume that $\overline{M} > 0$.

Step 1 : the test functions. For $\eta, \gamma > 0$ small parameters, we define

$$\varphi(t,x,y) = u(t,x) - u(t,y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2$$

and

$$\psi(t,x,y) = \xi(t,x) - \xi(t,y) - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2.$$

We denote by $\Phi(t,x,y) = \max(\varphi(t,x,y), \psi(t,x,y))$. For $x \geq y$, using (4.4.21) and (A0) we have

$$\varphi(t,x,y) \leq u_0(x) - u_0(y) + 2K_1 T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1 T - \gamma x^2 - \gamma y^2$$

$$\psi(t,x,y) \leq \xi_0(x) - \xi_0(y) + 2K_1 T - \frac{\eta}{T-t} - \gamma x^2 - \gamma y^2 \leq 2K_1 T - \gamma x^2 - \gamma y^2.$$

Therefore, we deduce

$$\lim_{|x|,|y| \to +\infty} \Phi(t,x,y) = -\infty.$$

Since φ, ψ are upper semi continuous, Φ reaches a maximum on Ω at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η and γ small enough

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \leq \Phi(\bar{t}, \bar{x}, \bar{y}), \\ \gamma|\bar{x}|, \gamma|\bar{y}| \to 0 \quad \text{as } \gamma \to 0. \end{cases}$$

Step 2 : $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. For instance, we assume that $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. In this case, we have that

$$\frac{\eta}{T} \leq u_0(\bar{x}) - u_0(\bar{y}) \leq 0,$$

where we have used the fact that u_0 is non increasing, and we get a contradiction. In the same way, we get a contradiction if $\Phi(\bar{t}, \bar{x}, \bar{y}) = \psi(\bar{t}, \bar{x}, \bar{y})$. The fact that $\bar{x} > \bar{y}$ is obtained directly using that $\Phi(\bar{t}, \bar{x}, \bar{y}) > 0$.

Step 3 : utilisation of the equation in the case $\Phi(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$. By duplication of the time variable and passing to the limit we get that

$$\frac{\eta}{T^2} \leq \frac{\eta}{(T-\bar{t})^2} \leq -M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \cdot |2\gamma \bar{x}|, \tag{4.4.24}$$

where we have used the fact that $\tilde{M}(u(\bar{t}, \bar{y}), [\xi(\bar{t}, \cdot)])(\bar{y}) \leq 0$. Using Remark 4.3.6, we have that $-M(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x})$ is bounded. Taking γ to zero, we get a contradiction in (4.4.24).

Step 4 : utilisation of equation in the case $\Phi(\bar{t},\bar{x},\bar{y}) = \psi(\bar{t},\bar{x},\bar{y})$. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned}\frac{\eta}{(T-\bar{t})^2} &\leq \tilde{L}(\bar{y},\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y})|2\gamma\bar{y}| - L(\bar{x},\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})|2\gamma\bar{x}| \\ &\leq 2M_0\left(|\gamma\bar{x}| + |\gamma\bar{y}|\right)\end{aligned}$$

where we have used the bounds on L and $\tilde{L}$ (see Remark 4.3.6). Taking γ to zero, we get a contradiction.

Proof of the lower bound. In order to prove our result, we will use the following lemma which proof is postponed.

Lemma 4.4.8. *For all $(t,x) \in [0,T]\times\mathbb{R}$, we have*

$$0 \leq \xi(t,x) - u(t,x) \leq 1. \tag{4.4.25}$$

Now we would like to prove that for all $\varepsilon > 0$,

$$\overline{M} = \sup_{(t,x,y)\in\Omega} \{\xi(t,y) - u(t,x) - (k_0+\varepsilon)(x-y) - 1\} \leq 0. \tag{4.4.26}$$

In fact, if (4.4.26) is true, then taking ε to 0 and using (4.4.25) we directly obtain the lower inequalities in (4.4.22) and (4.4.23). We argue by contradiction and assume that $\overline{M} > 0$.

Step 1 : the test function. For $\eta,\gamma > 0$ small parameters, we define

$$\varphi(t,x,y) = \xi(t,y) - u(t,x) - (k_0+\varepsilon)(x-y) - 1 - \frac{\eta}{T-t} - \gamma x^2.$$

Using (A0) and (4.4.21), we obtain that

$$\begin{aligned}\varphi(t,x,y) &\leq \xi_0(y) - u_0(x) + 2K_1T - (k_0+\varepsilon)(x-y) - 1 - \frac{\eta}{T-t} - \gamma x^2 \\ &\leq 1 + k_0(x-y) + 2K_1T - (k_0+\varepsilon)(x-y) - 1 - \frac{\eta}{T-t} - \gamma x^2 \\ &\leq 2K_1T - \gamma x^2 - \varepsilon(x-y).\end{aligned}$$

Therefore, we have that for $(t,x,y)\in\Omega$

$$\lim_{|x|,|y|\to+\infty} \varphi(t,x,y) = -\infty.$$

Since φ is upper semi continuous, φ reaches a maximum on Ω at a point that we denote by $(\bar{t},\bar{x},\bar{y})$. Classically we have for η and γ small enough

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \gamma|\bar{x}| \to 0 \quad \text{as } \gamma \to 0. \end{cases}$$

Step 2 : $\bar{t} > 0$ and $\bar{x} > \bar{y}$. By contradiction, assume first that $\bar{t} = 0$. Using (A0), we get a contradiction writing that

$$\frac{\eta}{T} < \xi_0(\bar{y}) - u_0(\bar{x}) - (k_0 + \varepsilon)(\bar{x} - \bar{y}) - 1 \leq 1 + k_0(\bar{x} - \bar{y}) - (k_0 + \varepsilon)(\bar{x} - \bar{y}) - 1 \leq 0.$$

If we assume that $\bar{x} = \bar{y}$ then, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we get that

$$0 < \xi(\bar{t}, \bar{x}) - u(\bar{t}, \bar{x}) - 1 - \frac{\eta}{T - \bar{t}} \leq 1 - 1 - \frac{\eta}{T} = -\frac{\eta}{T}.$$

This inequality yields a contradiction.

Step 3 : utilisation of the equation. By duplication of the time variable and passing to the limit we get that

$$\begin{aligned}\frac{\eta}{(T - \bar{t})^2} &\leq \tilde{M}(u(t, \bar{x}), [\xi(t, \cdot)])(\bar{x}) \cdot |2\gamma\bar{x} + k_0 + \varepsilon| - L(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |k_0 + \varepsilon| \\ &\leq -L(\bar{y}, \xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \cdot |k_0 + \varepsilon|,\end{aligned}$$

where we have used the fact that $\tilde{M}(u(\bar{t}, \bar{x}), [\xi(\bar{t}, \cdot)])(\bar{x}) \leq 0$. We replace L by its definition (4.3.16) and using (4.3.17), we have

$$\frac{\eta}{(T - \bar{t})^2} \leq 2V\left(N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y})\right)|k_0 + \varepsilon|. \tag{4.4.27}$$

Now we want to prove that $N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, \cdot)])(\bar{y}) \leq h_0$. Indeed, if it is true, we will get a contradiction in (4.4.27) because $V(h) = 0\ \forall h \leq h_0$. Let then $z > h_0$.

If $\bar{y} + z \geq \bar{x}$, then using that $u(\bar{t}, .)$ is non increasing, we get that

$$u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - \xi(\bar{t}, \bar{y}) < -k_0(\bar{x} - \bar{y}) - 1 < -1.$$

If $\bar{y} + z < \bar{x}$, using the fact that $\varphi(\bar{t}, \bar{x}, \bar{y} + z) \leq \varphi(\bar{t}, \bar{x}, \bar{y})$, we obtain

$$\xi(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) < -k_0 z \leq -1.$$

Using Lemma 4.4.8, we get that $u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y}) < -1$. We deduce that $I(u(\bar{t}, \bar{y} + z) - \xi(\bar{t}, \bar{y})) = 0$ for $z \geq h_0$ and so $N(\xi(\bar{t}, \bar{y}), [u(\bar{t}, .)])(\bar{y}) \leq h_0$.

$\square$

We now turn to the proof of Lemma 4.4.8.

Proof of Lemma 4.4.8. The proof is divided into several steps.

Step 1 : proof of the lower bound. We introduce

$$\overline{M} = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \{u(t, x) - \xi(t, x)\}.$$

We want to prove that $\overline{M} \leq 0$ and argue by contradiction by assuming that $\overline{M} > 0$.

Step 1.1 : the test function. Let η, γ be small parameters, and A a constant to be chosen later. We introduce

$$\varphi(t,x,y) = u(t,x) - \xi(t,y) - \frac{\eta}{T-t} - e^{At}\frac{(x-y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0,T]\times\mathbb{R}\times\mathbb{R}$ at $(\bar{t},\bar{x},\bar{y})$ and we have for η,γ small enough,

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \leq \varphi(\bar{t},\bar{x},\bar{y}), \\ \gamma\bar{x} \to 0 \quad \text{as } \gamma \to 0, \\ |\bar{x}-\bar{y}| \to 0 \quad \text{as } \varepsilon \to 0. \end{cases}$$

Step 1.2 : $\bar{t} > 0$ for ε small enough. We assume by contradiction that $\bar{t} = 0$. We have that

$$0 < u_0(\bar{x}) - \xi_0(\bar{y}) - \frac{\eta}{T} \leq k_0|\bar{x}-\bar{y}| - \frac{\eta}{T}.$$

Taking ε small enough, we get a contradiction.

Step 1.3 : utilisation of equation. By duplication of the time variable and passing to the limit and passing to the limit, we get setting $C_{A,\eta,\varepsilon} = \dfrac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\dfrac{(\bar{x}-\bar{y})^2}{2\varepsilon}$

$$\begin{aligned} C_{A,\eta,\varepsilon} &\leq \left|e^{A\bar{t}}\frac{x-\bar{y}}{\varepsilon}\right|\Big(\tilde{L}(\bar{y},\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) - M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})\Big) + o_\gamma(1) \\ &\leq \left|e^{A\bar{t}}\frac{x-\bar{y}}{\varepsilon}\right|\Big(\alpha\tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) - M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x})\Big) + o_\gamma(1), \end{aligned} \tag{4.4.28}$$

where we have used the fact that $V \geq 0$. We claim that

$$-M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x}) \leq \alpha|\bar{x}-\bar{y}| \quad \text{and} \quad \tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) \leq |\bar{x}-\bar{y}|. \tag{4.4.29}$$

Indeed, for $z > |\bar{x}-\bar{y}|$, using that $\xi(\bar{t},\cdot)$ is non increasing and that $\varphi(\bar{t},\bar{x},\bar{y}) > 0$, we have that

$$\xi(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) \leq \xi(\bar{t},\bar{y}) - u(\bar{t},\bar{x}) < 0.$$

Therefore, using the definition of E we obtain that (for ε small enough such that $|x-\bar{y}| \leq D$)

$$-M(u(\bar{t},\bar{x}),[\xi(\bar{t},\cdot)])(\bar{x}) = -\int_0^{|\bar{x}-\bar{y}|} E(\xi(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}))dz \leq \alpha|\bar{x}-\bar{y}|.$$

Similarly, using the fact that $u(\bar{t},\cdot)$ is non increasing, for all $z > |\bar{x}-\bar{y}|$, we have that

$$u(\bar{t},\bar{y}-z) - \xi(\bar{t},\bar{y}) \geq u(\bar{t},\bar{x}) - \xi(\bar{t},\bar{y}) > 0.$$

Therefore,

$$\tilde{K}(\xi(\bar{t},\bar{y}),[u(\bar{t},\cdot)])(\bar{y}) = \int_0^{|\bar{x}-\bar{y}|} \tilde{F}(u(\bar{t},\bar{y}-z)-\xi(\bar{t},\bar{y}))dz \le |\bar{x}-\bar{y}|.$$

This ends the proof of (4.4.29). Injecting (4.4.29) into (4.4.28), we get that

$$\frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{2\varepsilon} \le 2\alpha e^{A\bar{t}}\frac{(\bar{x}-\bar{y})^2}{\varepsilon} + o_\gamma(1).$$

Taking $A = 4\alpha$, we get a contradiction for γ small enough.

Step 2 : proof of the upper bound. We introduce

$$\overline{M} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \{\xi(t,x) - u(t,x) - 1\}.$$

We want to prove that $\overline{M} \le 0$. We argue by contradiction and assume that $\overline{M} > 0$.

Let η,γ be small parameters. We consider

$$\varphi(t,x,y) = \xi(t,x) - u(t,y) - 1 - \frac{\eta}{T-t} - \frac{(x-y)^2}{2\varepsilon} - \gamma x^2.$$

Classically, φ reaches a maximum on $[0,T]\times\mathbb{R}\times\mathbb{R}$ at $(\bar{t},\bar{x},\bar{y})$ and we have the following result for η and γ small enough

$$\begin{cases} 0 < \dfrac{\overline{M}}{2} \le \varphi(\bar{t},\bar{x},\bar{y}), \\ |\gamma\bar{x}| \to 0 \quad \text{as } \gamma \to 0, \\ |\bar{x}-\bar{y}| \to 0 \quad \text{as } \varepsilon \to 0. \end{cases} \tag{4.4.30}$$

As in the previous Step 1.2, we get that $\bar{t} > 0$.

By duplication of the time variable and passing to the limit we then get that

$$\frac{\eta}{(T-\bar{t})^2} \le \left(\tilde{M}(u(\bar{t},\bar{y}),[\xi(\bar{t},\cdot)])(\bar{y}) - L(\bar{x},\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})\right)\left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right| + o_\gamma(1) \tag{4.4.31}$$

$$\le 2V\left(N(\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x})\right)\left|\frac{\bar{x}-\bar{y}}{\varepsilon}\right| + o_\gamma(1), \tag{4.4.32}$$

where we have used the fact that $\tilde{M} \le 0$ and (4.3.17). We want to prove that

$$N(\xi(\bar{t},\bar{x}),[u(\bar{t},\cdot)])(\bar{x}) \le h_0.$$

In fact for all $z \ge h_0$, we have that $\bar{x}+z > \bar{y}$ for ε small enough, so using that $\varphi(\bar{t},x,\bar{y}) > 0$ we get that

$$u(\bar{t},\bar{x}+z) - \xi(\bar{t},\bar{x}) \le u(\bar{t},\bar{y}) - \xi(\bar{t},\bar{x}) < -1.$$

We deduce that $N(\xi(\bar{t},\bar{x}),[u(\bar{t},.)])(\bar{x}) = \int_0^{h_0} I(u(\bar{t},\bar{x}+z)-\xi(\bar{t},\bar{x}))dz \le h_0$. Using that $V(h) = 0$ for $h \le h_0$, we get a contradiction in (4.4.31) for γ small enough.

□

4.5 Effective Hamiltonian and effective flux-limiter

In this section we provide a justification for the definition of the effective Hamiltonian $\overline{H}$ provided in (4.2.4), we use the following proposition.

Proposition 4.5.1. *(Homogenization left and right of the perturbation) . Assume (A). Then for every $p \in [-k_0, 0]$, there exists a unique $\lambda \in \mathbb{R}$, such that there exists a bounded solution (w, χ) for $x \in \mathbb{R}$ of*

$$\begin{cases} M_p\left(w(x), [\chi]\right)(x)|p + w_x| = \lambda \\ \Big(\alpha K_p\left(\chi(x), [w]\right)(x) - 2V\Big(N_p\left(\chi(x), [w]\right)(x) + K_p\left(\chi(x), [w]\right)(x)\Big)\Big)|p + \chi_x| = \lambda \end{cases} \tag{4.5.1}$$

Moreover, for $p \in [-k_0, 0]$, we have $\lambda = \overline{H}(p) = -V\left(\frac{-1}{p}\right)|p|$.

Proof. We claim that for $\lambda = -|p|V\left(\frac{-1}{p}\right)$, the couple

$$(w, \chi) = \left(0, -\frac{p}{\alpha}V\left(\frac{-1}{p}\right)\right)$$

is a solution of (4.5.1).

-If $p = 0$, the result is obvious.

-If $p \in [-k_0, 0)$, since $\frac{-p}{\alpha}V\left(\frac{-1}{p}\right) + pz \geq 0$ if and only if $z \in [0, V\left(-1/p\right)/\alpha]$, then we have

$$M_p\left(w(x), [\chi]\right)(x) = \int_0^D E\left(-\frac{p}{\alpha}V\left(\frac{-1}{p}\right) + pz\right)dz = -V\left(\frac{-1}{p}\right), \tag{4.5.2}$$

we recall that $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Similarly, for all $z > 0$, we have $\frac{p}{\alpha}V\left(\frac{-1}{p}\right) - pz < 0$ if and only if $z < V\left(-1/p\right)/\alpha$, then

$$K_p\left(\chi(x), [w]\right)(x) = \int_0^D F\left(\frac{p}{\alpha}V\left(\frac{-1}{p}\right) - pz\right)dz = \frac{1}{\alpha}V\left(\frac{-1}{p}\right). \tag{4.5.3}$$

Finally, by definition we have that

$$N_p\left(\chi(x), [w]\right)(x) = \int_0^D I\left(\frac{p}{\alpha}V\left(\frac{-1}{p}\right) + pz\right)dz.$$

First, notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$, we have $\frac{1}{\alpha}V\left(\frac{-1}{p}\right) + \frac{1}{p} < 0$. Moreover, $\frac{p}{\alpha}V\left(\frac{-1}{p}\right) + pz > -1$ for $z < \frac{-1}{\alpha}V\left(\frac{-1}{p}\right) - \frac{1}{p}$. We distinguish two cases.

Case 1 : $\frac{-1}{\alpha}V\left(\frac{-1}{p}\right)-\frac{1}{p}\leq D$**.** In this case, we have

$$N_p\left(\chi(x),[w]\right)(x)=\frac{-1}{\alpha}V\left(\frac{-1}{p}\right)-\frac{1}{p}$$

and

$$N_p\left(\chi(x),[w]\right)(x)+K_p\left(\chi(x),[w]\right)(x)=-\frac{1}{p}. \tag{4.5.4}$$

Finally, using (4.5.2), (4.5.3), and (4.5.4), we obtain our desired result.

Case 2 : $\frac{-1}{\alpha}V\left(\frac{-1}{p}\right)-\frac{1}{p}>D$**.** In particular we have $-1/p\geq h_{max}$. Therefore, we have

$$N_p\left(\chi(x),[w]\right)(x)=D \quad \text{and} \quad K_p\left(\chi(x),[w]\right)(x)=\frac{V_{max}}{\alpha},$$

this implies that

$$N_p\left(\chi(x),[w]\right)(x)+K_p\left(\chi(x),[w]\right)(x)=D+\frac{V_{max}}{\alpha}>h_{max}.$$

Combining this result to (4.5.3), we obtain

$$\begin{aligned}\Big(\alpha K_p\left(\chi(x),[w]\right)(x)-2V\Big(N_p\left(\chi(x),[w]\right)(x)+K_p\left(\chi(x),[w]\right)(x)\Big)\Big)|p| &= -V_{max}|p| \\ &= -V\left(\frac{-1}{p}\right)\cdot|p|.\end{aligned} \tag{4.5.5}$$

Using (4.5.2) and (4.5.5), we obtain our desired result. The proof is now complete. □

4.6 Correctors for the junction

In order to obtain an homogenization result, we need to find the effective flux-limiter. That is why we consider the following cell problem : find $\lambda\in\mathbb{R}$ such that there exists a solution (w,χ) of the following Hamilton-Jacobi equation, for $x\in\mathbb{R}$,

$$\begin{cases} M\left(w(x),[\chi(\cdot)]\right)\cdot|w_x|=\lambda \\ L\left(x,\chi(x),[w(\cdot)]\right)(x)\cdot|\chi_x|=\lambda. \end{cases} \tag{4.6.1}$$

In this section we present a result of existence of correctors for the junction, which will be used for the proof of Theorem 4.3.2. We use the following notation : given $\overline{A}\in\mathbb{R}$, $\overline{A}\geq H_0$, we define two real numbers $\overline{p}_-$ and $\overline{p}_+$ defined by

$$\overline{H}(\overline{p}_+)=\overline{H}^+(\overline{p}_+)=\overline{H}(\overline{p}_-)=\overline{H}^-(\overline{p}_-)=\overline{A}. \tag{4.6.2}$$

Given the form of $\overline{H}$, there exists only one couple of real numbers satisfying (4.6.2).

Theorem 4.6.1 (Existence of global corrector for the junction)**.** *Assume (A).*

i) (General properties) There exists a constant $\overline{A} \in [H_0, 0]$ *such that there exists a solution* (w, χ) *of* (4.6.1) *with* $\lambda = \overline{A}$ *and such that there exists a constant* $C > 0$ *and a globally Lipschitz continuous function* m *such that for all* $x \in \mathbb{R}$,

$$|w(x) - m(x)| \leq C \quad \text{and} \quad |\chi(x) - m(x)| \leq C. \tag{4.6.3}$$

ii) (Bound from below at infinity) If $\overline{A} > H_0$, *then there exists a* $\gamma_0 > 0$ *such that for every* $\gamma \in (0, \gamma_0)$, *we have for all* $x \geq r + V_{max}/\alpha$ *and* $h \geq 0$,

$$\begin{aligned} w(x+h) - w(x) &\geq (\overline{p}_+ - \gamma)h, \\ \chi(x+h) - \chi(x) &\geq (\overline{p}_+ - \gamma)h \end{aligned} \tag{4.6.4}$$

and for $x \leq -r - V_{max}/\alpha$ *and* $h \geq 0$,

$$\begin{aligned} w(x-h) - w(x) &\geq (-\overline{p}_- - \gamma)h, \\ \chi(x-h) - \chi(x) &\geq (-\overline{p}_- - \gamma)h. \end{aligned} \tag{4.6.5}$$

(iii) (Rescaling) For $\varepsilon > 0$, *we set*

$$w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon \chi\left(\frac{x}{\varepsilon}\right),$$

then (up to a sub-sequence $\varepsilon_n \to 0$*) we have that* w^ε *and* χ^ε *converge locally uniformly towards a function* W *which satisfies*

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}(W_x) = \overline{A} & \text{for all } x \neq 0. \end{cases} \tag{4.6.6}$$

In particular, we have (with $W(0) = 0$*),*

$$W(x) = \overline{p}_+ x 1_{\{x>0\}} + \overline{p}_- x 1_{\{x<0\}}. \tag{4.6.7}$$

iv)(Monotonicity of the flux-limiter $\overline{A}$*) Let* $\phi_1, \phi_2 : \mathbb{R}^+ \times \mathbb{R} \to [0, 1]$ *be two functions satisfying (A6). Let* $\overline{A}_1$ *and* $\overline{A}_2$ *be their respective flux limiters given by Theorem 4.3.2. If, for all* $(t, x) \in \mathbb{R} \times \mathbb{R}$, *we have*

$$\phi_1(t, x) \leq \phi_2(t, x),$$

then

$$\overline{A}_1 \geq \overline{A}_2.$$

The proof of this theorem is postponed until Section 4.8.

4.7 Proof of convergence

This section is devoted to the proof of Theorem 4.3.2 which relies on the existence of correctors provided by Proposition 4.5.1 and Theorem 4.6.1. We will use the following lemmas, the first one being a direct consequence of Theorem 4.4.6.

Lemma 4.7.1. *(Barriers uniform in ε). Assume (A0) and (A). There exist a constant $K_1 > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}$, we have*

$$|u^\varepsilon(t,x) - u_0(x)| \leq K_1 t \quad \textit{and} \quad |\xi^\varepsilon(t,x) - \xi_0(x)| \leq K_1 t \tag{4.7.1}$$

The following lemma is a direct consequence of Theorem 4.4.7.

Lemma 4.7.2. *Assume (A0) and (A). Then the solution $(u^\varepsilon, \xi^\varepsilon)$ of (4.3.3) satisfies for all $t \geq 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$\begin{aligned} -k_0(x-y) - \varepsilon \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0, \\ -k_0(x-y) - \varepsilon \leq \xi^\varepsilon(t,x) - \xi^\varepsilon(t,y) \leq 0. \end{aligned} \tag{4.7.2}$$

Before passing to the proof of Theorem 4.3.2, let us mention that Theorem 4.3.5 is a direct consequence of this result joint to Theorem 4.3.2.

We now turn to the proof of Theorem 4.3.2.

Proof of Theorem 4.3.2. We introduce

$$\begin{aligned} \overline{u}(t,x) = \limsup_{\varepsilon \to 0}{}^* u^\varepsilon \quad \overline{\xi}(t,x) = \limsup_{\varepsilon \to 0}{}^* \xi^\varepsilon, \\ \underline{u}(t,x) = \liminf_{\varepsilon \to 0}{}_* u^\varepsilon \quad \underline{\xi}(t,x) = \liminf_{\varepsilon \to 0}{}_* \xi^\varepsilon, \end{aligned}$$

and

$$\overline{v} = \max\left(\overline{u}, \overline{\xi}\right) \quad \underline{v} = \min\left(\underline{u}, \underline{\xi}\right).$$

We want to prove that $\overline{v}$ is a sub-solution of (4.2.7) and that $\underline{v}$ is a super-solution of (4.2.7). Indeed, in this case, the comparison principle will imply that $\overline{v} \leq \underline{v}$. But by construction $\underline{v} \leq \overline{v}$, hence $\overline{v} = \underline{v} = u^0$, the unique solution of (4.2.7). This implies that $\overline{u} = \underline{u} = \overline{\xi} = \underline{\xi} = u^0$ and so u^ε and ξ^ε converge locally uniformly to u^0.

To prove that $\overline{v}$ is a sub-solution of (4.2.7), we argue by contradiction and assume that there is a point $(\bar{t}, \bar{x}) \in (0, +\infty) \times \mathbb{R}$ and a test function $\varphi \in C^1(J_\infty)$ such that

$$\begin{cases} \overline{v}(\bar{t},\bar{x}) = \varphi(\bar{t},\bar{x}), \\ \overline{v} \leq \varphi \quad \text{on} \quad Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}) & \text{with } \bar{r} > 0, \\ \overline{v} \leq \varphi - 2\eta \quad \text{outside} \quad Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}) & \text{with } \eta > 0, \\ \varphi_t(\bar{t},\bar{x}) + \overline{H}\left(\bar{x}, \varphi_x(\bar{t},\bar{x})\right) = \theta > 0, \end{cases} \tag{4.7.3}$$

where

$$\overline{H}\left(\bar{x},\varphi_x(\bar{t},\bar{x})\right)=\begin{cases}\overline{H}\left(\varphi_x(\bar{t},\bar{x})\right) & \text{if } \bar{x}\neq 0\\ F_{\overline{A}}\left(\varphi_x(\bar{t},0^-),\varphi_x(\bar{t},0^+)\right) & \text{if } \bar{x}=0.\end{cases}$$

We can assume that for ε small enough (up to changing φ at infinity), we have

$$u^\varepsilon,\xi^\varepsilon\leq\varphi-\eta \quad \text{outside} \quad Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}). \tag{4.7.4}$$

Using Lemmas 4.7.1 and 4.7.2 we get that the functions $\overline{u}$ and $\overline{\xi}$ satisfy for all $t>0$,

$$|\overline{u}(t,x)-u_0(x)|\leq K_1t \quad \text{and} \quad |\overline{\xi}(t,x)-\xi_0(x)|\leq K_1t \quad \text{for all } x\in\mathbb{R}, \tag{4.7.5}$$

and for $x\geq y$

$$\begin{cases}-k_0(x-y)\leq\overline{u}(t,x)-\overline{u}(t,y)\leq 0\\ -k_0(x-y)\leq\overline{\xi}(t,x)-\overline{\xi}(t,y)\leq 0.\end{cases} \tag{4.7.6}$$

We distinguish two cases.

Case 1 : $\bar{x}\neq 0$. We only consider the case $\bar{x}>0$, since the other case ($\bar{x}<0$) is treated in the same way. We define $p=\varphi_x(\bar{t},\bar{x})$, that according to (4.7.6), satisfies

$$-k_0\leq p\leq 0 \tag{4.7.7}$$

We choose $\bar{r}$ small enough so that $\bar{x}-2\bar{r}>0$. We introduce

$$\psi^\varepsilon(t,x)=\varphi(t,x)-\varepsilon\frac{p}{\alpha}V\left(\frac{-1}{p}\right).$$

We have the following lemma.

Lemma 4.7.3. $(\varphi,\psi^\varepsilon)$ *satisfies, in the viscosity sense, the inequality*

$$\begin{cases}\varphi_t+\tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x),\left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\varphi_x|\geq\frac{\theta}{2}\\ \psi^\varepsilon_t+\tilde{L}^\varepsilon\left(\frac{x}{\varepsilon},\frac{\psi^\varepsilon}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\psi^\varepsilon_x|\geq\frac{\theta}{2}\end{cases} \quad \text{on } Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}). \tag{4.7.8}$$

Proof of Lemma 4.7.3. For all $(t,x)\in Q_{\bar{r},\bar{r}}(\bar{t},\bar{x})$, we have for $\bar{r}$ small enough

$$\begin{aligned}\varphi_t(t,x)+\tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x),\left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|\varphi_x(t,x)|=&\varphi_t(\bar{t},\bar{x})+o_{\bar{r}}(1)\\ &+\tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x),\left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|p|\\ =&\theta+o_{\bar{r}}(1)\\ &+\tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t,x),\left[\frac{\psi^\varepsilon}{\varepsilon}(t,\cdot)\right]\right)(x)\cdot|p|\\ &-\overline{H}(p)\\ =:&\Delta\end{aligned}$$

where we have used for the first equality the regularity of the test function φ and the fact that the non-local operator $\tilde{M}^\varepsilon$ is bounded (see Remark 4.3.6) and (4.7.3) for the second equality.

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and $\bar{r}$ small enough we have that

$$\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t, x)}{\varepsilon} \leq pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1),$$

where we have used the fact that $\varphi \in C^1$. Now using the fact that $\tilde{E}$ is non increasing, we have

$$\tilde{E}\left(\frac{\psi^\varepsilon(t, x + \varepsilon.z) - \varphi(t, x)}{\varepsilon}\right) \geq \tilde{E}\left(pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right). \tag{4.7.9}$$

Moreover, we have that

$$pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1) \geq 0 \quad \text{iff} \quad z \leq \frac{1}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1).$$

We deduce that

$$\begin{aligned} \tilde{M}^\varepsilon\left(\frac{\varphi}{\varepsilon}(t, x), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot)\right]\right)(x) &\geq \int_0^D \tilde{E}\left(pz - \frac{p}{\alpha} V\left(\frac{-1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1)\right) dz \\ &\geq -V\left(-\frac{1}{p}\right) + o_{\bar{r}}(1) + o_\varepsilon(1). \end{aligned} \tag{4.7.10}$$

Using (4.7.9),(4.7.10) and the definition of $\overline{H}$, we have for $\bar{r}$ and ε small enough,

$$\Delta \geq \theta + o_{\bar{r}}(1) - V\left(\frac{-1}{p}\right)|p| + o_{\bar{r}}(1) + o_\varepsilon(1) + V\left(\frac{-1}{p}\right)|p| = \theta + o_{\bar{r}}(1) + o_\varepsilon(1) \geq \frac{\theta}{2}.$$

We now prove the second inequality in (4.7.8). Let us notice that for ε small enough, using the fact that the non-local operator $\tilde{K}^\varepsilon$ is bounded (see Remark 4.3.6) and the definition of ϕ, we have that

$$\phi\left(\frac{x}{\varepsilon} - \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t, x), \left[\frac{\varphi}{\varepsilon}(t, \cdot)\right]\right)(x)\right) = 1 \quad \text{for all } (t, x) \in Q_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}).$$

For all $(t,x) \in Q_{\bar r,\bar r}(\bar t,\bar x)$, we have for $\bar r$ small enough

$$\begin{aligned}
\psi^\varepsilon_t(t,x) + \tilde L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|\psi^\varepsilon_x(t,x)| = &\varphi_t(t,x)\\
&+ \tilde L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|p|\\
=&\theta + o_{\bar r}(1)\\
&+ \tilde L^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|p|\\
&- \overline{H}(p)\\
=:&\Delta'
\end{aligned}$$

If $p = 0$, we obtain directly our result. We then assume that $p \in [-k_0, 0)$. For all $D \geq z \geq 0$, and for ε and $\bar r$ small enough we have that

$$\frac{\varphi(t, x - \varepsilon z) - \psi^\varepsilon(t,x)}{\varepsilon} \leq -pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1).$$

Now, using the fact that $\tilde F$ is non increasing, we have that

$$\int_0^D \tilde F\left(-pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1)\right)dz \leq \tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)$$

which yields that

$$\frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1) \leq \tilde K^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x). \tag{4.7.11}$$

We now compute $\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)$. As above, and using the fact that $\tilde I$ is non decreasing, we have

$$\tilde N^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) \leq \int_0^D \tilde I\left(pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1)\right)dz. \tag{4.7.12}$$

We notice that thanks to assumption (A7), for all $p \in [-k_0, 0)$ we have

$$\frac{1}{p} + \frac{1}{\alpha}V\left(\frac{-1}{p}\right) < 0.$$

Using that $pz + \frac{p}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1) > -1$ if and only if $z < -\frac{1}{p} - \frac{1}{\alpha}V\left(\frac{-1}{p}\right) + o_{\bar r}(1) + o_\varepsilon(1)$, we have distinguish two cases.

First case : $-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1)\leq D$**.** In this case,

$$\begin{aligned}\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)&\leq\int_{0}^{D}\tilde{I}\left(pz+\frac{p}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1)\right)dz\\&\leq-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1).\end{aligned}\tag{4.7.13}$$

Then,

$$\begin{aligned}\Delta'&\geq\theta+o_{\bar{r}}(1)+\tilde{L}^{\varepsilon}\left(\frac{x}{\varepsilon},\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)|p|-\overline{H}(p)\\&\geq\theta+o_{\bar{r}}(1)+\alpha\tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\\&-2V\left(\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)+\tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\right)+V\left(\frac{-1}{p}\right)|p|\\&\geq\theta+o_{\bar{r}}(1)+V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1)\\&-2V\left(\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)+\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1)\right)+V\left(\frac{-1}{p}\right)|p|\\&\geq\theta+o_{\bar{r}}(1)+V\left(\frac{-1}{p}\right)|p|+o_{\bar{r}}(1)+o_{\varepsilon}(1)-2V\left(\frac{-1}{p}+o_{\bar{r}}(1)+o_{\varepsilon}(1)\right)|p|\\&+V\left(\frac{-1}{p}\right)|p|\\&\geq\theta+o_{\bar{r}}(1)+o_{\varepsilon}(1)\geq\frac{\theta}{2},\end{aligned}$$

where we have used the definition of $\tilde{L}^{\varepsilon}$ for the second inequality, (4.7.11) combined with assumption (A7) (see Remark 4.2.2) for the third inequality, (4.7.13) combined with the fact that V is non-decreasing for the fourth inequality and the fact V is a Lipschitz continuous function for the last inequality.

Second case : $-\frac{1}{p}-\frac{1}{\alpha}V\left(\frac{-1}{p}\right)+o_{\bar{r}}(1)+o_{\varepsilon}(1)>D$**.** In particular, by definition of D, we have $-1/p\geq h_{max}$ for ε and $\bar{r}$ small enough. Then using (4.7.11) and the definition of $\tilde{N}^{\varepsilon}$, we obtain

$$\begin{cases}\tilde{N}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\leq D\quad\text{and}\\\frac{V_{max}}{\alpha}+o_{\bar{r}}(1)+o_{\varepsilon}(1)\leq\tilde{K}^{\varepsilon}\left(\frac{\psi^{\varepsilon}}{\varepsilon}(t,x),\left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x).\end{cases}$$

Using assumption (A7) (see Remark 4.2.2) and the previous inequalities, we get, using the definition of $\tilde{L}^\varepsilon$, that

$$
\begin{aligned}
&\tilde{L}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) = \alpha \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) \\
&- 2V\left(\tilde{N}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x) + \tilde{K}^\varepsilon\left(\frac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\frac{\varphi}{\varepsilon}(t,\cdot)\right]\right)(x)\right) \\
&\geq V_{max} + o_{\bar{r}}(1) + o_\varepsilon(1) - 2V\left(D + \frac{V_{max}}{\alpha} + o_{\bar{r}}(1) + o_\varepsilon(1)\right) \\
&\geq -V_{max} + o_{\bar{r}}(1) + o_\varepsilon(1).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\Delta' \geq &\theta + o_{\bar{r}}(1) - V_{max}|p| + o_{\bar{r}}(1) + o_\varepsilon(1) + V\left(\frac{-1}{p}\right)|p| \\
\geq &\theta + o_{\bar{r}}(1) + o_\varepsilon(1) \\
\geq &\frac{\theta}{2},
\end{aligned}
$$

where we have used assumption (A4) ($V(h) = V_{max}\ \forall h \geq h_{max}$) and that $-1/p \geq h_{max}$. This ends the proof of Lemma 4.7.3.

$\square$

Getting a contradiction. Using (4.7.4), we have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{outside } Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}).$$

Using the comparison principle on bounded subsets for (4.3.3), we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{and} \quad \xi^\varepsilon \leq \psi^\varepsilon - \eta \quad \text{on } Q_{\bar{r},\bar{r}}(\bar{t},\bar{x}).$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \leq \varphi - \eta$ and $\overline{\xi} \leq \varphi - \eta$ on $Q_{\bar{r},\bar{r}}(\bar{t},\bar{x})$ and this contradicts the fact that $\overline{v}(\bar{t},\bar{x}) = \max\left(\overline{u}(\bar{t},\bar{x}), \overline{\xi}(\bar{t},\bar{x})\right)) = \varphi(\bar{t},\bar{x})$.

Case 2 : $\bar{x} = 0$. Using Theorem 4.4.5, we may assume that the test function has the following form

$$\varphi(t,x) = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad \text{on } Q_{\bar{r},2\bar{r}}(\bar{t},0), \tag{4.7.14}$$

where g is a C^1 function defined on $(0,+\infty)$. The last line in condition (4.7.3) then becomes

$$g'(\bar{t}) + F_{\overline{A}}(\overline{p}_-, \overline{p}_+) = g'(\bar{t}) + \overline{A} = \theta.$$

Let us consider (w,ζ) the solution of (4.6.1) provided by Theorem 4.6.1. We define

$$\varphi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(x) & \text{on } Q_{\bar{r},2\bar{r}}(\bar{t},0) \\ \varphi(t,x) & \text{outside } Q_{\bar{r},2\bar{r}}(\bar{t},0), \end{cases} \tag{4.7.15}$$

$$\psi^\varepsilon(t,x) = \begin{cases} g(t) + \zeta^\varepsilon(x) & \text{on } Q_{\bar r, 2\bar r}(\bar t, 0) \\ \varphi(t,x) & \text{outside } Q_{\bar r, 2\bar r}(\bar t, 0). \end{cases} \tag{4.7.16}$$

We have the following lemma,

Lemma 4.7.4. *$(\varphi^\varepsilon, \psi^\varepsilon)$ satisfies in the viscosity sence, for $\bar r$ and ε small enough on $Q_{\bar r, \bar r}(\bar t, 0)$,*

$$\begin{cases} \varphi^\varepsilon_t + \tilde M^\varepsilon \left(\dfrac{\varphi^\varepsilon}{\varepsilon}(t,x), \left[\dfrac{\psi^\varepsilon}{\varepsilon}(t,\cdot) \right] \right)(x) \cdot |\varphi^\varepsilon_x| \geq \dfrac{\theta}{2} \\ \psi^\varepsilon_t + \tilde L^\varepsilon \left(\dfrac{x}{\varepsilon}, \dfrac{\psi^\varepsilon}{\varepsilon}(t,x), \left[\dfrac{\varphi^\varepsilon}{\varepsilon}(t,\cdot) \right] \right)(x) \cdot |\psi^\varepsilon_x| \geq \dfrac{\theta}{2}. \end{cases} \tag{4.7.17}$$

Proof of Lemma 4.7.4. Let h be a test function touching φ^ε from below at $(t_1, x_1) \in Q_{\bar r, \bar r}(\bar t, 0)$, so we have

$$w\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(h(t_1, x_1) - g(t_1)\right)$$

and

$$w(y) \geq \frac{1}{\varepsilon}\left(h(t_1, \varepsilon y) - g(t_1)\right),$$

for y in a neighbourhood of $\dfrac{x_1}{\varepsilon}$. Since w does not depend on time, we have that

$$h_t(t_1, x_1) = g'(t_1).$$

Using that (w, ζ) is a solution of (4.6.1), we then deduce that

$$h_t(t_1,x_1) - g'(t_1) + \tilde M\left(w\left(\frac{x_1}{\varepsilon}\right), [\zeta]\right)\left(\frac{x_1}{\varepsilon}\right) \cdot |h_x(t_1,x_1)| \geq \overline{A},$$

which implies

$$h_t(t_1,x_1) + \tilde M\left(w\left(\frac{x_1}{\varepsilon}\right), [\zeta]\right)\left(\frac{x_1}{\varepsilon}\right) \cdot |h_x(t_1,x_1)| \geq \overline{A} + g'(t_1) \geq \frac{\theta}{2},$$

i.e.

$$h_t(t_1,x_1) + \tilde M^\varepsilon\left(\frac{\varphi^\varepsilon}{\varepsilon}(t_1,x_1), \left[\frac{\psi^\varepsilon}{\varepsilon}(t_1,\cdot)\right]\right)(x_1) \cdot |h_x(t_1,x_1)| \geq \frac{\theta}{2}. \tag{4.7.18}$$

Let f be a test function touching ψ^ε from below at $(t_2, x_2) \in Q_{\bar r, \bar r}(\bar t, 0)$. We have

$$\zeta\left(\frac{x_2}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(f(t_2, x_2) - g(t_2)\right)$$

and

$$\zeta(y) \geq \frac{1}{\varepsilon}\left(f(t_2, \varepsilon y) - g(t_2)\right)$$

for y in a neighbourhood of $\frac{x_2}{\varepsilon}$. Since ζ does not depend on time, we have that

$$f_t(t_2, x_2) = g'(t_2).$$

Therefore, using that (w, ζ) is a solution of (4.6.1), we get

$$f_t(t_2, x_2) - g'(t_2) + \tilde{L}\left(\frac{x_2}{\varepsilon}, \zeta\left(\frac{x_2}{\varepsilon}\right), [w]\right)\left(\frac{x_2}{\varepsilon}\right) \cdot |f_x(t_2, x_2)| \geq \overline{A},$$

which implies

$$f_t(t_2, x_2) + \tilde{L}\left(\frac{x_2}{\varepsilon}, \zeta\left(\frac{x_2}{\varepsilon}\right), [w]\right)\left(\frac{x_2}{\varepsilon}\right) \cdot |f_x(t_2, x_2)| \geq \overline{A} + g_t(t_2) \geq \frac{\theta}{2},$$

i.e.

$$f_t(t_2, x_2) + \tilde{L}^\varepsilon\left(\frac{x_2}{\varepsilon}, \frac{\psi^\varepsilon}{\varepsilon}(t_2, x_2), \left[\frac{\varphi^\varepsilon}{\varepsilon}(t_2, \cdot)\right]\right)(x_2) \cdot |f_x(t_2, x_2)| \geq \frac{\theta}{2}.$$

□

Getting the contradiction. We have that for ε small enough

$$\begin{aligned} u^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad &\text{on } Q_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \backslash Q_{\bar{r}, \bar{r}}(\bar{t}, 0) \\ \xi^\varepsilon + \eta \leq \varphi = g(t) + \overline{p}_- x 1_{\{x<0\}} + \overline{p}_+ x 1_{\{x>0\}} \quad &\text{on } Q_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \backslash Q_{\bar{r}, \bar{r}}(\bar{t}, 0). \end{aligned}$$

Using the fact that $w^\varepsilon, \zeta^\varepsilon \to W$ with $W(x) = \bar{p}_- x 1_{\{x<0\}} + \bar{p}_+ x 1_{\{x>0\}}$ (see Theorem 4.6.1), we deduce that for ε small enough, we have

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } Q_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \backslash Q_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

Combining this with (4.7.15) and (4.7.16), we get that

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{and} \quad \xi^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{outside } Q_{\bar{r}, \bar{r}}(\bar{t}, 0).$$

By the comparison principle on bounded subsets the previous inequality holds in $Q_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar{t}, 0)$, we obtain

$$\overline{u}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0) \quad \text{and} \quad \overline{\xi}(\bar{t}, 0) + \frac{\eta}{2} \leq \varphi(\bar{t}, 0)$$

which is a contradiction with the fact that $\overline{v}(\bar{t}, 0) = \max\left(\overline{u}(\bar{t}, 0), \overline{\xi}(\bar{t}, 0)\right) = \varphi(\bar{t}, 0)$.

□

4.8 Proof of the existence of correctors at the junction

This section contains the proof of Theorem 4.6.1. We proceed as in [FSZ17b, GIM15] and we will construct correctors on a truncated domain and then pass to the limit as the size of the domain goes to infinity.

For $l \in (r, +\infty)$, $r << l$ and $r \leq R << l$ we want to find $\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of

$$\left\{\begin{array}{ll} \left\{\begin{array}{l} G^1_R(x, w^{l,R}(x), [\chi^{l,R}], w^{l,R}_x) = \lambda_{l,R} \\ G^2_R(x, \chi^{l,R}(x), [w^{l,R}], \chi^{l,R}_x) = \lambda_{l,R} \end{array}\right. & \text{if } x \in (-l, l) \\ \left\{\begin{array}{l} \overline{H}^+(w^{l,R}_x) = \lambda_{l,R} \\ \overline{H}^+(\chi^{l,R}_x) = \lambda_{l,R} \end{array}\right. & \text{if } x = l \\ \left\{\begin{array}{l} \overline{H}^-(w^{l,R}_x) = \lambda_{l,R} \\ \overline{H}^-(\chi^{l,R}_x) = \lambda_{l,R} \end{array}\right. & \text{if } x = -l \end{array}\right. \tag{4.8.1}$$

with

$$G^1_R(x, w(x), [\chi], q) = \psi_R(x) M(w(x), [\chi])(x)|q| + (1 - \psi_R(x))\overline{H}(q), \tag{4.8.2}$$

$$G^2_R(x, \chi(x), [w], q) = \quad \psi_R(x) L\left(x, \chi(x), [w]\right)(x)|q| + (1 - \psi_R(x))\overline{H}(q). \tag{4.8.3}$$

Moreover, $\psi_R \in C^\infty$, $\psi_R : \mathbb{R} \to [0, 1]$, with

$$\psi_R \equiv \left\{\begin{array}{ll} 1 & \text{on } [-R, R] \\ 0 & \text{on } (-\infty, -R-1] \cup [R+1, +\infty), \end{array}\right. \quad \text{and } \psi_R(x) < 1 \ \forall x \notin [-R, R]. \tag{4.8.4}$$

As in the previous sections, to $G^{1,2}_R$ we associate $\tilde{G}^{1,2}_R$ which is defined in the same way but we replace the non-local operators M and L respectively by $\tilde{M}$ and $\tilde{L}$.

4.8.1 Comparison principle for a truncated problem

Proposition 4.8.1 (Comparison principle on a truncated domain). *Let us consider the following problem for $r < l_1 < l_2$ and $\lambda \in \mathbb{R}$, with $l_2 >> R$.*

$$\left\{\begin{array}{ll} \left\{\begin{array}{l} \tilde{G}^1_R(x, u(x), [\xi], u_x) \geq \lambda \\ \tilde{G}^2_R(x, \xi(x), [u], \xi_x) \geq \lambda \end{array}\right. & \text{if } x \in (l_1, l_2) \\ \left\{\begin{array}{l} \overline{H}^+(u_x) \geq \lambda \\ \overline{H}^+(\xi_x) \geq \lambda \end{array}\right. & \text{if } x = l_2 \end{array}\right. \tag{4.8.5}$$

and for $\varepsilon_0 > 0$,

$$\left\{\begin{array}{ll} \left\{\begin{array}{l} G^1_R(x, v(x), [\zeta], v_x) \leq \lambda - \varepsilon_0 \\ G^2_R(x, \zeta(x), [v], \zeta_x) \leq \lambda - \varepsilon_0 \end{array}\right. & \text{if } x \in (l_1, l_2) \\ \left\{\begin{array}{l} \overline{H}^+(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}^+(\zeta_x) \leq \lambda - \varepsilon_0 \end{array}\right. & \text{if } x = l_2 \end{array}\right. \tag{4.8.6}$$

Then if $v(l_1) \leq u(l_1)$ and $\zeta(l_1) \leq \xi(l_1)$, we have $v \leq u$ and $\zeta \leq \xi$ in $[l_1, l_2]$.

Proof. Like in [FSZ17b], the only new difficulty to prove this proposition is the comparison at l_2. But since near l_2, the system decouples itself, we can prooceed as in [GIM15, Proposition 4.1].

$\square$

Remark 4.8.2. *We have a similar result if we exchange the boundary conditions, that is to say for $l_1 < l_2 < -r$ and $l_2 < -R$, and if the Dirichlet condition is placed in $x = l_2$ and the following conditions are imposed at $x = l_1$,*

$$\begin{cases} \begin{cases} \overline{H}^-(u_x) \geq \lambda \\ \overline{H}^-(\xi_x) \geq \lambda \end{cases} & \text{if } x = l_1 \\ \begin{cases} \overline{H}^-(v_x) \leq \lambda - \varepsilon_0 \\ \overline{H}^-(\zeta_x) \leq \lambda - \varepsilon_0 \end{cases} & \text{if } x = l_1. \end{cases}$$

4.8.2 Existence of correctors on a truncated domain

Proposition 4.8.3 (Existence of correctors on a truncated domain)**.** *There exists a constant*
$\lambda_{l,R} \in \mathbb{R}$ such that there exists a solution $(w^{l,R}, \chi^{l,R})$ of (4.8.1) *for which there exists a constant C (depending only on k_0) and a Lipschitz continuous function $m^{l,R}$, such that*

$$\begin{cases} H_0 \leq \lambda_{l,R} \leq 0, \\ |w^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |\chi^{l,R}(x) - m^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \\ |m^{l,R}(x) - m^{l,R}(y)| \leq C|x - y| & \text{for all } x, y \in [-l, l], \\ |w^{l,R}(x) - \chi^{l,R}(x)| \leq C & \text{for all } x \in [-l, l], \end{cases} \tag{4.8.7}$$

with H_0 defined in (4.2.6)*.*

Proof. Classically, we consider the approximated truncated cell problem,

$$\begin{cases} \begin{cases} \delta v^\delta + G^1_R(x, v^\delta(x), [\zeta^\delta], v^\delta_x) = 0 \\ \delta \zeta^\delta + G^2_R(x, \zeta^\delta(x), [v^\delta], \zeta^\delta_x) = 0 \end{cases} & \text{if } x \in (-l, l) \\ \begin{cases} \delta v^\delta + \overline{H}^+(v^\delta_x) = 0 \\ \delta \zeta^\delta + \overline{H}^+(\zeta^\delta_x) = 0 \end{cases} & \text{if } x = l \\ \begin{cases} \delta v^\delta + \overline{H}^-(v^\delta_x) = 0 \\ \delta \zeta^\delta + \overline{H}^-(\zeta^\delta_x) = 0 \end{cases} & \text{if } x = -l. \end{cases} \tag{4.8.8}$$

Step 1 : construction of barriers. Using that $(0, 0)$ and $(C_0/\delta, C_0/\delta)$ are respectively obvious sub and super-solution of (4.8.8), with $C_0 = |\min_{p \in \mathbb{R}} \overline{H}_0(p)| = -H_0$ and that we have a comparison principle, we deduce that there exists a continuous viscosity solution (v^δ, ζ^δ) of (4.8.8) which satisfies

$$0 \leq v^\delta \leq \frac{C_0}{\delta} \quad \text{and} \quad 0 \leq \zeta^\delta \leq \frac{C_0}{\delta}. \tag{4.8.9}$$

Step 2 : control of the oscillations of v^δ and ζ^δ.

Lemma 4.8.4. *The functions v^δ and ζ^δ satisfy for all $x, y \in [-l, l]$, $x \geq y$,*

$$-k_0(x-y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{and} \quad -k_0(x-y) - 1 \leq \zeta^\delta(x) - \zeta^\delta(y) \leq 0 \tag{4.8.10}$$

Proof of Lemma 4.8.4. In the rest of the proof we will use the following notation

$$\Omega = \left\{(x, y) \in [-l, l]^2 \text{ s.t. } x \geq y\right\}.$$

Proof of the upper inequality. We want to prove that

$$\overline{M} = \sup_{(x,y)\in\Omega} \max\left(v^\delta(x) - v^\delta(y), \zeta^\delta(x) - \zeta^\delta(y)\right) \leq 0. \tag{4.8.11}$$

We argue by contradiction and assume that $\overline{M} > 0$. Since v^δ and ζ^δ are continuous and x, y belong to a compact, $\overline{M}$ is reached for a finite point that we denote by $(x, \bar{y}) \in \Omega$. Given that $\overline{M} > 0$, we deduce that $x \neq \bar{y}$. Therefore, we can use the viscosity inequalities for (4.8.8).

Let us for instance assume that $\overline{M} = v^\delta(x) - v^\delta(\bar{y})$, the other case is similar so we skip it. We distinguish 3 cases :

-If $(x, \bar{y}) \in (-l, l)$, we have

$$\begin{aligned} &\delta v^\delta(x) + G_R^1(x, v^\delta(x), [\zeta^\delta], 0) \leq 0 \\ &\delta v^\delta(\bar{y}) + \tilde{G}_R^1(\bar{y}, v^\delta(\bar{y}), [\zeta^\delta], 0) \geq 0. \end{aligned}$$

Combining these inequalities with the fact that $G_R^i(x, U, [\Xi], 0) = 0$ for $i = 1, 2$, we obtain

$$\delta\overline{M} \leq 0.$$

-If $x = l$ and $\bar{y} \in [-l, l)$, we obtain similarly

$$\delta\overline{M} \leq 0, \tag{4.8.12}$$

using the fact that $\overline{H}^+(0) = 0$.

-If $x \in (-l, l]$ and $\bar{y} = -l$, we obtain

$$\delta\overline{M} \leq H_0 < 0,$$

where we have used the fact that $\overline{H}^-(0) = H_0 < 0$.

For every value of $x, \bar{y}$ we obtain a contradiction, therefore $\overline{M} \leq 0$.

Proof of the lower inequalities. In order to proof these inequalities, we will use the following lemma which proof is postponed.

Lemma 4.8.5. *For all $x \in [-l, l]$, we have*

$$0 \leq \zeta^{\delta}(x) - v^{\delta}(x) \leq 1. \tag{4.8.13}$$

In order to prove (4.8.10), using Lemma 4.8.5 it is sufficient to prove that

$$\overline{M} = \sup_{(x,y)\in\Omega} \Big(\zeta^{\delta}(y) - v^{\delta}(x) - k_0(x-y) - 1 \Big) \leq 0. \tag{4.8.14}$$

We argue by contradiction and assume that $\overline{M} > 0$. Since Ω is compact and v^{δ} and ζ^{δ} are continuous, $\overline{M}$ is reached for a finite point that we denote by $(x, \bar{y}) \in \Omega$. Since $\overline{M} > 0$, we deduce that $x > \bar{y}$ (thanks to Lemma 4.8.5). Therefore, we can use the viscosity inequalities for (4.8.8). We distinguish 4 cases :

-If $x, \bar{y} \in (-l, l)$, we obtain

$$\begin{array}{l} \delta\zeta^{\delta}(\bar{y}) + G_R^2(\bar{y}, \zeta^{\delta}(\bar{y}), [v^{\delta}], -k_0) \leq 0 \\ \delta v^{\delta}(x) + \tilde{G}_R^1(x, v^{\delta}(x), [\zeta^{\delta}], -k_0) \geq 0, \end{array}$$

combining these inequalities and using the definition of $\overline{M}$, we obtain

$$\begin{aligned} \delta\overline{M} \leq \delta\zeta^{\delta}(\bar{y}) - \delta v^{\delta}(x) \leq \; & \tilde{G}_R^1(x, v^{\delta}(x), [\zeta^{\delta}], -k_0) \\ & -G_R^2(\bar{y}, \zeta^{\delta}(\bar{y}), [v^{\delta}], -k_0). \end{aligned} \tag{4.8.15}$$

Since the non-local operator $\tilde{M}$ is negative and that $\overline{H}(-k_0) = 0$ we deduce that

$$\tilde{G}_R^1(x, v^{\delta}(x), [\zeta^{\delta}], -k_0) \leq 0.$$

We now claim that $G_R^2(\bar{y}, \zeta^{\delta}(\bar{y}), [v^{\delta}], -k_0) \geq 0$. Using $\overline{H}(-k_0) = 0$ and (4.3.17), we get that

$$\begin{aligned} G_R^2(\bar{y}, \zeta^{\delta}(\bar{y}), [v^{\delta}], -k_0) = \; & L\left(\bar{y}, \zeta^{\delta}(\bar{y}), \left[v^{\delta}(\cdot)\right]\right)(\bar{y}) \cdot k_0\psi_R(\bar{y}) \\ \geq \; & -2k_0 V\left(N\left(\zeta^{\delta}(\bar{y}), \left[v^{\delta}(\cdot)\right]\right)(\bar{y})\right). \end{aligned} \tag{4.8.16}$$

Let us now prove that $N(\zeta^{\delta}(\bar{y}), [v^{\delta}(\cdot)])(\bar{y}) \leq h_0$. In fact, it is sufficient to prove that for all $z \in (h_0, D]$, we have

$$v^{\delta}(\bar{y} + z) - \zeta^{\delta}(\bar{y}) < -1. \tag{4.8.17}$$

First, if $z \geq x - \bar{y}$, using the fact that v^{δ} is non increasing and that $\overline{M} > 0$, we obtain

$$v^{\delta}(\bar{y} + z) - \zeta^{\delta}(\bar{y}) \leq v^{\delta}(x) - \zeta^{\delta}(\bar{y}) \leq -k_0(x - \bar{y}) - 1 < -1.$$

Second, in the case $z < x - \bar{y}$, using the fact that

$$\zeta^{\delta}(\bar{y} + z) - v^{\delta}(x) - k_0(x - \bar{y} - z) - 1 \leq \zeta^{\delta}(\bar{y}) - v^{\delta}(x) - k_0(x - \bar{y}) - 1,$$

and using Lemma 4.8.5 we deduce that

$$v^\delta(\bar{y}+z) - \zeta^\delta(\bar{y}) \le -k_0 z < -1. \tag{4.8.18}$$

This implies that $N(\zeta^\delta(\bar{y}), [v^\delta(\cdot)])(\bar{y}) \le h_0$. Using assumption (A3) ($V(h=0)$ if $h \le h_0$) and injecting this result in (4.8.16) we get that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \ge 0$. Using (4.8.15) we then get a contradiction.

-If $x \in (-l, l)$ and $\bar{y} = -l$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) \le 0 \\ \delta v^\delta(x) + \tilde{G}_R^1(x, v^\delta(x), [\zeta^\delta], -k_0) \ge 0. \end{array}$$

Using the fact that $\overline{H}^-(-k_0) = 0$ and that $\tilde{G}_R^1(x, v^\delta(x), [\zeta^\delta], -k_0) \le 0$ we obtain $\delta\overline{M} \le 0$.

-If $x = l$ and $\bar{y} \in (-l, l)$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \le 0 \\ \delta v^\delta(x) + \overline{H}^+(-k_0) \ge 0, \end{array}$$

using that $G_R^2(\bar{y}, \zeta^\delta(\bar{y}), [v^\delta], -k_0) \ge 0$ (see the first case) , and the fact that $\overline{H}^+(-k_0) < 0$, we directly obtain $\delta\overline{M} \le 0$.

-If $x = l$ and $\bar{y} = -l$, we obtain

$$\begin{array}{l} \delta\zeta^\delta(\bar{y}) + \overline{H}^-(-k_0) \le 0 \\ \delta v^\delta(x) + \overline{H}^+(-k_0) \ge 0, \end{array}$$

and so, we get $\delta\overline{M} \le 0$.

For every value of $x, \bar{y} \in [-l, l]$ we get a contradiction, therefore we have $\overline{M} \le 0$. This ends the proof of Lemma 4.8.4.

□

Step 3 : construction of a Lipschitz estimate. We want to construct a Lipschitz continuous function m^δ, such that there exists a constant $C > 0$ (independent of l and R) such that

$$\begin{cases} |v^\delta(x) - m^\delta(x)| \le C & \text{for all } x \in [-l, l], \\ |\zeta^\delta(x) - m^\delta(x)| \le C & \text{for all } x \in [-l, l], \\ |m^\delta(x) - m^\delta(y)| \le C|x-y| & \text{for all } x, y \in [-l, l]. \end{cases} \tag{4.8.19}$$

We define m^δ as an affine function in each interval of the form $[ih_0, (i+1)h_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ih_0) = v^\delta(ih_0) \quad \text{and} \quad m^\delta((i+1)h_0) = v^\delta((i+1)h_0).$$

Since m^δ and v^δ are non-increasing, and $|v^\delta((i+1)h_0) - v^\delta(ih_0)| \le k_0 h_0 + 1 = 2$, we deduce that for all $x \in [ih_0, (i+1)h_0]$,

$$-2 \le v^\delta((i+1)h_0) - m^\delta(ih_0) \le v^\delta(x) - m^\delta(x) \le v^\delta(ih_0) - m^\delta((i+1)h_0) \le 2 \quad (4.8.20)$$

and for all $x, y \in [-l, l]$,

$$|m^\delta(x) - m^\delta(y)| \le 2k_0|x-y|.$$

Now using Lemma 4.8.5, we have

$$|\zeta^\delta(x) - m^\delta(x)| \le 3.$$

Choosing $C = \max(2k_0, 3)$, we obtain (4.8.19).

Step 4 : passing to the limit as δ goes to 0. Using (4.8.9), Lemma 4.8.5 and (4.8.19), we deduce that there exists a subsequence $\delta_n \to 0$ such that

$$\begin{array}{ll} \delta_n v^{\delta_n}(0) \to -\lambda_{l,R} & \text{as } n \to +\infty, \\ \delta_n \zeta^{\delta_n}(0) \to -\lambda_{l,R} & \text{as } n \to +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) \to m^{l,R} & \text{as } n \to +\infty. \end{array}$$

The last convergence being locally uniform. Let us consider,

$$\overline{w}^{l,R} = \limsup_{\delta_n \to 0}{}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^{l,R} = \liminf_{\delta_n \to 0}{}_* (v^{\delta_n} - v^{\delta_n}(0))$$

and

$$\overline{\chi}^{l,R} = \limsup_{\delta_n \to 0}{}^* (\zeta^{\delta_n} - \zeta^{\delta_n}(0)) \quad \text{and} \quad \underline{\chi}^{l,R} = \liminf_{\delta_n \to 0}{}_* (\zeta^{\delta_n} - \zeta^{\delta_n}(0)).$$

Therefore, we have that $\lambda_{l,R}$, $\overline{w}^{l,R}$, $\underline{w}^{l,R}$, $\overline{\chi}^{l,R}$, $\underline{\chi}^{l,R}$ and $m^{l,R}$ satisfy

$$\begin{array}{r} H_0 \le \lambda_{l,R} \le 0, \\ |\overline{w}^{l,R} - m^{l,R}| \le C, \ |\underline{w}^{l,R} - m^{l,R}| \le C, \\ |\overline{\chi}^{l,R} - m^{l,R}| \le C, \ |\underline{\chi}^{l,R} - m^{l,R}| \le C, \\ |m_x^{l,R}| \le C, \end{array} \quad (4.8.21)$$

and thanks to Lemma 4.8.5, we have

$$|\underline{\chi}^{l,R} - \overline{w}^{l,R}|, \ |\overline{\chi}^{l,R} - \underline{w}^{l,R}| \le 1. \quad (4.8.22)$$

By stability of viscosity solutions, we have that $(\overline{w}^{l,R} - 2C, \overline{\chi}^{l,R} - 2C)$ and $(\underline{w}^{l,R}, \underline{\chi}^{l,R})$ are respectively a sub-solution and a super-solution of (4.8.1), and

$$\overline{w}^{l,R} - 2C \le \underline{w}^{l,R} \quad \text{and} \quad \overline{\chi}^{l,R} - 2C \le \underline{\chi}^{l,R}.$$

By Perron's method, we can construct a solution $(w^{l,R}, \chi^{l,R})$ of (4.8.1) and thanks to (4.8.21) and (4.8.22), $m^{l,R}$, $w^{l,R}$, $\chi^{l,R}$ and $\lambda_{l,R}$ satisfy (4.8.7).

The uniqueness of $\lambda_{l,R}$ is classical so we skip it. This ends the proof of Proposition 4.8.3.

□

Proof of Lemma 4.8.5. We separate the proof in two parts. This proof uses the vertex test function of the work of Imbert and Monneau [IM13, Theorem 3.2] to treat the comparison between v^δ and ζ^δ near $-l$ and l. In fact, we consider that we have a network composed of a single branch with two nodes (one in $-l$ and the other in l). Near $-l$ we consider an outgoing branch and near l we consider an incoming branch.

Step 1 : proof of $v^\delta(x) - \zeta^\delta(x) \leq 0$ for all $x \in [-l, l]$. We want to prove that

$$\overline{M} = \sup_{x \in [-l,l]} \Big(v^\delta(x) - \zeta^\delta(x) \Big) \leq 0.$$

We argue by contradiction and assume that $\overline{M} > 0$. Given that v^δ and ζ^δ are continuous, $\overline{M}$ is reached at a finite point that we denote by $x \in [-l, l]$. We distinguish 3 cases according to the position of x in the interval $[-l, l]$.

Case 1 : $x \in (-l, l)$. We define for ε a small parameter,

$$\varphi(x, y) = v^\delta(x) - \zeta^\delta(y) - \frac{(x-y)^2}{2\varepsilon} - \frac{1}{2}\left((x-x)^2 + (y-x)^2 \right).$$

Since $[-l, l]$ is compact and v^δ and ζ^δ are continuous functions, the function φ reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l, l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$, by classical arguments, we have that

$$\lim_{\varepsilon \to 0} M_\varepsilon = M, \quad \lim_{\varepsilon \to 0} |x_\varepsilon - y_\varepsilon| = 0, \quad \text{and} \quad (x_\varepsilon, y_\varepsilon) \to (x, x) \text{ as } \varepsilon \text{ goes to } 0. \tag{4.8.23}$$

We can also prove that

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.8.24}$$

Furthermore, for ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l, l)$, and using the viscosity inequalities we obtain

$$\begin{aligned} &\delta v^\delta(x_\varepsilon) + G_R^1(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - x)) \leq 0 \\ &\delta \zeta^\delta(y_\varepsilon) + \tilde{G}_R^2(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - x)) \geq 0, \end{aligned}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of $\overline{M}$, we obtain that

$$\begin{aligned} \delta\overline{M} \leq\ & \tilde{G}_R^2\left(y_\varepsilon, \zeta^\delta(y_\varepsilon), [v^\delta], p_\varepsilon - (y_\varepsilon - x)\right) - G_R^1\left(x_\varepsilon, v^\delta(x_\varepsilon), [\zeta^\delta], p_\varepsilon + (x_\varepsilon - x)\right) \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + ||\psi_R||_\infty ||\overline{H}'||_\infty \left(|y_\varepsilon - \bar{x}| + |x_\varepsilon - x|\right) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon - y_\varepsilon + x| \\ & - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon + x_\varepsilon - x| \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + ||\psi_R||_\infty ||\overline{H}'||_\infty \left(|y_\varepsilon - \bar{x}| + |x_\varepsilon - x|\right) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon| \\ & + (\alpha M_0|y_\varepsilon - x| + M_0|x_\varepsilon - x|) \\ \leq\ & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + o_\varepsilon(1) \\ & + \psi_R(y_\varepsilon)\alpha\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon).|p_\varepsilon| - \psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon).|p_\varepsilon| \end{aligned} \tag{4.8.25}$$

where we have replaced G_R^1 and $\tilde{G}_R^2$ by their definitions, used the fact that by definition $\overline{H}$ is a Lipschitz function and that that $V \geq 0$ for the second inequality, used Remark 4.3.6 for the third inequality and (4.8.23) for the last inequality.

We will compute the right part of the inequality in different steps.

1-Concerning the local operator.

$$\begin{aligned}\left|(\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon)\right| &\leq ||D\psi_R||_\infty |x_\varepsilon - y_\varepsilon||\overline{H}(p_\varepsilon)| \\ &\leq ||D\psi_R||_\infty V_{max}\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} \\ &= o_\varepsilon(1)\end{aligned} \tag{4.8.26}$$

where we have used the regularity of ψ_R for the first inequality, used the fact that by definition of $\overline{H}$, we have $|\overline{H}| \leq V_{max}|p|$ for the second inequality and used (4.8.24) for the last inequality.

2-Concerning the non-local operator M. We claim that $M(v^\delta(x_\varepsilon), [\zeta^\delta(\cdot)])(x_\varepsilon) \leq |x_\varepsilon - y_\varepsilon|$. To prove this, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) < 0.$$

Using the fact that ζ^δ is decreasing, that $x_\varepsilon + z \geq y_\varepsilon$ and that $M_\varepsilon > 0$, we obtain

$$\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon) \leq \zeta^\delta(y_\varepsilon) - v^\delta(x_\varepsilon) < 0.$$

Therefore we have

$$-\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon) = -\psi_R(x_\varepsilon)\int_0^{|x_\varepsilon - y_\varepsilon|} E(\zeta^\delta(x_\varepsilon + z) - v^\delta(x_\varepsilon))dz \tag{4.8.27}$$

$$\leq \alpha|x_\varepsilon - y_\varepsilon|. \tag{4.8.28}$$

In particular, this implies that

$$\left|\psi_R(x_\varepsilon)M(v^\delta(x_\varepsilon), [\zeta^\delta])(x_\varepsilon)\right||p_\varepsilon| \leq \alpha\frac{(x_\varepsilon - y_\varepsilon)^2}{\varepsilon} = o_\varepsilon(1). \tag{4.8.29}$$

3-Concerning the non-local operator $\tilde{K}$. We claim that $|\tilde{K}(\zeta\delta(y_\varepsilon), [v^\delta(\cdot)])(y_\varepsilon)| \leq |x_\varepsilon - y_\varepsilon|$. As before, it suffices to prove that for all $z > |x_\varepsilon - y_\varepsilon|$

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) > 0.$$

Using the fact that v^δ is decreasing, that $x_\varepsilon \geq y_\varepsilon - z$ and that $M_\varepsilon > 0$, we obtain

$$v^\delta(y_\varepsilon - z) - \zeta^\delta(y_\varepsilon) \geq v^\delta(x_\varepsilon) - \zeta^\delta(y_\varepsilon) > 0.$$

Therefore we have

$$\left|\psi_R(y_\varepsilon)\tilde{K}(\zeta^\delta(y_\varepsilon), [v^\delta])(y_\varepsilon)\right| \leq |x_\varepsilon - y_\varepsilon|. \tag{4.8.30}$$

Injecting (4.8.26), (4.8.27), and (4.8.30) into (4.8.25), we obtain $\delta\overline{M} \leq o_\varepsilon(1)$ and we get a contradiction for ε small enough.

Case 2 : $x = l$**.** In this case, we use the vertex test function introduced by Imbert and Monneau. We refer to [IM13] for a detailed description of the vertex test function, but for the readers convenience we recall the properties that we used to complete this proof. The vertex test function G^γ is associated to the single Hamiltonian $\overline{H}$. We fix $\gamma = \delta M/2$. It satisfies the following properties.

1. (Regularity)
$$G^\gamma \in C([-l,l]^2) \quad \begin{cases} G^\gamma(x,\cdot) \in C^1([-l,l]) & \text{for all} \quad x \in [-l,l] \\ G^\gamma(\cdot,y) \in C^1([-l,l]) & \text{for all} \quad y \in [-l,l]. \end{cases} \tag{4.8.31}$$
2. (Bound from below) $G^\gamma \geq 0 = G(0,0)$.
3. (Super-linearity) There exists $g : [0,+\infty) \to \mathbb{R}$ non-decreasing and such that for all $(x,y) \in [-l,l]^2$
$$g(|x-y|) \leq G^\gamma(x,y) \quad \text{and} \quad \lim_{a\to+\infty} \frac{g(a)}{a} = +\infty.$$
4. (Compatibility condition on the gradient)
$$\overline{H}(y, -G^\gamma_y(x,y)) - \overline{H}(x, G^\gamma_x(x,y)) \leq \gamma, \tag{4.8.32}$$
with for all $x \in [-l,l]$ and $p \in \mathbb{R}$,
$$\overline{H}(x,p) = \begin{cases} \overline{H}(p) & \text{if } x \in [-l,l) \\ \overline{H}^+(p) & \text{if } x = l. \end{cases} \tag{4.8.33}$$

We introduce the following test function, for $\varepsilon > 0$ a small parameter,
$$\varphi(x,y) = v^\delta(x) - \zeta^\delta(y) - \varepsilon G^\gamma\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) - \frac{1}{2}\left((x-x)^2 + (y-x)^2\right).$$
which like before reaches a maximum at a finite point $(x_\varepsilon, y_\varepsilon) \in [-l,l]$ and (4.8.23) remains true.

Using the viscosity equations, we have that
$$\begin{cases} \delta v^\delta(x_\varepsilon) + \overline{H}\left(x_\varepsilon, G^\gamma_x\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon}\right) + (x_\varepsilon - \bar{x})\right) \leq 0 \\ \delta\zeta^\delta(y_\varepsilon) + \overline{H}\left(y_\varepsilon, -G^\gamma_y\left(\frac{x_\varepsilon}{\varepsilon}, \frac{y_\varepsilon}{\varepsilon} - (y_\varepsilon - \bar{y})\right)\right) \geq 0. \end{cases}$$
Using the definition of $\overline{M}$ and combining the previous inequalities, we get that
$$\begin{aligned} \delta\overline{M} \leq & \overline{H}\left(y_\varepsilon, -G^\gamma_y\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} - (y_\varepsilon - \bar{y})\right)\right) - \overline{H}\left(x_\varepsilon, G^\gamma_x\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + (x_\varepsilon - \bar{x})\right) \\ \leq & \overline{H}\left(y_\varepsilon, -G^\gamma_y\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) - \overline{H}\left(x_\varepsilon, G^\gamma_x\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right) + o_\varepsilon(1), \end{aligned}$$
where we have used (4.8.23) combined with the fact that both $\overline{H}$ and $\overline{H}^+$ are Lipschitz continuous for the second inequality. Using the compatibility condition on the gradient of the vertex test function (4.8.32) we obtain
$$\delta\overline{M} \leq \gamma + o_\varepsilon(1),$$
and given that $\gamma = \delta\overline{M}/2$, we get a contradiction for ε small enough.

Case 3 : $x = -l$. This case is exactly like the previous one with the exception that the vertex test function must be adapted to treat the junction at $-l$. In particular, (4.8.33) is replaced by

$$\overline{H}(x,p) = \begin{cases} \overline{H}(p) & \text{if } x \in (-l,l] \\ \overline{H}^{-}(p) & \text{if } x = -l. \end{cases}$$

We skip the rest of the computation for this case.

In conclusion, we have $\overline{M} \le 0$ and for all $x \in [-l,l]$, $0 \le \zeta^\delta(x) - v^\delta(x)$.

Step 2 : proof of $\zeta^\delta(x) - v^\delta(x) \le 1$. We want to prove that

$$\overline{M} = \sup_{x\in[-l,l]} \Big(\zeta^\delta(x) - v^\delta(x) - 1\Big) \le 0.$$

We argue by contradiction and assume that $\overline{M} > 0$. Give that v^δ and ζ^δ are continuous, $\overline{M}$ is reached at a finite point that we denote by $x \in [-l,l]$. We distinguish 2 cases according to the position of x in the interval $[-l,l]$.

Case 1 : $x \in (-l,l)$. We define for ε a small parameter,

$$\varphi(x,y) = v^\delta(x) - \zeta^\delta(y) - 1 - \frac{(x-y)^2}{2\varepsilon} - \frac{1}{2}\left((x-x)^2 + (y-x)^2\right).$$

Using the same arguments as before, the test function reaches a maximum at a finite point that we denote by $(x_\varepsilon, y_\varepsilon) \in [-l,l]$. If we denote $M_\varepsilon = \varphi(x_\varepsilon, y_\varepsilon)$ (4.8.23) and (4.8.24) remain valid.

For ε small enough we have $x_\varepsilon, y_\varepsilon \in (-l,l)$, and using the viscosity inequalities we get that

$$\begin{array}{l} \delta\zeta^\delta(x_\varepsilon) + G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) \le 0 \\ \delta v^\delta(y_\varepsilon) + \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) \ge 0, \end{array}$$

with $p_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$. Combining these inequalities and using the definition of $\overline{M}$, we obtain

$$\begin{array}{rl} \delta\overline{M} \le & \tilde{G}_R^1(y_\varepsilon, v^\delta(y_\varepsilon), [\zeta^\delta], p_\varepsilon) - G_R^2(x_\varepsilon, \zeta^\delta(x_\varepsilon), [v^\delta], p_\varepsilon) \\ \le & (\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon) + 2\psi_R(x_\varepsilon) V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right).|p_\varepsilon|, \end{array} \tag{4.8.34}$$

where we have replaced G_R^2 and $\tilde{G}_R^1$ by their definition and used (4.3.17) and that $\tilde{M} \le 0$. We will compute the right part of (4.8.34) in different steps.

1-Concerning the local operator. Like before, we have

$$\left|(\psi_R(x_\varepsilon) - \psi_R(y_\varepsilon))\overline{H}(p_\varepsilon)\right| \le ||D\psi_R||_\infty |x_\varepsilon - y_\varepsilon||\overline{H}(p_\varepsilon)| = o_\varepsilon(1). \tag{4.8.35}$$

2-Concerning the non-local operator N. We claim that

$$N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon) \le h_0.$$

To prove this, it suffices to prove that for all $z \ge h_0$, we have

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) < -1.$$

Since $|x_\varepsilon - y_\varepsilon| \to 0$ as ε goes to 0, we have for all $z \ge h_0$ and ε small enough that $x_\varepsilon + z \ge y_\varepsilon$. Therefore, we get

$$v^\delta(x_\varepsilon + z) - \zeta^\delta(x_\varepsilon) \le v^\delta(y_\varepsilon) - \zeta^\delta(x_\varepsilon) < -1,$$

where we have used the fact that v^δ is decreasing for the first inequality and the fact that $M_\varepsilon > 0$ for the second inequality. This implies that

$$V\left(N(\zeta^\delta(x_\varepsilon), [v^\delta(\cdot)])(x_\varepsilon)\right) \le V(h_0) = 0. \tag{4.8.36}$$

Injecting (4.8.35) and (4.8.36) in (4.8.34), we obtain $\delta\overline{M} \le o_\varepsilon(1)$, and we get a contradiction for ε small enough.

Case 2 : $x = l$ of $x = -l$. Proceeding like in the previous step we obtain directly a contradiction by using the properties of the vertex test function.

This ends the proof of Lemma 4.8.5.

□

Proposition 4.8.6 (First definition of the flux limiter)**.** *The following limits exists (up to some sub-sequence),*

$$\begin{cases} \overline{A} = \lim\limits_{R\to+\infty} \overline{A}_R, \\ \overline{A}_R = \lim\limits_{l\to+\infty} \lambda_{R,l}. \end{cases}$$

Moreover, we have

$$H_0 \le \overline{A},\ \overline{A}_R \le 0. \tag{4.8.37}$$

Proof. This proposition is a direct consequence of (4.8.7). □

Proposition 4.8.7 (Control of the slopes on a truncated domain)**.** *Assume that l and R are big enough. Let $(w^{l,R}, \chi^{l,R})$ be the solution of (4.8.1) given by Proposition 4.8.3. We also assume up to a sub-sequence, that $\overline{A} = \lim\limits_{R\to+\infty} \lim\limits_{l\to+\infty} \lambda_{l,R} > H_0$. Then there exists $\gamma_0 > 0$ and a constant $C > 0$ (independent of l and R) such that for all $\gamma \in (0, \gamma_0)$ and for all $x \ge r + D$, $h \ge 0$ we have*

$$w^{l,R}(x+h) - w^{l,R}(x) \ge (\overline{p}_+ - \gamma)h - C \tag{4.8.38}$$

and

$$\chi^{l,R}(x+h) - \chi^{l,R}(x) \geq (\overline{p}_+ - \gamma)h - C. \tag{4.8.39}$$

Similarly, for all $x \leq -r - D$ *and* $h \geq 0$,

$$w^{l,R}(x-h) - w^{l,R}(x) \geq (-\overline{p}_- - \gamma)h - C \tag{4.8.40}$$

and

$$\chi^{l,R}(x-h) - \chi^{l,R}(x) \geq (-\overline{p}_- - \gamma)h - C. \tag{4.8.41}$$

Proof. We only do the proof of (4.8.38)-(4.8.39), since the proof of (4.8.40)-(4.8.41) is similar and we skip it. For $\mu > 0$, small enough, we denote by p_μ^+ the real number defined by

$$\overline{H}(p_\mu^+) = \overline{H}^+(p_\mu^+) = \lambda_{l,R} - \mu. \tag{4.8.42}$$

Using that

$$H_0 < \lambda_{l,R} \leq 0,$$

we deduce that p_μ^+ exists for μ small enough and $p_\mu^+ \in [-k_0, 0)$.

Let us now consider

$$\begin{cases} w^+ = p_\mu^+ x, \\ \chi^+ = p_\mu^+ x - \dfrac{p_\mu^+}{\alpha} V\left(\dfrac{-1}{p_\mu^+}\right), \end{cases}$$

that satisfy

$$\overline{H}(w_x^+) = \overline{H}^+(w_x^+) = \overline{H}(\chi_x^+) = \overline{H}^+(\chi_x^+) = \lambda_{l,R} - \mu \quad \text{for } x \in \mathbb{R}. \tag{4.8.43}$$

Let us consider $(w, \chi) = \left(0, -\dfrac{p_\mu^+}{\alpha} V\left(\dfrac{-1}{p_\mu^+}\right)\right)$ the correctors provided by Proposition 4.5.1 for $p = p_\mu^+$. Given the definition of w^+ and χ^+, we get

$$M(w^+(x), [\chi^+])(x) = M_{p_\mu^+}(w(x), [\chi])(x), \quad K(\chi^+(x), [w^+])(x) = K_{p_\mu^+}(\chi(x), [w])(x),$$

and

$$N(\chi^+(x), [w^+])(x) = N_{p_\mu^+}(\chi(x), [w])(x).$$

In particular this implies that

$$M(w^+(x), [\chi^+])(x) = -V\left(\frac{-1}{p_\mu^+}\right)$$

and

$$\alpha K(\chi^+(x),[w^+])(x) - 2V\Big(K(\chi^+(x),[w^+])(x) + N(\chi^+(x),[w^+])(x)\Big) = -V\left(\frac{-1}{p_\mu^+}\right).$$

Finally, given that the non-local operator K is bounded by D (see Remark 4.3.6), we have for all $x \in (r+D, l]$

$$\phi\Big(x - K(\chi^+(x),[w^+])(x)\Big) = 1.$$

Combining the previous results, we can see that the restriction of (w^+,χ^+) to $(r+D,l]$ satisfies

$$\begin{cases} \begin{cases} G_R^1(x,w^+(x),[\chi^+],w_x^+) = \overline{H}(p_\mu^+) = \lambda_{l,R} - \mu \\ G_R^2(x,\chi^+(x),[w^+],\chi_x^+) = \overline{H}(p_\mu^+) = \lambda_{l,R} - \mu \end{cases} & \text{if } x \in (r+D,l) \\ \begin{cases} \overline{H}^+(w_x^+) = \lambda_{l,R} - \mu \\ \overline{H}^+(\chi_x^+) = \lambda_{l,R} - \mu \end{cases} & \text{if } x = l \end{cases} \tag{4.8.44}$$

Let us introduce, for some $x_0 \in (r+D,l]$,

$$\begin{cases} g = w^{l,R} - w^{l,R}(x_0) \\ h = \chi^{l,R} - w^{l,R}(x_0), \end{cases} \quad \text{and} \quad \begin{cases} u = w^+ - w^+(x_0) - C - \dfrac{k_0}{\alpha}V_{max} \\ v = \chi^+ - w^+(x_0) - C - \dfrac{k_0}{\alpha}V_{max}, \end{cases} \tag{4.8.45}$$

with $C > 0$ the constant provided by Proposition 4.8.3. Then we have

$$g(x_0) = 0 \geq -C - \frac{k_0}{\alpha}V_{max} = u(x_0)$$

and

$$h(x_0) = \chi^{l,R}(x_0) - w^{l,R}(x_0) \geq -C \geq -C - \frac{k_0}{\alpha}V_{max} - \frac{p_\mu^+}{\alpha}V\left(\frac{-1}{p}\right) = v(x_0),$$

where we have used the fact that $p_\mu^+ \in [-k_0, 0)$ and $||V||_\infty \leq V_{max}$. Using that (g,h) is a solution of (4.8.5) and (u,v) is a solution of (4.8.6) (with $\varepsilon_0 = \mu$), joint to the comparison principle (Proposition 4.8.1), up to changing the value of the constant C, we get that

$$\begin{cases} w^{l,R}(x) - w^{l,R}(x_0) \geq p_\mu^+(x - x_0) - C \\ \chi^{l,R}(x) - \chi^{l,R}(x_0) \geq p_\mu^+(x - x_0) - C. \end{cases}$$

This implies that for all $h \geq 0$, and for all $x \in (r+D,l)$,

$$\begin{cases} w^{l,R}(x+h) - w^{l,R}(x) \geq p_\mu^+ h - C \\ \chi^{l,R}(x+h) - \chi^{l,R}(x) \geq p_\mu^+ h - C. \end{cases}$$

Finally, if we choose $\gamma_0 < |p_0 - \overline{p}_+|$, then we have

$$\overline{H}(\overline{p}_+ - \gamma) = \overline{H}^+(\overline{p}_+ - \gamma).$$

Choosing $\mu > 0$ such that
$$p_\mu^+ = \overline{p}_+ - \gamma.$$
we obtain (4.8.38)-(4.8.39).

$\square$

Proof of Theorem 4.6.1. The proof is performed in two steps.

Step 1 : proof of i) and ii) We want to pass to the limit as $l \to +\infty$ and then as $R \to +\infty$ on the solution of (4.8.1) given by Proposition 4.8.3. Using (4.8.3), there exists $l_n \to +\infty$, such that
$$m^{l_n,R} - m^{l_n,R}(0) \to m^R \quad \text{as } n \to +\infty,$$
the convergence being locally uniform. We also define
$$\overline{w}^R(x) = \limsup_{n\to+\infty}{}^* \left(w^{l_n,R} - w^{l_n,R}(0)\right), \quad \underline{w}^R(x) = \liminf_{n\to+\infty}{}_* \left(w^{l_n,R} - w^{l_n,R}(0)\right),$$
and
$$\overline{\chi}^R(x) = \limsup_{n\to+\infty}{}^* \left(\chi^{l_n,R} - \chi^{l_n,R}(0)\right), \quad \underline{\chi}^R(x) = \liminf_{n\to+\infty}{}_* \left(\chi^{l_n,R} - \chi^{l_n,R}(0)\right)$$
Thanks to (4.8.3), we know that these limits are finite and satisfy
$$m^R - C \le \underline{w}^R \le \overline{w}^R \le m^R + C. \quad \text{and} \quad m^R - C \le \underline{\chi}^R \le \overline{\chi}^R \le m^R + C.$$
By stability of viscosity solutions $(\overline{w}^R - 2C, \overline{\chi}^R - 2C)$ and $(\underline{w}^R, \underline{\chi}^R)$ are respectively a sub-solution and a super-solution of
$$\begin{cases} G_R^1(x, w^R(x), [\chi^R], w_x^R) = \overline{A}_R \\ G_R^2(x, \chi^R(x), [w^R], \chi_x^R) = \overline{A}_R. \end{cases} \tag{4.8.46}$$
Therefore, using Perron's method, we can construct a solution (w^R, χ^R) of (4.8.46), with m^R, $\overline{A}_R$, w^R and χ^R satisfying
$$\begin{cases} |m^R(x) - m^R(y)| \le C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ |w^R(x) - m^R(x)| \le C, |\chi^R(x) - m^R(x)| \le C & \text{for all } x \in \mathbb{R}, \\ |w^R(x) - \chi^R(x)| \le C & \text{for all } x \in \mathbb{R}, H_0 \le \overline{A}_R \le 0. \end{cases} \tag{4.8.47}$$
Using Proposition 4.8.7, if $\overline{A} > H_0$, we know that there exists a $\gamma_0 > 0$ and a constant $C > 0$ such that for all $\gamma \in (0, \gamma_0)$, for all $x \ge r + D$, and $h \ge 0$,
$$w^R(x+h) - w^R(x) \ge (\overline{p}_+ - \gamma)h - C \quad \text{and} \quad \chi^R(x+h) - \chi^R(x) \ge (\overline{p}_+ - \gamma)h - C.$$
Similarly, for all $x \le -r - D$ and $h \ge 0$,
$$w^R(x-h) - w^R(x) \ge (-\overline{p}_- - \gamma)h - C$$
and
$$\chi^R(x-h) - \chi^R(x) \ge (-\overline{p}_- - \gamma)h - C.$$
Proceeding like before, we pass to the limit as $R \to +\infty$ in order to build a solution (w, χ) of (4.6.1) with $\lambda = \overline{A}$ that satisfies (4.6.3), (4.6.4) and (4.6.5).

Step 2 : proof of iii). Let us now consider the rescaled functions $w^\varepsilon = \varepsilon w(x/\varepsilon)$ and $\chi^\varepsilon(x) = \varepsilon\chi(x/\varepsilon)$. Using (4.6.3), we have that

$$w^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \quad \text{and} \quad \chi^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon). \tag{4.8.48}$$

Therefore, there exists a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$, such that

$$w^{\varepsilon_n}, \chi^{\varepsilon_n} \to W \quad \text{locally uniformly as } n \to +\infty, \tag{4.8.49}$$

with $W(0) = 0$. Proceeding as in the proof of convergence (Section 4.7), away from the junction point, we have that W satisfies

$$\overline{H}(W_x) = \overline{A} \quad \text{for } x \neq 0.$$

This proves (4.6.6). Let us now prove (4.6.7).

For $x < 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \leq \overline{p}_- + \gamma,$$

where we have used (4.6.5). Therefore, we have $W_x = \overline{p}_-$ for $x < 0$, this equality remains valid if $\overline{A} = H_0$ (indeed, if $\overline{A} = H_0$, we have $\overline{p}_+ = \overline{p}_- = p_0 = W_x$).

For $x > 0$, we have for all $\gamma \in (0, \gamma_0)$, if $\overline{A} > H_0$,

$$W_x \geq \overline{p}_+ - \gamma,$$

where we have used (4.6.4). Therefore, we have that $W_x = \overline{p}_+$ for $x > 0$, this result is still valid if $\overline{A} = H_0$.

Combining these results, we obtain (4.6.7).

Step 3 : proof of v). In order to establish the monotonicity, we have to consider the approximated truncated cell problem (4.8.8). Let us consider $(v_1^\delta, \zeta_1^\delta)$ and $(v_2^\delta, \zeta_2^\delta)$ viscosity solutions of (4.8.8), respectively for ϕ_1 and ϕ_2, with $0 \leq \phi_1 \leq \phi_2$. First, using the fact that the non-local operator is negative, we have

$$G_R^{2,2}(x, [U], q) \leq G_R^{2,1}(x, [U], q),$$

with

$$G_R^{2,i}(x, [U], q) = \psi_R(x) L^i\left(x, \chi(x), [w]\right)(x)|q| + (1 - \psi_R(x))\overline{H}(q), \quad \text{for } i = 1, 2.$$

with

$$\begin{aligned} L^i\left(y, \Sigma(x), [U]\right)(x) &= \alpha K\left(\Sigma(x), [U]\right)(x) \\ &\quad - 2V\Big(N\left(\Sigma(x), [U]\right)(x) + K\left(\Sigma(x), [U]\right)(x)\Big)\phi_i\left(y - K\left(\Sigma(x), [U]\right)(x)\right). \end{aligned}$$

Therefore, we have

$$0 = \delta\zeta_1^\delta + G_R^{2,1}(x, \zeta_1^\delta(x), [v_1^\delta], \zeta_{1_x}^\delta) \geq \delta\zeta_1^\delta + G_R^{2,2}(x, \zeta_1^\delta(x), [v_1^\delta], \zeta_{1_x}^\delta),$$

meaning that $(v_1^\delta, \zeta_1^\delta)$ is a sub-solution of (4.8.8) with ϕ_2. The comparison principle and (4.8.9) imply that

$$\begin{cases} 0 \leq \delta v_1^\delta \leq \delta v_2^\delta \leq |H_0| \\ 0 \leq \delta\zeta_1^\delta \leq \delta\zeta_2^\delta \leq |H_0|. \end{cases}$$

Passing to the limit as $\delta \to 0$, we obtain

$$0 \geq \lambda_{l,R}^1 \geq \lambda_{l,R}^2 \geq H_0.$$

Passing to the limit as $l, R \to +\infty$, we get the result.

□

Theorem 4.8.8 (Effective flux limiter)**.** *Assume (A). We define the following set of functions,*

$$\mathcal{S} = \{(v,\zeta) \text{ s.t. } \exists \text{ a Lipschitz continuous function } m \text{ (with } m(0){=}0) \\ \text{and constant } C > 0 \text{ s.t. } ||v - m||_\infty, ||\zeta - m||_\infty \leq C\}.$$

Then we have

$$\overline{A} = \inf\{\lambda \in [H_0, 0] : \exists (v,\zeta) \in \mathcal{S} \text{ solution of } (4.6.1)\}. \tag{4.8.50}$$

Proof of Theorem 4.8.8. Up to a sub-sequence, let $\overline{A} = \lim_{R\to+\infty} \lim_{l\to+\infty} \lambda_{l,R}$. We want to prove that $\overline{A} = \inf E$, with

$$E = \{\lambda \in [H_0, 0] : \exists (v,\zeta) \in \mathcal{S} \text{ solution of } (4.6.1)\}.$$

We argue by contradiction and assume that there exists $\lambda \in E$ such that $\lambda < \overline{A}$. We denote by $(v^\lambda, \zeta^\lambda)$ a solution of (4.6.1) associated to λ. Arguing as in the proof of Theorem 4.6.1, Step 2, we deduce that the functions

$$v_\lambda^\varepsilon(x) = \varepsilon v^\lambda\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \zeta_\lambda^\varepsilon(x) = \varepsilon\zeta^\lambda\left(\frac{x}{\varepsilon}\right) \tag{4.8.51}$$

have a limit W^λ (with $W^\lambda(0) = 0$) which satisfies

$$\overline{H}(W_x^\lambda) = \lambda \quad \text{for } x \neq 0.$$

This means that for all $x > 0$, we have

$$W_x^\lambda \leq p_+^\lambda < \overline{p}_+ \quad \text{with } \overline{H}(p_+^\lambda) = \overline{H}^+(p_+^\lambda) = \lambda. \tag{4.8.52}$$

Similarly, for all $x < 0$, we have

$$W_x^\lambda \geq p_-^\lambda > \overline{p}_- \quad \text{with } \overline{H}(p_-^\lambda) = \overline{H}^-(p_-^\lambda) = \lambda. \tag{4.8.53}$$

These inequalities imply that for all $\gamma > 0$, there exists a constant $\tilde{C}_\gamma$ such that

$$v^\lambda(x), \zeta^\lambda(x) \le \begin{cases} (p_+^\lambda + \gamma)x + \tilde{C}_\gamma & \text{for } x > 0, \\ (p_-^\lambda - \gamma)x + \tilde{C}_\gamma & \text{for } x < 0. \end{cases} \tag{4.8.54}$$

Using Theorem 4.6.1 (ii), we have for γ small enough,

$$v^\lambda \le w \quad \text{and} \quad \zeta^\lambda \le \chi \quad \text{ for } |x| \ge \tilde{R}.$$

This implies that there exists a constant $C_{\tilde{R}}$ such that for all $x \in \mathbb{R}$, we have

$$v^\lambda(x) < w(x) + C_{\tilde{R}} \quad \text{and} \quad \zeta^\lambda(x) < \chi(x) + C_{\tilde{R}}.$$

Let us now introduce two functions (u, ξ) and (u^λ, ξ^λ), defined by

$$\begin{cases} u(t,x) = w(x) + C_{\tilde{R}} - \overline{A}t, \\ \xi(t,x) = \chi(x) + C_{\tilde{R}} - \overline{A}t, \end{cases} \quad \text{and} \quad \begin{cases} u^\lambda(t,x) = v^\lambda(x) - \lambda t, \\ \xi^\lambda(t,x) = \zeta^\lambda(x) - \lambda t. \end{cases}$$

Both functions are solutions of (4.3.3) (with $\varepsilon = 1$) and

$$u^\lambda(0,x) \le u(0,x) \quad \text{and} \quad \xi^\lambda(0,x) \le \xi(0,x).$$

Using the comparison principle (Proposition 4.4.4), we obtain

$$v^\lambda(x) - \lambda t \le w(x) - \overline{A}t + C_{\tilde{R}}.$$

Passing to the limit as t goes to infinity, we get $\overline{A} \le \lambda$, which is a contradiction. □

4.9 Link between the system of ODEs and the PDE

This section is devoted to the proof of Theorem 4.3.4, which is a direct application of our convergence result, Theorem 4.3.2 joint to the following result.

Theorem 4.9.1. *For $\varepsilon = 1$, (ρ, σ) defined by (4.2.2) and* (1.5.21) *is a discontinuous viscosity solution of the following equation*

$$\begin{cases} \rho_t + M\left(\rho(t,x), [\sigma(t,\cdot)]\right)(x) \cdot |\rho_x| = 0 \\ \sigma_t + L\left(x, \sigma(t,x), [\rho(t,\cdot)]\right)(x) \cdot |\sigma_x| = 0 \end{cases} \quad \textit{for } (t,x) \in (0, +\infty) \times \mathbb{R}. \tag{4.9.1}$$

The proof of Theorem 4.9.1 is given in Appendix 4.11. Let us use Theorem 4.9.1 to do the proof of Theorem 4.3.4.

Proof of Theorem 4.3.4. We recall that in Theorem 4.3.4 we have $u_0(x) = \xi_0^\varepsilon(x) = -x/h$. Let us begin by proving that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, we have

$$|\rho^\varepsilon(0,x) - u_0(x)| \le f(\varepsilon) \quad \text{and} \quad |\sigma^\varepsilon(0,x) - \xi_0^\varepsilon(x)| \le g(\varepsilon), \tag{4.9.2}$$

with $f(\varepsilon), g(\varepsilon) \to 0$ as ε goes to 0. Let us define a piece-wise affine function v satisfying

$$\rho^1(0,x) = v(x) \quad \text{for } x = U_i(0), \text{ for all } i \in \mathbb{Z}.$$

Given that for all $U_{i+1}(0) - U_i(0) \ge h_0$, we notice that v is k_0-Lipschitz continuous and by definition of $\rho^1(0,x)$, we have

$$\left|\rho^1(0,x) - v(x)\right| \le 1 \quad \text{for all } x \in \mathbb{R}.$$

Let us consider the integer $i_0 \in \mathbb{N}$ defined by $i_0 = \sup\{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \le -R\}$. Using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \le -R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \le U_{i_0}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} + \rho^1(0, U_{i_0}(0)) = -\frac{x}{h} + \frac{U_{i_0}(0)}{h} - i_0 - 1.$$

Let us now consider the integer $i_1 \in \mathbb{N}$ defined by

$$i_1 = \inf\{i \in \mathbb{Z}, \text{ s.t. } U_i(0) \ge R\}.$$

Now using the assumption that for all $i \in \mathbb{Z}$ such that $U_i(0) \ge R$ we have $U_{i+1}(0) - U_i(0) = h$, we deduce that for all $x \ge U_{i_1}(0)$

$$v(x) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} + \rho^1(0, U_{i_1}(0)) = -\frac{x}{h} + \frac{U_{i_1}(0)}{h} - i_1 - 1.$$

Moreover, we recall that for all $\varepsilon > 0$, we have $\rho^\varepsilon(0,x) = \varepsilon\rho^1(0, x/\varepsilon)$, this implies that for all $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0,x) - u_0(x)| &\le \left|\rho^\varepsilon(0,x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)\right| + \left|\varepsilon v\left(\frac{x}{\varepsilon}\right) - u_0(x)\right| \\ &\le \varepsilon + \varepsilon \max\left(\left|\frac{U_{i_1}(0)}{h} - i_1 - 1\right|, \left|\frac{U_{i_0}(0)}{h} - i_0 - 1\right|\right). \end{aligned} \tag{4.9.3}$$

Similarly, we have for all $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$,

$$\begin{aligned} |\rho^\varepsilon(0,x) - u_0(x)| &\le \left|\rho^\varepsilon(0,x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)\right| + \left|\varepsilon v\left(\frac{x}{\varepsilon}\right) - \varepsilon u_0\left(\frac{x}{\varepsilon}\right)\right| \\ &\le \varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} \left(|v(y) - u_0(y)|\right), \end{aligned} \tag{4.9.4}$$

where we have used the fact that $\varepsilon u_0(x/\varepsilon) = u_0(x)$. Combining (4.9.3) and (4.9.4) and choosing

$$\begin{aligned} f(\varepsilon) = \varepsilon + \varepsilon \max\Bigg(&\left|\frac{U_{i_0}(0)}{h} - i_0 - 1\right|, \\ &\max_{y \in [U_{i_0}(0), U_{i_1}(0)]} \left(|v(y) - u_0(y)|\right), \left|\frac{U_{i_1}(0)}{h} - i_1 - 1\right|\Bigg) \end{aligned}$$

we deduce the first inequality in (4.9.2) and proceeding in the same way we obtain the second inequality. Using (4.9.2), we deduce that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, we have

$$|(\rho^\varepsilon)^*(0,x) - u_0(x)| \le f(\varepsilon) + \varepsilon \quad \text{and} \quad |(\sigma^\varepsilon)^*(0,x) - \xi_0^\varepsilon(x)| \le g(\varepsilon) + \varepsilon. \tag{4.9.5}$$

Combining (4.9.2) and (4.9.5), we get

$$\begin{cases} u_0(x) - \max(f(\varepsilon), g(\varepsilon)) \le \rho^\varepsilon(0,x) \le u_0(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon \\ \xi_0^\varepsilon(x) - \max(f(\varepsilon), g(\varepsilon)) \le \sigma^\varepsilon(0,x) \le \xi_0^\varepsilon(x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon. \end{cases}$$

Using the fact that $(\rho^\varepsilon, \sigma^\varepsilon)$ is a viscosity solution of (4.3.3) and the comparison principle (Proposition 4.4.4) we deduce that (with $(u^\varepsilon, \xi^\varepsilon)$ the continuous solution of (4.3.3) associated to the initial condition $u_0(x) = \xi_0^\varepsilon(x) = -x/h$)

$$\begin{cases} u^\varepsilon(t,x) - \max(f(\varepsilon), g(\varepsilon)) \le \rho^\varepsilon(t,x) \le u^\varepsilon(t,x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon \\ \xi^\varepsilon(t,x) - \max(f(\varepsilon), g(\varepsilon)) \le \sigma^\varepsilon(t,x) \le \xi^\varepsilon(t,x) + \max(f(\varepsilon), g(\varepsilon)) + \varepsilon, \end{cases}$$

where we have used the fact that (4.3.3) is invariant by addition of constants to the solutions. Passing to the limit as $\varepsilon \to 0$ and using Theorem 4.3.2 we get that $\rho^\varepsilon, \sigma^\varepsilon \to u^0$ (the unique solution of (4.2.7) with (u_0, ξ_0^ε) for initial condition), which ends the proof of Theorem 4.3.4.

□

4.10 Analysis of system (4.3.1)

In this section we present some properties of the solution $(U_i, \Xi_i)_{i\in\mathbb{Z}}$ of

$$\begin{cases} \dot{U}_j(t) = \alpha\left(\Xi_j(t) - U_j(t)\right) \\ \\ \dot{\Xi}_j(t) = \alpha\left(U_j(t) - \Xi_j(t)\right) + 2V\left(U_{j+1}(t) - U_j(t)\right) \cdot \phi\left(U_j(t)\right). \end{cases} \tag{4.10.1}$$

We couple system (4.10.1) with an initial condition $(U_i(0), \Xi_i(0))_i$ that satisfy the following assumption.

(A0') (Initial conditions for (4.10.1)**).** For all $i \in \mathbb{Z}$,

$$0 \le \Xi_i(0) - U_i(0) \le \frac{V_{max}}{\alpha}, \quad U_{i+1}(0) - \Xi_i(0) \ge h_0, \quad \text{and} \quad U_{i+1}(0) - U_i(0) \le h_{max} \tag{4.10.2}$$

Proposition 4.10.1 (Bounds on the velocities of the vehicles)**.** *Assume (A) and (A0'), then the solution $(U_i, \Xi_i)_i$ of (4.10.1) satisfies for all $i \in \mathbb{Z}$*

$$0 \le \Xi_i(t) - U_i(t) \le \frac{V_{max}}{\alpha} \quad \textit{for all } t > 0. \tag{4.10.3}$$

Proof. Let us consider the equation satisfied by $\Xi_i - U_i$,

$$\begin{cases} \dfrac{d(\Xi_i - U_i)}{dt} = -2\alpha\left(\Xi_i - U_i\right) + 2V\left(U_{j+1} - U_j\right) \cdot \phi\left(U_j\right) \quad \text{for all } t > 0, \\ 0 \le \Xi_i(0) - U_i(0) \le \dfrac{V_{max}}{\alpha}. \end{cases}$$

Step 1 : proof of the upper bound in (4.10.3)**.** Using assumptions (A1), (A4), and (A6), we notice that $\Xi_i - U_i$ is a sub-solution of

$$\begin{cases} \dot{z} = -2\alpha z + 2V_{max}, \\ z(0) = \dfrac{V_{max}}{\alpha}. \end{cases} \tag{4.10.4}$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \leq z(t) = \frac{V_{max}}{\alpha} \quad \text{for all } t \geq 0.$$

Step 2 : proof of the lower bound in (4.10.3)**.** Using assumptions (A1), (A3), and (A6), we notice that $\Xi_i - U_i$ is a super-solution of

$$\begin{cases} \dot{z} = -2\alpha z, \\ z(0) = 0. \end{cases} \tag{4.10.5}$$

By comparison, we have

$$\Xi_i(t) - U_i(t) \geq z(t) = 0 \quad \text{for all } t \geq 0.$$

This ends the proof of Proposition 4.10.1.

$\square$

Proposition 4.10.2 (Conservation of the order in (4.10.1))**.** *Assume (A) and (A0'), then the solution* $(U_i, \Xi_i)_i$ *of* (4.10.1) *satisfies for all* $i \in \mathbb{Z}$*,*

$$U_{i+1}(t) - \Xi_i(t) \geq h_0 \quad \textit{for all } t > 0. \tag{4.10.6}$$

In particular, using Proposition 4.10.1, this result implies that

$$U_{i+1}(t) - U_i(t) \geq h_0 \quad \textit{and} \quad \Xi_{i+1}(t) - \Xi_i(t) \geq h_0 \quad \textit{for all } t > 0. \tag{4.10.7}$$

Proof. We will prove that for all $\delta > 0$ small, we have

$$U_{i+1}(t) - \Xi_i(t) \geq h_0 - \delta \quad \text{for all } t > 0.$$

Then passing to the limit as δ goes to 0 we will obtain (4.10.6).

Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf\{t, \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - \Xi_j(t) = h_0 - \delta\}.$$

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - \Xi_j(t^*) = h_0 - \delta$. By continuity, there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - \Xi_j(t_0) = h_0 \quad \text{and} \quad U_{j+1}(t) - \Xi_j(t) \in [h_0 - \delta, h_0] \quad \text{for all } t \in [t_0, t^*].$$

Using Proposition 4.10.1, in particular that $U_j \leq \Xi_j$, and assumption (A7) combined with Remark 4.2.2, we have that

$$\begin{aligned}\alpha(U_j - \Xi_j) + 2V\,(U_{j+1} - U_j) \cdot \phi(U_j) &\leq 2V\,(U_{j+1} - \Xi_j) \cdot \phi(\Xi_j)\\ &\leq 2V\,(h_0) \cdot \phi(\Xi_j) = 0.\end{aligned}$$

This implies that (U_j, Ξ_j) satisfies for all $t \in [t_0, t^*]$,

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j \leq 0, \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) \leq \Xi_j(t_0) \\ \Xi_j(t_0) = U_{j+1}(t_0) - h_0. \end{cases}$$

Therefore, we have for all $t \in [t_0, t^*]$

$$\Xi_j(t) \leq U_{j+1}(t_0) - h_0.$$

Using again Proposition 4.10.1, in particular that the functions $(U_i)_i$ are non-decreasing in time, we obtain that

$$\Xi_j(t^*) \leq U_{j+1}(t^*) - h_0,$$

which is a contradiction. This ends the proof of Proposition 4.10.2.

□

Proposition 4.10.3 (Maximal distance between two vehicles)**.** *Assume (A) and (A0'), then the solution $(U_i, \Xi_i)_i$ of* (4.10.1) *satisfies for all $i \in \mathbb{Z}$,*

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \textit{for all } t > 0. \tag{4.10.8}$$

In particular, using Proposition 4.10.1, we have that for all $i \in \mathbb{Z}$,

$$U_{i+1}(t) - \Xi_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} \quad \textit{for all } t > 0. \tag{4.10.9}$$

Proof. We will prove that for all $\delta > 0$ small, we have for all $i \in \mathbb{Z}$,

$$U_{i+1}(t) - U_i(t) \leq h_{max} + \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + \delta \quad \text{for all } t > 0. \tag{4.10.10}$$

Passing to the limit in the previous inequality as δ goes to 0, we will obtain (4.10.8).

Let $\delta > 0$, we argue by contradiction and assume there exists a time

$$t^* = \inf\left\{ t \text{ s.t. } \exists j \in \mathbb{Z} \text{ s.t. } U_{j+1}(t) - U_j(t) > h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta \right\}.$$

Let us consider $j \in \mathbb{Z}$ such that $U_{j+1}(t^*) - U_j(t^*) = h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta$. By continuity and (A0'), there exists a time $t_0 \in [0, t^*)$ such that

$$U_{j+1}(t_0) - U_j(t_0) = h_{max} \tag{4.10.11}$$

and

$$U_{j+1}(t) - U_j(t) \in \left[h_{max}, h_{max} + \frac{2r}{\phi_0} + \frac{3V_{max}}{2\alpha} + \delta\right] \quad \text{for all } t \in [t_0, t^*].$$

We distinguish three cases.

Case 1 : $U_j(t_0) \in [-r, r]$**.** The couple (U_j, Ξ_j) satisfy for all $t \in [t_0, t^*]$

$$\begin{cases} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{max} \cdot \phi(U_j), \end{cases} \quad \text{with} \quad \begin{cases} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \le \Xi_j(t_0) - U_j(t_0) \le \dfrac{V_{max}}{\alpha}. \end{cases} \tag{4.10.12}$$

In order to compare the distance $U_{j+1} - U_j$ when U_j is inside the perturbation, we consider the worst case scenario where the vehicle j advances at a speed $V_{max}\phi_0$ and $j+1$ advances at a speed V_{max}, until $U_j \ge r$ (meaning that the vehicle j is outside the perturbation). To be more exact, we notice that the couple (U_j, Ξ_j) is a super-solution of the following system

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(t_0) = U_{j+1}(t_0) - h_{max} \\ \zeta(t_0) = v(t_0). \end{cases} \tag{4.10.13}$$

Computing the solution of (4.10.13) we get

$$\begin{cases} v(t) = \dfrac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \dfrac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0) \\ \zeta(t) = -\dfrac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} + \dfrac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0) \end{cases} \tag{4.10.14}$$

By comparison, we obtain that

$$U_j(t) \ge v(t) = \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-t_0)} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t - t_0) + v(t_0). \tag{4.10.15}$$

Let $\hat{t} = \dfrac{1}{V_{\max}\phi_0}\left(\dfrac{V_{max}\phi_0}{2\alpha} + r - U_j(t_0)\right) + t_0$. Using (4.10.15), we have that $U_j(\hat{t}) \ge r$. We now prove that $\hat{t} < t^*$. In fact, for all $t \in [t_0, \hat{t}]$, we have

$$\begin{aligned} U_{j+1}(t) - U_j(t) &\le U_{j+1}(\hat{t}) - U_j(t_0) \le V_{max}(\hat{t} - t_0) + U_{j+1}(t_0) - U_j(t_0) \\ &= V_{max}\left(\frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(t_0)\right)\right) + U_{j+1}(t_0) - U_j(t_0) \\ &\le \frac{V_{max}}{2\alpha} + \left(\frac{r - U_j(t_0)}{\phi_0}\right) + h_{max} \\ &\le \frac{V_{max}}{2\alpha} + \frac{2r}{\phi_0} + h_{max}, \end{aligned}$$

where we have used Proposition 4.10.1 for the first line. From the previous inequality and the definition of t^*, we deduce that $\hat{t} < t^*$.

The couple (U_j, Ξ_j) satisfies for all $t \in [\hat{t}, t^*]$,

$$\left\{ \begin{array}{l} \dot{U}_j = \alpha(\Xi_j - U_j) \\ \dot{\Xi}_j = \alpha(U_j - \Xi_j) + 2V_{max}, \end{array} \right. \tag{4.10.16}$$

with

$$\left\{ \begin{array}{l} h_{max} \leq U_{j+1}(\hat{t}) - U_j(\hat{t}) \leq h_{max} + \dfrac{2r}{\phi_0} + \dfrac{V_{max}}{2\alpha} \\ 0 \leq \Xi_j(\hat{t}) - U_j(\hat{t}) \leq \dfrac{V_{max}}{\alpha}. \end{array} \right. \tag{4.10.17}$$

We can easily compute the explicit form of the solution of (4.10.17),

$$U_j(t) = \Big(\frac{V_{max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t})\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} - \frac{V_{max}}{2\alpha} + V_{max}(t - \hat{t}) + \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right)$$

and

$$\Xi_j(t) = \Big(\Xi_j(\hat{t}) - U_j(\hat{t}) - \frac{V_{max}}{\alpha}\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{max}}{2\alpha} + V_{max}(t - \hat{t}) + \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right).$$

Using Proposition 4.10.1, for all $t \in [\hat{t}, t^*]$, we have that

$$U_{j+1}(t) \leq V_{max}(t - \hat{t}) + U_{j+1}(\hat{t}). \tag{4.10.18}$$

Therefore, combining the previous results, we have for all $t \in [\hat{t}, t^*]$

$$\begin{aligned} U_{j+1}(t) - U_j(t) \leq & V_{max}(t - \hat{t}) + U_{j+1}(\hat{t}) - V_{max}(t - \hat{t}) - \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right) \\ & - \Big(\frac{V_{max}}{\alpha} - \Xi_j(\hat{t}) + U_j(\hat{t})\Big)\frac{e^{-2\alpha(t-\hat{t})}}{2} + \frac{V_{max}}{2\alpha} \\ \leq & U_{j+1}(\hat{t}) - \frac{1}{2}\left(\Xi_j(\hat{t}) + U_j(\hat{t})\right) + \frac{V_{max}}{2\alpha} \\ \leq & U_{j+1}(\hat{t}) - U_j(\hat{t}) + \frac{V_{max}}{2\alpha} \\ \leq & h_{max} + \frac{2r}{\phi_0} + \frac{V_{max}}{\alpha}, \end{aligned}$$

where we have used Proposition 4.10.1 for the second and third inequality and we have used (4.10.17) for the last inequality. The previous inequality remains valid for $t = t^*$ which gives us a contradiction.

Case 2 : $U_j(t_0) > r$. In this case, the couple (U_j, Ξ_j) satisfies system (4.10.16) for all $t \in (t_0, t^*]$, with the following initial conditions

$$\left\{ \begin{array}{l} U_j(t_0) = U_{j+1}(t_0) - h_{max} \\ 0 \leq \Xi_j(t_0) - U_j(t_0) \leq \dfrac{V_{max}}{\alpha}. \end{array} \right. \tag{4.10.19}$$

As above, the explicit solution of (4.10.16)-(4.10.19) has the following form,

$$U_j(t) = \Big(\frac{V_{max}}{\alpha} - \Xi_j(t_0) + U_j(t_0)\Big)\frac{e^{-2\alpha(t-t_0)}}{2} - \frac{V_{max}}{2\alpha} + V_{max}(t-t_0) + \frac{1}{2}\left(\Xi_j(t_0) + U_j(t_0)\right)$$

and

$$\Xi_j(t) = \Big(\Xi_j(t_0) - U_j(t_0) - \frac{V_{max}}{\alpha}\Big)\frac{e^{-2\alpha(t-t_0)}}{2} + \frac{V_{max}}{2\alpha} + V_{max}(t-t_0) + \frac{1}{2}\left(\Xi_j(t_0) + U_j(t_0)\right).$$

Arguing as above, we will obtain $U_{j+1}(t^*) - U_j(t^*) \le h_{max} + \dfrac{V_{max}}{2\alpha}$ which is a contradiction.

Case 3 : $U_j(t_0) < -r$. We treat this case in 3 steps.

Step 1 : left of the perturbation. We denote by

$$\hat{t} = \inf\left\{t \ge t_0 \text{ s.t. } U_j(t) = -r\right\}.$$

For all $t \in [t_0, \hat{t}]$, the couple (U_j, Ξ_j) satisfies (4.10.16)-(4.10.19) and therefore has the same form as the one presented in Case 2. In particular, for all $t \in [t_0, \hat{t}]$, we have

$$U_{j+1}(t) - U_j(t) \le h_{max} + \frac{V_{max}}{2\alpha}. \tag{4.10.20}$$

This implies that $\hat{t} < t^*$.

Step 2 : inside the perturbation. In the interval $[\hat{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (4.10.12) with the following initial condition

$$\begin{cases} U_{j+1}(\hat{t}) - U_j(\hat{t}) \le h_{max} + \dfrac{V_{max}}{2\alpha} \\ 0 \le \Xi_j(\hat{t}) - U_j(\hat{t}) \le \dfrac{V_{max}}{\alpha}. \end{cases}$$

The couple (U_j, Ξ_j) is a super-solution of

$$\begin{cases} \dot{v} = \alpha(\zeta - v) \\ \dot{\zeta} = \alpha(v - \zeta) + 2V_{max}\phi_0, \end{cases} \quad \text{with} \quad \begin{cases} v(\hat{t}) = U_{j+1}(\hat{t}) + h_{max} + \dfrac{V_{max}}{2\alpha} \\ \zeta(\hat{t}) = v(\hat{t}). \end{cases} \tag{4.10.21}$$

Computing the solution of (4.10.21), and by comparison, for all $t \in [\hat{t}, t^*]$, we have

$$U_j(t) \ge \frac{V_{max}\phi_0}{2\alpha} e^{-2\alpha(t-\hat{t})} - \frac{V_{max}\phi_0}{2\alpha} + V_{max}\phi_0(t-\hat{t}) + v(\hat{t}).$$

Let $\tilde{t} = \frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t})\right) + \hat{t}$. Using (4.10.15), we have that $U_j(\tilde{t}) \geq r$. We now prove that $\tilde{t} < t^*$. We recall that $U_j(\hat{t}) = -r$. In fact, for all $t \in [\hat{t}, \tilde{t}]$, we have

$$\begin{aligned}
U_{j+1}(t) - U_j(t) &\leq U_{j+1}(\tilde{t}) - U_j(\hat{t}) \leq V_{max}(\tilde{t} - \hat{t}) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\
&= V_{max}\left(\frac{1}{V_{\max}\phi_0}\left(\frac{V_{max}\phi_0}{2\alpha} + r - U_j(\hat{t})\right)\right) + U_{j+1}(\hat{t}) - U_j(\hat{t}) \\
&\leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max},
\end{aligned}$$

where we have used Proposition 4.10.1 for the first line. From the previous inequality and the definition of t^*, we deduce that $\tilde{t} < t^*$.

Step 3 : right of the perturbation. In the interval $[\tilde{t}, t^*]$, the couple (U_j, Ξ_j) satisfies (4.10.16), with the following initial condition

$$U_{j+1}(\tilde{t}) - U_j(\tilde{t}) \leq \frac{V_{max}}{\alpha} + \frac{2r}{\phi_0} + h_{max}.$$

Proceeding like before, we can prove that for all $t \in [\tilde{t}, t^*]$, we have

$$U_{j+1}(t) - U_j(t) \leq \frac{3V_{max}}{2\alpha} + \frac{2r}{\phi_0} + h_{max},$$

which gives us a contradiction for $t = t^*$. This ends the proof of Proposition 4.10.3.

□

4.11 Proof of Theorem 4.9.1

Before we give the proof of Theorem 4.9.1, we need the following result.

Lemma 4.11.1 (Link between the velocities)**.** *Assume (A). Let* $((U_j)_j, (\Xi_j)_j)$ *be the solution of (4.3.1) with an initial condition* $(U_j(0), \Xi_j(0))_j$ *satisfying (A0'). Then we have*

$$\dot{U}_j(t) = -M\left(u\left(t, U_j(t)\right), \left[\xi\left(t, \cdot\right)\right]\right)\left(U_j(t)\right) \tag{4.11.1}$$

and

$$\dot{\Xi}_j(t) = -L\left(\Xi_j(t), \xi\left(t, \Xi_j(t)\right), \left[u(t, \cdot)\right]\right)\left(\Xi_j(t)\right), \tag{4.11.2}$$

where u *and* ξ *are continuous functions such that*

$$\begin{cases} u(t,x) = \rho_*(t,x) = \rho(t,x) \text{ for } x = U_j(t),\ j \in \mathbb{Z}, \\ u \text{ is decreasing in } x, \end{cases} \tag{4.11.3}$$

$$\begin{cases} \xi(t,x) = \sigma_*(t,x) = \sigma(t,x) \text{ for } x = \Xi_j(t),\ j \in \mathbb{Z}, \\ \xi \text{ is decreasing in } x, \end{cases} \tag{4.11.4}$$

where ρ *and* σ *are defined respectively in* (4.2.2) *and* (4.3.2) *(with* $\varepsilon = 1$*).*

Proof. We drop the time dependence to simplify the presentation. Let $j \in \mathbb{Z}$. We recall that we chose $D = h_{max} + 3V_{max}/(2\alpha) + 2r/\phi_0$. Using the fact that $u(t, U_j(t)) = -(j+1)$ and (4.11.3), we have for all $z \in [0, +\infty)$,

$$\begin{cases} \xi(U_j + z) - u(U_j) > \xi(\Xi_j) - u(U_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ \xi(U_j + z) - u(U_j) \le 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Using Proposition 4.10.1, in particular that $\Xi_j - U_j \le D$, we have

$$\begin{aligned} &M\left(u\left(t, U_j(t)\right), \left[\xi\left(t, \cdot\right)\right]\right)\left(U_j(t)\right) = \int_0^D E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz \\ &= \int_0^{\Xi_j - U_j} E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz + \int_{\Xi_j - U_j}^D E\left(\xi\left(U_j + z\right) - u\left(U_j\right)\right) dz \\ &= -\alpha\left(\Xi_j - U_j\right). \end{aligned}$$

Combining this result with (4.3.1), we obtain (4.11.1). We now turn to the proof of (4.11.2).

We will begin by computing $K\left(\xi\left(t, \Xi_j(t)\right), [u(t, \cdot)]\right)\left(\Xi_j(t)\right)$. Using the fact that

$$\xi\left(t, \Xi(t)\right) = -(j+1)$$

and (4.11.4), we have for all $z \in [0, +\infty)$,

$$\begin{cases} u(\Xi_j - z) - \xi(\Xi_j) < u(U_j) - \xi(\Xi_j) = 0 & \text{if } z \in [0, \Xi_j - U_j) \\ u(\Xi_j - z) - \xi(\Xi_j) \ge 0 & \text{if } z \in [\Xi_j - U_j, +\infty). \end{cases}$$

Thanks to Proposition 4.10.1, this implies that

$$K\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right) = \int_0^{\Xi_j - U_j} F\left(u\left(\Xi_j - z\right) - \xi\left(\Xi_j\right)\right) dz = \Xi_j - U_j.$$

We now turn to the computation of $N\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right)$. We recall that thanks to Proposition 4.10.2, we have $U_{j+1} - \Xi_j \ge h_0$. In particular, we have that

$$\begin{cases} u(\Xi_j + z) - \xi(\Xi_j) > u(U_{j+1}) - \xi(\Xi_j) = -1 & \text{if } z \in [0, U_{j+1} - \Xi_j) \\ u(\Xi_j + z) - \xi(\Xi_j) \le -1 & \text{if } z \in [U_{j+1} - \Xi_j, +\infty). \end{cases}$$

Once more thanks to Proposition 4.10.3, we obtain

$$N\left(\xi\left(t, \Xi_j(t)\right), \left[u\left(t, \cdot\right)\right]\right)\left(\Xi_j(t)\right) = \int_0^{U_{j+1} - \Xi_j} I\left(u\left(\Xi_j + z\right) - \xi\left(\Xi_j\right)\right) dz = U_{j+1} - \Xi_j.$$

Combining the previous results with (4.3.16) and (4.3.1), we obtain (4.11.2).

□

Proof of Theorem 4.9.1. We remark that thanks to (4.11.3) and (4.11.4), we have for $x = U_j(t)$ and $y = \Xi_j(t),\ \ j \in \mathbb{Z}$,

$$\tilde{M}\left(\rho_*(t,x), [\sigma_*(t,\cdot)]\right)(x) = \tilde{M}\left(u(t,x), [\xi(t,\cdot)]\right)(x) \geq M\left(u(t,x), [\xi(t,\cdot)]\right)(x),$$

and

$$\tilde{L}\left(y, \sigma_*(t,y), [\rho_*(t,\cdot)]\right)(y) = \tilde{L}\left(y, \xi_*(t,y), [u(t,\cdot)]\right)(y) \geq L\left(y, \xi_*(t,y), [u(t,\cdot)]\right)(y).$$

Using Lemma 4.11.1, and Definition 4.4.1, we can see that (ρ_*, σ_*) is a discontinuous viscosity super-solution of (4.9.1). We obtain a similar result for (ρ^*, σ^*), therefore, (ρ, σ) is a discontinuous viscosity solution of (4.9.1).

□

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Chapitre 5

A comparison principle for Hamilton-Jacobi equation with moving in time boundary

M. Zaydan [1]

Abstract

In this chapter we consider an Hamilton-Jacobi equation on a moving in time domain. The boundary is described by a C^1 function. We show how we derive this equation from the work of [LLG98]. We only prove a comparison principle since the proof of other theoritical results can be found in [IM13]. At the end of the chapter, we consider a short homogenization result in order to reinforce the traffic flow interpretation of the equation.

5.1 Introduction

In this chapter, we consider an Hamilton-Jacobi equation posed on a moving in time domain. More precisely, the equation is posed in several interval of the real axis whose boundary (called "*junction points*") move in time. The junction points are denoted by $b_i\,(t) \in \mathbb{R}$ at time t and we set for $i \in \{1, ..., N+1\}$,

$$B_i = \{(t, x) \in (0, T) \times \mathbb{R}, \text{s.t. } b_{i-1}(t) < x < b_i(t)\}\,.$$

We will show in Section 5.2 that the considered equation can be obtained by a first order bus-vehicles interaction model, introduced in [LLG98], where authors assumed that buses represent a moving capacity restriction, i.e. the density of vehicles is reduced near the

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buses zones. In order to simplify the notations, let us first introduce the *flux limiting function*, (see [IM13]). For $i \in \{1, ..., N\}$, $t \in \mathbb{R}^+$ and $p = (p_1, p_2) \in \mathbb{R}^2$

$$F_{A_i}(t, p_1, p_2) = \max\Big(A_i(t), H_{i,i}^+(t, p_1), H_{i+1,i}^-(t, p_2)\Big)$$

where A_i is a locally lipschitz function and $H_{i,i}^+$ (resp. $H_{i+1,i}^-$) is the nondecreasing (resp. nonincreasing) part of the Hamiltonian $H_{i,i}$ (resp. $H_{i+1,i}$) whose definition is given later. The equation is given by

$$\begin{cases} u_t + H_i(u_x) = 0 & \text{if } (t,x) \in B_i, i = 1, .., N+1 \\ \dfrac{d}{dt}(u(t, b_i(t))) + F_{A_i}\Big(t, u_x^{i,-}(t,x), u_x^{i,+}(t,x)\Big) = 0 & \text{if } x = b_i(t), i = 1, ..., N \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{5.1.1}$$

where $u_t = \dfrac{\partial u}{\partial t}$ and $u_x = \dfrac{\partial u}{\partial x}$ denotes respectively the time and the space derivative. Moreover, we denote by

$$u_x^{i,+}(t, b_i(t)) = \lim_{\substack{(t,x)\to(t,b_i(t)) \\ x > b_i(t)}} u_x(t,x)$$
$$u_x^{i,-}(t, b_i(t)) = \lim_{\substack{(t,x)\to(t,b_i(t)) \\ x < b_i(t)}} u_x(t,x).$$

Equation (5.1.1) is quite similar to the one introduced by Imbert and Monneau in [IM13]. The difference here is that we consider a junction which moves in time. Stability, existence of solution and even the reduction of the class of test functions for (5.1.1) can be easily obtained adapting the proofs of these results in [IM13]. In this chapter, we prove a comparison principle for equation (5.1.1). We borrow the idea introduced in [BBCI16] and we use a localization procedure in order to insert the "good" test function in the next step of the proof. Let us now clarify the notations used in (5.1.1).

Assumptions and Notations (A).

- (A1) The functions $b_1, ..., b_N$ are time dependent differentiable functions such that $b_{i+1} > b_i$. We denote also by $b_0 = -\infty$ and $b_{N+1} = +\infty$. Moreover, we assume that for all $j \in \{1, ..., N\}$, b_j' is a locally lipschitz function.
- (A2) The Hamiltonians $H_1, ..., H_{N+1} : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions : for all $i \in \{1, ..., N+1\}$,

$$\begin{cases} H_i \text{ is continuous,} \\ H_i \text{ is superlinear i.e. } \displaystyle\lim_{|p|\to+\infty} \frac{H_i(p)}{|p|} = +\infty. \end{cases}$$

- (A3) For $i \in \{1, ..., N+1\}$ and for $k \in \{i, i+1\}$, $H_{k,i}(t,p) = H_k(p) - b_i'(t)p$. Morover, we assume that for all $i \in \{1, ..., N\}$, $k \in \{i, i+1\}$ and for all $t \in \mathbb{R}^+$, the Hamiltonian $H_{k,i}(t, \cdot)$ is quasi-convex. We denote by $H^+_{k,i}(t, \cdot)$ and $H^-_{k,i}(t, \cdot)$ respectively the non-decreasing and the non-increasing part of $H_{k,i}(t, \cdot)$.
- (A4) For all $i \in \{1, ..., N\}$,the flux limiter $A_i : [0,T] \to \mathbb{R}$ is a locally lipschitz function.

Main results. Our main goal is to prove a comparison principle for equation (5.1.1). In [IM13, BBCI16, IM14], a proof of comparison principle for (5.1.1) in the case where b_i=constant is done. In fact, they prove this result in a more general domain (such as a network, junction or two half spaces in $\mathbb{R}^N$) and more general Hamiltonians (depending on x and t). In [IM13], they prove a comparaison principle by replacing the classical penalization term $\frac{(x-y)^2}{2\varepsilon}$ by the new term $\varepsilon G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ where G is a vertex test function which allows to compare the Hamiltonians in different branches of the domain. As we mentioned above, our proof uses the idea introduced in [BBCI16] which keep the classical term $\frac{(x-y)^2}{2\varepsilon}$ and uses the fact that H^+ and H^- are respectively increasing and decreasing functions. Let us mention also the work [LS17] where the authors consider a Kirchoff-type Neumann condition at the junction and proved that its solution satisfy a comparison principle and then they proved that the flux-limited solutions reduce to Kirchoff-type viscosity solutions. Finally, concerning comparison principle for Hamilton-Jacobi equations with boundary conditions of Neumann type, let us cite [BL91, Gue16, Bar99, FIM12a, I$^+$91]. Combaining the comparison principle for (5.1.1) with Perron method, we obtain the following result

Theorem 5.1.1. *Assume (A) and that the initial datum u_0 is lipschitz continuous function. Then there exists a unique continuous viscosity solution u of* (5.1.1) *such that for all $T > 0$, there exists a constant $C_T > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}$,*

$$|u(t,x) - u_0(x)| \leq C_T.$$

The second result of this chapter is an homogenization result. We consider a macroscopic model describing the presence of a bus (or a large truck) and prove that the solution of the Hamilton-Jacobi formulation of this model converges towards the unique solution of equation (5.1.1) with particulary one Hamiltonian and one boundary function. As previous works [FSZ17b, FSZ17a, GIM15], the proof of convergence relies on the construction of suitable correctors. The difference here is that we don't consider a microscopic model since to our knowledge, no microscopic model considering the bus as a moving capacity constraint exist.

5.2 Traffic flow motivation and derivation of a Hamilton-Jacobi equation

5.2.1 A first order bus-vehicles interaction model

In this section, we show how we can obtain equation (5.1.1). To simplify the work and since the idea remains the same, we consider the case of one Hamiltonian H and one function b describing the bus trajectory. Before starting, we mention that our model was introduced in [LLG98] in order to study the interaction between buses and the surrounding traffic flow. Several papers about modeling the effect of buses on the traffic flow exists, see [LMP11, DL05, DMG14, CG07, GNPT07].

The idea is to consider the traffic flow on a single road where a bus is moving. In this model, we assume that the fundamental physical parameters of the model, i.e. the maximum density and the maximum mean velocity, don't depend on the position x if $x \neq b(t)$, i.e. the characteristics of the infrastructure don't change with the position far from the bus. The traffic flow is assumed to be described by a first order macroscopic model of the LWR type if the space variable $x \neq b(t)$. Bus should be considered as a moving capacity restriction from other drivers point of view. Authors in [LLG98] extended the notion of demand and supply introduced in [LK99] to the moving frame using the change of variables $\zeta = x - b(t)$. The model is given by

$$\begin{cases} \rho_t + (f(\rho))_x = 0 & \text{if } x \neq b(t) \\ \tilde{f}(t, \rho(t, x^-)) = \min\Big(B(t), \tilde{f}_D(t, \rho(t, x^-)), \tilde{f}_S(t, \rho(t, x^+))\Big) & \text{if } x = b(t) \end{cases} \tag{5.2.1}$$

where ρ is the density of vehicles at time t and position x, f is a stricly concave function (as Greenshield model [GCM+35]), reaching its unique maximum at a critical density ρ_c, describing the flow and $\tilde{f}(t,p) = f(p) - b'(t) \cdot p$. The function B is the limiter of the passing flux through the bus at time t. The definition of $\tilde{f}$ yields that for all t, the function $\tilde{f}(t, \cdot)$ reaches a unique maximum at a point denoted $\tilde{\rho}_c(t)$. The functions $\tilde{f}_D$ and $\tilde{f}_S$ are respectively the Demand and Supply functions defined as follows

$$\tilde{f}_D(t,p) = \begin{cases} \tilde{f}(t, \tilde{\rho}_c(t)) & \text{if } p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t,p) & \text{if } p < \tilde{\rho}_c(t) \end{cases}$$

and

$$\tilde{f}_S(t,p) = \begin{cases} \tilde{f}(t,p) & \text{if } p \geq \tilde{\rho}_c(t) \\ \tilde{f}(t, \tilde{\rho}_c(t)) & \text{if } p < \tilde{\rho}_c(t). \end{cases}$$

Before passing to the Hamilton-Jacobi formulation, let us present the two points below in order to clarify the model.

– The trajectory of the bus can be approximated by assuming that $b' = 0$ (bus-stops) or that b' is equal to the desired bus-speed V_b (if the bus enjoys special lanes) or is the minimum between the desired bus speed V_b and the local traffic speed, i.e.

$$b'(t) = \min\left(V_b, V\left(\rho\left(t, b(t)^+\right)\right)\right).$$

In this chapter we will only consider the second case i.e. when the velocity of b is V_b (see section 5.4). The case where $b' = 0$ reduces to the work [IM13]. In the case where the velocity of the bus depends on the density of vehicles, we will obtain a strongly coupled PDE-ODE system and we will have to introduce a good notion of solution for the system. In this case, we were not able to get a uniqueness result. Note that several paper like [LMP11, BCG12, BCG10, CG13, DMG14] considered the case where b depends on the density of vehicles but considered a different macroscopic model as this chapter.

– The second equation in (5.2.1) means that the passing flux through $x = b(t)$ is equal to the minimum between the upstream Demand, the downstream Supply and the flux limiter $B(t)$. Note that the flux at time t is limited only if $B(t) < \tilde{f}(t, \tilde{\rho}_c) = \max \tilde{f}(t, \cdot)$. The Demand function at the point $x = b(t)^-$ is the greatest possible outflow at that point and the Supply function at the point $x = b(t)^+$ is the greatest possible inflow at that point. Note that the passing flux through the bus is $\tilde{f}$ and not f. In fact, f describes the flux at a fix point x while the "real" passing flux throught the bus is equal to the flux assuming that the bus is fix minus the non-passing flux due to the variation of the position of b.

5.2.2 The Hamilton-Jacobi formulation

In order to derive the Hamilton-Jacobi equation, we proceed as in [IMZ13] considering the continous analogue of the discrete vehicles label defined by

$$\begin{cases} U^1(t,x) = g(t) - \int\limits_x^{b(t)} \rho(t,y)dy & \text{if } x < b(t) \\ U^2(t,x) = g(t) + \int\limits_{b(t)}^{x} \rho(t,y)dy & \text{if } x > b(t) \end{cases}$$

with

$$g(t) = -\int_0^t f\left(\rho\left(s, b(s)^-\right)\right) - b'(s)\rho\left(s, b(s)^-\right) ds.$$

Formally, we have the following equalities

$$\begin{aligned}U_t^1 &= g'(t) - \int_x^{b(t)} \rho_t(t,y)dy - b'(t)\rho\left(t, b(t)^-\right) \\ &= g'(t) + \int_x^{b(t)} \left(f\left(\rho\left(t,y\right)\right)\right)_y dy - b'(t)\rho\left(t, b(t)^-\right) \\ &= g'(t) - f\left(\rho\left(t,x\right)\right) + f\left(\rho\left(t, b(t)^-\right)\right) - b'(t)\rho\left(t, b(t)^-\right).\end{aligned}$$

Recalling the definition of g, we deduce that $U_t^1 + f\left(U_x^1\right) = 0$ if $x < b\left(t\right)$. Similary, we have $U_t^2 + f\left(U_x^2\right) = 0$ if $x > b(t)$. In fact, the last equality is true because $-g'(t)$ represents the passing flux at $b(t)$ which is equal to the outgoing flux at $b(t)$, i.e.

$$g'(t) = -f\left(\rho\left(t, b(t)^+\right)\right) + b'(t)\rho\left(t, b(t)^+\right).$$

We now set

$$u(t,x) = \begin{cases} -U^1\left(t,x\right) & \text{if } x < b\left(t\right) \\ -U^2\left(t,x\right) & \text{if } x > b\left(t\right) \end{cases}$$

and we define the Hamiltonian $H(p) = -f(-p)$. Then we deduce that we have

$$u_t + H\left(u_x\right) = 0 \quad \text{if } x \neq b(t).$$

The junction condition. Recalling the definition of U^1 and U^2, we have that

$$\frac{d}{dt}\left(u(t,b(t))\right) = -g'(t) = \min\Big(B\left(t\right), \tilde{f}_D\left(t, \rho\left(t, b\left(t\right)^-\right)\right), \tilde{f}_S\left(t, \rho\left(t, b\left(t\right)^+\right)\right)\Big).$$

Let $\tilde{H}\left(t,p\right) = H\left(p\right) - b'\left(t\right)p$ and $A\left(t\right) = -B\left(t\right)$. Denoting $\tilde{H}^+\left(t,\cdot\right)$ and $\tilde{H}^-\left(t,\cdot\right)$ respectively the non-deacreasing part and the non-increasing part of $\tilde{H}\left(t,\cdot\right)$, we deduce the following junction condition

$$\frac{d}{dt}u\left(t,b(t)\right) + \max\Big(A\left(t\right), \tilde{H}^+\left(t, u_x^-\left(t,b\left(t\right)\right)\right), \tilde{H}^-\left(t, u_x^+\left(t,b\left(t\right)\right)\right)\Big) = 0.$$

5.3 Comparison principle for (5.1.1)

In this section we present the main result of this chapter which is the comparison principle for (5.1.1). We give first the definition of viscosity solutions. As usual, we begin by introducing the class of test functions. For $T > 0$, set $B = (0,T) \times \mathbb{R}$.

Test functions. We denote by $\mathcal{C}^1(B)$ the class of test functions. If $\varphi \in \mathcal{C}^1(B)$, then

- φ is continuous.
- The restriction of φ on each B_i is C^1.
- For all $i = 1, ..., N$, the time dependent function $\varphi(t, b_i(t))$ is C^1 in time. Moreover,

$$\begin{aligned}\frac{d}{dt}\varphi(t, b_i(t)) &= \varphi_t^+(t, b_i(t)) + b_i'(t)\varphi_x^+(t, b_i(t)) \\ &= \varphi_t^-(t, b_i(t)) + b_i'(t)\varphi_x^-(t, b_i(t)).\end{aligned}$$

We recall the definition of the upper and lower semi-continuous envelopes u^* and u_* of a locally bounded function u on J_T,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \text{ and } u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

Definition 5.3.1. Assume (A) and let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$.

i) We say that u is a sub-solution (resp. super-solution) of (5.1.1) in $[0,T] \times \mathbb{R}$ if $u^*(0,x) \le u_0(x)$ (resp. $u_*(0,x) \ge u_0(x)$) and if for all test function $\varphi \in \mathcal{C}^1(B)$ touching u^* from above (resp. touching u_* from below) at $(t_0, x_0) \in B$, we have

$$\varphi_t + H_i(u_x) \le 0 \text{ (resp. } \ge 0) \text{ at } (t_0,x_0) \text{ if } (t_0,x_0) \in B_i$$

$$\frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}\left(t_0, u_x^{i,-}(t_0,x_0), u_x^{i,+}(t_0,x_0)\right) \le 0 \text{ (resp. } \ge 0) \text{ if } x_0 = b_i(t_0).$$

i) We say that u is a viscosity solution of (5.1.1) if u is a sub-solution and a super-solution of (5.1.1).

Theorem 5.3.1 (Reduction of test functions). *Assume (A). We fix $i \in \{1, ..., N\}$ and assume that*

$$A_i(t) \ge A_i^0(t) = \max\left(\min_{\mathbb{R}} H_{i,i}(t,\cdot), \min_{\mathbb{R}} H_{i+1,i}(t,\cdot)\right).$$

Let $t_0 \in (0,T)$ and let $p_{i,i}^{A_i(t_0)}$ and $q_{i+1,i}^{A_i(t_0)}$ two constant satisfying

$$\begin{cases} H_{i,i}\left(t_0, p_{i,i}^{A_i(t_0)}\right) = H_{i,i}^-\left(t_0, p_{i,i}^{A_i(t_0)}\right) = A_i(t_0) \\ H_{i+1,i}\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = H_{i+1,i}^+\left(t_0, q_{i+1,i}^{A_i(t_0)}\right) = A_i(t_0). \end{cases}$$

We consider the following Hamilton-Jacobi equation

$$u_t + H_k(u_x) = 0 \text{ for } (t,x) \in B_k, k = i, i+1. \tag{5.3.1}$$

- *Let $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ an upper semi-continuous sub-solution of* (5.3.1) *and satisfying*

$$u(t, b_i(t)) = \limsup_{(s,y)\to(t,b_i(t)), y>b_i(s)} u(s,y) = \limsup_{(s,y)\to(t,b_i(t)), y<b_i(s)} u(s,y). \tag{5.3.2}$$

If any test function φ *touching* u *from above at* $(t_0, b_i(t_0))$ *with*

$$\varphi(t,x) = g(t) + q_{i+1,i}^{A_i(t_0)}(x - b_i(t))\, 1_{\{x-b_i(t)>0\}} + p_{i,i}^{A_i(t_0)}(x - b_i(t))\, 1_{\{x-b_i(t)<0\}} \tag{5.3.3}$$

for some $g \in C^1(0,+\infty)$, *we have*

$$\frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}\left(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))\right) \le 0$$

then u *is a sub-solution of*

$$\frac{d}{dt}u(t, b_i(t)) + F_{A_i}\left(t, u_x^{i,-}(t, b_i(t)), u_x^{i,+}(t, b_i(t))\right) = 0 \quad \text{at } t_0.$$

– *Let* $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ *a lower semi-continuous super-solution of* (5.3.1). *If any test function* φ *touching* u *from below at* $(t_0, b_i(t_0))$ *with* φ *is defined as in* (5.3.3), *we have*

$$\frac{d}{dt}\varphi(t_0, b_i(t_0)) + F_{A_i}\left(t_0, \varphi_x^{i,-}(t_0, b_i(t_0)), \varphi_x^{i,+}(t_0, b_i(t_0))\right) \ge 0$$

then u *is a super-solution of*

$$\frac{d}{dt}u(t, b_i(t)) + F_{A_i}\left(t, u_x^{i,-}(t, b_i(t)), u_x^{i,+}(t, b_i(t))\right) = 0 \quad \text{at } t_0.$$

The proof of this theorem is similar to the proof of Theorem 2.7 in [IM13].

The next proposition is concerned with the supremum of sub-solutions. Such a result is used in the Perron process to construct solutions.

Proposition 5.3.2. *Assume (A). Let* $\mathcal{A}$ *be a nonempty set and let* $(u_a)_{a\in\mathbb{A}}$ *be a familly of sub-solutions of* (5.1.1) *on* $(0,T) \times \mathbb{R}$ *and satisfying* (5.3.2) *for all* $i \in \{1, ..., N\}$. *Let us assume that*

$$u = \sup_{a\in\mathcal{A}} u_a$$

is locally bounded on $(0,T) \times \mathbb{R}$. *Then* u *is a sub-solution of* (5.1.1) *on* $(0,T) \times \mathbb{R}$.

The proof is standard. The only new idea is to prove that u^* satisfies (5.3.2) in order to use the result of the preceding theorem. By Perron method, and the last proposition, we easily obtain the following result.

Theorem 5.3.3. *Assume (A) and that the initial datum* u_0 *is lipschitz continuous. Then there exists a viscosity solution* u *of* (5.1.1) *in* $[0,T) \times \mathbb{R}$ *and a constant* $C_T > 0$ *such that*

$$|u(t,x) - u_0(x)| \le C_T.$$

Theorem 5.3.4 (Comparison principle)**.** *Let $T > 0$. Assume that u_0 is a lipschitz continuous function. Let u be an upper semi-continuous sub solution and v be a lower semi-continuous super solution of (5.1.1), s.t. there exists a constant $K > 0$, s.t. for all $t \in [0, T]$, we have $u(t, x) \leq u_0(x) + Kt$ and $v(t, x) \geq u_0(x) - Kt$, then we have*

$$u(t, x) \leq v(t, x) \quad \textit{for all } (t, x) \in [0, T] \times \mathbb{R} \; .$$

As we mentioned before, we will adapt the idea introduced in [BBCI16]. The main difference here is the localization procedure in order to choose the good test function. Before starting the proof, we state the following useful remarks.

Remark 5.3.5. *We recall that for all $t > 0$, and for all $i \in \{1, ..., N+1\}$, $j \in \{1, ..., N\}$, the Hamiltonian $H_{i,j}(t, \cdot)$ is superlinear (see (A2)). Therefore, there exists a constant $C_t > 0$, such that for all $p \in \mathbb{R}$, we have $|p| \leq \max(C_t, H_{i,j}(t, p))$. We will denote by C_T the upper bound of C_t for $t \in [0, T]$.*

Remark 5.3.6. *There exists a constant $B_T > 0$ and a modulus of continuity w_T such that for all $t \in [0, T], p \in \mathbb{R}$ and for all $i \in \{1, ..., N+1\}$ and for $k = i, i+1$, we have*

$$\begin{cases} |H_{k,i}(t, p) - H_{k,i}(s, p)| \leq B_T |t - s| \cdot |p| \\ |H^+_{k,i}(t, p) - H^+_{k,i}(s, p)| \leq B_T \max(|t - s| \cdot |p|, w_T(|t - s|)) \\ |H^-_{k,i}(t, p) - H^-_{k,i}(s, p)| \leq B_T \max(|t - s| \cdot |p|, w_T(|t - s|)) \, . \end{cases}$$

Proof. The proof of these inequalities is very simple. We get the first line by the definition of the Hamiltonian $H_{k,i}$. To prove the second and the third lines, we simply use the continuity of the functions for $k = i, i+1$

$$\begin{cases} t \to \min_{\mathbb{R}} H_{k,i}(t, \cdot) \\ t \to p_0^{k,i}(t) = \max\{p \text{ s.t. } H_{k,i}(t, p) = \min_{\mathbb{R}} H_{k,i}(t, \cdot)\} \\ t \to q_0^{k,i}(t) = \min\{p \text{ s.t. } H_{k,i}(t, p) = \min_{\mathbb{R}} H_{k,i}(t, \cdot)\} \, . \end{cases}$$

□

Proof of Theorem 5.3.4. We introduce

$$M = \sup_{(t,x) \in [0,T) \times \mathbb{R}} \{u(t, x) - v(t, x)\} \, .$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$. Let L and R two constant such that $L < \min_{t \in [0,T]} b_1(t)$ and $R > \max_{t \in [0,T]} b_N(t)$. Let $\eta > 0$, we introduce

$$M_\eta = \sup_{\substack{t \in [0,T] \\ L \leq x \leq R}} \left\{ u(t, x) - v(t, x) - \frac{\eta}{T - t} \right\} . \tag{5.3.4}$$

Since we consider the maximum of an upper-semi continuous function on a compact set, we deduce that the maximum is reached at a point that we denote (t_η, x_η).

Case 1 : $M_\eta \leq 0$. Then we consider the following supremum

$$M_{\varepsilon,\alpha} = \sup_{\substack{t,s\in[0,T] \\ x,y\in\mathbb{R}}} \left\{ u(t,x) - v(s,y) - \frac{\eta}{T-t} - \frac{(x-y)^2}{2\varepsilon} - \frac{(t-s)^2}{2\varepsilon} - \alpha x^2 - \alpha y^2 \right\}$$

Classicaly, $M_{\varepsilon,\alpha} \geq M/2 > 0$ for η and α small enough. Moreover, the maximum is reached at (t,s,x,y) and $\alpha x \to 0$, $\alpha y \to 0$ as $\alpha \to 0$. We denote by $\bar{x}$ the common limit of x and y as ε goes to zero and by $\bar{t}$ the common limit of t and s as ε goes to zero. It's clear that $\bar{t} \neq 0$ since u_0 is Lipshitz. Moreover, taking ε to zero and using the upper-semi continuity property, we obtain that $u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) - \dfrac{\eta}{T-t} \geq M/2$, which implies that $\bar{x} \notin [L,R]$ because $M_\eta \leq 0$. We deduce that for ε small enough, either $x < b_1(t)$ and $y < b_1(s)$ or $x > b_N(t)$ and $y > b_N(s)$. Using the fact that u is a sub-solution and v is a super-solution, we obtain for $j = 1$ or $j = N+1$

$$\begin{cases} \dfrac{\eta}{(T-t)^2} + \dfrac{t-s}{\varepsilon} + H_j\left(\dfrac{x-y}{\varepsilon} + 2\alpha x\right) \leq 0 \\ \dfrac{t-s}{\varepsilon} + H_j\left(\dfrac{x-y}{\varepsilon} - 2\alpha y\right) \geq 0. \end{cases}$$

Subtracting the two inequalities and taking α to zero, we obtain a contradiction.

Case 2 : $M_\eta > 0$ and $x_\eta \neq b_i(t_\eta)$ for all $i \in \{1,...,N\}$. In this case, we consider

$$M_{\varepsilon,\alpha} = \sup_{\substack{t,s\in[0,T] \\ x,y\in\mathbb{R}}} \left\{ \begin{array}{l} u(t,x) - v(s,y) - \dfrac{\eta}{T-t} - \dfrac{(x-y)^2}{2\varepsilon} - \dfrac{(t-s)^2}{2\varepsilon} \\ -\alpha\left((x-x_\eta)^2 + (t-t_\eta)^2\right) \end{array} \right\}$$

Classicaly, $M_{\varepsilon,\alpha} \geq M_\eta > 0$. Moreover, the maximum is reached at (t,s,x,y) and we denote by $\bar{x}$ and $\bar{t}$ respectively the common limit of x and y and the common limit of t and s as ε goes to zero. Moreover, taking ε to zero, and using the upper-semi continuity, we obtain that

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) - \frac{\eta}{T-\bar{t}} - \alpha\left((\bar{x}-x_\eta)^2 + (\bar{t}-t_\eta)^2\right) \geq M_\eta. \tag{5.3.5}$$

If $\bar{x} \notin [L,R]$, we proceed as the case where $M_\eta \leq 0$. If not, then (5.3.5) and the definition of M_η implies that

$$M_\eta - \alpha\left((\bar{x}-x_\eta)^2 + (\bar{t}-t_\eta)^2\right) \geq M_\eta$$

which yields that $\bar{t} = t_\eta$ and $\bar{x} = x_\eta$. Writting the viscosity inequalities, we obtain that

$$\begin{cases} \dfrac{\eta}{(T-t)^2} + \dfrac{t-s}{\varepsilon} + 2\alpha(t-t_\eta) + H_j\left(\dfrac{x-y}{\varepsilon} + 2\alpha(x-x_\eta)\right) \leq 0 \\ \dfrac{t-s}{\varepsilon} + H_j\left(\dfrac{x-y}{\varepsilon} + 2\alpha(y-x_\eta)\right) \geq 0 \end{cases}$$

where j is the index such that $b_{j-1}(t_\eta) < x_\eta < b_j(t_\eta)$. Sending α to zero, we obtain a contradiction.

Case 3 : $M_\eta > 0$ **and there exists** $i_0 \in \{1, ..., N\}$ **s.t.** $x_\eta = b_{i_0}(t_\eta)$**.** We first introduce

$$M_{\nu,\alpha} = \sup_{\substack{t,s\in[0,T]\\ L\le x\le R}} \left\{ \begin{array}{l} u(t,x) - v(s, x + b_{i_0}(s) - b_{i_0}(t)) - \dfrac{\eta}{T-t} - \dfrac{(t-s)^2}{2\nu} \\ -\alpha\,(x - b_{i_0}(t))^2 - (t-t_\eta)^2 \end{array} \right\}$$

Classicaly, we have that

$$\begin{cases} M_{\nu,\alpha} \ge M_\eta \text{ and the maximum is reached at a point that we denote by } (t_\nu, s_\nu, x_\nu), \\ (t_\nu, s_\nu, x_\nu) \underset{\nu\to 0}{\longrightarrow} (t_\eta, t_\eta, x_\eta), \\ \alpha\,(x_\nu - b_{i_0}(t_\nu)) \underset{\alpha\to 0}{\longrightarrow} 0. \end{cases}$$

The second point implies that for ν small enough, $x_\nu \neq b_i(t_\nu)$ for all $i \neq i_0$.

We need the following lemma.

Lemma 5.3.7. *Let* $(\hat{t}, \hat{s}, \hat{x})$ *be the limit of* (t_ν, s_ν, x_ν) *as* α *goes to zero. We have that*

$$\limsup_{\nu\to 0} \left(\frac{(\hat{t}-\hat{s})^2}{2\nu} \right) = 0. \tag{5.3.6}$$

Proof. The proof is very simple and relies only on the upper-semi continuity property of the function. Since $M_{\nu,\alpha} \ge M_\eta$, taking α to zero, we obtain

$$u(\hat{t},\hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T-\hat{t}} - \frac{(\hat{t}-\hat{s})^2}{2\nu} - (\hat{t}-t_\eta)^2 \ge M_\eta > 0.$$

Then, taking ν to zero and recalling that $\lim_{\nu\to 0} |\hat{t}-\hat{s}| = 0$ implies that

$$\begin{aligned} M_\eta &\ge \limsup_{\nu\to 0} \left(u(\hat{t},\hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T-\hat{t}} \right) \\ &\ge \limsup_{\nu\to 0} \left(u(\hat{t},\hat{x}) - v(\hat{s}, \hat{x} + b_{i_0}(\hat{s}) - b_{i_0}(\hat{t})) - \frac{\eta}{T-\hat{t}} - \frac{(\hat{t}-\hat{s})^2}{2\nu} - (\hat{t}-t_\eta)^2 \right) \\ &\ge M_\eta. \end{aligned}$$

The last inequality implies that

$$\limsup_{\nu\to 0} \left(\frac{(\hat{t}-\hat{s})^2}{2\nu} + (\hat{t}-t_\eta)^2 \right) = 0$$

and in particular (5.3.6) is true. □

We now continue the proof. We have to distinguish two different cases :

Subcase $x_\nu \neq b_{i_0}(t_\nu)$. We define the new supremum,

$$M_{\nu,\alpha,\varepsilon} = \sup_{\substack{t,s\in[0,T]\\ L\leq x,y\leq R}} \left\{ \begin{array}{l} u(t,x) - v(s,y) - \dfrac{\eta}{T-t} - \dfrac{(t-s)^2}{2\nu} - \alpha\left(x - b_{i_0}(t)\right)^2 - (t-t_\eta)^2 \\ -G_\varepsilon\left(t,s,x,y\right) - \psi\left(t,s,x\right) \end{array} \right\}$$

with

$$\begin{cases} G_\varepsilon(t,s,x,y) = \dfrac{(x + b_{i_0}(s) - b_{i_0}(t) - y)^2}{2\varepsilon} \\ \psi(t,s,x) = (t-t_\nu)^2 + (s-s_\nu)^2 + (x - b_{i_0}(t) - x_\nu + b_{i_0}(t_\nu))^2 . \end{cases} \tag{5.3.7}$$

The maximum is reached at (t,s,x,y) and the fact that u_0 is lipschitz continuous, that b_{i_0} is a continuous function and the definition of G_ε, yields that

$$(t,s,x,y) \underset{\varepsilon\to 0}{\longrightarrow} (t_\nu, s_\nu, x_\nu, x_\nu + b_{i_0}(s_\nu) - b_{i_0}(t_\nu)) . \tag{5.3.8}$$

Equation (5.3.8) implies that for ε small enough,

$$\begin{cases} x \neq b_{i_0}(t) & \text{since } x_\nu \neq b_{i_0}(t_\nu) \\ y \neq b_{i_0}(s) & \text{since } y_\nu = x_\nu + b_{i_0}(s_\nu) - b_{i_0}(t_\nu) \neq b_{i_0}(s_\nu) \end{cases}$$

We now write the viscosity inequalities assuming that $x_\nu < b_{i_0}\left(t_\nu\right)$. The case where $x_\nu > b_{i_0}\left(t_\nu\right)$ is similar only replacing H_{i_0} by H_{i_0+1}. In order to simplify the notations, we will use the following notations :

$$\begin{cases} p_{\varepsilon,\nu,\alpha} = 2\alpha\left(x - b_{i_0}(t)\right) + 2\left(x - b_{i_0}(t) + b_{i_0}(t_\nu) - x_\nu\right) + \dfrac{x + b_{i_0}(s) - b_{i_0}(t) - y}{\varepsilon} \\ p_{\varepsilon,\nu} = \dfrac{x + b_{i_0}(s) - b_{i_0}(t) - y}{\varepsilon} \end{cases} \tag{5.3.9}$$

Using the fact that u is a sub solution of (5.1.1), and the definition of H_{i_0,i_0}, we deduce that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2\left(t - t_\eta\right) + 2\left(t - t_\nu\right) + H_{i_0,i_0}\left(t, p_{\varepsilon,\nu,\alpha}\right) \leq 0. \tag{5.3.10}$$

Using the fact that v is a supersolution of (5.1.1), we obtain

$$\frac{t-s}{\nu} + 2\left(s_\nu - s\right) + H_{i_0,i_0}\left(s, p_{\varepsilon,\nu}\right) \geq 0. \tag{5.3.11}$$

Combaining (5.3.10) and (5.3.11), we obtain

$$\frac{\eta}{(T-t)^2} + 2\left(t - t_\eta\right) + 2\left(t - t_\nu\right) + 2\left(s - s_\nu\right) \leq H_{i_0,i_0}\left(s, p_{\varepsilon,\nu}\right) - H_{i_0,i_0}\left(t, p_{\varepsilon,\nu,\alpha}\right). \tag{5.3.12}$$

The goal is to take first ε, then α and finally ν to zero. Using (5.3.10) and Remark 5.3.5, we deduce that there exists a constant $C_T > 0$ such that

$$|p_{\varepsilon,\nu,\alpha}| \leq \max\left(C_T, \frac{s-t}{\nu} + 2\left(t_\eta - t\right) + 2\left(t_\nu - t\right)\right) = C_{\nu,T} \tag{5.3.13}$$

which implies that

$$|p_{\varepsilon,\nu}| \leq C_{\nu,T} + o(\alpha) + o(\varepsilon). \tag{5.3.14}$$

Estimates (5.3.13) and (5.3.14) implies that $p_{\varepsilon,\nu,\alpha}$ and $p_{\varepsilon,\nu}$ converge as ε goes to zero (up to sub-sequence). Denoting by $\bar{p}_{\nu,\alpha} = \lim_{\varepsilon\to 0} p_{\varepsilon,\nu,\alpha}$ and by $\bar{p}_\nu = \lim_{\varepsilon\to 0} p_{\varepsilon,\nu}$ and taking ε to zero in (5.3.12), we obtain

$$\begin{aligned}\frac{\eta}{(T-t_\nu)^2} + 2(t_\nu - t_\eta) &\leq H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}) \\ &= H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) \\ &\quad + H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}).\end{aligned}$$

Recalling Remark 5.3.6 and using (5.3.13),(more precisely, we use (5.3.13) after taking ε to 0) , we deduce that

$$H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) - H_{i_0,i_0}(t_\nu, \bar{p}_{\nu,\alpha}) \leq B_T |t_\nu - s_\nu| \tilde{C}_{\nu,T}.$$

with $\tilde{C}_{\nu,T} = \max\left(C_T, \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu)\right)$. Therefore, we obtain

$$\frac{\eta}{(T-t_\nu)^2} + 2(t_\nu - t_\eta) \leq H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) + B_T |t_\nu - s_\nu| \tilde{C}_{\nu,T}.$$

First, we send α to zero to get that the limit of $H_{i_0,i_0}(s_\nu, \bar{p}_\nu) - H_{i_0,i_0}(s_\nu, \bar{p}_{\nu,\alpha}) = 0$ and then, recalling Lemma 5.3.7 and the definition of $\tilde{C}_{\nu,T}$, we send ν to zero to obtain a contradiction.

Subcase $x_\nu = b_{i_0}(t_\nu)$. In this case, we will use the following lemma

Lemma 5.3.8. *We have the following inequality*

$$-\frac{\eta}{(T-t_\nu)^2} + \frac{s_\nu - t_\nu}{\nu} + 2(t_\eta - t_\nu) \geq \max\left(\min_{\mathbb{R}} H_{i_0+1,i_0}(t_\nu, \cdot), \min_{\mathbb{R}} H_{i_0,i_0}(t_\nu, \cdot)\right).$$

Proof. We can assume that the maximum $M_{\nu,\alpha}$ is strict, (if not we add the term $-(t-t_\nu)^2 - (s-s_\nu)^2 - (x-x_\nu)^2)$.

We introduce the function $\psi : [0,T] \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = u(t, b_{i_0}(t)) - v(s_\nu, b_{i_0}(s_\nu)) - \frac{\eta}{T-t} - \frac{(t-s_\nu)^2}{2\nu} - (t-t_\eta)^2$$

This function reaches its strict maximum at t_ν. Let $\phi : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ defined as follows

$$\begin{aligned}\phi(t,x) = u(t,x) - v(s_\nu, b_{i_0}(s_\nu)) - \frac{\eta}{T-t} - \frac{(t-s_\nu)^2}{2\nu} - \alpha(x - b_{i_0}(t))^2 - (t-t_\eta)^2 \\ - L|x - b_{i_0}(t)|\end{aligned}$$

with $L > 0$ a constant such that for all $i \in \{1, ..., N+1\}$

$$\begin{cases} H_i\left(L\right) - 3K_T L > \dfrac{T}{\nu} + 2T \\ H_i\left(-L\right) - 3K_T L > \dfrac{T}{\nu} + 2T \end{cases} \tag{5.3.15}$$

with K_T an upper-bound of $\left|b_j'\right|$ on $[0, T]$ for all $j \in \{1, ..., N\}$. The constant L is well defined due to the superlinearity property of H_i(see (A2)).

The maximum of this function is reached at a point (t, x) with t close to s_ν (which implies that $t \neq 0$ and $t \neq T$). If $x \neq b_{i_0}(t)$, then writing the subsolution inequality, we obtain a contradiction using (5.3.15). We deduce that $x = b_{i_0}(t)$. Moreover, using that the strict maximum of ψ is reached at t_ν, we deduce that $t = t_\nu$ and $x = b_{i_0}(t_\nu)$. Writting the subsolution inequality, we obtain

$$\frac{\eta}{\left(T - t_\nu\right)^2} + \frac{t_\nu - s_\nu}{\nu} + 2\left(t_\nu - t_\eta\right) + \max\Big(A_{i_0}\left(t_\nu\right), H^+_{i_0,i_0}\left(t_\nu, -L\right), H^-_{i_0+1,i_0}\left(t_\nu, L\right)\Big) \leq 0.$$

The inequality above implies directly the desired result.

□

In order to introduce the new supremum $M'_{\nu,\alpha,\varepsilon}$, we will define two constant λ_1 and λ_2 whose existence is due to the preceding lemma and the properties of $H_{k,i}\left(t, \cdot\right)$ for $k = i, i+1$.

Definition 5.3.2. Let $p_0^{i_0+1,i_0}(t)$ and $q_0^{i_0,i_0}(t)$ the two functions defined in Remark 5.3.6. Let ν small enough such that $2\left(t_\eta - t_\nu\right) < \dfrac{\eta}{2T^2}$. We define λ_1 and λ_2 such that $\lambda_1 > p_0^{i_0+1,i_0}(t_\nu)$, $\lambda_2 < q_0^{i_0,i_0}(t_\nu)$ and

$$\begin{cases} -\dfrac{\eta}{\left(T - t_\nu\right)^2} + \dfrac{s_\nu - t_\nu}{\nu} + 2\left(t_\eta - t_\nu\right) < H^+_{i_0+1,i_0}\left(t_\nu, \lambda_1\right) < \dfrac{s_\nu - t_\nu}{\nu} - \dfrac{\eta}{2T^2} \\ -\dfrac{\eta}{\left(T - t_\nu\right)^2} + \dfrac{s_\nu - t_\nu}{\nu} + 2\left(t_\eta - t_\nu\right) < H^-_{i_0,i_0}\left(t_\nu, \lambda_2\right) < \dfrac{s_\nu - t_\nu}{\nu} - \dfrac{\eta}{2T^2}. \end{cases}$$

The existence of λ_1 and λ_2 is due to the quasi-convexity property of $H_{i_0+1,i_0}\left(t, \cdot\right)$ and $H_{i_0,i_0}\left(t, \cdot\right)$. We also have that

$$H^+_{i_0+1,i_0}\left(s_\nu, \lambda_1\right) < \frac{s_\nu - t_\nu}{\nu}.$$

In fact, using that $\displaystyle\lim_{p \to +\infty} \frac{H^+_{i_0+1,i_0}\left(t_\nu, p\right)}{p} = +\infty$, we deduce that there exists $C_T > 0$ such that

$$\lambda_1 \leq \max\left(C_T, \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2}\right)$$

and in particular

$$|\lambda_1| \leq \max\left(C_T, \frac{s_\nu - t_\nu}{\nu} - \frac{\eta}{2T^2}, p_0^{i_0+1,i_0}(t_\nu)\right).$$

Using the fact that the continuous function $p_0^{i_0+1,i_0}$ is bounded on $[0,T]$, we deduce using Remark 5.3.6 that

$$\lim_{\nu\to 0}\left(H^+_{i_0+1,i_0}(s_\nu,\lambda_1) - H^+_{i_0+1,i_0}(t_\nu,\lambda_1)\right) = 0$$

and that for ν small enough,

$$H^+_{i_0+1,i_0}(s_\nu,\lambda_1) < \frac{s_\nu - t_\nu}{\nu}.$$

Similary, we have also $H^-_{i_0,i_0}(s_\nu,\lambda_2) < \dfrac{s_\nu - t_\nu}{\nu}$.

Before defining $M'_{\nu,\alpha,\varepsilon}$, we recall the definition of function G_ε, see (5.3.7) and the notations used above, see (5.3.9). We set

$$M'_{\nu,\alpha,\varepsilon} = \sup_{\substack{t,s\in[0,T]\\ L\leq x,y\leq R}} \left\{ \begin{array}{l} u(t,x) - v(s,y) - \dfrac{\eta}{T-t} - \dfrac{(t-s)^2}{2\nu} - \alpha\,(x-b_{i_0}(t))^2 - (t-t_\eta)^2 \\ -G_\varepsilon(t,s,x,y) - \psi(t,s,x) - \varphi(x-b_{i_0}(t)) + \varphi(y-b_{i_0}(s)) \end{array} \right\}$$

with

$$\varphi(x) = \begin{cases} \lambda_1 x & \text{if } x \geq 0 \\ \lambda_2 x & \text{if } x < 0. \end{cases}$$

The maximum is reached at a point (t,s,x,y) and we have that

$$(t,s,x,y) \underset{\varepsilon\to 0}{\longrightarrow} (t_\nu, s_\nu, b_{i_0}(t_\nu), b_{i_0}(s_\nu)).$$

We distinguish three cases depending on the sign of $x - b_{i_0}(t)$.

If $x > b_{i_0}(t)$**.** If $y > b_{i_0}(s)$, we obtain the contradiction proceeding as in the case where $x_\nu \neq b_{i_0}(t_\nu)$. If $y \leq b_{i_0}(s)$, then using the fact that u is a subsolution, we obtain

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H_{i_0+1,i_0}(t, p_{\varepsilon,\nu,\alpha} + \lambda_1) \leq 0. \tag{5.3.16}$$

Using that $H_{i_0+1,i_0}(t,p) \geq H^+_{i_0+1,i_0}(t,p)$ and the fact that $p_{\varepsilon,\nu,\alpha} > 0$, and using (5.3.16), we deduce that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H^+_{i_0+1,i_0}(t,\lambda_1) \leq 0.$$

Sending ε to zero, we obtain a contradiction with the definition of λ_1.

If $x < b_{i_0}(t)$**.** We proceed as in the case where $x > b_{i_0}(t)$ using that $H_{i_0,i_0}(t,p) \geq H^-_{i_0,i_0}(t,p)$, that $p_{\varepsilon,\nu,\alpha} < 0$ and the definition of λ_2.

If $x = b_{i_0}(t)$**.** Using the fact that u is a subsolution, we obtain that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + F_{A_{i_0}}(t, p_{\varepsilon,\nu} + \lambda_2, p_{\varepsilon,\nu} + \lambda_1) \leq 0. \tag{5.3.17}$$

This time, we distinguish three cases depending on the sign of $y - b_{i_0}(s)$.

If $y > b_{i_0}(s)$**.** Note first that using (5.3.17), we have that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H^-_{i_0+1,i_0}(t, p_{\varepsilon,\nu} + \lambda_1) \le 0. \tag{5.3.18}$$

Using the fact that v is a super-solution, we have that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1) \ge 0. \tag{5.3.19}$$

We claim that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H^-_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1) \ge 0. \tag{5.3.20}$$

In order to obtain this inequality, we will prove that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H^+_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1) < 0. \tag{5.3.21}$$

If (5.3.21) is true, then combining it with (5.3.19), (5.3.20) will remain true. For ε small enough and using the fact that $p_{\varepsilon,\nu} < 0$, we have that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H^+_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1) \le \frac{t-s}{\nu} + 2(s_\nu - s) + H^+_{i_0+1,i_0}(s, \lambda_1) < 0.$$

In fact, the above inequality is true for ε small enough using the definition of λ_1. Finally, combaining (5.3.18) and (5.3.20), we deduce that

$$\begin{aligned}\frac{\eta}{(T-t)^2} + 2(t-t_\eta) + 2(t-t_\nu) + 2(s-s_\nu) \le\ & H^-_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1) \\ & - H^-_{i_0+1,i_0}(t, p_{\varepsilon,\nu} + \lambda_1).\end{aligned} \tag{5.3.22}$$

Recalling that $\lim_{p\to+\infty} \frac{H^+_{i+1,i}(t,p)}{p} = +\infty$ and $\lim_{p\to-\infty} \frac{H^-_{i+1,i}(t,p)}{p} = -\infty$, we deduce, using (5.3.21) and (5.3.18) that there exists a constant $C_T > 0$ such that

$$\begin{aligned}|p_{\varepsilon,\nu} + \lambda_1| &\le \max\Big(C_T, H^+_{i_0+1,i_0}(s, p_{\varepsilon,\nu} + \lambda_1), H^-_{i_0+1,i_0}(t, p_{\varepsilon,\nu} + \lambda_1)\Big) \\ &\le \max\left(C_T, \frac{s-t}{\nu} + 2(s-s_\nu), \frac{s-t}{\nu} + 2(t_\eta - t) + 2(t_\nu - t)\right).\end{aligned} \tag{5.3.23}$$

As in the case where $x_\nu \ne b_{i_0}(t_\nu)$, we take first ε to zero in (5.3.22), and then taking ν to zero, thanks to Remark 5.3.6 and Lemma 5.3.7, we obtain a contradiction.

If $y < b_{i_0}(s)$**.** Note first that using (5.3.17), we have that

$$\frac{\eta}{(T-t)^2} + \frac{t-s}{\nu} + 2(t-t_\eta) + 2(t-t_\nu) + H^+_{i_0,i_0}(t, p_{\varepsilon,\nu} + \lambda_2) \le 0.$$

As above, we can prove that

$$\frac{t-s}{\nu} + 2(s_\nu - s) + H^+_{i_0,i_0}(s, p_{\varepsilon,\nu} + \lambda_2) \ge 0.$$

and then we obtain the contradiction.

If $y = b_{i_0}(s)$. In this case, we have

$$\frac{t-s}{\nu} + 2(s_\nu - s) + F_{A_{i_0}}(s, \lambda_2, \lambda_1) \ge 0.$$

As above, we use the sub-solution inequality and the locally lipschitz property for A_{i_0} then we send first ε to zero and then ν to zero to obtain the contradiction.

□

5.4 A homogenization problem

The goal of this section is to prove that after rescaling, the solution of the Hamilton-Jacobi equation formulation of (5.4.1) below converges towards the unique solution of (5.1.1) including only one Hamiltonian and one function b. Most of the results are presented without much details since they can be found in previous works [FSZ17b, FSZ17a].

5.4.1 Presentation of the model

We consider the following model which modelize a moving capacity restriction (like a bus or more generally called "moving bottelneck") of the density of the vehicles, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\rho_t + (f(\rho)\phi(x - b(t)))_x + (g(\rho)(1 - \phi(x - b(t))))_x = 0 \tag{5.4.1}$$

where ρ is the density of vehicles, b represents the position of the bottelneck, f is the flux function outside the bottleneck region, g is the flux function in the bottelneck region and ϕ is a transition function. We make the following assumptions.

Assumptions (B).

– (B1) The flux function f is the Greenshields fundamental digram [GCM+35] given by

$$f(\rho) = \rho V_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

where V_{max} represents the maximal mean velocity of vehicls and ρ_{max} is the maximal density far from the bus.

– (B2) The flux function around the bus g is given by

$$g(\rho) = \rho V_{max}\left(1 - \frac{\rho}{\sigma_{max}}\right)$$

where σ_{max} is the maximal density around the bus. Moreover, $\sigma_{max} < \rho_{max}$.

– (B3) b is a linear function describing the trajectory of the bus and is defined by

$$b(t) = V_b t \quad \text{and we assume that } 0 < V_b < V_{max}.$$

– (B4) The function ϕ is a C^1 transition function and is given by

$$\phi(t) = \begin{cases} 0 & \text{if } x \in [-r, r] \\ 1 & \text{if } x < -r-1 \text{ ou } x > r+1. \end{cases}$$

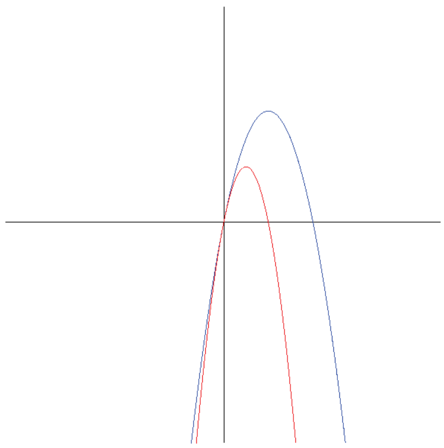

FIGURE 5.1 – Schematic representation of f (blue) and g (red) .

We assume that the initial density satisfies

$$0 \leq \rho(0, x) \leq \begin{cases} \rho_{max} & \text{if } |x| > r+1 \\ \sigma_{max} & \text{if } |x| \leq r+1. \end{cases}$$

5.4.2 Main result

Like in subsection 5.2.2, we will derive the Hamilton-Jacobi equation from model (5.4.1) by defining the analogue of the discrete vehicles label,

$$u(t, x) = h(t) - \int_0^x \rho(t, y)\, dy$$

where

$$h(t) = \int_0^t \left(f(\rho(s,0))\,\phi(-b(s)) + g(\rho(s,0))\,(1-\phi(-b(s))) \right) ds.$$

A simple computations yields to

$$u_t - f(-u_x)\,\phi(x-b(t)) - g(-u_x)\,(1-\phi(x-b(t))) = 0.$$

Setting $H(p) = -f(-p)$ and $F(p) = -g(-p)$ and recalling the definition of the function b (see assumption (B3)), we obtain the following Hamilton-Jacobi equation

$$u_t + H(u_x)\,\phi(x-V_b t) + F(u_x)\,(1-\phi(x-V_b t)) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

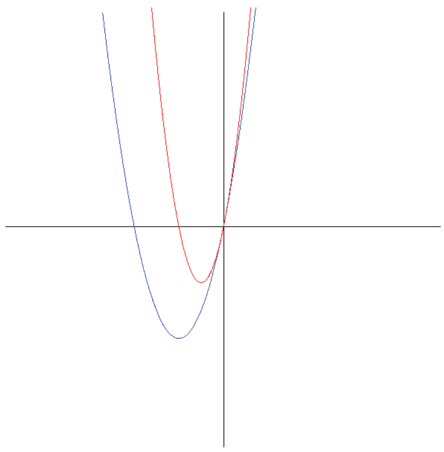

FIGURE 5.2 – Schematic representation of H (blue) and F (red) .

In order to introduce the convergence result, let us define the new Hamiltonians $\tilde{H}$ and $\tilde{F}$ defined as

$$\begin{cases} \tilde{H}(p) = H(p) - b'(t)\,p = H(p) - V_b p \\ \tilde{F}(p) = F(p) - b'(t)\,p = F(p) - V_b p. \end{cases}$$

Clearly, $\tilde{F} > \tilde{H}$ and we will use the following notations

$$\begin{cases} \tilde{H}_0 = \min_{\mathbb{R}} \tilde{H} \\ \tilde{F}_0 = \min_{\mathbb{R}} \tilde{F}. \end{cases}$$

The main result of this section is the following theorem. Let u^ε be the unique solution of

$$\begin{cases} u_t^\varepsilon + H(u_x^\varepsilon)\,\phi\left(\dfrac{x-V_b t}{\varepsilon}\right) + F(u_x^\varepsilon)\left(1-\phi\left(\dfrac{x-V_b t}{\varepsilon}\right)\right) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ u^\varepsilon(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{5.4.2}$$

We assume that the initial condition u_0 is a lipshtiz function satisfying

$$(A0)\begin{cases} -\rho_{max} \leq (u_0)_x \leq 0 & \text{if } |x| > r+1 \\ -\sigma_{max} \leq (u_0)_x \leq 0. & \text{if } |x| \leq r+1 \end{cases} \tag{5.4.3}$$

Theorem 5.4.1 (Junction condition by homogenization)**.** *Assume (B) and (A0). For $\varepsilon > 0$, let u^ε be the unique solution of* (5.4.2)*. Then there exists $A \in \left[\tilde{F}_0, 0\right]$ such that u^ε converges locally uniformly to the unique viscosity solution u^0 of the following equation*

$$\begin{cases} u_t + H(u_x) = 0 & \text{if } x \neq V_b t \\ \dfrac{d}{dt} u(t, V_b t) + \max\left(A, \tilde{H}^+\left(u_x^-(t, V_b t)\right), \tilde{H}^-\left(u_x^+(t, V_b t)\right)\right) = 0 & \text{if } x = V_b t \\ u(0, x) = u_0(x). \end{cases} \tag{5.4.4}$$

5.4.3 Viscosity solutions

In this subsection, we give the definition of viscosity solutions of equation (5.4.2) for $\varepsilon = 1$. We then study the space and time oscillations of the solution. The considered equation is given by

$$\begin{cases} u_t + H(u_x)\phi(x - V_b t) + F(u_x)(1 - \phi(x - V_b t)) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(0, x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{5.4.5}$$

Definition

We will introduce now the standard notion of viscosity solutions of equation (5.4.5).

Definition 5.4.1 (Viscosity solutions for (5.4.5))**.** Let $T > 0$. An upper semi-continuous function (resp. lower semi-continuous) $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of ((5.4.5)) on $[0, T] \times \mathbb{R}$, if $u(0, x) \leq u_0(x)$ (resp. $u(0, x) \geq u_0(x)$) and for all $(t, x) \in (0, T) \times \mathbb{R}$ and for all $\varphi \in C^1([0, T] \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point (t, x), we have

$$\varphi_t + H(\varphi_x)\phi(x - V_b t) + F(u_x)(1 - \phi(x - V_b t)) \leq 0 \ \ (\text{resp} \geq 0).$$

We say that a function u is a viscosity solution of ((5.4.5)) if u^* and u_* are respectively a sub-solution and a super-solution of ((5.4.5)).

5.4.4 Results for viscosity solutions of (5.4.5)

We begin by stating the comparison principle for (5.4.5) whose proof is standard [Bar13, CIL92].

Proposition 5.4.2 (Comparison principle for (5.4.5))**.** *Let u be a sub-solution of* (5.4.5) *and v be a super-solution of* (5.4.5)*. Let us also assume that there exists a constant $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$,*

$$u(t, x) \leq u_0(x - V_b t) + Kt \quad \textit{and} \quad -v(t, x) \leq -u_0(x - V_b t) + Kt. \tag{5.4.6}$$

Then we have $u(t,x) \leq v(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

Theorem 5.4.3. *Let $C_1 = (|H_0| + |F_0|)$ and $C_2 = \rho_{max} V_b$. There exists a unique viscosity solution of (5.4.5) such that*

$$u_0\left(x - V_b t\right) - C_2 t \leq u\left(t, x\right) \leq u_0\left(x - V_b t\right) + C_1 t.$$

Moreover, for all $x, y \in \mathbb{R}$ such that $x \geq y$ and for all $t, s \in [0,T]$ such that $t \geq s$, we have

$$\begin{aligned} -C_2\left(t-s\right) &\leq u(t,x) - u(s,x) \leq \left(C_1 + C_2\right)\left(t-s\right) \quad \textit{and} \\ -\rho_{max}\left(x-y\right) &\leq u(t,x) - u(t,y) \leq 0. \end{aligned}$$

In order to prove Theorem 5.4.3, we will study the following simpler equation since it's invariant by time translation.

$$\begin{cases} w_t - V_b w_x + H\left(w_x\right)\phi\left(x\right) + F\left(w_x\right)\left(1 - \phi\left(x\right)\right) = 0 \\ w(0,x) = u_0\left(x\right). \end{cases} \tag{5.4.7}$$

The unique solution u of (5.4.5) is given by

$$u\left(t,x\right) = w\left(t, x - V_b t\right)$$

where w is the unique viscosity solution of (5.4.7).

Remark 5.4.4. *The definition of viscosity solution of equation (5.4.7) is the same as Definition 5.4.1 i.e. replacing w_x and w_t respectively by φ_x and φ_t. Moreover, a comparison principle exists for (5.4.7).*

Lemma 5.4.5 (Existence of barriers for (5.4.7))**.** *The functions*

$$w^+(t,x) = u_0\left(x\right) + C_1 t \quad \textit{and} \quad w^-(t,x) = u_0(x) - C_2 t$$

are respectively super and sub-solutions of (5.4.7).

Proof. We will only prove that w^- is a sub-solution since the proof that w^+ is a super-solution is similar. Let φ a test function such that $w^- - \varphi$ reaches a maximum at (t_0, x_0). First, using the fact that w^- is a C^1 function in time, we have that $\varphi_t\left(t_0, x_0\right) = -\rho_{max} V_b$. Secondly, since u_0 is a lipschitz function, we deduce that $\left(u_0\right)_x\left(x_0^+\right)$ and $\left(u_0\right)_x\left(x_0^-\right)$ exists. Moreover, the fact that $w^- - \varphi$ reaches a maximum at (t_0, x_0) ensures that

$$\left(u_0\right)_x\left(x_0^+\right) \leq \varphi_x\left(t_0, x_0\right) \leq \left(u_0\right)_x\left(x_0^-\right).$$

We recall that we want to prove the following inequality at (t_0, x_0),

$$\varphi_t - V_b \varphi_x + H\left(\varphi_x\right)\phi\left(x\right) + F\left(\varphi_x\right)\left(1 - \phi\left(x\right)\right) \leq 0. \tag{5.4.8}$$

Inequality (5.4.8) is true since if $|x_0| < r + 1$, we have that $\varphi_x\left(t_0, x_0\right) \geq -\sigma_{max}$ which implies that $H\left(\varphi_x\left(t_0, x_0\right)\right) \leq 0$ and $F\left(\varphi_x\left(t_0, x_0\right)\right) \leq 0$. On the other hand, if $|x_0| \geq r+1$ then $\phi\left(x_0\right) = 1$ and $\varphi_x\left(t_0, x_0\right) \geq -\rho_{max}$, which implies that $H\left(\varphi_x\left(t_0, x_0\right)\right) \leq 0$. □

Applying Perron's method joint to the comparison principle, we obtain the following result.

Theorem 5.4.6 (Existence and uniqueness of viscosity solutions for (5.4.7))**.** *There exists a unique continuous solution w of* (5.4.7) *which satisfies*

$$u_0(x) - C_2 t \leq w(t,x) \leq u_0(x) + C_1 t.$$

5.4.5 Control of the oscillations for (5.4.7)

Proposition 5.4.7 (Control of the oscillations)**.** *Let $T > 0$. The unique solution w of* (5.4.7) *satisfies the following : for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t, s \in [0,T]$, $t \geq s$, we have*

$$-C_2 t \leq w(t,x) - w(s,x) \leq C_1(t-s) \quad \textit{and} \tag{5.4.9}$$

$$-\rho_{max}\,(x-y) \leq w(t,x) - w(t,y) \leq 0. \tag{5.4.10}$$

Proof. We begin by proving inequality (5.4.9). Let $h > 0$. We define $v\,(t,x) = w\,(t+h,x)$ and the goal is to prove that

$$w\,(t,x) - C_2 h \leq v\,(t,x) \leq w\,(t,x) + C_1 h. \tag{5.4.11}$$

All members of inequality (5.4.11) are viscosity solutions on $(0,+\infty)$ of (5.4.7) since equation (5.4.7) is invariant by time translation and by addition of constants. Using Lemma 5.4.5, we have that

$$w\,(0,x) - C_2 h \leq v\,(0,x) \leq w\,(0,x) + C_1 h.$$

The comparison principle for equation (5.4.7) implies directly that (5.4.11) is true.

We now turn to the proof of (5.4.10). In the rest of the proof we will use the following notation :

$$\Omega = \left\{(t,x,y) \in [0,T) \times \mathbb{R}^2 \text{ s.t. } x \geq y\right\}.$$

Proof of the upper inequality for the control of the space oscillations. We introduce,

$$M = \sup_{(t,x,y)\in\Omega} \left\{w(t,x) - w(t,y)\right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1 : the test function. For $\eta, \alpha > 0$, small parameters, we define

$$\varphi(t,x,y) = w(t,x) - w(t,y) - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

Classicaly, φ reaches a maximum at a point that we denote by $(\bar{t}, x, \bar{y}) \in \Omega$ and for η and α small enough, we have that

$$\begin{cases} 0 < \dfrac{M}{2} \leq \varphi(\bar{t}, x, \bar{y}), \\ \alpha|x|, \alpha|\bar{y}| \to 0 \text{ as } \alpha \to 0 \\ T > \bar{t} > 0 \\ \bar{x} > \bar{y}. \end{cases}$$

Step 2 : utilisation of the equation. By doubling the time variable and passing to the limit in this duplication parameter, we get that

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} \leq\ & 2\alpha\bar{x}V_b - H\left(2\alpha\bar{x}\right)\phi\left(\bar{x}\right) - F\left(2\alpha\bar{x}\right)\left(1 - \phi\left(\bar{x}\right)\right) \\ & + 2\alpha\bar{y}V_b + H\left(-2\alpha\bar{y}\right)\phi\left(\bar{y}\right) + F\left(-2\alpha\bar{y}\right)\left(1 - \phi\left(\bar{y}\right)\right). \end{aligned}$$

Passing to the limit as α goes to 0 and using the fact that $H(0) = F(0) = 0$, we obtain a contradiction.

Proof of the lower inequality for the control of the space oscillations We introduce

$$M = \sup_{(t,x,y)\in\Omega} \left\{ w(t,y) - w(t,x) - \rho_{max}(x-y) \right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1 : the test function. For $\eta, \alpha, \nu > 0$ small parameters, we define

$$\varphi(t,s,x,y) = w(t,y) - w(s,x) - \rho_{max}\left(x-y\right) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t} - \alpha x^2 - \alpha y^2.$$

The maximum of φ for $(t,s,x,y) \in [0,T]^2 \times \mathbb{R}^2$ such that $x \geq y$ reaches a maximum at a point that we denote by $(\bar{t}, \bar{s}, x, \bar{y})$ and for η,α and ν small enough, we have that

$$\begin{cases} 0 < \dfrac{M}{2} \leq \varphi(\bar{t}, \bar{s}, x, \bar{y}), \\ \alpha|x|, \alpha|\bar{y}| \to 0 \text{ as } \alpha \to 0 \\ T > \bar{t}, \bar{s} > 0 \\ \bar{x} > \bar{y}. \end{cases}$$

Step 2 : Utilisation of the equation Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined as follows

$$\psi(t,y) = w(\bar{s},\bar{x}) + \rho_{max}(\bar{x}-y) + \frac{(t-\bar{s})^2}{2\nu} + \frac{\eta}{T-t} + \alpha\bar{x}^2 + \alpha y^2.$$

Since $w - \psi$ reaches a maximum at $(\bar{t},\bar{y})$, we deduce using the control of the time oscillations of w (estimate (5.4.9)) that $\psi_t(\bar{t},\bar{y}) \geq -\rho_{max}V_b$. Denoting $p_\alpha = (-\rho_{max} + 2\alpha\bar{y})$ and recalling that w is a sub-solution of (5.4.7), we obtain that

$$-\rho_{max}V_b \leq \frac{\eta}{(T-t)^2} + \frac{\bar{t}-\bar{s}}{\nu} \leq V_b p_\alpha - H(p_\alpha)\phi(\bar{y}) - F(p_\alpha)(1-\phi(\bar{y})).$$

Sending α to zero and recalling that $H(-\rho_{max}) = 0$ and $F(-\rho_{max}) > 0$, we obtain that $-\rho_{max}V_b < -\rho_{max}V_b$ which yields to a contradiction. We deduce that $M \leq 0$ and the proof is complete.

□

5.5 Proof of convergence

The proof of convergence is based on the construction of correctors. Let λ be a constant greater than $\tilde{H}_0$. The definition of $\tilde{H}$ ensures the existence of two constants $\tilde{p}_+^\lambda$ and $\tilde{p}_-^\lambda$ such that

$$\begin{cases} \tilde{H}(\tilde{p}_+^\lambda) = \tilde{H}^+(\tilde{p}_+^\lambda) = \lambda \\ \tilde{H}(\tilde{p}_-^\lambda) = \tilde{H}^-(\tilde{p}_-^\lambda) = \lambda \end{cases}$$

where $\tilde{H}^+$ and $\tilde{H}^-$ are respectively the non-decreasing and the non-increasing part of $\tilde{H}$. For every $\lambda \geq \tilde{H}_0$, we define the following function

$$W^\lambda(t,x) = \tilde{p}_+^\lambda(x - V_bt)1_{\{x-V_bt>0\}} + \tilde{p}_-^\lambda(x-V_bt)1_{\{x-V_bt<0\}}.$$

Theorem 5.5.1. *There exists a unique constant $A \in \left[\tilde{F}_0, 0\right]$ such that there exists w solution of the following equation*

$$w_t + H(w_x)\phi(x - V_bt) + F(w_x)(1-\phi(x-V_bt)) = A$$

and such that $w^\varepsilon(t,x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ converges locally uniformly towards the function W^A.

Proof. We will not go into details because the proof is very similar to the proof of [FSZ17b, FSZ17a]. The idea is to construct a corrector on a truncated domain. We consider $l >> r$ and we want to find $\lambda_l \in \mathbb{R}$ such that there exists w^l solution of

$$\begin{cases} -V_b v_x^l + H\left(v_x^l\right)\phi\left(x\right) + F\left(v_x^l\right)\left(1-\phi\left(x\right)\right) = \lambda_l & \text{if } x \in (-l,l) \\ \tilde{H}^+\left(v_x^l\right) = \lambda_l & \text{if } x = l \\ \tilde{H}^-\left(v_x^l\right) = \lambda_l & \text{if } x = -l. \end{cases} \tag{5.5.1}$$

To do this, we consider the following approximated problem

$$\begin{cases} \delta v^{\delta,l} - V_b v_x^{\delta,l} + H\left(v_x^{\delta,l}\right)\phi\left(x\right) + F\left(v_x^{\delta,l}\right)\left(1-\phi\left(x\right)\right) = 0 & \text{if } x \in (-l,l) \\ \delta v^{\delta,l} + \tilde{H}^+\left(v_x^{\delta,l}\right) = 0 & \text{if } x = l \\ \delta v^{\delta,l} + \tilde{H}^-\left(v_x^{\delta,l}\right) = 0 & \text{if } x = -l. \end{cases} \tag{5.5.2}$$

We construct a unique solution $v^{\delta,l}$ of problem (5.5.2) such that

$$0 \le v^{\delta,l} \le \frac{\left|\tilde{H}_0\right|}{\delta}.$$

In particular, we remark also that $\delta v^{\delta,l}\left(0\right) \le \left|\tilde{F}_0\right|$. Then, as in the proof 5.4.5, we prove for all $x, y \in [-l,l]$ such that $x \ge y$

$$-\rho_{max}\left(x-y\right) \le v^{\delta,l}\left(x\right) - v^{\delta,l}\left(y\right) \le 0. \tag{5.5.3}$$

We can prove (5.5.3) only considering the sub-solution inequality using that

$$\begin{cases} \tilde{H}^+\left(0\right) = H\left(0\right) = F\left(0\right) = 0, \\ \tilde{H}^-\left(-\rho_{max}\right), F\left(-\rho_{max}\right) > H\left(-\rho_{max}\right) = 0. \end{cases}$$

Considering the function $v^{\delta,l}\left(x\right) - v^{\delta,l}\left(0\right)$ and passing to the limit as δ goes to zero (due to Arzelà-Ascoli Theorem), we obtain a solution of problem (5.5.1) where $\lambda_l = \lim\limits_{\delta\to 0} -\delta v^{\delta,l}\left(0\right)$.

The rest of the proof is the same as in [FSZ17b], and even simpler since the constructed solution of problem (5.5.1) is lipschitz so we don't need to consider lim sup, lim inf and the function m. Finally we obtain a unique constant A and a function v solution of

$$-V_b v_x + H\left(v_x\right)\phi\left(x\right) + F\left(v_x\right)\left(1-\phi\left(x\right)\right) = A, \quad x \in \mathbb{R}$$

such that $v^{\varepsilon}\left(x\right) = \varepsilon v\left(\frac{x}{\varepsilon}\right)$ converges locally uniformly towards the function $\tilde{p}_+^A 1_{\{x>0\}} + \tilde{p}_-^A 1_{\{x<0\}}$. The function $w\left(t,x\right) = v\left(x - V_b t\right)$ is the desired function of Theorem 5.5.1. $\square$

The following lemma is a direct result of Theorem 5.4.3 .

Lemma 5.5.2 (Uniform gradient bound)**.** *Assume (A0) and (B). Then the solution u^ε of (5.4.2) satisfies for all $t > 0$, for all $x, y \in \mathbb{R}$, $x \geq y$,*

$$-\rho_{max}(x-y) \leq u^\varepsilon(t,x) - u^\varepsilon(t,y) \leq 0.$$

We now turn to the proof of Theorem 5.4.1 .

Proof of Theorem 5.4.1. We introduce

$$\overline{u}(t,x) = \limsup_{\varepsilon\to 0}{}^* u^\varepsilon \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon.$$

We want to prove that $\overline{u}$ and $\underline{u}$ are respectively a sub-solution and a super-solution of (5.4.4). In this case, the comparison principle will imply that $\overline{u} \leq \underline{u}$. But, by construction, we have $\underline{u} \leq \overline{u}$, hence we will get $\underline{u} = \overline{u} = u^0$, the unique solution of (5.4.4).

Let us prove that $\overline{u}$ is a sub-solution of (5.4.4) (the proof for $\underline{u}$ is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$ and a point $(\bar{t}, x) \in (0, +\infty) \times \mathbb{R}$ such that for $\bar{r}, \eta > 0$ and $\theta > 0$

$$\begin{cases} \overline{u}(\bar{t},x) = \varphi(\bar{t},x) & \\ \overline{u} \leq \varphi & \text{on } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x) \\ \overline{u} \leq \varphi - 2\eta & \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x) \\ \varphi_t(\bar{t},x) + H(\varphi_x(\bar{t},x)) = \theta & \text{if } \bar{x} \neq V_b\bar{t} \\ \dfrac{d}{dt}\varphi(\bar{t},V_b\bar{t}) + \max\left(A, \tilde{H}^+\left(\varphi_x^-(\bar{t},\bar{x})\right), \tilde{H}^-\left(\varphi_x^+(\bar{t},\bar{x})\right)\right) = \theta & \text{if } \bar{x} = V_b\bar{t}. \end{cases} \tag{5.5.4}$$

Lemma 5.5.2 implies that the function $\overline{u}$ satisfies for all $t > 0$ and $x, y \in \mathbb{R}$, $x \geq y$,

$$-\rho_{max}(x-y) \leq \overline{u}(t,x) - \overline{u}(t,y) \leq 0. \tag{5.5.5}$$

First case : $x \neq V_b\bar{t}$. We choose r small enough such that $x \neq V_b t$ for all $(t,x) \in Q_{r,r}(\bar{t},\bar{x})$ and then we prove that φ is a super-solution of (5.4.2) on $Q_{r,r}(\bar{t},\bar{x})$ using the last inequality of (5.5.4), inequality (5.5.5) and the fact that

$$\phi\left(\frac{x - V_b t}{\varepsilon}\right) = 1.$$

.

Getting a contradiction. We have for ε small enough,

$$u^\varepsilon \leq \varphi - \eta \quad \text{outside } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x).$$

Using the comparison principle on bounded subsets we get

$$u^\varepsilon \leq \varphi - \eta \quad \text{on } \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x).$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \leq \varphi - \eta$ on $\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t},x)$ and this contradicts the fact that $\overline{u}(\bar{t},x) = \varphi(\bar{t},x)$.

Second case : $x = V_b\bar{t}$. In this case, using Theorem 5.3.1, the definition of the test function φ is given by

$$\varphi(t,x) = g(t) + \tilde{p}_+^A (x - V_b t) 1_{\{x - V_b t > 0\}} + \tilde{p}_-^A (x - V_b t) 1_{\{x - V_b t < 0\}}$$

with $g \in C^1(\mathbb{R}^+)$ and the last line in (5.5.4) becomes

$$g(\bar{t}) + A = \theta. \tag{5.5.6}$$

We define the perturbed test function φ^ε as

$$\varphi^\varepsilon(t,x) = \begin{cases} g(t) + w^\varepsilon(t,x) & \text{on } Q_{2\bar{r},2\bar{r}}(\bar{r}, V_b\bar{t}) \\ \varphi(t,x) & \text{outside } Q_{2\bar{r},2\bar{r}}(\bar{t}, V_b\bar{t}). \end{cases}$$

where w^ε is defined in Theorem 5.5.1. Using (5.5.6) and the definition of w, we prove that φ^ε satisfies in the viscosity sense

$$\varphi_t^\varepsilon + H(\varphi_x^\varepsilon)\,\phi\left(\frac{x - V_b t}{\varepsilon}\right) + F(\varphi_x^\varepsilon)\left(1 - \phi\left(\frac{x - V_b t}{\varepsilon}\right)\right) \geq \frac{\theta}{2} \quad \text{on } Q_{\bar{r},\bar{r}}(\bar{t}, V_b\bar{t}).$$

Getting the contradiction. We have that for ε small enough

$$u^\varepsilon + \eta \leq \varphi = g(t) + W^A(t,x) \quad \text{on } \mathcal{Q}_{2\bar{r},2\bar{r}}(\bar{t}, V_b\bar{t}) \backslash \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, V_b\bar{t}).$$

Using the fact that $w^\varepsilon \to W^A$, we have for ε small enough

$$u^\varepsilon + \frac{\eta}{2} \leq \varphi^\varepsilon \quad \text{on } \mathcal{Q}_{2\bar{r},2\bar{r}}(\bar{t}, V_b\bar{t}) \backslash \mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, V_b\bar{t}).$$

By the comparison principle on bounded subsets, the previous inequality holds in $\mathcal{Q}_{\bar{r},\bar{r}}(\bar{t}, V_b\bar{t})$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar{t}, V_b\bar{t})$, we obtain

$$\overline{u}(\bar{t}, V_b\bar{t}) + \frac{\eta}{2} \leq \varphi(\bar{t}, V_b\bar{t}) = \overline{u}(\bar{t}, V_b\bar{t}),$$

which is a contradiction.

□

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Chapitre 6

Conclusion et perspectives

Dans cette thèse, deux types de travaux ont été réalisés : le premier axe de travail était de considérer des modèles microscopiques de trafic routier et d'établir une connexion entre ces modèles et des modèles macroscopiques du genre de ceux qui sont introduits par Imbert et Monneau [IM13]. Une telle connexion va permettre de justifier rigoureusement les modèles macroscopiques du trafic routier. Le second axe était de considérer une équation d'Hamilton-Jacobi avec une jonction qui bouge en temps. Cette équation peut décrire la circulation des voitures sur une route avec la présence d'un véhicule particulier (plus lent que les voitures par exemple). On a prouvé l'existence et l'unicité d'une solution de viscosité pour cette EDP. On présente maintenant quelques extensions possibles des résultats de cette thèse.

Extensions possibles des résultats des chapitres 2-3 et 4. Dans ces trois chapitres, on a justifié l'équation introduite par Imbert et Monneau dans [IM13] pour un seul point de jonction à partir des modèles microscopiques. On présente maintenant deux extensions possibles de ces travaux :

- La première extension est de justifier l'équation d'Hamilton-Jacobi sur une route simple mais avec plusieurs points de jonction. Dans ce cas, c'est naturel de considérer un modèle microscopique avec n perturbations locales pour obtenir à l'échelle macroscopique n conditions de limitation du flux. En utilisant les techniques d'homogénéisation de cette thèse, on obtient toujours à l'échelle macroscopique une seule condition de jonction.
- La deuxième extension est de justifier l'équation d'Hamilton-Jacobi sur des réseaux. Forcadel et Salazar [FS] ont considéré le cas d'une seule route entrante et plusieurs routes sortantes. Une extension possible est de considérer un domaine plus général comme une jonction avec plusieurs routes entrantes et plusieurs routes sortantes.

Extensions possibles des résultats du chapitre 5. On rappelle que le résultat principal du chapitre 5 était le principe de comparaison pour une équation d'Hamilton-Jacobi sur un domaine qui varie en temps (5.1.1). On peut considérer les deux extensions suivantes :

- Pour simplifier l'idée, on considère le cas de deux fonctions b_1 et b_2 qui représentent les points de jonction. L'idée est de supposer qu'il existe un temps $t_0 > 0$ à partir duquel les fonctions b_1 et b_2 se séparent, c'est-à-dire

$$\begin{cases} b_1(t_0) = b_2(t_0) \text{ et} \\ b_1(t) > b_2(t) \text{ si } t > t_0. \end{cases}$$

Dans ce cas, la preuve que l'on a fait (voir preuve du principe de comparaison 5.3) présente un problème si $t_\eta = t_0$. Du point de vue trafic routier, on peut imaginer que la fonction b_1 décrit la trajectoire d'un bus qui arrive à un certain temps t_0 au point 0 et que $b_2(t) = 0$ représente un panneau de réduction de vitesse.
- Pour présenter la deuxième extension, on considère le cas d'une seule fonction b. Un cas intéressant consiste à supposer que la vitesse de b dépend de la dérivée de la fonction u,

$$b'(t) = V_b\left(-u_x\left(t, b(t)^+\right)\right). \tag{6.0.1}$$

On a montré (voir sous-section 5.2.1) que du point de vue trafic routier, la solution u de l'équation (5.1.1) vérifie $u_x = -\rho$ où ρ est la densité des véhicules et donc (6.0.1) signifie que

$$b'(t) = V_b\left(\rho\left(t, b(t)^+\right)\right)$$

où V_b est une fonction croissante et bornée. Ce modèle semble plus réel du point de vue trafic routier. Du point de vue mathématique, on doit introduire une notion convenable d'une solution du système (5.1.1)-(6.0.1) et dans ce cas, l'unicité de la solution reste une question ouverte.

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