Flux backgrounds, AdS/CFT and Generalized Geometry

Praxitelis Ntokos

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Praxitelis NTOKOS

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M.	Ignatios	ANTONIADIS	Examinateur
M.	Stephano	GIUSTO	Rapporteur
Mme	Mariana	GRAÑA	Directeur de thèse
M.	Alessandro	TOMASIELLO	Rapporteur

Abstract: The search for string theory vacuum solutions with non-trivial fluxes is of particular importance for the construction of models relevant for particle physics phenomenology. In the framework of the AdS/CFT correspondence, four-dimensional gauge theories which can be considered to descend from $\mathcal{N} = 4$ SYM are dual to ten-dimensional field configurations with geometries having an asymptotically AdS_5 factor. In this Thesis, we study mass deformations that break supersymmetry (partially or entirely) on the field theory side and which are dual to type IIB backgrounds with non-zero fluxes on the gravity side. The supergravity equations of motion constrain the parameters on the gauge theory side to satisfy certain relations. In particular, we find that the sum of the squares of the boson masses should be equal to the sum of the squares of the fermion masses, making these set-ups problematic for phenomenology applications.

The study of the supergravity duals for more general deformations of the conformal field theory requires techniques which go beyond the standard geometric tools. Exceptional Generalized Geometry provides a very elegant way to incorporate the supergravity fluxes in the geometry. We study AdS_5 backgrounds with generic fluxes preserving eight supercharges and we show that these satisfy particularly simple relations which admit a geometrical interpretation in the framework of Generalized Geometry. This opens the way for the systematic study of supersymmetric marginal deformations of the conformal field theory in the context of AdS/CFT.

Keywords: String Theory, Flux Compactifications, Supersymmetry, Supergravity, AdS/CFT Correspondence, (Exceptional) Generalized Geometry

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Contents

Introduction

One of the central research directions in modern theoretical physics is the attempt to connect string theory with particle physics phenomenology. This can be a very challenging task due to the large range of scales involved; the same theory is supposed to describe quantum gravity at the Planck scale ($\sim 10^{16}$ TeV) and also the Standard Model of particle physics which is currently experimentally accessible at the TeV scale. In general terms, the problems that appear have to do with the reduction of the large amount of symmetry which the theory possesses, and also with the existence of massless fields (moduli) which correspond to parameters of the theory that are left unfixed without some stabilization mechanism.

A very promising approach to the solution of these problems is the study of string theory backgrounds with non-trivial fluxes turned on. Flux compactifications play a key role both in the construction of phenomenologically-relevant models due to their potential to stabilize moduli, as well as in gauge/gravity duality where they realize duals of less symmetric gauge theories. Fluxes are also related to extended objects (branes) existing in string theory and which are also usually employed in the construction of models similar to the standard model or some supersymmetric extension of it.

A common way to obtain theories with interesting properties for phenomenology is to place D3-branes in flux compactifications (for reviews see [1, 2, 3]). D3-branes (or better stacks of them) "carry" on their world-volume gauge theories with $\mathcal{N} = 4$ supersymmetry. The gauge symmetry of theses theories is $U(N)$ if they are sitting at a regular point of the internal manifold or it can be a different gauge group (described by the so-called quiver diagrams [4]) if they are placed at singularities. The large amount of supersymmetry controls the UV behaviour of the theory, but on the other hand imposes very strong restrictions on the relations between the various parameters of the theory. A direct consequence of supersymmetry in our case is the so-called supertrace rule: the sum of the squares of the boson masses and the fermion masses are equal which is, of course, an obstacle necessary to overcome from the phenomenological perspective.

The situation is improved when non-trivial fluxes are taken into account. These can break supersymmetry to $\mathcal{N} = 1$ or even to $\mathcal{N} = 0$. More importantly, the supersymmetry-breaking terms that are generated are soft and hence they do not spoil the good renormalizability properties that supersymmetry provides. One may therefore hope to use these D3-brane set-ups to construct realistic theories of physics beyond-the-standard-model by avoiding the phenomenologically unviable supertrace rule. One of the main results included in this Thesis is that in fact this is not the case and that the supertrace rule persists even when supersymmetry is broken completely [5].

The gauge theory living on the D3-branes is conveniently studied by examining its supergravity dual in the framework of the AdS/CFT Correspondence. As argued by Polchinski and Strassler [6], adding mass terms for the three chiral fermions of $\mathcal{N} = 4$ SYM (and thus preserving $\mathcal{N} = 1$ supersymmetry) is dual to turning on non-normalizable modes for the three-form fluxes of type IIB supergravity. The bosonic mass terms on the other hand, are dual to the second order (in the polarization radius) term in the polarization potential of the D3-branes under the background five-form flux (which was present before the mass deformation). This polarization potential is again dictated by supersymmetry and one finds that the bosonic masses are determined by the fermionic ones so that the supertrace rule is still satisfied.

In this Thesis, we studied deformations of the $\mathcal{N} = 4$ SYM theory with generic fermion mass terms which break supersymmetry completely [7]. This theory, even it it not supersymmetric, has a memory of the original $SU(4)$ R-symmetry. This information can be combined with supergravity arguments in order to extract useful conclusions for the bosonic mass terms. The polarization potential dual to the boson masses can be decomposed in pieces corresponding to the trace and traceless part of the boson mass matrix. As we show, the latter is *not* fully determined by the fermion masses expressing the fact that supersymmetry has been broken. The former however is fixed by the supergravity equations of motion to be equal to the trace of the fermion mass matrix squared. This means that the supertrace rule is valid in the absence of supersymmetry and it rather expresses the fact that the theory has a UV conformal fixed point. Therefore, for gauge theories which have a holographic dual that is asymptotically AdS_5 , the sum of the squares of the boson masses and the fermion masses are equal, fact that consists a serious obstacle in obtaining standard model-like lagrangians.

For more general deformations of the $\mathcal{N} = 4$ SYM, one has to consider more complicated supergravity solutions. The study of generic backgrounds is a very ambitious goal since the supergravity equations of motion are highly non-linear. Relying on supersymmetry can improve the situation making the interplay between the geometry and the fluxes more controllable. In fact, there is a mathematical framework called generalized geometry which can incorporate the effect of the fluxes in purely geometric data.

In generalized geometry, the metric degrees of freedom are combined with those of the gauge fields into a generalized metric. Similarly, the vectors generating diffeomorphisms are combined with forms of various degree generating gauge transformations for the gauge fields to form generalized vectors. This geometric reformulation of backgrounds with fluxes gives a characterization that allows in principle to find new solutions, as well as to understand the deformations, which are the moduli of the lower dimensional theory. In the context of gauge/gravity duality, deformations of the background correspond to deformations of the dual gauge theory.

In this Thesis we focus on AdS_5 compactifications of type IIB and M-theory preserving eight supercharges [8]. These are dual to four-dimensional $\mathcal{N} = 1$ conformal field theories. The internal manifolds are respectively five and six-dimensional. The generalized tangent bundle combines the tangent bundle plus in the case of M-theory the bundle of two and five-forms, corresponding to the gauge symmetries of the three-form field and its dual six-form field, while in type IIB two copies of the cotangent bundle and the bundle of five-forms and the bundle of three-forms, corresponding respectively to the symmetries of the B-field and R-R 2-form field and their dual six-forms and the R-R 4-form. In both cases the generalized bundle transforms in the fundamental representation of $E_{6(6)}$, the U-duality group that mixes these symmetries.

Compactifications leading to backgrounds with eight supercharges in the language of (exceptional) generalized geometry are characterized [9] by two generalized geometric structures that describe the hypermultiplet and vector multiplet structures of the lower dimensional supergravity theory. When this theory is five-dimensional, the generalized tangent bundle has reduced structure group $USp(6) \subset USp(8) \subset E_{6(6)}$ [10], where $USp(8)$, the maximal compact subgroup of $E_{6(6)}$, is the generalized analogue of $SO(6)$, namely the structure group of the generalized tangent bundle equipped with a metric.

The integrability conditions on these structures required by supersymmetry were formulated in [11]. The "vector multiplet" structure is required to be generalized Killing, namely the generalized vector corresponding to this structure generates generalized diffeomorphisms (combinations of diffeomorphisms and gauge transformations) that leave the generalized metric invariant. The integrability condition for the hypermultiplet structure requires the moment maps for generic generalized diffeomorphisms to take a fixed value proportional to the cosmological constant of AdS. These conditions can be seen as a generalization of Sasaki-Einstein conditions: they imply that the generalized Ricci tensor is proportional to the generalized metric. They parallel the supersymmetry conditions obtained from five-dimensional gauged supergravity [12].

In this Thesis, we prove the integrability conditions for the generalized structures directly from the supersymmetry equations of type IIB and eleven dimensional supergravity. For that, the generalized structures are written in terms of $USp(8)$ bispinors. These are subject to differential and algebraic conditions coming from the supersymmetry transformation of the internal and external gravitino (plus dilatino in the case of type IIB). We show that the latter imply the integrability conditions for the generalized structure.

The content of this Thesis is organized as follows. In chapter 1, we introduce the main ideas of string theory that will be used in the following chapters where more specialized topics are developed. In chapter 2, we study mass deformations of the $\mathcal{N} = 4$ SYM theory which is realized on coincident D3-branes. Using representation-theory and supergravity arguments we show that the fermion masses completely determine the trace of the boson mass matrix, and in particular that the sums of the squares of the bosonic and fermionic masses are equal. In chapter 3, we move on to the study of generic supersymmetric backgrounds with fluxes and we present generalized geometry as the appropriate formalism to describe such backgrounds. Finally, in chapter 4 we write the supersymmetry conditions in the framework of exceptional generalized geometry for generic AdS_5 backgrounds preserving eight supercharges and we show how these follow directly from the Killing spinor equations. We conclude with a discussion of the results obtained in this Thesis and with the possible new research directions that these results could open. The Appendices contain technical information which is used throughout the main text.

This Thesis is based on the following papers

- [5]*D3-brane model building and the supertrace rule*
  I. Bena, M. Graña, S. Kuperstein, P. Ntokos, M. Petrini
  Physical Review Letters 116 (2016) no 14 141601
  arXiv:1510.07039
- [7] *Fermionic and bosonic mass deformations of N=4 SYM and their bulk supergravity dual*
  I. Bena, M. Graña, S. Kuperstein, P. Ntokos, M. Petrini
  JHEP 05 (2016) 149
  arXiv:1512.00011
- [8] *Generalized Geometric vacua with eight supercharges*
  M. Graña, P. Ntokos
  JHEP 08 (2016) 107
  arXiv:1605.06383

Chapter 1

Strings, fields and branes

The goal of this chapter is to provide the necessary string theory background on which the concepts developed in this Thesis will be based on. A proper presentation of the relevant material can be found in many textbooks and reviews (a standard reference is [13], [14]) and here we will restrict ourselves in providing rather schematic definitions and explanations of the objects that we will need later. Despite the narrowness that its name may indicate, string theory has to do with a lot of things: particles, strings, (mem)branes, fields, gauge theories, gravity, supersymmetry and extra dimensions. Actually, string theory provides a framework on which all these concepts are interrelated in a consistent and elegant way. In this first chapter, we will try to make this abundance of ideas in the theory clear having in mind the applications that will follow.

The structure of the chapter is as follows. In section 1.1 we introduce strings in the worldsheet perspective and derive their spectrum and their basic properties. Focusing on the massless spectrum in section 1.2, we describe the type II ten-dimensional supergravity theories which will be the "arena" of our computations for the biggest part of the Thesis. In section 1.3, we introduce M-theory which provides the natural framework for the description of dualities in the string theory web. Then, in section 1.4 we turn to vacuum solutions of type II supergravity theories and we explain the power of supersymmetry (when it is present) in describing such solutions. Finally, we introduce D-branes in 1.5 which play a major role in connecting string theory with gauge field theories. The most striking such connection is the AdS/CFT Correspondence which we introduce in 1.6.

1.1 Supersymmetric relativistic strings

The starting point of our discussion is the relatively simple case of classical relativistic bosonic strings. These are one-dimensional objects for which we will assume for the moment that they move in D-dimensional Minkowski space-time. Their "history", the *worldsheet*, is parametrized by coordinates $X^M(\tau,\sigma)$, $M = 0,1,\ldots,D-1$. The scale of this theory is set by the tension of the string $T = 1/2\pi\alpha'$ with $\alpha' = l_s^2 = 1/M_s^2$.

One can adopt two supplementary points of view when thinking about string theory. According to the first, string theory is considered to be a field theory defined on the two-dimensional worldsheet and the X^M are field-like degrees of freedom on this worldsheet. The other point of view is that the X^M are coordinates in the space-time where the strings live in and the string configurations are the dynamical degrees of freedom of the theory. Hence, in the quantum version, string theory is a second quantized theory according to the former point of view and a first quantized theory according to the latter.

From the worldsheet point of view, this theory has two gauge symmetries: two-dimensional reparametrization invariance and position-dependent rescalings of the two-dimensional metric which are called *Weyl* transformations. The solutions to their equation of motion are described by the center of mass modes x^M, p^M and oscillators $\tilde{\alpha}_n^M$ (left-moving) and α_n^M (right-moving) [1] where the level $n \in \mathbb{Z}^*$ is related to the mode of the Fourier expansion. In the quantum theory these become operators (creation and annihilation operators depending on the sign of n) which act on the vacuum state $|0; k>$ to produce the excited states of the theory. Among these, there is one with negative M^2, the bosonic string tachyon while the massless states are

- a symmetric tensor g_{MN} , to be identified with the graviton,
- an antisymmetric tensor B_{MN} , which is called the B-field and
- a scalar ϕ, which is called the dilaton. It turns out that the vacuum expectation value (vev) of the dilaton ϕ_0 can be identified with the string coupling g_s through

$$g_s = e^{\phi_0} \tag{1.1.1}$$

All the other states form a tower of massive modes with $\Delta M^2 = 4/\alpha'$.

On the passage from the classical theory of strings to the corresponding quantum mechanical version, there exists the possibility (as in any field quantization) that some of the symmetries become anomalous. Requiring the absence of such anomalies for the gauge symmetries of the theory has quite strong implications for the background in which the strings move in. The most dramatic one is the restriction of the space-time dimension to a specific integer number (critical dimension)! In the case of the bosonic sting one gets D=26.

Furthermore, one can also study strings moving in a curved background described by a space-time metric G_{MN}. This metric can be thought as a coherent state of the gravitons which string theory already contains. One can again find the conditions under which the quantum theory is still conformally invariant[2]. It turns out that the condition to be satisfied is the background to be Ricci flat, i.e. $R_{MN} = 0$. One can also consider more general backgrounds containing the other massless fields of the theory. Again, in order to "save" conformal invariance, we get a set of equations which can be derived from a field theory action. The latter is called an *effective action* because it concerns only the massless modes of the full string theory.

The spectrum of the theory that we have described so far suffers from two major drawbacks. First, it contains a state with negative mass squared (tachyon) which can

[1] At this point of the discussion, we have in mind closed strings, i.e. strings for which $X^M(\tau, \sigma) = X^M(\tau, \sigma + 2\pi)$. We will describe open strings later in this section.

[2] The renormalization procedure typically introduces a scale μ which ruins the conformal invariance. The latter is the residual symmetry when one fixes the worldsheet metric.

be a signal of instability for the vacuum state of the theory but more importantly it does not contain fermions! These problems are both solved if one introduces supersymmetry. There are five consistent *superstring* models: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. In this Thesis, we will mainly be concerned with the type II models which we briefly describe below.

A way to construct a superstring theory is to introduce worldsheet superpartners $\tilde{\psi}^M$ (left-moving) and ψ^M (right-moving) for the X^M 's making the worldsheet theory (1,1) supersymmetric. However, for fermions we can impose periodic (Ramond (R)) conditions or antiperiodic (Nevue-Schwarz (NS)) conditions. Again, in the quantum theory we have creators and annihilators (this time satisfying anticommutation relations) which act on the (tachyonic) vacuum $|0>$ to produce the excited states. It turns out that the (physical) massless states of the theory are described by a pair of creators, one coming from $\tilde{\psi}^M$ and the other from ψ^M. In the case of superstring, superconformal anomaly cancellation requires $D = 10$. Each of them can contribute with an R-mode or an NS-mode resulting in 4 possibilities which can be written schematically as

$$\begin{array}{llll} \text{R-R:} & \psi_R\tilde{\psi}_R|0>, & \text{NS-R:} & \psi_{NS}\tilde{\psi}_R|0> , \\ \text{NS-NS:} & \psi_{NS}\tilde{\psi}_{NS}|0>, & \text{R-NS:} & \psi_R\tilde{\psi}_{NS}|0> . \end{array} \tag{1.1.2}$$

A state $\psi_R^M|0>$ is a space-time fermion which transforms in the $\mathbf{16} = \mathbf{8}_s \oplus \mathbf{8}_c$ where the decomposition is in Weyl representations. To have space-time supersymmetry, we have to project out half of them. This can be done consistently through a procedure called *GSO projection* [15]. The GSO projection can be used in order to project out the tachyon which spoils the stability of bosonic string theory. Moreover, one can choose the GSO projection in order to remove the same or different **8** R-state resulting in type IIB or type IIA superstring theory respectively. Type II is because there are two gravitini in the spectrum in the NS-R and R-NS sector. We will not describe the full massless spectrum of the theory now (we will in the next section) but for the moment let us mention that the NS-NS sector is identical to that of the bosonic string.

As in the bosonic case, one can consider strings moving in a background made of all the massless fields of the theory. Respecting the symmetries of the theory again gives rise to a set of equations and again we can find an action from which these equations can be derived. This effective field theory will contain the graviton and will be supersymmetric. In other words, it will be a *supergravity.* We will describe in more detail the type II supergravities which are the low energy of the type II superstring theories in the next section.

Up to now, we have described the theory for closed strings in which case the local dynamics (in the worldsheet perspective) is supplemented with the periodicity conditions. For open strings instead, one has to impose specific boundary conditions for the motion of the endpoints. There are two kinds of these conditions, which are called *Dirichlet* and *Neumann* boundary conditions. Let us consider an open string which is parametrized by $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \pi$ moving in a flat background and assume that the $p+1$ coordinates of the $\sigma = 0$ endpoint satisfy Neumann boundary condition while the other $D-p-1$ satisfy Dirichlet. This is given by

$$\left.\frac{\partial X^M}{\partial \sigma}\right|_{\sigma=0} = 0, \quad M = 0, \cdots, p \qquad \left.\frac{\partial X^M}{\partial \tau}\right|_{\sigma=0} = 0, \quad M = p+1, \cdots, D-1 \tag{1.1.3}$$

The motion described by these conditions is that the particular endpoint is restricted to move in a $(p+1)$-dimensional plane[3]. The latter can be interpreted as an extended solitonic object in string theory usually called a Dp-brane (D for Dirichlet). The role of D-branes was underestimated in string theory until the celebrated work of Polchinski [16] who recognized that D-branes carry Ramond-Ramond charge. We will provide more details for D-branes in section 1.5.

A theory of open strings includes necessarily closed strings as well. This is because two open strings can combine to form a closed string. From this particular kind of interaction, we can understand that the closed string coupling constant g_s has to be related to the square of the open string coupling constant g^2_{open}.

We close this section by presenting a particular kind of symmetry, *T-duality*, which is a novel, purely stringy effect. As a simple but an illustrative example, we consider a bosonic closed string moving in a flat background but with one direction, say $\hat{X}$, being a circle of radius R[4]. In this case, the closed periodicity condition gets modified to

$$\hat{X}^M(\tau, \sigma + 2\pi) = \hat{X}^M(\tau, \sigma) + 2\pi R w \tag{1.1.4}$$

where w is an integer counting how many time the string is wrapped along the compact direction. It is called the *winding* number. In the set-up we consider, there is another integer n; it is related to the momentum of the string along the compact direction as

$$\hat{p} = \frac{n}{R} \tag{1.1.5}$$

This quantization condition follows the continuity of the "string waves" propagating along $\hat{X}$. It turns out that if we exchange the momentum number n with the winding number w, the spectrum of states has the same form as the original one but for compactification radius $R' = \alpha'/R$. This is the simplest example of T-duality.

We can study more interesting examples by considering d compact directions instead of one. In that case, the momenta and the windings can be packed in a $2d$ vector (n^a, w_a). One can keep the spectrum invariant now by a more extensive group of transformations, which turns out to be $SO(d,d,\mathbb{Z})$. This is the group of integer-valued matrices which leave invariant the "metric"

$$\eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \tag{1.1.6}$$

The group $SO(d,d,\mathbb{Z})$ is the T-duality group and is integer-valued because it applies to the full string theory spectrum. If we restrict our attention to the massless sector, it gets enhanced to its continuous version $SO(d,d)$. This group will appear very often in chapter 3 where we will introduce Generalized Geometry.

[3] Or more generally, in a $(p+1)$-dimensional submanifold of different topology.

[4] This is a very natural thing to do since $D = 26$ or $D = 10$ is too many directions for our world! Considering that some of the directions on which string theory (or any theory) is defined, are compact is called *compactification* of the theory.

1.2 Type II supergravities

In the previous section, we gave a brief overview of the basic elements of string theory adopting mostly the worldsheet point of view. For the supersymmetric models, we described the type II theories and we explained that the low energy limits of them correspond to the type II supergravity theories for which the space-time point of view is more natural. Let us now give the spectrum and the basic properties of the type IIA and IIB supergravities which correspond to the massless sector of the type IIA and IIB string theory respectively.

Let us start first with describing the spectrum. The NS-NS sector (see (1.1.2)) is common for the two theories and contains the metric g_{MN} , the B-field B_{MN} and the dilaton ϕ. The R-R sector for type IIA contains a 1-form $C_{(1)}$ and a 3-form $C_{(3)}$, while for type IIB a 0-form $C_{(0)}$, a 2-form $C_{(2)}$ and a 4-form $C_{(4)}$ constrained by the self-duality relation $\star dC_{(4)} = dC_{(4)}$. The fermionic sectors (NS-R and R-NS) contain two gravitinos (Ψ_1^M, Ψ_2^M) with the following chirality properties

$$\Gamma_{11}\Psi_1^M = \mp\Psi_1^M, \quad \Gamma_{11}\Psi_2^M = +\Psi_2^M \quad \text{type IIA/IIB} \tag{1.2.1}$$

and two dilatinos (λ^1, λ^2) with opposite chirality than the gravitinos. Both theories enjoy $\mathcal{N} = 2$ supersymmetry in ten dimensions which means that we have two ten-dimensional Majorana-Weyl supersymmetry parameters ϵ^1, ϵ^2 which have the same chirality properties as the gravitinos.

Since the main difference between type IIA and type IIB supergravity is in their chiral properties, it is convenient to describe both theories in the framework of the so-called *democratic formulation* [17]. According to this formulation, we extend the R-R sector of each theory by gauge potentials of appropriate degree. In this way, we actually double the R-R sector and so we have to impose a constraint in the field strengths in order to remain in the same theory. The procedure one follows is to first write down a pseudo-action[5], then derive the field equations from it and at the end impose the constraints.

In the case of type IIA, the additional potentials are $C_{(5)}, C_{(7)}$ and $C_{(9)}$[6] where the field strength for the last one is dual to the mass parameter (Romans mass) m_R for the massive IIA supergravity [18]. In the type IIB case, the additional potentials are $C_{(6)}$ and $C_{(8)}$. The bosonic part of the pseudo-action in the democratic formulation reads[7]

$$S_{\text{dem}} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{|g|}e^{-2\phi}\Big(R + \frac{1}{2}H^2 - 4(d\phi)^2 + \frac{1}{4}\sum_n \tilde{F}_{(2n)}^2\Big) \tag{1.2.2}$$

where $n = 0,1,\cdots,5$ for type IIA and $n = \frac{1}{2},\frac{3}{2},\cdots,\frac{5}{2}$ for type IIB. The above action includes the *improved* field strengths $\tilde{F}_{(2n)}$ which are given by

$$\tilde{F} = F - H\wedge C + m_R e^B, \quad H = dB, \quad F = dC \tag{1.2.3}$$

[5]It is called pseudo-action because the constraints are imposed on-shell.

[6]$C_{(9)}$ is non-dynamical since we can always remove its 10 degrees of freedom by a gauge transformation $C_{(9)} \to C_{(9)} + d\Lambda_{(8)}$.

[7]The kinetic terms for the field strengths are weighted as $\tilde{F}_{(n)}^2 = \dfrac{1}{n!}\tilde{F}_{M_1\cdots M_N}\tilde{F}^{M_1\cdots M_N}$.

where we are using a polyform[8] notation and

$$e^B = 1 + B + \frac{1}{2!} B \wedge B + \cdots \tag{1.2.4}$$

Of course, for the type IIB case one sets $m_R = 0$. The advantage of using $\tilde{F}$ instead of F is that the former stay invariant under the gauge transformations

$$B \to B + d\tilde{\Lambda}, \qquad C \to C + d\Lambda \wedge e^B - m_R \tilde{\Lambda} \wedge e^B \tag{1.2.5}$$

where $\tilde{\Lambda}$ is a one-form, $\Lambda = \Lambda_{\text{even}}/\Lambda_{\text{odd}}$ for type IIA/IIB and hence (1.2.2) is invariant as well. The price to pay is that the improved field strengths satisfy non-standard Bianchi identities, namely

$$dH = 0, \qquad d\tilde{F} = H \wedge \tilde{F} \tag{1.2.6}$$

The constant κ_{10} describes the gravitational coupling of localized objects to the supergravity (background) fields and it is given by

$$\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{(2\pi l_s)^8} \tag{1.2.7}$$

As already mentioned, (1.2.2) has to be supplemented with duality constraints which have to imposed on-shell. In the case of solutions with vanishing fermions (bosonic backgrounds) these take the form

$$\tilde{F}_{(2n)} = (-1)^{Int[n]} \star \tilde{F}_{(10-2n)} \tag{1.2.8}$$

Most of the applications with which will we will be concerned in this Thesis will be in the framework of type IIB supergravity. Therefore, for concreteness we are going to specialize from now on to the type IIB case while some of the results have analogues in the type IIA case.

Type II theories have $\mathcal{N} = 2$ supersymmetry in ten dimensions. The supersymmetry parameters are a pair of Majorana-Weyl spinors with the same chirality for type IIB which we write as a doublet

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}. \tag{1.2.9}$$

parametrizing 32 real supercharges. Since we are mainly interested in bosonic backgrounds, the supersymmetry transformations that give non-trivial information (see section 1.4) are those of the fermions which we write explicitly below

$$\delta\Psi_M = \nabla_M \epsilon - \frac{1}{4} \not{H}_M \sigma^3 \epsilon + \frac{e^\phi}{16} \left[(\not{F}_1 + \not{F}_5 + \not{F}_9)\Gamma_M (i\sigma^2) + (\not{F}_3 + \not{F}_7)\Gamma_M \sigma^1 \right] \epsilon \tag{1.2.10}$$

$$\delta\lambda = \left(\not{\partial}\phi - \frac{1}{2} \not{H} \sigma^3 \right) \epsilon - \frac{e^\phi}{8} \left[4(\not{F}_1 - \not{F}_9)(i\sigma^2) + 2(\not{F}_3 - \not{F}_7)\sigma^1 \right] \epsilon \tag{1.2.11}$$

[8]This actually means that we consider formal sums of forms defined on the 2^{10}-dimensional space $\bigoplus_{n=1}^{10} \wedge^n T^*$.

where $\tilde{\boldsymbol{F}}_{(n)} = \frac{1}{n!}\tilde{F}_{M_1\dots M_n}\hat{\Gamma}^{M_1\dots M_n}$ and σ^1,σ^2,σ^3 , the Pauli matrices acting on the doublet of type IIB spinors and we use hats for the ten-dimensional gamma matrices[9].

Returning back to the action (1.2.2), we can observe two things that seem quite peculiar from a general relativity perspective. The first is the overall factor $e^{-2\phi}$ that multiplies the action and the second is that the dilaton kinetic term seems to have the wrong sign. This is because we are in the so-called *string frame*. We can go the *Einstein frame* by defining the Einstein metric[10]

$$(g_E)_{MN} = (g_s e^{-\phi})^{1/2} g_{MN} \tag{1.2.12}$$

Applying this to (1.2.2) yields the standard Einstein-Hilbert term and a proper kinetic term for the dilaton. The new (physical) gravitational coupling is

$$\frac{1}{2\kappa^2} = \frac{1}{2g_s^2\kappa_{10}^2} \tag{1.2.13}$$

The democratic formulation which we have just described is very convenient for supergravity applications in Generalized Geometry which will be studied in chapters 3 and 4. However, for the applications in type IIB supergravity described in chapter 2 where we will need explicit solutions of the equations of motion, we will use the standard formulation. The IIB action in this case can be written as (in Einstein frame)[11]

$$S_{\rm st}^{\rm IIB} = -\frac{1}{2\kappa^2}\int d^{10}x\sqrt{|g|}\Big(R + \frac{1}{2}\frac{|d\tau|^2}{(Im\tau)^2} + \frac{g_s}{2}\frac{|G_3|^2}{Im\tau} + \frac{g_s^2}{4}\tilde{F}_5^2\Big) - \frac{ig_s^2}{8\kappa^2}\int\frac{C_4\wedge G_3\wedge G_3^*}{Im\tau} \tag{1.2.14}$$

Here, we have grouped together the R-R axion and the dilaton in the complex combination

$$\tau = C_0 + ie^{-\phi} \tag{1.2.15}$$

and the type IIB three-forms in the complex three-form

$$G_3 = F_3 - \tau H_3 \tag{1.2.16}$$

The self-duality constraint (1.2.8) does not apply any more for all the field strengths. However, one still has to impose

$$\star\,\tilde{F}_5 = \tilde{F}_5 \tag{1.2.17}$$

on the equations of motion. An interesting feature of the action (1.2.14) is that it is invariant under $SL(2,\mathbb{R})$ transformations of the form

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad G_3 \to \frac{G_3}{c\tau + d} \tag{1.2.18}$$

[9]Our conventions for gamma matrices and spinors are described in appendix B.

[10]Observe that for constant dilaton backgrounds the string and the Einstein frame metrics are the same.

[11]Here, we have also included a Chern-Simons term $-\frac{i}{2}\int\frac{C_4\wedge G_3\wedge G_3^*}{Im\tau} = \int C_4\wedge H_3\wedge F_3$ which is absent in the democratic formulation.

where a,b,c,$d \in \mathbb{R}$ and $ad - bc = 1$ and all the other fields are invariant. On the original field strengths F_3 and H_3, this symmetry is realized as the matrix representation

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \tag{1.2.19}$$

This $SL(2,\mathbb{R})$ symmetry is actually part of the U-duality group (see next section) and it will play an important role in chapters 3 and 4.

1.3 Supergravity in D=11

The supergravity theories we described in the previous section were formulated in ten space-time dimensions. We had good reasons for working in this particular dimension since these supergravities can be obtained as the low-energy limit of type II superstring theories which are known to have a consistent UV behaviour only for ten dimensions. However, it is a very natural question to ask which is the maximal dimensionality in which a consistent supergravity theory can be formulated. It turns out[12] [19] that the answer is $D = 11$ and the relevant supergravity was discovered by Cremmer, Julia and Scherk [20] and it has a privileged position in the catalogue of supergravity theories.

Gravitational theories do not have a good reputation for their renormalizability properties even if they are supersymmetric. In the case of ten-dimensional supergravities, string theory guarantees their UV completion and one can treat them as effective descriptions of string theory in the small curvature limit (compared to the string theory length scale). In the same sense, eleven-dimensional supergravity is considered to be the low-energy limit of a more fundamental eleven-dimensional theory called *M-theory* (Witten 1995). The complete description of M-theory is still lacking but the fundamental objects are assumed to be membranes while there are proposals that the underlying theory is a matrix theory [21].

Although we will not get in details, let us also mention that M-theory is considered to be the “master” theory which includes as limits and through a web of dualities all the five superstring models and eleven-dimensional supergravity. A particularly simple example which can also be seen at the supergravity level is that M-theory compactified on a circle gives type IIA sting theory. In this Thesis, we will use the terms M-theory and eleven-dimensional supergravity as synonymous and we will always mean the latter.

The field content of M-theory consists of

- The eleven-dimensional metric g_{MN} which plays the same role as in any extension of General Relativity.
- A fully antisymmetric three-form field A_{MNP} with the associated field strength $G = dA$. M-theory contains extended objects which couple to G electrically (M2-branes) and magnetically (M5-branes).

[12] The reason for that has to do with the dimensional reduction of the gravitino. Reducing on the 7-torus, ones gets a field content which cannot be packed in multiplets which involve only spins $s \leq 2$.

- A gravitino field Ψ_M^α which satisfies the Majorana condition. This is a 32-component spinor and completes the supergravity multiplet of the eleven-dimensional supergravity.

Let us first write the expression for the 11-dimensional action

$$S_{\rm M} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{|g|}\Big(R - \frac{1}{2}G^2\Big) - \frac{1}{12\kappa_{11}^2}\int G\wedge G\wedge A + S_{\rm F} \tag{1.3.1}$$

where we have written explicitly only the bosonic part since this is the relevant one for the rest of this Thesis. The associated equations of motion are

$$R_{MN} - \frac{1}{2}(G_{MPQR}G_N{}^{PQR} - \frac{1}{12}g_{MN}G^2) = 0 \tag{1.3.2}$$

and

$$d\star G + \frac{1}{2}G\wedge G = 0 \tag{1.3.3}$$

while the Bianchi identity for the four-form flux is simply

$$dG = 0 \tag{1.3.4}$$

The supersymmetry transformation rules for the bosons read

$$\delta g_{MN} = 2\bar\epsilon\,\Gamma_{(M}\Psi_{N)} \tag{1.3.5}$$

$$\delta A_{MNP} = -3\bar\epsilon\,\Gamma_{[MN}\Psi_{P]} \tag{1.3.6}$$

and for the gravitino (up to quadratic terms)

$$\delta\Psi_M = \nabla_M\epsilon + \frac{1}{288}\Big(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR}\Big)G_{NPQR}\,\epsilon \tag{1.3.7}$$

At the eleven-dimensional level, M-theory has also gauge symmetries. These are the usual diffeomorphisms present in any Einstein-type theory of gravity but also gauge transformations (the analogues of (1.2.5)) for the three-form potential:

$$A_{(3)} \to A_{(3)} + d\Lambda_{(2)} \tag{1.3.8}$$

Looking at (1.3.2) and (1.3.3), we see that they are invariant under the following global symmetry

$$g_{MN} \to e^{2\alpha}g_{MN}, \qquad A_{MNP} \to e^{3\alpha}A_{MNP} \tag{1.3.9}$$

Symmetries of this type have the common feature that the various fields transform with a factor of $e^{n\alpha}$ where n is the number of indices they carry[13] and they are usually called *trombone* symmetries.

The discussion of symmetries becomes much more interesting when one considers compactifications of M-theory on a d-torus. In this case, the theory "loses" part of its gauge symmetry due to the fact that it does not preserve the compactification ansatz or, in other words, it maps fields in the effective $(11-d)$-theory to degrees

[13]Note that this is a symmetry only at the level of the equations of motion and not at the level of the lagrangian.

of freedom out of it. As a result, some of the gauge (position-dependent in D=11) symmetries become global symmetries in the effective description (independent of the $11-d$ coordinates). We briefly describe below (following [22]) in what kind of groups this reorganization of global symmetries leads to.

In an M-theory compactification on a d-dimensional torus, one gets in the effective theory a metric, a number of one-forms and two-forms, and a bunch of scalar fields. Concentrating on the latter, we distinguish three types of them:

- d dilatonic scalars. These correspond to the diagonal elements of the metric g_{ii}, $i = 10-d,\cdots,11$.
- $\frac{1}{2}d(d-1)$ axionic scalars coming from the metric. These correspond to the elements of the metric g_{ij}, $i > j = 10-d,\cdots,11$.
- $n = \begin{pmatrix} d \\ 3 \end{pmatrix}$ axionic scalars coming from the three-form A. These correspond to the elements A_{ijk}, $i > j > k = 10-d,\cdots,11$.

If these are the only scalars in the effective theory (we will see below that one can consider additional scalars), then the global symmetry group can be expressed as the semi-direct product $GL(d,\mathbb{R}) \ltimes (\mathbb{R}^+)^n$. As mentioned earlier, (almost) all this symmetry comes from local symmetries at the eleven-dimensional level.

To be more concrete, the $SL(d,\mathbb{R})$ part of this symmetry comes from reparametrizations of the internal coordinates that preserve the torus structure. Then, the $\mathbb{R}^+$ factor in $GL(d,\mathbb{R}) = SL(d,\mathbb{R}) \times \mathbb{R}^+$ comes from a combination of the trombone symmetry (1.3.9) and scaling transformations which change the volume of the internal manifold. Regarding the $(\mathbb{R}^+)^n$ part of the global symmetry, this is due to the gauge transformation (1.3.8) when restricted to its action on the axions coming from A and with the requirement to preserve the compactification ansatz. The semi-direct product comes from the fact that the diffeomorphisms and the gauge transformations acting on A_{ijk} do not commute.

However, this is not the end of the discussion regarding the global symmetries of the compactified theory. It turns out that the effective theory (with the maximal number of scalars) coming from M-theory compactifications on a d-torus $d \leq 8$, has an $E_{d(d)}$ symmetry. In order to explain how this symmetry is related to the $GL(d,\mathbb{R}) \ltimes (\mathbb{R}^+)^n$, it is convenient to consider three different cases(for details see [22]):

- The case $d = 1,2$ is trivial since there is no $(\mathbb{R}^+)^n$ factor while the groups $GL(d,\mathbb{R})$ and $E_{d(d)}$ coincide.
- For $d = 3,4,5$, one gets $n = 1,4,10$ gauge axions respectively with their corresponding shift symmetries. The latter combine with the $GL(d,\mathbb{R})$ part to form the groups $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$, $SL(5,\mathbb{R})$ and $O(5,5)$ which are just different ways of writing $E_{3(3)}$, $E_{4(4)}$ and $E_{5(5)}$ respectively.
- The analysis for $d = 6,7,8$ is more complicated due to the fact that are additional scalars one can consider. These come from the dualization (in the $(11-d)$-external space) of the 1 three-from, 7 two-forms and 36 one-forms respectively.

Dualizing all these to scalar fields changes the global symmetries of the effective theory.[14]. A careful analysis of the algebra of transformations shows that the relevant global symmetry corresponds to the groups the exceptional groups $E_{6(6)}, E_{7(7)}$ and $E_{8(8)}$

We conclude that toroidal compactifications of M-theory are characterized by an $E_{d(d)}$ symmetry. Actually, it turns out that this symmetry has a much deeper meaning. It is related to the web of dualities that relate the five superstring theories mentioned earlier and it comes under the name *U-duality*. In type II language, it corresponds to transformations mixing the NS-NS and the R-R sector. In chapter 3, we will present a formalism in which supergravity compactifications can be described in a way where U-duality is manifest. This formalism is called Exceptional Generalized Geometry since it is based on the exceptional groups $E_{d(d)}$ and in chapter 4 we are going to apply it to study supersymmetric backgrounds both for type IIB and M-theory.

1.4 Supersymmetric vacuum solutions

In order to analyse the global symmetries of lower dimensional effective theories in the previous section, we were actually concerned with compactifications of M-theory on a torus. One can do the same thing for compactifications of type II supergravity; the analysis of the symmetries would be more intricate but the basic procedure to derive the spectrum and the action would be essentially the same and can be done in a rather straightforward way. Torus compactifications have several advantages from the computational point of view:

- Their topological and their differential-geometric properties are trivial. Since a torus is just a flat space endowed with periodicities, one can reorganize all the degrees of freedom in the action in a lower-dimensional language by just performing a Fourier expansion.

- The massless modes are very easily identified, they are just those that have no internal-coordinates-dependence. Therefore, one has just to split appropriately the indices in the various fields and then consider them as functions of only the external coordinates.

- Once the Kaluza-Klein reduction has been performed (keeping only the massless modes), the truncation is guaranteed to be consistent. This is because the mass of the modes is related to $U(1)^d$ isometry group of the d-torus and the massless modes correspond to singlets of it. Hence, they cannot source the higher Kaluza-Klein states and the reduction is consistent.

Despite the above computational advantages, torus-compactifications are characterized by a major drawback. They preserve 32 supercharges, so actually all the supersymmetry of type II string theory or M-theory survives the dimensional reduction. For compactifications down to four dimensions, this is $\mathcal{N} = 8$ which is just too much for any attempt to connect with particle physics phenomenology.

[14]Note that theory also loses some of its gauge symmetry due to the dualization.

In principle, one is interested in compactifications that partially break supersymmetry. The reasons that we still need some amount of supersymmetry preserved after the compactification are mainly two: **i)** The compactification scale is assumed to be close to the Planck scale and if supersymmetry is broken at that scale, corrections to the Higgs boson mass would lead to the well-known hierarchy problem. **ii)** From a more practical point of view, supersymmetric solutions are much easier to find and analyse due to the fact that the supersymmetry transformations ((1.2.10),(1.2.11) for type IIB and (1.3.7) for M-theory) contain first derivatives (in contrast to the equations of motion which are second order).

It seems like the whole problem now is just to pick the right manifold that preserves the desired amount of supersymmetry and then just do the dimensional reduction. Of course, the choice of the internal manifold should be consistent with the equations of motion or in our case with preserving the right amount of supersymmetry[15]. In practice, finding the class of allowed manifolds without any further assumptions is a quite complicated task, and therefore we first look at the "ground state" (vacuum) of the system which is a solution characterized by an especially large amount of symmetry.

In the vacuum, the higher-dimensional metric takes the block-diagonal form

$$ds^2 = e^{2A(y)}\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \tag{1.4.1}$$

where the external metric $\tilde{g}_{\mu\nu}(x)$ is required to admit the maximal amount of isometries, i.e. be Minkowski, de Sitter or anti-de Sitter, $g_{mn}(y)$ stands for the metric of the internal manifold the properties of which we would like to determined and $A(y)$ is the *warp factor*.

For concreteness, let us specialize to type IIB warped compactifications preserving 8 supercharges which will be the main interest of chapter 4. We will consider two different types of backgrounds: **i)** Minkowski vacua of the form $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times_w \mathcal{M}_6$ and **ii)** Anti-de Sitter vacua of the form $\mathcal{M}_{10} = AdS_5 \times_w \mathcal{M}_5$[16].

i)IIB 4-dim Minkowski vacua:

The vacuum expectation values (VEVs) of the fluxes should also respect the Poincaré symmetry of Minkowski space-time. This actually means that they can have either four legs or none on the external space and moreover that they should be independent of its coordinates. Combining this with the "democratic constraint" (1.2.8), we have [28] [17]

$$\tilde{F}_{(2n)} = F_{(2n)} + \mathrm{Vol}_4 \wedge \hat{F}_{(2n-4)}, \quad \hat{F}_{(2n-4)} = (-1)^{Int[n]} \star_6 F_{(10-2n)} \tag{1.4.2}$$

where now F have only internal components. The decomposition of the ten-dimensional supersymmetry parameters (1.2.9) is

$$\epsilon = \zeta_1^+ \otimes \Theta_1 + \zeta_2^+ \otimes \Theta_2 + \text{c.c.} \tag{1.4.3}$$

Here, the ζ_i^+ are four-dimensional Weyl spinors of positive chirality and serve as the supersymmetry parameters of the corresponding four-dimensional $\mathcal{N} = 2$ effective

[15] In fact, supersymmetry and the Bianchi identities imply the equations of motion [23],[24],[25].

[16] Supersymmetric warped compactifications do not have a good reputation for giving de Sitter vacua [26]. However, see also [27] for a recent and interesting possibility on this issue.

[17] Here, we slightly abuse notation. The F defined here refer to the internal components of $\tilde{F}$ while F in section 1.2 was the ten-dimensional exterior derivatives of the gauge fields.

supergravity. The Θ_i are doublets of Weyl spinors in six dimensions (with the same chirality for IIB):

$$\Theta_i = \begin{pmatrix} \eta_i^+ \\ \tilde{\eta}_i^+ \end{pmatrix} \tag{1.4.4}$$

and should be regarded as (position-dependent) expansion coefficients of the ten-dimensional fermions in terms of the four-dimensional ones.

The fact that the solution preserves the amount of supersymmetry parametrized by the ζ_i means that the supersymmetry variations of all the fields should vanish on this solution for every ζ_i. This requirement is trivial for the variations of the bosons since these are proportional to the VEVs of the fermions which vanish by four-dimensional Lorentz invariance. On the other hand, the supersymmetry variations of the fermions (Eqs. (1.2.10) and (1.2.11)) split in three equations corresponding to the external and internal components of the gravitino and to the dilatino variation. Using the decomposition of ten-dimensional gamma matrices as in (B.21) and after factoring out the ζ_i piece, we get[18]

$$(\partial\!\!\!/ A)\eta_i^+ - \frac{e^{\phi}}{4}(F\!\!\!/_1 + F\!\!\!/_3 + F\!\!\!/_5)\tilde{\eta}_i^+ = 0 \tag{1.4.5a}$$

$$(\partial\!\!\!/ A)\tilde{\eta}_i^+ + \frac{e^{\phi}}{4}(F\!\!\!/_1 - F\!\!\!/_3 + F\!\!\!/_5)\eta_i^+ = 0 \tag{1.4.5b}$$

$$\nabla_a\eta_i^+ - \frac{1}{4}H\!\!\!/_a\eta_i^+ + \frac{e^{\phi}}{8}(F\!\!\!/_1 + F\!\!\!/_3 + F\!\!\!/_5)\Gamma_a\tilde{\eta}_i^+ = 0 \tag{1.4.6a}$$

$$\nabla_a\tilde{\eta}_i^+ + \frac{1}{4}H\!\!\!/_a\tilde{\eta}_i^+ - \frac{e^{\phi}}{8}(F\!\!\!/_1 - F\!\!\!/_3 + F\!\!\!/_5)\Gamma_a\eta_i^+ = 0 \tag{1.4.6b}$$

$$(\partial\!\!\!/ \phi)\eta_i^+ - \frac{1}{2}H\!\!\!/\eta_i^+ - \frac{e^{\phi}}{2}(2F\!\!\!/_1 + F\!\!\!/_3)\tilde{\eta}_i^+ = 0 \tag{1.4.7a}$$

$$(\partial\!\!\!/ \phi)\tilde{\eta}_i^+ + \frac{1}{2}H\!\!\!/\tilde{\eta}_i^+ + \frac{e^{\phi}}{2}(2F\!\!\!/_1 - F\!\!\!/_3)\eta_i^+ = 0 \tag{1.4.7b}$$

These equations, known as the *Killing spinor equations*, provide the connection between the geometry and the flux configuration on the internal manifold for a given amount of supersymmetry preserved. In chapter 3, we will present a mathematical framework where these equations can acquire a purely geometrical meaning despite their complexity. For the moment, let us consider the simplest case where all the fluxes are set to zero. We then get for the internal spinor

$$\nabla_a\eta = 0 \tag{1.4.8}$$

As we explain in chapter 3, the fact that the internal manifold admits a covariantly constant spinor means that it is a *Calabi-Yau manifold*. Calabi-Yau manifolds are very well-studied objects in the mathematics and physics literature [29]. We will not provide many details for them since, in this Thesis, we will not make an analysis of the corresponding effective supergravity theories after the compactification. However,

[18] The indices a,b,c,$\cdots$ run from 1 to 6 while the indices m,n,$p\cdots$ run from 1 to 5.

let us mention that finding the set of massless (from the four-dimensional perspective) modes, or in other words the analogues of the Fourier zero modes of torus compactifications) can be done using the tools of complex and symplectic geometry (see section 3.1). Given that, one can derive the spectrum and the action of the effective supergravity theory for compactifications on Calabi-Yau 3-folds [30],[31],[32],[33].

Calabi-Yau manifolds admit a generalization when the condition (1.4.8) is weakened by the presence of fluxes. We will provide more details for this in chapter 3.

ii)IIB AdS_5 vacua:

We turn now to the study of AdS_5 vacua which are particularly important in string theory due to their role in the AdS/CFT Correspondence (see section 1.6). First, let us note that one cannot have an AdS vacuum with a fluxless configuration since the fluxes provide the necessary energy-momentum tensor to support a non-zero cosmological constant. However, there is a simple example with a connection to Calabi-Yau geometry which we will explain in section 3.1. But for the moment let us consider the general case where all the fluxes are allowed to have non-zero VEVs.

For backgrounds preserving eight supercharges, we parametrize[19] the ten-dimensional supersymmetry parameters ϵ_i as

$$\epsilon_i = \psi \otimes \chi_i \otimes u + \psi^c \otimes \chi_i^c \otimes u, \quad i = 1{,}2. \tag{1.4.9}$$

Here ψ stands for a complex spinor of $Spin(4{,}1)$ which represents the supersymmetry parameter in the corresponding five-dimensional supergravity theory, and satisfies the Killing spinor equation of AdS_5

$$\nabla_\mu \psi = \frac{m}{2}\rho_\mu \psi \tag{1.4.10}$$

where m is the curvature of the AdS[20]. (χ_1, χ_2) is a pair of (complex) sections of the $Spin$ bundle for the internal manifold. The two component complex object u fixes appropriately the reality and chirality properties of the ten-dimensional supersymmetry parameters ϵ_i (see (B.15)).

The fluxes now have to respect the $SO(4{,}2)$ symmetry of AdS_5 which forces them to have the form

$$\tilde{F}_{(2n)} = F_{(2n)} + \mathrm{Vol}_5 \wedge \hat{F}_{(2n-5)}, \quad \hat{F}_{(2n-5)} = (-1)^{Int[n]} \star_5 F_{(10-2n)} \tag{1.4.11}$$

where (1.2.8) was again used.

As previously, we insert the decompositions (1.4.9) and (1.4.11) in (1.2.10) and (1.2.11) and require the variations to vanish. After using (1.4.10), this gives rise to three equations corresponding to the external gravitino, internal gravitino and dilatino respectively:

$$\left[m - e^A (\partial\!\!\!/ A)\Gamma^6\Gamma_7 + i\frac{e^{\phi+A}}{4}\left((F\!\!\!/_1 + F\!\!\!/_5)\Gamma^6 - F\!\!\!/_3\right)\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{1.4.12}$$

[19]The decomposition of the ten-dimensional gamma matrices as well as that of the IIB spinors is described in detail in appendix B.

[20]Five-dimensional Minkowski solutions are described by taking appropriately the limit $m \to 0$.

$$\left[\nabla_m - \frac{1}{4}\not{H}_m\Gamma^6 + i\frac{e^\phi}{8}\left(\not{F}_1 + \not{F}_5 - \not{F}_3\Gamma^6\right)\Gamma_m\Gamma_7\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{1.4.13}$$

$$\left[(\not{\partial}\phi)\Gamma^6\Gamma_{(7)} + \frac{1}{2}\not{H}\Gamma_7 - \frac{ie^\phi}{2}\left(2\not{F}_1\Gamma^6 - \not{F}_3\right)\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{1.4.14}$$

where the Γ- matrices appearing in the above equations are built out of the five-dimensional ones as described in appendix B.

These Killing spinor equations were analysed in the framework of conventional differential geometry in [24]. In chapter 4, we will show that in the framework of Exceptional Generalized Geometry they can be written in a purely geometric form, but for the moment let us present the simplest case which has a nice description also in conventional geometry. As can be seen from (1.4.12), there are no non-trivial solutions with $m \neq 0$ for $F_i = 0$.

We consider the case where the only non-vanishing flux is F_5 and the two internal spinors are linearly dependent[21]

$$\chi_2 = i\chi_1 \equiv i\chi \tag{1.4.15}$$

which also has constant warp factor and dilaton. Taking $A = 0$ without loss of generality, we get from (1.4.13) and (1.4.12)

$$\nabla_m\chi = -\frac{im}{2}\gamma_m\chi \tag{1.4.16}$$

One can recognize in this expression the Killing spinor equation for a sphere. In fact, there is a larger class of manifolds which satisfy the above equations, but they are not spheres (but still Einstein). These manifolds are called Sasaki-Einstein and they are close relatives of Calabi-Yau manifolds. We will provide more details for Sasaki-Einstein manifolds and their relation to Calabi-Yau geometry in chapter 3. In chapter 4, we will see that the Killing spinor equations (1.4.12), (1.4.13) and (1.4.14) with generic fluxes can be written in Exceptional Generalized Geometry which allows for an interpretation of the underlying manifold as generalized Sasaki-Einstein.

Before closing this section, it is worth mentioning that the two kinds of backgrounds that we described in this section are actually related. To see this, we can observe that a four-dimensional Minkowski space can be embedded in AdS_5 as

$$ds^2_{AdS_5} = m^2r^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{dr^2}{m^2r^2} \tag{1.4.17}$$

Using this, we can write the ten-dimensional metric for warped AdS_5 compactifications as

$$ds^2 = e^{2A(y)}ds^2_{AdS_5} + ds^2_5(y) = (e^{2A(y)}m^2r^2)\eta_{\mu\nu}dx^\mu dx^\nu + \underbrace{\frac{e^{2A(y)}}{m^2}\frac{dr^2}{r^2} + ds^2_5(y)}_{ds^2_6} \tag{1.4.18}$$

[21]This is the only way the two spinors can be linearly dependent; for more details see appendix C of [24].

which is the form of the metric for warped Mink_4 compactifications with new warp factor $e^{2A}m^2r^2$. In chapter 3, we will see that this relation between Mink_4 and AdS_5 vacua is reflected in the internal geometry as a relation between Calabi-Yau and Sasaki-Einstein manifolds.

1.5 D-branes

In section 1.1, we introduced D-branes as hyper-surfaces on which the endpoints of (classical) open strings can end. Let us now move to the quantum theory and find the spectrum of these open strings considering for the moment that both endpoints lie on the same D-brane (satisfy identical boundary conditions). The effect of the boundary conditions on the mode expansion is that the center of mass mode is restricted to the D-brane hyper-surface while the momentum mode transverse to the brane vanishes. Regarding the oscillators, one finds that the left and the right moving modes are not independent but are related to each other. The ground state is again tachyonic while the massless modes (the ones which are obtained by acting on the ground state by lowest order oscillator modes) fall in two categories depending on whether they correspond to directions with Neumann or Dirichlet boundary conditions (see (1.1.3)). Specifically:

- Acting with oscillators carrying an an index in the first $p+1$ directions in (1.1.3) creates states corresponding to an abelian gauge field $\mathcal{A}_\mu(\xi), \mu = 0, \cdots, p$ where now ξ refers to the coordinates parametrizing the brane world-volume[22]. They have the interpretation of describing the longitudinal motion of the brane.
- Acting with oscillators with an index in the directions $i = p+1, \cdots, D-1$ of (1.1.3) creates states which are scalar with the respect to the Poincaré symmetry of the first $p+1$ directions. Therefore, one gets $D-p-1$ scalar fields $\Phi^i(\xi)$ which can be interpreted as fluctuations of the brane in the directions transverse to it.

From the interpretation of the fields living on a brane as coming from open string modes, it is natural to ask whether one can construct an action describing the dynamics of these fields in a similar way that the supergravity actions we presented in 1.2 were constructed from fields corresponding to closed string modes. However, the open string modes couple to the closed string modes through string interactions and therefore the effective description of a D-brane should include both. We will not get into the details of deriving the effective action that describes the dynamics of the massless modes but we will state the result and explain the features which are relevant for our purposes. This effective action consists of two parts

$$S_{\text{D}} = S_{\text{DBI}} + S_{\text{CS}} \tag{1.5.1}$$

$$S_{\text{DBI}} = -T_p \int d^{p+1}\xi \, e^{-\phi} \sqrt{-det([E_*]_{\mu\nu} + 2\pi\alpha' \mathcal{F}_{\mu\nu})} \tag{1.5.2}$$

[22]This will become more meaningful in a while. For the moment it is just $\xi^\mu = X^\mu$.

$$S_{\rm CS} = \mu_p \int \left[\sum C_{(n)} e^B\right]_* e^{2\pi\alpha' \mathcal{F}} \tag{1.5.3}$$

We should mention that the brane fields $\mathcal{A}_\mu$ and Φ^i appearing in (1.5.2) and (1.5.3) should be accompanied with their fermionic superpartners and there should exist a supersymmetric completion for the action. However, in this Thesis we will not be very explicit with this piece of the action[23].

Let us now explain the framework in which the above expressions appear and their meaning. First, we have specified at this point that we are we working in the framework of type II superstring theory and consequently the space-time is ten-dimensional. The coordinates $\xi^\mu, \mu = 0, \cdots ,p$ parametrize now the world-volume of the brane while the ten functions of $X(\xi)$ parametrize the (dynamical) position of the brane in space-time. The fields that appear in (1.5.2) and (1.5.3) fall in two categories:

- The world-volume fields which, as explained already, consist of a $U(1)$ gauge field $\mathcal{A}_\mu(\xi)$, $\mu = 0, \cdots ,p$ with field strength $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the $9-p$ scalar fields $\Phi^i(\xi)$, $i = p+1, \cdots 9$. The appearance of the latter in the action is implicit through the pullbacks $[\cdot]_*$ of the bulk fields (see below) while their identification with transverse motions of the brane is through $\delta x^i = 2\pi\alpha'\Phi^i$. These fields describe the dynamical degrees of freedom on the brane.

- The bulk fields $\{G_{MN}, B_{MN}, \phi, C_{(n)}\}$ which correspond to the closed string modes and which enter the world-volume action as external fields through the their pullback $[\cdot]_*$ on it. For example, in a gauge where $\xi^\mu = X^\mu$ we have

$$[B_*]_{\mu\nu} = B_{\mu\nu} + 2(2\pi\alpha')B_{i[\mu}\partial_{\nu]}\Phi^i + (2\pi\alpha')^2 B_{ij}\partial_\mu\Phi^i\partial_\nu\Phi^j \tag{1.5.4}$$

  and similarly for the other fields. The Dirac-Born-Infeld action (1.5.2) contains contributions only from the NS-NS sector where the following combination appears

$$E_{MN} = g_{MN} + B_{MN} \tag{1.5.5}$$

  On the other hand, the Chern-Simons action (1.5.3) contains the coupling of the D-brane to the R-R gauge potentials.

Depending on whether we consider type IIA or type IIB superstring theory, we get D-branes of different dimensionality. In particular, for type IIA we have D0, D2, D4, D6 and D8-branes while for type IIB D1, D3, D5, D7 and D9-branes. This is in agreement with (1.5.3) where the R-R gauge potentials are odd-dimensional for type IIA and even-dimensional for type IIB. This is in agreement with (1.5.3) where only the relevant R-R potentials for each theory appear.

The D-brane action also contains the tension T_p of the brane and the charge μ_p. The former describes describes the gravitational interaction of the brane with the background and the latter the electric coupling of it to the R-R gauge potentials. These are given by

$$T_p = \mu_p = \frac{2\pi}{g_s(2\pi l_s)^{p+1}} \tag{1.5.6}$$

[23] In section 1.6, we will give more details about the case of D3-branes.

D-branes can be viewed as higher-dimensional generalizations of charged particles in standard electromagnetism. This can be seen from their action where the DBI part (1.5.2) is similar to the gravitational worldline action $-m\int d\tau$ of a point particle while the CS piece (1.5.3) is the analogue of the electric coupling of a charged particle in an external electromagnetic field $q\int dx^\mu A_\mu$.

This last fact fits very well with the picture of the R-R gauge potentials as generalizations of the electromagnetic potential with their corresponding gauge transformations (1.2.5). In electromagnetism, one can "detect" the presence of a charged particle by studying the flux of the electric (and/or magnetic) field around it. The same happens here and one can see D-branes as (solitonic) solutions of the equations of motion derived from the type II supergravity action by going to the Einstein frame as explained in 1.2. For concreteness, let us present one such solution for a Dp-brane.

The ten-dimensional metric is sourced from the energy-momentum tensor of the D-brane. It takes the following "black brane" form

$$ds^2 = (Z(r))^{\alpha(p)}\eta_{\mu\nu}dx^\mu dx^\nu + (Z(r))^{\beta(p)}\delta_{ij}dy^i dy^j \tag{1.5.7}$$

where we have split the coordinates in brane directions x^μ and transverse directions y^i and $r = \sqrt{\delta_{ij}y^i y^j}$ parametrizes the distance from the brane. The exponents $\alpha(p)$ and $\beta(p)$ are just numbers depending on the dimensionality of the brane. The brane also sources the $(p+2)$-form field strength which has components

$$F_{\mu_1\cdots\mu_{p+1}i} = \epsilon_{\mu_1\cdots\mu_{p+1}i}\partial^i Z^{-1}(r) \tag{1.5.8}$$

and generally there is also a non-trivial dilaton profile. The entire solution is characterized by a single harmonic function which has the form

$$Z(r) = 1 + Qr^{p-7} \tag{1.5.9}$$

where Q is the charge of the D-brane. This can be seen by integrating the field strength on a sphere transverse to the brane at infinity:

$$\int_{S^{8-p}} \star F_{(p+2)} = Q \tag{1.5.10}$$

The charge Q can be identified with the constant μ_p in the Chern-Simons action (1.5.3). Moreover, the electric flux (1.5.10) for $F_{(p+2)}$ can alternatively be seen as a magnetic flux for $F_{(8-p)}$ sourced by a $(6-p)$-dimensional "magnetic monopole". The presence of both electric and magnetic sources leads to the quantization of the charge Q in a way similar to the Dirac quantization condition in ordinary electromagnetism.

The supergravity solution described above preserves exactly half of the 32 supersymmetries of the theory. States with this property are called *BPS states*. A result of this fact is that two D-branes which preserve the same 16 supercharges act no force on each other. This can be seen in a geometric way by considering a probe D-brane (a brane whose backreaction on the background can be neglected) in a background created by other D-branes. The potential of the probe brane is computed by inserting the values of the background fields in the probe D-brane action. It is a functional of the world-volume scalar fields which parametrize the transverse distance of the probe brane from the other branes. One finds then that this potential is constant owing to a cancellation of the DBI and CS pieces due to (1.5.6). We can therefore conclude that two D-branes can be in equilibrium since the attractive gravitational force coming from the DBI action and the repulsive electric force coming from the CS action cancel.

1.6 The AdS/CFT Correspondence

In the previous section, we saw that the world-volume of a Dp-brane contains an abelian gauge field $\mathcal{A}_\mu$ and $9-p$ scalar fields Φ^i. In fact, the DBI action (1.5.2) provides also the dynamics for these fields. One can see this by expanding the square root and keep only the lowest order terms in α'. Then, one obtains a Maxwell term $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ from the expansion of the square root and a kinetic term for the scalars coming from the pullback of the ten-dimensional metric on the D-brane world-volume. There is also a constant term which expresses the rest mass of the D-brane which is irrelevant for the purpose of this section and therefore we drop it.

One can obtain more interesting gauge theories with a richer field content by considering more copies of D-branes. We will explore this set-up in more detail in section 2.1, but for the moment let us present some basic facts which are necessary to introduce the AdS/CFT Correspondence. We focus therefore on the case of N parallel D3-branes sitting at the same point of the ten-dimensional space (coincident branes).

The $U(1)$ gauge symmetry in the case of a single brane gets now enhanced to a $U(N)$ gauge symmetry. This can be explained by the fact that an open string can have endpoints lying on different branes. These are charged under different $U(1)$ components of the gauge field corresponding to the different branes and moreover there should be transformations mixing them since they belong to the same string. Therefore, there is now a $U(N)$ gauge field $(A_\mu)^a{}_b$ with its corresponding non-abelian field strength. The scalar fields parametrizing the positions of the D3-branes now become also $N \times N$ matrices $(\Phi^i)^a{}_b$ transforming in the adjoint of $U(N)$. Although we have not been explicit with fermions, we should keep in mind that the D-brane action contains also a supersymmetric completion. It turns out that the fermionic content of the gauge theory living on the D3-branes is four Weyl fermions $(\lambda_\alpha)^a{}_b$ transforming in the adjoint of $U(N)$.

The field content we just described is the field content of $\mathcal{N}=4$ super Yang-Mills theory. A complete analysis shows that the interactions derived from the D-brane action are the right ones for $\mathcal{N}=4$ SYM. More precisely, comparing the dilaton factor in the DBI action with the standard normalization of the Yang-Mills action, we get a relation for the couplings on the two sides of the correspondence. Moreover, in the CS action there is a term of the form $\sim C_{(0)}\mathcal{F}\wedge\mathcal{F}$ providing the theta term for the gauge theory. The precise relations are

$$g_{YM}^2 = 4\pi g_s, \qquad \theta = 2\pi C_{(0)} \tag{1.6.1}$$

The $SU(4)$ R-symmetry of the gauge theory is identified with the $SO(6)$ global symmetry of the space transverse to the brane. Therefore, we see that the gauge theory living on the stack of N coincident D3-branes is $\mathcal{N}=4$ SYM.

Let us now consider the background solution in the presence of these D3-branes. The metric is given by (1.5.7) where the exponents for D3-branes are

$$\alpha = -\frac{1}{2}, \qquad \beta = +\frac{1}{2} \tag{1.6.2}$$

and the charge is given by $Q = 4\pi g_s N \alpha'^2$. Let us also take the so-called *near-horizon limit* $r \to 0$. However, since the transverse distance is identified with the scalar fields of the brane theory ($\delta x^i = 2\pi\alpha'\Phi$) and we want to keep the latter finite, we need also to send $\alpha' \to 0$. Therefore, the correct limit is

$$r \to 0, \qquad \alpha' \to 0, \qquad \frac{r}{\alpha'} \text{ fixed} \tag{1.6.3}$$

In this limit, the harmonic function $Z(r)$ becomes

$$Z(r) \to \frac{R^4}{r^4} \tag{1.6.4}$$

where

$$R^4 \equiv 4\pi g_s N \alpha'^2 \tag{1.6.5}$$

The metric finally becomes

$$ds^2 = (\frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2) + R^2 d\Omega_5^2 \tag{1.6.6}$$

which can be recognized as the metric of the product $AdS_5 \times S^5$ where both factors have the same radius R.

From the D3-brane action (1.5.2) and (1.5.3), one can also see that the coupling between the bulk fields and the brane fields is parametrized by α'. Therefore, in the limit (1.6.3), the interactions between the bulk theory and the gauge theory are turned off. However, the bulk theory was considered to be "sourced" by the branes which contain the gauge theory. The AdS/CFT Correspondence is actually the statement/conjecture that the bulk and the gauge theory are equivalent. Or more precisely, *type IIB string string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$*. It was first proposed by Maldacena in [34].

Let us discuss the space-time symmetries on the two sides of the correspondence. $\mathcal{N} = 4$ SYM is a four-dimensional conformally invariant theory and hence it enjoys conformal invariance in four dimensions characterized by the group $SO(4,2)$. This is precisely the isometry group of AdS_5. Actually, the $\mathcal{N} = 4$ SYM theory is characterized by *superconformal* symmetry which means that the 16 supersymmetries related to the four supertranslations have to be supplemented by other 16 supersymmetries related to the so-called superconformal generators. In total, it has 32 supercharges which is exactly the amount of supersymmetry of type IIB string theory. Finally, the R-symmetry of the gauge theory is $SU(4)$ and this is isomorphic to $SO(6)$, i.e. the isometry group of S^5, the internal manifold on the gravity side.

The gauge theory contains as independent parameters the gauge coupling g_{YM}^2, the theta angle θ and the number of colors N. These are given in terms bulk quantities through equations (1.6.1) and (1.6.5). A supergravity description for the bulk is valid when the string coupling constant g_s is small so that loop diagrams can be neglected and also when the radius of curvature is much greater than the string scale ($R \gg \sqrt{\alpha'}$). On the gauge theory side this means that the so-called *'t Hooft coupling* $\lambda \equiv g_{YM}^2 N$ is large and that $N \to \infty$. At large N, the 't Hooft coupling is the effective coupling of the gauge theory and therefore the supergravity regime (weak coupling) is dual to the strong coupling regime of the gauge theory.

Although we motivated the AdS/CFT Correspondence through the D3-brane set-up, we can observe that the basic statement of the correspondence makes no reference to D-branes. But then, since we cannot talk any more for the D-brane world-volume, where does the gauge theory live in? In order to answer this question, we have to look at the near-horizon limit (1.6.3) which actually probes the bulk field configuration in the region close to the brane. But since the brane fields are kept fixed in this procedure, this means that they are localized in the boundary of AdS_5 after the limit has been taken.

In fact, this bulk-boundary correspondence can be made more precise. The basic statement of AdS/CFT is that CFT operators are dual to supergravity solutions of the bulk fields. These are dual in the sense that the source in the CFT generating functional can be interpreted as the boundary value of the bulk field. The conformal dimension Δ of the CFT operator is related to the mass of the supergravity field. For example, for scalar fields this relation is

$$m^2 = \frac{\Delta(\Delta - 4)}{R^2} \tag{1.6.7}$$

The radial coordinate r of AdS has a very natural interpretation from the gauge theory perspective. This can be found by observing that the AdS metric (1.4.17) is invariant under

$$x_\mu \to \lambda x_\mu, \qquad r \to \frac{1}{\lambda} r \tag{1.6.8}$$

We therefore see that the radial coordinate can be identified with the typical energy scale on the dual field theory. The boundary $r \to \infty$ corresponds to the UV regime of the field theory while flowing to the IR corresponds to moving towards the interior of AdS.

Using AdS/CFT, we can also describe deformations of the original $\mathcal{N} = 4$ SYM theory. A deformation by a local operator $\mathcal{O}$ of conformal dimension Δ corresponds to turning on nonnormalizable modes for the dual field $\phi_{\mathcal{O}}$ on the gravity side. These have the asymptotic behaviour at large r

$$\phi_{\mathcal{O}}^{\text{non-norm}} \sim r^{\Delta - 4} \tag{1.6.9}$$

On the other hand, the vacuum expectation value of $\mathcal{O}$ corresponds to turning on normalizable modes modes for $\phi_{\mathcal{O}}$. Their asymptotic behaviour is

$$\phi_{\mathcal{O}}^{\text{norm}} \sim r^{-\Delta} \tag{1.6.10}$$

In the next chapter, we will be concerned with deformations of $\mathcal{N} = 4$ SYM which break supersymmetry first to $\mathcal{N} = 1$ and finally to $\mathcal{N} = 0$. We will see that the description of the theory in terms of the supergravity dual will provide very useful conclusions for the gauge theory.

Chapter 2

Mass deformations of $\mathcal{N} = 4$ SYM and their supergravity duals

The main focus of this chapter is to study deformations of the $\mathcal{N} = 4$ SYM by adding mass terms for the four fermions and the six scalars of the theory. The $\mathcal{N} = 4$ SYM theory can be realized on the world-volume of stacks of D3-branes and therefore, as explained in the last section of the previous chapter, AdS/CFT allows to study these theories by analysing the dual supergravity background. More specifically, adding mass terms for the fermions of the gauge theory corresponds to deforming the $AdS_5 \times S^5$ solution dual to the original $\mathcal{N} = 4$ theory with nonnormalizable modes for the type IIB three-form fluxes [6]. The supergravity equations of motion then relate the mass deformations of the fermions with those of the bosons. These relations should also have a "memory" of the $SU(4)$ R-symmetry of the $\mathcal{N} = 4$ theory, which we analyse using group theory arguments. Combining this information with results for special cases existing in the literature, we find a quite general constraint: the trace of the fermions mass matrix squared must be equal to the trace of the boson mass matrix.

This chapter is organized in the following way. In section 2.1, we explain that assembling parallel branes at the same point of a supergravity background can result in various new effects. One with particular importance for our purposes is a "polarization" mechanism for D-branes, in which lower dimensional branes can "expand" to higher dimensional ones in the presence of supergravity fluxes. This effect was employed by Polchinski and Strassler [6] to study the so-called $\mathcal{N} = 1^*$ theory which we briefly describe in section 2.2. In section 2.3, we explain the strategy for proving our main result for generic (nonsupersymmetric) bosonic and fermionic mass deformations. In sections 2.4 and 2.5 we use group theory to find the bosonic potential arising from the square of the fermionic masses using the isomorphism between $SO(6)$ and $SU(4)$. Although the group theory is well-known and most of Section 2.4 is a review, our final formulas in section 2.5 are new, as only their supersymmetric versions have so far appeared in the literature. In section 2.6 we explain how the bosonic masses appear in supergravity. This section contains the main observations of this chapter. In section 2.7 we recapitulate the main conclusions of our analysis and their relation to perturbative gauge theories. In subsection 2.7.2 we explain with explicit calculations why our results hold even when quantum corrections in the gauge theory are taken into account. Appendix A includes a summary of useful formulas for intertwining between $SO(6)$ and $SU(4)$ representations.

2.1 Myers effect

In the previous chapter, the physical picture we gave for D-branes is that of extended objects which, in a given supergravity background, couple to the fields of the NS-NS sector through their tension T and to those of the R-R sector through their electric charge μ. The precise form of these couplings is given by the sum (1.5.2) and (1.5.3) for a single Dp-brane. From the world-volume theory point of view, there is a $U(1)$ gauge field and $9-p$ scalars living on the brane corresponding to the transverse directions in space-time.

Let us take a closer look at the Chern-Simons piece (1.5.3). We see that in the absence of B-field and world-volume gauge field $\mathcal{A}$, the only contribution from the R-R sector comes from $C_{(p+1)}$ giving the natural (electric) coupling of a $(p+1)$-dimensional object to the corresponding $(p+2)$-dimensional field strength. However, when $B \neq 0$ or $\mathcal{F} \neq 0$, there are additional contributions from the R-R sector (for example of the form $B \wedge C_{(p-1)}$) indicating that the Dp-brane is charged under fields corresponding to lower-dimensional branes. This situation can be interpreted in terms of bound states of these lower dimensional branes [35] and the scalar fields can then be considered to be "composite", built out of the fields in the lower dimensional branes.

It is then natural to ask what is the description of this system in the dual picture; namely how the higher dimensional brane is described when one starts with a stack of branes of lower dimension. The answer was found by Myers in [36] and it resembles the familiar situation from electrostatics where a dielectric material gets polarized when external electric field is applied to it. In the present framework this means that collections of Dp-branes couple to R-R gauge fields $C_{(p'+1)}$ with $p' > p$, i.e. they acquire p'-brane dipole charge.

In this section, we will describe this phenomenon (following [36]) in a simple set-up where a number of D0-branes "polarize" into a (noncommutative) spherical D2-shell in a background of nontrivial $F_{(4)}$. In the next section, we will apply this idea to the situation relevant for AdS/CFT, i.e. the case where N parallel D3-branes are placed at the same point in transverse space. The Myers mechanism applies also there and the D3-branes acquire 5-brane dipole charge due to their magnetic coupling to the background R-R 3-form flux.

In order to proceed, one needs the generalization of (1.5.2) and (1.5.3) for the nonabelian case which is the relevant one here since this is the world-volume theory on N Dp-branes. This was constructed in [36] by demanding consistency with T-duality and the two pieces are given by

$$\tilde{S}_{\mathrm{DBI}} = -T_p \int d^{p+1}\xi \mathrm{Tr}\Big(e^{-\phi}\sqrt{-det([E_{\mu\nu} + E_{\mu i}(Q^{-1}-\delta)^{ij}E_{j\nu}]_* + 2\pi\alpha'\mathcal{F}_{\mu\nu})det(Q^i{}_j)}\Big) \tag{2.1.1}$$

and

$$\tilde{S}_{\mathrm{CS}} = \mu_p \int \mathrm{Tr}\Big([e^{i2\pi\alpha'\iota_\Phi\iota_\Phi}(\sum C_{(n)}e^B)]_* e^{2\pi\alpha'\mathcal{F}}\Big) \tag{2.1.2}$$

Let us pause for a moment to explain the above expressions which are important on their own right. The first obvious change compared to (1.5.2) and (1.5.3) is that the world volume gauge field $\mathcal{A}_\mu$ and scalars Φ^i now become hermitian $N \times N$ matrices in the adjoint of $U(N)$ corresponding to the combinations of branes on which an open string can end (Chan-Paton labels) [37]. The same holds for the background fields as a result of their functional dependence of Φ^i while the derivatives of the scalar fields in the pullbacks $[\cdot]_*$ are replaced with $U(N)$-covariant ones. Taking the trace is understood with a symmetrization over the gauge indices and ι_Φ denotes contraction in the directions transverse to the brane, i.e. $(\iota_\Phi C)_{i_1\cdots i_n} = \Phi^j C_{ji_1\cdots i_n}$. Moreover, we have

$$Q^i{}_j = \delta^i{}_j + i2\pi\alpha'[\Phi^i, \Phi_j] \tag{2.1.3}$$

and the i,j,k indices are raised and lowered with E_{ij} defined in (1.5.5) and its inverse E^{ij}.

In order to see how the above action attributes dielectric properties to the collection of D0-branes, it is necessary to compute their potential in a background with R-R fluxes. A convenient choice is a flat background where only $C_{(3)}$ is turned on such that the only nonzero components of $F_{(4)}$ are

$$F_{tijk} = -2f\epsilon_{ijk} \tag{2.1.4}$$

where f is a constant, $i,j,k \in \{1,2,3\}$[1] and we are working in a static gauge $t = \xi^0$. In such a situation, the contributions to the potential of the world-volume scalars come from the determinant $\det(Q^i{}_j)$ in (2.1.1) and the first term of the Taylor expansion of $C_{(3)}$ in (2.1.2) (the higher order terms vanish in our constant background but in general give the brane-analogue of a multipole expansion). The former is the usual quartic scalar interaction of a supersymmetric nonabelian gauge theory while the latter is a $U(N)$-invariant cubic interaction for the scalars. The explicit form is

$$V[\Phi] = 4\pi^2\alpha'^2\Big(-\frac{T_0}{4}\mathrm{Tr}([\Phi^i,\Phi^j]^2) - \frac{i}{3}\mu_0 \mathrm{Tr}(\Phi^i\Phi^j\Phi^k)F_{tijk}\Big) \tag{2.1.5}$$

with minima given by

$$[[\Phi^i,\Phi^j],\Phi^k] + if\epsilon_{ijk}[\Phi^j,\Phi^k] = 0 \tag{2.1.6}$$

An obvious solution to the above equation is given by diagonal (mutually-commuting) matrices

$$\Phi^i_{\mathrm{c}} = \frac{1}{2\pi\alpha'}\mathrm{diag}(x^i_1, \cdots, x^i_N) \tag{2.1.7}$$

which has the interpretation of N D0-brane sitting at the positions $x^i_1, \cdots, x^i_N$. The potential for this solution is obviously

$$V[\Phi^i_{\mathrm{c}}] = 0 \tag{2.1.8}$$

However, Eq. (2.1.6) has other solutions as well. In particular, it is satisfied when the matrices Φ^i form an N-dimensional representation of $SU(2)$ which we write as

$$\Phi^i_{\mathrm{nc}} = \frac{f}{2}\alpha^i_N, \quad \text{where} \quad [\alpha^i_N, \alpha^j_N] = 2i\epsilon^{ijk}\alpha^k_N \tag{2.1.9}$$

[1] Note that the rest of the coordinates $x^4, \cdots, x^9$ do not play any role in the discussion that follows.

Computing the potential for the case that this is the N-dimensional irreducible representation, we get

$$V[\Phi^i_{\rm nc}] = 4\pi^2\alpha'^2 f^4(N^3-N)\Big(\frac{T_0}{8}-\frac{\mu_0}{6}\Big) = -\frac{\pi^2 l_s^3 f^4}{6g}(N^3-N) \tag{2.1.10}$$

from which we see that the noncommutative solution is favoured energetically against the commutative one. It turns out that the above solution has the lowest potential and therefore corresponds to the ground state of the system. In matrix theory, it has the interpretation of the fuzzy sphere [38] and it corresponds to a D2-spherical shell with D0-branes bound to it.

In order to close this section, let us also try to give the physical picture of the comparison of the two solutions we just described, the commutative with zero potential and the noncommutative one with potential given by (2.1.10). As can be seen from this expression, the external electric field of $C_{(3)}$ tends to "expand" the system of N D0-branes by "increasing the noncommutativity" of the relevant coordinates (the electric potential is lower for $\Phi_{\rm nc}$). However, this increase of noncommutativity is obstructed by the gravitational attraction of the system (the potential due to the brane tension is higher for $\Phi_{\rm nc}$) which finally stabilizes the system. The net result is a configuration of point-like branes but with a sphere-like dipole moment. The analogy can be made more clear by comparing the second term of (2.1.5) with the standard expression for the potential energy of an electric dipole $U=-\vec{E}\cdot(q\,\vec{r}_{\rm rel})$ from ordinary electrostatics. In the latter case, the electric nature of the otherwise neutral dipole arises from the separation $\vec{r}_{\rm rel}$ of the two oppositely charged constituent particles. In the case of branes, these do not have internal structure and the corresponding dipole moment $\mu_0{\rm Tr}(\Phi^{[i}\Phi^j\Phi^{k]})$ arises from the noncommutativity of the coordinates of the constituent lower dimensional branes.

2.2 The $\mathcal{N}=1^\star$ theory

We now move to the main interest of this chapter which is to study (non)supersymmetric mass deformations of the $\mathcal{N}=4$ SYM in the framework of AdS/CFT. In section 1.6 we briefly presented the field content of $\mathcal{N}=4$ SYM in a way which makes the full supersymmetry of the theory manifest. However, in order to study deformations of the theory which break supersymmetry, it is more convenient to describe it in a $\mathcal{N}=1$ language. In this case, the six real adjoint scalars combine in three complex scalars Φ_I, $I=1,2,3$ as the lowest components of three $\mathcal{N}=1$ chiral multiplets. From the four $\mathcal{N}=4$ fermions one becomes the gaugino while the other three comprise the fermionic content of the chiral multiplets. In $\mathcal{N}=1$ language the superpotential of the theory reads

$$W=\frac{2\sqrt{2}}{3g^2_{YM}}\epsilon^{IJK}{\rm Tr}(\Phi_I\Phi_J\Phi_K) \tag{2.2.1}$$

The $\mathcal{N}=4$ SYM theory deformed with three chiral multiplet masses, known as the $\mathcal{N}=1^\star$ theory, is one of the most studied examples of supersymmetric confining gauge theory, as it shares some of the most interesting features of QCD: confinement, baryons and flux tubes. It is obtained from $\mathcal{N} = 4$ by adding to the superpotential arbitrary mass terms

$$\delta W = \frac{1}{g_{YM}^2} \mathrm{Tr}(m_1 \Phi_1^2 + m_2 \Phi_2^2 + m_3 \Phi_3^2) \tag{2.2.2}$$

Furthermore, since this theory has a conformal UV fixed point, it can be put on the lattice much easier than other four-dimensional gauge theories that one studies using the AdS/CFT correspondence, and hence can serve as an important benchmark for lattice gauge theory calculations.

The AdS/CFT dual of this theory has been spelled out by Polchinski and Strassler [6], who deformed $AdS_5 \times S^5$ with non-normalizable modes in the R-R and NS-NS three-form fluxes, corresponding to masses for the fermions in the three chiral multiplets. They argued that in the resulting geometry the D3-branes that source $AdS_5 \times S^5$ polarize via the Myers effect [36] into spherical shells with five-brane dipole charge, that are the holographic duals of the confining, screening and oblique vacua of the $\mathcal{N} = 1^\star$ theory [39].

In this section, we are going to review briefly the supergravity dual of the $\mathcal{N} = 1^\star$ theory following [6] with our main focus on the polarization potential of the D3-branes. This supergravity solution can be considered as a perturbation over the $AdS_5 \times S^5$, dual the original $\mathcal{N} = 4$ SYM theory. The solution for the undeformed background is given by:

$$ds^2 = Z^{-1/2} \eta_{\mu\nu} dy^\mu dy^\nu + Z^{1/2} \delta_{AB} dx^A dx^B \tag{2.2.3}$$

$$\tilde{F}_5 = d\chi_4 + \star d\chi_4, \qquad \chi_4 = \frac{1}{g_s Z} \mathrm{vol}_4 \tag{2.2.4}$$

The dilaton and the axion take the constant values

$$e^\phi = g_s, \qquad C_0 = \frac{\theta}{2\pi} \tag{2.2.5}$$

while all the other fields are vanishing.

In order to study perturbations linear in H_3 and F_3, it is more convenient to use the complex combination G_3 of the IIB three-form fluxes and the axio-dilaton τ defined in (1.2.16) and (1.2.15) respectively.

The solution for the non-normalizable modes of G_3 corresponding to the fermion masses was computed in [6] where the result was given in terms of two tensors T_3 and V_3 as

$$G_3 = \frac{c}{r^4} \left(T_3 - \frac{4}{3} V_3 \right) , \tag{2.2.6}$$

where $c = \zeta R^4 / g_s$. For the $\mathcal{N} = 1^*$ theory, the nonzero components of the tensor T_3 are given (in complex coordinates) by

$$T_{1\bar{2}\bar{3}} = m_1, \qquad T_{\bar{1}2\bar{3}} = m_2, \qquad T_{\bar{1}\bar{2}3} = m_3 \tag{2.2.7}$$

and is position independent while the tensor V_3 is given in terms of T_3 as

$$V_{ABC} = \frac{3}{r^2} x^D x_{[A} T_{BC]D} \ . \tag{2.2.8}$$

This solution and the existence of supersymmetry was enough to allow the authors of [6] to determine the full polarization potential of the D3-branes and to read off certain aspects of their physics. More precisely, in the limit when the number of five-branes is small, the polarization potential of the D3-branes is given by equation (62) of [6] and has three terms. The first term, proportional to the fourth power of the polarization radius, is a universal term that gives the difference between the mass of unpolarized D3-branes and the mass of a five-branes with all these D3-branes inside. The second term, proportional to the third power of the radius, represents roughly the polarization force that the RR and NSNS three-form perturbations exert on the five-brane shell. The third term, proportional to the square of the radius, is the potential felt by a probe D3-brane along what used to be the Coulomb-branch of the undeformed theory. This term comes from the backreaction of the three-forms dual to fermion masses on the metric, dilaton and five-form. In [6] the value of this term was guessed by using supersymmetry to complete the squares in the polarization potential. When the masses of the fermions in all the three chiral multiplets are equal, the value of this term was computed directly in supergravity by Freedman and Minahan [40] and found to be exactly the one guessed in [6].

2.3 Moving towards the $\mathcal{N}=0^\star$ theory

Our main goal for the rest of this chapter is to study the non-supersymmetric version of the Polchinski-Strassler story, and in particular to spell out a method to determine completely the D3-brane Coulomb branch potential (or the quadratic term in the polarization potential) for the $\mathcal{N}=4$ SYM theory deformed with a generic supersymmetry-breaking combination of fermion and boson masses. Many of the issues in the problem we are addressing have been touched upon in previous explorations, but when one tries to bring these pieces of the puzzle together one seems to run into contradictions. We will try to explain how these contradictions are resolved, and give a clear picture of what happens in the supergravity dual of the mass-deformed $\mathcal{N}=4$ theory.

As explained in [6], a fermion mass deformation of the $\mathcal{N}=4$ SYM field theory, $\lambda^i M_{ij}\lambda^j$, corresponds in the bulk to a combination of R-R and NS-NS three-form field strengths with legs orthogonal to the directions of the field theory, that transforms in the **10** of the $SU(4)$ R-symmetry group. The complex conjugate of the fermion mass, $M^\dagger$, corresponds to the complex conjugate combination transforming in the $\overline{\mathbf{10}}$. Since the dimension of these fields is 3, the normalizable and non-normalizable modes dual to them behave asymptotically as r^{-3} and r^{-1}.

The boson mass deformation in the field theory, $\phi^a\mathcal{M}_{ab}\phi^b$, can be decomposed into a term proportional to the trace of $\mathcal{M}$, which is a singlet under the $SU(4)\simeq SO(6)$ R-symmetry, and a symmetric traceless mass operator, which has dimension 2 and transforms in the $\mathbf{20'}$ of $SO(6)$. The traceless mass operator in the $\mathbf{20'}$ corresponds in the $AdS_5\times S^5$ bulk dual to a deformation of the metric, dilaton and the RR four-form potential that is an $L=2$ mode on the five-sphere, and whose normalizable and non-normalizable asymptotic behaviors are r^{-2} and $r^{-2}\log r$ [41]. On the other hand, the dimension of the trace operator is not protected, and hence, according to the standard lore, turning on this operator in the boundary theory does not correspond to deforming $AdS_5 \times S^5$ with a supergravity field[2], but rather with a stringy operator [42]. The anomalous dimension of this operator at strong coupling has consequently been argued to be of order $(g_s N)^{1/4}$.

On the other hand, there exist quite a few supergravity flows dual to field theories in which the sum of the squares of the masses of the bosons are not zero [43, 44, 45, 46, 47, 48, 49, 50, 51, 52], and none of these solutions has any stringy mode turned on, which seems to contradict the standard lore above. In the next sections of this chapter we would like to argue that the solution to this puzzle comes from the fact that the backreaction of the bulk fields dual to the fermions determines completely the singlet piece in the quadratic term of the Coulomb branch potential of a probe D3-brane. Therefore, the trace of the boson mass matrix that one reads off from the bulk will always be equal to the trace of the square of the fermion mass matrix.

This, in turn, indicates that in the presence of fermion masses, the stringy operator is not dual to the sum of the squares of the boson masses, but to the difference between it and the sum of the squares of the fermion masses. Mass deformations of the $\mathcal{N} = 4$ theory where the supertrace of the square of the masses is zero can therefore be described holographically by asymptotically-AdS supergravity solutions [43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. However, to describe theories where this supertrace is nonzero, one has to turn on "stringy" non-normalizable modes that correspond to dimension-$(g_s N)^{1/4}$ operators, which will destroy the AdS asymptotics.

To see this we begin by considering the backreaction of the three-form field strengths corresponding to fermion mass deformations on the metric, the dilaton and the four-form potential, which has been done explicitly for several particular choices of masses [40, 53]. This backreaction can give several terms that modify the action of a probe D3-brane, giving rise to a Coulomb-branch potential that is quadratic in the fermion masses and that transforms either in the **1** or in the **20′** of $SO(6)$. Furthermore, one can independently turn on non-normalizable modes in the **20′** of $SO(6)$ that correspond to deforming the Lagrangian with traceless boson bilinears, and that can also give rise to a Coulomb-branch potential. Since all these terms behave asymptotically as r^{-2} and transform in the same $SO(6)$ representation, disentangling the contributions of the non-normalizable modes from the terms coming from the backreaction of the three-forms can be quite nontrivial. For example, in equation (62) in [6], the Coulomb-branch potential appears to contain both contributions in the **1** and in the **20′** of $SO(6)$ coming from the backreaction of the fermion mass tensor T_{ijk}, and to have no non-normalizable contribution.

We will show that the backreaction of the modes dual to the fermion masses can only source terms in the D3-brane Coulomb-branch potential that are singlets under $SO(6)$, and hence the Coulomb-branch potential terms that transform in the **20′** of $SO(6)$ can only come from non-normalizable $L = 2$ (traceless) modes that one has to turn on separately from the fermion masses. Since the singlet term in the Coulomb-branch potential is the supergravity incarnation of the trace of the boson mass matrix, our result implies that in the bulk this boson mass trace is completely determined by the fermion masses: the sum of the squares of the boson masses will always be equal to the sum of the squares of the fermion masses.

[2] This is consistent with the fact that there are no perturbations around $AdS_5 \times S^5$ that are $SO(6)$ singlets and behave asymptotically as r^{-2} and $r^{-2} \log r$.

Our calculation establishes that asymptotically-AdS_5 solutions can only be dual to theories in which the sum of the squares of the boson masses is the same as the sum of the squares of the fermion masses. Theories where these quantities are not equal cannot by described holographically by such solutions.

From a field theory perspective this interpretation is very natural: the solutions that are asymptotically AdS_5 can only be dual to field theories that have a UV conformal fixed point, and therefore their masses and coupling constants should not run logarithmically in the UV (their beta-functions should be zero). At one loop this cannot happen unless the sum of the squares of the boson masses is equal to the sum of the squares of the fermion masses [54], which reduces the degree of divergence in the corresponding Feynman diagram and makes the beta-functions vanish.[3] Thus in perturbative field theory one inputs boson and fermion masses, and one cannot obtain a UV conformal fixed point unless the sums of their squares are equal; in contrast, in holography one inputs an asymptotically-AdS solution (dual to a conformal fixed point) and the non-normalizable modes corresponding to fermion masses, and obtains automatically the sum of the squares of the boson masses.

This understanding of how the sum of the squares of the boson masses appears in AdS-CFT also clarifies some hitherto unexplained miraculous cancellations. In the Pilch-Warner dual of the $\mathcal{N} = 2^\star$ theory [45], which from the $\mathcal{N} = 1$ perspective has a massless chiral multiplet and two chiral multiplets with equal masses, the only non-normalizable modes that were turned on in the UV were those corresponding to the fermion masses $M = \mathrm{diag}(m,m,0,0)$ and to a traceless ($L = 2$) boson bilinear of the form $\frac{m^2}{3}(|\phi_1|^2 + |\phi_2|^2 - 2|\phi_3|^2)$. Since the latter contains some tachyonic pieces one could have expected the potential for the field ϕ_3 to be negative, but in the full solution this potential came out to be exactly zero. Using the new understanding developed in this paper it is clear that this "miraculous cancellation" happens because the backreaction of the fields dual to fermion masses gives a non-trivial contribution to the trace of the boson mass,of the form $\frac{2m^2}{3}(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)$, and as a result the potential for ϕ_3 exactly cancels (see also [56] for a related discussion of some of this issues). The only way to create a tachyonic solution is to turn on a traceless ($L = 2$) boson bilinear whose coefficient is larger than $\frac{m^2}{3}$ [52].

One of the motivations for this work is the realization that the near-horizon regions of anti-branes in backgrounds with charges dissolved in fluxes have tachyonic instabilities [57, 58]. From the point of view of the AdS throat sourced by the anti-branes, this tachyon comes from a particular $L = 2$ bosonic mass term that is determined by the gluing of this throat to the surrounding region. Understanding the interplay between this mass mode and the fluxes of the near-brane region is crucial if one is to determine whether the tachyonic throat has any chance of supporting metastable polarized brane configurations of the type considered in the KPV probe analysis [59]. Preliminary results of this investigation have already appeared in [60].

[3] Note that this discussion only applies to asymptotically-AdS_5 backgrounds. The Klebanov-Strassler solution [55], which is not asymptotically-AdS_5, is dual to a field theory where the coupling constants run logarithmically.

2.4 Group theory for generic mass deformations

The goal of this section is to identify the $SO(6)$ representation of the fermionic and bosonic mass deformations. We begin by reviewing in detail the group theory behind the mass deformations because this will play an important role in our discussion.

2.4.1 Fermionic masses

The most general non-supersymmetric fermionic mass deformation of $\mathcal{N}=4$ SYM is given by the operator:[4,5]

$$\lambda^i M_{ij} \lambda^j \,, \tag{2.4.1}$$

where $\lambda^i, i = 1,...,4$ are the 4 Weyl fermions of the $\mathcal{N}=4$ theory, that in $\mathcal{N}=1$ language are the three fermions in chiral multiplets plus the gaugino. The mass matrix M is in the **10** of SU(4), which is the symmetric part of 4×4:

$$\mathbf{4} \times \mathbf{4} = \mathbf{6}_{\mathrm{a}} + \mathbf{10}_{\mathrm{s}} \,. \tag{2.4.2}$$

As noted in [6], this matrix in the **10** of $SU(4) \cong SO(6)$ can equivalently be encoded in an imaginary anti-self dual 3-form[6] T_{ABC}. The map between them for the $\mathcal{N}=1^*$ theory is given by (2.2.7) while for a generic mass matrix M will be given in the next section.

In the language of $\mathcal{N}=1$, one distinguishes a $U(1)_R \subset SU(4)_R$ that singles out the gaugino within the 4 fermions, or in other words the $SU(4)$ R-symmetry group is broken as:

$$SU(4)_R \to SU(3) \times U(1)_R \tag{2.4.3}$$

corresponding to the splitting of the fundamental index $\mathbf{4} = \mathbf{3} + \mathbf{1}$ ($i = \{I,4\}$). In this breaking, the fermionic mass matrix in the **10** decomposes as

$$\mathbf{10} = \mathbf{6} + \mathbf{3} + \mathbf{1} \ . \tag{2.4.4}$$

This corresponds to the breaking of M into the following pieces

$$M_{ij} = \begin{pmatrix} m_{IJ} & \hat{m}_I \\ \hat{m}_I^T & \tilde{m} \end{pmatrix} \tag{2.4.5}$$

where m_{IJ}, $\hat{m}_I$ and $\tilde{m}$ are respectively in the **6**, **3** and **1**.

2.4.2 Bosonic Masses

A generic 6×6 bosonic mass matrix $\mathcal{M}^2_{AB}$ has 21 components, coming from the symmetric piece in

$$(\mathbf{6} \times \mathbf{6})_s = \mathbf{1} + \mathbf{20}' \ . \tag{2.4.6}$$

[4] We use $i,j,k,\ldots = 1,2,3,4$ indices for the fermions (i.e. for the fundamental of $SU(4)$) and $A,B,C,\ldots = 1,\ldots,6$ for the bosons (fundamental $SO(6)$ representation).

[5] From now on, we omit the trace over the color indices to simplify notation.

[6] In our conventions the anti-self duality means $(\star_6 T)_{ABC} = \frac{1}{3!}\epsilon_{ABC}{}^{DEF} T_{DEF} = -iT^{ABC}$.

If bosonic masses come from the backreaction of the fermion masses on the supergravity fields, $\mathcal{M}^2$ should be of order M^2. The most naive guess is that they are related to the hermitian matrix $MM^\dagger$, which involves the following SU(4) representations:

$$\mathbf{10}\times\overline{\mathbf{10}} = \mathbf{1}+\mathbf{15}+\mathbf{84}\,. \tag{2.4.7}$$

From these very simple group-theory arguments one can immediately conclude that either our naive guess was too simple, or that the backreaction of the fermionic masses only generates the singlet (the trace) in the bosonic masses. However, since this goes against most people's intuition, particularly when there is some supersymmetry preserved, let us then push a bit further the possibility that our naive guess was wrong, or in other words that the bosonic masses are determined by fermionic ones, and see where it takes us.

The $\mathbf{20}'$ representation in (2.4.6), which is not in the product (2.4.7), appears instead in

$$\mathbf{10}\times\mathbf{10} = \mathbf{20}'_{\mathrm{s}}+\mathbf{35}_{\mathrm{s}}+\mathbf{45}_{\mathrm{a}}\,. \tag{2.4.8}$$

In terms of $SU(4)$, the $\mathbf{20}'_{\mathrm{s}}$ is one of the three 20-dimensional representations whose Young tableau and Dynkin label are:

$$\mathbf{20}' = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} = (0\,2\,0)\,. \tag{2.4.9}$$

There is an important caveat here: this representation is complex, and we therefore have to project out half of the components in order to get a real representation for the bosonic masses. As we will see in the next section this projection is directly related to the map between $SU(4)$ and $SO(6)$. A straightforward check that this representation is the one describing bosonic masses is to see what happens when $\mathcal{N}=1$ supersymmetry is preserved ($\hat{m}_I=\tilde{m}=0$ in (2.4.5)). The bosonic mass matrix should then be proportional to $mm^\dagger$ in

$$\mathbf{3}\times\overline{\mathbf{3}} = \mathbf{1}+\mathbf{8}\,. \tag{2.4.10}$$

The $\mathbf{1}$ representation is the one we discussed above, while the $\mathbf{8}$ representation indeed appears in $\mathbf{20}'$, with the right $U(1)_R$ charge, since for the breaking (2.4.3), we have [61]:

$$\mathbf{20}' = \overline{\mathbf{6}}(-4/3)+\mathbf{6}(4/3)+\mathbf{8}(0)\,. \tag{2.4.11}$$

From these group-theory arguments we conclude that if boson masses are generated by fermion masses at second order, then

$$\mathrm{Tr}\left(MM^\dagger\right) \to \mathrm{Tr}\left(\mathcal{M}^2\right) \tag{2.4.12}$$

while the other 20 components of $\mathcal{M}^2$ come from the product MM. Anticipating, we will see this map explicitly in the next section, from which we will conclude that only the former is true.

2.5 The explicit map between bosonic and fermionic mass matrices

In this section we will construct explicitly the maps (2.4.7) and (2.4.8), and the relationship between $SU(4)$ and $SO(6)$ representations. This will give the form of the possible terms in the supergravity fields that depend quadratically on fermion masses, which come from the backreaction of the fields dual to these masses. As shown in the previous section, the backreaction splits into two parts, corresponding to the $\mathbf{20'}$ and $\mathbf{1}$ representations.

To build a map between $SU(4)$ and $SO(6)$ one identifies the $\mathbf{6_a}$ representation of $SU(4)$ we have encountered above in (2.4.2) with the fundamental representation of $SO(6)$. The former is given by a 4×4 antisymmetric matrix, $\varphi^{\mathrm{T}} = -\varphi$, that transforms as $\varphi \to U\varphi U^{\mathrm{T}}$ under $U \in SU(4)$. The complex $\mathbf{6}$ can be further decomposed into two real representations, $\mathbf{6} = \mathbf{6}_+ + \mathbf{6}_-$, by imposing the duality condition:[7]

$$\star\,\varphi = \pm\varphi^\dagger \ , \tag{2.5.1}$$

where $(\star\varphi)^{ij} = \frac{1}{2}\epsilon^{ijkl}\varphi_{kl}$. In what follows we will use the following parametrization of $\mathbf{6}_+$:

$$\varphi = \begin{pmatrix} 0 & \bar{\Phi}_3 & -\bar{\Phi}_2 & -\Phi_1 \\ -\bar{\Phi}_3 & 0 & \bar{\Phi}_1 & -\Phi_2 \\ \bar{\Phi}_2 & -\bar{\Phi}_1 & 0 & -\Phi_3 \\ \Phi_1 & \Phi_2 & \Phi_3 & 0 \end{pmatrix} , \tag{2.5.2}$$

where the $\Phi_{1,2,3}$ are complex combinations of the six real scalars $\phi^{A=1,\dots,6}$ in the fundamental representation of $SO(6)$. We choose conventions such that $\Phi_I = \phi^I + i\,\phi^{I+3}$ for $I = 1,2,3$. This parametrization is convenient as it makes explicit the $\mathbf{6} \to \mathbf{3} + \overline{\mathbf{3}}$ decomposition and the relation with the three chiral multiplets of $\mathcal{N} = 4$. From (2.5.2) we find:

$$\varphi_{ij} = \sum_{A=1}^{6} G^A{}_{ij}\phi^A \qquad \text{or} \qquad \phi^A = \frac{1}{4}G^{A^{ij}}\varphi_{ji}\,, \tag{2.5.3}$$

where the six matrices G^A are antisymmetric self-dual matrices (sometimes referred as 't Hooft symbols, or generalized Weyl matrices) which intertwine between $SO(6)$ and $SU(4)$, and whose form and explicit properties we give in Appendix A, and $G^{A^{ij}} \equiv \overline{G}^A_{ji}$. An $SU(4)$ rotation given by a matrix U is related to an $SO(6)$ rotation by a matrix O via:[8]

$$U_i^{\,k}G^A{}_{kl}U_j^{\,l} = O^A_B G^B{}_{ij} \qquad \text{or} \qquad O^{AB} \equiv \frac{1}{4}G^A_{kl}U_j^{\,l}G^{B^{ji}}U_i^{\,k}\,. \tag{2.5.4}$$

Note that the action of $SO(6)$ is the same when $U \to -U$, and so, as expected, $SO(6) = SU(4)/\mathbb{Z}_2$.

With the help of t'Hooft matrices, we can work out the explicit map between the fermion mass matrix M_{ij} and an anti-self dual 3-form T_{ABC}. We get

$$T_{ABC} = -\frac{1}{2\sqrt{2}}\mathrm{Tr}\left(MG^A G^{B^\dagger} G^C\right) \ , \quad M_{ij} = \frac{1}{12\sqrt{2}}T_{ABC}(G^{A^\dagger}G^B G^{C^\dagger})_{ij} \ , \tag{2.5.5}$$

[7] The projection commutes with $SU(4)$ since ϵ^{ijkl} is an invariant tensor.

[8] The $SO(6)$ indices are raised with δ^{AB}.

where the trace in the first expression is over the $SU(4)$ indices and the numeric factors are chosen to reproduce (35) of [6] for a diagonal M. One can use the properties of the 't Hooft matrices in (A.2) and (A.3) to verify that T_{ABC} is indeed an anti-self-dual three-form.

In terms of the 3-form T, the different representations correspond to the following components:[9]

$$\begin{array}{rll} \mathbf{6}:\text{(1,2) primitive} \quad T_{I\bar{J}\bar{K}} = T^{\mathbf{6}}_{I\bar{J}\bar{K}} & , & \frac{1}{2}T_{I\bar{J}\bar{K}}\epsilon^{\bar{J}\bar{K}}{}_L = m_{IL} \\ \mathbf{3}:\text{(2,1) non-primitive} \quad T_{IJ\bar{K}} = T^{\mathbf{3}}_{IJ\bar{K}} & , & \frac{i}{2}T_{IJ\bar{K}}J^{J\bar{K}} = -\hat{m}_I \\ \mathbf{1}:\text{(3,0)} \quad T_{IJK} = T^{\mathbf{1}}_{IJK} & , & \frac{1}{6}T_{IJK}\epsilon^{IJK} = \tilde{m} \end{array} \tag{2.5.6}$$

where $J_{I\bar{J}}$ is the symplectic structure associated to the $SU(3)$ group. In our conventions it is just $J_{1\bar{1}} = J_{2\bar{2}} = J_{3\bar{3}} = i$.

Let us now discuss the bosonic masses, in the $\mathbf{20'}+\mathbf{1}$ representations of $SO(6)$. In terms of $SU(4)$, the $\mathbf{20'}$ representation is labelled by four indices and from its Young tableau (2.4.9) we learn that:

$$B_{ij,kl} = B_{kl,ij} = -B_{ji,kl} = -B_{ij,lk}\,. \tag{2.5.7}$$

Furthermore, the zero-trace condition

$$\epsilon^{ijkl}B_{ij,kl} = 0 \tag{2.5.8}$$

eliminates the singlet leaving only $\mathbf{20'}$ from $\mathbf{20'}\oplus\mathbf{1}$. Following our discussion we can decompose this complex $SU(4)$ representation into two real $SO(6)$ representations, $\mathbf{20'}_{\mathbb{C}} = \mathbf{20'}_+ + \mathbf{20'}_-$. This is achieved by requiring:

$$\overline{B_{ij,kl}} = \pm\frac{1}{4}\epsilon^{ijmn}B_{mn,pq}\epsilon^{pqkl}\,, \tag{2.5.9}$$

and we will use the choice $\mathbf{20'}_+$. The explicit map between the $\mathbf{20'}$ representations of $SU(4)$ and $SO(6)$ then works very similarly to (2.5.3):

$$V^{AB}_{\mathbf{20'}} = \frac{1}{4}G^{A^{ij}}B_{ij,kl}G^{B^{kl}}\,. \tag{2.5.10}$$

It is straightforward to verify that $V^{AB}_{\mathbf{20'}}$ is symmetric and real when $B_{ij,kl}$ satisfies (2.5.7) and (2.5.9) with the upper sign. Moreover, by using the fact that the 't Hooft matrices satisfy (A.2), one can see that the tracelessness of $V^{AB}_{\mathbf{20'}}$ is guaranteed by (2.5.8).

Now, given a fermionic mass matrix M, one can build the following matrix in the $\mathbf{20'}_+$:

$$B_{ij,kl} = \frac{1}{2}\left(M_{ik}M_{jl} - M_{il}M_{jk}\right) + \frac{1}{4}\epsilon_{ijpq}\epsilon_{rskl}\overline{M}^{pr}\overline{M}^{qs}\,. \tag{2.5.11}$$

[9] The primitive $\mathbf{6}$ and non-primitive $\bar{\mathbf{3}}$ pieces of a 3-form G are obtained as follows

$$\begin{aligned} G^{\mathbf{6}}_{I\bar{J}\bar{K}} &= G_{I\bar{J}\bar{K}} - J_{I[\bar{J}}G_{\bar{K}]L\bar{M}}J^{L\bar{M}} \\ G^{\bar{\mathbf{3}}}_{I\bar{J}\bar{K}} &= J_{I[\bar{J}}G_{\bar{K}]L\bar{M}}J^{L\bar{M}}\,. \end{aligned}$$

Here the first term is dictated by the Young tableau (2.4.9) and the second guarantees (2.5.9) with the $\mathbf{20}'_+$ choice. Furthermore, it is by construction traceless. One can add a trace to this, which, as discussed, should be built from $MM^\dagger$. We define

$$\widetilde{B}_{ij,kl} = -\frac{1}{2}\epsilon_{ijkl}\mathrm{Tr}\left(MM^\dagger\right) , \tag{2.5.12}$$

which in turn, using the properties listed in appendix A, implies that:

$$V_{\mathbf{1}}^{AB} \equiv \frac{1}{4}G^{Aij}\tilde{B}_{ij,kl}G^{Bkl} = \mathrm{Tr}\left(MM^\dagger\right)\delta^{AB} . \tag{2.5.13}$$

To summarize, the most general bosonic mass matrix produced by the backreaction of the fermionic masses is $V^{AB}_{\mathrm{quad.}}$, given by some linear combination of the $\mathbf{20}'$ and $\mathbf{1}$ contributions, $V^{AB}_{\mathbf{20}'}$ and $V^{AB}_{\mathbf{1}}$. The latter is related to the fermion masses as in (2.5.13), while the former is determined by (2.5.10) with (2.5.11). Out of this we can build a scalar $\phi^A V^{AB}_{\mathrm{quad.}}\phi^B$, or identifying the scalars ϕ^A with some local coordinates on the six-dimensional space x_A we get the “potentials"

$$V_{\mathbf{1}} \equiv x_A V_{\mathbf{1}}^{AB} x_B \ , \qquad V_{\mathbf{20}'} \equiv x_A V_{\mathbf{20}'}^{AB} x_B \ . \tag{2.5.14}$$

Let us now examine the form of these potentials for the simple example of a diagonal fermionic mass matrix:

$$M = \mathrm{diag}\left(m_1, m_2, m_3, m_4\right) , \tag{2.5.15}$$

which yields

$$\begin{aligned} V_{\mathbf{1}} \quad &= \quad (|m_1|^2+|m_2|^2+|m_3|^2+|m_4|^2)\left(x_1^2+\ldots+x_6^2\right) \qquad (2.5.16)\\ V_{\mathbf{20}'} \quad &= \quad \mathrm{Re}(m_2m_3+m_1m_4)(x_1^2-x_4^2)+\mathrm{Re}(m_1m_3+m_2m_4)(x_2^2-x_5^2)\\ &+ \quad \mathrm{Re}(m_1m_2+m_3m_4)(x_3^2-x_6^2)-2\,\mathrm{Im}(m_2m_3-m_1m_4)\,x_1x_4\\ &- \quad 2\,\mathrm{Im}(m_1m_3-m_2m_4)\,x_2x_5-2\,\mathrm{Im}(m_1m_2-m_3m_4)\,x_3x_6 \ . \end{aligned}$$

It is not hard to see that when the fourth fermionic mass is zero, and hence $\mathcal{N}=1$ supersymmetry is preserved, there is no combination of these two terms that can yield the $\mathcal{N}=1^\star$ supersymmetric bosonic mass potential

$$V_{\mathcal{N}=1^\star} = |m_1|^2\left(x_1^2+x_4^2\right)+|m_2|^2\left(x_2^2+x_5^2\right)+|m_3|^2\left(x_3^2+x_6^2\right) . \tag{2.5.17}$$

Hence, the bosonic mass matrix cannot be fully determined by the fermion mass matrix.

2.6 Mass deformations from supergravity

In this section, we will discuss how to get the bulk boson masses from the dual supergravity solution given by the full backreaction of the dual of the fermion masses on $AdS_5\times S^5$. The fully backreacted ten-dimensional (Einstein frame) metric is generically of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dy^\mu dy^\nu + ds_6^2 , \tag{2.6.1}$$

with the R-R four-form potential along space-time

$$C_4 = \alpha\, \mathrm{d}y^0 \wedge \ldots \wedge \mathrm{d}y^3 \ , \tag{2.6.2}$$

a dilaton ϕ and some internal 3-form fluxes that are combined into the complex form (1.2.16).

As explained in section 2.3 and as can be seen from the explicit flow solutions corresponding to mass deformations of $\mathcal{N} = 4$ theory that have been constructed explicitly [43, 45, 44] the boson masses can be read off from the quadratic terms in the D3 Coulomb-branch potential, given by:

$$V_{\mathrm{D3}} = \int d^4y\sqrt{g_\parallel} - \int C_4 = \int d^4y\,(e^{4A} - \alpha)\ , \tag{2.6.3}$$

where the warp factor and four-form potential are those of the fully backreacted solution. This computation is quite complicated for generic fermion masses, and was only obtained for some special choices, corresponding to the equal-mass $\mathcal{N} = 1^\star$ theory ($M = \mathrm{diag}(m,m,m,0)$) [40] and the supersymmetry-breaking-$SO(4)$-invariant $\mathcal{N} = 0^\star$ theory ($M = \mathrm{diag}(m,m,m,m)$) [53]. We will see how much of the quadratic term of V we can infer from these examples and from our group-theoretic arguments in the previous sections.

On the gravity side the fermionic mass deformation corresponds to the non-normalizable modes of the complex 3-form flux G_3 [44],[6]. As we argued in the previous sections, the **10** representation of the SU(4) fermion mass matrix M_{ij} is equivalent to the **10** of SO(6) corresponding to imaginary anti-self-dual 3-forms.

At first order in the mass perturbation the supergravity equations of motion are satisfied if the imaginary anti-self-dual 3-form $e^{4A}(\star_6 G_3 - iG_3)$ is closed and co-closed. One option is to set this to zero, i.e. to have G_3 be purely in the $\overline{\mathbf{10}}$ (imaginary self-dual), but this solution does not correspond to the dual of the $\mathcal{N} = 1^\star$ gauge theory.[10] The three-form flux has therefore both **10** and $\overline{\mathbf{10}}$ components, and has the r^{-1} behavior of a non-normalizable mode dual to the $\Delta = 3$ operator corresponding to the fermion masses. It has the same form as (2.2.6) which we repeat here for convenience

$$G_3 = \frac{c}{r^4}\left(T_3 - \frac{4}{3}V_3\right)\ , \tag{2.6.4}$$

T_3 is the imaginary anti-self-dual 3-form corresponding to the fermion masses and is now given by Eq. (2.5.5). V_3 is constructed from T_3 and combinations of the vector x^A as in (2.2.8), and it has both **10** and $\overline{\mathbf{10}}$ components.

At second order (quadratic in the fermionic masses) one has to solve for the dilaton, the metric and the 4-form potential, whose equations of motion depend quadratically on G_3, and this was only done for the special mass deformations discussed above [40, 53]; for supersymmetric unequal masses only the solution for the dilaton-axion is known [62]. Here we will not need the details of these solutions, but we note a few key points from which we will draw our conclusions.

The EOMs for the dilaton, warp factor and four-form potential have schematically the following structure:

$$\vec{\nabla}\cdot\vec{\nabla}\,(\text{Bosonic fields}) = (\text{3-form Fluxes})^2\ , \tag{2.6.5}$$

[10] On this solution, the D3-branes feel no force, which implies that the potential is zero.

Since the fluxes are known, a general solution for the bosonic fields has inhomogeneous and homogeneous parts.

For fluctuations around $AdS_5 \times S^5$, the homogenous part is a combination of harmonics of the sphere with different fall-offs in r. The quadratic term in (2.6.3) comes from modes with a r^{-2} fall off (the background warp factor $e^{4A_0} \sim r^4$), or in other words from modes which are dual to an operator of dimension $\Delta = 2$. Only the $\mathbf{20'}$ representation in the combination of metric and four-form potential that is relevant to compute (2.6.3) has this behavior [41]. It corresponds to the second harmonic on the five-sphere, and was referred in [6] as the $L = 2$ mode.

The inhomogeneous piece is sourced by quadratic combinations of the three-form fluxes, which transform in (2.4.7) and (2.4.8). Out of these, only the $\mathbf{1}$ and $\mathbf{20'}$ contribute to the masses of the bosons.

The corresponding pieces in the fields that give rise to these masses can then be schematically represented as:

$$\begin{aligned} \phi &\sim f^{\phi}_{\text{inhom.}}(r) V_{\mathbf{20'}} + g^{\phi}_{\text{inhom.}}(r) V_{\mathbf{1}} + h^{\phi}_{\text{hom.}}(r)\, U_{\mathbf{20'}} \\ g_{\|} &\sim f^{g}_{\text{inhom.}}(r) V_{\mathbf{20'}} + g^{g}_{\text{inhom.}}(r) V_{\mathbf{1}} + h^{g}_{\text{hom.}}(r)\, U_{\mathbf{20'}} \\ \alpha &\sim f^{RR}_{\text{inhom.}}(r) V_{\mathbf{20'}} + g^{RR}_{\text{inhom.}}(r) V_{\mathbf{1}} + h^{RR}_{\text{hom.}}(r)\, U_{\mathbf{20'}}\,, \end{aligned} \tag{2.6.6}$$

where the first two terms in each line correspond to inhomogeneous solutions, whose dependence on the fermionic mass we computed in the previous section (equation (2.5.16) for a diagonal mass matrix), and the last term is the contribution from the homogeneous solution whose angular dependence,

$$U_{\mathbf{20'}} \equiv x^A \mu^{\mathbf{20'}}_{AB} x^B\,, \tag{2.6.7}$$

is determined by 20 free parameters $\mu^{\mathbf{20'}}_{AB}$, that have the dimension of mass squared.[11] It is important to note that, unlike the components of $V_{\mathbf{20'}}$, the components of $U_{\mathbf{20'}}$ are *not* related in any direct way to the fermionic masses M_{ij}, but are determined in a given configuration by IR and UV boundary conditions.

With the solution for the metric and the 4-form potential at hand, one can compute the boson masses directly in supergravity, through (2.6.3). If one works in Einstein frame, this requires only the combination of warp factor and four-form potential $\Phi_- = e^{4A} - \alpha$, whose equation of motion has a right-hand side of the form (see (2.30) of [64]):

$$\Box\,(\Phi_-) \propto |\star_6 G_3 - iG_3|^2 + \ldots \propto |T_3|^2 + \ldots\,, \tag{2.6.8}$$

where the ... stand for the terms that are higher order in the mass deformation, and in the last step we have used (2.6.4) together with the duality properties $\star_6 T_3 = -iT_3$ and $\star_6 V_3 = -i\,(T_3 - V_3)$. The crucial observation is that V_3 drops out of the equation. The remaining piece, $|T_3|^2$, has no x-dependence and as a result is proportional to the singlet of the $\mathbf{10} \times \overline{\mathbf{10}}$ product. We see that out of the $\mathbf{20'}$ and the $\mathbf{1}$ parts in the inhomogeneous solution (2.6.6), only the latter contributes to the Φ_- equation[12].

[11] Only a subset of these are possible in a symmetric configuration. For example, when an $SO(3)$ symmetry is preserved ($M = \text{diag}(m,m,m,\tilde{m})$), there are only two invariant parameters [63].

[12] This fact was already noticed in [6, 65].

Furthermore, as we already mentioned, Φ_- unambiguously determines the r^2 part of the potential.

We therefore conclude that the quadratic piece in the bosonic potential is necessarily of the form:

$$V_{\rm D3}^{\rm quad.} = V_{\mathbf{1}} + U_{\mathbf{20'}} \,. \tag{2.6.9}$$

We emphasize once more that the 20 coefficients $\mu^{\mathbf{20'}}_{AB}$ in $U_{\mathbf{20'}}$ are added "by hand" and are fixed only by the boundary conditions. Furthermore, for the $\mathcal{N} = 1^\star$ theory ($m_4 = 0$) we know that this contribution has to be non-zero when the three masses of the chiral multiplets are different. This is obvious from the form of the $\mathcal{N} = 1^\star$ bosonic potential in (2.5.17), which has terms coming from both the **1** (trace) and the **20′** representations. Therefore, the solution dual to this theory must contain non-normalizable $L = 2$ modes.

We close this section by a short summary: when considering the supergravity dual of the mass-deformed $\mathcal{N} = 4$ theory, the backreaction of the fields dual to the fermion masses gives rise to perturbations in the dilaton, metric and 5-form flux proportional to $m_{\rm f}^2$, but these conspire to yield an overall zero contribution to the traceless part of the quadratic term of the polarization potential. That term therefore can arise only from the homogeneous traceless $L = 2$ modes that we referred to as $U_{\mathbf{20'}}$. This implies that in order to construct the supergravity dual of, say, $\mathcal{N} = 1^\star$ SYM theory one has to add "by hand" proper homogeneous **20′** UV modes in order to ensure that the bosonic masses will match the fermionic ones.

2.7 The trace of the bosonic and fermionic mass matrices

2.7.1 Constraints on the gauge theory from AdS/CFT

From the previous section we can arrive to another crucial observation. From (2.6.9) and the explicit form of the singlet (2.5.16) (or (2.5.13) for a generic mass matrix), we find

$$\begin{aligned} \mathrm{Tr}[\text{boson masses}^2] &= \mathrm{Tr}[\text{fermion masses}^2] \\ \mathrm{Tr}(\mathcal{M}^2) &= \mathrm{Tr}(MM^\dagger) = \mathrm{Tr}(mm^\dagger) + 2\,\hat{m}_I \bar{\hat{m}}^I + \tilde{m}^2 \,. \end{aligned} \tag{2.7.1}$$

This result establishes that only theories where the supertrace of the mass squared is zero can be described holographically by asymptotically-*AdS* solutions. The sum of the squares of the boson masses, which is an unprotected operator (also known as *the Konishi*) and has been argued to be dual to a stringy mode of dimension $(g_s N)^{1/4}$, can be in fact turned on without turning on stringy corrections, as one could have anticipated from the solutions of [43, 45, 44]. In the presence of fermion masses, what is dual to a stringy mode is not therefore the sum of the squares of the boson masses, but rather the mass super-trace (the difference between the sums of the squares of the fermion masses and the boson masses). Theories where this supertrace is zero can be described without stringy modes, but to describe theories where this supertrace is nonzero, one has to turn on "stringy" non-normalizable modes which destroy the *AdS* asymptotics.

One can also see the relation between this zero-supertrace condition and the existence of an asymptotically-AdS holographic dual from the dual gauge theory. Indeed, in a gauge theory where supersymmetry is broken by adding bosonic masses, there are no quadratic divergences, and the explicit breaking of supersymmetry is called soft. There are other soft supersymmetry-breaking terms that one can add to an $\mathcal{N} = 1$ Lagrangian, such as gaugino masses $\tilde{m}$, and trilinear bosonic couplings of the form

$$V_{\text{cubic}} = \frac{1}{2} c_{IJ}^{K} \phi^I \phi^J \bar{\phi}_K + \frac{1}{6} a_{IJK} \phi^I \phi^J \phi^K + h.c. \tag{2.7.2}$$

Similar to the quadratic terms discussed in the previous section, the bosonic cubic terms can also be read off by considering the action of probe D3-branes. They are proportional to the (3,0) and (2,1) imaginary anti-self-dual piece of the three form flux T_3 [36], which in turn are determined by the supersymmetry breaking fermionic masses $\hat{m}_I, \tilde{m}$ as in (2.5.6) [66]. One gets[13]

$$c_{IJ}^{K} = \delta_{[I}^{K} \hat{m}_{J]} \ , \qquad a_{IJK} = \tilde{m} \epsilon_{IJK} \ . \tag{2.7.3}$$

We see that supergravity constrains the masses and the couplings of the gauge theory to satisfy the equations (2.7.1) and (2.7.3). We will now explore the effect of these relations for the loop corrections on the gauge theory.

2.7.2 Quantum corrections in the gauge theory

Let us now forget for a moment AdS/CFT and focus solely on the gauge theory. We are interested in theories that descend from $\mathcal{N} = 4$ SYM and which contain supersymmetric masses for the the three chiral multiplets and also soft supersymmetry-breaking terms. Generically, the Lagrangian of such a theory (up to cubic terms) will contain[14]

$$\begin{aligned}
\mathcal{L}_{\text{susy}} + \mathcal{L}_{\text{soft}} = & -(mm^\dagger)_I{}^J \phi^I \bar{\phi}_J - \left(\frac{1}{2} m_{IJ} \psi^I \psi^J + h.c\right) \\
& - (m^2_{\text{soft}})_I{}^J \phi^I \bar{\phi}_J - \left(\frac{1}{2} b_{IJ} \phi^I \phi^J + \hat{m}_I \psi^I \lambda + \frac{1}{2} \tilde{m} \lambda\lambda + h.c.\right) \\
& - \left(\frac{1}{2} m_{IL} \epsilon^{LJK} \phi^I \bar{\phi}_J \bar{\phi}_K \right. \\
& \left. -\frac{1}{2} c_{IJ}^{K} \phi^I \phi^J \bar{\phi}_K - \frac{1}{6} a_{IJK} \phi^I \phi^J \phi^K + h.c.\right)
\end{aligned} \tag{2.7.4}$$

where in the $\mathcal{N} = 1$ notation ψ^I are the three fermions in the chiral multiplets and λ is the gaugino. In the above expression, the first and the third lines contain the supersymmetric terms coming from the superpotential (2.2.1) and (2.2.2) while the second and the fourth lines contain soft-supersymmetry breaking terms. Armed with this, one can compute the one-loop beta functions for all the coupling constants including the "non-standard soft supersymmetry breaking" terms $\hat{m}$ [68].

The authors of [68] considered general $\mathcal{N} = 1$ theories with a gauge multiplet $\{A_{\mu a}, \lambda_a\}$ and chiral multiplets $\{\phi_i, \psi_i\}$ where now the indices i,j,$k, \cdots$ transform in

[13] For exact normalizations see [67].

[14] Note that the coupling constant g_{YM} has been absorbed in a redefinition of the fields.

a representation R of the gauge group G and $a,b,c,\cdots$ are adjoint indices in G. They considered a superpotential of the form

$$W' = \frac{1}{6} Y^{ijk}\phi_i\phi_j\phi_k \tag{2.7.5}$$

where the coefficients Y^{ijk} are completely arbitrary and they also considered soft supersymmetry-breaking terms

$$\begin{aligned}\mathcal{L}'_{\text{soft}} =& (mm^\dagger + m^2_{\text{soft}})^i{}_j \phi_i\phi^j + \Big(\frac{1}{6}h^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}b^{ij}\phi_i\phi_j + \frac{1}{2}\tilde{m}\lambda_a\lambda_a + \text{h.c.}\Big) \\ &+ \Big(\frac{1}{2}r_i^{jk}\phi^i\phi_j\phi_k + \frac{1}{2}m^{ij}\psi_i\psi_j + \frac{1}{2}\hat{m}^{ia}\psi_i\lambda_a + \text{h.c.}\Big)\end{aligned} \tag{2.7.6}$$

where raising and lowering indices implies complex conjugation, e.g. $\phi_i^* = \phi^i$. Note that the supersymmetric masses have been included in the soft terms since they contribute in the same way in the Feynman diagrams.

Then, the one-loop beta functions for arbitrary values of the above couplings and representation R were computed to be[15]

$$16\pi^2\beta_g = g^3\Big(T(R) - 3C(G)\Big) \tag{2.7.7}$$

$$16\pi^2(\beta_m)_{ij} = P^k{}_i m_{kj} + P^k{}_j m_{ik} \tag{2.7.8}$$

$$16\pi^2(\beta_{\hat{m}})_{ia} = P^j{}_i\hat{m}_{ja} + g^2\Big(T(R) - 3C(G)\Big)\hat{m}_{ia} \tag{2.7.9}$$

$$\begin{aligned}16\pi^2(\beta_r)_i^{jk} =& \frac{1}{2}P^l{}_i r_l^{jk} + P^k{}_l r_i^{jl} + \frac{1}{2}r_i^{mn}Y_{lmn}Y^{ljk} + 2r_l^{mj}Y_{imn}Y^{kln} + 2g^2 r_l^{jk}C(R)^l{}_i \\ &+ 2g^2 r_l^{mj}(R_a)^k{}_i(R_a)^l{}_m - 2m_{lm}Y^{mnj}Y^{plk}Y_{npi} - 4g^2 m_{il}C(R)^l{}_m Y^{mjk} \\ &- 4g\sqrt{2}\Big(g^2C(G)\hat{m}^{ja}(R_a)^k{}_i + (R_a)^j{}_l Y^{lmk}Y_{mni}\hat{m}^{na}\Big) \quad + (k\leftrightarrow j)\end{aligned} \tag{2.7.10}$$

$$16\pi^2\beta_h^{ijk} = h^{ijl}P^k{}_l + Y^{ijl}\Big(h^{kmn}Y_{lmn} + 4g^2\tilde{m}C(R)^k{}_l\Big) + \text{ (cyclic permutations)} \tag{2.7.11}$$

$$\begin{aligned}16\pi^2\beta_b^{ij} =& b^{il}P^j{}_l + r^i_{lm}h^{jlm} + r_l^{im}r_m^{jl} - m_{kl}Y^{ilm}m_{mn}Y^{jnk} \\ &+ 4g^2\tilde{m}m^{ik}C(R)^j{}_k - 4g^2C(G)\hat{m}^{ia}\hat{m}^{ja} + (i\leftrightarrow j)\end{aligned} \tag{2.7.12}$$

$$16\pi^2\beta_{\tilde{m}} = 2g^2\Big(T(R) - 3C(G)\Big)\tilde{m} \tag{2.7.13}$$

$$\begin{aligned}16\pi^2(\beta_{(mm^\dagger+m^2_{\text{soft}})})^i{}_j =& \frac{1}{2}Y_{jpq}Y^{pqn}(mm^\dagger + m^2_{\text{soft}})^i{}_n + \frac{1}{2}Y^{ipq}Y_{pqn}(mm^\dagger + m^2_{\text{soft}})^n{}_j \\ &+ 2Y^{ipq}Y_{jpr}(mm^\dagger + m^2_{\text{soft}})^r{}_q + h_{jpq}h^{ipq} + r_j^{kl}r^i_{kl} + 2r^k_{jl}r^{il}_k \\ &- 4\Big(m^{kl}m_{lm} + \hat{m}_{ma}\hat{m}^{ka}\Big)Y^{imn}Y_{jkn} \\ &- 8g^2\Big(\tilde{m}\tilde{m}^*C(R)^i{}_j + m^{kl}m_{jk}C(R)^i{}_l + C(G)\hat{m}^{ia}\hat{m}_{ja} + (R_aR_b)^i{}_j\hat{m}_{ka}\hat{m}^{kb}\Big) \\ &- 4\sqrt{2}g\Big(Y^{iml}m_{mn}(R_a)^n{}_j\hat{m}_{la} + Y_{jml}m^{mn}(R_a)^i{}_n\hat{m}^{la}\Big)\end{aligned} \tag{2.7.14}$$

[15] Here, we have $g = g_{YM}$.

where

$$P^i{}_j = \frac{1}{2} Y^{ikl} Y_{jkl} - 2g^2 C(R)^i{}_j \tag{2.7.15}$$

and the group theory invariants are defined as

$$\mathrm{Tr}(R_a R_b) = T(R)\delta_{ab}, \quad f_{acd} f_{bcd} = C(G)\delta_{ab}, \quad C(R)^i{}_j = (R_a R_a)^i{}_j \tag{2.7.16}$$

with $(R_a)^i{}_j$ being the generators of G in the representation R and f_{abc} the structures constants. To summarize, for a superpotential of the form (2.7.5) and arbitrary values of the couplings in (2.7.6), the renormalization group evolution of these couplings at one loop is described by the beta functions given above.

Let us now specialize to our case, the $\mathcal{N} = 0^*$ theory. Since this descends from $\mathcal{N} = 4$ SYM (with gauge group $SU(N)$) which contains three chiral multiplets, the representation of the matter fields R is actually three copies of the adjoint representation of $SU(N)$. We can therefore split the index $i = (I, a_i)$ where i = 1,2,3 and $a_i = 1, \cdots N^2 - 1$. The generators in that case are written as

$$(R_a)^{(Ib)}{}_{(Jc)} = i\delta^I_J f_{abc} \tag{2.7.17}$$

This immediately gives $T(R) = 3C(G)$ and therefore from (2.7.7) we see that g does not run

$$\beta_g = 0 \tag{2.7.18}$$

The various couplings for the $\mathcal{N} = 0^*$ theory are

$$\begin{aligned}
Y^{(Ia)(Jb)(Kc)} &= \sqrt{2} g \epsilon^{IJK} f^{abc} \\
h^{(Ia)(Jb)(Kc)} &= -\sqrt{2} g \tilde{m} \epsilon^{IJK} f^{abc}, \\
r^{(Jb)(Kc)}_{(Ia)} &= \sqrt{2} g (m_{IL}\epsilon^{JKL} - 2i\delta^{[J}_I \hat{m}^{K]}) f^{abc}
\end{aligned} \tag{2.7.19}$$

and the mass parameters $m^{ij}, (mm^\dagger + m^2_{\text{soft}})^i{}_j$ and $\hat{m}_{ia}$ are diagonal in colour indices e.g. $\hat{m}_{(Ib)a} = \hat{m}_I \delta_{ab}$. From the equations (2.7.19), the first one expresses the $\mathcal{N} = 4$ superpotential (2.2.1), the second one expresses the relation between the (3,0) trilinear couplings (second of equations (2.7.3)) and the third one contains the supersymmetric trilinear couplings but also the soft (2,1) couplings (first of equations (2.7.3)). Inserting these expressions in the beta functions above, we find

$$\beta_m = \beta_{\hat{m}} = \beta_r = \beta_h = \beta_b = \beta_{\tilde{m}} = 0 \tag{2.7.20}$$

and

$$16\pi^2 (\beta_{(mm^\dagger + m^2_{\text{soft}})})^i{}_j = 4g^2 C(G) \Big(\underbrace{\mathrm{Tr}(mm^\dagger + m^2_{\text{soft}})}_{\mathrm{Tr}(\mathcal{M}^2)} - \underbrace{\mathrm{Tr}(mm^\dagger) - 2\hat{m}^k \hat{m}^k - \tilde{m}^2}_{\mathrm{Tr}(MM^\dagger)} \Big) \delta^i_j \tag{2.7.21}$$

We therefore find that if one uses the relation between the soft trilinear terms and the fermion masses (2.7.3), all the one loop beta functions except the one for the boson masses vanish exactly [54]. The one-loop beta functions for the boson mass trace vanishes if and only if the trace of the boson masses is equal to that of the fermions at tree level, which is precisely what happens for the $\mathcal{N} = 0^\star$ theories that have an asymptotically-AdS supergravity dual (Eq. (2.7.1)), and also for any gauge theory that has a UV conformal fixed point (such as the ones found on D3-branes at singularities). We have checked this for branes at a regular point of the internal manifold, and also for branes at $\mathbb{Z}_2$ and $\mathbb{Z}_3$ singularities.

The two-loop beta functions were computed in Refs. [69] and [70]. We find that for $D3$-branes at nonsingular points in the internal manifold, all of these beta functions again vanish when the supertrace of the square of the masses vanish (there might be additional regularization scheme-dependent conditions; for example, in Ref. [70] the mass of the fictitious "ϵ scalar" should be set to zero).

It is very likely that all beta functions vanish perturbatively at all loops. Indeed, the fermionic masses (2.5.6) are given by a constant (position independent) tensor, and therefore we do not expect them to run with the energy scale (corresponding to the radial distance away from the branes). Furthermore, since the trace of the bosonic masses is equal to the trace of the fermionic ones classically and at one and two loops, and the latter do not run, we expect this equality to hold at all loops. When the branes are placed in an $SO(3) \times SO(3)$ invariant background that has (1,2) but no (3,0) components, this expectation can also be confirmed by explicit calculations [67]: the theory on their world volume is simply $\mathcal{N} = 4$ broken to $\mathcal{N} = 1$ by the introduction of supersymmetric chiral multiplet masses, and it is broken to $\mathcal{N} = 0$ only by a certain traceless bosonic bilinear. Using some clever superspace tricks, this theory was shown in Ref. [71] to have vanishing beta functions at all loops.

Hence, the field theory computation of the one and two-loop beta functions confirms the results of our holographic analysis: Asymptotically-AdS solutions are dual to theories with UV conformal fixed points, and if one turns on the fermion masses, the sum of the squares of the boson masses is automatically determined to be equal to the sum of the squares of the fermion masses. Conversely, in perturbative field theory one can turn on arbitrary boson and fermion masses, but for a generic choice of masses the beta-functions will be non-zero and the theory will not have a UV conformal fixed point. These beta-functions only vanish when the sums of the squares of the fermion and boson masses are equal. We can graphically summarize this as two equivalent statements:

$$\begin{array}{rccc}
\text{SUPERGRAVITY:} & \text{Asympt-}AdS \;\Leftrightarrow\; \text{UV conformal} & \rightarrow & \sum m^2_{\text{boson}} = \sum m^2_{\text{fermion}} \\
\text{FIELD THEORY:} & \sum m^2_{\text{boson}} \neq \sum m^2_{\text{fermion}} & \rightarrow & \text{UV co}\cancel{\text{nform}}\text{al} \;\Leftrightarrow\; \text{Asympt-}\cancel{AdS}
\end{array}$$

Chapter 3

Supersymmetry and (Generalized) Geometry

As we explained in section 1.4, string theory compactifications that preserve some supersymmetry can be studied much more easily than non-supersymmetric ones due to the fact that the supersymmetry equations contain only first derivatives of the spinors while the equations of motion are second order differential equations. In this chapter, we are going to take a closer look to the restrictions imposed on the solutions by supersymmetry and more specifically, we are going to see how the latter determines the geometrical structure of the internal manifold.

Our final goal is to describe the Killing spinor equations ((1.4.5), (1.4.6),(1.4.7) and (1.4.12), (1.4.13),(1.4.14)) in purely geometrical terms. These expressions are completely general in the sense that any flux configuration respecting the isometries of the external space is allowed. Although our goal is to study the geometry in this general case, it is much easier to develop our tools for this study starting with simplified versions of the problem and then generalize them by relaxing our assumptions step by step.

The structure of this chapter follows the aforementioned generalization procedure. We start in section 3.1 by reviewing the general requirements that are imposed on the topology and the geometry of the internal manifold due to supersymmetry. The simplest cases arise for fluxless compactifications down to four dimensions where the internal manifold is described by the Calabi-Yau geometry, and five-dimensional compactifications with only five-form flux turned on in which case the internal manifold is Sasaki-Einstein.

In section 3.2, we make a first step towards the geometrization of the supergravity fluxes by introducing $O(d,d)$-generalized geometry. At this level of generalization, the fluxes from the NS-NS sector are incorporated in the geometry in the way we explain in subsection 3.2.1. Then, in 3.2.2 we describe how the Killing spinor equations can be written in a $O(d,d)$-generalized geometric way as the integrability conditions of pure spinors on the generalized tangent bundle.

On the other hand, fluxes from the R-R sector act as obstruction to the integrability of pure spinors and they cannot be given a geometric meaning in the context of $O(d,d)$-generalized geometry. One needs to generalize the geometry further and consider structures transforming in representations of the exceptional groups $E_{d(d)}$. The relevant geometry is thus called *Exceptional Generalized Geometry* (EGG). In section 3.3, we introduce the main ideas of this generalization providing more details for the particular case of $E_{6(6)}$. In the next chapter, we analyse the supersymmetry conditions in the language of EGG and we show how these follow from the Killing spinor equations.

Generalized geometry was introduced by Hitchin and his students [72],[73] in order to describe complex and symplectic geometry in a unifying framework. It was then realized that these are precisely the kind of geometries relevant for string compactifications with fluxes [74], [75][1]. The original references for Exceptional Generalized Geometry are [77] and [78].

3.1 Supersymmetry, topology and geometry

The Killing spinor equations that we encountered in section 1.4 are actually statements about the internal manifold with η or χ serving as the expansion coefficients of the ten-dimensional supersymmetry parameters in the solution under study (see (1.4.3) and (1.4.9)). Hence, when we write a Killing spinor equation for the internal manifold, it is always implied that such a mode expansion can be made for the ten-dimensional spinor which is a non-trivial requirement for the internal manifold $\mathcal{M}$. Having that in mind, we distinguish the superysmmetry constraints in two classes:

- *Topological* constraints on the manifold. The internal space should have the right topological properties that allow for the existence of a spinor field (satisfying certain reality and chirality conditions) in a well-defined way. More technically, the spinor implies a reduction of the *structure group*[2][79] of the tangent bundle of $\mathcal{M}$.

- *Differential* constraints on the manifold. Equations (1.4.6) and (1.4.13) actually describe the parallel transport of the internal spinor. This is a statement about the connection defined on $\mathcal{M}$ and in technical terms leads to a reduction of the *holonomy group*[3] of the internal space.

Let us now start discussing the simplest case of compactifications of type II supergravity in the absence of fluxes with (at least) a covariantly constant spinor (1.4.8) on the internal manifold. We will specialize to the case where the internal manifold is six-dimensional although most of the definitions and statements hold for all even-dimensional manifolds which admit a covariantly constant spinor.

[1]The deep relation between supersymmetric solutions and generalized geometry can also be seen at the ten-dimensional level [76].

[2]The structure group is the group of transition functions that are allowed in order to preserve some structure (or structures) on the overlap of two patches. The existence of well-defined tensors (tensors which take a constant value in every patch with a particular choice of frame) reduces the structure group from $GL(d,\mathbb{R})$ to a subgroup G of it. We refer to such a strucure as a G-structure.

[3]The holonomy group is related to the parallel transport of vector (or spinor) around a closed loop. Generically, the new vector differs from the original one by a $GL(d,\mathbb{R})$ element. The group of all such transformations is a subgroup of $GL(d,\mathbb{R})$ and is defined as the holonomy group.

From the spinor η, one can construct the following real two-form

$$\omega_{ab} = -2i\eta^{\dagger}\Gamma_{ab}\eta \tag{3.1.1}$$

which is well-defined on $\mathcal{M}$. The existence of ω reduces the structure group to $Sp(6,\mathbb{R})$. Furthermore, since η is covariantly constant, ω is closed:

$$d\omega = 0 \tag{3.1.2}$$

This actually implies the integrability of ω in the sense that one can define the so-called *Darboux coordinates* (q^i,p_i), $i = 1,2,3$ such that

$$\omega = dp_i \wedge dq^i \tag{3.1.3}$$

In such a case, we say that a *symplectic structure* is defined on $\mathcal{M}$.

The spinor η (and its charge conjugate $\eta^c = C^*\eta^*$) can be also used for the construction of another important three-form:

$$\Omega_{abc} = -2i\eta^{c\dagger}\Gamma_{abc}\eta \tag{3.1.4}$$

Ω is a complex three-form and in order to be well-defined, the structure group should be in $SL(3,\mathbb{C})$. The existence of Ω is related to the existence of an *almost complex structure* on $\mathcal{M}$, i.e. a tensor $I^a{}_b$ satisfying

$$I^a{}_c I^c{}_b = -\delta^a_b \tag{3.1.5}$$

Moreover, Ω is also closed

$$d\Omega = 0 \tag{3.1.6}$$

implying that I is integrable[4] (and hence becomes a *complex structure*) which actually means that one can define complex coordinates on $\mathcal{M}$. Splitting then the components of Ω and ω in holomorphic and anti-holomorphic, one has that that Ω is of type (3,0) and ω of type (1,1) and they satisfy the following *compatibility*[5] condition

$$\omega \wedge \Omega = 0 \tag{3.1.7}$$

The structure group of $\mathcal{M}$ then lies in the intersection of the symplectic and the complex structure[6]

$$SU(3) = Sp(6,\mathbb{R}) \cap SL(3,\mathbb{C}) \tag{3.1.8}$$

Complex manifolds which have a closed real two-form ω which is compatible with the complex structure are called *Kähler manifolds* and ω a Kähler form. In our case, we have in addition that Ω is closed and therefore the manifold is *Calabi-Yau*.

[4]Actually, the closure of Ω is a stronger condition than the integrability of I. A necessary and sufficient criterion for the latter is the vanishing of the *Nijenhuis tensor* the components of which are given by $(N_I)^a{}_{bc} = I^d{}_c\partial_d I^a{}_b - I^d{}_b\partial_d I^a{}_c + I^a{}_d\partial_b I^d{}_c - I^a{}_d\partial_c I^d{}_b$.

[5]Another equivalent statement would be that the symplectic structure can be obtained from the complex structure by lowering an index with the metric.

[6]The fact that the structure group is $SU(3)$ (instead of $U(3)$) is related to the fact that the normalization of Ω is fixed by $\omega^{d/2} = \frac{(d/2)!}{2^{d/2}}(-1)^{\frac{d(d/2+1)}{4}} i^{d/2}\Omega \wedge \bar{\Omega}$.

Calabi-Yau manifolds are Ricci-flat manifolds and therefore the Einstein equation for the internal manifold is satisfied in the absence of fluxes. As we mentioned in 1.4, the main advantage of compactifications on Calabi-Yau manifolds is the fact the set of *harmonic forms*[7] is well known. Using the so-called Hodge theorem, one can show that the space of harmonic p-forms is in one-to-one correspondence with the pth-cohomology. Information about the cohomology classes for a Kähler manifolds is encoded in what is known as the *Hodge diamond*. This is a diamond-like diagram which contains in its entries the dimensions of the various (p,q)-cohomology classes where $p(q)$ is the number of holomorphic(antiholomorphic) indices for each form. For a Calabi-Yau 3-fold, the information contained in its Hodge diamond is the following:

- There is 1 one-form (with representative 1), 1 (3,0)-form and 1 (0,3)-form (with representatives Ω and $\bar{\Omega}$) and 1 (3,3)-form which can be represented by the CY volume.
- There are $h^{1,1}$ (1,1)-forms and $h^{2,1}$ (2,1)-forms as well as another $h^{1,1}$ (2,2)-forms and $h^{2,1}$ (1,2)-forms following from the symmetries of the Hodge diamond.

We see that the CY 3-fold is characterized by two numbers of topological nature, $h^{1,1}$ and $h^{2,1}$. Given the above and choosing appropriate bases so that certain orthogonality properties are satisfied, one can expand any given form-field and deformations of the metric in this basis. One can then perform the dimensional reduction by integrating over the internal manifold in a similar fashion to the torus case, although more complicated.

We turn now to compactifications on odd-dimensional manifolds and we will be particularly interested in the case $d = 5$. Here, one cannot apply the same ideas that led to the concept of Calabi-Yau manifolds since one cannot have a complex or a symplectic structure. However, there is an analogous geometrical description which is derived directly from the Killing spinor equation (1.4.16) in a similar fashion that the Calabi-Yau conditions are derived from (1.4.8) which is the Killing spinor equation for Calabi-Yau manifolds.

At first sight, it might seem like (1.4.16) is "destroying" the Calabi-Yau condition (1.4.8). However, it turns out that the most natural way to understand the former is as a special case of the analysis for the latter. To see this, let $\mathcal{M}$ be a five-dimensional internal manifold and let $C(\mathcal{M})$ be the cone over it[8]:

$$ds_C^2 = dr^2 + r^2 ds_{\mathcal{M}}^2 \tag{3.1.9}$$

Let us also construct a Weyl spinor θ living on the cone from the spinor χ living on $\mathcal{M}$ in the following way:

$$\theta = \begin{pmatrix} \chi \\ -i\chi \end{pmatrix} \tag{3.1.10}$$

Taking the covariant derivative on the cone, we see that the extra piece of the spin connection cancels the flux term in (1.4.16) yielding

$$\nabla^{(C)}\theta = 0 \tag{3.1.11}$$

[7]A form is harmonic if it is both closed and coclosed. Forms that are harmonic on the internal manifold are massless if viewed as fields on the external space.

[8]For convenience, we have set $m = 1$ in (1.4.16).

and therefore the cone is a Calabi-Yau manifold! We are thus naturally led to the following definitions which can be given without involving supersymmetry.

- A manifold is *Sasaki* if the cone over it is Kähler.
- A manifold is *Sasaki-Einstein* if the cone over it is Calabi-Yau.

Sasaki-Einstein manifolds are Einstein manifolds (as the name suggests), a property which is inherited from the Ricci-flatness of the Calabi-Yau cone.

As we saw before, the Calabi-Yau conditions can be given either in spinor language (1.4.8) or in form language (3.1.2),(3.1.6) and (3.1.7). A similar story happens for Sasaki-Einstein (SE) manifolds which are characterized by a triplet of objects $(\xi, \omega_B, \Omega_B)$ (or equivalently $(\sigma, \omega_B, \Omega_B)$) which we are going to define through the cone construction.

The *Reeb vector* ξ is defined through the complex structure on the cone as

$$\xi = I(r\partial_r) \tag{3.1.12}$$

This has a vanishing cone direction $\xi^r = 0$ and therefore lies in the Sasaki-Einstein manifold. The *contact form* σ which is defined in the "dual manner"[9]:

$$\sigma = I(r^{-1}dr) \tag{3.1.13}$$

lies in the SE manifold and satisfies

$$\iota_\xi \sigma = 1 \tag{3.1.14}$$

Finally, the two-forms ω_B and Ω_B for Sasaki-Einstein manifolds can be defined through the cone construction as[10]

$$\omega_B = \frac{1}{r^2}\omega - \frac{1}{r} dr \wedge \sigma \tag{3.1.15}$$

and

$$\Omega_B = -\frac{i}{r^3}\iota_\xi \Omega \tag{3.1.16}$$

A Sasaki-Einstein manifold can be seen as a fibration over a Kähler-Einstein base defined by the Reeb vector. Then, ω_B becomes a (1,1)-form and Ω_B a (2,0)-form with respect to the complex structure defined on the base. ω_B is related to the contact structure σ through

$$d\sigma = 2\omega_B \tag{3.1.17}$$

The closure of the Calabi-Yau structures ω and Ω translate to

$$d\omega_B = 0 \tag{3.1.18}$$

and

$$d\Omega_B = 3i\sigma \wedge \Omega_B \tag{3.1.19}$$

[9]Note that in components, there is a minus sign due to the natural action of tensors on forms, i.e.: $(A \cdot j)_a = -A^b{}_a j_b$.

[10]We are assuming that the holomorphic (3,0)-form of the Calabi-Yau satisfies $\mathcal{L}_\xi \Omega = 3\Omega$ which is always possible in a non-compact Calabi-Yau.

while they also satisfy $\iota_\xi \omega_B = \iota_\xi \Omega_B = 0$. Finally, the compatibility condition (3.1.7) and the normalization of Ω imply

$$\omega_B \wedge \Omega_B = 0 \tag{3.1.20}$$

and

$$\omega_B \wedge \omega_B = \frac{1}{2} \Omega_B \wedge \bar{\Omega}_B \tag{3.1.21}$$

3.2 $O(d,d)$ Generalized Geometry

In the previous section, we saw that the main features of Calabi-Yau and Sasaki-Einstein manifolds can be conveniently described using the tools of complex and symplectic geometry. However, these geometries appeared in rather special cases of string theory backgrounds; the fluxless case for compactifications to four-dimensional Minkowski vacua and flux compactifications but with only special flux configuration (we considered the case where only F_5 was non-zero) for AdS_5 vacua. In order to study more general backgrounds, we need to generalize the geometric concepts we use.

3.2.1 Geometrizing the NS-NS degrees of freedom

The starting point of generalized geometry is the extension of the tangent bundle $T\mathcal{M}$ of the internal manifold to a *generalized tangent bundle* E in such a way that the elements of this bundle generate all of the bosonic symmetries of the theory (diffeomorphisms and gauge transformations). The generalized tangent bundle transforms in a given representation of the corresponding duality group acting on the symmetries. Following the historical path, we start by discussing the $O(d,d)$ generalized geometry, relevant to the NS-NS sector of type II theories compactified on d-dimensional manifolds. In section 3.3, we introduce $E_{d(d)}$ generalized geometry which encodes the full bosonic sector of type II theories compactified on a $(d-1)$-dimensional manifold, or M-theory on a d-dimensional geometry.

The NS-NS sector of type II supergravity contains the metric $g_{(mn)}$, the Kalb-Ramond field $B_{[mn]}$ and the dilaton ϕ. The symmetries of this theory are diffeomorphisms generated by vectors k and gauge transformations of the B-field which leave the $H = dB$ invariant and which are parametrized by one-forms λ. The latter corresponds to the restriction of the first transformation in (1.2.5) restricted on the d-dimensional internal manifold. The combined action of these symmetries can be thought to be generated by a single object

$$V = (k, \lambda) \ , \qquad k \in T\mathcal{M} \ , \ \lambda \in T^*\mathcal{M} \tag{3.2.1}$$

on the combined bundle $T\mathcal{M} \oplus T^*\mathcal{M}$. In fact, V is well-defined only in a patch of $\mathcal{M}$. If there is H-flux, the gauge potential can change in the overlap of two patches by a gauge transformation leaving the field strength invariant in a situation similar to the Dirac string. In order to construct a global section of the bundle, we need to consider

$$e^B V \equiv (k, \lambda + \iota_k B) \tag{3.2.2}$$

taking thus into account the non-trivial transformation of the B-field on the overlap of two patches. These *generalized vectors* belong to the *generalized tangent bundle*

$$E \simeq T\mathcal{M} \oplus T^*\mathcal{M} \tag{3.2.3}$$

where the isomorphism is provided by the e^B defined above. The structure group of this bundle can be reduced from $GL(2d)$ to $O(d,d)$ by observing that there exists an invariant metric defined by

$$\eta(V,V') \equiv \frac{1}{2}(\iota_k\lambda' + \iota_{k'}\lambda) \ . \tag{3.2.4}$$

The symmetries of this metric are:

- $k^a \to A^a{}_b k^b, \quad \lambda_a \to A_a{}^b\lambda_b$, where $A \in GL(d,\mathbb{R})$ and $A_c{}^a A^c{}_b = \delta^a_b$,
- $k^a \to k^a, \quad \lambda_a \to \lambda_a + B_{ab}k^b$, where $B \in \wedge^2 T^*\mathcal{M}$
- $k^a \to k^a + \beta^{ab}\lambda_b, \quad \lambda_a \to \lambda_a$, where $\beta \in \wedge^2 T\mathcal{M}$

All together, they form the group $O(d,d)$ which corresponds to the T-duality group of toroidal compactifications that we encountered in section 1.1. The first two of these symmetries are related to the fact that the components of metric and the B-field change on the overlap of two patches and therefore generalized diffeomorphisms parametrized by V have to take this fact into account. The meaning of the last transformation is more subtle since there is no "β-field" in the type II supergravity action. For more details for the interpretation of the β- transform in the context of non-geometric backgrounds, the reader is referred to [80],[81],[82],[83],[84],[85] and [86].

It is possible to extend many of the concepts of ordinary differential geometry on $T\mathcal{M}$ to analogues on E. The resulting geometry is called *Generalized Complex Geometry* (GCG) or $O(d,d)$-*generalized geometry*[11].

One of the key elements in this construction is the analogue of the Lie derivative. This is the so-called *Dorfman derivative* along a generalized vector V on another generalized vector V'[12]. It expresses the infinitesimal action of the symmetries encoded in V and is given by[13]

$$\mathbb{L}_V V' = (\mathcal{L}_k k', \mathcal{L}_k\lambda' - \iota_{k'}d\lambda) \tag{3.2.5}$$

where $\mathcal{L}$ is the ordinary Lie derivative. One can write this in a more $O(d,d)$-covariant way by embedding the ordinary derivative in a $O(d,d)$-covariant object through

$$D_M = (\partial_m, 0) \in E^* \tag{3.2.6}$$

where $m = 1,...,d$, while $M = 1,...,2d$. The Dorfman or generalized Lie derivative (3.2.5) takes the form

$$\mathbb{L}_V V' = (V\cdot D)V' - (D\times V)V' \tag{3.2.7}$$

where $\cdot$ and $\times$ stand respectively for the inner product and the projection to the adjoint representation between the vector and dual vector representations[14].

[11] For a more complete introduction to this with a focus on supergravity applications, see [87]. A nice review for generalized complex geometry with applications for D-branes is [88].

[12] By the Leibniz rule, it can be extended to arbitrary tensors constructed from E and E^*.

[13] Note that V and V' now are sections of E and therefore the Dorfman derivative takes into account the non-triviality of the B-field patching.

[14] Using explicit indices, $V\cdot D = V^M D_M$, $(D\times V) = D_M V^N|_{\text{adjoint}}$. In the $O(d,d)$ case, the latter is $(D\times V)_M{}^N = D_M V^N - \eta^{NP}\eta_{MQ}D_P V^Q$.

3.2.2 Supersymmetry in $O(d,d)$ Generalized Geometry

In section 3.1 we explained how the existence and the differential properties of a spinor (or more) on the internal manifold provide topological and geometrical properties of it. However, this could be done in a clearly geometric way only for rather special flux configurations; for fluxless backgrounds for $d = 6$ and only for five-form flux for $d = 5$. For more general backgrounds, the integrability conditions for the relevant structures fail to be satisfied and one loses the power of geometrical tools that have been developed for Calabi-Yau and Sasaki-Einstein manifolds. Generalized Geometry avoids the aforementioned failure by providing geometrical meaning for the fluxes and hence one can obtain integrability conditions for appropriately defined *generalized structures* which have a similar form to the previous ones but describe more general backgrounds.

Let us start with the case of $d = 6$ which is the relevant one for generalizing the concept of Calabi-Yau manifolds (or better Calabi-Yau 3-folds). Now, the internal spinor η in appearing in the Killing spinor equations (1.4.5) to (1.4.7) is not any more covariantly constant and therefore the CY forms ω and Ω are not closed. The first step therefore is to generalize these objects in their $O(d,d)$ counterparts which will have now (in general) indefinite rank. These are first built out of two internal spinors as

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\dagger}_\pm \;= \frac{1}{4}\sum_{k=0}^{6}\frac{1}{k!}\eta^{2\dagger}_\pm \Gamma_{a_1\dots a_k}\eta^1_+ \Gamma^{a_k\dots a_1} \tag{3.2.8}$$

where in the second equality a Clifford expansion has been performed. One can then identify these bispinors with (poly)forms through the *Clifford map*

$$\sum_{k=0}^{6}\frac{1}{k!}C^{(k)}_{a_1\dots a_k}\Gamma^{a_1\dots a_k} \qquad \longleftrightarrow \qquad \sum_{k=0}^{6}\frac{1}{k!}C^{(k)}_{a_1\dots a_k}\mathrm{d}x^{a_1}\wedge\ldots\wedge\mathrm{d}x^{a_k} \tag{3.2.9}$$

This map allows one to think of $\Phi^\pm$ either as (Weyl) spinors of $Spin(6,6)$ or as polyforms and change easily between the two pictures. As constructed these spinors are *pure* in the sense that they annihilated by half of the Cliff(6,6) gamma-matrices. In the form language this means that they are annihilated by half of the generalized vectors[15] where the (Clifford)[16] action of a generalized vector $V = (k,\lambda)$ on a polyform Φ is defined by

$$V\cdot\Phi = \iota_k\Phi + \lambda\wedge\Phi \tag{3.2.10}$$

In order to see how the CY structures embed inside the generalized structures, we have to set $\eta_1 = \eta_2$. Then, the pure spinors (3.2.8) reduce to

$$\Phi_+ = e^{-i\omega}, \qquad \Phi_- = -i\Omega \tag{3.2.11}$$

The geometrical significance of the pure spinors $\Phi^\pm$ can be seen in a more clear way if we introduce another piece of terminology. A *generalized almost complex structure*

[15]More technically, if the annihilator space of (the space of generalized vectors annihilating) Φ is of maximal rank, i.e. is six-dimensional.

[16]Note that generalized vectors can indeed be interpreted as $O(d,d)$-gamma matrices since acting on forms they satisfy $\{V,V'\} = 2\eta(V,V')$.

(GACS)[17] is defined as a tensor $\mathcal{I}^A{}_B$ on the generalized tangent bundle satisfying the following two properties

$$\mathcal{I}^A{}_C \mathcal{I}^C{}_B = -\delta^A_B, \qquad \mathcal{I}^C{}_A \eta_{CD} \mathcal{I}^D_B = \eta_{AB} \tag{3.2.12}$$

The first of these relations is the direct analogue of to (3.1.5) for the generalized tangent bundle while the second of them expresses the fact that the metric (3.2.4) is hermitian with respect to this generalized almost complex structure. The existence of a GACS implies the reduction of the structure group of E from $O(6,6)$ to $U(3,3)$ and it specifies a splitting of E (or better the complexification of it) into its $(\pm i)$-eigenbundles.

The relation of a pure spinor Φ and a GACS $\mathcal{I}$ is then established by identifying the annihilator space of Φ with the $(+i)$-eigenbundle of $\mathcal{I}$. Therefore, one obtains a one-to-one correspondence[18]

$$\text{pure spinor} \quad \Phi \quad \leftrightarrow \quad \mathcal{I} \quad \text{GACS} \tag{3.2.13}$$

We now turn to the integrability conditions for the generalized structures. The pure spinors $\Phi^{\pm}$ are formal sums of spinor bilinears constructed from the internal spinors η ((3.2.8) and (3.2.9)) and therefore one can use the conditions (1.4.5),(1.4.6) and (1.4.7) to derive differential conditions for $\Phi^{\pm}$. In the context of $\mathcal{N}=1$ vacua, this has been performed in [75] and it corresponds to setting $\eta_2^+ = \tilde{\eta}_2^+ = 0$ in the Killing spinor equations (see (1.4.3)). We will not provide here the technical details of this computation, but we will state the results and interpret them. The equations one gets for type IIB are[19]

$$(d + H\wedge)(e^{2A-\phi}\Phi_+) = e^{2A-\phi}\Big(dA \wedge \bar{\Phi}_+ - \frac{i}{16} e^{\phi+A} \star_6 \mathcal{F}_{RR}\Big) \tag{3.2.14}$$

and

$$(d + H\wedge)(e^{2A-\phi}\Phi_-) = 0 \tag{3.2.15}$$

where we have grouped together the following combination of R-R fluxes

$$\mathcal{F}_{RR} = F_1 - F_3 + F_5 \tag{3.2.16}$$

The norms of the $Spin(6,6)$ spinors satisfy

$$|\Phi^{\pm}| = e^A \tag{3.2.17}$$

Let us focus on the second of the two differential conditions (3.2.15) which seems to attribute integrability properties to the structure Φ_-. From the left-hand side, we that the NS-NS three-form flux H enters the geometry by providing a twisting for the exterior derivative:

$$d_H = d + H\wedge \tag{3.2.18}$$

[17] In string theory, the generalized almost complex structure appears naturally from the worldsheet point of view [89].

[18] There is a subtlety here since the one-to-one correspondence is modulo an overall factor which is related to the dilaton and the warp factor in supergravity applications. However, we will not be very precise with this factor in the context of $O(d,d)$-Generalized Geometry.

[19] Here the H-twisting appears with a different sign than in [75] due to the different conventions we use.

Perhaps, we can make the geometric picture more clear by observing that $e^{-B}de^{B} = d_H$ and defining the *dressed* pure spinors

$$\Phi^D = e^B \Phi \tag{3.2.19}$$

where the action of the exponential on forms is given by (1.2.4). Neglecting for the moment the scaling factor $e^{2A-\phi}$, (3.2.15) becomes

$$d\Phi_-^D = 0 \tag{3.2.20}$$

But this is very similar to the Calabi-Yau integrability condition (3.1.6). In fact, the closure of Φ_-^D implies that the GACS related to it through (3.2.13) is integrable[20] becoming now a generalized complex structure. According to the so-called *Generalized Darboux Theorem*[73], the existence of a generalized complex structure on a manifold implies that locally the manifold is equivalent to the product $\mathbb{C}^k \times \mathbb{R}^{d-2k}_{Sp}$ where $\mathbb{R}^{d-2k}_{Sp}$ is the standard symplectic space. Therefore, a generalized complex manifold interpolates between a complex and symplectic manifold. The integer $k \leq d/2$ is called the *rank* and is allowed to jump over certain points or planes. Manifolds which admit a pure spinor Φ that is closed $d\Phi = 0$ were called in [72] *Generalized Calabi-Yau*.

Let us now return back to the differential condition (3.2.15) or rather its dressed version (3.2.20). The e^B action (3.2.19) is inherited from the corresponding dressing of generalized vectors (3.2.2) and therefore it is natural to appear in the integrability conditions. The scaling factor in (3.2.15) is related on the one hand to the normalization of the internal spinors (see (3.2.17) or (4.2.4) for the five-dimensional case) and on the other hand to an additional $\mathbb{R}^+$ action on the generalized tangent bundle which we have neglected here (see [87] for details).

We therefore see that, according to our previous discussion, four-dimensional vacua $\mathcal{N} = 1$ vacua of type IIB supergravity with generic fluxes are H-twisted generalized Calabi-Yau manifolds. This means in particular that there is an (integrable) generalized complex structure on the "twisted" generalized tangent bundle spanned by the twisted generalized vectors (3.2.2).

Moving now to the other differential condition for the pure spinor Φ_+, we see that its integrability is obstructed from the presence of R-R fluxes through the combination $\mathcal{F}_{RR}$. This means that the GACS related to the pure spinor Φ_+ is not integrable. However, this is rather expected since, as we have already explained, in $O(d,d)$-generalized geometry only the NS-NS sector of type II supergravity is geometrized. In reference [90] where $\mathcal{N} = 2$ vacua were studied with NS-NS fluxes only, both pure spinors were shown to be twisted closed $d_H\Phi_\pm = 0$. The inclusion of the R-R fluxes in the geometry requires a further generalization to which we turn in the next section.

Before closing this section, let us briefly explain how $O(d,d)$-generalized geometry works for compactifications of type II theory down to AdS_5 (in this case $O(5,5)$-generalized geometry). The extension of the tangent bundle and the discussion of the symmetries for the metric η proceeds in exactly the same way as for the six-dimensional case. However, in odd dimensions one does not have a generalized almost complex

[20] Actually, the integrability of a GACS is a weaker condition than the closure of the corresponding pure spinor. A GACS is integrable if there exists a generalized vector V such that the pure spinor Φ associated with the GACS satisfies $d\Phi = V \cdot \Phi$.

structure and therefore the one-to-one correspondence between a GACS and an $O(5{,}5)$ pure spinor is not present here. Nevertheless, $O(5{,}5)$ pure spinors are well-defined and one can derive the differential conditions they satisfy from the Killing spinor equations for the internal manifold. We will not get into the details of this analysis, but let us mention some basic facts. The differential conditions have the *schematic* form[91][21]

$$d_H\Phi = dA \wedge \bar{\Phi} + m\Phi + \mathcal{F}_{RR} \tag{3.2.21}$$

where some terms may be absent depending on the precise form of the bispinor considered. We see that, similarly to the analysis for Mink_4 vacua, the R-R fluxes act generically as an obstruction for the integrability of Φ and there is also the "peculiar" term $dA \wedge \bar{\Phi}$ on the right-hand side. Furthermore, the AdS_5 mass parameter m also appears on the right-hand side in a term proportional to Φ. We will see in chapter 4 that in exceptional generalized geometry, m is embedded in an $SU(2)$ vector λ_a which breaks the R-symmetry of the theory $SU(2)_R \to U(1)_R$ and there is no term like $dA\wedge\bar{\Phi}$ (it has been incorporated in the geometry).

The fact that the supersymmetry conditions have a similar form in $O(6{,}6)$ and $O(5{,}5)$-generalized geometry gives us evidence that they could be described in a unified form for various dimensions. It turns out that is true in exceptional generalized geometry where the supersymmetry conditions have exactly the same form for various dimensions and the only thing that changes is the group theory formulae.[22].

3.3 Exceptional Generalized Geometry

The usefulness of generalized geometry in describing supersymmetric vacua in string compactifications, in the way we described in the previous section, is the fact that the structure group of the generalized tangent bundle at each point in $\mathcal{M}$, namely $O(d{,}d)$ coincides with the T-duality group of the massless sector of string theory. T-duality transforms the NS-NS sector and the R-R sector separately, i.e. does not mix them. However, the NS-NS and R-R sectors are the bosonic parts of the supersymmetric full string theory and we expect further symmetries mixing them to hold.

In order to include the gauge transformations of the R-R fields, or to do a generalized geometry for M-theory, one needs to extend the tangent bundle even further. Not surprisingly, the appropriate generalized bundle should transform covariantly under the group $E_{d(d)}$ [77, 78], which is the U-duality group of the massless sector of type II string theory (M-theory) when compactified on a $d-1$ (d) dimensional manifold. In the concrete applications in chapter 4, we will deal with compactifications of type IIB and M-theory down to five dimensions, and the relevant group is therefore $E_{6(6)}$. This extended version of generalized geometry is called *Exceptional Generalized Geometry* [93, 94]. In the following, we will introduce the main ideas of Exceptional Generalized Geometry for the type IIB case, and then we will present the analogous concepts for M-theory compactifications.

[21] The supersymmetry conditions for AdS_4 vacua were derived in [92] and they have a similar form to (3.2.21).

[22] In the next chapter we will study just AdS_5 vacua where the relevant group is $E_{6(6)}$, but the analysis is similar for AdS_4 vacua with the U-duality group being $E_{7(7)}$.

The generalized tangent bundle for type IIB decomposes as follows

$$E \simeq T\mathcal{M} \oplus (T^*\mathcal{M} \oplus T^*\mathcal{M}) \oplus \wedge^3 T^*\mathcal{M} \oplus (\wedge^5 T^*\mathcal{M} \oplus \wedge^5 T^*\mathcal{M}) \tag{3.3.1}$$

where the additional components $T^*\mathcal{M}$, $\wedge^3 T^*\mathcal{M}$ and the two copies of $\wedge^5 T^*\mathcal{M}$ correspond to the gauge transformations of C_2, C_4, C_6 and B_6, the dual of B_2 (one can also understand this in terms of the charges of the theory, namely D1, D3, D5 and NS5 -brane charges respectively). In the above expression, we have grouped together terms that transform as doublets under the $SL(2,\mathbb{R})$ symmetry of type IIB supergravity which we encountered in section 1.2.

The isomorphism implied in (3.3.1) is given by an element $e^{\mu} \in E_{6(6)}$, $\mu \in \mathfrak{e}_{6(6)}$ which can be constructed from the gauge fields of the theory in such way that the generalized vectors are well-defined in the overlap of two patches. This is in direct analogy with the $O(d,d)$ case where the only non-trivial gauge field is the B-field. The expression for μ in our case is given below in (3.3.6).

One can also here embed the derivative in a covariant object in E^*, such that its non-zero components are on $T^*\mathcal{M}$. The Dorfman derivative takes the same form as in the $O(d,d)$ case, namely (3.2.7). For its expression in terms of the $GL(5)$ decomposition of E in (3.3.1), namely the analogue of (3.2.5), see [93].

Finally, let us mention that a complete treatment of both $O(d,d)$ and $E_{d(d)}$ generalized geometry also includes the geometrization of the so-called trombone symmetry (see [93] for details). This is an additional $\mathbb{R}^+$ symmetry which exists in warped compactifications of M-theory and can be understood as a combination of the scaling symmetry in the eleven-dimensional theory (1.3.9) (and therefore is inherited also in type II) and constant shifts of the warp factor in the compactified theory. In order to incorporate the action of this symmetry for the supersymmetry conditions in chapter (4), we rescale appropriately our structures (see (4.1.10) below) where the appearance of the dilaton in the type IIB case reflects the fact that the dilaton can be interpreted as a contribution to the warp factor in an M-theory set-up.

Particular case of $E_{6(6)}$

Let us now specialize to the case of $E_{6(6)}$. The generalized tangent bundle E transforms in the fundamental **27** representation, whose decomposition is given in (3.3.1). In terms of representations of $GL(5) \times SL(2)$[23] this is

$$\mathbf{27} = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{5},\mathbf{2}) \oplus (\mathbf{10},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \; . \tag{3.3.2}$$

It will actually turn out to be convenient to use the $SL(6) \times SL(2)$ decomposition, where the two $SL(2)$ singlets are combined into a two-vector, while the two $SL(2)$ doublets are combined into a doublet of forms. Under $SL(6) \times SL(2)$ the fundamental (anti-fundamental) representation V (Z) of $E_{6(6)}$ therefore decomposes as

$$\mathbf{27} = (\overline{\mathbf{6}},\mathbf{2}) + (\mathbf{15},\mathbf{1}), \quad V = (V^i_{\ a}, V^{ab}) \tag{3.3.3a}$$

$$\overline{\mathbf{27}} = (\mathbf{6},\overline{\mathbf{2}}) + (\overline{\mathbf{15}},\mathbf{1}), \quad Z = (Z^a_{\ i}, Z_{ab}) \tag{3.3.3b}$$

[23] For details in the representations of $E_{6(6)}$ see appendix C.

where a,b,c,... run from 1 to 6 and i,j,k,... from 1 to 2.

The derivative embeds naturally in the anti-fundamental representation as[24]

$$D^i_m = D^i_6 = D_{mn} = 0, \qquad D_{m6} = e^{2\phi/3}\partial_m \tag{3.3.4}$$

where we use m,n,... for the coordinate indices on the internal manifold.

The adjoint representation splits under $SL(6) \times SL(2)$ as

$$\mathbf{78} = (\mathbf{35},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\overline{\mathbf{20}},\mathbf{2}), \quad \mu = (\mu^a{}_b, \mu^i{}_j, \mu^i_{abc}) \ . \tag{3.3.5}$$

In our conventions, the dilaton and gauge fields embed in this representation in the following way

$$\mu^1_{mn6} = e^{\phi} C_{mn} \tag{3.3.6a}$$

$$\mu^2_{mn6} = B_{mn} \tag{3.3.6b}$$

$$\mu^m{}_n = -\frac{\phi}{6}\delta^m_n \tag{3.3.6c}$$

$$\mu^6{}_6 = \frac{5\phi}{6} \tag{3.3.6d}$$

$$\mu^n{}_6 = -e^{\phi}(*C_4)^n \tag{3.3.6e}$$

$$\mu^i{}_j = \begin{pmatrix} -(\phi/2) & e^{\phi}C_0 \\ 0 & (\phi/2) \end{pmatrix} \tag{3.3.6f}$$

while the other components of μ vanish[25]. Note that the the gauge fields from the R-R sector carry an e^{ϕ} factor.

$E_{6(6)}$ for M-theory

We now turn to compactifications of eleven-dimensional supergravity down to AdS_5 and outline the M-theory counterpart of the of the above construction. The situation is similar to the type IIB case since the group of global symmetries remains the same, namely $E_{6(6)}$. However, the analogous formulae are more transparent since M-theory combines the degrees of freedom in a more compact form, avoiding thus the complications due to the $GL(5) \subset SL(6)$ embedding. In particular, the generalized tangent bundle is decomposed in this case as

$$E \simeq T\mathcal{M} \oplus \wedge^2 T^*\mathcal{M} \oplus \wedge^5 T^*\mathcal{M} \tag{3.3.7}$$

where the internal manifold $\mathcal{M}$ is now six-dimensional and the various terms correspond to momenta, M2- and M5-brane charges respectively. The latter can be dualized to a vector, and together with the first piece they form the $(\mathbf{6},\mathbf{2})$ piece in the split of the fundamental $\mathbf{27}$ representation under $SL(6) \times SL(2)$ given in (3.3.3). The

[24] The reason for the additional factor of $e^{2\phi/3}$ is related to the rescaling of the bispinors which will be introduced later, see (4.1.10).

[25] These other components of μ could have non-vanishing values in a different U-duality frame.

derivative is embedded in one of the two components of this doublet appearing in the anti-fundamental representation[26]

$$D_a^2 = \nabla_a, \qquad D_a^1 = D_{ab} = 0 \ . \tag{3.3.8}$$

The decomposition of the adjoint representation is given in (3.3.5), and the three-form gauge field C embeds in μ as

$$\mu^1_{abc} = -(\star C)_{abc} \tag{3.3.9a}$$

$$\mu^2_{abc} = \mu^i{}_j = \mu^a{}_b = 0. \tag{3.3.9b}$$

We therefore see that for both type IIB and M-theory, the effect of the gauge potentials can be incorporated in the twisting of the generalized tangent bundle by appropriately embedding them in an adjoint element e^μ of $E_{6(6)}$. This is similar to the dressing of pure spinors (3.2.19) by the NS-NS gauge potential e^B in $O(d,d)$-generalized geometry where the structure Φ was constructed as a bispinor in (3.2.8). In the next chapter, we will explore compactifications of type IIB and M-theory down to AdS_5 preserving eight supercharges with generic fluxes. We will show that one can construct appropriate bispinors $\mathcal{K}$ and $\mathcal{J}_a$ as the generalization of the pure spinors Φ. We will then prove that the dressed objects $K = e^\mu \mathcal{K}$ and $J_a = e^\mu \mathcal{J}_a$ need to satisfy certain integrability conditions which will be the analogues of the closure of the dressed pure spinors (3.2.20).

[26]Note that here D does not carry a rescaling factor in contrast to the type IIB case.

Chapter 4

Generalized Geometric vacua with eight supercharges

The analysis of the previous chapter revealed that in order to incorporate the effect of fluxes in geometric structures, one needs to extend the tangent bundle of the internal manifold appropriately to its generalized version in order to include the charges present in the theory (momenta, NS-brane and D-brane charges). In the case of $O(d,d)$-generalized geometry, the generalization of the tangent bundle includes only momenta and NS-brane charges; the latter corresponding to the gauge transformations of the B-field. Therefore, only the NS-NS sector is geometrized.

In this chapter, we will show that in exceptional generalized geometry we are able to describe the general case (AdS_5 vacua with generic fluxes preserving eight supercharges) in a purely geometric way. Specifically, we will write the Killing spinor equations relevant for AdS_5 compactifications purely in terms of the geometric structures $K = e^{\mu}\mathcal{K}$ and $J_a = e^{\mu}\mathcal{J}_a$ defined in the previous chapter which live in representations of the corresponding exceptional group. Here, μ contains all the gauge potentials (both from NS-NS and R-R sector, see (3.3.6)) and $\mathcal{K}$ and $\mathcal{J}_a$ can be constructed as bispinors of the internal spinors. We perform the analysis for both type IIB and M-theory compactifications.

We proceed as follows. In section 4.1 we introduce concretely the exceptional structures K and J_a focusing on the type IIB case. In subsection 4.1.1 we describe the compatibility conditions they should satisfy as a result of their group theory properties while in subsection 4.1.2 we give the integrability properties they should satisfy. Section (4.2) is devoted to the proof of the integrability conditions for the structures K and J_a which are constructed as bispinors. In section 4.3, we repeat the analysis for the M-theory case. Intermediate results which we make use of in the analysis are proved in section 4.4. Two key elements in the statement and the proof of the integrability conditions are the moment map for J_a and the Dorfman derivative along K. Their explicit calculation is given in sections 4.5 and 4.6 respectively. Appendix C contains the necessary group theory formulae to derive the results in this chapter.

4.1 Supersymmetry in Exceptional Generlaized Geometry

4.1.1 Backgrounds with eight supercharges

In the end of the previous chapter we mentioned briefly how the supergravity degrees of freedom can be packed into generalized geometric objects which belong to representations of the corresponding duality group. In this section, we focus on the case of backgrounds that have eight supercharges off-shell, and in the next subsection we show how the on-shell restriction (i.e., the requirement that the background preserves the eight supercharges) is written in the language of exceptional generalized geometry.

As we explained in section 3.1, backgrounds with off-shell supersymmetry are characterized in ordinary geometry by the existence of well-defined spinors, or in other words a reduction of the structure group of the tangent bundle from $SO(d)$ to subgroups of it singled out by the fact that they leave the well-defined spinors invariant. This means that the metric degrees of freedom can be encoded in objects that are invariant under the structure group, built out of bilinears of the spinors. For the familiar case of $SU(d/2)$ structures (like the case of Calabi-Yau), these objects are the Kähler 2-form ω and the holomorphic d/2-form Ω, satisfying certain compatibility conditions[1].

On-shell supersymmetry imposes differential conditions on the spinors, which are translated into differential conditions on the bilinears of spinors. In the absence of fluxes, the supersymmetric solutions involve an external Minkowski space, and the differential conditions lead to integrable structures on the internal space. In the case of M-theory compactifications down to five dimensions preserving eight supercharges, the internal manifold has to be Calabi-Yau, namely the Kähler 2-form and the holomorphic 3-form are closed.

Compactifications to AdS require on one hand some flux to support the curvature, and on the other hand the integrability conditions are weaker (they are usually referred to as weakly integrability conditions)[2]. The simplest example of compactifications to AdS_5 is that of type IIB, where the curvature is fully provided by the 5-form flux, and the internal space is Sasaki-Einstein (the simplest case being S^5). In section 3.1 we saw that Sasaki-Einstein manifolds are U(1)-fibrations over a Kähler-Einstein base (defined by a Kähler 2-form ω_B and a holomorphic 2-form Ω_B satisfying the compatibility condition) and a contact structure σ, satisfying[3]

$$d\sigma = 2m\,\omega_B, \qquad d\Omega_B = 3im\,\sigma \wedge \Omega_B \tag{4.1.1}$$

where m is at the same time the curvature of the internal space (more precisely, the Einstein condition is $R_{mn} = 4m^2 g_{mn}$), that of AdS_5, and give also the units of five-form flux. The integrability conditions on the structures for more general solutions were obtained in [24].

In M-theory there is no such a simple AdS_5 solution. The most well known solution is that of Maldacena and Nuñez [26], corresponding to the near horizon limit of M5-

[1] These were given in section 3.1 and we repeat them here for convenience:
$\omega \wedge \Omega = 0, \quad , \omega^{d/2} = \frac{(d/2)!}{2^{d/2}}(-1)^{\frac{d(d/2+1)}{4}} i^{d/2}\Omega \wedge \bar{\Omega}.$

[2] For full integrability all torsion classes are zero, while for weak integrability there is a torsion in a singlet representation of the structure group, proportional to the curvature of AdS.

[3] Here, we have restored the AdS_5 curvature m which we had set to 1 in chapter 3.

branes wrapped on holomorphic cycles of a Calabi-Yau 3-fold. More general solutions are studied in [95], and correspond topologically to fibrations of a two-sphere over a Kähler-Einstein base.

The effective five-dimensional gauged supergravity encodes the deformations of the background. When there is a G-structure, the moduli space of metric deformations is given by the deformations of the structures. Together with the moduli coming from the B-field and the R-R fields, they form, in the case of $\mathcal{N} = 2$ gauged supergravity, the hypermultiplets and vector multiplets of the effective theory.

In the generalized geometric language, metric degrees of freedom can also be encoded in bilinears of spinors (this time transforming under the the compact subgroup of the duality group, namely $USp(8)$ for the case of $E_{6(6)}$), and furthermore these can be combined with the degrees of freedom of the gauge fields such that the corresponding objects (called generalized structures[4]) transform in given representations of the $E_{d(d)}$ group. For eight supercharges in five dimensions the relevant generalized structures form a pair of objects (K, J_a), first introduced in [9]. In the next section we are going to give their explicit form, but for the moment let us explain their geometrical meaning.

The structure K transforms in the fundamental representation of $E_{6(6)}$ and it is a singlet under the $SU(2)$ R-symmetry group of the relevant effective supergravity theory. If K was to be built just as a bispinor (we will call that object $\mathcal{K}$, its explicit expression is given in (4.2.14)), then it would be a section of the right-hand side of (3.3.1) and it would not capture the non-trivial structure of the flux configuration on the internal manifold. Therefore, the proper generalized vector which transforms as a section of E is the dressed one

$$K = e^{\mu}\mathcal{K} \ . \tag{4.1.2}$$

This structure was called the V-structure (vector-multiplet structure) in [10] since it parametrizes the scalar fields of the vector multiplets in the effective theory.

The other algebraic structure, or rather an $SU(2)_R$ triplet of structures, describing the hypermultiplets (and thus called H-structure in [10]) is J_a, $a = 1,2,3$. It transforms in the adjoint of $E_{6(6)}$. As for K, we need the dressed object

$$J_a = e^{\mu}\mathcal{J}_a e^{-\mu} = e^{[\![\mu,\cdot]\!]}\mathcal{J}_a \tag{4.1.3}$$

where we are using $[\![\cdot,\cdot]\!]$ to denote the $\mathfrak{e}_{6(6)}$ adjoint action. These are normalized as[5]

$$\mathrm{Tr}(\mathcal{J}_a,\mathcal{J}_b) = 8\rho^2\delta_{ab} \tag{4.1.4}$$

where ρ will be related to the warp factor, and satisfy the $SU(2)$ algebra

$$[\![\mathcal{J}_a, \mathcal{J}_b]\!] = (4i\rho)\epsilon_{abc}\mathcal{J}_c \ . \tag{4.1.5}$$

As in Calabi-Yau compactifications where ω and Ω have to satisfy compatibility conditions to define a proper Calabi-Yau structure (see footnote 1), similar requirements apply here, and read

$$\mathcal{J}_a\,\mathcal{K} = 0 \ , \qquad c(\mathcal{K},\mathcal{K},\mathcal{K}) = 6\rho^3 \tag{4.1.6}$$

[4]In the case of $O(d,d)$ generalized geometry these are $Spin(d,d)$ pure spinors.

[5]We use the notation $\mathrm{Tr}(\cdot,\cdot)$ to denote the Killing form for $\mathfrak{e}_{6(6)}$.

where in the first expression we mean the adjoint action of $\mathcal{J}$ on $\mathcal{K}$, and in the second one c is the cubic invariant of $E_{6(6)}$. Since the above expressions are $E_{6(6)}$-covariant, they have exactly the same form if we replace $(\mathcal{K}, \mathcal{J}_a)$ with their dressed version (K, J_a).

4.1.2 Supersymmetry conditions

In the previous section we have introduced the generalized structures defining the backgrounds with eight supercharges off-shell, namely those that allow to define a five-dimensional (gauged) supergravity upon compactification. Here we discuss the integrability conditions that these backgrounds need to satisfy in order to preserve all eight supersymmetries leading to an AdS_5 geometry on the external space. The supersymmetry conditions were originally introduced in [11], and the relevant backgrounds called "exceptional Sasaki-Einstein" (the simplest case corresponding to Sasaki-Einstein manifolds). Here we will write the supersymmetry conditions in a slightly different way, and in section 4.2 we will use the fact that they are independent of the (generalized) connection[6] to choose a convenient one to verify them directly from the ten-dimensional supersymmetry conditions.

Compactifications to warped AdS_5 require, both in M-theory and in type IIB[7]

$$D\tilde{J}_a + \kappa\, \epsilon_{abc} \mathrm{Tr}(\tilde{J}_b, D\tilde{J}_c) = \lambda_a c(\tilde{K}, \tilde{K}, \cdot) \tag{4.1.7}$$

$$\mathbb{L}_{\tilde{K}} \tilde{K} = 0 \tag{4.1.8}$$

$$\mathbb{L}_{\tilde{K}} \tilde{J}_a = \frac{3i}{2} \epsilon_{abc} \lambda_b \tilde{J}_c \tag{4.1.9}$$

These equations involve the rescaled bispinors, which for type IIB are (the analogue expressions for M-theory are given in (4.3.1))

$$\tilde{K} = e^{-2\phi/3} K \ , \qquad \tilde{J}_a = e^{2A-2\phi} J_a \ , \tag{4.1.10}$$

where A is the warp factor and ϕ the dilaton. D is the derivative defined in (3.3.4), whose explicit index we have omitted, and corresponds to the direction missing in the cubic invariant[8]. The coefficient κ is related to the normalization of the structures and is given by

$$\frac{1}{\kappa} = i \|\tilde{J}_a\| \equiv i \sqrt{8 \mathrm{Tr}(\tilde{J}_a, \tilde{J}_a)} \tag{4.1.11}$$

and for type IIB is[9]

$$\kappa = -\frac{i}{4\sqrt{2}} e^{-3A+2\phi}. \tag{4.1.12}$$

Finally, λ_a are a triplet of constants related to the AdS_5 cosmological constant m by

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -2im \ . \tag{4.1.13}$$

[6]See [93] for details.

[7]In our conventions, (4.1.9) has a different sign in M-theory, see (4.3.3).

[8]To write this index explicitly we substitute $D \to D_M$, $c(\tilde{K}, \tilde{K}, \cdot) \to c_{MNP} \tilde{K}^N \tilde{K}^P$.

[9]Note that κ accounts for both the normalization of the internal spinors (see (4.2.4) below) and the rescalings (4.1.10) as can be seen by writing it as $\kappa = (8i\rho e^{2A-2\phi})^{-1}$.

Let us explain very briefly the meaning of these equations. For more details, see [10, 11]. The first equation which one can write in terms of the Dorfman derivative along a generic generalized vector,[10] implies that the moment map for the action of a generalized diffeomorphism along V takes a fixed value that involves the vector multiplet structure and the $SU(2)_R$ breaking parameters λ_a (AdS_5 vacua only preserve a $U(1)_R \in SU(2)_R$ [12, 96]), given by $\lambda_a J_a$. The second and third equation imply that $\tilde{K}$ is a generalized Killing vector of the background. Indeed, (4.1.8) implies that it leaves $\tilde{K}$ invariant, while (4.1.9) shows that the generalized diffeomorphism along $\tilde{K}$ amounts to an $SU(2)_R$ rotation of the J_a. This rotation does not affect the generalized metric which encodes all the bosonic degrees of freedom. Thus, the generalized vector $\tilde{K}$ was called "generalized Reeb vector" of the exceptional Sasaki-Einstein geometry.

As shown in [11], these conditions imply that these backgrounds are generalized Einstein, as the generalized Ricci tensor is proportional to the generalized metric.

We can compare these to the conditions coming from the five dimensional gauged supergravity [12]. More specifically, (4.1.9) corresponds to the hyperini variation, (4.1.8) corresponds to the gaugini, while (4.1.7) corresponds to a combination of the gravitini and the gaugini.

In the next section, we will give more details of the construction of H-and V structures in terms of internal spinors, and we show by an explicit calculation that AdS_5 compactifications preserving eight supercharges require conditions (4.1.7)-(4.1.9).

4.2 From Killing spinor equations to Exceptional Sasaki Einstein conditions

In this section we show that supersymmetry requires the integrability conditions (4.1.7)-(4.1.9).

4.2.1 The Reeb vector

Let us state again the basic conditions which we will require. We are interested in solutions of type IIB supergravity which

- respect the isometry group $SO(4,2)$ of AdS_5 and
- preserve 1/4 of the original supersymmetry, i.e. 8 supercharges.

According to the former condition, the ten-dimensional metric is written as (1.4.1) where $\tilde{g}_{\mu\nu}(x)$ is now the metric of AdS_5 and $g_{mn}(y)$ is the metric of the internal manifold, while the fluxes are of the form given in (1.4.11) so that they respect the democratic constraint (1.2.8) and where $F_{(n)}$ is purely an internal piece.

We start with the supersymmetry transformations of type IIB supergravity for the gravitino and the dilatino. These are given (in the democratic formulation) by (1.2.10) and (1.2.10) respectively.

[10]The expression is as follows

$$\kappa\, \epsilon_{abc} \mathrm{Tr} [\![\tilde{J}_b, \mathbb{L}_V \tilde{J}_c]\!] = \lambda_a c(\tilde{K}, \tilde{K}, V) \ .$$

The ansatz (1.4.9) gives the decomposition of the type IIB spinors (1.2.9) in terms of the external spinor ψ and the internal spinors χ_i. The spinor ψ is a Killing spinor for AdS_5 and therefore is required to satisfy (1.4.10) which involves the AdS_5 mass parameter m. The latter enters the Killing spinor equations for the internal spinors χ_i through the external component of the gravitino variation (1.2.10).

Combining the above information, the ten-dimensional Killing spinor equations translate into the conditions (1.4.12), (1.4.13) and (1.4.14). These equations contain all the information we need to derive the integrability conditions and we repeat them here for convenience:

$$\left[m - e^A(\partial\!\!\!/ A)\Gamma^6\Gamma_7 + i\frac{e^{\phi+A}}{4}\left((F\!\!\!/_1 + F\!\!\!/_5)\Gamma^6 - F\!\!\!/_3\right)\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{4.2.1}$$

$$\left[\nabla_m - \frac{1}{4}H\!\!\!/_m\Gamma^6 + i\frac{e^{\phi}}{8}\left(F\!\!\!/_1 + F\!\!\!/_5 - F\!\!\!/_3\Gamma^6\right)\Gamma_m\Gamma_{(7)}\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{4.2.2}$$

$$\left[(\partial\!\!\!/\phi)\Gamma^6\Gamma_{(7)} + \frac{1}{2}H\!\!\!/\Gamma_7 - \frac{ie^{\phi}}{2}\left(2F\!\!\!/_1\Gamma^6 - F\!\!\!/_3\right)\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0 \tag{4.2.3}$$

Now, let us mention some generic properties of IIB flux compactifications down to AdS_5 which are implied by the supersymmetry requirements. Although these statements can be proved without any reference to generalized geometry (as in [24]), we will postpone their proof until section 4.4.1 to see how nicely this formalism incorporates them. Here, we just state them.

The first property has to do with the norms of the internal spinors. From (4.4.8), we see that the two internal spinors have equal norms and from (4.4.11) that they scale as e^A:[11]

$$\chi_1^\dagger\chi_1 = \chi_2^\dagger\chi_2 \equiv \rho = \frac{e^A}{\sqrt{2}} \tag{4.2.4}$$

Moreover, (4.4.9) expresses the following orthogonality property

$$\chi_1^\dagger\chi_2 + \chi_2^\dagger\chi_1 = 0 \tag{4.2.5}$$

An important consequence of the supersymmetry conditions which will be crucial for the geometrical characterization of $\mathcal{M}$ is the existence of an isometry parametrized by a vector ξ [24], the so-called Reeb vector[12]. The components of ξ can be constructed from spinor bilinears as

$$\xi^m = \frac{1}{\sqrt{2}}(\chi_1^\dagger\gamma^m\chi_1 + \chi_2^\dagger\gamma^m\chi_2) \tag{4.2.6}$$

Actually, it turns out (see section 4.4.1) that ξ generates a symmetry of the full bosonic sector of the theory:

$$\mathcal{L}_\xi\{g, A, \phi, H, F_1, F_3, F_5\} = 0. \tag{4.2.7}$$

[11] Note that the ρ defined here is the same as the one appearing in the normalization condition of $\mathcal{J}$, Eq. (4.1.4).

[12] In the context of AdS/CFT, this isometry corresponds in the dual picture to the surviving R-symmetry of the $\mathcal{N} = 1$ gauge theory.

Using this, we can easily see that the Lie derivatives $\mathcal{L}_\xi \chi_i$ of the spinors satisfy the same equations (1.4.12) - (1.4.14) as the spinors themselves[13] and so they are proportional to them which means that they have definite charge. This charge is computed in appendix 4.4.1. From (4.4.31) we have

$$\mathcal{L}_\xi \chi_i = \frac{3im}{2} \chi_i \tag{4.2.8}$$

These conditions are very useful in proving the integrability conditions in the next section.

4.2.2 The H and V structures as bispinors

Let us now construct the H and V structures from the internal spinors, as appropriate $E_{6(6)}$ objects. For this, it is useful to decompose the group in its maximal compact subgroup $USp(8)$.[14]

The fundamental **27** (anti-fundamental $\overline{\mathbf{27}}$) representation is undecomposable, and corresponds to an antisymmetric 8×8 matrix $V^{\alpha\beta}$ ($Z^{\alpha\beta}$) which is traceless with respect to the symplectic form $C_{\alpha\beta}$ of $USp(8)$

$$\mathbf{27}\ , \quad V = V^{\alpha\beta}\ , \quad \text{such that } \mathrm{V}^{\alpha\beta}\mathrm{C}_{\alpha\beta} = 0 \tag{4.2.9}$$

The adjoint **78** representation corresponds to a symmetric 8×8 matrix and a fully antisymmetric rank 4 tensor

$$\mathbf{78} = \mathbf{36} + \mathbf{42}, \quad \mu = (\mu^{\alpha\beta}, \mu^{\alpha\beta\gamma\delta}) \tag{4.2.10}$$

The internal spinors (χ_1,χ_2) which are sections of $Spin(5) \cong USp(4)$, are combined into the following $USp(8)$ spinors

$$\theta_1 = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} \chi_1^c \\ \chi_2^c \end{pmatrix}. \tag{4.2.11}$$

In terms of the $USp(8)$ spinors θ_i, the normalization condition (4.2.4) implies

$$\theta_i^{*\alpha} \theta_{j,\alpha} = 2\rho\ \delta_{ij}. \tag{4.2.12}$$

Now, one can define the H and V structures as bispinors in a natural way. The triplet of H structures $\mathcal{J}_a$ are defined as

$$(\mathcal{J}_a)_\alpha^{\ \beta} = (\sigma_a)^{ij} \theta_{i,\alpha} \theta_j^{\star\beta} \tag{4.2.13}$$

where $\sigma_a = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Note that $\mathcal{J}_a$ have components only in the **36** piece of the **78**.

For the V structure, we have

$$\mathcal{K}^{\alpha\beta} = \mathcal{J}_0^{\alpha\beta} - \frac{1}{8} C^{\alpha\beta} C_{\delta\gamma} \mathcal{J}_0^{\gamma\delta}\ , \quad \text{with } (\mathcal{J}_0)_\alpha^{\ \beta} = \delta^{ij} \theta_{i,\alpha} \theta_j^{\star\beta} \tag{4.2.14}$$

---

[13] Here, note that the existence of the isometry is crucial for the Lie derivative to commute with the covariant one.

[14] Here, we just present some basic facts. More details are given in appendix C.

where $C^{\alpha\beta}$ is the charge conjugation matrix, which in our conventions is the symplectic form of $USp(8)$. Note that $\mathcal{K}$ is traceless by construction. From now on, we will drop the $USp(8)$ indices α, β in $\mathcal{K}$, $\mathcal{J}$.

The $\mathfrak{su}(2)$ algebra of the structures $\mathcal{J}_a$, Eq. (4.1.5), follows from the orthogonality and normalization of the spinors (4.2.12). Similarly we have

$$\mathcal{J}_a^2 = \mathcal{J}_0^2 = 2\rho\mathcal{J}_0 \tag{4.2.15a}$$

$$\mathcal{J}_0\mathcal{J}_a = \mathcal{J}_a\mathcal{J}_0 = 2\rho\mathcal{J}_a \tag{4.2.15b}$$

where ρ can also be related to the trace part of $\mathcal{J}_0$, namely

$$\rho = \frac{1}{4}\mathrm{Tr}[\mathcal{J}_0] \tag{4.2.16}$$

The fact that $\mathcal{J}_a$ and $\mathcal{J}_0$ commute translates in $E_{6(6)}$ language (by using (C.2.4)) into the compatibility condition (4.1.6).

In the following, it will turn out useful to have explicitly the $GL(5) \times SL(2)$ components of $\mathcal{K}$ and $\mathcal{J}_a$. For the former, using the decomposition of the **27** representation given in (3.3.2), we have:

$$\mathcal{K} = [\xi, (\zeta, \zeta_7), V, (R, R_7)] \ . \tag{4.2.17}$$

These can be organized in terms of a Clifford expansion as

$$\mathcal{K} = \frac{1}{2\sqrt{2}}\Big[i\xi_m\Gamma^{m67} + \zeta_m\Gamma^m + i\zeta_m^7\Gamma^{m7} + \frac{i}{2}V_{mn}\Gamma^{mn7}\Big] \tag{4.2.18}$$

where the various components can be obtained by taking appropriate traces with $\mathcal{K}$ [15]. In terms of bilinears involving the internal spinors χ_1 and χ_2 these components are

$$\begin{aligned}
\zeta^m &= \frac{1}{\sqrt{2}}(\chi_1^\dagger\gamma^m\chi_2 + \chi_2^\dagger\gamma^m\chi_1)\\
\zeta_7^m &= \frac{1}{\sqrt{2}}(-\chi_1^\dagger\gamma^m\chi_1 + \chi_2^\dagger\gamma^m\chi_2)\\
\xi^m &= \frac{1}{\sqrt{2}}(\chi_1^\dagger\gamma^m\chi_1 + \chi_2^\dagger\gamma^m\chi_2)\\
V^{mn} &= \frac{1}{\sqrt{2}}(\chi_1^\dagger\gamma^{mn}\chi_2 - \chi_2^\dagger\gamma^{mn}\chi_1)\\
R &= \frac{1}{\sqrt{2}}(\chi_1^\dagger\chi_1 - \chi_2^\dagger\chi_2)\\
R_7 &= \frac{1}{\sqrt{2}}(\chi_1^\dagger\chi_2 + \chi_2^\dagger\chi_1)
\end{aligned} \tag{4.2.19}$$

Note the absence of R and R_7 in the expansion (4.2.18). This is because these vanish as a consequence of the supersymmetry conditions that impose the two internal spinors to be orthogonal and have equal norm (see (4.2.4), (4.2.5)). Moreover, note that the

[15] For example, $\xi^m = \frac{1}{2\sqrt{2}}\mathrm{Tr}[\mathcal{K}\Gamma^{m67}]$.

vector component ξ of $\mathcal{K}$ appearing in the above expression is the Reeb vector given in (4.2.6).

For the particular case of Sasaki-Einstein manifolds, where $\chi_2 = i\chi_1$, also the one-forms ζ and ζ_7 are zero, while the two-form V corresponds to $*(\sigma \wedge \omega_B)$.[16] The holomorphic 2-form of the base Ω_B is instead embedded in J_a, to which we now turn.

The triplet $\mathcal{J}_a$ is in the $\mathbf{36}$ representation of $USp(8)$, which decomposes under $GL(5) \times SL(2)$ as

$$\mathbf{36} = (\mathbf{5},\mathbf{1}) + (\mathbf{10},\mathbf{1}) + (\mathbf{1},\mathbf{1}) + (\mathbf{10},\mathbf{2}) \ . \tag{4.2.20}$$

The Clifford expansion of $\mathcal{J}_a$ is [17]

$$\mathcal{J}_a = -\frac{1}{8}\Big[\mathcal{J}_a^{m6}\Gamma_{m6} + \frac{1}{2}\mathcal{J}_a^{mn}\Gamma_{mn} - \mathcal{J}_a^{7}\Gamma_7 + \frac{1}{2}\mathcal{J}_a^{mn6}\Gamma_{mn6} + \frac{1}{6}\mathcal{J}_a^{mnp}\Gamma_{mnp}\Big] \tag{4.2.21}$$

where each piece is given by the first terms in (C.3.5).

In particular, one can identify in the expansion (4.2.21) all possible spinor bilinears with non-zero charge under ξ[18]

$$\begin{array}{rcl} \mathcal{J}_+^{m6} &=& 4\chi_1^T\gamma^m\chi_2 \\ \mathcal{J}_+^{mn} &=& -2(\chi_1^T\gamma^{mn}\chi_1 + \chi_2^T\gamma^{mn}\chi_2) \\ \mathcal{J}_+^{mn6} &=& -2(\chi_1^T\gamma^{mn}\chi_1 - \chi_2^T\gamma^{mn}\chi_2) \\ \mathcal{J}_+^{mnp} &=& -4\chi_1^T\gamma^{mnp}\chi_2 \\ \mathcal{J}_+^{7} &=& 4i\chi_1^T\chi_2 \end{array} \tag{4.2.22}$$

where we have defined

$$\mathcal{J}_\pm = \mathcal{J}_1 \pm i\mathcal{J}_2 \ . \tag{4.2.23}$$

The components of $\mathcal{J}_-$ have exactly the same form with the replacement $\chi_i \to \chi_i^c$ and an overall minus sign in the above expressions.[19] On the other hand, $\mathcal{J}_3$ is neutral since it is constructed from two oppositely charged spinors (χ and $\chi^\dagger$). The explicit expressions for the related bilinears are

$$\begin{array}{rcl} \mathcal{J}_3^{m6} &=& 2(-\chi_1^\dagger\gamma^m\chi_2 + \chi_2^\dagger\gamma^m\chi_1) \\ \mathcal{J}_3^{mn} &=& 2(\chi_1^\dagger\gamma^{mn}\chi_1 + \chi_2^\dagger\gamma^{mn}\chi_2) \\ \mathcal{J}_3^{mn6} &=& 2(\chi_1^\dagger\gamma^{mn}\chi_1 - \chi_2^\dagger\gamma^{mn}\chi_2) \\ \mathcal{J}_3^{mnp} &=& 2(\chi_1^\dagger\gamma^{mnp}\chi_2 + \chi_2^\dagger\gamma^{mnp}\chi_1) \\ \mathcal{J}_3^{7} &=& 2i(-\chi_1^\dagger\chi_2 + \chi_2^\dagger\chi_1) \end{array} \tag{4.2.24}$$

Together with those coming from $\mathcal{K}$ (4.2.19), these form the set of spinor bilinears which are neutral under the Killing vector ξ. Moreover, note that expansions similar to (4.2.18) and (4.2.21) can be done for the rescaled bispinors $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{J}}$.

---

[16] The Reeb vector ξ and the contact structure σ satisfy $\iota_\xi\sigma = 1$.

[17] We use the notation $\mathcal{J}_a^{(I)} = \mathrm{Tr}[\mathcal{J}_a\Gamma^{(I)}]$, $a = 1,2,3$ where (I) is a collection of indices.

[18] Our notation is $\chi^T\gamma\chi' = \chi_\alpha\gamma^{\alpha\beta}\chi'_\beta$ and $\chi^\dagger\gamma\chi' = \chi^{*\alpha}\gamma_\alpha{}^\beta\chi'_\beta$ for a Cliff(5) element γ and two $Spin(5)$ spinors χ and χ'.

[19] For example, we have $\mathcal{J}_-^{m6} = -4\chi_1^{cT}\gamma^m\chi_2^c$.

4.2.3 Proof of the generalized integrability conditions

In this section we describe the general methodology used to prove the generalized integrability conditions (4.1.7)-(4.1.9) from the Killing spinor equations (1.4.12)-(1.4.14), while we relegate the details to sections 4.5 and 4.6.

Killing spinor equations

In order to use the supersymmetry conditions efficiently, we need to turn the Killing spinor equations (1.4.12)-(1.4.14) into equations on $\mathcal{J}_a$ and $\mathcal{J}_0$. This can be done easily by taking the complex conjugate and transpose of the former. From the equation coming from requiring that the variation of the external gravitino equation vanishes, Eq. (1.4.12), we get

External gravitino

$$m\mathcal{J}_\pm = \pm\mathcal{J}_\pm G^E \tag{4.2.25a}$$

$$m\mathcal{J}_3 = -\mathcal{J}_0 G^E \tag{4.2.25b}$$

$$m\mathcal{J}_0 = -\mathcal{J}_3 G^E \tag{4.2.25c}$$

where

$$G^E = e^A\left[(\partial\!\!\!/ A)\Gamma^6\Gamma_7 + i\frac{e^\phi}{4}\left((F\!\!\!/_1 + F\!\!\!/_5)\Gamma^6 - F\!\!\!/_3\right)\right] \tag{4.2.25d}$$

From the requirement that the variation of the internal component of the gravitino vanishes, Eq. (1.4.13), we get

Internal gravitino

$$\nabla_m\mathcal{J}_a = [\mathcal{J}_a, G_m^{IS}] + \{\mathcal{J}_a, G_m^{IA}\}, \quad a = 0,1,2,3 \tag{4.2.26a}$$

where

$$G_m^{IS} = -\frac{1}{4}H\!\!\!/_m\Gamma^6 + \frac{ie^\phi}{8}(F_{1,m} + F\!\!\!/_{3,m}\Gamma^6)\Gamma_7 - \frac{e^\phi}{8}(*F_5)\Gamma_{m6} \tag{4.2.26b}$$

$$G_m^{IA} = \frac{ie^\phi}{8}(F^p\Gamma_{mp} + \frac{F^{npq}}{3!}\Gamma_{mnpq}\Gamma^6)\Gamma_7 \tag{4.2.26c}$$

From requiring that the dilatino stays invariant, Eq. (1.4.14), we get

Dilatino

$$\mathcal{J}_a G^D = 0, \quad a = 0,1,2,3 \tag{4.2.27a}$$

where

$$G^D = \left[(\partial\!\!\!/\phi)\Gamma^6\Gamma_{(7)} - \frac{1}{2}H\!\!\!/\Gamma_7 + \frac{ie^\phi}{2}\left(2F\!\!\!/_1\Gamma^6 - F\!\!\!/_3\right)\right] \tag{4.2.27b}$$

Integrability conditions

Now, we are ready to prove the integrability conditions (4.1.7)-(4.1.9) for the H and V structures. These are given in terms of the dressed objects J_a, K, but it turns out to be more tractable to work with the undressed objects $\mathcal{J}$, $\mathcal{K}$, in particular since the gauge fields and the derivative satisfy

$$\tilde{\mu}\, D \equiv (\mu + \frac{2\phi}{3})D = 0 \tag{4.2.28}$$

where $\tilde{\mu}$ is an element of $\mathfrak{e}_{6(6)} \oplus \mathbb{R}^+$. The dilaton appears here due to the way it embeds in the $GL(5)$ piece in the adjoint action (see (3.3.6c), (3.3.6d)), and it reflects the fact that the (anti) fundamental representation is actually charged under the $\mathbb{R}^+$, i.e. we are working with objects which are dressed under the trombone (see (3.3.4) and (4.1.10)).

We will also use a crucial trick: the generalized integrability conditions stem from the generalized Lie derivative operation (3.2.7), which is independent of the generalized connection, as long as it is torsion free [93]. Thus, instead of embedding the partial derivative into the generalized derivative as in (3.3.4), we are going to embed the covariant derivative, namely we will use as generalized connection the ordinary Levi-Civita connection. We thus have

$$D_{m6} = e^{2\phi/3}\nabla_m \ . \tag{4.2.29}$$

J_a equations

Let us start with the moment map condition for the hyper-multiplet structure, Eq. (4.1.7), that we repeat here

$$D\tilde{J}_a + \kappa\, \epsilon_{abc}\mathrm{Tr}(\tilde{J}_b, D\tilde{J}_c) = \lambda_a c(\tilde{K},\tilde{K},\cdot) \tag{4.2.30}$$

When undressing J_a, each term on the left hand side contributes two terms, one where the derivative is acting on the naked $\mathcal{J}$, and another one with the derivative acting on μ. Acting on the whole equation by $e^{-\mu}$ to undress it, we get the twisted moment map densities M_a

$$\begin{aligned} M_a \equiv e^{-\mu}\left(D\tilde{J}_a + \kappa\, \epsilon_{abc}\mathrm{Tr}(\tilde{J}_b, D\tilde{J}_c)\right) = \\ D\tilde{\mathcal{J}}_a + [\![D\mu, \tilde{\mathcal{J}}_a]\!] + \kappa\epsilon_{abc}\mathrm{Tr}[\![\tilde{\mathcal{J}}_b, D\tilde{\mathcal{J}}_c]\!] + \kappa\epsilon_{abc}\mathrm{Tr}[\![\tilde{\mathcal{J}}_b, [\![D\mu, \tilde{\mathcal{J}}_c]\!]]\!] \end{aligned} \tag{4.2.31}$$

where in analogy with their twisted counterparts (4.1.10), we have defined the rescaled bispinors

$$\tilde{\mathcal{J}}_a = e^{2A-2\phi}\mathcal{J}_a \ , \qquad \tilde{\mathcal{K}} = e^{-2\phi/3}\mathcal{K} \ . \tag{4.2.32}$$

We are going to perform this calculation in $USp(8)$ basis, where the derivative D has components (cf. (C.3.2b))

$$D^{\alpha\beta} = \frac{ie^{2\phi/3}}{2\sqrt{2}}(\Gamma^{m67})^{\alpha\beta}\nabla_m \equiv (v^m)^{\alpha\beta}\nabla_m \tag{4.2.33}$$

where for later use we have defined the generalized vector v, which has only a vectorial component along direction of the generalized derivative. We then get that (4.2.31) reads, in $USp(8)$ basis

$$M_a = [\nabla_m \tilde{\mathcal{J}}_a, v^m] + ([\![\nabla_m \mu, \tilde{\mathcal{J}}_a]\!] v^m) + \mathrm{Tr}[\tilde{\mathcal{J}}_a G_m^{IS}] v^m - \mathrm{Tr}[(\nabla_m \mu)\tilde{\mathcal{J}}_a)] v^m \ . \qquad (4.2.34)$$

Here we have used the fact that the $\mathcal{J}_a$ contain only a **36** component (and thus the Killing form (C.2.11) just reduces to a matrix trace) and in the third and fourth terms we have used the $\mathfrak{su}(2)$ algebra (4.1.5). For the third term we also used the internal gravitino equation (4.2.26). The commutators $[\ ,\ ]$ and the traces are now understood as matrix commutators and traces respectively ($v^m \propto \Gamma^{m67}$). The second term means the action of the adjoint element $[\![\nabla_m \mu, \tilde{\mathcal{J}}_a]\!]$ on the fundamental v^m.

Although (4.2.34) seems not to be gauge-invariant (μ contains the gauge fields), this is not the case since the second and the fourth term together project onto the exterior derivative of the gauge fields, i.e. the fluxes. Using the internal and external gravitino equations (4.2.26) and (4.2.25) as well as the dilatino equations (4.2.27), we find (see section 4.5 for the details of this computation)

$$M_\pm = 0 \qquad (4.2.35a)$$

$$M_3 = (-2im)\rho e^{-4\phi/3}\mathcal{K} \qquad (4.2.35b)$$

We thus verify the $\pm$ components of the moment map equations (4.2.30), for the choice $\lambda_\pm = 0$, in agreement with (4.1.13). The third component M_3, should be, according to (4.2.30) and (4.1.13) proportional to the dual vector of K through the cubic invariant. Indeed, one can check using the explicit form of $\mathcal{K}$ in terms of spinors (4.2.14), as well as the spinor normalizations (4.2.15a) and the definition of the rescaled $\mathcal{K}$ (4.2.32) that

$$\left[c(\tilde{\mathcal{K}},\tilde{\mathcal{K}},\cdot)\right]^{\alpha\beta} = \rho e^{-4\phi/3}\mathcal{K}^{\alpha\beta} \ . \qquad (4.2.36)$$

We therefore verify the third component of the moment map equation with $\lambda_3 = -2im$, in accordance to (4.1.13).

K and compatibility equations

We rewrite here the integrability condition for K and the condition coming from requiring compatibility of the integrable H and V structures, Eqs (4.1.8) and (4.1.9)

$$\mathbb{L}_{\tilde{K}}\tilde{K} = 0 \qquad (4.2.37)$$

$$\mathbb{L}_{\tilde{K}}\tilde{J}_a = \frac{3i}{2}\epsilon_{abc}\lambda_b\tilde{J}_c \ . \qquad (4.2.38)$$

They both contain the Dorfman derivative along the (rescaled) twisted generalized vector $\tilde{K} = e^{-2\phi/3}K = e^{-2\phi/3}(e^\mu\mathcal{K})$. As before, it is convenient to split the contributions coming from the derivative of μ from the rest. Using the expression for the Dorfman derivative (3.2.7), one gets

$$e^{-\mu}\mathbb{L}_{\tilde{K}} = (\tilde{\mathcal{K}}\cdot v)\Big(\nabla + \nabla\mu\Big) - \Big(v \times (\nabla\tilde{\mathcal{K}} + (\nabla\mu)\tilde{\mathcal{K}})\Big) \qquad (4.2.39)$$

where the generalized vector v along the direction of the derivative D was defined in (4.2.33). The first and third term are the same as in $\mathbb{L}_{\tilde{\mathcal{K}}}$, while with the second and the fourth we define a twisted Dorfman derivative $\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}}$, namely

$$\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}} \equiv e^{-\mu}\mathbb{L}_{\tilde{K}} = \mathbb{L}_{\tilde{\mathcal{K}}} + (\tilde{\mathcal{K}}\cdot v)\nabla\mu - \, v \times ((\nabla\mu)\tilde{\mathcal{K}}) \ . \tag{4.2.40}$$

Using this twisted derivative, we can now rewrite the integrability conditions (4.2.37) and (4.2.38) as equations on the undressed structures $\mathcal{K}$ and $\mathcal{J}$ (or rather their rescaled versions $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{J}}$ defined in (4.2.32)) as follows

$$\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}}\tilde{\mathcal{K}} = 0 \tag{4.2.41}$$

$$\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}}\tilde{\mathcal{J}}_a = \frac{3i}{2}\epsilon_{abc}\lambda_b\tilde{\mathcal{J}}_c \tag{4.2.42}$$

These equations turn out to be very simple using the fact that the twisted Dorfman derivative along $\tilde{\mathcal{K}}$ on spinor bilinears actually reduces to the usual Lie derivative along the vector part of $\mathcal{K}$ [10], namely the Killing vector ξ defined in (4.2.6)

$$\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}} = \mathcal{L}_\xi \qquad \text{on bispinors .} \tag{4.2.43}$$

Let us show briefly why this is so. The derivative acting on a generic element can be split as in a differential operator, corresponding to the first term in (3.2.7), and the rest, which is an algebraic operator from the point of view of the element that it acts on:

$$\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}} = (\tilde{\mathcal{K}}\cdot v)\nabla + \mathcal{A} \tag{4.2.44}$$

The first piece reduces to the directional derivative along the Killing vector ξ. For the algebraic part, we decompose the operator $\mathcal{A}$, which acts in the adjoint, into the $USp(8)$ pieces

$$\mathcal{A} = \mathcal{A}|_{36} + \mathcal{A}|_{42} \tag{4.2.45}$$

and we have furthermore that $\mathcal{A}|_{36}$ can be viewed as an element of Cliff(6). We show in section 4.6 that supersymmetry implies that

$$\mathcal{A}|_{36} = \frac{1}{4}(\nabla_m\xi_n)\Gamma^{mn}, \qquad \mathcal{A}|_{42} = 0 \tag{4.2.46}$$

Now let us consider the action of $\widehat{\mathbb{L}}_{\mathcal{K}}$ on $\mathcal{K}$ and $\mathcal{J}_a$. These are respectively in the **27** and **36** of $USp(8)$, and combined they form the **63**, the representation of hermitean traceless bispinors, and thus we have simply

$$\mathcal{A}\,\mathcal{K} = \frac{1}{4}(\nabla_m\xi_n)[\Gamma^{mn},\mathcal{K}], \qquad [\![\mathcal{A},\mathcal{J}_a]\!] = \frac{1}{4}(\nabla_m\xi_n)[\Gamma^{mn},\mathcal{J}_a] \tag{4.2.47}$$

where the commutators are just gamma matrix commutators. Together with the directional derivative along ξ from the first term in (4.2.44), we conclude that $\widehat{\mathbb{L}}_{\tilde{\mathcal{K}}} = \mathcal{L}_\xi$.

Using this, it is very easy to show (4.2.41) and (4.2.42). Given that the $Spin(5)$ spinors have a definite charge under this action, Eq. (4.2.8), the $USp(8)$ spinors $\theta_{1,2}$ have charges $\pm(3im/2)$ and therefore the bispinors satisfy

$$\mathcal{L}_\xi\mathcal{J}_\pm = \pm 3im\mathcal{J}_\pm \quad \text{and} \quad \mathcal{L}_\xi\mathcal{J}_3 = \mathcal{L}_\xi\mathcal{K} = 0 \tag{4.2.48}$$

from which one can immediately verify (4.2.41) and (4.2.42).

Before closing this section, let us note that the fact that the twisted generalized Lie derivative along $\tilde{\mathcal{K}}$ reduces to an ordinary Lie derivative along its vector part is actually a generic feature of "generalized Killing vectors"[20]: it can be shown that if a generalized vector is such that the generalized Lie derivative along that vector on the objects defining the background –generalized metric for a generic background, and spinors or spinor bilinears for a supersymmetric one– vanishes, then the Dorfman derivative along such a generalized vector reduces to an ordinary Lie derivative along its vector component [97].

4.3 The M-theory analogue

The rescaled structures for M-theory are

$$\tilde{K} = K \ , \qquad \tilde{J}_a = e^{2A} J_a \ , \tag{4.3.1}$$

having the same form as for type IIB but with a vanishing dilaton.

Equations (4.1.7) and (4.1.8) have exactly the same form as in the type IIB case, with

$$\kappa = -\frac{i}{4\sqrt{2}} e^{-3A} = (8i\rho e^{2A})^{-1} \tag{4.3.2}$$

while (4.1.9) has a different sign in our conventions, i.e.

$$\mathbb{L}_{\tilde{K}} \tilde{J}_a = -\frac{3i}{2} \epsilon_{abc} \lambda_b \tilde{J}_c \tag{4.3.3}$$

where again $\lambda_1 = \lambda_2 = 0, \lambda_3 = -2im$. This sign difference is due to the fact the internal spinor has opposite charge compared to the type IIB case (cf. (4.4.39)).

The supersymmetry variation of the gravitino (up to quadratic terms) reads[21]

$$\delta\Psi_M = \nabla_M \epsilon + \frac{1}{288}\Big(\tilde{\Gamma}_M{}^{NPQR} - 8\delta_M^N \tilde{\Gamma}^{PQR}\Big) G_{NPQR}\epsilon \tag{4.3.4}$$

where $G = dC$ and ϵ is the eleven-dimensional (Majorana) supersymmetry parameter.

The eleven-dimensional metric is written again in the diagonal form

$$ds^2 = e^{2A(y)} \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu + g_{ab}(y) dy^a dy^b \tag{4.3.5}$$

where now the internal metric g_{ab}[22] is six-dimensional and the spinor decomposition ansatz for M-theory compactifications reads

$$\epsilon = \psi \otimes \theta + \psi^c \otimes \theta^c \tag{4.3.6}$$

[20] We thank C. Strickland-Constable for sharing this with us.

[21] We use tildes for the eleven-dimensional gamma-matrices (see appendix B).

[22] We use a,b,c,... to describe representations of the $GL(6)$ group of diffeomorphisms of the internal manifold. Moreover, we will suppress from now on the $SU(2)_R$ adjoint index a in J_a in order to avoid confusion with the $GL(6)$ ones.

where θ is a complex 8-component spinor on the internal manifold. Finally, the field strength G is allowed to have only internal components in order to respect the isometries of AdS_5.

There is again a vector field ξ which generates a symmetry of the full bosonic sector:

$$\mathcal{L}_\xi\{g, A, G\} = 0, \tag{4.3.7}$$

where ξ is now given by

$$\xi^a = \frac{i}{\sqrt{2}}\theta^\dagger\Gamma^{a7}\theta \tag{4.3.8}$$

One can construct the H and V structures in exactly the same way as for the type IIB case. In particular, the expressions (4.2.12) to (4.2.16) have exactly the same form where

$$\theta_1 = \theta\ , \qquad \theta_2 = \theta^c\ . \tag{4.3.9}$$

However, the θ_i are not constructed from two $Spin(5)$ spinors as in type IIB.

The decomposition of the supersymmetry variation (4.3.4) in external and internal pieces is similar to the type IIB case with the difference that here we do not have a dilatino variation. In terms of $\mathcal{J}_a$ and $\mathcal{J}_0$, we get the differential condition

$$\nabla_a\mathcal{J} = [\mathcal{J},G_a^{IS}] + \{\mathcal{J},G_a^{IA}\}, \quad \mathcal{J} = \mathcal{J}_\pm, \mathcal{J}_3, \mathcal{J}_0 \tag{4.3.10}$$

where

$$G_a^{IS} = -\frac{1}{36}G_{abcd}\Gamma^{bcd}, \qquad G_a^{IA} = -\frac{i}{12}(\star G)_{ab}\Gamma^{b7} \tag{4.3.11}$$

and the algebraic ones

$$m\mathcal{J}_\pm = \pm\mathcal{J}_\pm G^E \tag{4.3.12}$$

$$m\mathcal{J}_3 = -\mathcal{J}_0 G^E \tag{4.3.13}$$

$$m\mathcal{J}_0 = -\mathcal{J}_3 G^E \tag{4.3.14}$$

where now G^E is given by

$$G^E = e^A\left[(\partial\!\!\!/ A)\Gamma^7 + \frac{i}{12}(\star G)_{ab}\Gamma^{ab}\right] \tag{4.3.15}$$

The Clifford expansion for $\mathcal{K}$ is now

$$\mathcal{K} = \frac{1}{2\sqrt{2}}\Big[\zeta_a\Gamma^a + i\xi_a\Gamma^{a7} + \frac{i}{2}V_{ab}\Gamma^{ab7}\Big] \tag{4.3.16}$$

where the components correspond to the different pieces in the $SL(6)$ decomposition of the fundamental, Eq. (3.3.7), and correspond to the following spinor bilinears

$$\zeta^a = \frac{1}{\sqrt{2}}\theta^\dagger\Gamma^a\theta, \qquad V^{ab} = \frac{i}{\sqrt{2}}\theta^\dagger\Gamma^{ab7}\theta \tag{4.3.17}$$

and the vector ξ is the Killing vector defined in (4.3.8).

For the triplet $\mathcal{J}$, the expansion reads

$$\mathcal{J} = -\frac{1}{8}\Big[\frac{1}{2}\mathcal{J}^{ab}\Gamma_{ab} - \mathcal{J}^7\Gamma_7 + \frac{1}{6}\mathcal{J}^{abc}\Gamma_{abc}\Big] \tag{4.3.18}$$

where now the the analogue of the (4.2.20) split under $GL(6)$ is

$$\mathbf{36} = \mathbf{15} + \mathbf{1} + \mathbf{20} \tag{4.3.19}$$

The components of $\mathcal{J}_+$ are given by the following spinor bilinears, all charged under ξ

$$\mathcal{J}_+^{ab} = -2\theta^T\Gamma^{ab}\theta, \qquad \mathcal{J}_+^{abc} = -2\theta^T\Gamma^{abc}\theta, \qquad \mathcal{J}_+^7 = -2\theta^T\Gamma^7\theta \tag{4.3.20}$$

and the corresponding expressions for $\mathcal{J}_-$ are given by the replacement $\theta \to \theta^c$ and an overall minus sign. For $\mathcal{J}_3$, the analogous expressions are

$$\mathcal{J}_3^{ab} = 2\theta^\dagger\Gamma^{ab}\theta, \qquad \mathcal{J}_3^{abc} = 2\theta^\dagger\Gamma^{abc}\theta, \qquad \mathcal{J}_3^7 = 2\theta^\dagger\Gamma^7\theta \tag{4.3.21}$$

The procedure to prove the integrability conditions is the same as the one described in subsection 4.2.3 for type IIB. In particular, we again work with the undressed structures $\mathcal{K}$ and $\mathcal{J}$ and with the twisted moment map density and the twisted Dorfman derivative defined in (4.2.31) and (4.2.39) respectively for type IIB. We leave the details of this calculation to the appendices. The key point that the twisted Dorfman derivative along $\mathcal{K}$ reduces to the ordinary Lie derivative along ξ, Eq. (4.2.43), is also true here and from (4.4.39), we get

$$\mathcal{L}_\xi\mathcal{J}_\pm = \mp 3im\mathcal{J}_\pm \quad \text{and} \quad \mathcal{L}_\xi\mathcal{J}_3 = \mathcal{L}_\xi\mathcal{K} = 0 \tag{4.3.22}$$

4.4 Some constraints from supersymmetry

In this section we are going to prove some useful conditions that the spinor bilinears in (4.2.22), (4.2.24), (4.2.19), (4.3.20), (4.3.21), (4.3.17) and (4.3.8) satisfy and which serve as an intermediate step in order to derive the integrability conditions (4.1.7)-(4.1.9). The most important relations have also been stated and used before. Here we prove them. We split into the bilinears in type IIB, and those of M-theory.

4.4.1 Type IIB

Let us start by studying the vector ξ defined in (4.2.6). By tracing (4.2.26) with Γ_{n67}, we get

$$\nabla_m\xi_n = -\frac{1}{2}\zeta_7^p H_{mnp} + \frac{e^\phi}{4}\zeta^p F_{mnp} + \frac{e^\phi}{4}(*V)_{mnp}F^p + \frac{e^\phi}{4}V_{mn}(*F_5) \tag{4.4.1}$$

Since the right hand side is antisymmetric, we have $\nabla_{(m}\xi_{n)} = 0$ and therefore ξ is a Killing vector:

$$\mathcal{L}_\xi g = 0 \tag{4.4.2}$$

Actually ξ is more than an isometry. By taking $0 = \mathrm{Tr}[\mathcal{J}_0 G^D] = \mathrm{Tr}[\mathcal{J}_0 G^D\Gamma^7]$ from (4.2.27a), we obtain

$$\mathcal{L}_\xi\phi = \mathcal{L}_\xi C_0 = 0 \tag{4.4.3}$$

and by using the Bianchi identity for F_1 we get

$$\mathcal{L}_\xi F_1 = 0\ . \tag{4.4.4}$$

Moreover, by taking the trace of (4.2.25b), we get

$$\mathcal{L}_\xi A = 0 \; . \tag{4.4.5}$$

Using that $\mathrm{Tr}[\mathcal{J}_a G^D \Gamma^6] = \mathrm{Tr}[\mathcal{J}_a G^D \Gamma^{67}] = 0$, we get

$$\mathcal{J}_a^{mn}(*H)_{mn} = 0, \quad a = 1{,}2{,}3 \tag{4.4.6}$$

$$\mathcal{J}_a^{mn}(*F_3)_{mn} = 0, \quad a = 1{,}2{,}3 \tag{4.4.7}$$

By tracing (4.2.25c) with Γ^6 we also get that

$$R = 0 \tag{4.4.8}$$

Then, by tracing (4.2.25c) with Γ^{67} and using (4.4.7) with $a = 3$, we have

$$R_7 = 0 \tag{4.4.9}$$

The power of the warp factor in the norm of the spinors also comes from supersymmetry. By tracing (4.2.26) for $a = 0$, we get

$$\partial_m \rho = \frac{e^\phi}{4\sqrt{2}} V_{mn} F^n - \frac{e^\phi}{4\sqrt{2}} \zeta^n (*F_3)_{mn} \tag{4.4.10}$$

The right-hand side can be related to the warp factor by tracing (4.2.25b) with Γ_{m67} which yields[23]

$$\partial_m \rho - \rho \partial_m A = 0 \quad \Rightarrow \quad \rho = c\, e^A \tag{4.4.11}$$

and we chose $c = 1/\sqrt{2}$. Let us now show that the Lie derivative along ξ acting on the rest of the fluxes H,F_3 and F_5 vanishes. By tracing (4.2.26) for $a = 0$ with Γ_{n7} and antisymmetrizing over $[mn]$, we get

$$\nabla_{[m}\zeta^7_{n]} = -\frac{1}{2}\xi^p H_{mnp} \quad \Rightarrow \quad d(\iota_\xi H) = 0 \tag{4.4.12}$$

which by the Bianchi identity for H yields

$$\mathcal{L}_\xi H = 0 \tag{4.4.13}$$

The situation for F_3 is slightly more complicated due to the non-standard Bianchi identity it satisfies. By tracing (4.2.26) for $a = 0$ with Γ_n and antisymmetrizing over $[mn]$, we get

$$\nabla_{[m}\zeta_{n]} = -\frac{1}{4}(*V)_{pq[m}H_{n]}{}^{pq} - \frac{e^\phi}{4}\xi^p F_{mnp} - \frac{1}{2\sqrt{2}}\rho e^\phi (*F_3)_{mn} \tag{4.4.14}$$

We eliminate the H-term using $0 = \mathrm{Tr}[\mathcal{J}_0 G^D \Gamma_{mn67}]$ from (4.2.27) and we get

$$d\zeta = d\phi \wedge \zeta - e^\phi F_1 \wedge \zeta^7 - 2e^\phi \iota_\xi F_3 \tag{4.4.15}$$

[23]The integration constant is chosen so that it reproduces the standard value of the charge of the spinors, see (4.4.31).

Taking the exterior derivative of this expression, replacing again $\iota_\xi F_3$ from (4.4.15) and using (4.4.12), we get

$$d\iota_\xi F_3 + F_1 \wedge \iota_\xi H = 0 \tag{4.4.16}$$

The second term is equal to $\iota_\xi dF_3$ as can be seen from the RR Bianchi identities $dF_1 = 0$ and $dF_3 = H \wedge F_1$. Thus, (4.4.16) becomes simply

$$\mathcal{L}_\xi F_3 = 0 \tag{4.4.17}$$

In order to compute the the Lie derivative along ξ on F_5, we first need $\mathcal{L}_\xi \mathcal{J}_3^7$. By tracing (4.2.26) with Γ_7, we get for $a = 1,2,3$

$$\partial_m \mathcal{J}_a^7 = -\frac{1}{4}\mathcal{J}_a^{np67} H_{mnp} + \frac{ie^\phi}{8}\mathcal{J}_a^{np6} F_{mnp} + \frac{ie^\phi}{4}\mathcal{J}_{mp}^a F^p \tag{4.4.18}$$

and using $0 = \mathrm{Tr}[\mathcal{J}_a G^D \Gamma^{m6}]$ from (4.2.27), we get

$$\partial_m \mathcal{J}_a^7 = \mathcal{J}_a^7 \partial_m \phi + \frac{3ie^\phi}{8}\mathcal{J}_a^{np6} F_{mnp} - \frac{3ie^\phi}{4}\mathcal{J}_{mp}^a F^p \tag{4.4.19}$$

If we trace (4.2.25c) with Γ^{m6} and replace in the above equation for $a = 3$, we get

$$\partial_m \mathcal{J}_3^7 = \mathcal{J}_3^7 \partial_m(\phi - 3A) \quad \Rightarrow \quad \mathcal{L}_\xi \mathcal{J}_3^7 = 0 \tag{4.4.20}$$

where (4.4.3) and (4.4.5) were used. Now, it is easy to compute $\mathcal{L}_\xi F_5$. Taking the trace of (4.2.25b) with Γ_7 and using (4.4.11) gives

$$m\mathcal{J}_3^7 = -\frac{e^{\phi+2A}}{2\sqrt{2}}(*F_5) \tag{4.4.21}$$

Taking the Lie derivative along ξ on both sides and using (4.4.2), (4.4.3), (4.4.5) and (4.4.20), we get

$$\mathcal{L}_\xi F_5 = 0 \tag{4.4.22}$$

Finally, let us also state another relation which will be useful later. This is easily derived by tracing (4.2.26) for $a = 0$ with Γ_{mn7} and eliminating the H-term using $0 = \mathrm{Tr}[\mathcal{J}_0 G^D \Gamma_{n6}]$. We get

$$\nabla^m V_{mn} = V_{mn}\partial^m \phi - e^\phi \zeta_7^m (*F_3)_{mn} + \zeta^m (*H)_{mn} - \xi_n (*F_5) \tag{4.4.23}$$

The spinor charges

Here, we compute the charge q of the spinors χ_i under the U(1) generated by the Killing vector ξ. Actually, it turns out that it is more convenient to compute first $2q$, i.e. the charge of some charged spinor bilinear (we choose $\mathcal{J}_+^7$), and then divide by 2. In order to do that, we first need to derive some identities. Multiplying (B.29) with $(\mathcal{J}_a\Gamma_7)_{\beta\alpha}\mathcal{J}_0^{\delta\gamma}$ and using $\mathcal{J}_0\mathcal{J}_a = 2\rho\mathcal{J}_a$, we get for $a = 1,2,3$

$$\mathcal{J}_a^{ab}\mathrm{Tr}[\mathcal{J}_0\Gamma_{ab7}] = -16\mathrm{Tr}[\mathcal{J}_0\mathcal{J}_a\Gamma_7] + 8\rho\mathcal{J}_a^7 = -24\rho\mathcal{J}_a^7 \tag{4.4.24}$$

Actually, we can prove a stronger identity by rewriting this in terms of the 5-dimensional spinors χ_i, for which we use (4.2.22). We will need

$$\mathcal{J}_+^7 = 4iC_5^{\alpha\beta}\chi_\alpha^1\chi_\beta^2 \tag{4.4.25a}$$

$$\mathcal{J}_+^{m6} = 4(\gamma^m)^{\alpha\beta}\chi_\alpha^1\chi_\beta^2 \tag{4.4.25b}$$

$$\mathcal{J}_+^{mn} = -2(\gamma^{mn})^{\alpha\beta}(\chi_\alpha^1\chi_\beta^1 + \chi_\alpha^2\chi_\beta^2) \tag{4.4.25c}$$

and (see (4.2.19))

$$\xi_m = \frac{1}{\sqrt{2}}\gamma_m^{\alpha\beta}(\chi_\alpha^{1c}\chi_\beta^1 + \chi_\alpha^{2c}\chi_\beta^2) \tag{4.4.26a}$$

$$V_{mn} = \frac{1}{\sqrt{2}}\gamma_{mn}^{\alpha\beta}(\chi_\alpha^{1c}\chi_\beta^2 - \chi_\alpha^{2c}\chi_\beta^1)\ . \tag{4.4.26b}$$

Using (B.35) and the symmetry properties for gamma matrices in five dimensions, we can show

$$V_{mn}\mathcal{J}_+^{mn} = 4\xi_m\mathcal{J}_+^{m6} \tag{4.4.27}$$

Combining this with (4.4.24) for $a = +$ and using (4.4.11) we get

$$\xi_m\mathcal{J}_+^{m6} = -ie^A\mathcal{J}_+^7 \tag{4.4.28}$$

Now, we are ready to see how supersymmetry determines the spinor charges. If we trace (4.2.25a) with Γ^{m6} and replace in (4.4.19) for $a = \pm$, we get

$$\partial_m\mathcal{J}_\pm^7 = \mathcal{J}_\pm^7\partial_m(\phi - 3A) \mp 3me^{-A}\mathcal{J}_\pm^{m6} \tag{4.4.29}$$

If we contract with ξ^m, the first term drops out due to (4.4.3) and (4.4.5). For the second term, we get using (4.4.28)

$$\mathcal{L}_\xi\mathcal{J}_+^7 = 3im\mathcal{J}_+^7 \tag{4.4.30}$$

and therefore the charges of the spinors χ_i are

$$q = \frac{3im}{2} \tag{4.4.31}$$

4.4.2 M-theory

The Killing vector in M-theory is the bilinear (4.3.8). This is indeed Killing since (4.3.10) yields

$$\nabla_a\xi_b = -\frac{1}{6}G_{abcd}V^{cd} - \frac{1}{3\sqrt{2}}\rho(\star G)_{ab} \tag{4.4.32}$$

and the right-hand side is antisymmetric in a and b. Therefore

$$\mathcal{L}_\xi g = 0 \tag{4.4.33}$$

The trace of (4.3.13) immediately gives

$$\xi^a\partial_aA = \mathcal{L}_\xi A = 0 \tag{4.4.34}$$

Finally, we can compute dV by using (4.3.10) for $\mathcal{J}_0$ to get

$$dV = \iota_\xi G_4 \qquad \Longrightarrow \qquad \mathcal{L}_\xi G_4 = 0 \tag{4.4.35}$$

where the Bianchi identity for G_4 was used. We see that similarly to the type IIB case, ξ generates a symmetry of the full bosonic sector of the theory.

Let us also derive the warp factor dependence of the normalization of the spinors given by $\theta_i^{*\alpha}\theta_{j,\alpha} = 2\rho\,\delta_{ij}$. Taking the trace of (4.3.10) for $a = 0$ and eliminating G by taking the trace of (4.3.13), we find

$$\partial_m\rho - \rho\partial_m A = 0 \quad \Rightarrow \quad \rho = \frac{e^A}{\sqrt{2}} \tag{4.4.36}$$

where we have chosen the integration constant in the same way as for the IIB case. Another useful relation is found by tracing (4.3.10) with Γ^a, which yields

$$\nabla_a\zeta^a = \frac{1}{2}(\star G)_{ab}V^{ab} \tag{4.4.37}$$

Finally let us mention that the M-theory spinor has also definite charge under the action of ξ, i.e.

$$\mathcal{L}_\xi\theta = q\,\theta \tag{4.4.38}$$

Matching our conventions with those of [11], we find that

$$q = -\frac{3im}{2} \tag{4.4.39}$$

4.5 The moment map for J_a

4.5.1 Type IIB

In this section, we prove Eq. (4.1.7), which says that the moment map for the action of a generalized diffeomorphism is related to the dual vector associated to $\mathcal{K}$ (given by the cubic invariant of $E_{6(6)}$ $c(\mathcal{K},\mathcal{K},V)$). As explained in the main text, this condition can be written in terms of the twisted moment map density M_a which is given by (4.2.34) and we rewrite here for convenience:

$$M_a = [\nabla_m\tilde{\mathcal{J}}_a, v^m] + (\llbracket\nabla_m\mu, \tilde{\mathcal{J}}_a\rrbracket\, v^m) + \mathrm{Tr}[\tilde{\mathcal{J}}_a G^{IS}_m]v^m - \mathrm{Tr}[(\nabla_m\mu)\tilde{\mathcal{J}}_a)]v^m \tag{4.5.1}$$

where the second term means the action of $\llbracket\nabla_m\mu, \tilde{\mathcal{J}}_a\rrbracket$ on v^m while in the rest of the terms v^m is understood as an element of Cliff(6) and is given by $v^m = \frac{ie^{2\phi/3}}{2\sqrt{2}}\Gamma^{m67}$.

Let us compute the various terms in the above expression. The first term is computed by using (4.2.26) for $a = 1,2,3$. We give the result as a Clifford expansion

$$\begin{aligned}
[\nabla_m\tilde{\mathcal{J}}_a, \Gamma^{m67}] = {} & \Big[\frac{1}{16}\tilde{\mathcal{J}}^a_{mnpq7}H^{npq} - \frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_{mn6}F^n - \frac{1}{2}\tilde{\mathcal{J}}^a_{mn67}\partial^n A + \frac{1}{2}\tilde{\mathcal{J}}^a_{mn67}\partial^n\phi\Big]\Gamma^m \\
& + \Big[\frac{1}{8}\tilde{\mathcal{J}}^{np}_a H_{mnp} + \frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_{mn67}F^n - \frac{ie^\phi}{48}\tilde{\mathcal{J}}^a_{mnpq7}F^{npq} + \frac{1}{2}\tilde{\mathcal{J}}^a_{mn6}\partial^n A - \frac{1}{2}\tilde{\mathcal{J}}^a_{mn6}\partial^n\phi\Big]\Gamma^{m7} \\
& + \Big[\frac{1}{16}\tilde{\mathcal{J}}^a_{pqm}H_n{}^{pq} + \frac{ie^\phi}{16}\tilde{\mathcal{J}}^a_{mnp67}F^p - \frac{e^\phi}{16}\tilde{\mathcal{J}}^a_{mn}(*F_5) + \frac{1}{2}\tilde{\mathcal{J}}^a_{m6}\partial_n A - \frac{1}{2}\tilde{\mathcal{J}}^a_{m6}\partial_n\phi\Big]\Gamma^{mn7} \\
& + \Big[-\frac{1}{8}\tilde{\mathcal{J}}^a_{np6}H^{mnp} - \frac{ie^\phi}{4}\tilde{\mathcal{J}}^a_7 F_m - \frac{e^\phi}{4}\tilde{\mathcal{J}}^a_{m6}(*F_5) - \frac{1}{2}\tilde{\mathcal{J}}^a_{mn}\partial^n A + \frac{1}{2}\tilde{\mathcal{J}}^a_{mn}\partial^n\phi\Big]\Gamma^{m67}
\end{aligned} \tag{4.5.2}$$

where the derivatives of the dilaton and the warp factor appear as a result of the rescalings (4.2.32). The second and the fourth term in (4.5.1) are those that "twist" the moment map density. If we consider them separately they are not gauge invariant, however, their sum is, as it projects onto the fluxes. These terms are computed as follows. For the second term, it is more convenient to use the $SL(6)\times SL(2)$ basis. We first insert (3.3.6) and the $SL(6)\times SL(2)$ components of $\tilde{\mathcal{J}}_a$[24] in (C.1.5). We then use the resulting expression in (C.1.4) to compute the action on v^m and finally we transform it to the $USp(8)$ basis using (C.3.2b). For the fourth term in (4.5.1), we first transform $\nabla_m\mu$ to the $USp(8)$ basis using (C.3.4a) (exploiting the fact that the $\mathcal{J}_a$ do not have a **42** component) and then use (C.2.11). The combined result of these two terms is then[25]

$$\begin{aligned}
[\![\nabla_m\mu,\tilde{\mathcal{J}}_a]\!]\Gamma^{m67}-\mathrm{Tr}[\nabla_m\mu\tilde{\mathcal{J}}_a]\Gamma^{m67} &= \Big[-\frac{1}{24}\tilde{\mathcal{J}}^a_{mnpq7}H^{npq}-\frac{1}{4}\tilde{\mathcal{J}}^a_{mn67}\partial^n\phi\Big]\Gamma^m\\
&+\Big[\frac{ie^\phi}{24}\tilde{\mathcal{J}}^a_{mnpq7}F^{npq}+\frac{ie^\phi}{4}\tilde{\mathcal{J}}^a_{mn67}F^n\Big]\Gamma^{m7}\\
&+\Big[\frac{1}{4}\tilde{\mathcal{J}}^a_{m6}\partial_n\phi\Big]\Gamma^{mn7}\\
&+\Big[\frac{1}{8}\tilde{\mathcal{J}}^a_{np6}H_m{}^{np}-\frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_{np67}F_m{}^{np}+\frac{e^\phi}{4}\tilde{\mathcal{J}}^a_{m6}(*F_5)-\frac{ie^\phi}{4}\tilde{\mathcal{J}}^a_7F_m\Big]\Gamma^{m67}
\end{aligned}\tag{4.5.3}$$

Finally, the third term in (4.5.1) is computed directly from (4.2.26b) and the result reads

$$\mathrm{Tr}[\tilde{\mathcal{J}}_aG^{IS}_m]\Gamma^{m67}=\Big[-\frac{1}{8}\tilde{\mathcal{J}}^{np6}_aH_{mnp}+\frac{ie^\phi}{8}\tilde{\mathcal{J}}^7_aF_m+\frac{ie^\phi}{16}\tilde{\mathcal{J}}^{np67}_aF_{mnp}-\frac{e^\phi}{8}\tilde{\mathcal{J}}^a_{m6}(*F_5)\Big]\Gamma^{m67}\tag{4.5.4}$$

When adding (4.5.2), (4.5.3) and (4.5.4), the various terms organize themselves as coefficients of a Cliff(6) expansion. In the next step, we eliminate the H-field using the dilatino equation (4.2.27) by taking appropriate traces. More specifically, we use $\mathrm{Tr}[\mathcal{J}_aG_d\Gamma^m]=0$ for the Γ^m terms, $\mathrm{Tr}[\mathcal{J}_aG_d\Gamma^{m7}]=0$ for the Γ^{m7} terms, $\mathrm{Tr}[\mathcal{J}_aG_d\Gamma^{mn7}]=0$ for the Γ^{mn7} terms and $\mathrm{Tr}[\mathcal{J}_aG_d\Gamma^{m67}]=0$ for the Γ^{m67} terms. The result is

$$\begin{aligned}
\Big(\frac{ie^{2\phi/3}}{2\sqrt{2}}\Big)^{-1}M_a &= \Big[\frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_{mn6}F^n-\frac{ie^\phi}{16}\tilde{\mathcal{J}}^{np}_aF_{mnp}-\frac{1}{2}\tilde{\mathcal{J}}^a_{mn67}\partial^nA\Big]\Gamma^m\\
&+\Big[-\frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_{mn67}F^n-\frac{ie^\phi}{48}\tilde{\mathcal{J}}^a_{mnpq7}F^{npq}+\frac{1}{2}\tilde{\mathcal{J}}^a_{mn6}\partial^nA\Big]\Gamma^{m7}\\
&+\Big[-\frac{ie^\phi}{16}\tilde{\mathcal{J}}^a_{mnp67}F^p-\frac{ie^\phi}{16}\tilde{\mathcal{J}}^a_{pqm7}F_n{}^{pq}-\frac{e^\phi}{16}\tilde{\mathcal{J}}^a_{mn}(*F_5)+\frac{1}{2}\tilde{\mathcal{J}}^a_{m6}\partial_nA\Big]\Gamma^{mn7}\\
&+\Big[\frac{ie^\phi}{8}\tilde{\mathcal{J}}^a_7F_m+\frac{ie^\phi}{16}\tilde{\mathcal{J}}^{np67}_aF_{mnp}-\frac{e^\phi}{8}\tilde{\mathcal{J}}^a_{m6}(*F_5)-\frac{1}{4}\tilde{\mathcal{J}}^a_{mn}\partial^n\phi-\frac{1}{2}\tilde{\mathcal{J}}^a_{mn}\partial^nA\Big]\Gamma^{m67}
\end{aligned}\tag{4.5.5}$$

For $a=3$ we can find the relation between this and $\mathcal{K}$ by using the external gravitino equation (4.2.25c). Reading off the $\Gamma^m,\Gamma^{m7},\Gamma^{mn7}$ and Γ^{m67} components of

[24] These can be easily found using (C.3.5).

[25] Here, we mean $[\![\nabla_m\mu,\tilde{\mathcal{J}}_a]\!]\Gamma^{m67}=(\frac{ie^{2\phi/3}}{2\sqrt{2}})^{-1}[\![\nabla_m\mu,\tilde{\mathcal{J}}_a]\!]v^m$.

this equation, we see that the right-hand sides are exactly the brackets appearing in the above equation. Thus

$$\begin{aligned} M_3 &= -i\frac{me^{A-4\phi/3}}{2}\Big[\zeta_m\Gamma^m + i\zeta^7_m\Gamma^{m7} + \frac{i}{2}V_{mn}\Gamma^{mn7} + i\xi_m\Gamma^{m67}\Big] \\ &= -2im\rho e^{-4\phi/3}\mathcal{K} \end{aligned} \tag{4.5.6}$$

where in the last step we used (4.2.18). Following the same procedure for $a = \pm$ and using this time (4.2.25a), we get

$$M_\pm = 0 \tag{4.5.7}$$

These are exactly the conditions (4.2.35) which in turn imply the $\tilde{J}_a$ integrability condition (4.1.7).

4.5.2 M-theory

In this section, we will present the calculation leading to the integrability condition for the J_a for M-theory compactifications. The methodology is similar to the one for IIB described in the previous subsection. However the details are different due to the different $E_{6(6)}$ embedding of the derivative and the gauge field in M-theory (Eqs. (3.3.8) and 3.3.9). The general expression for the moment map density (4.2.31) now reads[26]

$$M = [\nabla_a\tilde{\mathcal{J}}, v^a] + (\llbracket\nabla_a\mu, \tilde{\mathcal{J}}\rrbracket\, v^a) + \mathrm{Tr}[\tilde{\mathcal{J}}G^{IS}_a]v^a - \mathrm{Tr}[(\nabla_a\mu)\tilde{\mathcal{J}})]v^a \tag{4.5.8}$$

where now

$$v^a = \frac{i}{2\sqrt{2}}\Gamma^{a7} \tag{4.5.9}$$

and G^I_a is given by (4.3.11).

The various terms are computed in exactly the same way as in type IIB so we just give the results here. The first term reads

$$\begin{aligned} [\nabla_a\tilde{\mathcal{J}}, \Gamma^{a7}] &= \Big[\frac{1}{72}\tilde{\mathcal{J}}^{bcd7}G_{abcd} - \frac{1}{2}\tilde{\mathcal{J}}^7\partial_a A\Big]\Gamma^a + \\ &+ \Big[-\frac{1}{36}\tilde{\mathcal{J}}^{bcd}G_{abcd} - \frac{1}{2}\tilde{\mathcal{J}}_{ab}\partial^b A\Big]\Gamma^{a7} \\ &+ \Big[\frac{i}{6}\tilde{\mathcal{J}}^7(\star G)_{ab} + \frac{1}{48}\tilde{\mathcal{J}}^{cd}G_{abcd} - \frac{1}{4}\tilde{\mathcal{J}}_{abc}\partial^c A\Big]\Gamma^{ab7} \end{aligned} \tag{4.5.10}$$

while the sum of the second and the fourth is simply

$$\llbracket\nabla_a\mu, \tilde{\mathcal{J}}\rrbracket\Gamma^{a7} - \mathrm{Tr}[\nabla_a\mu\tilde{\mathcal{J}}]\Gamma^{a7} = \Big[-\frac{i}{8}\tilde{\mathcal{J}}^7(\star G)_{ab}\Big]\Gamma^{ab7} \tag{4.5.11}$$

and the third gives

$$\mathrm{Tr}[\tilde{\mathcal{J}}G^I_a]\Gamma^{a7} = \Big[-\frac{1}{36}\tilde{\mathcal{J}}^{bcd}G_{abcd}\Big]\Gamma^{a7} \tag{4.5.12}$$

[26] As in the main text, we omit the $SU(2)$ index a with the understanding that $\tilde{\mathcal{J}} = \tilde{\mathcal{J}}_\pm, \tilde{\mathcal{J}}_3$.

For $M = M_\pm$, we see that the sum of (4.5.10), (4.5.11) and (4.5.12) vanishes by virtue of (4.3.12)[27]. Thus

$$M_\pm = 0 \tag{4.5.13}$$

For $M = M_3$, we follow the same procedure but this time using (4.3.14). The result is

$$\begin{aligned} M_3 &= -\frac{ime^A}{2}\Big[\zeta_a\Gamma^a + i\xi_a\Gamma^{a7} + \frac{i}{2}V_{ab}\Gamma^{ab7}\Big] \\ &= -2im\rho\mathcal{K} \end{aligned} \tag{4.5.14}$$

where we used (4.3.16). We this verify the M-theory moment map equation (4.1.7) where the rescaled structures are those of (4.3.1), are as in type IIB $\lambda_1 = \lambda_2 = 0$, and $\lambda_3 = -2im$.

4.6 The Dorfman derivative along K

4.6.1 Type IIB

The Dorfman derivative is a generalization of the usual Lie derivative for "generalized flows" parametrized by the $E_{6(6)}$ vector K. Here we show that the background is invariant under this flow.

The embedding of the derivative in the $E_{6(6)}$ object D, Eq. (4.2.33), picks a particular direction v in the space of generalized vectors. We start by showing Eq. (4.2.43), namely the fact that the (twisted) Dorfman derivative actually reduces to the Lie derivative along this direction.

As explained in the main text (see (4.2.44) and the discussion after that), the twisted Dorfman derivative can be split into a differential piece which is just the directional derivative along the Killing vector ξ, given in (4.2.6), namely

$$(\tilde{\mathcal{K}} \cdot v^m)\nabla_m = \xi^m\nabla_m \tag{4.6.1}$$

and an algebraic piece $\mathcal{A}$ in the adjoint of $E_{6(6)}$. We show that $\mathcal{A}$ satisfies the equations in (4.2.46). We start with the **36** piece which according to (4.2.39) reads

$$\mathcal{A}_{\alpha\beta} = (\tilde{\mathcal{K}} \cdot v^m)\nabla_m\mu_{\alpha\beta} - \big[\nabla_m\tilde{\mathcal{K}}, v^m\big]_{\alpha\beta} - \big[(\nabla_m\mu)\tilde{\mathcal{K}}, v^m\big]_{\alpha\beta} \tag{4.6.2}$$

where the commutators are just matrix commutators, $\nabla_m\mu_{\alpha\beta}$ in the first term is just the derivative of the **36** piece of μ interpreted as a Cliff(6) element, $((\nabla_m\mu)\tilde{\mathcal{K}})_{\alpha\beta}$ is the standard action[28] of $E_{6(6)}$ on the fundamental and (C.2.10a) was used for the projection in the adjoint.

The first and the third term in (4.6.2) twist the Dorfman derivative, so we are computing them together[29]. $\nabla_m\mu$ is computed just by inserting (3.3.6), in (C.3.4a)

[27] By taking the trace with Γ^a, Γ^{a7} and Γ^{ab7}.

[28] This term has contributions from both the **36** and the **42** components of μ.

[29] Similarly to the moment map equation described in the previous section, each of these terms is not gauge invariant but their sum is.

while we compete $(\nabla_m\mu)\tilde{\mathcal{K}}$ using (C.1.3) and then use (C.3.2a) to transform that to the $USp(8)$ basis. The result is

$$\begin{aligned}
(\tilde{\mathcal{K}}\cdot v^m)\nabla_m\mu - \big[(\nabla_m\mu)\tilde{\mathcal{K}}, v^m\big] &= \Big[-\frac{1}{6}\xi_m\partial_n\phi\Big]\Gamma^{mn} \\
&+ \Big[-\frac{1}{4}\zeta^p(*H)_{np} + \frac{e^\phi}{4}\zeta_7^p(*F_3)_{np} - \frac{e^\phi}{4}\xi_n(*F_5) + \frac{1}{12}V_{mn}\partial^m\phi\Big]\Gamma^{n6} \\
&+ \Big[-\frac{1}{8}\xi^p H_{mnp} + \frac{1}{6}\zeta_m^7\partial_n\phi\Big]\Gamma^{mn6} \\
&+ \Big[\frac{ie^\phi}{8}\xi^p F_{mnp} + \frac{i}{12}\zeta_m\partial_n\phi - \frac{ie^\phi}{4}\zeta_m^7 F_n\Big]\Gamma^{mn67}
\end{aligned} \tag{4.6.3}$$

where we have expressed the result in terms of the spinor bilinears ξ, ζ, ζ_7 and V defined in (4.2.18). Finally, the second term in (4.6.2) is easily computed by using (4.2.18):

$$\begin{aligned}
-[\nabla_m\tilde{\mathcal{K}}, v^m] &= \Big[\frac{1}{4}\nabla_m\xi_n + \frac{1}{6}\xi_m\partial_n\phi\Big]\Gamma^{mn} \\
&+ \Big[-\frac{1}{4}\nabla^m V_{mn} + \frac{1}{6}V_{mn}\partial^m\phi\Big]\Gamma^{n6} \\
&+ \Big[-\frac{1}{4}\nabla_m\zeta_n^7 - \frac{1}{6}\zeta_m^7\partial_n\phi\Big]\Gamma^{mn6} \\
&+ \Big[\frac{i}{4}\nabla_m\zeta_n + \frac{i}{6}\zeta_m\partial_n\phi\Big]\Gamma^{mn67}
\end{aligned} \tag{4.6.4}$$

where the derivatives of the dilaton appear due to the rescaling of $\tilde{\mathcal{K}}$ given in (4.1.10).

Collecting the pieces together, i.e. adding (4.6.3) and (4.6.4), we easily see that the terms proportional to Γ^{n6} cancel out due to (4.4.23), those proportional to Γ^{mn6} due to (4.4.12) and those proportional to Γ^{mn67} due to (4.4.15). The remaining terms in (4.6.1) are the sum of the first lines of (4.6.3) and (4.6.4) which is simply

$$\mathcal{A}|_{\mathbf{36}} = \frac{1}{4}(\nabla_m\xi_n)\Gamma^{mn}\ . \tag{4.6.5}$$

This is exactly the first equation in (4.2.46). Let us now look at $\mathcal{A}|_{\mathbf{42}}$, given by

$$\mathcal{A}|_{\alpha\beta\gamma\delta} = (\tilde{\mathcal{K}}\cdot v^m)\nabla_m\mu_{\alpha\beta\gamma\delta} - (v^m\times\nabla_m\tilde{\mathcal{K}})_{\alpha\beta\gamma\delta} - (v^m\times(\nabla_m\mu)\tilde{\mathcal{K}})_{\alpha\beta\gamma\delta} \tag{4.6.6}$$

where the **42** piece of the adjoint projection is given in (C.2.10b). The first term is computed by inserting (3.3.6) into (C.3.4b) while the third by using (C.2.10b). Using Fierz identities from appendix B, we get for the sum of these two terms

$$\begin{aligned}
\Big[(\tilde{\mathcal{K}}\cdot v^m)\nabla_m\mu - (v^m\times(\nabla_m\mu)\tilde{\mathcal{K}})\Big]_{\alpha\beta\gamma\delta} &= \Big[\frac{1}{2}\xi_m\partial_n\phi\Big]\Gamma^{m67}_{[\alpha\beta}\Gamma^{n67}_{\gamma\delta]} \\
&+ \Big[\frac{3}{4}\zeta^p(*H)_{np} - \frac{3e^\phi}{4}\zeta_7^p(*F_3)_{np} \\
&\quad + \frac{3e^\phi}{4}\xi_n(*F_5) - \frac{1}{4}V_{mn}\partial^m\phi\Big]\Gamma^n_{[\alpha\beta}\Gamma^6_{\gamma\delta]} \\
&+ \Big[\frac{3ie^\phi}{8}\xi^p F_{mnp} - \frac{3ie^\phi}{4}\zeta_m^7 F_n + \frac{i}{4}\zeta_m\partial_n\phi\Big]\Gamma^{mn7}_{[\alpha\beta}\Gamma^6_{\gamma\delta]} \\
&+ \Big[-\frac{3}{8}\xi^p H_{mnp} + \frac{1}{2}\zeta_m^7\partial_n\phi\Big]\Gamma^{mn7}_{[\alpha\beta}\Gamma^{67}_{\gamma\delta]}
\end{aligned} \tag{4.6.7}$$

containing only the fluxes. The second term in (4.6.6) is given by inserting (4.2.18) in (C.2.10b) and using again some Fierz identities from appendix B:

$$\begin{aligned}-(v^m \times \nabla_m \tilde{\mathcal{K}})_{\alpha\beta\gamma\delta} = &\Big[-\frac{1}{2}\xi_m\partial_n\phi\Big]\Gamma^{m67}_{[\alpha\beta}\Gamma^{n67}_{\gamma\delta]}\\ &+\Big[\frac{3}{4}\nabla^m V_{mn} - \frac{1}{2}V_{mn}\partial^m\phi\Big]\Gamma^{n}_{[\alpha\beta}\Gamma^{6}_{\gamma\delta]}\\ &+\Big[-\frac{3}{4}\nabla_m\zeta^7_n - \frac{1}{2}\zeta^7_m\partial_n\phi\Big]\Gamma^{mn7}_{[\alpha\beta}\Gamma^{67}_{\gamma\delta]}\\ &+\Big[\frac{3i}{4}\nabla_m\zeta_n + \frac{i}{2}\zeta_m\partial_n\phi\Big]\Gamma^{mn7}_{[\alpha\beta}\Gamma^{6}_{\gamma\delta]}\end{aligned} \tag{4.6.8}$$

If we insert now (4.6.7) and (4.6.8) in (4.6.6) and use (4.4.12), (4.4.15) and (4.4.23) (as for the **36** component), we get

$$\mathcal{A}|_{\mathbf{42}} = 0 \ , \tag{4.6.9}$$

which completes thus the proof of (4.2.46). Combining this with (4.6.1) and the fact that the $\mathcal{J}_a$ have only a **36** component we arrive at (4.2.43) as we explain in the main text.

4.6.2 M-theory

Let us now perform the same kind of calculation for the M-theory set-up. Although the details are different than in type IIB, the basic procedure to prove that the twisted Dorfman derivative along $\mathcal{K}$ is equal to the usual Lie derivative along the corresponding Killing vector is actually the same. The differential piece is again the directional derivative along ξ[30]

$$(\mathcal{K}\cdot v^a)\nabla_a = \xi^a\nabla_a \tag{4.6.10}$$

The **36** piece of the operator $\mathcal{A}$ is given by

$$\mathcal{A}_{\alpha\beta} = (\mathcal{K}\cdot v^a)\nabla_a\mu_{\alpha\beta} - \big[\nabla_a\mathcal{K}, v^a\big]_{\alpha\beta} - \big[(\nabla_a\mu)\mathcal{K}, v^a\big]_{\alpha\beta} \tag{4.6.11}$$

The first term together with the third is

$$(\mathcal{K}\cdot v^a)\nabla_a\mu - [(\nabla_a\mu)\mathcal{K}, v^a] = \Big[\frac{1}{24}\xi^d G_{abcd}\Big]\Gamma^{abc} + \Big[\frac{i}{8}V^{ab}(\star F)_{ab}\Big]\Gamma_7 \ . \tag{4.6.12}$$

while the second is

$$-[\nabla_a\mathcal{K}, v^{a7}] = \Big[\frac{1}{4}\nabla_a\xi_b\Big]\Gamma^{ab} + \Big[-\frac{1}{8}\nabla_a V_{bc}\Big]\Gamma^{abc} + \Big[-\frac{i}{4}\nabla_a\zeta^a\Big]\Gamma_7 \ . \tag{4.6.13}$$

It is straightforward to see using (4.4.35) and (4.4.37) that their sum is just

$$\mathcal{A}|_{\mathbf{36}} = \frac{1}{4}(\nabla_a\xi_b)\Gamma^{ab} \tag{4.6.14}$$

We finally show that $\mathcal{A}|_{\mathbf{42}} = 0$ also in M-theory. We have

$$\mathcal{A}_{\alpha\beta\gamma\delta} = (\mathcal{K}\cdot v^a)\nabla_a\mu_{\alpha\beta\gamma\delta} - (v^a\times\nabla_a\mathcal{K})_{\alpha\beta\gamma\delta} - (v^a\times(\nabla_a\mu)\mathcal{K})_{\alpha\beta\gamma\delta} \tag{4.6.15}$$

[30] We recall that $\tilde{\mathcal{K}} = \mathcal{K}$ for the M-theory case.

Similarly to type IIB

$$\Big[(\mathcal{K}\cdot v^a)\nabla_a\mu-(v^a\times(\nabla_a\mu)\mathcal{K})\Big]_{\alpha\beta\gamma\delta}=\Big[\frac{i}{16}V^{ab}(\star G)_{ab}\Big]\Gamma^{c}_{[\alpha\beta}\Gamma^{c7}_{\gamma\delta]}+\Big[-\frac{1}{8}\xi^d G_{abcd}\Big]\Gamma^{a7}_{[\alpha\beta}\Gamma^{bc7}_{\gamma\delta]} \tag{4.6.16}$$

where we have used (B.30) and (B.31) to simplify the terms proportional to V and (B.32) for the terms proportional to ξ. Using (4.3.16), we also get

$$-\Big[v^a\times\nabla_a\mathcal{K}\Big]_{\alpha\beta\gamma\delta}=\Big[-\frac{i}{8}\nabla_a\zeta^a\Big]\Gamma^{c}_{[\alpha\beta}\Gamma^{c7}_{\gamma\delta]}+\Big[-\frac{3}{8}\nabla_{[a}V_{bc]}\Big]\Gamma^{a7}_{[\alpha\beta}\Gamma^{bc7}_{\gamma\delta]} \tag{4.6.17}$$

where again the terms proportional to derivatives of ζ are absent because of (B.30) (B.31) while due to (B.32) only the exterior derivative of V appears. The sum of (4.6.16) and (4.6.8) vanishes using (4.4.35) and (4.4.37). We thus get

$$\mathcal{A}|_{\mathbf{42}}=0 \tag{4.6.18}$$

and therefore we verify (4.2.43) for M-theory as well.

Discussion of results and outlook

In this Thesis, we focused on the study of flux backgrounds in string theory compactifications. In chapter 2 we studied mass deformations of the $\mathcal{N} = 4$ SYM theory which, in the framework of AdS/CFT, are dual to supergravity solutions with non-trivial three-from fluxes, while in chapters 3 and 4 we explained how generic supersymmetric flux backgrounds can be given a geometrical meaning in the framework of generalized geometry.

The main result of chapter 2 was that gauge theories realized on the world-volume of stacks of D3-branes where supersymmetry is softly broken, should still satisfy the supertrace rule: the sum of the squares of the boson masses and the sum of the squares of the fermion masses should be equal. Since no light supersymmetric particles have been discovered so far, this result is not phenomenologically appealing. We obtained this result exploiting the fact that the theory descends for $\mathcal{N} = 4$ SYM and using arguments based on the AdS/CFT Correspondence.

Alternatively, all of the parameters in the Lagrangian (2.7.4) can be computed purely within $\mathcal{N} = 1$ supergravity with chiral fields including a hidden sector (moduli) on top of the observable sector (brane fields). After breaking supersymmetry spontaneously in the hidden sector via F terms (which can be done by turning on three-form fluxes), integrating out the moduli fields and taking the limit of infinite Planck mass while keeping the gravitino mass finite, one obtains a softly broken $\mathcal{N} = 1$ gauge theory for the visible sector. The parameters of the latter are given in terms of the F terms, the superpotential, and the Kähler potential of the original $\mathcal{N} = 1$ supergravity theory[31] [100, 101]. Comparing these with those obtained from the D3-brane action, one finds [65] that they all agree, except for the boson masses. Furthermore, it is only for nonscale supersymmetry breaking and zero supersymmetric masses that the supertrace obtained by the supergravity calculation is zero; generically, it is not. It would be interesting to understand why the supergravity calculation fails to reproduce this feature of the $D3$-brane action.

The expression for the soft parameters in terms of the fluxes is also expected to receive α' corrections coming from higher derivative terms in the ten-dimensional bulk and brane actions. These terms induce corrections to the Kähler potential of the four-dimensional $\mathcal{N} = 1$ supergravity theory that generically break the no-scale structure [102], and induce corrections to the soft masses [103] (and thus to their

[31] For D3-branes in CY compactifications, the Kähler potential is of sequestered form if the complex structure moduli are integrated out [65], as done in Kachru-Kallosh-Linde-Trivedi models [98] or in large volume scenarios [99].

trace). In the so-called ultralocal limit, where the coupling between the matter and the moduli in the Kähler potential has a particular form, the breaking of no-scale structure is not seen in the visible sector, and the soft terms do not get corrected. From our arguments in section 2.7.2, it is very likely that the full Kähler potential for $D3$-branes in Calabi-Yau manifolds falls into this category, and our result holds even when taking α' corrections into account. On the other hand, one might expect that nonperturbative corrections to the superpotential, which are usually invoked in string phenomenology scenarios, modify this result. Unfortunately, there is no way of analyzing this at the ten-dimensional level, as such corrections are modeled in the four-dimensional field theory only. One could thus try to compute the soft terms, including these types of corrections using $\mathcal{N}=1$ supergravity calculations, as discussed in the previous paragraph, and check to see whether the zero supertrace result still holds. However, it is hard to extract meaningful conclusions from such calculations: first, because these calculations fail already at tree level to reproduce the trace of soft masses found from the ten-dimensional equations of motion and, second, because to do these calculations correctly one would need to include the full dependence of the nonperturbative corrections on the moduli (particularly the unknown dependence on complex structure moduli).

It is worth stressing that our analysis also holds for D3-branes at orbifold singularities. Explicit tree-level and one-loop calculations for the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ model confirm our expectations. It would be interesting to see if this result extends also to other types of singularities and to other types of branes.

In chapter 3, we discussed backgrounds with more general flux configurations. We explained that Generalized Geometry provides the appropriate tools for the description of such backgrounds in the case that there is some supersymmetry preserved. We first introduced $O(d,d)$-generalized geometry in which only fluxes from the NS-NS sector acquire a geometrical meaning and in a second step we presented Exceptional Generalized Geometry where generic flux backgrounds (including both NS-NS and R-R fluxes) are described in a geometric language.

In chapter 4, we proved that the supersymmetry equations relevant for AdS_5 vacua with generic fluxes preserving eight supercharges in type IIB and M-theory compactifications translate into the integrability conditions (4.1.7),(4.1.8) and (4.1.9) in Exceptional Generalized Geometry. The integrability conditions involve generalized structures in the fundamental and adjoint representations of the $E_{6(6)}$ U-duality group. Although our calculations were performed for the particular case of AdS_5 compactifications, the integrability conditions are expected to be the same for other AdS_d vacua of type II (either IIA or IIB) and M-theory compactifications preserving eight supercharges, since these are described by vector and hypermultiplets. A particularly interesting case to analyse is that of AdS_4 vacua, where the relevant U-duality group is $E_{7(7)}$, with maximal compact subgroup $SU(8)$. The construction of the generalized structures from spinor bilinears is the same, and since our calculations were done in $USp(8)$ language, the extension to $SU(8)$ should be rather straightforward.

The description of AdS_5 vacua in exceptional generalized geometry has nice applications in AdS/CFT. The original example is the $AdS_5 \times S^5$ solution supported by five-form flux (in the type IIB case) which is dual to $\mathcal{N}=4$ SYM. Allowing for generic internal manifolds (and fluxes) but still preserving some supersymmetry corresponds to supersymmetric deformations on the field theory side. AdS vacua are dual to deformations that preserve conformal invariance on the gauge theory. Having a compact description of the internal geometry opens then the way for finding the supergravity dual of these deformations in a rather systematic way, as very recently shown in [104]. We will explore this direction further in future work.

Chapter A

't Hooft symbols

The explicit form of the 't Hooft matrices G^A_{ij} is

$$G^1 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \qquad G^2 = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \qquad G^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$
$$G^4 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \qquad G^5 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix} \qquad G^6 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{A.1}$$

Here $\sigma_{1,2,3}$ are the standard Pauli matrices and σ_0 is the 2×2 unit matrix. These matrices satisfy the following basis independent properties:

$$G^A_{ij}\delta_{AB}G^{B\,kl} = -2\left(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i\right) \,, \qquad \mathrm{Tr}\left(G^{A\dagger}G^B\right) = G^{A\,ij}G^B{}_{ji} = 4\delta_{AB}\,, \tag{A.2}$$

and

$$\begin{aligned} G^A_{ik}G^{B\dagger\,kj} + G^B_{ik}G^{A\dagger\,kj} &= 2\delta^{AB}\delta^j_i \\ i\epsilon_{ABCDEF}G^A{}_{ik_1}G^{B\,k_1k_2}G^C{}_{k_2k_3}G^{D\,k_4k_5}G^E{}_{k_5k_6}G^{F\,k_6j} &= \delta^j_i\,. \end{aligned} \tag{A.3}$$

Chapter B

Spinor conventions

In the paper we use spinors of $Spin(1{,}4)$ and $Spin(5)$ and $Spin(1{,}9)$ for type IIB, and $Spin(6)$ and $Spin(1{,}10)$ in M-theory. We give our conventions for all of them, explain their relations and provide some useful formulae for our calculations. In this section, all the indices are meant to be flat.

For five Euclidean dimensions, the gamma matrices are denoted by $\gamma^m, \quad m = 1, \dots 5$ and satisfy

$$(\gamma^m)^\dagger = \gamma^m \tag{B.1a}$$

$$(\gamma^m)^T = C_5 \gamma^m C_5^{-1} \tag{B.1b}$$

$$(\gamma^m)^* = D_5 \gamma^m D_5^{-1} \tag{B.1c}$$

where we take $D_5 = C_5$ and we have $\gamma^{12345} = 1_4$. An explicit construction for them is given by

$$\gamma^1 = \sigma^1 \otimes \sigma^0 \tag{B.2a}$$

$$\gamma^2 = \sigma^2 \otimes \sigma^0 \tag{B.2b}$$

$$\gamma^3 = \sigma^3 \otimes \sigma^1 \tag{B.2c}$$

$$\gamma^4 = \sigma^3 \otimes \sigma^2 \tag{B.2d}$$

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \tag{B.2e}$$

$$C_5 = \sigma^1 \otimes \sigma^2 \tag{B.2f}$$

For a spinor χ, the conjugate spinor is defined as

$$\chi^c = D_5^* \chi^* \tag{B.3}$$

and satisfies the properties

$$(\gamma^m \chi)^c = \gamma^m \chi^c, \qquad D_5^* D_5 = -1 \Rightarrow \chi^{cc} = -\chi \tag{B.4}$$

For the 5-dimensional external space, we have a Lorentzian version of the above. The intertwining relations for the gamma matrices ρ^μ are

$$(\rho^\mu)^\dagger = -A_{1,4} \rho^\mu A_{1,4}^{-1} \tag{B.5a}$$

$$(\rho^\mu)^T = C_{1,4} \rho^\mu C_{1,4}^{-1} \tag{B.5b}$$

$$(\rho^\mu)^* = -D_{1,4} \rho^\mu D_{1,4}^{-1} \tag{B.5c}$$

where $\mu = 0, \ldots, 4$, $\rho^{01234} = -i\,1_4$ and $D_{1,4} = -C_{1,4}A_{1,4}$. Explicitly we can take

$$\rho^0 = i\sigma^2 \otimes \sigma^0 \tag{B.6a}$$
$$\rho^i = \sigma^1 \otimes \sigma^i, \quad i = 1,2,3 \tag{B.6b}$$
$$\rho^4 = i\rho^0\rho^1\rho^2\rho^3 \tag{B.6c}$$
$$A_{1,4} = \rho^0 \tag{B.6d}$$
$$C_{1,4} = \rho^0\rho^2 \tag{B.6e}$$

The conjugate spinor is defined as

$$\psi^c = D^*_{1,4}\psi^* \tag{B.7}$$

and satisfies

$$(\rho^\mu\psi)^c = -\rho^\mu\psi^c, \qquad D^*_{1,4}D_{1,4} = -1 \Rightarrow \psi^{cc} = -\psi \tag{B.8}$$

Now, let us combine the above representations to construct a 10-dimensional Clifford algebra. We define

$$\hat{\Gamma}^\mu = \rho^\mu \otimes 1_4 \otimes \sigma^3, \quad \mu = 0, \ldots, 4 \tag{B.9a}$$
$$\hat{\Gamma}^{m+4} = 1_4 \otimes \gamma^m \otimes \sigma^1, \quad m = 1, \ldots, 5 \tag{B.9b}$$

The last factor is needed to allow for a chirality matrix in 10 dimensions:

$$\hat{\Gamma}_{(11)} = \hat{\Gamma}^0 \ldots \hat{\Gamma}^9 = 1_4 \otimes 1_4 \otimes \sigma^2 \tag{B.10}$$

The 10-dimensional interwiners are constructed as follows

$$A_{1,9} = -A_{1,4} \otimes 1_4 \otimes \sigma^3 \implies (\hat{\Gamma}^M)^\dagger = -A_{1,9}\hat{\Gamma}^M A^{-1}_{1,9} \tag{B.11a}$$
$$C_{1,9} = C_{1,4} \otimes C_5 \otimes \sigma^2 \implies (\hat{\Gamma}^M)^T = -C_{1,9}\hat{\Gamma}^M C^{-1}_{1,9} \tag{B.11b}$$
$$D_{1,9} = D_{1,4} \otimes D_5 \otimes \sigma^1 \implies (\hat{\Gamma}^M)^* = D_{1,9}\hat{\Gamma}^M D^{-1}_{1,9} \tag{B.11c}$$

A 10-dimensional spinor ϵ splits as

$$\epsilon = \psi \otimes \chi \otimes u \tag{B.12}$$

where u is acted upon by the Pauli matrices. For the conjugate spinor we have

$$\epsilon^c = D^*_{1,9}\epsilon^*, \qquad D^*_{1,9}D_{1,9} = 1 \Rightarrow \epsilon^{cc} = \epsilon \tag{B.13}$$

The type IIB Majorana-Weyl spinors ϵ_i are

$$\epsilon_i = \psi \otimes \chi_i \otimes u + \psi^c \otimes \chi^c_i \otimes u, \quad i = 1,2 \tag{B.14}$$

Their chirality and reality properties require

$$u = \sigma^2 u = \sigma^1 u^* \tag{B.15}$$

We construct now gamma matrices Γ^a, $a = 1, \dots 6$ for Cliff(6) from our representation for Cliff(5). We define

$$\Gamma^m = \begin{pmatrix} 0 & \gamma^m \\ \gamma^m & 0 \end{pmatrix}, \quad m = 1, \dots, 5, \qquad \Gamma^6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.16}$$

$$\Gamma_7 = i\Gamma^1 \dots \Gamma^6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad i\Gamma^{67} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.17}$$

The interwiner for Cliff(6) is

$$C = C^{\alpha\beta} = \begin{pmatrix} C_5 & 0 \\ 0 & C_5 \end{pmatrix}, \qquad C^{-1} = C_{\alpha\beta} = \begin{pmatrix} C_5^{-1} & 0 \\ 0 & C_5^{-1} \end{pmatrix} \tag{B.18}$$

which raises and lowers spinor indices as $\Gamma^{\alpha\beta} = C^{\alpha\gamma}\Gamma_\gamma{}^\beta$, $\Gamma_{\alpha\beta} = \Gamma_\alpha{}^\gamma C_{\gamma\beta}$. For any Cliff(6) element Γ, we have

$$\Gamma^{(n)}_{\beta\alpha} = -(-)^{Int[n/2]}\Gamma^{(n)}_{\alpha\beta} \tag{B.19}$$

while the reality properties read[1]

$$\Gamma_a^* = C\Gamma_a C^{-1} \tag{B.20}$$

With these Γ we can also construct ten-dimensional gamma matrices relevant for for compactifications to four dimensions as

$$\hat{\Gamma}^\mu = \rho^\mu \otimes 1_8, \quad \mu = 0, \dots, 3 \tag{B.21a}$$

$$\hat{\Gamma}^{m+3} = \rho^4 \otimes \Gamma^m, \quad m = 1, \dots, 6 \tag{B.21b}$$

The 6-dimensional gamma matrices act on USp(8) spinors θ_α, $\alpha = 1,..8$. In the main text, we use the following

$$\theta_1 = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} \chi_1^c \\ \chi_2^c \end{pmatrix} \tag{B.22}$$

satisfying

$$\theta^{*i\alpha} = (-i\sigma^2)^{ij} C^{\alpha\beta}\theta_{j\beta} \tag{B.23}$$

The eleven-dimensional gamma-matrices relevant for M-theory can be built directly from the six-dimensional ones Γ^a constructed above and from the ρ^μ of AdS_5 as follows

$$\tilde{\Gamma}^\mu = \rho^\mu \otimes \Gamma_7, \quad \mu = 0, \dots, 4 \tag{B.24a}$$

$$\tilde{\Gamma}^{a+4} = 1_4 \otimes \Gamma^a, \quad a = 1, \dots, 6 \tag{B.24b}$$

The relevant interwiners for eleven dimensions are

$$C_{1,10} = C_{1,4} \otimes C_6\Gamma_7 \implies (\tilde{\Gamma}^M)^T = -C_{1,10}\tilde{\Gamma}^M C_{1,10}^{-1} \tag{B.25a}$$

$$D_{1,10} = D_{1,4} \otimes D_6 \implies (\tilde{\Gamma}^M)^* = D_{1,10}\tilde{\Gamma}^M D_{1,10}^{-1} \tag{B.25b}$$

[1] All the C's defined in this section are antisymmetric, hermitian and unitary.

A spinor in eleven dimensions ϵ decomposes as

$$\epsilon = \psi \otimes \theta \tag{B.26}$$

while the conjugate spinor is given by

$$\epsilon^c = D^*_{1,10}\epsilon = \psi^c \otimes \theta^c \tag{B.27}$$

The Majorana property of the M-theory supersymmetry parameter requires then

$$\epsilon = \psi \otimes \theta + \psi^c \otimes \theta^c \tag{B.28}$$

We finish by giving some Fierz identities which are heavily used in our calculations

$$(\Gamma_{ab7})^{\alpha\beta}(\Gamma^{ab7})_{\gamma\delta} - 2(\Gamma_a)^{\alpha\beta}(\Gamma^a)_{\gamma\delta} + 2(\Gamma_{a7})^{\alpha\beta}(\Gamma^{a7})_{\gamma\delta} = 16\delta^{\alpha}_{[\gamma}\delta^{\beta}_{\delta]} + 2C^{\alpha\beta}C_{\gamma\delta} \tag{B.29}$$

$$\Gamma^{(a}_{[\alpha\beta}\Gamma^{b)7}_{\gamma\delta]} = \frac{1}{6}g^{ab}\Gamma^{c}_{[\alpha\beta}\Gamma^{c7}_{\gamma\delta]} \tag{B.30}$$

$$\Gamma^{[a}_{[\alpha\beta}\Gamma^{b]7}_{\gamma\delta]} = -\Gamma^{ab7}_{[\alpha\beta}C_{\gamma\delta]} = -\frac{i}{24}\epsilon_{abcdef}\Gamma^{cd7}_{[\alpha\beta}\Gamma^{ef7}_{\gamma\delta]} \tag{B.31}$$

$$\Gamma^{[a|7}_{[\alpha\beta}\Gamma^{bc]7}_{\gamma\delta]} = \Gamma^{a7}_{[\alpha\beta}\Gamma^{bc7}_{\gamma\delta]} + 2g^{a[b}\Gamma^{c]}_{[\alpha\beta}C_{\gamma\delta]} \tag{B.32}$$

$$\Gamma^{6}_{[\alpha\beta}\Gamma^{m}_{\gamma\delta]} = -\Gamma^{67}_{[\alpha\beta}\Gamma^{m7}_{\gamma\delta]} \tag{B.33}$$

$$\Gamma^{m67}_{[\alpha\beta}\Gamma^{np7}_{\gamma\delta]} + \Gamma^{mnp6}_{[\alpha\beta}C_{\gamma\delta]} = 2g^{m[n}\Gamma^{p]}_{[\alpha\beta}\Gamma^{6}_{\gamma\delta]} \tag{B.34}$$

$$\gamma^{\alpha\beta}_{mn}(\gamma^{mn})^{\gamma\delta} = 10C^{\alpha\beta}_5C^{\gamma\delta}_5 + 6\gamma^{\alpha\beta}_m(\gamma^m)^{\gamma\delta} + 8\gamma^{\alpha\gamma}_m(\gamma^m)^{\beta\delta} \tag{B.35}$$

Let us note that one can derive additional Fierz identities by exploiting the following Leibniz-like rule:

$$A_{[\alpha\beta}B_{\gamma\delta]} = C_{[\alpha\beta}D_{\gamma\delta]} \Longrightarrow (A\Gamma)_{[\alpha\beta}B_{\gamma\delta]} + A_{[\alpha\beta}(B\Gamma)_{\gamma\delta]} = (C\Gamma)_{[\alpha\beta}D_{\gamma\delta]} + C_{[\alpha\beta}(D\Gamma)_{\gamma\delta]} \tag{B.36}$$

for any antisymmetric elements A,B,C,D and Γ of Cliff(6).

Chapter C

E6 representation theory

The group $E_{6(6)}$ is a particular real form of the E_6 family of Lie groups. It is generated by 78 elements, out of which 36 are compact and 42 are not. It contains as subgroups $USp(8)$ and $SL(6) \times SL(2)$.

C.1 $SL(6) \times SL(2)$ decomposition

The vector representation V of $E_{6(6)}$ is 27-dimensional and splits under $SL(6)\times SL(2)$ as

$$\mathbf{27} = (\overline{\mathbf{6}}, \mathbf{2}) + (\mathbf{15}, \mathbf{1}), \quad V = (V^i_{\ a}, V^{ab}) \tag{C.1.1a}$$

while we will also need its dual

$$\overline{\mathbf{27}} = (\mathbf{6}, \overline{\mathbf{2}}) + (\overline{\mathbf{15}}, \mathbf{1}), \quad Z = (Z^a_{\ i}, Z_{ab}) \tag{C.1.1b}$$

The adjoint decomposes

$$\mathbf{78} = (\mathbf{35}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\overline{\mathbf{20}}, \mathbf{2}), \quad \mu = (\mu^a_{\ b}, \mu^i_{\ j}, \mu^i_{abc}) \tag{C.1.2}$$

and its action on the vector is given by

$$(\mu\, V)^i_{\ a} = -\mu^b_{\ a} V^i_b + \mu^i_{\ j} V^j_a + \frac{1}{2}\mu^i_{abc} V^{bc} \tag{C.1.3a}$$

$$(\mu\, V)^{ab} = \mu^a_{\ c} V^{cb} - \mu^b_{\ c} V^{ca} - \epsilon_{ij} (\star \mu^i)^{abc} V^j_c \tag{C.1.3b}$$

while on the dual vector by

$$(\mu\, Z)^a_i = \mu^a_{\ b} Z^b_i - \mu^j_{\ i} Z^a_j - \frac{1}{2}\epsilon_{ij} (\star \mu^j)^{abc} Z_{bc} \tag{C.1.4a}$$

$$(\mu\, Z)_{ab} = -\mu^c_{\ a} Z_{cb} + \mu^c_{\ b} Z_{ca} - \mu^i_{abc} Z^c_i \tag{C.1.4b}$$

where a,b,c,$\ldots$ run from 1 to 6 and i,j from 1 to 2.

The $\mathfrak{e}_{6(6)}$ algebra $[\![\mu, \nu]\!]$ is

$$[\![\mu, \nu]\!]^i_{\ j} = \mu^i_{\ k} \nu^k_j + \frac{1}{12}\mu^i_{abc}\epsilon_{jk} (\star \nu^k)^{abc} - (\mu \leftrightarrow \nu) \tag{C.1.5a}$$

$$[\![\mu,\nu]\!]^a_{\ b} = \mu^a_{\ c}\nu^c_{\ b} - \frac{1}{4}\mu^i_{bcd}\epsilon_{ij}(\star\nu^j)^{acd} - (\mu \leftrightarrow \nu) \tag{C.1.5b}$$

$$[\![\mu,\nu]\!]^i_{abc} = \mu^i_{\ j}\nu^j_{abc} - 3\mu^d_{\ [a}\nu^i_{bc]d} - (\mu \leftrightarrow \nu) \tag{C.1.5c}$$

The group $E_{6(6)}$ has a quadratic and a cubic invariant. Given a vector V and a dual vector Z, the quadratic invariant is

$$b(V,Z) = V^i_a Z^a_i + \frac{1}{2}V^{ab}Z_{ab} \tag{C.1.6}$$

while the cubic is given by

$$c(V,U,W) = \frac{1}{2\sqrt{2}}\epsilon_{ij}\left(V^{ab}U^i_aW^j_b + U^{ab}V^i_aW^j_b + W^{ab}V^i_aU^j_b\right) - \frac{1}{16\sqrt{2}}\epsilon_{abcdef}V^{ab}U^{cd}W^{ef} \tag{C.1.7}$$

where U,V and W are all in the fundamental. This allows to construct a dual vector from two vectors by "deleting" one of the vectors in the cubic invariant, namely

$$[c(V,U,\cdot)]^a_i = \frac{1}{2\sqrt{2}}\epsilon_{ij}\left(V^{ab}U^j_b + U^{ab}V^j_b\right) \tag{C.1.8a}$$

$$[c(V,U,\cdot)]_{ab} = \frac{1}{\sqrt{2}}\epsilon_{ij}V^i_{[a}U^j_{b]} - \frac{1}{8\sqrt{2}}\epsilon_{abcdef}V^{cd}U^{ef} \tag{C.1.8b}$$

C.2 $USp(8)$ decomposition

The other subgroup of $E_{6(6)}$ that we use is $USp(8)$. The **27** fundamental representations of $E_{6(6)}$ is irreducible under $USp(8)$, and encoded by an antisymmetric traceless tensor

$$V = V^{\alpha\beta} \tag{C.2.1}$$

with $V^\alpha{}_\alpha = 0$. The $USp(8)$ indices $\alpha, \beta, \ldots$ are raised and lowered with $C_{\alpha\beta}$ in (B.18) , which plays the role of $USp(8)$ symplectic invariant.

The adjoint decomposes as

$$\mathbf{78} = \mathbf{36} + \mathbf{42}, \quad \mu = (\mu^\alpha_{\ \beta}, \mu^{\alpha\beta\gamma\delta}) \tag{C.2.2}$$

with $\mu_{\alpha\beta} = \mu_{\beta\alpha}$, $\mu^{\alpha\beta\gamma\delta} = \mu^{[\alpha\beta\gamma\delta]}$ and $\mu^{\alpha\beta\gamma\delta}C_{\gamma\delta} = 0$. Furthermore, in our conventions we have

$$\mu^*_{\alpha\beta} = -\mu_{\alpha\beta}, \qquad \mu^*_{\alpha\beta\gamma\delta} = \mu_{\alpha\beta\gamma\delta} \tag{C.2.3}$$

The adjoint action is

$$(\mu V)^{\alpha\beta} = \mu^\alpha_{\ \gamma}V^{\gamma\beta} - \mu^\beta_{\ \gamma}V^{\gamma\alpha} - \mu^{\alpha\beta\gamma\delta}V_{\gamma\delta} \tag{C.2.4}$$

$$(\mu Z)^{\alpha\beta} = \mu^\alpha_{\ \gamma}Z^{\gamma\beta} - \mu^\beta_{\ \gamma}Z^{\gamma\alpha} + \mu^{\alpha\beta\gamma\delta}Z_{\gamma\delta} \tag{C.2.5}$$

and the $\mathfrak{e}_{6(6)}$ algebra is given by

$$[\![\mu,\nu]\!]_{\alpha\beta} = \mu_\alpha^{\ \gamma}\nu_{\gamma\beta} - \frac{1}{3}\mu_\alpha^{\ \gamma\delta\epsilon}\nu_{\gamma\delta\epsilon\beta} - (\mu \leftrightarrow \nu)\ . \tag{C.2.6a}$$

$$[\![\mu,\nu]\!]_{\alpha\beta\gamma\delta} = -4\mu_{[\alpha}{}^{\epsilon}\nu_{\beta\gamma\delta]\epsilon} - (\mu \leftrightarrow \nu) \tag{C.2.6b}$$

The quadratic and the cubic invariant of $E_{6(6)}$ take a particularly simple form in the $USp(8)$ basis

$$b(V,Z) = V^{\alpha\beta}Z_{\beta\alpha} \tag{C.2.7}$$

and

$$c(V,U,W) = V^{\alpha}_{\ \beta}U^{\beta}_{\ \gamma}W^{\gamma}_{\ \alpha} \tag{C.2.8}$$

and we also have

$$[c(V,U,\cdot)]^{\alpha\beta} = \frac{1}{2}(V^{\alpha}_{\ \gamma}V'^{\gamma\beta} - V^{\beta}_{\ \gamma}V'^{\gamma\alpha} - \frac{1}{4}C^{\alpha\beta}V^{\gamma\delta}V'_{\delta\gamma}) \tag{C.2.9}$$

In our calculations we also need the adjoint projection built out of a vector V and a dual vector Z. This is given by

$$(V \times Z)^{\alpha\beta} = 2V^{(\alpha}{}_{\gamma}Z^{|\gamma|\beta)} \tag{C.2.10a}$$

$$(V \times Z)^{\alpha\beta\gamma\delta} = 6\Big(V^{[\alpha\beta}Z^{\gamma\delta]} + V^{[\alpha}{}_{\epsilon}Z^{|\epsilon|\beta}C^{\gamma\delta]} + \frac{1}{3}(V^{\epsilon}{}_{\zeta}Z^{\zeta}{}_{\epsilon})C^{[\alpha\beta}C^{\gamma\delta]}\Big) \tag{C.2.10b}$$

Finally, the Killing form is

$$\mathrm{Tr}(\mu,\nu) = \mu^{\alpha\beta}\nu_{\alpha\beta} + \frac{1}{6}\mu^{\alpha\beta\gamma\delta}\nu_{\alpha\beta\gamma\delta} \tag{C.2.11}$$

C.3 Transformation between $SL(6) \times SL(2)$ and $USp(8)$

Our calculations involve objects which are more naturally described in the $SL(6) \times SL(2)$ basis (gauge fields and derivative) and others (spinors) which have a natural $USp(8)$ description. Therefore, it is useful to have explicit formulae for the transformation rules between them. For this purpose, we use the gamma matrices Γ^a defined in 6 dimensions. It's also useful to introduce two sets of them:

$$\Gamma^a_i = (\Gamma^a, i\Gamma^a\Gamma_7), \qquad i = 1,2 \tag{C.3.1}$$

The transformation rules for the vector (fundamental) and the dual vector (anti-fundamental) representation are

$$V^{\alpha\beta} = \frac{1}{2\sqrt{2}}(\Gamma^a_i)^{\alpha\beta}V^i_a + \frac{i}{4\sqrt{2}}(\Gamma_{ab7})^{\alpha\beta}V^{ab} \tag{C.3.2a}$$

$$Z^{\alpha\beta} = \frac{1}{2\sqrt{2}}(\Gamma^i_a)^{\alpha\beta}Z^a_i + \frac{i}{4\sqrt{2}}(\Gamma^{ab7})^{\alpha\beta}Z_{ab} \tag{C.3.2b}$$

and are easily inverted

$$V^i_a = \frac{1}{2\sqrt{2}}V^{\alpha\beta}(\Gamma^i_a)_{\beta\alpha}, \qquad V^{ab} = \frac{i}{2\sqrt{2}}V^{\alpha\beta}(\Gamma^{ab7})_{\beta\alpha} \tag{C.3.3a}$$

$$Z^a_i = \frac{1}{2\sqrt{2}}Z^{\alpha\beta}(\Gamma^a_i)_{\beta\alpha}, \qquad Z_{ab} = \frac{i}{2\sqrt{2}}Z^{\alpha\beta}(\Gamma_{ab7})_{\beta\alpha} \tag{C.3.3b}$$

For the adjoint representation we have[1]

$$\mu_{\alpha\beta} = \frac{1}{4}\Big[\mu^a{}_b(\Gamma_a{}^b)_{\alpha\beta} + i\epsilon_i{}^j\mu^i{}_j(\Gamma_7)_{\alpha\beta} + \frac{1}{6}\epsilon_i{}^j\mu^i_{abc}(\Gamma^{ab}\Gamma^c_j)_{\alpha\beta}\Big] \tag{C.3.4a}$$

$$\begin{aligned}\mu^{\alpha\beta\gamma\delta} = \frac{1}{8}\Big[&- \mu^a{}_b\Big((\Gamma^i_a)^{[\alpha\beta}(\Gamma^b_i)^{\gamma\delta]} - (\Gamma_{ac7})^{[\alpha\beta}(\Gamma^{cb7})^{\gamma\delta]}\Big)\\ &+ \mu^i{}_j(\Gamma^a_i)^{[\alpha\beta}(\Gamma^j_a)^{\gamma\delta]}\\ &+ i\mu^i_{abc}(\Gamma^a_i)^{[\alpha\beta}(\Gamma^{bc7})^{\gamma\delta]}\Big]\end{aligned} \tag{C.3.4b}$$

Their inverses are given by

$$\mu^a{}_b = -\frac{1}{4}\mu^{\alpha\beta}(\Gamma^a{}_b)_{\beta\alpha} - \frac{1}{16}\mu^{\alpha\beta\gamma\delta}(\Gamma^a_i)_{[\alpha\beta}(\Gamma^i_b)_{\gamma\delta]} \tag{C.3.5a}$$

$$\mu^i{}_j = -\frac{i}{4}\epsilon^i{}_j\mu^{\alpha\beta}(\Gamma_7)_{\beta\alpha} + \frac{1}{48}\mu^{\alpha\beta\gamma\delta}(\Gamma^i_a)_{[\alpha\beta}(\Gamma^a_j)_{\gamma\delta]} \tag{C.3.5b}$$

$$\mu^i_{abc} = -\frac{i}{4}\mu^{\alpha\beta}(\Gamma^i_a\Gamma_{bc7})_{\beta\alpha} + \frac{i}{8}\mu^{\alpha\beta\gamma\delta}(\Gamma^i_a)_{[\alpha\beta}(\Gamma_{bc7})_{\gamma\delta]} \tag{C.3.5c}$$

[1] $SL(2)$ indices are raised and lowered with δ_{ij}.

Bibliography

[1] G. Aldazabal, L. E. Ibanez, F. Quevedo, and A. M. Uranga, "D-branes at singularities: A Bottom up approach to the string embedding of the standard model," *JHEP* **08** (2000) 002, `hep-th/0005067`.

[2] R. Blumenhagen, B. Kors, D. Lust, and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," *Phys. Rept.* **445** (2007) 1–193, `hep-th/0610327`.

[3] J. P. Conlon, A. Maharana, and F. Quevedo, "Towards Realistic String Vacua," *JHEP* **05** (2009) 109, `0810.5660`.

[4] M. R. Douglas and G. W. Moore, "D-branes, quivers, and ALE instantons," `hep-th/9603167`.

[5] I. Bena, M. Graña, S. Kuperstein, P. Ntokos, and M. Petrini, "D3-brane model building and the supertrace rule," *Phys. Rev. Lett.* **116** (2016), no. 14 141601, `1510.07039`.

[6] J. Polchinski and M. J. Strassler, "The String dual of a confining four-dimensional gauge theory," `hep-th/0003136`.

[7] I. Bena, M. Graña, S. Kuperstein, P. Ntokos, and M. Petrini, "Fermionic and bosonic mass deformations of $\mathcal{N} = 4$ SYM and their bulk supergravity dual," *JHEP* **05** (2016) 149, `1512.00011`.

[8] M. Graña and P. Ntokos, "Generalized geometric vacua with eight supercharges," *JHEP* **08** (2016) 107, `1605.06383`.

[9] M. Grana, J. Louis, A. Sim, and D. Waldram, "E7(7) formulation of N=2 backgrounds," *JHEP* **07** (2009) 104, `0904.2333`.

[10] A. Ashmore and D. Waldram, "Exceptional Calabi–Yau spaces: the geometry of $\mathcal{N} = 2$ backgrounds with flux," `1510.00022`.

[11] A. Ashmore, M. Petrini, and D. Waldram, "The exceptional generalised geometry of supersymmetric AdS flux backgrounds," `1602.02158`.

[12] J. Louis, P. Smyth, and H. Triendl, "Supersymmetric Vacua in N=2 Supergravity," *JHEP* **08** (2012) 039, `1204.3893`.

[13] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 2007.

[14] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge University Press, 2007.

[15] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl. Phys.* **B122** (1977) 253–290.

[16] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `hep-th/9510017`.

[17] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls," *Class. Quant. Grav.* **18** (2001) 3359–3382, `hep-th/0103233`.

[18] L. J. Romans, "Massive N=2a Supergravity in Ten-Dimensions," *Phys. Lett.* **B169** (1986) 374.

[19] W. Nahm, "Supersymmetries and their Representations," *Nucl. Phys.* **B135** (1978) 149.

[20] E. Cremmer, B. Julia, and J. Scherk, "Supergravity Theory in Eleven-Dimensions," *Phys. Lett.* **B76** (1978) 409–412.

[21] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A Conjecture," *Phys. Rev.* **D55** (1997) 5112–5128, `hep-th/9610043`.

[22] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, "Dualization of dualities. 1.," *Nucl. Phys.* **B523** (1998) 73–144, `hep-th/9710119`.

[23] D. Lust and D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity," *JHEP* **02** (2005) 027, `hep-th/0412250`.

[24] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of type IIB supergravity," *Class. Quant. Grav.* **23** (2006) 4693–4718, `hep-th/0510125`.

[25] P. Koerber and D. Tsimpis, "Supersymmetric sources, integrability and generalized-structure compactifications," *JHEP* **08** (2007) 082, `0706.1244`.

[26] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," *Int. J. Mod. Phys.* **A16** (2001) 822–855, `hep-th/0007018`. [,182(2000)].

[27] U. Gran, J. B. Gutowski, and G. Papadopoulos, "On supersymmetric Anti-de-Sitter, de-Sitter and Minkowski flux backgrounds," `1607.00191`.

[28] M. Grana, R. Minasian, A. Tomasiello, and M. Petrini, "Supersymmetric backgrounds from generalized calabi-yau manifolds," *Fortsch. Phys.* **53** (2005) 885–893.

[29] P. Candelas and X. de la Ossa, "Moduli Space of Calabi-Yau Manifolds," *Nucl. Phys.* **B355** (1991) 455–481.

[30] S. Cecotti, S. Ferrara, and L. Girardello, "Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories," *Int. J. Mod. Phys.* **A4** (1989) 2475.

[31] S. Ferrara and S. Sabharwal, "Dimensional Reduction of Type II Superstrings," *Class. Quant. Grav.* **6** (1989) L77.

[32] M. Bodner, A. C. Cadavid, and S. Ferrara, "(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry," *Class. Quant. Grav.* **8** (1991) 789–808.

[33] M. Bodner and A. C. Cadavid, "Dimensional Reduction of Type Iib Supergravity and Exceptional Quaternionic Manifolds," *Class. Quant. Grav.* **7** (1990) 829.

[34] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* **38** (1999) 1113–1133, `hep-th/9711200`. [Adv. Theor. Math. Phys.2,231(1998)].

[35] E. Witten, "Bound states of strings and p-branes," *Nucl. Phys.* **B460** (1996) 335–350, `hep-th/9510135`.

[36] R. C. Myers, "Dielectric branes," *JHEP* **9912** (1999) 022, `hep-th/9910053`.

[37] J. E. Paton and H.-M. Chan, "Generalized veneziano model with isospin," *Nucl. Phys.* **B10** (1969) 516–520.

[38] B. de Wit, J. Hoppe, and H. Nicolai, "On the Quantum Mechanics of Supermembranes," *Nucl. Phys.* **B305** (1988) 545.

[39] R. Donagi and E. Witten, "Supersymmetric Yang-Mills theory and integrable systems," *Nucl.Phys.* **B460** (1996) 299–334, `hep-th/9510101`.

[40] D. Z. Freedman and J. A. Minahan, "Finite temperature effects in the supergravity dual of the N=1* gauge theory," *JHEP* **0101** (2001) 036, `hep-th/0007250`.

[41] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, "The mass spectrum of chiral n=2 d = 10 supergravity on s**5," *Phys. Rev.* **D32** (1985) 389.

[42] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[43] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c theorem," *Adv. Theor. Math. Phys.* **3** (1999) 363–417, `hep-th/9904017`.

[44] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, “The Supergravity dual of N=1 superYang-Mills theory,” *Nucl.Phys.* **B569** (2000) 451–469, `hep-th/9909047`.

[45] K. Pilch and N. P. Warner, “N=2 supersymmetric RG flows and the IIB dilaton,” *Nucl. Phys.* **B594** (2001) 209–228, `hep-th/0004063`.

[46] N. J. Evans and M. Petrini, “AdS RG flow and the superYang-Mills cascade,” *Nucl. Phys.* **B592** (2001) 129–142, `hep-th/0006048`.

[47] K. Pilch and N. P. Warner, “N=1 supersymmetric renormalization group flows from IIB supergravity,” *Adv. Theor. Math. Phys.* **4** (2002) 627–677, `hep-th/0006066`.

[48] A. Khavaev and N. P. Warner, “A Class of N=1 supersymmetric RG flows from five-dimensional N=8 supergravity,” *Phys. Lett.* **B495** (2000) 215–222, `hep-th/0009159`.

[49] F. Bigazzi, A. L. Cotrone, M. Petrini, and A. Zaffaroni, “Supergravity duals of supersymmetric four-dimensional gauge theories,” *Riv. Nuovo Cim.* **25N12** (2002) 1–70, `hep-th/0303191`.

[50] R. Apreda, D. E. Crooks, N. J. Evans, and M. Petrini, “Confinement, glueballs and strings from deformed AdS,” *JHEP* **05** (2004) 065, `hep-th/0308006`.

[51] C. N. Gowdigere and N. P. Warner, “Holographic Coulomb branch flows with N=1 supersymmetry,” *JHEP* **03** (2006) 049, `hep-th/0505019`.

[52] A. Buchel and C. Pagnutti, “Bulk viscosity of N=2* plasma,” *Nucl. Phys.* **B816** (2009) 62–72, `0812.3623`.

[53] M. Taylor, “Anomalies, counterterms and the N=0 Polchinski-Strassler solutions,” `hep-th/0103162`.

[54] I. Bena, M. Graña, S. Kuperstein, P. Ntokos, and M. Petrini, “D3-brane model building and the supertrace rule,” `1510.07039`.

[55] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities,” *JHEP* **08** (2000) 052, `hep-th/0007191`.

[56] C. Hoyos, S. Paik, and L. G. Yaffe, “Screening in strongly coupled N=2* supersymmetric Yang-Mills plasma,” *JHEP* **10** (2011) 062, `1108.2053`.

[57] I. Bena, M. Graña, S. Kuperstein, and S. Massai, “Tachyonic Anti-M2 Branes,” *JHEP* **06** (2014) 173, `1402.2294`.

[58] I. Bena, M. Graña, S. Kuperstein, and S. Massai, “Giant Tachyons in the Landscape,” *JHEP* **02** (2015) 146, `1410.7776`.

[59] S. Kachru, J. Pearson, and H. L. Verlinde, "Brane / flux annihilation and the string dual of a nonsupersymmetric field theory," *JHEP* **06** (2002) 021, `hep-th/0112197`.

[60] I. Bena and S. Kuperstein, "Brane polarization is no cure for tachyons," *JHEP* **09** (2015) 112, `1504.00656`.

[61] R. Slansky, "Group Theory for Unified Model Building," *Phys.Rept.* **79** (1981) 1–128.

[62] M. Graña and J. Polchinski, "Supersymmetric three-form flux perturbations on ads(5)," *Phys. Rev.* **D63** (2001) 026001, `hep-th/0009211`.

[63] F. Zamora, "Nonsupersymmetric SO(3) invariant deformations of N=1* vacua and their dual string theory description," *JHEP* **0012** (2000) 021, `hep-th/0007082`.

[64] S. B. Giddings, S. Kachru, and J. Polchinski, "Hierarchies from fluxes in string compactifications," *Phys.Rev.* **D66** (2002) 106006, `hep-th/0105097`.

[65] M. Graña, T. W. Grimm, H. Jockers, and J. Louis, "Soft supersymmetry breaking in calabi-yau orientifolds with d-branes and fluxes," *Nucl. Phys.* **B690** (2004) 21–61, `hep-th/0312232`.

[66] M. Graña, "D3-brane action in a supergravity background: The fermionic story," *Phys. Rev.* **D66** (2002) 045014, `hep-th/0202118`.

[67] I. Bena, J. Blåbäck, and D. Turton, "Loop corrections to the antibrane potential," `1602.05959`.

[68] I. Jack and D. Jones, "Nonstandard soft supersymmetry breaking," *Phys.Lett.* **B457** (1999) 101–108, `hep-ph/9903365`.

[69] S. P. Martin and M. T. Vaughn, "Two loop renormalization group equations for soft supersymmetry breaking couplings," *Phys. Rev.* **D50** (1994) 2282, `hep-ph/9311340`. [Erratum: Phys. Rev.D78,039903(2008)].

[70] I. Jack and D. R. T. Jones, "Soft supersymmetry breaking and finiteness," *Phys. Lett.* **B333** (1994) 372–379, `hep-ph/9405233`.

[71] A. Parkes and P. C. West, "Finiteness and Explicit Supersymmetry Breaking of the $N = 4$ Supersymmetric Yang-Mills Theory," *Nucl.Phys.* **B222** (1983) 269.

[72] N. Hitchin, "Generalized calabi-yau manifolds," *Quart. J. Math. Oxford Ser.* **54** (2003) 281–308, `math.dg/0209099`.

[73] M. Gualtieri, "Generalized complex geometry," `math.dg/0401221`.

[74] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," *JHEP* **08** (2004) 046, `hep-th/0406137`.

[75] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Generalized structures of n = 1 vacua," *JHEP* **11** (2005) 020, `hep-th/0505212`.

[76] A. Tomasiello, "Generalized structures of ten-dimensional supersymmetric solutions," *JHEP* **03** (2012) 073, `1109.2603`.

[77] C. M. Hull, "Generalised Geometry for M-Theory," *JHEP* **07** (2007) 079, `hep-th/0701203`.

[78] P. P. Pacheco and D. Waldram, "M-theory, exceptional generalised geometry and superpotentials," *JHEP* **09** (2008) 123, `0804.1362`.

[79] J. P. Gauntlett, D. Martelli, and D. Waldram, "Superstrings with intrinsic torsion," *Phys. Rev.* **D69** (2004) 086002, `hep-th/0302158`.

[80] P. Grange and S. Schafer-Nameki, "T-duality with H-flux: Non-commutativity, T-folds and G x G structure," *Nucl. Phys.* **B770** (2007) 123–144, `hep-th/0609084`.

[81] M. Grana, R. Minasian, M. Petrini, and D. Waldram, "T-duality, Generalized Geometry and Non-Geometric Backgrounds," *JHEP* **04** (2009) 075, `0807.4527`.

[82] D. Andriot, R. Minasian, and M. Petrini, "Flux backgrounds from Twists," *JHEP* **12** (2009) 028, `0903.0633`.

[83] C. M. Hull, "A Geometry for non-geometric string backgrounds," *JHEP* **10** (2005) 065, `hep-th/0406102`.

[84] C. M. Hull, "Doubled Geometry and T-Folds," *JHEP* **07** (2007) 080, `hep-th/0605149`.

[85] D. Andriot, O. Hohm, M. Larfors, D. Lust, and P. Patalong, "Non-Geometric Fluxes in Supergravity and Double Field Theory," *Fortsch. Phys.* **60** (2012) 1150–1186, `1204.1979`.

[86] D. Andriot and A. Betz, "β-supergravity: a ten-dimensional theory with non-geometric fluxes, and its geometric framework," *JHEP* **12** (2013) 083, `1306.4381`.

[87] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," *JHEP* **11** (2011) 091, `1107.1733`.

[88] P. Koerber, "Lectures on Generalized Complex Geometry for Physicists," *Fortsch. Phys.* **59** (2011) 169–242, `1006.1536`.

[89] U. Lindstrom, R. Minasian, A. Tomasiello, and M. Zabzine, "Generalized complex manifolds and supersymmetry," *Commun. Math. Phys.* **257** (2005) 235–256, `hep-th/0405085`.

[90] C. Jeschek and F. Witt, "Generalised G(2) - structures and type IIb superstrings," *JHEP* **03** (2005) 053, `hep-th/0412280`.

[91] A. Sim, "Exceptionally generalised geometry and supergravity, phd thesis (unpublished), imperial college,".

[92] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "A Scan for new N=1 vacua on twisted tori," *JHEP* **05** (2007) 031, `hep-th/0609124`.

[93] A. Coimbra, C. Strickland-Constable, and D. Waldram, "$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, Connections and M theory," `1112.3989`.

[94] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory," *JHEP* **03** (2014) 019, `1212.1586`.

[95] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, "Supersymmetric AdS(5) solutions of M theory," *Class. Quant. Grav.* **21** (2004) 4335–4366, `hep-th/0402153`.

[96] J. Louis and C. Muranaka, "Moduli spaces of AdS_5 vacua in N=2 supergravity," `1601.00482`.

[97] A. Coimbra and C. Strickland-Constable, "Supersymmetric Backgrounds, the Killing Superalgebra, and Generalised Special Holonomy," `1606.09304`.

[98] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, "De Sitter vacua in string theory," *Phys. Rev.* **D68** (2003) 046005, `hep-th/0301240`.

[99] J. P. Conlon, F. Quevedo, and K. Suruliz, "Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking," *JHEP* **08** (2005) 007, `hep-th/0505076`.

[100] V. S. Kaplunovsky and J. Louis, "Model independent analysis of soft terms in effective supergravity and in string theory," *Phys. Lett.* **B306** (1993) 269–275, `hep-th/9303040`.

[101] A. Brignole, L. E. Ibanez, and C. Munoz, "Soft supersymmetry breaking terms from supergravity and superstring models," *Adv. Ser. Direct. High Energy Phys.* **21** (2010) 244–268, `hep-ph/9707209`.

[102] K. Becker, M. Becker, M. Haack, and J. Louis, "Supersymmetry breaking and alpha-prime corrections to flux induced potentials," *JHEP* **06** (2002) 060, `hep-th/0204254`.

[103] L. Aparicio, M. Cicoli, S. Krippendorf, A. Maharana, F. Muia, and F. Quevedo, "Sequestered de Sitter String Scenarios: Soft-terms," *JHEP* **11** (2014) 071, `1409.1931`.

[104] A. Ashmore, M. Gabella, M. Graña, M. Petrini, and D. Waldram, "Exactly marginal deformations from exceptional generalised geometry," `1605.05730`.