Peeling and conformal scattering on the spacetimes of the general relativity
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Peeling et Scattering Conforme dans les Espaces-Temps de la Relativité Générale

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Chapter 1

Introduction

1.1 Une courte introduction à la relativité générale

La relativité générale a été découverte en 1915 par A. Einstein. Pendant plus d’un siècle d’existence, cette théorie a abouti à d’importantes résolutions grâce à son exactitude et à ses prévisions précises. La relativité générale nous a permis la compréhension de l’univers d’une façon plus profonde que la loi de Newton de la gravitation universelle. En particulier, elle nous a apporté les beaux modèles des espaces-temps de type trou noir en mathématiques et en physique, dont les observations récentes en astronomie gravitationnelle sont en train de prouver la validité.

Selon la relativité générale, la gravité est décrite de façon géométrique comme la courbure de l’espace-temps, qui est une variété Lorentzienne de dimension 4: \((\mathcal{M}, g)\). Les équations d’Einstein relient la courbure de l’espace-temps à l’énergie et à l’impulsion des matières et rayonnements qui sont présents dans l’Univers:

\[ G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \]

L’inconnue dans les équations est la métrique Lorentzienne \(g_{ab}\) qui est une forme bilinéaire symétrique non dégénérée de signature \((+−−−)\). Le tenseur d’Einstein \(G_{ab}\) est donné par

\[ G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \]

où \(R_{ab}\) et \(R\) sont la courbure de Ricci et la courbure scalaire de la métrique \(g_{ab}\). Le tenseur \(T_{ab}\) est le tenseur énergie-impulsion de la matière, \(\Lambda\) est la constante cosmologique, \(G\) est la constante gravitationnelle de Newton et \(c\) est la vitesse de la lumière dans le vide.


La métrique de Minkowski est la solution triviale dans le cas du vide, c’est à dire lorsque le tenseur
énergie-impulsion et la constante cosmologique sont nuls. Sur la variété

\[ \mathcal{M} = \mathbb{R}^4 \]

la métrique de Minkowski et donnée en coordonnées cartésiennes par

\[ g = dt^2 - dx^2 - dy^2 - dz^2. \]

L'espace-temps de Minkowski est plat, c'est-à-dire que le tenseur de courbure de Riemann est nul partout. L'espace tangent à \( \mathcal{M} \) en chaque point est \( \mathbb{R}^4 \) muni de la forme bilinéaire \( g \), on l'appelle l'espace vectoriel de Minkowski. La métrique de Minkowski est une métrique Lorentzienne de signature \((+ - - -)\). Pour chaque vecteur \( V \) dans l'espace tangent \( T_p \mathcal{M} \) en un point \( p \in \mathcal{M} \), on dit que \( V \) est

- temporel si \( g(V,V) > 0 \),
- spatial si \( g(V,V) < 0 \),
- isotrope ou de type lumière si \( g(V,V) = 0 \).

On dit que \( V \) est causal s'il n'est pas spatial. Une courbe est dite temporelle, spatiale, isotrope ou causale respectivement si son vecteur tangent est temporel, spatial, isotrope ou causal. Une hypersurface est dite temporelle, spatiale ou isotrope respectivement si son vecteur normal est spatial, temporel ou isotrope. Et on dit qu'une hypersurface est achronale (ou faiblement spatiale) si son vecteur normal est causal.

Dans le cas général, de l'énergie et de la matière sont présentes dans l'univers et apparaissent sous forme d'un tenseur énergie-impulsion dans le second membre des équations d'Einstein. Alors on obtient des espace-temps courbes \( \mathcal{M} \) munis d'une métrique Lorentzienne \( g \), dont la courbure de Riemann n'est pas nulle. La métrique étant non dégénérée, on peut trouver une base orthonormée au voisinage de chaque point de \( \mathcal{M} \); dans cette base, du fait de sa signature, la métrique \( g \) est décrite par la matrice

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

En conséquence, l'espace tangent en chaque point est une copie de l'espace vectoriel de Minkowski. On a donc des notions de causalité pour \( \mathcal{M} \) (vecteurs temporels, spatiaux, etc...) héritées de celles définies sur l'espace-temps de Minkowski.

L'espace-temps courbe que nous étudierons est celui de Kerr. Il est asymptotiquement plat, c'est-à-dire que la courbure de Riemann tend vers zéro à l'infini spatial. La métrique de Kerr est solution de l'équation d'Einstein dans le vide, elle décrit la géométrie d'un espace-temps vide autour d'un trou.
noir axialement symétrique non chargé et en rotation. L'espace-temps de Kerr décrit en coordonnées de Boyer-Lindquist est une variété de dimension 4:

\[ \mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega \]

munie de la métrique Lorentzienne

\[
g = \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4aMr \sin^2 \theta}{\rho^2} dtd\varphi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\varphi^2,
\]

\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2Mr + a^2, \]

\[ \sigma^2 = (r^2 + a^2)\rho^2 + 2Ma^2 \sin^2 \theta = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \]

où \( M > 0 \) est la masse du trou noir et \( a \neq 0 \) son moment angulaire par unité de masse. Si \( a = 0 \), \((\mathcal{M}, g)\) n’est autre que l’espace-temps de Schwarzschild, et si \( M = a = 0 \), \((\mathcal{M}, g)\) n’est autre que l’espace-temps de Minkowski (et \( \mathcal{M} \) devient simplement \( \mathbb{R}^4 \)).

1.2 L’approche conforme de l’analyse asymptotique


- la diffusion dans des géométries non stationnaires et la diffusion conforme,

- la stabilité non linéaire des solutions des équations d’Einstein,

- les champ quantiques,

- la diffusion inverse.

Le notions de points à l’infini $i_0, i^\pm$ et les hypersurfaces isotropes $\mathcal{I}^\pm$ sont présentés par R. Penrose [72] avec la compactification de l’espace-temps de Minkowski. Il a aussi donné la définition d’espace-temps asymptotiquement simple qui admet une compactification conforme régulière. Le principe de la *compactification conforme* est le suivant: soit $(\mathcal{M}, g)$ un espace-temps globalement hyperbolique, son compactifié conforme est une variété $(\hat{\mathcal{M}}, \hat{g})$, avec bord $\mathcal{I}$ qui satisfait les propriétés suivantes:

- L’intérieur de $\hat{\mathcal{M}}$ n’est pas autre que $\mathcal{M}$ et $\partial \hat{M} = \mathcal{I}$ correspond aux points à l’infini de $(\mathcal{M}, g)$.
- La relation entre le bord $\mathcal{I}$ et $(\mathcal{M}, g)$ est donnée par une fonction $\Omega$ qui est positive sur $\mathcal{M}$ et lisse sur $\hat{\mathcal{M}}$ et telle que $\Omega|_{\mathcal{I}} = 0$ et $d\Omega|_{\mathcal{I}} \neq 0$.
- La métrique $\hat{g}$ qui est donnée par $\Omega^2 g$ est une métrique Lorentzienne, lisse et non dégénérée sur $\hat{\mathcal{M}}$.

Une *équation conformément invariante* est une équation qui est définie sur $(\mathcal{M}, g)$, telle que pour toute solution $\Phi$ de cette équation, il existe un réel $s$ tel que le champ rééchelonné $\hat{\Phi} = \Omega^s \Phi$ est aussi une solution de cette équation sur $\mathcal{M}$ munie de la métrique conformément rééchelonnée $\hat{g}$.

Le peeling et la diffusion conforme sont deux aspects de l’analyse asymptotique pour les champs conformes dans le cadre de la compactification conforme. Dans cette thèse on va étendre le peeling à l’espace-temps de Kerr et construire une théorie de la diffusion conforme pour les champs spinoriels de masse nulle à l’aide de l’approche conforme.

1.2.1 Peeling


- En 1961: La propriété du peeling a été découverte par R. Sachs [74]. Il a étudié le champ gravitationnel linéarisé $\phi_{ABCD}$ en espace-temps plat (champ sans masse de spin 2) lorsqu’il n’est pas le mélange d’un rayonnement sortant et d’un rayonnement entrant. Par la méthode d’analyse du tenseur de Riemann, il a donné une formule covariante de la condition d’un rayonnement sortant. En d’autres termes, c’est la propriété que le champ gravitationnel ne contient pas de rayonnement entrant ou que sa partie entrante soit une impulsion de durée finie. Dans ce cas, il montre que les champs ont une propriété asymptotique générique, qu’il appelle peeling et qu’il décrit en termes des directions isotropes principales du champ et de leur alignement asymptotique le long des géodésiques isotropes sortantes. Plus en détail, la composante du champ spinoriel $\phi_{ABCD}$ qui se comporte comme $r^{-k-1} (k = 0, 1, \ldots 4)$ à l’infini le long des géodésiques nulles sortantes a $(4 - k)$ de ses directions isotropes principales alignées...
le long du générateur des géodésiques. Si on choisit une base spinorielle \( \{ o^A, \iota^A \} \) parallèle le long de la géodésique isotrope principale considérée et telle que \( o^A \tilde{o}^A \) soit le générateur de la géodésique, la composante \( \phi_{4-k} \) qui est obtenue en contractant \( \phi_{(ABCD)} \) avec \( 4-k \) \( i' \)s et \( k \) \( o' \)s, se comporte comme \( r^{-k-1} \) à l’infini le long de la géodésique.

- En 1963-1965: R. Penrose [71, 72] a étudié le peeling de Sachs pour tous les champs sans masse \( \phi_{(AB...F)} \) de spin \( n/2 \). Il a démontré que la propriété de peeling de ces champs est équivalente à la continuité à l’infini du champ conformément rééchelonné et il a utilisé le terme "peeling-off" pour désigner "falls-off" du champs à un ordre donné.

Pour la démonstration dans l’espace-temps plat, d’abord il a établi que les champs \( \phi_{(AB...F)} \) peuvent s’exprimer comme une dérivée \((n+1)\)ième d’un potentiel scalaire complexe \( \chi \), appelé potentiel de Hertz, qui est solution de l’équation des ondes \( \Box \chi = 0 \). En détail soit \( \mu^A'...\mu^F' \) sont spineurs constantes, on peut obtenir

\[
\phi_{(AB...F)} = \mu^A' \nabla_{AA'} \psi_{(B...F)} = ... = \mu^A'...\mu^F' \nabla_{AA'}...\nabla_{FF'} \chi.
\]

Ensuite il montre que le champs \( \phi_{(AB...DE...F)} \xi^D ... \xi^F \) (où il y a \( k \) spineurs \( \xi \) qui sont fixés et correspond à la direction nulle, \( k = 0, 1...n \)) se comporte comme \( r^{-k-1} \) à l’infini le long de la géodésique si et seulement si le champs \( \psi_{(B...F)} \xi^E ... \xi^F \) (où il y a \( k-1 \) spineurs \( \xi \), \( k = 1, 2...n \)) se comporte comme \( r^{-k} \) à l’infini le long de la géodésique, par l’induction cela est aussi équivalent à la condition raisonnable physiquement que le potentiel scalaire complexe \( \chi \) se comporte comme \( r^{-1} \) à l’infini le long de la géodésique (notons que la géodésique dans ce cas a juste la paramètre \( r \) qui peut changer, les autres paramètres sont gagnés).

Pour généraliser, il a donné la technique de conformalité et il a défini l’espace-temps asymptotique simple (voir dans la définition de compactification conforme ci-dessus) qui inclut deux catégories: l’espace-temps asymptotique plat et l’espace-temps asymptotique de Sitter. Il a prouvé que le champs asymptotiquement régulier (le champs qui existe sur toute la variété \( (\mathcal{M}, g) \) tel que son rééchelonné se prolonge à la frontière \( \mathcal{I} \) de la variété rééchelonné \( (\hat{\mathcal{M}}, \hat{g}) \)) a la propriété du peeling dans l’espace-temps asymptotique simple. En plus détail, sous la transformation conforme il a montré que

\[
\phi_{AB...CD...F} \mu^A ... \mu^B \mu^C \mu^D ... \xi^F = \Omega^{k+1} \hat{\phi}_{AB...CD...F} \hat{\mu}^A ... \hat{\mu}^C \hat{\xi}^D ... \hat{\xi}^F + O(\Omega^{k+2} \ln \Omega),
\]

où, \( \hat{\alpha} \) désigne le rééchelonné de \( \alpha \), \( O(f) \) désigne le comportement à l’infini qui est comme la fonction \( f \) à l’infini, et \( \Omega \) est équivalent à l’inverse de la paramètre \( r \) de la métrique \( g \) i.e \( \Omega \sim \hat{r} - \hat{r}_0 \sim r^{-1} \). À partir de la condition régulier du champs rééchelonné \( \hat{\phi}_{AB...F} \), on obtient que

\[
\phi_{AB...CD...F} \mu^A ... \mu^C \mu^D ... \xi^F = O(r^{-k-1}).
\]
Cependant, ce n’est pas facile de trouver les données initiales de l’équation d’Einstein pour obtenir les espaces-temps qui sont satisfaits des conditions *compactification conforme*, donc l’existence des espaces-temps asymptotiques simples n’est pas évidente.


En 2003, S. Klainerman- F. Nicolò [48] ont obtenu des hypothèses suffisantes de décroissance et de régularité sur les ensembles de données initiales pour les équations d’Einstein telles que les espace-temps correspondants (définis dans le complémentaire du domaine d’influence d’un ensemble compact) vérifient des propriétés de peeling compatibles avec la simplicité asymptotique. Il s’agit d’un raffinement de leur travail de 2002 [47] montrant que selon le type de décroissance imposé aux données, on obtient deux catégories de peeling: peeling faible et peeling fort, ce dernier correspondant à la description de Penrose des espaces-temps asymptotiquement simples.


Dans cette thèse, nous étudions le peeling selon la méthode de L. Mason et J-P. Nicolas et nous généralisons leurs travaux dans deux directions:
nous étendons les résultats à l’espace-temps de Kerr;

• en plus des champs scalaires et de Dirac linéaires, nous traitons aussi le cas d’une équation des ondes non linéaire.

Le principe de l’approche que nous adoptons est de combiner la technique de compactification conforme avec la méthode des champs de vecteurs. On décrit brièvement les étapes de la démarche:

• Construction de la compactification de Penrose de l’extérieur du trou noir. On limite l’étude à un voisinage $\Omega^+_{i_0}$ de l’infini spatial $i_0$ pour lequel on construit un feuilletage $\{\mathcal{H}_s\}_{0 \leq s \leq 1}$ dont la feuille $\mathcal{H}_1$ est la tranche d’espace $t = 0$ et la feuille $\mathcal{H}_0$ est l’infini isotrope futur.

• On obtient des expressions explicites des équations sur le compactifié et de la commutation d’opérateurs de dérivation (dérivées de Lie ou dérivées covariantes) avec ces équations. On obtient des lois de conservation exactes ou approchées pour les équations initiales et pour celles obtenues par commutations de dérivées successives.

• On calcule les énergies des champs sur les hypersurfaces spatiales $\mathcal{H}_s$ et à l’infini isotrope $\mathcal{H}_0$.

• On établit des estimations d’énergie dans les deux sens entre $\mathcal{H}_0$ et l’hypersurface spatiale $\mathcal{H}_1 = \Sigma_0$ sur laquelle les données sont spécifiées.

• On formule la définition du peeling à tous ordres et on obtient l’espace optimal des données initiales assurant un peeling à chaque ordre.

• On interprète les résultats obtenus en les comparant avec le cas plat et en observant que les classes de données assurant le peeling à un ordre donné, dans le cas plat et dans le cas de la métrique de Kerr, ont des propriétés de régularité et de décroissance à l’infini identiques. En d’autres termes, les propriétés assurant un peeling à un certain ordre ne sont pas plus restrictives en métriques de Kerr qu’en espace-temps de Minkowski, elles sont identiques.

1.2.2 Diffusion conforme

donsent les premières preuves rigoureuses de l’effet Hawking pour les champs quantiques lors d’un
Jin [43] a obtenu une théorie de la diffusion du champ de Dirac massif dans l’espace-temps de
Schwarzschild. En 1995 J-P. Nicolas [60] a développé une théorie de la diffusion pour les champs
de Dirac massifs dans des espaces-temps de type trou noir asymptotiquement de Sitter et pour les
champs sans masse dans le cas asymptotiquement plat. Il a également obtenu des résultats partiels
sur les profils asymptotiques pour une équation non linéaire de Klein-Gordon [61] sur des métriques
de type trous noirs sphériques générales. En 1999, I. Laba et A. Soffer [49] ont obtenu une diffusion
complète pour une équation non-linéaire de Schrödinger dans l’espace-temps de Schwarzschild. En
2001, D. Häfner [37] a construit une théorie de diffusion complète pour l’équation des ondes dans
des espaces-temps stationnaires, asymptotiquement plats, en utilisant la théorie de Mourre. En
La théorie de Mourre a permis de développer des théories de la diffusion en métrique de Kerr :
D. Häfner [40] a étudié les modes non superradiants de l’équation de Klein-Gordon en 2003 et D.
Häfner et J-P. Nicolas [39] ont obtenue une diffusion complète pour l’équation de Dirac sans masse en
de Mourre une théorie de la diffusion pour les champs de Dirac chargés, massif ou non et avec poten-

La diffusion conforme est une approche géométrique de la diffusion. La compactification conforme
de Penrose permet de comprendre la construction d’une théorie complète de la diffusion comme la
résolution d’un problème de Goursat avec les données initiales sur l’infini isotope passé ou futur \( \mathcal{I}^\pm \).
On interprète alors l’opérateur de diffusion comme un isomorphisme entre les espaces de données sur
\( \mathcal{I}^- \) et sur \( \mathcal{I}^+ \): il associe la restriction sur \( \mathcal{I}^+ \) de la solution rééchelonnée à sa restriction sur \( \mathcal{I}^- \).
Le principe de la diffusion conforme apparaît en 1967 avec P.D. Lax et S.R. Phillips et en fait, cette
théorie est développée en 1980 par F.G. Freidlander. Le problème de Goursat est le coeur de la
méthode. Sa résolution ne nécessite pas de travailler dans des cadres statiques ou stationnaires et la
diffusion conforme permet donc d’étendre les théories de diffusion à des situations non stationnaires.
Voici une description de l’histoire de la diffusion conforme:

des ondes en espace-temps plat. L’objet fondamental de leur théorie est un représentant en
translation, construit de façon spectrale, qu’ils ré-interprètent ensuite comme un profil asympto-
totique du champ le long des géodésiques isotropes sortantes.

- En 1980: F.G. Freidlander a utilisé la relation entre la théorie de Lax-Phillips et sa notion
de champ de radiation (il a donné la définition du champ de radiation en 1962 dans [28]) pour
construire la première théorie de diffusion conforme dans [29]. Il a étudié l’équation d’onde
dans un espace-temps statique asymptotiquement plat en résolvant le problème de Goursat.


En 2012: J. Joudioux [45] a obtenu une théorie de la diffusion conforme pour une équation d’onde non linéaire dans le même cadre non stationnaire.

En 2013-2016: J-P. Nicolas 2013 [65] et M. Mokdad 2016 [58] sont revenu au cas statique mais en étendant le cadre géométrique à des espaces-temps de type trou noir (Schwarzschild et Reissner-Nordstrøm de Sitter) pour construire des théories de diffusion conformes pour le champ scalaire linéaire et le champ de Maxwell. D’abord, ils ont trouvé des résultats pour la décroissance et après ils ont montré que ces résultats sont suffisants pour obtenir une théorie de diffusion conforme. Notons que dans leurs travaux ils ont appliqué le résultat de L. Hörmander pour résoudre le problème de Goursat.

En continuant à étudier la diffusion conforme selon la méthode de J-P. Nicolas et M. Mokdad, dans cette thèse on va obtenir une théorie de diffusion conforme pour le champ sans masse de spin $n/2$ dans l’espace-temps de Minkowski. On peut décrire la démarche selon les étapes suivante:

- Décrire les compactifications conformes (complète et partielle) de l’espace-temps de Minkowski et résoudre le problème de Cauchy.

- Calculer les énergies conformes sur les hypersurfaces $\Sigma_0$, $\mathcal{I}^+$ et $S_T$, où $S_T$ est une hypersurface spatiale qui tend vers $i^+$ quand $T$ tend vers l’infini. Après on obtient un résultat de décroissance dont on déduit une égalité d’énergie entre $\Sigma_0$ et $\mathcal{I}^+$. On obtient la définition de l’opérateur de trace sur l’infini isotrope $\mathcal{I}$ pour des données régulières à support compact et on étend cet opérateur par densité, à l’aide des estimations d’énergie, à l’ensemble des données d’énergie finie.

- Résoudre le problème de Goursat dans la compactification partielle conforme.

- Montrer que l’égalité des énergies est suffisant pour obtenir un opérateur diffusion conforme isométrique.
1.3 Contenu de la thèse

Dans un espace-temps compactifié \((\hat{M}, \hat{g})\) on considère une équation rééchelonnée \(E_{\hat{g}}(\hat{\psi}) = 0\) où \(\hat{\psi} = \Omega^{-1}\psi\). Dans cette thèse, on va utiliser deux équations rééchelonnées obtenues par transformation conforme:\[1\]

- La première est la forme rééchelonnée des équations de champ sans masse de spin \(n/2\) (dont l’opérateur de Dirac est un cas spécial)

\[
\hat{\nabla}^{AA'}\hat{\psi}_{AB...F} = \Omega^{-3}\nabla^{AA'}\psi_{AB...F}.
\]  
(1.1)

- La deuxième est la forme rééchelonnée de l’équation des ondes conforme

\[
\Box_{\hat{g}}\hat{\psi} + \frac{1}{6}\text{Scal}_{\hat{g}}\hat{\psi} = \Omega^{-3}\left(\Box_g\psi + \frac{1}{6}\text{Scal}_g\psi\right),
\]  
(1.2)

où \(\text{Scal}_g\) est la courbure scalaire de la métrique \(g\), elle est telle que

\[
\text{Scal}_{\hat{g}} = \Omega^{-2}\text{Scal}_g + 6\Omega^{-3}\Box_g\Omega.
\]  
(1.3)

où suivant les coordonnées locales \(\{x^a\}\) l’opérateur d’onde est donné par:

\[
\Box_g = \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det g|} g^{ab} \frac{\partial}{\partial x^b}\right).
\]  
(1.4)

Dans cette thèse chaque chapitre présente un nouveau résultat. On décrit brièvement le contenu des chapitres comme suit:

**Chapitre 2.** On établit le peeling pour les champs scalaires linéaires et non-linéaires en métrique de Kerr. Tout d’abord, on décrit la géométrie du block I (extérieur du trou noir) de l’espace-temps de Kerr dans les coordonnées étoile-Kerr, Kerr-étoile et Kruskal-Boyer-Lindquist. Après, on obtient la compactification de Penrose du block I. Ensuite, on décrit le voisinage \(\Omega^-\) de l’infini spatial \(i^-\), qui est obtenu en coupant le compactifié de Penrose par une hypersurface isotrope \(\mathcal{S}_{i^-}\), dans lequel on va travailler. Ce voisinage peut être feuilleté par des hypersurfaces \(\{\mathcal{H}_s\}_{0 < s \leq 1}\), où \(\mathcal{H}_s(0 < s \leq 1)\) sont des hypersurfaces spatiales et \(\mathcal{H}_0 = \mathcal{I}_{t_0}^-\) est une partie de l’infini isotrope. Pour définir un courant d’énergie satisfaisant une bonne loi de conservation approchée jusqu’à l’infini isotrope, on contracte le tenseur d’énergie-impulsion \(T_{ab}\) avec une modification du champ de vecteurs de Morawetz \(T^a\) dans l’espace-temps de Kerr.

On donne en détail les expressions de l’équation d’onde linéaire et non-linéaire, et les lois de conservation après un changement conforme de la métrique \(g\) en \(\hat{g}\). Ensuite, on calcule les énergies des champs scalaires linéaires et non-linéaires sur les hypersurfaces \(\mathcal{H}_s\) et \(\mathcal{S}_{t_0}\). Pour obtenir les lois de conservation à un ordre supérieur on commute dans les équations les dérivées de Lie selon les champs de vecteurs \(X_i \in A\):

\[
X_0 = \partial_t, \ X_1 = \partial_\varphi, \ X_2 = \sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi,
\]

\[\text{Pour la démonstration, voir [73] (Vol1 et Vol2) et aussi [65].}\]
\[ X_3 = \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi, \quad X_4 = \partial_R. \]

Notons que dans notre cas les deux champs de vecteurs \( X_2 \) et \( X_3 \) ne sont pas des champs de Killing mais ils sont utiles car les dérivées de Lie selon ces champs commutent avec l’opérateur de Laplace \( \Delta_{S^2} \) sur la sphère de dimension 2, et on peut voir que \([\mathcal{L}_{X_1}, \mathcal{L}_{X_i}] = \pm \mathcal{L}_{X_j}\) avec un signe + ou − respectivement pour \( \{i = 2, j = 3\} \) et \( \{i = 3, j = 2\} \).

Pour donner la définition de peeling on a besoin de contrôler des énergies en combinant les dérivées de Lie selon toutes les directions \( X_i \in \mathcal{A} \). Dans le cas linéaire, on a besoin d’une inégalité de type Poincaré. Dans l’étude du cas non-linéaire, il y a une difficulté supplémentaire pour le contrôle des termes non-linéaires, qui nécessitera l’utilisation d’inégalités de Hölder et de Sobolev.

Ensuite on va pouvoir comparer la nouvelle définition de peeling dans notre cas et celle du cas plat. La méthode de comparaison consiste à: premièrement construire la définition du peeling dans la compactification conforme complète de l’espace-temps de Minkowski, deuxièmement faire tendre les paramètres de masse et de moment angulaire du trou noir pour donner la définition du peeling dans la compactification partielle conforme, et enfin comparer les deux définitions au voisinage de \( i_0 \). On remarque que les données initiales qui assurent le peeling dans notre cas sont plus larges que celles obtenues dans le cas plat à partir de la compactification complète, mais analogues à celles obtenues à l’aide de la compactification partielle. Notre définition est plus générale que celle établie selon la description usuelle de R. Penrose [71, 72].

**Chapitre 3.** On va étudier le peeling du champ de Dirac dans l’espace-temps de Kerr en utilisant deux approches: la première utilise des "dérivées partielles" pour obtenir les estimations d’énergie de régularité supérieure, la seconde utilise des dérivées covariantes. On continue à travailler dans le même voisinage \( \Omega_{i_0}^+ \) de \( i_0 \) que précédemment. Tout d’abord, on va choisir une téttrade de Newman-Penrose sur l’espace-temps de Kerr compactifié, calculer les coefficients dans la dyade associée du spineur solution de l’équation de Dirac, puis obtenir l’expression de l’équation de Dirac en termes de ces composantes. Ensuite, on calcule les énergies du champ de Dirac sur les hypersurfaces \( \mathcal{H}_s \) et \( \mathcal{S}_{t_0} \).

Pour obtenir les lois de conservation d’ordre supérieur, dans la première approche on va commuter dans l’équation des opérateurs faisant intervenir directement les dérivées partielles des composantes du spineur selon les directions \( \partial_t, \partial_R, \partial_\varphi \) et une modification de l’opérateur de Dirac sur la sphère \( S^2 \) dotée \( D_{t, \omega} \). Dans la deuxième approche, on va commuter les dérivées covariantes selon toutes les directions \( X_i \in \mathcal{A} \).

Pour donner la définition du peeling, on va contrôler des énergies en combinant des dérivées du champs de Dirac selon toutes les directions. Notons que dans les deux approches, l’inégalité essentielle obtenue dans [54], permettant de contrôler la norme de la première composante du spineur par celle de certaines de ses dérivées angulaires dans le formalisme de Newman-Penrose, n’est plus vraie ici du fait que pour notre choix de téttrade, la famille de plans engendrés par les vecteurs angulaires complexes \( \{m^a, \bar{m}^a\} \) n’est pas intégrable. Il faut donc donner des nouvelles inégalités qui sont obtenues à partir des dérivées covariantes ou des dérivées partielles selon toutes les directions.
Ensuite on va interpréter notre définition en la comparant avec le peeling dans l’espace-temps plat. La démarche pour faire cette comparaison est similaire à celle qu’on a suivie pour les champs scalaires et la conclusion est analogue.

**Chapitre 4.** On va construire une théorie de la diffusion conforme pour les champs spinoriels de masse nulle dans l’espace-temps de Minkowski. D’abord on décrit les compactifications conformes de l’espace-temps de Minkowski, l’une complète réalisant l’espace-temps de Minkowski comme un ouvert relativement compact du cylindre d’Einstein, l’autre partielle ne construisant que les infinis isotropes. On résout le problème de Cauchy dans le cylindre d’Einstein, et on applique ce résultat pour résoudre le problème de Cauchy dans les espaces-temps compactifiés. Ensuite on va obtenir un résultat de décroissance, déduit de la régularité des solutions sur le cylindre d’Einstein. Ce résultat nous permettra, après avoir calculé les énergies sur les hypersurfaces $\Sigma_0, \mathcal{I}^+$ et $S_T$, où $S_T$ est une hypersurface spatiale qui tend vers $i^+$ quand $T$ tend vers l’infini, d’obtenir l’égalité des énergies entre l’hypersurface de Cauchy $\Sigma_0$ et l’infini isotrope. En effet, dans la compactification partielle conforme, il y a une difficulté pour obtenir l’égalité des énergies parce que $i^+$ est à l’infini, on doit donc montrer que l’énergie sur $S_T$ tend vers zero quand $T$ tend vers l’infini et c’est le résultat de décroissance qui implique cette propriété. On définit l’opérateur de la trace $\mathcal{T}^+$ (respectivement $\mathcal{T}^-$) sur l’infini isotrope $\mathcal{I}^+$ (respectivement sur $\mathcal{I}^-$), et on étend cet opérateur par densité à l’espace des données d’énergie finie, à l’aide de l’égalité d’énergie.

On va résoudre le problème de Goursat dans la compactification partielle conforme. À partir de l’égalité des énergies et les solutions des problèmes de Cauchy et de Goursat, on montre que l’opérateur de la trace est une isométrie, autrement dit un isomorphisme préservant les normes. L’opérateur de diffusion conforme est alors défini par $\mathcal{W} = \mathcal{T}^+ \circ (\mathcal{T}^-)^{-1}$. On conclut avec une discussion concernant les normes d’énergie sur l’infini isotrope lorsqu’on utilise la compactification partielle et une énergie associée à un vecteur temporel devenant tangent à l’infini isotrope, et la compactification complète et la translation temporelle le long du cylindre d’Einstein. La norme obtenue avec la compactification partielle est plus faible que l’autre et cette construction donne donc une théorie de la diffusion pour une classe plus large de données.
Chapter 2

Peeling for linear and non linear scalar fields on a Kerr background

2.1 Introduction

The property peeling was discovered for the first time by R. Sachs [74] for the field gravitation in the Minkowski spacetime since 1960 and then R. Penrose [71, 72] represented it using the conformal geometry language. He also argued that the peeling is still valid for the asymptotic simple spacetimes. However in the work of Penrose the initial data which ensures the peeling is not optimal.

Due to the work of J. Corvino [14] and J. Corvino-R. Schoen [15], we can think that black hole spacetimes such as Schwarzschild or Kerr are similar to asymptotically simple spacetimes in the neighbourhood of $i_0$. So we can understand that the peeling in these spacetimes is valid (of course in the neighbourhood of $i_0$). But to answer the natural question: what is the optimal initial data that ensures the peeling? It takes a long time and recently, the authors L. Mason and J-P. Nicolas [53, 54] have given the new definition of the peeling and have constructed the optimal initial data which ensures the peeling for the scalar field and some zero rest-mass field with spin $1/2$ and spin $1$ in the cases of Minkowski and Schwarzschild spacetimes. Their method is combining the conformal technique and the vector field method (estimate energy). In this chapter, using the method of L. Mason and J-P. Nicolas, we will generalize the definition of the peeling for the scalar linear and nonlinear fields on the Kerr spacetime.

The peeling of the scalar field can be described as the regularity of $r\psi$ at the null infinity $\mathcal{I}$ in a neighbourhood of $i_0$. To obtain the definition of the peeling, the main difference between the two case Schwarzschild and Kerr, is that there is neither symmetric sphere nor stationary of the Kerr spacetime. These later lead to the calculations for the energies, the conservation laws that become more complicated and also the control of the energies become more difficult.

Block I is a global hyperbolic spacetime so the Cauchy problem of the linear wave equation and also Dirac equation are well-posed (The well-posed of the nonlinear wave equation was solved by J-P. Nicolas [65]). However, since the spacelike infinity $i_0$ is singular so we can guarantee that the solutions are smooth at the null infinity $\mathcal{I}$ only if we choose the initial data support away far from
To construct the peeling model of scalar fields (and Dirac field in chapter 2) in the Kerr spacetime, first we need to understand that the Penrose compactification of the Block I of Kerr spacetime is obtained by using the star-Kerr and Kerr-star and Kruskal-Boyer-Lindquist coordinates. We can construct star-Kerr (resp. Kerr-star) coordinates by outgoing (resp. incoming) null geodesics
\[ V^\pm = \frac{r^2 + a^2}{\Delta} \partial_t \pm \frac{a}{\Delta} \partial_\varphi \]
and then we use the conformal mapping in the new coordinates to obtain the rescaled spacetimes. These works were completed by B. O’Neill [69], D. Häfner and J-P. Nicolas [39]. Then we need to choose a neighbourhood \( \Omega^+_t \) of spacelike infinity \( i_0 \), to do this we will generate the work of L.Mason and J-P. Nicolas [53, 54]. There are three boundaries of \( \Omega^+_t \): the two first boundaries \( \mathcal{H}_1 \) and \( \mathcal{J}^+_t \) are the part of \( \Sigma_0 \) and \( \mathcal{J}^+ \) respectively; the difficulty is the third boundary \( S^*_t \), which we need to choose null or spacelike hypersurface. This hypersurface is chosen naturally, since in the compactification of the exterior of black hole using the simple null geodesics (this was appeared in the work of D. Häfner [38]), the hypersurface \( \{ \hat{t} = \text{constant} \} \) is null.

We will use the Morawetz vector field to contract with the stress-energy tensor. This vector field was first introduced in the flat case by C. Morawetz [57] in the early 1960s for obtaining pointwise decay estimates. Then in Schwarzschild spacetime, it was used to find the pointwise decay rates for spherically symmetric equations by M. Dafermos and I. Rodnianski [16] and for studying the peeling by L. Mason and J-P. Nicolas [53, 54]. In our work we need to modify the Morawetz vector field in the rescaled spacetime of the star-Kerr spacetime, then the calculation of its Killing form will be done by the Lie derivative.

To obtain the conservation laws at higher order, we will commute the Lie derivatives along the five vector fields in \( \mathcal{A} = \{ X_i \ (i = 0, 1..4) \} \) in the original equation as follows:
\[
X_0 = \partial_t, \ X_1 = \partial_\varphi, \ X_2 = \sin^* \varphi \partial_\theta + \cot \theta \cos^* \varphi \partial_\varphi \\
X_3 = \cos^* \varphi \partial_\theta - \cot \theta \sin^* \varphi \partial_\varphi, \ X_4 = \partial_R.
\]
Note that in Kerr spacetime, the two vector fields \( X_2 \) and \( X_3 \) are not Killing as in the symmetric spherical spacetimes, but they are still useful to be commuted in the equation because of the nice properties: first they commute with the Laplacian operator on 2–sphere \( \Delta_{S^2} \) and second \( [\mathcal{L}_{X_i}, \mathcal{L}_{X_j}] = \pm \mathcal{L}_{X_j} \), the signal \( \pm \) respective on \( \{ i = 2, j = 3 \} \) or \( \{ i = 3, j = 2 \} \).

To give the definition of peeling we need to establish the inequalities of the energies on the hypersurfaces \( \mathcal{H}_1 \) and \( \mathcal{J}^+_t \). This work is done by the help of the basic inequalities: Poincaré’s inequality, Hölder’s inequality and Sobolev’s inequality.

We complete this chapter by the following organisation:
• **Section 2.** We talk about the geometric settings: the Kerr metric, the star-Kerr and Kerr-star and Kruskal-Boyer-Lindquist coordinates, the neighbourhood of spacelike infinity and the Morawetz vector field.

• **Section 3.** We construct the peeling model in the linear case. We calculate in details to give the equation expression in the rescaled coordinates \((^*_t, R, \theta, ^*_\varphi)\) and then commute the Lie derivatives along vector fields \(X_i \in \mathcal{A}\) into the original equation to obtain the conversation laws. After that, we make the necessary calculation to give the simpler equivalences of the energies of the scalar fields on the spacelike hypersurface \(\mathcal{H}_s\) and the null hypersurface \(\mathcal{I}^+_i\).

The general definition of the peeling is obtained by the energy estimate that contain the basic estimate and the high order estimate, note that we need to use the Poincaré’s inequality to control. At the end of this section, we will interpret our definition of the peeling by comparing it with the flat case: first, we construct the peeling definition for the flat spacetime by using full conformal compactification, second we constraint our model to give the peeling definition for the part conformal compactification, third we compare the classes of the initial data that ensures the peeling definitions in the neighbourhood of \(i_0\).

• **Section 4.** We construct the peeling model in the nonlinear case. We will use two types of the stress-energy tensor to obtain the two types of the energies and the conservation laws respectively, basically the calculations are similarly to those in the linear case. The main difficulty is arised by controlling the terms that contain the nonlinear part. To do this, we will use the Hölder’s inequality and Sobolev’s inequality (that is obtained from the Sobolev’s embedding). We will interpret our definition by the same way as in the linear case, note that we will reconstruct the definition of the peeling in the flat case by using the difference foliation of the full conformal compactification from the linear case.

• **Appendix.** In the end of this chapter, we will recall the divergence theorem, the dominant energy condition, the two basic inequalities including Poincaré’s inequality and Sobolev’s inequality and their application in our work and check out some calculations in details to prove Lemma 2.3.1.

**Remark.** Our results are given in a neighbourhood of spacelike infinity \(i_0\), where the difficulty is localized. Therefore the definition of the peeling is arised from the regularity on null infinity near \(i_0\). The regularity on full \(\mathcal{I}^+\) can be recovered from the addition that the data classes are in local Sobolev spaces on a Cauchy hypersurface.

**Notations**

1. We shall use the notation \(\lesssim\) to signify that the left-hand side is bounded above by a positive constant times the right-hand side for \(r\) is large enough, the constant is being independent of the parameters and functions appearing in the inequality.

2. We shall use the notation \(\simeq\) to signify that the left-hand side and similarly the right-hand
side can be controlled above and below by two constants times the right-hand side for \( r \) is large enough, the constants are being independent of the parameters and functions appearing in the two hand sides.

(3) We use the term "a function is of order \( k \) in \( R \)" in the sense that the simplest equivalent form for \( r \) is large enough if this function has a factor \( R^k \) but does not have a factor \( R^{k+1} \).

2.2 Geometric setting

2.2.1 The Kerr metric

In the terms of Boyer-Lindquist coordinates, Kerr’s spacetime is a manifold \( (M = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega, g) \) where the metric \( g \) has the form

\[
g = \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4aMr \sin^2 \theta}{\rho^2}dt d\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\varphi^2,
\]

(2.1)

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2Mr + a^2,
\]

\[
\sigma^2 = (r^2 + a^2)\rho^2 + 2Mra^2 \sin^2 \theta = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta,
\]

where \( M > 0 \) is the mass of the black hole and \( a \neq 0 \) its angular momentum per unit mass. If \( a = 0 \), \( \mathcal{M} \) reduces to the Schwarzschild spacetime, and if \( M = a = 0 \), \( \mathcal{M} \) reduces to the Minkowski spacetime.

Kerr spacetime is asymptotically flat, we see the coordinate \( r \) as a distance to the star when it is large enough, and as \( r \to +\infty \) the Kerr metric becomes nearly Minkowskian. Kerr’s spacetime has three isometries: the first is \( t \to t + \Delta t \) which expresses the time-invariance of the model since \( \partial_t \) is a Killing vector field, the second is \( \varphi \to \Delta \varphi \) which expresses the axial symmetry of the model since \( \partial_{\varphi} \) is a Killing vector field and the last \( t \to -t \), \( \varphi \to -\varphi \) which expresses the invariance of the model when we reverse both time and rotation.

Kerr metric has two types of singularities: the first is the set of points \( \{\rho^2 = 0\} \) (the equatorial ring \( \{ r = 0, \theta = \pi/2 \} \) of the \( \{\rho^2 = 0\} \) sphere) it is called a true curvature singularity; the second is the spheres where \( \Delta \) vanishes, they are mere coordinate singularities and also called horizons. Note that the horizons are null hypersurfaces since they are 3–dimension hypersurfaces but the restriction of Kerr metric on them are degenerate. We call the black hole which is the part of our spacetime lying beyond the horizon, we can go to the domain of the black hole only by the way that crosses the horizon with the speeds greater than the speed of the light. There are three types of Kerr spacetimes depending on the number of horizons:

- Slow Kerr spacetime for \( 0 < |a| < M \). \( \Delta \) has two real roots

\[
r_{\pm} = M \pm \sqrt{M^2 - a^2},
\]

so there are two horizons, the spheres \( \{ r = r_- \} \) and \( \{ r = r_+ \} \), they belong to two different sides of the hypersurface \( \{ r = M \} \). The sphere \( \{ r = r_- \} \) is called inner horizon and the sphere \( \{ r = r_+ \} \) is called outer horizon.
• Extreme Kerr spacetime for $|a| = M$. $M$ is then the double root of $\Delta$ and the sphere $\{r = M\}$ is only horizon.

• Fast Kerr spacetime for $|a| > M$. $\Delta$ has no real root and the spacetime has no horizon. So that it does not exists a black hole in this case.

Since the extreme case is unstable, we will consider the peeling in the slow Kerr spacetime case, it is usually considered as the generic model of a spacetime containing a rotating uncharged black hole. By having two horizons, the slow Kerr spacetime is separated into three connected domains called Boyer-Lindquist blocks:

• Block I: $B_I \{r > r_+\}$ is called Kerr’s exterior or the exterior of Kerr black hole. It is the "astronomical" and is the most reasonable, so it is more simple than the three blocks. In this region, the vectors $\partial/\partial r, \partial/\partial \theta, \partial/\partial \varphi$ are spacelike and for $r >> 1$, $\partial/\partial t$ is timelike, it is also future-oriented. But for $r$ satisfying that $g_{tt} < 0$, block I contains a region called the ergosphere where $\partial/\partial t$ is spacelike. The ergosphere is the toroidal domain around the outside horizon

$$\mathcal{E} = \{(t, r, \theta, \varphi) ; r_+ < r < M + \sqrt{M^2 - a^2 \cos^2 \theta}\}$$

Ergosphere is a consequence of rotation, it has an essential role in black hole and in principle, it enables the energy to be extracted from the black hole i.e the energy of a material particle which run from the domain $r >> 1$ into $\mathcal{E}$, can become negative (since $\partial_t$ is spacelike).

Since there is no timelike Killing vector field globally defined on Block I so it is not stationary. But the exterior of the ergosphere is absolutely stationary since $\partial/\partial t$ is unique (modulo a factor constant) timelike Killing vector field globally defined there. Also every point in block I, even inside the ergosphere, has a stationary neighbourhood.

Due to the lemma 2.4.4 in [69] along a future-oriented non spacelike curve, the quantity $a \varphi$ is increasing; this says that in $\mathcal{E}$ the effects of the rotation are forced in the direction of rotation of the black hole.

Beside block I admit a Cauchy hypersurface $\{t = constant\}$ so it is globally hyperbollic spacetime.

• Block II: $B_{II} \{r_- < r < r_+\}$ is the region between the outer and inner horizons (it only exists in the slow case); this region is total relativistic. Here $\partial/\partial r$ is timelike and $\partial/\partial t, \partial/\partial \theta, \partial/\partial \varphi$ are spacelike, this says that block II is not yet a spacetime although it has a time-orientable vector field.

---

1. A curve $\alpha$ is future-oriented if the tangent vector $V$ of $\alpha$ at each point $P \in \alpha$ is causal and $V(\tau) > 0$ where $\tau$ is a function of the time.

2. A Cauchy hypersurface is a spacelike hypersurface and intersects with every inextendible timelike curve at exactly one point.

3. See section 3.2.1 in chapter 2 for details.
It is a dynamic domain where the inertial frames are dragged towards the inner horizon (the time orientation implicit in this description is such that $\partial/\partial r$ is past pointing).

- Block III: $B_{III} \{ -\infty < r < r_- \}$ lies beyond the inner horizon. This region is not stationary and it has another ergosphere

$$E' = \left\{ (t, r, \theta, \varphi) ; M - \sqrt{M^2 - a^2 \cos^2 \theta} < r < r_- \right\},$$

besides it has the ring singularity \{ $r = 0, \theta = \pi/2$ \}. And the vector field $\partial_\varphi$ is spacelike in the two blocks I, II but it becomes timelike in a region $\mathfrak{F}$ in block III near the ring singularity, $\mathfrak{F}$ is called Carter time machine. Since block $B_{III}$ contain Carter time machine, for any two points $p, q$ in block III, there exists a timelike future-oriented curve from $p$ to $q$, this curve cross the time machine (see proposition 2.4.7 in [69]). Hence, block III is not only stationary, but also not causal\(^4\) (or vicious); this is differente from the two blocks I, II (in block I, II there exist no closed non spacelike curves so they are causal). This is the reason why the Cauchy problem is hard to make the sense\(^5\) on block III (so we usually consider the Cauchy problem on block I).

For $r \ll -1$, block III has the same information in Boyer-Lindquist coordinates as in block I (except that $-r$ is distance to the centre). However, since block III contains the ring singularity and time machine so that its gravity repels rather than attracts and also the region near block II is stranger than block II itself.

The Kerr spacetime has two types of the null geodesics that are principal null geodesics (PNG) and simple null geodesics (SNG), they represent light rays. Each type of geodesics has two families: one going towards the star that is called "incoming null geodesics" and the other goes out from the star is called "outgoing null geodesics". By using these geodesics we can construct star-Kerr and Kerr-star spacetimes that don't have the singularity on the horizons \{ $r = r_-$ \} and \{ $r = r_+$ \} respectively.

- (PNG). Since the Kerr spacetime has Petrov type D (see [69]) so the Weyl tensor has two double roots at each point. These roots are the principal null directions of the Weyl tensor and they are given by the two vector fields as follows

$$V^\pm = \frac{r^2 + a^2}{\Delta} \partial_t \pm \frac{a}{\Delta} \partial_\varphi.$$

Since $V^+$ and $V^-$ are repeated null directions of the Weyl tensor, by the Goldberg-sachs theorem (see [69]) their integral curves define geodesics shear-free null congruences. We shall refer to the integral curves of $V^+$ (resp. $V^-$) as the outgoing (resp. incoming) principal null geodesics.

\(^4\)A spacetime $M$ is causal if there exist no closed non spacelike curves in $M$.

\(^5\)This mean that the information propagated along timelike geodesic can be come back to a point where the solution is already determined, so can be given some incompatibility.
• (SNG). The simple null geodesics $W^\pm$ can be found from the fact that a geodesic can be understood as the projection of the hamiltonian flow on $\mathcal{M}$. We can see in D. Hafnër [38], there he has used the hamiltonian flow of the principal symbol of wave operator $\frac{1}{2} \Box_g$ to find the simple null geodesics.

The contrast to the PNG is the orthogonal plane of $\text{span} \left\{ W^+, W^- \right\}$ which is integrable\footnote{See [69] for the general definition of the integrable distribution.}, i.e., it is tangent to $2$–sphere $S^2$, while in the case of PNG the orthogonal plane of $\text{span} \left\{ V^+, V^- \right\}$ is not integrable i.e. it is not tangent to $2$–sphere $S^2$.

Since the quantities $\rho^2$ and $\sigma^2$ are positive on block I (in fact $\rho^2$ is positive on the whole spacetime, but $\sigma^2$ is negative in the time machine in block III), we denote $\rho = \sqrt{\rho^2}$ and $\sigma = \sqrt{\sigma^2}$. We will work on $\mathcal{B}_I = \mathbb{R}_t \times [r_+, +\infty] \times S^2_{\theta, \varphi}$ equipped with the metric (2.1), we denote $\Sigma$ the generic spacelike slice $:\Sigma = [r_+, +\infty] \times S^2_{\theta, \varphi}$ and $\Sigma_0 = \{ t = 0 \} \times \Sigma$.

2.2.2 Star-Kerr and Kerr-star coordinates

Kerr metric can be extended over horizons by star-Kerr and Kerr-star coordinates by using two families of the null geodesics. In this extension, the metric doesn’t have singularity on the horizons. Here, we recall the extension of the Kerr metric by using PNG. The other extension by using SNG we can be seen in [38].

Now the star-Kerr coordinates $(^*t, r, \theta, ^*\varphi)$ is constructed by outgoing principal null geodesics, that are the integral lines of the vector $V^+$ (formula (2.2)). The new coordinates $^*t$ and $^*\varphi$ have the form

$$^*t = t - r_*(r) \quad ^*\varphi = \varphi - \Lambda(r),$$

where the functions $r_*$ and $\Lambda$ are required to satisfy

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad \frac{d\Lambda}{dr} = \frac{a}{\Delta}.$$ 

The outgoing principal null geodesics now can be considered as the $r$ coordinate, that are parametrized by $s = -r$

$$\dot{r} = -1, \quad \dot{\theta} = 0, \quad \dot{^*t} = \dot{t} - \frac{dT}{dr} \dot{r} = 0, \quad \dot{^*\varphi} = \dot{\varphi} - \frac{d\Lambda}{dr} \dot{r} = 0.$$ 

Remark 2.2.1. Denote $^*K$ is star-Kerr spacetime and $(\partial_r)_{BL}$ and $(\partial_r)^*_{K}$ are the $r$ coordinate vector fields in Boyer-Lindquist and star-Kerr coordinate respectively. We have

$$V^+ = (\partial_r)^*_{K}$$

$$\partial_t = \partial_{^*t}, \quad \partial_\varphi = \partial_{^*\varphi},$$

$$(\partial_r)_{BL} = -\frac{r^2 + a^2}{\Delta} \partial_{^*t} + (\partial_r)^*_{K} - \frac{a}{\Delta} \partial_{^*\varphi}.$$
Star-Kerr coordinates are defined globally on block I. In star-Kerr coordinate system the Kerr metric takes the form
\[
g = \left(1 - \frac{2Mr}{\rho^2}\right) dt^* + \frac{4aMr \sin^2 \theta}{\rho^2} d^*t^*d^*\varphi - \frac{\sigma^2}{\rho^2} \sin^2 \theta d^* \varphi^2 \\
- \rho^2 d\theta^2 + 2d^*t^*d^r - 2a \sin^2 \theta d^* \varphi d^r.
\] (2.3)

The expression above shows that the metric \(g\) doesn’t have singularity, so it can be extended smoothly across the horizon \(\{r = r_+\}\). Besides, it does not degenerate there, since its determinant is
\[
det(g) = -\rho^4 \sin^2 \theta
\]
doesn’t vanish for \(r = r_+\). Thus this horizon can be added to block I as a smooth boundary.

We need also to understand the physical meaning of the horizon that we have just glued to block I. Actually it is the hypersurface
\[
\mathcal{H}^+ = \mathbb{R}^*t \times \{r = r_+\} \times S^2_{\theta,^*\varphi},
\]
which is reached along outgoing null geodesics as \(t \to -\infty\). This horizon is called "white hole horizon", from there the light rays (the outgoing principal null geodesics) will be appeared. White holes are natural features of the maximal extension of spacetime containing external black holes. It can be only reached as \(t \to -\infty\), which we also call the past horizon. It is a smooth hypersurface in the spacetime \((B_I \cup \mathcal{H}^-, g)\).

We now show that the past horizon \(\mathcal{H}^-\) is a null hypersurface. Indeed the metric constraint of \(g\) on hypersurfaces \(\{r = \text{constant}\}\) is
\[
g_{\{r=\text{constant}\}} = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{4aMr \sin^2 \theta}{\rho^2} d^*t^*d^*\varphi - \frac{\sigma^2}{\rho^2} \sin^2 \theta d^* \varphi^2 - \rho^2 d\theta^2,
\]
so it has the determinant
\[
det(g_r) = \rho^2 \Delta \sin^2 \theta
\]
this determinant vanishes for \(\Delta = 0\), hence it vanishes on \(\mathcal{H}^-\). Since \(g\) is not degenerated, it follows that one of the generators of \(\mathcal{H}^-\) is null, so \(\mathcal{H}^-\) is a null hypersurface.

Similarly star-Kerr coordinates, Kerr-star coordinates \(\left(^*t, r, \theta, ^*\varphi\right)\) are constructed by using the incoming principal null geodesics that are parametrized as the integral lines of \(V^-\). We have
\[
t^* = t + r_*\ , \ \varphi^* = \varphi + \Lambda(r)\ ,
\]
with the same function \(r_*\) and \(\Lambda\) as for star-Kerr coordinates. Consequently the incoming principal null geodesics can be considered as the \(r\) coordinates curves parametrized by \(s = r\).

**Remark 2.2.2.** Denote \(K^*\) is Kerr-star spacetime and \((\partial_r)_{BL}\) and \((\partial_r)_{K^*}\) are the \(r\) coordinate vector fields in Boyer-Lindquist and Kerr-star coordinate respectively. We have
\[
V^- = (\partial_r)_{K^*},
\]
Kerr-star coordinates are defined globally on block I. In Kerr-star coordinate system the Kerr metric takes the form

\[
g = \left(1 - \frac{2M}{\rho^2}\right) dt^*^2 + \frac{4aMr}{\rho^2} dt^*d\phi^* - \frac{\sigma^2}{\rho^2} \sin^2 \theta d\phi^* d\phi^* - \rho^2 \sin^2 \theta d\phi^* dr.
\]

(2.4)

This coordinate system allows us to add the horizon

\[
\mathcal{H}^+ = \mathbb{R}_t^* \times \{r = r_+\} \times S_{\theta^*, \phi^*}^2
\]

as a smooth null boundary to block I. This hypersurface is reached along incoming null geodesics; it is the horizon reached as \(t \to +\infty\) by light rays (the incoming null geodesics) falling into the black hole, so we call it as "black hole horizon" and also the future horizon.

Note that both the future and past horizons are reached as \(t\) goes to infinity, they depend on the time, so that they are not the origin horizons \(\mathbb{R}_t^* \times \{r_{\pm}\} \times S_{\theta^*, \phi^*}^2\) in Boyer-Lindquist coordinates, and also they do not contain any point of the origin horizons.

### 2.2.3 Kruskal-Boyer-Lindquist coordinates

Since the star-Kerr spacetime is obtained by adding the past horizon to block I and similarly the Kerr-star spacetime is obtained by adding the future horizon to block II, we now want to extend Kerr metric on both of these horizons. For this purpose we describe the third coordinates of Kerr spacetime that is a combination of star-Kerr and Kerr-star coordinates, it will be regular on both the future and the past horizons. Here the time and radial variables are replaced by

\[
U = e^{-\kappa^+ t^*}, \quad V = e^{\kappa^+ t^*}
\]

where \(\kappa_+\) is given by

\[
\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{M^2 - a^2}}{r_+ + a^2}
\]

is called the surface gravity at the outer horizon. The coordinate \(\theta\) does not change since it is regular on the three blocks (except on the axis). Finally the longitude function \(\varphi^\pm\) is chosen so that the principal null geodesics in the long horizons are our coordinate curves i.e on \(K^*\) (and \(^*K\)) \(\varphi^\pm\) is constant on integral curves \(\alpha\) of PNG \(V^\pm\) (for \(r = r_\pm\)), so that

\[
0 = d(\varphi^\pm \circ \alpha)/ds = \alpha'[\varphi^\pm] = V^\pm[\varphi^\pm] = (r_\pm^2 + a^2)\partial \varphi^\pm / \partial t^* + a \partial \varphi^\pm / \partial \varphi^*,
\]

so we can choose on \(K^*\)

\[
\varphi^\pm = \varphi^* - [a/(r_\pm^2 + a^2)] t^*;
\]
similarly on \( *K \) we can choose

\[
\varphi^\pm = \varphi - \frac{a}{(r_\pm^2 + a^2)} t.
\]

For compatibility on \( *K \cap K^* \), we average \( \varphi \) and \( \varphi^* \) and get

\[
\varphi^\pm = \frac{1}{2} \left( \varphi^* + \varphi - \frac{a}{r_\pm^2 + a^2} (t + t^*) \right) = \varphi - \frac{a}{r_\pm^2 + a^2} t.
\]

The functions \((U, V, \theta, \varphi^\pm)\) obtain an analytic coordinate system\( ^* \) on \( \mathcal{B}_I \cup \mathcal{H}^+ \cup \mathcal{H}^- \) (axis), which is called the Kruskal-Boyer-Lindquist (KBL) coordinates. In this coordinates, we have

\[
\mathcal{B}_I = \{0, +\infty \mid U \times 0, +\infty \mid V \times S_{\theta, \varphi^\pm}^2 ,
\]

\[
\mathcal{H}^+ = \{0\} \times [0, +\infty]_V \times S_{\theta, \varphi^\pm}^2 , \quad \mathcal{H}^- = [0, +\infty]_U \times \{0\} \times S_{\theta, \varphi^\pm}^2
\]
since \( t^* \) (resp. \( *t \)) is regular at \( \mathcal{H}^+ \) (resp. \( \mathcal{H}^- \)), takes all real valued on \( \mathcal{H}^+ \) (resp. \( \mathcal{H}^- \)) and tends to \(-\infty \) (resp. \( +\infty \)) at \( \mathcal{H}^- \) (resp. \( \mathcal{H}^+ \)). In the KBL coordinates, the Kerr metric takes the form

\[
g = -\frac{G_+^2 a^2 \sin^2 \theta}{4 \kappa_+ \rho^2} \left( r - r_-(r + r_+) \right) \left( \frac{\rho^2}{r^2 + a^2} + \frac{\rho_+^2}{r_+^2 + a^2} \right) (U^2 dV^2 + V^2 dU^2)
\]

\[
= -\frac{G_+ (r - r_-)}{2 \kappa_+ \rho^2} \left[ \frac{\rho^4}{(r^2 + a^2)^2} + \frac{\rho_+^4}{(r_+^2 + a^2)^2} \right] dU dV
\]

\[
- \frac{G_+ a \sin^2 \theta}{\kappa_+ \rho^2 (r_+^2 + a^2)} [\rho_+^2 (r - r_-) + (r^2 + a^2)(r + r_+)] (U dV - V dU) d\varphi^\pm
\]

\[
- \rho^2 d\theta - g_{\varphi \varphi} (d\varphi^\pm)^2 , \tag{2.5}
\]

where

\[
\rho_+^2 = r_+^2 + a^2 \cos^2 \theta , \quad G_+ = \frac{r - r_-}{UV} = e^{-2\kappa_+ r} |r - r_-| r_- / r_+ .
\]

The functions \( r \) and \( G \) are analytic and non-vanishing on \([0, +\infty]_U \times [0, +\infty]_V \). The expression \( (2.5) \) above shows that \( g \) is smooth on \( \mathcal{B}_I \cup \mathcal{H}^+ \cup \mathcal{H}^- \) and can be extended smoothly on \([0, +\infty]_U \times [0, +\infty]_V \times S_{\theta, \varphi^\pm}^2 \). The 2–sphere \( S_c^2 = \{U = V = 0\} \) that is the intersection of the future and past horizons is called the crossing sphere. It is a regular surface in the extended spacetime

\[
\mathcal{B}_I^{KBL} = \left( [0, +\infty]_U \times [0, +\infty]_V \times S_{\theta, \varphi^\pm}^2 , g \right)
\]

So that the horizon in KBL coordinates is a union of two smooth null boundaries \( \mathcal{H}^+ \cup S_c^2 \) and \( S_c^2 \cup \mathcal{H}^- \) and is given by

\[
\mathcal{H} = \mathcal{H}^- \cup S_c^2 \cup \mathcal{H}^+ = (\{0, +\infty \mid U \times \{0\} \times S_{\theta, \varphi^\pm}^2 \}) \cup (\{0\} \times [0, +\infty \mid V \times S_{\theta, \varphi^\pm}^2 \})
\]

The relation between the coordinate vector fields of KBL coordinates and of Boyer-Lindquist coordinates is given by

**Remark 2.2.3.**

\[
\partial_t = \kappa_+ ( -U \partial_U + V \partial_V ) - \frac{a}{r_+^2 + a^2} \partial_\varphi^\pm , \quad \partial_r = \kappa_+ \frac{r^2 + a^2}{\Delta} (U \partial_U + V \partial_V ) ,
\]

\[
\partial_\theta = \partial_\theta , \quad \partial_\varphi = \partial_\varphi^\pm .
\]

\footnote{See proposition 3.4.7 in [69].}
2.2.4 Penrose compactification

To study the asymptotic behavior of the fields on the exterior of the black hole by the geometric approach, we need to make this region compact. And then we will study the properties of the fields on the infinite boundaries of the space result. The compactification in our case is called the Penrose compactification.

The Penrose compactification of the exterior of Kerr black hole is obtained by combining two constructions: the first is compactification for Kerr-star spacetime $K^*$, the second is compactification for star-Kerr $^*K$ spacetime.

Future null infinity is defined as the set of points that are reached by outgoing principal null geodesics as $r \to +\infty$. Future null infinity is an abstract 3–surface, describing the congruence of outgoing principal null geodesics, can be given by using star-Kerr coordinates. To make the compactification of the exterior of the black hole in star-Kerr coordinates, we replace the variable $r$ by $R = 1/r$ in the formula of star-Kerr metric \( [2,3] \). The exterior of the black hole is now described

\[
\mathcal{B}_I = \mathbb{R} \times 0, \frac{1}{r_+} \left[ 0, \frac{1}{R} \right] \times S^2_{\theta, \varphi} ,
\]

with the conformal rescaled metric

\[
\hat{g} = \Omega^2 g_{*K} , \quad \Omega = R = 1/r .
\]
\[ \hat{g} = R^2 g_{\cdot K} = R^2 \left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \frac{4MaR \sin^2 \theta}{\rho^2} d\tau^2 + \sin^2 \theta d\phi^2 + \left( 1 + a^2 R^2 + \frac{2Ma^2 R \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\phi^2 - \left( 1 + a^2 R^2 \cos^2 \theta \right) d\theta^2 - 2d\tau dR + 2a \sin^2 \theta d\phi dR. \] (2.6)

Due to this expression, we can see that \( \hat{g} \) doesn’t have singularities on the hypersurfaces \( \{ R = 0 \} \) and \( \{ R = 1/r_+ \} \) so \( \hat{g} \) can be extended smoothly on the domain

\[ \mathbb{R}_{t^*} \times \left[ 0, \frac{1}{r_+} \right] \times S^{2}_{\theta, \phi}, \]

and the null hypersurface

\[ \mathcal{I}^+ := \mathbb{R}_{t^*} \times \{ R = 0 \} \times S^{2}_{\theta, \phi} \]

can be added to the rescaled spacetime as a smooth hypersurface, which describe the future null infinity. This hypersurface is indeed null because

\[ \hat{g}|_{R=0} = -d\theta^2 - \sin^2 \theta d\phi^2 \]

is degenerate (recall that \( \mathcal{I}^- \) is a 3—surface) and

\[ \det(\hat{g}) = -R^4 \rho^4 \sin^2 \theta \]

does not vanish for \( R = 0 \).

![Figure 2.2: Compactification block I by outgoing principal null geodesics.](image)

Similarly, using Kerr-star coordinates instead of star-Kerr coordinates, the exterior of the black hole is also given by

\[ \mathcal{B}_I = \mathbb{R}_{t^*} \times \left[ 0, \frac{1}{r_+} \right] \times S^{2}_{\theta, \phi^*}, \]
but with the conformal rescaled metric

\[ \hat{g}' = \Omega^2 g_K^\ast, \quad \Omega = R = 1/r. \]

\( \hat{g}' \) has the detailed form

\[
\hat{g}' = R^2 g_K^\ast = R^2 \left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \frac{4MaR \sin^2 \theta}{\rho^2} dt^* d\varphi^*
\]

\[ = \left( 1 + a^2 R^2 - \frac{2Ma^2 R \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\varphi^2 - (1 + a^2 R^2 \cos^2 \theta) d\theta^2 + 2 dt^* dR - 2a \sin^2 \theta d\varphi^* dR. \tag{2.7} \]

So we can also describe the past null infinity (the set of the points that are reached as \( r \rightarrow +\infty \) by incoming principal null geodesics) by

\[ \mathcal{I}^- = \mathbb{R}_t^* \times \{ R = 0 \} \times S^2_{\theta, \varphi^*}, \]

and we can add this hypersurface to the exterior of the black hole.

Therefore we obtain that the Penrose compactification of block I is the spacetime

\( \overline{(\mathcal{B}_I, \hat{g})}, \overline{\mathcal{B}_I} = \mathcal{B}_I \cup \mathcal{I}^+ \cup S^2_c \cup \mathcal{I}^- \cup \mathcal{I}^+ \cup \mathcal{I}^-, \)

where \( \hat{g} \) is defined by

\[ \hat{g} = \Omega^2 g, \quad \Omega = R = 1/r. \]

Note that the compactification spacetime is not compact since there are three "points" missing to the boundary: the first point \( i^+ \) is called future timelike infinity, it is the limit point of uniformly timelike curves as \( t \rightarrow +\infty \); the second point \( i^- \) is called past timelike infinity, it is symmetric to \( i^+ \) in the distant past; and the third point \( i_0 \) is called spacelike infinity, it is the limit point of uniformly spacelike curves as \( r \rightarrow +\infty \). Three "points" can be described as 2−spheres, they are singularities of the rescaled metric.
Finally, we serve the next sections by giving the inverse form of the rescaled metric $\hat{g}$

$$
\hat{g}^{-1} = -\frac{1}{\rho^2} \left( r^2 a^2 \sin^2 \theta \partial_t^2 + 2(r^2 + a^2) \partial_t \partial_R + 2ar^2 \partial_t \partial_\varphi + 2a \partial_R \partial_\varphi \right) - \frac{1}{\rho^2} \left( R^2 \Delta \partial_R^2 + r^2 \partial_\theta^2 + \frac{r^2}{\sin^2 \theta} \partial_\varphi^2 \right).
$$

\begin{equation}
(2.8)
\end{equation}

2.2.5 Neighbourhood of spacelike infinity

Since the peeling is only valid in a neighbourhood of spacelike infinity $i_0$, so to begin our work we need to choose a neighbourhood of $i_0$. For a given $^*t_0 < < -1$, we consider the domain as follows:

$$
\Omega_{^*t_0}^+ := \left\{ (^*t, R, \theta, ^*\varphi); ^*t \leq ^*t_0 + \hat{r} - r_*, \ 0 \leq t \leq +\infty \right\},
$$

where $\hat{r}$ is the parameter that appears in the compactification of block $\mathcal{B}_I$ by using the simple null geodesics (see Häfner [38]):

$$
\hat{r} = r_* + \int_{-\infty}^{r_*} \left( \sqrt{1 - \frac{a^2 \Delta(s)}{(r(s)^2 + a^2)^2}} - 1 \right) \, ds + a \sin \theta
$$

with $r_*$ is the Regge-Wheeler coordinate

$$
\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}.
$$

Now we show that $\Omega_{^*t_0}^+$ is a neighbourhood of $i_0$. Indeed, we can see that $i_0 \in \Omega_{^*t_0}^+$ and $\Omega_{^*t_0}^+$ is included in the compactification domain with two boundaries that are the parts of $\Sigma_0$ and $\mathcal{I}^+$; we
need to show that the third boundary cuts the two hypersurfaces $\Sigma_0$ and $\mathcal{I}^+$ at the finite points. The third boundary is given by:

$$S_{t_0} = \{ *t = *t_0 + \hat{r} - r_* \} .$$

- If $t = 0$ then $*t_0 + \hat{r} = 0$, hence $r$ is finite which in turn means that $S_{t_0}$ and $\Sigma_0$ intersect at a finite point.
- If $R = 0$ then $*t = *t_0 + a \sin \theta$ is finite which in turn means that $S_{t_0}$ and $\mathcal{I}^+$ intersect at a finite point.

So $\Omega_{t_0}^+$ is really a neighbourhood of $i_0$. Furthermore the third boundary $S_{t_0}$ is a null hypersurface. Indeed, if we set

$$f(*t, R, \theta, *\varphi) = *t - *t_0 - \hat{r} + r_* .$$

The 1–form conormal vector to the hypersurface $S_{t_0}$ is

$$\hat{\nabla}_a f dx^a = d\ast t + \frac{(a^2 + r^2)r^2}{\Delta} XdR - a \cos \theta d\theta ,$$

where

$$X = \sqrt{1 - \frac{a^2\Delta}{(r^2 + a^2)^2}} - 1 .$$

So we have

$$\rho^4 \hat{g}^{-1}(\hat{\nabla}_a f dx^a, \hat{\nabla}_b f dx^b) = r^2 a^2 + 2 \frac{(a^2 + r^2)^2 r^2}{\Delta} X + \frac{(a^2 + r^2)^2 r^2}{\Delta} X^2$$

$$= r^2 \left( a^2 + 2 \frac{(a^2 + r^2)^2}{\Delta} X + \frac{(a^2 + r^2)^2}{\Delta} X^2 \right) .$$

Since the positive solution of the equation

$$a^2 + 2 \frac{(a^2 + r^2)^2}{\Delta} x + \frac{(a^2 + r^2)^2}{\Delta} x^2 = 0$$

is

$$x = -1 + \sqrt{1 - \frac{a^2\Delta}{(a^2 + r^2)^2}} = X ,$$

so we can conclude that $\hat{g}^{-1}(\hat{\nabla}_a f dx^a, \hat{\nabla}_b f dx^b) = 0$, this means that the hypersurface $S_{t_0}$ is null\(^8\).

We can foliate $\Omega_{t_0}^+$ by the hypersurfaces

$$\mathcal{H}_s = \{ *t = -sr_*; *t \leq *t_0 + \hat{r} - r_* \} , \ 0 \leq s \leq 1 ,$$

- If $s = 1$ then $t = 0$, from which we get that the first boundary $\mathcal{H}_1$ is a part of the hypersurface $\Sigma_0 = \{ t = 0 \}$ inside $\Omega_{t_0}^+$.

\(^8\)We can see the construction and prove that $S_{t_0}$ is a null hypersurface in a more simple and natural way than above, if we work on the compactification by SNG, there $S_{t_0} = \{ \hat{t} = t - \hat{r} = *t_0 \}$ is really a null hypersurface.
If \( s = 0 \), we get that the second boundary \( \mathcal{H}_0 \) (also denoted by \( \mathcal{I}_{t_0}^+ \)) is a part of the hypersurface \( \mathcal{I}^+ \) inside \( \Omega_{t_0}^+ \). Besides, \( \mathcal{I}_{t_0}^+ \) can be understood as the limit of the hypersurface \( \mathcal{H}_s \) as \( s \) tends to zero. Indeed, for \( ^*t \) and \( (\theta, ^*\varphi) \) fixed, as \( s \) tends to zero then \( r_+ \) is constrained to tend towards \( +\infty \), thus we go to \( +\infty \) along an outgoing principal null geodesics and we approach the point \((^*t, R = 0, \theta, ^*\varphi) \in \mathcal{I}^+ \).

Figure 2.5: Foliation of the neighbourhood of spacelike infinity.

Now we show that \( \mathcal{H}_s \) \((0 < s \leq 1)\) are spacelike hypersurfaces. Indeed, if we set

\[
g(^*t, R, \theta, ^*\varphi) = ^*t + sr_+.
\]

The 1–form conormal vector to \( \mathcal{H}_s \) is

\[
\hat{\nabla}_a g dx^a = d^* t - \frac{s(a^2 + r^2)}{\Delta R^2} dR,
\]

so we have

\[
\rho^4 \hat{g}^{-1}(\hat{\nabla}_a g dx^a, \hat{\nabla}_b g dx^b) = r^2 a^2 \sin^2 \theta - 2 \frac{(a^2 + r^2)^2 r^2}{\Delta} s + \frac{(a^2 + r^2)^2 r^2}{\Delta} s^2
\]

\[
= r^2 \left( a^2 \sin^2 \theta - 2 \frac{(a^2 + r^2)^2}{\Delta} s + \frac{(a^2 + r^2)^2}{\Delta} s^2 \right).
\]

We consider the equation

\[
a^2 \sin^2 \theta - 2 \frac{(a^2 + r^2)^2}{\Delta} s + \frac{(a^2 + r^2)^2}{\Delta} s^2 = 0,
\]

it has two solutions

\[
s_1 = 1 - \sqrt{1 - \frac{a^2 \sin^2 \theta \Delta}{(a^2 + r^2)^2}} \text{ and } s_2 = 1 + \sqrt{1 - \frac{a^2 \sin^2 \theta \Delta}{(a^2 + r^2)^2}},
\]
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since $s_2 > 1$ and for $r$ is large enough, we have

$$s = -\frac{st}{r} > s_1 \simeq \frac{a^2 \sin^2 \theta R^2}{\sqrt{2}}.$$  

So we can conclude $\hat{g}^{-1}(\hat{\nabla}_a gdx^a, \hat{\nabla}_b gdx^b) < 0$, this means that $\mathcal{H}_s$ ($0 < s \leq 1$) are spacelike hypersurfaces.

We have the relations

$$\frac{dr_*}{dR} = -(r^2 + a^2)^{-1}\Delta, \quad dR|_{\mathcal{H}_s} = \frac{R^2 \Delta}{s(r^2 + a^2)} d^s t|_{\mathcal{H}_s} = \frac{r_* R^2 \Delta}{(r^2 + a^2)|^s t|^s t|_{\mathcal{H}_s} \text{ near } i^0}.$$  

The foliation $\mathcal{H}_s$ is not smooth because $r_*^{-1}$ is not a smooth function of $R$ at $R = 0$: although the first derivative is bounded and tends to 1, the second is logarithmically divergent as $R \to 0$.

With this foliation, we need a natural identifying vector field $\nu$ that satisfies $\nu(s) = 1$. We can choose

$$\nu = r^2 R^2 \frac{\Delta}{r^2 + a^2}|^s t|^{-1} \partial_R.$$  

The 4–volume measure $d\text{Vol}^4 = -(1 + a^2 R^2 \cos^2 \theta) d^s t R d^2 \omega$, against which the error terms will be integrated, is thus decomposed into the product of $ds$ and

$$\nu_d \text{Vol}^4|_{\mathcal{H}_s} = -r_* R^2 \frac{\Delta}{r^2 + a^2} \frac{1}{|^s t|^s t} (1 + a^2 R^2 \cos^2 \theta) d^s t d^2 \omega|_{\mathcal{H}_s}.$$  

The former being the measure along the integral lines of $\nu^a$ and the latter our 3–volume measure on each $\mathcal{H}_s$ and it can be written under the equivalent form as follows:

$$\nu_d \text{Vol}^4|_{\mathcal{H}_s} \simeq -\frac{1}{|^s t|^s t} d^s t d^2 \omega|_{\mathcal{H}_s}.$$  

2.2.6 The Morawetz vector field

To contract with the energy tensor (we will mention in the next section) to obtain the current energy, we need a timelike vector field everywhere near $i_0$. Here, we choose a modification of the Morawetz vector field on the Kerr background

$$T^a := \ast t^2 \partial_t - 2(1 + *tR) \partial_R,$$

the necessary properties of $T^a$ are given by following lemma:

Lemma 2.2.1. The vector field $T^a$ is everywhere timelike, future-oriented near $i_0$ and has the Killing form

$$\nabla(a T_b) dx^a dx^b$$

$$= 4 \left\{ (1 + \ast t R) M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) - \frac{2 M \ast t R}{\rho^2} \right\} d^s t^2 - 4 a \sin^2 \theta \ast t R d \ast \varphi$$

$$- 4 a \sin^2 \theta \left\{ 2(1 + \ast t R) M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) - 2 M \ast t R \frac{R}{\rho^2} + R \right\} d^s t d\varphi$$

$$+ 4 a^2 \cos^2 \theta (1 + \ast t R) R d\theta^2 + 4 a^2 \sin^2 \theta (1 + \ast t R) \left\{ R + \sin^2 \theta M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \right\} d\varphi^2.$$
\begin{proof}

First, we have
\[
\hat{g}_{ab} T^a T^b = R^2 \left( 1 - \frac{2Mr}{\rho^2} \right) t^4 + 4t^2(1 + tR)
\]
\[
= t^2 \left\{ 4(1 + tR) + t^2 R^2 \left( 1 - \frac{2Mr}{\rho^2} \right) \right\}.
\]

This latter vanishes for the two values of \( \pm tR \):
\[
\left( \pm tR \right)_ \pm = - \frac{1 \pm \sqrt{2M \rho}}{1 - \frac{2Mr}{\rho^2}}
\]
which get arbitrarily close to \(-2\), as \( R \) is small enough. And \( \hat{g}_{ab} T^a T^b \) is positive for \( \pm tR \) outside \( \left( \pm tR \right)_- \). In a small enough neighbourhood of \( i_0 \), we shall have
\[
-r_* < \pm t << -1, \text{ hence } - \frac{r_*}{r} < \pm t R << -R,
\]
and \( \pm tR \) therefore lives in an interval of the form \([-1 - \varepsilon, 0]\), where \( \varepsilon > 0 \) is as small as we wish it to be, since at infinity \( r_* \approx r \). Consequently, in a small enough neighbourhood of \( i_0 \), the vector \( T^a \) is uniformly timelike.

We also have \( T^a \) is future-oriented since the fact that
\[
T^a(t) = t^2 \partial_t - 2(1 + tR) \partial_R - 2(1 + tR) \partial_R (t + r_*)
\]
\[
= t^2 + 2(1 + tR) \frac{a^2 + r^2}{\Delta R^2} > 0
\]
in the neighbourhood \( \Omega^+_{i_0} \) of \( i_0 \).

Now we use Lie derivative to compute the Killing form of \( T^a \). We have
\[
\nabla_{(a} T_{b)} dx^a dx^b = \mathcal{L}_T \left( \hat{g}_{ab} dx^a dx^b \right) = \hat{g}_{ab} \mathcal{L}_T dx^a dx^b + \hat{g}_{ab} \left( \mathcal{L}_T dx^a \otimes dx^b + \mathcal{L}_T dx^b \otimes dx^a \right)
\]
and
\[
\mathcal{L}_T d^a t = 2 t^* d^a t, \quad \mathcal{L}_T d^a R = -2 R d^a t - 2^* t d^a R, \quad \mathcal{L}_T d\theta = \mathcal{L}_T d^a \varphi = 0.
\]

So that
\[
\nabla_{(a} T_{b)} dx^a dx^b
\]
\[
= 4 \left\{ (1 + tR) M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) - \frac{2M^2 t R}{\rho^2} \right\} d^* t^2 - 4a \sin^2 \theta d^* t d^* \varphi
\]
\[
- 4a \sin^2 \theta \left\{ 2(1 + tR) M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) - 2M^2 t R \frac{R}{\rho^2} + R \right\} d^* t d^* \varphi
\]
\[
+ 4a^2 \cos^2 \theta (1 + tR) R d^2 \theta^2 + 4a^2 \sin^2 \theta (1 + tR) \left\{ R + \sin^2 \theta M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \right\} d^* \varphi^2,
\]
this completes our proof.
\end{proof}
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2.3 The linear scalar fields

The linear scalar fields are the solutions of the wave equation

$$\Box_g \psi = 0,$$

where the operator d’Alembertian $\Box_g$ on the local coordinate $\{x^a\}$ is defined by (1.4)

$$\Box_g = \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^a} \left( \sqrt{|\det g|} g^{ab} \frac{\partial}{\partial x^b} \right).$$

The expression of the wave equation in the Boyer-Lindquist coordinate is

$$\frac{\sigma^2}{\Delta \rho^2} \partial_t^2 \psi + \frac{4aMr}{\Delta \rho^2} \partial_t \partial_\varphi \psi - \frac{1}{\rho^2} \partial_r (\Delta \partial_r \psi) - \frac{1}{\rho^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) - \rho^2 - 2Mr \frac{\sigma^2}{\Delta \rho^2 \sin^2 \theta} \partial_\varphi^2 \psi = 0,$$

which is equivalent to

$$\partial_t^2 \psi + \frac{4aMr}{\sigma^2} \partial_t \partial_\varphi \psi - \frac{\Delta}{\sigma^2} \partial_r (\Delta \partial_r \psi) - \frac{\Delta}{\sigma^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) - \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2 \theta} \partial_\varphi^2 \psi = 0.$$

Since the wave equation is hyperbolic symmetry, and since block I is a globally hyperbolic spacetime, by using Leray theorem the Cauchy problem of the system

$$\begin{cases}
\Box_g \psi = 0, \\
\psi|_{\Sigma_0} \in C^\infty_0 (\Sigma_0; \mathbb{C}), \\
\partial_t \psi|_{\Sigma_0} \in C^\infty_0 (\Sigma_0; \mathbb{C})
\end{cases}$$

has a unique solution $\psi \in C^\infty (B_I; \mathbb{C})$.

2.3.1 Wave equation for rescaled metric $\hat{g}$

We know that on the exterior of Kerr black hole $B_I$, the two properties are equivalent (see R. Penrose and W. Rindler [73], Vol 2):

1. $\psi \in C^\infty (B_I)$ satisfies wave equation $\Box_g \psi = 0$;

2. $\hat{\psi} = \Omega^{-1} \psi = r \psi \in C^\infty (B_I)$ satisfies wave equation $\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi} = 0$ for the rescaled metric $\hat{g}$ on $B_I$ and this solution can be extended to the boundaries $\mathcal{I}^{\pm}$ of $B_I$.

Now we find the expression of wave equation for the rescaled star-Kerr metric $\hat{g}$. First, due to the expression (1.3) of the rescaled scalar curvature and the expression (2.8) of the inverse metric $g^{-1}$, we have

$$\frac{1}{6} \text{Scal}_{\hat{g}} = \frac{1}{6} R^{-3} \Box_g R = \frac{1}{6} \frac{r^3}{\sqrt{|g|}} \partial_r \sqrt{|g|} g^{rr} \partial_r \frac{1}{r} = \frac{2Mr - a^2}{\rho^2}.$$

Second, we have the associated d’Alembertian for $\hat{g}$

$$\Box_{\hat{g}} = \frac{1}{\sqrt{|\hat{g}|}} \frac{\partial}{\partial x^a} \sqrt{|\hat{g}|} \hat{g}^{ab} \frac{\partial}{\partial x^b}.$$
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Calculating in detail, we get

$$\Box \hat{g} = -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_{t}^2 - \frac{r^2 + a^2}{\rho^2} \partial_{t} \partial_{R} - \frac{2ar^2}{\rho^2} \partial_{t} \partial_{\varphi} - \frac{a}{\rho^2} \partial_{R} \partial_{\varphi} - \frac{r^2}{\rho^2 \sin^2 \theta} \partial_{\varphi}^2$$

$$- \frac{r^2}{\rho^2 \sin \theta} \left( \partial_{R} R^2 (r^2 + a^2) \sin \theta \partial_{t} + \partial_{R} R^2 \sin \theta R^2 \Delta \partial_{R} + \partial_{R} a R^2 \sin \theta \partial_{\varphi} + \partial_{\theta} \sin \theta \partial_{\theta} \right)$$

$$= -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_{t}^2 - \frac{r^2 + a^2}{\rho^2} \partial_{t} \partial_{R} - \frac{2ar^2}{\rho^2} \partial_{t} \partial_{\varphi} - \frac{a}{\rho^2} \partial_{R} \partial_{\varphi} - \frac{r^2}{\rho^2 \sin^2 \theta} \partial_{\varphi}^2$$

$$- \frac{r^2}{\rho^2 \sin \theta} \left( (1 + a^2 R^2) \sin \theta \partial_{t} \partial_{R} + (R^2 - 2MR^3 + a^2 R^4) \sin \theta \partial_{R}^2 + aR^2 \sin^2 \theta \partial_{R} \partial_{\varphi} + \sin \theta \partial_{\theta}^2 \right)$$

$$- \frac{r^2}{\rho^2 \sin \theta} \left( 2a^2 R \sin \theta \partial_{t} + (2R - 6MR^2 + 4aR^3) \sin \theta \partial_{R} + 2aR \sin \theta \partial_{\varphi} + \cos \theta \partial_{\theta} \right)$$

$$= -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_{t}^2 - \frac{r^2 + a^2}{\rho^2} \partial_{t} \partial_{R} - \frac{2ar^2}{\rho^2} \partial_{t} \partial_{\varphi} - \frac{a}{\rho^2} \partial_{R} \partial_{\varphi} - \frac{r^2}{\rho^2 \sin^2 \theta} \partial_{\varphi}^2$$

$$- \frac{r^2}{\rho^2} \left( (1 + a^2 R^2) \partial_{t} \partial_{R} + (R^2 - 2MR^3 + a^2 R^4) \partial_{R}^2 + aR^2 \partial_{R} \partial_{\varphi} + \partial_{\theta}^2 \right)$$

$$- \frac{r^2}{\rho^2} \left( 2a^2 R \partial_{t} \partial_{R} + (2R - 6MR^2 + 4aR^3) \partial_{R} + 2aR \partial_{\varphi} + \cot \theta \partial_{\theta} \right)$$

$$= -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_{t}^2 - \frac{r^2 + a^2}{\rho^2} \partial_{t} \partial_{R} - \frac{2ar^2}{\rho^2} \partial_{t} \partial_{\varphi} - \frac{a}{\rho^2} \partial_{R} \partial_{\varphi}$$

$$- \frac{r^2}{\rho^2} \left( (1 + a^2 R^2) \partial_{t} \partial_{R} + (R^2 - 2MR^3 + a^2 R^4) \partial_{R}^2 + aR^2 \partial_{R} \partial_{\varphi} \right)$$

$$- \frac{r^2}{\rho^2} \left( 2a^2 R \partial_{t} \partial_{R} + (2R - 6MR^2 + 4aR^3) \partial_{R} + 2aR \partial_{\varphi} \right) - \frac{r^2}{\rho^2} \Delta S^2,$$

where $S^2$ is 2-sphere with the metric

$$d\omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2,$$

and the Laplacian operator on $S^2$ is

$$\Delta S^2 = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi \partial_\varphi = \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2.$$

So the wave equation for the rescaled metric $\hat{g}$ is

$$\left( \Box \hat{g} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi} = -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_{t}^2 \hat{\psi} - \frac{r^2 + a^2}{\rho^2} \partial_{t} \partial_{R} \hat{\psi} - \frac{2ar^2}{\rho^2} \partial_{t} \partial_{\varphi} \hat{\psi} - \frac{a}{\rho^2} \partial_{R} \partial_{\varphi} \hat{\psi}$$

$$- \frac{r^2}{\rho^2} \left( (1 + a^2 R^2) \partial_{t} \partial_{R} \hat{\psi} + (R^2 - 2MR^3 + a^2 R^4) \partial_{R}^2 \hat{\psi} + aR^2 \partial_{R} \partial_{\varphi} \hat{\psi} \right)$$

$$- \frac{r^2}{\rho^2} \left( 2a^2 R \partial_{t} \partial_{R} \hat{\psi} + (2R - 6MR^2 + 4aR^3) \partial_{R} \hat{\psi} + 2aR \partial_{\varphi} \hat{\psi} \right)$$

$$- \frac{r^2}{\rho^2} \Delta S^2 \hat{\psi} + 2 \frac{Mr - a^2}{\rho^2} \hat{\psi}$$

$$= 0. \tag{2.10}$$
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2.3.2 Commutation with vector fields

For higher order energy estimates we need to commute vector fields into the wave equation (2.10) (for metric $\hat{g}$). To do that, we use the five vector fields $X_i \in \mathcal{A}$ as follows:

$$X_0 = \partial_t, \; X_1 = \partial_\varphi, \; X_2 = \sin^* \varphi \partial_\theta + \cot \theta \cos^* \varphi \partial_\varphi, \; X_3 = \cos^* \varphi \partial_\theta - \cot \theta \sin^* \varphi \partial_\varphi, \; X_4 = \partial_R.$$  

Where $X_1, X_2$ and $X_3$ are vector fields tangent on 2–sphere $S^2$. The difference in the Kerr background from the symmetric spherical spacetimes is that the vector fields $X_2$ and $X_3$ are not Killing. But they are still useful to commute into the wave equation (2.10), since they commute with the Laplacian operator $\Delta_{S^2}$ and $[\mathcal{L}_{X_1}, \mathcal{L}_{X_2}] = \mathcal{L}_{X_3}, \; [\mathcal{L}_{X_1}, \mathcal{L}_{X_3}] = -\mathcal{L}_{X_2}$.

First, we commute the Lie derivative along the directions $X_0 = \partial_t$ and $X_1 = \partial_\varphi$ into the wave equation (2.10) (for rescaled metric $\hat{g}$) to get

$$\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_i} \hat{\psi} = 0 \; (i = 0, 1). \quad (2.11)$$

Second, we commute Lie derivative $\mathcal{L}_{X_i}$ ($i = 2, 3$) into the wave equation (2.10). For convenience, we will multiply the wave equation (2.10) with the factor $\rho^2/r^2$ and then we commute $\mathcal{L}_{X_i}$ into the new equation to get

$$\mathcal{L}_{X_2} \left( \frac{\rho^2}{r^2} \left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi} \right) = \frac{\rho^2}{r^2} \left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_2} \hat{\psi} - a^2 \sin 2\theta \sin^* \varphi \partial_t^2 \hat{\psi} + 2a \partial_t \mathcal{L}_{X_3} \hat{\psi} + \frac{a^2}{r^2} \partial_R \mathcal{L}_{X_3} \hat{\psi} + a R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} + 2a RL \mathcal{L}_{X_3} \hat{\psi}$$

$$= 0,$$

where $i \neq j$ and $i, j \in \{2, 3\}$. So we have

$$\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_2} \hat{\psi} = \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial_t^2 \hat{\psi} - 2a \partial_t \mathcal{L}_{X_3} \hat{\psi} - a^2 R^2 \partial_R \mathcal{L}_{X_3} \right)$$

$$- \frac{r^2}{\rho^2} \left( a R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} + 2a RL \mathcal{L}_{X_3} \hat{\psi} \right). \quad (2.12)$$

Similarly for $\mathcal{L}_{X_3}$

$$\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_3} \hat{\psi} = \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial_t^2 \hat{\psi} + 2a \partial_t \mathcal{L}_{X_2} \hat{\psi} + a^2 R^2 \partial_R \mathcal{L}_{X_2} \right)$$

$$+ \frac{r^2}{\rho^2} \left( a R^2 \partial_R \mathcal{L}_{X_2} \hat{\psi} + 2a RL \mathcal{L}_{X_2} \hat{\psi} \right). \quad (2.13)$$

Third, we commute the Lie derivative along the direction $X_4 = \partial_R$ into the equation (2.10) and note that $\mathcal{L}_{X_4}(\hat{\psi}) = \hat{\psi}_R$. Similarly as above, we will commute $\partial_R$ into the new equation that is obtained
by multiplying the wave equation (2.10) with the factor $\frac{\rho^2}{r^2}$, so we get

$$\partial_R \left( \frac{\rho^2}{r^2} \left( \Box \tilde{g} + \frac{1}{6} \text{Scal}_\tilde{g} \right) \tilde{\psi} \right) = \frac{\rho^2}{r^2} \left( \Box \tilde{g} + \frac{1}{6} \text{Scal}_\tilde{g} \right) \partial_R \hat{\psi} - 4a^2 R \partial_t \partial_R \hat{\psi} - 4a R \partial_R \partial_\varphi \hat{\psi}$$

$$- 2a^2 \partial_t \hat{\psi} - 2a \partial_\varphi \hat{\psi} - (4a^2 R^2 - 6M R^2 + 2R) \partial_R^2 \hat{\psi}$$

$$- (12aR^2 - 12MR + 2) \partial_R \hat{\psi} + 2(M - 2a^2 R) \hat{\psi},$$

then we have

$$\left( \Box \tilde{g} + \frac{1}{6} \text{Scal}_\tilde{g} \right) \partial_R \hat{\psi} = \frac{r^2}{\rho^2} \left( 4a^2 R \partial_t \partial_R \hat{\psi} + 4a R \partial_\varphi \partial_R \hat{\psi} + 2a^2 \partial_t \hat{\psi} + 2a \partial_\varphi \hat{\psi} \right)$$

$$+ \frac{r^2}{\rho^2} \left( (4a^2 R^2 - 6M R^2 + 2R) \partial_R^2 \hat{\psi} + (12aR^2 - 12MR + 2) \partial_R \hat{\psi} \right)$$

$$- 2(M - 2a^2 R) \hat{\psi} \right) \). \ (2.14)$$

### 2.3.3 Approximate conservation laws

We use the stress-energy tensor for the wave equation (for the rescaled metric $\tilde{g}$):

$$T_{ab}(\hat{\psi}) = T_{(ab)}(\hat{\psi}) = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \hat{\psi}^c \partial_c \hat{\psi} \partial_d \hat{\psi}$$

in order to define energies for which we establish estimates. We have

$$\nabla^a T_{ab}(\hat{\psi}) = \Box \hat{\psi} \partial_b \hat{\psi} = -2 \frac{Mr - a^2}{r^2} \hat{\psi} \partial_b \hat{\psi}$$

(2.15)

associated with the wave equation (2.10), so we obtain the conservation law for the wave equation

$$\nabla^a \left( T^b T_{ab}(\hat{\psi}) \right) = \Box \hat{\psi} T^b \partial_b \hat{\psi} + (\nabla^a T^b) T_{ab}(\hat{\psi})$$

$$= -2 \frac{Mr - a^2}{r^2} \hat{\psi} T^b \partial_b \hat{\psi} + (\nabla_a T_b) T^{ab}(\hat{\psi}) \). \ (2.16)$$

Now associating equality (2.15) for $L_{X_i} \hat{\psi} (i = 0, 1)$ and the equation (2.11), we obtain the conservation laws for the equation of $L_{X_i} (i = 0, 1)$ which is as follows

$$\nabla^a \left( T^b T_{ab}(L_{X_i} \hat{\psi}) \right) = -2 \frac{Mr - a^2}{r^2} (L_{X_i} \hat{\psi}) T^b \partial_b (L_{X_i} \hat{\psi}) + (\nabla_a T_b) T^{ab}(L_{X_i} \hat{\psi}) \). \ (2.17)$$

Associating equality (2.15) for $L_{X_3} \hat{\psi}$ and the equation (2.12), we obtain the conservation law for the equation of $L_{X_3}$ which is as follows

$$\nabla^a \left( T^b T_{ab}(L_{X_3} \hat{\psi}) \right) = \Box \hat{\psi} (L_{X_3} \hat{\psi}) T^b \partial_b (L_{X_3} \hat{\psi}) + \nabla^a T^b T_{ab}(L_{X_3} \hat{\psi})$$

$$= \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^2 \varphi \partial_t^2 - 2a \partial_\varphi \partial_t \hat{\psi} - a^2 R^2 \partial_R \hat{\psi} \right)$$

$$- aR^2 \partial_R \hat{\psi} - 2a R \partial_{L_{X_3}} \hat{\psi}$$

$$+ (\nabla_a T_b) T^{ab}(L_{X_3} \hat{\psi}) \). \ (2.18)$$
The following lemma gives us the equivalent expressions of the energies:

\[ \nabla^a \left( T^b_{ab}(\mathcal{L}_{X^a} \hat{\psi}) \right) = \Box \hat{\psi} (\mathcal{L}_{X^a} \hat{\psi}) T^b_b (\mathcal{L}_{X^a} \hat{\psi}) + \nabla^a T^b_{ab}(\mathcal{L}_{X^a} \hat{\psi}) \]

\[ = \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial_t^2 + 2a \partial_t \mathcal{L}_{X^2} \hat{\psi} + a^2 R^2 \partial_R \mathcal{L}_{X^2} \right) \]

\[ + a^2 \mathcal{L}_{X^2} \hat{\psi} \mathcal{L}_{X^2} \hat{\psi} + 2a R \mathcal{L}_{X^2} \hat{\psi} \hat{\psi} T^b_b (\mathcal{L}_{X^2} \hat{\psi}) \]

\[ + (\nabla_a T_b) T^{ab}(\mathcal{L}_{X^2} \hat{\psi}) \]  

Finally, associating equality (2.15) for \( \mathcal{L}_{X^2} \hat{\psi} = \partial_R \hat{\psi} \) and the equation (2.14), we obtain the conservation law for the equation of \( \mathcal{L}_{X^2} = \partial_R \) which is as follows

\[ \nabla^a \left( T^b_{ab}(\partial_R \hat{\psi}) \right) = \frac{r^2}{\rho^2} \left( 4a^2 R \partial_t \partial_R \hat{\psi} + 4a R \partial_{\varphi} \partial_R \hat{\psi} + 2a^2 \partial_t \hat{\psi} + 2a \partial_{\varphi} \hat{\psi} \right) T^b_b (\partial_R \hat{\psi}) \]

\[ + \frac{r^2}{\rho^2} \left( 4a^2 R^3 - 6MR^2 + 2R \right) \partial_R^2 \hat{\psi} \]

\[ + (12aR^2 - 12MR + 2) \partial_R \hat{\psi} \]

\[ - \left( 2(M - 2a^2 R) \hat{\psi} + 2 \frac{Mr - a^2}{r^2} \partial_R \hat{\psi} \right) T^b_b (\partial_R \hat{\psi}) \]

\[ + (\nabla_a T_b) T^{ab}(\partial_R \hat{\psi}) \]  

Where \((\nabla_a T_b) T^{ab}\) can be simply expressed by the following lemma (the proof can be found in appendix 2.5.2):

**Lemma 2.3.1.** We can express \((\nabla_a T_b) T^{ab}\) under the following simple form

\[ (\nabla_a T_b) T^{ab}(\hat{\psi}) = A_1 \hat{\psi}_t^2 + A_2 \hat{\psi}_t \hat{\psi}_R + A_3 \hat{\psi}_t \hat{\psi}_{\varphi} + A_4 \hat{\psi}_R^2 + A_5 \hat{\psi}_R \hat{\psi}_{\varphi} + A_6 \sin^2 \theta \hat{\psi}_g^2 + A_7 \hat{\psi}_{\varphi}^2 + A_8 |\nabla S^a \hat{\psi}|^2, \]

where the functions \(A_i (i = 1, 2, \ldots 8)\) are bounded.

### 2.3.4 The energy density

The energy density 3–form \(E(\hat{\psi})\) associated with \(T^a\) is given by

\[ E(\hat{\psi}) = T^a T_{ab}(\hat{\psi}) d^3 x^b = T^a T^b_a(\hat{\psi}) \partial_b \omega dVol^4. \]

For a hypersurface \(S\), we define the energy of the scalar field \(\hat{\psi}\) by

\[ \mathcal{E}_S(\hat{\psi}) = \int_S E(\hat{\psi}). \]

The following lemma gives us the equivalent expressions of the energies:
Lemma 2.3.2. The energies of $\psi$ on the hypersurfaces $\mathcal{H}_s$ and $\mathcal{I}_{t_0}^+$ have the following simpler equivalent expressions, for $r$ is large enough

$$
\mathcal{E}_{\mathcal{H}_s}(\psi) \simeq \int_{\mathcal{H}_s} \left( *t^2 \dot{\psi}_t^2 + \frac{R}{|t|} \dot{\psi}_R^2 + |\nabla_{S^2} \dot{\psi}|^2 \right) d^*t \wedge d^2\omega,
$$

$$
\mathcal{E}_{\mathcal{I}_{t_0}^+}(\psi) \simeq \int_{\mathcal{I}_{t_0}^+} \left( *t^2 \dot{\psi}_t^2 + |\nabla_{S^2} \dot{\psi}|^2 \right) d^*t \wedge d^2\omega
$$

where

$$
|\nabla_{S^2} \dot{\psi}|^2 = \dot{\psi}_\theta^2 + \frac{1}{\sin^2 \theta} \dot{\psi}_\varphi^2.
$$

And we can also check that the energy on the null hypersurface $\mathcal{E}_{\mathcal{S}_t}(\psi)$ is non negative.

Proof. We have

$$
T^aT_a^0 = *t^2 T_0^0 - 2(1 + *tR) T_1^0,
$$

where

$$
T_0^0 = \hat{g}^{0a}T_{0a} = -\frac{1}{\rho^2} \left( \frac{r^2a^2 \sin^2 \theta}{2} \dot{\psi}_t^2 - \frac{R^2 \Delta}{2} \dot{\psi}_R^2 - \frac{r^2}{2} \dot{\psi}_\theta^2 - \frac{r^2}{2 \sin^2 \theta} \dot{\psi}_\varphi^2 - a \dot{\psi}_R \dot{\psi}_\varphi \right)
$$

and

$$
T_1^0 = \hat{g}^{0a}T_{1a} = -\frac{1}{\rho^2} \left( (r^2 + a^2) \dot{\psi}_R^2 + r^2 a^2 \sin^2 \theta \dot{\psi}_t \dot{\psi}_R + ar^2 \dot{\psi}_R \dot{\psi}_\varphi \right).
$$

We also have

$$
T^aT_a^1 = *t^2 T_0^1 - 2(1 + *tR) T_1^1,
$$

where

$$
T_0^1 = -\frac{1}{\rho^2} \left( (r^2 + a^2) \dot{\psi}_t^2 + R^2 \Delta \dot{\psi}_t \dot{\psi}_R + a \dot{\psi}_t \dot{\psi}_\varphi \right)
$$

$$
+ MRa^2 \sin^2 \theta \frac{1}{\rho^4} \left( 1 - \frac{1}{\rho^2} \right) \left( \frac{r^2a^2 \sin^2 \theta}{2} \dot{\psi}_t^2 + \frac{R^2 \Delta}{2} \dot{\psi}_R^2 + r^2 \dot{\psi}_\theta^2 + \frac{r^2}{2 \sin^2 \theta} \dot{\psi}_\varphi^2 \right)
$$

$$
+ MRa^2 \sin^2 \theta \frac{1}{\rho^4} \left( 1 - \frac{1}{\rho^2} \right) \left( 2(r^2 + a^2) \dot{\psi}_t \dot{\psi}_R + 2ar^2 \dot{\psi}_t \dot{\psi}_\varphi + 2a \dot{\psi}_R \dot{\psi}_\varphi \right)
$$

and

$$
T_1^1 = -\frac{1}{\rho^2} \left( \frac{R^2 \Delta}{2} \dot{\psi}_R^2 - \frac{r^2a^2 \sin^2 \theta}{2} \dot{\psi}_t^2 - \frac{r^2}{2} \dot{\psi}_\theta^2 - \frac{r^2}{2 \sin^2 \theta} \dot{\psi}_\varphi^2 - ar^2 \dot{\psi}_t \dot{\psi}_\varphi \right).
$$

On the hypersurface $\mathcal{H}_s$, we have

$$
dR = \frac{\Delta R^2}{s(r^2 + a^2)} d^*t,
$$

so

$$
\mathcal{E}_{\mathcal{H}_s}(\psi) = \int_{\mathcal{H}_s} -T^aT_a^1 d^*t \wedge d^2\omega + T^aT_a^0 dR \wedge d^2\omega
$$

$$
= \int_{\mathcal{H}_s} \left( -T^aT_a^1 + \frac{\Delta R^2}{s(r^2 + a^2)} T^aT_a^0 \right) d^*t \wedge d^2\omega.
$$
Using the expressions of $T^a_b$ and $T^a_0$ from above, and noting that the component of $T^1_0$ that has the factor $(R/\rho^4) \approx R^5$ is not important, we obtain:

$$\rho^2 \left(-T^aT^a_0 + \frac{\Delta R^2}{s(r^2 + a^2)} T^aT^a_0 \right)$$

$$\simeq \left(*t^2(r^2 + a^2) + (1 + *tR)r^2a^2 \sin^2 \theta - \frac{*t^2r^2a^2 \sin^2 \theta}{2} \frac{\Delta R^2}{s(r^2 + a^2)} \right) \hat{\psi}_t^2$$

$$+ \left(- (1 + *tR)R^2 \Delta + \frac{\Delta R^2}{s(r^2 + a^2)} \left(\frac{*t^2 R^2 \Delta}{2} + 2(1 + *tR)(r^2 + a^2) \right) \right) \hat{\psi}_R^2$$

$$+ \left( (1 + *tR)r^2 + \frac{*t^2r^2}{2} \frac{\Delta R^2}{s(r^2 + a^2)} \right) \hat{\psi}_\theta^2 + \left(\frac{1}{\sin^2 \theta} (1 + *tR)r^2 + \frac{1}{\sin^2 \theta} \frac{*t^2r^2}{2} \frac{\Delta R^2}{s(r^2 + a^2)} \right) \hat{\psi}_\varphi^2$$

$$+ \left( *t^2 R^2 \Delta + 2(1 + *tR)r^2a^2 \sin^2 \theta \frac{\Delta R^2}{s(r^2 + a^2)} \right) \hat{\psi}_t \hat{\psi}_R + \left( *t^2 a + 2(1 + *tR)ar^2 \right) \hat{\psi}_t \hat{\psi}_\varphi$$

$$+ \left( *t^2 a + 2(1 + *tR)ar^2 \right) \frac{\Delta R^2}{s(r^2 + a^2)} \hat{\psi}_R \hat{\psi}_\varphi.$$
Chapter 2. Peeling for scalar fields on Kerr spacetime

therefore, for \(*t < -1\) and \(R\) is small enough, we can see that there exists a constant \(C_1 > 0\) such that

\[
-T^aT_a + \frac{\Delta R^2}{s(r^2 + a^2)} T^a T_a \geq C_1 \left( *t^2 \psi^2_{\text{R}} + \frac{R}{*t} \psi^2_{\text{R}} + |\nabla S^2 \psi|^2 \right) .
\]

Similarly, there exists a constant \(C_2 > 0\) such that

\[
-T^aT_a + \frac{\Delta R^2}{s(r^2 + a^2)} T^a T_a \leq C_2 \left( *t^2 \psi^2_{\text{R}} + \frac{R}{*t} \psi^2_{\text{R}} + |\nabla S^2 \psi|^2 \right) .
\]

Then, the energy of \(\hat{\psi}\) on \(\mathcal{H}_s\) has the simple equivalent expression as follows

\[
\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \simeq \int_{\mathcal{H}_s} \left( *t^2 \psi^2_{\text{R}} + \frac{R}{*t} \psi^2_{\text{R}} + |\nabla S^2 \psi|^2 \right) d^4x .
\]

If \(s \to 0\) then \(R \to 0\), so the the coefficient \(R/|*t|\) of \(\psi^2_{\text{R}}\) vanishes giving

\[
\mathcal{E}_{\mathcal{S}^+_{t_0}}(\hat{\psi}) \simeq \int_{\mathcal{S}^+_{t_0}} \left( *t^2 \psi^2_{\text{R}} + |\nabla S^2 \psi|^2 \right) d^4x .
\]

Finally, the energy of \(\hat{\psi}\) on the null hypersurface \(\mathcal{S}_{t_0}\) is non negative since the fact that

- \(\mathcal{S}_{t_0}\) is a null hypersurface, it is oriented by its future-pointing null vector field.
- The stress-energy tensor \(T_{ab}\) satisfies the dominant energy condition\(^9\) and the vector field Morawetz \(T^a\) is a future-oriented timelike vector field, so \(T_b^a T^b\) is a causal and future-pointing vector field.

\[
\square
\]

2.3.5 The energy estimates and the peeling

The basic estimate

To control the energies, we need the following lemma that is a consequence of the Poincaré’s inequality (see Lemma 2.5.1 in Appendix 2.5.4 of this chapter):

**Lemma 2.3.3.** For \(*t_0 < 0, |*t_0|\) large enough and for any smooth compactly supported initial data at \(t = 0\), the associated rescaled solution \(\hat{\psi}\) satisfies

\[
\int_{\mathcal{H}_s} \hat{\psi}^2 d^4x \leq \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) .
\]

Integrating on \(\Omega_{t_0}^{n_1,n_2}\) with \(n_1 < n_2\), by the conservation law (2.16) of \(\hat{\psi}\), we get

\[
\left| \mathcal{E}_{\mathcal{H}_{s_1}}(\hat{\psi}) + \mathcal{E}_{\mathcal{S}^{s_1,s_2}_{t_0}}(\hat{\psi}) - \mathcal{E}_{\mathcal{H}_{s_2}}(\hat{\psi}) \right| \\
\lesssim \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left( -2 \frac{Mr - a^2}{r^2} \hat{\psi} T^b \partial_b \hat{\psi} + (\nabla_a T_b) T^{ab}(\hat{\psi}) \right) \frac{1}{|*t|} d^4x d\omega d\sigma \\
\lesssim \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left( -2 \frac{Mr - a^2}{r^2} \hat{\psi} T^b \partial_b \hat{\psi} + (\nabla_a T_b) T^{ab}(\hat{\psi}) \right) \frac{1}{|*t|} d^4x d\omega d\sigma .
\]

\(^9\)see the dominant energy condition in Appendix 2.5.3 of this chapter.
where

\[
(\nabla_a T_b) T^{ab}(\hat{\psi}) = A_1 \hat{\psi}^2_t + A_2 \hat{\psi} \cdot t \hat{\psi}_R + A_3 \hat{\psi}_t \hat{\psi}_R + A_4 R \hat{\psi}_R^2 + A_5 \hat{\psi}_R \hat{\psi}_R \\
+ A_6 \sin^2 \theta \hat{\psi}_\theta^2 + A_7 \hat{\psi}^2_\varphi + A_8 |\nabla_{S^2} \hat{\psi}|^2,
\]

with \(A_i\) are bounded functions.

Now, we estimate the error term under the integral sign \(\int_{s_1}^{s_2} \int_{\mathcal{H}_s} \). Note that \(\frac{1}{|t|} \simeq \frac{1}{\sqrt{s}} \sqrt{\frac{R}{|t|}}\), we have

\[
|\text{Error}| := \left| \left( -2 \frac{Mr - a^2}{r^2} \hat{\psi} T^b \partial_b \hat{\psi} + (\nabla_a T_b) T^{ab}(\hat{\psi}) \right) \right| \frac{1}{|t|}
\]

\[
\lesssim \left( 2 R |t^2 \hat{\psi} \partial_t \hat{\psi} | + 4 R |\hat{\psi} \partial R \hat{\psi} | + A_1 |\hat{\psi}^2_t | + A_2 |\hat{\psi}_t \hat{\psi}_R | + A_3 |\hat{\psi}_t \hat{\psi}_R \hat{\psi}_R | \right) \frac{1}{|t|}
\]

\[
+ \left( A_4 |R \hat{\psi}_R^2 | + A_5 |\hat{\psi}_R \hat{\psi}_R \hat{\psi}_R | + A_6 |\sin^2 \theta \hat{\psi}_\theta^2 | + A_7 |\hat{\psi}^2_\varphi | + A_8 |\nabla_{S^2} \hat{\psi}|^2 \right) \frac{1}{|t|}
\]

\[
\lesssim 2 R \left( \hat{\psi}^2 + *t^2 \hat{\psi}^2_t \right) + \frac{4}{\sqrt{s}} \left( \hat{\psi}_R^2 + \frac{R}{|t|} \hat{\psi}_R^2 \right) + A_1 *t^2 \hat{\psi}_R^2 + A_2 \frac{R}{|t|} \hat{\psi}_R^2
\]

\[
+ A_3 \left( *t^2 \hat{\psi}_t \hat{\psi}_R + \hat{\psi}_\varphi^2 \right) + A_4 \frac{R}{|t|} \hat{\psi}_R^2 + A_5 \frac{R}{|t|} \hat{\psi}_R^2
\]

\[
+ \left( \max \{A_6, A_7\} \sin^2 \theta + A_8 \right) |\nabla_{S^2} \hat{\psi}|^2
\]

\[
\lesssim \frac{1}{\sqrt{s}} \left( *t^2 \hat{\psi}_R^2 + \frac{R}{|t|} \hat{\psi}_R^2 + |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}_T^2 \right),
\]

using the above lemma, we have

\[
\left| \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) + \mathcal{E}_{S^2_{t_0}}(\hat{\psi}) - \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \right|
\]

\[
\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{\mathcal{H}_s} \left( *t^2 \hat{\psi}_R^2 + \frac{R}{|t|} \hat{\psi}_R^2 + |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}_T^2 \right) d^*t d^2\omega ds
\]

\[
= \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) ds.
\]

Note that the function \(1/\sqrt{s}\) is integrable, then using the Gronwall’s inequality, we get the following result:

**Theorem 2.3.1.** For \(*t_0 < 0, |*t_0|\) large enough and for any smooth compactly supported initial data at \(t = 0\), the associated rescaled solution \(\hat{\psi}\) satisfies

\[
\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}),
\]

\[
\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) + \mathcal{E}_{S^2_{t_0}}(\hat{\psi}).
\]

The equations (2.11) (that are obtained by the Lie derivatives along the Killing vectors \(X_0\) and \(X_1\)) are the same as in the origin equation, so we can obtain a consequence of the above theorem as follows:
Corollary 2.3.1. For $t_0 < 0$, $|t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies

$$E_{\mathcal{F}^*_0} (\mathcal{L}_X \hat{\psi}) \lesssim E_{\mathcal{H}_1} (\mathcal{L}_X \hat{\psi}),$$

$$E_{\mathcal{H}_1} (\mathcal{L}_X \hat{\psi}) \lesssim E_{\mathcal{F}^*_0} (\mathcal{L}_X \hat{\psi}) + E_{S^*_0} (\mathcal{L}_X \hat{\psi}) (i = 0, 1).$$

High order estimate and the peeling

In this section, we consider the regularity at a higher order. First, for $\mathcal{L}_X = \partial_R \hat{\psi}$, by integrating the conservation law (2.20), we get

$$\left| E_{\mathcal{H}_s} (\hat{\psi}_R) + E_{S^2_t} (\hat{\psi}_R) - E_{\mathcal{H}_s} (\hat{\psi}_R) \right|$$

$$\lesssim \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| \frac{r^2}{\rho^2} \left( 4a^2 R^2 \partial_t \partial_R \hat{\psi} + 4aR \partial_t \partial_R \hat{\psi} + 2a^2 \partial_t \partial_\psi + 2a \partial_t \psi + 2a \partial_\psi \right) T^b \partial_b (\partial_R \hat{\psi}) \right| \frac{1}{|\psi|} \mathrm{d}^* t \mathrm{d}^2 \omega \mathrm{d}s$$

$$+ \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| \frac{r^2}{\rho^3} \left( 4a^2 R^3 - 6MR^2 + 2R \partial^2_\psi \hat{\psi} + (12a^2 R^2 - 12MR + 2) \partial_\psi \hat{\psi} \right) T^b \partial_b (\partial_R \hat{\psi}) \right| \frac{1}{|\psi|} \mathrm{d}^* t \mathrm{d}^2 \omega \mathrm{d}s$$

$$+ \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| \left( 2(M - 2a^2 R) \hat{\psi} + 2 \frac{M - a^2}{r^2} \partial_\psi \hat{\psi} \right) T^b \partial_b (\partial_R \hat{\psi}) + (\nabla a T_b) T^{ab} (\partial_R \hat{\psi}) \right| \frac{1}{|\psi|} \mathrm{d}^* t \mathrm{d}^2 \omega \mathrm{d}s$$

(2.21)

where

$$(\nabla a T_b) T^{ab} (\partial_R \hat{\psi}) = A_1 (\partial_R \hat{\psi})_2 \partial_t + A_2 (\partial_R \hat{\psi})_t (\partial_R \hat{\psi})_R + A_3 (\partial_R \hat{\psi})_t (\partial_R \hat{\psi})_\psi + A_4 (\partial_R \hat{\psi})_R + A_5 (\partial_R \hat{\psi})_R (\partial_R \hat{\psi})_\psi + A_6 \sin^2 \theta (\partial_R \hat{\psi})_g^2 + A_7 (\partial_R \hat{\psi})_R^2 + A_8 |\nabla s^2 (\partial_R \hat{\psi})|^2$$

and

$$T^b \partial_b (\partial_R \hat{\psi}) = \frac{1}{4} \partial_t \partial_R (\partial_R \hat{\psi}) - 2(1 + \frac{1}{4} R) \partial_R (\partial_R \hat{\psi}).$$

We can control all the terms on the right-hand side of inequality (2.21) in the same way as we have done in the basic estimate except the term that contains the partial derivative of the second order $\partial^2_R \hat{\psi}$. To control it, we need the help from its coefficient $4a^2 R^3 - 6MR^2 + 2R$ that is of order one in $R$, we can control this term as follows

$$\int_{\mathcal{H}_s} \left| (4a^2 R^3 - 6MR^2 + 2R) \partial^2_\psi \hat{\psi} T^b \partial_b (\partial_R \hat{\psi}) \right| \frac{1}{|\psi|} \mathrm{d}^* t \mathrm{d}^2 \omega$$

$$= \int_{\mathcal{H}_s} \left| (4a^2 R^3 - 6MR^2 + 2R) \partial^2_\psi \hat{\psi} \left( \frac{1}{4} \partial_t \partial_R (\partial_R \hat{\psi}) - 2(1 + \frac{1}{4} R) \partial_R (\partial_R \hat{\psi}) \right) \right| \frac{1}{|\psi|} \mathrm{d}^* t \mathrm{d}^2 \omega$$

$$\lesssim \int_{\mathcal{H}_s} \left( 2R (\partial_R \hat{\psi})_R^2 + \frac{1}{4} \partial_t (\partial_R \hat{\psi})_R^2 \right) \mathrm{d}^* t \mathrm{d}^2 \omega$$

$$\lesssim E_{\mathcal{H}_s} (\hat{\psi}_R).$$

Therefore the right-hand side of (2.21) can be controlled uniformly follows $s$ by the quantity

$$\int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( E_{\mathcal{H}_s} (\hat{\psi}_R) + E_{\mathcal{H}_s} (\hat{\psi}) \right) \mathrm{d}s.$$
By using the Gronwall’s inequality, we get the result

\[
\mathcal{E}_{\mathcal{J}^+_{t_0}}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{J}^+_{t_0}}(\psi) \lesssim \mathcal{E}_{\mathcal{H}_1}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{H}_1}(\psi),
\]

\[
\mathcal{E}_{\mathcal{H}_1}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{H}_1}(\psi) \lesssim \mathcal{E}_{\mathcal{J}^+_{t_0}}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{J}^+_{t_0}}(\psi) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\psi).
\]

So, we have

**Proposition 2.3.1.** For \(*t_0 < 0, |t_0|\) large enough and for any smooth compactly supported initial data at \(t = 0\), the associated rescaled solution \(\hat{\psi}\) satisfies

\[
\mathcal{E}_{\mathcal{J}^+_{t_0}}(\partial_R \hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_1}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{H}_1}(\psi),
\]

\[
\mathcal{E}_{\mathcal{H}_1}(\partial_R \hat{\psi}) \lesssim \mathcal{E}_{\mathcal{J}^+_{t_0}}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{J}^+_{t_0}}(\psi) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\partial_R \hat{\psi}) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\psi).
\]

By similar way, Proposition 2.3.1 can be generalized to obtain the following estimate at the order \(k\) for \(\mathcal{L}_{X_i}(\hat{\psi})\):

**Theorem 2.3.2.** For \(*t_0 < 0, |t_0|\) large enough and for any smooth compactly supported initial data at \(t = 0\), the associated rescaled solution \(\hat{\psi}\) satisfies

\[
\mathcal{E}_{\mathcal{J}^+_{t_0}}(\mathcal{L}_{X_i}(\hat{\psi})) \lesssim \sum_{q=0}^{p} \mathcal{E}_{\mathcal{H}_1}(\mathcal{L}_{X_i}^q(\hat{\psi})),
\]

\[
\mathcal{E}_{\mathcal{H}_1}(\mathcal{L}_{X_i}(\hat{\psi})) \lesssim \sum_{q=0}^{p} \left( \mathcal{E}_{\mathcal{J}^+_{t_0}}(\mathcal{L}_{X_i}^q(\hat{\psi})) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\mathcal{L}_{X_i}^q(\hat{\psi})) \right).
\]

Furthermore, we can extend to consider the regularity on the 2–sphere by controlling the energies of the Lie derivative \(\mathcal{L}_{X_i}(\hat{\psi})(i = 1, 2, 3)\). The control for the energy of \(\mathcal{L}_{X_i}\) has just been done by Corrollary 2.3.1. For the vector fields \(X_2\) and \(X_3\), integrating the conservation laws (2.18) and (2.19), we have

\[
\left| \mathcal{E}_{\mathcal{H}_1}(\mathcal{L}_{X_2}(\hat{\psi})) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\mathcal{L}_{X_2}(\hat{\psi})) - \mathcal{E}_{\mathcal{H}_2}(\mathcal{L}_{X_2}(\hat{\psi})) \right|
\]

\[
\lesssim \int_{s_1}^{s_2} \int_{H_s} \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin \phi \partial_t^2 - 2a\partial_t \mathcal{L}_{X_2}(\hat{\psi}) - a^2 R^2 \partial_R \mathcal{L}_{X_3}(\hat{\psi}) \right) \hat{\psi} \partial_b (\mathcal{L}_{X_2}(\hat{\psi})) \left| \frac{1}{|\ast t|} \right| \partial_t d^2 \omega ds
\]

\[
+ \int_{s_1}^{s_2} \int_{H_s} \left[ \frac{r^2}{\rho^2} \left( -a^2 R^2 \partial_R \mathcal{L}_{X_3}(\hat{\psi}) - 2a R \mathcal{L}_{X_3}(\hat{\psi}) \right) \hat{\psi} \partial_b (\mathcal{L}_{X_2}(\hat{\psi})) + (\nabla_a T_b) T^{ab} (\mathcal{L}_{X_2}(\hat{\psi})) \right] \left| \frac{1}{|\ast t|} \right| \partial_t d^2 \omega ds
\]

and

\[
\left| \mathcal{E}_{\mathcal{H}_1}(\mathcal{L}_{X_3}(\hat{\psi})) + \mathcal{E}_{\mathcal{S}^+_{t_0}}(\mathcal{L}_{X_3}(\hat{\psi})) - \mathcal{E}_{\mathcal{H}_2}(\math{L}_{X_3}(\hat{\psi})) \right|
\]

\[
\lesssim \int_{s_1}^{s_2} \int_{H_s} \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin \phi \partial_t^2 + 2a \partial_t \mathcal{L}_{X_3}(\hat{\psi}) + a^2 R^2 \partial_R \mathcal{L}_{X_3}(\hat{\psi}) \right) \hat{\psi} \partial_b (\mathcal{L}_{X_3}(\hat{\psi})) \left| \frac{1}{|\ast t|} \right| \partial_t d^2 \omega ds
\]

\[
+ \int_{s_1}^{s_2} \int_{H_s} \left[ \frac{r^2}{\rho^2} \left( a^2 R^2 \partial_R \mathcal{L}_{X_2}(\hat{\psi}) + 2a R \mathcal{L}_{X_2}(\hat{\psi}) \right) \hat{\psi} \partial_b (\mathcal{L}_{X_3}(\hat{\psi})) + (\nabla_a T_b) T^{ab} (\mathcal{L}_{X_3}(\hat{\psi})) \right] \left| \frac{1}{|\ast t|} \right| \partial_t d^2 \omega ds.
\]
In these cases, we notice that the term $\partial^2_t \psi$ appears, to control it we need to use the energy of $L_{X_0}(\hat{\psi}) = \hat{\psi}_t$ since
\[
\int_{s_1}^{s_2} \int_{H_s} |\partial^2_t \hat{\psi}|^2 \, dt \, d^2 \omega \leq E_{H_s}(\hat{\psi}_t).
\]
Therefore, we can control the energy of $L_{X_2}(\hat{\psi})$ by the the energies of $L_{X_3}(\hat{\psi})$ and $L_{X_0} \hat{\psi}$. Similarly, the energy of $L_{X_4}(\hat{\psi})$ can be controlled by the energies of $L_{X_2}(\hat{\psi})$ and $L_{X_0}(\hat{\psi})$. So that we can obtain the following theorem:

**Theorem 2.3.3.** For $*t_0 < 0$, $|*t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies
\[
\sum_{q=0}^p E_{\mathcal{S}}^{*}_{*t_0} (L^q_{X_{ij}} \hat{\psi}) \lesssim \sum_{q=0}^p E_{\mathcal{H}_1}(L^q_{X_{ij}} \hat{\psi}),
\]
\[
\sum_{q=0}^p E_{\mathcal{H}_1}(L^q_{X_{ij}} \hat{\psi}) \lesssim \sum_{q=0}^p \left( E_{\mathcal{S}}^{*}_{*t_0} (L^q_{X_{ij}} \hat{\psi}) + E_{S_{*t_0}}(L^q_{X_{ij}} \hat{\psi}) \right),
\]
where
\[
L^q_{X_{ij}} = L_{X_0} L_{X_1} \ldots L_{X_q} \text{ and } i_j \in \{0, 1, 2, 3\}.
\]

Associating Theorem 2.3.1 and Theorem 2.3.2 and Theorem 2.3.3 we obtain the general estimate at order $k$:

**Theorem 2.3.4.** For $*t_0 < 0$, $|*t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies
\[
\sum_{q=0}^p E_{\mathcal{S}}^{*}_{*t_0} (L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) \lesssim \sum_{q=0}^p E_{\mathcal{H}_1}(L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}),
\]
\[
\sum_{q=0}^p E_{\mathcal{H}_1}(L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) \lesssim \sum_{q=0}^p \left( E_{\mathcal{S}}^{*}_{*t_0} (L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) + E_{S_{*t_0}}(L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) \right).
\]

Now we give the definition of the peeling at order $k \in \mathbb{N}^*$:

**Definition 2.3.1.** A solution $\psi$ of the wave equation peels at order $k \in \mathbb{N}$ if the rescaled solution $\hat{\psi}$ satisfies
\[
\sum_{p=0}^k E_{\mathcal{S}}^{*}_{*t_0} \mathbb{P}(L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) < +\infty,
\]
where $\mathbb{P}$ is the polynomial with order $p$ of the basic term $L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}$. This means that for all $p \in \{0, 1 \ldots k\}$ we have for all $q \in \{0, 1 \ldots p\}$, $E_{\mathcal{S}}^{*}_{*t_0} (L^{p-q}_{X_4} L^q_{X_{ij}} \hat{\psi}) < +\infty$. 
We can see that the application of $L_{X_4}$ can be expressed

$$L_{X_4} \left( \frac{\hat{\psi}}{\partial t \hat{\psi}} \right) = \partial_R \left( \frac{\hat{\psi}}{\partial t \hat{\psi}} \right) = -\frac{r^2(a^2 + r^2)}{\Delta} \left( \partial_{t} + \partial_{r^*} + \frac{a}{r^2 + a^2} \partial_{\varphi} \right) \left( \frac{\hat{\psi}}{\partial t \hat{\psi}} \right)$$

$$= -\frac{r^2(a^2 + r^2)}{\Delta} \left( \partial_{r^*} + \frac{a}{r^2 + a^2} \partial_{\varphi} \frac{1}{\partial_{t^*}^2} \partial_{r^*} + \frac{a}{r^2 + a^2} \partial_{\varphi} \right) \left( \frac{\hat{\psi}}{\partial t \hat{\psi}} \right)$$

$$=: L \left( \frac{\hat{\psi}}{\partial t \hat{\psi}} \right).$$

Using the expression of the wave equation (2.10) (for $\hat{g}$), we can express $\partial_t^2$ follows the derivatives $\partial_{r^*}, \partial_{\varphi}$ and $\partial_t$ (the timelike derivative with order 1) and the Laplacian operator $\Delta S^2$, then $L$ is a modification of the following matrix that purely involves spacelike derivatives

$$\bar{L} = \begin{pmatrix} \partial_{r^*}^2 - \frac{2M(1-2MR)}{r^2} & \frac{1}{\partial_{r^*}} \\ \frac{1}{\partial_{r^*}} & \partial_{r^*} \end{pmatrix},$$

in the sense that: if $a = 0$, then $L = \bar{L}$.

The following theorem completely characterizes the largest class of the initial data that guarantees the peeling definition at order $k$. It follows directly from Theorem 2.3.4.

**Theorem 2.3.5.** The spaces of initial data $h^k(H_1)$ that guarantees the peeling of order $k$ obtained by completion of $C_0^\infty([-t_0, +\infty], S^2_\varphi) \times C_0^\infty([-t_0, +\infty], S^2_\varphi)$ in the norms

$$\left\| \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right\|_{h^k(H_1)}^2 = \sum_{p=0}^k \sum_{q=0}^p \mathcal{E}_{H_1} \left( L^{p-q} \mathcal{L}_{X_4}^q \left( \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right) \right)$$

where we have denoted by $\mathcal{E}_{H_1} \left( \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right)$ the energy $\mathcal{E}_{H_1}(\hat{\psi})$ where $\hat{\psi}$ is replaced by $\hat{\psi}_0$ and $\partial_t \hat{\psi} = \hat{\psi}_1$ is replaced by $\hat{\psi}_1$.

### 2.3.6 Interpretation

In this section, we will interpret our peeling definition by three steps: in the first step, we construct the peeling definition on the full conformal compactification of Minkowski spacetime; in the second step, we will constraint our model into the flat case to give the peeling definition on the partial conformal compactification; and in the third step, we will show that: on the flat case, the two classes of the initial data that guarantees the peeling are equivalent at order zero but at a higher order the class of the initial data that is constructed on the partial conformal compactification is slightly more large than the one constructed on the full conformal compactification. But this does not mean that our peeling definition is more general than that in the flat case. We can only conclude that the two definitions of the peeling in the flat case that we obtain from these ways, are more general than the one that was given by R.Penrose [71, 72].
We recall briefly the construction of the peeling on the flat spacetime obtained by the full conformal compactification. This work was done by the authors L. Mason and J-P. Nicolas \[53\]. We set
\[
\tau = \arctan(t - r) + \arctan(t + r) \quad \text{and} \quad \zeta = \arctan(t + r) - \arctan(t - r).
\]
By using the conformal transformation
\[
\hat{g} = \Omega^2 g, \quad \Omega = \frac{2}{\sqrt{1 + (t + r)^2} \sqrt{1 - (t - r)^2}}.
\]
the Minkowski spacetime can be embedded into the Einstein cylinder \[10\]
\[
\mathcal{M} = \{ |\tau| + \zeta \leq \pi, \zeta \geq 0, \omega \in S^2 \}.
\]
The rescaled metric on \(\mathcal{M}\) is detailed as follows
\[
\hat{g} = \Omega^2 g = d\tau^2 - d\zeta^2 - \sin^2 \zeta d\omega^2,
\]
and the volume form is now
\[
dVol^4 = \sqrt{|\hat{g}|} d\tau d\zeta d\omega^2 = \sin^2 \zeta d\tau d\zeta d^2 \omega = d\tau d\mu_{S^3},
\]
where \(d\mu_{S^3} = \sin^2 \zeta d\zeta d^2 \omega\) is the volume form of 3–sphere \(S^3\) with the Euclidian metric
\[
\sigma^2_{S^3} = d\zeta^2 + \sin^2 \zeta d\omega^2.
\]
The Killing vector field used for increasing the regularity in the energy estimates is
\[
K^a = \partial_\tau = \frac{1}{2} (1 + t^2 + r^2) \partial_t + tr \partial_r.
\]
The expression of the wave equation in the full conformal compactification is
\[
\Box \hat{\psi} + \hat{\psi} = (\partial_\tau^2 - \Delta_{S^3} + 1) \hat{\psi} = 0.
\]
This equation has the following stress-energy tensor:
\[
T_{ab} = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \left( < \nabla_c \hat{\psi}, \nabla_d \hat{\psi} > - \hat{\psi}^2 \right),
\]
which gives the conservation law with the zero error by contracting \(T_{ab}\) with \(K^a\)
\[
\nabla^a (K^a T_{ab}) = 0.
\]
On the other hand, we have
\[
K^a T_{ab} g^3 x^b = \hat{\psi} \nabla \hat{\psi} dVol^4 + \frac{1}{2} ( - \hat{\psi}^2 + |\nabla_{S^3} \hat{\psi}|^2 + \hat{\psi}^2 ) \partial_\tau dVol^4. \tag{2.22}
\]

\[^{10}\text{see detail in section 4.2.1 of chapter 3}\]
Denoting the level hypersurfaces of the function \( \tau \) by \( X_\tau = \{ \tau \} \times S^3 \) and the null infinity by \( \mathcal{I}^+ = \{ \tau + \zeta = \pi, \zeta \in ]0, \pi[ \} \), we can calculate the energy on \( X_\tau \) and \( \mathcal{I}^+ \) by the 3–form energy (2.22) on these hypersurfaces as follows

\[
E_{X_\tau}(\hat{\psi}) = \frac{1}{2} \int_{X_\tau} (\hat{\psi}_r^2 + |\nabla_{S^3} \hat{\psi}|^2 + \hat{\psi}^2) d\mu_{S^3},
\]

\[
E_{\mathcal{I}^+}(\hat{\psi}) = \frac{1}{\sqrt{2}} \int_{\mathcal{I}^+} \left( |\hat{\psi}_\tau - \hat{\psi}_\zeta|^2 + \frac{1}{\sin^2 \zeta} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3}.
\]

This work can be done similarly as in the proof of Lemma 3.3.1.

The energy of the linear scalar field is not conformally invariant, the following proposition (see L. Mason and J-P. Nicolas [53] for the proof) give us the equivalent expression of the energy in terms of \( \psi \)

**Proposition 2.3.2.** The energy of \( \psi \) on the hypersurface \( X_0 \) of the Einstein cylinder has the equivalent expression

\[
E_{X_0}(\hat{\psi}) \sim \int_{\Sigma_0} \left( (1 + r^2)(\partial_\tau \psi)^2 + (1 + r^2)(\partial_r \psi)^2 + (1 + r^2) \left| \frac{\nabla_{S^2} \psi}{r} \right|^2 + \psi^2 \right) r^2 dr d^2 \omega.
\]

Since on the Einstein cylinder, the conservation law has the zero error and the vector field \( \partial_\tau \) is Killing, we have the following energy equality

\[
E_{X_0}(\partial_\tau^k \hat{\psi}) = E_{\mathcal{I}^+}(\partial_\tau^k \hat{\psi}) \text{ with } k \in \mathbb{N},
\]

from which, we give the definition of the peeling at order \( k \) as follows:

**Definition 2.3.2.** A solution \( \psi \) of the wave equation peels at order \( k \in \mathbb{N} \) if the rescaled solution \( \hat{\psi} \) satisfies \( E_{\mathcal{I}^+}(\partial_\tau^k \hat{\psi}) < +\infty \). The optimal data initial space \( \mathfrak{h}^k(X_0) \) that guarantees the peeling definition at order \( k \), is the completion of \( C^\infty(X_0) \times C^\infty(X_0) \) on the norms

\[
\left\| \left( \begin{array}{c} \hat{\psi}_0 \\ \hat{\psi}_1 \end{array} \right) \right\|_{\mathfrak{h}^k(X_0)}^2 := E_{X_0} \left( \partial_\tau^k \left( \begin{array}{c} \hat{\psi}_0 \\ \hat{\psi}_1 \end{array} \right) \right),
\]

where we have denoted by \( E_{X_0} \left( \begin{array}{c} \hat{\psi}_0 \\ \hat{\psi}_1 \end{array} \right) \) the energy \( E_{X_0}(\hat{\psi}) \), where \( \hat{\psi} \) is replaced by \( \hat{\psi}_0 \) and \( \partial_\tau \hat{\psi} \) is replaced by \( \hat{\psi}_1 \).

Now we take the constraint of our model on the Minkowski spacetime. The energy of the solution on the hypersurface \( \mathcal{H}_1 \) is given by following proposition (see also L. Mason and J-P. Nicolas [53] for the proof):

**Proposition 2.3.3.** The energy of the solution \( \psi \) on the hypersurface \( \mathcal{H}_1 \) has the following simpler equivalent expression:

\[
E_{\mathcal{H}_1}(\hat{\psi}) \sim \int_{\{ t=0, r>-t_0 \}} \left( r^2 \hat{\psi}_r^2 + r^2 \hat{\psi}_\tau^2 + r^2 \left| \frac{\nabla_{S^2} \psi}{r} \right|^2 + \psi^2 \right) r^2 dr d^2 \omega.
\]
At the order zero, due to Propositions 2.3.2 and Proposition 2.3.3 we can see that the energies on \( X_0 \) and \( \mathcal{H}_1 \) are equivalent in a neighbourhood of spacelike \( i_0 \). Therefore we can conclude that the space of the initial data guarantees the peeling definition at the order zero on both two conformal compactification spacetimes that are the same.

At a higher order, we show that the initial data that guarantees the peeling, constructed on the partial conformal compactification is slightly larger than the one constructed on the full conformal compactification. Indeed, we consider the simple case at the order \( k = 1 \). On the hypersurface \( X_0 = \{ \tau = 0 \} \) we have

\[
\partial_\tau = \frac{1}{2}(1 + r^2)\partial_t, \quad d\mu_{S^3} = \sin^2 \zeta d\zeta d^2\omega = \frac{4r^2}{(1 + r^2)^2} \frac{2}{1 + r^2} dr d^2\omega.
\]

So we obtain the energy of \( \partial_\tau \hat{\psi} \) on the hypersurface \( X_0 \) of the full conformal compactification as follows

\[
\mathcal{E}_{X_0}(\partial_\tau \hat{\psi}) = \frac{1}{2} \int_{X_0} ((\partial_\tau^2 \hat{\psi})^2 + |\nabla_{S^3} \partial_\tau \hat{\psi}|^2 + \partial_\tau \hat{\psi}^2) \frac{4r^2}{(1 + r^2)^2} \frac{2}{1 + r^2} dr d^2\omega.
\]

Using the wave equation on the full conformal compactification

\[
\partial_\tau^2 \hat{\psi} - \hat{\psi} + \Delta_{S^3} \hat{\psi} = 0,
\]

we can replace the first term \( \partial_\tau^2 \hat{\psi} \) by \((1 - \Delta_{S^3})\hat{\psi} \). So to control the energy on \( t = 0 \), we need to control the term that is arisen by the elliptic operator \( \sqrt{1 - \Delta_{S^3}} \). This is equivalent to control independently over \( r^2 \partial_\tau \) and the angular derivatives on 2–sphere \( S^2 \).

On the other hand, on the hypersurface \( \Sigma_0 = \{ t = 0 \} \) of the partial conformal compactification we have

\[
\partial_t = \partial_t, \quad \partial_R = -r^2(\partial_t + \partial_\tau),
\]

so we obtain the energy at order one

\[
\mathcal{E}_{\mathcal{H}_1}(\partial_R \hat{\psi}) \simeq \int_{\{ t = 0, r > -t_0 \}} \left( r^2(\partial_t \partial_R \hat{\psi})^2 + \frac{1}{r^2} |(\partial_R^2 \hat{\psi})^2 + |\nabla_{S^2} \partial_R \hat{\psi}|^2 \right) dr d^2\omega,
\]

where we denote \( \hat{\psi} = r \psi \) to avoid confusion. Consider the first term

\[
\partial_t \partial_R \hat{\psi} = \frac{2}{1 + r^2} \partial_\tau \left\{ -r^2 \left( \frac{2}{1 + r^2} \partial_\tau + \partial_\tau \right) \right\} \left( \frac{2r}{1 + r^2} \hat{\psi} \right)
\]

\[
= -\frac{8r^3}{(1 + r^2)^3} \partial_\tau^2 \hat{\psi} - \frac{4r^3}{(1 + r^2)^2} \partial_\tau \partial_\tau \hat{\psi}
\]

\[
= \frac{8r^3}{(1 + r^2)^3} (1 - \Delta_{S^3}) \hat{\psi} - \frac{4r^3}{(1 + r^2)^2} \partial_\tau \partial_\tau \hat{\psi}.
\]

Therefore to control this energy we need to control \( r^2 \partial_\tau \) and the angular derivatives on 2–sphere \( S^2 \) but in a dependent way\(^{11}\) This is different from the full conformal compactification, then the

---

\(^{11}\)Since we have expressed the energy by the null vector field \( \partial_R \) so the action of \( \partial_\tau^2 \) that is arisen from \( \partial_t \partial_R \) or \( \partial_R^2 \) can not be understood as that of an elliptic operator as in the full conformal compactification case.
initial data space on the partial compactification is slightly larger than the one on the full conformal compactification at a higher order given. This does not mean that our peeling definition is more general than the one on the flat case (the reason is in the flat case, we have used the null derivative $\partial_r$ to control near $i_0$ instead of controlling all derivatives). We see only that the peeling definition which is constructed on the conformal compactification spacetimes (by energy estimate) is more general than the one that is constructed by Penrose [71, 72] due to the definition of R. Penrose, which only requires that the rescaled field is smooth on the whole Einstein cylinder. And we can conclude that our peeling definition on the Kerr spacetime is more general than the one that was given by Penrose.

2.4 The nonlinear case

The nonlinear scalar fields are the solutions of the nonlinear equation

$$\Box_g \psi + \psi^3 = 0. \tag{2.23}$$

The expression of the nonlinear equation in Boyer-Lindquist coordinates is

$$\partial_t^2 \psi + \frac{4aMr}{\sigma^2} \partial_t \partial_\varphi \psi - \frac{\Delta}{\sigma^2} \partial_r (\Delta \partial_r \psi) - \frac{\Delta}{\sigma^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) - \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2 \theta} \partial_\varphi^2 \psi + \frac{\rho^2 \Delta}{\sigma^2} \psi^3 = 0.$$

We can see in [63] that the Cauchy problem of the nonlinear wave equation is well-posed i.e with the initial data in $C_\infty^0(\Sigma_0; \mathbb{C})$, it has a unique solution $\psi \in C_\infty(\mathcal{B}_I; \mathbb{C})$.

Since $\text{Scal}_g = 0$, the modification of the nonlinear wave equation is conformally invariant. This means that the two following propositions are equivalent:

- $\psi \in C_\infty(\mathcal{B}_I)$ is a solution of (2.23),
- $\psi = \Omega^{-1} = r\psi \in C_\infty(\mathcal{B}_I)$ is a solution of the equation $\Box_\hat{g} \hat{\psi} + \frac{1}{6} \text{Scal}_\hat{g} \hat{\psi} + \hat{\psi}^3 = 0$ on $\mathcal{B}_I$, and this solution can be extended on the boundaries $\mathcal{J}_{\pm}$ of $\mathcal{B}_I$ (see [63]).

2.4.1 The equation for the rescaled metric $\hat{g}$ and the commutations

Calculating in details, we obtain the nonlinear wave equation for the rescaled Kerr metric $\hat{g}$ in the star-Kerr coordinates as follows:

$$\left(\Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}}\right) \hat{\psi} + \hat{\psi}^3 = -\frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial_t^2 \hat{\psi} - \frac{r^2 + a^2}{\rho^2} \partial_t \partial_r \hat{\psi} - \frac{2ar^2}{\rho^2} \partial_t \partial_\varphi \hat{\psi} - \frac{a}{\rho^2} \partial_R \partial_\varphi \hat{\psi}$$

$$- \frac{r^2}{\rho^2} \left( (1 + a^2 R^2) \partial_t \partial_R \hat{\psi} + (R^2 - 2MR^2 + a^2 R^4) \partial_R^2 \hat{\psi} \right)$$

$$- \frac{r^2}{\rho^2} \left( aR^2 \partial_R \partial_\varphi \hat{\psi} + 2a^2 R \partial_\varphi \hat{\psi} + (2R - 6MR^2 + 4aR^3) \partial_R \hat{\psi} \right.$$  

$$+ 2aR \partial_\varphi \hat{\psi} + \Delta \partial_\varphi^2 \psi - 2 \frac{Mr - a^2}{r^2} \psi \right) + \hat{\psi}^3$$

$$= 0. \tag{2.24}$$
Now similarly as in the linear case, in the purpose to estimate at a higher order, we will also commute the vector fields $X_i \in \mathcal{A}$ into the nonlinear wave equation (for rescaled metric $\hat{g}$). First, commuting the Lie derivative along the directions $X_0 = \partial_0$ and $X_1 = \partial_\varphi$ and noting that $\mathcal{L}_{X_0} \hat{\psi} = \hat{\psi}_t$ and $\mathcal{L}_{X_1} \hat{\psi} = \hat{\psi}_\varphi$, we get:

$$
\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi}_t + 3 \hat{\psi}^2 \hat{\psi}_t = 0, \tag{2.25}
$$

and

$$
\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi}_\varphi + 3 \hat{\psi}^2 \hat{\psi}_\varphi = 0 \tag{2.26}
$$

which are only different from the origin equation at the derivative of the nonlinear component.

Second, we commute the Lie derivative along the two vector fields $X_2$ and $X_3$ into the nonlinear wave equation. Recall that

$$
X_2 = \sin^* \varphi \partial_\theta + \cot \theta \cos^* \varphi \partial_\varphi, \quad X_3 = \cos^* \varphi \partial_\theta - \cot \theta \sin^* \varphi \partial_\varphi.
$$

Since $X_2$ and $X_3$ are not Killing on Kerr background, therefore the linear part will also be modified. For convenience, we will multiply the equation (2.24) with the factor $\rho^2/r^2$ and then we commute $\mathcal{L}_{X_2}$ and $\mathcal{L}_{X_3}$ into the new equation. We get:

$$
\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_2} \hat{\psi} = \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial_\varphi^2 \hat{\psi} + 2a \partial_\varphi \mathcal{L}_{X_3} \hat{\psi} + a^2 R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} \right)
$$

$$
+ \frac{r^2}{\rho^2} \left( a R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} + 2a R \mathcal{L}_{X_3} \hat{\psi} \right) - 3 \hat{\psi}^2 \mathcal{L}_{X_2} \hat{\psi} + \frac{a^2}{\rho^2} \sin 2\theta \sin^* \varphi \hat{\psi}^3. \tag{2.27}
$$

and

$$
\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \mathcal{L}_{X_3} \hat{\psi} = \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial_\varphi^2 \hat{\psi} - 2a \partial_\varphi \mathcal{L}_{X_3} \hat{\psi} - a^2 R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} \right)
$$

$$
- \frac{r^2}{\rho^2} \left( a R^2 \partial_R \mathcal{L}_{X_3} \hat{\psi} + 2a R \mathcal{L}_{X_3} \hat{\psi} \right) - 3 \hat{\psi}^2 \mathcal{L}_{X_2} \hat{\psi} + \frac{a^2}{\rho^2} \sin 2\theta \cos^* \varphi \hat{\psi}^3. \tag{2.28}
$$

And finally, we commute the Lie derivative along the direction $X_4 = \partial_R$ into the equation and also note that $\mathcal{L}_{X_4} \hat{\psi} = \hat{\psi}_R$. For convenience, we will multiply the equation (2.24) with the factor $\rho^2/r^2$ and then we commute $\partial_R$ into the new equation, we get:

$$
\left( \Box_{\hat{g}} + \frac{1}{6} \text{Scal}_{\hat{g}} \right) \hat{\psi}_R = \frac{r^2}{\rho^2} \left( 4a^2 R \partial_R \hat{\psi}_R + 4a R \partial_\varphi \hat{\psi}_R + 2a^2 \partial_\varphi \hat{\psi}_R + 2a \partial_\psi \hat{\psi}_R \right)
$$

$$
+ \frac{r^2}{\rho^2} \left( 4a^2 R^3 - 6MR^2 + 2R \right) \partial_R \hat{\psi}_R + (12a R^2 - 12MR + 2) \hat{\psi}_R
$$

$$
- 2(M - 2a^2 R) \hat{\psi}_R
$$

$$
- 3 \hat{\psi}^2 \hat{\psi}_R - \frac{a^2 \cos^2 \theta}{\rho^2} \hat{\psi}^3. \tag{2.29}
$$
Similarly to the linear case, we use a modification of the Morawetz vector field on the Kerr background to contract the stress-energy tensor

\[ T^a_{\partial a} := *t^2 \partial_{\cdot t} - 2(1 + *tR) \partial_R. \]

In constrast with the linear case, we use here two types of stress-energy tensors. The first type is associated with the linear Klein-Gordon equation \( \Box \hat{\psi} + \hat{\psi} = 0 \):

\[ T_{ab}(\hat{\psi}) = T_{(ab)}(\hat{\psi}) = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \left( < \partial_c \hat{\psi}, \partial_d \hat{\psi} > \right) \psi^2, \]

and the second type is associated with the non linear Klein-Gordon equation \( \Box \hat{\psi} + \hat{\psi} + \hat{\psi}^3 = 0 \)

\[ \bar{T}_{ab}(\hat{\psi}) = \bar{T}_{(ab)}(\hat{\psi}) = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \left( < \partial_c \hat{\psi}, \partial_d \hat{\psi} > \right) \psi^2 - \frac{1}{2} \psi^4. \]

From Lemma 2.2.1 and the form of the rescaled metric \( \hat{g}^{-1} \) (2.8), we obtain simpler expressions of \( \nabla_a T_b(\hat{\psi}) \) and \( \nabla_a \bar{T}_b(\hat{\psi}) \) by the following lemma (the proof can be found in Appendix 2.5.2).

**Lemma 2.4.1.** We can express \( \nabla_a T_b(\hat{\psi}) \) under the form

\[ \nabla_a T_b(\hat{\psi}) = A_1 \hat{\psi}_t^2 + A_2 \hat{\psi}_R \hat{\psi}_t + A_3 \hat{\psi}_t \hat{\psi}_R \partial \phi + A_4 R \hat{\psi}_R^2 + A_5 \hat{\psi}_R \hat{\psi}_\phi + A_6 \sin^2 \theta \hat{\psi}_t^2 + A_7 \hat{\psi}_t^2 + A_8 |\nabla_{S^2} \hat{\psi}|^2 + A_9 \hat{\psi}^2 \]

where the functions \( A_i (i = 1, 2 \ldots 9) \) are bounded, in addition \( A_9 \) is of order one in \( R \).

Similarly for the stress energy tensor \( \bar{T}_{ab} \), we have

\[ \nabla_a \bar{T}_b(\hat{\psi}) = B_1 \hat{\psi}_t^2 + B_2 \hat{\psi}_R \hat{\psi}_t + B_3 \hat{\psi}_t \hat{\psi}_R \partial \phi + B_4 R \hat{\psi}_R^2 + B_5 \hat{\psi}_R \hat{\psi}_\phi + B_6 \sin^2 \theta \hat{\psi}_t^2 + B_7 \hat{\psi}_t^2 + B_8 |\nabla_{S^2} \hat{\psi}|^2 + B_9 \hat{\psi}^2 + B_{10} \hat{\psi}^4 \]

where the functions \( B_i (i = 1, 2 \ldots 9) \) are bounded, in addition \( B_9, B_{10} \) are of order one in \( R \).

Now we use the stress tensor energy \( \bar{T}_{ab} \) and the vector field \( T^a \) to obtain the conservation law for the original equation (2.24):

\[ \nabla^a \left( T^b \bar{T}_{ab}(\hat{\psi}) \right) = \left( \Box_\hat{\psi} + \hat{\psi} + \hat{\psi}^3 \right) T^b \partial_b \hat{\psi} + \left( \nabla^a T^b \right) \bar{T}_{ab}(\hat{\psi}) \]

\[ = \left( \hat{\psi} - 2 \frac{M \tau - a^2}{r^2} \right) \left( t^2 \partial_{\cdot t} \hat{\psi} - 2(1 + *tR) \partial_R \hat{\psi} \right) + \left( \nabla_a T_b \right) \bar{T}_{ab}(\hat{\psi}) \]

(2.30)

where \( \nabla_a T_b \bar{T}_{ab} \) is given in Lemma 2.4.1.

For the conservation laws of \( L_0 \) \( (i = 0, 1 \ldots 4) \) and also for the high order of derivatives, we use the stress energy tensor \( T_{ab} \). Using equation (2.25) we have

\[ \nabla^a \left( T^b T_{ab}(\hat{\psi}_t) \right) = \left( \Box \hat{\psi}_t + \hat{\psi}_t^3 \right) T^b \partial_b \hat{\psi}_t + \left( \nabla^a T^b \right) T_{ab}(\hat{\psi}_t) \]

\[ = \left( \hat{\psi}_t - 2 \frac{M \tau - a^2}{r^2} \hat{\psi}_t - 3 \hat{\psi}_t^2 \right) \left( t^2 \partial_{\cdot t} \hat{\psi}_t - 2(1 + *tR) \partial_R \hat{\psi}_t \right) + \left( \nabla_a T_b \right) T_{ab}(\hat{\psi}_t) \]

(2.31)
and the same formula for the $\psi^*\varphi$ is obtained by using \((2.26)\)

\[
\nabla^a \left( T^{ab} T_{ab}(\dot{\psi}^*\varphi) \right) = \left( \Box_g \dot{\psi}^*\varphi + \ddot{\psi}^*\varphi \right) T^b \partial_b \dot{\psi}^*\varphi + \left( \nabla^a T^b \right) T_{ab}(\dot{\psi}^*\varphi)
\]

\[
= \left( \dot{\psi}^*\varphi - 2 \frac{Mr}{r^2} \dot{\psi}^*\varphi - 3 \ddot{\psi}^*\varphi \right) \left( t^2 \partial_t \dot{\psi}^*\varphi - 2(1 + *tR)\partial_R \dot{\psi}^*\varphi \right) + (\nabla_a T_b) T^{ab}(\dot{\psi}^*\varphi) .
\]  

(2.32)

The conservation laws for $L_{X_i}$ ($i = 2, 3, 4$) are more complicated, they corresponding to equations \((2.27)\) and \((2.28)\) and \((2.29)\) as follows:

\[
\nabla^a \left( T^{b} T_{ab}(L_{X_2} \dot{\psi}) \right) = \left( \Box_g (L_{X_2} \dot{\psi}) + L_{X_2} \ddot{\psi} \right) T^b \partial_b (L_{X_2} \dot{\psi}) + \left( \nabla^a T^b \right) T_{ab}(L_{X_2} \dot{\psi})
\]

\[
= \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial^2 \partial_t - 2a \partial_t L_{X_2} \dot{\psi} - a^2 R^2 \partial_R L_{X_2} - aR^2 \partial_R L_{X_3} \dot{\psi} - 2aR L_{X_3} \dot{\psi} \right) \dot{\psi} T^b \partial_b (L_{X_2} \dot{\psi})
\]

\[
+ \left( L_{X_2} \dot{\psi} - 3 \ddot{\psi} L_{X_2} \dot{\psi} + a^2 \frac{2}{\rho^2} \sin 2\theta \sin^* \varphi \dot{\psi}^3 - 2 \frac{Mr}{r^2} \frac{L_{X_2} \dot{\psi}}{\rho^2} \right) T^b \partial_b (L_{X_2} \dot{\psi})
\]

\[
+ (\nabla_a T_b) T^{ab}(L_{X_2} \dot{\psi})
\]

(2.33)

\[
\nabla^a \left( T^{b} T_{ab}(L_{X_3} \dot{\psi}) \right) = \left( \Box_g (L_{X_3} \dot{\psi}) + L_{X_3} \ddot{\psi} \right) T^b \partial_b (L_{X_3} \dot{\psi}) + \left( \nabla^a T^b \right) T_{ab}(L_{X_3} \dot{\psi})
\]

\[
= \frac{r^2}{\rho^2} \left( a^2 \sin 2\theta \sin^* \varphi \partial^2 \partial_t + 2a \partial_t L_{X_3} \dot{\psi} + a^2 R^2 \partial_R L_{X_2} + aR^2 \partial_R L_{X_3} \dot{\psi} + 2aR L_{X_3} \dot{\psi} \right) \dot{\psi} T^b \partial_b (L_{X_3} \dot{\psi})
\]

\[
+ \left( L_{X_3} \dot{\psi} - 3 \ddot{\psi} L_{X_3} \dot{\psi} + a^2 \frac{2}{\rho^2} \sin 2\theta \sin^* \varphi \dot{\psi}^3 + 2 \frac{Mr}{r^2} \frac{L_{X_3} \dot{\psi}}{\rho^2} \right) T^b \partial_b (L_{X_3} \dot{\psi})
\]

\[
+ (\nabla_a T_b) T^{ab}(L_{X_3} \dot{\psi})
\]

(2.34)

\[
\nabla^a \left( T^{b} T_{ab}(\dot{\psi} R) \right) = \left( \Box_g \dot{\psi} R + \ddot{\psi} R \right) T^b \partial_b (\dot{\psi} R) + \left( \nabla^a T^b \right) T_{ab}(\dot{\psi} R)
\]

\[
= \frac{r^2}{\rho^2} \left( 4a^2 R \partial_t \dot{\psi} R + 4aR \partial_t \varphi \dot{\psi} R + 2a^2 \partial_t \dot{\psi} + 2a \partial \varphi \dot{\psi} \right) T^b \partial_b (\dot{\psi} R)
\]

\[
+ \frac{r^2}{\rho^2} \left( (4a^2 R^3 - 6MR^2 + 2R) \partial_R \dot{\psi} R + (12aR^2 - 12MR + 2) \dot{\psi} \right) T^b \partial_b (\dot{\psi} R)
\]

\[
- \left( 2(M - 2a^2 R) \dot{\psi} + 2 \frac{Mr}{r^2} \dot{\psi} R \right) T^b \partial_b (\dot{\psi} R)
\]

\[
+ \left( \dot{\psi} R - 3 \ddot{\psi} R - \frac{ra^2 \cos^2 \theta}{\rho^2} \dot{\psi}^3 \right) T^b \partial_b (\dot{\psi} R) + (\nabla_a T_b) T^{ab}(\dot{\psi} R)
\]

(2.35)

where $(\nabla_a T_b) T^{ab}$ is given in Lemma 2.4.1

2.4.3 The energies and peeling

The energies

Now we obtain two energy types. One type that is associated with $\tilde{T}_{ab}$ and the other is associated with $T_{ab}$. The equivalent forms of the energies are given in the following lemma:
Lemma 2.4.2. The energies of the solution $\hat{\psi}$ of (2.24) associated with the stress-energy tensor $T_{ab}$ have equivalent forms

$$\hat{\mathcal{E}}_{\mathcal{H}_s}(\hat{\psi}) \simeq \int_{\mathcal{H}_s} \left( *t^2 \hat{\psi}^2_t + \frac{R}{|*t|} \hat{\psi}^2_R + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 + \hat{\psi}^4 \right) d^s t^2 d\omega,$$

$$\hat{\mathcal{E}}_{\mathcal{F}_{t0}^+}(\hat{\psi}) \simeq \int_{\mathcal{F}_{t0}^+} \left( *t^2 \hat{\psi}^2_t + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 + \hat{\psi}^4 \right) d^s t^2 d\omega,$$

and from the dominant energy condition (see Appendix [2.5.3]), the energy on the null hypersurface $S_{t}(\hat{\psi})$ is non negative.

The energies of $\hat{\psi}$, associated with the stress-energy tensor $T_{ab}$ have the equivalent forms

$$\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \simeq \int_{\mathcal{H}_s} \left( *t^2 (\hat{\psi})^2_t + \frac{R}{|*t|} (\hat{\psi})^2_R + |\nabla S^2 (\hat{\psi})|^2 + (\hat{\psi})^2 \right) d^s t^2 d\omega,$$

$$\mathcal{E}_{\mathcal{F}_{t0}^+}(\hat{\psi}) \simeq \int_{\mathcal{F}_{t0}^+} \left( *t^2 (\hat{\psi})^2_t + |\nabla S^2 \hat{\psi}|^2 + (\hat{\psi})^2 \right) d^s t^2 d\omega,$$

also from the dominant energy condition, the energy on the null hypersurface $S_{t}$ is non negative.

Proof. The proof is similar to the linear case. \qed

Remark 2.4.1. Note that the energies that are obtained by the stress-energy tensor $\hat{T}_{ab}$ do not define a norm due to the presence of the nonlinear term $\hat{\psi}^4$.

The basic estimate

Integrating the conservation law (2.30) for $\hat{\psi}$ on $\Omega_{t0}^{s_1,s_2}$, we obtain:

$$\int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| \left( 1 - \frac{2M}{r^2} \right) \hat{\psi} \left( *t^2 \partial_t \hat{\psi} - 2(1 + *tR)\partial_R \hat{\psi} \right) \right| d^s t^2 d\omega ds \leq \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| (\nabla_a T_b) \tilde{T}^{ab}(\hat{\psi}) \right| \frac{1}{|*t|} d^s t^2 d\omega ds$$

$$\leq \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| *t \hat{\psi} \partial_t \hat{\psi} \right| + \left| \hat{\psi} \partial_R \hat{\psi} \right| \frac{1}{|*t|} + \left| (\nabla_a T_b) \tilde{T}^{ab}(\hat{\psi}) \right| \frac{1}{|*t|} d^s t^2 d\omega ds$$

$$\leq \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| *t \hat{\psi} \partial_t \hat{\psi} \right| + \sqrt{s} \sqrt{\frac{R}{|*t|}} + \left| (\nabla_a T_b) \tilde{T}^{ab}(\hat{\psi}) \right| \frac{1}{|*t|} d^s t^2 d\omega ds$$

$$\leq \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left( \hat{\psi}^2 + *t^2 \hat{\psi}^2_t \right) + \frac{1}{\sqrt{s}} \left( \hat{\psi}^2 + \frac{R}{|*t|} \hat{\psi}^2_R \right) + \left| (\nabla_a T_b) \tilde{T}^{ab}(\hat{\psi}) \right| \frac{1}{|*t|} d^s t^2 d\omega ds. \quad (2.36)$$

Here we have used the equivalence

$$\frac{1}{|*t|} \simeq \frac{1}{\sqrt{s}} \sqrt{\frac{R}{|*t|}}.$$
And since this equivalence we can also control the third component of the later of (2.36) as follows

\[
\left| (\nabla_a T_b) T^{ab}(\hat{\psi}) \right| \frac{1}{|t|} \leq |A_1 \hat{\psi}_{tt}^2 + A_2 \hat{\psi}_{tt} \hat{\psi}_R + A_3 \hat{\psi}_{t} \hat{\psi}_{t} + A_4 R \hat{\psi}_R^2 + A_5 \hat{\psi}_R \hat{\psi}_{\varphi}| \frac{1}{|t|} \\
+ |A_6 \sin^2 \theta \hat{\psi}_\theta^2 + A_7 \hat{\psi}_\varphi^2 + A_8 |\nabla S^2 \hat{\psi}|^2 + A_9 \hat{\psi}_R^2| \frac{1}{|t|}
\]

\[
\lesssim (A_1 + A_3) \hat{\psi}_t^2 + \left( \frac{A_2}{\sqrt{s}} + \frac{A_2 R}{\sqrt{s} |t|} \hat{\psi}_R^2 \right) + A_4 \frac{R}{|t|} \hat{\psi}_R^2 \\
+ (A_3 + A_5 + A_7) \hat{\psi}_{\varphi}^2 + A_6 \sin^2 \theta \hat{\psi}_\theta^2 + A_8 |\nabla S^2 \hat{\psi}|^2 + A_9 \hat{\psi}_R^2
\]

\[
\lesssim \left( A_1 + \frac{1}{\sqrt{s}} A_2 + A_3 \right) *t^2 \hat{\psi}_t^2 + \frac{A_2 R}{\sqrt{s} |t|} \hat{\psi}_R^2 + A_4 \frac{R}{|t|} \hat{\psi}_R^2 \\
+ \left( (A_3 + A_5 + A_6 + A_7) \sin^2 \theta + A_8 \right) |\nabla S^2 \hat{\psi}|^2 + A_9 \hat{\psi}_R^2
\]

\[
\lesssim \frac{1}{\sqrt{s}} \left( *t^2 \hat{\psi}_t^2 + \frac{R}{|t|} \hat{\psi}_R^2 + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}_R^2 \right).
\]

So we conclude that the right-hand side of (2.36) can be controlled by \( \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) ds \). Using Gronwall’s inequality, we get the following result:

**Proposition 2.4.1.** For \( *t_0 < 0, |t_0| \) large enough and for any smooth compactly supported initial data at \( t = 0 \), the associated rescaled solution \( \hat{\psi} \) satisfies for all \( 0 \leq s_1 < s_2 \leq 1 \),

\[ \mathcal{E}_{\mathcal{H}_{s_1}}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_{s_2}}(\hat{\psi}) \]

\[ \mathcal{E}_{\mathcal{H}_{s_2}}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_{s_1}}(\hat{\psi}) + \mathcal{E}_{\mathcal{S}^{s_2}_t}(\hat{\psi}) \]

in particular for \( s_1 = 0, s_2 = 1 \) we get

\[ \mathcal{E}_{\mathcal{S}^{s_1}_t}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{H}_1}(\hat{\psi}) \]

\[ \mathcal{E}_{\mathcal{H}_1}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{S}^{s_1}_t}(\hat{\psi}) + \mathcal{E}_{\mathcal{S}^{s_1}_t}(\hat{\psi}) \]

where \( \mathcal{E}_{\mathcal{S}^{s_1}_t} = \mathcal{E}_{\mathcal{S}^{s_1}_t} \).

Now using the Sobolev’s inequality, we can control the integral of \( \hat{\psi}^6 \) on \( \mathcal{H}_s \) (see inequality (2.43) in Appendix 2.5.4)

\[
\left( \int_{\mathcal{H}_s} \hat{\psi}^6 d^*t d^2 \omega \right)^{1/3} \leq K \int_{\mathcal{H}_s} \left( (\partial_t(\hat{\psi}|_{\mathcal{H}_s})^2 + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 \right) d^*t d^2 \omega, \ 0 \leq s \leq 1
\]

where \( K \) is independent of \( s \). From \(*t = -sr_* \) on the hypersurface \( \mathcal{H}_s \), we have

\[
\partial_t(\hat{\psi}|_{\mathcal{H}_s}) = \hat{\psi}_{st} + \frac{r_* R^2 \Delta}{|t||r^2 + a^2|} \hat{\psi}_R \simeq \hat{\psi}_{st} + \frac{R}{|t|} \hat{\psi}_R,
\]
so that

$$
\left( \int_{\mathcal{H}_s} \hat{\psi}^6 d^* t d\omega \right)^{1/3} \lesssim \int_{\mathcal{H}_s} \left( \hat{\psi}^6 + \frac{R}{|t|} \hat{\psi}_R^2 \right)^{1/3} d^* t d\omega
$$

\[\leq \int_{\mathcal{H}_s} \left( \left( \hat{\psi}_t \right)^2 + 2 \frac{R^2}{|t|^2} \left( \hat{\psi}_R \right)^2 + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 \right) d^* t d\omega \]

\[\leq \int_{\mathcal{H}_s} \left( \left( \hat{\psi}_t \right)^2 + 2 \frac{R}{|t|} \left( \hat{\psi}_R \right)^2 + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 \right) d^* t d\omega \]

\[\lesssim \int_{\mathcal{H}_s} \left( \left( \hat{\psi}_t \right)^2 + \frac{R}{|t|} \left( \hat{\psi}_R \right)^2 + |\nabla S^2 \hat{\psi}|^2 + \hat{\psi}^2 \right) d^* t d\omega \]

\[\lesssim E_{\mathcal{H}_s}(\hat{\psi}). \]

From this inequality, we can control the component $\hat{\psi}_4$ in $\tilde{E}_{\mathcal{H}_s}(\hat{\psi})$

$$
\int_{\mathcal{H}_s} \hat{\psi}_4^4 d^* t d\omega \leq \frac{1}{2} \int_{\mathcal{H}_s} (\hat{\psi}^6 + \hat{\psi}^2) d^* t d\omega \lesssim \left( E_{\mathcal{H}_s}(\hat{\psi}) \right)^3 + E_{\mathcal{H}_s}(\hat{\psi}).
$$

So we can control the energy $\tilde{E}_{\mathcal{H}_s}(\hat{\psi})$, and hence the energy $E_{\mathcal{H}_s}(\hat{\psi})$ satisfies the following inequality:

$$
E_{\mathcal{H}_s}(\hat{\psi}) \lesssim \tilde{E}_{\mathcal{H}_s}(\hat{\psi}) \lesssim \left( E_{\mathcal{H}_s}(\hat{\psi}) \right)^3 + E_{\mathcal{H}_s}(\hat{\psi}). \quad (2.37)
$$

Therefore Proposition 2.4.1 has the following direct consequence:

**Corollary 2.4.1.** For $t_0 > 0$, $|t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies

$$
E_{\mathcal{S}_t^+}(\hat{\psi}) \lesssim \left( E_{\mathcal{H}_1}(\hat{\psi}) \right)^3 + E_{\mathcal{H}_1}(\hat{\psi}),
$$

$$
E_{\mathcal{H}_1}(\hat{\psi}) \lesssim \left( E_{\mathcal{S}_t^+}(\hat{\psi}) \right)^3 + E_{\mathcal{S}_t^+}(\hat{\psi}) + \tilde{E}_{\mathcal{S}_t^+}(\hat{\psi}).
$$
High order estimate and the peeling

First, we control the energy of $\hat{\psi}_R$. Integrating the conservation law (2.35), we obtain

$$
\left| \mathcal{E}_{H_s}(\hat{\psi}_R) + \mathcal{E}_{S^2_{s_0}}(\hat{\psi}_R) - \mathcal{E}_{H_2}(\hat{\psi}_R) \right| 
\lesssim \int_{s_1}^{s_2} \int_{H_s} \frac{r^2}{\rho^2} \left( 4a^2 R \partial_t \partial_R \hat{\psi} + 4aR \partial_r \partial_R \hat{\psi} + 2a^2 \partial_t \partial_R \hat{\psi} \right) T^b \partial_b(\partial_R \hat{\psi}) \left| \frac{1}{|*t|} \right| dt^2 \omega ds 
+ \int_{s_1}^{s_2} \int_{H_s} \frac{r^2}{\rho^2} \left( 2a \partial_r \partial_R \hat{\psi} + (4a^2 R^3 - 6MR^2 + 2R) \partial_R^2 \hat{\psi} \right) 
+ (12aR^2 - 12MR + 2) \partial_R \hat{\psi} \right) T^b \partial_b(\partial_R \hat{\psi}) \left| \frac{1}{|*t|} \right| dt^2 \omega ds 
+ \int_{s_1}^{s_2} \int_{H_s} \left( 2(M - 2a^2 R) \hat{\psi} + 2 \frac{Mr - a^2}{r^2} \partial_R \hat{\psi} \right) T^b \partial_b(\partial_R \hat{\psi}) \left| \frac{1}{|*t|} \right| dt^2 \omega ds 
+ \int_{s_1}^{s_2} \int_{H_s} \left( \hat{\psi}_R - 3 \hat{\psi}^2 \hat{\psi}_R - \frac{ra^2 \cos^2 \theta}{\rho^2} \hat{\psi}^3 \right) T^b \partial_b(\partial_R \hat{\psi}) \left| \frac{1}{|*t|} \right| dt^2 \omega ds 
+ \int_{s_1}^{s_2} \int_{H_s} (\nabla_a T_b) T^{ab}(\partial_R \hat{\psi}) \left| \frac{1}{|*t|} \right| dt^2 \omega ds ,
$$

where

$$(\nabla_a T_b) T^{ab}(\partial_R \hat{\psi}) = A_1(\partial_R \hat{\psi})^2_t + A_2(\partial_R \hat{\psi})^2_s + A_3(\partial_R \hat{\psi})^2_t (\partial_R \hat{\psi})_R + A_4(\partial_R \hat{\psi})^2_s (\partial_R \hat{\psi})_R 
+ A_5(\partial_R \hat{\psi}) (\partial_R \hat{\psi}) (\partial_R \hat{\psi})_R + A_6 \sin^2 \theta (\partial_R \hat{\psi})_R^2 + A_7(\partial_R \hat{\psi})_R^2 \partial_R \hat{\psi} + A_8 |\nabla S^2(\partial_R \hat{\psi})|^2 
+ A_9 (\partial_R \hat{\psi})^2$$

and

$$T^b \partial_b(\partial_R \hat{\psi}) = \ast t^2 \partial_t (\partial_R \hat{\psi}) - 2(1 + \ast R) \partial_R (\partial_R \hat{\psi}).$$

We may think that the right-hand side of (2.38) can be controlled in the same manner as for the basic estimate except for the following term which arises from the nonlinear part of the equation:

$$E = \int_{s_1}^{s_2} \int_{H_s} \left( -3 \hat{\psi}_R^2 - \frac{ra^2 \cos^2 \theta}{\rho^2} \hat{\psi}_R^3 \right) T^b \partial_b(\hat{\psi}_R) \left| \frac{1}{|*t|} \right| dt^2 \omega ds.$$
Using the Cauchy’s and Holder’s inequalities we have

\[ E = \int_{s_1}^{s_2} \int_{H_s} \left( -3\dot{\psi}^2 \dot{\psi}_R - \frac{ra^2 \cos^2 \theta}{\rho^2} \dot{\psi}^3 \right) \left( t^2 \partial_t \dot{\psi}_R - 2(1 + *tR) \partial_R \dot{\psi}_R \right) \left| \frac{1}{*t} \right| d^*t d^2\omega ds \]

\[ \lesssim \int_{s_1}^{s_2} \int_{H_s} \left( \frac{3}{2} \left( \dot{\psi}_R^4 + *t^2 (\partial_t \dot{\psi}_R)^2 \right) + \frac{ra^2 \cos^2 \theta}{2\rho^2} \left( \dot{\psi}^6 + *t^2 (\partial_t \dot{\psi}_R)^2 \right) \right) d^*t d^2\omega ds \]

\[ + \int_{s_1}^{s_2} \int_{H_s} 3(1 + *tR) \frac{1}{\sqrt{s}} \left( \dot{\psi}_R \dot{\psi}_R^2 + \frac{R}{*t} (\partial_R \dot{\psi}_R)^2 \right) d^*t d^2\omega ds \]

\[ + \int_{s_1}^{s_2} \int_{H_s} (1 + *tR) \frac{ra^2 \cos^2 \theta}{\rho^2} \frac{1}{\sqrt{s}} \left( \dot{\psi}^6 + \frac{R}{*t} (\partial_R \dot{\psi}_R)^2 \right) d^*t d^2\omega ds \]

\[ \lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \mathcal{E}_{H_s}(\dot{\psi}_R) ds + \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{H_s} \left( \dot{\psi}_R \dot{\psi}_R^2 + \dot{\psi}^6 \right) d^*t d^2\omega ds \]

\[ \lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \mathcal{E}_{H_s}(\dot{\psi}_R) ds + \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \int_{H_s} \dot{\psi}^6 d^*t d^2\omega \right) \left( \int_{H_s} \dot{\psi}^6 d^*t d^2\omega \right)^{1/3} ds \]

\[ + \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{H_s} \dot{\psi}^6 d^*t d^2\omega ds. \]

We know from the basic estimate case that we can control the norm of \( \dot{\psi}^6 \) on \( H_s \):

\[ \left( \int_{H_s} \dot{\psi}^6 d^*t d\omega \right)^{1/3} \lesssim \int_{H_s} \left( \dot{\psi}_\tau + \frac{R}{*t} \dot{\psi}_R \right)^2 + |\nabla_{S^2} \dot{\psi}|^2 + \dot{\psi}^2 d^*t d^2\omega \]

\[ \lesssim \mathcal{E}_{H_s}(\dot{\psi}) \lesssim \mathcal{E}_{H_s}(\dot{\psi}), \]

and similarly for the norm of \( \dot{\psi}_R^6 \)

\[ \left( \int_{H_s} \dot{\psi}_R^6 d^*t d\omega \right)^{1/3} \lesssim \mathcal{E}_{H_s}(\dot{\psi}_R), \]

so we have

\[ E \lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \mathcal{E}_{H_s}(\dot{\psi}_R) ds + \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H_s}(\dot{\psi}) \right)^2 \mathcal{E}_{H_s}(\dot{\psi}_R) ds \]

\[ + \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H_s}(\dot{\psi}) \right)^3 ds \]

\[ \lesssim \left( 1 + \left( \mathcal{E}_{H_1}(\dot{\psi}) \right)^2 \right) \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H_s}(\dot{\psi}_R) + \mathcal{E}_{H_s}(\dot{\psi}) \right) ds \]

\[ \lesssim \left( 1 + \left( \mathcal{E}_{H_1}(\dot{\psi}) \right)^2 \right) \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H_s}(\dot{\psi}_R) + \mathcal{E}_{H_s}(\dot{\psi}) \right) ds. \]

The inequality (2.38) gives the two inequalities

\[ \left| \mathcal{E}_{H_{s_1}}(\dot{\psi}_R) + \mathcal{E}_{S^{s_1,s_2}}(\dot{\psi}_R) - \mathcal{E}_{H_{s_2}}(\dot{\psi}_R) \right| \]

\[ \lesssim \left( 1 + \left( \mathcal{E}_{H_1}(\dot{\psi}) \right)^2 \right) \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H_s}(\dot{\psi}_R) + \mathcal{E}_{H_s}(\dot{\psi}) \right) ds \] (2.39)
and

\[ |\mathcal{E}_{H(1)}(\hat{\psi}_R) + \mathcal{E}_{S_t^{s_1} R}(\hat{\psi}_R) - \mathcal{E}_{H(2)}(\hat{\psi}_R)| \]

\[ \lesssim \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) \right)^2 \right) \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{H(1)}(\hat{\psi}_R) + \hat{\mathcal{E}}_{H(2)}(\hat{\psi}) \right) ds \]  

(2.40)

Using the Gronwall’s inequality for inequality (2.39) and (2.40), we get:

**Proposition 2.4.2.** For \( t_0 < 0, |t_0| \text{ large enough and for any smooth compactly supported initial data at } t = 0 \), the associated rescaled solution \( \hat{\psi} \) satisfies for all \( 0 \leq s_1 < s_2 \leq 1 \),

\[ \mathcal{E}_{H(1)}(\hat{\psi}_R) \lesssim \exp \left\{ \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) \right)^2 \right) (s_2 - s_1) \right\} \left( \mathcal{E}_{H(1)}(\hat{\psi}_R) + \hat{\mathcal{E}}_{H_1}(\hat{\psi}) \right) , \]

\[ \mathcal{E}_{H_1}(\hat{\psi}_R) \lesssim \exp \left\{ \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) \right)^2 \right) (s_2 - s_1) \right\} \times \left( \mathcal{E}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}_R) + \mathcal{E}_{H(1)}(\hat{\psi}_R) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) + \hat{\mathcal{E}}_{H_1}(\hat{\psi}) \right) \]

in particular for \( s_1 = 0, s_2 = 1 \)

\[ \mathcal{E}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}_R) \lesssim \exp \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) \right)^2 \right) \left( \mathcal{E}_{\mathcal{H}_1}(\hat{\psi}_R) + \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) \right) , \]

\[ \mathcal{E}_{H_1}(\hat{\psi}_R) \lesssim \exp \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) \right)^2 \right) \times \left( \mathcal{E}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}_R) + \mathcal{E}_{H_1}(\hat{\psi}_R) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) + \hat{\mathcal{E}}_{H_1}(\hat{\psi}) \right) \]

where \( S^{i_0} = S_t^{s_0} \).

Similarly, we can control the energies of \( \mathcal{L}_{X_1}(\hat{\psi}) = \hat{\psi}_t \) and \( \mathcal{L}_{X_1}(\hat{\psi}) = \hat{\psi}_\varphi \) from the conservation laws (2.31) and (2.32) to obtain the following result:

**Proposition 2.4.3.** For \( t_0 < 0, |t_0| \text{ large enough and for any smooth compactly supported initial data at } t = 0 \), the associated rescaled solution \( \hat{\psi} \) satisfies

\[ \mathcal{E}_{\mathcal{S}_t^{s_1} R}(\mathcal{L}_{X_1}(\hat{\psi})) \lesssim \exp \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) \right)^2 \right) \left( \mathcal{E}_{\mathcal{H}_1}(\mathcal{L}_{X_1}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) \right) , \]

\[ \mathcal{E}_{H_1}(\mathcal{L}_{X_1}(\hat{\psi})) \lesssim \exp \left( 1 + \left( \hat{\mathcal{E}}_{\mathcal{H}_1}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) \right)^2 \right) \times \left( \mathcal{E}_{\mathcal{S}_t^{s_1} R}(\mathcal{L}_{X_1}(\hat{\psi}) + \mathcal{E}_{H_1}(\mathcal{L}_{X_1}(\hat{\psi}) + \hat{\mathcal{E}}_{\mathcal{S}_t^{s_1} R}(\hat{\psi}) + \hat{\mathcal{E}}_{H_1}(\hat{\psi}) \right) \]

where \( i = 0, 1. \)
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Finally, to control the energies of $\mathcal{L}_{X_3} \dot{\psi}$ from the conservation law $[2.33]$ we need the help from the energy of $\dot{\psi}_t$ and $\mathcal{L}_{X_3} \dot{\psi}$. The same holds in controlling the energies of $\mathcal{L}_{X_3} \dot{\psi}$. Therefore we obtain:

**Proposition 2.4.4.** For $*t_0 < 0$, $|*t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies

$$\sum_{i=0}^{3} \mathcal{E}_{\mathcal{L}_{X_3}^i} (\mathcal{L}_{X_3} \dot{\psi}) \lesssim \exp \left( 1 + \left( \overline{\mathcal{E}}_{\mathcal{H}_1} (\dot{\psi}) \right)^2 \right) \left( \sum_{i=0}^{3} \mathcal{E}_{\mathcal{H}_1} (\mathcal{L}_{X_3} \dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{H}_1} (\dot{\psi}) \right),$$

$$\sum_{i=0}^{3} \mathcal{E}_{\mathcal{H}_1} (\mathcal{L}_{X_3} \dot{\psi}) \lesssim \exp \left( 1 + \left( \overline{\mathcal{E}}_{\mathcal{L}_{X_3}^i} (\dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{S}_{*t_0}} (\dot{\psi}) \right)^2 \right) \times \left( \sum_{i=0}^{3} \mathcal{E}_{\mathcal{L}_{X_3}^i} (\mathcal{L}_{X_3} \dot{\psi}) + \sum_{i=0}^{3} \mathcal{E}_{\mathcal{S}_{*t_0}} (\mathcal{L}_{X_3} \dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{L}_{X_3}^i} (\dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{S}_{*t_0}} (\dot{\psi}) \right).$$

By a similar way, Propositions 2.4.2 and 2.4.4 can be generalized to obtain the following estimate at order $k$

**Theorem 2.4.1.** For $*t_0 < 0$, $|*t_0|$ large enough and for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution $\hat{\psi}$ satisfies

$$\sum_{q=0}^{p} \mathcal{E}_{\mathcal{L}_{X_3}^q} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) \lesssim \exp \left( 1 + \sum_{q=1}^{p-1} \left( \mathcal{E}_{\mathcal{H}_1} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) \right)^2 + \left( \overline{\mathcal{E}}_{\mathcal{H}_1} (\dot{\psi}) \right)^2 \right) \times \left( \sum_{q=1}^{p} \mathcal{E}_{\mathcal{H}_1} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{H}_1} (\dot{\psi}) \right),$$

$$\sum_{q=0}^{p} \mathcal{E}_{\mathcal{H}_1} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) \lesssim \exp \left\{ 1 + \sum_{q=1}^{p-1} \left( \mathcal{E}_{\mathcal{L}_{X_3}^q} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) + \mathcal{E}_{\mathcal{S}_{*t_0}} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) \right)^2 + \left( \overline{\mathcal{E}}_{\mathcal{L}_{X_3}^q} (\dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{S}_{*t_0}} (\dot{\psi}) \right)^2 \right\} \times \left\{ \sum_{q=1}^{p} \left( \mathcal{E}_{\mathcal{L}_{X_3}^q} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) + \mathcal{E}_{\mathcal{S}_{*t_0}} (\mathcal{L}_{X_3}^{p-q} \mathcal{L}_{X_3}^q \dot{\psi}) \right) + \overline{\mathcal{E}}_{\mathcal{L}_{X_3}^q} (\dot{\psi}) + \overline{\mathcal{E}}_{\mathcal{S}_{*t_0}} (\dot{\psi}) \right\},$$

where

$$\mathcal{L}_{X_3}^q = \mathcal{L}_{X_3} \mathcal{L}_{X_3} \cdots \mathcal{L}_{X_3} , \quad i_j \in \{0, 1, 2, 3\} .$$

Now we give the definition of peeling at order $k \in \mathbb{N}^*$:
**Definition 2.4.1.** We say that a solution $\psi$ of the nonlinear wave equation peels at order $k$ if the rescaled solution $\hat{\psi}$ satisfies
\[
\sum_{p=0}^{k} E_{\mathcal{H}_s} \mathcal{P}(\mathcal{L}^{p-q}_{X_4} \mathcal{L}^{q}_{X_{ij}} \hat{\psi}) < +\infty,
\]
where $\mathcal{P}$ is the polynomial with order $p$ of the basic term $\mathcal{L}^{p-q}_{X_4} \mathcal{L}^{q}_{X_{ij}} \hat{\psi}$.

The following theorem completely characterizes the largest class of initial data that guarantees peeling at order $k$. It follows directly from Inequality 2.37, Corollary 2.4.1 and Proposition 2.4.1:

**Theorem 2.4.2.** The space of optimal initial data is completion of $\mathcal{C}^\infty_0([-t_0, +\infty] \times \mathbb{R} \times S^2_\omega)$ on the norm
\[
\left\| \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right\|_{\mathcal{L}^{\mathcal{H}_1}}^2 := \sum_{p=0}^{k} \sum_{q=0}^{p} E_{\mathcal{H}_1} \left( \mathcal{L}^{p-q}_{X_4} \mathcal{L}^{q}_{X_{ij}} \left( \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right) \right),
\]
where we have denoted by $E_{\mathcal{H}_1} \left( \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right)$ the energy $E_{\mathcal{H}_1}(\hat{\psi})$ where $\hat{\psi}$ is replaced by $\hat{\psi}_0$ and $\partial_t \hat{\psi} = \hat{\psi}_t$ is replaced by $\hat{\psi}_1$.

**2.4.4 Interpretation**

First, we construct the peeling definition for the nonlinear equation on the Minkowski spacetime. Using again the full conformal compactification, we embed the Minkowski spacetime into the Einstein cylinder then it is described by the domain
\[
\mathbb{M} = \{ |\tau| + \zeta \leq \pi, \zeta \geq 0, \omega \in S^2 \},
\]
with the rescaled metric
\[
\hat{g} = d\tau^2 - d\zeta^2 - \sin^2 \zeta d\omega^2.
\]
We foliate the domain $\mathbb{M}^+$ (where $\tau \geq 0$) in $\mathbb{M}$ by the hypersurfaces
\[
\mathcal{H}_s = \{ \tau = s(\pi - \zeta); 0 \leq s \leq 1 \}.
\]
We think that if $s = 0$ the hypersurface $\mathcal{H}_0$ coincide with the origin $\Sigma_0 = \{ \tau = 0 \}$ and if $s = 1$ the hypersurface $\mathcal{H}_1$ coincide with the future null infinity $\mathcal{I}^+$. If we set
\[
f(\tau, \xi, \omega) = \tau - s(\pi - \zeta)
\]
then the normal vector to the hypersurface $\mathcal{H}_s$ is
\[
\mathcal{N}^a \partial_a = \hat{g}^{ab} \nabla_b f = \hat{g}^{ab} (d\tau + sd\zeta) = \partial_\tau - s\partial_\zeta.
\]
We can choose a transversal vector $\mathcal{L}^a = \partial_{\tau}$. The energy of the solution on the hypersurface $\mathcal{H}_s$ is given by

$$\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) = \int_{\mathcal{H}_s} \hat{T}_{ab}(\hat{\psi}) K^a N^b (\mathcal{L}_{ab} \text{Vol}^4),$$

where $K^a$ is the conformal Killing vector filed $\partial_{\tau}$ and $\hat{T}_{ab}$ is the stress energy tensor

$$\hat{T}_{ab}(\hat{\psi}) = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \hat{g}^{cd} \partial_c \hat{\psi} \partial_d \hat{\psi} + \frac{1}{2} \hat{g}_{ab} \hat{\psi}^2 + \frac{1}{4} \hat{g}_{ab} \hat{\psi}^4.$$

Calculating in details, we obtain

$$\hat{\mathcal{E}}_{\mathcal{H}_s}(\hat{\psi}) = \frac{1}{2} \int_{\mathcal{H}_s} \left( \hat{\psi}_\tau^2 - 2s \hat{\psi}_\tau \hat{\psi}_\zeta + |\nabla_{S^3} \hat{\psi}|^2 + \hat{\psi}^2 + \frac{1}{2} \hat{\psi}^4 \right) d\mu_{S^3}$$

$$= \frac{1}{2} \int_{\mathcal{H}_s} \left( |\hat{\psi}_\tau - s \hat{\psi}_\zeta|^2 + (1 - s^2) \hat{\psi}_\zeta^2 + \frac{1}{\sin^2 \zeta} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 + \frac{1}{2} \hat{\psi}^4 \right) d\mu_{S^3}. $$

![Diagram](image.png)

Figure 2.6: A foliation for the full compactification of Minkowski spacetime

If we use the stress-energy tensor of the equation $\Box \hat{\psi} + \hat{\psi} = 0$

$$T_{ab}(\hat{\psi}) = \partial_a \hat{\psi} \partial_b \hat{\psi} - \frac{1}{2} \hat{g}_{ab} \hat{g}^{cd} \partial_c \hat{\psi} \partial_d \hat{\psi} + \frac{1}{2} \hat{g}_{ab} \hat{\psi}^2,$$

we obtain the energy

$$\mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) = \frac{1}{2} \int_{\mathcal{H}_s} \left( \hat{\psi}_\tau^2 - 2s \hat{\psi}_\tau \hat{\psi}_\zeta + |\nabla_{S^3} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3}$$

$$= \frac{1}{2} \int_{\mathcal{H}_s} \left( |\hat{\psi}_\tau - s \hat{\psi}_\zeta|^2 + (1 - s^2) \hat{\psi}_\zeta^2 + \frac{1}{\sin^2 \zeta} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3},$$

which can be given a norm of $\hat{\psi}$. 
Now we control the first energy as follows
\[ \tilde{\mathcal{E}}_{H_s}(\hat{\psi}) = \mathcal{E}_{H_s}(\hat{\psi}) + \frac{1}{4} \int_{H_s} \hat{\psi}^4 d\mu_{S^3} \leq \mathcal{E}_{H_s}(\hat{\psi}) + \frac{1}{8} \int_{H_s} (\hat{\psi}^6 + \hat{\psi}^2) d\mu_{S^3} \]
\[ \lesssim \mathcal{E}_{H_*(\hat{\psi})} + \int_{H_s} \hat{\psi}^6 d\mu_{S^3}. \]

The last term can be controlled by the second energy of \( \hat{\psi} \). Indeed, using the Sobolev’s inequality with the Euclidian metric \( \sigma^2_{S^3} \) on 3–sphere \( S^3 \) we have
\[
\left( \int_{H_s} \hat{\psi}^6 d\mu_{S^3} \right)^{1/3} \leq \int_{H_s} \left( \left( \partial_\xi \hat{\psi} \big|_{\tau=\pi-\xi} \right)^2 + \frac{1}{\sin^2 \xi} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3}
\]
\[
= \int_{H_s} \left( \left( \partial_\xi \hat{\psi} - s \partial_\tau \hat{\psi} \right)^2 + \frac{1}{\sin^2 \xi} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3}
\]
\[
\leq \int_{H_s} \left( (s \partial_\xi \hat{\psi} - \partial_\tau \hat{\psi})^2 + \left( 1 - s^2 \right) (\partial_\xi \hat{\psi})^2 + \frac{1}{\sin^2 \xi} |\nabla_{S^2} \hat{\psi}|^2 + \hat{\psi}^2 \right) d\mu_{S^3}
\]
\[ \lesssim \mathcal{E}_{H_*(\hat{\psi})}. \]

So we conclude
\[ \mathcal{E}_{H_s}(\hat{\psi}) \leq \tilde{\mathcal{E}}_{H_*(\hat{\psi})} \lesssim \mathcal{E}_{H_*(\hat{\psi})} + \left( \mathcal{E}_{H_*(\hat{\psi})} \right)^3, \quad 0 \leq s \leq 1. \quad (2.41) \]

Now we have the conservation law that is associated with \( \tilde{T}_{ab} \) for the rescaled equation \( \square \hat{\psi} + \hat{\psi} + \hat{\psi}^3 = 0 \) is given as follows
\[ \nabla^a (K^a \tilde{T}_{ab}(\hat{\psi})) = 0. \]

Integrating this conservation law on the whole domain \( M^+ \), by the partial integral formula we obtain
\[ \tilde{\mathcal{E}}_{\mathcal{I}^+}(\hat{\psi}) = \tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi}), \]

associated with the inequality (2.41) we have the following result:

**Proposition 2.4.5.** For any smooth solutions \( \hat{\psi} \) of the nonlinear wave equation associated with the smooth supported initial data, we have
\[ \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) \leq \mathcal{E}_{\Sigma_0}(\hat{\psi}) + \left( \mathcal{E}_{\Sigma_0}(\hat{\psi}) \right)^3, \]
\[ \mathcal{E}_{\Sigma_0}(\hat{\psi}) \lesssim \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) + \left( \mathcal{E}_{\mathcal{I}^+}(\hat{\psi}) \right)^3. \]

Now commuting the derivative \( \partial_\tau \) into rescaled equation we obtain
\[ \square \hat{\psi}_\tau + \hat{\psi}_\tau + 3 \hat{\psi}^2 \hat{\psi}_\tau = 0, \]
so the conservation law associated with \( \tilde{T}_{ab} \) is given by
\[ \nabla^a (K^a \tilde{T}_{ab}(\hat{\psi}_\tau)) = \left( \hat{\psi}^3 - 3 \hat{\psi}^2 \hat{\psi}_\tau \right) \hat{\psi}_\tau. \]
Integrating this conservation on the whole $\mathcal{M}^+$, we get

$$
\left| \tilde{\mathcal{E}}_{\mathcal{I}^+}(\hat{\psi}_\tau) - \tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi}_\tau) \right| \leq \int_0^1 \int_{\mathcal{H}_s} \left| \hat{\psi}^3 \hat{\psi}_\tau - 3\hat{\psi}^2 \hat{\psi}_\tau^2 \right| \nu_{\mathcal{H}_s} \text{dVol}_{\mathcal{H}_s} \text{d}s

\leq \int_0^1 \int_{\mathcal{H}_s} \left( \hat{\psi}_\tau^4 + \frac{3}{2} \hat{\psi}^4 + \frac{3}{2} \hat{\psi}_\tau^4 \right) (\pi - \zeta) \text{d}\mu_{\mathcal{S}_0} \text{d}s

\lesssim \int_0^1 \left( \tilde{\mathcal{E}}_{\mathcal{H}_s}(\hat{\psi}_\tau) + \tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi}) \right) \text{d}s,
$$

where $\nu = (\pi - \zeta) \partial_\tau$ is the identifying vector field of the hypersurface $\mathcal{H}_s$. Using the Gronwall’s inequality, we obtain the following proposition

**Proposition 2.4.6.** For any smooth solutions $\hat{\psi}$ of the nonlinear wave equation associated with the smooth supported initial data, we have

$$
\tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi}_\tau) \lesssim \tilde{\mathcal{E}}_{\mathcal{I}^+}(\hat{\psi}_\tau) + \tilde{\mathcal{E}}_{\mathcal{I}^+}(\hat{\psi}),
$$

$$
\tilde{\mathcal{E}}_{\mathcal{I}^+}(\hat{\psi}_\tau) \lesssim \tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi}_\tau) + \tilde{\mathcal{E}}_{\Sigma_0}(\hat{\psi})
$$

and in general, for all $k \geq 0$ we have

$$
\tilde{\mathcal{E}}_{\Sigma_0}(\partial_\tau^k \hat{\psi}) \lesssim \sum_{p=0}^k \tilde{\mathcal{E}}_{\mathcal{I}^+}(\partial_\tau^p \hat{\psi}),
$$

$$
\tilde{\mathcal{E}}_{\mathcal{I}^+}(\partial_\tau^k \hat{\psi}) \lesssim \sum_{p=0}^k \tilde{\mathcal{E}}_{\Sigma_0}(\partial_\tau^k \hat{\psi}).
$$

Associated with the inequality (2.41), we have the following consequence:

**Corollary 2.4.2.** For all $k \geq 0$, we have two inequalities

$$
\mathcal{E}_{\Sigma_0}(\partial_\tau^k \hat{\psi}) \lesssim \sum_{p=0}^k \mathcal{E}_{\mathcal{I}^+}(\partial_\tau^p \hat{\psi}) + \left( \mathcal{E}_{\mathcal{I}^+}(\partial_\tau^p \hat{\psi}) \right)^3,
$$

$$
\mathcal{E}_{\mathcal{I}^+}(\partial_\tau^k \hat{\psi}) \lesssim \sum_{p=0}^k \mathcal{E}_{\Sigma_0}(\partial_\tau^k \hat{\psi}) + \left( \mathcal{E}_{\Sigma_0}(\partial_\tau^k \hat{\psi}) \right)^3.
$$

Now we give the definition of the peeling at order $k$ as follows:

**Definition 2.4.2.** The solution of nonlinear wave equation has peel at order $k$ if the rescaled solution such that for all polynomial $\mathbb{P}$ with order $0 \leq p \leq k$ of basic term $\partial_\tau^p \hat{\psi}$, we have

$$
\sum_{p=0}^k \mathcal{E}_{\mathcal{I}^+}(\mathbb{P}(\partial_\tau^p \hat{\psi})) < +\infty.
$$

By Corollary 2.4.2, we obtain the largest class of initial data that guarantees peeling at order $k$:
Theorem 2.4.3. The space of optimal initial data \( h^k(\Sigma_0) \) that guarantees the definition peeling at order \( k \) is the completion of \( C^\infty(\Sigma_0) \times C^\infty(\Sigma_0) \) on the norm
\[
\left\| \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right\|^2_{h^k(\Sigma_0)} := \sum_{p=0}^k E_{\Sigma_0} \left( \partial_\tau \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right),
\]
where we have denoted by \( E_{H_1} \left( \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix} \right) \) the energy \( E_{H_1}(\hat{\psi}) \) where \( \hat{\psi} \) is replaced by \( \hat{\psi}_0 \) and \( \partial_\tau \hat{\psi} \) is replaced by \( \hat{\psi}_1 \).

Remark 2.4.2. The difference from the construction in the linear case is the fact that we need to give a foliation by the hypersurface leaves, which have the detailed relation expression between \( \tau \) and \( \xi \). This serves to control the nonlinear component when using Sobolev’s inequality.

Now we can continue to do similarly as in the linear case: taking the constraint on the flat spacetime of the energies obtained by the stress-energy tensor \( T_{ab} \) on the hypersurface \( \Sigma_0 \) in the Kerr case and then comparing the classes of the initial data of the energies constraint and the energies obtained directly on the flat spacetime. Then similarly as in the linear case, we can conclude that the initial data which guarantees the peeling definition on Kerr background is slightly large than the one on flat case at a higher order given (but the definition is not more general). As a consequence, our peeling definition is more general than the one that was given in the papers of Penrose [71, 72].

2.5 Appendix

2.5.1 The divergence theorem

The divergence theorem is a consequence of the Stokes’s theorem where we take the integral of the Hodge dual of a 1–form on a bounded domain:

Theorem 2.5.1. (The divergence theorem). Let \( \Omega \) a bounded open subset of the spacetime \( \mathcal{M} \) with piecewise \( C^1 \) boundary \( S \), \( l^a \) a vector field transverse to \( S \) and outgoing, \( n^a \) a normal vector field to \( S \) such that \( l_a n^a = 1 \). Let \( \alpha \) be a 1–form \( C^1 \) on \( \bar{\Omega} \), then
\[
-4 \int_S *\alpha_a dx^a = \int_\Omega \nabla_a \alpha^a d\text{Vol}^4 = \int_S \alpha_a n^a (l_\omega d\text{Vol}^4).
\]

And to calculate the energy in this chapter for the current energy quantity \( T_{ab} K^a \), we used the two formulas
\[
E_S(\psi) = \int_S T^b_a(\psi)K^a(\partial_b d\text{Vol}^4) = \int_S T_{ab}(\psi)K^a N^b(\mathcal{L}_\omega d\text{Vol}^4).
\]

The first form is a direct calculation from the 3–form energy
\[
T_{ab}(\psi)K^a dx^b = T^b_a(\psi)K^a(\partial_b d\text{Vol}^4)
\]
the second form is exactly the formula that appears in the divergence theorem.
2.5.2 Proof of Lemma [2.3.1]

In this section we will prove Lemma [2.3.1]. First of all we recall

\[ \nabla_{(a}T_{b)}dx^{a}dx^{b} = \left\{ (1 + tR)M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) - \frac{2M^*tR}{\rho^{2}} \right\} d^*t^2 - 4a \sin^2 \theta^* t d R d^* \varphi \]

\[ - 4a \sin^2 \theta \left\{ 2(1 + tR)M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) - 2M^*tR \rho^2 + R \right\} d^*t d^* \varphi \]

\[ + 4a^2 \cos^2 \theta(1 + tR)R d^2 + 4a^2 \sin^2 \theta(1 + tR) \left\{ R + \sin^2 \theta M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) \right\} d^* \varphi^2. \]

So in the expression of \( T^{ab} \), we are interested in the components having the factors \( \partial^2_t, \partial_t \partial_\varphi, \partial_R \partial_\varphi, \partial^2_\theta \) and \( \partial^2_\varphi \), so we can write

\[ T^{ab}dx^{a}dx^{b} := \hat{g}^{ac} \hat{g}^{bd}T_{cd} = \hat{g}^{ac} \hat{g}^{bd} \hat{\phi}_c \hat{\phi}_d - \frac{1}{2}\hat{g}^{ab} \hat{g}^{\hat{f}\hat{g}\hat{\phi}_f \hat{\phi}_g} \]

\[ = (\hat{g}^{0c} \hat{g}^{0d} \hat{\phi}_c \hat{\phi}_d) \partial^2_t + 2(\hat{g}^{0c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial_t \partial_\varphi + 2(\hat{g}^{1c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial_R \partial_\varphi \]

\[ + (\hat{g}^{22} \hat{g}^{22} \hat{\phi}_d) \partial^2_\theta + (\hat{g}^{3c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial^2_\varphi \]

\[ + \frac{\hat{g}^{fg} \hat{\phi}_f \hat{\phi}_g}{2} \left( \frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial^2_t + 2 \ar^2 \rho^2 \partial_t \partial_\varphi + 2 a \rho^2 \partial_R \partial_\varphi + \frac{r^2}{\rho^2} \partial^2_\theta + \frac{r^2}{\rho^2} \sin^2 \theta \partial^2_\varphi \right) \]

\[ = A + B \hat{g}^{fg} \hat{\phi}_f \hat{\phi}_g \]

where

\[ A = (\hat{g}^{0c} \hat{g}^{0d} \hat{\phi}_c \hat{\phi}_d) \partial^2_t + 2(\hat{g}^{0c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial_t \partial_\varphi + 2(\hat{g}^{1c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial_R \partial_\varphi \]

\[ + (\hat{g}^{22} \hat{g}^{22} \hat{\phi}_d) \partial^2_\theta + (\hat{g}^{3c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \partial^2_\varphi \]

and

\[ B = \frac{1}{2} \left( \frac{r^2 a^2 \sin^2 \theta}{\rho^2} \partial^2_t + 2 \ar^2 \rho^2 \partial_t \partial_\varphi + 2 a \rho^2 \partial_R \partial_\varphi + \frac{r^2}{\rho^2} \partial^2_\theta + \frac{r^2}{\rho^2} \sin^2 \theta \partial^2_\varphi \right). \]

So that

\[ (\nabla_{a}T_{b}) T^{ab} = \nabla_{(a}T_{b)}A + \hat{g}^{fg} \hat{\phi}_f \hat{\phi}_g \nabla_{(a}T_{b)}B. \]

The first term is:

\[ \nabla_{(a}T_{b)}A = 4 \left\{ (1 + tR)M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) - \frac{2M^*tR}{\rho^{2}} \right\} (\hat{g}^{0c} \hat{g}^{0d} \hat{\phi}_c \hat{\phi}_d) \]

\[ - 8a \sin^2 \theta \left\{ 2(1 + tR)M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) - 2M^*tR \rho^2 + R \right\} (\hat{g}^{0c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) \]

\[ - 8a \sin^2 \theta^* t(\hat{g}^{1c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d) + 4a^2 \cos^2 \theta(1 + tR)R(\hat{g}^{22} \hat{g}^{22} \hat{\phi}_d) \]

\[ + 4 \sin^2 \theta(1 + tR) \left\{ a^2 R + a^2 \sin^2 \theta M \frac{\partial}{\partial R} \left( \frac{R}{\rho^{2}} \right) \right\} (\hat{g}^{3c} \hat{g}^{3d} \hat{\phi}_c \hat{\phi}_d), \]
In which we can see clearly that the coefficients of $\psi_f \psi_g$ are bounded, moreover since $(R/\rho^2)'_R$ and $\hat{g}^{11}$ are of order greater than or equal one in $R$, hence the coefficient of $\hat{\psi}^2_R$ is of order one in $R$. And further

$$a^2 \cos^2 \theta (1 + tR) R (\hat{g}^{22} \hat{g}^{22} \hat{\phi}_\theta^2) + \sin^2 \theta (1 + tR) \left\{ a^2 R + a^2 \sin^2 \theta M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \right\} (\hat{g}^{33} \hat{g}^{33} \hat{\phi}_\varphi^2)$$

$$= a^2 (1 + tR) R \frac{r^4}{\rho^4} \left| \nabla S^2 \hat{\psi} \right|^2 + M a^2 \sin^2 \theta (1 + tR) \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \left( \frac{r^4}{\rho^4} \right) \left| \nabla S^2 \hat{\psi} \right|^2$$

$$- a^2 \sin^2 \theta (1 + tR) R \frac{r^4}{\rho^4} \hat{\psi}^2_\theta - M a^2 \sin^2 \theta (1 + tR) \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \left( \frac{r^4}{\rho^4} \right) \hat{\psi}^2_\varphi.$$ 

Therefore we can express $\nabla (a T_b) A$ under the form

$$\nabla (a T_b) A = D_1 \hat{\psi}^2_t + D_2 \hat{\psi} \cdot t \hat{\psi} R + D_3 \hat{\psi} \cdot t \hat{\psi} \cdot \psi + D_4 \hat{\psi} R \hat{\psi} \cdot \psi + D_5 \hat{\psi}^2_R + D_6 \sin^2 \theta \hat{\psi}^2_\theta + D_7 \left| \nabla S^2 \hat{\psi} \right|^2.$$

In the second term, we can see that $\nabla (a T_b) B$ is bounded due to the expression

$$\nabla (a T_b) B = \frac{1}{2} \left\{ \frac{r^2 a^2 \sin^2 \theta}{\rho^2} 4(1 + tR) \left[ R - M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \right] + 4tR^2 \left( 1 - \frac{2Mr}{\rho^2} \right) + 4R \right\}$$

$$+ \frac{a^2}{\rho^2} \left\{ -8(1 + tR) Ma \sin^2 \theta \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) + 8tMa \sin^2 \theta \frac{R}{\rho^2} - 4aR \sin^2 \theta \right\}$$

$$- \frac{4a^2 t \sin^2 \theta}{\rho^2} + \frac{2r^2 (1 + tR)}{\rho^2} a^2 R \cos^2 \theta$$

$$+ 2a^2 \frac{r^2}{\rho^2} (1 + tR) \left\{ R + M \sin^2 \theta \left[ R - M \frac{\partial}{\partial R} \left( \frac{R}{\rho^2} \right) \right] \right\}.$$

So that in the second term, we are interested only in the factor $\hat{g}^{fg} \hat{\phi}_f \hat{\phi}_g$, which can be expressed as follows

$$\hat{g}^{fg} \hat{\psi}_f \hat{\psi}_g = - \frac{1}{\rho^2} \left( r^2 a^2 \sin^2 \theta \hat{\psi}^2_t + 2(r^2 + a^2) \hat{\psi} \cdot t \hat{\psi} R + 2ar^2 \hat{\psi} \cdot t \hat{\psi} \cdot \psi + 2a \hat{\psi} R \hat{\psi} \cdot \psi \right)$$

$$- \frac{1}{\rho^2} \left( R^2 \Delta \hat{\psi}^2_R + r^2 \hat{\psi}^2_\theta + \frac{r^2}{\sin^2 \theta} \hat{\psi}^2_\varphi \right)$$

$$= C_1 \hat{\psi}^2_t + C_2 \hat{\psi} \cdot t \hat{\psi} R + C_3 \hat{\psi} \cdot t \hat{\psi} \cdot \psi + C_4 \hat{\psi} R \hat{\psi} \cdot \psi + C_5 \hat{\psi}^2_R + C_6 \left| \nabla S^2 \hat{\psi} \right|^2.$$

Associating with the expression of the first term, our lemma is proved.

**Remark 2.5.1.** Lemma [2.4.1] is a consequence of this proof and $\nabla (a T_b) \hat{g}^{ab}$ is of order one in $R$.

### 2.5.3 The dominant energy condition

We recall the dominant energy condition that was given in [65]:

**Proposition 2.5.1.** With the metric $g_{ab}$ has the signature $(+ --)$, the stress-energy tensor $T_{ab}$ satisfies the dominant energy condition: for every future-oriented causal vector field $V$, the vector
\(T_b^a V^b\) is itself causal and future-pointing. Another equivalent state is as follows: for all future-oriented causal vector fields \(V, W\), we have \(T_a^b V^a W^b \geq 0\). Here
\[
T_{ab} = \partial_a \partial_b - \frac{1}{2} g_{ab} g^{cd} \partial_c \partial_d.
\]

**Proof.** Let \(V\) be a future-oriented causal vector field, we can set:
\[
V = V^0 \partial_t + V', \quad V^0 \geq |V'|
\]
and set \(W^a = T_b^a V^b\).

We have
\[
W^0 = \partial_t \psi \nabla \psi - \frac{1}{2} ((\partial_t \psi)^2 - |\nabla \psi|^2) V^0
\]
\[
= \frac{1}{2} V^0 ((\partial_t \psi)^2 + |\nabla \psi|^2) + \partial_t \psi \nabla \psi
\]
\[
\geq \frac{1}{2} V^0 ((\partial_t \psi)^2 + |\nabla \psi|^2) - |\partial_t \psi||V'||\nabla \psi|
\]
\[
\geq \frac{1}{2} ((\partial_t \psi)^2 + |\nabla \psi|^2)(V^0 - |V'|) \geq 0.
\]

So if \(W^a\) is causal, it is future-oriented. We can verify the causality:
\[
g_{ab} W^a W^b = g_{ab} \left( \nabla^a \psi \nabla \psi - \frac{1}{2} \left< \nabla \psi, \nabla \psi > V^a \right> \right) \left( \nabla^b \psi \nabla \psi - \frac{1}{2} \left< \nabla \psi, \nabla \psi > V^b \right> \right)
\]
\[
= \left< \nabla \psi, \nabla \psi > - \left< \nabla \psi, \nabla \psi > V^2 \right> \right) + \frac{1}{4} \left< \nabla \psi, \nabla \psi > V^2 \right> \geq 0
\]
since \(V\) is causal.

**Remark 2.5.2.** Similarly the dominant energy condition is also valid if we consider the two difference of the stress-tensor energy
\[
T_{ab}(\psi) = T_{ab} = \psi_a \psi_b - \frac{1}{2} g_{ab} (g^{cd} \psi_c \psi_d - \psi^2),
\]
and
\[
\tilde{T}_{ab}(\psi) = \tilde{T}_{ab} = \psi_a \psi_b - \frac{1}{2} g_{ab} \left( g^{cd} \psi_c \psi_d - \psi^2 - \frac{1}{2} \psi^4 \right).
\]

**Remark 2.5.3.** In the spacetime \((\mathcal{M}, g)\) we can find an orthonormal basis in a neighbourhood of each point. In such basis, metric \(g\) is described by the following matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
so the tangent at each point is a copy of the Minkowski spacetime. Therefore the dominant energy condition is also valid in the neighbourhood of each point for all spacetimes.
2.5.4 Two basic inequalities

The first inequality is the Poincaré’s inequality

**Lemma 2.5.1.** Given \( *t_0 < 0 \), there exists a constant \( C > 0 \) such that for any \( f \in C_0^\infty(\mathbb{R}) \), we have

\[
\int_{-\infty}^{*t_0} (f(\ast t))^2 d\ast t \leq C \int_{-\infty}^{*t_0} \ast t^2 (f'(\ast t))^2 d\ast t.
\]

**Proof.** Integrating by parts, we have

\[
\int_{-\infty}^{*t_0} (f(\ast t))^2 d\ast t = \left[ (\ast t - \ast t_0) f(\ast t) \right]_{-\infty}^{*t_0} - \int_{-\infty}^{*t_0} (\ast t - \ast t_0) ff' d\ast t.
\]

The boundary term vanishes (recall that \( f \) is assumed to be compactly supported, which gets rid of the boundary term at \( -\infty \)) and using \( \ast t_0 < 0 \), we get

\[
\int_{-\infty}^{*t_0} (f(\ast t))^2 d\ast t \leq \int_{-\infty}^{*t_0} \ast t - \ast t_0 \left| f \right|^2 \left| f' \right| d\ast t
\]

which gives the result. \( \square \)

The second inequality is obtained by the Sobolev’s embedding. It is known (see [41]) that, when dealing with a three-dimensional Riemannian manifold with boundary \((M, g)\), the following inequality holds: there exist a constant \( K \), depending only on the dimension of the manifold, a real number \( \epsilon \), and a constant \( B \), such that, for any function in \( H^1(M) \), we have:

\[
||u||^2_{L^6(M)} \leq (K + \epsilon)||\nabla u||^2_{L^2(M)} + B||u||^2_{L^2(M)}; \quad (2.42)
\]

where the best value of the constant \( B \) satisfies the following inequality:

\[
B \geq (\text{Vol}(M, g))^{-2/3}.
\]

The inequality (2.42) is also valid for all the metrics \( \tilde{g} \) that are equivalent to \( g \) on \( M \).

In our case, since \( \ast \omega \) is the volume form associated with the cylinder \([-\infty, \ast t_0] \times S^2 \) and the metric \( \ast \omega^2 + d\omega^2 \). Then the Sobolev’s embedding from \( H^1([-\infty, \ast t_0] \times S^2) \) into \( L^6([-\infty, \ast t_0] \times S^2) \) gives us the Sobolev’s inequality

\[
\left( \int_{\mathcal{H}_s} \hat{\psi}^6 \ast \omega \right)^{1/3} \leq K \int_{[-\infty, \ast t_0] \times S^2} \left( (\partial_t(\hat{\psi}|_{\mathcal{H}_s}))^2 + |\hat{\nabla}_{S^2}\hat{\psi}|^2 + \hat{\psi}^2 \right) \ast \omega \quad (2.43)
\]
Chapter 3

Peeling for the Dirac field on a Kerr background

3.1 Introduction

Continuing to study the peeling problem, in this chapter we will generalize the peeling definition for the Dirac field in the Kerr spacetime. The peeling of the Dirac field can be described in terms of the principal null geodesics, i.e the components $\psi_1$ and $\psi_0$ of the Dirac field $\psi_A$ will be falling-off like $r$ and $r^2$ along the outgoing null geodesics respectively.

The peeling for the spinor field has been discussed initially by R. Sachs [74] in Minkowski spacetime for the spin $2$ zero rest-mass field $\psi_{ABCD}$ by the analytic method of Riemann tensor. After that R.Penrose [71, 72] has approached, by the conformal compactification method, to generalize the peeling definition of the spin $n/2$ zero rest-mass field in the asymptotic spacetimes. However, in the definition of R. Penrose, the initial data that guarantees the peeling is not optimal (it is only required that the initial data are smooth functions). L. Mason and J-P. Nicolas have generalized the peeling definition of the Dirac field on the Minkowski and Schwarzschild spacetimes, by using the vector field method and the conformal technique. They have also given the optimal initial data that guarantees the peeling, in which these data are a Sobolev space with an energy norm.

Of course, similarly as in chapter 1, to generalize the peeling definition on the Kerr background, the difficulty is that the Kerr spacetime is not static and not symmetric. Hence the compactification, the calculation and the control of the energies will become more complicated. We continue to work on the neighbourhood $\Omega_{t_0}$ of the spacelike infinity $i_0$ in block I, and the initial data has a support far away from $i_0$. We will approach the peeling of the Dirac field using two directions: the partial derivative and the covariant derivative. In both approaches, to give the peeling definition at a higher order, we need to commute the derivatives along the vector fields $\partial_t$, $\partial_R$ and the derivative operators on the 2–sphere $S^2$ into the equation to obtain the conservation laws and then we control the energies. In the Penrose compactification of block I by using the principal null geodesics, the span of the vector fields $\hat{m}^a$ and $\hat{m}^a$ is not integrable. So the inequality which controls the norm of $\hat{\psi}_0$ by the norm of the weight derivative $\hat{\partial}^a\hat{\psi}_0$ on the 2–sphere $S^2$ as in the Schwarzschild case.
(see [54]), is no longer true. Therefore, in both approaches we need to give a new way to control the energy of $\hat{\psi}_0$. In details, in the first approach we will define the operator $\mathcal{D}_{t,\omega}^{\ast}$ that is a modification of the symmetric derivative operator $\mathcal{D}_{\omega}$ on the 2–sphere $S^2$ by combining the derivative along the time direction $\partial_t$ and $\mathcal{D}_{\omega}$; this operator can be commuted conveniently to the Dirac’s system to obtain the conjugation system and plays an important role in our control. In the second approach, we will control the energy of $\hat{\psi}_0$ by combining the covariant derivatives along all the vector fields $X_i \in \mathcal{A}$, where $\mathcal{A}$ has been given in chapter 1.

We complete this chapter by the following organisation:

- **Section 2.** We give the geometric background: the spin structure for block I of the Kerr spacetime, a choice of the Newman-Penrose tetrad following the principal null geodesics, the calculations of the spin coefficients in details, and the calculations of the normal vector and transversal vector of the spacelike hypersurfaces $\mathcal{H}_s$ and the null hypersurface $\mathcal{I}^+_{t_0}$.

- **Section 3.** We give the Dirac’s equation and its consequence which is the Weyl’s equation in terms of the Newman-Penrose formalism and the spin coefficients. Then we give the expression of the Weyl’s equation on the Boyer-Lindquist coordinates and the rescaled coordinates. After that, we work on giving the simple equivalent expressions of the energies on the hypersurfaces $\mathcal{H}_s$ and $\mathcal{I}^+_{t_0}$.

- **Section 4.** Using the partial derivative approach, we commute the operators $\mathcal{D}_{\ast \varphi}$, $\mathcal{D}_R$ and $\mathcal{D}_{t,\omega}^{\ast}$ into the Weyl’s equation to obtain the conservation laws at a higher order. Then we control the energies to give the regularity at $\mathcal{I}^+_{t_0}$. Note that, at a higher order, we need to control the energies by combining all the derivatives. Hence we obtain the peeling definition.

- **Section 5.** Using the covariant derivative approach, we commute the covariant derivatives along the vector field $X_i \in \mathcal{A}$ into the Weyl’s equation to obtain the conservation laws at a higher order. Then we control the energies to give the peeling definition and similarly as in the partial derivative approach, we need to control the energies by combining all the covariant derivatives $\nabla_{X_i}$ at a higher order.

- **Section 6.** We will compare our peeling definition on the Kerr spacetime with the one that is constructed independently on the flat spacetime, by comparing the classes of the initial data that guarantees the peeling. The comparison method is similar to the one that we have done for the scalar fields.

- **Appendix.** In the appendix, the proofs of the lemmas and some calculations will be given. We also recall the compacted spin coefficient formalism and its application in our work.

**Notations**
Chapter 3. Peeling for Dirac field on Kerr spacetime

(1) We shall use the notation \( \lesssim \) to signify that the left-hand side is bounded above by a positive constant multiplied by the right-hand side, for \( r \) large enough; the constant is independent of the parameters and functions that appear in the inequality.

(2) We shall use the notation \( \simeq \) to signify that the left-hand side and similarly the right-hand side can be controlled above and below by two constants multiplied by the right-hand side, for \( r \) large enough; the constants are independent of the parameters and functions that appear in both sides.

(3) A function is said to be "bounded", if it is bounded from above and below, i.e its absolute value is bounded. The term "a function is of order \( k \) in \( R \)" means that for \( r \) large enough, the simple equivalent form of this function has a factor \( R^k \) but not \( R^{k+1} \), and similarly to "a function is of order \( k \) in \( \sin \theta \)".

(4) We use the formalism of abstract indices, 2–component spinor, Newman-Penrose and Geroch-Held-Penrose.

3.2 Geometric setting

3.2.1 Spin structure for block I

First, block I is time orientable since it has the non vanishing continuous timelike vector field \( \partial_t \) (recall that in Boyer-Lindquist coordinates \( \partial/\partial t \) is not everywhere timelike since it is spacelike in block II), and then the function \( t \) of Boyer-Lindquist coordinates can define a time function on block I. Since \( \partial_t \) is a timelike vector field on block I and

\[
\partial_t t = 1 > 0,
\]

so \( \partial_t \) is the future-oriented timelike vector field of block I.

Second, block I can be foliated by the level hypersurfaces \( \{ \Sigma_t \}_{t \in \mathbb{R}} \), where \( \Sigma_t = \{ t \} \times \Sigma \) is a Cauchy hypersurface\(^1\).

Therefore block I satisfies the "topological product" condition (see Geroch \(^{[33]}\)), more particular block I is globally hyperbolic. In dimension 4, block I admits a spin-structure (see R.P Geroch \(^{[32, 33, 34]}\) and E. Stiefel \(^{[76]}\)). We denote by \( \mathbb{S} \) (or \( \mathbb{S}^A \) in the abstract index formalism) the spin bundle over \( B_I \) and \( \mathbb{S} \) (or \( \mathbb{S}^A \)) which is the same bundle for the complex structure that is replaced by its conjugation. The dual bundles \( \mathbb{S}^* \) and \( \mathbb{S}^* \) will be denoted by \( \mathbb{S}_A \) and \( \mathbb{S}_A^* \) respectively. The complexified tangent bundle to \( B_I \) is described as the tensor product of \( \mathbb{S} \) and \( \mathbb{S} \), i.e.

\[
TB_I \otimes \mathbb{C} = \mathbb{S} \otimes \mathbb{S} \text{ or } T^a B_I \otimes \mathbb{C} = \mathbb{S}^A \otimes \mathbb{S}^A,
\]

and similarly

\[
T^* B_I \otimes \mathbb{C} = \mathbb{S}^* \otimes \mathbb{S}^* \text{ or } T^a B_I \otimes \mathbb{C} = \mathbb{S}_A \otimes \mathbb{S}_A^*.
\]

\(^{1}\) A Cauchy hypersurface is a spacelike hypersurface and intersects with every inextendible timelike curve at exactly one point.
An abstract tensor index $a$ is a combination of an unprimed spinor index $A$ and a primed spinor index $A'$ i.e $a = AA'$.

The spin bundle $\mathcal{S}$ admits a canonical sympletic form $\varepsilon_{AB}$, it is called the Levi-Civita symbol. We use $\varepsilon_{AB}$ to raise and lower spinor indices, note that $\varepsilon_{AB}$ is skew-symmetric, so the order in writing is important

$$\varepsilon^{AB} \kappa_B = \kappa^A, \quad \kappa^A \varepsilon_{AB} = \kappa_B.$$ 

Similarly, $\bar{\mathcal{S}}$ also admits a complex conjugate $\bar{\varepsilon}_{AB} = \bar{\varepsilon}_{A'B'} := \varepsilon_{A'B'}$, it plays a similar role on $\bar{\mathcal{S}}$. These sympletic structures are compatible with the metric i.e

$$g_{ab} = \varepsilon_{AB} \bar{\varepsilon}_{A'B'}.$$ 

### 3.2.2 The Newman-Penrose tetrad and spin coefficients

The Newman-Penrose formalism is described by the choice of a null tetrad, i.e. a set of four vector fields $l^a, n^a, m^a$ and $\bar{m}^a$, the first two vector fields are real and future-oriented, $\bar{m}^a$ is the complex conjugate vector field of $m^a$, and such that all four vector fields are null and $m^a$ is orthogonal to $l^a$ and $n^a$, as follows

$$l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0.$$ 

We will work in the normalization condition

$$l_a n^a = 1, \quad m_a \bar{m}^a = -1.$$ 

The set of the four vectors $\{l^a, n^a, m^a, \bar{m}^a\}$ in this condition is called the Newman Penrose normalization tetrad.

The Newman-Penrose tetrad at each point obtain a basis of the tangent space of our manifold, in other words this tetrad is a global section of the complexified principal bundle. The vectors $l^a$ and $n^a$ describe "dynamical" or scattering directions, i.e. along these directions the light rays cross the horizons and tend to infinity. The vector fields $m^a$ and $\bar{m}^a$ are spatial vectors, they generate rotations.

We can choose in $\mathcal{S}_A$ a spin-frame $\{o_A, \iota_A\}$ that is associated with the Newman-Penrose tetrad i.e such that

$$o^A \bar{o}^{A'} = l^a, \quad \iota^A \bar{\iota}^{A'} = n^a, \quad o^A \bar{\iota}^{A'} = m^a, \quad \iota^A \bar{o}^{A'} = \bar{m}^a, \quad o_A \iota^A = 1,$$ 

the last equality is obtained by the normalization condition.

In our work, we will use the normalized Newman-Penrose tetrad that was used by D.Häfner and J.P. Nicoals [39], where they use a simple modification of Kinnersley’s tetrad. In details, we choose

$$l^a \partial_a = \lambda V^+, \quad n^a \partial_a = \mu V^- \text{ with } \lambda = \mu,$$

and the two vectors $m^a$ and $\bar{m}^a$ are imposed from the orthogonal condition of the Newman Penrose normalized tetrad. This gives us

$$l^a \partial_a = \frac{1}{\sqrt{2} \Delta \rho^2} \left((r^2 + a^2)\partial_t + \Delta \partial_r + a \partial_\varphi\right), \quad (3.1)$$
\[ n^a \partial_a = \frac{1}{\sqrt{2\Delta \rho^2}} ((r^2 + a^2) \partial_t - \Delta \partial_r + a \partial \varphi) , \] (3.2)

\[ m^a \partial_a = \frac{1}{p \sqrt{2}} \left( ia \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right) , \text{ where } p = r + ia \cos \theta . \] (3.3)

The dual tetrad of a 1-form is

\[ l_a dx^a = \sqrt{\frac{\Delta}{2 \rho^2}} \left( dt - \frac{\rho^2}{\Delta} dr - a \sin^2 \theta d\varphi \right) , \] (3.4)

\[ n_a dx^a = \sqrt{2 \rho^2} \left( dt + \frac{\rho^2}{\Delta} dr - a \sin^2 \theta d\varphi \right) , \] (3.5)

\[ m_a dx^a = \frac{1}{p \sqrt{2}} \left( ia \sin \theta dt - \rho^2 \partial_\theta - i(r^2 + a^2) \sin \theta d\varphi \right) . \] (3.6)

In the rescaled coordinates \((*t, R, \theta, *\varphi)\) this Newman Prenrose tetrad takes the form

\[ \hat{l}^a \partial_a = - \sqrt{\frac{\Delta}{2 \rho^2}} \partial_R , \] (3.7)

\[ \hat{n}^a \partial_a = \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) \partial_t + a \partial \varphi + \frac{R^2 \Delta}{2} \partial_R \right) , \] (3.8)

\[ \hat{m}^a \partial_a = \frac{r}{p \sqrt{2}} \left( ia \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right) . \] (3.9)

Therefore we have the relation

\[ \hat{l}^a = r^2 l^a , \hat{n}^a = n^a , \hat{m}^a = rm^a . \]

In terms of the associated spin-frame \(\{o^A, \iota^A\}\), the corresponding rescaling is as follows

\[ \hat{o}^A = ro^A , \hat{\iota}^A = \iota^A . \]

We also give the dual of the rescaled tetrad as follows

\[ \hat{l}_a dx^a = \sqrt{\frac{\Delta}{2 \rho^2}} \left( d^* t - a \sin^2 \theta d^* \varphi \right) , \] (3.10)

\[ \hat{n}_a dx^a = \sqrt{\frac{\Delta}{2 \rho^2}} \left( R^2 d^* t - \frac{2 \rho^2}{\Delta} dR - a R^2 \sin^2 \theta d^* \varphi \right) , \] (3.11)

\[ \hat{m}_a dx^a = \frac{1}{p \sqrt{2}} \left( iaR \sin \theta d^* t - R \rho^2 d\theta - iR(r^2 + a^2) \sin \theta d^* \varphi \right) . \] (3.12)

So the 4-volume measure associated with the metric \(\hat{g}\) can be calculated by

\[ d\text{Vol}^4 = i \hat{l} \wedge \hat{n} \wedge \hat{m} \wedge \hat{m} = -(1 + a^2 R^2 \cos^2 \theta) d^* t \wedge dR \wedge d^2 \omega , \]

where \(d^2 \omega = \sin \theta d\theta d^* \varphi \).
The Newman-Penrose formalism is a powerful notation system for expressing the geometry of a spacetime in terms of complex null tetrad fields. The ordinary notations such as covariant derivative \( \nabla_a \) and curvature tensor \( R_{abcd} \) no longer appear, and moreover the index notation is not used, the equations are written out in full. We can now decompose the covariant derivative into directional covariant derivatives along the frame vectors. To each directional derivative corresponds a standard symbol

\[
D = l^a \nabla_a, \quad D' = n^a \nabla_a, \quad \delta = m^a \nabla_a, \quad \delta' = \bar{m}^a \nabla_a.
\]

Due to the following expression of \( g_{ab} \)

\[
g_{ab} = n_a l_b + l_a n_b - \bar{m}_a m_b - m_a \bar{m}_b,
\]

we can see that the first order derivative of the metric or the connection coefficients can be expressed by combining only derivatives of frame vectors along frame vectors. These derivatives are called spin coefficients. For a normalized tetrad, there are twelve spin coefficients defined as follows (see R. Penrose and W. Rindler [73], Vol 1, p. 226 – 228)

\[
\kappa = m^a Dl_a, \quad \rho = m^a \delta' l_a, \quad \sigma = m^a \delta l_a, \quad \tau = m^a D'l_a,
\]

\[
\varepsilon = \frac{1}{2} (n^a Dl_a + m^a D\bar{m}_a), \quad \alpha = \frac{1}{2} (n^a \delta' l_a + m^a \delta' \bar{m}_a),
\]

\[
\beta = \frac{1}{2} (n^a \delta l_a + m^a \delta \bar{m}_a), \quad \gamma = \frac{1}{2} (n^a D'l_a + m^a D'\bar{m}_a),
\]

\[
\pi = -\bar{m}^a Dn_a, \quad \lambda = -\bar{m}^a \delta' n_a, \quad \mu = -\bar{m}^a \delta n_a, \quad \nu = -\bar{m}^a D'\bar{n}_a.
\]

The spin coefficients can be calculated more simply by using the terms of the Ricci rotation coefficients (see S. Chandrasekha [10]). For this purpose, the frame vectors are denoted by

\[
l^a = e(1)^a, \quad n^a = e(2)^a, \quad m^a = e(3)^a, \quad \bar{m}^a = e(4)^a,
\]

and the dual 1–forms by

\[
l_a = e(1)_a, \quad n_a = e(2)_a, \quad m_a = e(3)_a, \quad \bar{m}_a = e(4)_a.
\]

And the components of tensors with respect to this frame and co-frame are denoted by light-face latin indices within brackets ( ) as follows

\[
R^{(a)}_{(b)(c)(d)} = R_{abcd} e^{(a)}_a e^{(b)}_b e^{(c)}_c e^{(d)}_d.
\]

The Ricci rotation coefficients are defined by

\[
\gamma^{(a)(b)(c)} = \frac{1}{2} \left[ \lambda^{(a)(b)(c)} + \lambda^{(c)(a)(b)} - \lambda^{(b)(c)(a)} \right],
\]

\[
\lambda^{(a)(b)(c)} = \left[ \frac{\partial}{\partial x^i} e^{(b)}_i - \frac{\partial}{\partial x^j} e^{(b)}_j \right] e^{(a)}_i e^{(c)}_j,
\]
and the expression of the spin-coefficients in terms of the $\gamma^{(a)(b)(c)}$ are

$$
\kappa = \gamma^{(3)(1)(1)} , \rho = \gamma^{(3)(1)(4)} , \varepsilon = \frac{1}{2} \left( \gamma^{(2)(1)(1)} + \gamma^{(3)(4)(1)} \right) ,
$$

$$
\sigma = \gamma^{(3)(1)(3)} , \mu = \gamma^{(2)(4)(3)} , \gamma = \frac{1}{2} \left( \gamma^{(2)(1)(2)} + \gamma^{(3)(4)(2)} \right) ,
$$

$$
\lambda = \gamma^{(2)(4)(4)} , \tau = \gamma^{(3)(1)(2)} , \alpha = \frac{1}{2} \left( \gamma^{(2)(1)(4)} + \gamma^{(3)(4)(4)} \right) ,
$$

$$
\nu = \gamma^{(2)(4)(2)} , \pi = \gamma^{(2)(4)(1)} , \beta = \frac{1}{2} \left( \gamma^{(2)(1)(3)} + \gamma^{(3)(4)(3)} \right) .
$$

For the tetrad Newman-Penrose normalization where we have chosen (3.1), (3.2) and (3.3). The calculation of the spin coefficients gives:

$$
\kappa = \sigma = \lambda = \nu = 0 ,
$$

$$
\tau = - \frac{ia \sin \theta}{\sqrt{2} \rho^2} , \pi = \frac{ia \sin \theta}{\sqrt{2} \rho^2} , \rho = \mu = - \frac{1}{p} \sqrt{\frac{\Delta}{2 \rho^2}} , \varepsilon = \frac{(r - M) \rho^2 - r \Delta}{2 \rho^2 \rho^2} ,
$$

$$
\alpha = \frac{ia \sin \theta}{\rho^2} - \frac{\cot \theta + a^2 \sin \theta \cos \theta}{2 \sqrt{2} \rho} , \beta = \frac{\cot \theta + a^2 \sin \theta \cos \theta}{2 \sqrt{2} \rho} ,
$$

$$
\gamma = \frac{(r - M) \rho^2 - r \Delta}{2 \rho^2 \rho^2} - \sqrt{\frac{\Delta}{2 \rho^2}} \frac{ia \cos \theta}{\rho^2} .
$$

To obtain the rescaled spin coefficients, we can calculate them from the original spin coefficients by adding the change obtained by the conformal mapping as in the following table (see R. Penrose and R. Rindler \[73\] Vol1)

<table>
<thead>
<tr>
<th>$\hat{\kappa}$</th>
<th>$\hat{\varepsilon}$</th>
<th>$\hat{\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega^{-3} \kappa$</td>
<td>$\Omega^{-2} \varepsilon$</td>
<td>$\Omega^{-1} (\pi + \delta' \omega)$</td>
</tr>
<tr>
<td>$\hat{\rho}$</td>
<td>$\hat{\alpha}$</td>
<td>$\hat{\lambda}$</td>
</tr>
<tr>
<td>$\Omega^{-2} (\rho - D \omega)$</td>
<td>$\Omega^{-1} (\alpha - \delta' \omega)$</td>
<td>$\lambda = \lambda$</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>$\hat{\beta}$</td>
<td>$\hat{\mu}$</td>
</tr>
<tr>
<td>$\Omega^{-2} \sigma$</td>
<td>$\Omega^{-1} \beta$</td>
<td>$\mu + \delta' \omega$</td>
</tr>
<tr>
<td>$\hat{\tau}$</td>
<td>$\hat{\gamma}$</td>
<td>$\hat{\nu}$</td>
</tr>
<tr>
<td>$\Omega \tau$</td>
<td>$\gamma - D' \omega$</td>
<td>$\Omega \nu$</td>
</tr>
</tbody>
</table>

where $\omega = \log \Omega = - \log r$. And we have

$$
D \omega = l^a \nabla_a \omega = \frac{1}{\sqrt{2 \Delta \rho^2}} \left( (r^2 + a^2) \partial_t + \Delta \partial_r + a \partial_\varphi \right) (\log r) = - \frac{1}{r} \sqrt{\frac{\Delta}{2 r^2}} ,
$$

$$
\delta' \omega = \bar{m}^a \nabla_a \omega = \frac{r}{\bar{p} \sqrt{2}} \left( -ia \sin \theta \partial_t + \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right) (\log r) = 0 ,
$$

$$
\delta \omega = m^a \nabla_a \omega = \frac{r}{p \sqrt{2}} \left( ia \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right) (\log r) = 0 ,
$$

$$
D' \omega = n^a \nabla_a \omega = \frac{1}{\sqrt{2 \Delta \rho^2}} \left( (r^2 + a^2) \partial_t - \Delta \partial_r + a \partial_\varphi \right) (\log r) = \frac{1}{r} \sqrt{\frac{\Delta}{2 r^2}} .
$$
Therefore, we have can calculate:

\[
\hat{\kappa} = \hat{\sigma} = \hat{\lambda} = \hat{\nu} = 0, \\
\hat{\tau} = -\frac{ia \sin \theta r}{\sqrt{2\rho^2}}, \quad \hat{\pi} = \frac{ia \sin \theta r}{\sqrt{2\rho^2}}, \\
\hat{\rho} = -\frac{r^2}{\bar{p}} \sqrt{\frac{\Delta}{2\rho^2}} + \frac{r}{\bar{p}} \sqrt{\frac{\Delta}{2\rho^2}} = -\frac{ria \cos \theta}{\bar{p}} \sqrt{\frac{\Delta}{2\rho^2}}, \\
\hat{\mu} = \sqrt{\frac{\Delta}{2\rho^2}} \left( R - \frac{1}{\bar{p}} \right), \\
\hat{\epsilon} = r^2 \left( \frac{\Delta}{2\rho^2} \right) = \frac{r^4(a^2 \cos^2 \theta + 2Mr) + r^3a^2}{2\rho^2 \sqrt{2\Delta \rho^2}}, \\
\hat{\alpha} = \frac{r}{\sqrt{2\rho}} \left( \frac{ia \sin \theta}{\bar{p}} - \frac{\cot \theta}{2} + \frac{a^2 \sin \theta \cos \theta}{2\rho^2} \right), \\
\hat{\beta} = \frac{r}{\sqrt{2\rho}} \left( \frac{\cot \theta}{2} + \frac{a^2 \sin \theta \cos \theta}{2\rho^2} \right), \\
\hat{\gamma} = \frac{\Delta}{2\rho^2} \left( \frac{ia \cos \theta}{\rho^2} + R \right). 
\]

(3.17) (3.18) (3.19) (3.20) (3.21) (3.22)

**Remark 3.2.1.** We can see that \(\hat{\tau}, \hat{\pi}, \hat{\mu}\) and \(\hat{\gamma}\) are of order one in \(R\); \(\hat{\rho}, \hat{\epsilon}, \hat{\alpha}\) and \(\hat{\beta}\) are of order zero in \(R\).

### 3.2.3 Neighbourhood of spacelike infinity

We continue to work in the neighbourhood \(\Omega_{*t_0}^+\) of \(i_0\), that is the domain we used in chapter 1. We recall briefly as follows: for a given \(*t_0 \ll -1\)

\[
\Omega_{*t_0}^+ := \left\{(*t, R, \theta, \varphi); \; \; *t \leq *t_0 + \hat{r} - r_*, \; 0 \leq t \leq +\infty \right\},
\]

where \(\hat{r}\) is the parameter that appears in the compactification of Kerr spacetime by using the simple null geodesics approach (see Häfner [38]):

\[
\hat{r} = r_* + \int_{-\infty}^{r_*} \left( \sqrt{1 - \frac{a^2 \Delta(s)}{(r(s)^2 + a^2)^2}} - 1 \right) ds + a \cos \theta,
\]

with \(r_*\) is the Regge-Wheeler coordinate

\[
\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}.
\]

We have foliated this neighbourhood by the hypersurfaces \(\{\mathcal{H}_s\}_{0 \leq s \leq 1}\) (which are spacelike except for \(\mathcal{H}_0\) which is null)

\[
\mathcal{H}_s = \{ *t = -sr_*; \; *t \leq *t_0 \}, \; 0 \leq s \leq 1.
\]

Now we will find the normal vector and transversal vector of \(\mathcal{H}_s\), we rewrite the hypersurface \(\mathcal{H}_s\) as follows

\[
\mathcal{H}_s = \{ *t + sr_* = 0; \; *t \leq *t_0 \}, \; 0 \leq s \leq 1.
\]
We put \( g(t, R, \theta, \varphi) = t + sr_* \). The 1-form conormal vector to \( H_s \) is
\[
\hat{\nabla}_b gdx^b = d^* t - \frac{s(a^2 + r^2)}{\Delta R^2} dR,
\]
so the gradient of function \( f \) is
\[
\hat{\nabla}^a g \partial_a = \hat{g}^{ab} \hat{\nabla}_b g = -\frac{1}{\rho^2} \left\{ r^2 \left( a^2 \sin^2 \theta - \frac{s(a^2 + r^2)^2}{\Delta} \right) \partial_t + (1 - s)(r^2 + a^2) \partial_R 
+ ar^2 \left( 1 - \frac{s(a^2 + r^2)}{\Delta} \right) \partial_\varphi \right\}.
\]
The normal vector of \( H_s \) and the gradient of function \( g \) are in the same direction. So that we choose the normal vector \( N^a \) of \( H_s \) by multiplying \( \hat{\nabla}^a f \) with the factor \( \frac{-\Delta R^2 \rho^2}{s(a^2 + r^2)} \left( 1 + a^2 R^2 - \frac{a^2 R^2 \Delta \sin^2 \theta}{s(a^2 + r^2)} \right)^{-1} \) to get
\[
N^a \partial_a = \partial_t + \beta \partial_R + \gamma \partial_\varphi,
\]
where
\[
\beta = \left( 1 - \frac{1}{s} \right) \Delta R^4 \left( 1 + a^2 R^2 - \frac{a^2 R^2 \Delta \sin^2 \theta}{s(a^2 + r^2)} \right)^{-1},
\]
\[
\gamma = a R^2 \left( 1 - \frac{\Delta}{s(a^2 + r^2)} \right) \left( 1 + a^2 R^2 - \frac{a^2 R^2 \Delta \sin^2 \theta}{s(a^2 + r^2)} \right)^{-1}.
\]
Now, the transversal vector of \( H_s \) is
\[
\mathcal{L}^a \partial_a = -\partial_R + \sigma \partial_\varphi.
\]
We can find the coefficient \( \sigma \) due to the condition \( \langle N^a, \mathcal{L}^a \rangle = 1 \), but \( \sigma \) is not important since when we contract \( \mathcal{L}^a \) with \( d\text{Vol}^4 \) on \( H_s \), we are interested only in \( \partial_R \), the contraction of \( \partial_\varphi \) with \( d\text{Vol}^4 \) is zero due to on the hypersurface \( H_s \)
\[
d^* t = \frac{s(a^2 + r^2)}{\Delta R^2} dR.
\]
The third hypersurface boundary of the neighbourhood is
\[
S_{*t_0} = \{ *t = *t_0 + \hat{r} - r_* \}.
\]
We set \( f(*t, R, \theta, \varphi) = *t - \hat{r} + r_* - *t_0 \), then the 1-form conormal vector to \( S_{*t_0} \) is
\[
\hat{\nabla}_b f dx^b = d^* t + \left( \frac{a^2 + r^2}{\Delta} \right) \left( \sqrt{1 - \frac{a^2 \Delta}{(r^2 + a^2)^2}} - 1 \right) dR - a \cos \theta d\theta.
\]
Since \( S_{*t_0} \) is a null hypersurface, so it contains its normal vector, which can be calculated
\[
N^a \partial_a = \hat{\nabla}^a f \partial_a = \hat{g}^{ab} \hat{\nabla}_b f
= -\frac{1}{\rho^2} \left\{ \left( r^2 a^2 \sin^2 \theta + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) \partial_t + (a^2 + r^2)(X + 1) \partial_R 
+ r^2 a \cos \theta \partial_\theta + a \left( r^2 + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) \partial_\varphi \right\},
\]
where
\[
X = \frac{a^2 R^2 - \frac{a^2 R^2 \Delta \sin^2 \theta}{s(a^2 + r^2)}}{1 + a^2 R^2 - \frac{a^2 R^2 \Delta \sin^2 \theta}{s(a^2 + r^2)}},
\]
and
\[
\rho^2 = \sqrt{a^2 \sin^2 \theta + \frac{(a^2 + r^2)^2 r^2}{\Delta} X}.
\]
where \( X := \sqrt{1 - \frac{a^2 \Delta}{(r^2 + a^2)^2}} - 1 \). The transversal vector has the form

\[
\hat{\mathcal{L}}^a \partial_a = \frac{\rho^2}{(a^2 + r^2)(X + 1)} \partial_t + \sigma_1 \partial_\varphi.
\]

The coefficient \( \sigma_1 \) can be calculated from the equality \( \hat{g}(\hat{N}^a, \hat{L}^a) = 1 \), but it is really not important since when we contract \( \hat{L}^a \) with \( \text{dvol}^4 \), we are only interested in \( \partial_t \), the contraction of \( \partial_\varphi \) with \( \text{dvol}^4 \) is zero due to on the hypersurface \( S_{t_0} \) we have the relation

\[
d^* t = -\frac{(a^2 + r^2)r^2}{\Delta} X dR + a \cos \theta d\theta.
\]

3.3 The Dirac field

3.3.1 Dirac’s and Weyl’s equations

The Dirac equation has the expression in terms of two-component spinors (sections of the bundles \( S^A, S_A, S^{A'} \) or \( S_A' \)) as follows

\[
\begin{align*}
\nabla^{AA'} \psi_A &= \mu \chi^{A'}, \\
\nabla^{AA'} \chi_{A'} &= \mu \psi^A, \quad \mu = \frac{m}{2}
\end{align*}
\]

(3.23)

where \( m \geq 0 \) is the mass of the field. In the massless case, equation (3.23) reduces to the Weyl anti-neutrino equation

\[
\nabla^{AA'} \psi_A = 0,
\]

(3.24)

due to the equation on \( \lambda \) (the Weyl neutrino equation)

\[
\nabla^{AA'} \lambda_{A'} = 0,
\]

de which is the complex conjugate of the anti-neutrino equation

\[
\nabla^{AA'} \bar{\lambda}_{A'} = 0.
\]

Equation (3.24) is the equation where we will study the peeling, which we simply call the Weyl’s equation.

The Weyl’s equation has a conserved current on a general curved spacetimes, defined by the future oriented null\(^2\) vector field

\[
J^a = \psi^A \bar{\psi}^{A'}.
\]

We denote by \( \psi_0 \) and \( \psi_1 \) the components of \( \psi_A \) in the spin-frame \( \{ o^A, \iota^A \} \):

\[
\psi_0 = \psi_A o^A, \quad \psi_1 = \psi_A \iota^A,
\]

\(^2\)The causality of \( J^a \) can be checked since \( J^a = |\psi_1|^2 o^a + |\psi_0|^2 n^a - \psi_1 \bar{\psi}_0 m^a - \psi_0 \bar{\psi}_1 \bar{m}^a \) and the normalization condition of the Newman-Penrose tetrad. The future oriented property can be verified since \( J^a(t) > 0 \).
Using the Newman-Penrose formalism, the Dirac’s equation can be expressed under the form (see for example [10])

\[
\begin{align*}
    n^a \partial_a \psi_0 - m^a \partial_a \psi_1 + (\mu - \gamma) \psi_0 + (\tau - \beta) \psi_1 &= \frac{m}{\sqrt{2}} \lambda_1^0, \\
    l^a \partial_a \psi_1 - m^a \partial_a \psi_0 + (\alpha - \pi) \psi_0 + (\varepsilon - \rho) \psi_1 &= \frac{m}{\sqrt{2}} \lambda_0^0, \\
    n^a \partial_a \lambda_0^0 - m^a \partial_a \lambda_1^0 + (\mu - \gamma) \lambda_0^0 + (\tau - \beta) \lambda_1^0 &= \frac{m}{\sqrt{2}} \psi_0, \\
    l^a \partial_a \lambda_1^0 - m^a \partial_a \lambda_0^0 + (\alpha - \pi) \lambda_0^0 + (\varepsilon - \rho) \lambda_1^0 &= \frac{m}{\sqrt{2}} \psi_0.
\end{align*}
\]

(3.25)

and the Weyl’s equation is simply as follows

\[
\begin{align*}
    n^a \partial_a \psi_0 - m^a \partial_a \psi_1 + (\mu - \gamma) \psi_0 + (\tau - \beta) \psi_1 &= 0, \\
    l^a \partial_a \psi_1 - m^a \partial_a \psi_0 + (\alpha - \pi) \psi_0 + (\varepsilon - \rho) \psi_1 &= 0.
\end{align*}
\]

(3.26)

Associating system (3.26) with the forms (3.1), (3.2) and (3.3) of Newman-Penrose formalism and the values of the spin coefficients (3.13), (3.14), (3.15) and (3.16), we obtain the form of the Weyl’s equation in Boyer-Lindquist coordinates as follows:

\[
\begin{align*}
    \frac{r^2 + a^2}{\sqrt{2} \Delta \rho^2} \partial_t \psi_0 - \sqrt{\frac{\Delta}{2 \rho^2}} \partial_r \psi_0 + \frac{a}{\sqrt{2} \Delta \rho^2} \partial_\varphi \psi_0 - \frac{ia \sin \theta}{p \sqrt{2}} \partial_t \psi_1 - \frac{1}{p \sqrt{2}} \partial_\theta \psi_1 - \frac{i}{p \sqrt{2} \sin \theta} \partial_\varphi \psi_1 \\
    - \frac{(r - M) \rho^2 + r \Delta}{2 \rho^2 \sqrt{2} \Delta \rho^2} \psi_0 - \left( \frac{\cot \theta}{2 \rho^2} + \frac{ia \sin \theta}{\sqrt{2} \rho^2} + \frac{a^2 \sin \theta \cos \theta}{2 \rho^2 \sqrt{2} \rho} \right) \psi_1 = 0,
\end{align*}
\]

\[
\begin{align*}
    \frac{r^2 + a^2}{\sqrt{2} \Delta \rho^2} \partial_t \psi_1 + \sqrt{\frac{\Delta}{\rho^2}} \left( \frac{1}{\rho^2} \right) \psi_0 + \frac{a}{\sqrt{2} \Delta \rho^2} \partial_\varphi \psi_1 + \frac{ia \sin \theta}{p \sqrt{2}} \partial_t \psi_0 - \frac{1}{p \sqrt{2}} \partial_\theta \psi_0 + \frac{i}{p \sqrt{2} \sin \theta} \partial_\varphi \psi_0 \\
    + \left( \frac{- \cot \theta}{2 \rho^2} + \frac{a^2 \sin \theta \cos \theta}{2 \rho^2 \sqrt{2} \rho} \right) \psi_0 + \left( \frac{(r - M) \rho^2 + r \Delta}{2 \rho^2 \sqrt{2} \Delta \rho^2} + \frac{ia \Delta \cos \theta}{\rho^2 \sqrt{2} \Delta \rho^2} \right) \psi_1 = 0
\end{align*}
\]

where \( p = r + ia \cos \theta \). From the two equations above, we get

\[
\frac{r^2 + a^2}{\sqrt{\Delta \rho^2}} D_t \psi + \sqrt{\frac{\Delta}{\rho^2}} \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) D_r \psi - \frac{\sqrt{2}}{r} D_\omega \psi + M \simeq 0,
\]

where \( \psi = \left( \begin{array}{c} \psi_0 \\ \psi_1 \end{array} \right) \) and \( M \) include the terms of lower order or the terms that have lower weights and

\[
D_\omega = \left( \begin{array}{cc} 0 & \frac{1}{\rho \sqrt{2}} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \\ \frac{1}{p \sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) & 0 \end{array} \right).
\]

Taking the constraint on the flat case we have

\[
D_t \psi + \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) D_r \psi - \frac{\sqrt{2}}{r} D_\omega \psi + \text{lower order terms} \simeq 0.
\]
The Weyl’s equation is hyperbolic symmetry, and since block I is globally hyperbolic spacetime, by using the Leray’s theorem the Cauchy problem

\[
\begin{aligned}
\nabla^{AA'} \psi_A &= 0, \\
\psi_A|_{\Sigma_0} &\in \mathcal{C}_0^\infty(\Sigma_0; S_A)
\end{aligned}
\]

has a unique solution \( \psi_A \in \mathcal{C}^\infty(\mathcal{B}_I; S_A) \).

### 3.3.2 Rescaling of the Weyl equation

The Weyl’s equation (3.24) is conformally invariant (see R. Penrose [73]) i.e the spinor field \( \psi_A \) is the solution of (3.24) on the exterior of the black hole \( \mathcal{B}_I \) if and only if the rescaled spinor field \( \hat{\psi}_A = \Omega^{-1} \psi_A = r \psi_A \) is the solution on the same domain of the rescaled equation

\[
\hat{\nabla}^{AA'} \hat{\psi}_A = 0.
\]

And the rescaled solution \( \hat{\psi}_A \) can be extended to the null infinity boundaries \( \mathcal{I}^\pm \) of \( \mathcal{B}_I \).

Decompose \( \psi_A \) onto the spin-frame \( \{o^A, \iota^A\} \) and \( \hat{\psi}_A \) onto the rescaled spin-frame \( \{\hat{o}^A, \hat{\iota}^A\} \), we have

\[
\psi_A = \psi_1 o_A - \psi_0 \iota_A, \\
\hat{\psi}_A = r \psi_A = r \psi_1 o_A - r \psi_0 \iota_A = \hat{\psi}_1 o_A - \hat{\psi}_0 Rl_A.
\]

Therefore we obtain the relations of the new components with the old components as follows

\[
\hat{\psi}_0 = r^2 \psi_0, \quad \hat{\psi}_1 = r \psi_1.
\]

Since \( D, D', \delta \) and \( \delta' \) are the directional derivatives \( l^a \nabla_a, n^a \nabla_a, m^a \nabla_a \) and \( \bar{m}^a \nabla_a \) respectively so they have the rescaled form as follows:

\[
\hat{D} = \hat{l}^a \partial_a = -\sqrt{\frac{\Delta}{2 \rho^2}} \partial_R, \tag{3.27}
\]

\[
\hat{D}' = \hat{n}^a \partial_a = \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) \partial_t + a \partial_\varphi + \frac{R^2 \Delta}{2} \partial_R \right), \tag{3.28}
\]

\[
\hat{\delta} = \hat{m}^a \partial_a = \frac{r}{p \sqrt{2}} \left( ia \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right), \tag{3.29}
\]

\[
\hat{\delta}' = \hat{\bar{m}}^a \partial_a = \frac{r}{\bar{p} \sqrt{2}} \left( -ia \sin \theta \partial_t + \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right). \tag{3.30}
\]

Applying the expression (3.26) of the equation \( \nabla^{AA'} \psi_A = 0 \), we obtain the rescaled of the Weyl’s equation by using the rescaled directional derivatives and the spin coefficients as follows:

\[
\begin{aligned}
(\hat{D}' + \hat{\mu} - \hat{\gamma}) \hat{\psi}_0 - (\hat{\delta} - \hat{\tau} + \hat{\beta}) \hat{\psi}_1 &= 0, \\
(\hat{D} + \hat{\varepsilon} - \hat{\rho}) \hat{\psi}_1 - (\hat{\delta}' - \hat{\alpha} + \hat{\pi}) \hat{\psi}_0 &= 0.
\end{aligned}
\]
This given us the system in Geroch-Held-Penrose (GHP) formalism

\[
\begin{aligned}
\hat{p}'\hat{\psi}_0 - \hat{\delta}'\hat{\psi}_1 &= -\hat{\mu}\hat{\psi}_0 - \hat{\tau}\hat{\psi}_1, \\
\hat{p}\hat{\psi}_1 - \hat{\delta}\hat{\psi}_0 &= \hat{\rho}\hat{\psi}_1 + \hat{\pi}\hat{\psi}_0
\end{aligned}
\]  
(3.31)

where $\hat{p}$, $\hat{p}'$, $\hat{\delta}$ and $\hat{\delta}'$ are the weighted differential operators of the GHP formalism

\[
\begin{aligned}
\hat{p}'\hat{\psi}_0 &= (\hat{D}' - \hat{\gamma})\hat{\psi}_0, \quad \hat{\delta}'\hat{\psi}_1 = (\hat{\delta} + \hat{\beta})\hat{\psi}_1, \\
\hat{p}\hat{\psi}_1 &= (\hat{D} + \hat{\varepsilon})\hat{\psi}_1, \quad \hat{\delta}\hat{\psi}_0 = (\hat{\delta} - \hat{\alpha})\hat{\psi}_0.
\end{aligned}
\]

Using the rescaled directional derivatives from (3.27) to (3.30) and the values of the rescaled spin coefficients from (3.17) to (3.22), we have:

\[
\begin{aligned}
\hat{p}' &= \sqrt{\frac{2}{2\rho^2}} \left( (r^2 + a^2)\partial_t + a\partial_\varphi + \frac{R^2\Delta}{2} \partial_R \right) + \frac{1}{\sqrt{2\Delta\rho^2}} \left( \Delta \left( R + \frac{ia \cos \theta}{\rho^2} \right) - \frac{(r - M\rho^2 - r\Delta)}{\rho^2} \right), \\
\hat{\delta} &= \frac{r}{\rho\sqrt{2}} \left( i \sin \theta \partial_t + \partial_\theta + \frac{r}{\sin \theta} \partial_\varphi \right) + \frac{r}{\sqrt{2\rho^2}} \left( \frac{\cot \theta}{2} + \frac{a^2 \sin \theta \cos \theta}{2\rho^2} \right), \\
\hat{p} &= -\sqrt{\frac{\Delta}{2\rho^2}} \partial_R + \frac{r^2(r - M\rho^2 - r\Delta)}{2\rho^2\sqrt{2\Delta\rho^2}}, \\
\hat{\delta}' &= \frac{r}{\rho\sqrt{2}} \left( -i \sin \theta \partial_t + \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi \right) + \frac{r}{\rho\sqrt{2}} \left( -\frac{ia \sin \theta}{\rho} + \frac{\cot \theta}{2} - \frac{a^2 \cos \theta \sin \theta}{2\rho^2} \right).
\end{aligned}
\]

Now we can give the simpler expression of the Weyl’s system by using the terms of the matrix and the derivative operators. In this sense, we set

\[
M_t = \begin{pmatrix}
\sqrt{\frac{2}{\Delta\rho^2}(r^2 + a^2)} & 0 \\
0 & 0
\end{pmatrix}, \quad M_R = \begin{pmatrix}
R \sqrt{\frac{\Delta}{2\rho^2}} & 0 \\
0 & -\sqrt{\frac{\Delta}{2\rho^2}}
\end{pmatrix}, \quad M_{\varphi} = \begin{pmatrix}
\frac{a}{\sqrt{\frac{\Delta}{2\rho^2}}} & 0 \\
0 & 0
\end{pmatrix},
\]

\[
D_{t,\omega} = \begin{pmatrix}
0 \\
\frac{r}{\rho\sqrt{2}} \left( i \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \\
\frac{r}{\rho\sqrt{2}} \left( i \sin \theta \partial_t + \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right)
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
\sqrt{\frac{\Delta}{2\rho^2}} \left( 2R + \frac{ia \cos \theta}{\rho^2} - \frac{1}{\rho} \right) - \frac{(r - M\rho^2 - r\Delta)}{2\rho^2\sqrt{2\Delta\rho^2}} \\
\frac{a^2 r \cos \theta \sin \theta}{2\sqrt{2\rho^2}} \\
\frac{r \sin \theta}{\rho\sqrt{2\rho^2}} \left( 1 - \frac{ia \cos \theta}{2\rho} \right) - \frac{M r^3 - a^2 r^3 \sin^2 \theta - M a^2 r^2 \cos^2 \theta}{2\rho^2\sqrt{2\Delta\rho^2}} + \frac{iar \cos \theta}{\rho} \sqrt{\frac{\Delta}{2\rho^2}}
\end{pmatrix},
\]

where $D_{t,\omega}$ is a modification\(^3\) of the symmetric derivative operator on 2–sphere $S^2$

\[
D_{\omega} = \begin{pmatrix}
0 \\
\frac{r}{\rho\sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \\
\frac{r}{\rho\sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right)
\end{pmatrix}.
\]

\(^3\)In the sense that if $a = 0$, then we have the consistency $D_{t,\omega} = D_{\omega}$.
So the expression of the Weyl’s system in the matrix form is

\[ M_t D_t \hat{\psi} + M_R D_R \hat{\psi} - D_{t, \omega} \hat{\psi} + M_\varphi D_\varphi \hat{\psi} + P \hat{\psi} = 0 \]  

(3.35)

where

\[ \hat{\psi} := \begin{pmatrix} \hat{\psi}_0 \\ \hat{\psi}_1 \end{pmatrix}, \]

and the action of the partial derivative operator \( D_X \) along a vector field \( X \) on \( \hat{\psi} \) is

\[ D_X \hat{\psi} := \begin{pmatrix} \partial_X \hat{\psi}_0 \\ \partial_X \hat{\psi}_1 \end{pmatrix}. \]

The conjugate system of the Weyl’s system can be also expressed in the matrix form by conjugating the matrix coefficients of the equation (3.35) as follows

\[ M_t D_t \hat{\psi} + M_R D_R \hat{\psi} - \bar{D}_{t, \omega} \hat{\psi} + M_\varphi D_\varphi \hat{\psi} + \bar{P} \hat{\psi} = 0 \]  

(3.36)

which is equivalent to

\[ M_t D_t \hat{\psi} + M_R D_R \hat{\psi} - \bar{D}_{t, \omega} \hat{\psi} + M_\varphi D_\varphi \hat{\psi} + P \hat{\psi} = Q \hat{\psi}, \]  

(3.37)

where

\[ Q = P - \bar{P} = \begin{pmatrix} -\frac{ia \cos \theta}{\rho^2} \sqrt{\frac{2\Delta}{\rho^4}} & -\frac{\sqrt{2}ia \sin \theta}{\rho^2} \\ -\frac{ia^3 r \cos^2 \theta \sin \theta}{\sqrt{2} \rho^4} & -2a^2 r \cos^2 \theta \sqrt{\frac{\Delta}{2 \rho^4}} \end{pmatrix} = \begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix} \]

with the components \( A_0, A_1, A_2 \) and \( A_3 \) are of order nonnegative in \( \rho \).

### 3.3.3 Energy

The equivalent form of the energy of the Dirac field is given by the following lemma:

**Lemma 3.3.1.** We have the following simple equivalent expressions of the energies on \( H_s, 0 \leq s \leq 1 \)

\[ \mathcal{E}_{H_s}(\hat{\psi}) \simeq \int_{H_s} \left( \frac{R}{|r| t} |\hat{\psi}_0|^2 + |\hat{\psi}_1|^2 \right) d^4t \sqrt{d^2\omega}, \]

and we can show that the energy on the null hypersurface \( S_{t_0} \) is non negative.

**Proof.** The conserved current for the rescaled Weyl’s equation is

\[ \hat{j}^a = \hat{\psi}^A \tilde{\psi}^\Lambda \hat{\psi}^\Lambda + g^{ab} \hat{j}_b = |\hat{\psi}_1|^2 \tilde{\phi}^a + |\hat{\psi}_0|^2 \tilde{\phi}^a - \hat{\psi}_1 \tilde{\psi}_0 \tilde{m}^a + \hat{\psi}_1 \tilde{\psi}_0 \hat{m}^a. \]

The energy of the Dirac field on a spacelike hypersurface \( S \) is defined as

\[ \mathcal{E}_S := -4 \int_S \hat{j}^a \sqrt{\text{d}^4d} = \int_S \hat{g}(\mathcal{N}^a, \hat{j}^a) \mathcal{L}^a \sqrt{\text{d}Vol}, \]
Now the energy of $\hat{\psi}$ on $H_s$ takes the form:

$$\mathcal{E}_{H_s}(\hat{\psi}) = \int_{H_s} \hat{g}(N^a, J^a) L^a \, d\text{Vol}^4$$

$$= \int_{H_s} \hat{g}(N^a, J^a) (-\partial_R + \sigma_t \partial_\varphi) \, d\text{Vol}^4$$

$$= \int_{H_s} \hat{g}(N^a, J^a) (1 + a^2 R^2 \cos^2 \theta) d^* t d^2 \omega.$$

First, we calculate the scalar product $\langle \partial_t, \hat{n}^a \partial_a \rangle$ and $\langle \partial_{\varphi}, \hat{n}^a \partial_a \rangle$ as follows:

$$\langle \partial_t, \hat{n}^a \partial_a \rangle = \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) < \partial_t, \partial_t > + a < \partial_t, \partial_{\varphi} > + \frac{R^2 \Delta}{2} < \partial_t, \partial_R > \right)$$

$$= \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) R^2 \left( 1 - \frac{2Mr}{\rho^2} \right) + \frac{2Ma^2 R \sin^2 \theta}{\rho^2} - \frac{R^2 \Delta}{2} \right)$$

$$= \sqrt{\frac{\Delta}{2 \rho^2}} R^2,$$

and

$$\langle \partial_{\varphi}, \hat{n}^a \partial_a \rangle = \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) < \partial_{\varphi}, \partial_t > + a < \partial_{\varphi}, \partial_{\varphi} > + \frac{R^2 \Delta}{2} < \partial_{\varphi}, \partial_R > \right)$$

$$= \sqrt{\frac{2}{\Delta \rho^2}} \left( (r^2 + a^2) \frac{2MaR \sin^2 \theta}{\rho^2} - a \sin^2 \theta \left( 1 + a^2 R^2 + \frac{2Ma^2 R \sin^2 \theta}{\rho^2} \right) \right)$$

$$+ a \sin^2 \theta \frac{R^2 \Delta}{2} \right)$$

$$= - \sqrt{\frac{\Delta}{2 \rho^2}} R^2 a \sin^2 \theta.$$

Now we calculate the scalar product $\hat{g}(N^a, J^a) = \langle N^a \partial_a, J^a \partial_a \rangle$ as follows:

$$\langle N^a \partial_a, J^a \partial_a \rangle = \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, |\hat{\psi}|^2 \hat{n}^a \partial_a + |\hat{\psi}|^2 \hat{\partial}^a \partial_a - \hat{\psi}_1 \hat{\psi}_0 \hat{m}^a \partial_a - \hat{\psi}_1 \hat{\psi}_0 \hat{m}^a \partial_a \rangle$$

$$= \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 \rangle + \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 \hat{m}^a \partial_a - \hat{\psi}_1 \hat{\psi}_0 \hat{m}^a \partial_a \rangle$$

$$+ \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 - \hat{\psi}_1 \hat{\psi}_0 \rangle$$

$$- \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 \rangle - \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 \rangle.$$

Where the scalar product coefficients of the first two terms are

$$\langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{\psi}_1 \hat{\psi}_0 \rangle = \sqrt{\frac{\Delta}{2 \rho^2}} (1 - a \sin^2 \theta \gamma),$$

and

$$\langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{n}^a \rangle = \sqrt{\frac{\Delta}{2 \rho^2}} R^2 (1 - a \sin^2 \theta \gamma) + \sqrt{\frac{2}{\Delta \rho^2}} (-\theta^2 + a^2 + a^2 \sin^2 \theta) \beta$$

$$= \sqrt{\frac{\Delta}{2 \rho^2}} R^2 \left\{ 1 - a \sin^2 \theta \gamma - \frac{2\beta \rho^2}{R^2 \Delta} \right\}.$$
Two later terms of \( \langle N^a, J^a \rangle \) can be calculated in combining as follows:

\[
\langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \hat{m}^a \rangle > \psi_1 \bar{\psi}_0 + \langle \partial_t + \beta \partial_R + \gamma \partial_{\varphi}, \check{m}^a \rangle > \bar{\psi}_1 \psi_0 \\
= 2R \left\{ \langle \partial_t, \hat{m}^a \rangle > \psi_1 \bar{\psi}_0 + \langle \gamma \partial_{\varphi}, \hat{m}^a \rangle > \bar{\psi}_1 \psi_0 \right\} \\
= 2R \left\{ \frac{1}{R \rho \sqrt{2}} \left( ia \sin \theta R^2 \left( 1 - \frac{2M \rho}{\rho^2} \right) + ia \sin \theta \frac{2MR}{\rho^2} \right) \bar{\psi}_1 \psi_0 \\
+ \gamma \frac{R \rho \sqrt{2}}{R \rho \sqrt{2}} \left( ia \sin \theta \frac{2MaR \sin^2 \theta}{\rho^2} - i \sin \theta \left( 1 + a^2 R^2 + \frac{2Ma^2 R \sin^2 \theta}{\rho^2} \right) \right) \bar{\psi}_1 \psi_0 \right\} \\
= \frac{\Delta}{2 \rho^2} 2R \left\{ \frac{ia \sin \theta R}{\sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 + \frac{i \sin \theta (1 + a^2 R^2) \gamma}{R \sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right\},
\]

here we are not interested in \( \partial_R \) since \( \langle \partial_R, \hat{m}^a \rangle > = \langle \partial_R, \check{m}^a \rangle > = 0 \).

Therefore we obtain:

\[
\langle N^a, J^a \rangle = \sqrt{\frac{\Delta}{2 \rho^2}} (1 - a \sin^2 \theta \gamma) |\psi_1|^2 + \sqrt{\frac{\Delta}{2 \rho^2}} R^2 \left( 1 - a \sin^2 \theta \gamma - \frac{2 \beta \rho^2}{R^2 \Delta} \right) |\bar{\psi}_0|^2 - \\
- \sqrt{\frac{\Delta}{2 \rho^2}} 2R \left\{ \frac{ia \sin \theta R}{\sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 + \frac{i \sin \theta (1 + a^2 R^2) \gamma}{R \sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right\}.
\]

Now due to the inequalities:

\[
2R \left\{ \frac{ia \sin \theta R}{\sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right\} \leq 2 \left| \frac{ia \sin \theta R}{\sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right| \leq R |\bar{\psi}_1|^2 + \frac{a^2 \sin^2 \theta \gamma}{\Delta} |\bar{\psi}_0|^2,
\]

\[
2R \left\{ \frac{i \sin \theta (1 + a^2 R^2) \gamma}{R \sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right\} \leq \left| \frac{i \sin \theta (1 + a^2 R^2) \gamma}{R \sqrt{\Delta \rho^2}} \bar{p} \bar{\psi}_1 \psi_0 \right| \leq |\gamma| \left( \frac{1}{R} |\bar{\psi}_1|^2 + \frac{(1 + a^2 R^2)^2 \sin^2 \theta}{\Delta R} |\bar{\psi}_0|^2 \right),
\]

we obtain

\[
\langle N^a, J^a \rangle \geq \sqrt{\frac{\Delta}{2 \rho^2}} \left( 1 - a \sin^2 \theta \gamma - R - \frac{|\gamma|}{R} \right) |\bar{\psi}_1|^2 \\
+ \sqrt{\frac{\Delta}{2 \rho^2}} R^2 \left( 1 - a \sin^2 \theta \gamma - \frac{2 \beta \rho^2}{R^2 \Delta} - \frac{a^2 \sin^2 \theta \gamma}{\Delta} - \frac{|\gamma| \rho^2 (1 + a^2 R^2)^2 \sin^2 \theta}{\Delta R} \right) |\bar{\psi}_0|^2
\]

and

\[
\langle N^a, J^a \rangle \leq \sqrt{\frac{\Delta}{2 \rho^2}} \left( 1 - a \sin^2 \theta \gamma + R + \frac{|\gamma|}{R} \right) |\bar{\psi}_1|^2 \\
+ \sqrt{\frac{\Delta}{2 \rho^2}} R^2 \left( 1 - a \sin^2 \theta \gamma - \frac{2 \beta \rho^2}{R^2 \Delta} + \frac{a^2 \sin^2 \theta \gamma}{\Delta} + \frac{|\gamma| \rho^2 (1 + a^2 R^2)^2 \sin^2 \theta}{\Delta R} \right) |\bar{\psi}_0|^2.
\]
Chapter 3. Peeling for Dirac field on Kerr spacetime

Note that the factor $\gamma$ is equivalent to $R^2$ and the factor $\beta$ is equivalent to $(1 - \frac{1}{s}) R^2$. So that in the formula of the energy of $\psi$ on $\mathcal{H}_s$, the coefficient of $\dot{\psi}_1$ is equivalent to 1 and the coefficient of $\dot{\psi}_0$ is equivalent to $(\frac{2}{s} - 1) R^2$. Then we use the inequalities

$$1 \leq \frac{2}{s} - 1 \leq \frac{2}{s},$$

and the equivalence

$$\frac{R^2}{s} \simeq \frac{R}{|s| t},$$

to obtain the simple equivalent expression of the energy of $\dot{\psi}$ on $\mathcal{H}_s$:

$$\mathcal{E}_{\mathcal{H}_s}(\dot{\psi}) \simeq \int_{\mathcal{H}_s} \left( \frac{R}{|s| t} |\dot{\psi}_0|^2 + |\dot{\psi}_1|^2 \right) \text{d}^4 t \text{d}^2 \omega.$$

The energy of $\dot{\psi}$ on $S_{st}$ takes the form:

$$\mathcal{E}_{S_{st}}(\dot{\psi}) = \int_{S_{st}} \dot{g}(\hat{N}^a, \dot{j}^a) \hat{L}^a \text{dVol}^4$$

$$= \int_{S_{st}} \dot{g}(\hat{N}^a, \dot{j}^a) \left( \frac{\rho^2}{(a^2 + r^2)(X + 1)} \partial_t + \sigma_1 \partial_{\varphi} \right) \text{dVol}^4$$

$$= \int_{S_{st}} \dot{g}(\hat{N}^a, \dot{j}^a) \frac{\rho^2}{(a^2 + r^2)(X + 1)} (1 + a^2 R^2 \cos^2 \theta) \text{dR} \text{d}^2 \omega.$$

Similarly to the calculation of the energy on $\mathcal{H}_s$, we have the decomposition:

$$< \hat{N}^a \partial_a, \dot{\partial}^a > = < \hat{N}^a \partial_a, |\dot{\psi}_1|^2 \hat{l}^\alpha \partial_\alpha + |\dot{\psi}_0|^2 \hat{n}^a \partial_a - \dot{\psi}_1 \tilde{\psi}_0 \tilde{n}^a \partial_a - \dot{\psi}_1 \tilde{\psi}_0 \tilde{m}^a \partial_a >$$

$$= < \hat{N}^a \partial_a, \hat{l}^a \partial_a > + 2 \Re \left\{ < \hat{N}^a \partial_a, \hat{m}^a \partial_a > |\dot{\psi}_1| \tilde{\psi}_0 \right\}.$$

The coefficients can be calculated as follows:

$$< \hat{N}^a \partial_a, \hat{l}^a \partial_a > = \frac{1}{\rho^2} \sqrt{\Delta} \left\{ - \left( r^2 a^2 \sin^2 \theta + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) + a^2 \sin^2 \theta \left( r^2 + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) \right\}$$

$$= \frac{1}{\rho^2} \sqrt{\Delta} \frac{(a^2 + r^2) X}{\Delta} (-\rho^2) = - \sqrt{\frac{\Delta}{2 \rho^2}} \frac{(a^2 + r^2) X}{\Delta},$$

$$= \sqrt{\frac{\Delta}{2 \rho^2}} \frac{a^2 r^2}{(a^2 + r^2)(X + 2)},$$

$$< \hat{N}^a \partial_a, \hat{n}^a \partial_a > = - \frac{1}{\rho^2} \left\{ \left( r^2 a^2 \sin^2 \theta + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) \frac{1}{r^2} \sqrt{\Delta} \right\}$$

$$+ (a^2 + r^2)(X + 1)(-r^2 + a^2) + a^2 \sin^2 \theta) \sqrt{\frac{2}{\Delta \rho^2}}$$

$$- a \left( r^2 + \frac{(a^2 + r^2)^2 r^2}{\Delta} X \right) \frac{a \sin^2 \theta}{r^2} \sqrt{\frac{\Delta}{2 \rho^2}} \right\}$$

$$= \sqrt{\frac{2}{\Delta \rho^2}} (a^2 + r^2) \left( \frac{X}{2} + 1 \right),$$
\[ < \hat{N}^a \partial_a, \hat{m}^a \partial_a > = - \frac{r}{p \sqrt{2} \rho^2} \left\{ \left( r^2 a^2 \sin^2 \theta + \frac{(r^2 + a^2)^2 r^2}{\Delta} X \right) \frac{ia \sin \theta}{r^2} \left( 1 - \frac{2Mr}{\rho^2} \right) \right. \\
+ \left. \frac{i}{\sin \theta} \left( r^2 a^2 \sin^2 \theta + \frac{(r^2 + a^2)^2 r^2}{\Delta} X \right) \frac{2MaR \sin^2 \theta}{\rho^2} \right. \\
+ \left. a(r^2 + \frac{(a^2 + r^2)r^2}{\Delta} X) \right\} \left[ \frac{ia \sin \theta}{r^2} \left( 1 + \frac{a^2 \sin^2 \theta}{r^2} \right) \right. \\
\left. \left. + \frac{ar}{p \sqrt{2}} (i \sin \theta - \cos \theta) \right\} \right. \\
\leq \sqrt{\frac{\Delta}{2\rho^2}} \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} |\hat{\psi}_1|^2 + \frac{(2Mr \sin^2 \theta)}{2}\sqrt{\frac{2\rho^2}{\Delta}} |\hat{\psi}_0|^2 \\
= \sqrt{\frac{\Delta}{2\rho^2}} \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} |\hat{\psi}_1|^2 + \sqrt{\frac{2}{\Delta \rho^2}} \frac{(a^2 + r^2)(X + 2)}{2} |\hat{\psi}_0|^2. \\
\] 

Therefore we have 

\[ < \hat{N}^a, \hat{L}^a > \geq \sqrt{\frac{\Delta}{2\rho^2}} \left( \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} - \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} \right) |\hat{\psi}_1|^2 \\
+ \sqrt{\frac{2}{\Delta \rho^2}} (a^2 + r^2) \left( \frac{X}{2} + 1 - \frac{X + 2}{2} \right) |\hat{\psi}_0|^2 \\
\geq 0 \]

\[ < \hat{N}^a, \hat{L}^a > \leq \sqrt{\frac{\Delta}{2\rho^2}} \left( \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} + \frac{a^2 r^2}{(a^2 + r^2)(X + 2)} \right) |\hat{\psi}_1|^2 \\
+ \sqrt{\frac{2}{\Delta \rho^2}} (a^2 + r^2) \left( \frac{X}{2} + 1 + \frac{X + 2}{2} \right) |\hat{\psi}_0|^2 \\
= \sqrt{\frac{\Delta}{2\rho^2}} \frac{2a^2 r^2}{(a^2 + r^2)(X + 2)} |\hat{\psi}_1|^2 \\
+ \sqrt{\frac{2}{\Delta \rho^2}} (a^2 + r^2)(X + 2) |\hat{\psi}_0|^2. \\
\] 

Since we have \(-1 < X \leq 0\), the coefficients of \(|\hat{\psi}_1|^2\) and \(|\hat{\psi}_0|^2\) are bounded and non negative. So the energy of the Dirac field on the hypersurface \(S_t\) is non negative. \(\square\)
Remark 3.3.1. Note that the energy on $S_{\cdot t}$ is the flux of $\mathbf{J}$ through $S_{\cdot t}$. This is clearly non negative since $\mathbf{J}$ is future-oriented null vector field and $S_{\cdot t}$ is oriented by its future-pointing null normal vector field.

3.4 Approach using partial derivatives

3.4.1 The conservation laws of Dirac’s system

The conservation law of the original system (3.35) is simple as follows

$$\nabla^{AA'} (\tilde{\psi}_A \tilde{\psi}_{A'}) = 2\mathbb{R} \left\{ \left( \nabla^{AA'} \tilde{\psi}_A \right) \tilde{\psi}_{A'} \right\} = 0. \quad (3.38)$$

To obtain the conservation laws at a higher order, we need to commute the vector fields into (3.35). Here, we will do this work for the derivatives at the order one, the conservation laws at a higher order will be done similarly. By using the partial derivatives, we commute $D_{\cdot t}, D_R, D_{\cdot \varphi}$ and the modification $D_{t, \omega}$ of the symmetric derivative operator on 2–sphere into (3.35).

First, commuting $D_{\cdot t}, D_{\cdot \varphi}$ and $D_R$ into the system (3.35), we obtain

$$M_{\cdot t} D_{\cdot t}^2 \dot{\psi} + M_R D_R D_{\cdot t} \dot{\psi} - D_{t, \omega} D_{\cdot t} \dot{\psi} + M_{\cdot \varphi} D_{\cdot \varphi} D_{\cdot t} \dot{\psi} + PD_{\cdot t} \dot{\psi} = -[D_{\cdot t}, M_{\cdot t} D_{\cdot t}] \dot{\psi} - [D_{\cdot t}, M_R D_R] \dot{\psi} + [D_{\cdot t}, D_{t, \omega}] \dot{\psi} - [D_{\cdot t}, M_{\cdot \varphi} D_{\cdot \varphi}] \dot{\psi} - (D_{\cdot t} P) \dot{\psi} = 0, \quad (3.39)$$

$$M_{\cdot t} D_{\cdot t} D_{\cdot \varphi} \dot{\psi} + M_R D_R D_{\cdot \varphi} \dot{\psi} - D_{t, \omega} D_{\cdot \varphi} \dot{\psi} + M_{\cdot \varphi} D_{\cdot \varphi}^2 \dot{\psi} + PD_{\cdot \varphi} \dot{\psi} = -[D_{\cdot \varphi}, M_{\cdot t} D_{\cdot t}] \dot{\psi} - [D_{\cdot \varphi}, M_R D_R] \dot{\psi} + [D_{\cdot \varphi}, D_{t, \omega}] \dot{\psi} - [D_{\cdot \varphi}, M_{\cdot \varphi} D_{\cdot \varphi}] \dot{\psi} - (D_{\cdot \varphi} P) \dot{\psi} = 0. \quad (3.40)$$

Since (see Appendix 3.7.5)

$$[D_R, D_{t, \omega}] \dot{\psi} \simeq D_{t, \omega} \dot{\psi},$$

we have

$$M_{\cdot t} D_{\cdot t} D_R \dot{\psi} + M_R D_R^2 \dot{\psi} - D_{t, \omega} D_R \dot{\psi} + M_{\cdot \varphi} D_{\cdot \varphi} D_R \dot{\psi} + PD_R \dot{\psi} = -[D_R, M_{\cdot t} D_{\cdot t}] \dot{\psi} - [D_R, M_R D_R] \dot{\psi} + [D_R, D_{t, \omega}] \dot{\psi} - [D_R, M_{\cdot \varphi} D_{\cdot \varphi}] \dot{\psi} - (D_R P) \dot{\psi} \simeq -[D_R, M_{\cdot t} D_{\cdot t}] \dot{\psi} - [D_R, M_R D_R] \dot{\psi} + [D_R, D_{t, \omega}] \dot{\psi} - [D_R, M_{\cdot \varphi} D_{\cdot \varphi}] \dot{\psi} - (D_R P) \dot{\psi} = 0 \quad (3.41)$$

where the Lie brackets $[\cdot, \cdot]$ are understood as the error terms, they are the parts that are not commuted. In the first two equations above, the right-hand sides are both zero, this can be understood due to the fact that the coefficient functions of the Weyl’s system does not contain the variables $^t t$ and $^\varphi \varphi$, so the derivatives $\partial_{\cdot t}$ and $\partial_{\cdot \varphi}$ are commuted to this system. In the third equation of $D_R$, using the expression of the matrix coefficients (3.32),(3.33),(3.34), we can give the simple expressions of the coefficients on the right-hand side of (3.41) as follows

$$[D_R, M_{\cdot t} D_{\cdot t}] = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} D_{\cdot t}, \quad [D_R, M_R D_R] = \begin{pmatrix} B_2 & 0 \\ 0 & B_3 \end{pmatrix} D_R,$$
where the components $B_i (i = 1, 2, \ldots, 9)$ are of an order nonnegative in $R$ and furthermore $B_2$ is of order one in $R$ (see Appendix 3.7.5 for detail).

Second, the commutator of the operator $D_{t,\omega}$ is more complicate

$$M_tD_{t}D_{t,\omega}\hat{\psi} + M_RD_{t}D_{t,\omega}\hat{\psi} - D_{t,\omega}^2\hat{\psi} + M_\varphi D_{t,\omega}\hat{\psi} + PD_{t,\omega}\hat{\psi}$$

$$= -[D_{t,\omega}, M_tD_{t}][\hat{\psi}] - [D_{t,\omega}, M_RD_{t}][\hat{\psi}] - [D_{t,\omega}, M_\varphi D_{t,\omega}][\hat{\psi}] - (D_{t,\omega}P)[\hat{\psi}].$$

The action of $D_{t,\omega}^2$ on $\hat{\psi}$ is understood as the action after multiplying two matrix i.e

$$D_{t,\omega}^2 = D_{t,\omega}D_{t,\omega}^\dagger = \begin{pmatrix} \tilde{\partial}\tilde{\partial}' & 0 \\ 0 & \tilde{\partial}'\tilde{\partial} \end{pmatrix},$$

where

$$D_{t,\omega} = \begin{pmatrix} 0 & \tilde{\partial} \\ \tilde{\partial}' & 0 \end{pmatrix}$$

with

$$\tilde{\partial} = \frac{r}{p\sqrt{2}} \left( ia\sin\theta\partial_t + \partial_\varphi + \frac{i}{\sin\theta}\partial_\theta + \frac{\cot\theta}{2} \right),$$

$$\tilde{\partial}' = \frac{r}{p\sqrt{2}} \left( -ia\sin\theta\partial_t + \partial_\varphi - \frac{i}{\sin\theta}\partial_\theta + \frac{\cot\theta}{2} \right).$$

We can see that the action of $D_{t,\omega}$ on a matrix is

$$D_{t,\omega} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\partial} \\ \tilde{\partial}' & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \tilde{\partial}C & \tilde{\partial}D \\ \tilde{\partial}'A & \tilde{\partial}'B \end{pmatrix},$$

so after applying the operator $D_{t,\omega}$ on the system, we get that the lines of the resulting system are interchanged from the original system. So that we can also understand the operator $D_{t,\omega}^2$ as the effect of $D_{t,\omega}$ on $D_{t,\omega}$ after interchanging two line of $D_{t,\omega}$. Therefore, we can rewrite the equation (3.42) as

$$M_tD_{t}D_{t,\omega}^\dagger\hat{\psi} + M_RD_{t}D_{t,\omega}^\dagger\hat{\psi} - D_{t,\omega}^2\hat{\psi} + M_\varphi D_{t,\omega}^\dagger\hat{\psi} + PD_{t,\omega}^\dagger\hat{\psi}$$

$$= -[D_{t,\omega}^\dagger, M_tD_{t}][\hat{\psi}] - [D_{t,\omega}^\dagger, M_RD_{t}][\hat{\psi}] - [D_{t,\omega}^\dagger, M_\varphi D_{t,\omega}][\hat{\psi}] - (D_{t,\omega}^\dagger P)[\hat{\psi}].$$

where $D_{t,\omega}^\dagger$ is the matrix $D_{t,\omega}$ after interchanging its lines. Using again the expressions of the matrix coefficients (3.32), (3.33), (3.34), we can give the simple expression of the coefficients on the right-hand side of (3.43) as follows

$$[D_{t,\omega}^\dagger, M_tD_{t}] = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} D_{t},$$

$$[D_{t,\omega}^\dagger, M_RD_{t}] \simeq \begin{pmatrix} D_2 & 0 \\ 0 & D_3 \end{pmatrix} D_R - M_RD_{t,\omega}^\dagger.$$
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\begin{align*}
[D^*_t,\omega M^*_\varphi D^*_r] &= \begin{pmatrix} D_4 & 0 \\ 0 & 0 \end{pmatrix} D^*_r, \\
\imath D^*_{t,\omega} P &= \begin{pmatrix} D_5 & D_6 \\ D_7 & D_8 \end{pmatrix},
\end{align*}

where the components \(D_i (i = 1, 2...8)\) are of order nonnegative in \(R\) (see Appendix 3.7.5 for more details). By comparison to the conjugate equation (3.37), we can write

\begin{equation}
M_{t}D_{t}D^*_{t,\omega}\hat{\psi} + MRD_{R}D^*_{t,\omega}\hat{\psi} - \bar{D}_{t,\omega}D^*_{t,\omega}\hat{\psi} + M_{t}D_{\varphi}D^*_{t,\omega}\hat{\psi} + PD^*_{t,\omega}\hat{\psi} = \left\{ -QD^*_{t,\omega}\hat{\psi} - [D^*_{t,\omega}, M_{t}D_{t}]\hat{\psi} - [D^*_{t,\omega}, MRD_{R}]\hat{\psi} - \frac{1}{2} \left[ D^*_{t,\omega}, M_{t}D_{\varphi}\right]\hat{\psi} \right\} + QD^*_{t,\omega}\hat{\psi},
\end{equation}

(3.44)

where the components in the sign \{\} is the error that is arisen by commuting the operator \(D_{t,\omega}\) into the original system (3.35).

To obtain the conservation laws we define the operators:

\begin{align*}
\mathcal{D}_{t}\hat{\psi}_{A} &= \partial_{t}\hat{\psi}_{1}\hat{o}_{A} - \partial_{t}\hat{\psi}_{0}\hat{i}_{A}, \\
\mathcal{D}_{R}\hat{\psi}_{A} &= \partial_{R}\hat{\psi}_{1}\hat{o}_{A} - \partial_{R}\hat{\psi}_{0}\hat{i}_{A}, \\
\mathcal{D}_{\varphi}\hat{\psi}_{A} &= \partial_{\varphi}\hat{\psi}_{1}\hat{o}_{A} - \partial_{\varphi}\hat{\psi}_{0}\hat{i}_{A}, \\
\mathcal{D}_{t,\omega}\hat{\psi}_{A}' &= \mathcal{D}_{t,\omega}^{*}\hat{\psi}_{A} = \bar{\mathcal{D}}\hat{\psi}_{1}\hat{o}_{A} - \bar{\mathcal{D}}\hat{\psi}_{0}\hat{i}_{A},
\end{align*}

where \(\mathcal{D}_{t,\omega}\) is the modification\(^4\) of the symmetric derivative operator on 2–sphere

\begin{equation}
\mathcal{D}_{\omega}\hat{\psi}_{A'} = \bar{\mathcal{D}}\hat{\psi}_{1}\hat{o}_{A'} - \bar{\mathcal{D}}\hat{\psi}_{0}\hat{i}_{A'},
\end{equation}

with

\begin{equation}
\bar{\mathcal{D}} = \frac{r}{p\sqrt{2}} \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\varphi} + \frac{\cot \theta}{2} \right), \\
\bar{\mathcal{D}} = \frac{r}{p\sqrt{2}} \left( \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\varphi} + \frac{\cot \theta}{2} \right).
\end{equation}

From the property \(\hat{i}_{A'}^{A'} = -i_{A'}^{A'}\hat{o}_{A} = -1\), we can obtain the conservation laws by multiplying the matrix \((\mathcal{D}_{X}\hat{\psi}_{A})_0 (\mathcal{D}_{X}\hat{\psi}_{A})_1\) on the left-hand side by the error components and then take the real part. The equations (3.39) and equation (3.40) are obtained by commuting \(\mathcal{D}_{t}\) and \(\mathcal{D}_{\varphi}\) into the rescaled Weyl’s equation respectively, where the errors are zero so the conservation laws are also zero:

\begin{equation}
\hat{\nabla}^{AA'} \left[ \left( \mathcal{D}_{t}\hat{\psi}_{A} \right) \left( \mathcal{D}_{t}\hat{\psi}_{A} \right) \right] = \hat{\nabla}^{AA'} \left[ \left( \mathcal{D}_{\varphi}\hat{\psi}_{A} \right) \left( \mathcal{D}_{\varphi}\hat{\psi}_{A} \right) \right] = 0.
\end{equation}

(3.45)

The equation (3.41) is obtained by commuting \(\mathcal{D}_{R}\) into the rescaled Weyl’s equation, the conservation

\(^4\)In the sense that if \(a = 0\), then we have the consistency \(\mathcal{D}_{t,\omega} = \mathcal{D}_{\omega}.\)
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The equation (3.44) is obtained by commuting $\hat{D}_{t,\omega}$ into the rescaled Weyl’s equation, the conservation law is

$$
\hat{\nabla}^{AA'} \left[ \left( \hat{D}_{R} \hat{\psi}_A \right) \left( \hat{D}_{R} \hat{\psi}_{A'} \right) \right] 
\simeq -2 \Re \left\{ \left( \frac{\partial R \hat{\psi}_0}{\partial R \hat{\psi}_1} \right) \left[ \left( \begin{array}{cc} B_1 & 0 \\ 0 & 0 \end{array} \right) \partial_{t,\omega} \hat{\psi}_A + \left( \begin{array}{cc} B_2 & 0 \\ 0 & B_3 \end{array} \right) \partial_R \hat{\psi}_A \right] \right\} 
- 2 \Re \left\{ \left( \frac{\partial R \hat{\psi}_0}{\partial R \hat{\psi}_1} \right) \left[ -D_{t,\omega} \hat{\psi}_A + \left( \begin{array}{cc} B_4 & 0 \\ 0 & 0 \end{array} \right) \partial_{t,\omega} \hat{\psi}_A + \left( \begin{array}{cc} B_5 & B_6 \\ B_7 & B_8 \end{array} \right) \hat{\psi}_A \right] \right\} 
- 2 \Re \left\{ \left( B_3 \partial_R \hat{\psi}_1 - \bar{\delta}' \hat{\psi}_0 + B_7 \hat{\psi}_0 + B_8 \hat{\psi}_1 \right) \partial_R \hat{\psi}_0 \right\}.
$$

(3.46)

The equation (3.44) is obtained by commuting $\hat{D}_{t,\omega}$ into the rescaled Weyl’s equation, the conservation law is

$$
\hat{\nabla}^{AA'} \left[ \left( \hat{D}_{t,\omega} \hat{\psi}_A \right) \left( \hat{D}_{t,\omega} \hat{\psi}_{A'} \right) \right] 
\simeq -2 \Re \left\{ \left( \bar{\delta} \hat{\psi}_0 \bar{\delta} \hat{\psi}_1 \right) \left[ \left( \begin{array}{cc} A_0 & A_1 \\ A_2 & A_3 \end{array} \right) D_{t,\omega} \hat{\psi}_A' + \left( \begin{array}{cc} D_1 & 0 \\ 0 & 0 \end{array} \right) \partial_{t,\omega} \hat{\psi}_A \right] \right\} 
- 2 \Re \left\{ \left( \bar{\delta} \hat{\psi}_0 \bar{\delta} \hat{\psi}_1 \right) \left[ \left( \begin{array}{cc} D_2 & 0 \\ 0 & D_3 \end{array} \right) \partial_R \psi_A' - M_R D_{t,\omega} \hat{\psi}_A' + \left( \begin{array}{cc} D_4 & 0 \\ 0 & 0 \end{array} \right) \partial_{t,\omega} \hat{\psi}_A \right] \right\} 
- 2 \Re \left\{ \left( \bar{\delta} \hat{\psi}_0 \bar{\delta} \hat{\psi}_1 \right) \left( D_5 \ D_6 \ D_7 \ D_8 \right) \hat{\psi}_A \right\} 
\simeq -2 \Re \left\{ \left( A_0 \bar{\delta}' \hat{\psi}_0 + A_1 \bar{\delta} \hat{\psi}_1 + D_1 \partial_{t,\omega} \hat{\psi}_0 + D_2 \partial_R \hat{\psi}_0 - \frac{R^2}{\sqrt{2}} \bar{\delta}' \hat{\psi}_0 
+ D_4 \partial_{t,\omega} \hat{\psi}_0 + D_5 \hat{\psi}_0 + D_6 \hat{\psi}_1 \right) \bar{\delta}' \hat{\psi}_0 \right\} 
- 2 \Re \left\{ \left( A_2 \bar{\delta}' \hat{\psi}_0 + A_3 \bar{\delta} \hat{\psi}_1 + D_3 \partial_R \hat{\psi}_1 - \frac{1}{\sqrt{2}} \bar{\delta} \hat{\psi}_1 + D_7 \hat{\psi}_0 + D_8 \hat{\psi}_1 \right) \bar{\delta} \hat{\psi}_1 \right\}.
$$

(3.47)

3.4.2 Energy estimates and peeling

Now we consider the solution $\hat{\psi}_A \in C^\infty(\bar{B}_I, S_A)$ of the Weyl’s equation with compactly supported initial data. Integrating on $\Omega_{t_0}^{s_1,s_2}$ by the conservation laws (3.38) and (3.45) we obtain

$$
E_{H_{s_1}} (\hat{\psi}) + E_{S_{s_1}^{s_2}} (\hat{\psi}) = E_{H_{s_2}} (\hat{\psi}) \text{ for any } 0 \leq s_1 < s_2 \leq 1,
$$

(3.48)

$$
E_{H_{s_1}} (\hat{D}_{t,\omega} \hat{\psi}) + E_{S_{s_1}^{s_2}} (\hat{D}_{t,\omega} \hat{\psi}) = E_{H_{s_2}} (\hat{D}_{t,\omega} \hat{\psi}) \text{ for any } 0 \leq s_1 < s_2 \leq 1,
$$

(3.49)

and

$$
E_{H_{s_1}} (\hat{D}_{t,\omega} \hat{\psi}) + E_{S_{s_1}^{s_2}} (\hat{D}_{t,\omega} \hat{\psi}) = E_{H_{s_2}} (\hat{D}_{t,\omega} \hat{\psi}) \text{ for any } 0 \leq s_1 < s_2 \leq 1.
$$

(3.50)
For $\mathcal{D}_R \hat{\psi}$, integrating the conservation law (3.46) on $\Omega_{s_{1, s_{2}}^t}$, we get

$$\left| \mathcal{E}_{\mathcal{H}_{s_{1}}}(\mathcal{D}_R \hat{\psi}) + \mathcal{E}_{\mathcal{S}_{s_{1, s_{2}}^t}}(\mathcal{D}_R \hat{\psi}) - \mathcal{E}_{\mathcal{H}_{s_{2}}}(\mathcal{D}_R \hat{\psi}) \right| =$$

$$\leq \int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} 2 \left| B_1 \partial_t \hat{\psi}_0 + B_2 \partial_R \hat{\psi}_0 - \tilde{\partial} \hat{\psi}_1 + B_4 \partial_r \hat{\psi}_0 + B_5 \hat{\psi}_0 + B_6 \hat{\psi}_1 \right| \left| \partial_R \hat{\psi}_0 \right| \frac{1}{|*t|} d^* t d\omega^2 d s$$

$$+ \int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} 2 \left| B_3 \partial_R \hat{\psi}_1 - \tilde{\partial}' \hat{\psi}_0 + B_7 \hat{\psi}_0 + B_8 \hat{\psi}_1 \right| \left| \partial_R \hat{\psi}_1 \right| \frac{1}{|*t|} d^* t d\omega^2 d s . \quad (3.51)$$

Last for $\mathcal{D}_{t, \omega} \hat{\psi}$, integrating the conservation law (3.47) on $\Omega_{s_{1, s_{2}}^t}$, we get

$$\left| \mathcal{E}_{\mathcal{H}_{s_{1}}}(\mathcal{D}_{t, \omega} \hat{\psi}) + \mathcal{E}_{\mathcal{S}_{s_{1, s_{2}}^t}}(\mathcal{D}_{t, \omega} \hat{\psi}) - \mathcal{E}_{\mathcal{H}_{s_{2}}}(\mathcal{D}_{t, \omega} \hat{\psi}) \right| =$$

$$\leq \int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} 2 \left| A_0 \tilde{\partial}' \hat{\psi}_0 + A_1 \tilde{\partial} \hat{\psi}_1 + D_1 \partial_t \hat{\psi}_0 + D_2 \partial_R \hat{\psi}_0 - \frac{R^2}{\sqrt{2}} \tilde{\partial}' \hat{\psi}_0 
+ D_4 \partial_r \hat{\psi}_0 + D_5 \hat{\psi}_0 + D_6 \hat{\psi}_1 \right| \left| \tilde{\partial}' \hat{\psi}_0 \right| \frac{1}{|*t|} d^* t d\omega^2 d s$$

$$+ \int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} 2 \left| A_2 \tilde{\partial}' \hat{\psi}_0 + A_3 \tilde{\partial} \hat{\psi}_1 + D_3 \partial_R \hat{\psi}_1 - \frac{1}{\sqrt{2}} \tilde{\partial} \hat{\psi}_1 + D_7 \hat{\psi}_0 + D_8 \hat{\psi}_1 \right| \left| \tilde{\partial} \hat{\psi}_1 \right| \frac{1}{|*t|} d^* t d\omega^2 d s . \quad (3.52)$$

Now we will control the right-hand sides of the inequalities (3.51), (3.52) by the energies of $\mathcal{D}_R \hat{\psi}$, $\mathcal{D}_{t, \omega} \hat{\psi}$, $\mathcal{D}_{t, \omega} \hat{\psi}$ and $\hat{\psi}$ on the hypersurface $\mathcal{H}_{s}$. To do this, we need the following lemma to control the energies of $\partial_t \hat{\psi}_0$, $\tilde{\partial}' \hat{\psi}_0$ and $\hat{\psi}_0$ on $\mathcal{H}_{s}$:

**Lemma 3.4.1.** If we put

$$||\partial_t \hat{\psi}_0||^2 = \int_{\mathcal{H}_{s}} \left| \partial_t \hat{\psi}_0 \right|^2 d^* t d\omega^2 , \quad ||\tilde{\partial}' \hat{\psi}_0||^2 = \int_{\mathcal{H}_{s}} \left| \tilde{\partial}' \hat{\psi}_0 \right|^2 d^* t d\omega^2$$

and

$$||\hat{\psi}_0||^2 = \int_{\mathcal{H}_{s}} \left| \hat{\psi}_0 \right|^2 d^* t d\omega^2 ,$$

then the first and third norms can be controlled uniformly in $s$ by the energies of $\mathcal{D}_{t, \omega} \hat{\psi}$, $\mathcal{D}_{t, \omega} \hat{\psi}$, $\mathcal{D}_R \hat{\psi}$ and $\hat{\psi}$ on $\mathcal{H}_{s}$; and the second norm can be controlled uniformly in $s$ by the energies of $\mathcal{D}_R \hat{\psi}$ and $\hat{\psi}$ on $\mathcal{H}_{s}$.

The proof of this lemma will be given in Appendix [3.7.2]. Returning to our control, since $B_2$ is of order one in $R$, the term containing the coefficient $B_2$ can be controlled as follows

$$\int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} B_2 \left| \partial_R \hat{\psi}_0 \right| \left| \partial_R \hat{\psi}_0 \right| \frac{1}{|*t|} d^* t d\omega^2 d s$$

$$\leq \int_{s_{1}}^{s_{2}} \int_{\mathcal{H}_{s}} \left| \partial_R \hat{\psi}_0 \right|^2 R \frac{1}{|*t|} d^* t d\omega^2 d s$$

$$\leq \int_{s_{1}}^{s_{2}} \mathcal{E}_{\mathcal{H}_{s}}(\mathcal{D}_R \hat{\psi}) d s$$

$$\leq \int_{s_{1}}^{s_{2}} \frac{1}{\sqrt{s}} \mathcal{E}_{\mathcal{H}_{s}}(\mathcal{D}_R \hat{\psi}) d s .$$
The terms containing the coefficients $B_1$, $B_4$, $B_5$, $B_6$, $B_7$ and $D_2$ and the term $-\partial \psi_1 \partial_R \psi_0$ can be controlled in the same manner, due to the fact that these coefficients are of order nonnegative in $R$ and to the equivalence

$$\frac{1}{|*t|} \simeq \frac{1}{\sqrt{s}} \sqrt{R} |*t|,$$

when $R$ is small enough. For instance we control

$$\int_{s_1}^{s_2} \int_{\mathcal{H}_s} B_1 \left| \partial_t \hat{\psi}_0 \right| \left| \partial_R \hat{\psi}_0 \right| \frac{1}{|*t|} d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \int_{\mathcal{H}_s} \left| \partial_t \hat{\psi}_0 \right| \left| \partial_R \hat{\psi}_0 \right| \frac{1}{\sqrt{s}} \sqrt{R} |*t| d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{\mathcal{H}_s} \left| \partial_t \hat{\psi}_0 \right| \left| \partial_R \hat{\psi}_0 \right| \sqrt{R} |*t| d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{\mathcal{H}_s} \left( \left| \partial_t \hat{\psi}_0 \right|^2 + \frac{R}{|*t|} \left| \partial_R \hat{\psi}_0 \right|^2 \right) d^* t d\omega^2 \frac{d}{ds}$$

using Lemma \[3.4.1\] we can control this latter as follows

$$\int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{t,\omega}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{\tau}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_R) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \right) \frac{d}{ds}.$$

The resting terms can be controlled more simply since the coefficients are of order nonnegative in $R$ and due to Lemma \[3.4.1\] for instance we control the term containing the coefficient $D_5$ as follows

$$\int_{s_1}^{s_2} \int_{\mathcal{H}_s} D_5 \left| \hat{\psi}_0 \right| \left| \bar{\partial} \hat{\psi}_0 \right| \frac{1}{|*t|} d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{\mathcal{H}_s} \left( \left| \hat{\psi}_0 \right|^2 + \left| \bar{\partial} \hat{\psi}_0 \right|^2 \right) d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{t,\omega}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{\tau}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_R) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \right) \frac{d}{ds},$$

and the term having the coefficient $D_7$ as follows

$$\int_{s_1}^{s_2} \int_{\mathcal{H}_s} D_7 \left| \hat{\psi}_0 \right| \left| \bar{\partial} \hat{\psi}_1 \right| \frac{1}{|*t|} d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{\mathcal{H}_s} \left( \left| \hat{\psi}_0 \right|^2 + \left| \bar{\partial} \hat{\psi}_1 \right|^2 \right) d^* t d\omega^2 \frac{d}{ds}$$

$$\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{t,\omega}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_{\tau}) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}_R) + \mathcal{E}_{\mathcal{H}_s}(\hat{\psi}) \right) \frac{d}{ds}.$$
So that the right-hand side of the inequalities \[3.51\], \[3.52\] is controlled by the same quantity
\[
\int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \mathcal{E}_{\mathcal{H}_s}(\mathcal{D}_{t,\omega} \dot{\psi}) + \mathcal{E}_{\mathcal{H}_s}(\mathcal{D}_{t,\varphi} \dot{\psi}) + \mathcal{E}_{\mathcal{H}_s}(\mathcal{D}_R \dot{\psi}) + \mathcal{E}_{\mathcal{H}_s}(\dot{\psi}) \right) \, ds.
\]

The same holds if we consider the high-order of the partial derivatives\footnote{This is done more clearly by using the covariant approach, see formula \[3.60\].}. So if we take the sum
\[
\sum_{l+p+q=0}^k \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right)
\]
and then use the Gronwall’s inequality we get the following theorem:

**Theorem 3.4.1.** For any smooth compactly supported data on \( \mathcal{H}_1 \), the associated solution \( \dot{\psi}_A \) of Dirac’s system satisfies
\[
\sum_{l+p+q=0}^k \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right) \lesssim \sum_{l+p+q=0}^k \mathcal{E}_{\mathcal{H}_1} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right) + \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right).
\]

Note that, by the similar way if we use the derivative operator \( \mathcal{D}_t \) on 2–sphere instead of \( \mathcal{D}_{t,\omega} \), we can also obtain the energy estimates, but our controls need to add the time derivative \( \mathcal{D}_{t,\omega} \) as the following theorem

**Theorem 3.4.2.** For any smooth compactly supported data on \( \mathcal{H}_1 \), the associated solution \( \dot{\psi}_A \) of Dirac’s system satisfies
\[
\sum_{l+m+p+q=0}^k \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right) \lesssim \sum_{l+m+p+q=0}^k \mathcal{E}_{\mathcal{H}_1} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right) + \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\omega}^l \mathcal{D}_t^p \mathcal{D}_R^q \dot{\psi}_A \right).
\]

Now we define the peeling of the Dirac field at order \( k \in \mathbb{N}^* \) as follows:

**Definition 3.4.1.** We say that a solution \( \psi \) of the Dirac’s equation peels at order \( k \in \mathbb{N} \) if
\[
\sum_{l+p+q=0}^k \mathcal{E}_{\mathcal{S}_{s_0}} \left( \mathcal{D}_{t,\varphi}^l \mathcal{D}_R^p \mathcal{D}_R^q \dot{\psi}_A \right) < +\infty.
\]
Due to Theorem 3.4.1, we can find the optimal initial data space that guarantees the peeling at the order $k$:

\textbf{Theorem 3.4.3.} The initial data space $\mathfrak{h}^k(\mathcal{H}_1)$ that guarantees the definition of the peeling at the order $k$ is the completion of $C_0^\infty([-\ast t_0, +\infty[\times S^2_\omega)$ on the norm

$$||\hat{\psi}||_{\mathfrak{h}_k^1(\mathcal{H}_1)} := \left( \sum_{l+p+q=0}^k \mathcal{E}_{\mathcal{H}_1}(\partial_{\ast t,\omega}^l \partial_{\ast \varphi}^p \partial_{R}^q \hat{\psi}_A) \right)^{1/2}.$$ 

\section{3.5 Approach using covariant derivatives}

In this section, we will approach to the peeling problem by using the covariant derivatives. The covariant derivatives take us to obtain the conservation laws with the calculations that are more brief than the approach where we use the partial derivatives. We will use the vector fields in $X_i \in \mathcal{A}$ as follows:

$$X_0 = \partial_{\ast t} , X_1 = \partial_{\ast \varphi} , X_2 = \sin^\ast \varphi \partial_\theta + \cot \theta \cos^\ast \varphi \partial_{\ast \varphi} ,$$

$$X_3 = \cos^\ast \varphi \partial_\theta - \cot \theta \sin^\ast \varphi \partial_{\ast \varphi} , X_4 = \partial_R ,$$

where $X_1, X_2$ and $X_3$ are vectors tangent to 2-sphere, on Kerr spacetime $X_1$ is Killing but $X_2$ and $X_3$ are not Killing.

\subsection*{3.5.1 Curvature spinors}

Given a spacetime $(\mathcal{M}, g)$ with a spin structure and equipped with the Levi-Civitta connection, we recall that the Riemann tensor $R_{abcd}$ can be decomposed as follows (see eq.(4.6.1) pp.231 in R. Penrose and W. Rindler [73] Vol1):

$$R_{abcd} = X_{ABCD} \varepsilon^A_{B'} \varepsilon^C_{D'} + \Phi_{AB'C'D'} \varepsilon^A_{B'} \varepsilon^C_{D'} + \tilde{X}_{AB'C'D'} \varepsilon_{AB} \varepsilon_{CD} ,$$

where $X_{ABCD}$ is a complete contraction of the Riemann tensor in its primed spinor indices

$$X_{ABCD} = \frac{1}{4} R_{abcd} \varepsilon^A_{B'} \varepsilon^C_{D'} ,$$

and $\Phi_{ab} = \Phi_{(ab)}$ is the trace-free part of the Ricci tensor multiplied by $-1/2$:

$$2\Phi_{ab} = 6\Lambda g_{ab} - R_{ab} , \quad \Lambda = \frac{1}{24} \text{Scal}_g .$$

We set

$$P_{ab} = \Phi_{ab} - \Lambda g_{ab} ,$$

$$X_{ABCD} = \Psi_{ABCD} + \Lambda (\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC}) , \quad \Psi_{ABCD} = X_{(ABCD)} = X_{A(BCD)} .$$
Under a conformal rescaling \( \hat{g} = \Omega^2 g \) we have (see R. Penrose and W. Rindler [73], Vol 2)

\[
\hat{\Psi}_{ABCD} = \Psi_{ABCD}, \quad \hat{\Lambda} = \Omega^{-2} \Lambda + \frac{1}{4} \Omega^{-3} \Box \Omega, \quad \Box = \nabla^a \nabla_a ,
\]

\[
\hat{P}_{ab} = P_{ab} - \nabla_b \Upsilon_a + \Upsilon_{AB'} \Upsilon_{BA'}, \quad \text{with} \quad \Upsilon_a = \Omega^{-1} \nabla_a \Omega = \nabla_a \log \Omega.
\]

**Lemma 3.5.1.** For the rescaled star-Kerr metric we have

\[
\hat{\Lambda} = \frac{Mr - a^2}{2 \rho^2}.
\]

And in the expression of \( \hat{\Phi}_{ab} \mathrm{d}x^a \mathrm{d}x^b \), the coefficients \( \hat{\Phi}_{ab} \) are bounded, moreover the coefficients \( \hat{\Phi}_{a3} \) are of order one in \( \sin^2 \theta \).

**Proof.** Since the Kerr metric is Ricci flat hence \( \Phi_{ab} = P_{ab} = R_{ab} = \Lambda = 0 \). First, to find \( \hat{\Lambda} \), we use the expression (1.3) of the rescaled scalar curvature and the expression (2.8) of the inverse metric \( g^{-1} \) to get

\[
\hat{\Lambda} = \frac{1}{4} R^{-3} \Box g = \frac{1}{4} \frac{r^3}{\sqrt{|g|}} \partial_r \left( \sqrt{|g|} g^{rr} \partial_r \frac{1}{r} \right) = \frac{Mr - a^2}{2 \rho^2}.
\]

Second, to check that \( \hat{\Phi}_{ab} \) are bounded, we have to check the boundedness in \( R \) (i.e. there is no spatial angular singularity \( r \) which appears after using the conformal mapping) and the boundedness in \( \sin^2 \theta \) (i.e. there is no angular singularity which appears in the coefficient of \( \partial^2 \phi \) in the expression of \( \hat{g}^{-1} \)). First of all we have

\[
\hat{\Phi}_{ab} = \hat{P}_{ab} + \hat{\Lambda} \hat{g}_{ab} = \Upsilon_{AB'} \Upsilon_{BA'} - \nabla_b \Upsilon_a + \frac{Mr - a^2}{2 \rho^2} \hat{g}.
\]

Due to \( (Mr - a^2)/\rho^2 \approx R/2 \) and the boundedness of the coefficients of the rescaled metric \( \hat{g} \), we need to consider only the boundedness of the coefficients in the first two terms in the above expression of \( \hat{\Phi}_{ab} \). Since \( \Omega = R = 1/r \), we have

\[
\Upsilon_a \mathrm{d}x^a = -\frac{\mathrm{d}r}{r}.
\]

We need to determine its spinor components. We do this in the dyad \{\( o^A, \nu^A \)\}, denoting \( x^0 = t \), \( x^1 = r \), \( x^2 = \theta \), \( x^3 = \varphi \). We have:

\[
\Upsilon_{AA'} = \frac{-1}{r} g^1_{AA'} = -\frac{1}{r} g^{11} \varepsilon_{AB} \varepsilon_{A'B'} g_{BB'} = \frac{\Delta}{r \rho^2 \varepsilon_{AB} \varepsilon_{A'B'} g_{BB'}}
\]

and

\[
g_{BB'}^1 = \begin{pmatrix} n_1 & -m_1 \\ -m_1 & l_1 \end{pmatrix} = \sqrt{\frac{\rho^2}{2 \Delta}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It follows that

\[
\Upsilon_{AA'} = -\frac{1}{r} \sqrt{\frac{\Delta}{2 \rho^2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
The non zero components of \( \alpha_{ab} := \Upsilon_{AB} \Upsilon_{BA'} \) are therefore
\[
\alpha_{00'00'} = \Upsilon_{00'} \Upsilon_{00'} = \alpha_{11'11'} = \Upsilon_{11'} \Upsilon_{11'} = \frac{\Delta}{2r^2 \rho^2},
\]
\[
\alpha_{01'10'} = \Upsilon_{00'} \Upsilon_{11'} = \alpha_{10'01'} = \Upsilon_{11'} \Upsilon_{00'} = -\frac{\Delta}{2r^2 \rho^2}.
\]
So that we have
\[
\Upsilon_{AB} \Upsilon_{BA'} dx^a dx^b = \frac{\Delta}{2r^2 \rho^2} (l_a l_b + n_a n_b - m_a \tilde{m}_b - \tilde{m}_a m_b) dx^a dx^b
\]
\[
= \frac{\Delta}{2\rho^2} (R^2 \dot{i}_a \dot{b} + R^{-2} \dot{n}_a \dot{n}_b - \dot{m}_a \tilde{m}_b - \tilde{m}_a m_b) dx^a dx^b.
\]
(3.54)

From the expression of the Newman Penrose dual tetrad \((3.10), (3.11), (3.12)\), we have
\[
R^2 \dot{i}_a \dot{b} + R^{-2} \dot{n}_a \dot{n}_b = \frac{\Delta}{\rho^2} \left( R^2 d^a t^2 + \frac{\rho^4 + (r^2 + a^2)^2}{\Delta^2 R^2} dR^2 + a^2 R^2 \sin^2 \theta d^* \varphi^2 \right)
\]
\[
+ \frac{\Delta}{\rho^2} \left( -2\frac{\rho^2}{\Delta} d^* t dR - 2aR^2 \sin^2 \theta d^* t d^* \varphi + \frac{2a \sin^2 \theta (r^2 + a^2)}{\Delta} dR d^* \varphi \right)
\]
and
\[
\dot{m}_a \tilde{m}_b + \tilde{m}_a m_b
\]
\[
= \frac{1}{\rho^2} \left( a^2 R^2 \sin^2 \theta d^* t^2 + \frac{a^2 \sin^2 \theta (r^2 + a^2)^2}{\Delta^2 R^2} dR^2 + R^2 \rho^4 d\theta^2 + R^2 (r^2 + a^2)^2 \sin^2 \theta d^* \varphi^2 \right)
\]
\[
+ \frac{1}{\rho^2} \left( -2a^2 \sin^2 \theta (r^2 + a^2) \frac{\Delta}{\Delta} d^* t dR - 2aR^2 (r^2 + a^2) \sin^2 \theta d^* t d^* \varphi
\]
\[
+ \frac{2a \sin^2 \theta (r^2 + a^2)^2}{\Delta} dR d^* \varphi \right) .
\]
(3.56)

We can think that in the above expressions, the coefficients are bounded except if the coefficient of \( dR^2 \) contains the factor \( 1/R^2 \), but it will vanish when we calculate the combination \((\Upsilon_{AB} \Upsilon_{BA'} - \nabla_b \Upsilon_a) dx^a dx^b\). Indeed
\[
\nabla_b \Upsilon_a dx^a dx^b = \nabla_b \left( -\frac{dr}{r} \right) dx^b = \frac{dr^2}{r^2} + \frac{1}{r} \Gamma^1_{ab} dx^a dx^b = \frac{dR^2}{R^2} + \frac{1}{r} \Gamma^1_{ab} dx^a dx^b.
\]
(3.57)

Using the equations \((3.54), (3.55), (3.57)\) the coefficient of \( dR^2 \) in \( \dot{\Phi}_{ab} \) is
\[
\frac{\Delta}{2\rho^2} \frac{\Delta}{\rho^2} \frac{\rho^4 + (r^2 + a^2)^2}{\Delta^2 R^2} - \frac{1}{R^2} = a^4 \sin^4 \theta + 2a^2 \sin^2 \theta \frac{2r^2 a^2}{2\rho^4 R^2} \approx a^2 \sin^2 \theta \text{ as } r \to +\infty
\]
so that it is bounded.

Now we need to consider the component \( \frac{1}{r} \Gamma^1_{ab} dx^a dx^b \) that appears in the expression of \( \nabla_b \Upsilon_a dx^a dx^b \).

Among the Christoffel symbols
\[
\Gamma^1_{ab} = \frac{1}{2} g^{1c} \left( \partial g_{ac} \partial x^b - \partial g_{bc} \partial x^a \partial x^c \right) = -\frac{\Delta}{2\rho^2} \left( \partial g_{a1} \partial x^b + \partial g_{b1} \partial x^a - \partial g_{ab} \partial x^r \right),
\]
for $r$ is large enough we have

$$\Gamma^1_{00} = \frac{\Delta}{2 \rho^2} \left( \frac{\rho^2 - 2Mr}{\rho^2} \right)^{\prime} \approx \frac{M}{r^2}, \quad \Gamma^1_{03} = \frac{aM \sin^2 \theta \Delta}{\rho^2} \left( \frac{r}{\rho^2} \right)^{\prime} \approx \frac{aM \sin^2 \theta}{r^2},$$

$$\Gamma^1_{11} = -\frac{\rho^2}{2\Delta} \left( \frac{\Delta}{\rho^2} \right)^{\prime} \approx -\frac{M}{r^2}, \quad \Gamma^1_{12} = \frac{\Delta}{2\rho^2} \left( \frac{\rho^2}{\Delta} \right)^{\prime} \approx \frac{a^2 \sin 2\theta}{2r^4},$$

$$\Gamma^1_{22} = -\frac{\Delta r}{\rho^2} \approx -r, \quad \Gamma^1_{33} = -\frac{\Delta \sin^2 \theta}{2\rho^2} \left( \frac{\sigma^2}{\rho^2} \right)^{\prime} \approx -\sin^2 \theta r.$$ 

and

$$dt = d^* t + \frac{r^2 + a^2}{\Delta} dr = d^* t - \frac{r^2 + a^2}{R^2 \Delta} dR.$$ 

Therefore, we see that all the coefficients of $\nabla_b \gamma_a dx^a dx^b$ in star-Kerr coordinates ($* t, R, \theta, * \varphi$) are bounded except maybe the coefficient of $d^* \tau dR$ which involves $\Gamma^1_{00} dt^2$ and $\Gamma^1_{11} dr^2$ and have unbounded coefficients in the star-Kerr coordinates. However they appear in $\nabla_b \gamma_a dx^a dx^b$ in a combination that cancels unbounded contributions for $r$ is large enough:

$$\frac{1}{r} \left( \frac{\Delta}{2 \rho^2} \left( \frac{\rho^2 - 2Mr}{\rho^2} \right)^{\prime} \right)_r \left( d^* t + \frac{r^2 + a^2}{\Delta} dr \right)^2 - \frac{\rho^2}{2\Delta} \left( \frac{\Delta}{\rho^2} \right)^{\prime}_r dr^2$$

$$= \frac{1}{r} \left( \frac{\Delta}{2 \rho^2} \left( \frac{\Delta}{\rho^2} - \frac{a^2 \sin^2 \theta}{\rho^2} \right)^{\prime} \right)_r \left( d^* t + \frac{\rho^2 + a^2 \sin^2 \theta}{\Delta} dr \right)^2 - \frac{\rho^2}{2\Delta} \left( \frac{\Delta}{\rho^2} \right)^{\prime}_r dr^2$$

$$\approx \frac{1}{r} \left( \frac{\Delta}{2 \rho^2} \left( \frac{\Delta}{\rho^2} \right)^{\prime} \right)_r \left( d^* t + \frac{\rho^2}{\Delta} dr \right)^2 - \frac{\rho^2}{2\Delta} \left( \frac{\Delta}{\rho^2} \right)^{\prime}_r dr^2$$

$$= \frac{1}{r} \left( \frac{\Delta}{2 \rho^2} \left( \frac{\Delta}{\rho^2} \right)^{\prime} \right)_r \left( d^* t^2 + 2\frac{\rho^2}{\Delta} d^* t dr \right)$$

$$\approx \frac{1}{r} \left( \frac{\Delta}{2 \rho^2} \left( \frac{\Delta}{\rho^2} \right)^{\prime} \right)_r \left( d^* t^2 - 2\frac{\rho^2}{R^2 \Delta} d^* tdR \right).$$

From this equivalence we can see that the coefficient of $dR d^* t$ is bounded.

Finally from all the calculations above we can also see clearly that the coefficients $\hat{\Phi}_{a3}$ are of order one in $\sin^2 \theta$. 

\[
\square
\]

### 3.5.2 Commutation of covariant derivatives

Commuting the covariant derivative $\hat{\nabla}_V$ along a vector field $V^a$ into the Weyl’s equation, we have:

$$0 = V^a \hat{\nabla}_a \hat{\nabla}^{BB'} \hat{\psi}_B = -V^a \hat{\nabla}^B \varepsilon^{BC'} \hat{\nabla}^C \psi^C$$

$$= -\varepsilon^{BC'} (V^a \Delta_{ac} \psi^C + \hat{\nabla}_C \hat{\psi}_B) - (\hat{\nabla}_c V^a) \hat{\nabla}_a \psi^C)$$

$$= -\varepsilon^{BC'} V^a \Delta_{ac} \psi^C + \hat{\nabla}^{BB'} (\hat{\nabla}_V \psi_B) - (\hat{\nabla}^b V^a) \hat{\nabla}_a \psi_B,$$
where $\Delta_{ab} = \nabla_a \nabla_b - \nabla_b \nabla_a$. The first and third terms of the right-hand side can be calculated more explicitly

$$V^a \dot{\Delta}_{ac} \hat{\psi}^C = V^a \left[ \dot{\varepsilon}_{A'C'} \hat{X}_{ACE}^C + \dot{\varepsilon}_{C'A'} \hat{\Phi}_{A'C'}^C \right] \hat{\psi}^E.$$  

The symmetries of the Riemann tensor imply that

$$\dot{X}_{ACE}^C = 3 \dot{\varepsilon}_{AE} = 3 \frac{Mr - a^2}{2\rho^2} \dot{\varepsilon}_{AE},$$

whence

$$-\varepsilon^B'C' V^a \dot{\varepsilon}_{AC'} \hat{X}_{ACE}^C \hat{\psi}^E = \varepsilon^B'C' \frac{3}{2\rho^2} \dot{\varepsilon}_{AE} = -V_{C'A'} \dot{\Phi}^{EB'C'A'} \hat{\psi}^E = \frac{3}{2\rho^2} V_{AB'} \hat{\psi}^E. $$

The term involving $\dot{\Phi}_{ab}$ can be written

$$-\varepsilon^B'C' V^a \dot{\varepsilon}_{AB} \hat{\Phi}_{AC'}^C \hat{\psi}^E = -V_{C'A'} \dot{\Phi}^{EB'C'A'} \hat{\psi}^E = -V_a \dot{\Phi}^{BB'} \hat{\psi}_B.$$

It follows that the equation satisfied by $\hat{\nabla}_V \hat{\psi}_B$ is

$$\hat{\nabla}^{BB'} \left( \hat{\nabla}_V \hat{\psi}_B \right) = \left( \hat{\nabla}^b V^a \right) \hat{\nabla}_a \hat{\psi}_B + V_a \dot{\Phi}^{bb'} \hat{\psi}_B - \frac{3}{2\rho^2} V^{bb'} \hat{\psi}_B. \quad (3.58)$$

In the expression above, we see that it includes two components: the first one $\left( \hat{\nabla}^b V^a \right) \hat{\nabla}_a \hat{\psi}_B$ is of order one in the covariant derivative and the second one $V_a \dot{\Phi}^{bb'} \hat{\psi}_B - \frac{3}{\rho^2} V_b \hat{\psi}_B$ is of zero order in the covariant derivative. For the first component, the following lemma gives its performances (its proof can be found in Appendix [3.7.3])

**Lemma 3.5.2.** For a vector field $X^a_i \in A = \{X_0 = \partial_t, X_1, X_2, X_3, X_4 = \partial_R\}$ the component is of order one in the covariant derivative i.e $(\hat{\nabla}^b X^a_i) \hat{\nabla}_a \hat{\psi}_B$ that can be expressed as a linear combination of $\hat{\nabla}_V \hat{\psi}_B (X_i \in A)$ with the coefficients that are bounded.

For the second component, for $X^a_i \in A = \{X_0 = \partial_R, X_1, X_2, X_3, X_4 = \partial_t\}$, we have

$$X_{ia} \dot{\Phi}^{bb'} \hat{\psi}_B - \frac{3}{\rho^2} X^{bb'}_i \hat{\psi}_B = X_{ia} \dot{\Phi}^{bb'} \hat{\psi}_B - \frac{3}{\rho^2} X^{bb'}_i \hat{\psi}_B = X_{ia} \dot{\Phi}^{bb'} \hat{\psi}_B - \frac{3}{\rho^2} X^{bb'}_i \hat{\psi}_B. \quad (3.64)$$

From Lemma [3.5.1] and the expressions [(2.0), (2.8)] of $\hat{g}$ and $\hat{g}^{-1}$, we can see that the coefficients $X^a_i \dot{\Phi}^{bb'} \hat{\phi}_a$ are bounded (the angular singularity $1/\sin^2 \theta$ of $\hat{g}^{bb'}$ will disappear since it go along with $\dot{\Phi}_{a3}$ that are of order one in $\sin^2 \theta$), except the one with the angular singularity $\cot \theta \hat{g}^{33} \dot{\Phi}_{33}$, but it is not a problem since it go along with $(\partial_{\varphi})^b$ which is of order one in $\sin \theta$ (we can see in the matrix expression (3.64) of $(\partial_{\varphi})^a$ below). The boundedness of $\frac{Mr - a^2}{\rho^2}$ is clear, since it is equivalent to $MR$ as $r$ is large enough.
The spinor $\hat{\nabla}_X \psi_A$ thus can be expressed as follows

$$\hat{\nabla}^{AA'} \left( \hat{\nabla}_X \phi_A \right) = \left[ \hat{\nabla}^{AA'}, \hat{\nabla}_X \right] \psi_A = \sum_{j=0}^{4} T_j (\partial_a)^a \hat{\nabla}_X \phi_A + P_i (\partial_a)^a \psi_A - 3 \frac{Mr - a^2}{\rho^2} \left( X_i \right)^a \phi_A \tag{3.59}$$

where $T_j$ are bounded.

From this equality, the equation that is satisfied by the higher order radial derivatives is obtained by means of the commutator expansion:

$$\hat{\nabla}^{AA'} \left( \hat{\nabla}_X^k \phi_A \right) = \left[ \hat{\nabla}^{AA'}, \hat{\nabla}_X^k \right] \psi_A = \sum_{p=0}^{k-1} \hat{\nabla}_X^{k-p-1} \left[ \hat{\nabla}^{AA'}, \hat{\nabla}_X \right] \hat{\nabla}_X^p \phi_A$$

$$= \sum_{p=0}^{k-1} \hat{\nabla}_X^{k-p-1} \left( T_j (\partial_a)^a \hat{\nabla}_X \hat{\nabla}_X^p \phi_A \right) + \sum_{p=0}^{k-1} \hat{\nabla}_X^{k-p-1} \left( P_i (\partial_a)^a \phi_A - 3 \frac{Mr - a^2}{\rho^2} \left( X_i \right)^a \hat{\nabla}_X^p \phi_A \right). \tag{3.60}$$

Now we have the values of Infeld-van der Waerden matrix that are:

$$\hat{g}_{0}^{AA'} = \begin{pmatrix} \hat{n}_0 & -\hat{\tilde{m}}_0 \\ -\hat{\tilde{m}}_0 & \hat{\tilde{m}}_0 \end{pmatrix} = \begin{pmatrix} R^2 \sqrt{\frac{\Delta}{2\rho^2}} & \frac{\rho a \sin \theta}{\rho^2} \\ -\frac{p a \sin \theta}{\rho^2} & -\sqrt{\frac{\Delta}{2\rho^2}} \end{pmatrix}, \tag{3.61}$$

$$\hat{g}_{1}^{AA'} = \begin{pmatrix} \hat{n}_1 & -\hat{\tilde{m}}_1 \\ -\hat{\tilde{m}}_1 & \hat{\tilde{m}}_1 \end{pmatrix} = -\sqrt{\frac{2\rho^2}{\Delta}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.62}$$

$$\hat{g}_{2}^{AA'} = \begin{pmatrix} \hat{n}_2 & -\hat{\tilde{m}}_2 \\ -\hat{\tilde{m}}_2 & \hat{\tilde{m}}_2 \end{pmatrix} = \frac{\rho^2}{\rho^2 \sqrt{\frac{\Delta}{2\rho^2}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.63}$$

$$\hat{g}_{3}^{AA'} = \begin{pmatrix} \hat{n}_3 & -\hat{\tilde{m}}_3 \\ -\hat{\tilde{m}}_3 & \hat{\tilde{m}}_3 \end{pmatrix} = \frac{a \sin \theta}{r^2} \sqrt{\frac{\Delta}{2\rho^2}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho^2}{\rho^2 \sqrt{\frac{\Delta}{2\rho^2}}} \end{pmatrix}. \tag{3.64}$$

Therefore we can see that the components of the matrix $\hat{g}_i^{AA'} (i = 0, 1, 2, 3)$ are bounded and we thus set

$$\hat{g}_i^{AA'} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

where $a_i, b_i, c_i$ and $d_i$ are bounded. It follows that the action of the partial derivative along the coordinates on the field spinor can be expressed as follows

$$(\partial x^i)^a \phi_A = -a_i \hat{\Psi}_0 o^{A'} - b_i \hat{\Psi}_0 t^{A'} + c_i \hat{\Psi}_1 o^{A'} + d_i \hat{\Psi}_1 t^{A'} \tag{3.65}$$

with $x^i = 0, R, \theta, \varphi$.

For the vector fields $(V_2)^a, (V_3)^a$ which have the angular singularity $\cot \theta \partial_{\varphi}$, we can see the
boundedness of the coefficients in their expressions since \((\partial \cdot \varphi)^n\) is of order one in \(\sin \theta\). So we can also obtain the equalities similarly to (3.65).

Using equations (3.59) and (3.65), we can obtain the conservation law of \(\tilde{\nabla}_X \tilde{\psi}_A\) as follows:

**Lemma 3.5.3.** The conservation law at first order has the simple form as follows

\[
\tilde{\nabla}^{AA'} \left( (\tilde{\nabla}_X \tilde{\psi}_A)(\tilde{\nabla}_X \tilde{\psi}_{A'}) \right) =
\]

\[
= 2 \Re \left( A_1(\tilde{\nabla}_X \tilde{\psi}_0)(\tilde{\nabla}_X \tilde{\psi}_0) + A_2 \tilde{\psi}_0(\tilde{\nabla}_X \tilde{\psi}_1) + A_3(\tilde{\nabla}_X \tilde{\psi}_1)(\tilde{\nabla}_X \tilde{\psi}_1) + A_4 \tilde{\psi}_1(\tilde{\nabla}_X \tilde{\psi}_1) \right) +
\]

\[
+ 2 \Re \left( A_5 \tilde{\psi}_0(\tilde{\nabla}_X \tilde{\psi}_1) + A_6 \tilde{\psi}_1(\tilde{\nabla}_X \tilde{\psi}_0) + A_7(\tilde{\nabla}_X \tilde{\psi}_0)(\tilde{\nabla}_X \tilde{\psi}_1) + A_8(\tilde{\nabla}_X \tilde{\psi}_1)(\tilde{\nabla}_X \tilde{\psi}_0) \right)
\]

where \(A_i (i = 1, 2...8)\) are bounded, furthermore \(A_1\) is of order one in \(R\).

**Proof.** Due to (3.65), the coefficients \(A_i (i = 1, 2...8)\) are clearly bounded, we need only to prove that the coefficient \(A_1\) is of order one in \(R\). Indeed, using the expression (3.65) of \((\partial x^i)^n\), we can obtain the action of \((\partial x^i)^n\) on the product of two spin 1/2 field \(\tilde{\psi}_A\) and \(\tilde{\phi}_{A'}\) as following

\[
(\partial x^i)^n \tilde{\psi}_A \tilde{\phi}_{A'} = -a_i \tilde{\psi}_0 \tilde{\phi}_0 + b_i \tilde{\psi}_0 \tilde{\phi}_1 + c_i \tilde{\psi}_1 \tilde{\phi}_0 - d_i \tilde{\psi}_1 \tilde{\phi}_1.
\]

We are interested in the coefficient of the first term \(a_i\). If \(i = 0\) or \(i = 3\), we can see that in the matrix (3.61), (3.64): \(a_0\) and \(a_3\) are of order greater than or equal one in \(R\). If \(i = 2\), we can see that in the matrix (3.63): \(a_2 = 0\) so the first term disappears. Finally, the case \(i = 1\) is the trickiest but we will see that in the proof of Lemma 3.5.2 in Appendix 3.7.3, the coefficient of the terms that contain \(\partial_R \otimes X_i\) is of order one in \(R\). This completes our proof.

\[\square\]

**Remark 3.5.1.** Note that for the Killing vector fields \(X_1 = \partial_\varphi\) and \(X_4 = \partial_t\), the terms are product of two covariant components (for example \((\tilde{\nabla}_X \tilde{\psi}_0)(\tilde{\nabla}_X \tilde{\psi}_0)\), which will disappear.

### 3.5.3 Energy estimates and peeling

First, we give the basic inequality to control the energy of \(\tilde{\psi}_0\) on \(\mathcal{H}_s\) (its proof will be given in Appendix 3.7.4) as follows:

**Lemma 3.5.4.** We can control the energy of \(\tilde{\psi}_0\) on \(\mathcal{H}_s\) by the energies of the covariant derivatives along all the directions \(X_i \in \mathcal{A}\) and \(\tilde{\psi}\) on \(\mathcal{H}_s\) uniformly in \(s\). In more details, we have

\[
\int_{\mathcal{H}_s} |\tilde{\psi}_0|^2 d^2*td\omega \lesssim \sum_{i=0}^4 \mathcal{E}_{\mathcal{H}_s}(\tilde{\nabla}_X \tilde{\psi}) + \mathcal{E}_{\mathcal{H}_s}(\tilde{\psi}).
\]  

(3.66)
The difficulty is to control the term containing the coefficient $A_1$, which is of order one in $R$

$$
\int_{s_1}^{s_2} \int_{H_s} \left| A_1(\hat{\nabla}_X, \hat{\Psi})_0(\hat{\nabla}_X, \hat{\Psi})_0 \right| \frac{1}{|t|} d^* t d^2 \omega d s \lesssim \int_{s_1}^{s_2} \int_{H_s} \left| (\hat{\nabla}_X, \hat{\Psi})_0(\hat{\nabla}_X, \hat{\Psi})_0 \right| \frac{R}{|t|} d^* t d^2 \omega d s
$$

$$
\lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{H_s} \left( \left| \hat{\psi}_0 \right|^2 + \left| (\hat{\nabla}_X, \hat{\Psi})_0 \right|^2 \right) d^* t d^2 \omega d s
$$

The term containing the coefficient $A_1$, can be controlled since $A_1$ is of order one in $R$

$$
\int_{s_1}^{s_2} \int_{H_s} \left| A_1(\hat{\nabla}_X, \hat{\Psi})_0(\hat{\nabla}_X, \hat{\Psi})_0 \right| \frac{1}{|t|} d^* t d^2 \omega d s \lesssim \int_{s_1}^{s_2} \int_{H_s} \left| (\hat{\nabla}_X, \hat{\Psi})_0(\hat{\nabla}_X, \hat{\Psi})_0 \right| \frac{R}{|t|} d^* t d^2 \omega d s
$$

The terms containing the coefficients $A_3, A_4$, can be controlled similarly, for instance

$$
\int_{s_1}^{s_2} \int_{H_s} \left| A_3(\hat{\nabla}_X, \hat{\Psi})_1 \right| \frac{1}{|t|} d^* t d^2 \omega d s \lesssim \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \int_{H_s} \left( \left| \hat{\psi}_1 \right|^2 + \left| (\hat{\nabla}_X, \hat{\Psi})_1 \right|^2 \right) d^* t d^2 \omega d s
$$

The terms containing the coefficients $A_5, A_6, A_7, A_8$, can be controlled similarly, for instance

$$
\int_{s_1}^{s_2} \int_{H_s} \left| A_5(\hat{\nabla}_X, \hat{\Psi})_1 \right| \frac{1}{|t|} d^* t d^2 \omega d s \lesssim \int_{s_1}^{s_2} \int_{H_s} \left| \hat{\psi}_0(\hat{\nabla}_X, \hat{\Psi})_1 \right| \frac{1}{\sqrt{s}} \int_{H_s} \sqrt{R} \frac{1}{|t|} d^* t d^2 \omega d s
$$

The difficulty is to control the term containing the coefficient $A_2$, so we use Lemma 3.5.4

$$
\int_{s_1}^{s_2} \int_{H_s} \left| A_2(\hat{\nabla}_X, \hat{\Psi})_0 \right| \frac{1}{|t|} d^* t d^2 \omega d s \lesssim \int_{s_1}^{s_2} \int_{H_s} \left| \hat{\psi}_0(\hat{\nabla}_X, \hat{\Psi})_1 \right| \frac{1}{\sqrt{s}} \int_{H_s} \sqrt{R} \frac{1}{|t|} d^* t d^2 \omega d s
$$

$$
\lesssim \int_{s_1}^{s_2} \int_{H_s} \left( \left| \hat{\psi}_0 \right|^2 + \left| (\hat{\nabla}_X, \hat{\Psi})_0 \right|^2 \right) d^* t d^2 \omega d s
$$

$$
\lesssim \int_{s_1}^{s_2} \int_{H_s} \left( \sum_{j=0}^{4} \mathcal{E}_{H_s}(\hat{\nabla}_X, \hat{\psi}) + \mathcal{E}_{H_s}(\hat{\psi}) + \mathcal{E}_{H_s}(\hat{\nabla}_X, \hat{\Psi}) \right) d s
$$

$$
\lesssim \int_{s_1}^{s_2} \int_{H_s} \left( \sum_{j=0}^{4} \mathcal{E}_{H_s}(\hat{\nabla}_X, \hat{\psi}) + \mathcal{E}_{H_s}(\hat{\psi}) \right) d s
$$
Therefore we can conclude that the right-hand side of (3.67) can be controlled by the quantity
\[ \int_{s_1}^{s_2} \frac{1}{\sqrt{s}} \left( \sum_{j=0}^{4} E_{H_j}(\hat{\nabla}_{X_j} \hat{\psi}) + E_{H_5}(\hat{\psi}) \right) \, ds . \]

The function \( 1/\sqrt{s} \) is integrable on \([0, 1]\), so we can apply the Gronwall’s inequality to obtain:

**Theorem 3.5.1.** For any smooth compactly supported data on \( H_1 \), the associated solution \( \hat{\psi}_A \) of Dirac’s system satisfies
\[
\sum_{p=0}^{k} E_{S_{x^+} t_0} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) \lesssim \sum_{p=0}^{k} E_{H_1} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right),
\]
\[
\sum_{p=0}^{k} E_{H_1} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) \lesssim \sum_{p=0}^{k} \left\{ E_{S_{x^+} t_0} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) + E_{S_{x^+} t_0} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) \right\}
\]

where
\[ \hat{\nabla}_{X_{i_j}} \hat{\psi} = \hat{\nabla}_{X_{i_0}} \hat{\nabla}_{X_{i_1}} ... \hat{\nabla}_{X_{i_p}} \hat{\psi} , \, X_{i_j} \in A . \]

Now we define the peeling of Dirac field at order \( k \in \mathbb{N}^* \) in the term of the covariant derivative as follows:

**Definition 3.5.1.** We say that a solution \( \psi \) of Dirac’s equation peels at order \( k \) if
\[
\sum_{p=0}^{k} E_{x^+ t_0} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) < +\infty .
\]

Due to Theorem 3.5.1, we can find the optimal initial data space that guarantees the peeling at order \( k \) by the following theorem:

**Theorem 3.5.2.** The spaces of initial data \( h^k(H_1) \) that guarantees the definition peeling at order \( k \) is the completion of \( C_0^\infty([-t_0, +\infty[ \times S^2 \) on the norm
\[
||\hat{\psi}_A||_{h^k(H_1)} = \left\{ \sum_{p=0}^{k} E_{H_1} \left( \hat{\nabla}_{X_{i_j}} \hat{\psi} \right) \right\}^{1/2}.
\]

### 3.6 Interpretation

In this section we will interpret our peeling definition by comparing the classes of initial data that guarantees the peeling for Dirac fields on the flat spacetime and on the Kerr spacetime. Note that, our comparison is done in a neighbourhood of the spacelike infinity \( i_0 \), which is similar to the one that we have done for the scalar fields.
First, we construct briefly the peeling definition on the flat spacetime. Recall that the Minkowski spacetime can be embedded conformally into the Einstein cylinder with the conformal factor
\[ \Omega = \frac{2}{\sqrt{1 + (t + r)^2 \sqrt{1 + (t - r)^2}}} \]
And the vector field used to define the peeling at a higher order is
\[ \partial_\tau = \frac{1}{2} \left( (1 + t^2 + r^2) \partial_t + 2tr \partial_r \right). \] (3.68)
We can see that \( \partial_\tau \) is a conformal Killing vector field of the Minkowski spacetime. We denote the hypersurface \( t = 0 \) by \( \Sigma_0 \) and \( X_0 = \{ \tau = 0 \} \) and choose the Newman-Penrose tetrad on the Minkowski spacetime as follows:
\[
\begin{align*}
   l &= \frac{1}{\sqrt{2}} (\partial_t + \partial_r), \\
   n &= \frac{1}{\sqrt{2}} (\partial_t - \partial_r), \\
   m &= \frac{1}{\sqrt{2}} (\partial_\theta + \frac{i}{\sin \theta} \partial_\phi),
\end{align*}
\]
The energy of the rescaled Dirac field on \( X_0 \) is given by
\[
\mathcal{E}_{X_0}(\hat{\psi}_A) = \int_{X_0} \hat{\nu}^A \hat{\psi}_A \bar{\psi}_A^A d\mu_{X_0} = \int_{\Sigma_0} \Omega^{-1} \nu^A \Omega^{-2} \psi_A \bar{\psi}_A^A \Omega^3 d\mu_{\Sigma_0} = \int_{\Sigma_0} \nu^A \psi_A \bar{\psi}_A^A d\mu_{\Sigma_0} = \mathcal{E}_{\Sigma_0}(\psi),
\]
where
\[
\nu^A = \partial_t = \frac{1}{\sqrt{2}} (l + n)
\]
is the unit normal vector to the hypersurface \( \Sigma_0 \). Therefore the energy on a spacelike hypersurface is conformally invariant. Calculating in details, we get
\[
\mathcal{E}_{X_0}(\hat{\psi}_A) = \mathcal{E}_{\Sigma_0}(\psi_A) = \frac{1}{\sqrt{2}} \int_{\Sigma_0} (|\psi_0|^2 + |\psi_1|^2) r^2 dr d^2 \omega. \] (3.69)
Intergrating the conservation law of Dirac equation on the Einstein cylinder, we have
\[
\mathcal{E}_{X_0}(\hat{\psi}_A) = \mathcal{E}_{P^+}(\hat{\psi}_A). \]
Since \( \partial_\tau \) is a Killing vector field, we can commute \( \partial_\tau^k \) into the Dirac equation to obtain
\[
\mathcal{E}_{X_0}(\partial_\tau^k \hat{\psi}_A) = \mathcal{E}_{P^+}(\partial_\tau^k \hat{\psi}_A). \]
From this energy equality, we give the definition of the peeling at the order \( k \) as follows:

**Definition 3.6.1.** A solution \( \psi_A \) of the Weyl’s equation is said to peels at order \( k \in \mathbb{N} \) if the rescaled solution \( \hat{\psi}_A \) is such that \( \mathcal{E}_{P^+}(\partial_\tau^k \hat{\psi}_A) < +\infty \). The optimal initial data space that guarantees the definition peeling at order \( k \) is the completion of \( C^\infty(X_0) \) on the norm
\[
||\hat{\psi}_A||_{h^k(X_0)} = \left( \mathcal{E}_{X_0}(\partial_\tau^k \hat{\psi}_A) \right)^{1/2}.
\]
\(^6\text{see more details in Section 4.2.1 of chapter 3}\)
Now we take the constraint of our model on the Minkowski spacetime. On the hypersurface $\mathcal{H}_1 \subset \{t = 0\}$, we have the relation
\[ d^* t = -dr_* = \frac{-r^2 + a^2}{\Delta}dr. \]

Using Lemma 3.3.1, we obtain the equivalent expression for the energy of the solution on the hypersurface $\mathcal{H}_1$ on the partial conformal compactification as follows
\[ E_{\mathcal{H}_1}(\hat{\psi}) \simeq \int_{\{t=0,r>^*t_0\}} \left( \frac{R}{|t\psi_0|^2 + |\psi_1|^2} \right) d^*td^2\omega \]
\[ \simeq \int_{\{t=0,r>^*t_0\}} \left( |\psi_0|^2 + |\psi_1|^2 \right) \frac{r^2 + a^2}{\Delta} rdrd^2\omega \]
\[ \simeq \int_{\{t=0,r>^*t_0\}} \left( |\psi_0|^2 + |\psi_1|^2 \right) rdrd^2\omega. \quad (3.70) \]

At order zero, due to (3.69) and (3.70) we can see that in the neighbourhood of spacelike infinity $i_0$, the energies of the solution on $X_0$ and $\mathcal{H}_1$ are equivalent. So the initial data that guarantees the peeling definition at order zero are the same on both two conformal compactification spacetimes.

At a higher order, we will show that the initial data that guarantees the peeling on the partial conformal compactification is slightly larger than the one on the full conformal compactification. For simplicity, we will compare the classes of the initial data at the first order of regularity.

Using (3.69), the equivalent expression of the energy at the first order of the solution on the hypersurface $X_0$ can be obtained as
\[ E_{X_0}(\partial_t \hat{\psi}_A) = E_{X_0} \left( \frac{1 + r^2}{2} \partial_t \hat{\psi}_A \right) = E_{X_0} \left( \frac{(1 + r^2)^2}{4} \partial_t \hat{\psi}_A \right) \]
\[ = \frac{1}{\sqrt{2}} \int_{\Sigma_0} \left( |\partial_t \psi_0|^2 + |\partial_t \psi_1|^2 \right) \frac{1 + r^2}{2} rdrd^2\omega. \quad (3.71) \]

Using the expression of the Weyl’s equation on the Minkowski spacetime
\[ \partial_t \psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_r \psi - \frac{\sqrt{2}}{r} D_\omega \psi + \text{lower order terms}, \]

to estimate the energy $E_{X_0}(\partial_t \hat{\psi}_A)$ we need to control the energies of $(1 + r^2)\partial_t \psi$ and $\frac{1 + r^2}{r} D_\omega \psi$ independently.

Now we take the constraint of our model that is constructed by using the covariant derivatives on the Minkowski spacetime. At the first order, the quantity defining the space of initial data on $\mathcal{H}_1$ is
\[ \sum_{i=0}^4 \mathcal{E}_{\mathcal{H}_1} (\hat{\nabla}_{X_i} \hat{\psi}) + \mathcal{E}_{\mathcal{H}_1} (\hat{\psi}), \]
where we have
\[ \hat{\nabla}_R \hat{\psi} = \hat{D}_R \hat{\psi} - \hat{\kappa} \hat{\psi}_0 \hat{\partial}_A + \hat{\pi} \hat{\psi}_0 \hat{i}_A. \]
The energy of $\mathcal{D}_R(\hat{\psi}_A)$ on the hypersurface $\mathcal{H}_1 \subset \Sigma_0$ is

$$
\mathcal{E}_{\mathcal{H}_1}(\mathcal{D}_R\hat{\psi}_A) = \int_{\{t=0, r>-\ast t_0\}} \left( \frac{1}{r^2} |\partial_R \hat{\psi}_0|^2 + |\partial_R \hat{\psi}_1|^2 \right) r^2 dr d^2\omega
$$

$$
= \int_{\{t=0, r>-\ast t_0\}} \left( |\partial_R \psi_0|^2 + |\partial_R \psi_1|^2 \right) r^4 dr d^2\omega.
$$

(3.72)

Now using the equation satisfied by $\psi_A$ in components we have

$$
\frac{r^2 + a^2}{\sqrt{\Delta \rho^2}} D_t \psi + \sqrt{\frac{\Delta}{\rho^2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \partial_r - \frac{\sqrt{2}}{r} D_\omega \psi + \text{lower order terms} = 0,
$$

so that

$$
D_R \psi = -r^2 \left( \frac{r^2 + a^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_\varphi \right) \psi
$$

$$
\simeq - \begin{pmatrix} 2r^2 \partial_r \psi_0 \\ 0 \end{pmatrix} + r D_\omega \psi + a \partial_\varphi \psi + \text{lower order terms}.
$$

Therefore to estimate the energy $\mathcal{E}_{\mathcal{H}_1}(\mathcal{D}_R\hat{\psi}_A)$, we need to control the energies of $r^2 \partial_r \hat{\psi}_0$ and $r D_\omega \hat{\psi}$ dependently.

In (3.71) and (3.72), the coefficients of the derivatives of $\psi_0$ and $\psi_1$ are equivalent respectively in the neighbourhood of spacelike infinity $i_0$. But on the partial conformal compactification, we get a gain since we only control $r^2 \partial_r \psi_0$ and not $r^2 \partial_r \psi_1$. Furthermore, the controls of $r^2 \partial_r \psi_0$ and $D_\omega \psi$ are dependent, since we use the null vector field $\partial_R$ to control, this is different from the full conformal compactification. Therefore the initial data that guarantees the peeling at order one on the partial conformal compactification is slightly larger than the one on the full conformal compactification at a higher order given. So we conclude that the peeling on Kerr background which is valid for a class of the initial data is slightly larger than the one on the flat spacetime. This does not mean that our definition of the peeling is more general than the one on the flat case (the reason is in the flat case we have determined the relation of the null derivative $\partial_\tau$ to control near $i_0$ instead of controlling all derivatives), we can only conclude that our peeling definition is more general than the one that was given in Penrose’s papers [71, 72].

3.7 Appendix

3.7.1 Compacted spin coefficient formalism

We say that a scalar $\eta$ has weight $\{r', r; t', t\}$ if under a rescaling of the spin-frame by nowhere vanishing scalar fields $\lambda$ and $\mu$,

$$
\sigma^A \mapsto \lambda \sigma^A, \quad \iota^A \mapsto \mu \iota^A,
$$

it transforms $\eta$ as follows

$$
\eta \mapsto \lambda^{r'} \mu^r \lambda^{t'} \mu^t.
$$
So we can see, $\psi_0 = \psi_A o^A$ has weight $\{\lambda, 0; 0, 0\}$ and $\psi_1 = \psi_A v^A$ has weight $\{0, \mu; 0, 0\}$. If the spin-frame is normalized, then to preserve the normalization it is required that $\mu = 1/\lambda$ and then only two numbers are necessary $p = r' - r$ and $q = t' - t$. So in the normalization case, a scalar is then said to have weight $\{p, q\}$ or equivalent to have boost weight $\frac{1}{2}(p + q)$ and spin weight $\frac{1}{2}(p - q)$. Not all scalars have a weight and the derivatives along the Newman-Penrose formalism $l^a \partial_a, n^a \partial_a, m^a \partial_a$ and $\bar{m}^a \partial_a$ do not transform weighted scalars into weighted scalars. The compacted spin coefficient formalism combine these derivatives with unweighted spin coefficients to give the weighted derivative operators denoted $\breve{\partial}, \bar{\partial}, \bar{\partial}'$. These weight derivatives which act on a weighted scalars $\eta$ of weight $\{r', r; t', t\}$ is defined by

$$ \breve{\partial} \eta := (l^a \partial_a - r' \varepsilon - r \gamma' - t' \bar{\varepsilon} - t \bar{\gamma}') \eta, $$
$$ \bar{\partial} \eta := (m^a \partial_a - r' \beta - r \alpha' - t' \bar{\alpha} - t \bar{\beta}') \eta, $$
$$ \bar{\partial}' \eta := (\bar{m}^a \partial_a - r' \alpha - r \beta' - t' \bar{\beta} - t \bar{\alpha}') \eta, $$
$$ \breve{\partial}' \eta := (n^a \partial_a - r' \gamma - r \varepsilon' - t' \bar{\gamma} - t \bar{\varepsilon}') \eta. $$

Then we get the weighted scalars as follows

- $\breve{\partial} \eta$ has weight $\{r' + 1, r; t' + 1, t\}$,
- $\breve{\partial}' \eta$ has weight $\{r', r + 1; t', t + 1\}$,
- $\bar{\partial} \eta$ has weight $\{r' + 1, r; t', t + 1\}$,
- $\bar{\partial}' \eta$ has weight $\{r', r + 1; t' + 1, t\}$.

In the normalization case, we have the relations between the spin coefficients

$$ \kappa = -\nu', \rho = -\mu', \sigma = -\lambda', \tau = -\pi', \varepsilon = -\gamma', \alpha = -\beta', $$
$$ \kappa' = -\nu, \rho' = -\mu, \sigma' = -\lambda, \tau' = -\pi, \varepsilon' = -\gamma, \alpha' = -\beta $$

so that the weighted derivatives can be expressed as

$$ \breve{\partial} \eta := (l^a \partial_a + p \gamma' + q \bar{\gamma}') \eta, $$
$$ \bar{\partial} \eta := (m^a \partial_a - p \beta + q \bar{\beta}') \eta, $$
$$ \bar{\partial}' \eta := (\bar{m}^a \partial_a + p \beta' - q \bar{\beta}) \eta, $$
$$ \breve{\partial}' \eta := (n^a \partial_a - p \gamma - q \bar{\gamma}) \eta. $$

If $\{m^a, \bar{m}^a\}$ is integrable, we can consider $m^a$ and $\bar{m}^a$ as the tangent vector fields on 2–sphere $\mathcal{S}$ (a simple example is the 2–sphere $S^2$ with radius $R$ in Minkowski spacetime). Consider a function with the weight scalar $f = a^D f_{(A...DE...F)} o^A ... o^D \iota^E ... \iota^H$ where $f_{(A...DE...F)} \in S(A...DE...F)$. The application of the operators $\bar{\partial}' \bar{\partial}$ and $\bar{\partial} \bar{\partial}'$ on $f$ can be calculated as follows (see eqs.(4.15.54) and (4.15.56) pp.297 in R. Penrose and W. Rindler [73] Vol1)
Proposition 3.7.1. Suppose that $f$ has $j+s$ indices $A\ldots D$ and $j-s$ indices $E\ldots H$ where $-j \leq s \leq j$ ($j, s$ are integral or half-integral) then we have

$$\bar{\partial}'\partial f = -(j + s + 1)(j - s)\frac{1}{2}R^{-2}f, \quad (3.73)$$

$$\bar{\partial}\partial f = -(j - s + 1)(j + s)\frac{1}{2}R^{-2}f, \quad (3.74)$$

Applying this proposition for the component $\hat{\psi}$ that has weight $\{1, 0\}$ we can see that

$$\int_{S^2} |\bar{\partial}'\hat{\psi}|^2d^2\omega = -\int_{S^2} \hat{\psi}_0\bar{\partial}\partial\hat{\psi}_0d^2\omega$$

$$= \int_{S^2} \left(\frac{1}{2} - \frac{1}{2} + 1\right) \left(\frac{1}{2} + \frac{1}{2}\right)\frac{1}{2}R^{-2}|\hat{\psi}_0|^2d^2\omega$$

$$= \int_{S^2} \frac{R^{-2}}{2}\hat{\psi}_0^2d^2\omega. \quad (3.75)$$

So we obtain

$$\int_{S^2} |\hat{\psi}|^2d^2\omega \lesssim \int_{S^2} |\bar{\partial}'\hat{\psi}|^2d^2\omega. \quad (3.76)$$

This is the basic inequality that we have used to estimate the energies in this chapter (of course it is only valid for the case such that $\{\hat{m}^a, \check{m}^a\}$ is integrable).

3.7.2 Proof of Lemma 3.4.1

We will prove Lemma 3.4.1 by the aid of the Weyl’s system. Indeed, we think clearly that the norms of $\hat{\psi}_1$, $\partial_R\hat{\psi}_1$, and $\bar{\partial}\hat{\psi}_1$ can be controlled by the energies of $\hat{\psi}$, $\check{\partial}_R\hat{\psi}$ and $\check{\partial}_{t,\omega}\hat{\psi}$ on $\mathcal{H}_s$ respectively. And from the second equation of the equivalent Weyl’s system $(3.35)$, we have

$$-\frac{1}{\sqrt{2}}\partial_R\hat{\psi}_1 - \bar{\partial}'\hat{\psi}_0 + \frac{2R^2a^2\cos\theta\sin\theta}{2\sqrt{2}}\hat{\psi}_0 + \left(\frac{M}{2\sqrt{2}} + \frac{ia\cos\theta}{\sqrt{2}}\right)\hat{\psi}_1 \simeq 0$$

so that

$$\bar{\partial}'\hat{\psi}_0 \simeq -\frac{1}{\sqrt{2}}\partial_R\hat{\psi}_1 + \frac{R^2a^2\cos\theta\sin\theta}{2\sqrt{2}}\hat{\psi}_0 + \left(\frac{M}{2\sqrt{2}} + \frac{ia\cos\theta}{\sqrt{2}}\right)\hat{\psi}_1.$$

From this equivalence and the fact that the norm of $\partial_R\hat{\psi}_1$ can be controlled by the energy of $\hat{\psi}$ on $\mathcal{H}_s$, we can conclude that the norm of $\bar{\partial}'\hat{\psi}_0$ on $\mathcal{H}_s$ can be controlled uniformly in $s$ by the norms of $\check{\partial}_R\hat{\psi}$ and $\hat{\psi}$ on $\mathcal{H}_s$.

From the first equation of the Weyl’s system $(3.35)$, we have

$$\sqrt{2}\partial_t\hat{\psi}_0 + \frac{R^2}{2}\partial_R\hat{\psi}_0 - \bar{\partial}\hat{\psi}_1 + a\sqrt{2}R^2\partial_\varphi\hat{\psi}_0 + \frac{R}{\sqrt{2}}\hat{\psi}_0 - \frac{Ria\sin\theta}{\sqrt{2}}\hat{\psi}_1 \simeq 0$$

so that

$$\sqrt{2}\partial_t\hat{\psi}_0 \simeq -\frac{R^2}{2}\partial_R\hat{\psi}_0 + \bar{\partial}\hat{\psi}_1 - a\sqrt{2}R^2\partial_\varphi\hat{\psi}_0 - \frac{R}{\sqrt{2}}\hat{\psi}_0 + \frac{Ria\sin\theta}{\sqrt{2}}\hat{\psi}_1.$$
We see that the norms of \( R\psi_0, R\partial_\varphi \psi_0 \) and \( R\partial_R \psi_0 \) can be controlled by the energies of \( \psi, \partial_\varphi \psi \) and \( \partial_R \hat{\psi} \) on \( \mathcal{H}_s \); the norm of \( \partial \hat{\psi}_1 \) can be controlled by the energy of \( \partial_t \omega \hat{\psi} \) on \( \mathcal{H}_s \). So that we conclude that the norm of \( \partial_t \hat{\psi}_0 \) can be controlled uniformly in \( s \) by the energies of \( \partial_t \omega \hat{\psi}, \partial_\varphi \hat{\psi}, \partial_R \hat{\psi} \) and \( \hat{\psi} \) on \( \mathcal{H}_s \).

Now for the norm of \( \hat{\psi}_0 \), we consider the expression of \( \partial \hat{\psi}_0 \)
\[
\partial \hat{\psi}_0 = \frac{r}{\tilde{p} \sqrt{2}} \left\{ -ia \sin \theta \partial_t \hat{\psi}_0 + \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \hat{\psi}_0 \right\}.
\]
Since the span of \( \frac{\tilde{p} \sqrt{2}}{r} \partial \right\} \left\{ -ia \sin \theta \partial_t \hat{\psi}_0 + \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \hat{\psi}_0 \right\}.
\]

From this inequality and the control of the norms of \( \partial_t \hat{\psi}_0 \) and \( \partial \hat{\psi}_0 \), we conclude that the norm of \( \hat{\psi}_0 \) can be also controlled uniformly in \( s \) by the energies of \( \partial_t \omega \hat{\psi}, \partial_\varphi \hat{\psi}, \partial_R \hat{\psi} \) and \( \hat{\psi} \) on \( \mathcal{H}_s \).

### 3.7.3 Proof of Lemma 3.5.2

We have
\[
(\nabla^b X^a) \partial_a \otimes \partial_b = \left( g^{bc} \partial_c X^a + \hat{g}^{bd} \hat{\Gamma}^a_{de} X^c \right) \partial_a \otimes \partial_b. \tag{3.76}
\]
First, dealing with Killing vector fields \( X_0 = \partial_t \) or \( X_1 = \partial_\varphi \) is the same, so for instance, the equation for \( X_1 \) is as follows
\[
(\nabla^b X^a) \partial_a \otimes \partial_b = \left( \hat{g}^{bd} \hat{\Gamma}^a_{d3} \right) \partial_a \otimes \partial_b,
\]
where the coefficients are
\[
\hat{g}^{bd} \hat{\Gamma}^a_{d3} = \frac{1}{2} \hat{g}^{bd} \hat{g}^{ae} \left( \frac{\partial \hat{g}_{de}}{\partial x^\varphi} + \frac{\partial \hat{g}_{3e}}{\partial x^d} - \frac{\partial \hat{g}_{d3}}{\partial x^e} \right) = \frac{1}{2} \hat{g}^{bd} \hat{g}^{ae} \left( \frac{\partial \hat{g}_{3e}}{\partial x^d} - \frac{\partial \hat{g}_{d3}}{\partial x^e} \right).
\]
In this equality, the symmetry of the index \( d \) and \( e \) gives us a consequence that is we will get the pair of coefficients with opposite signs. And then after contracting \( (\nabla^b X^a) \partial_a \otimes \partial_b \) with \( \nabla_a \hat{\psi}_B \), we will get zero. The result also hold for \( X_0 = \partial_t \). So we have
\[
(\nabla^b X^a) \nabla_a \hat{\psi}_B = (\nabla^b X^a) \nabla_a \hat{\psi}_B = 0.
\]

Second, for the equations with the vector fields \( X_2^a, X_3^a \) that are tangent on 2–sphere \( S^2 \), we prove that the quantity \( (\nabla^b X^a) \partial_b \otimes \partial_a \) can be written as a linear combinations of \( \nabla X_i \hat{\psi}_B \), \( X_i \in \mathcal{A} \) with the coefficients that are bounded and that are of order greater than one in \( R \). For instance the
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The equation for $X_2$ is as follows

\[
\hat{\nabla}^a X_2^b = \hat{g}^{ad} \partial_d X^b + \hat{g}^{ad} \hat{R}^b_{\, de} X^c \\
= \hat{g}^{a3} \cos^* \varphi \partial_a \otimes \partial_\theta - \hat{g}^{a3} \cot \theta \sin^* \varphi \partial_a \otimes \partial_\varphi - \hat{g}^{a2} \cos^* \varphi \partial_a \otimes \partial_\varphi \\
+ \frac{1}{2} \hat{g}^{ad} \hat{g}_{be} \partial_{\theta e} \sin^* \varphi \partial_a \otimes \partial_b + \frac{1}{2} \hat{g}^{ad} \hat{g}_{be} \left( \frac{\partial \hat{g}_{3e}}{\partial x^d} - \frac{\partial \hat{g}_{3d}}{\partial x^e} \right) \cot \theta \cos^* \varphi \partial_a \otimes \partial_b \\
= \hat{g}^{a3} \partial_t \otimes \left( \cos^* \varphi \partial_\theta - \cot \theta \sin^* \varphi \partial_\varphi \right) + \hat{g}^{a13} \partial_R \otimes \left( \cos^* \varphi \partial_\theta - \cot \theta \sin^* \varphi \partial_\varphi \right) \\
- \frac{r^2 \cos^* \varphi}{\rho^2 \sin^2 \theta} \left( \partial_\varphi \otimes \cot \theta \cot \theta \otimes \partial_\varphi \right) + \left( \frac{\cot \theta}{\sin^2 \theta} + \frac{1}{2} \frac{r^2}{\rho^2 \sin^4 \theta} \right) \frac{r^2}{\rho^2} \sin^* \varphi \partial_\varphi \otimes \partial_\varphi \\
+ \left( \frac{1}{2} \hat{g}^{a1} \hat{g}^{b3} \frac{\partial \hat{g}_{13}}{\partial \theta} \partial_a \otimes \partial_b + \frac{1}{2} \hat{g}^{a3} \hat{g}^{b1} \frac{\partial \hat{g}_{31}}{\partial \theta} \partial_a \otimes \partial_b \right) \sin^* \varphi + \frac{1}{2} \hat{g}^{a3} \hat{g}^{b3} \frac{\partial \hat{g}_{33}}{\partial \theta} \sin^* \varphi \partial_a \otimes \partial_b , \quad (3.77)
\]

where the indices $a, b$ in the last line are not both equal to 3. Calculating in details, we have

\[
\frac{\cot \theta}{\sin^2 \theta} + \frac{1}{2} \frac{r^2}{\rho^2} \frac{\partial \hat{g}_{33}}{\partial \theta} = \frac{a^2 r^2}{\rho^2} \cot \theta + \frac{2MRa^2}{\rho^2} \left( \frac{r^2 + a^2}{\rho^2} + 1 \right) \sin^3 \theta \cos \theta
\]

replace it in equation $(3.77)$ we have

\[
\hat{\nabla}^a X_2^b = \hat{g}^{ad} \partial_d X^b + \hat{g}^{ad} \hat{R}^b_{\, de} X^c \\
= -\hat{g}^{a3} \partial_t \otimes X_3^b - \hat{g}^{a13} \partial_R \otimes X_3^b - \frac{r^2 \cos^* \varphi}{\rho^2 \sin^2 \theta} \left( \partial_\varphi \otimes \cot \theta \cot \theta \otimes \partial_\varphi \right) \\
+ \left( \frac{a^2 r^2}{\rho^2} \cot \theta + \frac{2MRa^2}{\rho^2} \left( \frac{r^2 + a^2}{\rho^2} + 1 \right) \sin^3 \theta \cos \theta \right) \frac{r^2}{\rho^2} \sin^* \varphi \partial_\varphi \otimes \partial_\varphi \\
+ \left( \frac{1}{2} \hat{g}^{a0} \hat{g}^{b0} \frac{\partial \hat{g}_{00}}{\partial \theta} \partial_a \otimes \partial_b + \frac{1}{2} \hat{g}^{a0} \hat{g}^{b3} \frac{\partial \hat{g}_{03}}{\partial \theta} \partial_a \otimes \partial_b \right) \sin^* \varphi + \frac{1}{2} \hat{g}^{a3} \hat{g}^{b3} \frac{\partial \hat{g}_{33}}{\partial \theta} \sin^* \varphi \partial_a \otimes \partial_b . \quad (3.78)
\]

Note that

\[
\partial_\theta = \sin^* \varphi X_2 + \cos^* \varphi X_3 \,, \quad \cot \theta \partial_\varphi = \cos^* \varphi X_2 + \sin^* \varphi X_3 ,
\]

these equalities are useful since the equation $(3.77)$ contains terms having $\partial_\theta$ and the factors $\hat{g}^{33}(\partial \hat{g}_{33}/\partial \theta)(d = 0, 1, 3)$ with the angular singularity $\cot \theta \partial_\varphi$. Then, from equation $(3.78)$, after contracting $\hat{\nabla}^a X_2^b$ with $\hat{\nabla}_a \hat{\psi}_B$, we can write the result as a linear combination of $\hat{\nabla}_X \hat{\psi}_B \,(X_i \in \mathcal{A})$. The boundedness of the coefficients is clear since the coefficients in the result of the linear combination that does not have the angular singularity $\cot \theta$ and the spatial singularity $r$. Further, we can
see in the above expression \eqref{3.78} that the coefficients of the terms containing \( \partial_R \otimes X_i \) are of order greater than or equal one in \( R \) since \( \partial \hat{g}_{00}/\partial \theta \), \( \partial \hat{g}_{03}/\partial \theta \) and \( \hat{g}^{11}, \hat{g}^{13} \) are of order larger than or equal one in \( R \).

**Remark 3.7.1.** We can understand that the no angular singularity in the expression of \((\hat{\nabla}^a X^b_2)\hat{\nabla}_b \hat{\psi}_B\) due to the fact that \( X^b_2 \) belong to the smooth section i.e \( \Gamma(TS^2) \hookrightarrow \Gamma(TM) \) and then \( \hat{\nabla}^a X^b_2 \) belong to \( \Gamma(T^a \mathcal{M} \otimes TM) \) which is also a smooth section.

Third, for the equation with the vector field \( X^a_i = \partial_R \). Due to \eqref{3.76}, we have

\[
\hat{\nabla}^a X^b_1 = \left( \hat{g}^{ad} \partial_d X^b + \hat{g}^{ad} \hat{\Gamma}^b_{dc} X^c \right)
\]

\[
= \frac{1}{2} \hat{g}^{ad} \frac{\partial \hat{g}_{de}}{\partial R} \partial_d \otimes \partial_e.
\]

Because the indices \( d \) and \( e \) are symmetric, after contracting \( \hat{\nabla}^a X^b_1 \) with \( \hat{\nabla}_b \hat{\psi}_B \), we get

\[
(\hat{\nabla}^a X^b_1)\hat{\nabla}_b \hat{\psi}_B = \frac{1}{2} \hat{g}^{ad} \frac{\partial \hat{g}_{de}}{\partial R} \partial_d \otimes \partial_e (\hat{\nabla}_c \hat{\psi}_B).
\]

Using the expression of the metric \( \hat{g} \), we can see that \( \partial \hat{g}_{de}/\partial R \) is either of order one in \( R \) or disappears. So the coefficients of \((\hat{\nabla}^a X^b_1)\hat{\nabla}_b \hat{\psi}_B\) is of the order one in \( R \). The angular singularity can not also appear because \( \partial \hat{g}_{d3}/\partial R \) is of order one in \( \sin^2 \theta \). Furthermore, from the equality

\[
\partial_\theta = \sin^* \varphi X_2 + \cos^* \varphi X_3,
\]

we can write \((\hat{\nabla}^a X^b_1)\hat{\nabla}_b \hat{\psi}_B\) (\( X_i \in \mathcal{A} \)).

### 3.7.4 Proof of Lemma \(3.5.4\)

We will prove Lemma \(3.5.4\) by the aid from the first equation of the Weyl’s system and the covariant derivatives of the Dirac field. Note that we use the symbol \( \hat{\nabla}_X \dot{\Psi} \) in the sense that the covariant derivative act on the full spinor field. First, we have

\[
\frac{\sqrt{2} p}{r} \dot{m}^a + \frac{\sqrt{2} \bar{p}}{r} \dot{\bar{m}}^a = 2 \partial_\theta = 2(\sin \varphi X_2^a + \cos \varphi X_3^a)
\]

\[
\Rightarrow \frac{\sqrt{2} p}{r} \left( \hat{\nabla}_{\dot{m}}^a \dot{\Psi} \right)_1 + \frac{\sqrt{2} \bar{p}}{r} \left( \hat{\nabla}_{\dot{\bar{m}}}^a \dot{\Psi} \right)_1 = 2 \left( \sin \varphi \left( \hat{\nabla}_{X_2} \dot{\Psi} \right)_1 + \cos \varphi \left( \hat{\nabla}_{X_3} \dot{\Psi} \right)_1 \right)
\]

\[
\Leftrightarrow \frac{\sqrt{2} p}{r} \left( \dot{\hat{\psi}}_1 + \hat{\beta} \dot{\hat{\psi}}_0 - \hat{\mu} \dot{\hat{\psi}}_0 \right) + \frac{\sqrt{2} \bar{p}}{r} \left( \dot{\bar{\hat{\psi}}}_1 + \hat{\alpha} \dot{\bar{\hat{\psi}}}_1 \right) = 2 \left( \sin \varphi \left( \hat{\nabla}_{X_2} \dot{\Psi} \right)_1 + \cos \varphi \left( \hat{\nabla}_{X_3} \dot{\Psi} \right)_1 \right)
\]

\[
\Leftrightarrow 2 \partial_\theta \dot{\hat{\psi}}_1 + \left( \frac{\sqrt{2} p}{r} \dot{\hat{\beta}} + \frac{\sqrt{2} \bar{p}}{r} \dot{\hat{\alpha}} \right) \dot{\hat{\psi}}_1 - \frac{\sqrt{2} p}{r} \dot{\hat{\mu}} \dot{\hat{\psi}}_0 = 2 \left( \sin \varphi \left( \hat{\nabla}_{X_2} \dot{\Psi} \right)_1 + \cos \varphi \left( \hat{\nabla}_{X_3} \dot{\Psi} \right)_1 \right)
\]

\[
\Leftrightarrow 2 \partial_\theta \dot{\hat{\psi}}_1 = - \left( \frac{\sqrt{2} p}{r} \dot{\hat{\beta}} + \frac{\sqrt{2} \bar{p}}{r} \dot{\hat{\alpha}} \right) \dot{\hat{\psi}}_1 + \frac{\sqrt{2} p}{r} \dot{\hat{\mu}} \dot{\hat{\psi}}_0 + 2 \left( \sin \varphi \left( \hat{\nabla}_{X_2} \dot{\Psi} \right)_1 + \cos \varphi \left( \hat{\nabla}_{X_3} \dot{\Psi} \right)_1 \right). \tag{3.79}
\]

Second, we have

\[
\frac{\sqrt{2} p}{r} \dot{m}^a - \frac{\sqrt{2} \bar{p}}{r} \dot{\bar{m}}^a = 2 \left( \frac{i}{\sin \theta} \partial_\varphi + ia \sin \theta \partial_t \right).
\]
It gives us two equations in coordinates

\[
\frac{\sqrt{2}p}{r} \left( \nabla_{\hat{n}^a} \hat{\Psi} \right)_0 - \frac{\sqrt{2}p}{r} \left( \nabla_{\hat{n}^a} \hat{\Psi} \right)_0 = 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_0 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_0 \right)
\]

\[
\Leftrightarrow \frac{\sqrt{2}p}{r} \left( -\hat{\partial} \hat{\psi}_0 - \hat{\epsilon} \hat{\psi}_0 \right) - \frac{\sqrt{2}p}{r} \left( \hat{\rho} \hat{\psi}_1 - \hat{\delta} \hat{\psi}_0 - \hat{\alpha} \hat{\psi}_0 \right) = 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_0 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_0 \right)
\]

\[
\Leftrightarrow -2 \left( ia \sin \theta \partial_t + \frac{i}{\sin \theta} \partial_\varphi \right) \hat{\psi}_0 - \left( \frac{\sqrt{2}p}{r} \hat{\epsilon} - \frac{\sqrt{2}p}{r} \hat{\alpha} \right) \hat{\psi}_0 - \frac{\sqrt{2}p}{r} \hat{\rho} \hat{\psi}_1
\]

\[
= 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_0 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_0 \right),
\]

(3.80)

where \( \left( \frac{\sqrt{2}p}{r} \hat{\epsilon} - \frac{\sqrt{2}p}{r} \hat{\alpha} \right) \) is of order zero in \( R \). And

\[
\frac{\sqrt{2}p}{r} \left( \nabla_{\hat{n}^a} \hat{\Psi} \right)_1 - \frac{\sqrt{2}p}{r} \left( \nabla_{\hat{n}^a} \hat{\Psi} \right)_1 = 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_1 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_1 \right)
\]

\[
\Leftrightarrow \frac{\sqrt{2}p}{r} \left( \hat{\delta} \hat{\psi}_1 + \hat{\beta} \hat{\psi}_0 - \hat{\mu} \hat{\psi}_0 \right) - \frac{\sqrt{2}p}{r} \left( \hat{\delta} \hat{\psi}_1 + \hat{\alpha} \hat{\psi}_0 \right) = 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_1 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_1 \right)
\]

\[
\Leftrightarrow 2 \left( ia \sin \theta \partial_t + \frac{i}{\sin \theta} \partial_\varphi \right) \hat{\psi}_1 + \left( \frac{\sqrt{2}p}{r} \hat{\beta} - \frac{\sqrt{2}p}{r} \hat{\alpha} \right) \hat{\psi}_1 - \frac{\sqrt{2}p}{r} \hat{\mu} \hat{\psi}_0
\]

\[
= 2 \left( \frac{i}{\sin \theta} \left( \nabla_\varphi \hat{\Psi} \right)_1 + ia \sin \theta \left( \nabla_t \hat{\Psi} \right)_1 \right).
\]

(3.81)

For the components of \( \nabla_R \hat{\Psi} \), we have

\[
\left( \nabla_R \hat{\Psi} \right)_0 = \left( \nabla_R \hat{\Psi} \right)_0 \hat{\partial}^A = -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \hat{\psi}_A \right)_0 \hat{\partial}^A
\]

\[
= -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \left( \hat{\psi}_1 \hat{\partial}_A - \hat{\psi}_0 \hat{\partial}_A \right) \right) \hat{\partial}^A = -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \hat{\psi}_1 \hat{\partial}_A - \hat{\partial} \hat{\psi}_0 \hat{\partial}_A - \hat{\partial} \hat{\psi}_0 \hat{\partial}_A \right)
\]

\[
= -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \hat{\psi}_1 - \hat{\partial} \hat{\psi}_0 - \hat{\epsilon} \hat{\psi}_0 \right),
\]

(3.82)

similarly

\[
\left( \nabla_R \hat{\Psi} \right)_1 = -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \hat{\psi}_1 + \hat{\partial} \hat{\psi}_0 - \hat{\pi} \hat{\psi}_0 \right) = -\sqrt{\frac{2\rho^2}{\Delta}} \left( \hat{\partial} \hat{\psi}_1 + \hat{\partial} \hat{\psi}_0 + \hat{\epsilon} \hat{\psi}_0 \right).
\]

(3.83)

Now we will transform the first equation of the Weyl’s system into the form of the covariant derivatives, we have:

\[
\sqrt{2} \left( \partial_t + a R^2 \partial_\varphi + \frac{R^2}{2} \partial_\theta \right) \hat{\psi}_0 + \frac{1}{\sqrt{2}} \left( \left( R + ia \cos \theta R^2 \right) - R^2 \right) \hat{\psi}_0 -
\]

\[
- \frac{1}{\sqrt{2}} \left( ia \sin \theta \partial_t + \frac{i}{\sin \theta} \partial_\varphi \right) \hat{\psi}_1 - \frac{1}{\sqrt{2}} \partial_\theta \hat{\psi}_1 - \cot \theta \hat{\psi}_1 + \frac{a^2 R^2 \sin \theta \cos \theta}{2 \sqrt{2}} \hat{\psi}_1 \simeq -R \hat{\psi}_0 - \hat{\psi}_1
\]

\[
\Rightarrow 4 \left( \partial_t + a R^2 \partial_\varphi + \frac{R^2}{2} \partial_\theta \right) \hat{\psi}_0 - 2 \left( ia \sin \theta \partial_t + \frac{i}{\sin \theta} \partial_\varphi \right) \hat{\psi}_1 - 2 \partial_\theta \hat{\psi}_1 - \cot \theta \hat{\psi}_1
\]

\[
\simeq -2(\sqrt{2} + 1)R \hat{\psi}_0 - 2\sqrt{2} \hat{\psi}_1.
\]
Multiplying the later with $\sin \theta$ then using the equalities \((3.79), (3.80), (3.81), (3.82), (3.83)\) we obtain
\[
4\sin \theta \partial_t \hat{\psi}_0 + 4aR^2 \sin \theta \partial_\varphi \hat{\psi}_0 + 2R^2 \sin \theta \partial_R \hat{\psi}_0 - 2\left(ia \sin^2 \theta \partial_t + i \partial_\varphi \right) \hat{\psi}_1 - \\
- 2 \sin \theta \partial_\theta \hat{\psi}_1 - \cos \theta \hat{\psi}_1 \simeq 2(\sqrt{2} + 1)R \sin \theta \hat{\psi}_0 - 2\sqrt{2} \sin \theta \hat{\psi}_1
\]
\[
\Rightarrow 4 \sin \theta \left(1 - a^2 R^2 \sin^2 \theta \right) \partial_t \hat{\psi}_0 + 4aR^2 \sin \theta \left( a \sin^2 \theta \partial_t + \partial_\varphi \right) \hat{\psi}_0 + 2R^2 \sin \theta \partial_R \hat{\psi}_0 - \\
- 2 \left(ia \sin^2 \theta \partial_t + i \partial_\varphi \right) \hat{\psi}_1 - 2 \sin \theta \partial_\theta \hat{\psi}_1 - \cos \theta \hat{\psi}_1 \simeq 2(\sqrt{2} + 1)R \sin \theta \hat{\psi}_0 - 2\sqrt{2} \sin \theta \hat{\psi}_1
\]
\[
\Rightarrow 4 \sin \theta \partial_t \hat{\psi}_0 + 4aR^2 \sin \theta \left( a \sin^2 \theta \partial_t + \partial_\varphi \right) \hat{\psi}_0 + 2R^2 \sin \theta \partial_R \hat{\psi}_0 - \\
- 2 \left(ia \sin^2 \theta \partial_t + i \partial_\varphi \right) \hat{\psi}_1 - 2 \sin \theta \partial_\theta \hat{\psi}_1 - \cos \theta \hat{\psi}_1 \simeq 2(\sqrt{2} + 1)R \sin \theta \hat{\psi}_0 - 2\sqrt{2} \sin \theta \hat{\psi}_1
\]
\[
\Rightarrow 4 \sin \theta \partial_t \hat{\psi}_0 + 2R^2 \sin \theta \left( \left( \nabla_R \hat{\Psi} \right)_0 \right) - \hat{\psi}_0 - \cos \theta \hat{\psi}_1 - \\
- 4aR^2 \sin \theta \left\{ \left( \left( \nabla_\varphi \hat{\Psi} \right)_1 \right) + a \sin^2 \theta \left( \nabla_t \hat{\Psi} \right)_0 \right\}
\]
\[
- i \sin \theta \left( \frac{\sqrt{2p}}{r} \hat{\beta} - \frac{\sqrt{2\rho}}{r} \hat{\alpha} \right) \hat{\psi}_0 - i \sin \theta \frac{\sqrt{2p}}{r} \hat{\rho} \hat{\psi}_1
\]
\[
- 2 \left\{ \left( i \left( \nabla_\varphi \hat{\Psi} \right)_1 \right) + ia \sin^2 \theta \left( \nabla_t \hat{\Psi} \right)_1 \right\}
\]
\[
- \sin \theta \left( \frac{\sqrt{2p}}{r} \hat{\beta} - \frac{\sqrt{2\rho}}{r} \hat{\alpha} \right) \hat{\psi}_1 + \sin \theta \frac{\sqrt{2p}}{r} \hat{\mu} \hat{\psi}_0
\]
\[
- \sin \theta \left\{ - \left( \frac{\sqrt{2p}}{r} \hat{\beta} + \frac{\sqrt{2\rho}}{r} \hat{\alpha} \right) \hat{\psi}_1 + \frac{\sqrt{2p}}{r} \hat{\mu} \hat{\psi}_0
\]
\[
+ 2 \left( \sin \varphi \left( \hat{\nabla}_{X_2} \hat{\Psi} \right)_1 \right) + \cos \varphi \left( \hat{\nabla}_{X_3} \hat{\Psi} \right)_1 \right\}
\]
\[
\simeq 2(\sqrt{2} + 1)R \sin \theta \hat{\psi}_0 - 2\sqrt{2} \sin \theta \hat{\psi}_1.
\] (3.84)

Due to equation (3.84), we can see that the energy of $\sin \theta \partial_t \hat{\psi}_0$ on $\mathcal{H}_s$ can be controlled uniformly in $s$ by the energies of the covariant derivatives $\hat{\nabla}_{X_i} \hat{\Psi}$ along all the vector fields $X_i \in \mathcal{A}$ and $\hat{\psi}$ on $\mathcal{H}_s$. On the other hand, from equality (3.83), we have
\[
\left( \hat{\nabla}_R \hat{\Psi} \right)_1 \simeq \hat{\nabla} \hat{\psi}_0 + \tilde{\rho} \hat{\psi}_1 \simeq
\]
\[
\simeq \frac{1}{\sqrt{2}} \left( -ia \sin \theta \partial_t + \partial_\theta \right) \hat{\psi}_0 + \frac{1}{\sqrt{2}} \left( - \frac{i a \sin \theta}{\rho} \cot \theta + \frac{a^2 \cos \theta \sin \theta}{2\rho^2} \right) \hat{\psi}_0 + \tilde{\rho} \hat{\psi}_1.
\]

Using the inequality (3.75) in Appendix 3.7.1 we have
\[
\int_{S^2} \left| \hat{\psi}_0 \right|^2 d^2\omega \lesssim \int_{S^2} \frac{1}{\sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \hat{\psi}_0 d^2\omega.
\]

Therefore we can conclude that the energy of $\hat{\psi}_0$ on $\mathcal{H}_s$ can be controlled uniformly in $s$ by the energies of the covariant derivative $\hat{\nabla}_{X_i} \hat{\Psi}$ ($X_i \in \mathcal{A}$) and $\hat{\psi}$ on $\mathcal{H}_s$.

**Remark 3.7.2.** We can think that the two Lemma 3.4.1 and Lemma 3.5.4 are of the same essence (although the last lemma is slightly weaker control), since we can control the norm of $\hat{\psi}_0$ by the
derivatives along all the directions. The first lemma expresses this essence in the partial derivative term and the second lemma expresses it in the covariant derivative term.

3.7.5 Some calculations

We check out the calculations of the Lie bracket. For any two functions $A$ and $B$ we have

$$[A \partial_x, B \partial_y] = A \partial_x (B \partial_y) - B \partial_y (A \partial_x) = A(\partial_x B) \partial_y - B(\partial_y A) \partial_x.$$ 

The expressions (3.32), (3.33), (3.34) can be written under the following simpler equivalences (in the sense that are of the same of order in $R$)

$$M_{*t} \simeq \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_R \simeq \begin{pmatrix} \frac{R^2}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}, \quad M_{*\varphi} \simeq \begin{pmatrix} \sqrt{2}aR^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P \simeq \begin{pmatrix} \frac{R}{\sqrt{2}} & \frac{-ia \sin \theta R}{2\sqrt{2}} \\ \frac{a^2 \sin \theta \cos \theta R^2}{2\sqrt{2}} & \frac{M}{2\sqrt{2}} - \frac{ia \cos \theta}{\sqrt{2}} \end{pmatrix}.$$ 

So we have

$$[D_R, D_{*t,\omega}] = \begin{pmatrix} 0 \\ \partial_R \left( \frac{r}{p} \right) \frac{\bar{p}}{r} \partial_{\theta} \end{pmatrix} \simeq D_{*t,\omega},$$

$$[D_R, M_{*t} D_{*t}] = (D_R M_{*t}) D_{*t} \simeq \begin{pmatrix} \partial_R \sqrt{2} \\ 0 \end{pmatrix} D_{*t} \simeq \begin{pmatrix} 0 \\ 0 \end{pmatrix} D_{*t}$$

similarly

$$[D_R, M_R D_R] \simeq \begin{pmatrix} \sqrt{2R} & 0 \\ 0 & 0 \end{pmatrix} D_R, \quad [D_R, M_{*\varphi} D_{*\varphi}] \simeq \begin{pmatrix} 2\sqrt{2}aR & 0 \\ 0 & 0 \end{pmatrix} D_{*\varphi},$$

$$D_R P \simeq \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{a^2 \sin \theta \cos \theta R}{\sqrt{2}} \end{pmatrix}.$$ 

So that we can conclude that the coefficients $B_i (i = 1, 2...8)$ are of non negative order in $R$ and furthermore $B_2$ is of order one in $R$.

For checking the calculations of $\partial_{*t,\omega}$, we need to keep $p, \bar{p}$ and $\rho^2$ in the simpler equivalences of the expressions since they contain $\cos \theta$. We have

$$M_{*t} \simeq \begin{pmatrix} \frac{r}{\rho^2} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_R \simeq \begin{pmatrix} \frac{R}{\sqrt{2}\rho^2} & 0 \\ 0 & \frac{-r}{\sqrt{2}\rho^2} \end{pmatrix}, \quad M_{*\varphi} \simeq \begin{pmatrix} aR \frac{2}{\rho^2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$P \simeq \begin{pmatrix} \frac{(M_r-a^2)+(1-MR)a^2 \cos^2 \theta}{2\rho^2 \sqrt{2}\rho^2} & \frac{-ia r \sin \theta}{\sqrt{2}\rho^2} \left( 1 - \frac{ia \cos \theta}{2\rho} \right) \\ \frac{a^2 \cos \theta \sin \theta \sin \theta}{2\sqrt{2}\rho^2} & \frac{M_r-a^2 \sin^2 \theta - MA^2 \cos^2 \theta}{2\rho^2 \sqrt{2}\rho^2} + \frac{ia \cos \theta \cos^2 \theta}{\bar{p} \sqrt{2}\rho^2} \end{pmatrix}.$$
Therefore we have

\[ [D^*_{t,\omega}, M_t D_t] \simeq \left( \begin{array}{cc} \frac{1}{\sqrt{2}} \partial_\theta & 0 \\ \frac{1}{\sqrt{2}} \partial_\theta & 0 \end{array} \right) \left( \begin{array}{cc} r \sqrt{\frac{2}{\rho^2}} & 0 \\ 0 & 0 \end{array} \right) D_t = \left( \partial_\theta \left( r \sqrt{\frac{2}{\rho^2}} \right) & 0 \\ 0 & 0 \right) D_t \]

\[ = \left( \begin{array}{cc} -ra^2 \sin 2\theta \sqrt{\rho^2} & 0 \\ \sqrt{2} \rho^2 & 0 \end{array} \right) D_t \simeq \left( \begin{array}{cc} -a^2 \sin 2\theta R & 0 \\ \sqrt{2} & 0 \end{array} \right) D_t . \]

Similarly

\[ [D^*_{t,\omega}, M_R D_R] = (D^*_{t,\omega} M_R) D_R - M_R[D_R, D^*_{t,\omega}] \]

\[ \simeq \left( \begin{array}{cc} -a^2 \sin 2\theta \sqrt{\rho^2} & 0 \\ \frac{4\rho^2}{a^2 \sin 2\theta \sqrt{\rho^2}} & 0 \end{array} \right) D_R - M_R D^*_{t,\omega} , \]

\[ [D^*_{t,\omega}, M_{\varphi} D_{\varphi}] \simeq \left( \begin{array}{cc} -a^2 \sin 2\theta R \sqrt{\rho^2} & 0 \\ \frac{2\rho^2}{a^2 \sin 2\theta R \sqrt{\rho^2}} & 0 \end{array} \right) D_{\varphi} . \]

For the case of $D^*_{t,\omega} P$, we note that the components of $P$ have same form $f(\theta)/g(\theta)$ with the order of the function $f$ in $R$ is less than or equal that of $g$ and

\[ \partial_\theta \left( \frac{f(\theta)}{g(\theta)} \right) = \frac{\partial_\theta f(\theta)}{g(\theta)} - \frac{f(\theta)\partial_\theta g(\theta)}{g^2(\theta)} . \]

Since we take the derivative along $\partial_\theta$, so it does not change the order in $R$ of $\partial_\theta f(\theta)$ and $\partial_\theta g(\theta)$ in comparison to $f$ and $g$ respectively. Then the order in $R$ of $\partial_\theta(f(\theta)/g(\theta))$ is non negative. So that we can also conclude that the coefficients $D_i(i = 1, 2, ..., 8)$ are of order non negative in $R$.
Chapter 4

Conformal scattering for spin $n/2$ zero rest-mass fields on Minkowski spacetime

4.1 Introduction

Conformal scattering is a geometric approach to the scattering theory. The basic idea is to solve the Goursat problem on the null infinity hypersurfaces $\mathcal{I}^\pm$, which was started by R. Penrose [70]. After that the conformal scattering theory has been constructed and developed by F. Friedlander [29, 30], J. Baez, I. Segal and S. Zhou [9], L. Hörmander [42], and recently by L. Mason and J-P. Nicolas [52], J. Joudioux [45], J-P. Nicolas [66] and M. Mokdad [58].

For purpose, to obtain a conformal scattering theory for the spinor field with a higher helicity spin, in this chapter we will construct a scattering theory for the spin $n/2$ zero rest-mass field $\phi_{(AB...F)}$ (this is a solution of the massless equation $\nabla^{A'}\phi_{AB...F} = 0$) on the Minkowski spacetime. As a consequence, we obtain the conformal scattering theory for the linear gravity (the case $n = 4$) in the flat spacetime.

To construct the conformal scattering on the Minkowski spacetime for spin $n/2$ zero rest-mass fields, our work is based on solving the Cauchy problem and Goursat problem and using the energy equality to show that the trace operator is isometric.

The Cauchy problem will be solved as follows: first, we will solve the Cauchy problem on the Einstein cylinder by showing that the evolution system is symmetric hyperbolic, then using the Leray’s theorem it has a unique solution, and hence we show that the constraint system is conserved along the evolution solution; second, we apply the solution of the Cauchy problem on the Einstein cylinder to solve the Cauchy problem on the conformal compactification spacetimes.

After calculating the energies of the solution of the Cauchy problem, we will obtain the energy equality. Note that the energy equality in the part conformal compactification is more difficult to obtain than in the full conformal compactification since the timelike infinity point $i^+$ is infinite, here we can not use the divergence theorem directly. To solve this difficulty, we need to obtain a decay result of the components of the spin $n/2$ zero rest-mass field, then we will approach $i^+$ by the spacelike hypersurface $S_T$ as $T$ tends $+\infty$ and show that the energy on this hypersurface tends to...
zero by using the decay result. The energy equality will show that the energies of the restriction of the solution of the Cauchy problem on the null infinity hypersurfaces $I^\pm$ are finite, so we can define the trace operator $T^\pm$ on $I^\pm$; and also due to the energy equality, we can see that the trace operator is one to one and has a closed range.

To show that the trace operator is surjective, we need to solve the Goursat problem. The Goursat problem will be solved in the partial conformal compactification in the following steps: the first one is solve the Goursat problem on the future $I^+(S)$ of the spacelike hypersurface $S$ (which is chosen such that it pass strictly on the part of the support) by using the result that is generalization of L’Hörmander [42]; the second one is to solve the Cauchy problem on the past $I^-(S)$ of $S$ with the initial data that is the constraint of the solution in the first step on $S$; the solution of the Goursat problem is the union of the solutions of the two steps. To define the trace operator on $\Sigma_0$, we need to show that the energy of the Goursat solution on $\Sigma_0$ is finite, this work is again done by using the energy equality.

We complete this chapter by the following organisation:

- **Section 2.** Describe the geometry of the Minkowski spacetime: the conformal compactification spacetimes (full and partial), the Newman-Penrose tetrad and the calculation of the spin coefficients.

- **Section 3.** We express the spin $n/2$ zero rest-mass field $\phi_{AB...F}$ and the massless equation $\nabla^{AA'} \phi_{AB...F}$ in the conformal compactification spacetimes, then we establish a decay result.

- **Section 4.** We will solve the Cauchy problem on the Einstein cylinder and then apply this result to solve the Cauchy problem on the conformal compactification spacetimes.

- **Section 5.** We give the calculation of the energies and use the decay result to obtain the energy equality. We define the trace operator $T^+$ (res. $T^-$) on the null infinity hypersurface $I^+$ (res. $I^-$) and extend this trace operator on the Sobolev space obtained by the energy norm. Then we show that this trace operator is injective and has a closed range.

- **Section 6.** To show that the trace operator is surjective, we solve the Goursat problem on the partial conformal compactification. After that, we obtain the definition of the conformal scattering operator $W := T^+ \circ (T^-)^{-1}$. Then we give some conclusion about the norm on $I^+$.

- **Appendix.** In the appendix, we will develop the spinor form of the commutators, find a non-trivial solution of the constraint system, give the way that generalise the result of L.Hörmander [42] for the spin $n/2$ zero rest-mass field $\phi_{AB...F}$ and give the detailed calculations for the Goursat problem.

**Notations**
We use the formalisms of abstract indices, \((n+1)\)-component spinor, Newman-Penrose and Geroch-Held-Penrose.

We use the notation \(n_a n_b \ldots n_c \ l_d \ldots \ l_f\) to sum the \(C_n^k\) components where each component has \(k\) factors \(n_i\) and \(n-k\) factors \(l_j\) \((i,j \in \{a,b\ldots f\})\).

### 4.2 Geometric setting

In this section we will construct the conformal structure of the Minkowski spacetime by using two types: the full conformal and the partial conformal. We recall that the Minkowski metric in the spherical coordinates has the form:

\[ g = dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \]  

The Newman-Penrose tetrad normalization for the metric \(g\) can be chosen as

\[ l^a = \frac{1}{\sqrt{2}} (\partial_t + \partial_r), \quad n^a = \frac{1}{\sqrt{2}} (\partial_t - \partial_r), \quad m^a = \frac{1}{r\sqrt{2}} \left( \partial_\vartheta + \frac{i}{\sin \theta} \partial_\varphi \right), \]

they are associated with the spin-frame \(\{o^A, \iota^A\}\) by

\[ l^a = o^A o^A', \quad n^a = \iota^A \iota^A', \quad m^a = o^A \iota^A'. \]

The volume form associated with metric \(g\) is

\[ dVol^4_g = r^2 \sin \theta dt dr d\theta d\varphi = r^2 dt dr d\omega. \]

#### 4.2.1 The full conformal compactification

We choose the advanced and retarded coordinates: \(u = t - r, \ v = t + r\), then we put

\[ p = \arctan u, \quad q = \arctan v, \]

\[ \tau = p + q = \arctan(t-r) + \arctan(t+r), \]

\[ \zeta = q - p = \arctan(t+r) - \arctan(t-r). \]

Choosing the conformal factor

\[ \Omega = \frac{2}{\sqrt{1+u^2}\sqrt{1+v^2}} = \frac{2}{\sqrt{1+(t-r)^2}\sqrt{1+(t+r)^2}}, \]

we obtain the rescaled metric

\[ \hat{g} = \Omega^2 g = d\tau^2 - d\zeta^2 - \sin^2 \zeta d\omega^2, \]  

and the full conformal compactification of the Minkowski spacetime is described by the domain

\[ \hat{M} = \{|\tau| + \zeta \leq \pi, \ \zeta \geq 0, \ \omega \in S^2\}. \]

The full conformal metric \(\hat{g}\) can be extended analytically to the whole Einstein cylinder \(\mathcal{C} = \mathbb{R}_\tau \times S^3_{\zeta, \theta, \varphi}\). The full conformal boundary of Minkowski spacetime can be described as follows:
• The future and fast null infinities are

\[ \mathcal{I}^+ = \{ (\tau, \zeta, \omega); \tau + \zeta = \pi, \zeta \in [0, \pi[, \omega \in S^2 \} , \]

\[ \mathcal{I}^- = \{ (\tau, \zeta, \omega); \tau - \zeta = \pi, \zeta \in [0, \pi[, \omega \in S^2 \} , \]

which are smooth null hypersurfaces for \( \hat{g} \).

• The future and past timelike infinities are

\[ i^\pm = \{ (\tau = \pm \pi, \zeta = 0, \omega); \omega \in S^2 \} , \]

which are smooth points for \( \hat{g} \).

• The spacelike infinity is

\[ i_0 = \{ (\tau = 0, \zeta = \pi, \omega); \omega \in S^2 \} , \]

which is also a smooth point for \( \hat{g} \).

The hypersurface \( \{ t = 0 \} \) in the Minkowski spacetime is described by the 3--sphere \( S^3 = \{ \tau = 0 \} \) excluding the point \( i_0 \) on the Einstein cylinder i.e \( S^3 = \{ t = 0 \} \cup i_0 \).

![Diagram](image_url)

Figure 4.1: The full conformal compactification of the Minkowski spacetime.

We can choose the the Newman-Penrose tetrad normalization as follows:

\[ \hat{\iota}^a = \frac{1}{\sqrt{2}} \left( \partial_\tau + \partial_\zeta \right) , \hat{\iota}^a = \frac{1}{\sqrt{2}} \left( \partial_\tau - \partial_\zeta \right) , \hat{m}^a = \frac{1}{\sqrt{2} \sin \zeta} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi \right) . \]
We can calculate
\[
\partial_t = (1 + \cos \tau \cos \zeta) \partial_r - \sin \tau \sin \zeta \partial_\zeta ,
\]
\[
\partial_r = -\sin \tau \sin \zeta \partial_\tau + (1 + \cos \tau \cos \zeta) \partial_\zeta ,
\]
So that we can think that the vector field $\partial_t$ is normal to the null hypersurface $I^\pm$ and tends to zero at $i^\pm$. We also have
\[
\hat{l}^a = \frac{1}{2} l^a, \quad \hat{n}^a = \frac{1}{2} n^a .
\]
In terms of the associated spin-frame, these later correspond to the rescaling
\[
\hat{o}_A = \frac{\sqrt{1 + v^2}}{\sqrt{1 + u^2}} o_A, \quad \hat{\iota}_A = \frac{\sqrt{1 + u^2}}{\sqrt{1 + v^2}} \iota_A ,
\]
where
\[
\Omega_1 = \frac{2}{1 + u^2}, \quad \Omega_2 = \frac{2}{1 + v^2} .
\]
Since $\hat{l}^a \hat{n}_a = l^a n_a = 1$, we have the relation of the dual $\{ o_A, \iota_A \}$ and its rescaling $\{ \hat{o}_A, \hat{\iota}_A \}$ is
\[
\hat{o}_A = \Omega_1 o_A, \quad \hat{\iota}_A = \Omega_2 \iota_A .
\]
The volume form associated with the rescaled metric $\hat{g}$ is
\[
d\text{Vol}^4_\hat{g} = \sqrt{|\hat{g}|} d\tau d\zeta d^2 \omega = \sin^2 \zeta d\tau d\zeta d^2 \omega = d\tau d\mu_{S^3} ,
\]
where $d\mu_{S^3} = \sin^2 \zeta d\zeta d^2 \omega$ is the volume form of 3–sphere $S^3$ with the Euclidian metric
\[
\sigma^2_{S^3} = d\zeta^2 + \sin^2 \zeta d\omega^2 .
\]
We can calculate the spin coefficients by using the terms of the Ricci notation coefficients (see chapter 2) and give the result as follows:
\[
\hat{\kappa} = \hat{\varepsilon} = \hat{\sigma} = \hat{\gamma} = \hat{\lambda} = \hat{\tau} = \hat{\nu} = \hat{\pi} = 0 ,
\]
\[
\hat{\rho} = \hat{\mu} = \frac{\cot \zeta}{\sqrt{2}}, \quad \hat{\alpha} = -\hat{\beta} = -\frac{\cot \theta}{2\sqrt{2} \sin \zeta} .
\]

### 4.2.2 A partial conformal compactification

We choose the new variables $u = t - r$, $R = 1/r$ (here $u$ is also called retard time variable). Then we obtain the following expression for the rescaled metric $\hat{g}$ by using the conformal factor $\Omega' = R$
\[
\hat{g} = R^2 g = R^2 du^2 - 2 dudR - d\omega^2 ,
\]
which can be extended as an analytic metric on the domain $\mathbb{R}_u \times [0, +\infty] \times S^2_{\theta, \varphi}$. So we can add to the Minkowski spacetime the boundary $\mathbb{R}_u \times \{ R = 0 \} \times S^2_{\theta, \varphi}$. As $r$ goes to $+\infty$, a point on this boundary $(u = u_0, R = 0, \theta = \theta_0, \varphi = \varphi_0)$ is reached along an outgoing radial null geodesic
\[
\gamma_{u_0, \theta_0, \varphi_0}(r) = (t = r + u_0, r, \theta = \theta_0, \varphi = \varphi_0) ,
\]
then there is a one to one correspondence between the outgoing radial null geodesics and the points on the boundary. So this boundary describes the future null infinity $\mathcal{I}^+$

$$
\mathcal{I}^+ = \mathbb{R}_u \times \{ R = 0 \} \times S^2_\omega .
$$

And similarly we can use an advanced time variable $v = t + r$, which allows us to construct the past null infinity $\mathcal{I}^-$. We also denote the points at infinity by $i^+, i^-$ and $i_0$

- The future (res. past) timelike infinity point $i^+$ (res. $i^-$) defined as the limit point of uniformly timelike curves as $t$ tend to $+\infty$ (res. $-\infty$) is

$$
i^\pm = \{ (u = \pm \infty, R = 0, \omega); \ \omega \in S^2 \} .
$$

- The spacelike infinity point $i_0$ defined as the limit point of uniformly spacelike curves as $r$ tend to $+\infty$ is

$$
i_0 = \{ (u = \mp \infty, R = 0, \omega); \ \omega \in S^2 \} .
$$

The null infinity hypersurface $\mathcal{I}^\pm$ are the same null infinity hypersurfaces in the full conformal compactification, but the difference from the full conformal compactification is that the points $i^\pm$ and $i_0$ are infinite.

Now the partial conformal compactification can be described by the domain

$$
\hat{\mathcal{M}} = \mathcal{M} \cup \mathcal{I}^\pm .
$$

We make the following choice of the Newman-Penrose tetrad normalization

$$
\hat{l}^a = -\frac{1}{\sqrt{2}} \partial_R, \quad \hat{n}^a = \sqrt{2} \left( \partial_u + \frac{R^2}{2} \partial_R \right), \quad \hat{m}^a = \frac{1}{\sqrt{2}} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right).
$$

We can think that

$$
\hat{l}^a = r^2 l^a, \quad \hat{n}^a = n^a, \quad \hat{m}^a = rm^a.
$$

In terms of the associated spin-frame, these later correspond to the rescaling

$$
\hat{\sigma}^A = r o^A, \quad \hat{\iota}^A = \iota^A, \quad \hat{o}_A = o_A, \quad \hat{i}_A = R i_A.
$$

The volume form associated with the rescaled metric $\hat{g}$ is

$$
d\text{Vol}_\hat{g}^4 = (\Omega')^4 d\text{Vol}_g^4 = R^2 dt dr d\omega = -dt dR d\omega .
$$

We can calculate the spin coefficients by using the terms of the Ricci notation coefficients (see chapter 2) and give the result as follows:

$$
\hat{k} = \hat{\epsilon} = \hat{\sigma} = \hat{\lambda} = \hat{\tau} = \hat{\nu} = \hat{\kappa} = \hat{\rho} = \hat{\mu} = 0 , \quad (4.6)
$$

$$
\hat{\gamma} = \frac{R}{\sqrt{2}}, \quad \hat{\alpha} = -\hat{\beta} = -\frac{\cot \theta}{2\sqrt{2}} . \quad (4.7)
$$
Chapter 4. Conformal scattering on Minkowski spacetime

4.3 The spin $n/2$ zero rest-mass field

From the total symmetry of $\phi_{AB\ldots F}^{\text{n index}} = \phi_{(AB\ldots F)}^{\text{n index}}$, we can obtain the expression of the spin $n/2$ zero rest-mass field:

$$
\phi_{AB\ldots F} = \phi_n o_{A0B\ldots 0F} - \phi_{n-1}(t_{A0B\ldots 0F} + \ldots + o_{A0B\ldots nF})
+ \ldots + (-1)^k \phi_{n-k} \sum_{k \text{ terms}} t_{A^{lB\ldots lC}} o_{D\ldots EF} + \ldots + (-1)^n \phi_0 t^{aB\ldots nF},
$$

(4.8)

where $\phi_r = \phi_{00\ldots 0\frac{r}{n-r \text{ terms}} 11\ldots 1}$ (0 $\leq r \leq n$). So the weight function $\phi_r$ has the weight $(n-r, r; 0, 0)$ or simply $(p = n - r, q = r)$. And we can calculate

$$
\phi_{AB\ldots F}^{\ell} \phi_{A'B'\ldots F'} = |\phi_{n-k}|^2 \sum_{k \text{ terms}}^{n} n_{a_n b_{n-1} c_{n-k}} l_{d\ldots f} + A,
$$

(4.9)

where $A$ is the sum of the components that contain $m_a$ or $\bar{m}_a$.

Using the Geroch-Held-Penrose formalism, we can rewrite the massless equation $\nabla^A\phi_{AB\ldots F} = 0$ under the form (see eq.(4.12.44) in [73] Volume 1)

$$
\begin{cases}
\n\phi_r - \partial' \phi_{r-1} &= -(r - 1) \lambda \phi_{r-2} + r \pi \phi_{r-1} + (n - r + 1) \rho \phi_r - (n - r) \kappa \phi_{r+1}, \\
\n\phi'_{r} - \partial \phi_{r+1} &= (n - r - 1) \sigma \phi_{r+2} - (n - r) \tau \phi_{r+1} - (r + 1) \mu \phi_r + r \nu \phi_{r-1},
\end{cases}
$$

(4.10)

where $r = 1, 2\ldots n$ in the first equation and $r = 0, 1\ldots n - 1$ in the second equation, and

$$
\begin{align*}
\n\phi_r &= (l^a \partial_a - (n - 2r) \varepsilon) \phi_r, \\
\n\partial' \phi_{r-1} &= (\bar{m}^a \partial_a - (n - 2r + 2) \alpha) \phi_{r-1}, \\
\n\phi'_{r} &= (n^a - (n - 2r) \gamma) \phi_{r}, \\
\n\partial \phi_{r+1} &= (m^a \partial_a - (n - 2r - 2) \beta) \phi_{r+1}
\end{align*}
$$

are the effects of the weight derivatives on the weight functions.

On the full conformal compactification

In section 2, we have proved that the rescaled spin-frame $\{\hat{o}_A, \hat{i}_A\}$ is given by

$$
\hat{o}_A = \Omega_1 o_A, \quad \hat{i}_A = \Omega_2 i_A
$$
where

\[
\hat{\phi}_{AB...F} = \Omega^{-1}\phi_{AB...F} = \Omega_1^{-1}\Omega_2^{-1}\phi_{AB...F}
\]

\[
= \Omega_1^{-1}\Omega_2^{-1}\phi_n o_{A0B...OF} - \Omega_1^{-1}\Omega_2^{-1}\phi_{n-1}(\iota_A o_{B...OF} + ... + o_{A0B...t_F})
\]

\[
+ ... + \Omega_1^{-1}\Omega_2^{-1}(-1)^k\phi_{n-k}\sum_{k \text{ terms}}^{k \text{ terms}} \Omega_2^{-1}\phi_{n-k-1}(\iota_A o_{B...OF} + ...
\]

\[
+ ... + \Omega_1^{-1}\Omega_2^{-1}(-1)^n\phi_0\iota_A t_{B...t_F}
\]

\[
= \Omega_1^{-1}\phi_n \hat{o}_A \hat{o}_B...\hat{o}_F - \Omega_1^{-1}\Omega_2^{-1}\phi_{n-1}(\iota_A \hat{o}_B...\hat{o}_F + ... + \hat{o}_A \hat{o}_B...\hat{i}_F)
\]

\[
+ ... + \Omega_1^{-1}\Omega_2^{-1}(-1)^k\phi_{n-k}\sum_{k \text{ terms}}^{k \text{ terms}} \Omega_2^{-1}\phi_{n-k-1}(\iota_A \hat{o}_B...\hat{o}_F + ...
\]

\[
+ ... + \Omega_1^{-1}\Omega_2^{-1}(-1)^n\phi_0\iota_A \hat{i}_B...\hat{i}_F.
\]

Comparing the coefficients, we get

\[
\hat{\phi}_{n-k} = \Omega_1^{-1-k}\Omega_2^{-1-(n-k)}(-1)^k\phi_{n-k} , 0 \leq k \leq n .
\]  

(4.11)

Remplacing the spin coefficients \([4.3]\) and \([4.4]\) into the expression \([4.10]\), we get the expression of the massless equation in \(\hat{M}\) as follows:

\[
\left\{
\begin{array}{ll}
\frac{1}{\sqrt{2}}(\partial_r + \partial_\xi)\hat{\phi}_r - \frac{1}{\sqrt{2}\sin \xi} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\phi + (n - 2r + 2) \cot \theta \right) \hat{\phi}_{r-1}
& = (n - r + 1) \frac{\cot \xi}{2} \hat{\phi}_r ,
\end{array}
\right.
\]  

(4.12)

where \(r = 1, 2...n\) in the first equation and \(r = 0, 1...n - 1\) in the second equation.

On the partial conformal compactification

We have \(\Omega' = 1/r\) and the rescaled spin-frame \(\{\hat{o}_A, \iota_A\}\) is given by

\[
\hat{o}_A = o_A , \hat{i}_A = \Omega' \iota_A = R \iota_A ,
\]

so we have

\[
\hat{\phi}_{AB...F} = (\Omega')^{-1}\phi_{AB...F}
\]

\[
= \Omega^{-1}\phi_n o_{A0B...OF} - (\Omega')^{-1}\phi_{n-1}(\iota_A o_{B...OF} + ... + o_{A0B...t_F})
\]

\[
+ ... + (\Omega')^{-1}(-1)^k\phi_{n-k}\sum_{k \text{ terms}}^{k \text{ terms}} \phi_{n-k-1}(\iota_A o_{B...OF} + ...
\]

\[
+ ... + (\Omega')^{-1}(-1)^n\phi_0\iota_A t_{B...t_F}
\]

\[
= r\phi_n \hat{o}_A \hat{o}_B...\hat{o}_F - r^2\phi_{n-1}(\iota_A \hat{o}_B...\hat{o}_F + ... + \hat{o}_A \hat{o}_B...\hat{i}_F)
\]

\[
+ ... + r^{1+k}(-1)^k\phi_{n-k}\sum_{k \text{ terms}}^{k \text{ terms}} \phi_{n-k-1}(\iota_A \hat{o}_B...\hat{o}_F + ...
\]

\[
+ ... + r^{1+n}(-1)^n\phi_0\iota_A \hat{i}_B...\hat{i}_F.
\]
Comparing the coefficients, we get
\[ \hat{\phi}_{n-k} = r^{1+k} \phi_{n-k}, \quad 0 \leq k \leq n. \] (4.13)

Remplacings the spin coefficients (4.6) and (4.7) into the expression (4.10), we get the expression of the massless equation in \( \hat{M} \) as follows:
\[
\begin{aligned}
&\left\{-\frac{1}{\sqrt{2}} \partial R \hat{\phi}_r - \frac{1}{\sqrt{2}} \left( \partial_\theta - i \frac{\sin \theta}{\sin \varphi} \partial_\varphi + (n - 2r + 2) \cot \theta \right) \hat{\phi}_{r-1} \right. \\
&\left. \left( \sqrt{2} \partial_u + \frac{R^2}{\sqrt{2}} \partial_R - (n - 2r) \frac{R}{\sqrt{2}} \right) \hat{\phi}_r - \frac{1}{\sqrt{2}} \left( \partial_\theta + i \frac{\sin \theta}{\sin \varphi} \partial_\varphi - (n - 2r - 2) \cot \theta \right) \hat{\phi}_{r+1} \right) = 0
\end{aligned}
\] (4.14)

where \( r = 1, 2, \ldots n \) in the first equation and \( r = 0, 1, \ldots n - 1 \) in the second equation.

**Decay of the spin \( n/2 \) zero rest-mass field \( \phi_{AB...F} \)**

**Proposition 4.3.1.** There exists two constants \( C_{k}^{\pm} \) such that
\[
\lim_{t \to \pm \infty} t^{n+2} \phi_{n-k} = C_{k}^{\pm}.
\]

In other words, all of the components of spin \( n/2 \) zero rest mass field \( \phi_{AB...F} \) decays as \( 1/t^{n+2} \) along the integral line of \( \partial_t \). As a direct consequence of this decay result, on the partial conformal compactification, we have
\[
\lim_{t \to \pm \infty} t^{n+2} \hat{\phi}_{n-k} = C_{k}^{\pm}.
\]

**Proof.** On the full conformal compactification, we have: \( \hat{\phi}_{AB...F}(i^+) = \lim_{t \to +\infty} \Omega^{-1} \phi_{AB...F} \), so that we can put
\[
C_{k}^{+} = \lim_{t \to +\infty} \Omega_1^{-(n-k)} \Omega_2^{-k} \Omega^{-1} \phi_{n-k}
= \lim_{t \to +\infty} \left( \frac{\sqrt{1 + (t - r)^2}}{\sqrt{2}} \right)^{n-k} \left( \frac{\sqrt{1 + (t + r)^2}}{\sqrt{2}} \right)^k \frac{\sqrt{1 + (t - r)^2}}{\sqrt{2}} \phi_{n-k}.
= \lim_{t \to +\infty} t^{n+2} \phi_{n-k}.
\]

Similarly, we can show that there exists constants \( C_{k}^{-} \) such that
\[
C_{k}^{-} = \lim_{t \to -\infty} t^{n+2} \phi_{n-k} = C_{k}^{-}.
\]

The last equations in Proposition 4.3.1 are a direct consequence of the decay result above and the equations (4.13)
4.4 The Cauchy problem

In this section, we consider the Cauchy problem of the massless equation \( \nabla^A A^B \psi_{AB...F} = 0 \). First, we will show that it is well-posed in the whole Einstein cylinder \( \mathcal{C} = \mathbb{R} \times S^3_{\zeta,\theta,\varphi} \) and then as a consequence, we obtain that it is also well-posed in the conformal compactification spacetimes \( \hat{\mathcal{M}} \) and \( \hat{\mathcal{M}}' \).

Consider the Cauchy problem of the rescaled massless equation with the initial data on \( S^3 = \{ \tau = 0 \} \) in \( \mathcal{C} \)

\[
\begin{align*}
\nabla^A A^B \hat{\phi}_{AB...F} &= 0, \\
\hat{\phi}_{AB...F}|_{S^3} &= \hat{\psi}_{AB...F} \in C^\infty(S^3, \mathcal{S}_{(AB...F)}) \cap D,
\end{align*}
\]

where \( D \) is the constraint space on \( \Sigma_0 \), which can be also understood as the projection space of \( \nabla^A A^B \hat{\phi}_{AB...F} = 0 \) on the future-oriented timelike vector \( T^a = \partial_\tau \) at \( \tau = 0 \)

\[
D = \left\{ \hat{\phi}_{AB...F} \in L^2(\Sigma_0, \mathcal{S}_{(AB...F)}) : \left( T^a \nabla^Z A^B \hat{\phi}_{ZAC...F} \right) |_{\tau = 0} = 0 \right\}.
\]

This problem can be stated as follows:

**Proposition 4.4.1. (Cauchy problem)** The Cauchy problem for the rescaled massless equation \((4.15)\) in \( \mathcal{C} \) is well-posed i.e for any \( \hat{\psi}_{AB...F} \in C^\infty(S^3, \mathcal{S}_{(AB...F)}) \cap D \) there exists a unique \( \hat{\phi}_{AB...F} \) solution of \( \nabla^A A^B \hat{\phi}_{AB...F} = 0 \) such that

\[
\hat{\phi}_{AB...F} \in C^\infty(\mathcal{C}, \mathcal{S}_{(AB...F)}) ; \hat{\phi}_{AB...F}|_{\tau=0} = \hat{\psi}_{AB...F}.
\]

**Proof.** First, we show that the system \((4.15)\) can split into the constraint equation

\[
T^a \nabla^Z A^B \hat{\phi}_{ZAC...F} = 0
\]

and a symmetric hyperbolic evolution system. Indeed, \((4.15)\) can be expressed as a set of \(2n\) scalar equations on the spin components of \( \hat{\phi}_{AB...F} \) (see equation \((4.12)\))

\[
\begin{align*}
\frac{1}{\sqrt{2}}(\partial_\tau + \partial_\zeta) \hat{\phi}_r - \frac{1}{\sqrt{2}\sin \zeta} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + (n - 2r + 2) \cot \theta \right) \hat{\phi}_{r-1} &= (n - r + 1) \cot \frac{\theta}{2} \hat{\phi}_r \\
& \quad \text{with } 1 \leq r \leq n, \\
\frac{1}{\sqrt{2}}(\partial_\tau - \partial_\zeta) \hat{\phi}_r - \frac{1}{\sqrt{2}\sin \zeta} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - (n - 2r - 2) \cot \theta \right) \hat{\phi}_{r+1} &= -(r + 1) \cot \frac{\theta}{2} \hat{\phi}_r \\
& \quad \text{with } 0 \leq r \leq n - 1.
\end{align*}
\]

From the equations of the above system, we will keep the first and the last equation which correspond to \( r = n \) and \( r = 0 \) respectively. Then we obtain \((n - 1)\) equations which are a consequence of adding \((n - 1)\) couples of the equations of the above system corresponding to \( r = 1, 2, \ldots, n - 1 \) respectively.
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So we get the evolution system with \((n + 1)\) equations as follows

\[
\begin{cases}
\frac{1}{\sqrt{2}} (\partial_{\tau} + \partial_{\zeta}) \hat{\varphi}_n - \frac{1}{\sqrt{2} \sin \zeta} \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\varphi} + (2 - n) \cot \frac{\theta}{2} \right) \hat{\varphi}_{n-1} = \frac{\cot \theta}{2} \hat{\varphi}_n , \\
\sqrt{2} \partial_{\tau} \hat{\varphi}_r - \frac{1}{\sqrt{2} \sin \zeta} \left( \partial_{\theta} - \frac{i}{\sin \theta} \partial_{\varphi} + (n - 2r + 2) \cot \frac{\theta}{2} \right) \hat{\varphi}_{r-1} = (n - 2r) \cot \theta \hat{\varphi}_r \\
-\frac{1}{\sqrt{2} \sin \zeta} \left( \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\varphi} - (n - 2r - 2) \cot \frac{\theta}{2} \right) \hat{\varphi}_{r+1} = (n - 2r) \cot \theta \hat{\varphi}_r \\
\frac{1}{\sqrt{2}} (\partial_{\tau} - \partial_{\zeta}) \hat{\varphi}_0 - \frac{1}{\sqrt{2} \sin \zeta} \left( \partial_{\theta} + \frac{i}{\sin \theta} \partial_{\varphi} + (2 - n) \cot \frac{\theta}{2} \right) \hat{\varphi}_1 = -\frac{\cot \theta}{2} \hat{\varphi}_0 
\end{cases}
\] (4.16)

where \(r = 1, 2 \ldots n - 1\).

We can rewrite the evolution system above under the matrix form by putting

\[
\Phi = \begin{pmatrix} \hat{\varphi}_n \\ \hat{\varphi}_{n-1} \\ \vdots \\ \hat{\varphi}_0 \end{pmatrix}
\]

The effect of the derivative operator on \(\Phi\), can be understood as the effect on each components of \(\Phi\), for instance

\[
\partial_{\tau} \Phi = \begin{pmatrix} \partial_{\tau} \hat{\varphi}_n \\ \partial_{\tau} \hat{\varphi}_{n-1} \\ \vdots \\ \partial_{\tau} \hat{\varphi}_0 \end{pmatrix}
\]

So that the matrix coefficient with \(\partial_{\tau}\) and \(\partial_{\zeta}\) respectively are

\[
A = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \ldots & 0 & 0 \\
0 & \sqrt{2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \sqrt{2} & 0 \\
0 & 0 & \ldots & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

which are \((n + 1) \times (n + 1)\)–matrix diagrams. And the matrix coefficient with \(\partial_{\theta}\) and \(\partial_{\varphi}\) respectively are

\[
C = \begin{pmatrix}
0 & \frac{1}{\sqrt{2} \sin \zeta} & \ldots & 0 & 0 \\
-\frac{1}{\sqrt{2} \sin \zeta} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \frac{1}{\sqrt{2} \sin \zeta} \\
0 & 0 & \ldots & \frac{1}{\sqrt{2} \sin \zeta} & 0
\end{pmatrix}
\]
and

\[
D = \begin{pmatrix}
0 & \frac{i}{\sqrt{2}} \sin \zeta \sin \theta & \ldots & 0 & 0 \\
-\frac{i}{\sqrt{2}} \sin \zeta \sin \theta & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & \frac{i}{\sqrt{2}} \sin \zeta \sin \theta \\
0 & 0 & \ldots & -\frac{i}{\sqrt{2}} \sin \zeta \sin \theta & 0
\end{pmatrix}.
\]

So that we obtain the matrix form of the evolution system as

\[
A \partial_\tau \Phi + B \partial_\zeta \Phi + C \partial_\theta \Phi + D \partial_\varphi \Phi + H \Phi = 0
\]

which is equivalent to

\[
\partial_\tau \Phi + A^{-1} B \partial_\zeta \Phi + A^{-1} C \partial_\theta \Phi + A^{-1} D \partial_\varphi \Phi + A^{-1} H \Phi = 0,
\]

where \(A, B, C, D\) are given above and \(H\) is the matrix of zero order terms. The coefficients of \(C\) and \(D\) are singular at \(\xi = 0, \pi\) and \(\theta = 0, \pi\). These are coordinates singularities due to the choice of spherical coordinates on \(S^3\). The spherical symmetry entails that these are not authentic singularities. Since \(A, B, C, D\) are Hermitian and \(A\) is diagonal we can easily check that \(A^{-1} B, A^{-1} C, A^{-1} D\) are also Hermitian. Therefore the evolution system \((4.16)\) is a symmetric hyperbolic system. By using Leray’s theorem, with the initial data \(\hat{\psi}_{AB...F} \in C^\infty(S^3, S(AB...F))\) the evolution system \((4.16)\) has a unique solution \(\hat{\phi}_{AB...F} \in C^\infty(E, S(AB...F))\). We will show that this solution is the solution of the original system \((4.15)\) by checking that the constraints system is conserved under the evolution solution. Indeed, if we put

\[
\hat{\nabla}_{A}^{\tau} \hat{\phi}_{ZAC...F} = \hat{\Xi}_{AA'C...F},
\]

we can understand the constraint system on \(\Sigma_{\tau} = \{\tau = \text{constant}\}\) by projecting \(\hat{\nabla}^{AA'} \hat{\psi}_{AB...F}\) on \(T^a\) to get

\[
\Lambda_{C...F} = T^z \hat{\Xi}_{zC...F}.
\]

We can obtain the spinor form of the evolution system by projecting of \(\hat{\nabla}^{AA'} \hat{\phi}_{AB...F} = 0\) on \((T^a)\perp\) to get

\[
\hat{E}_{AA'C...F} = \hat{\Xi}_{AA'C...F} - (T^z \hat{\Xi}_{ZZ'C...F}) T_a.
\]

Since \(\hat{\phi}_{AB...F}\) is a solution of the evolution system so

\[
\hat{E}_{AA'C...F} = \hat{\Xi}_{AA'C...F} - (T^z \hat{\Xi}_{ZZ'C...F}) T_a = 0,
\]

hence

\[
\hat{\nabla}^{AA'} \Xi_{AA'C...F} = \hat{\nabla}^{AA'} \left( (T^z \hat{\Xi}_{ZZ'C...F}) T_a \right).
\]

We can express (see eq.(5.8.1) in Vol.1 or see equation (4.37) in Appendix 4.7.1 of this chapter)

\[
\hat{\nabla}^{AA'} \Xi_{AA'C...F} = \hat{\nabla}^{AA'} \hat{\nabla}^{Z}_{A'A'} \hat{\phi}_{ZAC...F} = \hat{\nabla}^{A'(A \hat{\nabla}^{Z})_{A'} \phi}_{ZAC...F} = \hat{\phi}_{AZM(C...K\hat{\psi}_{F})^{AZM} = 0,
\]
where $\Psi_{ABCD}$ is the Weyl conformal spinor of the Einstein metric (4.2), it disappears since $\Psi_{ABCD} = \Psi_{ABCD}$ ($\Psi_{ABCD}$ is invariant under the conformal operator) and $\Psi_{ABCD} = 0$ in the Minkowski spacetime. So we have

\[
0 = \hat{\nabla} AA' \left( (T^z \hat{\Xi}_{zC...F}) T_a \right) \\
= T_a \hat{\nabla}^a (T^z \hat{\Xi}_{zC...F}) + (T^z \hat{\Xi}_{zC...F}) \hat{\nabla}^a T_a \\
= \partial_\tau (T^z \hat{\Xi}_{zC...F}) + (T^z \hat{\Xi}_{zC...F}) \hat{\nabla}^a T_a \\
= \frac{1}{\sqrt{2}} (\hat{D} + \hat{D}') (T^z \hat{\Xi}_{zC...F}) + (T^z \hat{\Xi}_{zC...F}) \hat{\nabla}^a T_a. \tag{4.18}
\]

Because (see eq.(4.5.26) in Vol.1) we have:

\[
\hat{D} \hat{o}_A = \hat{\varepsilon} \hat{o}_A - \hat{k} \hat{i}_A = 0
\]

and similarly

\[
\hat{D} \hat{i}_A = -\hat{\varepsilon} \hat{i}_A + \hat{\pi} \hat{o}_A = 0,
\]

\[
\hat{D}' \hat{o}_A = \hat{\gamma} \hat{o}_A - \hat{\tau} \hat{i}_A = 0,
\]

\[
\hat{D}' \hat{i}_A = -\hat{\gamma} \hat{o}_A + \hat{\nu} \hat{i}_A = 0
\]

so the effect of $\hat{D} + \hat{D}'$ on the full spinor $(T^z \hat{\Xi}_{zC...F})$ can be understood as the effect on the weight functions only. And due to $v = \hat{\nabla}^a T_a = 0$ (since $\partial_\tau$ is a Killing vector field), by projecting the equation (4.18) on the spin-frame $\{\hat{o}_A, \hat{i}_A\}$, we get its scalar form as follows

\[
\frac{1}{\sqrt{2}} \partial_\tau \hat{\Lambda} = 0
\]

where $\hat{\Lambda}$ is the matrix components of $T^z \hat{\Xi}_{zC...F}$. This equation has a unique solution and because $\hat{\Lambda}|_{\tau=0} = 0$ so the solution is zero for all $\tau$ i.e

\[
T^{ZZ'} \hat{\nabla}_A^Z \hat{\phi}_{ZAC...F} = 0 \text{ for all } \tau
\]

so that the constraints system is conserved.

\[\square\]

**Remark 4.4.1.** It is not completely clear that all the finite energy data for the evolution system can be approached by smooth compact supported data satisfying the constraint system. The existence of the non-trivial solution of the constraint system is not clearly seen, it can be found in Appendix 4.7.2.

Immediately the Cauchy problem is well-posed in the full conformal compactification $\hat{\mathbb{M}}$:

**Corollary 4.4.1.** The solution of the system (4.15) in the full conformal compactification $\hat{\mathbb{M}}$ is the constraint of the solution of the system (4.15) in $\mathbb{C}$ on $\hat{\mathbb{M}}$.

We need to solve the Cauchy problem in the partial conformal compactification $\hat{\mathbb{M}}$ to give the solution for constructing our theory. Because $i_0$ is infinite, so we need to suppose that the support of the initial data is compact. Then we can state the Cauchy problem on the partial conformal compactification as follows:
Corollary 4.4.2. The Cauchy problem of the system (4.15) in \( \hat{\mathbb{M}} \) with the initial data \( \hat{\psi}_{AB...F} \in C_0^\infty(\Sigma_0, \hat{S}_{(AB...F)}) \cap \mathcal{D} \) is well-posed i.e for any \( \hat{\psi}_{AB...F} \in C_0^\infty(\Sigma_0, \hat{S}_{(AB...F)}) \cap \mathcal{D} \) there exists a unique \( \hat{\phi}_{AB...F} \) solution of \( \hat{\nabla}^{AA'}\hat{\phi}_{AB...F} = 0 \) such that

\[
\hat{\phi}_{AB...F} \in C^\infty(\hat{\mathbb{M}}, \hat{S}_{(AB...F)}); \quad \hat{\phi}_{AB...F}\big|_{t=0} = \hat{\psi}_{AB...F},
\]

where we also denote by \( \mathcal{D} \) the constraint space on \( \Sigma_0 = \{ t = 0 \} \) in \( \hat{\mathbb{M}} \).

Proof. Using the full conformal mapping, we can transform the domain \( \hat{\mathbb{M}} \) into the Einstein cylinder. Now the initial data \( \hat{\psi}_{AB...F} = \Omega^{-1} \Omega' \hat{\psi}_{AB...F} \) is zero in the neighbourhood of \( i_0 \) which is a smooth point on the cylinder, so we extend the initial data which is zero in the rest of the support. Now we can apply Proposition 4.4.1, then the solution will be the restriction of that of the Cauchy problem in \( \mathcal{C} \) on \( \hat{\mathbb{M}} \).

\[ \square \]

4.5 Energies

We say that \( S \) is a spacelike hypersurface with the future-oriented unit normal vector field \( \nu^a \) in the Minkowski spacetime \( \mathbb{M} \). We define the current conserved energy by

\[
J_a = \phi_{AB...F} \bar{\phi}_{AB'...F'} \tau^{b} \tau^{c} \ldots \tau^{f} = \left( |\phi_{n-k}|^2 \sum_{k=0}^{n} \hat{n}_a \hat{n}_b \ldots \hat{n}_c \hat{l}_d \ldots \hat{l}_f + A \right) \tau^{b} \tau^{c} \ldots \tau^{f},
\]

where \( \tau^{a} \) are timelike vector fields, which doesn’t change when we changing the metric by using the conformal mapping. For convenience we give

\[
\phi_{AB...F} \bar{\phi}_{AB'...F'} = |\phi_{n}|^2 \left( \hat{n}_a \hat{l}_b \ldots \hat{l}_f + |\phi_0|^2 \hat{n}_a \ldots \hat{n}_f \right) \\
+ \sum_{k=1}^{n-1} |\phi_{n-k}|^2 \left( \hat{n}_a \hat{n}_b \ldots \hat{n}_c \hat{l}_d \ldots \hat{l}_f + \hat{l}_a \hat{n}_b \ldots \hat{n}_c \hat{n}_d \hat{l}_e \ldots \hat{l}_f \right) + A,
\]

(4.19)

where \( A \) is the sum of the components that contain \( m^a \) or \( \bar{m}^a \). Note that when we calculate the energies in the conformal compactification spacetimes, the sum \( A \) will be vanished from the normalization condition of Newman-Penrose tetrad.

Since \( S \) is a spacelike hypersurface, we can choose the transversal vector to \( S \) is also \( \nu^a \). The energy of the field \( \phi_{AB...F} \) is defined on \( S \) by

\[
\mathcal{E}_S(\phi_{AB...F}) = \int_S J_a \nu^a (\nu^a d\text{Vol}^4) = \int_S J_a \nu^a d\mu_S.
\]

(4.20)

Now we set the conformality as

\[
\hat{g} := \Omega^2 g, \quad \hat{\phi}_{AB...F} := \Omega^{-1} \phi_{AB...F},
\]
the current conserved energy is now given by

\[ \hat{J}_a = \hat{\phi}_{AB...F} \hat{\phi}_{A'B'...F'} \tau^b \tau^c ... \tau^f , \]

the unit normal vector to \( S \) for \( \hat{g} \) is now

\[ \hat{\nu}^a = \Omega^{-1} \nu^a , \]

and if we denote by \( \mu_S \) (resp. \( \hat{\mu}_S \)) the measure induced on \( S \) by \( g \) (resp. \( \hat{g} \)), then

\[ \hat{\mu}_S = \Omega^3 \mu_S . \]

The energy of the rescaled field on \( S \) is

\[ \hat{E}_S(\hat{\phi}_{AB...F}) = \int_S \hat{J}_a \hat{\nu}^a d\hat{\mu}_S = \int_S \Omega^{-2} J_a \Omega^{-1} \nu^a \Omega^3 d\mu_S = E_S(\phi_{AB...F}) . \]

Therefore, if the vector fields \( \tau \) don’t change, then the energy on a spacelike hypersurface is conformally invariant.

**On the full conformal compactification**

We choose the vectors \( \tau^b, \tau^c...\tau^f \) as follows

\[ \tau^b = \tau^c = ... = \tau^f = \partial_\tau = \frac{1}{\sqrt{2}} (\hat{l}_a + \hat{n}_a) . \]

The unit normal vector to the hypersurface \( \Sigma_0 \) is

\[ \hat{\nu}^a_{\Sigma_0} = \partial_\tau = \frac{1}{\sqrt{2}} (\hat{l}_a + \hat{n}_a) . \]

Combining with the expression \([4.19]\), we can calculate

\[ \hat{J}_a \hat{\nu}^a_{\Sigma_0} = \left( \frac{1}{\sqrt{2}} \right)^n \left| \hat{\phi}_{n-k} \right|^2 \sum_{k=0}^n \hat{\nu}_a \hat{\nu}_b...\hat{\nu}_c \hat{l}_d...\hat{l}_f \left( \hat{l}^b + \hat{n}^b \right) \left( \hat{l}^c + \hat{n}^c \right) ...(\hat{l}^f + \hat{n}^f) \left( \hat{l}^a + \hat{n}^a \right) \]

\[ = \left( \frac{1}{\sqrt{2}} \right)^n \sum_{k=0}^n C^k_n \left| \hat{\phi}_{n-k} \right|^2 , \]

so

\[ \hat{E}_{\Sigma_0}(\hat{\phi}_{AB...F}) = \left( \frac{1}{\sqrt{2}} \right)^n \int_{\Sigma_0} \sum_{k=0}^n C^k_n \left| \hat{\phi}_{n-k} \right|^2 d\mu_{S^3} . \] (4.21)

Since we have the definition of trace operator \( \hat{\phi}_{AB...F} |_{\mathcal{I}^+} \) on the null infinity hypersurface \( \mathcal{I}^+ \), so we can define and calculate the energy on this hypersurface. The normal vector to the null infinity \( \mathcal{I}^+ \) is

\[ \hat{N}^a_{\mathcal{I}^+} = \hat{n}^a = \frac{1}{\sqrt{2}} (\partial_\tau - \partial_\zeta) , \]
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so the transversal vector to the null infinity $\mathcal{I}^+$ is

$$\hat{\mathcal{L}}_{\mathcal{I}^+}^a = \sqrt{2}\partial_\tau.$$ 

Using again the expression (4.19), we can calculate

$$\hat{J}_a\hat{\mathcal{N}}_+^a = \left(\frac{1}{\sqrt{2}}\right)^{n-1} \left(|\hat{\phi}_{n-k}|^2 \sum_{k=0}^{n} \hat{n}_a \hat{n}_b \ldots \hat{n}_c \hat{l}_d \ldots \hat{l}_f \right) (\hat{l}^b + \hat{n}^b)(\hat{l}^c + \hat{n}^c)(\hat{l}^f + \hat{n}^f)\hat{n}^a$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n-1} \sum_{k=0}^{n-1} C_{n-1}^{k} |\hat{\phi}_{n-k}|^2,$$

so

$$\hat{\mathcal{E}}_{\mathcal{I}^+}(\hat{\phi}_{AB\ldots F}) = \int_{\mathcal{I}^+} \hat{J}_a\hat{\mathcal{N}}_+^a (\hat{\mathcal{L}}_{\mathcal{I}^+}^a d\text{Vol}_g^4)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n-2} \int_{\mathcal{I}^+} \sum_{k=0}^{n-1} C_{n-1}^{k} |\hat{\phi}_{n-k}|^2 d\mu_{S^3}.$$ 

On the partial conformal compactification

We choose the vectors $\tau^b, \tau^c, \ldots, \tau^f$ as follows

$$\tau^b = \tau^c = \ldots = \tau^f = \partial_t = \partial_u = \frac{1}{\sqrt{2}}(\hat{n}^a + R^2 \hat{i}^a).$$

The unit normal vector to the hypersurface $\Sigma_0$ is

$$\hat{n}^a_0 = r\partial_r = \frac{r}{\sqrt{2}}(\hat{n}^a + R^2 \hat{i}^a).$$

We have

$$\tau^b \tau^c \ldots \tau^f = \left(\frac{1}{\sqrt{2}}\right)^{n-1} (\hat{n}^b + R^2 \hat{i}^b)(\hat{n}^c + R^2 \hat{i}^c)(\hat{n}^f + R^2 \hat{i}^f)$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n-1} \left(\hat{b} \hat{c} \hat{d} \ldots \hat{f} + R^2(n-1) \hat{b} \hat{c} \hat{d} \ldots \hat{f} \right)_{\text{n-1 terms}}$$

$$+ \left(\frac{1}{\sqrt{2}}\right)^{n-1} \left(\sum_{k=1}^{n-1} R^2(k-1) \hat{b} \hat{c} \hat{d} \ldots \hat{f} \right)_{\text{k-1 terms}}$$

$$+ \left(\frac{1}{\sqrt{2}}\right)^{n-1} \left(\sum_{k=1}^{n-1} R^2(k-1) \hat{b} \hat{c} \hat{d} \ldots \hat{f} \right)_{\text{k terms}}$$

and using the expression (4.19)

$$\hat{\phi}_{AB\ldots F} \hat{\phi}_{A'B'\ldots F'} = |\hat{\phi}_n|^2 \hat{\phi}_{n-1} \hat{\phi}_{n-1} + |\hat{\phi}_n|^2 \hat{\phi}_{n-1} \hat{\phi}_{n-1}$$

$$+ \sum_{k=1}^{n-1} |\hat{\phi}_{n-k}|^2 \left(\hat{n}_a \hat{n}_b \ldots \hat{n}_c \hat{l}_d \ldots \hat{l}_f \right)_{\text{k terms}}$$

$$+ \sum_{k=1}^{n-1} |\hat{\phi}_{n-k}|^2 \left(\hat{n}_a \hat{n}_b \ldots \hat{n}_c \hat{l}_d \ldots \hat{l}_f \right)_{\text{n-k terms}} + A,$$
we can obtain
\[ \hat{J}_a = \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \sum_{k=0}^{n-1} R^{2k} |\hat{\phi}_{n-k}|^2 \hat{l}_a + \sum_{k=1}^n R^{2(k-1)} |\hat{\phi}_{n-k}|^2 \hat{n}_a \right), \]

where \( A \) vanishes due to the normalization condition of the Newman-Penrose tetrad. Now we can calculate

\[ \hat{J}_a \hat{v}_\Sigma_0^a = \left( \frac{1}{\sqrt{2}} \right)^n \left( \sum_{k=0}^{n-1} R^{2k} C_{n-1}^k |\hat{\phi}_{n-k}|^2 \hat{l}_a + \sum_{k=1}^n R^{2(k-1)} C_{n-1}^{k-1} |\hat{\phi}_{n-k}|^2 \hat{n}_a \right) (\hat{n}^a + R^2 \hat{l}^a) \]

\[ = \left( \frac{1}{\sqrt{2}} \right)^n \int_{\Sigma_0} \left( \sum_{k=0}^{n-1} R^{2k} C_{n-1}^k |\hat{\phi}_{n-k}|^2 + \sum_{k=1}^n R^{2(k-1)} C_{n-1}^{k-1} |\hat{\phi}_{n-k}|^2 \right) dr d\omega. \quad (4.22) \]

The normal vector to the null infinity hypersurface \( \mathcal{I}^+_T = \mathcal{I}^+ \cap \{ u \leq T \} \) is

\[ \hat{N}^a_{\mathcal{I}^+_T} = \hat{n}^a = \sqrt{2} \partial_u, \]

so the transversal vector to the null infinity hypersurface \( \mathcal{I}^+_T = \mathcal{I}^+ \cap \{ u \leq T \} \) is

\[ \hat{L}^a_{\mathcal{I}^+_T} = -\frac{1}{\sqrt{2}} \partial_R. \]

With the supported compact initial data on \( \Sigma_0 \), the solution \( \hat{\phi}_{AB...F} \) has the support on \( \mathcal{I}^+ \) far away from \( i_0 \) (that is a consequence of the finite propagation speed). So we can calculate

\[ \hat{J}_a \hat{N}^a_{\mathcal{I}^+_T} = \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \sum_{k=0}^{n-1} R^{2k} |\hat{\phi}_{n-k}|^2 \hat{l}_a + \sum_{k=1}^n R^{2(k-1)} |\hat{\phi}_{n-k}|^2 \hat{n}_a \right) \hat{n}^a \]

\[ = \left( \frac{1}{\sqrt{2}} \right)^{n-1} |\hat{\phi}_{n}|^2 \text{ (since on } \mathcal{I}^+_T; R^{2k} = 0 \text{ with } k \geq 1). \]

Therefore

\[ \hat{E}_{\mathcal{I}^+_T}(\hat{\phi}_{AB...F}) = \int_{\mathcal{I}^+_T} \hat{J}_a \hat{N}^a_{\mathcal{I}^+_T} (\hat{L}^a_{\mathcal{I}^+_T} dVol^4) \]

\[ = \left( \frac{1}{\sqrt{2}} \right)^n \int_{\mathcal{I}^+_T} |\hat{\phi}_{n}|^2 dud^2\omega, \]

and we can define

\[ \hat{E}_{\mathcal{I}^+_T}(\hat{\phi}_{AB...F}) = \lim_{T \to +\infty} \hat{E}_{\mathcal{I}^+_T}(\hat{\phi}_{AB...F}). \quad (4.23) \]
To obtain the energy equality in the partial conformal compactification, we need to define another hypersurface

\[ S_T = \left\{ (t, r, \omega) \in \mathbb{R}_t \times \mathbb{R}_r \times S^2 \mid t = T + \sqrt{1 + r^2} \right\}, \]

which tends to \( i^+ \) as \( T \) tends to \(+\infty\). Since the initial data has a compact support on \( \Sigma_0 \), we obtain a closed form of the hypersurfaces \( \Sigma_0, \mathcal{I}_T^+ \) and \( S_T \).

![Diagram](image_url)

Figure 4.2: The spacelike hypersurface \( S_T \) on the partial conformal compactification.

Now the conormal vector to the hypersurface \( S_T \) is

\[ \hat{N}_a dx^a = dt - \frac{r}{\sqrt{1 + r^2}} dr, \]

so that the unit normal vector to \( S_T \) is

\[
\hat{N}^a \partial x^a = r^2 \left( \partial_t + \frac{r}{\sqrt{1 + r^2}} \partial_r \right) = r^2 \partial_t - \frac{r}{\sqrt{1 + r^2}} \partial_R \\
= \frac{r^2}{\sqrt{2}} \left( R^2 \hat{t}^a + \hat{r}^a \right) + \frac{r \sqrt{2}}{\sqrt{1 + r^2}} \hat{r}^a \\
= \left( \frac{1}{\sqrt{2}} + \frac{r \sqrt{2}}{\sqrt{1 + r^2}} \right) \hat{t}^a + \frac{r^2}{\sqrt{2}} \hat{r}^a. \tag{4.24}
\]

The transversal vector satisfies \( g(\hat{N}^a, \hat{L}^a) = 1 \), is

\[ \hat{L}^a = \frac{1 + r^2}{1 + 2r^2} \left( \partial_t - \frac{r}{\sqrt{1 + r^2}} \partial_r \right), \]

so the contraction of \( \hat{L}^a \) into the volume form for \( \hat{g} \) is

\[ \hat{L}^a \wedge \text{dvol}_{\hat{g}} = \frac{1 + r^2}{1 + 2r^2} R^2 \left( \text{d}r + \frac{r}{\sqrt{1 + r^2}} \text{d}t \right) \text{d}^2 \omega. \]

Since on \( S_T \) we have

\[ \text{d}t = \frac{r}{\sqrt{1 + r^2}} \text{d}r, \]
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Therefore
\[ \hat{\mathcal{L}}^a \, d\text{Vol}_g^4 = \frac{1 + r^2}{1 + 2r^2} R^2 \left( 1 + \frac{r^2}{1 + r^2} \right) dr^2 \omega = R^2 dr^2 \omega. \]

Now we can calculate
\[
\tau^b \tau^c ... \tau^f = \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \hat{\Phi}^b R^2 \hat{\Phi}^c (\hat{\Phi}^d + R^2 \hat{\Phi}^e) ... (\hat{\Phi}^f + R^2 \hat{\Phi}^g) \right)
\]
\[
= \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \hat{\Phi}^b \hat{\Phi}^c ... \hat{\Phi}^f + R^{2(n-1)} \hat{\Phi}^b \hat{\Phi}^c ... \hat{\Phi}^f \right) 
+ \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \sum_{k=1}^{n-1} R^{2(k-1)} \hat{\Phi}^b \hat{\Phi}^c ... \hat{\Phi}^f \right) 
+ \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \sum_{k=1}^{n-1} R^{2k} \hat{\Phi}^b \hat{\Phi}^c ... \hat{\Phi}^f \right),
\]

and using the expression \(4.19\)
\[
\hat{\phi}_{AB...F} \hat{\phi}_{A'B'...F'} = \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f \right) + \left( \frac{1}{\sqrt{2}} \right)^n \left( \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f \right)
+ \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \sum_{k=1}^{n-1} R^{2k} C_{n-1}^{k-1} \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f \right)
+ A,
\]

we obtain
\[
< \hat{j}^a, \hat{\mathcal{N}}^a > \hat{\mathcal{N}}^a = \Phi_{AB...F} \Phi_{A'B'...F'} \tau^b \tau^c ... \tau^f \hat{\mathcal{N}}^a
\]
\[
= \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r \sqrt{2}}{\sqrt{1 + r^2}} \right) \sum_{k=1}^{n-1} R^{2(k-1)} C_{n-1}^{k-1} \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f
\]
\[
+ \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r \sqrt{2}}{\sqrt{1 + r^2}} \right) \sum_{k=0}^{n-1} R^{2k} C_{n-1}^{k} \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f,
\]

where \(A\) vanishes due to the normalization condition of the Newman-Penrose tetrad. Associating with \(\hat{\mathcal{L}}^a \, d\text{Vol}_g^4\), we can calculate the normalization of \(\hat{\phi}_{AB...F}\) on \(S_T\) as follows
\[
\hat{\mathcal{E}}_T (\hat{\phi}_{AB...F}) = \int_{S_T} < j^a, \mathcal{N}^a > (\hat{\mathcal{L}}^a \, d\text{Vol}_g^4)
\]
\[
= \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r \sqrt{2}}{\sqrt{1 + r^2}} \right) \sum_{k=1}^{n-1} R^{2k} C_{n-1}^{k-1} \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f dr^2 \omega
\]
\[
+ \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \sum_{k=0}^{n-1} R^{2k} C_{n-1}^{k} \hat{\Phi}^a \hat{\Phi}^b \hat{\Phi}^c \hat{\Phi}^d \hat{\Phi}^e \hat{\Phi}^f dr^2 \omega. \quad (4.25)
\]

The equality energy

**Proposition 4.5.1.** In the conformal compactification spacetimes we have the same result about the equality of the energies of the spin \(n/2\) zero rest-mass fields on the hypersurfaces \(\Sigma_0\) and \(\mathcal{I}^+\) as
follows:

\[ \mathcal{E}_{\mathcal{S}^+}(\hat{\phi}_{AB...F}) = \hat{\Sigma}_0(\hat{\phi}_{AB...F}) , \]  
(4.26)

\[ \mathcal{E}_{\mathcal{S}_*}(\hat{\phi}_{AB...F}) = \hat{\Sigma}_0(\hat{\phi}_{AB...F}) . \]  
(4.27)

**Proof.** Because the vector fields \( \partial_t \) and \( \partial_\tau \) are Killing, so we can obtain the conservation laws:

\[ \hat{\nabla}^a \hat{j}_a = \hat{\nabla}^a \hat{j}_a = 0 . \]

For the full conformal compactification, we just need to integrate the conservation law \( \hat{\nabla}^a \hat{j}_a = 0 \) on the cone which is formed by the hypersurfaces \( \Sigma_0 \) and \( \mathcal{S}^+ \) to obtain the equality of the energies. For the partial conformal compactification, integrating the conservation law \( \hat{\nabla}^a \hat{j}_a = 0 \) on the domain which is formed by the hypersurfaces \( \Sigma_0, \mathcal{S}_* \) and \( S_T \), we obtain

\[ \hat{\mathcal{E}}_{\mathcal{S}_*}(\hat{\phi}_{AB...F}) + \hat{\mathcal{E}}_{S_T}(\hat{\phi}_{AB...F}) = \hat{\Sigma}_0(\hat{\phi}_{AB...F}) . \]

Taking the limit as \( T \) tend to \(+\infty\) for the equality above, we get

\[ \hat{\mathcal{E}}_{\mathcal{S}^+}(\hat{\phi}_{AB...F}) + \lim_{T\to+\infty} \hat{\mathcal{E}}_{S_T}(\hat{\phi}_{AB...F}) = \hat{\Sigma}_0(\hat{\phi}_{AB...F}) . \]

Due to the decay result of the components \( \hat{\phi}_{n-k} \) in Proposition 4.3.1 and the form (4.25) of the energy on \( S_T \), we have

\[
\begin{align*}
\lim_{T\to+\infty} \hat{\mathcal{E}}_{S_T}(\hat{\phi}_{AB...F}) &= \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r\sqrt{2}}{\sqrt{1+r^2}} \right) \sum_{k=1}^{n} R^{2k} C_{n-k-1} |\hat{\phi}_{n-k}|^2 d\omega d^2\omega \\
&+ \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \sum_{k=0}^{n} R^{2k} C_{n-k-1} \left| \hat{\phi}_{n-k} \right|^2 d\omega d^2\omega \\
&\leq \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r\sqrt{2}}{\sqrt{1+r^2}} \right) \sum_{k=1}^{n} R^{2k} C_{n-k-1} \left| \frac{r^{k+1}}{t(r)^{n+2}} \right|^2 d\omega d^2\omega \\
&+ \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \sum_{k=0}^{n} R^{2k} C_{n-k-1} \left| \frac{r^{k+1}}{t(r)^{n+2}} \right|^2 d\omega d^2\omega \\
&= \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \left( \frac{1}{\sqrt{2}} + \frac{r\sqrt{2}}{\sqrt{1+r^2}} \right) \sum_{k=1}^{n} C_{n-k-1} \left| \frac{r}{t(r)^{n+2}} \right|^2 d\omega d^2\omega \\
&+ \lim_{T\to+\infty} \int_{S_T} \left( \frac{1}{\sqrt{2}} \right)^{n-1} \sum_{k=0}^{n} C_{n-k-1} \left| \frac{r}{t(r)^{n+2}} \right|^2 d\omega d^2\omega ,
\end{align*}
\]
where \( t(r) = T + \sqrt{1 + r^2} \). We can control the right-hand side of the inequality above as follows

\[
\text{Right-hand side} \leq C \lim_{T \to +\infty} \int_{S^2} \int_{r=0}^{+\infty} \frac{1}{t(r)^4} \, dr \, d\omega^2
\]

\[
\leq 2\pi C \lim_{T \to +\infty} \int_{r=0}^{+\infty} \frac{1}{t(r)^4} \, dr
\]

\[
= 2\pi C \lim_{T \to +\infty} \left( \int_{r=0}^{T} \frac{1}{t(r)^4} \, dr + \int_{r=T}^{+\infty} \frac{1}{r^4} \, dr \right)
\]

\[
\leq 2\pi C \lim_{T \to +\infty} \left( \int_{r=0}^{T} \frac{1}{T^4} \, dr + \int_{r=T}^{+\infty} \frac{1}{r^4} \, dr \right)
\]

\[
= 2\pi C \lim_{T \to +\infty} \frac{2}{3T^3} = 0,
\]

because \( t(r) = T + \sqrt{1 + r^2} \) so \( t(r) > r \) and \( t(r) > T \). So that

\[
\lim_{T \to +\infty} \hat{E}_{ST}(\hat{\phi}_{AB...F}) = 0.
\]

Therefore we can obtain the same result about the equality of the energies for the partial conformal compactification

\[
\hat{\mathcal{E}}_{\mathcal{I}^+}(\hat{\phi}_{AB...F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB...F}).
\]

\( \square \)

**Remark 4.5.1.** We cut the partial compactification \( \hat{\mathbb{M}} \) by a spacelike hypersurface \( \mathcal{S} \), suppose that \( \mathcal{S} \) intersects \( \mathcal{I}^+ \) at \( Q \). Then we also have the energy equality

\[
\hat{\mathcal{E}}_{\mathcal{I}^+,Q}(\hat{\phi}_{AB...F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB...F}),
\]

where \( \mathcal{I}^+,Q \) is the future part of \( Q \) in \( \mathcal{I}^+ \).

As a consequence of the energy equality, the energy on the null infinity \( \mathcal{I}^+ \) of the spin \( n/2 \) zero rest-mass field is finite. We can now define the trace operator:

**Definition 4.5.1.** Since the Cauchy problem of the system (4.15) is well-posed in \( \hat{\mathbb{M}} \) and the energy of spin \( n/2 \) zero rest-mass field on null infinity \( \mathcal{I}^+ \) is finite we can give the definition of the trace operator \( \mathcal{T}^+ \) from \( C_0^\infty(\Sigma_0, S_{(AB...F)}) \) to \( C^\infty(\mathcal{I}^+, \mathbb{C}) \) as follows

\[
\mathcal{T}^+: C_0^\infty(\Sigma_0, S_{(AB...F)}) \to C^\infty(\mathcal{I}^+, \mathbb{C})
\]

\[
\hat{\psi}_{AB...F} \mapsto \hat{\phi}_n|_{\mathcal{I}^+}.
\]

An other consequence of the energy equality is that we can extend the domain of the trace operator, where the trace operator is one to one and has closed range.
Corollary 4.5.1. We extend the trace operator

\[ \mathcal{T}^+: \mathcal{H}_0 = L^2(\Sigma_0, \mathbb{S}_{(AB...F)}) \longrightarrow \mathcal{H}^+ = L^2(\mathcal{I}^+, \mathbb{C}) \]

\[ \hat{\phi}_{AB...F}|_{\Sigma_0} \longmapsto \hat{\phi}_n|_{\mathcal{I}^+} \]

where \( \mathcal{H}_0 = L^2(\Sigma_0, \mathbb{S}_{(AB...F)}) \) is the closed space of \( \mathcal{C}_0^\infty(\Sigma_0, \mathbb{S}_{AB...F}) \) in the energy norm

\[ ||\hat{\phi}_{AB...F}|_{\Sigma_0}||^2 = \mathcal{E}_{\Sigma_0}(\hat{\phi}_{AB...F}) \]

\[ = \left( \frac{1}{\sqrt{2}} \right)^n \int_{\Sigma_0} \left( \sum_{k=0}^{n-1} R^{2k} C_{n-1}^k |\hat{\phi}_{n-k}|^2 + \sum_{k=1}^n R^{2k} C_{n-1}^{k-1} |\hat{\phi}_{n-k}|^2 \right) d\mathcal{r}^2 \omega , \]

and similarly \( \mathcal{H}^+ = L^2(\mathcal{I}^+, \mathbb{C}) \) is the closed space of \( \mathcal{C}_0^\infty(\mathcal{I}^+, \mathbb{C}) \) in the energy norm

\[ ||\hat{\phi}_n|_{\mathcal{I}^+}||^2 = \hat{\mathcal{E}}_{\mathcal{I}^+}(\hat{\phi}_{AB...F}) = \left( \frac{1}{\sqrt{2}} \right)^n \int_{\mathcal{I}^+} |\hat{\phi}_n|^2 d\mathcal{u}^2 \omega . \]

The trace operator in the new domains is one to one and has closed range.

**Proof.** It is clear that \( \mathcal{T}^+ \) is one to one from the equality energy. And from the equality energy we also have \( \mathcal{T}^+ \) transforms a Cauchy sequence to another, so the domain image \( \mathcal{T}^+(L^2(\Sigma_0, \mathbb{S}_{(AB...F)})) \) is closed. \( \square \)

### 4.6 The Goursat problem and the conformal scattering operator

Now we will show that the trace operator is surjective, it is enough to prove that the image of \( \mathcal{T}^+ \) is dense in \( \mathcal{H}^+ \). To do that we show \( \mathcal{C}_0^\infty(\mathcal{I}^+, \mathbb{C}) \subset \text{Im} \mathcal{T}^+ \), this is equivalent to solve Goursat problem with the initial data in \( \mathcal{C}_0^\infty(\mathcal{I}^+, \mathbb{C}) \).

**The Goursat problem**

Now we consider the Goursat problem in the part compactification \( \hat{\mathcal{M}} \)

\[
\begin{cases}
\hat{\nabla}^{AA'} \hat{\phi}_{AB...F} = 0, \\
\hat{\phi}_n|_{\mathcal{I}^+} = \hat{\psi}_n \in \mathcal{C}_0^\infty(\mathcal{I}^+, \mathbb{C}), \\
\hat{\phi}_{AB...F}|_{\mathcal{I}^+} = \hat{\psi}_{AB...F} \in \mathcal{D}_{\mathcal{I}^+}
\end{cases}
\]

here \( \mathcal{D}_{\mathcal{I}^+} \) is the constraint space on \( \mathcal{I}^+ \).

We recall the expression of the massless equation \( \hat{\nabla}^{AA'} \hat{\phi}_{AB...F} = 0 \) in the partial conformal compactification \( \hat{\mathcal{M}} \) (see eq. (4.14))

\[
\begin{cases}
-\frac{1}{\sqrt{2}} \partial_R \hat{\phi}_r - \frac{1}{\sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + (n - 2r + 2) \cot \frac{\theta}{2} \right) \hat{\phi}_{r-1} = 0, \\
\left( \sqrt{2} \partial_u + \frac{R}{\sqrt{2}} \partial_R - (n - 2r) \frac{R}{\sqrt{2}} \right) \hat{\phi}_r - \frac{1}{\sqrt{2}} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - (n - 2r - 2) \cot \frac{\theta}{2} \right) \hat{\phi}_{r+1} = 0
\end{cases}
\]
where \( r = 1, 2 \ldots n \) in the first equation and \( r = 0, 1 \ldots n - 1 \) in the second one. Since the constraint system on \( \mathcal{I}^+ \) is the projection of the equation \( \hat{\nabla}^A \hat{\phi}_{AB \ldots F} = 0 \) on the null normal vector \( \hat{n}^a \), the constraint on the null infinity hypersurface \( \mathcal{I}^+ \) is that of the second equation of the system above on \( \mathcal{I}^+ \)

\[
\sqrt{2} \partial_u \hat{\phi}_r|_{\mathcal{I}^+} - \frac{1}{\sqrt{2}} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - (n - 2r - 2) \frac{\cot \theta}{2} \right) \hat{\phi}_{r+1}|_{\mathcal{I}^+} = 0.
\]

Therefore on \( \mathcal{I}^+ \), we have

\[
\hat{\phi}_r|_{\mathcal{I}^+}(u) = \hat{\phi}_r|_{\mathcal{I}^+}(-\infty) + \frac{1}{2} \int_{-\infty}^{u} \left( \partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - (n - 2r - 2) \frac{\cot \theta}{2} \right) \hat{\phi}_{r+1}|_{\mathcal{I}^+}(s) ds,
\]

where \( r = 0, 1 \ldots n - 1 \). So from the initial data \( \hat{\psi}_n \in C_0^\infty(\mathcal{I}^+, \mathbb{C}) \) (its support away from \( i^+ \) and \( i^0 \)), we can find the other components to obtain the full spinor field \( \hat{\psi}_{AB \ldots F} := \hat{\phi}_{AB \ldots F}|_{\mathcal{I}^+} \). And we can think that its support is far away from \( i^+ \).

To solve the Goursat problem, we choose a spacelike hypersurface \( S \) in \( \hat{\mathcal{M}} \) such that it crosses \( \mathcal{I}^+ \) strictly in the past of the support of the initial data \( \hat{\psi}_n \), we denote the point of intersection of \( S \) and \( \mathcal{I}^+ \) by \( Q \). The Goursat problem will be solved following two step:

**Step one:** We solve the Goursat problem in the future \( \mathcal{I}^+(S) \) of \( S \). On the partial compactification we have (see equation \( 4.35 \) in Appendix 4.7.1 of this chapter)

\[
2\hat{\nabla}_A \hat{\nabla}^A \hat{\phi}_{AB \ldots F} = \nabla_{\hat{Q}B \ldots F} F = 0,
\]

so the Goursat problem on the future \( \mathcal{I}^+(S) \) has a problem consequence as follows

\[
\begin{align*}
\hat{\square} \hat{\phi}_{AB \ldots F} &= 0, \\
\hat{\phi}_{AB \ldots F}|_{\mathcal{I}^+ Q} &= \hat{\psi}_{AB \ldots F}|_{\mathcal{I}^+ Q} \in C_0^\infty(\mathcal{I}^+ Q, S_{(AB \ldots F)}), \\
\hat{\nabla}^A \hat{\phi}_{AB \ldots F}|_{\mathcal{I}^+ Q} &= 0.
\end{align*}
\]

(4.29)

where \( \mathcal{I}^+ Q \) is the future part of \( Q \) in the null infinity hypersurface \( \mathcal{I}^+ \). Here we apply the general result of the paper of L.Hörmander, this system has a unique solution (the general result of Lörmander will be given in Appendix 4.7.3).

Now we show that this solution is also a solution of the system \( 4.28 \) by proving that \( \hat{\nabla}^A \hat{\psi}_{AB \ldots F} = 0 \) and using again the general result of L.Hörmander. First, the components of \( \hat{\nabla}^A \hat{\psi}_{AB \ldots F} \) i.e the restrictions of the components of \( \hat{\nabla}^A \hat{\phi}_{AB \ldots F} \) on the hypersurface \( \mathcal{I}^+ Q \) are both zero. Indeed, if we set

\[
\Xi^{A'}_{B \ldots F} := \nabla^A \hat{\phi}_{AB \ldots F},
\]

then \( \Xi^{A'}_{B \ldots F} \) is symmetric in the indices \( B \ldots F \), and we have

\[
\Xi^{A'}_{B \ldots F}|_{\mathcal{I}^+ Q} = i_{A'} \Xi^{A'}_{B \ldots F}|_{\mathcal{I}^+ Q} = \hat{\nabla}^A \hat{\phi}_{AB \ldots F}|_{\mathcal{I}^+ Q} = 0,
\]
so all the components of $\Xi'_{B...F}|_{\mathcal{J}^+,Q}$ on $\mathcal{J}^+,Q$ are zero. For the components of $\Xi''_{B...F}|_{\mathcal{J}^+,Q}$, using the equation

$$\Box \hat{\phi}_{AB...F} = \frac{1}{2} \nabla_{AK'} \nabla^{K'} \hat{\phi}_{KB...F} = \frac{1}{2} \nabla_{AK'} \Xi'_{B...F} = \frac{1}{2} \Theta_{AB...F} = 0,$$

we have

$$\Theta_1_{B...F} = \Theta_0_{B...F} = 0,$$

where $\Theta_1_{B...F}$ and $\Theta_0_{B...F}$ are obtained by the differential equations which are of order one in the components of $\Xi'_{B...F}$ and $\Xi''_{B...F}$ (We can see the expression in detail of these equations in Appendix 4.7.4). So, taking the constraint of these equations on $\mathcal{J}^+,Q$ we obtain the restrictive equations of the components of $\Xi'_{B...F}$ and $\Xi''_{B...F}$ on $\mathcal{J}^+,Q$. Since all the components of $\Xi''_{B...F}$ are zero on $\mathcal{J}^+,Q$, we can obtain the Cauchy problem of the system of differential equations of order one, where the unknowns are only the restrictions of the components of $\Xi''_{B...F}$ on $\mathcal{J}^+,Q$:

$$\begin{cases}
\Theta_1_{B...F}|_{\mathcal{J}^+} = 0, \\
\Xi''_{B...F}|_{\mathcal{V}(P)} = 0
\end{cases}$$

(4.30)

where $\mathcal{V}(P)$ is the neighborhood of the point $P$ chosen to belong to $\mathcal{J}^+,Q$, near $i^+$ and not belonging to the support of $\hat{\phi}_{AB...F}$. Since the Cauchy problem has a unique solution, we have the components of $\Xi''_{B...F}|_{\mathcal{J}^+,Q}$ are also both zero. Therefore we have that the restrictions of the components of $\hat{\phi}_{AB...F}$ on $\mathcal{J}^+,Q$ are both zero (see Appendix 4.7.4 of this chapter for detail).

Now we have (see equation (4.36) in the appendix of this chapter)

$$0 = \nabla^{AA'} \Box \hat{\phi}_{AB...F} = \frac{1}{2} \nabla^{AA'} \nabla_{AK'} \Xi'_{B...F} = \frac{1}{4} \Box \Xi_{B...F} + \frac{1}{2} \nabla^{A'} K' \Xi'_{B...F},$$

raising the indices $B...F$, we obtain the system

$$\begin{cases}
\hat{\xi}_{AB...F} + \Box^{A'} K' \Xi_{AB...F} = 0, \\
The restrictions of all the components of $\Xi_{AB...F}$ on $\mathcal{J}^+,Q$ = 0
\end{cases}$$

(4.31)

with

$$\Box^{A'} K' \Xi_{B...F} = \hat{X}^{A'} K' Q' B Q' B Q' B + \hat{\Phi}^{A'} K' Q' B Q' B + \cdots + \hat{\Phi}^{A'} K' Q' F B Q' ,$$

where $\hat{X}_{ABCD}$ and $\hat{\Phi}_{ABCD}$ are the curvature spinor. Due to the same calculations in chapter 2, the rescaled scalar curvature $\hat{\Lambda} = 0$. Moreover the Weyl conformal spinor $\Psi_{ABCD} = 0$ in the Minkowski spacetime, therefore

$$\hat{X}_{ABCD} = \hat{\Psi}_{ABCD} + \hat{\Lambda}(\hat{\xi}_{AC} \hat{\xi}_{BD} + \hat{\xi}_{AC} \hat{\xi}_{BD}) = \Psi_{ABCD} + \hat{\Lambda}(\hat{\xi}_{AC} \hat{\xi}_{BD} + \hat{\xi}_{AC} \hat{\xi}_{BD}) = 0.$$

The components of $\hat{\Phi}_{ABCD}$ are $C^\infty$ due to the following formula (see L. Mason and J-P. Nicolas [74])

$$\hat{\Phi}_{ab} dx^a dx^b = \frac{1}{2} (R^2 du^2 - 2 du dR + d\omega^2).$$
Using the general result of L. Hörmander, we get $\Xi A'B...F = 0$ and then $\Xi A'_{B...F} = \hat{\nabla} AA' \hat{\phi}_{AB...F} = 0$. So the solution of the system (4.29) is a solution of the system (4.28). For convenience, we denote by $\hat{\phi}^1_{AB...F}$ the solution of this step.

**Step two:** We need to extend the solution of the Goursat problem on future $\mathcal{I}^+ (\mathcal{S})$ down to $\Sigma_0$. This is equivalent to solve the Cauchy problem in the past $\mathcal{I}^- (\mathcal{S})$ of $\mathcal{S}$:

\[
\begin{aligned}
\hat{\nabla} AA' \hat{\phi}_{AB...F} &= 0, \\
\hat{\phi}_{AB...F} |_{\mathcal{S}} &= \hat{\phi}^1_{AB...F} |_{\mathcal{S}}.
\end{aligned}
\]  

(4.32)

![Figure 4.3: Embedding the domain $\mathcal{I}^- (\mathcal{S})$ into the Einstein cylinder.](image)

Due to the conformal transformations, we can embedd the domain $\mathcal{I}^- (\mathcal{S})$ into the Einstein cylinder. We extend $\mathcal{S}$ to the spacelike hypersurface $\mathcal{O}$ and the initial data $\hat{\phi}_{AB...F} |_{\mathcal{S}}$ is zero in the rest of the support, we can now consider the equivalent Cauchy problem:

\[
\begin{aligned}
\hat{\nabla} AA' \hat{\phi}_{AB...F} &= 0, \\
\hat{\phi}_{AB...F} |_{\mathcal{S}} &= \hat{\phi}^1_{AB...F} |_{\mathcal{S}}, \\
\hat{\phi}_{AB...F} |_{\mathcal{O}-\mathcal{S}} &= 0.
\end{aligned}
\]  

(4.33)

As a consequence of Proposition 4.15, this Cauchy problem is well-posed, we denote its solution by $\hat{\phi}^2_{AB...F}$ and the solution of this step by $\hat{\phi}^2_{AB...F}$. We can obtain from the divergence theorem

\[
\hat{\mathcal{E}}_{\mathcal{S}}(\hat{\phi}_{AB...F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB...F}),
\]  

(4.34)

and since the energy of the spin $n/2$ zero rest-mass field is invariant under the conformal transformations, we have

\[
\hat{\mathcal{E}}_{\mathcal{S}}(\hat{\phi}_{AB...F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB...F}).
\]
Due to the energy equality (see Remark 4.5.1), we obtain

\[ \hat{\mathcal{E}}_{\mathcal{I}}(\hat{\phi}_{AB\ldots F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB\ldots F}) = \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}_{AB\ldots F}) \]

Therefore the energy of the solution on the hypersurface \( \Sigma_0 \) is finite and we can define the trace operator as the constraint of the solution of the Cauchy problem \( (4.32) \) on \( \Sigma_0 \).

Finally, the solution of the Goursat problem is the union of the solutions of two step above

\[ \hat{\phi}_{AB\ldots F} = \begin{cases} 
\hat{\phi}_{AB\ldots F}^1 & \text{in the domain } \mathcal{I}^+(S), \\
\hat{\phi}_{AB\ldots F}^2 & \text{in the domain } \mathcal{I}^-(S). 
\end{cases} \]

We can summarize everything that we have just done above, by the following proposition:

**Proposition 4.6.1.** (Goursat problem) The Goursat problem for the rescaled massless equation

\[ \hat{\nabla}^{AA'}\hat{\phi}_{AB\ldots F} = 0 \text{ in } \hat{\mathcal{M}} \]

is well-posed i.e for any \( \hat{\psi}_n \in C^\infty_0(\mathcal{I}^+) \) and \( \hat{\psi}_{AB\ldots F} \in \mathcal{D}_{\mathcal{I}}^+ \) there exists a unique \( \hat{\phi}_{AB\ldots F} \) solution of \( \hat{\nabla}^{AA'}\hat{\phi}_{AB\ldots F} = 0 \) such that

\[ \hat{\phi}_{AB\ldots F} \in C^\infty(\hat{\mathcal{M}}, \mathbb{S}_{(AB\ldots F)}); \hat{\phi}_n|_{\mathcal{I}^+} = \hat{\psi}_n \text{ and } \hat{\phi}_{AB\ldots F}|_{\mathcal{I}^+} = \hat{\psi}_{AB\ldots F}. \]

Furthermore, the energy norm of the constraint of the solution \( \hat{\phi}_{AB\ldots F}|_{\Sigma_0} \) on \( \Sigma_0 \) is finite.

**Conformal scattering operator**

Since the Goursat problem is resolved, we have the trace operator \( \mathcal{T}^+(\hat{\psi}_{AB\ldots F}) = \hat{\phi}_n|_{\mathcal{I}^+} \) of the solution of the massless equation is surjective and using Corollary 4.5.1, we can think that this trace operator is isometric from \( \mathcal{H}_0 = L^2(\Sigma_0, \mathbb{S}_{(AB\ldots F)}) \) onto \( \mathcal{H}^+ = L^2(\mathcal{I}^+, \mathbb{C}) \). So we can give the definition of the conformal scattering operator as follows:

**Definition 4.6.1.** We define similarly the isometric operators \( \mathcal{T}^+ \) and \( (\mathcal{T}^+)^{-1} \):

\[ \mathcal{T}^- (\hat{\phi}_{AB\ldots F}|_{\Sigma_0}) = \hat{\phi}_0|_{\mathcal{I}^-} \text{ and } (\mathcal{T}^-)^{-1} (\hat{\phi}_0|_{\mathcal{I}^-}) = \hat{\phi}_{AB\ldots F}|_{\Sigma_0}, \]

then scattering operator \( \mathcal{W} := \mathcal{T}^+ \circ (\mathcal{T}^-)^{-1} \) is an isometric operator that associates the past scattering \( \mathcal{H}^- \) (the definition of \( \mathcal{H}^- \) is the same \( \mathcal{H}^+ \)) data to the future scattering \( \mathcal{H}^+ \) data.

**Conclude**

The energy equality in Proposition 4.5.1 is the basis to prove that the trace operator is an isometry from \( \mathcal{H}_0 \) onto \( \mathcal{H}^+ \). Obtaining this equality in the full conformal compactification is simple, because the past and future infinite points \( i^\pm \) are finite on the Einstein cylinder. But in the partial conformal compactification, we get a difficulty since the past and future null infinite points are infinite; to solve this difficulty, we need to obtain a decay result for the spin \( n/2 \) zero rest-mass field and approach to \( i^+ \) (similarly \( i^- \)) by a spacelike hypersurface \( S_T \). This explains why the norm of the field on the
future null infinite hypersurface $\mathcal{I}^+$ in the full conformal compactification is stronger than the one in the partial conformal compactification

$$\hat{E}_{\mathcal{I}^+} (\hat{\phi}_{AB...F}) = \left( \frac{1}{\sqrt{2}} \right) ^{n-2} \int _{\mathcal{I}^+} \sum _{k=0} ^{n-1} C _{n-1} ^{k} |\hat{\phi}_{n-k}|^2 d\mu_S$$

$$\gg \hat{E}_{\mathcal{I}^+} (\hat{\phi}_{AB...F}) = \left( \frac{1}{\sqrt{2}} \right) ^n \int _{\mathcal{I}^+} |\hat{\phi}_n|^2 d\mu _{3} \omega ,$$

in the sense that the norm in full conformal compactification is the sum of all components of the spin $n/2$ zero rest-mass field and the norm in the partial conformal compactification is only one component.

Our construction is only valid on the plat spacetime with the result decay can be obtained simply since the Minkowski spacetime can be fully embedded into the Einstein cylinder by the full conformal compactification. It is no longer true for a more general case such as Schwarzschild spacetime, because the spin $n/2$ zero rest-mass equations have not solution on these spacetimes.

### 4.7 Appendix

#### 4.7.1 Spinor form of commutators

In this section, we will give the spinor form of the commutators $\Delta ^{ab} = \nabla ^{[a} \nabla ^b]$ which have been done in R. Penrose [73] for the $\Delta _{ab} = \nabla _{[a} \nabla _b]$. First, due to the anti-symmetric property of $\Delta ^{ab}$ we can write

$$\Delta ^{ab} = 2 \nabla ^{[a} \nabla ^b] = \varepsilon ^{A'B'} \Box ^{AB} + \varepsilon ^{AB} \Box ^{A'B'} ,$$

where

$$\Box ^{AB} = \nabla ^X (A \nabla _B)_X ' , \quad \Box ^{A'B'} = \nabla _X (A' \nabla _{B'})_X .$$

Now we have

$$\Delta ^{ab} = g ^{ac} g ^{bd} \Delta _{cd} ,$$

and the effect of $\Delta _{ab}$ on the spinor form $\kappa ^C$ has the form

$$\Delta _{ab} \kappa ^C = \{ \varepsilon ^{A'B'} X ^{ABE} _C + \varepsilon ^{AB} \Phi ^{A'B'} _{EC} \} \kappa ^E ,$$

where $X ^{ABCD}$ and $\Phi ^{ABC'D'}$ are the curvature spinors in the expression of the Riemann tensor $R _{abcd}$ (We can see this decomposition by equation (3.53) in Chapter 2). So we obtain

$$\Delta _{ab} \kappa ^C = \varepsilon ^{AC} \varepsilon ^{A'C'} \varepsilon ^{BD} \varepsilon ^{B'D'} \Delta _{cd} \kappa ^C = \{ \varepsilon ^{A'B'} X ^{AB} _E C + \varepsilon ^{AB} \Phi ^{A'B'} _{E C} \} \kappa ^E ,$$

which by symmetrizing and skew-symmetrizing over $AB$, yeilds the equations

$$\Box ^{AB} \kappa ^C = X ^{AB} _E C \kappa ^E , \quad \Box ^{A'B'} \kappa ^C = \Phi ^{A'B'} _{E C} \kappa ^E .$$
Conformal scattering on Minkowski spacetime

Similarly, we can obtain the formula of the primed spin-vectors

\[ \Delta^{a'b'C'} = \left\{ \varepsilon^{AB} \hat{X}^{A'B'}_{\ E'} C' + \varepsilon^{A'B'} \Phi^{AB}_{\ E'} C' \right\} \tau^{E'}, \]

\[ \Box^{a'b'C'} = \Phi^{AB}_{\ E'} C' \tau^{E'}, \Box^{A'B'} C' = \hat{X}^{A'B'}_{\ E'} C' \tau^{E'}. \]

Lowering the index \( C \) (or \( C' \)), we also get

\[ \Box^{AB}_C C = X^{ABE}_{\ C\ K_E}, \ \Box^{A'B'}_C C = \Phi^{A'B'} E_{\ C\ K_E}, \]

\[ \Box^{AB}_{C'} C' = \Phi^{AB}_E C' \tau^{E'}, \ \Box^{A'B'}_{C'} C' = \hat{X}^{A'B'} E_{\ C'\ K_E}. \]

For the action of \( \Box^{AB} \) and \( \Box^{A'B'} \) on many index spinors, we expand them by a sum of outer products of spin vectors and use the properties above. Now we will give the demonstrations of the two expressions that we used in the Goursat problem. Frist, we have

\[
\hat{\nabla}_{ZA} \hat{\nabla}^{A'} \hat{\phi}_{AB...F} = \hat{\varepsilon}^{AM} \hat{\nabla}_{ZA} \hat{\nabla}^{A'} \hat{\phi}_{AB...F} = \hat{\varepsilon}^{AM} \left( \hat{\nabla}^A \hat{Z} \hat{\nabla}^M \hat{A'} \hat{\phi}_{AB...F} + \hat{\nabla}^A \hat{Z} \hat{\nabla}^M \hat{A'} \hat{\phi}_{AB...F} \right) \\
= \hat{\varepsilon}^{AM} \left( \frac{1}{2} \hat{\varepsilon}^{ZM} \hat{\phi}_{AB...F} + \hat{\nabla}^A \hat{Z} \hat{\nabla}^M \hat{A'} \hat{\phi}_{AB...F} \right) \\
= \frac{1}{2} \hat{\nabla}^Z \hat{\phi}_{AB...F} + \hat{\nabla}^Z \hat{\phi}_{AB...F} - \hat{\nabla}^Z \hat{\phi}_{AB...F} - \hat{\nabla}^Z \hat{\phi}_{AB...F} \\
= \frac{1}{2} \hat{\nabla}^Z \hat{\phi}_{AB...F},
\]

because in our case the curvature spinor \( \hat{\phi}^{ABC} \) disappears

\[ \hat{X}_{ABC} = \hat{\Psi}_{ABC} + \hat{A} (\hat{\varepsilon}_{AC} \hat{\varepsilon}_{BD} + \hat{\varepsilon}_{AD} \hat{\varepsilon}_{BC}) = \hat{\Psi}_{ABC} + \hat{A} (\hat{\varepsilon}_{AC} \hat{\varepsilon}_{BD} + \hat{\varepsilon}_{AD} \hat{\varepsilon}_{BC}) = 0. \]

Second

\[
\hat{\nabla}^{A'} \hat{\nabla}_{AK'} \Xi_{B...F} = - \hat{\varepsilon}^{A'M} \hat{\nabla}^{A'} \hat{\nabla}_{A'} \Xi_{B...F} \\
= - \hat{\varepsilon}^{A'M} \left( \hat{\nabla}^{A'} \hat{\nabla}_{A'} \Xi_{B...F} + \hat{\nabla}^{A'} \hat{\nabla}_{A'} \Xi_{B...F} \right) \\
= - \hat{\varepsilon}^{A'M} \left( \frac{1}{2} \hat{\nabla}^{A'M} \hat{\phi}_{B...F} + \hat{\nabla}^{A'M} \hat{\phi}_{B...F} \right) \\
= \frac{1}{2} \hat{\nabla}^{A'} \hat{\phi}_{B...F} + \hat{\nabla}^{A'} \hat{\phi}_{B...F} - \hat{\nabla}^{A'} \hat{\phi}_{B...F} - \hat{\nabla}^{A'} \hat{\phi}_{B...F} \\
= \frac{1}{2} \hat{\nabla}^{A'} \hat{\phi}_{B...F} + \hat{\nabla}^{A'} \hat{\phi}_{B...F} - \hat{\nabla}^{A'} \hat{\phi}_{B...F} - \hat{\nabla}^{A'} \hat{\phi}_{B...F}.
\]
And the expression which we used in the Cauchy problem
\[ \hat{\nabla}^A' \hat{\nabla}_Z' A' \hat{\phi}_{ZAC...F} = \hat{\nabla}^A'(A' \hat{\nabla}^Z_\Lambda' \hat{\phi}_{ZAC...F} ) = \Box^{AZ} \hat{\phi}_{AZC...F} = -\hat{X}^{AZM} A' \phi_{MZC...F} - \hat{X}^{AZM} Z \phi_{AMC...F} - \cdots - \hat{X}^{AZM} F \phi_{AZC...M} = -(n-1) \hat{\phi}_{AZM(C...K} \hat{\psi}^{F)}_A Z^M (4.37) \]

since \( \hat{X}^{A(ZM)}_A = 0 \) and \( \hat{X}^{(AZM)}_C = \hat{\psi}^{AZM}_C \).

Note that if we define the wave operator by using the spinor form as following
\[ \Box = \varepsilon^{MN} \varepsilon_{M'N'} \nabla^M_{\Lambda} \nabla^N_{N'} = \nabla_a \nabla^a, \]
then we can obtain
\[ \Box = \varepsilon^{MN} \nabla_{N'M'} \nabla^N_{N'} = \varepsilon^{MN} \left( \nabla_{N'M} \nabla^N_{N'} + \nabla_{N'M'} \nabla^N_{N} \right) = \varepsilon^{MN} \left( \frac{1}{2} \varepsilon_{MN} \Box + \Box_{MN} \right) = \Box - \Box_{M}^{M}. \]
Similarly
\[ \Box = \varepsilon^{M'N'} \left( -\frac{1}{2} \varepsilon^{M'N'} \Box + \Box_{M'}^{M'} \right) = -\Box + \Box_{M'}^{M'.} \]
Therefore the wave operators that appear in the first and the second expressions and the original wave operator that acts on \( \Phi = (\phi_1, \phi_2, \ldots, \phi_n) \), which are of the same modulo the derivation terms of order less than or equal one.

**4.7.2 Non-trivial solutions of the constraint system on \( C^\infty_0(\Sigma_0) \)**

As we have seen, the spin \( n/2 \) zero rest-mass field equations
\[ \nabla^{AA'} \phi_{AB...F} = 0 \]
are an overdetermined system, which can be split into an evolution part and a spacelike constraint that is preserved by the evolution. This constraint system is analogous to an elliptic equation, it is therefore not clear that it admits smooth compactly supported solutions. Penrose [72] shows that in the Minkowski spacetime any solution of the spin \( n/2 \) zero rest-mass field, at least locally can be obtained from a scalar potential (also called Hertz-type potential) \( \chi \) satisfying the wave equation \( \Box \chi = 0 \). The construction is as follows: let \( \phi_{(AB...F)} \) be a solution of the spin \( n/2 \) zero rest-mass field equation and \( \mu_{A'} \) a spinor which is chosen to constant throughout Minkowski spacetime. Then at least locally we can find a spin \( (n-1)/2 \) zero rest-mass field \( \psi_{(B...F)} \) satisfying the spin \( (n-1)/2 \) zero rest-mass equation \( \nabla^{BB'} \psi_{B...F} = 0 \) and
\[ \phi_{AB...F} = \alpha_{A'} \nabla^{A'}_A \psi_{B...F}. \]
Continuing this process we get the scalar potential $\chi$ as follows

$$
\phi_{AB...F} = \alpha_{A'}\beta_{B'}...\gamma_{F'}\nabla_{A'}\nabla_{B'}...\nabla_{F'}\chi
$$

where $\chi$ satisfying $\Box\chi = 0$ and $\alpha_{A'}, \beta_{B'}...\gamma_{F'}$ are given constant spinor.

Conversely we show that any solution $\chi$ satisfying the wave equation gives rise to a solution $\phi_{AB...F}$ of equation (4.38) via a choice of constant spinors $\alpha_{A'}, \beta_{B'}...\gamma_{F'}$. Indeed, we have

$$
\nabla Z' A_{A'}\nabla_{A'}\nabla_{B'}...\nabla_{F'}\chi = \alpha_{A'}\beta_{B'}...\gamma_{F'}\left(\frac{1}{2}e^{Z'A'}\Box + \Box Z'A'\right)\nabla_{B'}...\nabla_{F'}\chi
$$

$$
= \alpha_{A'}\beta_{B'}...\gamma_{F'}\frac{1}{2}e^{Z'A'}\nabla_{B'}...\nabla_{F'}\Box\chi
$$

$$
= 0 .
$$

Here, due to our work on the Minkowski spacetime, all the curvatures disappear, then $\Box Z'A' = 0$ and the wave operator can be commuted with the derivatives, for instance $[\Box, \nabla_{B'}] = 0$.

Therefore $\phi_{(AB...F)} := \alpha_{A'}\beta_{B'}...\gamma_{F'}\nabla_{A'}\nabla_{B'}...\nabla_{F'}\chi$ is a solution of the equation (4.38). This shows that in the Minkowski spacetime:

a) If the energy of the initial data on $\Sigma_0$ is finite, then the energy of the solution on the space slices $\Sigma_T = \{t = T\}$ are also finite (by energy estimate). Since $\overline{C_0^\infty(\Sigma_T)} = L^2(\Sigma_T)$, we can consider the wave equation $\Box\chi = 0$ with the compactly supported initial data, then we get a unique solution which is smooth compactly supported in space due to the finite propagation speed property. So that, there are solutions of spin $n/2$ zero rest-mass equation (4.38) that are smooth and compactly supported in space.

b) As a consequence of a), the constraint equations admit smooth compactly supported solutions and non-trivial finite energy solutions. It is not completely clear that all finite energy solutions can be obtained from a Hertz-type potential since the construction is merely local. Hence we shall work on the constrained subspace $\mathcal{H}_0 = \overline{C_0^\infty(\Sigma_0, S_{(AB...F)})} \cap D(L^2(\Sigma_0, S_{AB...F}))$.

Since the equation (4.38) is conformally invariant, the same property is valid on the conformal compactifications (full and partial) of the Minkowski spacetime.

### 4.7.3 Generalisation of L.Hörmander’s result

In this part we will extend the result of L.Hörmander [42] for $n$–dimension. The result of L.Hörmander has been extended in the minor modifications for the conditions of the metric and the coefficients of the terms with order one in the wave operator by J-P. Nicolas [64] for the scalar wave equation. And then he has given the way to apply these results to solve the Goursat problem of the scalar wave equation in the Schwarzschild spacetime (see [66]). Here we will show how the Goursat
problem is valid for the spin $n/2$ field (i.e. $n$–dimension) in the future $\mathcal{I}^{+}(S)$ of $S$ in $\hat{\mathbb{M}}$ (recall that $S$ is the spacelike hypersurface in $\hat{\mathbb{M}}$ such that it pass $\mathcal{I}^{+}$ strictly in the past of the support data).

Now we consider the future $\mathcal{I}^{+}(S)$ in $\hat{\mathbb{M}}$ and we cut off the future $\mathcal{V}$ of a point in $\hat{\mathbb{M}}$ lying in the future of the past of the support of the Goursat data. We obtain the resulting spacetime $\hat{\mathcal{M}}$, then we extend $\hat{\mathcal{M}}$ as a cylindrical globally hyperbolic spacetime $(\mathbb{R}_{t} \times S^{3}, \hat{g})$ where $g_{\mid \hat{\mathcal{M}}} = \hat{g}_{\mid \hat{\mathcal{M}}}$. We also extend the part of $\mathcal{I}^{+}$ inside $\mathcal{I}^{+}(S) - \mathcal{V}$ as a null hypersurface $\mathcal{C}$ that is the graph of a Lipschitz function over $S^{3}$ and the data by zero on the rest of the extended hypersurface.

Now, our work will be based on the strategy of the [64] to generalise the Goursat problem of the equation spin $n$ field

$$\Box \phi_{A\ldots F} = 0$$

in the resulting spacetime $(\mathcal{M} = \mathbb{R}_{t} \times S^{3}, g)$. Due to the result of E. Stiefel [76], $(\mathcal{M} = \mathbb{R}_{t} \times S^{3}, g)$ is parallelizable (i.e. it admit a continuous global frame in the sense that the tangent space at each point has a basis), so we can translate this equation into the scalar form in choosing the spin-frame $\{o,\iota\}$ for $\mathcal{M}$ (such that in this spin-frame the Newman-Penrose tetrad is $C^{\infty}$) as follows

$$P \Phi + L_{1} \Phi = 0,$$

where

$$P = \begin{pmatrix}
\Box & 0 & \ldots & 0 \\
0 & \Box & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \Box
\end{pmatrix}$$

is the $(n+1) \times (n+1)$–matrix diagram,

$$\Phi = \begin{pmatrix}
\phi_{0} \\
\phi_{1} \\
\vdots \\
\phi_{n}
\end{pmatrix}$$

is the components of $\phi_{A\ldots F}$ on the spin-frame and

$$L_{1} = \begin{pmatrix}
L_{00} & L_{01} & \ldots & L_{0n} \\
L_{10} & L_{11} & \ldots & L_{1n} \\
\ldots & \ldots & \ldots & \ldots \\
L_{n0} & L_{n1} & \ldots & L_{nn}
\end{pmatrix}$$

is the $(n+1) \times (n+1)$–matrix where the components are the operators that have the coefficients $C^{\infty}$

$$L_{ij} = b_{ij}^{\alpha} \partial_{t} + b_{ij}^{\alpha} \partial_{\alpha} + c^{ij}.$$
If we define the norm of the vector by the maximal element, we can see that

\[ 0 = \sum_{i=0}^{n} \partial_{t} \tilde{\phi}_i (P \Phi + L_1 \Phi)_i = \sum_{i=0}^{n} \partial_{t} \tilde{\phi}_i (P \phi_i + (L_1 \Phi)_i) \]

and

\[ \sum_{i=0}^{n} \partial_{t} \tilde{\phi}_i (L_1 \Phi)_i \leq ||\partial_{t} \Phi|| ||L_1 \Phi|| \leq ||\partial_{t} \Phi|| (||\nabla \Phi|| + ||\partial_{r} \Phi|| + ||\Phi||), \]

these are everything needed to control the energies in the first step of the proof of theorem 2 in [64]. Then the conditions of the metric $g$ and the coefficients in $L_1$ satisfies $H_1$ and $H_2$ in [64]. Therefore we can do the same to the strategy of [64] to solve the Goursat problem for the system (4.40).

**Theorem 4.7.1.** With the initial data $(\phi_i, \partial_{t} \phi_i) \in C^\infty_0 (C) \times C^\infty_0 (C)$ for all $i = 1, 2...n$, the equation (4.40) has a unique solution $\Phi = (\phi_1, \phi_2..., \phi_n)$:

\[ \phi_i \in C(\mathbb{R}; H^1(S^3)) \cap C^1(\mathbb{R}; L^2(S^3)) \text{ for all } i = 1, 2...n. \]

Then by local uniqueness and causality, using the finite propagation speed, the solution $\Phi$ vanishes in $\mathcal{I}^+(S) - \mathfrak{M}$, so the Goursat problem that we are studying has a unique smooth solution in the future of $S$, that is the restriction of $\Phi$ to $\mathfrak{M}$.
4.7.4 Detailed calculations for the Goursat problem

We have the expression of the spinor field $\Xi^A'_{(B\ldots F)}$ as follows:

$$
\Xi^A'_{(B\ldots F)} = \sum_{k=0}^{n-1} (-1)^k \Xi^{1'}_{n-k} \hat{\alpha}^A_{D\ldots C} \hat{\delta}^D_{n-k} \hat{\gamma} = \sum_{k=0}^{n-1} (-1)^k \Xi^{0'}_{n-k} \hat{\alpha}^A_{D\ldots C} \hat{\delta}^D_{n-k} \hat{\gamma}.
$$

The effect of $\hat{\nabla}_{ZA'}$ on the full spinor field can be understood as

$$
\hat{\nabla}_a \Xi = (\hat{D}\Xi)\hat{n}_a + (\hat{D}'\Xi)\hat{i}_a - (\hat{\delta}\Xi)\hat{m}_a - (\hat{\delta}'\Xi)\hat{m}_a.
$$

In the partial conformal compactification $\hat{M}$, we have the twelve values of the spin coefficients which are:

$$
\hat{k} = \hat{\sigma} = \hat{\lambda} = \hat{\tau} = \hat{\nu} = \hat{\pi} = \hat{\rho} = \hat{\mu} = \hat{\epsilon} = 0,
$$

$$
\hat{\gamma} = -\frac{R}{\sqrt{2}}, \hat{\beta} = -\hat{\alpha} = \frac{\cot \theta}{2\sqrt{2}}.
$$

The effect of the covariant derivative on the spin-frame $\{\hat{\alpha}_A, \hat{i}_A\}$ gives (see eq.(4.5.26) in Vol.1):

$$
\hat{D}\hat{\alpha}_A = \hat{\varepsilon}\hat{\alpha}_A - \hat{\kappa}\hat{i}_A = 0, \quad \hat{D}\hat{i}_A = -\hat{\varepsilon}\hat{i}_A + \hat{\kappa}\hat{\alpha}_A = 0,
$$

$$
\hat{d}'\hat{\alpha}_A = \hat{\alpha}\hat{\alpha}_A - \hat{\rho}\hat{i}_A = -\frac{\cot \theta}{2\sqrt{2}} \hat{\alpha}_A, \quad \hat{d}'\hat{i}_A = -\hat{\alpha}\hat{i}_A + \hat{\lambda}\hat{\alpha}_A = \frac{\cot \theta}{2\sqrt{2}} \hat{i}_A,
$$

$$
\hat{d}\hat{\alpha}_A = \hat{\beta}\hat{\alpha}_A - \hat{\sigma}\hat{i}_A = \frac{\cot \theta}{2\sqrt{2}} \hat{\alpha}_A, \quad \hat{d}\hat{i}_A = -\hat{\beta}\hat{i}_A + \hat{\mu}\hat{\alpha}_A = -\frac{\cot \theta}{2\sqrt{2}} \hat{i}_A,
$$

$$
\hat{D}'\hat{\alpha}_A = \hat{\gamma}\hat{\alpha}_A - \hat{\tau}\hat{i}_A = -\frac{R}{\sqrt{2}} \hat{\alpha}_A, \quad \hat{D}'\hat{i}_A = -\hat{\gamma}\hat{i}_A + \hat{\nu}\hat{\alpha}_A = \frac{R}{\sqrt{2}} \hat{i}_A.
$$

Similarly on the dual conjugation spin-frame $\{\hat{\alpha}'_{A'}, \hat{i}'_{A'}\}$ we have:

$$
\hat{D}\hat{\alpha}'_{A'} = 0, \quad \hat{D}\hat{i}'_{A'} = 0,
$$

$$
\hat{d}'\hat{\alpha}'_{A'} = -\frac{\cot \theta}{2\sqrt{2}} \hat{\alpha}'_{A'}, \quad \hat{d}'\hat{i}'_{A'} = \frac{\cot \theta}{2\sqrt{2}} \hat{i}'_{A'},
$$

$$
\hat{d}\hat{\alpha}'_{A'} = \frac{\cot \theta}{2\sqrt{2}} \hat{\alpha}'_{A'}, \quad \hat{d}\hat{i}'_{A'} = -\frac{\cot \theta}{2\sqrt{2}} \hat{i}'_{A'},
$$

$$
\hat{D}'\hat{\alpha}'_{A'} = -\frac{R}{\sqrt{2}} \hat{\alpha}'_{A'}, \quad \hat{D}'\hat{i}'_{A'} = \frac{R}{\sqrt{2}} \hat{i}'_{A'}.
$$
Therefore we obtain the detailed expression of $\hat{\nabla}_{ZA'} \Xi_{A'}^{(B...F)}$ as follows:

$$\hat{\nabla}_{ZA'} \Xi_{A'}^{(B...F)} = (\hat{D} \Xi_{A'}^{(B...F)}) \hat{n}_a + (\hat{D}' \Xi_{A'}^{(B...F)}) \hat{m}_a - (\hat{\delta} \Xi_{A'}^{(B...F)}) \hat{m}_a - (\hat{\delta}' \Xi_{A'}^{(B...F)}) \hat{n}_a$$

$$= \hat{D} \left( \sum_{k=0}^{n-1} (-1)^k \Xi_{n-1-k}^{1'} \hat{A}' \hat{i}_{B...iC} \hat{i}_{D...iF} \right) \hat{i}_A \hat{i}_{A'}$$

$$- \hat{D}' \left( \sum_{k=0}^{n-1} (-1)^k \Xi_{n-1-k}^{0'} \hat{A}' \hat{i}_{B...iC} \hat{i}_{D...iF} \right) \hat{A} \hat{o}_{A'}$$

$$+ \hat{\delta} \left( \sum_{k=0}^{n-1} (-1)^k \Xi_{n-1-k}^{1'} \hat{A} \hat{i}_{B...iC} \hat{i}_{D...iF} \right) \hat{i} \hat{A} \hat{i}_{A'}$$

$$- \hat{\delta}' \left( \sum_{k=0}^{n-1} (-1)^k \Xi_{n-1-k}^{1'} \hat{A} \hat{i}_{B...iC} \hat{i}_{D...iF} \right) \hat{A} \hat{i} \hat{i}_{A'}$$

$$= \sum_{k=0}^{n-1} (-1)^k \left( -\hat{D} \Xi_{n-1-k}^{1'} + \hat{\delta} \Xi_{n-1-k}^{0'} + \frac{n-2-2k}{2\sqrt{2}} \cot \theta \Xi_{n-1-k}^{0'} \right) \hat{i}_A \hat{i}_{B...iC} \hat{i}_{D...iF}$$

$$+ \sum_{k=0}^{n-1} (-1)^k \left\{ \frac{(2k+2-n)R}{\sqrt{2}} \Xi_{n-1-k}^{0'} + \frac{2k-n}{2\sqrt{2}} \cot \theta \right\} \hat{A} \hat{i}_{B...iC} \hat{i}_{D...iF}$$

So the equation $\hat{\nabla}_{ZA'} \Xi_{B...F} = 0$ is equivalent to the system

$$\begin{cases} 
- \hat{D} \Xi_{n-1-k}^{1'} + \hat{\delta} \Xi_{n-1-k}^{0'} + \frac{n-2-2k}{2\sqrt{2}} \cot \theta \Xi_{n-1-k}^{0'} = 0, \\
- \hat{D}' + \frac{(2k+2-n)R}{\sqrt{2}} \Xi_{n-1-k}^{0'} + \frac{2k-n}{2\sqrt{2}} \cot \theta \Xi_{n-1-k}^{1'} = 0
\end{cases}$$

for all $k = 0, 1...n-1$. Taking the constrain of this system on $\mathcal{I}^+$, we get only the constraint of the second equations

$$-\sqrt{2} \partial_a \Xi_{n-1-k}^{0'} \big|_{\mathcal{I}^+} = 0,$$

for all $k = 0, 1...n-1$. Integrating these equations we get $\Xi_{n-1-k}^{0'} \big|_{\nu(p)} = \text{constant}$ so that the Cauchy problem with the initial condition $\Xi_{n-1-k}^{0'} \big|_{\nu(p)} = 0$ has a unique solution which is zero.
Bibliography


Peeling et scattering conforme dans les espaces-temps de la relativité générale


Mots-clés: Relativité générale, problème de Goursat, équation des ondes linéaire et non linéaire, équation de Dirac, peeling, diffusion conforme, technique conforme, méthode des champs de vecteurs, formalismes de Newman-Penrose et de Geroch-Held-Penrose.

Peeling and conformal scattering on the spacetimes of the general relativity

Abstract: This work explores two aspects of asymptotic analysis in general relativity: peeling and conformal scattering. On the one hand, the peeling is constructed for linear and nonlinear scalar fields as well as Dirac fields on Kerr spacetime, which is non-stationary and merely axially symmetric. This generalizes the work of L. Mason and J-P. Nicolas (2009, 2012). The vector field method (geometric energy estimates) and the conformal technique are developed. They allow us to formulate the definition of the peeling at all orders and to obtain the optimal space of initial data which guarantees these behaviours. On the other hand, a conformal scattering theory for the spin-$n/2$ zero rest-mass equations on Minkowski spacetime is constructed. Using the conformal compactifications (full and partial), the spacetime is completed with two null hypersurfaces representing respectively the past and future end points of null geodesics. The asymptotic behaviour of fields is then obtained by solving the Cauchy problem for the rescaled equation and considering the traces of the solutions on these hypersurfaces. The invertibility of the trace operators, that to the initial data associate the future or past asymptotic behaviours, is obtained by solving the Goursat problem on the conformal boundary. The conformal scattering operator is then obtained by composing the future trace operator with the inverse of the past trace operator.

Keywords: General relativity, Goursat problem, linear and nonlinear wave equation, Dirac equation, peeling, conformal scattering, conformal technique, vector field method, Newman-Penrose and Geroch-Held-Penrose formalisms.