Analyse de stabilité de systèmes à coefficients dépendant du retard

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Thèse de doctorat

Introduction

1 Systèmes à des coefficients dépendant du retard

Des systèmes de temporisation, connus également sous le nom de systèmes héréditaires, ou de systèmes avec des séquelles, ont été rencontrés dans les disciplines scientifiques et techniques en grande partie en raison du temps nécessaire pour transporter le matériel ou l'information. Par exemple, dans les communications, la transmission de données est toujours accompagnée d'un certain délai. Dans les systèmes économiques, le retard apparaît depuis que les décisions et les effets sur le marché, tels que la politique d'investissement, les fluctuations de prix, les cercles commerciaux sont séparés par un certain intervalle de temps.

Une caractéristique distincte des systèmes de temporisation est que leur évolution dépend de l'information de l'histoire passée avec une mémoire sélective. De tels systèmes peuvent être modélisés en utilisant des équations différentielles sur des espaces abstraits ou fonctionnels, ou sur des anneaux d'opérateurs. Le moyen le plus populaire consiste à décrire les systèmes de temporisation comme des équations différentielles fonctionnelles (FDE).

Des exemples de systèmes de temporisation modélisés comme des FDE peuvent être trouvés dans un large éventail d'applications, y compris les systèmes de contrôle, la physique, la biologie, la chimie, l'économie ainsi que les systèmes de transport.

Les systèmes de temporisation fournissent une simplification raisonnable de modèles plus complexes de systèmes dimensionnels infinis découlant d'équations différentielles partielles (EDP). Il est souvent possible de simplifier les systèmes décrits avec les EDP pour les systèmes à retard temporel avec des variables d'état à dimension finie lorsque seul le comportement des systèmes à certains points spatiaux est intéressant. Dans certains cas, il est possible de réduire les EDP hyperboliques pour retarder les systèmes de type neutre représentés par des équations différentielles et différentielles couplées appropriées. Beaucoup de ces modèles sont liés au phenonmena de propagation sans perte existant dans l'ingénierie thermique, hydraulique et électronique. Un délai a également été introduit délibérément dans la conception de contrôle pour améliorer la stabilité ou la performance.

La stabilité est un concept fondamental des systèmes dynamiques et est à la fois théorique et importance pratique. Il indique le comportement qualitatif d'un modèle et joue un rôle central dans la théorie du contrôle puisque de nombreuses tâches de contrôle peuvent être transformées en problème de stabilisation. Les retards peuvent avoir des effets complexes sur la stabilité du système: de «faibles» retards peuvent déstabiliser certains systèmes, tandis que de «longs» retards peuvent en stabiliser d'autres. De plus, la stabilité du système peut changer plusieurs fois à mesure que le retard augmente.

La stabilité des systèmes de temporisation LTI dépend exactement de racines de l'équation caractéristique associée. Le système est asymptotiquement stable si et seulement si toutes les racines de l'équation caractéristique associée sont situées sur le gauche moitié plan complexe. Cette observation conduit aux approches basées sur le spectre. En raison du fait que l'équation caractéristique des systèmes de temporisation n'est pas polynomiale, le calcul de la racine la plus caractéristique est un problème difficile. Plusieurs approches pour calculer numériquement les racines les plus caractéristiques existent, mais un calcul intensif est requis. L'approche D-décomposition fournit une méthode efficace pour déterminer le nombre de racines caractéristiques sur le demi-plan complexe droit pour un domaine de paramètre donné sans connaître les emplacements exacts des racines caractéristiques. L'idée principale de l'approche D-decomposition est de séparer l'espace des paramètres en sous-régions disjointes. Dans chaque sous-région, le nombre de racines instables est constant. À la limite des sous-régions, des racines caractéristiques imaginaires apparaissent. En analysant les comportements de ces racines caractéristiques imaginaires par rapport à une petite variation de paramètres, la stabilité du système peut être déterminée pour différentes sous-régions dans le domaine des paramètres.

Les méthodes τ -decomposition peuvent être vues comme un cas particulier des méthodes D-decomposition car le paramètre impliqué dans ce cas est le délai τ. le τ-les méthodes de décomposition se déroulent comme suit: en commençant par une valeur de retarder τ^l que l'on connaisse le nombre de racines caractéristiques sur le demi plan complexe droit (habituellement $\tau^l = 0$), on balaie un intervalle de retard d'intérêt (τ^l, τ^u) et identifier tous les retards τ_k, $k = 1, 2, \cdots, N-1$ pour lesquels il existe des racines caractéristiques sur l'axe imaginaire. Ces valeurs de retard sont appelées textit delays critiques et le la fréquence des racines imaginaires correspondantes sont appelées fréquences de croisement. en identifiant la direction que ces racines traversent l'axe imaginaire, on peut déterminer le changement du nombre de racines planes complexes de la moitié droite alors que τ passe par chaque τ_k. Ainsi, on peut diviser (τ_0, τ_N) en sous-intervalles (τ_{k-1}, τ_k) et le nombre de demi-plans droits dans chaque sous-intervalle est constant et peut être déterminé explicitement. En particulier, les sous-intervalles de retard pour que les systèmes soient stables peuvent être calculés.

Il y a deux ingrédients clés des méthodes τ -decomposition. Le premier est de identifier tous les retards critiques et les racines caractéristiques imaginaires correspondantes. le la seconde consiste à caractériser la direction de croisement de ces racines au fur et à mesure que le retard augmente.

Fait intéressant, pour certains systèmes, le retard peut également apparaître dans les coefficients du système. Des systèmes de retard avec cette caractéristique ont été rencontrés, par exemple, dans la dynamique des populations avec les structures d'âge, le modèle de la mouche, les modèles hématopoïétiques, la stella dynamo, ainsi que l'analyse du taux de convergence des systèmes de contrôle. Il existe différentes raisons pour que les coefficients du système dépendent du délai. Au fur et à mesure que l'information, la substance ou l'énergie est transmise, leur quantité ou leur amplitude peut généralement diminuer avec le temps en raison de la dissipation, ce qui fait que leur influence dépend du délai.

Les coefficients dépendants du retard peuvent aussi résulter de la linéarisation d'un système de retard non linéaire autour d'un point d'équilibre, la localisation de l'équilibre pouvant dépendre du retard. Des systèmes de temps de retard avec des coefficients dépendants du retard peuvent émerger d'équations différentielles partielles.

Il existe un lien entre les systèmes avec des coefficients dépendant du retard et un contrôle de retour basé sur le retard temporel. Une façon intuitive d'exploiter le retard dans le retour est d'approximer les dérivées de la sortie mesurée en utilisant la sortie retardée. Le retour de sortie est très commun dans la pratique de l'ingénierie, souvent en raison de la difficulté de mesurer toutes les variables d'état. Lorsqu'un retour statique de la sortie n'est pas suffisant pour stabiliser le système ou pour assurer des performances satisfaisantes, il peut être souhaitable d'utiliser les dérivées de la sortie pour la rétroaction. Cette stratégie est bien incorporée dans le schéma de contrôle PID extrêmement populaire. Différentes méthodes d'ajustement des gains pour un contrôle PID ont été proposées dans la littérature, voir par exemple et les références qui y figurent. Le schéma de différences finies est souvent utilisé pour approcher les dérivés de sortie. Par exemple, l'approximation du premier ordre peut s'écrire

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau}, \tag{1}$$

où τ représente une valeur de retard positive. Il est facile de voir que (1) conduit à retarder les coefficients dépendants dans les systèmes en boucle fermée alors que τ apparaît dans le dénominateur. Cette idée peut être étendue à l'approximation de dérivées d'ordre supérieur via un polynôme d'interpolation.

Malheureusement, la plupart des méthodes de stabilité proposées pour les systèmes avec des coefficients sans retard ne se prêtent pas directement aux systèmes avec des coefficients dépendant du retard. Malgré la littérature abondante sur les systèmes à retardement, les publications sur l'analyse systématique de la stabilité des systèmes avec des coefficients dépendant du retard sont rares. Suivant l'idée de l'approche τ -decomposition, Beretta et Kuang ont présenté une méthode efficace pour effectuer une telle analyse de stabilité pour des systèmes avec des coefficients dépendant du retard basés sur le graphique de certaines fonctions. Leur travail traite des systèmes avec un seul retard soumis à certaines restrictions et ne s'applique pas aux systèmes avec des racines caractéristiques répétées sur l'axe imaginaire. Cette méthode a été appliquée à plusieurs modèles de dynamique hématopoïétique et à la bifurcation de Hopf de la dynamique de production de cellules sanguines.

Cette thèse est consacrée à l'analyse de stabilité de systèmes avec des coefficients dépendant du retard. Les objectifs sont:

- développer des méthodes efficaces pour une analyse de stabilité précise des systèmes avec un seul retard ou des retards proportionnels;
- acquérir une meilleure compréhension du lien entre la stabilité des systèmes avec des coefficients sans retard et dépendant du retard;
- appliquer les méthodes d'analyse de stabilité proposées aux modèles rencontrés dans diverses disciplines scientifiques ainsi qu'à la conception de systèmes de contrôle.

L'analyse de stabilité dans cette thèse est inspirée par le travail de Beretta et Kuang dans un article de 2002. Bien que ce document aborde les systèmes avec un seul délai sous certaines restrictions, nos résultats s'appliquent à une classe plus générale de systèmes selon des hypothèses moins restrictives. Les contributions et la nouveauté peuvent être résumées comme suit. Nous avons généralisé l'approche τ -decomposition pour les systèmes dépendant du délai coefficients. Des méthodes d'analyse de la stabilité sont développées pour les systèmes avec des retards proportionnés. Nous dérivons les critères de direction de traversée qui s'appliquent à racines caractéristiques imaginaires avec possiblement des multiplicités. Un «principe de séparation» est révélé par les critères de direction de franchissement proposés: la direction de racines caractéristiques imaginaires dépend du produit de deux termes. Le premier terme reflète la direction de croisement des racines imaginaires lorsque les coefficients du système sont fixé. Le deuxième terme est vraiment unique sur le type spécial de systèmes de retard étudié dans cette thèse: il dépend de la façon dont les coefficients sont paramétrés par le délai, et devient constant pour les systèmes avec des coefficients sans retard.

Une perspective à deux paramètres des systèmes avec des coefficients dépendant du retard est proposée. La nouvelle perspective a les avantages suivants: 1) Il donne un aperçu géométrique du problème de stabilité et établit un lien entre le système sans retard et les systèmes avec coefficients dépendants du retard; 2) La vision géométrique nous a motivés à dériver des résultats plus généraux qui peuvent être facilement interprété. Le principe de séparation susmentionné devient également assez clair du point de vue géométrique; 3) Notre analyse suggère que l'approche à deux paramètres peut être considérée comme un cadre avec lequel certains outils conçus pour analyser des systèmes à coefficients fixes peuvent être commodément exploités pour résoudre le type particulier de systèmes de retard considérés dans cette thèse.

En tant qu'application, nous avons démontré que les systèmes avec des coefficients dépendant du retard peuvent résulter de schémas de contrôle à rétroaction qui utilisent des signaux de sortie retardés pour approximer les dérivées de sortie. Nous recherchons des intervalles de temps qui atteignent une vitesse de convergence prédéfinie du système en boucle fermée. Par la suite, les méthodes d'analyse de stabilité développées dans cette thèse sont adaptées pour faire face à ce problème particulier.

2 Systèmes avec un seul délai

Dans ce chapitre, nous étudions les systèmes avec coefficients dépendants du retard et un seul retard. L'équation caractéristique correspondante peut s'écrire

$$P(\lambda,\tau)+Q(\lambda,\tau)e^{-\lambda\tau}=0 \tag{2}$$

où $P(\lambda,\tau)$ et $Q(\lambda,\tau)$ sont polynomiaux dans λ et continus par rapport à λ. Les systèmes avec cette fonctionnalité sont rencontrés dans divers modèles dans la pratique. Étant donné un intervalle d'intérêt de retard, noté $\mathscr{I}$, notre objectif est de trouver tous les sous-intervalles de sorte que le système soit asymptotiquement stable si et seulement si le retard est contenu dans ces sous-intervalles. Nous donnons d'abord une définition précise de la classe de systèmes considérée dans ce chapitre, suivie d'un ensemble d'hypothèses sur lesquelles repose notre analyse. Nous montrons comment décomposer $\mathscr{I}$ en sous-ensembles disjoints de telle sorte qu'un nombre fixe de textit fonctions de fréquence et de *fonctions d'angle de phase* peut être défini dans chaque sous-intervalle. Nous identifions toutes les paires critiques $(j\omega,\tau)$ en fonction des conditions exprimées avec les fonctions de fréquence et d'angle de phase.

Les critères de direction de croisement sont dérivés, ce qui détermine si les racines caractéristiques sur l'axe imaginaire deviennent stables ou instables lorsque τ balaie certains délais critiques. Notre analyse a assoupli certaines des hypothèses d'un travail antérieur de Beretta et Kuang. Alors que le critère de direction de croisement de Beretta et de Kuang n'utilise que la dérivée de premier ordre de certaines fonctions, nous montrons qu'un critère plus général basé sur l'analyse d'ordre supérieur peut être dérivé. L'analyse d'ordre supérieur suggère une corrélation géométrique possible entre les fonctions d'angle de phase et le nombre de racines caractéristiques instables. Dans certaines conditions supplémentaires, une telle corrélation permet de déterminer le nombre de racines caractéristiques instables sur la base des valeurs des fonctions d'angle de phase sans calculer les paires critiques ainsi que la direction de croisement racine correspondante. Cette observation motive le développement d'un cadre géométrique pour l'analyse de stabilité présentée au chapitre 5.

La méthode proposée partitionne le domaine de délai d'intérêt pour le retard dans les sous-intervalles de sorte que l'ampleur condition donne un nombre fixe de solutions de fréquences ω as fonctions du délai τ dans chaque sous-intervalle. La partition est faite en résolvant des équations polynomiales. Avec chaque fonction de fréquence, une fonction d'angle de phase est associée. Les paires critiques peuvent être identifiées en fonction de la valeur des fonctions d'angle de phase. Les critères de croisement racine sont dérivés, qui utilise l'information des dérivées d'ordre supérieur des racines caractéristiques par rapport au retard lorsque les dérivées d'ordre inférieur disparaissent. Pour être précis, le critère de direction de franchissement peut être exprimé comme suit:

$$\begin{aligned}\operatorname{sgn}\left(\left(\tfrac{d}{d\tau}\right)^l\Re(\lambda)(\tau^*))\right) &= \operatorname{sgn}\left(\partial_\omega F(\omega_k^{(i)}(\tau^*),\tau^*)\right)\\ &\quad\times\operatorname{sgn}\left(\left(\tfrac{d}{d\tau}\right)^l\theta_k^{(i)}(\tau^*)\right).\end{aligned} \tag{3}$$

pour $l=1,2,\cdots,n_d$, où $\theta_k^{(i)}$, $\omega_k^{(i)}$ sont la fonction de phase et la fonction de fréquence, respectivement, associées à une paire critique, et n_d est l'ordre le plus bas de la dérivée non nulle de la fonction de phase.

Les conditions de traversée sont composées de deux facteurs. Le premier facteur est déjà connu pour les systèmes à coefficients fixes et le deuxième facteur dépendant de la fonction d'angle de phase résulte du fait que les coefficients du système sont paramétrés par le retard. L'analyse suggère une corrélation intéressante entre la valeur des fonctions d'angle de phase et la variation du nombre de racines caractéristiques instables. Cette observation motive le développement d'une perspective géométrique différente de l'analyse de stabilité dans les chapitres suivants.

3 Systèmes avec des délais commensurables

Dans ce chapitre, la stabilité des systèmes avec des délais proportionnels et des coefficients dépendant du retard est étudiée selon la méthode de décomposition τ. L'objectif principal est d'étendre les résultats du chapitre 3 aux systèmes avec des retards correspondants. Nous considérons les systèmes avec des équations caractéristiques de la forme:

$$\sum_{i=0}^{M} P_i(\lambda,\tau)e^{-i\lambda\tau} = 1, \tag{4}$$

et chaque $P_i(\lambda,\tau)$ est un polynôme de λ et continu par rapport à τ.

Pour les systèmes avec un seul retard, la condition d'amplitude de l'existence de racines caractéristiques imaginaires a motivé l'introduction de la fonction $F(\omega,\tau)$, une fonction polynomiale dans ω. Les racines de cette fonction polynomiale dans ω capturent les fréquences de croisement potentielles des racines caractéristiques. Ensuite, les directions de croisement correspondant à une paire critique $(j\omega,\tau)$ peuvent être déterminées par la dérivée partielle de cette fonction polynomiale et la dérivée de la fonction d'angle de phase à cette paire critique. En utilisant le lemme de Schur-Cohn généralisé, nous sommes capables de donner une définition généralisée de $F(\omega,\tau)$, qui est toujours un polynôme dans ω et qui s'applique maintenant aux systèmes avec des délais proportionnels. Pour les équations caractéristiques avec un seul retard, cette nouvelle définition est réduite à celle du chapitre 3. Nous montrons aussi que l'idée géométrique qui a conduit à la définition de $F(\omega,\tau)$ dans le chapitre 3 peut être généralisée pour le cas de retard proportionnel.

Avec la fonction polynomiale $F(\omega,\tau)$ ainsi définie, nous suivons une procédure similaire à celle du chapitre 3. Nous énonçons tout d'abord un ensemble d'hypothèses et discutons de leurs implications. Le domaine de retard $\mathscr{I}$ est alors décomposé en plusieurs sous-intervalles disjoints $\mathscr{I}^{(i)}$. Dans chaque $\mathscr{I}^{(i)}$, un nombre fixe de fonctions de fréquence sont définies. Nous montrons que dans certaines conditions qui sont réalistes en pratique, une fonction d'angle de phase unique peut être associée à chaque fonction de fréquence. Puis, comme dans le cas du retard unique, les paires critiques de systèmes ayant des retards proportionnels peuvent être identifiées en suivant chaque fonction d'angle de phase. Avec les notions et les fonctions du Chapitre 3 généralisées et adaptées au cas de retard commensurable, nous démontrons que les critères de direction de croisement racine du Chapitre 3 peuvent être étendus au type plus général de systèmes considéré ici. Le chapitre se termine par deux exemples numériques illustratifs. Le premier se rapporte à l'analyse de la stabilité α et le deuxième démontre le calcul de la vitesse critique d'un véhicule automobile avec un retard dans la génération de la force du pneu.

La méthode d'analyse de stabilité proposée dans ce chapitre traite des systèmes avec des retards et des coefficients proportionnels en fonction du délai. Cette méthode suit l'approche généralisée de τ -decomposition. La méthode partitionne l'intervalle de délai d'intérêt dans les sous-intervalles disjoints de sorte qu'une magnitude généralisée condition donne un nombre fixe de solutions de fréquences ω as fonctions du délai τ dans chaque sous-intervalle. Nous avons fourni des conditions pour que les racines imaginaires apparaissent à certaines valeurs de retard critiques, suivies d'un critère pour identifier les fréquences de croisement. Notre analyse montre que les résultats développés dans le dernier chapitre pour les systèmes avec un seul retard peuvent être largement étendus aux systèmes avec des retards proportionnés. Tout comme le cas du retard unique, le critère de la racine croisée reflète le «principe de séparation»: la direction de croisement des racines caractéristiques sur l'axe imaginaire dépend de deux facteurs, l'un étant «classique» dans le sens où il existe pour des systèmes fixes. l'autre est nouveau, ce qui reflète la monotonie des fonctions d'angle de phase aux paires critiques. Dans le chapitre suivant, cette observation intéressante sera expliquée d'un point de vue géométrique.

4 Une perspective à deux paramètres

Nous passons en revue l'analyse de stabilité des systèmes avec des coefficients dépendant du retard d'un point de vue à deux paramètres. Le paramètre dans les coefficients du système et celui dans l'état sont considérés comme deux variables différentes, notées r et q respectivement. Le fait que les coefficients du système dépendent du délai signifie simplement $r = q$. Pour comprendre la stabilité du système original, il suffit d'analyser la stabilité du système à deux paramètres dans l'espace des paramètres r - q, puis d'imposer la restriction $r = q = \tau$. L'idée de base est illustrée avec des systèmes avec un seul retard. L'extension au cas de retard commensurable est discutée ultérieurement. Les critères de direction de croisement racine sont d'abord dérivés sous la forme la plus générale en exploitant l'idée géométrique sous-jacente à la perspective à deux paramètres. Ces critères nous permettent de tirer profit de quelques méthodes d'analyse de stabilité performantes développées à l'origine pour des systèmes à coefficients fixes. Par exemple, la série Puiseux peut être facilement appliquée pour développer une méthode d'analyse complète. Pour des racines caractéristiques simples sur l'axe imaginaire, nous simplifions ces critères et récupérons les résultats développés dans les chapitres précédents sous des hypothèses moins restrictives. Au fur et à mesure que nous développons notre théorie, la dynamique de la population avec une structure de stade est prise comme exemple pour illustrer l'idée principale.

Pour les systèmes avec coefficients dépendants du retard, nous considérons le paramètre de délai dans le coefficient du système et le paramètre de délai dans l'état comme deux variables notées r et q, respectivement, sous réserve de la restriction $r = q = \tau$ Nous avons défini les courbes de retard critique, qui séparent le domaine r - q en régions de stabilité disjointes, dans chaque région le nombre de racines instables est constant et le changement de stabilité ne peut se produire que sur les courbes limites de ces régions de stabilité. Le point de vue géométrique établit une connexion entre les problèmes plus classiques où les coefficients du système sont indépendants du retard et l'analyse de la stabilité des systèmes avec des coefficients dépendant du retard. La perspective à deux paramètres fournit un aperçu géométrique du problème, ce qui nous permet de dériver des résultats plus généraux concernant les directions de croisement des racines, applicables aux racines caractéristiques avec multiplicité. Il établit également une connexion entre les problèmes plus classiques où les coefficients du système sont indépendants du retard et l'analyse de stabilité des systèmes avec des coefficients dépendant du retard. Par conséquent, nous pouvons facilement appliquer certains outils développés à l'origine pour les systèmes de retard à coefficients fixes aux systèmes avec des coefficients dépendant du retard. L'analyse confirme notre conjecture dans le chapitre 3 que la direction de croisement des racines caractéristiques imaginaires dépend partiellement de la monotonie des fonctions d'angle de phase et que la différentiabilité des fonctions d'angle de phase n'est pas essentielle. La corrélation entre le nombre de racines instables et la position des fonctions d'angle de phase peut être facilement interprétée du point de vue à deux paramètres.

5 Systèmes de contrôle basés sur un schéma de différence de retard

En pratique d'ingénierie, la différence de retard est souvent utilisée pour approcher les dérivées des signaux de sortie pour le contrôle de retour, conduisant à un système en boucle fermée avec un retard à la fois dans les états et dans les coefficients du système. Dans ce contexte, il est important de trouver toutes les valeurs de retard contenues dans un certain intervalle qui garantissent la stabilité exponentielle du système en boucle fermée soumis à une loi de contrôle basée sur le retard.

Nous avons d'abord considéré un schéma de retour qui n'utilise que la dérivée de premier ordre de la sortie approchée par un schéma de différences finies. Après avoir spécifié la loi de contrôle basée sur une telle approximation par différence finie, nous avons dérivé l'équation caractéristique du système en boucle fermée linéarisée. Ensuite, il est montré qu'en décalant la variable dans l'équation caractéristique, la condition pour la stabilité exponentielle avec le taux de décroissance α est équivalente à une condition pour la stabilité asymptotique.

Les méthodes d'analyse de stabilité développées dans les chapitres précédents exigent que les coefficients du système soient continus dans τ. Cependant, du fait que le retard apparaît comme un dénominateur dans les coefficients du fait du schéma de différences finies, les coefficients ne sont pas bornés car τ se rapproche de 0. Nous proposons donc quelques méthodes pratiques pour calculer une valeur inférieure positive. lié τ^l pour notre test τ -sweeping. La limite inférieure τ^l est choisie de telle sorte que le nombre de racines caractéristiques instables du système à boucle fermée décalé est connu pour $\tau \in (0, \tau^l]$.

Une fois que la procédure de contrôle et d'analyse de stabilité devient claire pour un retour basé sur un seul retard, nous avons présenté un schéma d'approximation généralisé pour les dérivées d'ordre supérieur de la sortie pour stabiliser une chaîne d'intégrateur. L'idée est d'approximer l'historique de sortie par interpolation polynomiale et de remplacer les dérivées réelles de la sortie par les dérivées de la polynôme. Nous dérivons une borne sur l'erreur d'approximation, ce qui nous permet de calculer la borne inférieure τ^l pour notre test τ -sweep.

Enfin, la procédure de conception et d'analyse proposée a été appliquée à plusieurs problèmes pratiques. Les résultats montrent qu'un schéma de contrôle basé sur des dérivées de sortie approchées peut surpasser celui basé sur les dérivées de sortie exactes. Ceci confirme l'observation que le retard peut présenter un effet stabilisateur dans certaines situations.

Nous avons abordé l'analyse de stabilité pour les schémas de contrôle qui utilisent la différence finie pour approcher les dérivées des signaux de sortie. Le délai est traité comme un paramètre de conception. Etant donné un intervalle de retard borné d'intérêt de la forme $(0, \tau^u]$, nous proposons une méthode pour trouver tous les sous-intervalles des valeurs de retard contenues dans cet intervalle de sorte que le système soit exponentiellement stable. α-stability, connue sous le nom de stabilité α Il est montré qu'après avoir déplacé la variable de Laplace, la stabilité α du système de contrôle est équivalente à la stabilité asymptotique d'une nouvelle équation caractéristique avec des coefficients dépendant du retard Pour analyser la stabilité de l'équation caractéristique la plus récente, nous avons proposé quelques méthodes pour calculer une borne inférieure positive τ^l pour l'intervalle de retard de telle sorte que l'analyse de stabilité doit seulement être effectuée dans $[\tau^l, \tau^u]$.

Par conséquent, nous sommes en mesure d'appliquer les résultats dans les chapitres précédents sur l'analyse de stabilité du système de retard avec des coefficients dépendant du temps. La procédure d'analyse de stabilité est illustrée par deux exemples qui sont motivés par le contrôle de suivi de trajectoire d'un robot mobile et le contrôle de tangage d'un aéronef, respectivement. Les résultats montrent qu'un retard plus important dans l'approximation par différence finie peut en effet améliorer les performances de contrôle en termes de vitesse de convergence exponentielle de la trajectoire ainsi que la rapidité de réponse aux signaux de référence.

6 Points de vue

Des systèmes de temporisation avec des coefficients dépendants du retard apparaissent dans diverses disciplines scientifiques et techniques. Cette thèse contribue à l'analyse de stabilité des systèmes avec cette particularité.

La méthode développée dans cette thèse suit une approche généralisée de τ-decomposition. L'idée de cette approche est de balayer le paramètre de retard à travers un intervalle d'intérêt, d'identifier tous les retards critiques et les fréquences de croisement correspondantes et de déterminer les directions de croisement des racines caractéristiques sur l'axe imaginaire. Ensuite, le nombre de racines caractéristiques instables pour différentes valeurs de retard peut être facilement déterminé.

Nous avons d'abord considéré les systèmes avec un seul retard. L'intervalle de retard est d'abord décomposé en sous-intervalles disjoints, à l'intérieur desquels un nombre fixe de fréquences peut être défini en fonction du retard et chaque fonction de fréquence est associée à une fonction d'angle de phase. Les retards critiques et les fréquences de croisement sont identifiés en fonction des fonctions d'angle de phase et des fonctions de fréquence. Les critères qui déterminent la direction de croisement des racines caractéristiques sur l'axe imaginaire sont proposés, ce qui permet d'exploiter les dérivées d'ordre supérieur des racines caractéristiques par rapport au retard lorsque les dérivées d'ordre inférieur disparaissent. Ces résultats sont ensuite étendus aux systèmes avec des retards proportionnés.

Notre analyse montre que la direction de croisement d'une racine imaginaire dépend du produit de deux termes distincts. Le premier terme a déjà été découvert dans la littérature pour les systèmes à retard à coefficients fixes. Le second terme est lié à la monotonie de la fonction d'angle de phase aux retards critiques, ce qui est unique pour les systèmes avec des coefficients dépendant du retard. De plus, la corrélation entre la position des fonctions d'angle de phase et le nombre de racines instables est également suggérée par les critères de croisement des racines.

Pour acquérir une meilleure compréhension de ces résultats, une approche à deux paramètres est proposée pour fournir un aperçu géométrique du problème. Le paramètre de retard dans les coefficients du système et dans l'état sont considérés comme deux variables différentes, disons r et q. La stabilité du système peut alors être déterminée en considérant d'abord différentes régions de stabilité dans l'espace des paramètres r - q, puis en imposant la restriction $r = q = \tau$. Lorsque r et q entrent et quittent différentes régions de stabilité dans l'espace r - q le long de la ligne de 45, la stabilité du système peut changer. Ce point de vue à deux paramètres conduit à une interprétation plus intuitive des résultats de stabilité précédemment obtenus par une approche analytique. Avec l'approche à deux paramètres, des méthodes d'analyse de stabilité plus générales sont développées plus loin qui s'appliquent aux systèmes avec des racines caractéristiques imaginaires répétées possibles sous des hypothèses relâchées. Bien que les critères de croisement des racines puissent être exprimés en utilisant les dérivées de la fonction d'angle de phase, l'analyse de la perspective à deux paramètres montre clairement que la différentiabilité des fonctions d'angle de phase n'est pas essentielle. La monotonie de ces fonctions aux retards critiques est suffisante pour déterminer les directions de croisement des racines.

nous avons montré que le système avec des coefficients dépendant du retard peut provenir d'un schéma de contrôle qui utilise une sortie retardée pour approcher les dérivées de sortie. Les systèmes en boucle fermée résultants ont le paramètre de retard dans le dénominateur, et les méthodes d'analyse développées dans cette thèse ne peuvent pas être appliquées immédiatement en raison de cette singularité à $\tau = 0$. Pour résoudre ce problème, plusieurs méthodes pratiques d'estimation d'une borne inférieure positive pour le test de balayage τ sont proposées, qui consistent principalement à résoudre certaines équations polynomiales. Par la suite, les méthodes d'analyse de stabilité proposées dans les chapitres précédents sont appliquées pour trouver tous les intervalles de retard qui garantissent une vitesse de convergence désirée de la trajectoire du système en boucle fermée.

Il y a plusieurs directions à suivre dans le futur.

Premièrement, nous avons seulement considéré le système nominal sans prendre en compte les incertitudes dans les paramètres du système. Une analyse de stabilité robuste pour les systèmes comportant des incertitudes sera importante à la fois en théorie et en pratique. La méthode traditionnelle τ -decomposition est difficile à appliquer pour la raison suivante. Pour dire la stabilité pour une valeur de retard donnée, disons τ_0, il faut que tous les retards critiques inférieurs à τ_0 soient identifiés et que le comportement de croisement racine soit analysé à chacun de ces retards critiques. C'est une tâche assez formidable puisque le nombre de retards critiques et les comportements de stabilité du système à ces retards peuvent dépendre des paramètres incertains d'une manière complexe. De l'avis de l'auteur, la perspective à deux paramètres discutée au chapitre 5 suggère une approche plus réaliste de ce problème. Selon cette perspective, l'espace des paramètres $r - q$ introduit dans le chapitre 5 est séparé par des courbes de retard critiques en sous-régions disjointes, à l'intérieur desquelles le nombre de racines caractéristiques instables est invariant. Par conséquent, on peut déterminer la stabilité du système par rapport à des paramètres incertains en analysant la taille de ces courbes de retard critiques qui peuvent varier sous la perturbation de paramètres incertains. Il apparaît que si les courbes limites de chaque sous-région dans le plan $r - q$ ne passent jamais par le point $(r, q) = (\tau_0, \tau_0)$ sous toutes les variations de paramètres possibles, alors stabilité / instabilité robuste peut être déduit pour le système avec la valeur de retard τ_0. D'autre part, ces courbes limites, à savoir la courbe des retards critiques, sont déterminées par un ensemble d'équations algébriques, ce qui permet d'approcher l'étude de la stabilité robuste en analysant la variation des racines des équations algébriques sous perturbation paramétrique.

Deuxièmement, le type de délai considéré dans cette thèse est limité à un seul retard ou plusieurs retards proportionnels générés par un seul paramètre. Il serait intéressant de généraliser les résultats pour les systèmes avec plusieurs paramètres de retard indépendants. Dans ce but, on peut analyser la stabilité du système dans un espace de paramètres dimensionnel supérieur en généralisant l'approche à deux paramètres proposée dans cette thèse.

Une autre direction consiste à étendre l'analyse de stabilité aux systèmes non linéaires en exploitant les approches basées sur les valeurs propres. Lorsque le système linéarisé a des racines caractéristiques sur l'axe imaginaire, le système non linéaire correspondant peut toujours être asymptotiquement stable. On peut extraire le collecteur central du système non linéaire sur la base de la valeur propre imaginaire et de l'espace propre de la dynamique linéarisée. Il suffit alors d'analyser la stabilité du système complet en ne considérant la dynamique que sur le collecteur central, qui est un objet de dimension finie. Pour l'introduction et l'application de variétés invariantes générales d'équations différentielles ordinaires.

UNIVERSITÉ PARIS SACLAY

DOCTORAL THESIS

Stability Analysis of Systems with Delay-Dependent Coefficients

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in the

Laboratoire des Signaux et Systemes

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Declaration of Authorship

I, Chi JIN, declare that this thesis titled, "Stability Analysis of Systems with Delay-Dependent Coefficients" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

"*I seem to have been only like a boy playing on the sea-shore and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.*"

Issac Newton

Université Paris Saclay

Abstract

Doctor of Philosophy

Stability Analysis of Systems with Delay-Dependent Coefficients

by Chi JIN

Systems with delay-dependent coefficients have been encountered in various applications of science and engineering. Despite the rich literature on time-delay systems, there are few results concerning stability analysis of systems with delay-dependent coefficients. This thesis is devoted to the stability analysis of this class of systems.

Stability analysis methods are developed based on the corresponding characteristic equation following a generalized τ-decomposition approach. Given a delay interval of interest, we are able to identify all the critical delay values contained in this interval for which the characteristic equation admits roots on the imaginary axis of the complex plane. Various root crossing direction criteria are proposed to determine whether these characteristic roots move toward the left or the right half complex plane as the delay parameter goes through these critical delay values. The number of unstable characteristic roots for any given delay can thus be determined. Our analysis covers systems with a single delay or commensurate delays under certain assumptions. The root crossing direction criteria developed in this thesis can be applied to characteristic roots with multiplicity, or characteristic roots whose locus parameterized by the delay is tangent to the imaginary axis. As an application, it is demonstrated that systems with delay-dependent coefficients can arise from control schemes that use delayed output to approximate its derivatives for stabilization. The stability analysis methods developed in this thesis are tailored and applied to find the delay intervals that achieve a demanded convergence rate of the closed-loop system.

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Last but most importantly, I owe very much to my parents for their full support of my decisions and for all the sacrifice they have made so that I can have the luxury to pursue my own dream.

Contents

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List of Symbols

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	positive reals
$\mathbb{R}_+^*$	non-negative reals
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_+$	the right half complex plane
$\mathbb{C}_-$	the left half complex plane
$\mathbb{D}$	the closed unit disk in the complex plane
$\partial\mathbb{D}$	the unit circle in the complex plane
$\Re(\cdot)$	the real part of a complex number
$\Im(\cdot)$	the imaginary part of a complex number
$\mathrm{ord}(\cdot)$	the degree of a polynomial

Dedicated to Paris

Chapter 1

Introduction

1.1 Systems with Time-Delay

Time-delay systems, known also as hereditary systems, or systems with after effects, have been encountered in both scientific and engineering disciplines largely due to the time needed to transport material or information. For instance, in communications, data transmission is always accompanied by certain time-delay. In economic systems, delay appears since decisions and market effects, such as investment policy, price fluctuations, trade circles are separated by some time interval.

A distinct feature of time-delay systems is that their evolution depends on the information of the past history with a selective memory. Such systems may be modeled using differential equations on abstract [1] or functional spaces [8], or over rings of operators [2] . The most popular way is to describe time-delay systems as functional differential equations (FDE). The study of FDEs has a long history, which can be traced back to the enlightening era, associated with such great names as Euler, Bernoulli, Lagrange, Laplace. A number of monographs covering different aspects of time-delay systems are available in the literature, including [108], [111] and [8]. The definition and some basic properties of FDEs will be discussed in Chapter 2.

Examples of time-delay systems modeled as FDEs can be found in a wide range of applications including control systems [57], Physics [18], Biology [42], Chemistry [43] Economics [44] as well as transportation systems [79]. In mechanical engineering for instance, the metal cutting process on a lathe can be described as

$$m\ddot{y}(t)+c\dot{y}(t)+ky(t)=-F_t(f+y(t)-y(t-\tau)),$$

which is related to the regenerative chattering phenomenon. Another example from mechanical engineering is the dynamics of internal combustion engines [57], which is governed by the following equation:

$$J\dot{\omega}(t) = T_i(t-\tau_i) - T_f(t) - T_{load}(t),$$

where T_i is the output torque of the engine, which is delayed by τ_i seconds due to engine cycle delays. The time delay in the two equations above are due to the rotational motion of mechanical components.

Time-delay systems provides reasonable simplification of more complex models of infinite dimensional systems arising from partial differential equations (PDE). It is often possible to simplify systems described with PDEs to time-delay systems with finite dimensional state variables when only the behavior of the systems at certain spatial points is of interest. In some cases, it is possible to reduce hyperbolic PDEs to delay systems of neutral type represented by appropriate coupled differential and difference equations. A lot of these models are related with the lossless propagation phenonmena existing in thermal, hydraulic and electronic engineering. For examples with respect to these models, see ,e.g., [114], [107] and [107].

Time-delay has also been purposely introduced in control design for enhancing stability or performance, which is not a new idea as pointed out in [32]. Early contributions in this direction can be traced to the work of Smith [69], [70]. Later, the proportional minus delay controller was proposed in [71], which exhibited good noise attenuation capabilities. Delay-based controllers can use an averaged derivative to replace the actual derivative of the output. Such kind of controllers have been applied to mechanical systems in [72], [73]. Other examples include canceling out uncertainties [104], tuning active vibration absorbers [105], stabilizing unstable periodic trajectories [106]. For further report on the favorable effect of delay in controllers, see for instance [76], [81], [82]. In a series of recent papers [75], [76], [77], [78] , systematic methods for determining the parameters of Proportional-Retarded or proportional-integral-retarded controllers are proposed. By using the elimination technique developed in [62], the real part of the rightmost root can be rendered as small as possible, thereby achieving a maximal exponential convergence rate. Dominant pole placement of a triple real root using delayed PID control has also been studied in [80].

1.2 Stability Analysis of Time-Delay Systems

Stability is a fundamental concept of dynamical systems and is of both theoretical and practical importance. It tells the qualitative behavior of a model and plays a central role in control theory since many control tasks can be transformed into a stabilization problem. Time-delay can have complex influences on system stability: "small" delays may destabilize some systems, while "large" delays may stabilize others. Moreover, system stability may switch for several times as the delay increases [31].

There is a rich literature on stability analysis of time-delay systems, as a result of the research efforts dedicated to time-delay systems over the past decades. Most of the proposed methods for stability analysis can be categorized into two classes: the Lyapunov-based methods and the spectrum-based methods. The Lyapunov-based methods consist in constructing a Lyapunov function or a Lyapunov functional for proving stability, by revoking the Razumikhin theorem or the Lyapunov-Krazovski theorem, respectively [8] . It may seem easier to use the Razumikhin theorem, since only a Lyapunov function is required. However such simplicity comes at the cost of very conservative stability conditions because such a Lyapunov function does not necessarily exist even if the system is asymptotically stable. In general, a Lyapunov functional needs to be constructed for a more accurate stability analysis. As a matter of fact, it is proved that for linear time-invariant systems with discrete and distributed delays, asymptotic stability is equivalent to the existence of a quadratic Lyapunov functional [88]. Various techniques for constructing Lyapunov functionals using LMIs are summarized in [58], [31] and [87].

Most results in the literature use matrices for presenting the Lyapunov functional and formulate the stability condition as LMIs, which can be efficiently solved as a convex programming problem. However, the infinite dimensional nature of the Lyapunov functional means any presentation using a small number of decision variables will in general introduce conservativeness. By exploiting the idea of the finite-element method, in [54] Gu proposed a discretized Lyapunov functional approach for stability analysis, which is guaranteed to find a Lyapunov functional for a large class of asymptotically stable linear systems if the discretization resolution is sufficiently high [60]. Recently, some different functional approximation approaches have been proposed using polynomial bases and SOS techniques [83], [84], [85], [86]. These methods also apply to the construction of the invertible positive-definite Lyapunov operators, and thus open the way to full-state feedback for LTI delay systems via convex optimization. Nevertheless, the computational burden grows very rapidly as the number of decision variables increases. Moreover, the optimization based approach does not always provide insight into the link between system structures and stability.

On the other hand, the stability of LTI time-delay systems depends exactly on the roots of the associated characteristic equation. The system is asymptotically stable if and only if all the roots of the associated characteristic equation are located on the left half complex plane. This observation leads to the spectrum-based approaches. Due to the fact that the characteristic equation of time-delay systems are not polynomials, computing the right most characteristic root is a challenging problem. Several approaches for numerically computing the rightmost characteristic roots exist [10], but intensive computation is required. The

D-decomposition approach provides an efficient method of determining the number of characteristic roots on the right complex half plane [7] for a given parameter domain without knowing the exact locations of characteristic roots. The main idea of *D*-decomposition approach is to separate the parameter space into disjoint sub-regions. In each sub-region, the number of unstable roots is constant. On the boundary of the sub-regions, imaginary characteristic roots appear. By analyzing the behaviors of these imaginary characteristic roots with respect to small variation of parameters, the system stability can be determined for different sub-regions in the parameter domain.

The τ-decomposition methods [22] can be viewed as a special case of the *D*-decomposition methods as the parameter involved in this case is the delay τ. The τ-decomposition methods roughly proceed as follows: starting with one value of delay τ^l that one knowns the number of characteristic roots on the right half complex plane (usually $\tau^l = 0$), one sweeps through an delay interval of interest (τ^l, τ^u) and identify all delays τ_k, $k = 1, 2, \cdots, N-1$ for which there are characteristic roots on the imaginary axis. These delay values are referred to as *critical delays* and the frequency of the corresponding imaginary roots are called crossing frequencies.By identifying the direction these roots cross the imaginary axis, one may determine the change of the number of right half complex plane roots as τ goes through each τ_k. Thus, one may divide (τ_0, τ_N) into subintervals (τ_{k-1}, τ_k), and the number of right half plane roots within each subinterval is constant and can be explicitly determined. Especially, the subintervals of delay for the systems to be stable can be computed.

There are two key ingredients of the τ-decomposition methods. The first one is to identify all the critical delays and corresponding imaginary characteristic roots. The second is to characterize the crossing direction of these roots as the delay increases. Regarding the first problem, various approaches are available, including the bilinear transformations and related methods [67], the matrix pencil based methods [13], [33], and elimination technique based on a generalized Schur-Cohn theorem [21]. These algebraic techniques lead to matrix eigenvalue problems or polynomial equations, which can be solved efficiently. Geometric methods have appeared in [55], [56], which apply to systems with multiple independent delay parameters. The complexity of the second issue depends on whether the imaginary root of concern is simple or repeated. For simple imaginary characteristic roots, its local behavior with respect to the delay can be determined by differentiation of the characteristic equation and revoking the implicit function theorem. It is also possible to directly working on the matrices that define the time-delay system and use Jacobi's formula to compute the root crossing directions [10]. For imaginary roots with multiplicities, perturbation based methods have been proposed in [39], [40]. The recent work in [39] shows that the asymptotic behavior of imaginary roots with multiplicities can be completely characterized by the Newton-Puiseux series. A general frequency-sweeping approach has also been proposed to compute the increase of unstable roots as τ passes through critical delays. A more detailed introduction and discussion of these results will be presented in Chapter 2.

There are many other methods of analyzing the stability of time delay systems. For instance, it is possible to determine the number of characteristic roots with a positive real component by the integration of some function along a contour in the complex plane. Interested reader may refer to [109] and the references therein. Another approach is to approximate time-delay systems with a finite dimensional one by discretizing the time history of the state variables. Various techniques pertaining to this discretization approach are presented in the monograph [110].

1.3 Systems with delay dependent coefficients

Interestingly, for some systems the delay may also appear in the system coefficients. Delay systems with this feature have been encountered in, for instance, the population dynamics with age structures [16], the blowfly model [36], the hematopoietic models [20], the stella dynamo [18], as well as convergence rate analysis of control systems [35]. There are various reasons for the system coefficients to be delay-dependent. As the information, substance or energy is being transmitted, their quantity or magnitude may in general decrease over time due to dissipation, causing their influence to be delay-dependent. Here we provide several examples with more details. The source and dissipative process of a stella dynamos in [18] can be described by the following equations:

$$\begin{cases} \dot{B}_\phi(t) = c_1 e^{-c_2 T_0} A(t - T_0) - c_2 B_\phi(t), \\ \dot{A}(t) = c_3 e^{-c_2 T_1} B_\phi(t - T_1) - c_2 A(t), \end{cases}$$

where B_ϕ is the strength of toroidal field, A is the strength of poloidal field, and c_1, c_2, c_3, T_0, T_1 are positive constants. The characteristic equation of the above system can be easily obtained as

$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1 c_3 e^{-c_2\tau} e^{-\tau\lambda} = 0, \tag{1.1}$$

where $\tau = T_0 + T_1$.

Delay-dependent coefficients can also result from linearization of a nonlinear time-delay system about some equilibrium point, as the location of the equilibrium my depend on the delay. Consider the model of hematopoietic stem cell dynamics given in [20]. The model is nonlinear, and possesses two equilibria. The linearized equation in the neighborhood of the nonzero equilibrium has the following characteristic equation

$$\lambda + A(\tau) - B(\tau)e^{-\lambda\tau} = 0,$$

where A, B are nonlinear functions of τ.

Time-delay systems with delay-dependent coefficients may emerge from partial differential equations. For instance, the age-structured Hematopoietic Stem Cells in [116] are given by:

$$\begin{cases} \partial_t \tilde{r}(t,a) + \partial_a \tilde{r}(t,a) = -(\tilde{\delta} + \tilde{\beta}(C(t)))\tilde{r}(t,a), \text{ for } a > 0, t > 0, \\ \partial_t \tilde{p}(t,a) + \partial_a \tilde{p}(t,a) = -\tilde{\gamma}\tilde{p}(t,a), \text{ for } 0 < a < \tilde{\tau}, t > 0, \\ \partial_t r(t,a) + \partial_a r(t,a) = -(\delta + \beta(C(t)))r(t,a), \text{ for } a > 0, t > 0, \\ \partial_t p(t,a) + \partial_a p(t,a) = -\gamma p(t,a), \text{ for } 0 < a < \tau, t > 0. \end{cases}$$

where $r(t,a)$ is the density of resting healthy cells at time t and age a, $\tilde{r}(t,a)$ denotes the density of resting unhealthy cells, $p(t,a)$ the density of proliferating healthy cells and $\tilde{p}(t,a)$ the density of proliferating healthy cells. The boundary condition for all $t > 0$ is given by

$$\begin{cases} \tilde{r}(t,0) = 2(1-\tilde{K})\tilde{p}(t,\tilde{r}), \\ \tilde{p}(t,0) = \tilde{\beta}(C(t))\tilde{x}(t) + 2\tilde{K}\tilde{p}(t,\tilde{\tau}), \\ r(t,0) = 2p(t,\tau), \\ p(t,0) = \beta(C(t))x(t). \end{cases}$$

Using the method of characteristics [113] and following similar arguments as those in [117] as well as [113], the partial differential equation can be reduced to a delay-difference equation with delay-dependent coefficients [115]:

$$\begin{cases} \dot{\tilde{x}}(t) = -[\tilde{\delta} + \tilde{\beta}(x(t)+\tilde{x}(t))]\tilde{x}(t) + 2(1-\tilde{K})e^{-\tilde{\gamma}\tilde{\tau}}\tilde{u}(t-\tau), \\ \tilde{u}(t) = \tilde{\beta}(x(t)+\tilde{x}(t))\tilde{x}(t) + 2\tilde{K}e^{-\tilde{\gamma}\tilde{\tau}}\tilde{u}(t-\tilde{\tau}), \\ \dot{x}(t) = -[\delta + \beta(x(t)+\tilde{x}(t))]x(t) + 2e^{-\gamma\tau}\beta(x(t-\tau)+\tilde{x}(t-\tau))x(t-\tau). \end{cases} \tag{1.2}$$

The model of cell density in a generic compartment in [113] is described by a different set of PDEs but can also be reduced to a system with delay-dependent coefficients after the method of characteristics is applied.

There is a link between systems with delay-dependent coefficients and feedback control based on time delay. An intuitive way of exploiting time delay in feedback is to approximate the derivatives of the measured output using delayed output. Output feedback is very common in engineering practice, often due to the difficulty of measuring all the state variables. When a static feedback of the output is not sufficient for stabilizing the system or to ensure satisfactory performance, it may be desirable to use the derivatives of the output for feedback. This strategy is well embodied in the extremely popular PID control scheme [45]. Various methods for tuning the gains for a PID control have been proposed in the literature, see for instance [46] and the references therein. The finite-difference scheme is often used to approximate the output derivatives. For instance, the first-order approximation can be written as

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau}, \tag{1.3}$$

where τ represents some positive delay value. It is easy to see that (1.3) leads to delay dependent coefficients in the closed-loop systems as τ appears in the denominator. This idea can be extended to the approximation of higher order derivatives via interpolation polynomial, as shown in [32], where the stabilization of a chain of integrators is addressed. A rescaling technique is adopted in [32] to prevent system coefficients from depending on the delay. However, for more general systems the rescaling technique can not be applied and the approximation scheme will in general lead to systems with delay-dependent coefficients.

Fast convergence to the reference point is critical for a controller to achieve satisfactory performance. For a delay-based controller, it is useful to find a range of delay values such that the closed-loop system is α-stable, which means the system trajectory converges to the reference point with a pre-specified exponential rate α. The analysis of α-stability can lead to systems with delay-dependent coefficients. For instance, consider the characteristic equation with a single delay:

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{1.4}$$

where $P(\tau)$ and $Q(\tau)$ are polynomials that do not depend on τ. It is shown in [8] that α-stability means that the real part of all characteristic roots of (1.4) must be less than α. Therefore the α-stability of (1.4) is equivalent to the asymptotic stability of a system associated with the following 'shifted' characteristic equation:

$$P(\lambda - \alpha) + Q(\lambda - \alpha)e^{\alpha\tau}e^{-\lambda\tau} = 0. \tag{1.5}$$

We see that the delay now appears in the system coefficients.

Unfortunately, most stability methods proposed for systems with delay-free coefficients do not lend themselves directly to systems with delay-dependent coefficients. Despite the rich literature on time-delay systems, the publications on systematic stability analysis of systems with delay-dependent coefficients are rare. Following the idea of the τ-decomposition approach, Beretta and Kuang [14] presented an effective method of performing such a stability analysis for systems with delay dependent coefficients based on the graph of some functions. Their work deals with systems with a single delay subject to certain restrictions and does not apply to systems with repeated characteristic roots on the imaginary axis. This method has been applied to several hematopoietic dynamics model in [20] and the Hopf bifurcation of blood cell production dynamics in [19]. Some extension of [14] can be found in [15] for the analysis of stage structured predator-prey models.

1.4 Research Objectives and Contribution

This thesis is devoted to stability analysis of systems with delay dependent coefficients. The objectives are:

- develop efficient methods for precise stability analysis of systems with a single delay or commensurate delays;
- acquire a deeper understanding of the connection between stability of systems with delay-free and delay-dependent coefficients;
- apply the proposed stability analysis methods to models encountered in various scientific disciplines as well as the design of control systems.

The stability analysis in this thesis is inspired by the work of Beretta and Kuang [14]. While[14] addresses systems with a single delay under some restrictions, our results apply to more general class of systems under less restrictive assumptions. The contributions and novelty can be summarized as follows. We generalized the τ-decomposition approach for systems with delay-dependent coefficients. Stability analysis methods are developed for systems with commensurate delays. We derive crossing direction criteria that apply to imaginary characteristic roots with possibly multiplicities. A 'separation principle' is revealed by the proposed crossing direction criteria: the crossing direction of imaginary characteristic roots depends on the product of two terms. The first term reflects the crossing direction of imaginary roots when the system coefficients are fixed. The second term is truly unique about the special type of delay systems studied in this thesis: it depends on how the coefficients are parameterized by the delay, and becomes constant for systems with delay-free coefficients.

A two-parameter perspective of systems with delay-dependent coefficients is proposed. The new perspective has the following advantages: 1) It provides geometric insight into the stability problem, and establishes a link between system with delay-free and systems with delay-dependent coefficients; 2)The geometric insight motivated us to derive more general results that can be easily interpreted. The aforementioned separation principle also becomes quite clear from the geometric point of view; 3) Our analysis suggests that the two-parameter approach can be regarded as a framework with which some tools designed for analyzing systems with fixed-coefficients can be conveniently exploited for solving the special type of delay systems considered in this thesis.

As an application, we demonstrated that systems with delay-dependent coefficients can result from feedback control schemes that use delayed output signals for approximating output derivatives. We seek for time-delay intervals that achieve a pre-specified convergence speed of the closed loop system. Subsequently, the stability analysis methods developed in this thesis are tailored to cope with this particular problem.

1.5 Notation

The following notation will be used in this paper. For a function $f(x_1,\cdots,x_n)$ of multiple arguments $x_1,\cdots,x_n$, $\partial^l_{x_i} f$ denotes the lth partial derivative of function f with respect to the argument x_i. Suppose $F(x_1,x_2)$ is a function of two arguments, we may write $F_{x_2}(x_1)$ to emphasize that F is regarded as a function of x_1 while x_2 is viewed as a parameter. The characteristic equations of time-delay systems will be denoted as

$$D(\lambda,\tau)=0,$$

where λ is the Laplace variable and τ is the delay parameter. Suppose $(\omega,\tau)\in\mathbb{R}^*_+\times\mathbb{R}^*_+$ satisfies

$$D(j\omega,\tau)=0,$$

then $(j\omega,\tau)$ is referred to as a *critical pair* and we say ω is a *crossing frequency* and τ is a *critical delay*. Characteristic roots with a positive real part are called unstable characteristic roots. We use $N^u(\tau)$ to denote the number of unstable characteristic roots for a given delay value τ.

Chapter 2

Preliminaries and Literature Review

2.1 Chapter Overview

The objective of this chapter is to introduce some preliminary results and techniques pertaining to the development in later chapters. We are focused on the publications that are most closely related to this thesis. It is not intended to present a complete overview of the literature. We first present the basic notion of functional differential equations, including the definition, the uniqueness, existence and stability of solutions. After introducing the characteristic equation associated with the linearized time-delay systems, we discuss some important properties of the characteristic roots. Next, we present a general procedure of the τ-decomposition approach, which has been widely applied for stability analysis of time-delay systems with fixed coefficients. There are two key issues of this approach: the first one is to identify the critical delay values and the corresponding frequencies of imaginary characteristic roots; the second is to determine whether theses characteristic roots becomes stable or unstable as the delay parameter goes through these critical delays. Various techniques that address these problems are presented and discussed. There are few publications devoted to stability analysis of systems with delay-dependent coefficients. In this regard, we mainly discuss a work by Berreta and Kuang [14], which motivated the development of the theories in this thesis.

2.2 Functional Differential Equations

Time-delay systems can be described using functional differential equations. Let $\mathscr{C}([a,b],\mathbb{R}^n)$ be the Banach space of continuous functions that map the interval $[a,b]$ to $\mathbb{R}^n$. In this thesis we will be concerned with systems with bounded delays, therefore the state of a time-delay system considered in this thesis is a function mapping $[-\tau,0]$ to $\mathbb{R}^n$. In this case, we will simply use $\mathscr{C}$ to denote $\mathscr{C}([-\tau,0],\mathbb{R}^n)$ as the state-space. For any positive number T and continuous function $\phi \in \mathscr{C}([t_0-\tau,t_0+T])$, we use ϕ_t to denote a segment of the function ϕ

defined as $\phi_t(s) = \phi(t+s)$, $s \in [-\tau, 0]$. For any real function ϕ defined in a bounded interval, let $||\phi||$ be its supremum norm .

The general form of an autonomous retarded functional differential equation is

$$\dot{x}(t) = f(x_t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathscr{C} \mapsto \mathbb{R}^n$. We stipulate that the origin of the state space $x_t = 0$ is an equilibrium of (2.1). The solution of (2.1) is defined for $t \in [0, +\infty)$ if the functional $f(x_t)$ is globally lipschiz, i.e., there exists some positive number L such that $||f(\phi_1) - f(\phi_2)|| < L||\phi_1 - \phi_2||$, $\forall t \in [0, +\infty)$ and $\phi_1, \phi_2 \in \mathscr{C}$.

For system (2.1), the equilibrium $x_t = 0$ is said to be stable if for any $t_0 \geq 0$ and any $\varepsilon > 0$, there exists a positive number δ such that $||x_{t_0}|| < \delta$ implies $||x(t)|| < \varepsilon$ for all $t \geq t_0$. It is said to be asymptotically stable if it is stable and $\lim_{t \to +\infty} x(t) = 0$ for all initial conditions. We say (2.1) is α-stable if for any initial condition x_0 and $t \geq 0$, the trajectory satisfies

$$||x_t|| < ke^{-\alpha t}||x_0|| \tag{2.2}$$

for some positive numbers k and some given real number α. We say (2.1) is globally exponentially stable if it is α-stable for some $\alpha > 0$.

In this thesis we only consider autonomous RFDEs with discrete delays. We linearize (2.1) about the origin, which leads to a linear time-delay system of the following form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \tag{2.3}$$

where $A_i \in \mathbb{R}^{n \times n}$, $i = 0, 1, \cdots, m$, are real matrices and $0 < \tau_1 < \tau_2, \cdots, < \tau_m$ represents the time-delays. It is known that for linear systems of the form (2.3), asymptotic stability is equivalent to exponential stability. Moreover, the asymptotic stability of the linear system (2.3) implies the asymptotic stability of (2.1), although the converse is not necessarily true.

Now consider linear neutral type time-delay systems of the following form:

$$\tfrac{d}{dt}(x(t) + \sum_{k=1}^{m} H_k x(t - \tau_k)) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k), \tag{2.4}$$

where $x(t) \in \mathbb{R}^n$ and $0 < \tau_1 < \tau_2, \cdots, < \tau_m$ represent the time-delays. The forward solution of (2.4) exists uniquely. The stability notions of the RFDEs apply also to the neutral type systems. However, for the neutral type system (2.4), asymptotic stability does not imply exponential stability [59], [10].

2.3 Characteristic Roots

2.3.1 Characteristic Equations

Denoting $\overrightarrow{\tau} = (\tau_1, \tau_2, \cdots, \tau_m)$, the characteristic matrix of (2.3) is defined as

$$\Delta(\lambda, \overrightarrow{\tau}) = \lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda \tau_i}, \tag{2.5}$$

and the characteristic matrix of (2.4) is defined as

$$\Delta(\lambda, \overrightarrow{\tau}) = \lambda \left(I + \sum_{k=1}^{m} H_k e^{-\lambda \tau_k} \right) - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda \tau_i} \tag{2.6}$$

The characteristic function $D(\lambda, \overrightarrow{\tau})$ of either (2.3) or (2.4) is defined as

$$D(\lambda, \overrightarrow{\tau}) = \det(\Delta(\lambda, \overrightarrow{\tau})), \tag{2.7}$$

where $\Delta(\lambda, \overrightarrow{\tau})$ is the characteristic matrix associated with (2.3) or (2.4). In this thesis we are concerned with system with commensurate delays generated by a single delay parameter. In this case, the characteristic equation can be written as

$$D(\lambda, \tau) = \sum_{i=0}^{M} P_i(\lambda) e^{-i\lambda \tau} = 0. \tag{2.8}$$

If $\lambda \in \mathbb{C}$ satisfies the characteristic equation:

$$D(\lambda, \overrightarrow{\tau}) = 0, \tag{2.9}$$

we say λ is a characteristic root. The characteristic roots are critical for stability analysis because time-delay systems of the form (2.3) or (2.4) is exponential stable if and only if all the characteristic roots of the associated characteristic equations are located on $\mathbb{C}_-$ and no sequence of characteristic roots approaches the imaginary axis. The spectral abscissa $\mathbf{c}$ is defined as

$$\mathbf{c} = \sup\{\Re(\lambda) : D(\lambda, \overrightarrow{\tau}) = 0\}. \tag{2.10}$$

As a matter of fact, asymptotic growth rate of the solution x_t satisfies (2.2), where α can be any number larger than the spectral abscissa $\mathbf{c}$.

Contrary to finite dimensional systems, time-delay systems in general have infinite number of characteristic roots since $D(\lambda, \vec{\tau})$ is a transcendental function. Consequently, in comparison with finite-dimensional systems, it is significantly more involved to determine the stability of a time-delay system based on the characteristic roots, which cannot be easily obtained.

2.3.2 Properties of the Characteristic Roots

We first discuss the retarded type system (2.3). Although the number of characteristic roots of (2.3) is infinite, given any vertical line in the complex plane, the characteristic roots lying to the right of such a line are finite. Especially, there are only a finite number of characteristic roots in any vertical strip of the complex plane given by

$$\{\lambda \in \mathbb{C} | a < \Re(\lambda) < b\}.$$

Each characteristic root of (2.3) is continuous with respect to τ, therefore since the number of characteristic roots located to the right of any vertical line in the complex plane is finite, it is easy to see that the abscissa $\mathbf{c}$ depends on τ continuously.

On the other hand, the situation with the neutral type system (2.4) is more subtle. Let

$$D_d(\lambda, \tau) = \det(I + \sum_{k=1}^{m} H_k e^{-\lambda \tau_k})$$

be the characteristic equation of the difference equation

$$x(t) + \sum_{i=1}^{m} H_k x(t - \tau_k) = 0.$$

Suppose ξ satisfies $D_d(\xi, \tau) = 0$, then there exists a sequence of characteristic roots of (2.4) $\{\lambda_n\}_{n \geq 1}$ such that

$$\lim_{n \to \infty} \Re(\lambda_n) = \Re(\xi), \ \lim_{n \to \infty} \Im(\lambda_n) = \infty.$$

Therefore, contrary to the retarded time-delay systems, the characteristic roots of a neutral type system contained in a given a vertical strip of the complex plane can be infinite. Although each individual characteristic root of the neutral type system is continuous with respect to τ, such a continuous dependence does not carry over to the abscissa $\mathbf{c}$. Therefore special precaution should be taken when applying some of the stability methods, such as

$\tau-$sweeping, for stability analysis. Let $\mathbf{c}_d$ be the abscissa of $D_d(\lambda,\tau)$, and define

$$\mathbf{C}(\overrightarrow{\tau}) = \lim_{\varepsilon\to 0^+} \mathbf{c}^{\varepsilon}(\overrightarrow{\tau}),$$

where $\mathbf{c}^{\varepsilon} = \sup\{\mathbf{c}(\overrightarrow{\tau}+\delta\overrightarrow{\tau}) | \delta\overrightarrow{\tau} \in \mathbb{R}^m \text{ and } ||\delta\overrightarrow{\tau}|| \le \varepsilon\}$. The following result can be found in [10]:

Proposition 2.1. *For every $\overrightarrow{\tau} \in (\mathbb{R}_+)^m$, we have*

$$\boldsymbol{C}(\overrightarrow{\tau}) = \max\left(\boldsymbol{C}_d(\overrightarrow{\tau}), \boldsymbol{c}(\overrightarrow{\tau})\right).$$

Moreover, $\boldsymbol{C}(\overrightarrow{\tau})$ is continuous in $\overrightarrow{\tau}$.

The above proposition means that if all the characteristic roots of $D_d(\lambda,\overrightarrow{\tau})$ have a real component smaller than some real number ξ for all positive delay parameters, then given any number $\xi_1 > \xi$, the characteristic roots of $D(\lambda,\overrightarrow{\tau})$ located to the left of the vertical line $\Re(\lambda) = \xi'$ depend on τ continuously.

2.4 The τ-Decomposition Method

Due to the difficulty in computing the characteristic roots of time-delay systems, it is desirable to determine system stability without the exact knowledge of the abscissa of the characteristic equation. When the characteristic roots close to $\mathbb{C}_+$ depend on τ continuously, the *D*-decomposition approach becomes a convenient and powerful tool for stability analysis [6], [22].

The idea of this approach is to decompose the parameter space into disjoint sub-regions. In the interior of each sub-region, the system has no imaginary characteristic roots, therefore the number of unstable roots, i.e., roots in $\mathbb{C}_+$ is constant in each sub-regions. The boundary of each sub-regions are those parameter points with which the characteristic equation admits roots on the imaginary axis, referred to also as the imaginary characteristic roots. To determine the system stability for any given parameter point p_1, one starts with some parameter point p_0 , for which the number of unstable roots is known. For instance, the parameter point p may include the delay parameter, so that when the delay is set to 0, the unstable roots can be easily determined since the characteristic equation is reduced to a polynomial. Imagine that the parameter point p moves along a continuous path γ in the parameter space that connects p_0 and p_1. The point p crosses the boundaries of different sub-regions in the parameter spaces as it leaves one sub-region and enters another. When p lies on some boundary surface, some characteristic roots appear on the imaginary axis. By determining whether these imaginary characteristic roots move to $\mathbb{C}_+$ or $\mathbb{C}_-$, the change in the number of unstable roots as p crosses the boundary can be obtained.

When the parameter p in the D-decomposition is the delay parameter, it is also know as the τ-decomposition method. In this thesis we analyze systems with one delay parameter τ. Given a delay interval of interest denoted as $\mathscr{I}$, the τ-decomposition method proceeds as follows. First, a series of delay values are identified, for which the characteristic equation admits roots on the imaginary axis. These delay values are referred to as the critical delays and arranged in the ascending order:

$$\tau_1 < \tau_2 < \cdots < \tau_L.$$

Following the notion in [39], suppose τ^* is a critical delay and $j\omega^*$ is an imaginary characteristic root corresponding to τ^*, we will refer to $(j\omega^*, \tau^*)$ as a *critical pair*, and ω^* as a *crossing frequency*. The second key step is to analyze the local behavior of the imaginary root of each critical pair $(j\omega^*, \tau^*)$. More precisely, one has to determine how many characteristic roots will move toward $\mathbb{C}_+$ through the point $j\omega^*$ as τ sweeps through τ^*. This step is referred to as the root crossing direction analysis. Subsequently, the number of unstable roots in each interval (τ_i, τ_{i+1}) can be determined. According to the aforementioned procedure, there are two key issues of the τ-decomposition method. The first one is to identify all the critical pairs. The second one is the root crossing direction analysis at these critical pairs. Some techniques related to these two issues will be introduced in the ensuing sections. There are many other methods available in the literature, and it is not our intention to give an extensive coverage of them. Instead, we focus mainly on those methods that are closely related to the analysis in this thesis.

2.4.1 Identifying Critical Pairs: the Single-Delay Case

To better illustrate the ideas of different methods, we first consider systems with a single delay. The characteristic equation can be written as

$$D(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0. \tag{2.11}$$

When λ lies on the imaginary axis, i.e., $\lambda = j\omega$ for some $\omega \in \mathbb{R}$, $e^{-\lambda\tau}$ lies on the unit circle of the complex plane $\partial\mathbb{D}$. This is an important property exploited in most techniques that identify imaginary characteristic roots.

The bilinear transformation based method [74] replaces $e^{-\lambda\tau}$ with $\frac{1-\lambda T}{1+\lambda T}$ and then (2.11) can be transformed to

$$(1+\lambda T)P(\lambda)+(1-\lambda T)Q(\lambda)=0,$$

which is a polynomial in both λ and T and therefore easier to deal with than a quasi-polynomial. It is easy to see that λ is an imaginary characteristic roots of (2.11) if and only if it is a root of the last equation for some positive T. One may replace λ with $j\omega$ in the last equation and further eliminate the variable T to derive an polynomial equation of ω. Then all the real solutions in ω are the frequencies of imaginary characteristic roots corresponding to certain delay values.

It is possible to directly work on (2.11) to obtain the crossing frequencies of the imaginary characteristic roots. For imaginary λ, since $e^{-\lambda\tau}$ is on the unit circle $\mathbb{D}$ we must have $|P(\lambda)|=|Q(\lambda)|$. In [3] , the following polynomial is introduced:

$$F(\omega)=P(j\omega)P(-j\omega)-Q(j\omega)Q(-j\omega). \tag{2.12}$$

It is clear that if ω is a real root of $F(\omega)$, then $\lambda=j\omega$ must be an imaginary characteristic root of (2.11). Let $\omega_1<\omega_2<\cdots<\omega_H$ be the non-negative solution of

$$F(\omega)=0, \tag{2.13}$$

then each ω_i, $1\leq i\leq H$ is a crossing frequency. Corresponding to each ω_i, there is a sequence of critical delays τ_{km}, $m=1,2,\cdots$, satisfying:

$$\tau_{km}=\frac{1}{\omega_k}\angle\left(-\frac{P(j\omega_k)}{Q(j\omega_k)}\right)+\frac{2\pi m}{\omega_k}, \tag{2.14}$$

where $\angle(\cdot)$ is the phase angle of a complex number restricted to the interval $[0,2\pi)$.

When it comes to systems with commensurate delays, the simple magnitude condition $|P(j\omega)|=|Q(j\omega)|$ at some crossing frequency ω is no longer available. Nevertheless, thanks to the next result from the matrix theory, one may still define a polynomial $F(\omega)$ to identify all crossing frequencies.

2.4.2 Identifying Critical Pairs: the Commensurate-Delay Case

First associate (2.8) with the following function:

$$\hat{D}(\lambda, x) = \sum_{k=0}^{M} P_k(\lambda) x^k, \tag{2.15}$$

where x can be a scalar or a matrix.

Let $\mathscr{H}$ be the Schur's hermitian form associated with (2.8) defined as

$$\begin{aligned} \mathscr{H}(\lambda, X) = & \sum_{k=1}^{M} |P_0 x_k + P_1 x_{k+1} + \ldots + P_{M-k} x_M|^2 \\ & - \sum_{k=1}^{M} |\overline{P_M} x_k + \overline{P_{M-1}} x_{k+1} + \ldots + \overline{P_k} x_M|^2 \end{aligned} \tag{2.16}$$

where $X = col(x_1, x_2, \ldots, x_M)$. The hermitian form $\mathscr{H}$ can be expressed as

$$\mathscr{H}(\lambda) = X^T \mathrm{H}(\lambda) X \tag{2.17}$$

where

$$\begin{aligned} \mathrm{H}(\lambda, \tau) = & \quad \hat{Q}(\lambda, S)^H \hat{Q}(\lambda, S) \\ & -\hat{D}(\lambda, S)^H \hat{D}(\lambda, S), \end{aligned} \tag{2.18}$$

and $\hat{Q}(\lambda, S) = \sum_{k=0}^{M} \overline{P_k} S^{M-k}$,

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is an $M \times M$ shift matrix. Now the function F can be defined for the commensurate-delay systems as

$$F(\omega) = -\det(\mathrm{H}(j\omega)). \tag{2.19}$$

The following generalized Schur-Cohn lemma is given in [21].

Lemma 2.1.

$$F(\omega) = -|P_M(j\omega)|^{2M} \prod_{i,k=1}^{M} (1 - z_i \overline{z_k}), \tag{2.20}$$

where z_i, $i = 1, 2, \ldots, M$ are the roots of the polynomial $\hat{D}(\lambda, x)$ in x.

A necessary and sufficient condition for $j\omega^*$ to be an imaginary characteristic root is that $\hat{D}(j\omega^*, x)$ has roots in x on the unit circle $\mathbb{D}$ of the complex plane. Therefore, $j\omega^*$ being an imaginary characteristic root implies that $F(\omega^*) = 0$. However, the converse is not true. It is possible that all z_i's defined in the last lemma do not lie on $\mathbb{D}$ when $\omega = \omega^*$ but we still have $F(\omega^*) = 0$ because there may exist some z_k, z_l such that $z_1\overline{z_k} = 1$. These discussions lead us to the following result:

Proposition 2.2. *If $\lambda = j\omega$ is an imaginary characteristic root of (2.8), then the following must hold:*

$$F(\omega) = 0. \tag{2.21}$$

The polynomial $F(\omega)$ defined in (2.19) was first introduced in [13] for identifying crossing frequencies. Once the imaginary characteristic roots are known, the critical delay values can be easily computed.

2.4.3 A Matrix Pencil Based Approach

Sometimes it is easier to work directly on the state-space representation of the system instead of the characteristic equation. For this purpose, some matrix pencil based methods have been proposed. In this subsection we present the method proposed in [33]. Consider the following matrix pencil:

$$\Lambda(x) = xW + U,$$

where $M, N \in \mathbb{R}^{(2Mn^2)\times(2Mn^2)}$ are given by:

$$W = \begin{pmatrix} I_{n^2} & 0 & \cdots & 0 & 0 \\ 0 & I_{n^2} & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & I_{n^2} & 0 \\ 0 & 0 & \cdots & 0 & B_m \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & -I_{n^2} & 0 & \cdots & 0 \\ 0 & 0 & -I_{n^2} & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & I_{n^2} \\ B_{-m} & 0 & \cdots & 0 & B_M \end{pmatrix}.$$

and B_{-k}, $k = 1, \cdots, M$ are defined as:

$$B_{-k} = I_n \otimes A_k^T,\ B_i = A_i \otimes I_n,\ B_0 = A \oplus A^T. \tag{2.22}$$

The operators $\otimes$ and $\oplus$ are the Kronecker product and sum. (see, e.g., [102]).

It is shown in [33] that $\lambda = j\omega^*$ is an imaginary characteristic if and only if there exits some complex number $z^* \in \partial\mathbb{D}$ such that

$$\det(\Lambda(z^*)) = 0, \tag{2.23}$$

and

$$\det(A_0 + \sum_{i=1}^{M} A_i (z^*)^i - j\omega^* I) = 0$$

are both satisfied.

2.5 Root crossing direction analysis

2.5.1 Simple Characteristic Roots

Given a critical pair $(j\omega^*, \tau^*)$, if $\lambda = j\omega^*$ is a simple root, then $\partial_\lambda D(\lambda, \tau)|_{(j\omega^*,\tau^*)} \neq 0$. By the implicit function theorem [9], in a neighborhood of $(j\omega^*, \tau^*)$, the characteristic root λ is a function of τ denoted here as $\lambda(\tau)$. We have

$$\frac{d\lambda(\tau^*)}{d\tau} = -\frac{\partial_\tau D(\lambda, \tau)}{\partial_\lambda D(\lambda, \tau)}\bigg|_{(j\omega^*,\tau^*)}. \tag{2.24}$$

If $\operatorname{sgn}(\Re(\frac{d}{d\tau}\lambda(\tau^*))) = 1$, the characteristic root crosses the imaginary axis and moves towards $\mathbb{C}_+$ as τ increases through τ^*. If this term is -1, the characteristic root moves towards $\mathbb{C}_-$ and becomes stable. When the right hand side of the last equation is zero, higher order analysis is necessary, which is reported in [12] and briefly introduced in [10]. For the characterization and analysis of crossing roots with multiplicity, several different approaches are available in the literature, including those based on the perturbation theory and Newton-Puiseux series [27][39], as well as the geometrical approach [29][30].

In the monograph [10], the following class of delay systems with parameterized coefficients and delays are discussed:

$$\dot{x} = A_0(p)x(t) + \sum_{i=1}^{m} A_i(p)x(t - \tau_i(p)). \tag{2.25}$$

Suppose $\lambda = j\omega^*$ is a simple characteristic root of $D(\lambda, p^*)$ and let $\lambda(p)$ be the trajectory of the characteristic root that passes through $j\omega^*$ as p passes through p^*. Then using the Jacobi's formula and some properties of left and right eigenvalues of rank one matrices, the following equation is derived

$$\partial_{p_i}\lambda(p) = \frac{v_0^* \cdot \partial_{p_i} M \cdot u_0}{v_0^* \cdot \partial_\lambda M \cdot u_0}, \tag{2.26}$$

where v_0^* and u_0 are the left or right eigenvectors of $M(j\omega^*, \tau^*)$.

It is worth mentioning that the equation (2.25) can actually represent a large class of systems with delay-dependent coefficients, although such type of systems are not discussed in [10]. For instance, if one sets $p = \tau$, $\tau_i = ip$, then (2.25) represents systems with commensurate delays. On the other hand, suppose $p = col\{p_1, \cdots, p_m\}$ and $\tau_i(p) = p_i$, then (2.25) represents systems with m independent delays. Therefore the formula (2.26) can actually be applied to systems with delay-dependent coefficients.

It is possible that the real part of $\lambda'(\tau)$ is zero at some critical pair. In this situation, higher order derivatives of the characteristic root with respect to τ needs to be computed in order to determine the root crossing direction. This type of analysis has been reported in [12].

Although it is quite straightforward to use the formula (2.24) or (2.26) to determine the crossing direction of simple characteristic roots, these formula do not provide deep insight into this problem. The right hand side of (2.24) or (2.26) rely on λ and τ in a complicated way, which does not reveal how the crossing direction may vary for different critical pairs.

In [3], an interesting relationship between the function $F(\omega)$ defined in (2.12) and the crossing direction of characteristic roots are derived. Let $(j\omega^*, \tau^*)$ be a critical pair of the characteristic equation (2.11), $\lambda(\tau)$ be the roots of (2.11) in a neighborhood of $(j\omega^*, \tau^*)$, then

$$\operatorname{sgn}\left(\Re(\lambda'(\tau^*))\right) = \operatorname{sgn}(F'(\omega^*)). \tag{2.27}$$

According to the last equation, as τ sweeps through τ^* from left, a pair of imaginary roots $\pm j\omega^*$ crosses the imaginary axis toward the right half complex plane if $F'(\omega^*) > 0$. This pair of roots move toward the left half complex plane if $F'(\omega^*) < 0$.

Two important *invariance properties* now follow from (2.14) and (2.27). The former shows that the crossing frequency ω_k is invariant with respect to a shift of $2\pi/\omega_k$ in the delay. The crossing direction of each characteristic root at $\lambda = j\omega_k$ is independent of the corresponding delay, as indicated in (2.27).

This further implies a simple root crossing pattern in the way characteristic roots with different frequencies cross the imaginary axis. It is easy to see

$$\operatorname{sgn}(F'(\omega_k)) = -\operatorname{sgn}(F'(\omega_{k+1})), \tag{2.28}$$

therefore the crossing direction of each two neighboring imaginary roots $j\omega_k$ and $j\omega_{k+1}$ always have the opposite crossing directions. However, we note that this pattern of alternating crossing directions does not always hold for systems with commensurate delays. Using this property, it is easy to see that the roots crossing toward one side of the imaginary axis more often than toward the other side. If the characteristic equation (2.11) admit at most a finite number of roots on the right half plane, then it can be deduces that the imaginary roots must cross toward the right more frequently, otherwise for large delays, the number of characteristic roots lying to the right of the imaginary axis would fall below zero. It is then claimed that there exits some positive number T^* such that the system (2.11) remains unstable for all $\tau > T^*$ and no stability switches will occur if τ further increases from T^*. However, their argument is based implicitly on the conditions that 1) the roots of (2.11) on the right half complex plane is finite; 2)all roots rely on the delay in a continuous way, which are not necessarily true given the assumptions in [3]. This issue is identified in [37] and the following assumption is introduced in addition to those in [3]:

$$\limsup \left\{ \left| \frac{Q(\lambda)}{P(\lambda)} \right| : |z| \to \infty, \Re(z) \geq 0 \right\} \leq K < 1.$$

To the author's best knowledge, the crossing analysis based on the function $F(\omega)$ has not been extended to systems with commensurate delays in the literature.

2.5.2 Repeated Characteristic Roots

The frequency-sweeping framework developed in [39] provides a general method for comprehensive stability analysis of multiple roots on the imaginary axis. Recall the characteristic equation for systems with commensurate delays:

$$D(\lambda, \tau) = \sum_{i=0}^{N} p_i(\lambda) e^{-i\lambda\tau} = 0, \tag{2.29}$$

where each $p_i(\lambda)$ is a polynomial. Corresponding to the characteristic equation, the following function is also defined:

$$\hat{D}(\lambda,z) = \sum_{i=0}^{N} p_i(\lambda) z^i = 0.$$

Sweeping through $\omega \geq 0$, for each $\lambda = j\omega$ suppose the equation above admits N solutions in z. Denote these solutions as $z_i(j\omega)$, $i = 1, \cdots, N$. Then the graph of $\Gamma_i(\omega) = |z_i(j\omega)|$ is referred to as a *frequency-sweeping curve*(FSC).

As τ increases through some critical delay $\tau_{\alpha k}$, where $\tau_{\alpha k}$ is given in (2.14), the increase of characteristic roots on the right half complex plane in a small neighborhood of λ_α is just the same as the quantity $NF_{z_\alpha}(\tau_{\alpha k})$ defined as

$$NF_{z_\alpha}(\tau_{\alpha k}) = N_{z_\alpha}(\tau+\varepsilon) - N_{z_\alpha}(\tau-\varepsilon), \tag{2.30}$$

where $N_{z_\alpha}(\tau)$ is the number of the FSCs: $\Gamma_i(\omega)$, $i = 1, \cdots, N$ that satisfy 1) $z_i(\omega) = \exp(-\lambda_\alpha \tau_\alpha)$, 2) $\Gamma_i(\omega) > 1$, and ε is an arbitrarily small positive number. In other words, the characteristic roots crossing the imaginary axis is associated with the corresponding frequency-sweeping curves crossing the horizontal line 1. Using this property, it is shown that the crossing of characteristic roots on the imaginary axis with multiplicity has similar invariance properties as the systems with just a single delay and simple imaginary characteristic roots. Furthermore, the system stability can be analyzed completely, in the sense that the eventual number of unstable characteristic roots as $\tau \to +\infty$ can be easily determined.

As mentioned in [39], when the critical pair is not regular, which includes the case of multiple characteristic roots on the imaginary axis, it is necessary to use the Puiseux series to analyze the asymptotic behaviors of these roots. Here we give a very rough idea about how Puiseux series can come into play. The characteristic equation (2.29) can be expanded at each critical pair (λ_0, τ_0) as

$$F(\Delta\lambda, \Delta\tau) = 0,$$

where $\Delta\lambda = \lambda - \lambda_0$, $\Delta\tau = \tau - \tau_0$ and $F(\Delta\lambda, \Delta\tau)$ is a series obtained through the Taylor expansion. The critical pair (λ_0, τ_0) may not be regular, in the sense that $\partial_\lambda D(\lambda_0, \tau_0) = 0$ or $\partial_\tau D(\lambda_0, \tau_0) = 0$. Let n be the number such that $\partial_\lambda^i D(\lambda_0, \tau_0) = 0$, for $i = 1, ..., n-1$ and $\partial_\lambda^n D(\lambda_0, \tau_0) \neq 0$. Also let g be such a number that $\partial_\tau^i D(\lambda_0, \tau_0) = 0$, for $i = 1, ..., g-1$ and $\partial_\tau^g D(\lambda_0, \tau_0) \neq 0$. Then there exits a positive number v such that the sequence $F(\Delta\lambda, \Delta\tau)$

determines the following v Puiseux series:

$$\Delta\lambda = \sum_{i=g_i}^{\infty} C_{ki}(\Delta\tau)^{\frac{i}{n_k}},\ k = 1,\cdots,v,$$

and $n_1 + \cdots n_v = n$. Conversely, we can also express the increase of τ from τ_0 as a series of $\Delta\lambda$, known as the duel Puiseux series:

$$\Delta\tau = \sum_{i=v_i}^{\infty} D_{ki}(\Delta\lambda)^{\frac{i}{g_k}}, k = 1,\cdots,v,$$

where $g_1 + \cdots + g_v = g$. Then it easy to see that the curves of $(\lambda(\tau),\tau)$ in a small neighbourhood of (λ_0,τ_0) may have several branches. The local behaviors of these branches are fully characterized by these Puiseux series. The Puiseux series can be obtained based on the Newton polygon. A constructive algorithm for computing the Puiseux series can be found in [38].

As mentioned in [40][41], the interest in characterizing the algebraic/geometric multiplicities corresponding to characteristic roots on the imaginary axis is emphasized, since such multiplicities characterize the local behavior of imaginary characteristic roots. An constructive approach in investigating the multiplicity of crossing imaginary roots is proposed in [28] through a class of functional confluent Vandermonde matrices. A sharper bound on the multiplicity of imaginary characteristic roots is established.

2.6 Systems with Delay-Dependent Coefficients

There are few publications that address specifically systems with delay-dependent coefficients. A notable exception is the work of Berreta and Kuang reported in [14], where a systematic stability analysis method is presented for systems with delay-dependent coefficients and a single delay. Following a generalized τ-decomposition approach, they extended the results in [3] for characteristic equations of the following form:

$$D(\lambda,\tau) = P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0. \tag{2.31}$$

The same definition of F in (2.12) is resumed, except that now $F(\omega,\tau)$ in general depends explicitly on τ. Consider a delay interval of interest denoted as $\mathscr{I}$. Functions $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are polynomials in λ and analytic in τ. In the sequel, we may write $D_\tau(\lambda)$ instead of $D(\lambda,\tau)$ to regard it as function of λ with τ as a parameter.

The authors assume that each positive root of

$$F(\omega,\tau)=0, \tag{2.32}$$

in ω, denoted as $\omega_k(\tau)$, $k=1,\cdots,L$ is defined and differentiable in $\mathscr{I}$. If for some $\omega^*>0$, $j\omega^*$ is an imaginary root of $D_\tau(\lambda)$, then there exits some k such that $\omega_k(\tau^*)=\omega^*$. Moreover, it must also hold that

$$\angle P(j\omega,\tau)-\angle Q(j\omega,\tau)=-\omega\tau+\pi+2l\pi \tag{2.33}$$

where l is an integer. The last condition is transformed into the following equation:

$$S_{k,l}(\tau):=\tau-\frac{\theta_k(\tau)+2l\pi}{\omega_k(\tau)}=0, \tag{2.34}$$

where

$$\theta_k(\tau)=\angle\left(-\frac{P(j\omega_k(\tau),\tau)}{Q(j\omega_k(\tau),\tau)}\right) \tag{2.35}$$

is a differentiable function under some assumptions. Then the root crossing direction criteria (2.27) can be modified as

$$\Re(\lambda'(\tau))|_{\tau=\tau^*}=\operatorname{sgn}(\partial_\omega F(\omega_k(\tau^*),\tau^*))\operatorname{sgn}(S'_{k,n}(\tau^*)). \tag{2.36}$$

These results show that the invariance properties indicated in [3] no longer hold when the system coefficients depend on the delay. At any given critical delay τ^*, the roots of $D_{\tau^*}(\lambda)$ crosses the imaginary axis at $\pm j\omega_k(\tau^*)$, which is in general different for different τ^*. Since a critical delay τ^* must satisfy (2.34), a series of constant shifts from τ^* in general does not produce a series of critical delays. The invariance of cross direction of critical delay does not hold as well. Indeed, comparing (2.36) with (2.27), it is clear that the crossing direction of a characteristic root on the imaginary axis depends also on an extra term, namely $\operatorname{sgn}(S'_{k,n}(\tau^*))$. Therefore it is totally possible that the crossing directions of imaginary roots associated with the frequency function $\omega_k(\tau)$ may switch at various critical delays.

The method developed in [14] has been applied to several hematopoietic dynamics model in [20] and the hopf bifurcation of blood cell production dynamics in [19].

Chapter 3

Stability Analysis of Systems with a Single Delay

3.1 Chapter Overview

In this chapter we study systems with delay-dependent coefficients and a single delay. Systems with this feature are encountered in various models in practice. Given a delay interval of interest, denoted as $\mathscr{I}$, our objective is to find all the subintervals such that the system is asymptotically stable if and only if the delay is contained in these subintervals. We first give a precise definition of the class of systems considered in this chapter, followed by a set of assumptions which our analysis relies on. We show how to decompose $\mathscr{I}$ into disjoint subsets such that a fixed number of *frequency functions* and *phase angle functions* can be defined in each subinterval. We identify all critical pairs $(j\omega, \tau)$ based on conditions expressed with the frequency and phase angle functions.

Crossing direction criteria are derived, which determines whether the characteristic roots on the imaginary axis will become stable or unstable as τ sweeps through some critical delays. Our analysis relaxed some of the assumptions of an earlier work of Beretta and Kuang [14]. While the crossing direction criterion of Beretta and Kuang utilizes just the first order derivative of certain functions, we show that a more general criterion based on higher-order analysis can be derived. The higher-order analysis suggests a possible geometric correlation between the phase angle functions and the number of unstable characteristic roots. Under some additional conditions, such a correlation makes it possible to determine the number of unstable characteristic roots based on the values of phase angle functions without computing the critical pairs as well as the corresponding root crossing direction. This observation motivates the development of a geometric framework for the stability analysis presented in Chapter 5. A Part of this chapter has been published in [61].

3.2 Problem Statement

In this chapter, we analyze time-invariant systems with a single delay parameter ranging in a given interval $\mathscr{I} = [\tau^l, \tau^u]$, with the restriction $0 \leq \tau^l < \tau^u$. The lower bound τ^l and the upper bound τ^u denote the minimal and maximal delay of interest, respectively. After linearization, the system dynamics is represented by the following characteristic equation:

$$D(\lambda,\tau) = P(\lambda,\tau) + Q(\lambda,\tau)e^{-\tau\lambda} = 0, \tag{3.1}$$

where $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are continuous in τ and are polynomials of λ with real coefficients for each given $\tau \in \mathscr{I}$. In some context, we may write $P_\tau(\lambda)$ and $Q_\tau(\lambda)$ instead of $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ when they are regarded as functions (polynomial in this case) of λ for a given τ. The same convention is also used for other functions of two independent variables with τ as one of them. For example, we may write $D_\tau(\lambda)$ instead of $D(\lambda,\tau)$ to emphasize that we are considering D as a function of λ while τ is viewed as a parameter. We say $(j\omega^*,\tau^*)$ is a *critical pair* and τ^* is a *critical delay*, if $\omega^* \geq 0$, $\tau^* \in \mathscr{I}$ and (3.1) holds with $(\lambda,\tau) = (j\omega^*,\tau^*)$.

We will develop in this chapter a systematic method for determining all the subinterval contained in the given $\mathscr{I}$ such that all the characteristic roots of (3.1) is on $\mathbb{C}_-$ when τ is restricted in these subintervals.

Systems (3.1) can arise from a state-space equation with feedback mechanism:

$$\begin{aligned}\dot{x}(t) &= A(\tau)x(t) + B(\tau)u(t)\\ u(t) &= Cx(t-\tau)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $A(\tau)$, $B(\tau)$ are matrices of appropriate dimensions. Then its characteristic equation involves at most a single delay and can be written as (3.1). This becomes obvious once the pair (A,B) is transformed into the Kalman controllable canonical form. We note that (3.1) is more general than the retarded type functional differential equations, it can also present equations of the neutral type. For the definition of functional differential equations, reader may refer to [10],

3.2.1 Assumptions

The solutions of (3.1) with λ on the imaginary axis plays a key role in the generalized τ-decomposition approach. When λ is restricted to the imaginary axis, (3.1) becomes

$$D(j\omega,\tau) = 0, \tag{3.2}$$

where ω is real. For (3.2) to hold with some real ω, it is necessary for the norm of $P(j\omega,\tau)$ and $Q(j\omega,\tau)$ to be the same, which leads to

$$F(\omega,\tau) = 0. \tag{3.3}$$

where

$$F(\omega,\tau) = P(j\omega,\tau)P(-j\omega,\tau) - Q(j\omega,\tau)Q(-j\omega,\tau). \tag{3.4}$$

The condition (3.3) is necessary but not sufficient for $j\omega$ to be a characteristic root of $D(\lambda,\tau)$. Another condition required is the match between the phase angles of $P(j\omega,\tau)$ and $Q(j\omega,\tau)$, which will be addressed later.

We will restrict ourselves to systems that satisfy the following four assumptions:

Assumption I. For all $\tau \in \mathscr{I}$, P_τ satisfies

$$\mathrm{ord}(P_\tau) = n. \tag{3.5}$$

Moreover,

$$\lim_{\omega\to\infty} \left| \frac{Q_\tau(j\omega)}{P_\tau(j\omega)} \right| < 1. \tag{3.6}$$

Assumption II. No $(\omega,\tau) \in \mathbb{R}_+ \times \mathscr{I}$ satisfies

$$\begin{aligned} P(j\omega,\tau) &= 0, \\ Q(j\omega,\tau) &= 0, \end{aligned}$$

simultaneously.

Assumption III. Any critical pair $(j\omega^*,\tau^*)$ must satisfy

$$\partial_\omega F(\omega^*,\tau^*) \neq 0. \tag{3.7}$$

Furthermore, the distance between any two critical delays is bounded from zero.

Assumption IV. There are only a finite number of pairs $(\omega,\tau) \in \mathbb{R} \times \mathscr{I}$ that simultaneously satisfy (3.3) and

$$\partial_\omega F(\omega,\tau) = 0. \tag{3.8}$$

These four assumptions are less restrictive than typical in the literature either stated explicitly or implicitly. Assumption I above requires the leading coefficient of P_τ not to vanish for all $\tau \in \mathscr{I}$, and

$$\mathrm{ord}(Q_\tau) \leq n. \tag{3.9}$$

For time-delay systems of retarded type, (3.9) is satisfied with strict inequality. When (3.9) is an equality, the time-delay system is of neutral type, and (3.6) requires the absolute value of its leading coefficient is strictly less than that of P_τ. Systems of neutral type involve some surprising subtleties. See [4] for an example for systems with single delay, and [59] [10] for more comprehensive coverage.

Assumption II is less restrictive than the counterpart in [14] which is

$$P(j\omega,\tau) + Q(j\omega,\tau) \neq 0 \text{ for all } (\omega,\tau) \in \mathbb{R}^2. \tag{3.10}$$

Indeed, the two complex equations in Assumption II are equivalent to four real equations with two real "unknowns" ω and τ. Obviously, cases that violate this assumption are degenerate and rare. On the other hand, the set

$$\{P(j\omega,\tau) + Q(j\omega,\tau) \mid (\omega,\tau) \in \mathbb{R}^2\}$$

is a region in the complex plane, and (3.10) requires this region not to include the origin, which is more restrictive.

Regarding assumption III, we will later show that condition (3.7) guarantees that

$$\partial_\lambda D(\lambda,\tau)\big|_{(j\omega^*,\tau^*)} \neq 0 \tag{3.11}$$

for any critical pair $(j\omega^*,\tau^*)$. Then it follows that in a small neighborhood of $(j\omega^*,\tau^*)$, a characteristic root is a well-defined function of τ, denoted here as $\lambda(\tau)$. the remaining part of the assumption means that the graph of $\lambda(\tau)$ is on the imaginary axis only at one point $\lambda^* = \lambda(\tau^*)$ in this neighborhood. A more restrictive assumption is to assume $\Re(\lambda'(\tau)) \neq 0$, which is implicitly assumed in most works of this nature, including [14]. Assumption IV is also quite natural, and is satisfied by almost all practical cases.

In [14], it is implicitly assumed that the number of real roots, $\pm\omega_k, k = 1,2,\cdots,m$, of $F_\tau(\omega)$ remains constant when τ is within the delay interval of interest $\mathscr{I}$, and they are continuously differentiable functions of τ. With our relaxed assumptions, these are no longer true. Especially, the real roots may suddenly emerge or disappear as the delay τ increases within $\mathscr{I}$. It is therefore essential to understand the structure of this solution set in order to solve the stability problem. This will be discussed in the next section.

3.3 Stability Analysis

The main idea for stability analysis here is along the line of so-called τ-decomposition method, with the main idea described in the introduction. The validity of the method is based on the fact that there exists a constant $c > 0$ for any closed interval of τ such that all roots of $D_\tau(\lambda)$ with $\Re(\lambda) > -c$ vary continuously as τ changes. This is true under Assumption I, see, e.g., [59] and [10].

The critical aspects of the stability analysis are: (i) identifying the values of τ such that there is at least one root of $D_\tau(\lambda)$ on the imaginary axis, as well as the corresponding imaginary roots, and (ii) determining whether these imaginary roots move from the left-half plane to the right-half plane, or vise versa, or return to the original side as τ increases through these values. In this section, we will consider the first aspect, and describe the process of stability analysis assuming we know the answer to the second aspect. In the next section, we will describe some methods of accomplishing the second aspect.

To accomplish the first aspect stated in the last paragraph, it is useful to introduce the notation

$$S(\lambda,\tau) = -\frac{P(\lambda,\tau)}{Q(\lambda,\tau)}e^{\tau\lambda}, \tag{3.12}$$

whenever

$$Q(\lambda,\tau) \neq 0. \tag{3.13}$$

Then

$$S(j\omega,\tau) = W(\omega,\tau)e^{j\theta(\omega,\tau)}, \tag{3.14}$$

where

$$\begin{aligned} W(\omega,\tau) &= \left|\frac{P(j\omega,\tau)}{Q(j\omega,\tau)}\right|, && (3.15)\\ \theta(\omega,\tau) &= \angle P(j\omega,\tau) - \angle Q(j\omega,\tau) + \omega\tau + \pi. && (3.16) \end{aligned}$$

When $\lambda = j\omega$ is on the imaginary axis, we note that (3.2) is equivalent to the following two conditions

$$\begin{aligned} W(\omega,\tau) &= 1, \qquad (3.17)\\ \theta(\omega,\tau) &= 2r\pi, \text{ for some integer } r, \qquad (3.18) \end{aligned}$$

provided that (3.13) holds. Let

$$\mathscr{W} = \{(\tau,\omega) \mid \tau \in \mathscr{I}, \omega \in \mathbb{R}, F(\omega,\tau) = 0\}, \qquad (3.19)$$

then $(\tau,\omega) \in \mathscr{W}$ if and only if (τ,ω) satisfies (3.13) and (3.17) in view of Assumption II. Therefore, an effective approach to determine all (τ,ω) satisfying (3.2) is to first determine the set $\mathscr{W}$, and then choose from $\mathscr{W}$ those (τ,ω) that also satisfy (3.18).

Since $F(\omega,\tau) = 0$ is a necessary condition for ω to be a crossing frequency, it is easy to see the real roots of $F_\tau(\omega)$ will play a critical role in the analysis of system stability switch. To keep track of these real roots, we will examine the function $F(\omega,\tau) = F_\tau(\omega)$ more closely. First note that $F_\tau(\omega)$ is a polynomial of ω^2. To see this, one only needs to notice that for any real ω, $F_\tau(\omega)$ must be an even function. Consequently the coefficients of all terms with an odd power of $F_\tau(\omega)$ must be zero. Denoting $\alpha = \omega^2$, we can rewrite the function F as:

$$\begin{aligned} \hat{F}(\alpha,\tau) &= F(\omega,\tau), \qquad (3.20)\\ \alpha &= \omega^2. \qquad (3.21) \end{aligned}$$

Therefore, a solution of

$$\hat{F}(\alpha,\tau) = 0 \qquad (3.22)$$

will provide n solutions α_k, $k = 1,2,\ldots,n$. Without loss of generality, let α_k, $k = 1,2,\ldots,n_p$, $n_p \le n$, be the only real and positive solutions. Then, all the real solutions of (3.3) are $\pm\omega_k$, $k = 1,2,\ldots,n_p$, where $\omega_k = \sqrt{\alpha_k}$. In general, the number of positive real roots n_p depends on τ. In order to understand this dependence, let $\tau^{(i)}$, $i = 1,2,\ldots,K-1$ be the set of all $\tau \in \mathscr{I}$ such that (ω,τ) simultaneously satisfies (3.3) and (3.8) for some $\omega \in \mathbb{R}_+$ (recall this set is indeed finite according to Assumption IV). We agree to order $\tau^{(i)}$ in ascending order

$$\tau^{(1)} < \tau^{(2)} < \cdots < \tau^{(K-1)}.$$

We will also write $\tau^{(0)} = \tau^l$ and $\tau^{(K)} = \tau^u$. Then, we may partition $\mathscr{I}$ into K subintervals

$$\mathscr{I}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}],\ i = 1,2,\ldots,K. \qquad (3.23)$$

We note that the boundary points of $\mathscr{I}^{(i)}$ can not be critical delays due to Assumption III. In Chapter 5 we will investigate the more complicated situation where these boundary points can be critical delays. The interior of $\mathscr{I}^{(i)}$ is denoted as $\mathscr{I}_o^{(i)} = (\tau^{(i-1)}, \tau^{(i)})$. Then the structure of the set $\mathscr{W}$ may be very clearly described in the following proposition.

Proposition 3.1. *For a given i, the number of real roots of $F_\tau(\omega)$ are the same for all $\tau \in \mathscr{I}_o^{(i)}$, and they are all simple. These real simple roots are continuous functions of τ, and may be expressed as $\pm\omega_k^{(i)}(\tau)$, $k = 1, 2, \ldots, m(i)$, where $m(i) \leq n$, and $\omega_k^{(i)}(\tau) > 0$ for all $\tau \in \mathscr{I}_o^{(i)}$.*

Proof. For a fixed i, by definition, for all $\tau \in \mathscr{I}_o^{(i)}$, any $\omega \in \mathbb{R}$ that satisfies

$$F_\tau(\omega) = 0 \tag{3.24}$$

must satisfy

$$F'_\tau(\omega) = \partial_\omega F(\omega, \tau) \neq 0, \tag{3.25}$$

from which we conclude that all real roots of $F_\tau(\omega)$ are simple. As $F_\tau(\omega)$ is an even function of ω, we can also conclude that the $-\omega$ is also a roots if ω is a real root, and $\omega = 0$ is not a root (otherwise, it cannot be simple). To show the invariance of the number of real solutions within $\mathscr{I}_o^{(i)}$, let $\tau^* \in \mathscr{I}_o^{(i)}$, and let ω_k^*, $k = 1, 2, \ldots, m$ be the only real roots of $F_{\tau^*}(\omega)$. By the continuity of roots with respect to coefficients[9], we may define m continuous functions $\omega_k(\tau)$, $k = 1, 2, \ldots, m$ in $\mathscr{I}_o^{(i)}$, $\omega_k(\tau^*) = \omega_k^*$, and each $\omega_k(\tau)$ is a root of $F_\tau(\omega)$. The proof is complete if we show that all $\omega_k(\tau)$ are real in $\mathscr{I}_o^{(i)}$ as this also implies that $\omega_k(\tau)$ are simple roots of $F_\tau(\omega)$.

For a given k, let

$$\tau_M = \sup\{\tau_a \in \mathscr{I}_o^{(i)} \mid \omega_k(\tau) \in \mathbb{R} \text{ for all } \tau \in [\tau^*, \tau_a]\}.$$

By continuity, $\omega_k(\tau_M)$ is real. We will show $\tau_M = \tau^{(i)}$. If not, for arbitrarily small $\varepsilon > 0$, $\tau_{M+\varepsilon} \in \mathscr{I}_o^{(i)}$ and $\omega_k(\tau_M + \varepsilon)$ is not real, which can be made arbitrarily close to $\omega_k(\tau_M)$ with sufficiently small ε. But this means that its complex conjugate $\bar{\omega}_k(\tau_M + \varepsilon)$ is also a root of the polynomial with real coefficients $F_{\tau_{M+\varepsilon}}(\omega)$ and arbitrarily close to $\omega_k(\tau_M)$. The continuity of roots with respect to the coefficients means that $\omega_k(\tau_M)$ cannot be a simple root of $F_{\tau_M}(\omega)$, which contradicts the first part of this proposition that we have already proven. Similarly, we can show that $\omega_k(\tau)$ is real for all $\tau \in (\tau^{(i-1)}, \tau^*)$, and the proof is complete. □

As τ moves rightward from a point in $\mathscr{I}_o^{(i)}$, some, say m, real roots, and $2l$ complex roots of $F_\tau(\omega)$ may merge to form a multiple root as τ reaches $\tau^{(i)}$, and some, say $2k$, become complex while $m + 2l - 2k$ roots remain real as τ enters $\mathscr{I}_o^{(i+1)}$. The most common scenarios are either two real roots merge and become complex, or two complex roots merge and become real as τ moves from $\mathscr{I}_o^{(i)}$ to $\mathscr{I}_o^{(i+1)}$ through $\tau^{(i)}$.

A real root of $F_\tau(\omega)$ in $\mathscr{I}_o^{(i)}$, say $\omega_k^{(i)}(\tau)$, $k \leq m(i)$, that does not merge with other roots at $\tau^{(i)}$ remains real, and becomes $\omega_l^{(i+1)}$ for some $l \leq m(i+1)$ as τ moves from $\mathscr{I}_o^{(i)}$ to $\mathscr{I}_o^{(i+1)}$ through $\tau^{(i)}$.

For a given i and k, as $\omega_k^{(i)}$ depends on τ continuously in $\mathscr{I}_o^{(i)}$, we will require $\angle P(j\omega_k^{(i)}(\tau),\tau)$ and $\angle Q(j\omega_k^{(i)}(\tau),\tau)$ to be continuous functions of τ. This means that

$$\theta_k^{(i)}(\tau) = \theta(\omega_k^{(i)}(\tau),\tau),\ k = 1,2,\cdots,m(i) \tag{3.26}$$

are continuous functions of τ within $\mathscr{I}_o^{(i)}$, and will be known as the phase functions. On the other hand, this continuity requirement means that the values of $\angle P(j\omega_k^{(i)}(\tau),\tau)$, $\angle Q(j\omega_k^{(i)}(\tau),\tau)$ and $\theta_k^{(i)}(\tau)$ may not be restricted to any 2π range. Furthermore, if $\omega_k^{(i)}(\tau)$ and $\omega_l^{(i)}(\tau)$ merge at, say, $\tau^{(i)}$, and we extend the definition of $\theta_k^{(i)}(\tau)$ and $\theta_l^{(i)}(\tau)$ to $\tau^{(i)}$ by continuity, then it is possible that

$$\theta_k^{(i)}(\tau) - \theta_l^{(i)}(\tau) = 2\pi r,$$

for some integer $r \neq 0$ even though

$$\omega_k^{(i)}(\tau^{(i)}) = \omega_l^{(i)}(\tau^{(i)}). \tag{3.27}$$

Going through each interval $\mathscr{I}^{(i)}$ and each curve $\omega_k^{(i)}(\tau)$, we may identify all $\tau = \tau_l$ such that

$$\theta_k^{(i)}(\tau_l) = 2\pi r,\ r \text{ integer}, \tag{3.28}$$

for some k if $\tau_l \in \mathscr{I}^{(i)}$. Notice, the ends of the intervals, $\tau^{(i)}$, $i = 0,1,\cdots,K$ should also be included. We will order such τ_l in an ascending order

$$\tau^l \leq \tau_1 < \tau_2 < \cdots < \tau_L \leq \tau^u.$$

Each τ_l is known as a *critical delay*. For each given τ_l, it is possible that more than one k satisfies (3.28), and we denote the corresponding $\omega_k^{(i)}(\tau_l) \geq 0$ as ω_{lh}, $h = 1,2,\cdots,H_l$. Therefore, we can identify all the pairs (ω_{lh},τ_l), $h = 1,2,\cdots,H; l = 1,2,\cdots,L$, that satisfy (3.2).

It is worth mentioning that a simple imaginary root $j\omega$ of $D_\tau(j\omega)$ may be a repeated root of $F_\tau(\omega)$. However, the converse is not true, as indicated by the following result.

Lemma 3.1. *Any pair $(\omega^*,\tau^*) \in \mathbb{R}\times\mathscr{I}$ that satisfies Assumption III must also satisfy (3.11).*

Proof. At (ω^*, τ^*)

$$\begin{aligned} F &= \bar{P}P - \bar{Q}Q = 0, \\ e^{-\tau\lambda} &= -P/Q = -\bar{Q}/\bar{P}, \\ \partial_\lambda D &= \partial_\lambda P + \partial_\lambda Q e^{-\tau\lambda} - \tau Q e^{-\tau\lambda} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_\omega F &= 2\Re\left(j\bar{P}\partial_\lambda P - j\bar{Q}\partial_\lambda Q\right) \\ &= -2\Im\left(\bar{P}\partial_\lambda P - \bar{Q}\partial_\lambda Q\right) \\ &= -2\Im\left(\bar{P}\partial_\lambda P - \bar{P}\frac{\bar{Q}}{\bar{P}}\partial_\lambda Q + \tau\bar{P}P\right) \\ &= -2\Im\left(\bar{P}\partial_\lambda P + \bar{P}e^{-\tau\lambda}\partial_\lambda Q - \tau\bar{P}Q^{-\tau\lambda}\right) \\ &= -2\Im\left(\bar{P}\partial_\lambda D\right). \end{aligned} \tag{3.29}$$

The above indicates that $\partial_\omega F(\omega^*, \tau^*) \neq 0$ implies (3.11). □

To see that the converse of the above lemma may not hold, note that the proof above shows that $\partial_\omega F(\omega^*, \tau^*) = 0$ only implies that $\partial_\lambda D(j\omega^*, \tau^*)$ is parallel to $P(j\omega^*, \tau^*)$, which does not necessarily mean $\partial_\lambda D(j\omega^*, \tau^*) = 0$.

Now we will describe the representation of the second aspect we mentioned at the beginning of this section, i.e., the movement of the imaginary roots. For a given pair (ω_{lh}, τ_l) that satisfies (3.2), a sufficiently small $\varepsilon > 0$, and any $\tau \in (\tau_l, \tau_l + \varepsilon)$, there is a unique solution λ_{lh}^+ of (3.1) in the neighborhood of $j\omega_{lh}$. Assumption III and continuity means that $\Re(\lambda_{lh}^+)$ must be nonzero, and have the same sign for any $\tau \in (\tau_l, \tau_l + \varepsilon)$. Similarly, let λ_{lh}^- be the unique solution of (3.1) in the neighborhood of $j\omega_{lh}$ corresponding to a given $\tau \in (\tau_l - \varepsilon, \tau_l)$, then $\Re(\lambda_{lh}^-)$ must have the same sign for all such τ. We define

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \frac{\mathrm{sgn}\left(\Re(\lambda_{lh}^+)\right) - \mathrm{sgn}(\Re(\lambda_{lh}^-))}{2}. \tag{3.30}$$

If $\mathrm{Inc}(\omega_{lh}, \tau_l) = 1$, a root of $D_\tau(\lambda)$ moves from the left-half plane to the right-half plane crossing the imaginary axis at $j\omega_{lh}$ as τ increases from $\tau_l - \varepsilon$ to $\tau_l + \varepsilon$. On the other hand, if $\mathrm{Inc}(\omega_{lh}, \tau_l) = -1$, then the root moves from the right-half plane to the left-half plane as τ increases from $\tau_l - \varepsilon$ to $\tau_l + \varepsilon$. If $\mathrm{Inc}(\omega_{lh}, \tau_l) = 0$, the root moves towards the imaginary axis, touching it at $j\omega_{lh}$, then return to the same half plane without crossing the imaginary axis.

We also define

$$\mathrm{Inc}(\tau_l) = 2\sum_{h=1}^{H_l} \mathrm{Inc}(\omega_{lh}, \tau_l). \tag{3.31}$$

Then, as τ increases from $\tau_l - \varepsilon$ to $\tau_l + \varepsilon$, there is a net increase of $\mathrm{Inc}(\tau_l)$ roots on the right-half plane. Notice, $\omega_{lh} > 0$, $h = 1, 2, \ldots, H_l$ only accounts for the roots on the upper half of the imaginary axis, and the coefficient 2 in front of the summation sign in (3.31) accounts for the fact that the roots of $D_\tau(\lambda)$ are symmetric to the real axis.

Let the number of right-half plane roots of $D_\tau(\lambda)$ be $N^u(\tau)$. Then, for any $\tau \in \mathscr{I}$, $\tau \neq \tau_l$, $l = 1, 2, \ldots, L$, we have

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L_\tau} \mathrm{Inc}(\tau_l), \tag{3.32}$$

where $L_\tau = \max\{l \mid \tau_l < \tau\}$.

If $\tau^l = 0$, as $D_{\tau^l}(\lambda)$ is a polynomial, $N^u(\tau^l)$ is easily obtained. If $\tau^l > 0$, $N^u(\tau^l)$ may be obtained by a method covered in [5] or [3] (but notice the correction [4]). If there are imaginary roots for $D_{\tau^l}(\lambda)$, $N^u(\tau^l)$ should not count these imaginary roots, and $\mathrm{Inc}(\omega_{1h}, \tau^l)$ should be defined as,

$$\mathrm{Inc}(\omega_{1h}, \tau^l) = \begin{cases} 1, & \text{if } \operatorname{sgn}\left(\Re(\lambda_{1h}^+)\right) = 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.33}$$

instead. Obviously, $N^u(\tau)$ remains the same in the interval (τ_l, τ_{l+1}) for any given l. The system is stable for all $\tau \in (\tau_l, \tau_{l+1})$ if $N^u(\tau) = 0$ for any $\tau \in (\tau_l, \tau_{l+1})$.

3.4 Crossing Direction Conditions

In the last section, a general procedure of determining the subintervals of τ in $\mathscr{I}$ such that $D_\tau(\lambda)$ is stable has been developed. In this section we shall address in this section a key step of that procedure, which is to determine each term $\mathrm{Inc}(\omega_{lh}, \tau_l)$ and thus the crossing direction of every imaginary characteristic root.

Given a critical pair $(j\omega_{lh}, \tau_l)$ for $D(\lambda, \tau)$, in theory it is possible to continue the trajectory of this root $\lambda(\tau)$ locally as τ further increases and thereby determine the local behavior of this root for τ in a neighborhood of τ_l. For this purpose, some numberical methods need to be employed, which may be cumbersome to implement in practice. We aim to put forth a simple analytical method that not only facilitates numerical implementation, but also motivates deeper investigation into this problem in later chapters.

The simplest case is when

$$\Re\left(\lambda_{lh}'(\tau)\right)_{\tau=\tau_l} \neq 0, \tag{3.34}$$

where, $\lambda_{lh}(\tau)$ is the implicit function defined by (3.1) in the neighborhood of $(j\omega_{lh}, \tau_l)$ provided that $\lambda_{lh}(\tau)$ is differentiable at τ_l. This can be guaranteed by requiring $D(\lambda, \tau)$ to be differentiable w.r.t τ at $(j\omega_{lh}, \tau_l)$ [9]. Indeed, provided that (3.34) is satisfied, it is easy to see

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \mathrm{sgn}\left(\Re\left(\lambda_{lh}'(\tau_l)\right)\right), \tag{3.35}$$

if $\tau_l > \tau^l$. On the other hand, if $\tau_l = \tau^l$, we have

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \max\left\{0, \mathrm{sgn}\left(\Re\left(\lambda_{lh}'(\tau_l)\right)\right)\right\}. \tag{3.36}$$

If (3.34) is violated, and $D(\lambda, \tau)$ is differentiable to a sufficiently high order at $(j\omega_{lh}, \tau_l)$, we may express $\mathrm{Inc}(\omega_{lh}, \tau_l)$ using higher order derivatives. Suppose

$$\begin{aligned}
\Re\left(\frac{d^k\lambda(\tau)}{d\tau^k}\right)_{\tau=\tau_l} &= 0,\ k = 1, 2, \ldots, m-1, \\
\Re\left(\frac{d^m\lambda(\tau)}{d\tau^m}\right)_{\tau=\tau_l} &\neq 0.
\end{aligned}$$

Then, if $\tau_l > \tau^l$, then

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \begin{cases} \mathrm{sgn}\left(\Re\left(\frac{d^m\lambda(\tau_l)}{d\tau^m}\right)\right), & \text{if } m \text{ is odd,} \\ 0, & \text{if } m \text{ is even.} \end{cases} \tag{3.37}$$

If $\tau_l = \tau^l$, on the other hand, then

$$\mathrm{Inc}(\omega_{lh}, \tau_l) = \max\left\{0, \mathrm{sgn}\left(\Re\left(\frac{d^m\lambda(\tau_l)}{d\tau^m}\right)\right)\right\}. \tag{3.38}$$

An explicit expression of $\mathrm{sgn}\left(\Re\left(\lambda_{lh}'(\tau_l)\right)\right)$ will be given first and the high-order analysis will be performed later. The expression is similar to that given in [14], but the derivation here is more succinct.

Theorem 3.1. *Let $(\omega^*, \tau^*) \in \mathbb{R} \times \mathscr{I}$ satisfy (3.2) and (3.7). Then (3.1) defines λ as a differentiable function of τ in a sufficiently small neighborhood of $(j\omega^*, \tau^*)$, and*

$$sgn\left(\Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau^*}\right) = sgn\left(\partial_\omega F(\omega, \tau)\right)_{\substack{\tau=\tau^* \\ \omega=\omega^*}} \cdot sgn\left(\frac{d_F\theta(\omega(\tau), \tau)}{d\tau}\right)_{\substack{\tau=\tau^* \\ \omega=\omega^*}} \tag{3.39}$$

where

$$\frac{d_F\theta}{d\tau} = \partial_\omega\theta\frac{d_F\omega}{d\tau} + \partial_\tau\theta$$

is the total derivative of $\theta(\omega,\tau)$ with respect to τ when ω is considered as a function of τ defined implicitly by (3.3) in a sufficiently small neighborhood of (ω^*,τ^*), and $\frac{d_F\omega}{d\tau}$ is the derivative of the function $\omega(\tau)$ so defined.

Proof. Lemma 3.1 and Assumption III indicate that $\partial_\lambda D(\lambda,\tau)$ is defined and non-zero in a neighborhood of $(j\omega^*,\tau^*)$. Therefore, the equation (3.1), or equivalently

$$S(\lambda,\tau) = 1, \tag{3.40}$$

defines λ as a differentiable function of τ in a small neighborhood of τ^* in view of the implicit function theorem. By differentiating (3.40), we obtain

$$\partial_\lambda S\frac{d\lambda}{d\tau} + \partial_\tau S = 0,$$

from which

$$\frac{d\lambda}{d\tau} \quad = \quad -\,\partial_\tau S\big/\,\partial_\lambda S = -\,\partial_\tau S\overline{(\partial_\lambda S)}\Big/\,|\partial_\lambda S|^2.$$

But, at $\lambda = j\omega^*$,

$$\begin{aligned}
\partial_\lambda S(\lambda,\tau) &= \frac{1}{j}\partial_\omega S(j\omega,\tau) \\
&= \frac{1}{j}\left[(\partial_\omega W)e^{j\theta} + j(\partial_\omega\theta)We^{j\theta}\right] \\
&= -j\frac{1}{W}\partial_\omega W + \partial_\omega\theta.
\end{aligned}$$

In the last step, (3.40) has been used. Similarly, we may obtain:

$$\partial_\tau S = \frac{1}{W}\partial_\tau W + j\partial_\tau\theta.$$

Therefore,

$$\begin{aligned}\operatorname{sgn}\left(\Re\left(\frac{d\lambda}{d\tau}\right)\right) &= -\operatorname{sgn}\left(\Re\left(\left(\frac{1}{W}\partial_\tau W + j\partial_\tau\theta\right)\right.\right.\\ &\quad \left.\left.\times\left(\partial_\omega\theta + j\frac{1}{W}\partial_\omega W\right)\right)\right)\\ &= \operatorname{sgn}\left(\frac{\partial_\omega W\partial_\tau\theta - \partial_\tau W\partial_\omega\theta}{W}\right). \end{aligned} \tag{3.41}$$

When ω is a function of τ defined implicitly by (3.3), or equivalently by (3.17), we have:

$$\frac{d_F\omega}{d\tau} = -\,\partial_\tau W/\partial_\omega W = -\,\partial_\tau F/\partial_\omega F. \tag{3.42}$$

In view of $|Q(\omega^*,\tau^*)| = |P(\omega^*,\tau^*)|$, it is easy to show that

$$\left.\frac{1}{W}\partial_\omega W\right|_{\substack{\tau=\tau^*\\ \omega=\omega^*}} = \left.\frac{1}{|P|^2}\partial_\omega F\right|_{\substack{\tau=\tau^*\\ \omega=\omega^*}}. \tag{3.43}$$

A substitution of (3.41) by (3.42) and (3.43) yields

$$\operatorname{sgn}\left(\Re\left(\frac{d\lambda}{d\tau}\right)\right) = \operatorname{sgn}\left(\frac{1}{|P|^2}\partial_\omega F\left(\frac{d_F\omega}{d\tau}\partial_\omega\theta + \partial_\tau\theta\right)\right),$$

from which (3.39) can be easily derived. □

We now make a useful observation about the first factor in (3.39).

Proposition 3.2. *For any given i and k, the quantity*

$$sgn\left(\partial_\omega F(\omega,\tau)\right)_{\omega=\omega_k^{(i)}(\tau)} \tag{3.44}$$

remains constant for all $\tau \in \mathscr{I}_o^{(i)}$.

Proof. Due to the continuity of $\partial_\omega F(\omega,\tau)$, in order for $\partial_\omega F(\omega_k^{(i)}(\tau),\tau)$ to change sign, it must first vanish, which violates the definition of $\mathscr{I}_o^{(i)}$. □

The above proposition indicates that the first factor in the expression on the right hand side of (3.39) only needs to be checked once for each curve $\omega_k^{(i)}(\tau)$ within the interval $\mathscr{I}_o^{(i)}$.

Next, we will provide an explicit expression for the second factor.

Proposition 3.3. *If (ω, τ) satisfies (3.2),*

$$\begin{aligned}\frac{d_F\theta}{d\tau} &= \frac{1}{|P|^2}\left(P_r\frac{d_F P_i}{d\tau} - P_i\frac{d_F P_r}{d\tau} - Q_r\frac{d_F Q_i}{d\tau} + Q_i\frac{d_F Q_r}{d\tau}\right)\\ &\quad +\tau\frac{d_F\omega}{d\tau} + \omega,\end{aligned}$$

where the subscripts r and i represent the real and imaginary part of the quantities respectively, and the total derivatives may be calculated by

$$\frac{d_F\phi}{d\tau} = \partial_\omega\phi\frac{d_F\omega}{d\tau} + \partial_\tau\phi,$$

where ϕ may be P_r, P_i, Q_r and Q_i, and

$$\frac{d_F\omega}{d\tau} = -\partial_\tau F/\partial_\omega F.$$

Proof. From

$$S = We^{-j\theta} = \frac{Pe^{j\omega\tau}}{Q}, \tag{3.45}$$

by taking total derivative with respect to τ, with $\omega(\tau)$ implicitly defined by (3.3), and noticing

$$W(\omega(\tau), \tau) = 1 \text{ for all } \tau,$$

we obtain:

$$-j\frac{d_F\theta}{d\tau}We^{-j\theta} = \frac{d}{d\tau}\left(\frac{P}{Q}\right)e^{j\omega\tau} + j\left(\tau\frac{d_F\omega}{d\tau} + \omega\right)\frac{Pe^{j\omega\tau}}{Q}.$$

Solving the above and (3.45) for $d_F\theta/d\tau$, we obtain

$$\frac{d_F\theta}{d\tau} = \frac{1}{j}\left(\frac{1}{P}\frac{d_F P}{d\tau} - \frac{1}{Q}\frac{d_F Q}{d\tau}\right) + \tau\frac{d_F\omega}{d\tau} + \omega. \tag{3.46}$$

In view of $|P|^2 = |Q|^2$, the expression in the parentheses in (3.46) can be written as

$$\frac{1}{P}\frac{d_F P}{d\tau} - \frac{1}{Q}\frac{d_F Q}{d\tau} = \frac{\bar{P}}{P\bar{P}}\frac{d_F P}{d\tau} - \frac{\bar{Q}}{Q\bar{Q}}\frac{d_F Q}{d\tau} = \frac{\bar{P}\frac{d_F P}{d\tau} - \bar{Q}\frac{d_F Q}{d\tau}}{P\bar{P}}.$$

A substitution of (3.46) by the above leads to the proposition. $\square$

Since the following definition is well defined due to Proposition 3.2:

$$\mathrm{sgn}_k^{(i)} = \mathrm{sgn}\big(\partial_\omega F(j\omega_k^{(i)}(\tau),\tau)\big),\ \forall\tau\in(\tau^{(i-1)},\tau^{(i)}),$$

we can rewrite (3.39) as

$$\mathrm{sgn}\left(\Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau^*}\right) = \mathrm{sgn}_k^{(i)}\cdot\mathrm{sgn}(\theta_k^{(i)}(\tau^*)), \tag{3.47}$$

where i, k satisfy $\lambda(\tau^*) = j\omega_k^{(i)}(\tau^*)$ and $\tau^*\in\mathscr{I}^{(i)}$. The last equation reflects an interesting 'separation principle': the root crossing direction is determined by the product of two distinct terms. The first term $\mathrm{sgn}_k^{(i)}$ alone is sufficient to determine the root crossing direction for system with fixed coefficients. The second term is what is particular about systems with delay-dependent coefficients. When the system coefficients are delay-free, the second term is always 1, thus the root crossing direction is reduced to (2.27). One may wonder if there exist some essential reasons why the crossing direction criterion can be decomposed into two terms which are not necessarily related. In Chapter V, we will review the problem from a geometric point of view and give an intuitive interpretation of this formula.

3.4.1 A Summary of the Stability Analysis Procedure

Here we summarize the procedure of the proposed stability analysis.

Step 1. Solve (3.3) together with (3.8) subject to $\omega\geq 0$, $\tau\in\mathscr{I}$ to obtain $\tau^{(i)}$, $i=0,\cdots,K$. $\mathscr{I}$ is thus decomposed into each sub-interval: $\mathscr{I}^{(i)} = [\tau^{(i-1)},\tau^{(i)}]$.

Step 2. In each $\mathscr{I}^{(i)}$, compute the real roots of $F_\tau(\omega)$, and thus obtain the frequency functions $\omega_k^{(i)}(\tau)$, $k=1,\cdots,m(i)$. Solve (3.28) to find all the critical delay value τ_i, $i=1,\cdots,L$ and thus the set of critical delays.

Step 3. Compute $\mathrm{Inc}(\tau_i)$ for each critical delay τ_i using the root crossing direction formula (3.47) together with (3.31).

Step 4. Now for any interval (τ_i,τ_{i+1}) we can arbitrarily pick a delay value r' in it and compute $N^u(r')$ via (3.32), then it follows that for all τ in (τ_i,τ_{i+1}), the number of unstable roots is equal to $N^u(r')$.

3.5 Invariance Properties

3.5.1 General Situations

When $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are independent of τ, $F(\omega,\tau)$ is independent of τ, the curves $\omega_k^{(i)}(\tau)$ become constants, and $d_F\theta/d\tau = \omega =$ constant. As a result, the crossing direction given in (3.39) is independent of delay. This fact is well-known in the literature on single or commensurate delay systems with delay-independent coefficients, and have been stated either implicitly [5] or explicitly [24] as the *invariance property*.

More generally, for systems with delay-dependent coefficient polynomials discussed in this chapter, we may still identify delay intervals where the crossing direction is invariant provided $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are continuously differentiable with respect to τ. Indeed, for a given subinterval $\mathscr{I}_o^{(i)} = (\tau^{(i-1)}, \tau^{(i)})$, and frequency curve $\omega_k^{(i)}(\tau)$, we may identify all the delay values $\tau_{kl}^{(i)}, l = 1,2,\ldots,L-1$, $\tau^{(i-1)} < \tau_{k1}^{(i)} < \tau_{k2}^{(i)} < \cdots < \tau_{k,L-1}^{(i)} < \tau^{(i)}$, such that $(d_F\theta/d\tau)_{\tau=\tau_{kl}^{(i)}} = 0$. Let $\tau_{k0}^{(i)} = \tau^{(i-1)}$, $\tau_{k,L}^{(i)} = \tau^{(i)}$. Then, we may conclude, by continuity, that the crossing direction at the curve $\omega_k^{(i)}(\tau)$ remains invariant for all $\tau \in (\tau_{k,l-1}^{(i)}, \tau_{kl}^{(i)}), l = 1,2,\ldots,L$. Note that the intervals for invariant crossing direction $(\tau_{k,l-1}^{(i)}, \tau_{kl}^{(i)})$ are different for different frequency curves in general.

However, this is a very conservative way of identifying delay intervals that preserves the invariance properties of root crossing directions because, roughly speaking, it depends just on the monotonicity of some phase function $\theta_k^{(i)}(\tau)$, which is not a necessity for preserving the crossing directions of imaginary characteristic roots associated with the frequency function $\omega_k^{(i)}(\tau)$.

3.6 Numerical Examples

In this section, we present three examples to illustrate the method developed in this chapter. These examples may not be solved using the method in [14] due to the need to divide the interval or due to violation of (3.10). Assumption I-IV can be verified at different steps of the analysis for these models.

Example 1. We first consider the stellar dynamos model (1.1) mentioned in Chapter 1. The system characteristic equation is

$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1c_3e^{-c_2\tau}e^{-\tau\lambda} = 0. \tag{3.48}$$

The parameters are set as: $c_1 = -10$, $c_2 = 2$, $c_3 = 3$. We are concerned with the stability of the system for $\tau \in \mathscr{I} = [0,2]$. We verify that Assumptions I and II indeed hold. The other assumptions can be verified as we carry out the analysis. The function F in this case is

$$F(\omega,\tau) = \omega^4 + 2c_2^2\omega^2 + c_2^4 - c_1^2c_3^2e^{-2c_2\tau}, \tag{3.49}$$

Only one pair of parameters $(\omega, \tau) = (0, \tau^{(1)})$ simultaneously satisfies (3.3) and (3.8), where

$$\tau^{(1)} = -\frac{1}{2c_2}\ln(\frac{c_2^4}{c_1^2 c_3^2}) \approx 1.006.$$

Therefore, Assumption IV is satisfied and the interval $\mathscr{I}$ is partitioned into two subintervals $\mathscr{I}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$, $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, where $\tau^{(0)} = 0$, $\tau^{(2)} = 2$. There is one positive real root $\omega_1^{(1)}(\tau)$ of $F_\tau(\omega)$ for $\tau \in (0, \tau^{(1)})$. As τ reaches $\tau^{(1)}$, this solution merges with the negative solution $-\omega_1^{(1)}(\tau)$, and they become complex as τ enters $\mathscr{I}^{(2)}$, and $F_\tau(\omega)$ does not have any real solution for τ in $\mathscr{I}^{(2)}$. In this case, we have

$$\omega_1^{(1)}(\tau) = \sqrt{|c_1 c_3| e^{-c_2\tau} - c_2^2}.$$

Corresponding to $\omega = \omega_1^{(1)}(\tau)$, $\theta_1^{(1)}(\tau)$ defined in (3.26) is plotted against τ in Figure 3.1b. It can be seen that the curve intersects the horizontal line 2π at $\tau_1 \approx 0.2748$ and $\tau_2 \approx 0.5314$. Therefore, $H_1 = 1$, $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 3.6490$, and $H_2 = 1$, $\omega_{21} = \omega_1^{(1)}(\tau_2) \approx 2.5228$. Now it is easy to verify that Assumption III also holds.

It can be verified that $\partial_\omega F(\omega_1^{(1)}(\tau), \tau) > 0$ for $\tau = 0.5$, and the above inequality holds for all $\tau \in \mathscr{I}_o^{(1)}$ according to Proposition 3.2. It can be easily calculated that

$$\frac{d}{d\tau}\theta_1^{(1)}(\tau_1) > 0, \ \frac{d}{d\tau}\theta_1^{(1)}(\tau_2) < 0,$$

which are also obvious from Figure 3.1b. Therefore, we conclude from (3.39) that a pair of characteristic roots cross the imaginary axis from the left-half plane to the right-half plane as τ increases through τ_1, and this pair of characteristic roots return to the left-half plane as τ further increases through τ_2. In other words, $\mathrm{Inc}(\omega_{11}, \tau_1) = 1$, and $\mathrm{Inc}(\omega_{21}, \tau_2) = -1$. Some simple calculation shows that the system is asymptotically stable for $\tau = 0$. A plot of $N^u(\tau)$ is shown in Figure 3.1c, from which we conclude that the system is stable for $\tau \in [0, \tau_1) \cup (\tau_2, \tau^u]$; it is unstable for $\tau \in (\tau_1, \tau_2)$. Simulation is carried out to verify the stability of the system for delay values $\tau = 0.5, 1.5, 2.5$, respectively. Let $x(t)$ satisfy the differential equation

$$\ddot{x}(t) + 2c_2\dot{x}(t) + c_2^2 x(t) - c_1 c_3 e^{-c_2\tau} x(t-\tau) = 0$$

corresponding to the characteristic equation (3.48). In Figure 3.1d we observe that the trajectory of $x(t)$ converges to zero when $\tau = 0.5, 2.5$, and diverges when $\tau = 1.5$. These results are consistent with our stability analysis.

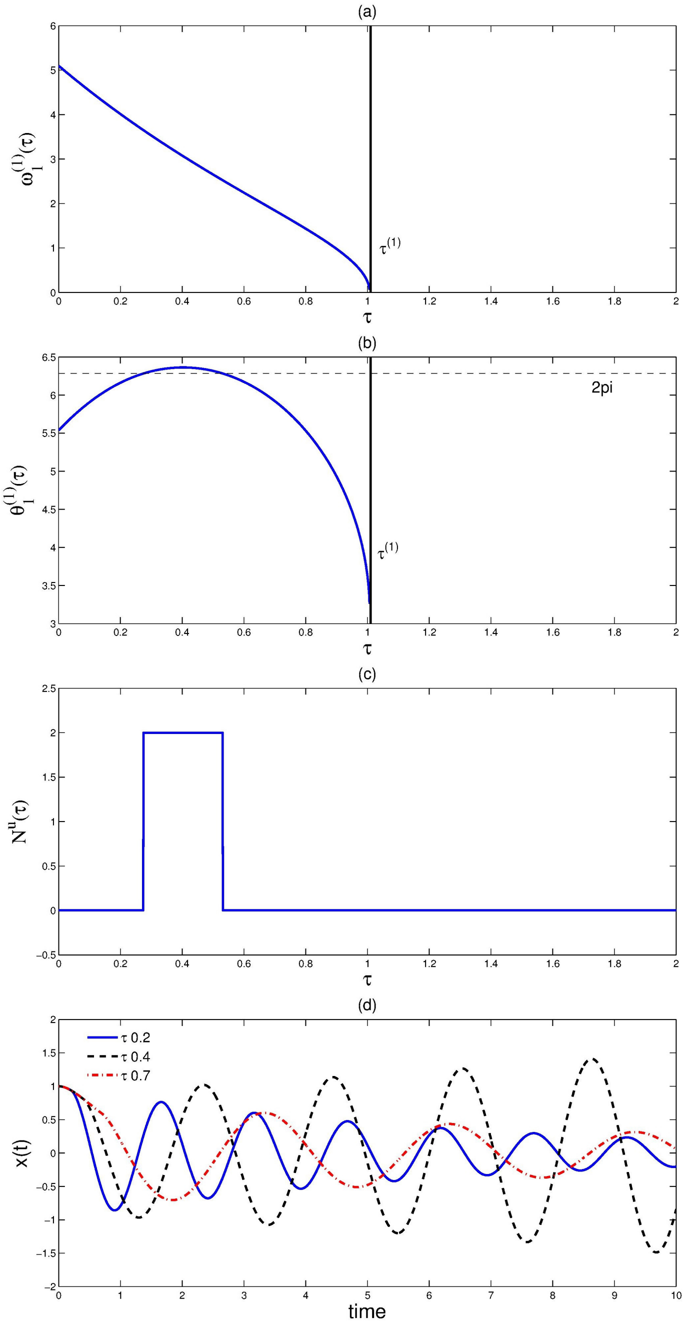

FIGURE 3.1: Stability analysis of the stellar dynamos. (a)The graph of frequency function. (b)The graph of the phase angle function. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 2π at τ_1 ,τ_2, giving two critical delays. (c)The number of unstable roots. (d)The time-response of the system with various delay values.

Example 2. Consider the following characteristic equation representing the population dynamics in [16]:

$$\lambda^2 + a\lambda + c + (b(\tau)\lambda + d(\tau))e^{-\lambda\tau} = 0, \tag{3.50}$$

where functions $b(\tau)$ and $d(\tau)$ take the following form:

$$b(\tau) = k_1 e^{-m\tau},\ d(\tau) = k_2 e^{-m\tau}.$$

The parameters are set as:

$$a = 2, c = 1, k_1 = 4, k_2 = 2, m = 3.5.$$

We analyze the stability of the system for $\mathscr{I} = [0, 2.5]$. Our assumptions can be verified to hold for this example as we carry out the analysis. The details of the verification is omitted here. The function F in this case is

$$F(\omega,\tau) = \omega^4 + (a^2 - b^2(\tau) - 2c)\omega^2 + c^2 - d^2(\tau). \tag{3.51}$$

Solving (3.3) and (3.8) together for $(\omega,\tau) \in \mathbb{R}_+ \times \mathscr{I}$, we obtain two pairs of solutions: $(\omega,\tau) \approx (0, 1.981)$, $(\omega,\tau) \approx (0.720, 2.391)$. The interval $\mathscr{I}$ is thus partitioned into three subintervals $\mathscr{I}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$, $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, $\mathscr{I}^{(3)} = [\tau^{(2)}, \tau^{(3)}]$, where $\tau^{(0)} = 0$, $\tau^{(1)} \approx 1.981$, $\tau^{(2)} \approx 2.391$, $\tau^{(3)} = 2.5$. The polynomial $F_\tau(\omega)$ has one positive real root, namely $\omega_1^{(1)}(\tau)$, in the interval $(\tau^{(0)}, \tau^{(1)})$ and two positive roots, namely $\omega_1^{(2)}(\tau)$ and $\omega_2^{(2)}(\tau)$, in the interval $(\tau^{(1)}, \tau^{(2)})$. It has no real root for $\tau \in (\tau^{(2)}, \tau^{(3)})$. We have the following expressions:

$$\begin{aligned}
\omega_1^{(1)}(\tau) &= 2^{-1/2}\sqrt{(b^2(\tau) + 2c - a^2) + \Delta^{1/2}(\tau)},\ \tau \in \mathscr{I}^{(1)},\\
\omega_1^{(2)}(\tau) &= 2^{-1/2}\sqrt{(b^2(\tau) + 2c - a^2) + \Delta^{1/2}(\tau)},\ \tau \in \mathscr{I}^{(2)},\\
\omega_2^{(2)}(\tau) &= 2^{-1/2}\sqrt{(b^2(\tau) + 2c - a^2) - \Delta^{1/2}(\tau)},\ \tau \in \mathscr{I}^{(2)},
\end{aligned}$$

where $\Delta(\tau) = (b^2(\tau) + 2c - a^2)^2 - 4(c^2 - d^2(\tau))$. We observe that $\pm\omega_2^{(2)}(\tau)$ emerge as a pair of real roots of $F_\tau(\omega)$ at $\tau = \tau^{(1)}$ and $\omega_2^{(2)}(\tau^{(1)}) = 0$. As τ approaches $\tau^{(2)}$ from the left, the solution $\omega_1^{(2)}(\tau)$ merges with $\omega_2^{(2)}(\tau)$. These two roots become complex as τ increases beyond $\tau^{(2)}$. The corresponding phase functions $\theta_1^{(1)}(\tau)$, $\theta_1^{(2)}(\tau)$, $\theta_2^{(2)}(\tau)$ are plotted against τ in Figure3.2b. These curves intersect the horizontal line 0 at $\tau_1 \approx 0.7576$ and $\tau_2 \approx 2.1745$. Therefore, $H_1 = 1$, $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 2.7556$ and $H_2 = 1$, $\omega_{21} = \omega_1^{(2)}(\tau_2) \approx 1.1837$.

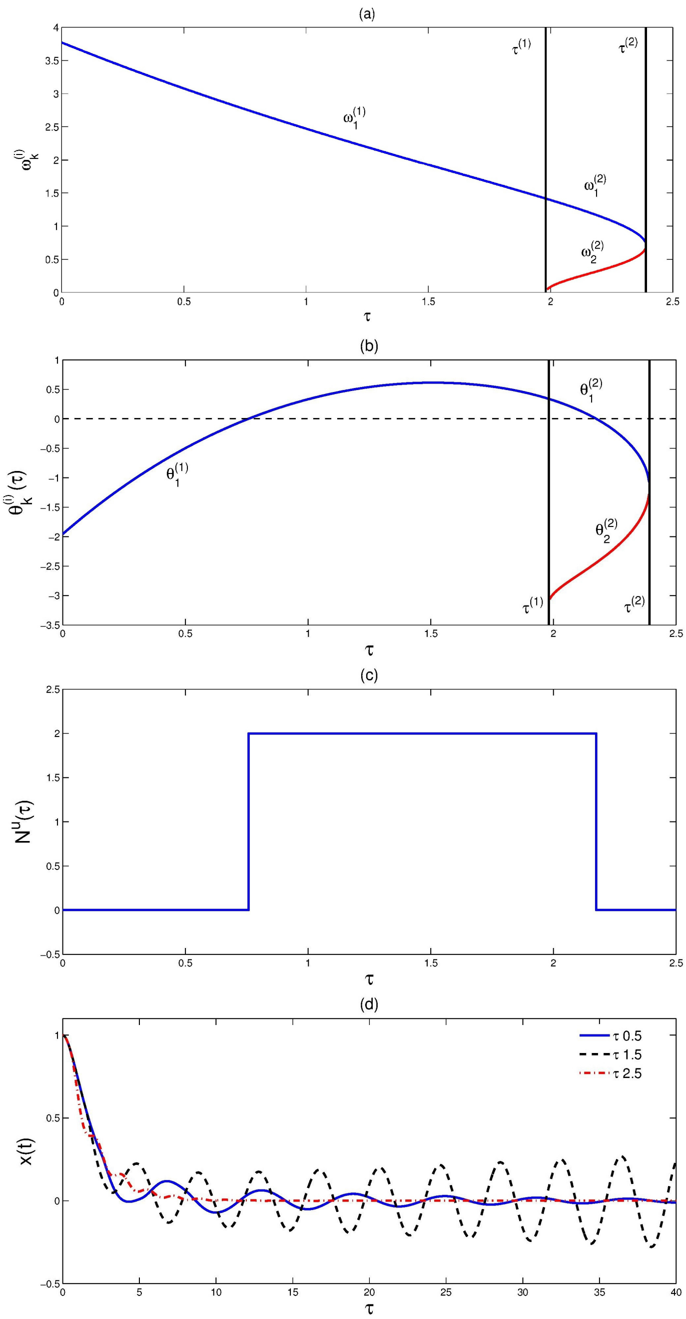

FIGURE 3.2: The stability analysis of the population dynamics (3.50). (a)The graphs of frequency functions. (b)The graphs of the phase angle functions. The graph of $\theta_1^{(1)}(\tau)$ and $\theta_1^{(2)}(\tau)$ crosses the horizontal line 0 at τ_1 and τ_2 respectively, giving two critical delays. (c)The number of unstable roots. (d)The time-response of the system with various delay values.

It can be verified that

$$\partial_\omega F(\omega_1^{(1)}(1),1) > 0,\ \partial_\omega F(\omega_1^{(2)}(2),2) > 0,$$

therefore $\partial_\omega F(\omega_1^{(1)}(\tau),\tau) > 0$ for $\tau \in (\tau^{(0)},\tau^{(1)})$ and $\partial_\omega F(\omega_1^{(2)}(\tau),\tau) > 0$ for $\tau \in (\tau^{(1)},\tau^{(2)})$. Computation shows that

$$\frac{d}{d\tau}\theta_1^{(1)}(\tau_1) > 0,\ \frac{d}{d\tau}\theta_1^{(2)}(\tau_2) < 0,$$

which also follows from the graph of phase functions plotted in Figure3.2b. We deduce by using (3.39) that a pair of characteristic roots cross the imaginary axis from the left-half plane to the right-half plane as τ increases through τ_1. As τ further increases, a of characteristic roots cross the imaginary axis from the right-half plane to the left-half plane as τ increases through τ_2. Consequently, we have $\mathrm{Inc}(\omega_{11},\tau_1) = 1$ and $\mathrm{Inc}(\omega_{21},\tau_2) = -1$. It is easy to verify that (3.50) is asymptotically stable for $\tau = 0$. Therefore, we conclude that the system is asymptotically stable for $\tau \in [0,\tau_1) \cup (\tau_2, 2.5]$; it is unstable for $\tau \in (\tau_1,\tau_2)$. The plot of $N^u(\tau)$ is given in Figure3.2c. Simulation results are shown in Figure3.2d based on the following differential equation corresponding to the characteristic equation (3.50):

$$\ddot{x}(t) + a\dot{x}(t) + cx(t) + b(\tau)\dot{x}(t-\tau) + d(\tau)x(t-\tau) = 0.$$

Example 3. Consider a system with the following characteristic equation for $\mathscr{I} = [0,1]$:

$$\lambda^2 + 4 + ((1-2e^{-2\tau})\lambda + 1 - 4e^{-2\tau})e^{-\lambda\tau} = 0. \tag{3.52}$$

We notice that $P(j\omega,\tau) + Q(j\omega,\tau) = 0$ when $\tau = \frac{1}{2}\ln(2)$ and $\omega = \sqrt{3}$. Therefore Condition (3.10) in [14] is not satisfied. However we can verify that all of our assumptions are satisfied. We have

$$F(\omega,\tau) = \omega^4 - (4e^{-4\tau} - 4e^{-2\tau} + 9)\omega^2 + 12 + 16e^{-4\tau} + 8e^{-2\tau}.$$

We find no $(\omega,\tau) \in \mathbb{R}_+ \times \mathscr{I}$ simultaneously satisfies (3.3) and (3.8), which means $\mathscr{I}^{(1)} = \mathscr{I}$. There are two positive roots of $F_\tau(\omega)$ for all $\tau \in \mathscr{I}^{(1)}$, therefore $\omega_1^{(1)}(\tau)$, $\omega_2^{(1)}(\tau)$ are defined in $\mathscr{I}^{(1)}$. With the corresponding phase functions plotted in Figure3.3b, we observe that $\theta_1^{(1)}(\tau)$ intersects the horizontal line 0 at $\tau_1 \approx 0.1982$ and $\theta_2^{(1)}(\tau)$ intersects the horizontal line 2π at $\tau_2 \approx 0.6933$. We also have $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 1.4945$ and $\omega_{12} = \omega_2^{(1)}(\tau_2) \approx 2.2656$. Computation shows that $\partial_\omega F(\omega_{11},\tau_1) < 0$ and $\partial_\omega F(\omega_{12},\tau_2) > 0$. From Figure3.3b, it is easy to see $\frac{d}{d\tau}\theta_1^{(1)}(\tau_1) > 0$ and $\frac{d}{d\tau}\theta_2^{(1)}(\tau_2) > 0$. Accordingly we can

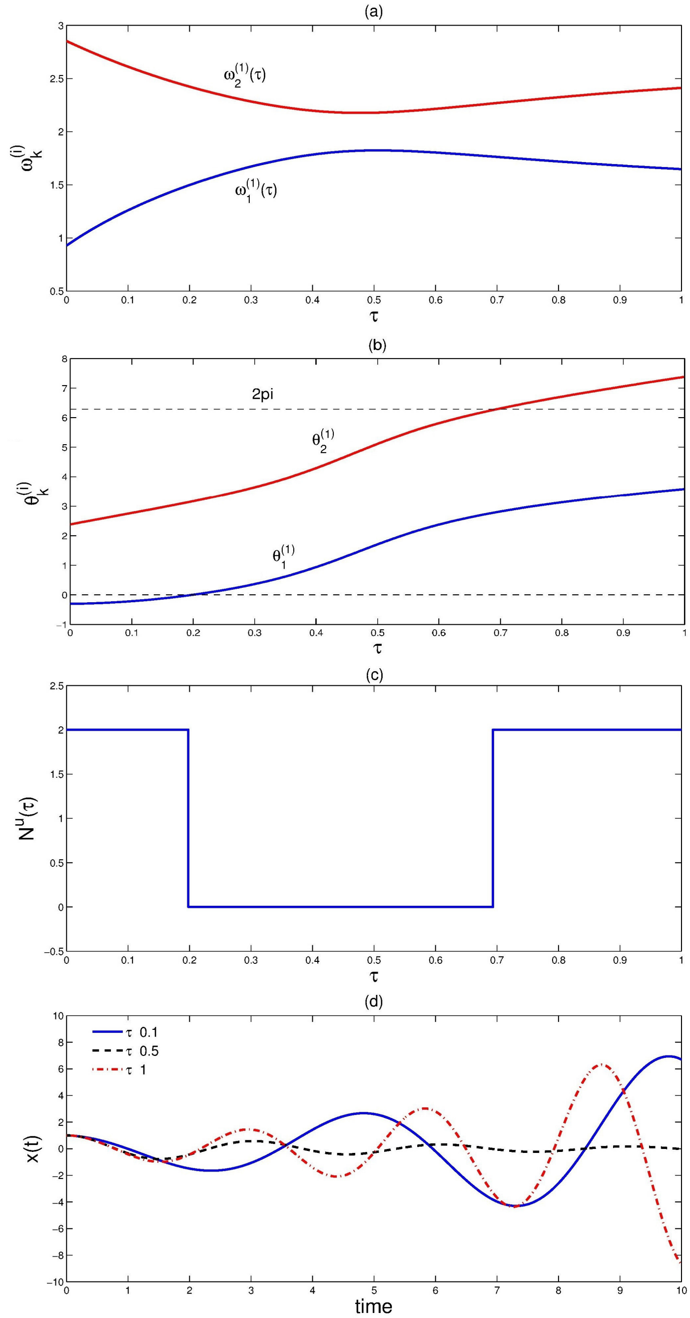

FIGURE 3.3: Stability analysis of the system (3.52). (a)The graphs of frequency functions. (b)The graphs of the phase angle functions. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at τ_1 and the graph of $\theta_2^{(1)}(\tau)$ crosses 2π at τ_2. Therefore two critical delays can be identified. (c)The number of unstable roots. (d)The time-response of the system with various delay values.

deduce that the characteristic root $j\omega_{11}$ moves toward the left-half plane and the characteristic root $j\omega_{12}$ moves towards the right-half plane as τ increases and sweeps through τ_1 and τ_2 respectively. The system has two unstable characteristic roots for $\tau = 0$, therefore it is asymptotically stable for $\tau \in (\tau_1, \tau_2)$ and unstable for $\tau \in [0, \tau_1) \cup (\tau_2, 1]$. This example shows that the method developed here is useful even if it is not necessary to divide $\mathscr{I}$ into subintervals. Simulation results based on the following differential equation

$$\ddot{x}(t) + 4x(t) + (1 - 2e^{-2\tau})\dot{x}(t-\tau) + (1 - 4e^{-2\tau})x(t-\tau) = 0$$

are given in Fig.3.3d, which confirm our theoretical analysis.

3.7 Toward Differentiation-Free Analysis

The formula (3.47) does not provide any information about the root crossing direction at a critical pair $(j\omega_k^{(i)}(\tau^*), \tau^*)$ if the first order derivative of the phase angle function is zero at τ^*. Here we give an extended version of Theorem 3.1 , which takes into account of higher order derivatives of the phase angle function.

Theorem 3.2. *In Theorem (3.1), further suppose n_d is a positive integer such that the following two conditions hold:*
1) $\partial_\tau^{n_d} P(\lambda, \tau)|_{\tau=\tau^*}$ and $\partial_\tau^{n_d} Q(\lambda, \tau)|_{\tau=\tau^*}$ are defined,
2) $(\frac{d}{d\tau})^l \theta_k^{(i)}(\tau^*) = 0$, for $l = 1, \cdots, n_d - 1$. Then for $l = 1, 2, \cdots, n_d$, the following holds:

$$\begin{aligned} sgn\left((\tfrac{d}{d\tau})^l \Re(\lambda)(\tau^*))\right) = sgn\left(\partial_\omega F(\omega_k^{(i)}(\tau^*), \tau^*)\right) \\ \times sgn\left(\left(\tfrac{d}{d\tau}\right)^l \theta_k^{(i)}(\tau^*)\right). \end{aligned} \tag{3.53}$$

The proof is given in Section 3.8. The above theorem implies that there might exist a correlation between the crossing direction of imaginary roots and how an associated phase angle crosses $2l\pi$ horizontal lines. In the last theorem, suppose $F(j\omega^*(\tau^*), \tau^*)$ is positive, then the pair of characteristic roots $\pm j\omega^*$ crosses toward right half complex plane if and only if the phase angle function $\theta_k^{(i)}(\tau)$ crosses the $2l\pi$ horizontal line from the lower side to the upper side at $\tau = \tau^*$. This pair of characteristic roots crosses toward left if and only if the phase angle function crosses the $2l\pi$ horizontal line from above to below. If $F(j\omega^*(\tau^*), \tau^*)$ is negative, the root crossing direction is reversed. Such a correlation suggests that as τ sweeps through an interval $[\tau_a, \tau_b] \in \mathscr{I}^{(i)}$, the increase in the number of unstable roots, $N^u(\tau_a) - N^u(\tau_b)$ depends on the value of phase angle functions only at τ_a and τ_b and how the phase angle function behave for $\tau \in (\tau_a, \tau_b)$ is actually irrelevant. This claim is stated formally in the next theorem:

Assumption V. For any given i, k and any critical pair $(j\omega_k^{(i)}(\tau^*),\tau^*)$, $\tau^*\in\mathscr{I}^{(i)}$, the number n_d defined in Theorem (3.2) exists.

Theorem 3.3. *Let Assumption I-V hold. Suppose both τ_a and τ_b are contained in $\mathscr{I}^{(i)}$ for some given i and are not critical delays. The following holds:*

$$N^u(\tau_b)-N^u(\tau_a)=2\sum_k sgn_k^{(i)}\left(\left\lfloor\frac{\theta_k^{(i)}(\tau_b)}{2\pi}\right\rfloor-\left\lfloor\frac{\theta_k^{(i)}(\tau_a)}{2\pi}\right\rfloor\right). \tag{3.54}$$

Proof. Let the pair (i,j) be fixed. Let $2l\pi<2(l+1)\pi<\cdots<2(l+n'-1)\pi$ be all the horizontal lines that lie between $\theta_k^{(i)}(\tau_a)$ and $\theta_k^{(i)}(\tau_b)$. Then we must have

$$n'_k=\left|\left\lfloor\frac{\theta_k^{(i)}(\tau_b)}{2\pi}\right\rfloor-\left\lfloor\frac{\theta_k^{(i)}(\tau_a)}{2\pi}\right\rfloor\right|. \tag{3.55}$$

As τ sweeps through $[\tau_a,\tau_b]$, a number of critical pairs $(j\omega_k^{(i)}(\tau_h^*),\tau_h^*)$, $h=1,\cdots,n''$ associated with the phase angle function $\theta_k^{(i)}(\tau)$ may appear, when $\theta_k^{(i)}(\tau)$ crosses horizontal lines $2(l+v)\pi$ for some integer $v=v_1,v_2,\cdots,v_{n''}$. We classify these critical pairs be classified according to the corresponding horizontal lines: we say a critical delay τ^* belong in a set $\mathscr{T}_v(i,k)$ if $\theta_k^{(i)}(\tau^*)=2(l+v)\pi$.

As τ sweeps from left to right, suppose the graph of $\theta_k^{(i)}(\tau)$ crosses some $2(l+v)\pi$ line from below to above for n_{v1} times, from above to below for n_{v2} times and it touches this horizontal line without crossing it for n_{v3} times, then we must have:

$$n_{v1}-n_{v2}=\mathrm{sgn}(\theta_k^{(i)}(\tau_b)-\theta_k^{(i)}(\tau_a)). \tag{3.56}$$

To see this, one only need to notice that we must have $n_{v1}-n_{v2}$ equals either 1 or -1 if $2(l+v)\pi$ lies between $\theta_k^{(i)}(\tau_a)$ and $\theta_k^{(i)}(\tau_b)$, otherwise $n_{v1}=n_{v2}=0$ (see Figure 3.4 for illustration).

In view of (3.53), if $0\le v\le n'-1$ we must have

$$\sum_{\tau^*\in\mathscr{T}_v(i,k)}\mathrm{Inc}(\omega_k^{(i)}(\tau^*),\tau^*) = \mathrm{sgn}_k^{(i)}(n_{v1}-n_{v2}) \tag{3.57}$$

$$= \mathrm{sgn}_k^{(i)}\mathrm{sgn}(\theta_k^{(i)}(\tau_b)-\theta_k^{(i)}(\tau_a)). \tag{3.58}$$

Therefore,

$$\begin{aligned} N^u(\tau_b) - N^u(\tau_a) &= 2\sum_k \sum_{v=v_1}^{v=v_{n''}} \sum_{\tau^* \in \mathscr{T}_v(i,k)} \mathrm{Inc}(\omega_k^{(i)}(\tau^*), \tau^*) \\ &= 2\sum_k \sum_{v=0}^{n'_k - 1} \mathrm{sgn}_k^{(i)} \mathrm{sgn}\big(\theta_k^{(i)}(\tau_b) - \theta_k^{(i)}(\tau_a)\big) \\ &= 2\sum_k \mathrm{sgn}_k^{(i)} n'_k \mathrm{sgn}(\theta_k^{(i)}(\tau_b) - \theta_k^{(i)}(\tau_a)). \end{aligned} \tag{3.59}$$

The proof is completed after n'_k in the last equation is replaced with the following expression resulted from (3.55):

$$n'_k \cdot \mathrm{sgn}\left(\theta_k^{(i)}(\tau_b) - \theta_k^{(i)}(\tau_a)\right) = \left\lfloor \frac{\theta_k^{(i)}(\tau_b)}{2\pi} \right\rfloor - \left\lfloor \frac{\theta_k^{(i)}(\tau_a)}{2\pi} \right\rfloor. \tag{3.60}$$

□

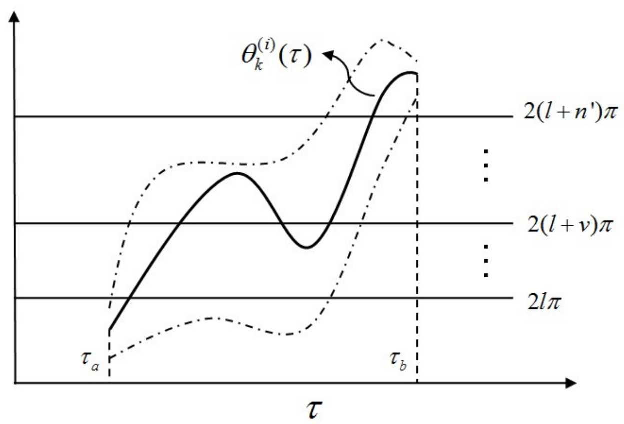

FIGURE 3.4: Correlation between the phase angle functions and number of unstable roots.

One may further ask whether the equation (3.55) still hold if the number n_d in Theorem (3.2) does not exist, or if the function $\theta_k^{(i)}(\tau)$ is not differentiable at τ^*. In Chapter 5, we will use some different techniques for the stability analysis, which will give affirmative answers to these questions.

The geometrical correlation between the position of phase angle functions and the number of unstable roots suggests that we may be able to determine system stability even if there are significant uncertainties in the system coefficients. Suppose that given the bound of the uncertainty in the system coefficients, the number $N^u(\tau_a)$ for some given τ_a is known. Suppose also we can derive an upper and lower bound of the possible value of each $\theta_k^{(i)}(\tau)$. In Figure 3.4 , the solid line depicts a phase angle function corresponding to the nominal coefficients and its real value is contained in a 'tube' marked by the solid lines. According to the equation (3.55), the width of the tube for $\tau \in (\tau_a, \tau_b)$ is actually irrelevant to system stability at $\tau = \tau_b$. As long as the tube is sufficiently narrow at τ_b such that the tube does not intersect any $2l\pi$ horizontal line at τ_b for some integer l, then the number of unstable roots is just the same as that of the nominal system. It is hard to claim such a robust property by directly applying the τ-sweeping method because it requires the analysis of all imaginary roots that appear as τ sweeps through $[\tau_a, \tau_b]$ and their crossing directions for all possible values of the uncertain coefficients, which is apparently a formidable task.

3.8 Proof of Theorem 3.2

Proposition 3.4. *Suppose $h : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a C^∞ function, $y : \mathbb{R} \to \mathbb{R}$ is a C^∞ function. Let $\left(\frac{d}{dt}\right)^k y$ denote the kth derivative of function y. Then we have*

$$\left(\tfrac{d}{dt}\right)^k h(y(t_1),t_1) = \left(\partial_y h \cdot (\tfrac{d}{dt})^k y + U_{h,k}(Y_k)\right)_{t=t_1}, \tag{3.61}$$

where $Y_k := (y, \frac{d}{dt}y, \left(\frac{d}{dt}\right)^2 y, ..., \left(\frac{d}{dt}\right)^{k-1} y)$, $U_{h,k}$ is a polynomial of (the elements of) Y_k with coefficients depending only on the partial derivatives of h.

Example: Let us compute the first and second order derivative of some function $h(y(t),t)$ w.r.t t, where all the symbols take the same meaning as in the last proposition, we have

$$\tfrac{d}{dt}h = \partial_y h \cdot \tfrac{d}{dt}y + \partial_t h,$$

$$\left(\tfrac{d}{dt}\right)^2 h = \partial_y h(\tfrac{d}{dt})^2 y + \partial_y^2 h \cdot (\tfrac{d}{dt}y)^2 + 2\partial_y h \partial_t h \cdot \tfrac{d}{dt}y + \partial_t^2 h.$$

Therefore $U_{h,1}(y) = \partial_t h$ and $U_{h,2}(y, \frac{d}{dt}y) = \partial_y^2 h \cdot (\frac{d}{dt}y)^2 + 2\partial_y \partial_t h \cdot \frac{d}{dt}y + \partial_t^2 h$.

Proposition 3.5. *Let n_1 be some positive integer, Suppose $h : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a C^{n_1} function. Further suppose $x \in C^{n_1} : \mathbb{R} \to \mathbb{R}$, $y \in C^{n_1} : \mathbb{R} \to \mathbb{R}$ and $\left(\frac{d}{dt}\right)^k x(t_1) = 0$ for some constant t_1, $1 \le k < n_1$. Then we have for $1 \le i \le n_1$*

$$\begin{aligned} & \left(\tfrac{d}{dt}\right)^i h(x(t),y(t),t) \mid_{t=t_1} \\ = \quad & \partial_x h \cdot \left(\tfrac{d}{dt}\right)^i x|_{t=t_1} + \left(\tfrac{d}{dt}\right)^i h(x(t_1),y(t),t) \mid_{t=t_1} \end{aligned}$$

The last two propositions can be easily shown using mathematical induction. We omit the proofs here.

Proposition 3.6. *Let $h, g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be C^∞ functions. Suppose there exist C^∞ function $x, y, a, b : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$ and some constant number t_1 the following hold:*

$$h(x(t), y(t), t) = 0, g(a(t), b(t), t) = 0 \tag{3.62}$$

$$y(t_1) = b(t_1) \tag{3.63}$$

$$h(x(t_1), \cdot, \cdot) = g(a(t_1), \cdot, \cdot) \tag{3.64}$$

Let $n_1 \geq 1$ be such a number that $(\frac{d}{dt})^k a(t_1) = 0$ for $0 \leq k < n_1$ and t_1 is some constant. Suppose in addition $\partial_x h/\partial_y h|_{t=t_1}$ as well as $\partial_a g/\partial_b g|_{t=t_1}$ exist and are not real. Then it follows that for $1 \leq k \leq n_1$

$$\left(\tfrac{d}{dt}\right)^k x|_{t=t_1} = \frac{\Re(\partial_a g(p_1)\overline{j\partial_y h(p_2)})}{\Re(\partial_x h(p_2)\overline{j\partial_y h(p_2)})} \left(\tfrac{d}{dt}\right)^k a|_{t=t_1}, \tag{3.65}$$

where $p_1 = (a(t_1), \omega(t_1), t_1)$ and $p_2 = (x(t_1), y(t_1), t_1)$ Moreover, for $0 < k < n_1$ it holds

$$\left(\tfrac{d}{dt}\right)^k y|_{t=t_1} = \left(\tfrac{d}{dt}\right)^k b|_{t=t_1} \tag{3.66}$$

Proof. We prove by induction. First consider the case $k = 1$. We have

$$(\partial_x h \cdot \tfrac{d}{dt}x + \partial_y h \cdot \tfrac{d}{dt}y + \partial_t h)\ |_{t=t_1} = 0 \tag{3.67}$$

$$(\partial_a g \cdot \tfrac{d}{dt}a + \partial_b g \cdot \tfrac{d}{dt}b + \partial_t g)\ |_{t=t_1} = 0 \tag{3.68}$$

it follows that

$$\tfrac{d}{dt}x(t_1) = -\left.\frac{\Re(j\partial_y h\overline{\partial_t h})}{\Re(j\partial_y h\overline{\partial_x h})}\right|_{t=t_1}$$

$$\tfrac{d}{dt}y(t_1) = -\left.\frac{\Re(j\partial_x h\overline{\partial_t h})}{\Re(j\partial_x h\overline{\partial_y h})}\right|_{t=t_1}$$

$$\tfrac{d}{dt}a(t_1) = -\left.\frac{\Re(j\partial_b g\overline{\partial_t g})}{\Re(j\partial_b g\overline{\partial_a g})}\right|_{t=t_1}$$

$$\tfrac{d}{dt}b(t_1) = -\frac{\Re(j\partial_a g\overline{\partial_t g})}{\Re(j\partial_a g\overline{\partial_b g})}\bigg|_{t=t_1}$$

since $\partial_y h|_{t=t_1} = \partial_b g|_{t=t_1}$ and $\partial_t h|_{t=t_1} = \partial_t g|_{t=t_1}$ due to (3.63) as well as (3.64), then clearly (3.65) and (3.66) hold.

If $n_1 = 1$ we have done (for in this case k ≤ 1). If $n_1 > 1$, we prove that given (3.65) and (3.66) hold for $k < i$, then (3.65) and (3.66) also hold for $k = i$, $1 < i \leq n_1$. Since $\left(\frac{d}{dt}\right)^k a = 0$ for $k \leq i-1$, it follows from Proposition 3.5 that

$$\left(\tfrac{d}{dt}\right)^i g|_{t=t_1} = \partial_a g \cdot \left(\tfrac{d}{dt}\right)^i a\ |_{t=t_1} + \left(\tfrac{d}{dt}\right)^i g(a(t_1), b(t), t)\ |_{t=t_1}$$

Let use denote

$$\hat{g}(\cdot,\cdot) = g(a(t_1),\cdot,\cdot),\ \hat{h}(\cdot,\cdot) = h(x(t_1),\cdot,\cdot)$$

Applying Proposition 3.4 to the second term on the R.H.S of the last equation we derive

$$\left(\tfrac{d}{dt}\right)^i g|_{t=t_1} = \left(\partial_a g \cdot \left(\tfrac{d}{dt}\right)^i a + \partial_b g \cdot \left(\tfrac{d}{dt}\right)^i b + U_{\hat{g},i}(Y_{g,i})\right)\ |_{t=t_1} = 0 \tag{3.69}$$

where $Y_{g,i} = (\left(\frac{d}{dt}\right)^i b, \ldots, \left(\frac{d}{dt}\right)^{i-1} b)$. The same argument shows that

$$\left(\tfrac{d}{dt}\right)^i h|_{t=t_1} = \left(\partial_x h \cdot \left(\tfrac{d}{dt}\right)^i x + \partial_y h \cdot \left(\tfrac{d}{dt}\right)^i y + U_{\hat{h},i}(Y_{h,i})\right)\ |_{t=t_1} = 0 \tag{3.70}$$

where $Y_{h,i} = (\left(\frac{d}{dt}\right)^i y, \ldots, \left(\frac{d}{dt}\right)^{i-1} y)$. We derive from (3.69) that

$$\begin{aligned}\left(\tfrac{d}{dt}\right)^i a(t_1) &= -\frac{\Re(j\partial_b g\overline{U_{\hat{g},i}(Y_{g,i})})}{\Re(j\partial_b g\overline{\partial_a g})}\bigg|_{t=t_1} \\ \left(\tfrac{d}{dt}\right)^i b(t_1) &= -\frac{\Re(j\partial_a g\overline{U_{\hat{g},i}(Y_{g,i})})}{\Re(j\partial_a g\overline{\partial_b g})}\bigg|_{t=t_1}\end{aligned} \tag{3.71}$$

Equality (3.64) implies $\hat{g} \equiv \hat{h}$ and thus $U_{\hat{h},i}(\cdot) = U_{\hat{g},i}(\cdot)$. The fact that $\left(\frac{d}{dt}\right)^l y(t_1) = \left(\frac{d}{dt}\right)^l b(t_1)$ for $1 \leq l \leq i-1$ (we remind readers this has already been assumed to be true when we stared mathematical induction for $k = i$) implies $Y_{h,i}(t_1) = Y_{g,i}(t_1)$. We thus have $U_{\hat{h},i}(Y_{h,i})|_{t=t_1} = U_{\hat{g},i}(Y_{g,i})|_{t=t_1}$, which combined with (3.70) leads to

$$\begin{aligned}\left(\tfrac{d}{dt}\right)^i x(t_1) &= -\frac{\Re(j\partial_y h \cdot \overline{U_{\hat{g},i}(Y_{g,i})})}{\Re(j\partial_y h\overline{\partial_x h})}\bigg|_{t=t_1} \\ \left(\tfrac{d}{dt}\right)^i y(t_1) &= -\frac{\Re(j\partial_x h \cdot \overline{U_{\hat{g},i}(Y_{g,i})})}{\Re(j\partial_x h\overline{\partial_y h})}\bigg|_{t=t_1}\end{aligned} \tag{3.72}$$

Noticing $\partial_y h|_{t=t_1} = \partial_b g|_{t=t_1}$, it is clear from (3.71) and (3.72) that (3.65) holds. If $i < n_1$, we have $\left(\frac{d}{dt}\right)^i a(t_1) = 0$ (recall the setup of the propostion), which implies $\left(\frac{d}{dt}\right)^i x(t_1) = 0$. Consequently we have

$$\left(\tfrac{d}{dt}\right)^i y(t_1) = -U_{\hat{h},i}(Y_{h,i})/\partial_y h|_{t=t_1} = -U_{\hat{g},i}(Y_{g,i})/\partial_b g|_{t=t_1} = \left(\tfrac{d}{dt}\right)^i b(t_1).$$

Thus by mathematical induction, the proposition is proved. □

We are now ready to provide the proof of Theorem 3.2.

Proof. Proof of Theorem (3.2). We first remind the reader that n_d is such a number that

$$\left(\tfrac{d}{d\tau}\right)^l \theta_k^{(i)}(\tau^*) = 0,$$

for $0 < l < n_d$. For notational simplicity, we denote $f(\tau) = \omega_k^{(i)}(\tau)$ and $\omega_k^{(i)}(\tau^*) = \omega^*$. By Assumption III, we have

$$\partial_\omega F(f(\tau^*), \tau^*) \neq 0,$$

which together with Proposition 3.1 in the appendix imply that $\partial_\lambda D(jf(\tau^*), \tau^*) \neq 0$. Then by the implicit function theorem, $D(\lambda, \tau) = 0$ determines λ as a C^{n_d} function of τ in a neighborhood of τ^*. Let $\lambda(\tau) = x(\tau) + jy(\tau)$, then $x, y \in C^{n_d}$ are real functions defined in a neighborhood of $\tau = \tau^*$, $\lambda = \lambda(\tau^*)$. We define a function $h(x, y, \tau)$ as

$$h(x, y, \tau) := D(x + jy, \tau) \tag{3.73}$$

then clearly we have $h(x(\tau), y(\tau), \tau) = D(\lambda(\tau), \tau) = 0$. On the other hand, by definition we have

$$P_0(jf(\tau), \tau) + P_1(jf(\tau), \tau)\exp(j\theta_k^{(i)}(\tau) - jf(\tau)\tau) = 0 \tag{3.74}$$

To facilitate notation, we denote

$$a(\tau) = \theta_k^{(i)}(\tau) \tag{3.75}$$

Further define $g(a, f, \tau) = P_0(jf, \tau) + P_1(jf, \tau)e^{j(a - f\tau)}$, we have

$$g(a(\tau), f(\tau), \tau) = 0 \tag{3.76}$$

and $g(\cdot, \cdot, \cdot)$ is locally C^{n_d}. Noticing $e^{ja(\tau^*)} = 1$ and $x(\tau^*) = 0$, it is easy to verify that

$$g(a(\tau^*), \cdot, \cdot) = h(x(\tau^*), \cdot, \cdot)$$

Now it follows from Proposition 3.6 that for $k \le n_d$

$$\left(\tfrac{d}{dt}\right)^k x(t_1) = \frac{\Re(\partial_a g(a(\tau^*), \omega^*, \tau^*)\overline{j\partial_y h(0,\omega^*,\tau^*)})}{\Re(\partial_x h(0,\omega^*,\tau^*)\overline{j\partial_y h(0,\omega^*,\tau^*)})}\left(\tfrac{d}{dt}\right)^k a \,|_{\tau=\tau^*} \tag{3.77}$$

Since at $\tau = \tau^*$ it holds that $\partial_a g(a(\tau^*), \omega^*, \tau^*) = jP_1 e^{-j\omega^*\tau^*}$,$\partial_x h(0,\omega^*,\tau^*) = \partial_\lambda D(j\omega^*,\tau^*)$, $\partial_y h(0,\omega^*,\tau^*) = j\partial_\lambda D(j\omega^*,\tau^*)$, from (3.77) we obtain

$$\begin{aligned} \left(\tfrac{d}{dt}\right)^k \Re(\lambda(\tau^*)) &= \frac{\Re(jP_1(j\omega^*,\tau^*)e^{-j\omega^*\tau^*}\overline{\partial_\lambda D(j\omega^*,\tau^*)})}{|\partial_\lambda D(j\omega^*,\tau^*)|^2}\left(\tfrac{d}{dt}\right)^k a(\tau^*) \\ &= \frac{\Re(P_0(j\omega^*,\tau^*)\overline{j\partial_\lambda D(j\omega^*,\tau^*)})}{|\partial_\lambda D(j\omega^*,\tau^*)|^2}\left(\tfrac{d}{dt}\right)^k a(\tau^*). \end{aligned} \tag{3.78}$$

From the equation (3.29) we know

$$2\Re\left(j\partial_\lambda D(j\omega^*,\tau^*)\overline{P_0(j\omega^*,\tau^*)}\right) = \partial_\omega F(\omega^*,\tau^*).$$

By definition we also have $(\frac{d}{d\tau})^l a(\tau^*) = \left(\frac{d}{dt}\right)^l \theta_k^{(i)}(\tau^*)$, for $l \le n_d$. Substitute these into (3.78), (3.53) is proved. □

3.9 Chapter Summary

A method of stability analysis for time-delay systems with coefficients depending on the delay has been developed. The method is an extension of the one given in [14] to more general cases. The method partitions the delay domain of interest for the delay into subintervals so that the magnitude condition yields a fixed number of solutions of frequencies ω as functions of the delay τ within each subinterval. With each frequency function a phase angle function is associated. Critical pairs can be identified based on the value of the phase angle functions. Root crossing criteria are derived, which utilizes the information of higher order derivatives of characteristic roots with respect to the delay when the lower order derivatives vanish. The crossing conditions are comprised of two factors. The first factor is already known for systems with fixed coefficients and the second factor depending on the phase angle function results from the fact that the system coefficients are parameterized by the delay. Analysis suggests an interesting correlation between the value of phase angle functions and the change of the number of unstable characteristic roots. This observation motivates the development of a different geometric perspective of stability analysis in Chapter 5.

Chapter 4

Systems with Commensurate Delays

4.1 Chapter Overview

In this chapter, stability of systems with commensurate delays and delay-dependent coefficients is studied along the line of the τ-decomposition approach. The main objective is to extend the results in Chapter 3 to systems with commensurate delays.

For systems with a single delay, the magnitude condition for the existence of imaginary characteristic roots motivated the introduction of function $F(\omega,\tau)$, a polynomial function in ω . The roots of this polynomial function in ω capture potential crossing frequencies of the characteristic roots. Then the crossing directions corresponding to a critical pair $(j\omega,\tau)$ can be determined by the partial derivative of this polynomial function and the derivative of the phase angle function at this critical pair. By using the generalized Schur-Cohn lemma [21], we are able to give a generalized definition of $F(\omega,\tau)$, which is still a polynomial in ω and but now applies to systems with commensurate delays. For characteristic equations with just a single delay, this new definition is reduced to the one in Chapter 3. We also show that the geometric idea that led to the definition of $F(\omega,\tau)$ in Chapter 3 can be generalized for the commensurate-delay case.

With the polynomial function $F(\omega,\tau)$ thus defined, we follow a similar procedure as we did in Chapter 3. We first state a set of assumptions and discuss their implications. The delay domain $\mathscr{I}$ is then decomposed into several disjoint sub-intervals $\mathscr{I}^{(i)}$'s. Within each $\mathscr{I}^{(i)}$, a fixed number of frequency functions are defined. We show that under some conditions that are realistic in practice, a unique phase angle function can be associated with each frequency function. Then similar to the single-delay case, the critical pairs of systems with commensurate delays can be identified by tracking each phase angle functions. With the notions and functions from Chapter 3 generalized and tailored for the commensurate-delay case, we prove that the root crossing direction criteria in Chapter 3 can be extended to the more general type of systems considered here. The chapter is concluded with two illustrative numerical examples. The first one pertains to α-stability analysis and the second one demonstrates the computation of the critical speed of an automobile vehicle with time delay in tire force generation.

4.2 Systems with Commensurate Delays

Consider time-delay systems with characteristic equations of the following form:

$$D(\lambda,\tau)=\sum_{k=0}^{M}P_k(\lambda,\tau)e^{-k\lambda\tau}=0, \tag{4.1}$$

where each $P_k(\lambda,\tau)$ is continuous in τ and is a polynomial of λ with real coefficients for any given $\tau\in\mathscr{I}$, where $\mathscr{I}=[\tau^l,\tau^u]$ is the delay interval of interest and $N^u(\tau^l)$ is supposed to be known. Our objective is to find all the sub-intervals contained in $\mathscr{I}$ for which (4.1) is asymptotically stable. We may write $D(\lambda,\tau)$ as $D_\tau(\lambda)$ and $P_i(\lambda,\tau)$ as $P_{i\tau}(\lambda)$ when λ is viewed as the argument and τ is regarded as a parameter of these functions.

For systems represented by a state equation of the form

$$\dot{x}(t)=A_1(\tau)x(t)+A_2(\tau)x(t-\tau),$$

where A_1 and A_2 are matrices of appropriate dimensions, commensurate delays may appear in the corresponding characteristic equation even though there is only a single delay in the original state-space equation. Characteristic equations with delay-dependent coefficients may also result from the $\alpha-$stability analysis of systems with delay-free parameters. For instance, consider the following characteristic equation [13], [23]:

$$\lambda+e^{-\tau\lambda}+e^{-2\tau\lambda}=0. \tag{4.2}$$

The characteristic equation is said to be $\alpha-$stable if the real part of all its roots is smaller than $-\alpha$. Replacing λ with $\lambda-\alpha$ in (4.2), we obtain

$$\lambda-\alpha+e^{\alpha\tau}e^{-\tau\lambda}+e^{2\alpha\tau}e^{-2\tau\lambda}=0, \tag{4.3}$$

It is easy to see that the $\alpha-$stability of (4.2) is equivalent to the asymptotic stability of characteristic equation (4.3), which has delay-dependent coefficients.

Define

$$\hat{D}(\lambda,\tau,x)=\sum_{k=0}^{M}P_k(\lambda,\tau)x^k, \tag{4.4}$$

where x can be a scalar or a square matrix of any dimension. Notice that $\hat{D}(\lambda,\tau,x)$ can be a matrix or scalar depending on x. We denote $\hat{D}_{\lambda,\tau}(x)=\hat{D}(\lambda,\tau,x)$ when we view λ, τ as two parameters and x as the variable. By definition we have

$$\hat{D}_{j\omega,\tau}(e^{-j\omega\tau})=\hat{D}(j\omega,\tau,e^{-j\omega\tau}).$$

Introduce the following Hermitian matrix:

$$\begin{aligned}\mathrm{H}(\lambda,\tau)= &\quad \hat{Q}(\lambda,\tau,S)^H\hat{Q}(\lambda,\tau,S)\\ &-\hat{D}(\lambda,\tau,S)^H\hat{D}(\lambda,\tau,S),\end{aligned} \tag{4.5}$$

where $\hat{Q}(\lambda,\tau,S)=\sum_{k=0}^{M}\overline{P_k}S^{M-k}$ and

$$S=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}$$

is an $M\times M$ shift matrix. Define a function F as

$$F(\omega,\tau)=-\det\left(\mathrm{H}(j\omega,\tau)\right), \tag{4.6}$$

We may write $F_\tau(\omega)$ instead of $F(\omega,\tau)$ when it is regarded as a polynomial in ω. We claim that

$$F(\omega,\tau)=0 \tag{4.7}$$

is a necessary condition for $(j\omega,\tau)$ to be a critical pair, i.e., for $(\omega,\tau)\in\mathbb{R}_+^*\times\mathscr{I}$ to satisfy

$$D(j\omega,\tau)=0. \tag{4.8}$$

This readily follows from the following expression of F, which has already been discussed in Chapter 2:

$$F(\omega,\tau)=-|P_M(j\omega,\tau)|^{2M}\prod_{i,k=1}^{M}(1-z_i\overline{z_k}), \tag{4.9}$$

where z_i, $i=1,2,\ldots,M$ are the roots of the polynomial $\hat{D}_{j\omega,\tau}(x)$. Equation (4.7) can be viewed as a generalization of the magnitude condition proposed in [61] for systems with a single delay. The geometric interpretation is given in the appendix of this chapter.

We introduce a set of standing assumptions for systems with commensurate delays considered in this dissertation.

Assumption I. For all $\tau \in \mathscr{I}$, $P_{0\tau}$ satisfies $\mathrm{ord}(P_{0\tau}) = n$ and

$$\mathrm{ord}(P_{i\tau}) \le n,\ i = 1, 2, \cdots, M.$$

Furthermore, all the roots of following equation

$$s^M + c_{1\tau} s^{M-1} + c_{2\tau} s^{M-2} + \cdots + c_{M-1,\tau} s + c_M = 0$$

satisfy $|s| < 1$, where

$$c_{i\tau} = \lim_{\lambda \to \infty} \frac{P_{i\tau}(\lambda)}{P_{0\tau}(\lambda)}.$$

Assumption II. No $(\omega, \tau) \in \mathbb{R} \times \mathscr{I}$ satisfies (4.7) and

$$P_M(j\omega, \tau) = 0 \tag{4.10}$$

simultaneously.

Assumption III. There are only a finite number of (ω, τ) in $\mathbb{R}_+ \times \mathscr{I}$ that simultaneously satisfy (4.7) and

$$\partial_\omega F(\omega, \tau) = 0. \tag{4.11}$$

Moreover, $\mathrm{ord}(F_\tau(\omega))$ is constant for $\tau \in \mathscr{I}$.

Assumption IV. Suppose that $(j\omega^*, \tau^*)$ is a critical pair, i.e., $(\omega^*, \tau^*) \in \mathbb{R}_+ \times \mathscr{I}$ satisfies (4.8), then (4.11) does not hold for $\omega = \omega^*$ and $\tau = \tau^*$. Furthermore, each $P_k(\lambda, \tau)$, $k = 0, \ldots, M$ is differentiable in a neighborhood of $(j\omega^*, \tau^*)$.

Assumption V Only a finite triplets of the form $(\omega, \tau, x) \in \mathbb{R} \times \mathscr{I} \times \mathbb{C}$ satisfy simultaneously (4.7) as well as

$$\begin{cases} \hat{D}(j\omega, \tau, x) & = 0, \\ \partial_x \hat{D}(j\omega, \tau, x) & = 0. \end{cases}$$

Let Φ_τ be the set of all the τ values appearing in these triples. It is further assumed that Φ_τ does not contain any critical delay.

Assumption I implies that all the roots of (4.1) with real parts larger than $-c$, where c is some positive constant, vary continuously as τ changes [59]. We will later prove that Assumption II together with Assumption IV ensures that if $\lambda = j\omega^*$ is an imaginary root for some critical delay τ^*, it must be simple. Consequently λ must be locally a continuous function of τ according to the implicit function theorem. It will also be clarified later that Assumption V is needed to ensure that the uniqueness of the unit solution of $\hat{D}(j\omega(\tau), \tau, x) = 0$ in x is defined except for some specific τ values, where $\omega(\tau)$ is a root of the polynomial $F_\tau(\omega)$. As a consequence, we are able to associate with each frequency function a unique phase angle function. Notice that this assumption is automatically satisfied if $M = 1$. It can be expected that these assumptions should hold for general systems in practice. For instance, Assumption II requires two real variables do not satisfy three real equations at the same time. Assumption V essentially requires five real equations to admit a finite number of solutions in four real numbers. These conditions can be satisfied in general except for some degenerated cases.

4.3 Stability analysis

4.3.1 Identifying imaginary roots

We develop a generalized τ-decomposition approach for systems with commensurate delays. We start with one value of delay τ^l for which one knows the the number of roots of the characteristic equation on the right half pane. Let the delay parameter τ sweep through an interval of interest $\mathscr{I} = [\tau^l, \tau^u]$, one can identify all critical delay values, for which equation (4.1) admits at least one root on the imaginary axis. We arrange the critical delay values in ascending order as:

$$\tau^l \leq \tau_1 < \tau_2 < \ldots < \tau_L \leq \tau^u. \tag{4.12}$$

Due to Assumption I, the characteristic roots close to the imaginary axis is continuous with respect to τ, therefore the number of unstable roots can not change for τ in each interval (τ_k, τ_{k+1}), $k = 1, \ldots, L-1$. Once the crossing direction of characteristic roots on the imaginary axis as τ goes through τ_k is computed, one can determine the change of the number of roots on the right half plane. We will provide a criterion to determine the crossing directions of the imaginary roots and show how system stability can be conveniently determined based on the graphs of the phase angle functions.

We address the first critical aspect of our analysis, that is to identify all critical delay values for which the characteristic equation admits imaginary roots. Another key aspect is to determine how these imaginary roots move as τ increases and sweeps through these critical values, which will be discussed subsequently.

Recall that if $\lambda = j\omega$ is an imaginary root of $D_\tau(\lambda)$, ω must be a real root of $F_\tau(\omega)$. We first show that $\mathscr{I}$ can be decomposed into disjoint intervals in which the number of real roots of $F_\tau(\omega)$ is invariant. Let

$$\tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(K-1)}$$

be exactly all the τ value contained in all the pairs $(\omega, \tau) \in \mathbb{R}_+ \times \mathscr{I}$ that simultaneously satisfy (4.6) and (4.11). We also write $\tau^{(0)} = \tau^l$ and $\tau^{(K)} = \tau^u$ and then decompose $\mathscr{I}$ into K subintervals

$$\mathscr{I}^{(i)} = [\tau^{(i-1)}, \tau^i],\ i = 1, \ldots, K. \tag{4.13}$$

It has been shown in Proposition 3.1 that for any given index i, $F_\tau(\omega)$ admits a fixed number of real roots for all τ in the interior of $\mathscr{I}^{(i)}$. These roots are simple and can be regarded as continuous functions of τ, denoted as $\omega_k^{(i)}(\tau)$, $k = 1, 2, ..., m(i)$. The definition of these functions is extended to $\mathscr{I}^{(i)}$ by requiring them to be continuous at $\tau^{(i-1)}$ and $\tau^{(i)}$.

Proposition 4.1. *Suppose $(\omega^*, \tau^*) \in \mathbb{R} \times \mathscr{I}$ satisfies (4.7) but does not satisfy (4.11). Furthermore, assume $\tau^* \notin \Phi_\tau$, then $\hat{D}_{j\omega^*,\tau^*}(x)$ admits a unique root on $\partial\mathbb{D}$, which is simple.*

Proof. Let $x = z_l$, $l = 1, 2, \ldots, M$ be the roots of $\hat{D}_{j\omega,\tau}(x)$ for (ω, τ) in some neighborhood of (ω^*, τ^*). In this neighborhood, Assumption V together with $\tau^* \notin \Phi_\tau$ ensures that $\hat{D}'_{\omega\tau}(z_l) \neq 0$ for each l. Therefore by the implicit function theorem, each z_l is locally a continuous function of (ω, τ) denoted as $z_l(\omega, \tau)$ and is differentiable in ω. We first prove the existence of a solution on the unit disk. Suppose such a solution on $\partial\mathbb{D}$ does not exist, then it follows from $F(\omega^*, \tau^*) = 0$ and (4.9) that there exist two roots z_i, z_k of $\hat{D}_{j\omega^*,\tau^*}(x)$, with $i, k \leq M$ and $i \neq k$ such that $\overline{z_i}(\omega^*, \tau^*) z_k(\omega^*, \tau^*) = 1$. It is implied by (4.9) that $F(\omega, \tau)$ can be decomposed as

$$F(\omega, \tau) = g_1(\omega, \tau)(1 - z_l\overline{z_k})(1 - z_k\overline{z_l}),$$

where g_1 is a differentiable function at (ω^*, τ^*) and the arguments of z_l, z_k are omitted for brevity. The last equality further implies that 4.11 holds at (ω^*, τ^*), which leads to a contradiction. Therefore there must exist at least one z_l on $\partial\mathbb{D}$. To see the uniqueness, suppose there exist two solutions z_l, z_k, $l \neq k$ both on $\partial\mathbb{D}$. Then using (4.9), we can locally express F

as

$$F(\omega,\tau) = g_2(\omega,\tau)(1 - z_l\overline{z_l})(1 - z_k\overline{z_k}),$$

where g_2 is differentiable at (ω^*,τ^*) and the arguments of z_l, z_k are again suppressed. This further implies that 4.11 holds at (ω^*,τ^*), which is again a contradiction. □

In light of the above proposition, for given i and k, we introduce function $z_k^{(i)}(\tau)$ as the unique solution of

$$\hat{D}(j\omega_k^{(i)}(\tau),\tau,x) = 0, \tag{4.14}$$

for x on $\partial\mathbb{D}$, for any $\tau \in (\tau^{(i-1)},\tau^{(i)}) - \Phi_\tau$. Since the roots of $\hat{D}_{j\omega,\tau}(x)$ is continuous with respect to ω and τ, we can extend the definition of $z_k^{(i)}(\tau)$ to the entire $\mathscr{I}^{(i)}$ by requiring the function $z_k^{(i)}(\tau)$ to be continuous on $\mathscr{I}^{(i)}$. Further define:

$$\theta_k^{(i)}(\tau) = \angle z_k^{(i)}(\tau) + \omega_k^{(i)}(\tau)\tau, \tag{4.15}$$

where $\angle z_k^{(i)}(\tau)$ is a continuous function in $\mathscr{I}^{(i)}$, which measures the phase angel of the complex number $z_k^{(i)}(\tau)$. Notice, the value of $\angle z_k^{(i)}(\tau)$ is not restricted to any 2π interval. The following proposition is obvious from this definition:

Proposition 4.2. *Suppose $\tau^* \in \mathscr{I}^{(i)}$, then $(j\omega^*,\tau^*)$ is a critical pair if and only if there exist some integers k such that $\omega^* = \omega_k^{(i)}(\tau^*)$ and*

$$\theta_k^{(i)}(\tau^*) = 2r\pi,\ r \text{ integer}. \tag{4.16}$$

Going through each interval $\mathscr{I}^{(i)}$ and each curve $\omega_k^{(i)}(\tau)$, we may identify all $\tau = \tau_l$, $l = 1,2,\cdots,L$ such that (4.16) holds for some integer k . For each given τ_l, it is possible that more than one k satisfies (4.16), and we denote the corresponding $\omega_k^{(i)}(\tau_l) \geq 0$ as ω_{lh}, $h = 1,2,\cdots,H_l$. In this way we can identify all the critical pairs $(j\omega_{lh},\tau_l)$, $h = 1,2,\cdots,H_l; l = 1,2,\cdots,L$.

Suppose $\tau^* \in \mathscr{I}^{(i)}$ is a critical delay, then it is easy to see that the derivative of $\theta_k^{(i)}(\tau)$, $k = 1,2,...,m(i)$ exits. Indeed we have

$$\begin{aligned} \frac{d\omega_k^{(i)}(\tau^*)}{d\tau} &= -\left.\frac{\partial_\tau F}{\partial_\omega F}\right|_{\substack{\omega=\omega^*\\ \tau=\tau^*}}, \\ \frac{dz_k^{(i)}(\tau^*)}{d\tau} &= -\left.\frac{\partial_\tau \hat{D}}{\partial_x \hat{D}}\right|_{\substack{\lambda=j\omega^*\\ \tau=\tau^*}}. \end{aligned} \tag{4.17}$$

Then by differentiating both sides of the relation:

$$\exp\left(j\theta_k^{(i)}(\tau) - j\omega_k^{(i)}(\tau)\tau\right) = z_k^{(i)}(\tau),$$

at $\tau = \tau^*$, we have

$$\frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = \frac{1}{j}\frac{dz_k^{(i)}(\tau^*)}{d\tau}\left(z_k^{(i)}(\tau^*)\right)^{-1} + \frac{d\left(\omega_k^{(i)}(\tau^*)\tau^*\right)}{d\tau}. \tag{4.18}$$

Higher order derivatives can be obtained by differentiating the last equation iteratively.

4.3.2 Counting unstable roots

We will first show that each imaginary root of $D_\tau(\lambda)$ corresponding to some critical delay is locally a differentiable function of τ. For this purpose, we introduce the following formula, which will also be useful for determining the crossing direction of the imaginary roots.

Proposition 4.3. *Let $(j\omega^*, \tau^*)$ be a critical pair. Let $x = z_k^*$, $k = 1, \ldots, M$ be all the roots of $\hat{D}_{j\omega^*,\tau^*}(x)$. Without loss of generality, let z_1^* be the unique root of $\hat{D}_{j\omega^*,\tau^*}(x)$ on $\partial\mathbb{D}$. Then the following holds:*

$$\partial_\omega F(\omega^*, \tau^*) = -2c|P_M(\lambda,\tau)|^{2M}\Re\left(\overline{j\partial_\lambda D(\lambda,\tau)} \cdot \sum_{i=1}^{M} iP_i(\lambda,\tau)e^{-i\lambda\tau}\right)_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}}, \tag{4.19}$$

where

$$c = \left|\sum_{i=1}^{M} iP_i(j\omega^*,\tau^*)e^{-ij\omega^*\tau^*}\right|^{-2} \times \prod_{\substack{i,k=1 \\ (i,k)\neq(1,1)}}^{M} (1 - z_i^*\overline{z_k^*}). \tag{4.20}$$

The proof is given in the appendix of this chapter. This proposition together with Assumption IV indicates that given a critical pair $(j\omega^*, \tau^*)$, we must have $\partial_\lambda D(j\omega^*, \tau^*) \neq 0$. Therefore by the implicit function theorem, (4.8) determines λ as a differentiable function of τ, for (λ, τ) in a neighborhood of $(j\omega^*, \tau^*)$. We shall write this function as $\lambda(\tau)$. To count the number of unstable roots for a given delay value, we introduce some quantities as follows. If $\tau^* \neq \tau^l$, define

$$\mathrm{Inc}(\omega^*, \tau^*) = \lim_{\varepsilon\to 0^+} \frac{\Re(\lambda(\tau^*+\varepsilon)) - \Re(\lambda(\tau^*-\varepsilon))}{2}. \tag{4.21}$$

If $\tau^* = \tau^l$, which implies $\tau^* = \tau_1$, define instead

$$\mathrm{Inc}(\omega^*, \tau^*) = \max\{0, \lim_{\varepsilon \to 0^+} \Re(\lambda(\tau^* + \varepsilon))\}. \tag{4.22}$$

The limits in the definitions above exist because Assumption IV guarantees that in some neighborhood of τ^*, $\Re(\lambda(\tau))$ is continuous and equals 0 if and only $\tau = \tau^*$. Now let the number of right half plane roots of $D_\tau(\lambda)$ be $N^u(\tau)$. For any $\tau \in \mathscr{I}$, $\tau \neq \tau_l$, $l = 1, 2, \ldots, L$, it is easy to see the following relation holds:

$$N^u(\tau) = N^u(\tau^l) + \sum_{l=1}^{L_\tau} \mathrm{Inc}(\tau_l), \tag{4.23}$$

where $L_\tau = \max\{l \mid \tau_l < \tau\}$ and

$$\mathrm{Inc}(\tau_l) = 2 \sum_{h=1}^{H_l} \mathrm{Inc}(\omega_{lh}, \tau_l) \tag{4.24}$$

It is clear that the quantity $\mathrm{Inc}(\omega^*, \tau^*)$ indicates the cross direction of the characteristic root at the critical pair $(j\omega^*, \tau^*)$, i.e., towards $\mathbb{C}_+$ or $\mathbb{C}_-$ these imaginary roots moves as τ sweeps through a critical delay τ^*.

In the case $\Re(\lambda'(\tau_l)) \neq 0$, $\mathrm{Inc}(\omega^*, \tau^*)$ satisfies

$$\mathrm{Inc}(\omega^*, \tau^*) = \begin{cases} \mathrm{sign}\big(\Re(\lambda'(\tau^*))\big) & \text{if } \tau^* \neq \tau^l \\ \max\{\mathrm{sign}\big(\Re(\lambda'(\tau^*))\big), 0\} & \text{if } \tau^* = \tau^l \end{cases} \tag{4.25}$$

We next provide a formula to compute the right hand side of (4.25).

4.3.3 Root Crossing Direction Analysis

Proposition 4.4. *Let $(j\omega^*, \tau^*)$ be a critical pair. Let i, k be such numbers that $\tau^* \in \mathscr{I}^{(i)}$ and $\omega_k^{(i)}(\tau^*) = \omega^*$. Then (4.1) defines λ as a differentiable function of τ in a sufficiently small neighborhood of $(j\omega^*, \tau^*)$. Let n_d be such a number that $(\frac{d}{d\tau})^l \theta_k^{(i)}(\tau^*) = 0$, for $l = 1, 2, \cdots, n_d - 1$, then the following holds:*

$$\begin{aligned} sgn\left(\Re\left((\tfrac{d}{d\tau})^l \lambda(\tau^*)\right)\right) &= (-1)^{N_x(j\omega^*, \tau^*)} sgn\left(\partial_\omega F(\omega^*, \tau^*)\right) \\ &\quad \times sgn\left((\tfrac{d}{d\tau})^l \theta_k^{(i)}(\tau^*)\right),\ l = 1, 2, \cdots, n_d. \end{aligned} \tag{4.26}$$

where $N_x(j\omega^*, \tau^*)$ is the number of roots of $\hat{D}_{j\omega^*, \tau^*}(x)$ that are outside the unit disk $\mathbb{D}$.

Proof. Let $\omega(\tau)$ be the differentiable function implicitly defined by (4.7) in some neighborhood of τ^*, which satisfies $\omega(\tau^*) = \omega^*$. For notational convenience, denote

$$a(\tau) = \theta_k^{(i)}(\tau). \tag{4.27}$$

With Assumption III, Proposition 4.3 implies that $\partial_\lambda D(j\omega^*, \tau^*) \neq 0$. Then by the implicit function theorem, $D(\lambda, \tau) = 0$ determines λ as a differentiable function of τ in a neighborhood of τ^*. Let $\lambda(\tau) = x(\tau) + jy(\tau)$, then x, y are real differentiable functions in a neighborhood of $\tau = \tau^*$, $\lambda = \lambda(\tau^*)$. We define a function $h(x, y, \tau)$ as

$$h(x, y, \tau) = D(x + jy, \tau), \tag{4.28}$$

then clearly in a neighborhood of $\tau = \tau^*$ we have

$$h(x(\tau), y(\tau), \tau) = 0. \tag{4.29}$$

Define $g(a, \omega, \tau) = \hat{D}(j\omega, \tau, \exp(ja - j\omega\tau))$ then it is easy to see

$$g(a(\tau), \omega(\tau), \tau) = 0, \tag{4.30}$$

for τ in some neighborhood of τ^*. Noticing the definition (4.27) and $x(\tau^*) = 0$, the following is obvious:

$$g(a(\tau^*), \cdot, \cdot) = h(x(\tau^*), \cdot, \cdot) = h(0, \cdot, \cdot). \tag{4.31}$$

Now it follows from Proposition 3.6 in Chapter 3 that for $k \leq n_d$

$$(\tfrac{d}{d\tau})^k x(\tau^*) = \frac{\Re(j\partial_y h(p_1)\overline{\partial_a g(p_2)})}{\Re(j\partial_y h(p_1)\overline{\partial_x h(p_1)})} (\tfrac{d}{d\tau})^k a(\tau^*). \tag{4.32}$$

where $P_1 = (0, \omega^*, \tau^*)$, $P_2 = (a(\tau^*), \omega^*, \tau^*)$.

Since at $\tau = \tau^*$, $\lambda = j\omega^*$ it holds that $\partial_a g(p_2) = j\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*}$, $\partial_y h(p_1) = j\partial_\lambda D$, $\partial_x h = \partial_\lambda D$, we obtain from (4.32)

$$\begin{aligned} \frac{d\Re(\lambda(\tau^*))}{d\tau} &= \frac{\Re(j\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*} \cdot \overline{\partial_\lambda D})}{|\partial_\lambda D|^2} a' \Big|_{\substack{\tau=\tau^* \\ \lambda=j\omega^*}} \\ &= \frac{-\Re(\sum_{i=1}^M iP_i e^{-ij\omega^*\tau^*} \cdot \overline{j\partial_\lambda D})}{|\partial_\lambda D|^2} a' \Big|_{\substack{\tau=\tau^* \\ \lambda=j\omega^*}}. \end{aligned} \tag{4.33}$$

Proposition 4.3 gives

$$\partial_\omega F(\omega^*,\tau^*) = -2c|P_M|^{2M}\Re(\sum_{i=1}^{M} iP_i e^{-ij\omega^*\tau^*}\cdot \overline{j\partial_\lambda D})_{\substack{\tau=\tau^*\\ \lambda=j\omega^*}},$$

where functions are evaluated at $\tau=\tau^*$, $\lambda=j\omega^*$, and it is easy to see

$$\operatorname{sgn}(c) = (-1)^{N_x(\omega^*,\tau^*)}.$$

By definition we also have $(\frac{d}{d\tau})^k a(\tau^*) = (\frac{d}{d\tau})^k \theta_k^{(i)}(\tau^*)$. Substitute these expressions into (4.33) and notice that $x(\tau)$ is differentiable at τ^*, (4.26) is proved. □

The following can be easily shown using a continuity argument:

Proposition 4.5. *For fixed index i and k, the quantity $\partial_\omega F(j\omega_k^{(i)}(\tau),\tau)$ does not change sign for $\tau\in(\tau^{(i-1)},\tau^{(i)})$.*

In view of the last proposition, we introduce the following notation:

$$\operatorname{sgn}_k^{(i)} = \partial_\omega F(\omega_k^{(i)}(\tau),\tau), \forall\tau\in(\tau^{(i-1)},\tau^i). \tag{4.34}$$

Now Equation (4.26) can be rewritten as

$$\operatorname{sgn}\left(\Re\left((\tfrac{d}{d\tau})^l\lambda(\tau^*)\right)\right) = (-1)^{N_x(j\omega^*,\tau^*)}\cdot\operatorname{sgn}_k^{(i)}\cdot\operatorname{sgn}\left((\tfrac{d}{d\tau})^l\theta_k^{(i)}(\tau^*)\right), \quad l=1,2,\cdots,n_d, \tag{4.35}$$

The last theorem establishes an interesting link between system stability and the phase angle curves. In any given interval $\mathscr{I}^{(i)}$, following the graph of each $\theta_k^{(i)}(\tau)$, one can identify τ^* as a critical delay if the graph of $\theta_k^{(i)}(\tau)$ intersects any horizontal line located at $2r\pi$ for some integer r and conclude that $\pm j\omega_k^{(i)}(\tau^*)$ is a pair of imaginary roots corresponding to τ^*. Whether this pair of roots become stable or unstable as τ increases depends partially on how the graph of $\theta_k^{(i)}(\tau)$ crosses $2r\pi$, namely from below to above or vise versa. It also depends on the quantity $\operatorname{sgn}_k^{(i)}$ as well as $N_x(j\omega_k^{(i)}(\tau^*),\tau^*)$.

Regarding the last two factors, we have the following observation. First, the sign of the quantity $\partial_\omega F(\omega_k^{(i)}(\tau),\tau)$ is invariant in $(\tau^{(i-1)},\tau^{(i)})$. Second, for any interval $U\in\mathscr{I}^{(i)}$ such that $U\cap\Phi_\tau=\phi$ (recall the set Φ_τ is defined in Assumption V), the quantity $N_x(\omega_k^{(i)}(\tau),\tau)$ is also invariant over U. To see this is indeed true, first notice that for the roots of $\hat{D}_{j\omega,\tau}(x)$ to enter or leave the unite disk, it must first lie on $\partial\mathbb{D}$ since the roots of $\hat{D}_{j\omega,\tau}(x)$ is continuous with respect to the parameters. However, Proposition 4.1 indicates that the root of $\hat{D}_{j\omega,\tau}(x)$

on $\partial\mathbb{D}$ exists uniquely and is simple given $\omega = \omega_k^{(i)}(\tau)$ and $\tau \in U$. Therefore we conclude that $N_x(\omega_k^{(i)}(\tau), \tau)$ must be constant on U. In the case when $\mathscr{I}^{(i)}$ is disjoint with Φ_τ, the quantity $N_x(\omega_k^{(i)}(\tau), \tau)$ is constant on $\mathscr{I}^{(i)}$. The following is obviously true:

Proposition 4.6. *If Φ_τ is empty, then $N_x(j\omega_k^{(i)}(\tau), \tau)$ is constant for τ in the interior of $\mathscr{I}^{(i)}$.*

The first order derivative $\frac{d\theta_k^{(i)}(\tau^*)}{d\tau}$ can be determined simply based on the graphs of the phase functions. Here we provide a formula to compute it:

$$\frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = -\left.\frac{\sum_{l=0}^{M} \frac{d_F P_l}{d\tau} e^{lj\omega^*\tau^*}}{\sum_{l=1}^{M} lP_l e^{lj\omega^*\tau^*}}\right|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}}, \tag{4.36}$$

where

$$\left.\frac{d_F P_l}{d\tau}\right|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}} = j\partial_\lambda P_l \frac{d\omega_k^{(i)}}{d\tau} + \left.\partial_\tau P_l\right|_{\substack{\lambda=j\omega^* \\ \tau=\tau^*}}. \tag{4.37}$$

This formula can be readily derived by differentiating the left hand side of (4.14) and using (4.18). Higher order derivatives can be obtained by differentiating the last equation iteratively. In practice however, one can determine the derivatives of the phase angle function simply based on its graph in the same spirit of [14].

4.3.4 Numerical example for the commensurate-delay case

4.3.5 Example 1

We consider the characteristic equation (4.3) with the delay interval

$$\mathscr{I} = [\tau^l, \tau^u] = [0, 0.8],$$

and $\alpha = 1.5$. Here the upper bound of the interval does not have any particular meaning and can be replaced by any other positive numbers. Of cause, one needs to check that the assumptions hold for the delay interval of interest, which is true for the one chosen here. By definition, we have

$$P_0 = \lambda - \alpha,\ P_1 = e^{\alpha\tau},\ P_2 = e^{2\alpha\tau}.$$

The expression of function F can be obtained using (4.5) and (4.7) as

$$F = -\omega^4 + a_1\omega^2 + a_2,$$

where

$$
\begin{aligned}
a &= e^{\tau\lambda},\ a_1 = 2a^4 + a^2 - \frac{9}{2}, \\
a_2 &= -a^8 + a^6 + \frac{15}{2}a^4 + \frac{9}{4}a^2 - \frac{81}{16}.
\end{aligned}
$$

By solving (4.7) and (4.11) together for $(\omega,\tau) \in \mathbb{R}_+ \in \mathscr{I}$, we find that $\mathscr{I}$ can be decomposed into $\mathscr{I}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$, $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$ and $\tau^{(0)} = 0$, $\tau^{(1)} \approx 0.4045$, $\tau^{(2)} = 0.8$. Polynomial $F_\tau(\omega)$ has one positive solution in $\mathscr{I}_o^{(1)}$ and two positive solutions in $\mathscr{I}_o^{(2)}$. Consequently functions $\omega_1^{(1)}(\tau)$, $\omega_2^{(1)}(\tau)$, $\omega_2^{(2)}(\tau)$ are well defined in the corresponding intervals as plotted Fig.4.1a. The associated phase curves are plotted in Fig.4.1b. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at $\tau_1 = 0.2368$ and the graphs of $\theta_1^{(2)}(\tau)$, $\theta_2^{(2)}(\tau)$ cross the horizontal line 2π at $\tau_2 = 0.6878$, $\tau_3 = 0.6976$ respectively. By definition, we have $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 2.9010$, $\omega_{12} = \omega_2^{(2)}(\tau_2) \approx 5.4195$, $\omega_{22} = \omega_1^{(2)}(\tau_3) \approx 5.6381$.

We verify that the following holds:

$$
\begin{aligned}
&\partial_\omega F(\omega_1^{(1)}(\tau),\tau) < 0,\ N_x(\omega_1^{(1)}(\tau),\tau) = 1,\ \forall\tau \in \mathscr{I}_o^{(1)}, \\
&\partial_\omega F(\omega_1^{(2)}(\tau),\tau) < 0,\ N_x(\omega_1^{(2)}(\tau),\tau) = 1,\ \forall\tau \in \mathscr{I}_o^{(2)}, \\
&\partial_\omega F(\omega_2^{(2)}(\tau),\tau) > 0,\ N_x(\omega_2^{(2)}(\tau),\tau) = 0,\ \forall\tau \in \mathscr{I}_o^{(2)}.
\end{aligned}
$$

Therefore, we conclude from (4.26) that $\mathrm{Inc}(\omega^*,\tau^*) = 1$ at all the three crossing points. In other words, the the imaginary roots $\pm j\omega_{11}$, $\pm j\omega_{12}$, $\pm j\omega_{22}$ all move toward the right half plane as the delay value increases and sweeps through τ_1, τ_2, τ_3, respectively. It can be verified that $N^u(0) = 0$, then the number of unstable roots can be easily computed using (4.23) and is plotted against τ in Fig.4.1c. It is clear that System (4.2) is $\alpha-$stable with $\alpha = 1.5$ for $\tau \in [0,\tau_1)$ and not $\alpha-$stable for $\tau \in [\tau_1, 0.8]$.

4.3.6 Example 2: Vehicle lateral dynamics

Stability of the lateral dynamics is one of the critical issues in the design and control of automobile vehicles. The most widely used vehicle lateral model is a second order linear system, which does not take into account any delay effect [101]. It is known that as the vehicle speed increases, the pair of eigenvalues of such a model will move toward the imaginary axis. Understeering vehicles remain stable regardless of the velocity. On the other hand, there exists a critical speed for an oversteering vehicle, above which the vehicle dynamics becomes unstable. Therefore it is important to know the critical speed of a vehicle for the purpose of vehicle design.

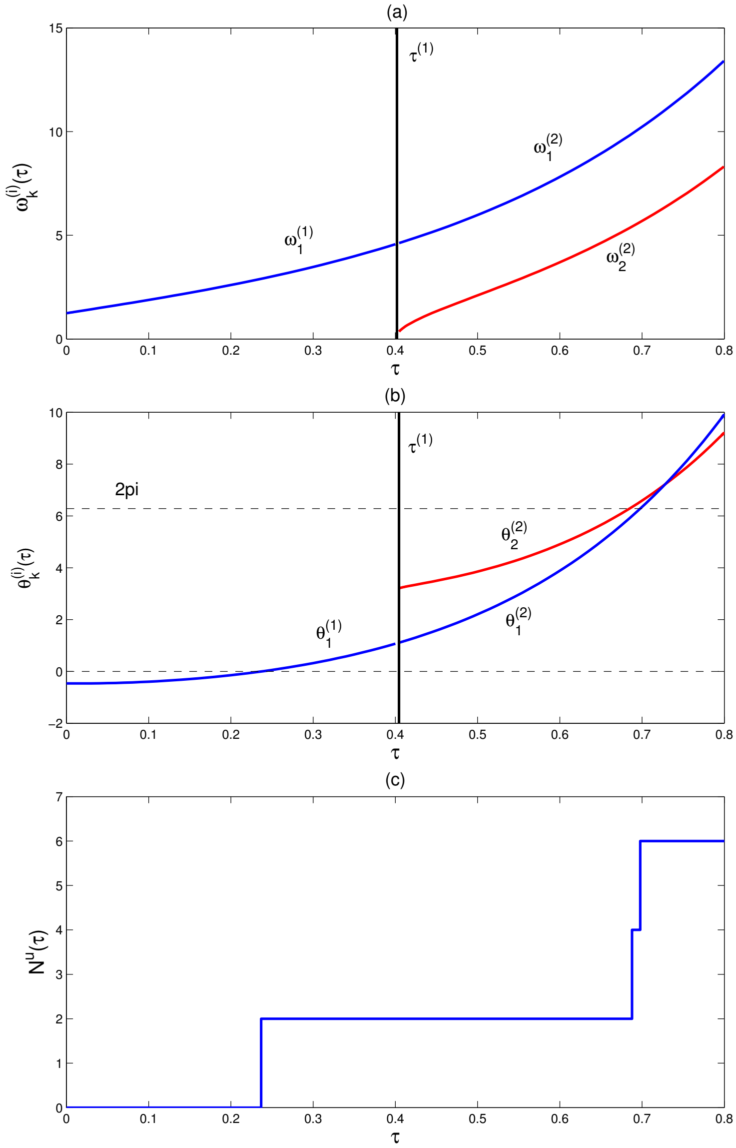

FIGURE 4.1: Stability analysis of the system 4.3. (a)The graphs of frequency functions. (b)The graphs of phase angle functions. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at τ_1. The graph of $\theta_2^{(2)}(\tau)$ and $\theta_1^{(2)}(\tau)$ crosses the 2π horizontal line at τ_2 and τ_3 respectively. Therefore three critical delays exist within $\mathscr{I}$. (c)The number of unstable roots.

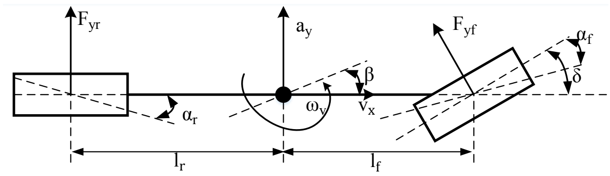

FIGURE 4.2: A single track model

From road test, the time delay in the generation of vehicle tire forces becomes significant, which in some situations can lead to instability. Since the commonly used vehicle model does not capture such a delay effect, they may not accurately predict vehicle stability at high speeds. Some researchers attempted to approximate the delay as first order dynamics for the ease of analysis. With the theoretical results developed in this chapter, we are able to take into account the time-delay effect of tire forces and describe the vehicle lateral dynamics as a functional differential equation.

The single track model for vehicle lateral dynamics can be found in, e.g.,[101] and is illustrated in Fig.4.2. Let ω, β be the yaw rate and lateral slip angle of vehicle, respectively. Let v be the longitudinal vehicle speed, which is viewed as a fixed parameter in the differential equation. The angle from the wheel plane to the wheel center velocity is defined as the wheel slip angle. In the linear model, assuming the front wheel steering angle is zero, the front wheel slip angle α_f can be computed as

$$\alpha_f = \beta + \frac{\omega l_f}{v},$$

and the rear wheel slip angle α_r can be computed as

$$\alpha_r = \beta - \frac{\omega l_r}{v},$$

where l_f, l_r are the distance between the front wheel axis to vehicle center of mass and the rear wheel axis to vehicle center of mass, respectively. Let the static lateral tire force at the front and rear axles be denoted as F_{yf}, F_{yr}, respectively. We have the following expression:

$$F_{yl} = -\alpha_l C_l, \; l = f, r,$$

where C_l is known as the tire lateral stiffness. In the dynamic model, there is a delay in tire force generation, therefore we use the following equation instead:

$$F_{yl}(t) = -\alpha_l(t - \tau(v))C_l, \; l = f, r,$$

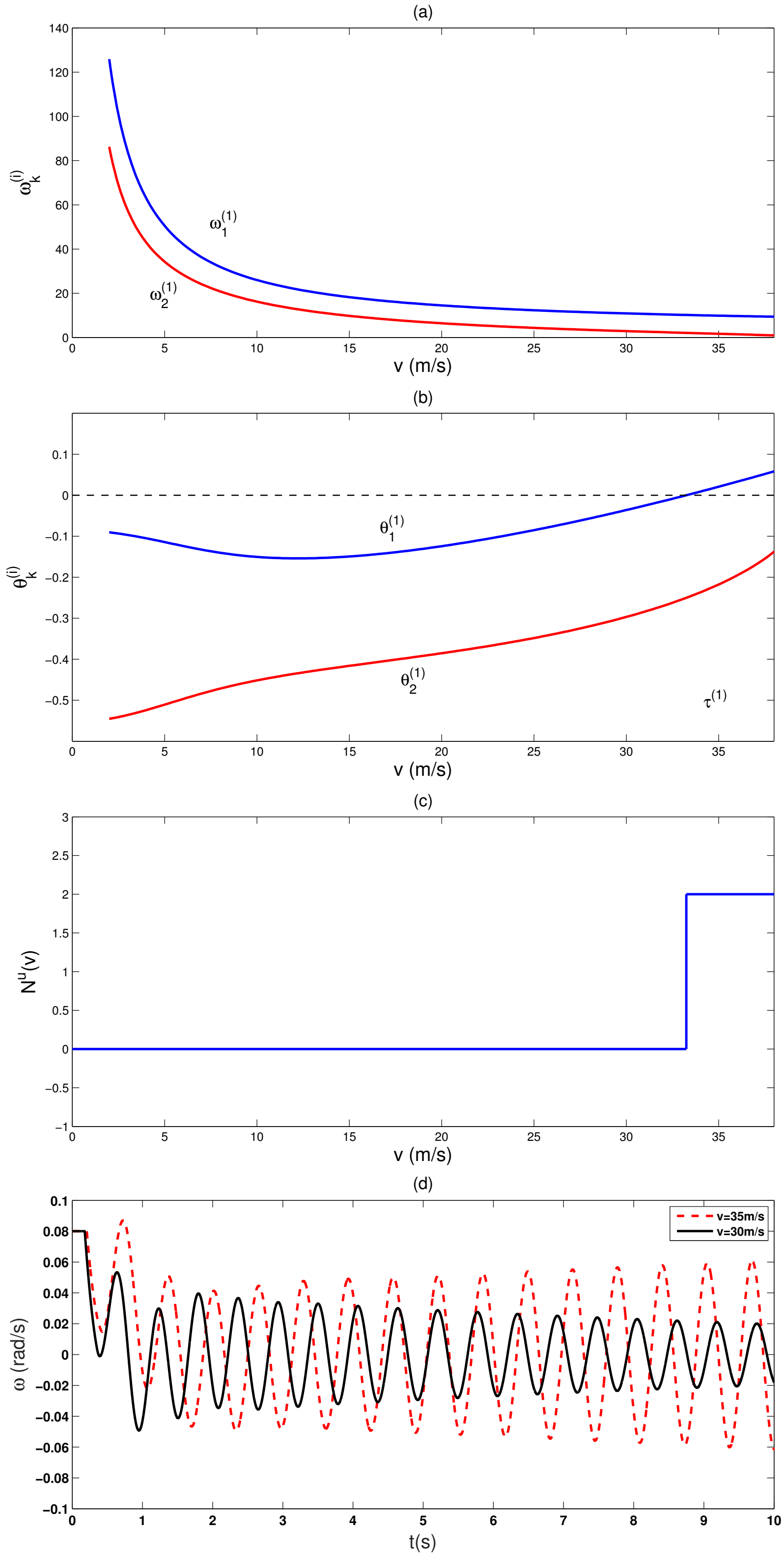

FIGURE 4.3: Stability analysis of an automobile vehicle. (a)The graphs of frequency functions. (b)The graphs of the phase angle functions. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at v_1, which means v_1 is the critical vehicle speed. (c)The number of unstable roots. (d)The time response of the yaw rate for different vehicle speeds.

and the delay is a function of the longitudinal velocity v: $\tau(v) = k_d v$. Clearly, the higher the speed, the more significant the tire force delay is. Then using the Newton-Euler formula, the FDE that governs the lateral vehicle dynamics can be derived as

$$\begin{aligned}
\dot{\omega}(t) &= \frac{l_f^2 C_f + l_r^2 Cr}{I_z v}\omega(t-k_d v) + \frac{l_f C_f - l_r C_r}{I_z}\beta(t-k_d v), \\
\dot{\beta}(t) &= -\omega(t) + \frac{l_f C_f - l_r C_r}{m v^2}\omega(t-k_d v) + \frac{C_f + C_r}{m v}\beta(t-k_d v).
\end{aligned}$$

The value of the model parameters are listed as follows: $m = 1376$, $l_f = 1.4$, $l_r = 1.2$, $J_z = 1900$, $C_f = -140000$, $C_r = -120000$, $k_d = 0.0059$.

The longitudinal velocity interval of interest is set as

$$\mathscr{I}_v = [v^l, v^u] = [2m/s, 38m/s].$$

Notice the differential equation becomes singular as $v \to 0^+$, therefore we set the lower bound of $\mathscr{I}_v$ to be slightly larger than 0. From engineering practice we know that the vehicle dynamics is asymptotically stable for $v \leq 2m/s$, although in theory finding the lower bound τ^l such that the vehicle dynamics remains stable for $v \in (0, \tau^l)$ is itself an issue to be solved. This Problem will be addressed in Chapter 6.

The characteristic equation can now be computed as

$$\lambda^2 + \left(\frac{424.31\lambda}{v} - 27.36\right)e^{-\lambda\tau(v)} + \frac{43437}{v^2}e^{-2\lambda\tau(v)} = 0, \tag{4.38}$$

where $\tau = k_d v$. The function $F(\omega, v)$ is derived according to (4.6), which is an 8th order polynomial. Its detailed expression is omitted here. By solving (4.7) and (4.11) together for $(\omega, v) \in \mathbb{R}_+ \in \mathscr{I}$, we find that there exists no such solutions. Therefore $\mathscr{I}_v$ is just equal to $\mathscr{I}_v^{(1)} = [v^{(0)}, v^{(1)}]$, and $v^{(0)} = v^l = 2m/s$, $v^{(1)} = v^u = 38m/s$. Polynomial $F_v(\omega)$ has two real solutions in $\mathscr{I}^{(1)}$, namely $\omega_1^{(1)}(v)$ and $\omega_2^{(1)}(v)$, the graphs of which are plotted in Fig.4.3.a. The associated phase curves are plotted in Fig.4.3.b. The graph of $\theta_1^{(1)}(v)$ crosses the horizontal line 0 at $v_1 = 33.2498m/s$ and it is clear that

$$\tfrac{d}{dv}\theta_1^{(1)}(v_1) > 0.$$

By definition, we have $\omega_{11} = \omega_1^{(1)}(v_1) \approx 10.218$. We verify that the following holds:

$$\partial_\omega F(\omega_1^{(1)}(v), v) < 0,\ N_x(\omega_1^{(1)}(v), v) = 1,\ \forall \tau \in \mathscr{I}_o^{(1)}, \tag{4.39}$$

Therefore, we conclude from (4.26) that $\mathrm{Inc}(j\omega_1^{(1)}(\tau_1), \tau_1) = 1$, which means that the pair of characteristic roots $\lambda = \pm j\omega_1^{(1)}(\tau_1)$ move toward the right half plane as the delay value increases and sweeps through τ_1. Notice that $N^u(\tau^l) = 0$. The number of unstable roots can be computed using (4.23) and is plotted against τ in Fig.4.3.c. It is clear that the characteristic equation (4.38) is asymptotically stable for $\tau \in [0, v_1)$ and unstable for $\tau \in (v_1, v^u]$. It is easy to verify that if the time delay is neglected in the model, then the vehicle dynamics remains stable for v up to $40m/s$. Therefore, after the delay effect is taken into account, the predicted safe speed is reduced.

4.4 Chapter Summary

A method of stability analysis for systems with commensurate delays and coefficients depending on the delay is presented following the generalized τ-decomposition approach. The method partitions the delay interval of interest into disjoint subintervals so that a generalized magnitude condition yields a fixed number of solutions of frequencies ω as functions of the delay τ within each subinterval. We provided conditions for imaginary roots to appear at some critical delay values, followed by a criterion to identify crossing frequencies. Our analysis shows that the results developed in the last chapter for systems with a single delay can be largely extended to systems with commensurate delays. Just as the single-delay case, the root crossing direction criterion reflects the 'separation principle': the crossing direction of characteristic roots on the imaginary axis depends on two factors, one is 'classical' in the sense that it exists for systems with fixed coefficients and the other is new, which reflects the monotonicity of the phase angle functions at the critical pairs. In the next chapter, this interesting observation will be explained from a geometric point of view.

4.5 APPENDIX

4.5.1 Proof of Proposition 4.3

Noticing $\frac{d}{dx}\hat{D}_{j\omega^*,\tau^*}(x)|_{x=z_k} \neq 0$, $i = 1, \ldots, M$, by the implicit function theorem the roots of $\hat{D}_{j\omega,\tau}(x)$ are differentiable functions in (ω, τ) defined in a neighborhood of (ω^*, τ^*). Denoting these functions as $z_k = z_k(\omega, \tau)$ for each k, we have $z_k(\omega^*, \tau^*) = z_k^*$. We will first show

$$\partial_\omega F(\omega^*, \tau^*) = 2ac|P_M|^{2M}\left|\sum_{i=1}^{M} iP_i e^{-ij\omega^*\tau^*}\right|^2, \tag{4.40}$$

where $a = \Re\left(\partial_\omega z_1 \cdot \overline{z_1}\right)_{\substack{\omega=\omega^* \\ \tau=\tau^*}}$. This can be derived through the following computation:

$$\begin{aligned}
& \partial_\omega F(\omega^*, \tau^*) \\
= & -\partial_\omega \left(|P_M|^{2M}\right) \prod_{i,k=1}^{M} (1 - z_i \overline{z_k})_{\substack{\omega=\omega^* \\ \tau=\tau^*}} \\
& -|P_M|^{2M} \partial_\omega \left(\prod_{i,k=1}^{M} (1 - z_i \overline{z_k}) \right)_{\substack{\omega=\omega^* \\ \tau=\tau^*}} \\
= & -|P_M|^{2M} \partial_\omega (1 - z_1 \overline{z_1}) \prod_{\substack{i,k=1 \\ (i,k)\neq(1,1)}}^{M} (1 - z_i \overline{z_k})_{\substack{\omega=\omega^* \\ \tau=\tau^*}}.
\end{aligned} \tag{4.41}$$

On the other hand, it is easy to verify

$$\partial_\omega (z_1 \overline{z_1}) = 2\Re\left(\partial_\omega z_1 \overline{z_1}\right).$$

A substitution of the last equality into (4.41) leads to (4.40).

In a neighborhood of (ω^*, τ^*) we have

$$\sum_{i=0}^{M} P_i(j\omega, \tau) z_1^i(\omega, \tau) = 0. \tag{4.42}$$

Denote $b = \Re\left(\partial_\omega z_1 \cdot \overline{j z_1}\right)_{\substack{\omega=\omega^* \\ \tau=\tau^*}}$, it is easy to verify

$$\partial_\omega z_1(\omega^*, \tau^*) = a z_1(\omega^*, \tau^*) + j b z_1(\omega^*, \tau^*). \tag{4.43}$$

Differentiating both sides of (4.42) w.r.t ω, and noticing $z_1(\omega^*, \tau^*) = e^{-j\omega^*\tau^*}$, we have

$$\begin{aligned}
0 &= j \sum_{i=0}^{M} \partial_\lambda P_i z_1^i + \sum_{i=0}^{M} i P_i z_1^i \cdot (a + jb) \\
&= j \sum_{i=0}^{M} \left(\partial_\lambda P_i z_1^i - i\tau^* P_i z_1^i\right) + \sum_{i=0}^{M} i P_i z_1^i \cdot (a + jb + j\tau^*) \\
&= j \partial_\lambda D + \sum_{i=0}^{M} i P_i z_1^i \cdot (a + jb + j\tau^*),
\end{aligned}$$

where functions are evaluated at at $\omega = \omega^*$, $\lambda = j\omega^*$ and $\tau = \tau^*$. It is easy to deduce from the previous equality

$$a = -\left.\frac{\Re\left(\overline{j\partial_\lambda D} \cdot \sum_{i=1}^{M} i P_i z_1^i\right)}{|\sum_{i=1}^{M} i P_i z_1^i|^2}\right|_{\substack{\lambda=j\omega^* \\ \omega=\omega^*, \tau=\tau^*}}. \tag{4.44}$$

The right hand side of (4.44) is well-defined because (4.40) implies that the term $\sum_{i=0}^{M} iP_i z_1^i$ in the last equation is not zero. Equality (4.19) is proved after (4.44) is plugged into (4.40).

4.5.2 Geometric Meaning of the $F(\omega,\tau) Function$

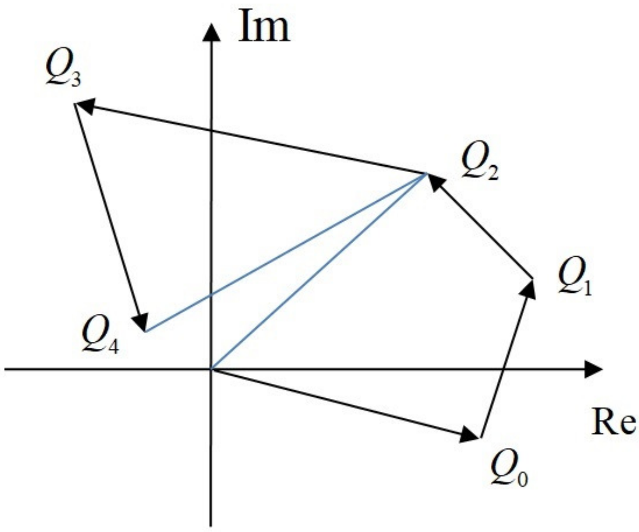

FIGURE 4.4: Geometry of Schur's Hermitian form

For a clear presentation, we omit τ in the system coefficients.

Denote

$$Q_l = \sum_{k=0}^{l} P_k(\lambda) z^k, \; l = 0, \cdots, M, \tag{4.45}$$

and $\overrightarrow{Q_k Q_{k+1}} = Q_{k+1} - Q_k$, $k = 0, \cdots, M$. Recall the Hermitian form $\mathscr{H}(\lambda, x)$ defined as

$$\begin{aligned}\mathscr{H}(\lambda, X) = & \sum_{k=1}^{M} |P_0 x_k + P_1 x_{k+1} + \ldots + P_{M-k} x_M|^2 \\ & - \sum_{k=1}^{M} |\overline{P_M} x_k + \overline{P_{M-1}} x_{k+1} + \ldots + \overline{P_k} x_M|^2,\end{aligned}$$

where $x = (z, z^2, \cdots, z^M)$ and $z \in \partial\mathbb{D}$. It admits a very interesting geometric interpretation as follows. Noticing

$$\begin{aligned} & |P_0 x_k + P_1 x_{k+1} + \ldots + P_{M-k} x_M| \\ = \; & |z^k||P_0 + P_1 z + \ldots + P_{M-k} z_{M-k}| \\ = \; & |\overrightarrow{Q_0 Q_{M-k}}| \end{aligned}$$

and

$$\begin{aligned} & |\overline{P_M} x_k + \overline{P_{M-1}} x_{k+1} + \ldots + \overline{P_k} x_M| \\ = \; & |z^k||\overline{P_M} + \overline{P_{M-1}} z^1 + \ldots + \overline{P_k} z^{M-k}| \\ = \; & |\overrightarrow{Q_M Q_k}| \end{aligned}$$

Then it follows that

$$\mathscr{H}(\lambda, X) = \sum_{k=1}^{M} |\overrightarrow{Q_0 Q_k}|^2 - |\overrightarrow{Q_M Q_k}|^2. \tag{4.46}$$

If $z_0 \in \partial\mathbb{D}$ is such that

$$\sum_{k=0}^{M} P_k(\lambda) z_0^k = 0,$$

it is obvious that $Q_0 = Q_M$ and hence $|Q_0 Q_k| = |Q_M Q_k|$ for each k, therefore (see Fig.4.4 for illustration)

$$\mathscr{H}(\lambda, x_0) = x_0^T \mathrm{H}(\lambda) x_0 = 0,$$

which gives $\det(H(\lambda)) = 0$. For in the single delay case, in Chapter 3 we have defined $F(\omega, \tau) = |P(j\omega, \tau)|^2 - |Q(j\omega, \tau)|^2$, which is the same as (4.46) when $M = 1$.

Chapter 5

A Two-Parameter Approach for Stability Analysis

5.1 Chapter Overview

We review the stability analysis of systems with delay-dependent coefficients from a two-parameter point of view. The parameter in system coefficients and that in the state are regarded as two different variables, denoted as r and q respectively. The fact that system coefficients depend on the delay simply means $r = q$. To understand the stability of the original system, it suffices to analyze the stability of the two parameter system in the r-q parameter space, then impose the restriction $r = q = \tau$. The basic idea is illustrated with systems with a single delay. The extension to commensurate-delay case is discussed subsequently. Root crossing direction criteria are first derived in the most general form by exploiting the geometric idea underlying the two-parameter perspective. These criteria allow us to take advantage of some powerful stability analysis methods developed originally for systems with fixed coefficients. For instance, the Puiseux series can be readily applied for developing a complete analysis method. For simple characteristic roots on the imaginary axis, we simplify these criteria and recover the results developed in previous chapters under less restrictive assumptions. As we develop our theory, the population dynamics with a stage structure is taken as an example to illustrate the main idea. The materials of this chapter have been partially published in [34].

5.2 Systems with a Single Delay

We review single-delay systems with the following characteristic equation:

$$D(\lambda,\tau) = P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{5.1}$$

where $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are polynomials in λ and continuous in τ. Once the insight of the two-parameter approach becomes clear for systems with a single delay, we will discuss its extension to systems with commensurate delays. Let the function $F(\omega,\tau)$ be defined as in (3.4). We now state a set of standing assumptions for (5.1) that hold throughout this chapter.

Assumption I. For all $\tau \in \mathscr{I}$, there exits $n \geq 0$ such that the order of polynomial P_τ equals n. The following also holds for $\tau \in \mathscr{I}$:

$$\lim_{\omega\to\infty} \left| \frac{Q_\tau(j\omega)}{P_\tau(j\omega)} \right| < 1. \tag{5.2}$$

Assumption II. No $(\omega,\tau) \in \mathbb{R}_+ \times \mathscr{I}$ satisfies

$$\begin{cases} P(j\omega,\tau) & = & 0, \\ Q(j\omega,\tau) & = & 0, \end{cases}$$

simultaneously.

Assumption III. There are only a finite number of pairs (ω,τ) in $\mathbb{R}_+ \times \mathscr{I}$ that simultaneously satisfy

$$F(\omega,\tau) = 0, \tag{5.3}$$

as well as

$$\partial_\omega F(\omega,\tau) = 0. \tag{5.4}$$

Define $\mathscr{T}_F$ as the set of the $\tau \in \mathscr{I}$ that appear in such pairs.

Assumption IV. For all $\tau \in \mathscr{I}$, $\lambda = 0$ is not a characteristic root of $D_\tau(\lambda)$. We define $\mathscr{T}_c$ as the set of critical delays. Equivalently, $\mathscr{T}_c$ is the set of all $\tau \in \mathscr{I}$ for which $D_\tau(\lambda)$ admits imaginary roots. Further assume that any bounded delay interval contains at most a finite number of critical delays.

Assumption I-IV should hold for general systems with characteristic equations of the form (5.1) except for some degenerated cases.

In Chapter 3, we required $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ to be differentiable at all critical delays. It is also assumed there that if $j\omega^*$ is an imaginary root of $D_{\tau^*}(\lambda)$ for some critical delay τ^*, then $F'_{\tau^*}(\omega^*) \neq 0$. In this chapter we will develop more general results that do not require these assumptions.

Our objective is to determine the delay intervals contained in $\mathscr{I}$ for which system (5.1) is asymptotically stable. For this purpose, We will investigate how imaginary roots migrate as τ increases and goes through the critical delay from a two-parameter perspective .

5.3 Stability Analysis

This section introduces some notation used in this chapter. Most of these terms and functions have already been defined in previous chapters, which will be reviewed here briefly to make this chapter self-contained.

We arrange the elements of $\mathscr{T}_c$, namely all the critical delays $\tau_1, \tau_2, \cdots, \tau_L$, in an ascending order as

$$\tau^l \leq \tau_1 < \tau_2 < \cdots < \tau_L \leq \tau^u.$$

As discussed in Chapter 3, Assumption I guarantees that there exists $c > 0$ such that any characteristic root of $D_\tau(\lambda)$ with real part greater than $-c$ varies continuously with τ. Consequently, as τ goes through $\mathscr{I}$, the number of characteristic roots on the right half complex plane remains constant in any interval that contains no critical delay.

Recall the set $\mathscr{T}_F$ defined in Assumption III. We agree to order the elements of $\mathscr{T}_F \cup \{\tau^l, \tau^u\}$ in an ascending order

$$\tau^{(0)} < \tau^{(1)} < \cdots < \tau^{(K)}. \tag{5.5}$$

We note that the two sets $\mathscr{T}_F$ and $\{\tau^l, \tau^u\}$ may have common elements given the relaxed assumptions in this chapter. Then, we may partition $\mathscr{I}$ into K subintervals:

$$\mathscr{I}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}],\ i = 1, 2, \ldots, K. \tag{5.6}$$

As discussed in Chapter 3, the number of real roots of $F(\omega, \tau)$ in ω are constant for all τ in the interior of $\mathscr{I}^{(i)}$, and they are all simple. These roots are continuous functions of τ in $\mathscr{I}^{(i)}$ and denoted as $\omega_k^{(i)}(\tau)$, $k = 1, 2, \cdots, m(i)$, and $m(i)$ is the number of real roots of $F_\tau(\omega)$ for $\tau \in \mathscr{I}^{(i)}$. The graph of $\omega_k^{(i)}(\tau)$ are referred to as the *frequency curves*.

We further introduce a set $\Omega_F(\tau)$ that collects all different values of the real roots of $F_\tau(\omega)$. More precisely, we define:

$$\Omega_F(\tau) = \bigcup_{k=1,2,\cdots m(i)} \{\omega_k^{(i)}(\tau)\} \ \text{ if } \tau \in \mathscr{I}^{(i)}. \tag{5.7}$$

Since $\partial_\omega F(\omega_k^{(i)}(\tau), \tau) \neq 0$ for any τ in the interior of $\mathscr{I}^{(i)}$, we denote:

$$\text{sgn}_k^{(i)} = \text{sgn}\left(\partial_\omega F(\omega_k^{(i)}(r), r)\right), \forall r \in (\tau^{(i-1)}, \tau^{(i)}). \tag{5.8}$$

For a given index pair (i,k), the phase function is defined as:

$$\begin{aligned}\theta_k^{(i)}(\tau) &= \angle P(j\omega_k^{(i)}(\tau),\tau) - \angle Q(j\omega_k^{(i)}(\tau),\tau) \\ &\quad + \omega_k^{(i)}(\tau)\tau + \pi.\end{aligned} \tag{5.9}$$

Here the quantity $\angle P(j\omega_k^{(i)}(\tau),\tau)$ measures the phase angle of $P(j\omega_k^{(i)}(\tau),\tau)$ and is continuous on $\mathscr{I}^{(i)}$. Similar properties also apply to the quantity $\angle Q(j\omega_k^{(i)}(\tau),\tau)$. These two quantities are well defined in view of Assumption II. Consequently, each function $\theta_k^{(i)}(\tau)$ is continuous on $\mathscr{I}^{(i)}$ and its range is not limited to any interval of 2π. The graph of each $\theta_k^{(i)}(\tau)$ is referred to as a *phase curve*.

An imaginary number $j\omega^*$ is a characteristic root of $D_{\tau^*}(\lambda)$ and τ^* is some critical delay if and only if there exist some i, k such that

$$\theta_k^{(i)}(\tau^*) = 2l\pi, \ l \text{ integer} \tag{5.10}$$

and $\omega^* = \pm\omega_k^{(i)}(\tau^*)$. Therefore one can obtain $\mathscr{T}_c$, the set of all critical delays by solving (5.10) in each interval $\mathscr{I}^{(i)}$.

We will borrow the characteristic equation of the population model (3.50) for illustration here. However, a different set of parameters are chosen. The characteristic equation is

$$\lambda^2 + a\lambda + c + (b(\tau)\lambda + d(\tau))e^{-\lambda\tau} = 0 \tag{5.11}$$

where $a = 0.8$, $b = 2.5$, $c = 0.12e^{-m_j\tau}$, $d = 0.2e^{-m_j\tau}$, $m_j = 0.1192$. We set $\mathscr{I} = [0,14]$. We note that these parameters are not taken from some real biological system. They are set as such to serve our illustration purpose. It will later become clear that with this set of parameters, Assumption III in Chapter 1 is violated. Therefore, stability analysis based on the root crossing criterion in Chapter 1 becomes problematic for this set of model parameters.

Simple computation shows:

$$P(\lambda,\tau) = \lambda^2 + a\lambda + c, \ Q(\lambda,\tau) = b(\tau)\lambda + d(\tau),$$

$$\begin{aligned}F(\omega,\tau) &= (c-\omega^2)^2 + (a\omega)^2 - (b(\tau)\omega)^2 - d(\tau)^2 \\ &= \omega^4 + \omega^2(a^2 - b^2(\tau) - 2c) + c^2 - d^2(\tau).\end{aligned} \tag{5.12}$$

Using (5.12), (5.4) can be written as

$$\partial_\omega F(\omega,\tau) = 4\omega^3 - 2\omega(a^2 - b^2(\tau) - 2c).$$

Solving (5.3) together with (5.4) for $(\omega, \tau) \in \mathbb{R} \times \mathscr{I}$ we obtain two solutions in τ, namely $\tau \approx 6.160$ and $\tau \approx 9.762$. By definition $\tau^{(0)} = 0$, $\tau^{(1)} \approx 6.160$ and $\tau^{(2)} \approx 9.762$. Therefore $\mathscr{I}$ can be decomposed into $\mathscr{I}^{(1)} = [0, \tau^{(1)}]$, $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, $\mathscr{I}^{(3)} = [\tau^{(2)}, \tau^{u}]$. From (5.12) we further obtain

$$\omega_1^{(1)}(\tau) = 2^{-\frac{1}{2}}\sqrt{b^2(\tau) + 2c - a^2 + \Delta^{\frac{1}{2}}(\tau)}, \tau \in \mathscr{I}^{(1)},$$
$$\omega_1^{(2)}(\tau) = 2^{-\frac{1}{2}}\sqrt{b^2(\tau) + 2c - a^2 + \Delta^{\frac{1}{2}}(\tau)}, \tau \in \mathscr{I}^{(2)},$$
$$\omega_2^{(2)}(\tau) = 2^{-\frac{1}{2}}\sqrt{b^2(\tau) + 2c - a^2 - \Delta^{\frac{1}{2}}(\tau)}, \tau \in \mathscr{I}^{(2)},$$

where $\Delta(\tau) = (b^2(\tau) + 2c - a^2)^2 - 4(c^2 - d^2(\tau))$. The graph of $\omega_1^{(1)}(\tau)$, $\omega_1^{(2)}(\tau)$, $\omega_2^{(2)}(\tau)$ are plotted in Fig.5.1a. We have function $\theta_1^{(1)}(\tau)$ defined in $\mathscr{I}^{(1)}$ and functions $\theta_1^{(2)}(\tau)$, $\theta_2^{(2)}(\tau)$ defined in $\mathscr{I}^{(2)}$. The graphs of these functions are plotted in Fig. 5.1b. We find the graph of $\theta_1^{(1)}(\tau)$ intersects the horizontal line located at 0 at $\tau_1 \approx 0.878$. The graph of $\theta_2^{(2)}(\tau)$ intersects the horizontal line located at 2π at $\tau_2 = \tau^{(2)} \approx 9.762$. By definition the set of critical delays is $\mathscr{T}_c = \{\tau_1, \tau_2\}$.

5.3.1 Counting Unstable Roots

Let $\mathscr{B}_\delta(\lambda)$ be an open ball centered at λ in the complex plane with radius $\delta > 0$. Suppose $j\omega^*$ is an imaginary characteristic root of $D_{\tau^*}(\lambda)$ with multiplicity $\mu \geq 1$ for some critical delay $\tau^* \in \mathscr{I}$. The continuous dependence of the characteristic roots on τ means that for any sufficiently small positive number δ, one can find $\varepsilon(\delta) > 0$ such that for any $\Delta\tau$ with an absolute value smaller than $\varepsilon(\delta)$, $D_{\tau^*+\Delta\tau}(\lambda)$ has exactly μ roots within $\mathscr{B}_\delta(j\omega^*)$. We will investigate how these imaginary roots migrate as τ goes through a small neighborhood of τ^*. To make this problem precise, we define $N^u(\tau, \mathscr{B}_\delta(j\omega^*))$ as the number of roots of $D_\tau(\lambda)$ contained in $\mathscr{B}_\delta(j\omega^*) \cap \mathbb{C}_+$. Recall that the set $\mathscr{T}_c$ collects exactly all critical delays and the set $\Omega_F(\tau)$ is defined in (5.7). For each critical pair $(j\omega^*, \tau^*)$, define the one-sided increment of unstable roots as

$$\mathrm{Inc}^+(\omega^*, \tau^*) \quad = \quad \lim_{\varepsilon \to 0^+} N^u\big(\tau^* + \varepsilon, \mathscr{B}_\delta(j\omega^*)\big) - N^u\big(\tau^*, \mathscr{B}_\delta(j\omega^*)\big) \tag{5.13}$$

The the one-sided increment $\mathrm{Inc}^+(\omega^*, \tau^*)$ counts the number of characteristic roots that enters the right half complex plane from $j\omega^*$ as τ increases from τ^*. We note that in the definition δ is made sufficiently small such that $\mathrm{Inc}^+(\omega^*, \tau^*)$ is independent of δ.

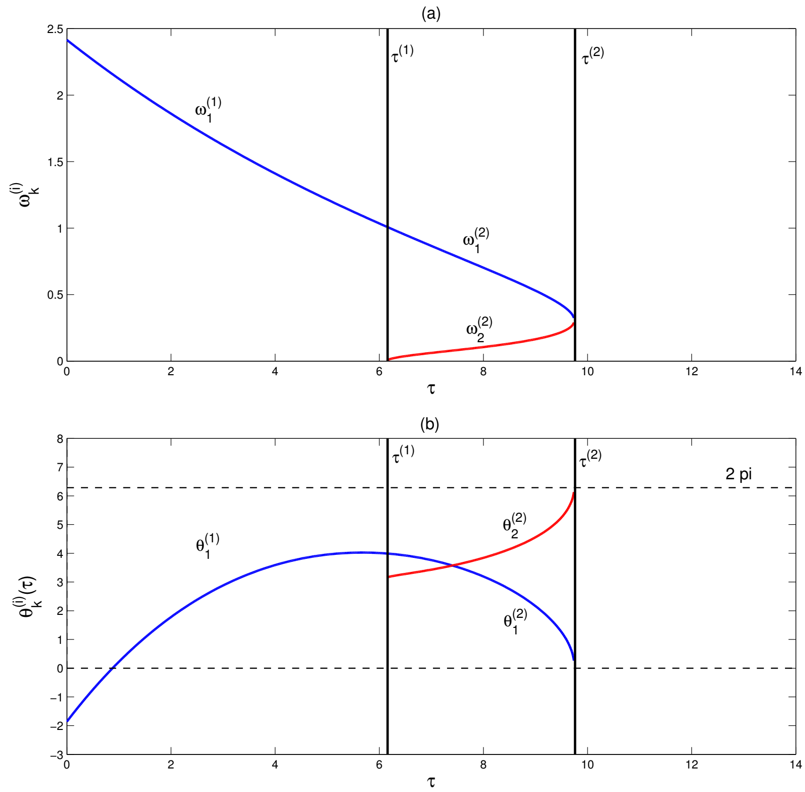

FIGURE 5.1: The graphs of frequency and phase functions associated with the population model (5.11). (a)The graphs of frequency functions. (b)The graphs of phase angle functions. The graph of $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at τ_1. The graph of $\theta_1^{(2)}(\tau)$ and $\theta_2^{(2)}(\tau)$ meets the horizontal line 0 and 2π, respectively, when $\tau = \tau_2$. Therefore two critical delays exist within $\mathscr{I}$.

Now we consider the two-sided increment in the number of unstable roots for each critical pair $(j\omega^*,\tau^*)$ denoted as $\mathrm{Inc}(\omega^*,\tau^*)$. If $\tau^* \neq \tau^l$, define

$$\mathrm{Inc}(\omega^*,\tau^*) \quad = \quad \frac{1}{2}\left(\lim_{\varepsilon\to 0^+} N^u\big(\tau^*+\varepsilon,\mathscr{B}_\delta(j\omega^*)\big) - N^u\big(\tau^*-\varepsilon,\mathscr{B}_\delta(j\omega^*)\big)\right). \quad (5.14)$$

As τ is swept from left through a neighborhood of some critical delay value: $[\tau^*-\varepsilon,\tau^*+\varepsilon]$, the quantity $\mathrm{Inc}(\omega^*,\tau^*)$ is just equal to the increment in the number of unstable characteristic roots that appear in a small vicinity of $j\omega^*$. If $\lambda = j\omega^*$ is a simple characteristic root, then $\mathrm{Inc}(\omega^*,\tau^*) = 1$ means this root crosses the imaginary axis towards $\mathbb{C}_+$. It moves towards $\mathbb{C}_-$ if $\mathrm{Inc}(\omega^*,\tau^*) = -1$. Otherwise this root merely touches the imaginary axis but does not cross it. Therefore, for simple characteristic roots on the imaginary axis, $\mathrm{Inc}(\omega^*,\tau^*)$ serves as an indicator of the crossing direction of this imaginary root. On the other hand, a repeated root $j\omega^*$ can be considered as a cluster of different characteristic roots which merge at $\tau = \tau^*$. As τ increases from τ^*, these roots will move along different branches of curves in the complex plane. Consequently the quantity $\mathrm{Inc}(\omega^*,\tau^*)$ does not reflect the crossing direction of any specific characteristic roots. Nevertheless, we will follow the convention of the simple roots case and continue to refer to the formula that determines the quantity $\mathrm{Inc}(\omega^*,\tau^*)$ as root crossing direction criteria.

In the case $\tau^* = \tau^l$ the previous definition of $\mathrm{Inc}(\omega^*,\tau^*)$ becomes problematic because there is no valid delay value smaller than τ^l. Therefore we use the one-sided increment instead:

$$\mathrm{Inc}(\omega^*,\tau^l) \quad = \quad \mathrm{Inc}^+(\omega^*,\tau^l), \quad (5.15)$$

The total increment of unstable roots as τ increases and goes through τ^* , which satisfies:

$$\mathrm{Inc}(\tau^*) \quad = \quad \sum_{\omega^*\in\Omega_F(\tau^*)} 2\mathrm{Inc}(\omega^*,\tau^*) \quad (5.16)$$

The coefficient 2 in (5.16) is due to the symmetry of imaginary roots about the real axis.

We shall determine the number of unstable roots for any given $\tau \in \mathscr{I}$ denoted as $N^u(\tau)$. It is easy to see the following relation holds for any τ that is not a critical delay:

$$N^u(\tau) = N^u(\tau^l) + \sum_{k=1}^{L(\tau)} \mathrm{Inc}(\tau_k), \quad (5.17)$$

where $L(\tau)$ is the index of the largest critical delay that satisfies $\tau_{L(\tau)} < \tau$.

5.4 A Two-Parameter Perspective

In this section we will provide a formulation to compute the quantity $\mathrm{Inc}(\omega^*, \tau^*)$ for each critical pair $(j\omega^*, \tau^*)$. For this purpose, we will first present a novel two-parameter perspective of systems with delay-dependent coefficients to gain some geometric insight.

5.4.1 Critical Delay Curves

Consider the characteristic equation

$$\tilde{D}(\lambda, r, q) = P(\lambda, r) + Q(\lambda, r)e^{-\lambda q} = 0, \tag{5.18}$$

where $q, r \in \mathscr{I}$ are two independent parameters. Then equation (5.1) becomes equivalent to (5.18) if we impose the restriction $q = r = \tau$. We denote

$$\tilde{D}_{rq}(\lambda) = \tilde{D}(\lambda, r, q)$$

For any $r \in \mathscr{I}^{(i)}$ that satisfies $\omega_k^{(i)}(r) \neq 0$, define the *the critical delay curves* as:

$$\tau_k^{(i)}(r) = -\frac{\theta_k^{(i)}(r)}{\omega_k^{(i)}(r)} + r. \tag{5.19}$$

5.4.2 Stability Regions in the Two-Parameter Plane

In (5.18) let the parameter $r \in \mathscr{I}$ be fixed. It is easy to see that a necessary and sufficient condition for $j\omega^*$, $\omega^* > 0$ to be a characteristic root of $\tilde{D}_{rq}(\lambda)$ is that the following equations

$$\omega^* = \omega_k^{(i)}(r), \tag{5.20}$$

$$q = \tau_k^{(i)}(r) + 2l\pi/\omega_k^{(i)}(r), \; l \text{ integer}, \tag{5.21}$$

hold together for some i, k.

Consider the square region $\mathscr{I} \times \mathscr{I}$ on the r-q parameter plane. Our analysis so far has shown that as the parameter point (r, q) moves in this region, (5.18) admits some imaginary roots if and only if (r, q) lies on one of the critical delay curves. Therefore the critical delay curves split the parameter domain on the r-q plane into sub-regions; within the interior of each sub-region the number of unstable roots is invariant.

In Fig.5.2 the blue and red curves are the critical delay curves of the population model (5.11). The blue ones correspond to the graphs of a family of functions $\tau_k^{(i)}(\tau) + 2l\pi/\omega_k^{(i)}(\tau)$ parameterized by integer l and $i = 1, 2$, $k = 1$. The red curves corresponds to the graphs of a

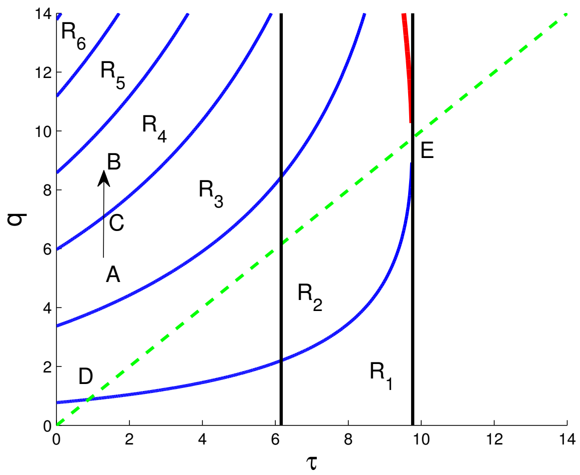

FIGURE 5.2: The critical delay curves of the population model (5.11). The characteristic equation admits imaginary roots if and only if the parameter point (r,q) is on these curves. Within each region enclosed by these curves, the number of unstable characteristic roots is a constant.

family of functions $\tau_2^{(2)}(\tau)+2l\pi/\omega_2^{(2)}(\tau)$ parameterized by the integer l. Now consider two points A and B, which forms a vertical line segment. A, B are in two different but adjacent parameter regions denoted by R_3, R_4 respectively. Let the parameter (r,q) start moving from A vertically towards B, or in other words we fix r and increase q continuously. When the point reaches a boundary curve at Point C some imaginary roots appear. By monitoring whether this imaginary root move towards $\mathbb{C}_+$ or $\mathbb{C}_-$ as q further increases one can determine how the the number of unstable roots changes as the parameter crosses the boundary curve and enters R_4 from R_3. As discussed in Chapter 2, the moving direction of the imaginary roots are captured by the differential information of Function F with the parameter (r,q) fixed at the crossing point. To be more precise, let r in (5.18) be fixed and suppose $\lambda = j\omega^*$ is a simple root of (5.18) for $q=q^*$. Then for (λ,q) in a neighbourhood of $(j\omega^*,q^*)$, (5.18) defines λ as a differentiable function of q denoted by $\lambda(q)$. According to (2.27), we have

$$\operatorname{sgn}\left(\Re\left(\frac{d\lambda(q^*)}{dq}\right)\right) = \operatorname{sgn}\big(\partial_\omega F(\omega^*,r)\big). \tag{5.22}$$

The original system (5.1) is equivalent to (5.18) with restriction $r=q=\tau$. Therefore, as the delay value in (5.1) sweeps through $\mathscr{I}$, on the r-q plane, the parameter point (r,q) moves along the 45 degree dashed green line in Fig.5.2 and thus enters or leaves different stability regions of the r-q plane. The dashed green line intersects the critical delay curves at Point D and E, therefore that at these points imaginary roots appear. This conclusion is consistent with our previous analysis based on (5.10). Then the crossing direction of these imaginary roots may be determined by applying (5.22) and taking into account whether the 45 degree line crosses the critical delay curves from below or from above.

Suppose $(j\omega_k^{(i)}(\tau^*), \tau^*)$ is a critical pair. It is easy to see from (5.19) and (5.21) that if derivative of the phase function $\frac{d}{d\tau}\theta_k^{(i)}(\tau^*) > 0$, the critical delay curve satisfies $\frac{d}{d\tau}\hat{\tau}_k^{(i)}(\tau^*) < 1$, therefore the parameter (r,q) enters some stability region in the r-q space from below as (r,q) moves along the 45 degree line. Furthermore, the crossing direction of this characteristic root as (r,q) enters a stability region from below may be determined based on the term $\mathrm{sgn}((\partial_\omega F(j\omega_k^{(i)}(\tau^*), \tau^*))$. On the other hand, if derivative of the phase function $\frac{d}{d\tau}\theta_k^{(i)}(\tau^*) < 0$, the parameter (r,q) enters some stability region in the r-q space from above as (r,q) moves along the 45 deg line, consequently the crossing direction of the characteristic root is reversed. This observation gives an interpretation of the 'separation principle' we obtained in previous chapters, which is, the crossing direction of an imaginary root is determined by the product of $\mathrm{sgn}(\partial_\omega F(j\omega_k^{(i)}(\tau^*), \tau^*))$ and $\mathrm{sgn}(\frac{d}{d\tau}\theta_k^{(i)}(\tau^*))$.

In the next section, we will exploit this geometric idea to carry out more complicated stability analysis.

5.5 Crossing Direction Criteria

In this section we derive formulas to compute the increment of unstable roots $\mathrm{Inc}(\omega^*, \tau^*)$ defined for each critical pairs $(j\omega^*, \tau^*)$ of (5.1) by considering (5.18) with the restriction $r = q = \tau$. We need some notation for the formulation of the main theorem.

5.5.1 Some Additional Notation

To every $\tau \in (\tau^{(i-1)}, \tau^{(i)}]$ we assign a number $\tau_- \in (\tau^{(i-1)}, \tau)$ such that the interval $[\tau_-, \tau)$ does not contain any critical delays of $D_\tau(\lambda)$. Similarly, we also assign a numbers $\tau_+ \in (\tau, \tau^{(i)})$ to every $\tau \in [\tau^{(i-1)}, \tau^{(i)})$ such that the interval $(\tau, \tau_+]$ does not contain any critical delay of $D_\tau(\lambda)$.

Suppose $\tau^* \in [\tau^{(i-1)}, \tau^{(i)})$ is a critical delay, we assign to the index i a number i' such that

$$i' = \begin{cases} i-1, & \text{if } \tau^* = \tau^{(i-1)}, \\ i, & \text{otherwise.} \end{cases} \tag{5.23}$$

Then we can introduce a set $\mathscr{K}_+(\tau^*)$ which collects each k that satisfies the following two conditions:

$$\theta_k^{(i)}(\tau^*) = 2l\pi, \ l \text{ integer}, \tag{5.24}$$

$$\lim_{\tau \to \tau^{*+}} \operatorname{sgn}\big(\theta_k^{(i)}(\tau) - \theta_k^{(i)}(\tau^*)\big) = 1. \tag{5.25}$$

Similarly, if $\tau^* \neq \tau^l$ define $\mathscr{K}_-(\tau^*)$ as the set of each k' that satisfies the following two conditions:

$$\theta_{k'}^{(i')}(\tau^*) = 2l\pi,\ l \text{ integer}, \tag{5.26}$$

$$\lim_{\tau \to \tau^{*-}} \operatorname{sgn}\big(\theta_{k'}^{(i')}(\tau) - \theta_{k'}^{(i')}(\tau^*)\big) = 1. \tag{5.27}$$

We note that there is a one-to-one correspondence between the function $\tau_k^{(i)}(\cdot)$ and $\theta_k^{(i)}(\cdot)$. The event that point (τ,τ) crosses some curve $\tau_k^{(i)}(\cdot)$ in the r-q parameter plane is associated with the event that the phase curve $\theta_k^{(i)}(\tau)$ crosses some horizontal $2l\pi$ line for some integer l. Moreover, the two crossing directions are also related. As τ increase, the point $(r,q) = (\tau,\tau)$ crosses some stability region boundary curve of the form $\tau_k^{(i)}(\tau) - 2l/\omega_k^{(i)}(\tau)$ in the r-q parameter plane at the point (τ^*,τ^*) from above to below (from below to above) if and only if the curve $\theta_k^{(i)}(\tau)$ crosses some $2l\pi$ horizontal line at $\tau = \tau^*$ from above to below (from below to above), for some integer l. More precisely, in view of (5.19) we have the following equivalence:

Proposition 5.1. *Equation (5.25) is equivalent to*

$$\lim_{r \to \tau^{*+}} sgn\big(r - \tau_k^{(i)}(r) - \theta_k^{(i)}(\tau^*)/\omega_k^{(i)}(r)\big) = 1,$$

and (5.27) is equivalent to

$$\lim_{r \to \tau^{*-}} sgn\big(r - \tau_{k'}^{(i')}(r) - \theta_{k'}^{(i')}(\tau^*)/\omega_{k'}^{(i')}(r)\big) = 1.$$

Therefore $\mathscr{K}_+(\tau^*)$ is just the set that collects all index k such that on the $r-q$ plane some branches of the critical delay curves $\tau_k^{(i)}(r) + \theta_k^{(i)}(\tau^*)/\omega_k^{(i)}(r)$ approach the 45 degree line $r = q$ from below as $r \to \tau^{*+}$. On the other hand, $\mathscr{K}_-(\tau^*)$ is just the set that collects all index k such that on the $r-q$ plane some branches of the critical delay curves $\tau_k^{(i')}(r) + \theta_k^{(i')}(\tau^*)/\omega_k^{(i')}(r)$ approach the 45 degree line $r = q$ from below as $r \to \tau^{*-}$.

We further decompose each set $\mathscr{K}_+(\tau^*)$ or $\mathscr{K}_-(\tau^*)$ into subsets by the frequency of different imaginary eigenvalues. Define

$$\mathscr{K}_+(\omega^*,\tau^*) = \{k \in \mathscr{K}_+(\tau^*) | \omega_k^{(i)}(\tau^*) = \omega^*\},$$

$$\mathscr{K}_-(\omega^*,\tau^*) = \{k \in \mathscr{K}_-(\tau^*) | \omega_k^{(i')}(\tau^*) = \omega^*\}.$$

Returning to the population model (5.11), consider Point $D = (\tau_1,\tau_1)$ in Fig.5.2. Since at this point Ω_F contains only one element, namely $\omega_1^{(1)}(\tau_1)$ and at Point D the curve of

$\theta_1^{(1)}(\tau)$ has a tangent with positive slope (refer to Fig. 5.1), we deduce that $\mathscr{K}_-(\tau_1) = \mathscr{K}_-(\omega_1^{(1)}(\tau_1), \tau_1) = \phi$ and $\mathscr{K}_+(\tau_1) = \mathscr{K}_+(\omega_1^{(1)}(\tau_1), \tau_1) = \{1\}$. On the other hand, since as $\tau \to \tau_2^+$, $\theta_1^{(2)}(\tau) \to 0^+$ and $\theta_2^{(2)}(\tau) \to 2\pi^-$, we deduce $\mathscr{K}_-(\tau_2) = \{1\}$. There are no phase curve defined for $\tau > \tau^{(2)}$, which means $\mathscr{K}_+(\tau_2)$ is empty.

5.5.2 General Results

Consider the following characteristic equation with the delay parameter in the system coefficients fixed at $\tau^* \geq 0$:

$$\tilde{D}(\lambda, r \equiv \tau^*, \tau) = 0 \tag{5.28}$$

Obviously, any critical pair $(j\omega^*, \tau^*)$ of (5.1) is also a critical pair of (5.28). For characteristic equation (5.28), we will use $\mathrm{Inc}_{r=\tau^*}(\omega, \tau)$ or $\mathrm{Inc}^+_{r=\tau^*}(\omega, \tau)$ to denote the two-sided or one-sided increment of number of unstable roots defined for some critical pair $(j\omega, \tau)$. In other words, $\mathrm{Inc}^+_{r=\tau^*}(\omega, \tau)$ and $\mathrm{Inc}_{r=\tau^*}(\omega, \tau)$ are defined according to (5.13) and (5.14), respectively, based on the characteristic equation with fixed coefficients (5.28) rather than (5.1).

Lemma 5.1. *Suppose $\tau^* \in [\tau^{(i-1)}, r^{(i)}) - \{\tau^l\}$ and $\lambda = j\omega^*$, $\omega^* > 0$ is a characteristic root of $D_{\tau^*}(\lambda)$ defined in (5.1). Let $i' = i - 1$ if $\tau^* = \tau^{(i-1)}$, otherwise let $i' = i$. The following holds*

$$\begin{aligned} Inc(\omega^*, \tau^*) = &\sum_{k \in \mathscr{K}_+(\omega^*, \tau^*)} Inc_{r=\tau_+^*}(\omega_k^{(i)}(\tau_+^*), \tau_+^*) \\ &- \sum_{k \in \mathscr{K}_-(\omega^*, \tau^*)} Inc_{r=\tau_-^*}(\omega_k^{(i')}(\tau_-^*), \tau_-^*). \end{aligned} \tag{5.29}$$

Proof. To prepare for the proof, let us first make some geometric constructions illustrated in Fig.5.3. Let $\mathscr{B}_\delta(j\omega^*)$ be a ball on the complex plane which contain no other roots of the characteristic function $D_\tau(\lambda)$ in (5.1) except $\lambda = j\omega^*$ when $\tau = \tau^*$. Suppose the multiplicity of this root is m^*. Let $O = (\tau^*, \tau^*)$ be a point on the $r - q$ plane. Construct a square $\mathscr{C}_\varepsilon(O)$ centered at O with side length ε. Le δ be sufficiently small such that, for any two different elements ω_a^*, ω_b^* in $\Omega^*(\tau^*)$, $\mathscr{B}_\delta(j\omega_a^*) \cap \mathscr{B}_\delta(j\omega_b^*) = \{\phi\}$. Then choose $\varepsilon(\delta) > 0$, such that for all $(r, q) \in \mathscr{C}_\varepsilon(O)$, the characteristic function $\tilde{D}_{rq}(\lambda)$ defined in (5.18) has exactly m^* roots in $\mathscr{B}_\delta(j\omega^*)$, where m^* is the multiplicity of $j\omega^*$ for $\tau = \tau^*$. Since $\tau_k^{(i)}(\tau) + 2l_k/\omega_k^{(i)}(\tau)$ are continuous functions for all k, we also make ε sufficiently small such that the square $\mathscr{C}_\varepsilon(O)$ intersects neither the graphs of function$\tau_k^{(i')}(\tau) + 2l_k\pi/\omega_k^{(i')}(\tau)$, for each k not in $\mathscr{K}_-(\tau^*)$ nor the graphs of function $\tau_k^{(i)}(\tau) + 2l_k\pi/\omega_k^{(i)}(\tau)$ for each k not in $\mathscr{K}_+(\tau^*)$.

Pick a number $0 < \varepsilon_1 < \frac{1}{2}\varepsilon$ and on the $r-q$ plane define $A = (\tau^* - \varepsilon_1, \tau^* - \varepsilon_1)$, $B = (\tau^* - \varepsilon_1, \tau^* - \varepsilon/2)$, $C = (\tau^* + \varepsilon_1, \tau^* - \varepsilon/2)$, $D = (\tau^* + \varepsilon_1, \tau^* + \varepsilon_1)$. Let ε_1 be sufficiently small such that $\tau^* - \varepsilon/2 < \min_{k \in \mathscr{K}_-(\omega^*,\tau^*)} \tau_k^{(i)}(\tau^* - \varepsilon_1) + 2l_k\pi/\omega_k^{(i)}(\tau^* - \varepsilon_1)$ if $\mathscr{K}_-(\omega^*,\tau^*)$ is not empty and $\tau^* - \varepsilon/2 < \min_{k \in \mathscr{K}_+(\omega^*,\tau^*)} \tau_k^{(i)}(\tau^* + \varepsilon_1) + 2l_k\pi/\omega_k^{(i)}(\tau^* + \varepsilon_1)$ if $\mathscr{K}_+(\omega^*,\tau^*)$ is not empty.

Now on the $r-q$ plane let the parameter point $P = (r,q)$ move along the path $ABCD$ from A to D. Due to the continuity of $\tau_k^{(i)}(\cdot)$ and $\omega_k^{(i)}(\cdot)$ for all k, P crosses the graphs of function $\tau_k^{(i)}(\tau) + 2l_k\pi/\omega_k^{(i)}(\tau)$ for each $k \in \mathscr{K}_-(\omega^*,\tau^*)$ as it moves from A to B, and l_k is the integer satisfying

$$\tau^* = \tau_k^{(i)}(\tau^*) + 2l_k\pi/\omega_k^{(i)}(\tau^*).$$

Let these intersections be $P_1, P_2, \cdots P_{K_-}$, K_- is just the number of elements in $\mathscr{K}_-(\omega^*,\tau^*)$. As P moves from C to D it crosses the graphs of $\tau_k^{(i)}(\tau) + 2l_k\pi/\omega_k^{(i)}(\tau)$, for each $k \in \mathscr{K}_+(\omega^*,\tau^*)$ at points say $Q_1, Q_2, \cdots Q_{K_+}$, where K_+ is just the cardinality of $\mathscr{K}_+(\omega^*,\tau^*)$. It is easy to see that δ and $\varepsilon(\delta)$ is selected in such a way that as P is confined in $\mathscr{C}_\varepsilon(O)$, $\tilde{D}_{rq}(\lambda)$ admits imaginary roots in $\mathscr{B}_\delta(j\omega^*)$ if and only if $P = P_l$, $1 \le l \le K_-$ or $P = Q_l$, $1 \le l \le K_+$. Revoke the definition of Inc_{fix} at each of these crossing points, the theorem follows. $\square$

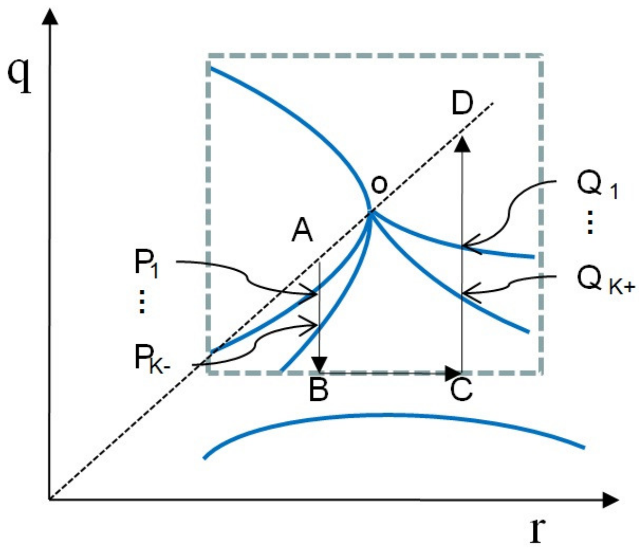

FIGURE 5.3: Illustration for the proof of Theorem 5.1.The blue curves represent the critical delay curves, namely the graphs of $\tau_k^{(i)}(r) + \theta_k^{(i)}(\tau^*)/\omega_k^{(i)}(r)$ for some appropriate index i, k. As the parameter point (r,q) moves along the path $A-B-C-D$, characteristic roots appear on the imaginary axis when (r,q) meets these critical delay curves.

Theorem 5.1. *Suppose* $\tau^* \in [\tau^{(i-1)}, r^{(i)}) - \{\tau^l\}$ *and* $\lambda = j\omega^*$, $\omega^* > 0$ *is a characteristic root of* $D_{\tau^*}(\lambda)$ *defined in (5.1). Let* $i' = i-1$ *if* $\tau^* = \tau^{(i-1)}$, *otherwise let* $i' = i$. *The following holds*

$$Inc(\omega^*, \tau^*) = \sum_{k \in \mathscr{K}_+(\omega^*, \tau^*)} sgn_k^{(i)} - \sum_{k \in \mathscr{K}_-(\omega^*, \tau^*)} sgn_k^{(i')}, \tag{5.30}$$

where the quantity $\mathrm{sgn}_k^{(i)}$ is defined in (5.8).

Proof. It follows from (5.22) that

$$\mathrm{Inc}_{r=\tau_+^*}(\omega_k^{(i)}(\tau_+^*), \tau_+^*) = \mathrm{sgn}\left(\partial_\omega F(j\omega_k^{(i)}(\tau_+^*), \tau_+^*)\right) = \mathrm{sgn}_k^{(i)}, \tag{5.31}$$

and

$$\mathrm{Inc}_{r=\tau_-^*}(\omega_k^{(i)}(\tau_-^*), \tau_-^*) = \mathrm{sgn}\left(\partial_\omega F(j\omega_k^{(i)}(\tau_-^*), \tau_-^*)\right) = \mathrm{sgn}_k^{(i')}. \tag{5.32}$$

□

Corollary 5.1. *Suppose* $\tau^* \in [\tau^{(i-1)}, r^{(i)}) - \{\tau^l\}$ *is a critical delay. Let* $i' = i-1$ *if* $\tau^* = \tau^{(i-1)}$, *otherwise let* $i' = i$. *The following holds*

$$Inc(\tau^*) = \sum_{k \in \mathscr{K}_+(\tau^*)} 2\, sgn_k^{(i)} - \sum_{k \in \mathscr{K}_-(\tau^*)} 2\, sgn_k^{(i')} \tag{5.33}$$

This corollary is a direct consequence of Theorem 5.1 and (5.16). The following results can be derived from Corollary 5.1. We omit the proof.

So far we have proposed a formula that determines $\mathrm{Inc}(\omega^*, \tau^*)$ for each critical pair $(j\omega^*, \tau^*)$ as long as $\tau^* \neq \tau^l$. However, if τ^l is a critical delay and $j\omega^*$ is a corresponding imaginary characteristic root with multiplicity, then these criteria can not be applied because the number τ_-^* can no longer be defined as there is no valid delay value smaller than τ^l. Although it might be possible to artificially extend the domain of τ to contain τ^l in its interior, a more satisfactory solution is to derive some formula also for the one-sided increment $\mathrm{Inc}^+(\omega^*, \tau^*)$, since by definition we have $\mathrm{Inc}(\omega^*, \tau^l) = \mathrm{Inc}^+(\omega^*, \tau^l)$ if $(j\omega^*, \tau^l)$ is a critical pair.

Proposition 5.2. *For a critical pair* $(j\omega^*, \tau^*)$, *we have*

$$N^u(\tau^{*+}) - N^u(\tau^*) = \sum_{\omega^* \in \Omega(\tau^*)} Inc^+(j\omega^*, \tau^*),$$

and

$$Inc^{+}(j\omega^*,\tau^*) = -Inc_{r=\tau^*}(j\omega^*,\tau^*) + \sum_{k\in\mathscr{K}^{+}(\omega^*,\tau^*)} sgn_k^{(i)}. \tag{5.34}$$

As pointed out in Theorem 1 of [39], to obtain each term $\mathrm{Inc}_{r=\tau^*}(j\omega^*,\tau^*)$, one can first compute the corresponding Puisuex series at the critical pair $(j\omega^*,\tau^*)$ for the characteristic equation (5.28), and $\mathrm{Inc}_{r=\tau^*}(j\omega^*,\tau^*)$ is identical to the number of values in $\mathbb{C}_+$ of the Puiseux series. This technique has been briefly introduced in Chapter 2.

The proof of Proposition 5.2 follows an idea very similar to the one behind the proof of Lemma 5.1. One can draw a path *O*-*B*-*C*-*D* in the (r,q) parameter space, where $O=[\tau^*,\tau^*]$, $B=[\tau^*,\tau^*-\varepsilon]$, $C=[\tau^*+\varepsilon,\tau^*-\varepsilon]$, $D=[\tau^*+\varepsilon,\tau^*+\varepsilon]$, and ε is a sufficiently positive number. As the parameter (r,q) leaves from Point O and moves along the path OB, the increment of unstable roots is just $-\mathrm{Inc}_{r=\tau^*}(j\omega^*,\tau^*)$ by definition. No characteristic roots crosses the imaginary axis as (r,q) moves along BC. Finally, as the parameter moves along CD, the number of unstable roots generated is equal to the second term of the the right hand side of (5.34).

5.5.3 Simplification for Simple Characteristic Roots

Proposition 5.3. *Suppose* $\tau^*\in[\tau^{(i-1)},\tau^{(i)})-\{\tau^l\}$, *and* $\lambda=j\omega_k^{(i)}(\tau^*)$ *is a simple characteristic root of* $D_{\tau^*}(\lambda)$. *Denote* $\omega^*=\omega_k^{(i)}(\tau^*)$. *For* τ *in a neighborhood of* τ^* *this root is a function of* τ *written as* $\lambda(\tau)$. *If* $\tau^*=\tau^{(i-1)}$, *set* $i'=i-1$, *otherwise let* $i'=i$. *Let* k' *be the index that satisfies (5.26). We have the following criterion concerning the crossing direction of the imaginary root* $\lambda(\tau^*)$:

$$\lim_{\varepsilon\to 0^+} sgn\big(\Re(\lambda(\tau))\big)\big|_{\tau=\tau^*-\varepsilon}^{\tau=\tau^*+\varepsilon} = 2Inc(\omega^*,\tau^*), \tag{5.35}$$

$$\begin{aligned} Inc(\omega^*,\tau^*) = \quad & \frac{1}{2}sgn_k^{(i)} \times \Big(sgn\big(\theta_k^{(i)}(\tau_+^*)-\theta_k^{(i)}(\tau^*)\big) \\ & -sgn\big(\theta_{k'}^{(i')}(\tau_-^*)-\theta_{k'}^{(i')}(\tau^*)\big)\Big). \end{aligned} \tag{5.36}$$

In the above proposition, if $\tau^*\neq\tau^{(i-1)}$, we must have $i=i'$ and $k=k'$, then (5.36) clearly shows that the crossing direction of the imaginary root $j\omega^*$ associated with the frequency function $\omega_k^{(i)}(\tau)$ is determined by the monotonicity of the phase function $\theta_k^{(i)}(\tau)$ at τ^* as well as the quantity $\mathrm{sgn}_k^{(i)}$. On the other hand, if in the last proposition $\tau^*=\tau^{(i-1)}$, it is easy to see $\theta_k^{(i)}(\tau^*) = \theta_{k'}^{(i')}(\tau^*) + 2l\pi$, for some integer l. Therefore one can piece $\theta_k^{(i)}(\tau)$ together with $\theta_{k'}^{(i')}(\tau) + 2l\pi$ at τ^* to form one continuous phase function, say $\theta(\tau)$. Then from (5.36) we can again deduce that the crossing direction is determined by the quantity $\mathrm{sgn}_k^{(i)}$ as well as the monotonicity of $\theta_k^{(i)}(\tau)$ at τ^*.

Proof. $\tau^* \in \mathscr{I}_o^{(i)}$ implies that $\lambda = j\omega^*$ is not a repeated root as indicated by (5.22) since we have $\partial_\lambda D(j\omega^*, \tau^*) \neq 0$. Therefore for sufficiently small positive number δ there exists only one root in $\mathscr{B}_\delta(j\omega^*)$ for τ in a sufficiently small neighbor hood of τ^*. Therefore $\lambda(\tau)$ can be locally defined. Equation (5.35) is then obvious, we only prove (5.36). For $\Delta\tau \in [0, \varepsilon]$, where ε is sufficiently small, there are only four possible situations. The first two are i) $\theta_k^{(i)}(\tau^* - \Delta\tau) < 2l_k\pi$, $\theta_k^{(i)}(\tau^* + \Delta\tau) > 2l_k\pi$; ii)$\theta_k^{(i)}(\tau^* - \Delta\tau) > 2l_k\pi$, $\theta_k^{(i)}(\tau^* + \Delta\tau) > 2l_k\pi$. The rest two can be obtained by taking the inverse sign of inequality in case i) and case ii). The condition of case i) implies that $\mathscr{K}_-(\omega^*, \tau^*) = \{\phi\}$, $\mathscr{K}_+(\omega^*, \tau^*) = \{k\}$. It follows from Proposition 5.3 that $\mathrm{Inc}(\omega^*, \tau^*) = \mathrm{sgn}(i, k)$ and this is consistent with (5.36). The condition of case ii) implies that $\mathscr{K}_-(\omega^*, \tau^*) = \{\phi\}$, $\mathscr{K}_+(\omega^*, \tau^*) = \{\phi\}$. Once again from Proposition 5.3 we have $\mathrm{Inc}(\omega^*, \tau^*) = 0$. It is straightforward to check that (5.36) produces the same result. The proof for the rest two situations follows the same line of argument and is therefore omitted. $\square$

Proposition 5.4. *Suppose $\tau^* \in [\tau^{(i-1)}, \tau^{(i)})$ and $j\omega_k^{(i)}(\tau^*) = j\omega^*$ is a simple imaginary characteristic root of $D_{\tau^*}(\lambda)$. In a neighborhood of $(j\omega^*, \tau^*)$, the characteristic equation (5.1) implicitly determines λ as a function of τ denoted as $\lambda(\tau)$. It follows that*

$$\lim_{\tau \to \tau^{*+}} sgn(\Re(\lambda(\tau))) = sgn\big(\theta_k^{(i)}(\tau_+^*) - \theta_k^{(i)}(\tau^*)\big) sgn_k^{(i)}. \tag{5.37}$$

Assume that there exists a positive integer n_d such that $(\frac{d}{d\tau})^{n_d} P(j\omega, \tau)$ and $(\frac{d}{d\tau})^{n_d} Q(j\omega, \tau)$ exist at $(j\omega^*, \tau^*)$. Furthermore, assume that n_d satisfies $(\frac{d}{d\tau})^l \theta_k^{(i)}(\tau^*) = 0$, for $1 \le l < n_d$ and $(\frac{d}{d\tau})^{n_d} \theta_k^{(i)}(\tau^*) \neq 0$. Then the following holds:

$$sgn\left(\left(\tfrac{d}{d\tau}\right)^l \Re(\lambda(\tau^*))\right) = sgn\left(\left(\tfrac{d}{d\tau}\right)^l \theta_k^{(i)}(\tau^*)\right) \times sgn_k^{(i)}, \tag{5.38}$$

for $1 \le l \le n_d$.

It is worth mentioning that Proposition 5.4 is reduced to the first-order root-crossing criteria given in Chapter 3 when $l = 1$ in Formula (5.38).

Proof. It is easy to see that for $1 \le k \le m(i)$, $\omega_k^{(i)}(\tau)$ is locally $\mathbf{C}^{n_d}$ by the inverse function theorem. Then it is easy to see that $\tau_k^{(i)}(\cdot)$'s and $\theta_k^{(i)}(\cdot)$'s are also locally $\mathbf{C}^{n_d}$.
Since $\lambda = j\omega^*$ is an unrepeated root of (5.18) when $q = r = \tau^*$, for (r,q) in a neighborhood of (τ^*,τ^*), λ can be viewed as a continuous function in (r,q) denoted by $\hat{\lambda}(r,q)$. Since $\partial_\lambda D_r(\lambda,r,q) \neq 0$ at $\lambda = j\omega^*$, $r = q = \tau^*$ as implied by (5.22) and also because $D_r(\lambda,r,q)$ is locally $\mathbf{C}^{n_d}$ in both q and r, it follows from the implicit function theorem that $\hat{\lambda}(r,q)$ is C^{n_d} in q. On the other hand λ as a root of (5.1) is locally a function of τ only, therefore we also denote $\lambda(\tau) = \hat{\lambda}(\tau,\tau)$.
The number τ^* being a critical delay implies that there exists an integer l such that

$$\theta_k^{(i)}(\tau^*) = 2l\pi$$

Denote

$$E_k^{(i)}(q) = q - \tau_k^{(i)}(r) - \frac{2l\pi}{\omega_k^{(i)}(r)} \tag{5.39}$$

and thus $E_k^{(i)}(\tau^*) = 0$. Denote the graph of function $\tau_k^{(i)}(r) + 2l\pi/\omega_k^{(i)}(r)$ as γ, which is a curve on the $r-q$ plane and intersects the line $q = r$ at point $C = (\tau^*,\tau^*)$. Refer to Fig.5.4 for illustration. Let ε be a sufficiently small variable and let the vertical line $r = \tau^* + \varepsilon := r_1(\varepsilon)$ intersect γ and the line $q = r$ at Point A, Point B respectively. Set $A_r = B_r = r_1$, $A_q = \tau_k^{(i)}(r_1) + 2l\pi/\omega_k^{(i)}(r_1)$ and $B_q = r_1$, then it follows that $A = (A_r, A_q)$, $B = (B_r, B_q)$. Further denote $AB = B_q - A_q$. Apply (5.22) at Point A, we deduce that there is a positive number $c_1(\varepsilon)$ such that

$$\partial_q \Re\big(\hat{\lambda}(r,q)\big)_{(r,q)=A} = c_1 \partial_\omega F(\omega_k^{(i)}(r),r)|_{(r,q)=A} \tag{5.40}$$

Using Taylor expansion and noticing $\hat{\lambda}(r_1,A_q)$ has zero real part, we have

$$\Re\big(\hat{\lambda}(r,q)\big)_{(r,q)=B} = c_1 \partial_\omega F(\omega_k^{(i)}(r),r)|_{(r,q)=A} \cdot AB + o(AB) \tag{5.41}$$

Combine the last equality with $\operatorname{sgn}(AB) = \operatorname{sgn}(\theta_k^{(i)}(\tau_+^*) - 2l_k\pi)$, (5.37) follows. Further noticing $E_k^{(i)}(\tau^* + \varepsilon) = AB$ as well as $E_k^{(i)}(\tau^*) = 0$ and using Taylor expansion we also have

$$AB = c_2 \left(\frac{d}{dq}\right)^{n_d} E_k^{(i)}(\tau^*)\varepsilon^{n_d} + o(\varepsilon^{n_d})$$

where $c_2 > 0$. Substitute the last equality into (5.41) and further noticing $\mathrm{sgn}\big(\partial_\omega F(\omega_k^{(i)}(r),r)\big) = \mathrm{sgn}_k^{(i)}$ is constant for r in a neighborhood of r^*, it follows that

$$\Re(\lambda(\tau^*+\varepsilon)) = c_3 F(\omega_k^{(i)}(r^*),r^*)\left(\frac{d}{dq}\right)^{n_d} E_k^{(i)}(\tau^*)\varepsilon^{n_d} + o(\varepsilon^{n_d}) \tag{5.42}$$

where $c_3 > 0$. Since $E_k^{(i)}(\tau)\omega_k^{(i)}(\tau) + 2l\pi = \theta_k^{(i)}(\tau)$, it is straight forward to verify that there exists a positive number c_4 such that

$$\left(\frac{d}{dq}\right)^{n_d} E_k^{(i)}(\tau^*) = c_4\left(\frac{d}{d\tau}\right)^{n_d}\theta_k^{(i)}(\tau^*) \tag{5.43}$$

Substitute (5.43) into (5.42), (5.38) is thus proved. □

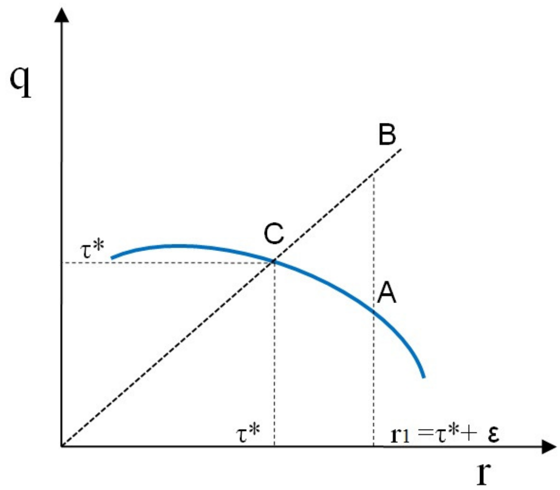

FIGURE 5.4: Illustration for the proof of Proposition 5.4

5.5.4 A Summary of the Stability Analysis Procedure

The results on the imaginary root crossing problem in the last section allows us to find all the delay intervals for which system (5.1) is asymptotically stable in a systematic way. The proposed analysis steps are listed as follows.

Step 1. Solve (5.3) together with (5.4) subject to $\omega > 0$, $\tau \in \mathscr{I}$ to obtain $\tau^{(i)}$, $i = 0,\cdots,K$ and thus the set $\mathscr{T}_F$ defined in Assumption III. $\mathscr{I}$ is thus decomposed into each $\mathscr{I}^{(i)} = [\tau^{(i-1)},\tau^{(i)}]$. In each sub-interval $\mathscr{I}^{(i)}$, solve for the real roots of $F_\tau(\omega)$, and thus obtain the frequency functions $\omega_k^{(i)}(\tau)$, $k = 1,\cdots,m(i)$. Then the crossing frequency set $\Omega_F(\tau)$ is known for each τ.

Step 2. In each $\mathscr{I}^{(i)}$, solve (5.10) to find all the critical delay values τ_i, $i = 1, \cdots, L$ and thus the critical delay set $\mathscr{T}_c$.
Step 3, If τ^l is not a critical delay, jump to Step 4. Otherwise compute $\mathrm{Inc}(\tau^l)$. There are two possible situations. Case 1, τ^l does not belong in $\mathscr{T}_F$, which implies that all the corresponding imaginary roots are simple. Therefore $\mathrm{Inc}(\tau^l)$ can be obtained using (5.37) and (5.16). Case 2, τ^l belongs in $\mathscr{T}_F$. Then $\mathrm{Inc}(\tau^l)$ can be computed using (5.34).
Step 4. Compute $\mathrm{Inc}(\tau_i)$ for each $\tau_i > \tau^l$. One can in general use (5.33) together with (5.16). However in the case where τ_i is not in $\mathscr{T}_F$, (5.36) is simpler to apply for computing $\mathrm{Inc}(\tau_i)$.
Step 5. Now for any interval (τ_i, τ_{i+1}) we can arbitrarily pick a delay value τ' in it and compute $N^u(\tau')$ via (5.17), then it follows that for all τ in (τ_i, τ_{i+1}), the number of unstable roots is equal to $N^u(\tau')$.

5.6 Extension to Systems with Commensurate Delays

In this section we discuss the extension of the previous results to commensurate delay systems studied in Chapter 4. The characteristic equation of such type of systems can be written as

$$D(\lambda, \tau) = \sum_{k=0}^{M} P_i(\lambda, \tau) e^{-k\lambda\tau} = 0. \tag{5.44}$$

Also recall the following definition:

$$\hat{D}(\lambda, \tau, x) = \sum_{k=0}^{M} P_i(\lambda, \tau) x$$

Here we retain Assumption I-III as well as Assumption V in Chapter IV, while Assumption IV in Chapter **??** is not required. With Assumption IV discarded, we are able to deal with multiple imaginary characteristic roots.

Let the function $F(\omega, \tau)$, the frequency functions as well as the phase functions be defined as in Chapter IV. The rest of notations introduced in this chapter takes the same definition as for systems with a single delay except that now the frequency functions and the phase functions are now defined for the commensurate-delay systems.

The two-parameter perspective also applies to systems with commensurate delays. The equation (5.44) is just equivalent to

$$D(\lambda, r, q) = \sum_{k=0}^{M} P_i(\lambda, r) e^{-k\lambda q} = 0, \tag{5.45}$$

with the restriction

$$q = r = \tau. \tag{5.46}$$

Therefore, we can analysis the stability of system (5.45) in the r-q plane, and then impose the restriction $q = r = \tau$, just as what we did for the single-delay situation. The critical delay curve $\tau_k^{(i)}(r)$ associated with the frequency curve $\omega_k^{(i)}(r)$ can be defined in the same way as in (5.19), namely

$$\tau_k^{(i)}(r) = -\frac{\theta_k^{(i)}(r)}{\omega_k^{(i)}(r)} + r.$$

For some given index i, k, it is easy to see that $(j\omega_k^{(i)}(r), r, q)$ satisfies (5.44) if and only if the condition (5.21) is satisfied. Then the geometric idea described for single delay systems readily apply to commensurate delay systems. Therefore Lemma 5.1 can be shown to hold for the commensurate delay case using the same argument. For commensurate-delay systems, the formula 5.30 needs to be corrected as

$$\begin{aligned}\mathrm{Inc}(\omega^*, \tau^*) = &\sum_{k \in \mathscr{K}_+(\tau^*)} (-1)^{N_x(j\omega_k^{(i)}(\tau^{*+}), \tau^{*+})} \cdot \mathrm{sgn}_k^{(i)} \\ &- \sum_{k \in \mathscr{K}_-(\tau^*)} (-1)^{N_x(j\omega_k^{(i')}(\tau^{*-}), \tau^{*-})} \cdot \mathrm{sgn}_k^{(i')},\end{aligned} \tag{5.47}$$

where $N_x(j\omega_k^{(i)}(\tau^{*+}), \tau^{*+})$, denotes the number of roots of

$$\hat{D}(\omega_k^{(i)}(\tau), \tau, x) = 0$$

in x located outside $\mathbb{D}$ as $\tau \to \tau^{*-}$. and $N_x(j\omega_k^{(i)}(\tau^{*-}), \tau^{*-})$, denotes the number of roots of

$$\hat{D}(j\omega_k^{(i')}(\tau), \tau, x) = 0$$

in x located outside $\mathbb{D}$ as $\tau \to \tau^{*-}$.

The number $N_x(j\omega_k^{(i)}(\tau^{*+}), \tau^{*+})$ can be computed in the following way. Let $U \in \mathscr{I}^{(i)}$ be an interval that contains τ^* and satisfies 1)$U \cup \mathscr{T}_c = \{\tau^*\}$, 2)$U \cup \Phi_\tau = \phi$, where Φ_τ is defined in Assumption V of Chapter 4. Then one can pick any number $\tau' \in U$ and $\tau' > \tau^*$, then it is easy to see that

$$N_x(j\omega_k^{(i)}(\tau^{*+}),\tau^{*+}) = N_x(j\omega_k^{(i)}(\tau'),\tau').$$

To compute the number $N_x(j\omega_k^{(i')}(\tau^{*-}),\tau^{*-})$, let $V \in \mathscr{I}^{(i')}$ be an interval that contains τ^* and satisfied 1)$V \cup \mathscr{T}_c = \{\tau^*\}$, 2)$V \cup \Phi_\tau = \phi$. We pick any number $\tau' \in V$ and $\tau' < \tau^*$, then it follows that

$$N_x(j\omega_k^{(i')}(\tau^{*-}),\tau^{*-}) = N_x(j\omega_k^{(i')}(\tau'),\tau').$$

As statede in Proposition 4.6, for any fixed i, k, the number $N_x(j\omega_k^{(i)}(\tau),\tau)$ is a constant for τ in the interior of $\mathscr{I}^{(i)}$ if the set Φ_τ is empty.

The equation (5.47) can be simplified for simple imaginary characteristic roots, with which one can easily recover the root crossing direction criterion given in Chapter 4. The argument is very close to the one we have made for the single delay case and therefore omitted here. In particular, the equation (5.48) can be modified for systems with commensurate delays as

$$\begin{aligned}\mathrm{Inc}(\omega^*,\tau^*) = \quad & \frac{1}{2}(-1)^{N_x(j\omega^*,\tau^*)}\mathrm{sgn}_k^{(i)} \times \Big(\mathrm{sgn}\big(\theta_k^{(i)}(\tau_+^*) - \theta_k^{(i)}(\tau^*)\big) \\ & -\mathrm{sgn}\big(\theta_{k'}^{(i')}(\tau_-^*) - \theta_{k'}^{(i')}(\tau^*)\big)\Big).\end{aligned} \tag{5.48}$$

5.7 Numerical Examples

5.7.1 Stability of the Population Model

In the previous analysis of the population model (5.11) we have already decomposed $\mathscr{I}$ into $\mathscr{I}^{(i)}$, $i = 1,2,3$ and we know $\tau^{(0)} = 0$, $\tau^{(1)} \approx 6.160$, $\tau^{(2)} \approx 9.762$, and $\tau^{(3)} = \tau^u = 14$. We also have $\mathscr{T}_c = \{\tau_1,\tau_2\}$ and $\tau_1 \approx 0.878$, $\tau_2 = \tau^{(2)}$. It is easy to check $N^u(0) = 0$. We pick arbitrarily a number in $(\tau^{(1)},\tau_1)$ as τ_{1-}. and a number in $(\tau_1,\tau^{(2)})$ as τ_{1+}. From the lower diagram of Fig. 5.1 it is easy to see

$$\mathrm{sgn}(\theta_1^{(1)}(\tau_{1-}) - \theta_1^{(1)}(\tau_1)) = -1,\ \mathrm{sgn}(\theta_1^{(1)}(\tau_{1+}) - \theta_1^{(1)}(\tau_1)) = 1.$$

Simple computation shows $\mathrm{sgn}_1^{(1)} = 1$. It thus follows from (5.36) that

$$\mathrm{Inc}(\tau_1) = 2\mathrm{Inc}(j\omega_1^{(1)}(\tau_1^*),\tau_1) = 2.$$

In other words, the pair of characteristic roots $\pm j\omega_1^{(1)}(\tau_1)$ crosses the imaginary axis towards $\mathbb{C}_+$ as τ increases beyond τ_1. We can make the same conclusion about the crossing direction of this pair of imaginary root by invoking Proposition 5.4 and noticing $\frac{d}{d\tau}\theta_1^{(1)}(\tau_1) > 0$.

We now consider the critical delay τ_2. We have already shown that $\mathscr{K}_-(j\omega_2^{(2)}(\tau_2),\tau_2) = \{1\}$. It can be verified that $\mathrm{sgn}_1^{(2)} = 1$. Applying (5.33) we have

$$\begin{aligned}\mathrm{Inc}(\tau^*) &= \sum_{k\in\mathscr{K}_+(\tau^*)} 2\ \mathrm{sgn}_k^{(i)} - \sum_{k\in\mathscr{K}_-(\tau^*)} 2\ \mathrm{sgn}_k^{(i')} \\ &= -2\mathrm{sgn}_1^{(2)} \\ &= -2. \end{aligned} \tag{5.49}$$

We deduce $\mathrm{Inc}(\tau_2) = 2\mathrm{Inc}(j\omega_2^{(2)}(\tau_2),\tau_2) = -2$. Therefore the pair of roots $\pm j\omega_2^{(2)}(\tau_2)$ cross the imaginary axis towards $\mathbb{C}_-$ as τ increases and sweeps through τ_2. Consequently, we conclude that the system is asymptotically stable for $\tau \in [0,\tau_1)\cup(\tau_2,14]$ and has two unstable characteristic roots for $\tau \in (\tau_1,\tau_2)$.

5.8 Chapter Summary

A two-parameter approach is taken in this chapter. For systems with delay-dependent coefficients, we view the delay parameter in the system coefficient and the delay parameter in the state as two variables denoted as r and q, respectively, subject to the restriction $r = q = \tau$. We defined the critical delay curves, which separate the r-q parameter domain into disjoint stability regions, in each region the number of unstable roots are constant and stability switch can only occur on the boundary curves of these stability regions. The geometric point of view establishes a connection between the more classical problems where system coefficients are delay-independent and stability analysis of systems with delay-dependent coefficients. The two-parameter perspective provides geometric insight into the problem, which allows us to derive more general results concerning the root crossing directions, applicable to characteristic roots with multiplicity. It also establishes a connection between the more classical problems where system coefficients are delay-independent and stability analysis of systems with delay-dependent coefficients. Consequently, we can readily apply some tools originally developed for delay systems with fixed-coefficients to systems with delay-dependent coefficients. The analysis confirms our conjecture in Chapter 3 that the crossing direction of imaginary characteristic roots depends partially on the monotonicity of the phase angle functions, and differentiability of phase angle functions are not essential. The correlation between the number of unstable roots and the position of phase angle functions can be easily interpreted from the two-parameter point of view.

Chapter 6

Stability Analysis of Control Systems Subject to Delay-Based Feedback

6.1 Chapter Overview

In engineering practice, delay-difference is often used to approximate the derivatives of output signals for feedback control, leading to a closed-loop system with delay both in the states and in the system's coefficients. In this context, it is important find all the delay values contained in some interval that guarantee the exponential stability of the closed-loop system subject to a delay-based control law.

We first consider a feedback scheme that uses only the first order derivative of the output approximated by a finite-difference scheme. After specifying the control law based on such a finite-difference approximation, we derive the characteristic equation of the linearized closed-loop system. Then it is shown that by shifting the variable in the characteristic equation, the condition for exponential stability with decay rate α is equivalent to a condition for just asymptotic stability.

The stability analysis methods developed in previous chapters require the system coefficients to be continuous in τ. However, due to the fact that the delay appears as a denominator in the coefficients as a result of the finite-difference scheme, the coefficients are not bounded as τ approaches 0. Therefore we propose some convenient methods to compute a positive lower bound τ^l for our τ-sweeping test. The lower bound τ^l is selected in such a way that the number of unstable characteristic roots of the shifted closed loop system is known for $\tau \in (0, \tau^l]$.

Once the control design and stability analysis procedure becomes clear for feedback based on a single delay, we present a generalized approximation scheme for higher order derivatives of the output, which has been used in [32] for stabilizing a chain of integrator. The idea is to approximate the output history through polynomial interpolation and replace the actual derivatives of the output with the derivatives of the polynomial. We derive a bound on the approximation error, which allows us to compute the lower bound τ^l for our τ-sweeping test.

Finally the proposed design and analysis procedure is applied to several practical problems. The results shows that a control scheme based on approximated output derivatives may outperform that based on the exact output derivatives. This confirms the observation that delay can exhibit stabilizing effect in certain situations.

The materials of this chapter have been partially published in [35].

6.2 Control Scheme Based on a Finite Difference Scheme

There are several distinct control design philosophies. The constructive control design philosophy consists in transforming systems into certain forms that facilitate the design of control law or the construction of control Lyapunov functions. For various design techniques following this idea, readers may refer to [93]-[100] . Very often these design techniques require full-state feedback and extension to output feedback may require the design of observers. Another control design philosophy is first choose a control law with fixed structure but several control parameters to be tuned. For instance, in the series work [75]-[78] , the authors considered feedback laws with a proportional-integral-retarded output feedback structure. The corresponding control parameters are the feedback gains and the time delay. In this chapter, we will be concerned with the second design philosophy. A controller based on the derivatives of the output is assumed to exist in the first place. Such a controllers may be designed using the aforementioned methods based on state-feedback, since the derivatives of the output can be considered as state variables, as suggested by the Byrnes-Isidori canonical form [94]. Then we approximate the derivatives of the output using the delayed signal of the output and thus derive a controller with the time delay as a control parameter to be tunned for system stability.

6.2.1 Motivating Example

We consider a standard robot path following problem [52] with some simplification. As illustrated in Fig.6.1, a unicycle traveling at a constant speed V is required to follow a straight path. The robot is assumed to be non-holonomic, so the direction of its translational velocity is always along its heading direction. The control input u is the derivative of its yaw rate, which reflects the yaw moment applied to the robot. It is easy to see the linearized dynamics of the system is described by

$$\begin{cases} \dot{e} = V\theta \\ \dot{\theta} = \omega \\ \dot{\omega} = u \\ y = (\omega \;\; e)^T, \end{cases} \tag{6.1}$$

where e stands for the lateral tracking error, θ is the heading angle of the robot and ω is the yaw rate. Signal y is the output vector measured by the on-board sensors.

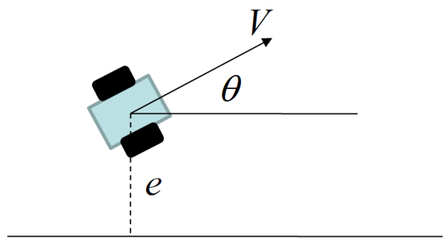

FIGURE 6.1: Illustration of the robot path-following problem

One can choose positive coefficients k_0, k_1, k_2 and a linear feedback

$$u = -k_0\omega - k_1 e - k_2 V\theta,$$

which stabilizes the system. In practice θ may not be convenient to measure, therefore we suppose it is not a component of the output. Noticing $\theta = V^{-1}\dot{e}$ and $\dot{e} \approx \frac{e(t)-e(t-\tau)}{\tau}$ for small τ, we choose instead the following control law utilizing the delayed output:

$$u = -(k_0 \;\; k_1)y - (0 \;\; k_2)\frac{y(t) - y(t-\tau)}{\tau}. \tag{6.2}$$

It can be shown that if system (6.1) can be stabilized by the following control law for some fixed number k_0, k_1, k_2:

$$u = -(k_0 \;\; k_1)y - (0 \;\; k_2)\dot{y}, \tag{6.3}$$

then it can also be stabilized by (6.2) for sufficiently small delay. However, since the delay appears as a denominator, it can not be too small. Otherwise, the noise contained in the measurement of y will be greatly amplified by the feedback and thus severely deteriorates the performance of the closed-loop system. On the other hand, an excessively large value of τ may cause slow convergence, strong oscillation, or even instability. Therefore it is practically important to find a set of delay values for (6.2) such that the closed loop system is exponentially stable with some pre-specified decay rate. Then one has the freedom to choose the delay value within this set that also optimizes other performance indexes.

6.2.2 A Finite Difference Scheme

We consider linear time-invariant systems of the form

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{cases} \tag{6.4}$$

where $x \in \mathbb{R}^n$ is the system state vector and $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}^m$ is the output vector of the system that is available for control feedback. Matrices A, B, C are of appropriate dimensions.

Suppose a control law of the following form is first designed for stabilization:

$$u(t) = K_0 y(t) + K_1 \dot{y}(t), \tag{6.5}$$

where K_0, K_1 are the gain matrices. We assume the signal $\dot{y}(t)$ is not fully available to the controller and hence approximated using the finite difference scheme:

$$\dot{y}(t) = \frac{y(t) - y(t-\tau)}{\tau}. \tag{6.6}$$

As a result, (6.5) becomes:

$$u(t) = K_0 y(t) + K_1 \frac{y(t) - y(t-\tau)}{\tau}. \tag{6.7}$$

For any time instant $t \geq 0$, we denote the trajectory of the closed-loop system consisted of (6.4)-(6.7) as x_t, which is a function defined as:

$$x_t(s) = x(t+s), \; s \in [-\tau, 0]. \tag{6.8}$$

We will be concerned with the convergence rate of x_t. For any positive number α, We say system (6.4)-(6.7) is $\alpha-$stable, if there exists some positive number c such that

$$||x_t|| < ce^{-\alpha t},$$

where $||x_t||$ denotes the supreme norm of the function x_t. Let τ^u be the maximal delay value under consideration, we will propose a method that can be used to find all the sub-intervals in $(0, \tau^u]$ such that the closed-loop system is α-stable if τ takes value in these sub-intervals.

We introduce two standing assumptions:
Assumption 1. In (6.5) K_1 satisfies $K_1CB = 0$.

Assumption 1 ensures that control signal $u(t)$ does not explicitly appear on the right hand side of (6.5), which will result in an algebraic loop. Consequently, the system (6.4)-(6.5) is guaranteed to be well-posed.

Assumption 2. None of the eigenvalues of the system (6.4)-(6.5) has a real component equal to $-\alpha$.

The state vector x in (6.4) does not necessarily contain only the states of the open-loop plant, it can also include the states of an augmented system introduced for achieving dynamic output-feedback. Accordingly, the output y may be a function of the states of both the original plant and the augmented part. Therefore, although the feedback in (6.5) takes the form of a PD control, a variety of feedback scheme can be converted into this form. For instance, for an SISO plant with the output $y \in \mathbb{R}$, the classical PID controller can be constructed by first introducing an extra state σ satisfying

$$\dot{\sigma}(t) = y(t),$$

where $y(t)$ is the measured output of the plant. First define $u = K_I\sigma + v$ and then v becomes the new control input, which is defined according to the PD scheme as: $v = K_P y + K_D \dot{y}$. We have thus constructed a PID control law.

As noted in [32], when the open-loop system possesses more than a pair of imaginary roots, then it is necessary to introduce multiple delays in order to stabilize. The control feedback based on commensurate delays will be addressed in Section 6.5.

6.3 Stability Analysis of the Delay-difference Scheme

6.3.1 Characteristic Equation and Stability

Let $G(s) = 0$ be the characteristic equation of the open-loop system (6.4) with $u \equiv 0$, then $G(s)$ is a polynomial in s of degree n. There exist polynomials G_{u1}, G_{u2} such that the characteristic equation of the control system consisting of (6.4)-(6.5) takes the form

$$G_c(s) = G(s) + G_{u1}(s) + G_{u2}(s)s = 0, \tag{6.9}$$

where $G_{u1}(s)$ and $G_{u2}(s)s$ are generated by the term $K_0 y$ and $K_1 \dot{y}$ in the control feedback (6.5), respectively. An implication of Assumption 1 is that the degree of G_{u2} must be smaller than $n-1$. After the delayed feedback (6.7) is applied, the corresponding characteristic equation of the closed loop system becomes

$$G(s)+G_{u1}(s)+G_{u2}(s)\frac{1-e^{-s\tau}}{\tau}=0. \tag{6.10}$$

It is proved in [8] that the closed loop system (6.4)-(6.7) is α-stable if and only if each root of (6.10) in s has a real part smaller than $-\alpha$. We perform a shift of variable $s \mapsto \lambda-\alpha$ in (6.10) and obtain

$$P_\alpha(\lambda,\tau)+Q_\alpha(\lambda,\tau)e^{-\lambda\tau}=0, \tag{6.11}$$

where

$$\begin{aligned} P_\alpha(\lambda,\tau) &= G(\lambda-\alpha)+G_{u1}(\lambda-\alpha)+G_{u2}(\lambda-\alpha)\tau^{-1}, \\ Q_\alpha(\lambda,\tau) &= -\tau^{-1}e^{\alpha\tau}G_{u2}(\lambda-\alpha). \end{aligned}$$

It is easy to see that α-stability of the closed loop system with control law (6.7) is equivalent to the asymptotic stability of the characteristic equation (6.11).

Our objective is equivalent to computing all the subintervals contained in $(0,\tau^u]$ that guarantee that all the the roots of (6.11) in λ are located on $\mathbb{C}_-$ if τ belongs in these subintervals. We use the quantity $N^u(\tau)$ to denote the number of characteristic roots of (6.11) on the right half complex plane if $\tau>0$. If $\tau=0$, it is understood as the number of characteristic roots of (6.9) with real parts larger than $-\alpha$.

6.3.2 The Generalized τ-decomposition Approach

Here we briefly review the generalized τ-decomposition approach we have taken in the previous chapters. One first starting with one value of delay τ^l for which one knowns the number of characteristic roots on $\mathbb{C}+$, and sweeps τ through the delay interval of interest $\mathscr{I}=[\tau^l,\tau^u]$ of delays, and identify all critical delays τ_l, $l=1,2,\ldots,L$. Thus, one may divide $\mathscr{I}$ into subintervals (τ_{l-1},τ_l), and the number of right-half plane roots within each subinterval is constant. Next, for each critical delay τ_l, the characteristic roots on the imaginary axis are located and their crossing directions are determined using the root crossing criteria we developed in previous chapters. Subsequently, the number of unstable characteristic roots can be determined for each subinterval (τ_{l-1},τ_l).

The generalized $\tau-$decomposition approach requires the system coefficients to be continuous on the delay interval $\mathscr{I}$. However, for the system considered in this paper, the coefficients are not bounded on $\mathscr{I}$ as τ appears in the denominator. For this reason we can not simply set $\tau^l = 0$. It is pointed out in [32] that there exists some positive numbers τ' such that the characteristic equation has no imaginary characteristic root for $\tau \in (0, \tau')$ and $N^u(\tau') = N^u(0)$. Then τ' can be viewed as an lower bounded of the delay interval for the τ-sweeping test. Therefore we can simply set $\tau^l = \tau'$. The computation of such a number will be addressed next.

6.4 A Lower Bound for the Delay Interval

We can rewrite (6.11) as

$$D_\alpha(\lambda,\tau) = G_{c\alpha}(\lambda) + Q(\lambda)R(\lambda,\tau) = 0, \tag{6.12}$$

where

$$\begin{aligned} G_{c\alpha}(\lambda) &= G_c(\lambda-\alpha),\ Q(\lambda) = G_{u2}(\lambda-\alpha), \\ R(\lambda,\tau) &= \frac{1-e^{(-\lambda+\alpha)\tau}}{\tau} - \lambda + \alpha. \end{aligned} \tag{6.13}$$

and the definition of the function G_c is given in (6.9).

In the ensuing analysis, we will view $G_{c\alpha}(s)$ as the 'nominal' part of the characteristic equation and the term $Q(s)R(s,\tau)$ as the perturbation. It is known in [32] that for any real c, all the roots of (6.10) with real part larger than c converge to the roots of (6.5) with real parts also larger that c. As consequence, we are able to show that for sufficiently small τ, $N^u(\tau) = N^u(0)$.

Lemma 6.1. *For any $\omega \in \mathbb{R}$, the term $R(j\omega,\tau)$ defined in (6.13) can be bounded as*

$$|R(j\omega,\tau)| \le \tilde{R}(\omega,\tau),\ \forall \tau \ge 0,\ i = 1,2, \tag{6.14}$$

where

$$\tilde{R}(\omega,\tau) = \frac{1}{2}(\omega^2+\alpha^2)\tau e^{\alpha\tau} \tag{6.15}$$

Proof. Denote

$$R_0(\lambda,\tau) = 1 - e^{(-\lambda+\alpha)\tau} - (\lambda-\alpha)\tau,$$

then we have $R(\lambda,\tau)=\tau^{-1}R_0(\lambda,\tau)$ as well as

$$\begin{aligned}\partial_\tau R_0(\lambda,\tau) &= (\lambda-\alpha)(e^{-(\lambda-\alpha)\tau}-1), &(6.16)\\ \partial_\tau^2 R_0(\lambda,\tau) &= -(\lambda-\alpha)^2 e^{-(\lambda-\alpha)\tau}, &(6.17)\end{aligned}$$

and thus

$$\begin{aligned}R_0(\lambda,\tau) &= \int_0^\tau \partial_\tau R_0(\lambda,\mu)d\mu\\ &= \int_0^\tau \big(\partial_\tau R_0(\lambda,0)+\int_0^{\mu_0}\partial_\tau^2 R_0(\lambda,\mu_1)d\mu_1\big)d\mu_0\\ &= \tau\partial_\tau R_0(\lambda,0)+\int_0^\tau\int_0^{\mu_0}\partial_\tau^2 R_0(\lambda,\mu_1)d\mu_1 d\mu_0\\ &= \int_0^\tau(\tau-\mu)\partial_\tau^2 R_0(\lambda,\mu)d\mu. &(6.18)\end{aligned}$$

In deriving (6.18) we used $\partial_\tau R(\lambda,0)=0$ which follows from (6.16). The expression (6.18) combined with (6.17) yields

$$\begin{aligned}|R(j\omega,\tau)| &< \max_{\mu\in[0,\tau]}|\partial_\tau^2 R_0(j\omega,\mu)|\tfrac{1}{2}\tau\\ &= \max_{\mu\in[0,\tau]}|(j\omega-\alpha)^2 e^{-(j\omega-\alpha)\mu}|\tfrac{1}{2}\tau\\ &= \tfrac{1}{2}(\omega^2+\alpha^2)e^{\alpha\tau}\tau=\tilde{R}(\omega,\tau), &(6.19)\end{aligned}$$

for $\tau>0$. The lemma is thus proved. □

Proposition 6.1. *Let $\hat{\tau}$ be the positive number that satisfies*

$$\hat{\tau}e^{\alpha\hat{\tau}}=\hat{z}, \tag{6.20}$$

where

$$\begin{aligned}\hat{z} &= \inf_{\substack{\omega\geq 0\\ Q(j\omega)(\omega^2+\alpha^2)\neq 0}} 2J(\omega), &(6.21)\\ J(\omega) &= \frac{|G_{c\alpha}(j\omega)|}{|Q(j\omega)|(\omega^2+\alpha^2)}. &(6.22)\end{aligned}$$

Given Assumption 2 and $0<\tau^l<\hat{\tau}$, the characteristic equation (6.11) admits no imaginary characteristic root for $\tau\in(0,\tau^l]$.

Proof. We first notice that the function $\tau\exp(\alpha\tau)$ is monotonic. Then the equations (6.15), (6.20) as well as (6.21) together indicate that for all $(\omega,\tau)\in\mathbb{R}\times[0,\hat{\tau})$ we must have

$$\begin{aligned}|G_{c\alpha}(j\omega)| &\geq \frac{1}{2}|Q(j\omega)|(\omega^2+\alpha^2)\hat{z}\\ &= |Q(j\omega)\tilde{R}(\omega,\hat{\tau})|\\ &> |Q(j\omega)R(j\omega,\tau)|.\end{aligned}$$

Therefore

$$G_{c\alpha}(j\omega)+Q(j\omega)R(j\omega,\tau)=0$$

can not hold for any $(\omega,\tau)\in\mathbb{R}\times(0,\hat{\tau})$, then the proposition follows. □

We now show that $\hat{\tau}$ can be computed by solving a pair of polynomial equations. To this end, define $E_0:\mathbb{R}\times\mathbb{R}_+^*\mapsto\mathbb{R}$ and $E:\mathbb{R}_+^*\times\mathbb{R}_+^*\mapsto\mathbb{R}$ as:

$$E_0(\omega,z)=|G_{c\alpha}(j\omega)|^2-z|Q(j\omega)(\omega^2+\alpha^2)|^2, \tag{6.23}$$

$$E(w,z)=E_0(\sqrt{w},z). \tag{6.24}$$

It is easy to see that for any fixed z, $E_0(\omega,z)$ must be a polynomial in ω^2. Therefore the function $E(w,z)$ is a polynomial in w.

Proposition 6.2. *Let S_z be the set of z values appearing in the pairs $(w,z)\in\mathbb{R}\times\mathbb{R}_+$ that satisfy the following polynomial equations:*

$$\begin{cases}E(w,z)=0,\\ \partial_w E(w,z)=0.\end{cases} \tag{6.25}$$

If S_z is empty, then $\hat{\tau}$ defined in (6.20) equals $+\infty$. Otherwise the following holds:

$$\hat{\tau}e^{\alpha\hat{\tau}}=2\sqrt{\min S_z}. \tag{6.26}$$

Proof. If S_z is not empty, define $z^*=\min S_z$, otherwise $z^*=+\infty$. We must have $z^*>0$ because the equation $E_0(\omega,0)=0$ does not admit any real solution due to Assumption 2. It then follows from Proposition 3.1 that the number of real solution of equation $E_0(\omega,z)=0$ in ω remains 0 for $z\in[0,z^*)$. We can thus deduce that $E_0(\omega,z)>0, \forall(\omega,z)\in\mathbb{R}\times[0,z^*)$ and $E_0(\omega^*,z^*)=0$ for some real ω^*, which implies $4z^*=\hat{z}^2$, where $\hat{z}$ is defined in (6.21). The proof is completed after comparing (6.20) with (6.26). □

The last proposition indicates that we can obtain $\hat{\tau}$ by solving first (6.25) and then (6.26). Since the former one is a set of polynomial equations in w and z, there exist algorithms to find the numerical values of its roots efficiently. One can use standard Mathematical software, such as Matlab and Mathematica to solve this equation. The left hand side of (6.26) is a monotonic function, therefore once $\min S_z$ is known, $\hat{\tau}$ can be easily computed.

We now propose a second method for finding a valid value of τ^l, which does not require the knowledge of $\hat{\tau}$. One can pick a small positive number z_1 that satisfies

$$z_1 < \lim_{\omega\to+\infty} \frac{|G_{c\alpha}(j\omega)|^2}{|Q(j\omega)|^2\omega^4},$$

The right hand side of the last inequality cannot be 0 since the degree of $Q(j\omega)\omega$ must be smaller than the degree of $G_{c\alpha}$ due to our assumption on the relative degree. Consider the following polynomial equation in w:

$$E(w,z_1) = 0. \tag{6.27}$$

If (6.27) admits no positive solution in w, it is clear that

$$|G_{c\alpha}(j\omega)|^2 > z_1|Q(j\omega)|(\omega^2+\alpha^2)^2,\ \forall\omega\in\mathbb{R}.$$

Therefore we must have $4z_1 < \hat{z}^2$ and can conclude that it suffices for τ^l to satisfy

$$\tau^l \exp(\alpha\tau^l) \leq 2\sqrt{z_1}. \tag{6.28}$$

Equation (6.27) has only one variable and therefore is simpler to solve than (6.25). However, to use the second method we need a sufficiently small initial guess of z_1. If (6.27) admits any non-negative real solution in w for some given z_1, we have to repeatedly reduce the value of z_1 until this equation admits no solution in $w \geq 0$.

6.5 Commensurate-Delay Feedback

In general, it may become necessary to use high order derivatives of the output to construct the feedback. Therefore, it is reasonable to consider high order derivatives approximated using multiple delays. Multiple delays can also be useful for approximating just the first order derivative using linear regression for its capability of ameliorating the noise in the measurement.

For notational simplicity, we assume $y \in \mathbb{R}$. The control law using high order derivatives of the output can be written as

$$u(t) = \sum_{l=0}^{m} k_l (\tfrac{d}{dt})^l y(t), \tag{6.29}$$

where m is smaller than the relative degree of the output y with respect to the input u. This condition ensures that each $(\frac{d}{dt})^l y(t)$ can be regarded as a state variable [94], as a result the last control law does not induce an algebraic loop.

To generalize the approximation scheme 6.6 for higher order time derivatives of the output, we follow the idea proposed in [32]. For any given initial time t_0, we use a polynomial function $y_p(\cdot)$ to approximate the time history of the output $y(t)$, $t \in [t_0 - m\tau, t_0]$, which interpolates the function $y(\cdot)$ at m past instants: $t, t-\tau, t-2\tau, \cdots, t-m\tau$, i.e.,

$$y_p\left(t_0 - l\frac{\tau}{N}\right) = y\left(t_0 - l\frac{\tau}{N}\right),\ l = 1,2,\cdots,m. \tag{6.30}$$

The polynomial $y_p(t)$ can be expressed as

$$y_p(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_m(t-t_0)^m$$

where the coefficients satisfy

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & (-\tau)^{-1} & & \\ & & \ddots & \\ & & & (-\tau)^{-m} \end{pmatrix} T_m^{-1} \begin{pmatrix} y(t_0) \\ y(t_0-\tau) \\ \vdots \\ y(t_0-m\tau). \end{pmatrix} \tag{6.31}$$

and

$$T_m = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & m & m^2 & \dots & m^m \end{pmatrix}$$

The derivatives of $y(t)$ at t_0 can now be approximated as

$$(\tfrac{d}{dt})^l y(t_0) \approx (\tfrac{d}{dt})^l y_p(t_0) = l! c_l, \quad 1 \le l \le m. \tag{6.32}$$

Since $t_0 \geq 0$ is arbitrary, a combination of (6.31) and (6.32) shows that (6.29) can be approximated with the following control law using commensurate delays [32]:

$$u_\tau(t) = \begin{pmatrix} k_0 & k_1\frac{1}{-\tau} & k_2\frac{2!}{(-\tau)^2} & \cdots & k_m\frac{m!}{(-\tau)^m} \end{pmatrix} T_m^{-1} \begin{pmatrix} y(t) \\ y(t-\tau) \\ \vdots \\ y(t-m\tau) \end{pmatrix}. \tag{6.33}$$

Assuming a stable internal dynamics, to analyze the closed loop system subject to the control law (6.29), one only need to consider a characteristic equation of the following form:

$$G_c(s) = G(s) + G_u(s)\sum_{i=0}^{m} k_i s^m = 0, \tag{6.34}$$

where $G(s)$, $G_u(s)$ are polynomial equations. Then to analyze the closed-loop system subject to the delayed feedback control (6.33), we only need to consider the following characteristic equation:

$$G_{c\tau} = 0, \tag{6.35}$$

where

$$G_{c\tau}(s) = G_c(s) + G_u(s) \begin{pmatrix} k_0 & k_1\frac{1}{-\tau} & k_2\frac{2!}{(-\tau)^2} & \cdots & k_m\frac{m!}{(-\tau)^m} \end{pmatrix} T_m^{-1} \begin{pmatrix} 1 \\ e^{-s\tau} \\ \vdots \\ e^{-ms\tau} \end{pmatrix}. \tag{6.36}$$

To α-stability analysis, one may further perform a variable shift: $s \mapsto \lambda - \alpha$ in the last equation, which will lead to a characteristic equation with commensurate delays:

$$\sum_{i=0}^{m} P_i(\lambda,\tau)e^{-i\lambda\tau} = 0. \tag{6.37}$$

The detailed expression of each $P_i(\lambda,\tau)$ can be easily derived and is omitted here. The interpolation-based approximation method can be view as a generalization of the delay-difference scheme for approximating the first order derivative. Similarly, we arrive at a closed loop system with the delay parameter also appearing in the denominators of system coefficients. It is worth mentioning that when the open loop system is a chain of integrator, it is possible to use a rescaling technique to transform the closed loop system such that the delay no longer exists in system coefficients [32]. This technique also played a crucial role in the study of the so-called peaking phenomena [31], [93]. However, it may not be applicable to systems with more general structures.

We next propose a method to compute a lower bound τ^l for the τ-sweeping test. It is required to satisfy $N^u(\tau) = N^u(0)$, for $\tau \in (0, \tau^l]$, and $N^u(0)$ is the number of α-unstable roots of the characteristic equation (6.34). When α-stability is concerned, we need to assume no characteristic root of (6.34) have a real component equal to $-\alpha$.

Denoting

$$\hat{R}(s,\tau) = \frac{1}{\tau}\left(e^{-\tau\lambda} - \sum_{i=0}^{m} \frac{(-\tau s)^i}{i!}\right),$$

the characteristic equation (6.36) can be written as

$$\begin{aligned} G_{c\tau}(s) \quad &= \quad G(s) + G_u(s)\sum_{l=i}^{m} k_l s^l \\ &\quad - G_u(s) T_m^{-1} \begin{pmatrix} k_1 & k_2 \frac{2!}{-\tau} & \cdots & k_m \frac{m!}{(-\tau)^{m-1}} \end{pmatrix} \begin{pmatrix} \hat{R}(s,\tau) \\ \vdots \\ \hat{R}(s,m\tau) \end{pmatrix}. \end{aligned} \tag{6.38}$$

Since the α-stability is concerned, we perform the variable shift in (6.38): $\lambda = s + \alpha$, and obtain the shifted characteristic equation of the closed-loop system subject to the feedback using commensurate delays:

$$D_\alpha(\lambda,\tau) = G_{c\alpha}(\lambda) + Q(\lambda)\begin{pmatrix} R(\lambda,\tau) \\ \vdots \\ R(\lambda,m\tau) \end{pmatrix}, \tag{6.39}$$

where $G_{c\alpha}(\lambda) = G_c(\lambda-\alpha)$, $R(\lambda,\tau) = \hat{R}(\lambda-\alpha,\tau)$ and

$$Q(\lambda) = -G_u(\lambda-\alpha)T_m^{-1}\begin{pmatrix} k_1 & k_2 \frac{2!}{-\tau} & \cdots & k_m \frac{m!}{(-\tau)^{m-1}} \end{pmatrix}. \tag{6.40}$$

We suppress the explicit dependence of $Q(\lambda)$ and $R_l(\lambda,\tau)$ on α to simplify the notation. To determine an upper bound τ^l such that the stability of $D_\alpha(\lambda,\tau)$ is identical to $G_{c\alpha}(\lambda)$ for $\tau \in (0,\tau^l)$, we first establish a bound for each the $R(\lambda,\tau)$ resulted from the approximation error of the approximation scheme.

It follows from the Taylor theorem that

$$R(\lambda,\tau) = \frac{1}{\tau}\int_0^\tau -\frac{d^{m+1}e^{(\alpha-\lambda)\mu_1}}{d\mu_1^{m+1}}(\tau-\mu_1)^m d\mu_1.$$

Therefore, for any real ω and $\tau > 0$

$$\begin{aligned}|R(j\omega,\tau)| &= \left|\frac{1}{\tau}\int_0^{\tau} -\frac{d^{m+1}e^{(\alpha-j\omega)\mu_1}}{d\mu_1^{m+1}}(\tau-\mu_1)^m d\mu_1\right| \\ &< \frac{1}{\tau}\int_0^{\tau} |j\omega-\alpha|^{m+1}e^{\alpha\mu_1}(\tau-\mu_1)^m d\mu_1 \\ &= \frac{1}{(m+1)!}|j\omega-\alpha|^{m+1}\tau^m e^{\tau\alpha}, \end{aligned} \tag{6.41}$$

Using the inequality (6.41), we obtain

$$\begin{aligned}\left|\sum_{l=1}^{m} k_l \frac{l!}{(\tau)^{l-1}} R(j\omega,l\tau)\right|^2 &\le \sum_{i,h=1}^{m} |R(j\omega,i\tau)||R(j\omega,h\tau)|k_l k_h \frac{l!h!}{\tau^{l+h-2}} \\ &\le m\sum_{l=1}^{m} |R(j\omega,l\tau)|^2 k_l^2 \frac{l!^2}{\tau^{2l-2}} \\ &= \frac{m(\omega^2+\alpha^2)^{m+1}}{(m+1)!^2}\sum_{l=1}^{m} l!^2 k_l^2 e^{2l\tau}\tau^{2(m+1-l)}. \end{aligned}$$

Now it is clear that we can construct some function $\tilde{Q}(\omega,\tau)$ non-negative for $(\omega,\tau)\in\mathbb{R}\times\mathbb{R}_+$, which is polynomial in ω and satisfies

$$\left|G_u(j\omega-\alpha)T_m^{-1}\sum_{l=1}^{m} k_l \frac{l!}{\tau^{l-1}} R(j\omega,l\tau)\right| \le |\tilde{Q}(\omega,\tau)|,$$

Moreover, $|\tilde{Q}(\omega,\tau)|$ can be made monotonically increasing in $\tau\in[0,+\infty]$ for any given real ω. In particular, we can simply choose

$$\tilde{Q}(\omega,\tau) = \left|G_u(j\omega-\alpha)T_m^{-1}\right|\sqrt{\frac{m(\omega^2+\alpha^2)^{m+1}}{(m+1)!^2}\sum_{l=1}^{m} l!^2 k_l^2 e^{2l\tau}\tau^{2(m+1-l)}}. \tag{6.42}$$

Now define

$$E_0(\omega,\tau) = |G_{c\alpha}(j\omega)|^2 - |\tilde{Q}(\omega,\tau)|^2$$

it is clear that $E(\omega,\tau)$ is a polynomial of ω^2 for fixed τ and is decreasing as $\tau\ge 0$ increases. Define

$$E(w,\tau) = E_0(\sqrt{w},\tau),$$

the following holds:

Proposition 6.3. *Let τ' be any fixed positive number such that $\lim_{w\to+\infty}E(w,\tau')\to+\infty$, and the equation*

$$E(w,\tau') = 0 \tag{6.43}$$

does not admit non-negative real roots in w, then $D_\alpha(\lambda,\tau)$ does not admit any imaginary root in λ. Equivalently, all the roots of the characteristic function $G_{c\tau}(s)$ given in (6.36) corresponding to the closed-loop system are located to the left of the vertical line $\Re(s) = -\alpha$.

Proof. Since for any real τ', $\lim_{w\to+\infty} E(w,\tau') \to +\infty$, the equation (6.43) must admit some non-negative root in w if $|G_{c\alpha}(j\omega)|^2 - |\tilde{Q}(j\omega,\tau)|^2 \le 0$ for some real ω. Therefore, the condition in the proposition ensures that

$$|G_{c\alpha}(j\omega)| > |\tilde{Q}(\omega,\tau')|, \forall\omega \in \mathbb{R}.$$

Since for any fixed real ω, $|\tilde{Q}(j\omega,\tau')|$ is increasing with respect to τ, the last inequality further leads to

$$|G_{c\alpha}(j\omega)| > |\tilde{Q}(\omega,\tau)|,\ \forall(\omega,\tau) \in \mathbb{R}\times(0,\tau').$$

Consequently, for all $(\omega,\tau) \in \mathbb{R}\times(0,\tau')$ the following holds:

$$\begin{aligned}|G_{c\alpha}(j\omega)| &> |\tilde{Q}(\omega,\tau)| \\ &\ge \left| G_u(j\omega-\alpha)T_m^{-1}\sum_{l=1}^{m} k_l \frac{l!}{(-\tau)^{l-1}} R(j\omega,l\tau)\right|,\end{aligned}$$

which implies that the shifted characteristic equation $D_\alpha(\lambda,\tau) = 0$ in (6.39) can not have any imaginary root for $\tau \in (0,\tau')$ and thus the proposition. □

Once the number τ' in the last proposition is found, one can simply set $\tau^l = \tau'$. The number τ' can be found via line search: one starts with an initial guess of τ' and check if (6.43) has any non-negative real root in w. If any such root exits, it means the guessed value of τ' is too large, then a smaller value of τ' is chosen and whole procedure is repeated until no root of (6.43) in w can be found in $[0,+\infty)$.

6.6 Numerical Examples

Example I. We first apply the proposed stability analysis method to the unicycle model (6.1). We set the maximal delay value under consideration as $\tau^u = 0.5$.

We start with designing the control law of the form (6.5). Various tools for controlling LTI systems can be employed. For instance, one can use LQR or LMI based techniques to optimally determine the control parameters. Since the specific method for determining control parameters is irrelevant to our stability analysis procedure, we simply set the eigenvalues of the closed loop system as

$$s_1 = -2,\ s_2 = -1.5 + 4j,\ s_3 = -1.5 - 4j.$$

Accordingly we obtain the following control parameters:

$$k_0 = 5,\ k_1 = 97/4,\ k_2 = 73/2.$$

and the characteristic equation of the closed loop system is:

$$s^3 + k_0 s^2 + k_1 s + k_2 = 0. \tag{6.44}$$

From the real part of the eigenvalues we deduce that the decay rate of the control system using the exact output derivative is 1.5. Suppose we require that when the delay-difference approximation is used for feed-back control, the decay rate is no less than 1, we therefore set $\alpha = 1$.

We derive the explicit expression of (6.10) as:

$$s^3 + a_0 s^2 + a_1 s + a_2(\tau) + a_3(\tau) e^{-s\tau} = 0, \tag{6.45}$$

where $a_0 = k_0 - 3,\ a_1 = 3 - 2k_0,\ a_2 = k_0 + k_2 + k_1\tau^{-1} - 1$ and $a_3 = -e^{\tau}k_1/\tau$. We first determine the value of τ^l as the lower bound for the τ-sweeping test. We have

$$G_{c\alpha}(\lambda) = (\lambda - 1)^3 + k_0(\lambda - 1)^2 + k_1(\lambda - 1) + k_2, \tag{6.46}$$

Using the expression above, the function $E(w,z)$ can be derived as

$$\begin{aligned} 0 &= w^3 + w^2\big(4k_0 - 2k_1 + (k_0 - 3)^2 - k_1^2 c_1^2 - 6\big) + \\ &\quad \omega^2\big((6 - 2k_0)(k_0 - k_1 + k_2 - 1) + (k_1 - 2k_0 + 3)^2 - \\ &\quad 2k_1^2 c_1^2\big) + (k_0 - k_1 + k_2 - 1)^2 - k_1^2 z \end{aligned}$$

We solve the equation (6.27) with $z_1 = 0.016$ and find no solution in $w \in [0, +\infty)$. We check that (6.28) holds with $\tau^l = 0.07$ and $z_1 = 0.016$. Therefore, we set $\tau^l = 0.07$ and analyze the stability of (6.11) with the delay interval $\mathscr{I} = [0.07, 0.5]$.

We follow the procedure outlined in Section 3.4.1. Computation shows that

$$\begin{cases} F(\omega,\tau) &= \omega^6 + (a_0^2 - 2a_1)\omega^4 + (a_1^2 - 2a_0 a_2)\omega^2 \\ &\quad + a_2^2 - a_3^2, \\ \partial_\omega F(\omega,\tau) &= 6\omega^5 + 4(a_0^2 - 2a_1)\omega^3 + 2(a_1^2 - 2a_0 a_2)\omega. \end{cases} \tag{6.47}$$

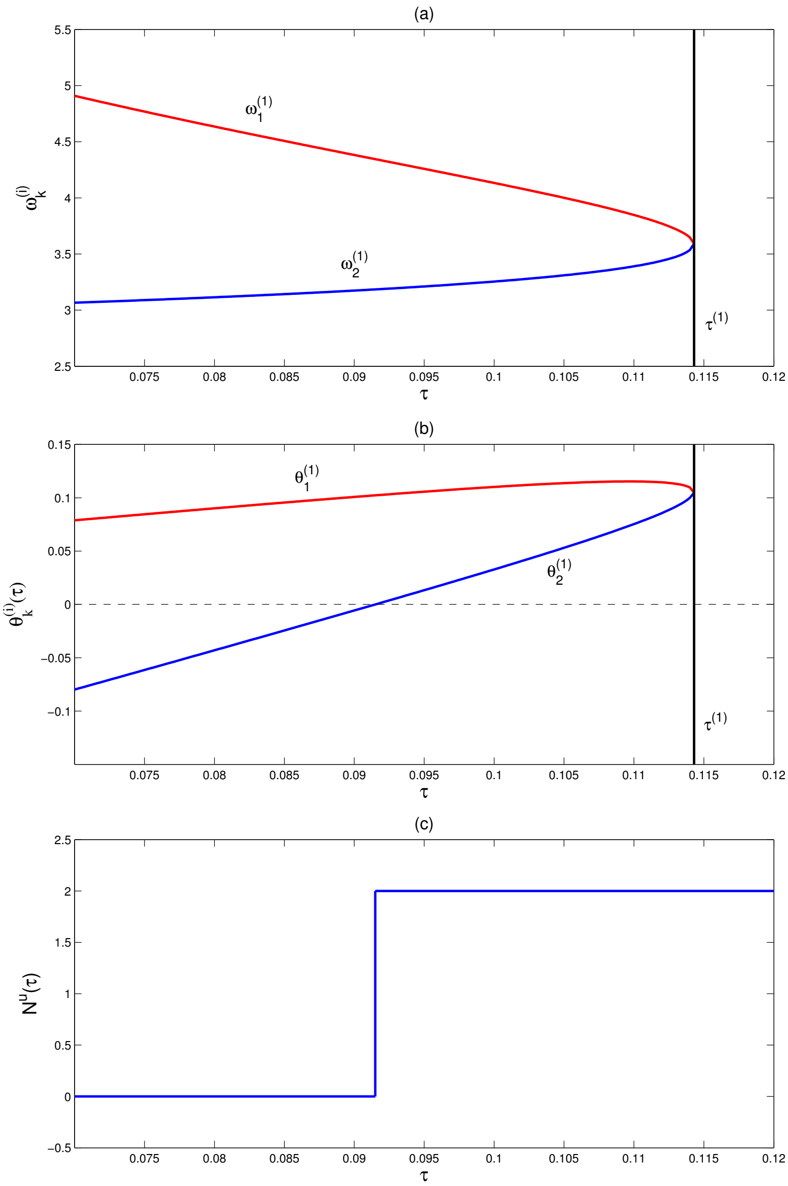

FIGURE 6.2: Stability analysis of the mobile robot. (a)The graphs of frequency functions. (b)The graphs of phase angle functions. The graph of $\theta_2^{(1)}(\tau)$ crosses the horizontal line 0 at τ_1, giving one critical delay within $\mathscr{I}$. (c)The number of α-unstable roots.

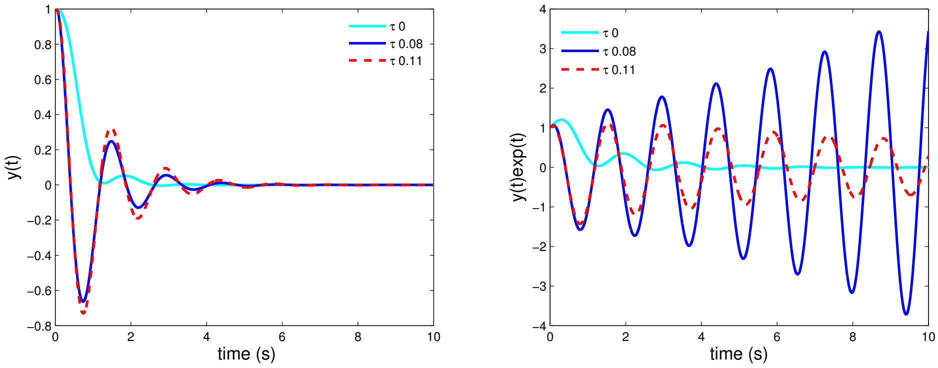

FIGURE 6.3: Simulation results of the mobile robot. (a)The time response of the lateral tracking error. (b)The lateral tracking error weighted by the function e^t. The convergence of the weighted tracking error indicates α-stability for $\alpha = 1$.

Solving the previous two equations together using MATLAB for τ with the constraint $\omega \in \mathbb{R}$ and $\tau \in \mathscr{I}$, we find only one solution $\tau^{(1)} \approx 0.1143$. The set $\mathscr{I}$ is thus decomposed into two intervals:

$$\mathscr{I} = \mathscr{I}^{(1)} \cup \mathscr{I}^{(2)},$$

where $\mathscr{I}^{(1)} = [\tau^l, \tau^{(1)}]$ and $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^u]$. It can be verified that $m(1) = 2$ and $m(2) = 0$. The values of $\omega_1^{(1)}(\tau)$ and $\omega_2^{(1)}(\tau)$ for various $\tau \in \mathscr{I}^{(1)}$ are computed numerically based on (6.47) and the graph of these two functions are depicted in Fig.6.2 a. The corresponding phase functions $\theta_1^{(1)}(\tau)$ and $\theta_2^{(1)}(\tau)$ are plotted in Fig.6.2 b.

We observe that the only intersection between the phase curves corresponding to $\theta_i^{(1)}(\tau)$, $i = 1,2$ and the horizontal lines whose value equals $2l\pi$ for some integer l takes place only at $\tau_1 \approx 0.0915$ when $l = 0$. Therefore two imaginary roots $\pm j\omega_{11}$ appear when $\tau = \tau_1$. By definition we have:

$$\omega_{11} = \omega_2^{(1)}(\tau_1) \approx 3.1846.$$

We can verify

$$\partial_\omega F(\omega_2^{(1)}(\tau_1), \tau_1) > 0, \ \frac{d\theta_2^{(1)}(\tau_1)}{d\tau} > 0.$$

and the derivative of $\theta_2^{(1)}(\tau_1)$ is positive, which can also be observed from Fig.6.2 b. Therefore, it can be concluded that the pair of imaginary roots $\pm j\omega_{11}$ cross the imaginary axis toward right half complex plane as τ goes through τ_1 increasingly. No root of (6.11) crosses

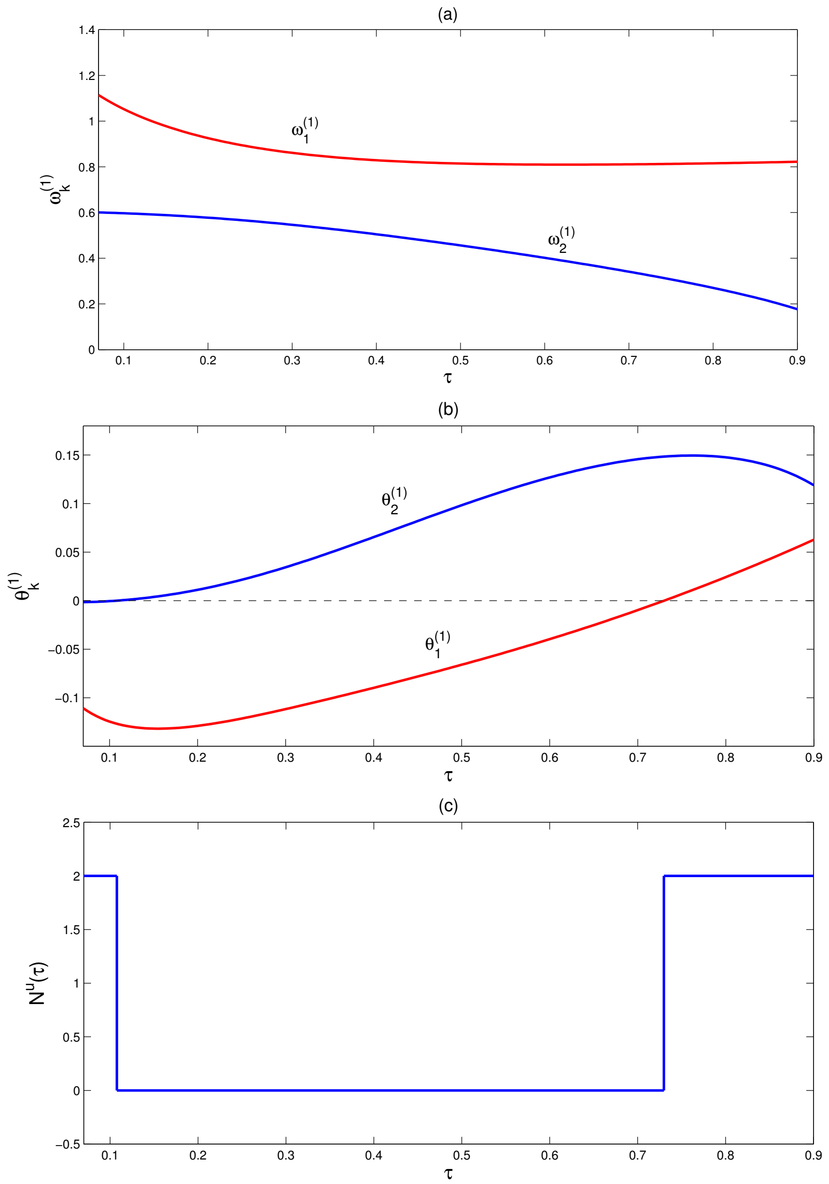

FIGURE 6.4: Stability analysis of the aircraft example. (a)The graphs of frequency functions. (b)The graphs of phase angle functions. The graph of $\theta_2^{(1)}(\tau)$ and $\theta_1^{(1)}(\tau)$ crosses the horizontal line 0 at τ_1 and τ_2, respectively, giving two critical delays within $\mathscr{I}$. (c)The number of α-unstable roots.

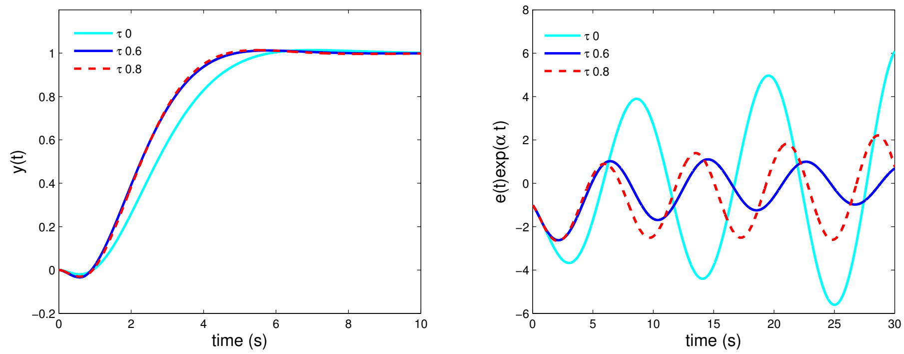

FIGURE 6.5: Simulation results of the aircraft example corresponding to a step input.

Simulation results of the aircraft example corresponding to a step input. Left: the time response of the output $y(t)$ with respect to various delay values. Right: The output signal tracking error weighted by the function $e^{\alpha t}$ with $\alpha = 0.73$. The convergence of the weighted tracking error indicates α-stability.

the imaginary roots in Interval $\mathscr{I}^{(2)}$ as $m(2) = 0$. Clearly we have $N^u(\tau) = 0$ for $\tau \in (0, \tau_1)$ and $N^u(\tau) = 2$ for $\tau \in (\tau_1, \tau^u]$. Therefore we claim that for $\alpha = 1$ the closed-loop system is α-stable for $\tau \in (0, \tau_1)$ and α-unstable for $\tau \in (\tau_1, 0.5]$.

Simulation is carried out using Simulink. We investigate the evolution of the signal $e^{\alpha t}y(t)$ over time by setting τ as $0s$, $0.08s$ and $0.11s$ respectively. It is clear that if the system is α-stable, then $e^{\alpha t}y(t)$ must be convergent, otherwise it diverges. It is shown in the right plot of Fig.6.5 that the weighed output response corresponds to $\tau = 0.08s$ converges and that corresponding to $\tau = 0.11s$ divergences. These results are all consistent with our previous analysis.

Example II. Consider a plant with the following transfer function:

$$y(s) = \frac{b_1 s + 1}{s^3 + a_2 s^2 + a_3 s} u(s), \tag{6.48}$$

which models the pitch dynamics of an aircraft [49]. The parameters are chosen as

$$b_1 = -0.4, \ a_2 = 3, \ a_3 = 2.75.$$

To regulate the output $y(t)$ to a given constant set point r, a PD control feedback is first designed as

$$u(t) = k_D \dot{e}(t) + k_p e(t), \tag{6.49}$$

where $e(t) = y(t) - r$. Then (6.49) is replaced by

$$u(t) = k_D \frac{e(t) - e(t-\tau)}{\tau} + k_p e(t). \tag{6.50}$$

The control diagram is depicted in Fig.6.6.

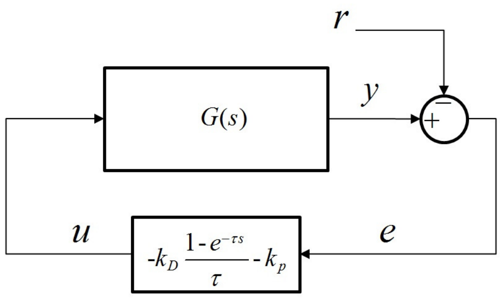

FIGURE 6.6: The control diagram of Example II.

Then characteristic equation (6.9) of the closed-loop system (6.48)(6.49) is derived as

$$s^3 + (a_2 + b_1 k_D)s^2 + (a_3 + k_D + b_1 k_p)s + k_p = 0. \tag{6.51}$$

We set the control parameter as $k_D = 0.5$ and $k_p = 1.15$. The three characteristic roots of (6.51) are computed as

$$\begin{aligned} s_1 &\approx -1.3852,\ s_2 \approx -0.7074 - 0.5743i, \\ s_3 &\approx -0.7074 + 0.5743i. \end{aligned}$$

We thus deduce that with the exact PD control law (6.49), the exponential decay rate of the closed-loop trajectory is approximately 0.7074. We will investigate whether it is possible to use the finite difference approximation to achieve a faster decay rate. To this end, we set $\alpha = 0.73$. It is apparent that the system is not α-stable if the delay value is close to zero.

Now we consider the characteristic equation (6.11). Simple computation shows:

$$\begin{aligned} P_\alpha(j\omega,\tau) &= 0.2164 - 0.64\omega^2 + (0.3007\omega - \omega^3)j, \\ Q_\alpha(j\omega,\tau) &= \tau^{-1}(-0.646e^{0.73\tau} + 0.2e^{0.73\tau}\omega j). \end{aligned}$$

Using these expressions, $F(\omega,\tau)$ and $E(w,z)$ can be obtained through straight-forward computation. The exact expressions of these functions are omitted here.

To compute τ^l, we apply Proposition 6.2 and solve Equation (6.25). We found the smallest z appearing in the pairs $(w,z) \in \mathbb{R}_+ \times \mathbb{R}_+$ that satisfy (6.25) is $z \approx 6.3802/4 \times 10^{-3}$. Then solving (6.26), we obtain $\hat{\tau} \approx 0.0719$ and therefore set τ^l as 0.07. Assuming the maximal delay of interest is $\tau^u = 0.9$, we have $\mathscr{I} = [0.07, 0.9]$.

Solving $\partial_\omega F(\omega,\tau) = 0$ together with $F(\omega,\tau) = 0$, we find no solution pair $(\omega,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+$. Therefore $\mathscr{I}^{(1)} = \mathscr{I}$. We find that $F_\tau(\omega)$ admits two positive roots, namely $\omega_1^{(1)}(\tau)$ and $\omega_2^{(1)}(\tau)$ for $\tau \in \mathscr{I}^{(1)}$. The graph of these two functions are presented in Fig.6.4 a. The functions $\theta_i^{(1)}(\tau)$, $i = 1,2$ are determined accordingly, and their graphs are given in Fig.6.4 b. The graphs of $\theta_2^{(1)}(\tau)$, $\theta_1^{(1)}(\tau)$ intersect horizontal line 0 at $\tau_1 = 0.1077$ and $\tau_2 = 0.7298$, respectively. Computation shows

$$\omega_{11} = \omega_2^{(1)}(\tau_1) \approx 0.5957, \ \omega_{21} = \omega_1^{(1)}(\tau_2) \approx 0.8119.$$

We can further verify:

$$\mathrm{sgn}_2^{(1)} = \partial_\omega F(\omega_2^{(1)}(\tau_1), \tau_1) < 0, \ \theta_2^{(1)}(\omega_2^{(1)}(\tau_1), \tau_1) < 0, \tag{6.52}$$

as well as

$$\mathrm{sgn}_1^{(1)} = \partial_\omega F(\omega_1^{(1)}(\tau_2), \tau_2) > 0, \ \theta_1^{(1)}(\omega_2^{(1)}(\tau_2), \tau_2) > 0. \tag{6.53}$$

We can deduce the followings. For small τ, (6.11) is not stable since $N^u(0) = 2$. As τ sweeps through $\mathscr{I}$, a pair of imaginary characteristic root of (6.11) first appears at $\tau = \tau_1$ with the value $\pm j\omega_{11}$. According to (6.52) and (3.47), this pair of roots move to the left half plane as τ increases from τ_1, and thus become stable. A second pair of imaginary characteristic roots of (6.11), namely $\pm j\omega_{21}$, appear at $\tau = \tau_2$. According to (6.53) and (3.47), they move to the right half plane and become unstable as τ further increases from τ_2. We can now conclude that for $\alpha = 0.73$, the closed-loop system (6.48)(6.50) is α-stable for $\tau \in (\tau_1, \tau_2)$ and is α-unstable for $\tau \in (0, \tau_1) \cup (\tau_2, 0.9]$.

The output of the closed-loop system (6.48)(6.50) corresponding to the unit step input is given in the left plot of Fig.6.5. We also inspect the weighed tracking error $e(t)\exp(\alpha t)$ and plot its graph in the right plot of Fig.6.5. Obviously the system being $\alpha-$stable means the weighed error must converge to 0. The weighed error converges, though slowly, to zero when $\tau = 0.6$ and diverges when $\tau = 0.8$. This observation is consistence with our analysis. It is worth mentioning that with the fixed feedback gain, the exponential convergence rate $\alpha = 0.73$ can neither be achieve using the exact output derivative or with any delay value smaller than τ_1. Therefore, for this example and the chosen control gain k_s, k_p, the delay in the feedback can indeed improve the control performance. This can also be concluded from the step response of $y(t)$. Compared with the control feedback using $\dot{y}(t)$, the finite difference approximation with $\tau = 0.6$ or $\tau = 0.8$ significant reduces the raising time of the system, as shown in Fig.6.5.

6.7 Chapter Summary

We have addressed the stability analysis for control schemes that use finite difference to approximate the derivatives of output signals. The delay is treated as a design parameter. Given any bounded delay interval of interest of the form $(0, \tau^u]$, we propose a method to find all the subintervals of delay values contained in this interval such that the system is exponentially stable with guaranteed convergence rate α, known as the α-stability. It is shown that after shifting the Laplace variable, the α-stability of the control system is equivalent to the asymptotic stability of a new characteristic equation with delay-dependent coefficients. To analyze the stability of the later characteristic equation, we proposed some methods to calculate a positive lower bound τ^l for the delay interval such that the stability analysis only needs to be carried out in $[\tau^l, \tau^u]$. Consequently we are able to apply the results in previous chapters on stability analysis of delay system with delay-dependent coefficients. The stability analysis procedure is illustrated with two examples that are motivated by mobile robot path-following control and aircraft pitch control, respectively. The results show that a larger delay in the finite-difference approximation may indeed improve the control performance in terms of the exponential convergence rate of the trajectory as well as response rapidity to reference signals.

Chapter 7

Conclusions and Perspectives

Time-delay systems with delay-dependent coefficients appear in various scientific and engineering disciplines. This thesis contributes to the stability analysis of systems with this special feature.

The method developed in this thesis follow a generalized τ-decomposition approach. The idea of this approach is to sweep the delay parameter through an interval of interest, identify all the critical delays and the corresponding crossing frequencies and determine the crossing directions of the characteristic roots on the imaginary axis. Then the number of unstable characteristic roots for different delay values can be easily determined.

We first considered systems with a single delay. The delay interval is first decomposed into disjoint sub-intervals, within which a fixed number of frequencies can be defined as functions of the delay and each frequency function is associated with a phase angle function. Critical delays and crossing frequencies are identified based on the phase angle functions and the frequency functions. Criteria that determine the crossing direction of characteristic roots on the imaginary axis are proposed, which is able to exploit the higher order derivatives of the characteristic roots with respect to the delay when the lower order derivatives vanish. These results are subsequently extended to systems with commensurate delays.

Our analysis shows that the crossing direction of an imaginary root depends on the product of two separate terms. The first term has already been discovered in the literature for time-delay systems with fixed coefficients. The second term is related to the monotonicity of the phase angle function at critical delays, which is unique for systems with delay-dependent coefficients. Moreover, the correlation between the position of the phase angle functions and the number of unstable roots is also suggested by the root crossing criteria.

To acquire a deeper understanding of these results, a two-parameter approach is proposed to provide geometric insight into the problem. The delay parameter in system coefficients and in the state are regarded as two different variables, say r and q. Then system stability can be determined by first considering different stability regions in the r-q parameter space and then imposing the restriction $r = q = \tau$. As r and q enters and leaves different stability regions in the r-q space along the 45 degree line, system stability may switch. This two-parameter point of view leads to more intuitive interpretation of the stability results formerly obtained through an analytical approach. With the two-parameter approach, more general stability analysis methods are further developed that apply to systems with possibly repeated imaginary characteristic roots under relaxed assumptions. Although the root crossing criteria can be expressed using the derivatives of the phase angle function, the analysis from the two-parameter perspective makes it clear that the differentiability of phase angle functions is not essential. The monotonicity of these functions at critical delays are sufficient for determining the root crossing directions.

we showed that system with delay dependent coefficients can arise from a control scheme that uses delayed output to approximate the output derivatives. The resulted closed loop systems have the delay parameter in the denominator, and the analysis methods developed in this thesis cannot be applied immediately due to this singularity at $\tau = 0$. To resolve this issue, several convenient methods for estimating a positive lower bound for the τ-sweeping test are proposed, which consist mainly in solving some polynomial equations. Subsequently, the stability analysis methods proposed in previous chapters are applied to find all delay intervals that guarantee a desired convergence speed of the trajectory of the closed loop system.

There are various directions worth pursuing in the future.

Firstly, we have only considered the nominal system without taking into account uncertainties in system parameters. A robust stability analysis for systems with uncertainties will be important in both theory and practice. The traditional τ-decomposition method is difficult to apply for the following reason. To tell the stability for a given delay value, say τ_0, it requires all critical delays less than τ_0 to be identified and the root crossing behavior analyzed at each of these critical delay. This is a quite formidable task since the number of critical delays and the system stability behaviors at these delays can depend on the uncertain parameters in a complex manner. In the author's opinion, the two-parameter perspective discussed in Chapter 5 suggests a more feasible approach to this problem. According to this perspective, the $r-q$ parameter space introduced in Chapter 5 is separated by critical delay curves into disjoint sub-regions, within which the number of unstable characteristic roots is invariant. Consequently, one can determine system stability with respect to uncertain parameters by analyzing how large these critical delay curves may vary under the perturbation of uncertain parameters. It is apparent that if the boundary curves of each sub-region in the $r-q$ plane never passes through the point $(r,q) = (\tau_0, \tau_0)$ under all possible parameter variation, then robust stability/instability can be deduced for the system with the delay value τ_0. On the other hand, these boundary curves, namely the curve of critical delays, are determined by a set of algebraic equations, which means the study of robust stability may be approached by analyzing the variation of roots of algebraic equations under parameter perturbation.

Second, the type of delay considered in this thesis is restricted to a single delay or multiple commensurate delays generated by a single parameter. It would be interesting to generalize the results for systems with multiple independent delay parameters. For this purpose, one may analyze system stability in a higher dimensional parameter space as a generalization of the two-parameter approach proposed in this thesis.

Another direction is to extend the stability analysis to nonlinear systems by exploiting the eigenvalue based approaches. When the linearized system has characteristic roots on the imaginary axis, the corresponding nonlinear system can still be asymptotically stable. One may extract the center manifold of the nonlinear system based on the imaginary eigenvalue and eigenspace of the linearized dynamics. Then it suffices to analyze the stability of the complete system by considering the dynamics only on the center manifold, which is a finite dimensional object. For introduction and applications of general invariant manifolds of ordinary differential equations, see for instance [89], [90]. For Stability analysis of nonlinear time-delay systems on center manifolds, interested readers can refer to [91], [92].

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ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Analyse de stabilité de systèmes à coefficients dépendant du retard

Mots clés : système de retard, analyse de stabilité, analyse de domaine fréquentiel

Résumé : Des systèmes à coefficients dépendant du retard ont été rencontrés dans diverses applications de la science et de l'ingénierie. Malgré la littérature abondante sur les systèmes de temporisation, il y a peu de résultats concernant l'analyse de stabilité des systèmes avec des coefficients dépendant du retard.

Cette thèse est consacrée à l'analyse de stabilité de cette classe de systèmes. Les méthodes d'analyse de la stabilité sont développées à partir de l'équation caractéristique correspondante suivant une approche généralisée tau-décomposition. Étant donné un intervalle d'intérêt de retard, nous sommes capables d'identifier toutes les valeurs de retard critique contenues dans cet intervalle pour lesquelles l'équation caractéristique admet des racines sur l'axe imaginaire du plan complexe.

Le critère de direction de croisement des racines sont proposées pour déterminer si ces racines caractéristique se déplacent vers le plan . complexe demi-gauche ou demi-droite lorsque le paramètre de retard passe par ces valeurs de retard critique. Le nombre de racines caractéristiques instables pour un retard donné peut ainsi être déterminé. Notre analyse comprend les systèmes avec un seul retard ou des retards proportionnés sous certaines hypothèses. Le critère de direction de croisement des racines développés dans cette thèse peut être appliqués aux multiple racines caractéristiques, ou aux racines caractéristiques dont la position paramétrée par le retard est tangent à l'axe imaginaire. En tant qu'application, il est démontré que les systèmes avec des coefficients dépendant du retard peuvent provenir de schémas de contrôle qui utilisent une sortie retardée pour approcher ses dérivés pour la stabilisation. Les méthodes d'analyse de stabilité développées dans cette thèse sont adaptées et appliquées pour trouver les intervalles de retard qui atteignent un taux de convergence demandé du système en boucle fermée.

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ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Title : Stability analysis of systems with delay-dependent coefficients

Keywords : time-delay systems, stability analysis, frequency-domain analysis

Abstract : Systems with delay-dependent coefficients have been encountered in various applications of science and engineering. Despite the rich literature on time-delay systems, there are few results concerning stability analysis of systems with delay-dependent coefficients. This thesis is devoted to the stability analysis of this class of systems.

Stability analysis methods are developed based on the corresponding characteristic equation following a generalized tau-decomposition approach. Given a delay interval of interest, we are able to identify all the critical delay values contained in this interval for which the characteristic equation admits roots on the imaginary axis of the complex plane. Various root crossing direction criteria are proposed to determine whether these characteristic roots move toward the left or the right half complex plane as the delay parameter goes through these critical delay values. The number of unstable characteristic roots for any given delay can thus be determined. Our analysis covers systems with a single delay or commensurate delays under certain assumptions. The root crossing direction criteria developed in this thesis can be applied to characteristic roots with multiplicity, or characteristic roots whose locus parameterized by the delay is tangent to the imaginary axis. As an application, it is demonstrated that systems with delay-dependent coefficients can arise from control schemes that use delayed output to approximate its derivatives for stabilization. The stability analysis methods developed in this thesis are tailored and applied to find the delay intervals that achieve a demanded convergence rate of the closed-loop system.

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