Some problems of direct and indirect stabilization of wave equations with locally boundary fractional damping or with localised Kelvin-Voigh

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Quelques problèmes de stabilisation directe et indirecte d'équations d'ondes par des contrôles de type fractionnaire frontière ou de type Kelvin-Voight localisé

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RÉSUMÉ

Cette thèse est consacrée à l'étude de la stabilisation directe et indirecte de différents systèmes d'équations d'ondes avec un contrôle frontière de type fractionnaire ou un contrôle local viscoélastique de type Kelvin-Voight. Nous considérons, d'abords, la stabilisation de l'équation d'ondes multidimensionnel avec un contrôle frontière fractionnaire au sens de Caputo. Sous des conditions géométriques optimales, nous établissons un taux de décroissance polynomial de l'énergie de système. Ensuite, nous nous intéressons à l'étude de la stabilisation d'un système de deux équations d'ondes couplées via les termes de vitesses, dont une seulement est amortie avec contrôle frontière de type fractionnaire au sens de Caputo. Nous montrons différents résultats de stabilités dans le cas 1-d et N-d. Finalement, nous étudions la stabilité d'un système de deux équations d'ondes couplées avec un seul amortissement viscoélastique localement distribué de type Kelvin-Voight.

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Mots Clés

C_0-**semi-groupe, Dérivée Fractionnaire, Stabilité Forte, Stabilité Exponentielle, Stabilité Polynomiale, Optimalité, Méthode Spectrale, Conditions Géométriques.**

ABSTRACT

This thesis is devoted to study the stabilisation of the system of waves equations with one boundary fractional damping acting on apart of the boundary of the domain and the stabilisation of a system of waves equations with locally viscoelastic damping of Kelvin-Voight type. First, we study the stability of the multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. Second, we study the stability of the system of coupled one-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Next, we study the stability of the system of coupled multi-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Finally, we study the stability of the multidimensional waves equations with locally viscoelastic damping of Kelvin-Voight is applied for one equation around the boundary of the domain.

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Keywords

C_0-**semigroupe, Fractional Derivative, Strong Stability, Exponential Stability, Polynomial Stability, Optimality, Spectrale Method, Geometric Condition.**

TABLE DES MATIÈRES

List of symbols

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non negative real numbers.
$\mathbb{R}^*$	The set of non zero real numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^*$	The set of non zero natural numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{Z}^*$	The set of non zero integer numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Q}_+$	The set of non negative rational numbers.
$\mathbb{Q}^*$	The set of non zero rational numbers.
$\mathbb{C}$	The set of complex numbers.
i	The imaginary unit.
$\Re$	The real part.
$\Im$	The imaginary part.
L^p	The Lebesgue space.
H^m	The sobolev space.
C^0	The space of continuous function.
C^1	the space of continuously differentiable functions.
$\lvert\cdot\rvert$	The modulus.
$\lVert\cdot\rVert$	The norm.
max	The maximum.
min	The minimum.
sup	The supreme.
inf	The infimum.
$f_y = \partial_y f$	The partial derivative of f with respect of y.
$f_{yy} = \partial_{yy} f$	The second partial derivative of f with respect of y.
$\partial_t^{\alpha,\eta}$	Fractional Derivative.
$sign$	The sign function or signum function.

Asymptotic Notation

Let f and g be two functions defined on some subset of the real numbers, we define

- $f(x) = O(g(x))$ as $x \to \infty$ to mean that there exists a positive number M and a real numbers x_0 such that

$$|f(x)| \leq M\,|g(x)| \quad \forall\ x \geq x_0.$$

- $f(x) = o(g(x))$ as $x \to \infty$ to mean that

$$\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0.$$

- $f \sim g$ as $x \to \infty$ to mean that

$$\lim_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| = 1.$$

INTRODUCTION

Cette thèse est consacrée à l'étude de la stabilisation directe et indirecte d'équations d'ondes par des contrôles de type fractionnaire frontière ou de type Kelvin-Voight localisé. Dans la première partie, nous considérons la stabilisation d'une équation d'ondes avec un contrôle fractionnaire au sens de Caputo agissant sur une partie du bords. D'abords, en combinant le critère général d'Arendt Batty avec le théorème de Holmgren, on montre la stabilité forte du notre système dont l'absence de la compacité de la résolvent et sans aucune condition géométrique considéré sur le domaine. Puis, on montre que notre système n'est uniformement stable en général. Donc, on éspère qu'il y a une décroissance énérgetique polynomial pour des données initiales assez régulières. Pour ce but, en appliquant la méthode frequentielle combinée avec une méthode de multiplicateur et, on suppose que la région du bords contrôlée satsifait la "Geometric Control Conditon" (GCC) et en utilisant la décroissance exponentielle des équations d'ondes avec un controle standard, on montre une décroissance énergetique polynomiale qui depends de l'ordre de la dérivée fractionnaire.

La deuxième partie est consacrée à l'étude de la stabilisation d'un système de deux équations d'ondes couplées fortement sous l'effet d'un seul amortissement frontière de type fractionnaire appliqué à une seul équation est considéré. Dans ce cas, la stabilité du système est influencé par la nature arithmétique de la quotient de vitesse de propagation des ondes et par la nature algébrique du terme de couplage. Par conséquence, différents résultats de stabilité polynomial sont établis qui depends de l'ordre de la dérivée fractionnaire.

La troisième partie est consacrée à l'étude de la stabilisation d'un système multidimensionel de deux équations d'ondes couplées fortement sous l'effet d'un seul amortissememnt frontière de type fractionnaire appliqué à une seule équation. Tout d'abords, on suppose que la région controlé du bords vérifie "Multiplier Geometric Condition" (MGC), en combinant le cryitère général d'Arendt-Batty avec une méthode de multiplicateur, on montre la stabilité forte du notre système, sous l'égalité de la vitesse des ondes et une condition sur le terme de couplage, dont l'absence de la compacité de la résolvente. Puis, la stabilité non exponentielle est prouvé dans la deuxième partie. D'où, une stabilitée polynomiale est espérée pour des données initiales assez régulière en appliquant la méthode fréquentielle et la condition (MGC), qui depends de l'ordre de la dérivée fractionnaire.

Dans la quatrième partie, dans le cas de la stabilité d'un système de deux équations d'ondes couplées, un amortissement viscoélastique locallement distribué de type Kelvin-Voight est appliqué à une seule équation. D'abords, d'après un théorème d'Hormander et une bonne continuation d'une résultat sur l'estimation de Carelman et par conséquent la stabilité forte du système est assurée. Puis, la stabilité non exponentielle est prouvée dans la deuxième partie et finalement, une décroissance polynomial de l'énergie du système est etablie.

In this thesis, we study the stabilization of the system of wave equations with locally boundary fractional dissipation law. Also, we study the stability of coupled wave equations with one viscoelastic damping around the boundary Γ of type Kelvin-Voight. This ph.D thesis is divided into 4 parts. In part 1, we consider a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. First, combining a general criteria of Arendt and Batty with Holmgren's theorem, we show the strong stability of our system in the absence of the compactness of the resolvent and without any additional geometric conditions. Next, we show that our system is not uniformly stable in general, since it is the case of the interval. Hence, we look for a polynomial decay rate for smooth initial data of our system by applying a frequency domain approach combined with a multiplier method. Indeed, by assuming that the boundary control region satisfies the Geometric Control Condition (GCC) and by using the exponential decay of the wave equation with a standard damping, we establish a polynomial energy decay rate for smooth solutions, which depends on the order of the fractional derivative.

In part 2, we study the stability of one-dimensional coupled wave equation via one order terms with one boundary fractional damping acting on a part of the boundary of the domain. The stability of our system is influenced by the arithmetic nature of the wave propagation velocity quotient and by the algebraic nature of the coupling term. Consequently, different results of the polynomial stability are established, which depends on the order of the fractional derivative.

In part 3, we study the stability of multidimensional coupled wave equation via one order terms with one boundary fractional damping acting on a part of the boundary of the domain. First, by combining a general criteria of Arendt and Batty with a multiplier method, we show the strong stability of our system under the equality of speed propagations and some conditions on the coupling parameter term, in the absence of the compactness of the resolvent under the multiplier geometric condition denoted by (MGC). Next, under the equality of speed propagations and another condition on the coupling parameter term, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain under the multiplier geometric condition, which depends on the order of the fractional derivative.

In part 4, we study the stability of a system of two coupled wave equations on one locally viscoelastic damping of type Kelvin-Voight applied for one equation around the boundary Γ. First, the strong stability of the system is ensured using a Hormander Theorem and a mild continuation of Carelman estimation. Next, the nonuniform stability is proved. Finally, an optimal polynomial energy decay rate of system is established.

Thesis overview

This thesis is devoted to study the stabilization of the system of waves equations with one boundary fractional damping acting on apart of the boundary of the domain and the stabilization of a system of waves equations with locally viscoelastic damping of Kelvin-Voight type. First, we study the stability of the multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. Second, we study the stability of the system of coupled one-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Next, we study the stability of the system of coupled multi-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Finally, we study the stability of the multidimensional waves equations with locally viscoelastic damping of Kelvin-Voight is applied for one equation around the boundary of the domain.

Let Ω be a bounded set in $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2 where Γ_0 and Γ_1 are open subsets of Γ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, Γ_1 is non empty and $\nu = (\nu_1, \ldots, \nu_d)$ is the outward unit normal along the boundary Γ.

Definition 0.0.1. *We say that Γ satisfies the Geometric Control Condition named **GCC**, if every ray of geometrical optics, starting at any point $x \in \Omega$ at time $t = 0$, hits Γ_1 in finite time T.*

Definition 0.0.2. *We say that the Multiplier Geometric Control Condition **MGC** holds if there exist $x_0 \in \mathbb{R}^d$ and a positive constant $m_0 > 0$ such that*

$$m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad m \cdot \nu \geq m_0 \quad \text{on} \quad \Gamma_1.$$

This dissertation is divided into five chapters.

<u>**Chapter 1**</u> : In this Chapter, firstly, we present some well known results on a C_0-semigroup, including some theorems on strong, exponential and polynomial stability of a C_0-semigroup. Secondly, we define the fractional derivative operator in the sense of Caputo and we present some physical interpretations. Next, we define two different types of geometric conditions and we present some models that satisfy or do not satisfy these conditions. Finally, we present an appendix that contains almost all the secondary calculations used in this Thesis.

<u>**Chapter 2**</u> : In this Chapter, suppose that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and Γ_1 is non empty. We consider the

multidimensional wave equation

$$
\begin{aligned}
u_{tt} - \Delta u &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (0.0.1)\\
u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, && (0.0.2)\\
\frac{\partial u}{\partial \nu} + \gamma \partial_t^{\alpha,\eta} u &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+. && (0.0.3)
\end{aligned}
$$

The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see [20]) of order α with respect to the time variable and is defined by

$$
\partial_t^{\alpha,\eta} \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds, \quad 0 < \alpha < 1, \ \eta \geq 0. \tag{0.0.4}
$$

The system (0.0.1)-(0.0.3) is considered with initial conditions

$$
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{where} \quad x \in \Omega. \tag{0.0.5}
$$

In [46], B. Mbodje considered a 1-d wave equation with boundary fractional damping acting on a part of the boundary of the domain :

$$
\left\{
\begin{aligned}
\partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0, \qquad 0 < x < 1, \quad t > 0,\\
u(0,t) &= 0,\\
\partial_x u(1,t) + \gamma \partial_t^{\alpha,\eta} u(1,t) &= 0, \qquad 0 < \alpha < 1, \quad \eta \geq 0,\\
u(x,0) &= u_0(x),\\
\partial_t u(x,0) &= v_0(x).
\end{aligned}
\right. \tag{0.0.6}
$$

Theorem 0.0.3. *Let μ be a function defined by*

$$
\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \quad \xi \in \mathbb{R} \quad \text{and} \quad \alpha \in]0,1[,
$$

then the relation between the input "U" and the output "O" of the following system

$$
\varphi_t(\xi,t) + \left(|\xi|^2 + \eta\right) \varphi(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}, \ \eta \geq 0 \quad et \quad t \in \mathbb{R}_0^+,
$$

$$
\varphi(\xi,0) = 0,
$$

$$
O(t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \mu(\xi) \varphi(\xi,t) d\xi,
$$

is given by

$$
O(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-\eta(t-\tau)}}{(t-\tau)^\alpha} \frac{dU}{d\tau}(\tau) d\tau.
$$

Firstly, B. Mbodje used theorem 0.0.3, to reformulate system (0.0.6). Next, he proved that system (0.0.6) is not uniformly stable; In other words, its energy has no exponential decay rate. However, using LaSalle's invariance principle, he proved that the system (0.0.6) is strongly stable for usual initial data. Secondly, he established a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data.

In this Chapter, our main interest is to generalize the results of [46] by considering the multi-dimensional case and by improving the polynomial energy day rate.

Firstly, we reformulate system (0.0.1)-(0.0.3) into an augmented system. For this aim, we need the following results

Theorem 0.0.4. *Let μ be the function defined by*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-d}{2}}, \quad \xi \in \mathbb{R}^d \quad \textit{and} \quad 0 < \alpha < 1,$$

then the relation between the 'input' U and the 'output' O of the following system

$$\partial_t \omega(\xi,t) + (|\xi|^2+\eta)\omega(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^d, \quad t > 0,$$

$$\omega(\xi,0) = 0,$$

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi,$$

is given by

$$O(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-\eta(t-\tau)}}{(t-\tau)^\alpha} \frac{dU}{d\tau}(\tau)d\tau.$$

Now, using Theorem 0.0.4, system (0.0.1)-(0.0.5) recast into the following augmented model :

$$u_{tt} - \Delta u = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{0.0.7}$$

$$u = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \tag{0.0.8}$$

$$\partial_t \omega(\xi,t) + (|\xi|^2+\eta)\omega(\xi,t) - \mu(\xi)\partial_t u(x,t) = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \ \xi \in \mathbb{R}^d, \tag{0.0.9}$$

$$\frac{\partial u}{\partial \nu} + \gamma\kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ \tag{0.0.10}$$

where γ is a positive constant, $\eta \geq 0$ and $\kappa = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$.

Moreover, system (0.0.7)-(0.0.10) is considered with the following initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \omega(\xi,0) = 0, \quad x \in \Omega, \ \xi \in \mathbb{R}^d. \tag{0.0.11}$$

Our main interest is the existence, uniqueness and regularity of the solution to this system. We define the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^d)$$

equipped with the following inner product

$$((u,v,\omega),(\tilde{u},\tilde{v},\tilde{\omega}))_{\mathcal{H}} = \int_{\Omega}\left(v\bar{\tilde{v}} + \nabla u \nabla \bar{\tilde{u}}\right)dx + \tilde{\gamma}\kappa\int_{\mathbb{R}^d}\omega(\xi)\bar{\tilde{\omega}}(\xi)d\xi,$$

where $\tilde{\gamma} = \gamma\,|\Gamma_1|$ and $H^1_{\Gamma_0}(\Omega)$ is given by

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega), \quad u = 0 \ \text{ on } \ \Gamma_0\right\}.$$

The energy of the solution of system is defined by :

$$E(t) = \frac{1}{2}\|(u,u_t,w)\|^2_{\mathcal{H}}.$$

For smooth solution, a direct computation gives

$$E'(t) = -\tilde{\gamma}\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|w(\xi,t)|^2 d\xi.$$

Then, system (0.0.7)-(0.0.11) is dissipative in the sense that its energy is a nonincreasing function of the time variable t.
Now, we define the linear unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \left\{\begin{array}{l} U = (u,v,\omega)^\top \in \mathcal{H};\ \Delta u \in L^2(\Omega),\ v \in H^1_{\Gamma_0}(\Omega),\quad |\xi|\omega \in L^2(\mathbb{R}^d), \\ -(|\xi|^2+\eta)\omega + v|_{\Gamma_1}\mu(\xi) \in L^2(\mathbb{R}^d),\ \dfrac{\partial u}{\partial \nu}|_{\Gamma_1} = -\gamma\kappa\displaystyle\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi)d\xi \end{array}\right\}$$

and

$$\mathcal{A}(u,v,\omega)^\top = (v,\Delta u,-(|\xi|^2+\eta)\omega + v|_{\Gamma_1}\mu(\xi))^\top.$$

By denoting $v = u_t$ and $U_0 = (u_0,v_0,w_0)^\top$, system (0.0.7)-(0.0.11) can be rewritten as an abstract linear evolution equation on the space $\mathcal{H}$

$$U_t = \mathcal{A}U, \quad U(0) = U_0. \tag{0.0.12}$$

It easy to check that operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$, and consequently generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lumer-Phillips' theorem (see [42, 51]). Then the solution of the evolution equation (0.0.12) admits the following representation :

$$U(t) = e^{t\mathcal{A}}U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.12) . Hence, the semi-group theory allows to show the next existence and uniqueness result :

Theorem 0.0.5. *For any initial data $U_0 \in \mathcal{H}$, the problem* (0.0.12) *admits a unique weak* *solution* $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. *Moreover, if* $U_0 \in D(\mathcal{A})$*, then the problem* (0.0.12) *admits a unique strong solution* $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.

Secondly, we study the strong stability of system (0.0.7)-(0.0.11) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [58] or the spectrum decomposition theory of Benchimol [17] are not applicable in this case. We use then a general criteria of Arendt-Batty [9] and without any additional geometric condition, following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

Theorem 0.0.6. *Assume that* $\eta \geq 0$. *Then, the* C_0-*semigroup* $(e^{t\mathcal{A}})_{t\geq 0}$ *is strongly stable on the energy space* $\mathcal{H}$*, i.e., for any* $U_0 \in \mathcal{H}$*, we have*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

To prove Theorem 0.0.6, we need the following two lemmas.

Lemma 0.0.7. *Assume that* $\eta \geq 0$. *Then, for all* $\lambda \in \mathbb{R}$*, we have*

$$\ker (i\lambda I - \mathcal{A}) = \{0\}.$$

Remark that for $\eta = 0$, the operator $-\mathcal{A}$ is not invertible. Consequently, we prove the following lemma

Lemma 0.0.8. *If* $\eta > 0$*, for all* $\lambda \in \mathbb{R}$*, we have*

$$\mathrm{R}\,(i\lambda I - \mathcal{A}) = \mathcal{H}$$

while if $\eta = 0$*, for all* $\lambda \in \mathbb{R}^\star$*, we have*

$$\mathrm{R}\,(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Then, following a general criteria of Arendt-Batty (see [9]), the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, from Lemma 0.0.7 we directly deduce that $\mathcal{A}$ has no pure imaginary eigenvalues. Next, using Lemma 0.0.8, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$.

Now, our aim is to show that system (0.0.7)-(0.0.11) is not uniformly (*i.e.* exponentially) stable

in general, since it is already the case for $\Omega = (0,1)$ as shown below.
Our result is the following.

Theorem 0.0.9. *Assume that $d = 1$. The semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.*

This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. For this aim, we prove that then there exists a constant $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{|k|\geq k_0}$, for k large enough, a subsequence of eigenvalues satisfied the following asymptotic behavior

$$\lambda_k = i(k+\frac{1}{2})\pi + i\frac{\gamma \sin\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} - \frac{\gamma\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} + O\left(\frac{1}{k^{2-\alpha}}\right),$$

Then a decay of polynomial type is hoped. Hence, we consider the case where $\eta > 0$ and under the (GCC) condition. For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [18] (see also [14, 15, 40]) combined with a multiplier method and using the exponentially decay of the problem of wave equation with standard boundary damping on Γ_1 :

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) &= \quad 0, & x \in \Omega, \quad t>0, \\ \qquad \varphi(x,t) &= \quad 0, & x\in\Gamma_0, \quad t>0, \\ \quad \partial_\nu\varphi(x,t) &= -\varphi_t(x,t), & x\in\Gamma_1, \quad t>0. \end{cases} \tag{0.0.13}$$

Define the auxiliary space $\mathcal{H}_a = H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$ and the auxiliary unbounded linear operator $\mathcal{A}_a$ by

$$D(\mathcal{A}_a) = \left\{\Phi = (\varphi,\psi)\in\mathcal{H}_a : \quad \Delta\varphi\in L^2(\Omega); \quad \psi\in H^1_{\Gamma_0}(\Omega); \quad \frac{\partial\varphi}{\partial\nu} = -\psi \quad \text{on} \quad \Gamma_1\right\}$$

$$\mathcal{A}_a(\varphi,\psi) = (\psi,\Delta\varphi)\,.$$

Then, we introduce the following hypothesis :

$$(H): \text{ the problem (0.0.13) is uniformly stable in the energy space } H^1_{\Gamma_0}(\Omega)\times L^2(\Omega).$$

Now, we present the main result of this Chapter.

Theorem 0.0.10. *Assume that $\eta > 0$ and that the condition (H) holds. Then, for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution U of (0.0.12) satisfies the following estimation*

$$E(t,U) \leq C\frac{1}{t^{\frac{1}{1-\alpha}}}\|U_0\|^2_{D(\mathcal{A})}. \tag{0.0.14}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

To proof this theorem, by tacking $\ell = 2-2\alpha$, the polynomial energy decay $E(t)$ holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{H1}$$

and

$$\sup_{|\lambda|\in\mathbb{R}} \frac{1}{|\lambda|^{\ell}}\|(i\lambda I-\mathcal{A})^{-1}\| < +\infty \tag{H2}$$

are satisfied. Condition $(H1)$ is already proved in theorem 0.0.6. We will prove condition $(H2)$ using an argument of contradiction. For this purpose, suppose that $(H2)$ is false, then there exist a real sequence (λ_n), with $|\lambda_n| \to +\infty$ and a sequence $(U^n) \subset D(\mathcal{A})$, verifying the following conditions

$$\|U^n\|_{\mathcal{H}} = \|(u^n, v^n, \omega^n)\|_{\mathcal{H}} = 1 \tag{0.0.15}$$

and

$$\lambda_n^{\ell}(i\lambda_n - \mathcal{A})U^n = (f_1^n, f_2^n, f_3^n) \to 0 \quad \text{in} \quad \mathcal{H}. \tag{0.0.16}$$

In the following, we will check the condition $(H2)$ by finding a contradiction with (0.0.15) such as $\|U\|_{\mathcal{H}} = o(1)$. Consequently, condition $(H2)$ holds, and the energy of smooth solution of system (0.0.7)-(0.0.11) decays polynomial to zero as t goes to infinity. Finally, using the density of the domain $D(\mathcal{A})$ in $\mathcal{H}$, we can easily prove that the energy of weak solution of the system (0.0.7)-(0.0.11) decays to zero as t goes to infinity.

Chapter 3 : In this chapter, we study the stability of a system of coupled wave equations in one dimensional case with a fractional damping acting on a part of the boundary of the domain. We consider the coupled wave equations

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & \text{on} \quad]0,1[\times]0,+\infty[, \\ y_{tt} - ay_{xx} - bu_t = 0 & \text{on} \quad]0,1[\times]0,+\infty[\end{cases} \tag{0.0.17}$$

where $(x,t) \in]0,1[\times]0,+\infty[$, $a>0$ and $b \in \mathbb{R}^*$.
This system is subjected to the boundary conditions

$$\begin{cases} u(0,t) = 0 & \text{in} \quad]0,+\infty[, \\ y(0,t) = y(1,t) = 0 & \text{in} \quad]0,+\infty[, \\ u_x(1,t) = -\gamma\partial_t^{\alpha,\eta}u(1,t) & \text{in} \quad]0,+\infty[\end{cases} \tag{0.0.18}$$

where $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to time variable. It is defined in equation (0.0.4). The system (0.0.17), (0.0.18) is considered with initial conditions

$$\begin{cases} u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \text{where} \quad x\in]0,1[, \\ y(x,0)=y_0(x), \quad y_t(x,0)=y_1(x) \quad \text{where} \quad x\in]0,1[. \end{cases} \tag{0.0.19}$$

First, using Theorem 0.0.3, for $d=1$, we reformulate system (0.0.17)-(0.0.19) into an augmented model defined by

$$\begin{aligned} u_{tt}-u_{xx}+by_t &= 0, \quad \text{on} \quad (0,1)\times\mathbb{R}^+,, && (0.0.20)\\ y_{tt}-ay_{xx}-bu_t &= 0, \quad \text{on} \quad (0,1)\times\mathbb{R}^+, && (0.0.21)\\ \omega_t(\xi,t)+(|\xi|^2+\eta)\omega(\xi,t)-u_t(1,t)\mu(\xi) &= 0, \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^+, && (0.0.22)\\ y(0,t)=y(1,t)=u(0,t) &= 0, && (0.0.23)\\ u_x(1,t)+\gamma\kappa\int_{\mathbb{R}}\mu(\xi)\omega(\xi,t)d\xi &= 0, && (0.0.24) \end{aligned}$$

This system is subject to the boundary conditions

$$\begin{aligned} u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), && (0.0.25)\\ y(x,0)=y_0(x), \quad y_t(x,0)=y_1(x), && (0.0.26)\\ \omega(\xi,0)=0. && (0.0.27) \end{aligned}$$

where $\kappa=\frac{\sin(\alpha\pi)}{\pi}$. Now, using a semigroup approach, we establish well-posedness result for the problem (0.0.20)-(0.0.27). First the energy of this system is given by

$$E(t)=\frac{1}{2}\left(\int_0^1\left(|u_t|^2+|y_t|^2+|u_x|^2+a|y_x|^2\right)dx+\gamma\kappa\int_{-\infty}^{+\infty}|\omega|^2d\xi\right).$$

Then a straightforward computation gives

$$E'(t)=-\gamma\kappa\int_{-\infty}^{+\infty}(\xi^2+\eta)|\omega|^2d\xi\leq 0.$$

Thus, the system (0.0.20)-(0.0.27) is dissipative in the sense that its energy is nonincreasing with respect to the time t. Next, we define the Hilbert space

$$\mathcal{H}=H_L^1(]0,1[)\times L^2(]0,1[)\times H_0^1(]0,1[)\times L^2(]0,1[)\times L^2(\mathbb{R}),$$

endowed with inner product

$$\left\langle U,\widetilde{U}\right\rangle=\int_0^1\left(u_x\bar{\tilde{u}}_x+v\bar{\tilde{v}}+ay_x\bar{\tilde{y}}_x+z\bar{\tilde{z}}\right)dx+\gamma\kappa\int_{-\infty}^{+\infty}\omega\bar{\tilde{\omega}}d\xi,$$

for all $U=(u,v,y,z,\omega)^\top\in\mathcal{H}$ and $\widetilde{U}=(\tilde{u},\tilde{v},\tilde{y},\tilde{z},\tilde{\omega})\in\mathcal{H}$, where $H_L^1(]0,1[)$ is the Sobolev

space defined by
$$H_L^1(]0,1[) = \left\{ u \in H^1(\Omega), \qquad u(0) = 0 \right\}.$$
Finally, we define the unbounded linear operator $\mathcal{A}$ by

$$D(A) = \left\{ \begin{array}{l} U = (u,v,y,z,\omega)^\top \in \mathcal{H};\ u \in H^2\left(]0,1[\right) \cap H_L^1(]0,1[), \\ y \in H^2\left(]0,1[\right) \cap H_0^1(]0,1[), v \in H_L^1(]0,1[),\ z \in H_0^1\left(]0,1[\right), \\ -\left(\xi^2+\eta\right)\omega + v\left(1\right)\mu(\xi) \in L^2\left(\mathbb{R}\right), \\ u_x(1) + \gamma\kappa \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi = 0,\ |\xi|\omega \in L^2\left(\mathbb{R}\right). \end{array} \right\},$$

and
$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - bz \\ z \\ ay_{xx} + bv \\ -\left(\xi^2+\eta\right)\omega + v(1)\mu(\xi) \end{pmatrix}$$
If $U = (u, u_t, y, y_t, \omega)^T$ is a regular solution of system (0.0.20)-(0.0.27), then we rewrite this system as the following evolution equation
$$\left\{ \begin{array}{l} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{array} \right. \tag{0.0.28}$$

where $U_0 = (u_0, u_1, y_0, y_1, \omega)^\top$. It is known that operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lumer-Phillips' theorem (see [42, 51]). Then, the solution to the evolution equation (0.0.28) admits the following representation :
$$U(t) = e^{t\mathcal{A}}U_0, \quad t \geq 0,$$
which leads to the well-posedness of (0.0.28) . Hence, semi-group theory allows to show the next existence and uniqueness results :

Theorem 0.0.11. *For any initial data $U_0 \in \mathcal{H}$, the problem (0.0.28) admits a unique weak solution $U(t) \in C^0\left(\mathbb{R}^+, \mathcal{H}\right)$. Moreover if $U_0 \in D(\mathcal{A})$ then the problem (0.0.28) admits a unique strong solution $U(t) \in C^1\left(\mathbb{R}^+, \mathcal{H}\right) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

Secondly, we study the strong stability of system (0.0.20)-(0.0.27) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [58] or the spectrum decomposition theory of Benchimol [17] are not applicable in this case. We use then a general criteria of Arendt-Battay [9], following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A})\cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result :

Theorem 0.0.12. *Assume that $\eta \geq 0$. Then the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e., for any $U_0 \in \mathcal{H}$ we have*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

To proof Theorem 0.0.12, we need the following two lemmas

Lemma 0.0.13. *Assume that $\eta \geq 0$ and b satisfying the following condition*

$$b^2 \neq \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)\pi^2}{(a+1)(k_1^2+k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}. \tag{C}$$

Then, for all $\lambda \in \mathbb{R}$, we have

$$\ker\left(i\lambda I - \mathcal{A}\right) = \{0\}.$$

We remark that for $\eta = 0$, the operator $-\mathcal{A}$ it's not invertible and consequently, we proove

Lemma 0.0.14. *If $\eta > 0$, for all $\lambda \in \mathbb{R}$, we have*

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}$$

while if $\eta = 0$, for all $\lambda \in \mathbb{R}^\star$, we have

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}.$$

Consequently, following a general criteria of Arendt-Batty see [9], the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma\left(\mathcal{A}\right)\cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, from Lemma 0.0.13 we directly deduce that $\mathcal{A}$ has non pure imaginary eigenvalues. Next, using Lemma 0.0.14, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A})\cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A})\cap i\mathbb{R} = \{0\}$ if $\eta = 0$.
Now, our aim is to show that system (0.0.20)-(0.0.27) is not uniformly (*i.e.* exponentially) stable in general since it is already the case for $\Omega = (0,1)$ as shown below. Our result is the following

Theorem 0.0.15. *Assume that $d = 1$. The semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.*

This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. Then, we proved that, for $a = 1$ and b satisfying the condition (C), then there exists a constant $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{|k|\geq k_0}$, for k large enough, a subsequence of eigenvalues satisfied the following asymptotic behavior

$$\lambda_k = \frac{ik\pi}{2} + \frac{c_1}{k^{1-\alpha}} + \frac{ic_2}{k^{1-\alpha}} + O\left(\frac{1}{k^{1-\alpha}}\right)$$

for

$$c_1 = \frac{\gamma(-1)^k\left(\cos(b) - (-1)^k\right)\left(\cos\left(\frac{\pi}{2}(1-\alpha)\right)\right)}{2^\alpha k^{1-\alpha}} \quad \text{and} \quad c_2 = \frac{\gamma(-1)^k\left(\cos(b) - (-1)^k\right)\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{2^\alpha k^{1-\alpha}}.$$

Then a decay of polynomial type is hoped. For that purpose, we will use a frequency domain approach, namely we will use theorem 2.4 of [18] (see also [14, 15, 40]).

Theorem 0.0.16. *Suppose that $\eta > 0$, $a = 1$ and b satisfies the condition (C). Then, for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution of U of (0.0.28) satisfies the following estimation*

$$E(t,U) \leq \frac{C}{t^{\ell(\alpha)}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0,$$

where

$$\ell(\alpha) = \begin{cases} \frac{1}{3-\alpha} & \text{if} \quad a = 1 \quad \text{and} \quad b = k\pi \\ \frac{1}{1-\alpha} & \text{if} \quad a = 1 \quad \text{and} \quad b \neq k\pi \end{cases}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

Theorem 0.0.17. *Assume that $\eta > 0$, $a \neq 1$ and b satisfies condition (C). If ($a \in \mathbb{Q}$ and b small enough) or $\sqrt{a} \in \mathbb{Q}$, then for all $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution U of (0.0.28) satisfies the following estimation*

$$E(t,U) \leq \frac{C}{t^{\frac{1}{3-\alpha}}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

Chapter 4 : In this chapter, we study the stabilization of the system of multidimensional wave equation defined by :

$$\begin{aligned}
u_{tt} - \Delta u + by_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (0.0.29)\\
y_{tt} - a\Delta y - bu_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (0.0.30)\\
u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, && (0.0.31)\\
y &= 0, \quad \text{on} \quad \Gamma \times \mathbb{R}^+, && (0.0.32)\\
\frac{\partial u}{\partial \nu} + \gamma \partial_t^{\alpha,\eta} u &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, && (0.0.33)
\end{aligned}$$

where ν is the unit outward normal vector along the boundary Γ_1, γ is a positive constant involved in the boundary control, $a > 0$ and $b \in \mathbb{R}_*$. The notation $\partial_t^{\alpha,\eta}$ stands the generalized Caputo's fractional derivative see [20] of order α with respect to the time variable and is defined in equation (0.0.4) The system (0.0.29)-(0.0.33) is considered with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{where} \quad x \in \Omega, \tag{0.0.34}$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad \text{where} \quad x \in \Omega, \tag{0.0.35}$$

First, using Theorem 0.0.4, we reformulate our system into the augmented model

$$\begin{aligned}
u_{tt} - \Delta u + by_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, &(0.0.36)\\
y_{tt} - a\Delta y - bu_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, &(0.0.37)\\
\partial_t \omega(\xi,t) + \left(|\xi|^2 + \eta\right)\omega(\xi,t) - \mu(\xi)\partial_t u(x,t) &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \ \xi \in \mathbb{R}^d, &(0.0.38)\\
u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, &(0.0.39)\\
y &= 0, \quad \text{on} \quad \Gamma \times \mathbb{R}^+, &(0.0.40)\\
\frac{\partial u}{\partial \nu} + \gamma\kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ &(0.0.41)
\end{aligned}$$

where γ is a positive constant, $\eta \geq 0$ and $\kappa = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$. Finally, system (0.0.36)-(0.0.41) is considered with the following initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{where} \quad x \in \Omega, \tag{0.0.42}$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad \text{where} \quad x \in \Omega, \tag{0.0.43}$$
$$\omega(\xi,0) = 0 \qquad \text{where} \quad \xi \in \mathbb{R}^d. \tag{0.0.44}$$

Our main interest is the existence, uniqueness and regularity of the solution of this system. We define the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^N), \tag{0.0.45}$$

equipped with following inner product

$$\left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}} = \int_\Omega \left(v\bar{\tilde{v}} + \nabla u \nabla \bar{\tilde{u}} + z\bar{\tilde{z}} + a\nabla y \nabla \bar{\tilde{y}} \right) dx + \tilde{\gamma}\kappa \int_{\mathbb{R}^d} \omega(\xi)\bar{\tilde{\omega}}(\xi) d\xi$$

where $\tilde{\gamma} = \gamma\,|\Gamma_1|$ $U = (u,v,y,z,\omega), \widetilde{U} = (\tilde{u},\tilde{v},\tilde{y},\tilde{z},\tilde{\omega}) \in \mathcal{H}$ and $H^1_{\Gamma_0}(\Omega)$ is given by

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega), \quad u = 0 \ \text{ on } \ \Gamma_0 \right\}.$$

The energy of the solution of system (0.0.36)-(0.0.44) is defined by :

$$E(t) = \frac{1}{2} \left\| (u, u_t, y, y_t, \omega) \right\|_{\mathcal{H}}^2 . \tag{0.0.46}$$

For smooth solution, a direct computation gives

$$E'(t) = -\gamma\kappa \int_{\mathbb{R}^d} \left(|\xi|^2 + \eta \right) |\omega(\xi, t)|^2 d\xi. \tag{0.0.47}$$

Then, system (0.0.36)-(0.0.44) is dissipative in the sense that its energy is a non-increasing function of the time variable t. Now, we define the linear unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega)^\top \in \mathcal{H};\ \Delta u \in L^2(\Omega), \quad y \in H^2(\Omega) \cap H_0^1(\Omega), \\ v \in H_{\Gamma_0}^1(\Omega), \quad z \in H_0^1(\Omega), \quad -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1} \mu(\xi) \in L^2(\mathbb{R}^d), \\ \dfrac{\partial u}{\partial \nu} + \gamma\kappa \displaystyle\int_{\mathbb{R}^d} \mu(\xi)\omega(\xi) d\xi = 0 \ \text{ on } \ \Gamma_1, \quad |\xi|\omega \in L^2(\mathbb{R}^d) \end{array} \right\} \tag{0.0.48}$$

and

$$\mathcal{A} \begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - bz \\ z \\ a\Delta y + bv \\ -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1} \mu(\xi). \end{pmatrix}$$

By denoting $v = u_t$ and $z = y_t$ and $U_0 = (u_0, v_0, y_0, z_0, \omega_0)^\top$, system (0.0.36)-(0.0.44) can be written as an abstract linear evolution equation on the space $\mathcal{H}$

$$\left\{ \begin{array}{l} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{array} \right. \tag{0.0.49}$$

It is known that operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lumer-Phillips' theorem (see [42, 51]). Then the solution to the evolution equation (0.0.28) admits the following representation :

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.49) . Hence, semi-group theory allows to show the next existence and uniqueness results :

Theorem 0.0.18. *For any initial data $U_0 \in \mathcal{H}$, the problem (0.0.49) admits a unique weak solution $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. Moreover if $U_0 \in D(\mathcal{A})$ then the problem (0.0.49) admits a unique strong solution $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

Secondly, we study the strong stability of system (0.0.36)-(0.0.44) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [58] or the spectrum decomposition theory of Benchimol [17] are not applicable in this case. We use then a general criteria of Arendt-Battay [9] combining with a specifying multiplier and under the **MGC** condition, following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

Theorem 0.0.19. *Assume that $\eta \geq 0$,$a = 1$ and b is small enough. Then the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e., for any $U_0 \in \mathcal{H}$ we have*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

To proof theorem 0.0.19, we need the following two lemmas

Lemma 0.0.20. *Assume that $\eta \geq 0$, $a = 1$ and b is small enough. Then, for all $\lambda \in \mathbb{R}$, we have*

$$\ker\left(i\lambda I - \mathcal{A}\right) = \{0\}.$$

We remark that for $\eta = 0$, the operator $-\mathcal{A}$ it's not invertible and consequently, we proofed

Lemma 0.0.21. *Suppose that $a = 1$ and b is small enough. Then, if $\eta > 0$, for all $\lambda \in \mathbb{R}$, we have*

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}$$

while if $\eta = 0$, for all $\lambda \in \mathbb{R}^{\star}$, we have

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}.$$

Then, following a general criteria of Arendt-Batty see [9], the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma\left(\mathcal{A}\right) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, from Lemma 0.0.20 we directly deduce that $\mathcal{A}$, for $a = 1$ and b small enough, has non pure imaginary eigenvalues. Next, using Lemma 0.0.21, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$.
The nonuniform stability is proved in **Chapter 3**. Then, a decay of polynomial type is hoped. For this aim, we consider the case where $\eta > 0$, $a = 1$ and b small enough. For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [18] combining with a multiplier method and the **MGC** condition. Now, we present the main result

Theorem 0.0.22. *Assume that $\eta > 0$, $a = 1$ and b small enough. Then, for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the energy* *of the strong solution of* (0.0.49) *satisfies the following estimation*

$$E(t,U) \leq C\frac{1}{t^{\frac{1}{1-\alpha}}}\|U_0\|^2_{D(\mathcal{A})}.$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

Chapter 5 : In this chapter, we consider the following two wave equations coupled with a viscoelastic damping around the boundary Γ :

$$\begin{cases} \rho_1(x)u_{tt} - \mathrm{div}(a_1(x)\nabla u + b(x)\nabla u_t) + \alpha y_t = 0 & \text{in} \quad \Omega\times\mathbb{R}^+, \\ \rho_2(x)y_{tt} - \mathrm{div}(a_2(x)\nabla y) - \alpha u_t = 0 & \text{in} \quad \Omega\times\mathbb{R}^+, \end{cases} \tag{0.0.50}$$

with the following initial conditions :

$$u(\cdot,0) = u_0\left(\cdot\right),\ y(\cdot,0) = y_0\left(\cdot\right),\ u_t(\cdot,0) = u_1\left(\cdot\right),\ y_t(\cdot,0) = y_1\left(\cdot\right) \quad \text{in } \Omega, \tag{0.0.51}$$

and the following boundary conditions :

$$u\left(x,t\right) = y\left(x,t\right) = 0 \quad \text{on } \Gamma\times\mathbb{R}^+. \tag{0.0.52}$$

The functions $\rho_1, \rho_2, a_1, a_2, b \in L^\infty(\Omega)$ such that

$$\rho_1(x) \geq \rho_1,\ \ \rho_2(x) \geq \rho_2,\ \ a_1(x) \geq a_1,\ \ a_2(x) \geq a_2$$

and α is a real constant number.

The local viscoelastic damping is a natural phenomena of bodies which have one part made of viscoelastic material, and the other is made of elastic material. There are a few number of publications concerning the wave equation with local viscoelastic damping.

First, using a semi-group approach, we establish well-posedness result for the system Kelvin Voight with viscoelastic damping $\mathrm{div}(b(x)\nabla u_t)$ be applied around the boundary Γ. For this aim, we define the energy of system (0.0.50)-(0.0.52)by :

$$E(t) = \frac{1}{2}\int_\Omega \left(\rho_1(x)|u_t|^2 + \rho_2(x)|y_t|^2 + a_1(x)|\nabla u|^2 + a_2(x)|\nabla y|^2\right)dx. \tag{0.0.53}$$

Then a straightforward computation gives

$$E'(t) = -\int_\Omega b(x)|\nabla u_t|^2 dx \leq 0.$$

Thus, the system (0.0.50)-(0.0.52) is dissipative in the sense that its energy is nonincreasing with respect to the time t. For any $\gamma > 0$ we define the $\gamma-$neighborhood O_γ of the boundary

Γ as follows

$$O_\gamma = \{x \in \Omega \ : \quad |x - y| \leq \gamma, \quad y \in \Gamma\}. \tag{0.0.54}$$

More precisely, we assume that

$$\begin{cases} \rho_1(x) \geq \rho_1 > 0, \ \rho_2(x) \geq \rho_2 > 0, \ a_1(x) \geq a_1 > 0, \ a_2(x) \geq a_2 > 0 \quad \text{for all } x \in \Omega, \\ \\ b(x) \geq b_0 > 0 \quad \text{for all } x \in O_\gamma. \end{cases}$$

Next, we define the Hilbert space

$$\mathcal{H} = \left(H_0^1(\Omega) \times L^2(\Omega)\right)^2$$

endowed with the inner product

$$\left\langle U, \widetilde{U} \right\rangle = \int_\Omega \left(a_1 \nabla u \cdot \nabla \bar{\tilde{u}} + a_2 \nabla y \cdot \nabla \bar{\tilde{y}} + \rho_1 v \bar{\tilde{v}} + \rho_2 z \bar{\tilde{z}}\right) dx,$$

for all $U = (u, v, y, z)^\top \in \mathcal{H}$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})^\top \in \mathcal{H}$. Finally, we define the unbounded linear operator $\mathcal{A}$ by

$$\begin{aligned} D(\mathcal{A}) = \Big\{ &(u, v, y, z) \in \mathcal{H} : \ \operatorname{div}(a_1(x) \nabla u + b(x) \nabla v) \in L^2(\Omega), \\ &\qquad \operatorname{div}(a_2(x) \nabla y) \in L^2(\Omega) \quad \text{and} \quad v, z \in H_0^1(\Omega) \Big\}, \end{aligned} \tag{0.0.55}$$

and

$$\mathcal{A} \begin{pmatrix} u \\ v \\ y \\ z \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{1}{\rho_1(x)} \left(\operatorname{div}(a_1(x) \nabla u + b(x) \nabla v) - \alpha z\right) \\ z \\ \dfrac{1}{\rho_2(x)} \left(\operatorname{div}(a_2(x) \nabla y) + \alpha v\right) \end{pmatrix}.$$

By denoting $v = u_t$, $z = y_t$ and $U_0 = (u_0, v_0, y_0, z_0, w_0)^\top$, system (0.0.50)-(0.0.52) can be written as an abstract linear evolution equation on the space $\mathcal{H}$

$$U_t = \mathcal{A} U, \quad U(0) = U_0. \tag{0.0.56}$$

It is known that, under the hypothesis (H), the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lumer-Phillips' theorem (see [42, 51]). Then the solution of the evolution equation (0.0.56) admits the following representation :

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.52) . Hence, the semi-group theory allows to show the next existence and uniqueness result :

Theorem 0.0.23. *For any initial data $U_0 \in \mathcal{H}$, the problem* (0.0.56) *admits a unique weak solution $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. Moreover if $U_0 \in D(\mathcal{A})$ then the problem* (0.0.56) *admits a unique strong solution $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

Secondly, we study the strong stability of system (0.0.50)-(0.0.52) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [58] or the spectrum decomposition theory of Benchimol [17] are not applicable in this case. We use then a general criteria of Arendt-Battay [9] combining with a unique continuation result based on a Carleman estimate , following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

Theorem 0.0.24. *Under hypothesis* (H)*, the C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of* (0.0.56) *satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 0.0.24, we need the following two lemmas.

Lemma 0.0.25. *Under hypothesis* (H)*, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \qquad \forall\ \lambda \in \mathbb{R}.$$

Lemma 0.0.26. *Under hypothesis* (H)*, we have $i\lambda I - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}$.*

Next, we show that the system (0.0.50)-(0.0.52) is not exponentially stable. Throughout, this part, we assume that

$$a_1, a_2, \rho_1, \rho_2 \in \mathbb{R}^+ \quad \text{and} \quad b \in \mathbb{R}^+_* \tag{H'}$$

Then, we have the following result

Theorem 0.0.27. *Under hypothesis* (H')*, the system* (0.0.50)-(0.0.52) *is not uniformly stable in the energy space $\mathcal{H}$.*

This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. Then, we proved that then there exists a constant $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{|k|\geq k_0}$, for

k large enough, a subsequence of eigenvalues satisfied the following asymptotic behavior :

$$\lambda_k = i\sqrt{\frac{a}{\rho_2}}\mu_k - \frac{\alpha^2}{2b\rho_2\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right), \tag{0.0.57}$$

where μ_k, is the eigenvalues of $-\Delta$ i.e.

$$\begin{cases} -\Delta\varphi_k = \mu_k^2\varphi_k & \text{in} \quad \Omega, \\ \varphi_k = 0 & \text{on} \quad \Gamma. \end{cases}$$

Then a decay of polynomial type is hoped. Now, we assume that

$$a_1, a_2, \rho_1, \rho_2, b \in C^{1,1}(\overline{\Omega}). \tag{C1}$$

Also, we assume the following supplementary conditions.
There exists a function $q \in C^1(\Omega, \mathbb{R}^N)$ and $0 < \alpha < \beta < \gamma$, such that

$$\partial_j q_k = \partial_k q_j, \quad \operatorname{div}(a_1 a_2 \rho_2 q), \operatorname{div}(a_1 a_2 \rho_1 q) \in C^{0,1}(\Omega_\beta) \quad \text{and} \quad q = 0 \quad \text{on} \quad O_\alpha, \tag{C2}$$

There exists a constant $\sigma_1 > 0$, such that

$$2a_2\partial_j(a_{1k}q_k) + a_1(q_k\partial_j a_2 + q_j\partial_k a_2) + \left[a_1\left(\frac{a_2}{\rho_2}q\nabla\rho_2 - q\nabla a_2\right)\right] I \geq \sigma_1 I, \quad \forall x \in \Omega_\beta. \tag{C3}$$

There exists a constant $\sigma_2 > 0$, such that

$$2a_1\partial_j(a_{2k}q_k) + a_2(q_k\partial_j a_1 + q_j\partial_k a_1) + \left[a_2\left(\frac{a_1}{\rho_1}q\nabla\rho_1 - q\nabla a_1\right)\right] I \geq \sigma_2 I, \quad \forall x \in \Omega_\beta. \tag{C4}$$

There exists a constant $M > 0$ such that for all $v \in H_0^1(\Omega)$, we have

$$|(q\cdot\nabla v)\,\nabla b - (q\cdot\nabla b)\nabla v| \leq M\sqrt{b}\,|\nabla v|, \quad \forall x \in \Omega_\beta. \tag{C5}$$

Theorem 0.0.28. *Assume that conditions* (H), (C1) − (C6) *are satisfied. Then for all initial data $U_0 \in D(\mathcal{A})$, then there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution U of* (0.0.56) *satisfyies the following estimation :*

$$E(t, U) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.0.58}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t, U)$ converges to zero as t goes to infinity.

For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [18] (see also [14, 15, 40]) combining with a multiplier method. Then, we have showing that, firstly

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{on} \quad O_\gamma.$$

Next, we have showing that

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{on} \quad \Omega_\beta.$$

Hence, we obtain

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{on} \quad \Omega.$$

In conclusion, using equations (0.0.57) and (0.0.58), an optimal polynomial energy decay of type $\frac{1}{t}$ is obtained.

Aperçu de la thèse

Cette thèse est consacrée à l'étude de la stabilisation directe et indirecte de différents systèmes d'équations d'ondes avec un contrôle frontière de type fractionnaire ou un contrôle local viscoélastique de type Kelvin-Voight. Nous considérons, d'abords, la stabilisation de l'équation d'ondes multidimensionnel avec un contrôle frontière fractionnaire au sens de Caputo. Sous des conditions géométriques optimales, nous établissons un taux de décroissance polynomial de l'énergie de système. Ensuite, nous nous intéressons à l'étude de la stabilisation d'un système de deux équations d'ondes couplées via les termes de vitesses, dont une seulement est amortie avec contrôle frontière de type fractionnaire au sens de Caputo. Nous montrons différents résultats de stabilités dans le cas 1-d et N-d. Finalement, nous étudions la stabilité d'un système de deux équations d'ondes couplées avec un seul amortissement viscoélastique localement distribué de type Kelvin-Voight.

Cette thèse est divisée en cinq chapitres.

Chapitre 1. Dans ce chapitre, nous rappelons quelques définitions et théorèmes concernant la théorie de semi-groupe et l'analyse spectrale. Ainsi, nous présentons et discutons les conditions géométriques et les méthodes utilisées dans cette thèse pour obtenir notre résultats de la stabilité.

Chapitre 2. Ce chapitre est consacré à la stabilisation d'une équation d'onde sous l'action d'un amortissement de type fractionnaire au sens de Caputo. Soit Ω un ouvert borné non vide dans $\mathbb{R}^d$, $d \geq 2$, ayant une frontière Γ de classe C^2 composée de deux morceaux : Γ_0 la partie encastrée et Γ_1 la partie où on applique un amortissement de type fractionnaire où $\nu = (\nu_1, \nu_2, \ldots, \nu_N)$ est le vecteur normal unitaire extérieur à Γ. On considère l'équation suivante :

$$\left\{\begin{array}{llll} u_{tt} - \Delta u & = & 0 & \text{dans} \quad \Omega \times \mathbb{R}^+ \\ u & = & 0 & \text{sur} \quad \Gamma_0 \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial \nu} & = & -\gamma \partial_t^{\alpha,\eta} u & \text{sur} \quad \Gamma_1 \times \mathbb{R}^+ \end{array}\right. \tag{0.0.59}$$

où γ est une constante strictement positive et $\nu = (\nu_1, \nu_2, \ldots, \nu_N)$ est le vecteur normal unitaire extérieur à Γ. La notation $\partial_t^{\alpha,\eta}$ signifie la généralisation de la dérivée fractionnaire au sens de Caputo d'ordre α par rapport à la variable t et elle est définie par

$$\partial_t^{\alpha,\eta} \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds. \tag{0.0.60}$$

Les conditions initiales sont données par

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{tels que} \quad x\in\Omega.$$

Dans [46], Mbodje a étudié la stabilisation du système (0.0.59) dans le cas monodimensionnel. D'abords, il a reformulé le système en utilisant la même méthode du (Mbodje and Montseny) qui est basée sur le fait que la relation entrée-sortie par une équation de diffusion qui réalise l'opérateur de dérivée fractionnaire, définie par le théorème suivant

Théorème 0.0.29. *Soit μ la fonction définie par*

$$\mu(\xi)=|\xi|^{\frac{2\alpha-1}{2}}, \quad \xi\in\mathbb{R} \quad et \quad \alpha\in]0,1[,$$

alors la relation entre l'entrée "U" et la sortie "O" du système suivant

$$\varphi_t(\xi,t)+\left(|\xi|^2+\eta\right)\varphi(\xi,t)-U(t)\mu(\xi)=0, \quad \xi\in\mathbb{R},\ \eta\geq 0 \quad et \quad t\in\mathbb{R}_0^+,$$

$$\varphi(\xi,0)=0,$$

$$O(t)=\frac{\sin(\alpha\pi)}{\pi}\int_{-\infty}^{+\infty}\mu(\xi)\varphi(\xi,t)d\xi,$$

est donnée par

$$O(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{e^{-\eta(t-\tau)}}{(t-\tau)^\alpha}\frac{dU}{d\tau}(\tau)d\tau.$$

Toutefois, en utilisant la principe de l'invariance de LaSalle, il a prouvé que son système est fortement stable. Puis, il a demontré que l'énergie de la solution du son système décroît polynomialement comme $\frac{1}{t}$ vers 0 pour des données initiales assez régulières.
Dans le cas multidimensionnel, on a généralisé le travail du Mbodje et on a amélioré le taux de la décroissance énergétique polynomiale. Tout d'abord, on a reformulé le système (0.0.59) en utilisant le théorème suivant

Théorème 0.0.30. *Soit μ la fonction définie par*

$$\mu(\xi)=|\xi|^{\frac{2\alpha-d}{2}}, \quad \xi\in\mathbb{R}^d \quad et \quad \alpha\in]0,1[,$$

alors la relation entre l'entrée "U" et la sortie "O" du système suivant

$$\omega_t(\xi,t)+\left(|\xi|^2+\eta\right)\omega(\xi,t)-U(t)\mu(\xi)=0, \quad \xi\in\mathbb{R}^d,\ \eta\geq 0 \quad et \quad t\in\mathbb{R}_0^+,$$

$$\omega(\xi,0)=0,$$

$$O(t)=\frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}\int_{\mathbb{R}^d}\mu(\xi)\varphi(\xi,t)d\xi,$$

est donnée par

$$O(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{e^{-\eta(t-\tau)}}{(t-\tau)^\alpha}\frac{dU}{d\tau}(\tau)d\tau.$$

En utilisant le théorème 0.0.30, le système (0.0.59) sera écrit sous la forme d'un modèle augmenté situé ci-dessous

$$\begin{aligned}
u_{tt}-\Delta u &= 0, \quad \text{dans} \quad \Omega\times\mathbb{R}^+, && (0.0.61)\\
u &= 0, \quad \text{sur} \quad \Gamma_0\times\mathbb{R}^+, && (0.0.62)\\
\partial_t\omega(\xi,t)+(|\xi|^2+\eta)\omega(\xi,t)-\mu(\xi)\partial_t u(x,t) &= 0, \quad \text{sur} \quad \Gamma_1\times\mathbb{R}^+,\ \xi\in\mathbb{R}^d, && (0.0.63)\\
\frac{\partial u}{\partial\nu}+\gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi,t)d\xi &= 0, \quad \text{sur} \quad \Gamma_1\times\mathbb{R}^+ && (0.0.64)
\end{aligned}$$

où $\gamma>0$, $\eta\geq 0$ et $\kappa=\frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$ et le système (0.0.61)-(0.0.64) est considéré par les conditions initiales suivantes

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad \omega(\xi,0)=0,\quad x\in\Omega,\quad \xi\in\mathbb{R}^d. \qquad (0.0.65)$$

On définit l'espace de Hilbert

$$\mathcal{H}=H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\mathbb{R}^d)$$

muni du produit scaliare suivant

$$\left(U,\tilde{U}\right)=\int_\Omega \nabla u\nabla\bar{\tilde{u}}+v\bar{\tilde{v}}dx+\tilde{\gamma}\kappa\int_{\mathbb{R}^d}\omega\bar{\tilde{\omega}}d\xi.$$

où $\tilde{\gamma}=\gamma|\Gamma_1|$ et $H^1_{\Gamma_0}(\Omega)$ est définit par

$$H^1_{\Gamma_0}(\Omega)=\{u\in H^1(\Omega);\quad u_{|_{\Gamma_0}}=0\}$$

L'énergie de la solution du système est définie par

$$E(t)=\frac{1}{2}\|(u,u_t,\omega)\|^2_{\mathcal{H}}.$$

Pour une solution régulière, par un calcul direct on obtient

$$E'(t)=-\tilde{\gamma}\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega(\xi,t)|^2d\xi.$$

Donc, le système (0.0.61)-(0.0.64) est dissipatif au sens que l'énergie est décroissante en fonction de la variable t. Maintenant, on définit l'opérateur linéaire non borné $\mathcal{A}$ par

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, \omega)^{\top} \in \mathcal{H};\ \Delta u \in L^2(\Omega),\ v \in H^1_{\Gamma_0}(\Omega),\quad |\xi|\omega \in L^2(\mathbb{R}^d), \\ -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1}\mu(\xi) \in L^2(\mathbb{R}^d),\ \dfrac{\partial u}{\partial \nu}|_{\Gamma_1} = -\gamma\kappa \displaystyle\int_{\mathbb{R}^d} \mu(\xi)\omega(\xi)d\xi \end{array} \right\} \tag{0.0.66}$$

and

$$\mathcal{A}(u, v, \omega)^{\top} = (v, \Delta u, -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1}\mu(\xi))^{\top}$$

On note $v = u_t$ et $U_0 = (u_0, v_0, \omega_0)$, le système (0.0.61)-(0.0.64) peut être écrire comme une équation d'évolution linéaire sur l'espace $\mathcal{H}$

$$U_t = \mathcal{A}U, \quad U(0) = U_0. \tag{0.0.67}$$

On a demontré que l'opérateur $\mathcal{A}$ est m-disssipatif sur $\mathcal{H}$ et par conséquent, d'après Lumer-Philipps (voir [42, 51]), il engendre un C_0-semi-groupe de contractions $e^{t\mathcal{A}}$. Donc la solution de l'équation d'évolution (0.0.67) admet la représentation suivante :

$$U(t) = e^{t\mathcal{A}}U_0, \quad \forall t \geq 0,$$

ce qui mène que le système (0.0.67) est bien posé. Par conséquent, la théorie du semi-groupe permet du démontrer le résultat de l'existence et l'unicité suivant :

Théorème 0.0.31. *Pour toute donnée initiale $U_0 \in \mathcal{H}$, le problème (0.0.67) admet une solution unique faible $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. De plus, si $U_0 \in D(\mathcal{A})$ donc le problème (0.0.67) admet une solution unique forte $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

Puis on a étudié la stabilité forte du notre système, au sens que l'énergie converge vers zéro lorsque t tends vers l'infini, dont l'absence de la compacité de la résolvante de l'opérateur $\mathcal{A}$ et sans aucune condition géométrique sur le domaine Ω. Donc le principe de l'invariance de Lasalle [58] et la décomposition spectrale de Benchimol [17] ne seront pas appliquées dans ce cas. Pour cela, en utilisant le critère général d'Arendt-Batty [9], on montre que un C_0-semi-groupe de contractions $e^{t\mathcal{A}}$ dans un espace de Banach est fortement stable, si $\mathcal{A}$ n'admet pas des valeurs propres imaginaires pures et $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contient un ensemble dénombrable d'éléments. Par conséquent, on a obtenu le résultat de stabilité suivant

Théorème 0.0.32. *Supposons que $\eta \geq 0$. Alors, le C_0-semi-groupe $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ est fortement stable dans l'espace d'énergie $\mathcal{H}$, i.e. pour tout $U_0 \in \mathcal{H}$, on a*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

Pour la démonstration de ce Théorème, on a besoin de démontrer les deux Lemmes suivantes

Lemma 0.0.33. *Supposons que $\eta \geq 0$. Alors, pour tout $\lambda \in \mathbb{R}$, on a*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

On a remarqué que, pour $\eta = 0$, l'opérateur $-\mathcal{A}$ n'est pas inversible et par conséquent, on aura

Lemma 0.0.34. *Si $\eta > 0$, pour tout $\lambda \in \mathbb{R}$, on a*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}$$

tandis que, si $\eta = 0$, pour tout $\lambda \in \mathbb{R}^*$, on a

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Alors, d'après le Lemme (0.0.33), on déduit directement que l'opérateur $\mathcal{A}$ n'admet pas des valeurs propres imaginaires pures. Puis, en utilisant le Lemme (0.0.34) et le théorème de Banach fermé on conclut que $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ si $\eta > 0$ et $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ si $\eta = 0$.
Maintenant, on va étudier le genre de cette stabilité. Pour ce but, on a démontré que le système (0.0.61)-(0.0.64) n'est pas uniformément stable c-à-d n'est pas exponentiellement stable dans le cas $\Omega = [0,1]$. Notre résultat est le suivant

Théorème 0.0.35. *Supposons que $d = 1$. Le semi-groupe de contractions $e^{t\mathcal{A}}$ n'est pas uniformément stable dans l'espace de l'énergie $\mathcal{H}$.*

Ce résultat est dû au fait qu'une branche des valeurs propres du l'opérateur $\mathcal{A}$ s'approche de l'axe imaginaire. Pour cela, on a trouvé qu'il existe $k_0 \in \mathbb{N}^*$ et une suite $(\lambda_k)_{|k| \geq k_0}$ des valeurs propres simples de l'opérateur et qui satisfait le comportement asymptotique suivant

$$\lambda_k = i\left(k + \frac{1}{2}\right)\pi + i\frac{\gamma \sin\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} - \frac{\gamma \cos\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} + O\left(\frac{1}{k^{2-\alpha}}\right).$$

Alors une décroissance de type polynomiale est éspérée. Pour ce but, on considère le cas $\eta > 0$ et sous la condition (GCC) sur le bords, on utilise la méthode frequentielle [18] (voir aussi [14, 15, 40]) combinée avec une méthode de multiplicateur et on utilise la décroissance exponentielle des équations d'ondes avec un amortissement standard sur Γ_1 :

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) = 0, & x \in \Omega, \quad t > 0, \\ \varphi(x,t) = 0, & x \in \Gamma_0, \quad t > 0, \\ \partial_\nu \varphi(x,t) = -\varphi_t(x,t), & x \in \Gamma_1, \quad t > 0. \end{cases} \tag{0.0.68}$$

On définit l'espace auxilière $\mathcal{H}_a = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ et l'opérateur auxiliaire linéaire non-borné

$\mathcal{A}_a$ définie par

$$D(\mathcal{A}_a) = \left\{ \varPhi = (\varphi, \psi) \in \mathcal{H}_a : \quad \Delta\varphi \in L^2(\Omega); \quad \psi \in H^1_{\Gamma_0}(\Omega); \quad \frac{\partial \varphi}{\partial \nu} = -\psi \quad \text{on} \quad \Gamma_1 \right\}.$$

$$\mathcal{A}_a(\varphi, \psi) = (\psi, \Delta\varphi)\,.$$

Et on introduit la condition suivante

(H) : Le problème (0.0.68) est uniformément stable dans l'espace de l'énergie $\mathcal{H}_a = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$.

Bardos et al. in [13], ont demontré que la condition (H) est vérifiée sous la condition (GCC). Maintenant, je présente le résultat principale du ce chapitre

Théorème 0.0.36. *Supposons que $\eta > 0$ et que la condition (H) soit vérifiée. Donc, pour toute donnée initiale $U_0 \in D(\mathcal{A})$, il existe une constante $C > 0$ indépendante de U_0, tel que l'énergie de la solution forte U de (0.0.67) satisfait l'estimation suivante*

$$E(t, U) \leq C \frac{1}{t^{\frac{1}{1-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.0.69}$$

En particulier, pour $U_0 \in \mathcal{H}$ l'énergie converge vers zéro lorsque t tends vers l'infini.

Pour la preuve de ce théorème, en prenant $\ell = 2 - 2\alpha$, la décroissance énergétique polynomiale (0.0.69) sera vérfiée si les conditions suivantes

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{$H1$}$$

et

$$\sup_{|\lambda| \in \mathbb{R}} \frac{1}{|\lambda|^{\ell}} \|(i\lambda I - \mathcal{A})^{-1} < +\infty. \tag{$H2$}$$

sont satisfaites. La condition $(H1)$ est déjà vérifiée dans le théorème (0.0.32). Donc, on montre $(H2)$ par un raisonnement de contradiction. Dans ce but, on suppose que $(H2)$ est fausse, donc il existe une suite réel (λ_n) avec $\lambda_n \to +\infty$ et la suite $U^n \in D(\mathcal{A})$, vérifie les deux conditions suivantes

$$\|U^n\|_{\mathcal{H}} = \|(u^n, v^n, \omega^n)\|_{\mathcal{H}} = 1 \tag{0.0.70}$$

et

$$\lambda_n^{\ell}\,(i\lambda_n - \mathcal{A})\, U^n = (f_1^n, f_2^n, f_3^n) \to 0 \quad \text{in} \quad \mathcal{H}. \tag{0.0.71}$$

En prouvant plusieurs estimation, on a demontré que $\|U\| = o(1)$, ce qui conterdit (0.0.70). Par conséquent la condition $(H2)$ est vérifiée et que l'énergie de la solution régulière du système (0.0.61)-(0.0.64) décroît polynomialement vers zéro lorsque $t \to +\infty$. Finalement, en utilisant la densitée du domaine $D(\mathcal{A})$ dans $\mathcal{H}$, on peut démontrer que l'énergie de la solution faible du système (0.0.61)-(0.0.64) décroît vers zéro lorsque $t \to +\infty$.

Chapitre 3. Dans ce chapitre, on a étudié la stabilisation d'un système des équations d'ondes, monodimensionel, couplées fortement sous l'action d'un amortissement au bords de type fractionnaire au sens de Caputo appliqué à une seule équation considéré. On considère le système suivant

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & sur \quad]0,1[\times]0,+\infty[, \\ y_{tt} - ay_{xx} - bu_t = 0 & sur \quad]0,1[\times]0,+\infty[\end{cases} \tag{0.0.72}$$

où $(x,t) \in]0,1[\times]0,+\infty[$, $a > 0$ et $b \in \mathbb{R}^*$. Ce système est considéré par les conditions aux bords suivants

$$\begin{cases} u(0,t) = 0 & \text{dans} \quad]0,+\infty[, \\ y(0,t) = y(1,t) = 0 & \text{dans} \quad]0,+\infty[, \\ u_x(1,t) = -\gamma\partial_t^{\alpha,\eta}u(1,t) & \text{dans} \quad]0,+\infty[\end{cases} \tag{0.0.73}$$

où $\partial_t^{\alpha,\eta}$ est définie dans l'équation (0.0.60). En utilisant le Théorème 0.0.29, on reformule notre système comme

$$\begin{aligned} u_{tt} - u_{xx} + by_t &= 0, \quad \text{dans} \quad (0,1)\times\mathbb{R}^+, & (0.0.74) \\ y_{tt} - ay_{xx} - bu_t &= 0, \quad \text{dans} \quad (0,1)\times\mathbb{R}^+, & (0.0.75) \\ \partial_t\omega(\xi,t) + \left(|\xi|^2+\eta\right)\omega(\xi,t) - \mu(\xi)\partial_t u(1,t) &= 0, \quad \text{sur} \quad \mathbb{R}^+,\ \xi\in\mathbb{R}, & (0.0.76) \\ u(0) &= 0, \quad \text{sur} \quad \mathbb{R}^+, & (0.0.77) \\ y(0) = y(1) &= 0, \quad \text{sur} \quad \mathbb{R}^+, & (0.0.78) \\ u_x(1) + \gamma\kappa\int_{\mathbb{R}}\mu(\xi)\omega(\xi,t)d\xi &= 0, \quad \text{sur} \quad \mathbb{R}^+ & (0.0.79) \end{aligned}$$

où γ est une constante positive, $\eta \geq 0$ and $\kappa = \frac{2\sin(\alpha\pi)}{\pi}$. Finalement, le système (0.0.74)-(0.0.79) est considéré avec les conditions initiales

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad &\text{où} \quad x\in\Omega, & (0.0.80) \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad &\text{où} \quad x\in\Omega, & (0.0.81) \\ \omega(\xi,0) = 0 \qquad &\text{où} \quad \xi\in\mathbb{R}^d. & (0.0.82) \end{aligned}$$

On définit l'espace

$$H_L^1(0,1) = \{u \in H^1(0,1); \quad u(0) = 0\}$$

et l'espace d'énergie

$$\mathcal{H} = H_L^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

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$$\left\langle U, \widetilde{U} \right\rangle = \int_0^1 \left(u_x \bar{\tilde{u}}_x + v\bar{\tilde{v}} + a y_x \bar{\tilde{y}}_x + z\bar{\tilde{z}} \right) dx + \gamma\kappa \int_{-\infty}^{+\infty} \omega\bar{\tilde{\omega}} d\xi,$$

pour tout $U = (u, v, y, z, \omega)^\top \in \mathcal{H}$ et $\widetilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\omega}) \in \mathcal{H}$. Puis on définit l'opérateur lineaire non-borné $\mathcal{A}$ par

$$D(A) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega)^\top \in \mathcal{H};\ u \in H^2\left(]0,1[\right) \cap H_L^1(]0,1[), \\ y \in H^2\left(]0,1[\right) \cap H_0^1(]0,1[), v \in H_L^1(]0,1[),\ z \in H_0^1\left(]0,1[\right), \\ -\left(\xi^2+\eta\right)\omega + v\left(1\right)\mu(\xi) \in L^2\left(\mathbb{R}\right), \\ u_x(1) + \gamma\kappa \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi = 0,\ |\xi|\omega \in L^2\left(\mathbb{R}\right). \end{array} \right\} \tag{0.0.83}$$

et

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - bz \\ z \\ a y_{xx} + bv \\ -\left(\xi^2+\eta\right)\omega + v(1)\mu(\xi) \end{pmatrix}. \tag{0.0.84}$$

Si $U = (u, v, y, z, \omega)$ est une solution régulière du système (0.0.74)-(0.0.82), donc on peut écrire ce système comme une équation d'évolution

$$\left\{ \begin{array}{rcl} U'(t) & = & \mathcal{A}U(t), \\ U(0) & = & U_0 \in \mathcal{H}. \end{array} \right. \tag{0.0.85}$$

On a demontré que l'opérateur $\mathcal{A}$ est m-disssipatif sur $\mathcal{H}$ et par conséquence, d'après Lumer-Philipps (see [42, 51]), il engendre un C_0-semi-groupe de contractions $e^{t\mathcal{A}}$. Donc la solution de l'équation d'évolution (0.0.85) admet la représentation suivante :

$$U(t) = e^{t\mathcal{A}}U_0, \quad \forall t \geq 0,$$

ce qui mène que le système (0.0.67) est bien posé. Par conséquent, la théorie du semi-groupe permet du démontrer le résultat de l'éxistence et l'unicité suivant :

Théorème 0.0.37. *Pour toute donnée initiale $U_0 \in \mathcal{H}$, le problème (0.0.67) admet une solution unique faible $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. De plus, si $U_0 \in D(\mathcal{A})$ donc le problème (0.0.67) admet une* *solution unique forte* $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.

Puis, on a étudié la stabilité forte du système (0.0.74)-(0.0.82) et on a trouvé le résultat suivant

Théorème 0.0.38. *Supposons que* $\eta \geq 0$ *et le coefficient* b *satisfait la condition suivante*

$$b^2 \neq \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)\pi^2}{(a+1)(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}. \tag{C}$$

Alors, le C_0-*semi-groupe* $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ *est fortement stable dans l'espace d'énergie* $\mathcal{H}$*, i.e. pour tout* $U_0 \in \mathcal{H}$*, on a*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

Comme le résolvante n'est pas compact, on va utiliser le critère générale d'arendt-Batty, et on a trouvé que pour ($\eta > 0$ et $\lambda \in \mathbb{R}$) ou ($\eta = 0$ et $\lambda \in \mathbb{R}^*$) alors $R(I - \lambda\mathcal{A}) = \mathcal{H}$, et pour $\eta = 0$ et $\lambda = 0$, on a démontré que $0 \notin \rho(\mathcal{A})$. Par conséquent, en utilisant le théorème de Banach fermé, on aura que si $\eta > 0$ alors $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ et si $\eta = 0$ alors $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$.
Puis, on va étuder le genre de cette stabilité. On a trouvé que si le paramètre de couplage b vérifie la condition (C), en utilisant une approche spectrale, la décroissance de l'énergie du système (0.0.74)-(0.0.82) n'est pas uniforme. De plus, si $b \notin \pi\mathbb{Z}$, $\exists k_0 \in \mathbb{N}^*$ suffisamment large tel que

$$\sigma(\mathcal{A}) \supset \sigma_0 \cup \sigma_1$$

où

$$\sigma_0 = \{\lambda_k\}_{k\in J}, \quad \sigma_1 = \{\lambda_k\}_{|k|\geq k_0}, \quad \sigma_0 \cap \sigma_1 = \emptyset.$$

De plus, J est un enemble fini et

$$\lambda_k = \frac{ik\pi}{2} + \frac{c_1}{k^{1-\alpha}} + \frac{ic_2}{k^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right)$$

pour

$$c_1 = \frac{\gamma(-1)^k \left(\cos(b) - (-1)^k\right)\left(\cos\left(\frac{\pi}{2}(1-\alpha)\right)\right)}{2^\alpha k^{1-\alpha}} \quad \text{et} \quad c_2 = \frac{\gamma(-1)^k \left(\cos(b) - (-1)^k\right)\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{2^\alpha k^{1-\alpha}}.$$

De plus, pour le cas où $a \neq 1$, on admet la même démarche que le cas où $a = 1$, on aura

$$\lambda_m = i\left(m + \frac{1}{2}\right)\pi + o(1), \quad \text{et}\backslash\text{ou} \quad \lambda_n = in\pi\sqrt{a} + o(1).$$

Après l'étude spectrale, alors une décroissance de type polynomiale est espérée. Pour cela, on utilise le théorème de Borichev et Tomilov (méthode frequentielle) et en utilisant la théorie de des équations différentielles ordinaires et si le paramètre de couplage b vérifie la condition (C), on montre que la décroisssance polynomiale de l'énergie du système (0.0.74)-(0.0.82) est fortement influencé par la nature du paramètre du couplage b (ainsi avec des conditions supplémentaires sur b) et par la propriété arithmétique du rapport de vitesse de propagation des ondes (a). On aura les théorèmes principaux suivantes

Théorème 0.0.39. *Supposons que $\eta > 0$, $a = 1$ et b satisfait la condition (C), donc pour tout $U_0 \in D(\mathcal{A})$ il existe une constante $C_1 > 0$ indépendante de U_0 tel que l'énergie de la solution forte du système (0.0.85), admet la décroissance polynomiale suivante*

$$E(t) \leq \frac{C_1}{t^{\ell(\alpha)}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0,$$

où

$$\ell(\alpha) = \begin{cases} \frac{1}{3-\alpha} & si \quad b \in \pi\mathbb{Z}, \\ \frac{1}{1-\alpha} & si \quad b \notin \pi\mathbb{Z}. \end{cases}$$

Théorème 0.0.40. *Supposons que $\eta > 0$, $a \neq 1$ et b satisfait la condition (C). Si ($a \in \mathbb{Q}$ et b suffisamment petit) ou $\sqrt{a} \in \mathbb{Q}$, donc pour toute $U_0 \in D(\mathcal{A})$ il existe $C > 0$ indépendante de U_0 tel que l'energie de la solution forte du système (0.0.85) admet la décroisance polynomiale suivante*

$$E(t) \leq \frac{C_2}{t^{\frac{1}{3-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0.$$

Chapitre 4. Dans ce chapitre, on a généralizé les travaux du chapite 2. On a étudié la stabilisation d'un système des équations d'ondes, multidimensionel, couplées fortement sous l'action d'un amortissement au bords de type fractionnaire au sens de Caputo appliqué à une seule équation considéré. On considère un ouvert, borné Ω dan $\mathbb{R}^d$ tel que $d \geq 2$ et $\Gamma = \partial\Omega$ de classe C^2 qui vérifie $\Gamma = \Gamma_0 \cup \Gamma_1$ et $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ et on considère le système suivant :

$$\begin{cases} u_{tt} - \Delta u + b y_t = 0 & \text{sur} \quad \Omega \times]0, +\infty[, \\ y_{tt} - a\Delta y - b u_t = 0 & \text{sur} \quad \Omega \times]0, +\infty[\end{cases} \tag{0.0.86}$$

où $(x, t) \in \Omega \times]0, +\infty[$, $a > 0$ et $b \in \mathbb{R}^*$. Ce système sera complété par les conditions initiales suivantes

$$\begin{cases} u = 0 & \text{sur} \quad \Gamma_0 \times]0, +\infty[, \\ y = 0 & \text{sur} \quad \Gamma \times]0, +\infty[, \\ \dfrac{\partial u}{\partial \nu} = -\gamma \partial_t^{\alpha,\eta} u(1, t) & \text{sur} \quad \Gamma_1 \times]0, +\infty[\end{cases} \tag{0.0.87}$$

où $\partial_t^{\alpha,\eta}$ est définie dans l'équation (0.0.60). En utilisant le Théorème 0.0.29, on reformule notre système comme

$$\begin{aligned} u_{tt}-\Delta u+by_t &= 0, \quad \text{dans} \quad \Omega\times\mathbb{R}^+, && (0.0.88)\\ y_{tt}-a\Delta y-bu_t &= 0, \quad \text{dans} \quad \Omega\times\mathbb{R}^+, && (0.0.89)\\ \partial_t\omega(\xi,t)+\left(|\xi|^2+\eta\right)\omega(\xi,t)-\mu(\xi)\partial_t u(1,t) &= 0, \quad \text{sur} \quad \mathbb{R}^+,\ \xi\in\mathbb{R}^d, && (0.0.90)\\ u &= 0, \quad \text{sur} \quad \Gamma_0\times\mathbb{R}^+, && (0.0.91)\\ y &= 0, \quad \text{sur} \quad \Gamma\times\mathbb{R}^+, && (0.0.92)\\ \frac{\partial u}{\partial\nu}+\gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi,t)d\xi &= 0, \quad \text{sur} \quad \mathbb{R}^+ && (0.0.93) \end{aligned}$$

où γ est une constante positive, $\eta\geq 0$ and $\kappa=\frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$. Finalement, le système (0.0.88)-(0.0.93) est considéré avec les conditions initiales

$$\begin{aligned} u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{où} \quad x\in\Omega, && (0.0.94)\\ y(x,0)=y_0(x), \quad y_t(x,0)=y_1(x) \quad \text{où} \quad x\in\Omega, && (0.0.95)\\ \omega(\xi,0)=0 \qquad \text{où} \quad \xi\in\mathbb{R}^d. && (0.0.96) \end{aligned}$$

On définit l'espace de Hilbert

$$\mathcal{H}=H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times H^1_0(\Omega)\times L^2(\Omega)\times L^2(\mathbb{R}^d).$$

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$$\left(U,\tilde{U}\right)=\int_\Omega\left(\nabla u\nabla\bar{\tilde{u}}+v\bar{\tilde{v}}+z\bar{\tilde{z}}+a\nabla y\nabla\bar{\tilde{y}}\right)dx+\tilde{\gamma}\kappa\int_{\mathbb{R}^d}\omega\bar{\tilde{\omega}}d\xi.$$

où $\tilde{\gamma}=\gamma|\Gamma_1|$ et $H^1_{\Gamma_0}(\Omega)$ est définie par

$$H^1_{\Gamma_0}(\Omega)=\{u\in H^1(\Omega); \quad u=0 \quad \text{sur} \quad \Gamma_0\}$$

L'énergie de la solution du système est définit par

$$E(t)=\frac{1}{2}\|(u,u_t,y,y_t,\omega)\|^2_{\mathcal{H}}.$$

Pour une solution régulière et par un calcul direct on obtient

$$E'(t)=-\tilde{\gamma}\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega(\xi,t)|^2d\xi.$$

Donc, le système (0.0.88)-(0.0.91) est dissipatif au sens que l'énergie est décroissante en fonction de la variable t. Maintenant, on définit l'opérateur linéaire non borné $\mathcal{A}$ par

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega)^\top \in \mathcal{H};\ \Delta u \in L^2(\Omega), \quad y \in H^2(\Omega) \cap H_0^1(\Omega), \\ v \in H^1_{\Gamma_0}(\Omega), \quad z \in H_0^1(\Omega), \quad -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1}\mu(\xi) \in L^2(\mathbb{R}^d), \\ \dfrac{\partial u}{\partial \nu} + \gamma\kappa \displaystyle\int_{\mathbb{R}^d} \mu(\xi)\omega(\xi) d\xi = 0 \ \text{ sur } \ \Gamma_1, \quad |\xi|\omega \in L^2(\mathbb{R}^d) \end{array} \right\} \tag{0.0.97}$$

et

$$\mathcal{A} \begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - bz \\ z \\ a\Delta y + bv \\ -(|\xi|^2 + \eta)\omega + v|_{\Gamma_1}\mu(\xi). \end{pmatrix}$$

On note $v = u_t$, $z = y_t$ et $U_0 = (u_0, v_0, y_0, z_0, \omega_0)$, le système (0.0.88)-(0.0.96) peut être écrire comme une équation d'évolution linéaire sur l'espace $\mathcal{H}$

$$U_t = \mathcal{A}U, \quad U(0) = U_0. \tag{0.0.98}$$

On a demontré que l'opérateur $\mathcal{A}$ est m-disssipatif sur $\mathcal{H}$ et par conséquent, d'après Lumer-Philipps, il engendre un C_0-semi-groupe de contractions $e^{t\mathcal{A}}$. Donc la solution de l'équation d'évolution (0.0.98) admet la représentation suivante :

$$U(t) = e^{t\mathcal{A}}U_0, \quad \forall t \geq 0,$$

ce qui mène que le système (0.0.98) est bien posé. Par conséquent, la théorie du semi-groupe permet du démontrer le résultat de l'éxistence et l'unicité suivant :

Théorème 0.0.41. *Pour toute donnée initiale $U_0 \in \mathcal{H}$, le problème (0.0.67) admet une solution unique faible $U(t) \in C^0(\mathbb{R}^+, \mathcal{H})$. De plus, si $U_0 \in D(\mathcal{A})$ donc le problème (0.0.67) admet une solution unique forte $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.*

Puis on a étudié la stabilité forte du notre système, au sens que l'énergie converge vers zéro lorsque t tends vers l'infini, dont l'absence de la compacité de la résolvante de l'opérateur $\mathcal{A}$ et avec une condition géométrique notée (MGC) sur le domaine Ω. Donc la méthode classique de l'invariance de Lasalle et la décomposition spectrale de Benchimol ne seront pas appliquées dans ce cas. Pour cela, en utilisant le critère général d'Arendt-Batty, on montre que un C_0-semi-groupe de contractions $e^{t\mathcal{A}}$ dans un espace de Banach est fortement stable, si $\mathcal{A}$ n'admet pas des valeurs propres imaginaires pures et $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contient un ensemble dénombrable d' éléments. Par conséquent, on a obtenue le résultat de stabilité suivant

Théorème 0.0.42. *Supposons que $\eta \geq 0$, $a = 1$ et b suffisamment petit. Donc le C_0-semi-groupe $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ est fortement stable dans l'espace d'énergie $\mathcal{H}$, i.e. pour tout $U_0 \in \mathcal{H}$, on a*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

Pour la démonstration de ce théorème, on a besoin de démontrer les deux lemmes suivantes

Lemma 0.0.43. *Supposons que $\eta \geq 0$, $a = 1$ et b suffisamment petit. Donc, pour tout $\lambda \in \mathbb{R}$, on a*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

On a remarqué que pour $\eta = 0$ l'opérateur $-\mathcal{A}$ n'est pas inversible et par conséquent, on aura

Lemma 0.0.44. *Si $\eta > 0$, pour tout $\lambda \in \mathbb{R}$, on a*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}$$

tandis que, si $\eta = 0$, pour tout $\lambda \in \mathbb{R}^*$, on a

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Alors, d'après le Lemme (0.0.43), on déduit directement que l'opérateur $\mathcal{A}$ n'admet pas des valeurs propres imaginaires pures. Puis, en utilisant le Lemme (0.0.44) et le théorème de Banach fermé on conclut que $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ si $\eta > 0$ et $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ si $\eta = 0$.
Ensuite, la stabilité non-exponentielle (i.e. non uniforme) du système (0.0.88)-(0.0.96) est déjà prouvé dans le chapitre 2. Alors une décroissance de type polynomiale est éspérée. Pour ce but, on considère le cas $\eta > 0$, $a = 1$ et b suffisamment petit et le domaine Ω satisfait les conditions (MGC). Donc on a obtenu le résultat suivant

Théorème 0.0.45. *Supposons que $\eta > 0$, $a = 1$ et b suffisamment petit. Donc, pour toute donnée initiale $U_0 \in D(\mathcal{A})$ il existe une constante $C > 0$ indépendante de U_0, telle que l'énergie de la solution forte U de (0.0.98) satisfaite l'estimation suivante*

$$E(t, U) \leq C \frac{1}{t^{\frac{1}{1-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Chapitre 5 : Dans ce chapitre, on considère un système des équations d'ondes couplées sous l'action d'un seul amortissement viscoélastique autour de Γ, définit par :

$$\begin{cases} \rho_1(x)u_{tt} - \mathrm{div}(a_1(x)\nabla u + b(x)\nabla u_t) + \alpha y_t = 0 & \text{dans} \quad \Omega \times \mathbb{R}^+, \\ \rho_2(x)y_{tt} - \mathrm{div}(a_2(x)\nabla y) - \alpha u_t = 0 & \text{dans} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{0.0.99}$$

avec les conditions initiales suivantes

$$u(\cdot,0)=u_0\left(\cdot\right),\ y(\cdot,0)=y_0\left(\cdot\right),\ u_t(\cdot,0)=u_1\left(\cdot\right),\ y_t(\cdot,0)=y_1\left(\cdot\right)\quad \text{dans}\quad \Omega, \tag{0.0.100}$$

et les conditions aux bords suivantes

$$u\left(x,t\right)=y\left(x,t\right)=0\quad \text{sur}\Gamma\times\mathbb{R}^{+}. \tag{0.0.101}$$

Les fonctions $\rho_1,\rho_2,a_1,a_2,b\in L^\infty(\Omega)$ telles que

$$\rho_1(x)\geq\rho_1,\ \ \rho_2(x)\geq\rho_2,\ \ a_1(x)\geq a_1,\ \ a_2(x)\geq a_2$$

et α est une constante réelle.
L'amortissement viscoélastique local est un phénomène naturel de corps qui ont une partie en matériau viscoélastique et l'autre est en matériau élastique. Il existe un certain nombre de publications concernant l'équation des ondes avec l'amortissement viscoélastique local.
D'abords, en utilisant la théorie de semi-groupe, on montre que le système (0.0.99)-(0.0.101) est bien posé avec le control $\operatorname{div}(b(x)\nabla u_t)$ appliqué sur le bords Γ. Pour ce but, on définit l'énergie du système (0.0.99)-(0.0.101) par

$$E(t)=\frac{1}{2}\int_\Omega\left(\rho_1(x)|u_t|^2+\rho_2(x)|y_t|^2+a_1(x)|\nabla u|^2+a_2(x)|\nabla y|^2\right)dx. \tag{0.0.102}$$

Donc, par un calcul direct on montre que

$$E'(t)=-\int_\Omega b(x)|\nabla u_t|^2dx\leq 0.$$

D'où le système (0.0.99)-(0.0.101) est dissipatif au sens que l'énergie est décroissante par rapport au temps t. Pour tout $\gamma>0$, on définit le $\gamma-$voisinage O_γ de bords Γ par

$$O_\gamma=\{x\in\Omega\ \ :\quad |x-y|\leq\gamma,\quad y\in\Gamma\}. \tag{0.0.103}$$

Plus précisément, on suppose que

$$\begin{cases}\rho_1(x)\geq\rho_1>0,\ \ \rho_2(x)\geq\rho_2>0,\ \ a_1(x)\geq a_1>0,\ \ a_2(x)\geq a_2>0\quad \text{pour tout } x\in\Omega,\\ \\ b(x)\geq b_0>0\quad \text{pour tout } x\in O_\gamma.\end{cases}$$

Puis, On définit l'espace de Hilbert $\mathcal{H}$ par :

$$\mathcal{H}=\left(H_0^1(\Omega)\times L^2(\Omega)\right)^2$$

muni du produit scalaire

$$\left\langle U,\widetilde{U}\right\rangle=\int_{\Omega}\left(a_1\nabla u\cdot\nabla\bar{\tilde{u}}+a_2\nabla y\cdot\nabla\bar{\tilde{y}}+\rho_1 v\bar{\tilde{v}}+\rho_2 z\bar{\tilde{z}}\right)dx,$$

pour tout $U=(u,v,y,z)^\top\in\mathcal{H}$ et $\tilde{U}=(\tilde{u},\tilde{v},\tilde{y},\tilde{z})^\top\in\mathcal{H}$. Finalement, on définit l'opérateur linéaire non-borné $\mathcal{A}$ par

$$\begin{aligned}D(\mathcal{A})=\Big\{&(u,v,y,z)\in\mathcal{H}:\ \operatorname{div}(a_1(x)\nabla u+b(x)\nabla v)\in L^2(\Omega),\\&\operatorname{div}(a_2(x)\nabla y)\in L^2(\Omega)\quad\text{et}\quad v,z\in H_0^1(\Omega)\Big\}\end{aligned}\tag{0.0.104}$$

et

$$\mathcal{A}\begin{pmatrix}u\\v\\y\\z\end{pmatrix}=\begin{pmatrix}v\\\dfrac{1}{\rho_1(x)}\left(\operatorname{div}(a_1(x)\nabla u+b(x)\nabla v)-\alpha z\right)\\z\\\dfrac{1}{\rho_2(x)}\left(\operatorname{div}(a_2(x)\nabla y)+\alpha v\right)\end{pmatrix}.$$

En dénotant $v=u_t$, $z=y_t$ et $U_0=(u_0,v_0,y_0,z_0)^\top$, le système (0.0.99)-(0.0.101) s'écrit comme une équation d'évolution abstraite linéaire dans l'espace $\mathcal{H}$

$$U_t=\mathcal{A}U,\quad U(0)=U_0.\tag{0.0.105}$$

On sait que, sous la condition (H), l'opérateur $\mathcal{A}$ est m-dissipatif sur $\mathcal{H}$ et par conséquent, il engendre un C_0-semigroup de contraction $e^{t\mathcal{A}}$ d'après le théorème de Lummer-Philips. Donc la solution de l'équation d'évolution (0.0.105) admets la représentation suivante :

$$U(t)=e^{t\mathcal{A}}U_0,\quad t\geq 0.$$

ce qui implique que le système (0.0.99)-(0.0.101) est bien posé. Donc, la théorie de semi-groupe permette de montrer le résultat d'existence et d'unicité suivante :

Théorème 0.0.46. *Pour toute donnée initiale $U_0\in\mathcal{H}$, le problème (0.0.105) admet une solution unique faible $U(t)\in C^0(\mathbb{R}^+,\mathcal{H})$. De plus, si $U_0\in D(\mathcal{A})$, donc le problème (0.0.105) admet une solution unique forte $U(t)\in C^1(\mathbb{R}^+,\mathcal{H})\cap C^0(\mathbb{R}^+;D(\mathcal{A}))$.*

Puis, on a étudié la stabilité forte du système (0.0.99)-(0.0.101) aus sens que l'énergie décroît vers zéro lorsque t tends vers l'infinie pour toute donnée initiale dans $\mathcal{H}$. Il est facile de voir que la résolvente de l'opérateur $\mathcal{A}$ n'est pas compacte, d'où la méthode classique de l'invariance de LaSalle ou la théorie de la décomposition spectrale de Benchimol ne seront pas appliquées dans ce cas. Par conséquent, en utilisant un critère général d'Arendt-Batty [9] combiné avec un résultat de continuation basée sur les estimations de Carelman voir [28], on démontre que l'opérateur $\mathcal{A}$ n'admet pas des valeurs propres imaginaires pures et $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contient seulement un ensemble dénombrable d'élements. Par conséquent, on va démontrer le résultat de stabilité suivant :

Théorème 0.0.47. *Sous l'hypothèse* (H)*, le* C_0*-semi-groupe* $e^{t\mathcal{A}}$ *est fortement stable dans* $\mathcal{H}$ *; i.e. pour tout* $U_0 \in \mathcal{H}$*, la solution de* (0.0.105) *satisfaite*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

Pour démontrer ce théorème, on a besoin de démontrer ses deux Lemmes.

Lemme 0.0.48. *Sous l'hypothèse* (H)*, on a*

$$\ker\left(i\lambda I - \mathcal{A}\right) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

Lemme 0.0.49. *Sous l'hypothèse* (H)*,* $i\lambda I - \mathcal{A}$ *est surjective pour tout* $\lambda \in \mathbb{R}$*.*

Puis, on démontre que le système (0.0.99)-(0.0.101) n'est pas uniformément stable i.e. exponentiellement table. Pour cela, on suppose que

$$a_1, a_2, \rho_1, \rho_2 \in \mathbb{R}^+ \quad \text{et} \quad b \in \mathbb{R}_*^+. \tag{H'}$$

Donc, on aura le résultat suivant

Théorème 0.0.50. *Sous l'hypothèse* (H')*, le système* (0.0.99)-(0.0.101) *n'est pas uniformément stable dans l'espace d'énérgie* $\mathcal{H}$*.*

Ce résultat est dû au fait que, une sous suite des valeurs propres de l'opérateur $\mathcal{A}$ s'approche de l'axe des imaginaires. Donc, on montre qu'il existe une constante $k_0 \in \mathbb{N}^*$ et une suite $(\lambda_k)_{|k|\geq k_0}$, pour k assez grand. On a trouvé une sous suite de ses valeurs propres satisfiat la direction asymptotique suivante

$$\lambda_k = i\sqrt{\frac{a}{\rho_2}}\mu_k - \frac{\alpha^2}{2b\rho_2\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right), \tag{0.0.106}$$

où μ_k est la valeure propre du $-\Delta$, i.e.

$$\begin{cases} -\Delta\varphi_k = \mu_k^2\varphi_k & \text{dans} \quad \Omega, \\ \varphi_k = 0 & \text{sur} \quad \Gamma. \end{cases}$$

Donc, une stabilitée polynomiale est éspérée. Maintenant, on suppose que

$$a_1, a_2, \rho_1, \rho_2, b \in C^{1,1}(\overline{\Omega}). \tag{C1}$$

Ainsi, on suppose les conditions supplémentaires suivantes.
Il existe une fonction $q \in C^1(\Omega, \mathbb{R}^N)$ et $0 < \alpha < \beta < \gamma$, tels que

$$\partial_j q_k = \partial_k q_j, \quad \operatorname{div}(a_1 a_2 \rho_2 q), \operatorname{div}(a_1 a_2 \rho_1 q) \in C^{0,1}(\Omega_\beta) \quad \text{et} \quad q = 0 \quad \text{sur} \quad O_\alpha, \tag{C2}$$

Il existe une constante $\sigma_1 > 0$, telle que

$$2a_2 \partial_j(a_{1k} q_k) + a_1(q_k \partial_j a_2 + q_j \partial_k a_2) + \left[a_1 \left(\frac{a_2}{\rho_2} q \nabla \rho_2 - q \nabla a_2\right)\right] I \geq \sigma_1 I, \quad \forall x \in \Omega_\beta. \tag{C3}$$

Il existe une constante σ_2, telle que

$$2a_1 \partial_j(a_{2k} q_k) + a_2(q_k \partial_j a_1 + q_j \partial_k a_1) + \left[a_2 \left(\frac{a_1}{\rho_1} q \nabla \rho_1 - q \nabla a_1\right)\right] I \geq \sigma_2 I, \quad \forall x \in \Omega_\beta. \tag{C4}$$

Il existe une constante $M > 0$, tel que pour tout $v \in H_0^1(\Omega)$, on a

$$|(q \cdot \nabla v)\, \nabla b - (q \cdot \nabla b) \nabla v| \leq M \sqrt{b}\, |\nabla v|, \quad \forall x \in \Omega_\beta. \tag{C5}$$

Théorème 0.0.51. *On suppose que les conditions* $(\mathrm{H}), (\mathrm{C1}) - (\mathrm{C6})$ *soit vérifiées. Donc, pour toute donnée initial* $U_0 \in D(\mathcal{A})$*, il existe une constante* $C > 0$ *indépendante de* U_0*, telle que l'énergie de la solution forte* U *de* (0.0.105) *satisfaite l'estimation suivante :*

$$E(t, U) \leq C \frac{1}{t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.0.107}$$

En particulier, pour $U_0 \in \mathcal{H}$*, l'énergie* $E(t, U)$ *converge vers zéro lorsque* t *tends vers l'infini.*

Dans ce but, on va appliquer la méthode fréquentielle, voir le théorème 2.4 de [18] combinée avec une méthode de multiplicateur. Donc, premièrement, on démontre que

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{sur} \quad O_\gamma,$$

Puis, on prouve que

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{sur} \quad \Omega_\beta.$$

Donc, on obtient

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{sur} \quad \Omega.$$

Finalement, en utilisant (0.0.106) et (0.0.107), on obtient la décroissance énergetique polynomiale optimale de la solution forte de (0.0.105).

CHAPITRE 1

PRELIMINARIES

As the analysis done in this P.H.D thesis local on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters. we refer to [15], [16], [18], [21], [22], [24], [26],[27], [29], [31], [32], [40], [42], [43], [51], [52], [54], [55].

1.1 Bounded and Unbounded linear operators

We start this chapter by give some well known results abound bounded and undounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{C}$, and H will always denote a Hilbert space equipped with the scalar product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $\|\cdot\|_H$.

A linear operator $T : E \longmapsto F$ is a transformation which maps linearly E in F, that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall\ u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

Definition 1.1.1. *A linear operator $T : E \to F$ is said to be bounded if there exists $C \geq 0$ such that*

$$\|Tu\|_F \leq C\|u\|_E \quad \forall\ u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 1.1.2. *A bounded operator $T \in \mathcal{L}(E,F)$ is said to be compact if for each sequence $(x_n)_{n\in\mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a subsequence which converges in F.*

The set of all compact operators from E into F is denoted by $\mathcal{K}(E,F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E,E)$.

Definition 1.1.3. *Let $T \in \mathcal{L}(E,F)$, we define*

- *Range of T by*

$$\mathcal{R}(T) = \{Tu : \ u \in E\} \subset F.$$

- *Kernel of T by*

$$\ker(T) = \{u \in E : \ Tu = 0\} \subset E.$$

Theorem 1.1.4. (***Fredholm alternative***) *If $T \in \mathcal{K}(E)$, then*

- $\ker(I-T)$ *is finite dimension,* $(I$ *is the identity operator on* $E)$.
- $\mathcal{R}(I-T)$ *is closed.*
- $\ker(I-T) = 0 \Leftrightarrow \mathcal{R}(I-T) = E.$

Definition 1.1.5. *An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.*

$$T : D(T) \subset E \longmapsto F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E. If $D(T) = E$ then $T \in \mathcal{L}(E,F)$.

Definition 1.1.6. *Let $T : D(T) \subset E \longmapsto F$ be an unbounded linear operator.*

- *The range of T is defined by*

$$\mathcal{R}(T) = \{Tu : \ u \in D(T)\} \subset F.$$

- *The Kernel of T is defined by*

$$\ker(T) = \{u \in D(T) : \ Tu = 0\} \subset E.$$

- *The graph of T is defined by*

$$\mathrm{G}(T) = \{(u, Tu) : \ u \in D(T)\} \subset E \times F.$$

Definition 1.1.7. *A map T is said to be closed if $\mathrm{G}(T)$ is closed in $E \times F$. The closedness of* *an unbounded linear operator T can be characterize as following*

$$\text{if } u_n \in D(T) \text{ such that } u_n \to u \text{ in } E \text{ and } Tu_n \to v \text{ in } F, \text{ then } u \in D(T) \text{ and } Tu = v.$$

Definition 1.1.8. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator.*

- *The resolvent set of T is defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \ \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- *The resolvent of T is defined by*

$$R(\lambda, T) = \left\{(\lambda I - T)^{-1} : \ \lambda \in \rho(T)\right\}.$$

- *The spectrum set of T is the complement of the resolvent set in $\mathbb{C}$, denoted by*

$$\sigma(T) = \mathbb{C}/\rho(T).$$

Definition 1.1.9. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. we can split the spectrum $\sigma(T)$ of T into three disjoint sets, given by*

- *The ponctuel spectrum of T is define by*

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) \neq \{0\}\}$$

  in this case λ is called an eigenvalue of T.

- *The continuous spectrum of T is define by*

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} \ \ \ker(\lambda I - T) = 0, \ \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded}\right\}.$$

- *The residual spectrum of T is define by*

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \ \ker(\lambda I - T) = 0 \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}.$$

Definition 1.1.10. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator and let λ be an eigevalue of A. A non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue value λ, if there exists $n \in \mathbb{N}^*$ such that*

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

If $n = 1$, then e is called an eigenvector.

Definition 1.1.11. *Let $T : D(T) \subset E \longmapsto F$ be a closed unbounded linear operator. We say that T has compact resolvent, if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.*

Theorem 1.1.12. Let $(T, D(T))$ be a closed unbounded linear operator on H then the space $\left(D(T), \|\cdot\|_{D(T)}\right)$ where $\|u\|_{D(A)} = \|Tu\|_H + \|u\|_H \ \forall\ u \in D(T)$ is banach space .

Theorem 1.1.13. *Let $(T, D(T))$ be a closed unbounded linear operator on H then, $\rho(T)$ is an open set of $\mathbb{C}$.*

1.2 Semigroups, Existence and uniqueness of solution

In this section, we start by introducing some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t &= AU, \quad t > 0, \\ U(0) &= U_0, \end{cases} \tag{1.2.1}$$

where A is the infinitesimal generator of a C_0-semigroup $S(t)$ over a Hilbert space H. Lets start by basic definitions and theorems.
Let $(X, \|\cdot\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 1.2.1. *A family $(S(t))_{t\geq 0}$ of bounded linear operators in X is called a strong continous semigroup (in short, a C_0-semigroup) if*

- $S(0) = I$ *(I is the identity operator on X).*
- $S(t+s) = S(t)S(s), \quad \forall\ t, s \geq 0.$
- *For each $u \in H$, $S(t)u$ is continous in t on $[0, +\infty[$.*

Sometimes we also denote $S(t)$ by e^{tA}.

Definition 1.2.2. *For a semigroup $(S(t))_{t\geq 0}$, we define an linear operator A with domain $D(A)$ consisting of points u such that the limit*

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

exists. Then A is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$.

Proposition 1.2.3. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup in X. Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded. Moreover, if $M = 1$ then $(S(t))_{t\geq 0}$ is said to be a C_0-semigroup of contractions.

Definition 1.2.4. *An unbounded linear operator $(A, D(A))$ on H, is said to be dissipative if*

$$\Re \langle Au, u \rangle_H \leq 0, \quad \forall\ u \in D(A).$$

Definition 1.2.5. *An unbounded linear operator $(A, D(A))$ on X, is said to be m-dissipative if*

- *A is a dissipative operator.*
- *$\exists\ \lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - A) = X$.*

Theorem 1.2.6. *Let A be a m-dissipative operator, then*

- *$\mathcal{R}(\lambda I - A) = X, \quad \forall\ \lambda > 0$.*
- *$]0, \infty[\ \subseteq \rho(A)$.*

Theorem 1.2.7. (***Hille-Yosida***) *An unbounded linear operator $(A, D(A))$ on X, is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ if and only if*

- *A is closed and $\overline{D(A)} = X$.*
- *The resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}^+$, and for all $\lambda > 0$,*

$$\left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} \leq \lambda^{-1}.$$

Theorem 1.2.8. (***Lummer-Phillips***) *Let $(A, D(A))$ be an unbounded linear operator on X, with dense domain $D(A)$ in X. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if it is a m-dissipative operator.*

Theorem 1.2.9. *Let $(A, D(A))$ be an unbounded linear operator on X. If A is dissipative with $\mathcal{R}(I - A) = X$, and X is reflexive then $\overline{D(A)} = X$.*

Corollary 1.2.10. *Let $(A, D(A))$ be an unbounded linear operator on H. A is the infinitesimal generator of a C_0-semigroup of contractions if and only if A is a m-dissipative operator.*

Theorem 1.2.11. *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

Theorem 1.2.12. (***Hille-Yosida***) *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$.*

1. *For $U_0 \in D(A)$, the problem (1.2.1) admits a unique strong solution*

$$U(t) = S(t)U_0 \in C^0\left(\mathbb{R}^+, D(A)\right) \cap C^1\left(\mathbb{R}^+, H\right).$$

2. *For $U_0 \in H$, the problem (1.2.1) admits a unique weak solution*

$$U(t) \in C^0\left(\mathbb{R}^+, H\right).$$

1.3 Stability of semigroup

In this section we start by itroducing some definion about strong, exponential and polynomial stability of a C_0-semigroup. Then we collect some results about the stability of C_0-semigroup.

Let $(X, \|\cdot\|_X)$ be a Banach space, and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 1.3.1. *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. We say that the C_0-semigroup $(S(t))_{t\geq 0}$ is*

- *Strongly stable if*

$$\lim_{t\to+\infty} \|S(t)\,u\|_X = 0, \quad \forall\ u \in X.$$

- *Uniformly stable if*

$$\lim_{t\to+\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$

- *Exponentially stable if there exist two positive constants M and ϵ such that*

$$\|S(t)\,u\|_X \leq M e^{-\epsilon t}\|u\|_X, \quad \forall\ t > 0,\ \forall\ u \in X.$$

- *Polynomially stable if there exist two positive constants C and α such that*

$$\|S(t)\,u\|_X \leq C t^{-\alpha}\|u\|_X, \quad \forall\ t > 0,\ \forall\ u \in X.$$

Proposition 1.3.2. *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on X. The following statements are equivalent*

- *$(S(t))_{t\geq 0}$ is uniformly stable.*
- *$(S(t))_{t\geq 0}$ is exponentially stable.*

First, we look for the necessary conditions of strong stability of a C_0-semigroup. The result was obtained by Arendt and Batty.

Theorem 1.3.3. (***Arendt and Batty***) *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues.*
- *$\sigma(A) \cap i\mathbb{R}$ is countable.*

Then $S(t)$ is strongly stable.

Remark 1.3.4. *If the resolvent $(I-T)^{-1}$ of T is compact, then $\sigma(T) = \sigma_p(T)$. Thus, the state of Theorem 1.3.3 lessens to $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.*

Next, when the C_0-semigroup is strongly stabe, we look for the necessary and suffient conditions of exponential stability of a C_0-semigroup. In fact, exponential stability results are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them . In this thesis we will review only two methods. The first method is a frequency domain approach method was obtained by Huang-Pruss.

Theorem 1.3.5. (***Huang-Pruss***)*Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. $S(t)$ is uniformly stable if and only if*

- $i\mathbb{R} \subset \rho(A)$.
- $\sup\limits_{\beta\in\mathbb{R}} \| (i\beta I - A)^{-1} \|_{\mathcal{L}(H)} < +\infty$.

The second one, is a classical method based on the spectrum analysis of the operator A.

Definition 1.3.6. *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$.*

- *The growth bound of A is define by*

$$\omega_0(A) = \inf\left\{\omega \in \mathbb{R} :\ \exists\ N_\omega \in \mathbb{R} \text{ such that } \forall\ t \geq 0 \text{ we have } \|S(t)\| \leq N_\omega e^{\omega t}\right\}.$$

- *The spectral bound of A is define by*

$$s(A) = \sup\left\{\Re(\lambda) :\ \lambda \in \sigma(A)\right\}.$$

Proposition 1.3.7. *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Then $(S(t))_{t\geq 0}$ is uniformly exponentially stable if and only if its growth bound $\omega_0(A) < 0$.*

Proposition 1.3.8. *Let $(A, D(A))$ be an unbounded linear operator on H. Assume that A is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Then, we have*

$$s(A) \leq \omega_0(A).$$

Corollary 1.3.9. *Let $(A, D(A))$ be an unbounded linear-operator on H. Assume that $s(A) = 0$, then $(S(t))_{t\geq 0}$ is not uniformly exponentially stable.*

In the case when the C_0-semigroup is not exponentialy stable we look for a polynomial one. In general, polynomial stability results also are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them . In this thesis we will review only one method. The first method is a frequency domain approach method was obtained by Batty, A.Borichev and Y.Tomilov, Z. Liu and B. Rao.

Theorem 1.3.10. (***Batty , A.Borichev and Y.Tomilov, Z. Liu and B. Rao.***)*Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on H. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $\ell > 0$ the following conditions are equivalent*

1. $\displaystyle\lim_{|\lambda|\to+\infty} \sup \frac{1}{\lambda^{\ell}} \| (\lambda I - A)^{-1} \|_{\mathcal{L}(H)} < +\infty.$
2. $\displaystyle\|S(t)\,U_0\|_H \leq \frac{C}{t^{\ell^{-1}}}\ \|U_0\|_{D(A)} \quad \forall\ t > 0,\ U_0 \in D(A),$ *for some* $C > 0$.

1.4 Fractional Derivative Control

In this part, we introduce the necessary elements for the good understanding of this manuscript. It includes a brief reminder of the basic elements of the theory of fractional computation as well as some examples of applications of this theory in this scientific field. The concept of fractional computation is a generalization of ordinary derivation and integration to an arbitrary order. Derivatives of non-integer order are now widely applied in many domains, for example in economics, electronics, mechanics, biology, probability and viscoelasticity.
A particular interest for fractional derivation is related to the mechanical modeling of gums and rubbers. In short, all kinds of materials that preserve the memory of previous deformations in particular viscoelastic. Indeed, the fractional derivation is introduced naturally. There exists a many mathematical definitions of fractional order integration and derivation. These definitions do not always lead to identical results but are equivalent for a wide large of functions. We introduce the fractional integration operator as well as the two most definitions of fractional derivatives, used, namely that Riemann-Liouville and Caputo, by giving the most important properties of the notions. Fractional systems appear in different fields of research. However, the progressive interest in their applications in the basic and applied sciences. It can be noted that for most of the domains presented (automatic, physics, mechanics of continuous media). The fractional operators are used to take into account memory effects. We can mention the works that reroute various applications of fractional computation.
In physics, on of the most remarkable applications of fractional computation in physics was in the context of classical mechanics. Riewe, has shown that the Lagrangien of the motion of temporal derivatives of fractional orders leads to an equation of motion with friction forces and nonconservative are essential in macroscopic variational processing such as friction. This

result are remarkable because friction forces and non conservative forces are essential in the usual macroscopic variational processing and therefore in the most advances methods classical mechanics. Riewe, has generalized the usual Lagrangian variation which depends on the fractional derivatives in order to deal with the usual non-conservative forces. On the another hand, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivative. The definition of the fractional order derivation is based on that of a fractional order integration, a fractional order derivation takes on a global character in contrast to an integral derivation. It turns out that the derivative of a fractional order of a function requires the knowledge of $f(t)$ over the entire interval $]a,b[$, where in the whole case only the local knowledge of f around t is necessary. This property allows to interpret fractional order systems as long memory systems, the whole systems being then interpretable as systems with short memory. Now, we give the definition of the fractional derivatives in the sense of Riemann-Liouville as well as some essential properties.

Definition 1.4.1. *The fractional integral of order $\alpha > 0$, in sense Riemann-Liouville is given by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a.$$

Definition 1.4.2. *The fractional derivative of order $\alpha > 0$, in sens of Riemann-Liouville of a function f defined on the interval $[a,b]$ is given by*

$$D_{RL,a}^\alpha(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad n = [\alpha]+1, \ t > a. \tag{1.4.1}$$

In particular, if $\alpha = 0$, then

$$D_{RL,a}^0 f(t) = I_a^0 f(t) = f(t).$$

If $\alpha = n \in \mathbb{N}$, then

$$D_{RL,a}^n f(t) = f^{(n)}(t).$$

Moreover, if $0 < \alpha < 1$, then $n = 1$, then

$$D_{RL,a}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds, \quad t > a.$$

Example 1.4.3. *Let $\alpha > 0$, $\gamma > -1$ and $f(t) = (t-a)^\gamma$, then*

$$I_a^\alpha f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha},$$

$$D_{RL,a}^\alpha f(t) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}.$$

In particular, if $\gamma = 0$ *and* $\alpha > 0$, *then* $D^{\alpha}_{RL,a}(C) = C\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$.

The derivatives of Riemann-Liouville have certain disadvantages when attempting to model real world phenomena. The problems studied require a definition of the fractional derivatives allowing the use of the physically interpretable initial conditions including $y(0), y'(0)$, etc. There shortcomings led to an alternative definition of fractional derivatives that satisfies these demands in the last sixties. It was introduced by Caputo. In fact, Caputo and Minardi used this definition in their work on viscoelasticity.
Now, we give the definition of the fractional derivatives in the sense of Caputo as well as some essential properties.

Definition 1.4.4. *The fractional derivative of order* $\alpha > 0$, *in sens of Caputo, defined on the interval* $[a, b]$, *is given by*

$$D^{\alpha}_{C,a} f(t) = D^{\alpha}_{RL,a}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k\right), \tag{1.4.2}$$

where

$$n = \begin{cases} [\alpha] + 1 & \text{if} \quad \alpha \notin \mathbb{N}, \\ \alpha & \text{if} \quad \alpha \in \mathbb{N}^*. \end{cases}$$

In particular, where $0 < \alpha < 1$, *the relation* (1.4.2) *take the form*

$$\begin{aligned} D^{\alpha}_{C,a} f(t) &= D^{\alpha}_{C,a}\left([f(t) - f(a)]\right) \\ &= I^{1-\alpha}_{a} f'(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-a)^{-\alpha} f'(s) ds. \end{aligned}$$

If $\alpha \in \mathbb{N}$, *then* $f^{(n)}(t)$ *and* $D^{\alpha}_{C,a}(t)$ *coincides i.e.*

$$D^{\alpha}_{C,a} f(t) = f^{(n)}(t).$$

Example 1.4.5. *Let* $\alpha > 0$ *and* $f(t) = (t-a)^{\gamma}$ *where* $\gamma > -1$. *Then*

$$D^{\alpha}_{Ca} f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}.$$

In particular, if $\gamma = 0$ *and* $\alpha > 0$, *then* $D^{\alpha}_{Ca} C = 0$.

1.5 Geometric Condition

In this section, we present two different types on the geometric conditions.

Definition 1.5.1. *We say that the multiplier control condition* **MGC** *holds if there exist* $x_0 \in \mathbb{R}^d$ *and a positive constant* $m_0 > 0$ *such that*

$$m \cdot \nu \leq 0 \quad on \quad \Gamma_0 \quad and \quad m \cdot \nu \geq m_0 \quad on \quad \Gamma_1,$$

with $m(x) = x - x_0$*, for all* $x \in \mathbb{R}^d$*.* □

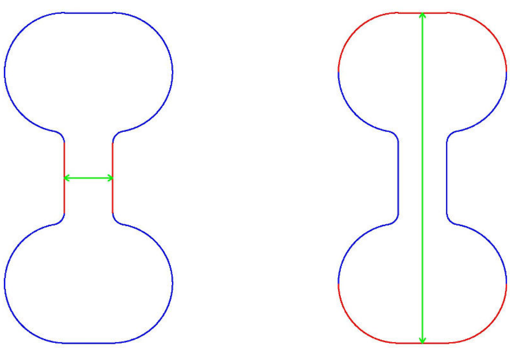
We recall the Geometric Control condition **GCC** introduced by Bardos, Lebeau and Rauch [13] :

Definition 1.5.2. *We say that* Γ *satisfies the geometric condition named* ***GCC****, if every ray of geometrical optics, starting at any point* $x \in \Omega$ *at time* $t = 0$*, hits* Γ_1 *in finite time* T*.*

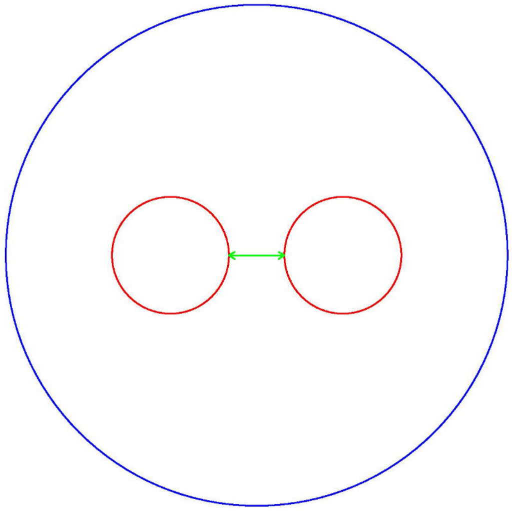
Remark 1.5.3. *In [13], Bardos* et al. *proved that* (H) *holds if* Γ *is smooth (of class* C^∞*),* $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ *and the* **GCC** *condition. For less regular domains, namely of class* C^2*,* (H) *holds if the vector field assumptions described in [33] (see (i),(ii),(iii) of Theorem 1 in [33]) hold. Moreover, in Theorem 1.2 of [34] the authors prove that* (H) *holds for smooth domains under weaker geometric conditions than in [33] (without (ii) of Theorem 1). Finally, it is easy to see that the multiplier control condition* **MCC** *implies that the vector field assumptions described in [33] are satisfied and therefore the condition* (H) *holds if* **MCC** *holds.* □

Now, we present a models satisfies and doesn't satisfy (GCC) and (MGC) conditions.

•**Models does not satisfies** (GCC) :

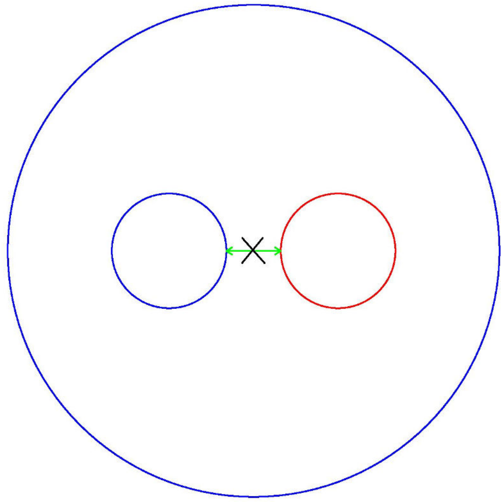


This model does not satisfy the GCC boundary condition.

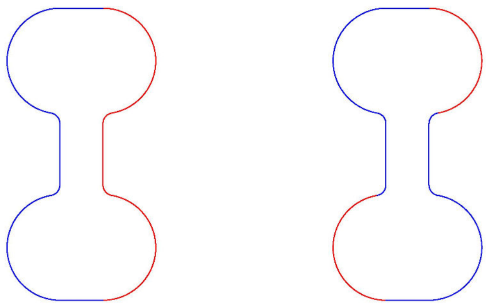


This model does not satisfy the GCC boundary condition.

•**Models satisfies** (GCC) **boundary conditions** :

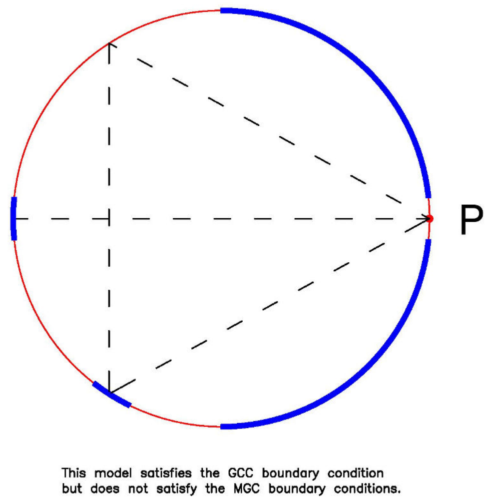


This model satisfies the GCC boundary condition.



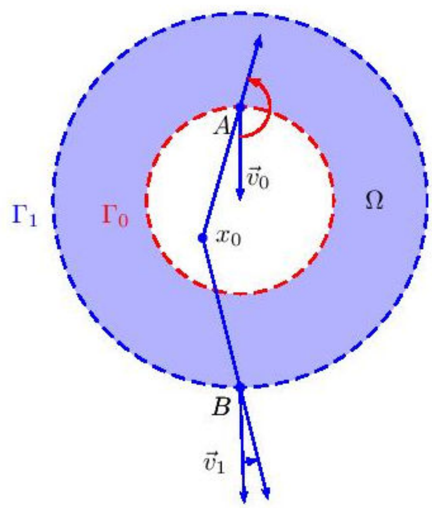
This model satisfies the GCC boundary condition.

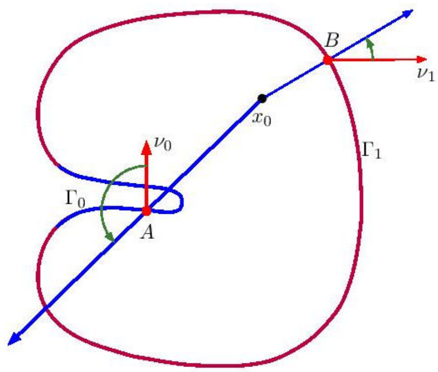
- **Models satisfy** (GCC) **but does not satisfy** (MGC) **condition**



Remark 1.5.4. *In this figure, We take an open arc Υ in the boundary that contains a half-circumference and let P denote the midpoint of Υ. For ε sufficiently small denote γ_ε the closed arc centered at P with length less ε. For a ray to miss $\Upsilon\backslash P$ at must hit P as does the equilateral triangle with vertex P. Let θ denote the union of two open arcs centered respectively at the antipodal of P and one of the other vertices of the equilateral triangle. Let $\Gamma_1 = (\Upsilon \cup \theta)\backslash\gamma_\varepsilon$ and $\Gamma_0 = \partial\Omega\backslash\Gamma_1$, then the condition* **GCC** *holds.*

•**Models satisfies** (MGC) **and** (GCC) **boundary conditions** :





This model satisfies the **MGC** boundary condition without $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

1.6 Appendix

Theorem 1.6.1. *Let μ be the function defined by*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-d}{2}}, \quad \xi \in \mathbb{R}^d \quad \text{and} \quad 0 < \alpha < 1. \tag{1.6.1}$$

The relation between the "imput" U and the "output" O of the following system

$$\partial_t \omega(\xi,t) + \left(|\xi|^2 + \eta\right)\omega(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \text{ and } \eta \geq 0, \tag{1.6.2}$$

$$\omega(\xi,0) = 0, \tag{1.6.3}$$

$$O(t) = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi \tag{1.6.4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U. \tag{1.6.5}$$

Proof:
Step 1. Take $\eta = 0$, the from equation (1.6.2) and (1.6.3), we have

$$\omega(\xi,t) = \int_0^t \mu(\xi)e^{-|\xi|^2(t-\tau)}U(\tau)d\tau. \tag{1.6.6}$$

Then from equations (1.6.4) and (1.6.6), we get

$$O(t) = \delta \int_{\mathbb{R}^d} |\xi|^{2\alpha-d}\left[\int_0^t e^{-|\xi|^2(t-\tau)}U(\tau)d\tau\right]d\xi. \tag{1.6.7}$$

where $\delta = \dfrac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$. Next, using the spherical coordinates defined by,

$$\left\{\begin{array}{lcl} \xi_1 & = & \rho\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{d-3})\sin(\phi_{d-2})\sin(\phi_{d-1}), \\ \xi_2 & = & \rho\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{d-3})\sin(\phi_{d-2})\cos(\phi_{d-1}), \\ \xi_3 & = & \rho\sin(\phi_1)\sin(\phi_2)\cdots\sin(\phi_{d-3})\cos(\phi_{d-2}), \\ \xi_4 & = & \rho\sin(\phi_1)\sin(\phi_2)\cdots\cos(\phi_{d-3}), \\ \vdots & & \\ \xi_{d-1} & = & \rho\sin(\phi_1)\cos(\phi_2), \\ \xi_d & = & \rho\cos(\phi_1). \end{array}\right. \tag{1.6.8}$$

where, $\rho = |\xi| = \sqrt{\sum_{i=1}^d |\xi_i|^2}$, $\phi_j \in [0,\pi]$ if $1 \leq j \leq d-2$ and $\phi_{d-1} \in [0,2\pi]$. The jacobian J is defined by

$$J=\rho^{d-1}\prod_{j=1}^{d-2}\sin^{d-1-j}(\phi_j). \tag{1.6.9}$$

Since the integrating is a function which depends only on $|\xi|=\rho$, thus we can integrate on all the angles and the calculation reduces that of a simple integral on the positive real axis. Then, from equations (1.6.7)-(1.6.9) we get

$$O(t)=\delta\int_0^{+\infty}\rho^{2\alpha-1}\prod_{j=1}^{d-2}\left(\int_0^{\pi}\sin^{d-1-j}(\phi_j)d\phi_j\right)\int_0^{2\pi}d\phi_{d-1}\left[\int_0^t e^{-\rho^2(t-\tau)}U(\tau)d\tau\right]d\rho. \tag{1.6.10}$$

By induction, it easy to see that

$$\prod_{j=1}^{d-2}\left(\int_0^{\pi}\sin^{d-1-j}(\phi_j)d\phi_j\right)\int_0^{2\pi}d\phi_{d-1}=\frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}. \tag{1.6.11}$$

Inserting equation (1.6.11) in equation (1.6.10), we get

$$O(t)=\frac{\sin(\alpha\pi)}{\pi}\int_0^t 2\left[\int_0^{+\infty}\rho^{2\alpha-1}e^{-\rho^2(t-\tau)}d\rho\right]U(\tau)d\tau. \tag{1.6.12}$$

Thus,

$$O(t)=\frac{\sin(\alpha\pi)}{\pi}\int_0^t\left[(t-\tau)^{-\alpha}\Gamma(\alpha)\right]U(\tau)d\tau. \tag{1.6.13}$$

Using the fact that $\dfrac{\sin(\alpha\pi)}{\pi}=\dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ in equation, we obtain

$$O(t)=\int_0^t\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}U(\tau)d\tau. \tag{1.6.14}$$

It follows that, from equation (1.6.14) we have

$$O=I^{1-\alpha}U. \tag{1.6.15}$$

Step 2. By simply effecting the following change of function

$$\omega(\xi,t):=e^{-\eta t}\varphi(\xi,t)$$

in equations (1.6.2) and (1.6.4), we directly obtain

$$\partial_t\omega(\xi,t)+\left(|\xi|^2+\eta\right)\omega(\xi,t)-U(t)\mu(\xi)=0,\quad \xi\in\mathbb{R}^N,t\in\mathbb{R}^+\text{ and }\eta\geq 0, \tag{1.6.16}$$

$$\omega(\xi,0)=0, \tag{1.6.17}$$

$$O(t) = \delta e^{-\eta t} \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi. \tag{1.6.18}$$

Hence, from **Step 1**, (1.6.16)-(1.6.18) yield the desired result

$$O(t) = e^{-\eta t} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} e^{\eta\tau} U(\tau)d\tau.$$

The proof has been completed.

Lemma 1.6.2. *For all $\lambda \in \mathbb{R}$ and $\eta > 0$, we have*

$$A_1 = \int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha-d}}{|\lambda|+\eta+|\xi|^2} d\xi = c\left(|\lambda|+\eta\right)^{\alpha-1} \quad \textit{and} \quad A_3 = \left(\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha-d}}{(|\lambda|+\eta+|\xi|^2)^2}\right)^{\frac{1}{2}} = \tilde{c}\left(|\lambda|+\eta\right)^{\frac{\alpha}{2}-1}$$

where $c, \tilde{c}$ are two positive constants given by

$$c = \frac{d\pi^{\frac{d}{2}+1}}{2\Gamma\left(\frac{d}{2}+1\right)\sin(\alpha\pi)} \quad \textit{and} \quad \tilde{c} = \left(\frac{d\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_1^{+\infty} \frac{(y-1)^\alpha}{y^2} dy\right)^{\frac{1}{2}}. \tag{1.6.19}$$

Proof:
Calculation of A_1 :
Using the hyper-spherical coordinates in A_1, we get

$$A_1 = \int_0^{+\infty} \frac{\rho^{2\alpha-1}}{|\lambda|+\eta+\rho^2} \prod_{j=1}^{d-2}\left(\int_0^\pi \sin^{d-1-j}(\phi_j)d\phi_j\right) \int_0^{2\pi} d\phi_{d-1}d\rho. \tag{1.6.20}$$

Using equation (1.6.11) in equation (1.6.20), we get

$$A_1 = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \int_0^{+\infty} \frac{\rho^{2\alpha-1}}{|\lambda|+\eta+\rho^2} d\rho. \tag{1.6.21}$$

Let $x = \rho^2$ in equation (1.6.21), we obtain

$$A_1 = \frac{d\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_0^{+\infty} \frac{x^{\alpha-1}}{|\lambda|+\eta+x} dx. \tag{1.6.22}$$

Let $y = \dfrac{x}{|\lambda|+\eta} + 1$ in equation (1.6.22), we get

$$A_1 = \frac{d\pi^{\frac{d}{2}}(|\lambda|+\eta)^{\alpha-1}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_1^{+\infty} \frac{(y-1)^{\alpha-1}}{y} dy. \tag{1.6.23}$$

Let $z = \dfrac{1}{y}$ in equation (1.6.23), we get

$$A_1 = \frac{d\pi^{\frac{d}{2}}(|\lambda|+\eta)^{\alpha-1}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1}dz. \tag{1.6.24}$$

Then from equation (1.6.24), we get

$$A_1 = \frac{d\pi^{\frac{d}{2}}(|\lambda|+\eta)^{\alpha-1}}{2\Gamma\left(\frac{d}{2}+1\right)} B(1-\alpha,\alpha) = \frac{d\pi^{\frac{d}{2}}(|\lambda|+\eta)^{\alpha-1}}{2\Gamma\left(\frac{d}{2}+1\right)} \Gamma(1-\alpha)\Gamma(\alpha). \tag{1.6.25}$$

Using the fact that $\Gamma(1-\alpha)\Gamma(\alpha) = \dfrac{\pi}{\sin(\alpha\pi)}$ in equation (1.6.25), we get

$$A_1 = c\,(|\lambda|+\eta)^{\alpha-1}\,, \tag{1.6.26}$$

where c is defined in (1.6.19).
Calculation of A_3 :
Using the hyper-spherical coordinates and the same arguments of A_1, we get

$$A_3^2 = \frac{d\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \int_0^{+\infty} \frac{\rho^{2\alpha-1}}{(|\lambda|+\eta+\rho^2)^2} d\rho. \tag{1.6.27}$$

Let $y = \dfrac{x}{|\lambda|+\eta} + 1$ in equation (1.6.27), we get

$$A_3^2 = \frac{d\pi^{\frac{d}{2}}\,(|\lambda|+\eta)^{\alpha-2}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_1^{+\infty} \frac{(y-1)^{\alpha}}{y^2} dy. \tag{1.6.28}$$

Its clear that $\displaystyle\int_1^{+\infty} \frac{(y-1)^{\alpha-1}}{y^2} dy < +\infty$ for $\alpha \in]0,1[$, then we get

$$A_3 = \tilde{c}\,(|\lambda|+\eta)^{\frac{\alpha}{2}-1}\,,$$

where $\tilde{c}$ is defined in (1.6.19)

Lemma 1.6.3. *If* $\lambda \in D = \{\lambda \in \mathbb{C};\ \ \Re(\lambda)+\eta > 0\} \cup \{\lambda \in \mathbb{C};\ \Im(\lambda) \neq 0\}$, *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\eta)^{\alpha-1}.$$

Proof: Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}.$$

We have

$$|f_\lambda(\xi)| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\Re(\lambda)+\eta+|\xi|^2} \\ or \\ \dfrac{\mu^2(\xi)}{|\Im(\lambda)|+\eta+|\xi|^2} \end{cases}$$

Then the function f_λ is integrable. Moreover,

$$|f_\lambda(\xi)| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0+\eta+|\xi|^2} & \text{for all} \quad \Re(\lambda) \geq \eta_0 > -\eta. \\ \dfrac{\mu^2(\xi)}{\eta_1+|\xi|^2} & \text{for all} \quad |\Im(\lambda)| \geq \eta_1 > 0. \end{cases}$$

Them from Theorem 1.16.1 in [60] , the function $f_\lambda : D \to \mathbb{C}$ is holomorphe. For a real $\lambda > -\eta$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2} d\xi &= \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+\xi} dx && \text{with} \quad \xi^2 = x \\
&= (\lambda+\eta)^{\alpha-1} \int_1^{+\infty} y^{-1}(y-1)^{\alpha-1} dy && \text{with} \quad x = (\lambda+\eta)(y-1) \\
&= (\lambda+\eta)^{\alpha-1} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1} dz && with \quad yz = 1 \\
&= (\lambda+\eta)^{\alpha-1} B(1-\alpha,\alpha) \\
&= (\lambda+\eta)^{\alpha-1} \Gamma(1-\alpha)\Gamma(\alpha) \\
&= \frac{\pi}{\sin(\alpha\pi)} (\lambda+\eta)^{\alpha-1}.
\end{aligned}$$

The proof has been completed.

CHAPITRE 2

STABILIZATION OF MULTIDIMENSIONAL WAVE EQUATION WITH LOCALLY BOUNDARY FRACTIONAL DISSIPATION LAW

2.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 open subsets of Γ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ_1 is non empty. We consider the multidimensional wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (2.1.1)\\ u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, && (2.1.2)\\ \frac{\partial u}{\partial \nu} + \gamma \partial_t^{\alpha,\eta} u &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ && (2.1.3) \end{aligned}$$

where ν is the unit outward normal vector along the boundary Γ_1 and γ is a positive constant involved in the boundary control. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative see [20] of order α with respect to the time variable and is defined by

$$\partial_t^{\alpha,\eta} \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds, \quad 0 < \alpha < 1, \ \eta \geq 0.$$

The system (2.1.1)-(2.1.3) is considered with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{where} \quad x \in \Omega. \qquad (2.1.4)$$

The fractional derivative operator of order α, $0<\alpha<1$, is defined by

$$[D^\alpha f](t)=\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\frac{df}{d\tau}(\tau)d\tau. \tag{2.1.5}$$

The fractional differentiation is inverse operation of fractional integration that is defined by

$$[I^\alpha f](t)=\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau, \quad 0<\alpha<1. \tag{2.1.6}$$

From equations (2.1.5), (2.1.6), clearly

$$[D^\alpha f]=I^{1-\alpha}Df. \tag{2.1.7}$$

Now, we present marginally distinctive forms of (2.1.5) and (2.1.6). These exponentially modified fractional integro-differential operators an will be denoted by us follows

$$[D^{\alpha,\eta} f](t)=\int_0^t \frac{(t-\tau)^{-\alpha}e^{-\eta(t-\tau)}}{\Gamma(1-\alpha)}\frac{df}{d\tau}(\tau)d\tau \tag{2.1.8}$$

and

$$[I^{\alpha,\eta} f](t)=\int_0^t \frac{(t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}}{\Gamma(\alpha)}f(\tau)d\tau. \tag{2.1.9}$$

Note that the two operators D^α and $D^{\alpha,\eta}$ differ just by their Kernels. $D^{\alpha,\eta}$ is merely Caputo's fractional derivative operator, expect for its exponential factor. Thus, similar to identity (2.1.7), we do have

$$[D^{\alpha,\eta} f]=I^{1-\alpha,\eta}Df. \tag{2.1.10}$$

The order of our derivatives is between 0 and 1.

The boundary fractional damping of the type $\partial_t^{\alpha,\eta}u$ where $0<\alpha<1$, $\eta\geq 0$ arising from the material property has been used in several applications such as in physical, chemical, biological, ecological phenomena. For more details we refer the readers to [46], [47], [10], [11], [12] and [44]. In theoretical point of view, fractional derivatives involves singular and non-integrable kernels ($t^{-\alpha}$, $0<\alpha<1$). This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

There are a few number of publications concerning the stabilization of distributed systems with fractional damping. In [46], B. Mbodje considered a $1-d$ wave equation with boundary fractional damping acting on a part of the boundary of the domain :

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) &=& 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) &=& 0, & \\ \partial_x u(1,t) + \gamma \partial_t^{\alpha,\eta} u(1,t) &=& 0, & 0 < \alpha < 1, \quad \eta \geq 0, \\ u(x,0) &=& u_0(x), & \\ \partial_t u(x,0) &=& v_0(x). & \end{cases} \tag{2.1.11}$$

Firstly, he proved that system (2.1.11) is not uniformly stable, on other words its energy has no exponential decay rate. However, using LaSalle's invariance principle, he proved that system (2.1.11) is strongly stable for usual initial data. Secondly, he established a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. In this paper, our main interest is to generalize the results of [46] by considering the multidimensional case and by improving the polynomial energy day rate. Then, we study the stabilization of the system of multidimensional wave equation with boundary fractional damping (2.1.1)-(2.1.3). In a first step, a general criteria of Arendt and Batty [9] and with the help of Holmgren's theorem we show the strong stability of system (2.1.1)-(2.1.3), but for the simple example like the case when $\Omega = (0,1)$ we show that our system is not uniformly stable, since the corresponding spatial operator has a sequence of eigenvalues that approach the imaginary axis. Hence, we are interested in proving a weaker decay of the energy, for that purpose, we will apply a frequency domain approach (see [18, 15, 40]) based on the growth of the resolvent on the imaginary axis. More precisely, we will give sufficient conditions that garantee the polynomial decay of the energy of our system (for smooth initial data). For this aim, by assuming that the boundary control region satisfy the Geometric Control Condition (GCC) and by using the exponential decay of the wave equation with the standard damping

$$\partial_\nu u(x,t) + u_t(x,t) = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+^\star$$

we establish a polynomial energy decay rate for smooth solutions, which depends on the order of the fractional derivative. More precisely, we show that the energy of smooth solution of system (2.1.1)-(2.1.3) converges to zero, as t goes to infinity, as $\frac{1}{t^{\frac{1}{1-\alpha}}}$.

In [23], Zhang and Dai considered the multidimensional wave equation with boundary source term and fractional dissipation defined by

$$\begin{cases} u_{tt} - \Delta u &=& 0, & x \in \Omega \quad t > 0, \\ u &=& 0, & x \in \Gamma_0 \quad t > 0, \\ \dfrac{\partial u}{\partial \mu} + \partial_t^\alpha u &=& |u|^{m-1} u, & x \in \Gamma_1 \quad t > 0, \\ u(x,0) &=& u_0, & x \in \Omega, \\ u_t(x,0) &=& u_1(x), & x \in \Omega \end{cases} \tag{2.1.12}$$

where $m > 1$. They proved by Fourrier transforms and the Hardy-Littelwood-Sobolev inequality the exponential stability for sufficiently large initial data.
In [2], Benaissa and al. considered the Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type defined by

$$\left\{\begin{array}{rcll}\varphi_{tt}(x,t)+\varphi_{xxxx}(x,t) & = & 0, & \text{in }]0,L[\times]0,+\infty[,\\ \varphi(0,t)=\varphi_x(0,t) & = & 0, & \text{in }]0,+\infty[,\\ \varphi_{xx}(L,t) & = & 0, & \text{in }]0,+\infty,\\ \varphi_{xxx}(L,t) & = & \gamma\partial_t^{\alpha,\eta}\varphi(L,t), & \text{in }]0,+\infty[\end{array}\right. \tag{2.1.13}$$

where $0 < \alpha < 1$, $\eta \geq 0$ and $\gamma > 0$. They proved, under the condition $\eta = 0$, by a spectral analysis, the non uniform stability. On the other hand, for $\eta > 0$, they also proved that the energy of system (4.1.17) decay as time goes to infinity as $\frac{1}{t^{\frac{1}{1-\alpha}}}$.

This chapter as organized as follows : In Subsection 2.2.1, we reformulate the system (2.1.1)-(3.1.3) into an augmented system by coupling the wave equation with a suitable diffusion equation and we prove the well-posedness of our system by semigroup approach. In the subsection 2.2.2, combining a general criteria of Arendt and Batty with Holmgren's theorem we show that the strong stability of our system in the absence of the compactness of the resolvent and without any additional geometric conditions. In subsection 2.2.3, We show that our system is not uniformly stable in general, since it is the case of the interval, more precisely we show that an infinite number of eigenvalues approach the imaginary axis. In Section 2.3, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method. Indeed, by assuming that the boundary control region satisfy the Geometric Control Condition (GCC) and by using the exponential decay of the wave equation with a standard damping, we establish a polynomial energy decay for smooth solution as type $\frac{1}{t^{\frac{1}{1-\alpha}}}$.

2.2 Well-Posedness and Strong Stability

In this section, we will study the strong stability of system (2.1.1)-(2.1.3) in the absence of the compactness of the resolvent and without any additional geometric conditions on the domain Ω. First, we will study the existence, uniqueness and regularity of the solution of our system.

2.2.1 Augmented model and well-Posedness

Firstly, we reformulate system (2.1.1)-(2.1.3) into an augmented system. For this aim, we need the following results

Proposition 2.2.1. *Let μ be the function defined by*

$$\mu(\xi)=|\xi|^{\frac{2\alpha-d}{2}},\quad \xi\in\mathbb{R}^d \quad \textit{and} \quad 0<\alpha<1. \tag{2.2.1}$$

Then the relation between the 'input' U and the 'output' O of the following system

$$\partial_t\omega(\xi,t)+(|\xi|^2+\eta)\omega(\xi,t)-U(t)\mu(\xi)=0,\quad \xi\in\mathbb{R}^d,\quad t>0, \tag{2.2.2}$$

$$\omega(\xi,0)=0, \tag{2.2.3}$$

$$O(t)=\frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi,t)d\xi, \tag{2.2.4}$$

is given by

$$O=I^{1-\alpha,\eta}U=D^{\alpha,\eta}U \tag{2.2.5}$$

where $D^{\alpha,\eta}$ and $I^{1-\alpha,\eta}$ are given by (4.1.11) and (4.1.12) respectively.

Proof:[proof] See theorem 1.6.1 in chapter 1.

Now, using Proposition 2.2.1, system (2.1.1)-(2.1.4) may be recast into the following augmented model :

$$\begin{aligned} u_{tt}-\Delta u &= 0, && \text{in} \quad \Omega\times\mathbb{R}^+, && (2.2.6)\\ u &= 0, && \text{on} \quad \Gamma_0\times\mathbb{R}^+, && (2.2.7)\\ \partial_t\omega(\xi,t)+(|\xi|^2+\eta)\omega(\xi,t)-\mu(\xi)\partial_t u(x,t) &= 0, && \text{on} \quad \Gamma_1\times\mathbb{R}^+,\ \xi\in\mathbb{R}^d, && (2.2.8)\\ \frac{\partial u}{\partial\nu}+\gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi,t)d\xi &= 0, && \text{on} \quad \Gamma_1\times\mathbb{R}^+ && (2.2.9) \end{aligned}$$

where γ is a positive constant, $\eta\geq 0$ and $\kappa=\frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$. Finally, system (2.2.6)-(2.2.9) is considered with the following initial conditions

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad \omega(\xi,0)=0,\quad x\in\Omega,\ \xi\in\mathbb{R}^d. \tag{2.2.10}$$

Our main interest is the existence, uniqueness and regularity of the solution to this system. We define the Hilbert space

$$\mathcal{H}=H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\mathbb{R}^d), \tag{2.2.11}$$

equipped with the following inner product

$$((u,v,\omega),(\tilde{u},\tilde{v},\tilde{\omega}))_{\mathcal{H}}=\int_{\Omega}\left(v\bar{\tilde{v}}+\nabla u\nabla\bar{\tilde{u}}\right)dx+\tilde{\gamma}\kappa\int_{\mathbb{R}^d}\omega(\xi)\bar{\tilde{\omega}}(\xi)d\xi$$

where $\tilde{\gamma}=\gamma\,|\Gamma_1|$, and $H^1_{\Gamma_0}(\Omega)$ is given by

$$H^1_{\Gamma_0}(\Omega)=\left\{u\in H^1(\Omega),\quad u=0\ \text{ on }\ \Gamma_0\right\}.$$

The energy of the solution of system is defined by :

$$E(t)=\frac{1}{2}\|(u,u_t,w)\|^2_{\mathcal{H}}. \tag{2.2.12}$$

For smooth solution, a direct computation gives

$$E'(t)=-\gamma\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|w(\xi,t)|^2d\xi. \tag{2.2.13}$$

Then, system (2.2.6)-(2.2.10) is dissipative in the sense that its energy is a nonincreasing function of the time variable t. Now, we define the linear unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A})=\left\{\begin{array}{l}U=(u,v,\omega)^\top\in\mathcal{H};\ \Delta u\in L^2(\Omega),\ v\in H^1_{\Gamma_0}(\Omega),\quad |\xi|\omega\in L^2(\mathbb{R}^N),\\ -(|\xi|^2+\eta)\omega+v|_{\Gamma_1}\mu(\xi)\in L^2(\mathbb{R}^d),\ \dfrac{\partial u}{\partial\nu}\,|_{\Gamma_1}\ =-\gamma\kappa\displaystyle\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi)d\xi\end{array}\right\} \tag{2.2.14}$$

and

$$\mathcal{A}(u,v,\omega)^\top=(v,\Delta u,-(|\xi|^2+\eta)\omega+v|_{\Gamma_1}\mu(\xi))^\top.$$

By denoting $v=u_t$ and $U_0=(u_0,v_0,w_0)^\top$, system (2.2.6)-(2.2.10) can be written as an abstract linear evolution equation on the space $\mathcal{H}$

$$U_t=\mathcal{A}U,\quad U(0)=U_0. \tag{2.2.15}$$

It is known that operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lumer-Phillips' theorem (see [42, 51]). Then the solution to the evolution equation (2.2.15) admits the following representation :

$$U(t)=e^{t\mathcal{A}}U_0,\quad t\geq 0,$$

which leads to the well-posedness of (2.2.15) . Hence, semi-group theory allows to show the next existence and uniqueness results :

Theorem 2.2.2. *For any initial data $U_0\in\mathcal{H}$, the problem (2.2.15) admits a unique weak solution $U(t)\in C^0(\mathbb{R}^+,\mathcal{H})$. Moreover if $U_0\in D(\mathcal{A})$ then the problem (2.2.15) admits a unique* *strong solution* $U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A}))$.

2.2.2 Strong Stability of the system

In this subsection, we study the strong stability of system (2.2.6)-(2.2.10) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle [58] or the spectrum decomposition theory of Benchimol [17] are not applicable in this case. We use then a general criteria of Arendt-Battay [9], following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

Theorem 2.2.3. *Assume that $\eta \geq 0$. Then the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e., for any $U_0 \in \mathcal{H}$ we have*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

First we need to prove the following lemmas :

Lemma 2.2.4. *Assume that $\eta \geq 0$. Then, for all $\lambda \in \mathbb{R}$, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof: Let $U \in D(\mathcal{A})$ and let $\lambda \in \mathbb{R}$, such that

$$\mathcal{A}U = i\lambda U. \tag{2.2.16}$$

Equivalently, we have

$$v = i\lambda u, \tag{2.2.17}$$
$$\Delta u = i\lambda v, \tag{2.2.18}$$
$$-(|\xi|^2+\eta)\omega + v|_{\Gamma_1}\mu(\xi) = i\lambda\omega. \tag{2.2.19}$$

Next, a straightforward computation gives

$$\Re\,(\mathcal{A}U, U)_{\mathcal{H}} = -\gamma\kappa \int_{\mathbb{R}^d} (|\xi|^2+\eta)|\omega|^2 d\xi. \tag{2.2.20}$$

Then, using (2.2.16) and (2.2.20) we deduce that

$$\omega = 0 \quad \text{a.e. in} \quad \mathbb{R}^d. \tag{2.2.21}$$

It follows, from (2.2.14) and (2.2.19), that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \Gamma_1. \tag{2.2.22}$$

Thus, by eliminating v, the system (2.2.17)-(2.2.19) implies that

$$\lambda^2 u + \Delta u = 0 \quad \text{in} \quad \Omega, \tag{2.2.23}$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{2.2.24}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1. \tag{2.2.25}$$

Now we distinguish two cases :

Case 1. $\lambda = 0$. A straightforward computation gives $u = 0$ and consequently, $U = 0$.

Case 2. $\lambda \neq 0$. Then, using (2.2.17) and (2.2.22) we deduce that $u = 0$ on Γ_1. Therefore, using Holmgren's theorem, we deduce that $u = 0$ and consequently, $U = 0$.

Lemma 2.2.5. *Assume that $\eta = 0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0 \in \sigma(\mathcal{A})$.*

Proof: First, let $\varphi_k \in H^1_{\Gamma_0}(\Omega)$ be an eigenfunction of the following problem

$$\begin{cases} -\Delta \varphi_k = \mu_k^2 \varphi_k, & \text{in} \quad \Omega, \\ \varphi_k = 0, & \text{on} \quad \Gamma_0, \\ \dfrac{\partial \varphi_k}{\partial \nu} = 0, & \text{on} \quad \Gamma_1 \end{cases} \tag{2.2.26}$$

such that

$$\|\varphi_k\|^2_{H^1_{\Gamma_0}(\Omega)} = \int_\Omega |\nabla \varphi_k|^2.$$

Next, define the vector $F = (\varphi_k, 0, 0) \in \mathcal{H}$. Assume that there exists $U = (u, v, w) \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F.$$

It follows that

$$v = -\varphi_k \quad \text{in} \quad \Omega, \quad |\xi|^2 \omega + \mu(\xi) v = 0 \quad \text{on} \quad \Gamma_1 \tag{2.2.27}$$

and

$$\begin{cases} \Delta u = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} + \gamma \kappa \displaystyle\int_{\mathbb{R}^d} \mu(\xi) \omega(\xi) d\xi = 0, & \text{on} \quad \Gamma_1. \end{cases} \tag{2.2.28}$$

From (2.2.27), we deduce that $\omega(\xi) = |\xi|^{\frac{2\alpha - d - 4}{2}} \varphi_k|_{\Gamma_1}$. We easily check that, for $\alpha \in]0, 1[$, the function $\omega(\xi) \notin L^2\left(\mathbb{R}^d\right)$. So, the assumption of the existence of U is false and consequently the operator $-\mathcal{A}$ is not invertible.

Lemma 2.2.6. *Assume that ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^\star$). Then, for any $f \in L^2(\Omega)$, the following problem*

$$\begin{cases} \lambda^2 u + \Delta u &= f, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} + (\lambda^2 c_1 + i\lambda c_2)u &= 0, \quad \text{on} \quad \Gamma_1, \end{cases} \tag{2.2.29}$$

where

$$c_1(\lambda,\eta) = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi \quad \text{and} \quad c_2(\lambda,\eta) = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu^2(\xi)(|\xi|^2+\eta)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi \tag{2.2.30}$$

admits a unique solution $u \in H^1_{\Gamma_0}(\Omega)$.

Proof: First, it is easy to check that, if ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^\star$), then, for $\alpha \in]0,1[$, the coefficients $c_1(\lambda,\eta)$ and $c_2(\lambda,\eta)$ are well defined. Moreover, if $\eta > 0$ and $\lambda = 0$ then, using Lax-Milgram 's theorem we deduce that system (2.2.29) admits a unique solution $u \in H^1_{\Gamma_0}(\Omega)$. Now, assume that $\eta \geq 0$ and $\lambda \in \mathbb{R}^\star$ and let us consider the following problem

$$\begin{cases} -\Delta u &= f, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} + (\lambda^2 c_1 + i\lambda c_2)u &= 0, \quad \text{on} \quad \Gamma_1 \end{cases} \tag{2.2.31}$$

Let $u = u_1 + iu_2$, $f = f_1 + if_2$ and we separate the real and the imaginary part of (2.2.31), we obtain

$$\begin{cases} -\Delta u_1 &= f_1 \quad \text{in} \quad \Omega, \\ -\Delta u_2 &= f_2 \quad \text{in} \quad \Omega, \\ u_1 = u_2 &= 0 \quad \text{on} \quad \Gamma_0, \\ \dfrac{\partial u_1}{\partial \nu} + \lambda^2 c_1 u_1 - \lambda c_2 u_2 &= 0 \quad \text{on} \quad \Gamma_1, \\ \dfrac{\partial u_2}{\partial \nu} + \lambda^2 c_1 u_2 + \lambda c_2 u_1 &= 0 \quad \text{on} \quad \Gamma_1. \end{cases} \tag{2.2.32}$$

Next, we give a variational formulation of (2.2.32). For this aim, find $(u_1,u_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ such that

$$a((u_1,u_2),(\varphi_1,\varphi_2)) = L((\varphi_1,\varphi_2)), \qquad \forall(\varphi_1,\varphi_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega), \tag{2.2.33}$$

where

$$\begin{aligned} a((u_1,u_2),(\varphi_1,\varphi_2)) &= \int_\Omega (\nabla u_1 \nabla \varphi_1 + \nabla u_2 \nabla \varphi_2)\, dx + c_1 \int_{\Gamma_1} \left(\lambda^2 u_1 \varphi_1 + \lambda^2 u_2 \varphi_2\right) d\Gamma_1 \\ &\quad + c_2 \int_{\Gamma_1} (\lambda u_1 \varphi_2 - \lambda u_2 \varphi_1)\, d\Gamma_1, \end{aligned}$$

and

$$L((\varphi_1,\varphi_2)) = \int_\Omega (f_1\varphi_1 + f_2\varphi_2)\, dx.$$

It is clear that the bilinear form a is continuous and coercive on the space $\left(H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)\right)^2$ and the linear form L is continuous on the space $H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$. Consequently, by Lax-Milligram's theorem, the variational problem (2.2.33) admits a unique solution $(u_1,u_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$. By choosing appropriated test functions in (2.2.33), we see that (u_1,u_2) satisfies (2.2.32) and therefore problem (2.2.31) admits a unique solution $u \in H^1_{\Gamma_0}(\Omega)$. In addition, we have (see [37])

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}. \tag{2.2.34}$$

It follows, from the compactness of the embedding $H^1_{\Gamma_0}(\Omega) \subset L^2(\Omega)$, that the inverse operator $(-\Delta)^{-1}$ defined in (2.2.31) is compact in $L^2(\Omega)$. Then applying $(-\Delta)^{-1}$ to (2.2.29), we get

$$\left(\lambda^2(-\Delta)^{-1} - I\right) u = (-\Delta)^{-1} f. \tag{2.2.35}$$

In addition, the same computation in (2.2.23)-(2.2.25) shows that $\ker\left(\lambda^2(-\Delta)^{-1} - I\right) = \{0\}$. Then, following Fredholm's alternative (see [19]), the equation (2.2.35) admits a unique solution.

Lemma 2.2.7. *If $\eta > 0$, for all $\lambda \in \mathbb{R}$, we have*

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}$$

while if $\eta = 0$, for all $\lambda \in \mathbb{R}^\star$, we have

$$\mathrm{R}\left(i\lambda I - \mathcal{A}\right) = \mathcal{H}.$$

Proof: We give the proof in the case $\eta > 0$, the proof of the second statement is fully similar. Let $\lambda \in \mathbb{R}$ and $F = (f,g,h)^\top \in \mathcal{H}$, then we look for $U = (u,v,\omega)^\top \in D(\mathcal{A})$ solution of

$$(i\lambda I - \mathcal{A})U = F. \tag{2.2.36}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v & = f, \quad \text{in} \quad \Omega, \\ i\lambda v - \Delta u & = g, \quad \text{in} \quad \Omega, \\ i\lambda\omega + (|\xi|^2+\eta)\omega - v|_{\Gamma_1}\mu(\xi) & = h, \quad \text{on} \quad \Gamma_1. \end{cases}$$

As before, by eliminating v and ω from the above system and using the fact that

$$\partial_\nu u + \gamma\kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi)d\xi = 0 \quad \text{on} \quad \Gamma_1,$$

we get the following system :

$$\begin{cases} \lambda^2 u + \Delta u & = -g - i\lambda f, \qquad \text{in} \quad \Omega, \\ u & = 0, \qquad \text{on} \quad \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right) u & = -i\lambda c_1 f + c_2 f + I_h^1 + I_h^2, \quad \text{on} \quad \Gamma_1, \end{cases} \tag{2.2.37}$$

where c_1, c_2 is defined in equation (2.2.30) and I_h^1, I_h^2 are given by

$$I_h^1(\lambda,\eta) = i\lambda\gamma\kappa \int_{\mathbb{R}^d} \frac{h(\xi)\mu(\xi)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi \quad \text{and} \quad I_h^2(\lambda,\eta) = -\gamma\kappa \int_{\mathbb{R}^d} \frac{h(\xi)\mu(\xi)(|\xi|^2+\eta)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi.$$

It easy to check that, for $h \in L^2\left(\mathbb{R}^d\right)$ and $\alpha \in]0,1[$, the integrals I_h^1 and I_h^2 are will defined. First, let $\varphi_h \in H^1_{\Gamma_0}(\Omega)$ be defined by

$$\begin{cases} -\Delta\varphi_h & = & 0 & \text{in} \quad \Omega, \\ \varphi_h & = & 0 & \text{on} \quad \Gamma_0, \\ \dfrac{\partial \varphi_h}{\partial \nu} & = & I_h^1 + I_h^2 & \text{on} \quad \Gamma_1. \end{cases} \tag{2.2.38}$$

Then setting $\tilde{u} = u + \varphi_h$ in (2.2.38), then we get

$$\begin{cases} \lambda^2\tilde{u} + \Delta\tilde{u} & = & \lambda^2\varphi_h - (g - i\lambda f) & \text{in} \quad \Omega, \\ \tilde{u} & = & 0 & \text{on} \quad \Gamma_0, \\ \dfrac{\partial \tilde{u}}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right)\tilde{u} & = & -i\lambda c_1 f + c_2 f + \left(\lambda^2 c_1 + i\lambda c_2\right)\varphi_h & \text{on} \quad \Gamma_1. \end{cases} \tag{2.2.39}$$

Next, let $\theta \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ such that

$$\theta = 0 \quad \text{and} \quad \frac{\partial\theta}{\partial\nu} = -i\lambda c_1 f + c_2 f + \left(\lambda^2 c_1 + i\lambda c_2\right)\varphi_h \in H^{\frac{1}{2}}(\Gamma_1) \text{ on } \Gamma_1. \tag{2.2.40}$$

Then setting $\chi = \tilde{u} - \theta$, we get

$$\begin{cases} \lambda^2\chi + \Delta\chi & = & \lambda^2\varphi_h - \lambda^2\theta - \Delta\theta - (g - i\lambda f) & \text{in} & \Omega, \\ \chi & = & 0 & \text{on} & \Gamma_0, \\ \dfrac{\partial\chi}{\partial\nu} + \left(\lambda^2 c_1 + i\lambda c_2\right)\chi & = & 0 & \text{on} & \Gamma_1. \end{cases} \tag{2.2.41}$$

Using Lemma 2.2.6, problem (2.2.41) has a unique solution $\chi \in H^1_{\Gamma_0}(\Omega)$ and therefore problem (2.2.37) has a unique solution $u \in H^1_{\Gamma_0}(\Omega)$. By defining $v = i\lambda u - f$ in Ω and

$$\omega = \frac{h(\xi)}{i\lambda + |\xi|^2 + \eta} + \frac{i\lambda u|_{\Gamma_1}\mu(\xi)}{i\lambda + |\xi|^2 + \eta} - \frac{f|_{\Gamma_1}\mu(\xi)}{i\lambda + |\xi|^2 + \eta}$$

we deduce that $U = (u, v, \omega)$ belongs to $D(\mathcal{A})$ and is solution of (2.2.36). This completes the proof.

Proof of Theorem 2.2.3. Following a general criteria of Arendt-Batty see [9], the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, from Lemma 2.2.4 we directly deduce that $\mathcal{A}$ has non pure imaginary eigenvalues. Next, using Lemmas 2.2.5 and 2.2.7, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. The proof is thus completed.

2.2.3 Non Uniform Stability

The aim of this section is to show that system (2.2.6)-(2.2.10) is not uniformly (*i.e.* exponentially) stable in general since it is already the case for $\Omega = (0, 1)$ as shown below. Our result is the following

Theorem 2.2.8. *Assume that $d = 1$. The semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.*

This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. For this aim, let $\lambda \in \mathbb{C}$ and $U = (u, v, \omega)^\top \in D(\mathcal{A})$ be such that $\mathcal{A}U = \lambda U$. Equivalently we have

$$\begin{cases} v & = & \lambda u, \\ u_{xx} & = & \lambda v, \\ -(|\xi|^2 + \eta)\omega + v(1)\mu(\xi) & = & \lambda\omega. \end{cases}$$

Since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$, for same $\alpha_0 > 0$ large enough. By eliminating v and ω from the above system and using the fact that

$$u_x(1) + \gamma\kappa \int_{\mathbb{R}} \mu(\xi)\omega(\xi)d\xi = 0,$$

we get the following system :

$$\begin{cases} \lambda^2 u - u_{xx} & = & 0, \\ u(0) & = & 0, \\ u_x(1) + \gamma\lambda(\lambda+\eta)^{\alpha-1}u(1) & = & 0. \end{cases} \tag{2.2.42}$$

We have the following asymptotic behavior

Proposition 2.2.9. *There exist $k_0 \in \mathbb{N}^*$ and a sequence $(\lambda_k)_{|k|\geq k_0}$ of simple eigenvalues of $\mathcal{A}$ and satisfying the following asymptotic behavior :*

$$\lambda_k = i(k+\frac{1}{2})\pi + i\frac{\gamma\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} - \frac{\gamma\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}k^{1-\alpha}} + O\left(\frac{1}{k^{2-\alpha}}\right), \tag{2.2.43}$$

for k large enough.

Proof:

The general solution of (2.2.42) is given by

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}. \tag{2.2.44}$$

Thus the boundary conditions may be written as the following system

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h_1(\lambda)e^{\lambda} & h_2(\lambda)e^{-\lambda} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.2.45}$$

where

$$h_1(\lambda) = \lambda + \gamma\lambda(\lambda+\eta)^{\alpha-1} \quad \text{and} \quad h_2(\lambda) = -\lambda + \gamma\lambda(\lambda+\eta)^{\alpha-1}.$$

Hence a non-trivial solution u of system (2.2.42) exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = \det M(\lambda)$, then we have

$$f(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{(\lambda+\eta)^{1-\alpha}}, \tag{2.2.46}$$

where

$$f_0(\lambda) = e^{\lambda} + e^{-\lambda} \quad \text{and} \quad f_1(\lambda) = \gamma\left(e^{\lambda} - e^{-\lambda}\right). \tag{2.2.47}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$. It easy to check that the roots of f_0 are given by

$$\lambda_k^0 = i\mu_k, \quad k \in \mathbb{Z}, \tag{2.2.48}$$

where $\mu - K = i\left(k+\frac{1}{2}\right)\pi$. Using Rouché's theorem, we deduce that $f(\lambda)$ admits an infinity of simple roots in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$ denoted by λ_k, with $|k| \geq k_0$, for k_0 large enough, such that

$$\lambda_k = i\mu_k + o(1) \quad \text{as} \quad k \to +\infty. \tag{2.2.49}$$

Equivalently we have

$$\lambda_k = i\mu_k + \varepsilon_k \quad \text{where} \quad \lim_{|k|\to+\infty} \varepsilon_k = 0. \tag{2.2.50}$$

Using (2.2.47), we get

$$f_0(\lambda_k) = 2i(-1)^k\varepsilon_k + O(\varepsilon_k^2), \tag{2.2.51}$$

$$f_1(\lambda_k) = 2i\gamma(-1)^k + O(\varepsilon_k^2), \tag{2.2.52}$$

$$\frac{1}{(\lambda_k+\eta)^{1-\alpha}} = \frac{\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{k^{1-\alpha}\pi^{1-\alpha}} - \frac{i\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{k^{1-\alpha}\pi^{1-\alpha}} + O\left(\frac{1}{k^{2-\alpha}}\right). \tag{2.2.53}$$

Next, by inserting (2.2.51)-(2.2.53) in the identity $f(\lambda) = 0$ and keeping only the terms of order $\frac{1}{k^{1-\alpha}}$, we find after a simplification

$$\varepsilon_k = -\frac{\gamma\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{k^{1-\alpha}\pi^{1-\alpha}} + i\frac{\gamma\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{k^{1-\alpha}\pi^{1-\alpha}} + O\left(\frac{1}{k^{2-\alpha}}\right). \tag{2.2.54}$$

From equation (2.2.54), we have

$$|k|^{1-\alpha}\Re(\lambda_k) \approx -\frac{\gamma\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{\pi^{1-\alpha}}.$$

Inserting equation (2.2.54) in (2.2.50), we get the desired equation (2.2.43). This implies that the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.

<u>**Numerical Validation.**</u> The asymptotic behavior λ_k in equation (2.2.43) can be numerically validated. For instance, with $\alpha = 0.5, \eta = 1$ and $\gamma = 1$ then from equation (2.2.43) we have

$$\lim_{k\to+\infty} \sqrt{k}\Re(\lambda_k) = -\frac{\sqrt{2}}{2\sqrt{\pi}} \approx -0.398942.$$

The table below confirms this behavior.

k	400	500	600	700	800	900	1000
$\sqrt{k}\Re(\lambda_k)$	-0.39874	-0.398781	-0.398808	-0.398827	-0.398842	-0.398853	-0.398862

2.3 Polynomial stability under Geometric Control Condition

This section is devoted to the study of the polynomial stability of system (2.2.6)-(2.2.10) in the case $\eta > 0$ and under appropriated geometric conditions. For that purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [18] (see also [14, 15, 40]) that we partially recall.

Theorem 2.3.1. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\ell > 0$ the following conditions are equivalent*

$$\|(is - A)^{-1}\| = O(|s|^{\ell}),\ s \to \infty, \tag{2.3.1}$$

$$\|T(t)A^{-1}\| = O(t^{-1/\ell}),\ t \to \infty. \tag{2.3.2}$$

As the condition $i\mathbb{R} \subset \rho(A)$ was already checked in Theorem 2.2.3, it remains to prove that condition (2.3.1) holds. This is made with the help of a multiplier method under some geometric conditions on the boundary of the domain and by using the exponential decay of an auxiliary problem. Firstly, like as [1, 49], we consider the following auxiliary problem, namely the wave equation with standard boundary damping on Γ_1 :

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi(x,t) = 0, & x \in \Omega, \quad t > 0, \\ \varphi(x,t) = 0, & x \in \Gamma_0, \quad t > 0, \\ \partial_\nu \varphi(x,t) = -\varphi_t(x,t), & x \in \Gamma_1, \quad t > 0. \end{cases} \tag{2.3.3}$$

Define the auxiliary space $\mathcal{H}_a = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and the auxiliary unbounded linear operator $\mathcal{A}_a$ by

$$D(\mathcal{A}_a) = \left\{ \Phi = (\varphi, \psi) \in \mathcal{H}_a : \quad \Delta\varphi \in L^2(\Omega); \quad \psi \in H^1_{\Gamma_0}(\Omega); \quad \frac{\partial\varphi}{\partial\nu} = -\psi \quad \text{on} \quad \Gamma_1 \right\}$$

$$\mathcal{A}_a(\varphi, \psi) = (\psi, \Delta\varphi)\,.$$

We then introduce the following condition :

(H) : the problem (2.3.3) is uniformly stable in the energy space $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$.

Secondly, we recall the Geometric Control condition **GCC** introduced by Bardos, Lebeau and Rauch [13] :

Definition 2.3.2. *We say that Γ satisfies the geometric condition named **GCC**, if every ray of geometrical optics, starting at any point $x \in \Omega$ at time $t = 0$, hits Γ_1 in finite time T.*

We also recall the multiplier control condition **MCC** in the following definition :

Definition 2.3.3. *We say that the multiplier control condition* **MCC** *holds if there exist* $x_0 \in \mathbb{R}^d$ *and a positive constant* $m_0 > 0$ *such that*

$$m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad m \cdot \nu \geq m_0 \quad \text{on} \quad \Gamma_1,$$

with $m(x) = x - x_0$, *for all* $x \in \mathbb{R}^d$. □

Next, we present the main result of this section.

Theorem 2.3.4. *Assume that* $\eta > 0$ *and that the condition (H) holds. Then, for all initial data* $U_0 \in D(\mathcal{A})$, *there exists a constant* $C > 0$ *independent of* U_0, *such that the energy of the strong solution* U *of* (4.2.14) *satisfies the following estimation*

$$E(t,U) \leq C \frac{1}{t^{\frac{1}{1-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}. \tag{2.3.4}$$

In particular, for $U_0 \in \mathcal{H}$, *the energy* $E(t,U)$ *converges to zero as* t *goes to infinity.*

As announced in Theorem 2.3.1, by tacking $\ell = 2 - 2\alpha$, the polynomial energy decay (2.3.4) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{$H1$}$$

and

$$\sup_{|\lambda| \in \mathbb{R}} \frac{1}{|\lambda|^\ell} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty \tag{$H2$}$$

are satisfied. Condition $(H1)$ is already proved in Theorem 2.2.3. We will prove condition $(H2)$ using an argument of contradiction. For this purpose, suppose that $(H2)$ is false, then there exist a real sequence (λ_n), with $|\lambda_n| \to +\infty$ and a sequence $(U^n) \subset D(\mathcal{A})$, verifying the following conditions

$$\|U^n\|_{\mathcal{H}} = \|(u^n, v^n, \omega^n)\|_{\mathcal{H}} = 1 \tag{2.3.5}$$

and

$$\lambda_n^\ell (i\lambda_n - \mathcal{A})U^n = (f_1^n, f_2^n, f_3^n) \to 0 \quad \text{in} \quad \mathcal{H}. \tag{2.3.6}$$

For simplicity, we drop the index n. Detailing equation (2.3.6), we get

$$i\lambda u - v \;=\; \frac{f_1}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad H^1_{\Gamma_0}(\Omega), \tag{2.3.7}$$

$$i\lambda v - \Delta u \;=\; \frac{f_2}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\Omega), \tag{2.3.8}$$

$$(i\lambda + |\xi|^2 + \eta)\omega - v|_{\Gamma_1}\mu(\xi) \;=\; \frac{f_5}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^N). \tag{2.3.9}$$

Note that U is uniformly bounded in $\mathcal{H}$. Then, taking the inner product of (2.3.6) with U in

$\mathcal{H}$, we get

$$-\gamma\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2d\xi=\Re\left((i\lambda I-\mathcal{A})U,U\right)_{\mathcal{H}}=\frac{o(1)}{\lambda^\ell}. \tag{2.3.10}$$

Inserting equation (2.3.7) in (2.3.8), we get

$$\lambda^2u+\Delta u=-\frac{f_2}{\lambda^\ell}-\frac{if_1}{\lambda^{\ell-1}}. \tag{2.3.11}$$

Lemma 2.3.5. *Assume that $\eta>0$. Then the solution $(u,v,w)\in D(\mathcal{A})$ of (2.3.7)-(2.3.9) satisfies the following asymptotic behavior estimation*

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \tag{2.3.12}$$

$$\|\partial_\nu u\|_{L^2(\Gamma_1)} = \frac{o(1)}{\lambda^{1-\alpha}}, \tag{2.3.13}$$

$$\|u\|_{L^2(\Gamma_1)} = \frac{o(1)}{\lambda}. \tag{2.3.14}$$

Proof: Using equations (2.3.5) and (2.3.7), we deduce directly the first estimation (2.3.12). Now, from the boundary condition

$$\partial_\nu u+\gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi)d\xi=0 \quad \text{on} \quad \Gamma_1$$

we get

$$\left|\frac{\partial u}{\partial\nu}\right|\leq\gamma\kappa\left(\int_{\mathbb{R}^d}\frac{\mu^2(\xi)}{|\xi|^2+\eta}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2d\xi\right)^{\frac{1}{2}}. \tag{2.3.15}$$

Then, combining equation (2.3.10) and equation (2.3.15), we obtain the desired estimation (2.3.13). Finally, multiplying equation (2.3.9) by $(i\lambda+|\xi|^2+\eta)^{-1}\mu(\xi)$, integrating over $\mathbb{R}^d$ with respect to the variable ξ and applying Cauchy-Shwartz inequality, we obtain

$$A_1\left|v|_{\Gamma_1}\right|\leq A_2\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2d\xi\right)^{\frac{1}{2}}+\frac{1}{|\lambda|^\ell}A_3\left(\int_{\mathbb{R}^d}|f_3(\xi)|^2d\xi\right)^{\frac{1}{2}}, \tag{2.3.16}$$

where,

$$A_1=\int_{\mathbb{R}^N}\frac{|\mu(\xi)|^2}{(|\lambda|+|\xi|^2+\eta)}d\xi,\quad A_2=\left(\int_{\mathbb{R}^N}\frac{|\mu(\xi)|^2}{|\xi|^2+\eta}d\xi\right)^{\frac{1}{2}} \quad \text{and} \quad A_3=\left(\int_{\mathbb{R}^N}\frac{|\mu(\xi)|^2}{(|\lambda|+|\xi|^2+\eta)^2}d\xi\right)^{\frac{1}{2}}.$$

Using Lemma 1.6.2. we have

$$A_1=c\left(|\lambda|+\eta\right)^{\alpha-1} \quad \text{and} \quad A_3=\tilde{c}\left(|\lambda|+\eta\right)^{\frac{\alpha}{2}-1} \tag{2.3.17}$$

where c and $\tilde{c}$ are two positive constants. Inserting equation (2.3.10) and (2.3.17) in equation

(2.3.16) and using the fact that $\ell = 2-2\alpha$, we get

$$\|v\|_{L^2(\Gamma_1)} = o(1). \tag{2.3.18}$$

It follows, from (2.3.7), that equation (2.3.14) holds. The proof has been completed.

Lemma 2.3.6. *Assume that $\eta > 0$. Then the solution $(u, v, w) \in D(\mathcal{A})$ of (2.3.7)-(2.3.9) satisfies the following asymptotic behavior estimation*

$$\int_\Omega |\lambda u|^2 dx - \int_\Omega |\nabla u|^2 dx = \frac{o(1)}{\lambda^\ell}. \tag{2.3.19}$$

Proof: Multiplying equation (2.3.11) by $\bar{u}$, using Green formula and Lemma 2.3.5 we get equation (2.3.19).

Lemma 2.3.7. *Assume that the condition (H) holds and let u be a solution of problem (2.3.11). Then, for any $\lambda \in \mathbb{R}$, the solution $\varphi_u \in H^1(\Omega)$ of system*

$$\begin{cases} -(\lambda^2+\Delta)\varphi_u = u & in \quad \Omega, \\ \varphi_u = 0 & on \quad \Gamma_0, \\ \dfrac{\partial \varphi_u}{\partial \nu} + i\lambda\varphi_u = 0 & in \quad \Gamma_1. \end{cases} \tag{2.3.20}$$

satisfying the following estimate

$$\|\lambda\varphi_u\|_{L^2(\Gamma_1)} + \|\nabla\varphi_u\|_{L^2(\Omega)} + \|\lambda\varphi_u\|_\Omega \lesssim \|u\|_{L^2(\Omega)}. \tag{2.3.21}$$

Proof: First, by Huang-Pruss Theorem (see [25]-[29]-[52]), the exponential stability of system (2.3.3) implies that the resolvent of the auxiliary operator $\mathcal{A}_a$ is uniformly bounded on the imaginary axis i.e. there exists $M > 0$ such that

$$\|(i\lambda I - \mathcal{A}_a)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq M < +\infty. \tag{2.3.22}$$

for all $\lambda \in \mathbb{R}$. Now, since $u \in L^2(\Omega)$, then the pair $(0, u)$ belong $\mathcal{H}_a$ and from (2.3.22), then there exists a unique solution $(\varphi_u, \psi_u) \in D(\mathcal{A}_a)$ such that $(i\lambda - \mathcal{A}_a)(\varphi_u, \psi_u) = (0, u)^\top$ i.e.

$$i\lambda\psi_u - \Delta\varphi_u = u, \tag{2.3.23}$$
$$i\lambda\varphi_u = \psi_u \tag{2.3.24}$$

and such that

$$\|(\varphi_u, \psi_u)\|_{\mathcal{H}} \leq M\|u\|_{L^2(\Omega)}. \tag{2.3.25}$$

From equations (2.3.23)-(2.3.25), we get

$$\|\nabla\varphi_u\|_{L^2(\Omega)} + \|\lambda\varphi_u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}. \tag{2.3.26}$$

So, multiplying the first equation of (2.3.20) by $\lambda\bar{\varphi}_u$, using Green formula and using the third equation of (2.3.20), we get

$$-\lambda\int_\Omega |\lambda\varphi_u|^2 dx + \lambda\int_\Omega |\nabla\varphi_u|^2 dx + i\int_{\Gamma_1} |\lambda\varphi_u|^2 d\Gamma_1 = \lambda\int_\Omega u\bar{\varphi}_u dx. \tag{2.3.27}$$

By tacking the imaginary part of equation (2.3.27) and using Cauchy-Shwartz inequality, we deduce from (2.3.26), that

$$\|\lambda\varphi_u\|_{L^2(\Gamma_1)} \lesssim \|u\|_{L^2(\Omega)}. \tag{2.3.28}$$

Finally, Combining equations (2.3.26) and (2.3.28) we obtain the desired equation (2.3.21). the proof is thus complete.

Lemma 2.3.8. *Assume that $\eta > 0$ and condition (H) holds. Then the solution $(u, v, \omega) \in D(\mathcal{A})$ of (2.3.7)-(2.3.9) satisfies the following asymptotic behavior estimation*

$$\int_\Omega |\lambda u|^2 dx = o(1). \tag{2.3.29}$$

Proof: First, multiplying equation (2.3.11) by $\bar{\varphi}_u$ and using Green formula, we obtain

$$\int_\Omega u(\lambda^2\bar{\varphi}_u + \Delta\bar{\varphi}_u)dx + \int_{\Gamma_1}\left(\frac{\partial u}{\partial\nu}\bar{\varphi}_u - \frac{\partial\bar{\varphi}_u}{\partial\nu}u\right)d\Gamma = -\int_\Omega\left(\frac{f_2}{\lambda^\ell} + i\frac{f_1}{\lambda^{\ell-1}}\right)\bar{\varphi}_u dx. \tag{2.3.30}$$

It follows from (2.3.20) that

$$\int_\Omega |u|^2 dx = \int_{\Gamma_1}\left((\partial_\nu u)\bar{\varphi}_u + i\lambda\bar{\varphi}_u u\right)d\Gamma + \int_\Omega\left(\frac{f_2}{\lambda^\ell} + i\frac{\lambda f_1}{\lambda^\ell}\right)\bar{\varphi}_u dx. \tag{2.3.31}$$

Firstly, using equation (2.3.21), (2.3.12) and the fact that $\|f_2\|_{L^2(\Omega)} = o(1)$, we get

$$\int_\Omega \frac{f_2}{\lambda^\ell}\bar{\varphi}_u dx = \frac{o(1)}{\lambda^{\ell+2}}. \tag{2.3.32}$$

On the other hand, multiplying the first equation of (2.3.20) by f_1 and integrating, we get

$$\int_\Omega \lambda^2 f_1\bar{\varphi}_u dx = \int_\Omega \nabla f_1\cdot\nabla\bar{\varphi}_u dx + i\int_{\Gamma_1}\lambda f_1\bar{\varphi}_u d\Gamma - \int_\Omega u f_1 dx. \tag{2.3.33}$$

So, using (2.3.13), (2.3.14) and the fact that $\|f_1\|_{H_{\Gamma_0}(\Omega)^1(\Omega)} = o(1)$, we obtain, from (2.3.33) that

$$\int_\Omega \lambda^2 f_1\bar{\varphi}_u dx = \frac{o(1)}{\lambda}. \tag{2.3.34}$$

Secondly, using (2.3.13),(2.3.14) and (2.3.21), we get

$$\int_{\Gamma_1} (\partial_\nu u)\, \bar{\varphi}_u d\Gamma_1 = \frac{o(1)}{\lambda^{3-\alpha}} \quad \text{and} \quad \int_{\Gamma_1} \lambda u \bar{\varphi}_u d\Gamma = \frac{o(1)}{\lambda^2}. \tag{2.3.35}$$

Inserting equations (2.3.32), (2.3.34) and (2.3.35) in (2.3.31) and use the fact that $\ell = 2 - 2\alpha$, we get

$$\int_\Omega |\lambda u|^2 dx = o(1). \tag{2.3.36}$$

The proof is thus complete.

Proof of Theorem 2.3.4. Using (2.3.19) and (2.3.29), we get

$$\int_\Omega |\nabla u|^2 dx = o(1).$$

It follows, from (2.3.10) and (2.3.19), that $\|U\|_{\mathcal{H}} = o(1)$ which is a contradiction with (2.3.5). Consequently condition $(H2)$ holds and the energy of smooth solution of system (2.2.6)-(2.2.10) decays polynomial to zero as t goes to infinity. Finally, using the density of the domain $D(\mathcal{A})$ in $\mathcal{H}$, we can easily prove that the energy of weak solution of system (2.2.6)-(2.2.10) decays to zero as t goes to infinity. The proof has been completed.

Conclusion

We have studied the stabilization of multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. Non uniform stability is proved and a polynomial energy decay rate of type $\frac{1}{t^{\frac{1}{1-\alpha}}}$ is established. In view of the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$ see equation (2.2.43), we deduce that the optimal energy decay rate of type $\frac{1}{t^{\frac{2}{1-\alpha}}}$. This question still be open.

CHAPITRE 3

THE INFLUENCE OF THE COEFFICIENTS OF A SYSTEM OF WAVE EQUATIONS COUPLED BY VELOCITIES WITH ONE FRACTIONAL DAMPING ON ITS INDIRECT BOUNDARY STABILIZATION

3.1 Introduction

In this chapter, we consider the one-dimensional coupled wave equations defined by :

$$\begin{cases} u_{tt} - u_{xx} + by_t = 0 & on \quad]0,1[\times]0,+\infty[, \\ y_{tt} - ay_{xx} - bu_t = 0 & on \quad]0,1[\times]0,+\infty[\end{cases} \tag{3.1.1}$$

where $(x,t) \in]0,1[\times]0,+\infty[$, $a > 0$ and $b \in \mathbb{R}^*$. This system is subject to the boundary conditions

$$\begin{cases} u(0,t) = 0 & \text{in} \quad]0,+\infty[, \\ y(0,t) = y(1,t) = 0 & \text{in} \quad]0,+\infty[, \\ u_x(1,t) = -\gamma \partial_t^{\alpha,\eta} u(1,t) & \text{in} \quad]0,+\infty[\end{cases} \tag{3.1.2}$$

where $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds.$$

The system (3.1.1), (3.1.2) is considered with initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{where} \quad x \in]0,1[, \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{where} \quad x \in]0,1[. \end{cases} \tag{3.1.3}$$

The fractional derivative operator of oder α such that $\alpha \in]0,1[$ is defined by

$$[D^\alpha f](t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{df}{d\tau}(\tau) d\tau. \tag{3.1.4}$$

The fractional differentiation is inverse operation of fractional integration that is defined by

$$[I^\alpha f](t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad 0 < \alpha < 1. \tag{3.1.5}$$

From equations (3.1.4), (3.1.5), clearly

$$[D^\alpha f] = I^{1-\alpha} D f. \tag{3.1.6}$$

Now, we present marginally distinctive forms of (3.1.4) and (3.1.5). These exponentially modified fractional integro-differential operators an will be denoted by us follows

$$[D^{\alpha,\eta} f](t) = \int_0^1 \frac{(t-\tau)^{-\alpha} e^{-\eta(t-\tau)}}{\Gamma(1-\alpha)} \frac{df}{d\tau}(\tau) d\tau \tag{3.1.7}$$

and

$$[I^{\alpha,\eta} f](t) = \int_0^t \frac{(t-\tau)^{\alpha-1} e^{-\eta(t-\tau)}}{\Gamma(\alpha)} f(\tau) d\tau. \tag{3.1.8}$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and non-integrable kernels ($t^{-\alpha}$, $0 < \alpha < 1$). This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

In the last year, fractional differential equations have become popular among scientists in order to model various stable physical phenomena with a slow decay rate, say that are not uniformly stable (i.e. are not of exponential type) .

It has been shown (see [45] and [47]) that, as ∂_t forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations. Boundary dissipations of fractional order or, in general, of convolution type are not only important from the theoretical point of view but also for applications. They naturally arise in physical, chemical, biological, ecological phenomena see for example [50], [57] and references therein. They are used to describe memory and hereditary properties of various materials and processes. For example, in viscoelasticity, due to the nature of the material micro-structure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzman assumption, we end up with a stress-strain relationship defined by a time convolution. Viscoelastic response occurs in a variety of

materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [10]- [11]- [12] and [44]). In our case, the fractional dissipations may come from a viscoelastic surface of the beam or simply describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see [46] and [47]).
In [46], Mbodje investigate the asymptotic behavior of solutions with the system

$$\begin{cases} \partial_t^2 u(x,t) - \partial_x^2 u(x,t) &= 0, \qquad x \in]0,1[, \quad t>0, \\ u(0,t) &= 0. \\ \partial_x u(1,t) &= -\gamma \partial_t^{\alpha,\eta} u(1,t), \quad \alpha \in]0,1[, \quad \eta \geq 0, \\ u(x,0) &= u_0(x), \\ \partial_t u(x,0) &= v_0(x), \end{cases} \tag{3.1.9}$$

He also proved that the associated semigroup is not exponentially stable, but only strongly asymptotically and the solution of this system will decay, as times goes to infinity, as $\frac{1}{t}$.
In this chapter, we obtain the stabilization of the coupled 1-D wave equations by means of a boundary viscoelastic damper, the action of which is to cause a feedback (frictional) force opposite in direction of the fractional derivative of the position of a boundary point. More precisely, we are interested im finding a rate of decay for the energy of such a feedback system.
The plan of this chapter is as follows :
In section 3.2, first we show that the system (3.1.1)-(3.1.3) can be replaced by an augmented model by coupling the wave equation with a suitable diffusion equation for can reformulate into an evolution equation and we deduce the well-posedness property of the problem by the semigroup approach. Second, using a criteria of Arendt-Batty we show that the augmented model is strongly stable in absence of compactness of the resolvent under a condition on b. In section 3.3, we show that the augmented model is non uniformly stable i.e.(non exponential), this result is due to the fact that a subsequence of eigenvalues is due to the imaginary axis. In section 3.4, we show the polynomial energy decay rate of type $\frac{1}{t^{\frac{1}{3-\alpha}}}$ if $a=1$ and $b=k\pi$, $\frac{1}{t^{\frac{1}{1-\alpha}}}$ if $a=1$ and $b \neq k\pi$, $\frac{1}{t^{\frac{1}{3-\alpha}}}$ if $a \neq 1$ and ($a \in \mathbb{Q}$ and b small enough) or $\sqrt{a} \in \mathbb{Q}$.

3.2 Well-Posedness and Strong Stability

3.2.1 Well-Posedness

In this subsection, using a semigroup approach, we establish the well-posedness result for the problem (3.1.1)-(3.1.3). We are in position to reformulate system (3.1.1)-(3.1.3) into the augmented model for can be reformulated into the well-known operator theoretic form : $U_t(t) = \mathcal{A}U(t)$. For this goal (see [46]), let μ be the function defined by $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ where $\xi \in \mathbb{R}$ and

$\alpha \in]0,1[$. Using theorem 0.0.3, the solution of the system

$$\partial_t \omega(\xi,t) + (\xi^2+\eta)\omega(\xi,t) - u_t(1,t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}, \eta \geq 0, t > 0, \tag{3.2.1}$$

$$\omega(\xi,0) = 0, \tag{3.2.2}$$

$$u_x(1) = \frac{\sin(\alpha\pi)}{\pi}\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi,t)d\xi \tag{3.2.3}$$

verifying that

$$u_x(1,t) = I^{1-\alpha,\eta}u_t = D^{\alpha,\eta}u_t. \tag{3.2.4}$$

Then the system (3.1.1)-(3.1.2) is equivalently to the augmented model defined by

$$\begin{aligned}
u_{tt} - u_{xx} + by_t &= 0, \quad \text{on} \quad (0,1)\times\mathbb{R}^+,, & (3.2.5)\\
y_{tt} - ay_{xx} - bu_t &= 0, \quad \text{on} \quad (0,1)\times\mathbb{R}^+, & (3.2.6)\\
\omega_t(\xi,t) + (|\xi|^2+\eta)\omega(\xi,t) - u_t(1,t)\mu(\xi) &= 0, \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^+, & (3.2.7)\\
y(0,t) = y(1,t) = u(0,t) &= 0, & (3.2.8)\\
u_x(1,t) + \gamma\kappa\int_{\mathbb{R}} \mu(\xi)\omega(\xi,t)d\xi &= 0, & (3.2.9)\\
\omega(\xi,0) &= 0. & (3.2.10)
\end{aligned}$$

where γ is a positive constant, $\eta \geq 0$ and $\kappa = \frac{\sin(\alpha\pi)}{\pi}$. Our main interest is the existence, uniqueness and regularity of the solution of this system. First the energy of this system is given by

$$E(t) = \frac{1}{2}\left(\int_0^1 \left(|u_t|^2 + |y_t|^2 + |u_x|^2 + a|y_x|^2\right)dx + \gamma\kappa\int_{-\infty}^{+\infty} |\omega|^2 d\xi\right).$$

Then a straightforward computation gives

$$E'(t) = -\gamma\kappa\int_{-\infty}^{+\infty} (\xi^2+\eta)|\omega|^2 d\xi \leq 0.$$

Thus, the system (3.2.5)-(3.2.10) is dissipative in the sense that its energy is non increasing with respect to the time t. Next, we define the Hilbert space

$$\mathcal{H} = H_L^1(]0,1[) \times L^2(]0,1[) \times H_0^1(]0,1[) \times L^2(]0,1[) \times L^2(\mathbb{R}),$$

endowed with inner product

$$\left\langle U, \widetilde{U}\right\rangle = \int_0^1 \left(u_x\bar{\tilde{u}}_x + v\bar{\tilde{v}} + ay_x\bar{\tilde{y}}_x + z\bar{\tilde{z}}\right)dx + \gamma\kappa\int_{-\infty}^{+\infty} \omega\bar{\tilde{\omega}}d\xi,$$

for all $U = (u, v, y, z, \omega)^\top \in \mathcal{H}$ and $\widetilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\omega}) \in \mathcal{H}$, where $H_L^1(]0,1[)$ is the Sobolev space defined by

$$H_L^1(]0,1[) = \left\{u \in H^1(\Omega), \qquad u(0) = 0\right\}.$$

Finally, we define the unbounded linear operator $\mathcal{A}$ by

$$D(A) = \left\{\begin{array}{l} U = (u, v, y, z, \omega)^\top \in \mathcal{H};\ u \in H^2\left(]0,1[\right) \cap H_L^1(]0,1[), \\ y \in H^2\left(]0,1[\right) \cap H_0^1(]0,1[), v \in H_L^1(]0,1[),\ z \in H_0^1\left(]0,1[\right), \\ -\left(\xi^2 + \eta\right)\omega + v\left(1\right)\mu(\xi) \in L^2\left(\mathbb{R}\right), \\ u_x(1) + \gamma\kappa \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi = 0,\ |\xi|\omega \in L^2\left(\mathbb{R}\right). \end{array}\right\},$$

and

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - bz \\ z \\ ay_{xx} + bv \\ -\left(\xi^2 + \eta\right)\omega + v(1)\mu(\xi) \end{pmatrix}$$

If $U = (u, u_t, y, y_t, \omega)^T$ is a regular solution of system (3.2.5)-(3.2.10), then we rewrite this system as the following evolution equation

$$\left\{\begin{array}{l} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{array}\right. \tag{3.2.11}$$

where $U_0 = (u_0, u_1, y_0, y_1, \omega)^\top$.

Proposition 3.2.1. *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof: For all $U = (u, v, y, z, \omega) \in D\left(\mathcal{A}\right)$, we have

$$\Re\left(\langle \mathcal{A}U, U\rangle\right) = -\gamma\kappa \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\omega|^2 d\xi \leq 0, \tag{3.2.12}$$

which implies that $\mathcal{A}$ is dissipative. Now, let $F = (f_1, f_2, f_3, f_4, f_5)^\top$, we prove the existence of $U = (u, v, y, z, \omega)^\top \in D(\mathcal{A})$, unique solution of the equation

$$(I - \mathcal{A})U = F. \tag{3.2.13}$$

Equivalently, we have the following system

$$u-v = f_1, \tag{3.2.14}$$
$$v-u_{xx}+bz = f_2, \tag{3.2.15}$$
$$y-z = f_3, \tag{3.2.16}$$
$$z-ay_{xx}-bv = f_4, \tag{3.2.17}$$
$$(1+\xi^2+\eta)\omega - v(1)\mu(\xi) = f_5. \tag{3.2.18}$$

From (3.2.18) and (3.2.14), we get

$$\omega = \frac{f_5}{1+\xi^2+\eta}+\frac{u(1)\mu(\xi)}{1+\xi^2+\eta}-\frac{f_1(1)\mu(\xi)}{1+\xi^2+\eta}. \tag{3.2.19}$$

Combining equations (3.2.14) and (3.2.16) in (3.2.15) and (3.2.17), we get

$$u-u_{xx}+by = f_1+f_2+bf_3, \tag{3.2.20}$$
$$y-ay_{xx}-bu = -bf_1+f_3+f_4, \tag{3.2.21}$$

with the boundary conditions

$$u(0)=0, \quad u_x(1) = -\gamma\kappa\int_{-\infty}^{+\infty}\mu(\xi)\omega(\xi)d\xi \quad \text{and} \quad y(0)=y(1)=0. \tag{3.2.22}$$

Let $\phi = (\varphi_1,\varphi_2)\in H_L^1(]0,1[)\times H_0^1(]0,1[)$ be the test function. Multiplying equations (3.2.20) and (3.2.21) respectively by φ_1 and φ_2 respectively, we obtain

$$\int_0^1 u\varphi_1 dx + \int_0^1 u_x\varphi_{1x}dx - [u_x\varphi_1]_0^1 + b\int_0^1 y\varphi_1 dx = \int_0^1 F_1\varphi_1 dx, \tag{3.2.23}$$
$$\int_0^1 y\varphi_2 dx + a\int_0^1 y_x\varphi_{2x}dx - b\int_0^1 u\varphi_2 dx = \int_0^1 F_2\varphi_2 dx \tag{3.2.24}$$

where $F_1 = f_1+f_2+bf_3$ and $F_2=-bf_1+f_3+f_4$. Using (3.2.9) and (3.2.22), we get

$$-[u_x\varphi_1]_0^1 = \varphi_1(1)M_1 + u(1)\varphi(1)M_2 + f_1(1)\varphi(1)M_2 \tag{3.2.25}$$

where

$$M_1 = \gamma\kappa\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_5}{1+\eta+\xi^2}d\xi \quad \text{and} \quad M_2 = \gamma\kappa\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{1+\eta+\xi^2}d\xi. \tag{3.2.26}$$

Using the fact $f_5\in L^2(]-\infty,+\infty[)$ and $\alpha\in]0,1[$, it easy to check that $M,N<+\infty$. Adding equations (3.2.23) and (3.2.24), we obtain

$$a\left((u,y),(\varphi_1,\varphi_2)\right) = L\left(\varphi_1,\varphi_2\right), \quad \forall(\varphi_1,\varphi_2)\in H_l^1(]0,1[)\times H_0^1(]0,1[), \tag{3.2.27}$$

where

$$\begin{aligned} a\left((u,y),(\varphi_1,\varphi_2)\right) &= \int_0^1 u\varphi_1 dx + \int_0^1 u_x\varphi_{1x}dx + \int_0^1 y\varphi_2 dx + a\int_0^1 y_x\varphi_{2x}dx \\ & u(1)\varphi_1(1)M_2 + b\int_0^1 y\varphi_1 dx - b\int_0^1 u\varphi_2 dx \end{aligned} \tag{3.2.28}$$

and

$$L(\varphi_1,\varphi_2) = \int_0^1 F_1\varphi_1 dx + \int_0^1 F_2\varphi_2 dx - M_1\varphi_1(1) + \varphi_1(1)f_1(1)M_2. \tag{3.2.29}$$

Using Lax-Milligram, we deduce that there exists $(u,y)\in H_L^1(]0,1[)\times H_0^1(]0,1[)$ unique solution of the variational problem (3.2.27). Applying the classical elliptic regularity we deduce that $U=(u,v,y,z,\omega)\in D(A)$.

From proposition 3.2.1, we have have the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lummer-Phillips theorem (see [42] and [51]). Then the solution to the evolution equation (3.2.11) admits the following representation :

$$U(t) = e^{t\mathcal{A}}U_0, \quad t\geq 0,$$

which leads to the well-posedness of (3.2.11). Hence, semi-group theory allows to show the next existence and uniqueness results :

Theorem 3.2.2. *For any $U_0\in\mathcal{H}$, problem (3.2.11) admits a unique weak solution*

$$U(t)\in C^0(\mathbb{R}^+,\mathcal{H}).$$

Moreover, if $U_0\in D(\mathcal{A})$, then

$$U(t)\in C^1(\mathbb{R}^+,\mathcal{H})\cap C^0(\mathbb{R}^+,D(\mathcal{A})).$$

3.2.2 Strong Stability

In this subsection, we study the strong stability of system (3.2.5)-(3.2.10) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$. It is easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle or the spectrum decomposition theory of Benchimol are not applicable in this case. We use then a general criteria of Arendt-Batty, following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A})\cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result.

Theorem 3.2.3. *The semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable on the energy space $\mathcal{H}$*

i.e. $\lim_{t\to+\infty}\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}=0$ *for all* $U_0\in\mathcal{H}$, *if and only if*

$$b^2\neq\frac{(k_1^2-ak_2^2)(ak_1^2-k_2^2)\pi^2}{(a+1)(k_1^2+k_2^2)},\quad\forall k_1,k_2\in\mathbb{N}^*,k_2<k_1. \tag{C}$$

For the proof of Theorem 3.2.3, we need the following two lemmas.

Lemma 3.2.4. *Suppose that* $\eta\geq 0$ *and* b *satisfying condition* (C)*. Then, for all* $\lambda\in\mathbb{R}$*, we have*

$$\ker(i\lambda I-\mathcal{A})=\{0\}.$$

Proof: Let $\lambda\in\mathbb{R}$ and $U=(u,v,y,z,\omega)\in D(\mathcal{A})$ such that

$$\mathcal{A}U=i\lambda U. \tag{3.2.30}$$

Using equation (3.2.12), we get

$$-\gamma\kappa\int_{-\infty}^{+\infty}(\xi^2+\eta)|\omega|^2d\xi=\Re\left(\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\right)=0,$$

then

$$\omega=0. \tag{3.2.31}$$

Now, detailing (3.2.30), we get $v=i\lambda u$, $z=i\lambda y$, $v(1)=0$ and using equations (3.2.9), we obtain the following system

$$\begin{aligned}\lambda^2u+u_{xx}-i\lambda by&=0, &(3.2.32)\\ \lambda^2y+ay_{xx}+i\lambda bu&=0, &(3.2.33)\\ u(0)=u(1)=u_x(1)=y(0)=y(1)&=0. &(3.2.34)\end{aligned}$$

Remark that $\lambda\neq 0$, indeed if $\lambda=0$ it easy to check that $U=0$. Combining equations (3.2.32) and (3.2.33), we get

$$\begin{aligned}au_{xxxx}+Bu_{xx}+Cu&=0, &(3.2.35)\\ u(0)&=0, &(3.2.36)\\ u(1)&=0, &(3.2.37)\\ u_x(1)&=0, &(3.2.38)\\ u_{xx}(0)&=0, &(3.2.39)\\ u_{xx}(1)&=0 &(3.2.40)\end{aligned}$$

where $B = \lambda^2(a+1)$ and $C = \lambda^2(\lambda^2 - b^2)$. The characteristic equation is given by

$$P(t) = at^4 + Bt^2 + C. \tag{3.2.41}$$

Setting $P_1(m) = am^2 + Bm + C$, the discriminant of P_1 is given by $\Delta = \lambda^4(a-1)^2 + 4ab^2\lambda^2 > 0$, then P_1 has two distinct reel roots m_1 and m_2 given by

$$m_1 = \frac{-B - \sqrt{\Delta}}{2a} \quad \text{and} \quad m_2 = \frac{-B + \sqrt{\Delta}}{2a}.$$

It is clear that $m_1 < 0$. As $B^2 - \Delta = 4a\lambda^2(\lambda^2 - b^2)$, the sign of m_2 depends following to the value of $\lambda^2 - b^2$. Therefore, we distinguish the three case.
Case 1 : $(\lambda^2 < b^2)$. For this case, clearly that $m_2 > 0$. Setting $t_1 = \sqrt{-m_1}$ and $t_2 = \sqrt{m_1 2}$, then P has 4 roots $it_1, -it_1, t_2$ and $-t_2$. Then, we find the general solution of (3.2.35) given by

$$u(x) = c_1 e^{it_1 x} + c_2 e^{-itx} + c_3 e^{t_2 x} + c_4 e^{-t_2 x}. \tag{3.2.42}$$

Using equations (3.2.36) and (3.2.39) in (3.2.42), we get

$$(t_1^2 + t_2^2)(c_3 + c_4) = 0. \tag{3.2.43}$$

In the other hand, $t_1^2 + t_2^2 = \frac{-\sqrt{\Delta}}{a} \neq 0$, then from (3.2.36) and (3.2.43) we get

$$u(x) = 2ic_1 \sin(t_1 x) + 2c_3 \sinh(t_2 x). \tag{3.2.44}$$

Assume that $u \neq 0$, using (3.2.37),(3.2.38) and (3.2.40), we get

$$\begin{pmatrix} \sin(t_1) & \sinh(t_2) \\ t_1 \cos(t_1) & t_2 \cosh(t_2) \\ -t_1^2 \sin(t_1) & t_2^2 \sinh(t_2) \end{pmatrix} \begin{pmatrix} c \\ c' \end{pmatrix} = 0. \tag{3.2.45}$$

Therefore the rank of the previous matrix is one, that means that we have

$$t_1 t_2 \left(t_2 \cos(t_1) \sinh(t_2) + t_1 \sin(t_1) \cosh(t_2)\right) = 0, \tag{3.2.46}$$
$$\left(t_1^2 + t_2^2\right) \sin(t_1) \sinh(t_2) = 0, \tag{3.2.47}$$
$$t_2 \sin(t_1) \cosh(t_2) - t_1 \cos(t_1) \sinh(t_2) = 0. \tag{3.2.48}$$

Using the fact $t_1^2 + t_2^2 \neq 0$, then from (3.2.47) we have

$$\sin(t_1) \sinh(t_2) = 0. \tag{3.2.49}$$

Suppose that $\sinh(t_2) = 0$ then $t_2 = ik\pi$ for all $k \in \mathbb{Z}$, then $m_2 = -(k\pi)^2 < 0$ its not possible.

Then from (3.2.49), $\sin(t_1) = 0$ then $\cos(t_1) = \pm 1$, then from (3.2.46) and (3.2.48), we get $t_2 \sinh(t_2) = 0$, its not possible. Therefore, for this case (3.2.42) admits only trivial solution, then $U = 0$.

Case 2 $(\lambda^2 = b^2)$: For this case, it easy to check that $m_2 = 0$ and $t_1 = \sqrt{-m_1} = |b|\sqrt{\frac{a+1}{a}}$. Then P has 2 simple roots $it_1, -it_1$ and 0 as a double root. Then we find the general solution of (3.2.35) by

$$u(x) = c_1 e^{it_1 x} + c_2 e^{-it_1 x} + c_3 + c_4 x. \tag{3.2.50}$$

Using equations (3.2.36)-(3.2.40) and the fact $t_1 \neq 0$, then the general solution of (3.2.35) is given by

$$u(x) = 2ic_1 \sin(t_1 x). \tag{3.2.51}$$

Suppose that $c_1 \neq 0$, Using (3.2.37) and (3.2.38) in (3.2.50) we get $\cos(t_1) = \sin(t_1) = 0$, That is not possible. Then (3.2.50) admits only trivial solution.

Case 3 $(\lambda^2 > b^2)$: For this case clearly that $m_2 < 0$. Setting $t_1 = \sqrt{-m_1}$ and $t_2 = \sqrt{-m_2}$, then P has four roots $it_1, -it_1, it_2, -it_2$ and the general solution of (3.2.35) is given by

$$u(x) = c_1 e^{it_1 x} + c_2 e^{-it_1 x} + c_3 e^{it_2 x} + c_4 e^{-it_2 x}. \tag{3.2.52}$$

Using equations (3.2.36) and (3.2.39) and the fact that $t_1^2 - t_2^2 = m_2 - m_1 \neq 0$, then the general solution in equation (3.2.52) is given by

$$u(x) = 2i \sin(t_1 x) + 2ic_3 \sin(t_2 x). \tag{3.2.53}$$

Using equations (3.2.37), (3.2.38) and (3.2.40), we get the following system

$$\begin{pmatrix} \sin(t_1) & \sin(t_2) \\ t_1 \cos(t_1) & t_2 \cos(t_2) \\ -t_1^2 \sin(t_1) & -t_2^2 \sin(t_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = 0. \tag{3.2.54}$$

Assume that $u \neq 0$, therefore the rank of system (3.2.54) is one. That means that we have

$$t_2 \sin(t_1)\cos(t_2) - t_1 \cos(t_1)\sin(t_2) = 0, \tag{3.2.55}$$

$$(t_2^2 - t_1^2)\sin(t_1)\sin(t_2) = 0, \tag{3.2.56}$$

$$t_1 t_2 (t_1 \cos(t_2)\sin(t_1) - t_2 \cos(t_1)\sin(t_2)) = 0,. \tag{3.2.57}$$

Using the fact $t_2^2 - t_1^2 \neq 0$, then from equation (3.2.56) if $\sin(t_1) = 0$ then $\cos(t_1) = \pm 1$. Using (3.2.55) and the fact that $(t_1, t_2) \neq (0, 0)$, we get $\sin(t_2) = 0$. Then there exists $k_1, k_2 \in \mathbb{N}^*$,

$k_1 < k_2$ (remember that $t_1 > t_2 > 0$) such that $t_1 = k_1\pi$ and $t_2 = k_2\pi$. Hence,

$$(k_1^2 + k_2^2)\pi^2 = t_1^2 + t_2^2 = -m_1 - m_2 = \frac{\lambda^2(a+1)}{a}. \tag{3.2.58}$$

and

$$k_1^2 k_2^2 \pi^4 = t_1 t_2^2 = m_1 m_2 = \frac{\lambda^2(\lambda^2 - b^2)}{a} \tag{3.2.59}$$

Eliminating λ^2 in (3.2.58) and (3.2.59), we get

$$b^2 = \frac{(ak_1^2 - k_2^2)(k_1^2 - ak_2^2)}{(a+1)(k_1^2 + k_2^2)}\pi^2, \quad \forall k_1, k_2 \in \mathbb{N}^*, k_2 < k_1.$$

Under this condition (3.2.35) admits a non trivial solution. Conversely if (C) holds, λ is not an eigenvalue of $\mathcal{A}$.

Lemma 3.2.5. *Assume that $\eta = 0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0 \in \sigma(\mathcal{A})$.*

Proof: Define the vector $F = (\sin(x), 0, 0, 0, 0) \in \mathcal{H}$. Assume that there exists $U = (u, v, y, z, \omega) \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F,$$

it follows that

$$v = -\sin x \quad \text{in} \quad]0,1[, \quad |\xi|^2\omega + \sin(1)\mu(\xi) = 0. \tag{3.2.60}$$

From (3.2.60), we deduce that $\omega(\xi) = |\xi|^{\frac{2\alpha-5}{2}}\sin(1) \notin L^2(\mathbb{R})$. So, the assumption of the existence of U is false and consequently, the operator $-\mathcal{A}$ is not invertible.

Lemma 3.2.6. *Assume that $(\eta > 0,\ \lambda \in \mathbb{R})$ or $(\eta = 0,\ \lambda \in \mathbb{R}^*)$. Then, for any $h, g \in L^2((0,1))$, the following problem*

$$\begin{cases} \lambda^2 u + u_{xx} - i\lambda b y & = & h, \quad in \quad (0,1), \\ \lambda^2 y + a y_{xx} + i\lambda b u & = & g, \quad in \quad (0,1), \\ u(0) & = & 0, \\ y(0) = y(1) & = & 0, \\ u_x(1) + \left(\lambda^2 c_1 + i\lambda c_2\right) u(1) & = & 0 \end{cases} \tag{3.2.61}$$

where

$$c_1(\lambda,\eta) = \gamma\kappa \int_{\mathbb{R}} \frac{\mu^2(\xi)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi \quad and \quad c_2(\lambda,\eta) = \gamma\kappa \int_{\mathbb{R}} \frac{\mu^2(\xi)(|\xi|^2+\eta)}{|\lambda|^2 + (|\xi|^2+\eta)^2} d\xi \tag{3.2.62}$$

admits a unique solution $(u, y) \in H_L^1((0,1)) \times H_0^1((0,1))$.

Proof: First, it easy to check that, if ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), then, for $\alpha \in]0,1[$, the coefficients $c_1(\lambda,\eta)$ and $c_2(\lambda,\eta)$ are well defined. Moreover, if $\eta > 0$ and $\lambda = 0$ then, using Lax-Milligram's theorem we deduce that system (3.2.61) admits a unique solution $(u,y) \in {H_L}^1((0,1)) \times H_0^1((0,1))$.
Next, let us consider the following problem

$$\begin{cases} -u_{xx} + i\lambda b y & = h, \text{in} \quad (0,1), \\ -y_{xx} - i\lambda b u & = g, \text{in} \quad (0,1), \\ u(0) & = 0, \\ y(0) = y(1) & = 0, \\ u_x(1) + (\lambda^2 c_1 + i\lambda c_2)\, u(1) & = 0. \end{cases} \tag{3.2.63}$$

Next, we give a variational formulation of (3.2.63). For this aim, find $(u_1,u_2) \in H_L((0,1))^1 \times H_0^1(\Omega)$ such that

$$a((u,y),(\varphi,\psi)) = L((\varphi,\psi)) \tag{3.2.64}$$

where

$$a((u,y),(\varphi,\psi)) = \int_0^1 (u_x\varphi_x + y_x\psi_x)dx + (\lambda^2 c_1 + i\lambda c_2)u(1)\varphi(1) \\ + i\lambda b \int_0^1 (y\varphi - u\psi)\, dx$$

and

$$L((\varphi,\psi)) = \int_0^1 (h\varphi + g\psi)\, dx.$$

Define the operator,

$$\mathcal{L}\mathcal{U} = \begin{pmatrix} -u_{xx} + i\lambda b y \\ -a y_{xx} - i\lambda b u \end{pmatrix}, \qquad \forall \mathcal{U} = (u,y)^\top \in H_L^1(0,1) \times H_0^1(0,1)$$

Then $\mathcal{L}$ is an isomorphism from $H_L^1(\Omega) \times H_0^1(\Omega)$ into $H_L^1(\Omega) \times H_0^1(\Omega)$. Using the compactness embedding from $L^2(\Omega) \times L^2(\Omega)$ into $(H_L^1(\Omega))' \times H^{-1}(\Omega)$ and $H_L^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$, we deduce that $\mathcal{L}^{-1}$ is compact from $L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$. Then applying $\mathcal{L}^{-1}$ to (3.2.61), we get

$$\left(\lambda^2 \mathcal{L}^{-1} - I\right) U = \mathcal{L}^{-1} f. \tag{3.2.65}$$

The same computation in Lemma 3.2.4 shows $\ker(\lambda^2\mathcal{L}^{-1} - I) = \{0\}$ for b small enough. Then following Fredholm's alternative, the equation (3.2.61) admits a unique solution.

Lemma 3.2.7. *If $\eta > 0$, for all $\lambda \in \mathbb{R}$, we have*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}$$

while if $\eta = 0$, for all $\lambda \in \mathbb{R}^*$, we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof: We give the proof in the case $\eta > 0$, the proof of the second statement is fully similar. Let $\lambda \in \mathbb{R}$ and $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, then we look for $U = (u, v, y, z, \omega)^\top \in D(\mathcal{A})$ solution of

$$(i\lambda U - \mathcal{A})U = F. \tag{3.2.66}$$

Equivalently, we have

$$\begin{aligned}
i\lambda u - v &= f_1, && (3.2.67)\\
i\lambda v - u_{xx} + bz &= f_2, && (3.2.68)\\
i\lambda y - z &= f_3, && (3.2.69)\\
i\lambda z - a y_{xx} - bv &= f_4, && (3.2.70)\\
i\lambda \omega + (\xi^2 + \eta)\omega - v(1)\mu(\xi) &= f_5. && (3.2.71)
\end{aligned}$$

As before, by eliminating v and ω from the above system and using the fact that

$$u_x(1) + \gamma\kappa \int_{\mathbb{R}} \mu(\xi)\omega(\xi)d\xi = 0,$$

we get the following system

$$\begin{aligned}
\lambda^2 u + u_{xx} - i\lambda b y &= -f_2 - i\lambda f_1 - b f_3, && (3.2.72)\\
\lambda^2 y + a y_{xx} + i\lambda b u &= -f_4 - i\lambda f_3 + b f_1, && (3.2.73)\\
u(0) &= 0, && (3.2.74)\\
y(0) = y(1) &= 0, && (3.2.75)\\
u_x(1) + (\lambda^2 c_1 + i\lambda c_2)u(1) &= -i\lambda c_1 f_1(1) + c_2 f_1(1) + I^1_{f_5} + I^2_{f_5}. && (3.2.76)
\end{aligned}$$

where c_1, c_2 is defined in equation (3.2.62) and $I^1_{f_5}, I^2_{f_5}$ are given by

$$I^1_{f_5}(\lambda, \eta) = i\lambda\gamma\kappa \int_{\mathbb{R}} \frac{f_5(\xi)\mu(\xi)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi \quad \text{and} \quad I^2_{f_5}(\lambda, \eta) = -\gamma\kappa \int_{\mathbb{R}} \frac{f_5(\xi)\mu(\xi)(|\xi|^2 + \eta)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi.$$

It easy to check that, for $f_5 \in L^2(\mathbb{R})$ and $\alpha \in]0, 1[$, the integrals $I^1_{f_5}$ and $I^2_{f_5}$ are will defined.

First, let $(\varphi_{f_5},\psi_{f_5})\in H_L^1(\Omega)\times H_0^1(\Omega)$ are defined by

$$\begin{cases}-(\varphi_5)_{xx}+i\lambda b\psi_5 &= 0, & \text{in} \quad (0,1)\\ -(\psi_5)_{xx}-i\lambda b\varphi_5 &= 0, & \text{in} \quad (0,1)\\ \varphi_5(0) &= 0, & \\ \psi_5(0)=\psi_5(1) &= 0, & \\ (\varphi_5)_x(1) &= I_{f_5}^1+I_{f_5}^2. & \end{cases} \tag{3.2.77}$$

Then setting $\tilde{u}=u+\varphi_{f_5}$ and $\tilde{y}=y+\psi_{f_5}$ in (3.2.72)-(3.2.76), we get

$$\begin{cases}\lambda^2\tilde{u}+\tilde{u}_{xx}-i\lambda b\tilde{y} &= \lambda^2\varphi_5-f_2-i\lambda f_1-bf_3 & \text{in} \quad (0,1),\\ \lambda^2\tilde{y}+a\tilde{y}_{xx}+i\lambda b\tilde{u} &= \lambda^2\psi_5-f_4-i\lambda f_3+bf_1 & \text{in} \quad (0,1),\\ \tilde{u}(0) &= 0, & \\ \tilde{y}(0)=\tilde{y}(1) &= 0, & \\ \tilde{u}_x(1)+(\lambda^2c_1+i\lambda c_2)\tilde{u}(1) &= -i\lambda c_1f_1(1)+c_2f_1(1)+(\lambda^2c_1+i\lambda c_2)\varphi_{f_5}(1). & \end{cases} \tag{3.2.78}$$

Next, let $\theta\in H^2((0,1))\cap H_L^1((0,1))$, such that

$$\theta(1)=0,\quad \theta_x(1)=-i\lambda c_1f_1(1)+c_2f_1(1)+(\lambda^2c_1+i\lambda c_2)\varphi_{f_5}(1).$$

Then setting $\chi=\tilde{u}-\theta$ in (3.2.78), we get the following problem

$$\begin{cases}\lambda^2\chi+\chi_{xx}-i\lambda b\tilde{y} &= \lambda^2\varphi_{f_5}-\lambda^2\theta-\theta_{xx}-f_2-i\lambda f_1-bf_3 & \text{in} \quad (0,1),\\ \lambda^2\tilde{y}+a\tilde{y}_{xx}+i\lambda b\chi &= \lambda^2\psi_{f_5}-i\lambda b\theta-f_4-i\lambda bf_3+bf_1 & \text{in} \quad (0,1),\\ \chi(0) &= 0, & \\ \tilde{y}(0)=\tilde{y}(1) &= 0, & \\ \chi_x(1)+(\lambda^2c_1+i\lambda c_2)\,\chi(1) &= 0. & \end{cases} \tag{3.2.79}$$

Using Lemma 3.2.6, problem (3.2.79) has a unique solution $(\chi,\tilde{y})\in H_L^1((0,1))\times H_0^1(\Omega)$ and therefore problem (3.2.72)-(3.2.76) has a unique solution $(u,y)\in H_l^1((0,1))\times H_0^1(\Omega)$. By defining $v=i\lambda u-f_1$, $z=i\lambda y-f_3$ in $(0,1)$ and

$$\omega=\frac{h(\xi)}{i\lambda+|\xi|^2+\eta}+\frac{i\lambda u_{|\Gamma_1}\mu(\xi)}{i\lambda+|\xi|^2+\eta}-\frac{f_{|\Gamma_1}\mu(\xi)}{i\lambda+|\xi|^2+\eta}.$$

We deduce that $U=(u,v,y,z,\omega)$ belongs to $D(\mathcal{A})$ and is solution of (3.2.66). This completes the proof.

Proof of Theoreme 3.2.3 Following a general criteria of Arendt-Batty see [9], the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma(\mathcal{A})\cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, using Lemma 3.2.4, we directly deduce that $\mathcal{A}$ has non pure imaginary eigenvalues. Next, using Lemmas 3.2.5 and 3.2.7, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. The proof is thus completed.

3.3 Non Uniformly Stable

3.3.1 Non Uniform Stability for $a = 1$

In this section, we assume that $a = 1$, $\eta > 0$ and b verified condition (C) reduced to

$$b \neq \frac{\pi}{\sqrt{2}} \frac{k_1^2 - k_2^2}{\sqrt{k_1^2 + k_2^2}}. \tag{$C1$}$$

Our goal is to show that system (3.2.5)-(3.2.10) is not exponentially stable. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is due to the imaginary axis.

Theorem 3.3.1. *Assume that $a = 1$ and b satisfies condition* (C1). *Then the semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.*

For the proof of Theorem 3.3.1, we aim to show that an infinite of eigenvalues of $\mathcal{A}$ approach the imaginary axis. we can distinguish two cases. We determine the characteristic equation satisfied by the eigenvalues of $\mathcal{A}$. For the aim, let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and let $U = (u, v, y, z, \omega)^\top \in D(\mathcal{A})$ be an associated eigenvector such $\|U\|_{\mathcal{H}} = 1$. Then

$$\begin{aligned}
v &= \lambda u, && (3.3.1)\\
u_{xx} - bz &= \lambda v, && (3.3.2)\\
z &= \lambda y, && (3.3.3)\\
y_{xx} + bv &= \lambda z, && (3.3.4)\\
-(|\xi|^2 + \eta)\omega + v(1)\mu(\xi) &= \lambda\omega. && (3.3.5)
\end{aligned}$$

Inserting equations (3.3.1), (3.3.3) in (3.3.2), (3.3.4) and (3.3.5), we get

$$\begin{aligned}
\lambda^2 u - u_{xx} + b\lambda y &= 0, && (3.3.6)\\
\lambda^2 y - y_{xx} - b\lambda u &= 0, && (3.3.7)\\
(\lambda + |\xi|^2 + \eta)\omega - \lambda u(1)\mu(\xi) &= 0. && (3.3.8)
\end{aligned}$$

Lemma 3.3.2. *Let $\lambda \in \mathbb{C}$ be a eigenvalue of $\mathcal{A}$. Then $\Re(\lambda)$ is bounded.*

Proof: Multiplying equations (3.3.6) and (3.3.7) by $\overline{\lambda u}$ and $\overline{\lambda y}$ respectively, and integrating over $(0,1)$, we get

$$\lambda\left(\|\lambda u\|^2+\|u_x\|^2\right)+b|\lambda|^2\int_0^1 y\bar{u}dx = \bar{\lambda}u_x(1)\bar{u}(1), \tag{3.3.9}$$

$$\lambda\left(\|\lambda y\|^2+\|y_x\|^2\right)-b|\lambda|^2\int_0^1 u\bar{y}dx = 0,. \tag{3.3.10}$$

Adding equations (3.3.9), (3.3.10) , we get

$$\lambda\left(\|\lambda u\|^2+\|\lambda y\|^2+\|u_x\|^2+\|y_x\|^2\right)+2ib|\lambda|^2\Im\left(\int_0^1 y\bar{u}dx\right)=\bar{\lambda}u_x(1)\bar{u}(1). \tag{3.3.11}$$

Multiplying equation (3.3.8) by $\gamma\kappa\bar{\omega}$, integrate over $\mathbb{R}$ and using (3.2.9), we get

$$\lambda\gamma\kappa\|\omega\|^2+\gamma\kappa\int_{-\infty}^{+\infty}(|\xi|^2+\eta)|\omega|^2d\xi+\lambda u(1)\bar{u}_x(1)=0. \tag{3.3.12}$$

Inserting equation (3.3.12) in (3.3.11), and using the fact $\|U\|_{\mathcal{H}}=1$ we get

$$\bar{\lambda}=-\gamma\kappa\int_{-\infty}^{+\infty}(|\xi|^2+\eta)|\omega|^2d\xi+2ib|\lambda|^2\Im\left(\int_0^1 \bar{y}udx\right). \tag{3.3.13}$$

Consequently, we have

$$\Re(\lambda)=-\gamma\kappa\int_{-\infty}^{+\infty}(|\xi|^2+\eta)|\omega|^2d\xi<0.$$

The proof is thus complete.

Proposition 3.3.3. *There exists $k_0\in\mathbb{N}^*$ sufficiently large such that*

$$\sigma(\mathcal{A})\supset\sigma_0\cup\sigma_1,$$

where

$$\sigma_0=\{\lambda_k\}_{k\in J},\quad \sigma_1=\{\lambda_k\}_{|k|\geq k_0},\quad \sigma_0\cap\sigma_1=\emptyset.$$

Moerover, J is a finite set, and

$$\lambda_k=\frac{ik\pi}{2}+\frac{c_1}{k^{1-\alpha}}+\frac{ic_2}{k^{1-\alpha}}+o\left(\frac{1}{k^{1-\alpha}}\right),$$

where, for $b\neq k\pi$

$$c_1=\frac{\gamma(-1)^k(\cos(b)-(-1)^k)\cos(\frac{\pi}{2}(1-\alpha))}{2^\alpha\pi^{1-\alpha}},\quad c_2=\frac{\gamma(-1)^{k+1}(\cos b-(-1)^k)\sin(\frac{\pi}{2}(1-\alpha))}{2^\alpha\pi^{1-\alpha}}.$$

Proof: Using equations (3.3.6)-(3.3.8), we get

$$\begin{aligned} \lambda^2 u - u_{xx} + b\lambda y &= 0, & (3.3.14)\\ \lambda^2 y - y_{xx} - b\lambda u &= 0, & (3.3.15)\\ (\lambda+\xi^2+\eta)\omega - \lambda u(1)\mu(\xi) &= 0,. & (3.3.16)\end{aligned}$$

From equation (3.3.15), we have

$$\begin{aligned} u &= \frac{1}{b\lambda}\left(\lambda^2 y - y_{xx}\right), & (3.3.17)\\ u_{xx} &= \frac{1}{b\lambda}\left(\lambda^2 y_{xx} - y_{xxxx}\right),. & (3.3.18)\end{aligned}$$

Inserting equations (3.3.17) and (3.3.18) in equation (3.3.15), we obtain

$$y_{xxxx} - 2\lambda^2 y_{xx} + \lambda^2(\lambda^2+b^2)y = 0. \tag{3.3.19}$$

Using equation (3.3.16), we easy to check that

$$u_x(1) = -\gamma\lambda(\lambda+\eta)^{\alpha-1}u(1). \tag{3.3.20}$$

Using equations (3.3.17) and (3.3.20), we get

$$\lambda^2 y_x(1) - \gamma\lambda(\lambda+\eta)^{\alpha-1}y_{xx}(1) - y_{xxx}(1) = 0. \tag{3.3.21}$$

Finally, using the fact $y(0)=y(1)=u(0)=0$, and (3.3.21) we get the following system

$$\begin{aligned} y_{xxxx} - 2\lambda^2 y_{xx} + \lambda^2(\lambda^2+b^2)y &= 0, & (3.3.22)\\ y(0) &= 0, & (3.3.23)\\ y(1) &= 0, & (3.3.24)\\ y_{xx}(0) &= 0, & (3.3.25)\\ \lambda^2 y_x(1) - \gamma\lambda(\lambda+\eta)^{\alpha-1}y_{xx}(1) - y_{xxx}(1) &= 0,. & (3.3.26)\end{aligned}$$

We consider the characteristic equation, we get

$$t^4 - 2\lambda^2 t^2 + \lambda^2(\lambda^2+b^2) = 0. \tag{3.3.27}$$

The general solution of equation (3.3.19) is given by

$$y(x) = \sum_{i=1}^{4} c_i e^{t_i x}, \tag{3.3.28}$$

where $t_1(\lambda) = \sqrt{\lambda^2 + ib\lambda}$, $t_2(\lambda) = -t_1(\lambda)$, $t_3(\lambda) = \sqrt{\lambda^2 - ib\lambda}$ and $t_4(\lambda) = -t_3(\lambda)$. Here and below, for simplicity we denote $t_i(\lambda)$ by t_i. Thus the boundary conditions may be written as the following system

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{-t1} & e^{t3} & e^{-t3} \\ t_1^2 & t_1^2 & t_3^2 & t_3^2 \\ g_{1,\lambda}(t_1)e^{t_1} & g_{2,\lambda}(t_1)e^{-t_1} & g_{3,\lambda}(t_3)e^{t_3} & g_{4,\lambda}(t_3)e^{-t_3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0, \tag{3.3.29}$$

where

$$\begin{cases} g_{1,\lambda}(t_1) = \lambda^2 t_1 - \gamma\lambda(\lambda+\eta)^{\alpha-1} t_1^2 - t_1^3, \\ g_{2,\lambda}(t_1) = -\lambda^2 t_1 - \gamma\lambda(\lambda+\eta)^{\alpha-1} t_1^2 + t_1^3, \\ g_{3,\lambda}(t_3) = \lambda^2 t_3 - \gamma\lambda(\lambda+\eta)^{\alpha-1} t_3^2 - t_3^3, \\ g_{4,\lambda}(t_3) = -\lambda^2 t_3 - \gamma\lambda(\lambda+\eta)^{\alpha-1} t_3^2 + t_3^3 \end{cases}$$

Step 1. We start by the expansion of t_1 and t_3 :

$$t_1 = \lambda + \frac{ib}{2} + \frac{b^2}{8\lambda} - \frac{i|b|^3}{16\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \tag{3.3.30}$$

$$t_3 = \lambda - \frac{ib}{2} + \frac{b^2}{8\lambda} + \frac{i|b|^3}{16\lambda^2} + O\left(\frac{1}{\lambda^3}\right). \tag{3.3.31}$$

The determinant of $M(\lambda)$ is given by

$$f(\lambda) = (t_1^2 - t_3^2)\left((g_2 - g_4)e^{-(t_1+t_3)} + (g_1 - g_3)e^{t_1+t_3} + (g_3 - g_2)e^{t_3 - t_1} + (g_4 - g_1)e^{t_1 - t_3}\right), \tag{3.3.32}$$

where

$$\begin{cases} g_2 - g_4 = \lambda^2(t_3 - t_1) + \gamma\lambda(\lambda+\eta)^{\alpha-1}(t_3^2 - t_1^2) + t_1^3 - t_3^3, \\ g_1 - g_3 = \lambda^2(t_1 - t_3) + \gamma\lambda(\lambda+\eta)^{\alpha-1}(t_3^2 - t_1^2) + t_3^3 - t_1^3, \\ g_3 - g_2 = \lambda^2(t_1 + t_3) + \gamma\lambda(\lambda+\eta)^{\alpha-1}(t_1^2 - t_3^2) - (t_1^3 + t_3^3), \\ g_4 - g_1 = -\lambda^2(t_1 + t_3) + \gamma\lambda(\lambda+\eta)^{\alpha-1}(t_1^2 - t_3^2) + t_1^3 + t_3^3. \end{cases} \tag{3.3.33}$$

Using (3.3.30) and (3.3.31), we get

$$\begin{cases} e^{-(t_3+t_1)} = e^{-2\lambda}\left(1 - \dfrac{b^2}{4\lambda} + \dfrac{b^4}{32\lambda^2} + O\left(\dfrac{1}{\lambda^3}\right)\right), \\ e^{t_1+t_3} = e^{2\lambda}\left(1 + \dfrac{b^2}{4\lambda} + \dfrac{b^4}{32\lambda^2} + O\left(\dfrac{1}{\lambda^3}\right)\right), \\ e^{t_3-t_1} = e^{-ib}\left(1 + \dfrac{ib^3}{8\lambda^2} + O\left(\dfrac{1}{\lambda^3}\right)\right), \\ e^{t_1-t_3} = e^{ib}\left(1 - \dfrac{ib^3}{8\lambda^2} + O\left(\dfrac{1}{\lambda^3}\right)\right). \end{cases} \tag{3.3.34}$$

Similarly, using (3.3.25) and (3.3.26) in (3.3.33), we get

$$\begin{cases} g_2 - g_4 &= 2ib\lambda^2 - 2ib\gamma\lambda^{1+\alpha} - 2ib\gamma\eta(\alpha-1)\lambda^\alpha + \dfrac{ib^3}{4} + O\left(\dfrac{1}{\lambda^{1-\alpha}}\right), \\ g_1 - g_3 &= -2ib\lambda^2 - 2ib\gamma\lambda^{1+\alpha} - 2ib\gamma\eta(\alpha-1)\lambda^\alpha - \dfrac{ib^3}{4} + O\left(\dfrac{1}{\lambda^{1-\alpha}}\right), \\ g_3 - g_2 &= 2ib\gamma\lambda^{1+\alpha} + b^2\lambda + 2ib\gamma\eta(\alpha-1)\lambda^\alpha + O\left(\dfrac{1}{\lambda^{1-\alpha}}\right), \\ g_4 - g_1 &= 2ib\gamma\lambda^{1+\alpha} - b^2\lambda + 2ib\gamma\eta(\alpha-1)\lambda^\alpha + O\left(\dfrac{1}{\lambda^{1-\alpha}}\right). \end{cases} \tag{3.3.35}$$

Using (3.3.34) and (3.3.35) in (3.3.32), we get

$$f(\lambda) = A + B + C + D, \tag{3.3.36}$$

where

$$\begin{cases} A &= -4b^2\lambda^3 e^{-2\lambda}\left(1 - \dfrac{\gamma}{\lambda^{1-\alpha}} + O\left(\dfrac{1}{\lambda}\right)\right), \\ B &= 4b^2\lambda^3 e^{-2\lambda}\left(1 + \dfrac{\gamma}{\lambda^{1-\alpha}} + O\left(\dfrac{1}{\lambda}\right)\right), \\ C &= -4\gamma b^2\lambda^{2+\alpha} e^{-ib}\left(1 + O\left(\dfrac{1}{\lambda^\alpha}\right)\right), \\ D &= -4\gamma b^2\lambda^{2+\alpha} e^{ib}\left(1 + O\left(\dfrac{1}{\lambda^\alpha}\right)\right). \end{cases} \tag{3.3.37}$$

Using (3.3.37), we find the following asymptotic expansion

$$S(\lambda) = \frac{f(\lambda)}{4b^2\lambda^3} = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\alpha}} + \frac{O(1)}{\lambda}, \tag{3.3.38}$$

where

$$f_0(\lambda) = e^{2\lambda} - e^{-2\lambda} \quad \text{and} \quad f_1(\lambda) = \gamma\left(e^{2\lambda} + e^{-2\lambda} - (e^{ib} + e^{-ib})\right). \tag{3.3.39}$$

Step 2. We look at the roots of $S(\lambda)$. It easy to check that the root of f_0 is given by

$$\lambda_k^0 = i\mu_k, \quad k \in \mathbb{Z}, \tag{3.3.40}$$

where $\mu_k = \frac{1}{2}k\pi$. Since the real part of λ is bounded (see Proposition 3.3.2), then with help of Rouché's Theorem, and λ large enough, we show that the roots of S are close of those of f_0. In the other words, there exists a sequence λ_k of roots of f such that

$$\lambda_k = i\mu_k + o(1) \quad \text{as} \quad k \to +\infty. \tag{3.3.41}$$

This implies that the C_0-semigroup of contraction $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$. On the other hand, we will find the real part of the eigenvalues λ_k for $b \neq k\pi$.

Step 3. From step 2, we can write

$$\lambda_k = i\mu_k + \varepsilon_k \quad \text{where} \quad \varepsilon_k = o(1). \tag{3.3.42}$$

Consequently, it follows from (3.3.39) and (3.3.42) that

$$f_0(\lambda_k) = 4(-1)^k\varepsilon_k + O(\varepsilon_k^2), \tag{3.3.43}$$

$$f_1(\lambda_k) = \frac{2\gamma\left((-1)^k - \cos(b) + O\left(\varepsilon_k^2\right)\right)}{i^{1-\alpha}k^{1-\alpha}\pi^{1-\alpha}}, \tag{3.3.44}$$

and

$$\frac{1}{i^{1-\alpha}} = \cos\left(\frac{\pi}{2}(1-\alpha)\right) - i\sin\left(\frac{\pi}{2}(1-\alpha)\right). \tag{3.3.45}$$

It follows that , from (3.3.43)-(3.3.45) where $b \neq k\pi$,

$$\varepsilon_k = \frac{(-1)^k\gamma\left(\cos(b) - (-1)^k\right)\cos\left(\frac{\pi}{2}(1-\alpha)\right)}{2^\alpha k^{1-\alpha}\pi^{1-\alpha}} - i\frac{(-1)^k\gamma\left(\cos(b) - (-1)^k\right)\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{2^\alpha k^{1-\alpha}\pi^{1-\alpha}} + o\left(\frac{1}{k^{2-\alpha}}\right). \tag{3.3.46}$$

3.3.2 Non Uniform Stability for $a \neq 1$

In this subsection, we assume that $a \neq 1$, $\eta > 0$ and b verified condition (C). Our goal is to show that (3.2.5)-(3.2.10) is not exponentially stable. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is due to the imaginary axis.

Theorem 3.3.4. *Assume that $a \neq 1$ and b satisfies the condition* (C). *Then, the semigroup of contractions $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.*

For the proof of Theorem 3.3.4, we aim to show that an infinite of eigenvalues of $\mathcal{A}$ approach the imaginary axis. We determine the characteristic equation satisfied by the eigenvalues of $\mathcal{A}$. For the aim, let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and let $U = (u, v, y, z, \omega)^\top \in D(\mathcal{A})$ be an associated eigenvector such that $\|U\|_{\mathcal{H}} = 1$. Then

$$v = \lambda u, \tag{3.3.47}$$

$$u_{xx} - bz = \lambda v, \tag{3.3.48}$$

$$z = \lambda y, \tag{3.3.49}$$

$$y_{xx} + bv = \lambda z, \tag{3.3.50}$$

$$-(|\xi|^2 + \eta)\omega + v(1)\mu(\xi) = \lambda\omega. \tag{3.3.51}$$

Inserting equations (3.3.47), (3.3.49) in (3.3.48), (3.3.50) and (3.3.51), we get

$$\lambda^2 u - u_{xx} + b\lambda y = 0, \tag{3.3.52}$$

$$\lambda^2 y - a y_{xx} - b\lambda u = 0, \tag{3.3.53}$$

$$(\lambda + |\xi|^2 + \eta)\omega - \lambda u(1)\mu(\xi) = 0. \tag{3.3.54}$$

Lemma 3.3.5. *Let $\lambda \in \mathbb{C}$ be a eigenvalue of $\mathcal{A}$. Then, $\Re(\lambda)$ is bounded.*

Proof: Multiplying equations (3.3.52) and (3.3.53) by $\bar{u}$ and $\bar{y}$ respectively, and integrating over $(0,1)$, we get

$$\|\lambda u\|^2 + \|u_x\|^2 - u_x(1)\bar{u}(1) + b\lambda \int_0^1 y\bar{u}dx = 0, \tag{3.3.55}$$

$$\|\lambda y\|^2 + a\|y_x\|^2 - b\lambda \int_0^1 u\bar{y}dx = 0,. \tag{3.3.56}$$

Adding (3.3.55), (3.3.56) and using (3.2.9), we get

$$\|\lambda u\|^2 + \|\lambda y\|^2 + \|u_x\|^2 + a\|y_x\|^2 + \gamma\kappa\bar{u}(1)\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi + 2ib\lambda\Im\left(\int_0^1 y\bar{u}dx\right) = 0. \tag{3.3.57}$$

Remark that $\lambda \neq 0$, indeed if $\lambda = 0$ it easy to check that $U = 0$. Multiplying equation (3.3.54) by $\bar{\omega}$, integrate over $\mathbb{R}$ and using the fact $\lambda \neq 0$, we get

$$\|\omega\|^2 + \frac{1}{\lambda}\int_{-\infty}^{+\infty}(|\xi|^2 + \eta)|\omega|^2 d\xi = \bar{u}(1)\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi. \tag{3.3.58}$$

Inserting equation (3.3.58) in (3.3.57), and using the fact $\|U\|_{\mathcal{H}} = 1$ we get

$$\bar{\lambda} = -\gamma\kappa\int_{-\infty}^{+\infty}(|\xi|^2 + \eta)|\omega|^2 d\xi - 2ib|\lambda|^2\Im\left(\int_0^1 y\bar{u}dx\right). \tag{3.3.59}$$

Consequently, we have

$$\Re(\lambda) = -\gamma\kappa\int_{-\infty}^{+\infty}(|\xi|^2 + \eta)|\omega|^2 d\xi < 0.$$

The proof is thus complete.

Proposition 3.3.6. *Assume $a \neq 1$, $\eta > 0$ and b satisfying condition (C). There exists a constant $N \in \mathbb{N}$ such that*

$$\sigma(\mathcal{A}) \supset \sigma_0 \cup \sigma_1,$$

where

$$\sigma_1 = \{\lambda_m\}_{m\in\mathbb{Z}^*, |m|\geq N} \cup \{\lambda_n\}_{n\in\mathbb{Z}^*, |n|\geq N},$$

Moreover, J is finite set, and

$$\lambda_m = i\left(m+\frac{1}{2}\right)\pi + o(1) \quad and/or \quad \lambda_n = in\pi\sqrt{a} + o(1).$$

Proof: From equations (3.3.52) and (3.3.53), we have

$$\begin{aligned} u &= \frac{1}{b\lambda}(\lambda^2 y - a y_{xx}), \qquad (3.3.60)\\ u_{xx} &= \frac{1}{b\lambda}\left(\lambda^2 y_{xx} - a y_{xxxx}\right) \qquad (3.3.61)\end{aligned}$$

Inserting equations (3.3.60) and (3.3.61) in equation (3.3.52), we obtain

$$a y_{xxxx} - (a+1)\lambda^2 y_{xx} + \lambda^2(\lambda^2+b^2)y = 0. \qquad (3.3.62)$$

Using the same fact in the case $a=1$, then we get the following system

$$\begin{aligned} a y_{xxxx} - (a+1)\lambda^2 y_{xx} + \lambda^2(\lambda^2+b^2)y &= 0, \qquad (3.3.63)\\ y(0) &= 0, \qquad (3.3.64)\\ y(1) &= 0, \qquad (3.3.65)\\ y_{xx}(0) &= 0, \qquad (3.3.66)\\ \lambda^2 y_x(1) - a\gamma\lambda(\lambda+\eta)^{\alpha-1}y_{xx}(1) - a y_{xxx}(1) &= 0,. \qquad (3.3.67)\end{aligned}$$

We consider the characteristic equation defined by

$$at^4 - (a+1)\lambda^2 t^2 + \lambda^2(\lambda^2+b^2) = 0. \qquad (3.3.68)$$

The general solution of equation (3.3.68) is given by

$$y(x) = \sum_{i=1}^{4} c_i e^{r_i x}. \qquad (3.3.69)$$

where

$$\begin{cases} r_1(\lambda) = \sqrt{\dfrac{(a+1)\lambda^2 + \sqrt{(a-1)^2\lambda^4 - 4ab^2\lambda^2}}{2a}}, \quad r_2(\lambda) = -r_1(\lambda), \\ \\ r_3(\lambda) = \sqrt{\dfrac{(a+1)\lambda^2 - \sqrt{(a-1)^2\lambda^4 - 4ab^2\lambda^2}}{2a}}, \quad r_4(\lambda) = -r_3(\lambda). \end{cases}$$

Here and bellow, for simplicity we denote $r_i(\lambda)$ by r_i. Thus the boundary conditions may be written as the following system

$$M(\lambda)C(\lambda)=\begin{pmatrix}1 & 1 & 1 & 1\\ e^{r1} & e^{-r1} & e^{r3} & e^{-r3}\\ r_1^2 & r_1^2 & r_3^2 & r_3^2\\ f_1(r_1)e^{r1} & f_2(r_1)e^{-r1} & f_3(r_3)e^{r3} & f_4(r_3)e^{-r3}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix}=0, \tag{3.3.70}$$

where

$$\begin{cases} f_1(r_1) &=& \lambda^2 r_1 - a\gamma\lambda(\lambda+\eta)^{\alpha-1}r_1^2 - ar_1^3,\\ f_2(r_1) &=& -\lambda^2 r_1 - a\gamma\lambda(\lambda+\eta)^{\alpha-1}r_1^2 + ar_1^3,\\ f_3(r_3) &=& \lambda^2 r_3 - a\gamma\lambda(\lambda+\eta)^{\alpha-1}r_3^2 - ar_3^3,\\ f_4(r_3) &=& -\lambda^2 r_3 - a\gamma\lambda(\lambda+\eta)^{\alpha-1}r_3^2 + ar_3^3 \end{cases} \tag{3.3.71}$$

Step 1. We start by the expansion of r_1 and r_3 :

$$r_1 = \lambda - \frac{b^2}{2(a-1)\lambda} - \frac{b^4(5a-1)}{8(a-1)^3\lambda^3} + O\left(\frac{1}{\lambda^5}\right) \tag{3.3.72}$$

$$r_3 = \frac{\lambda}{\sqrt{a}} + \frac{\sqrt{a}b^2}{2(a-1)\lambda} - \frac{a^2b^4(a-5)}{8\sqrt{a}(a-1)^3\lambda^3} + O\left(\frac{1}{\lambda^5}\right). \tag{3.3.73}$$

The determinant of $M(\lambda)$ is given by

$$\begin{aligned} f(\lambda) = \left(r_1^2 - r_3^2\right)\Big((f_2(r_1)-f_4(r_3))e^{-(r_1+r_3)} + (f_1(r_1)-f_3(r_3))e^{r_1+r_3}\\ (f_3(r_3)-f_2(r_1))e^{r_3-r_1} + (f_4(r_3)-f_1(r_1))e^{r_1-r_3}\Big), \end{aligned} \tag{3.3.74}$$

where

$$\begin{cases} f_2(r_1)-f_4(r_3) &=& \lambda^2(r_3-r_1) + a\gamma\lambda(\lambda+\eta)^{\alpha-1}\left(r_3^2-r_1^2\right) + a\left(r_1^3-r_3^3\right),\\ f_1(r_1)-f_3(r_3) &=& \lambda^2(r_1-r_3) + a\gamma\lambda(\lambda+\eta)^{\alpha-1}\left(r_3^2-r_1^2\right) + a\left(r_3^3-r_1^3\right),\\ f_3(r_3)-f_2(r_1) &=& \lambda^2(r_1+r_3) + a\gamma\lambda(\lambda+\eta)^{\alpha-1}\left(r_1^2-r_3^2\right) - a\left(r_1^3+r_3^3\right),\\ f_4(r_3)-f_1(r_1) &=& -\lambda^2(r_1+r_3) + a\gamma\lambda(\lambda+\eta)^{\alpha-1}\left(r_1^2-r_3^2\right) + a\left(r_1^3+r_3^3\right). \end{cases} \tag{3.3.75}$$

Using equations (3.3.72) and (3.3.73), we get

$$\left\{\begin{array}{lll}
e^{-(r_1+r_3)} & = & e^{-\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\left(1-\dfrac{b^2(\sqrt{a}-1)}{2(a-1)\lambda}+\dfrac{b^4(\sqrt{a}-1)^2}{8(a-1)^2\lambda^2}+O\left(\dfrac{1}{\lambda^3}\right)\right),\\[2ex]
e^{(r_1+r_3)} & = & e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\left(1+\dfrac{b^2(\sqrt{a}-1)}{2(a-1)\lambda}+\dfrac{b^4(\sqrt{a}-1)^2}{8(a-1)^2\lambda^2}+O\left(\dfrac{1}{\lambda^3}\right)\right),\\[2ex]
e^{r_3-r_1} & = & e^{\lambda\left(-1+\frac{1}{\sqrt{a}}\right)}\left(1+\dfrac{b^2(\sqrt{a}+1)}{2(a-1)\lambda}+\dfrac{b^4(\sqrt{a}+1)^2}{8(a-1)^2\lambda^2}+O\left(\dfrac{1}{\lambda^3}\right)\right),\\[2ex]
e^{r_1-r_3} & = & e^{\lambda\left(1-\frac{1}{\sqrt{a}}\right)}\left(1-\dfrac{b^2(\sqrt{a}+1)}{2(a-1)\lambda}+\dfrac{b^4(\sqrt{a}+1)^2}{8(a-1)^2\lambda^2}+O\left(\dfrac{1}{\lambda^3}\right)\right).
\end{array}\right. \tag{3.3.76}$$

Similarly, using equations (3.3.72) and (3.3.73) in (3.3.75), we get

$$\left\{\begin{array}{lll}
f_2(r_1)-f_4(r_3) & = & \lambda^3(a-1)+\gamma(1-a)\lambda^{2+\alpha}+a\gamma\eta(\alpha-1)\lambda^{1+\alpha}\\
& & +\dfrac{b^2(1-2\sqrt{a}-3a)\lambda}{2(a-1)}+O(\lambda^\alpha)\\[2ex]
f_1(r_1)-f_3(r_3) & = & \lambda^3(1-a)+\gamma(1-a)\lambda^{2+\alpha}+a\gamma\eta(\alpha-1)\lambda^{1+\alpha}\\
& & +\dfrac{b^2(-1+2\sqrt{a}+3a)\lambda}{2(a-1)}+O(\lambda^\alpha)\\[2ex]
f_3(r_3)-f_2(r_1) & = & \lambda^3(1-a)+\gamma(a-1)\lambda^{2+\alpha}+a\gamma\eta(1-\alpha)\lambda^{1+\alpha}\\
& & +\dfrac{b^2(-1-2\sqrt{a}+3a)\lambda}{2(a-1)}+O(\lambda^\alpha)\\[2ex]
f_4(r_3)-f_1(r_1) & = & \lambda^3(1-a)+\gamma(a-1)\lambda^{2+\alpha}+a\gamma\eta(1-\alpha)\lambda^{1+\alpha}\\
& & +\dfrac{b^2(1+2\sqrt{a}-3a)\lambda}{2(a-1)}+O(\lambda^\alpha).
\end{array}\right. \tag{3.3.77}$$

Using equations (3.3.76) and (3.3.77) in (3.3.74), we get

$$f(\lambda)=A+B+C+D, \tag{3.3.78}$$

where

$$\left\{\begin{array}{lll} A & = & \dfrac{\lambda^5(a-1)^2}{a} e^{-\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\left(1-\dfrac{\gamma}{\lambda^{1-\alpha}}-\dfrac{b^2(\sqrt{a}-1)}{2(a-1)\lambda}+O\left(\dfrac{1}{\lambda^{2-\alpha}}\right)\right), \\ B & = & \dfrac{\lambda^5(a-1)^2}{a} e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\left(-1-\dfrac{\gamma}{\lambda^{1-\alpha}}-\dfrac{b^2(\sqrt{a}-1)}{2(a-1)\lambda}+O\left(\dfrac{1}{\lambda^{2-\alpha}}\right)\right), \\ C & = & \dfrac{\lambda^5(a-1)^2}{a} e^{\lambda\left(-1+\frac{1}{\sqrt{a}}\right)}\left(-1+\dfrac{\gamma}{\lambda^{1-\alpha}}-\dfrac{b^2(\sqrt{a}+1)}{2(a-1)\lambda}+O\left(\dfrac{1}{\lambda^{2-\alpha}}\right)\right), \\ D & = & \dfrac{\lambda^5(a-1)^2}{a} e^{\lambda\left(1-\frac{1}{\sqrt{a}}\right)}\left(1+\dfrac{\gamma}{\lambda^{1-\alpha}}-\dfrac{b^2(\sqrt{a}+1)}{2(a-1)\lambda}+O\left(\dfrac{1}{\lambda^{2-\alpha}}\right)\right). \end{array}\right. \tag{3.3.79}$$

Using equation (3.3.79), we find the following asymptotic expansion

$$h(\lambda)=\frac{af(\lambda)}{(a-1)^2\lambda^5}=f_0(\lambda)+\frac{f_1(\lambda)}{\lambda^{1-\alpha}}+\frac{f_2(\lambda)}{\lambda}+O\left(\frac{1}{\lambda^{2-\alpha}}\right), \tag{3.3.80}$$

where

$$\left\{\begin{array}{lll} f_0(\lambda) & = & e^{-\lambda\left(1+\frac{1}{\sqrt{a}}\right)}-e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}-e^{\lambda\left(\frac{1}{\sqrt{a}}-1\right)}+e^{\lambda\left(1-\frac{1}{\sqrt{a}}\right)}, \\ f_1(\lambda) & = & \gamma\left(-e^{-\lambda\left(1+\frac{1}{\sqrt{a}}\right)}-e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}+e^{\lambda\left(\frac{1}{\sqrt{a}}-1\right)}+e^{\lambda\left(1-\frac{1}{\sqrt{a}}\right)}\right), \\ f_2(\lambda) & = & \dfrac{-b^2}{2(a-1)}\left((\sqrt{a}-1)\left(e^{-\lambda\left(1+\frac{1}{\sqrt{a}}\right)}+e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\right)\right. \\ & & \left.+(\sqrt{a}+1)\left(e^{\lambda\left(\frac{1}{\sqrt{a}}-1\right)}+e^{\lambda\left(1-\frac{1}{\sqrt{a}}\right)}\right)\right) \end{array}\right. \tag{3.3.81}$$

Step 2. We look at the roots of $h(\lambda)$. It easy to check that

$$f_0(\lambda)=e^{\lambda\left(1+\frac{1}{\sqrt{a}}\right)}\left(\left(1+e^{-2\lambda}\right)\left(-1+e^{-\frac{2\lambda}{\sqrt{a}}}\right)\right). \tag{3.3.82}$$

Then the roots of f_0 is given by

$$\lambda_m^0=i\left(m+\frac{1}{2}\right)\pi \quad \text{and/ or} \quad \lambda_n^0=in\pi\sqrt{a} \qquad m,n\in\mathbb{Z}.$$

Since the real part of λ is bounded, then with help of Rouch'é's Theorem, and λ large enough, we show that the roots of $h(\lambda)$ are close of those of $f_0(\lambda)$. In the other words, there exists a sequence λ_k of roots of f such that

$$\lambda_m = i\left(m+\frac{1}{2}\right)\pi + o(1) \quad \text{and/or} \quad \lambda_n = in\pi\sqrt{a} + o(1) \quad \text{as} \quad m,n \to +\infty. \tag{3.3.83}$$

This implies that the C_0-semigroup of contraction $e^{t\mathcal{A}}$ is not uniformly stable in the energy space for $a \neq 1$, $\eta > 0$ and b verified (C).

3.4 Polynomial Stability

3.4.1 Polynomial Stability for $a = 1$

In this subsection, assume that $a = 1$ and $\eta > 0$ and b satisfy the condition (C1), we study the asymptotic behavior of solution of system (3.2.5)-(3.2.10) for 2 distinguish cases ($b = k\pi$ and $b \neq k\pi$) for all $k \in \mathbb{Z}$. Our main result is the following theorem.

Theorem 3.4.1. *Assume that $\eta > 0$, $a = 1$ and b satisfies condition* (C). *Then, for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution U of satisfies the following estimation*

$$E(t) \leq \frac{C_1}{t^{\ell(\alpha)}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0, \tag{3.4.1}$$

where

$$\ell(\alpha) = \begin{cases} \frac{1}{3-\alpha} & \text{if } a = 1 \quad \text{and } b = k\pi, \\ \frac{1}{1-\alpha} & \text{if } a = 1 \quad \text{and } b \neq k\pi, \end{cases}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t, U)$ converges to zero at t goes to infinity.

Following Borichev and Tomilov, a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space verify

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{$H1$}$$

and

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{|\lambda|^{\ell}}\|(i\lambda I - \mathcal{A})^{-1}\| < +\infty. \tag{$H2$}$$

Condition $(H1)$ was already proved, we will prove $(H3)$ using an argument of contradiction. Suppose that $(H2)$ is false, then there exist a real sequence (λ_n) and a sequence $U^n = (u^n, v^n, y^n, z^n, \omega^n) \in D(\mathcal{A})$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \quad \|U^n\| = \|(u^n, v^n, y^n, z^n, \omega^n)\| = 1, \tag{3.4.2}$$

$$\lambda_n^{\ell}(i\lambda_n I - \mathcal{A})U^n = (f_1^n, g_1^n, f_2^n, g_2^n, f_3^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}. \tag{3.4.3}$$

Detailing equation (3.4.3), we get

$$i\lambda_n u^n - v^n = \frac{f_1^n}{\lambda_n^\ell} \longrightarrow 0 \quad \text{in} \quad H_L^1(0,1), \tag{3.4.4}$$

$$i\lambda_n v^n - u_{xx}^n + bz_n = \frac{g_1^n}{\lambda_n^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(0,1), \tag{3.4.5}$$

$$i\lambda_n y^n - z^n = \frac{f_2^n}{\lambda_n^\ell} \longrightarrow 0 \quad \text{in} \quad H_0^1(0,1), \tag{3.4.6}$$

$$i\lambda_n z^n - y_{xx} - bv^n = \frac{g_2^n}{\lambda_n^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(0,1), \tag{3.4.7}$$

$$i\lambda_n \omega^n + (\xi^2+\eta)\omega^n - v^n(1)\mu(\xi) = \frac{f_3^n(\xi)}{\lambda_n^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}). \tag{3.4.8}$$

Multiply in $\mathcal{H}$ equation (3.4.3) by the uniformly bounded sequence $U^n = (u^n, v^n, y^n, z^n, \omega^n)$, we get

$$-\gamma\kappa \int_{-\infty}^{+\infty} (\xi^2+\eta)|\omega^n|^2 d\xi = \Re\left(\langle (i\lambda_n I - \mathcal{A})U^n, U^n\rangle_{\mathcal{H}}\right) = \frac{o(1)}{\lambda_n^\ell}. \tag{3.4.9}$$

For the simplicity, we drop the index n.

Lemma 3.4.2. *Assume that $a = 1$, $\eta > 0$ and b satisfies the condition* (C1)*. Then, the solution (u, v, y, z, ω) of* (3.4.4)-(3.4.8) *satisfies the following estimations*

$$\begin{cases} \|u\| = \dfrac{O(1)}{\lambda}, & \|y\| = \dfrac{O(1)}{\lambda}, \\ |u(1)| = \dfrac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}, & |u_x(1)| = \dfrac{o(1)}{\lambda^{\frac{\ell}{2}}}. \end{cases} \tag{3.4.10}$$

Proof: From equations (3.4.4), (3.4.6) and (3.4.2), it easy to check that

$$\|u\| = \frac{O(1)}{\lambda} \quad \text{and} \quad \|y\| = \frac{O(1)}{\lambda}.$$

Multiplying equation (3.4.8) by $(i\lambda + |\xi|^2 + \eta)^{-1}\mu(\xi)$, tacking the absolute values of both sides, integrating over $\mathbb{R}$ with respect to the variable ξ and applying Cauchy-Shwartz inequality, we obtain

$$A_1|v(1)| \leq A_2\left((|\xi|^2+\eta)|\omega|^2 d\xi\right)^{\frac{1}{2}} + \frac{1}{|\lambda|^\ell} A_3 \left(\int_{\mathbb{R}} |f_3(\xi)| d\xi\right)^{\frac{1}{2}} \tag{3.4.11}$$

where

$$A_1 = \int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{|\lambda| + |\xi|^2 + \eta} d\xi, \quad A_2 = \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{|\xi|^2+\eta} d\xi\right)^{\frac{1}{2}} \quad \text{and} \quad A_3 = \left(\int_{\mathbb{R}} \frac{|\mu(\xi)|^2}{(|\lambda| + |\xi|^2+\eta)^2} d\xi\right)^{\frac{1}{2}}.$$

For $\alpha \in]0,1[$, from lemma 1.6.2, we have

$$A_1 = c_2(|\lambda|+\eta)^{\alpha-1} \quad A_2 = c_2 \quad \text{and} \quad A_3 = c_3(|\lambda|+\eta)^{\frac{\alpha}{2}-1}, \tag{3.4.12}$$

where, c_1, c_2 and c_3 in $\mathbb{R}$. Then, we deduce that from equation (3.4.11) and (3.4.12)

$$|v(1)| = \frac{o(1)}{\lambda^{\frac{\ell}{2}-1+\alpha}}. \tag{3.4.13}$$

Using equations (3.4.4) and (3.4.13), we obtain

$$|u(1)| = \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}. \tag{3.4.14}$$

Furthermore, we have

$$\begin{aligned} |u_x(1)| &\leq \gamma\kappa \left|\int_{-\infty}^{+\infty} \mu(\xi)\omega(\xi)d\xi\right| \\ &\leq \gamma\kappa \left|\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-\frac{1}{2}}\mu(\xi)(\xi^2+\eta)^{\frac{1}{2}}\omega(\xi)d\xi\right| \\ &\leq \gamma\kappa \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\omega|^2 d\xi\right)^{\frac{1}{2}}. \end{aligned} \tag{3.4.15}$$

For $\alpha \in]0,1[$, we have

$$\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)^{-1}|\mu(\xi)|^2\right)^{\frac{1}{2}} < +\infty \tag{3.4.16}$$

Using equations (3.4.9) and (3.4.16) in (3.4.15), we obtain

$$|u_x(1)| = \frac{o(1)}{\lambda^{\frac{\ell}{2}}}. \tag{3.4.17}$$

Lemma 3.4.3. *Assume that $a = 1$, $\eta > 0$ and b satisfies the condition* (C1). *Then, For all $h \in W^{1,\infty}(0,1)$, the solution (u, v, y, z, ω) of* (3.4.4)-(3.4.8) *satisfies the following estimation we have*

$$\begin{aligned} -\int_0^1 h'|\lambda u|^2 dx - \int_0^1 h'|u_x|^2 dx - \int_0^1 h'|\lambda y|^2 dx - \int_0^1 h'|y_x|^2 dx + h(1)|y_x(1)|^2 \\ -h(0)|y_x(0)^2| - h(0)|u_x(0)|^2 = \frac{o(1)}{\lambda^{\ell+2\alpha-2}} + \frac{O(1)}{\lambda}. \end{aligned} \tag{3.4.18}$$

Proof: Substitute v and z in equations (3.4.4) and (3.4.6) by (3.4.5) and (3.4.7), we get

$$\lambda^2 u + u_{xx} - i\lambda b y = -\frac{g_1 + i\lambda f_1 + b f_2}{\lambda^\ell}, \tag{3.4.19}$$

$$\lambda^2 y + y_{xx} + i\lambda b u = -\frac{g_2 + i\lambda f_2 - b f_1}{\lambda^\ell}. \tag{3.4.20}$$

Multiplying equation (3.4.19) by $2h\bar{u}_x$, integrate by part and using Lemma 3.4.2, we get

$$-\int_0^1 h'|\lambda u|^2 dx - \int_0^1 h'|u_x|^2 dx - h(0)|u_x(0)|^2 + 2i\lambda b \int_0^1 h y_x \bar{u} dx = \frac{o(1)}{\lambda^{\ell-2+2\alpha}} + \frac{O(1)}{\lambda}. \tag{3.4.21}$$

Multiplying equation (3.4.20) by $2h\bar{y}_x$, integrate by part and using lemma 3.4.2, we get

$$-\int_0^1 h'|\lambda y|^2 dx - \int_0^1 h'|y_x|^2 dx + h(1)|y_x(1)|^2 - h(0)|y_x(0)|^2 + 2i\lambda b \int_0^1 h \bar{y}_x u dx = \frac{o(1)}{\lambda^\ell}. \tag{3.4.22}$$

Adding equations (3.4.21) , (3.4.22) and tacking the reel part we get equations (3.4.18). Then the proof has been completed.

Lemma 3.4.4. *Assume that $a = 1$, $\eta > 0$ and b satisfies the condition* (C1). *Then, the solution (u, v, y, z, ω) of (3.4.19)-(3.4.20) satisfies the following estimations*

$$|y_x(1)|^2 = \frac{o(1)}{\lambda^{\delta(\alpha)}}, \tag{3.4.23}$$

where

$$\delta = \begin{cases} \ell - 2 + 2\alpha & \text{if} \quad b \notin \pi\mathbb{Z}, \\ \ell + 2\alpha - 6 & \text{if} \quad b \in \pi\mathbb{Z}. \end{cases}$$

Proof: Let $Y = (u, u_x, y, y_x)$, then system (3.4.19) and (3.4.20), could be written as

$$Y_x = BY + F, \tag{3.4.24}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda^2 & 0 & i\lambda b & 0 \\ 0 & 0 & 0 & 1 \\ -i\lambda b & 0 & -\lambda^2 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ -\dfrac{g_1 + i\lambda f_1 + b f_2}{\lambda^\ell} \\ 0 \\ -\dfrac{g_2 + i\lambda f_2 - b f_1}{\lambda^\ell} \end{pmatrix}. \tag{3.4.25}$$

Using Ordinary Differential Equation Theory, the solution of equation (3.4.24) is given by

$$Y(x) = e^{Bx} Y_0 + \int_0^x e^{B(x-z)} F(z) dz. \tag{3.4.26}$$

The solution on 1 is given by

$$Y(1) = e^B Y_0 + \int_0^1 e^{B(1-z)} F(z) dz. \tag{3.4.27}$$

Equivalently, we get

$$e^{-B} Y(1) = Y_0 + e^{-B} \int_0^1 e^{B(1-z)} F(z) dz, \tag{3.4.28}$$

where

$$e^B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{11} & b_{23} & b_{13} \\ -b_{13} & -b_{14} & b_{11} & b_{12} \\ -b_{23} & -b_{13} & b_{21} & b_{11} \end{pmatrix} \quad \text{and} \quad e^{-B} = \begin{pmatrix} b_{11} & -b_{12} & b_{13} & -b_{14} \\ -b_{21} & b_{11} & -b_{23} & b_{13} \\ -b_{13} & b_{14} & b_{11} & -b_{12} \\ b_{23} & -b_{13} & -b_{21} & b_{11} \end{pmatrix},$$

where (b_{ij}) is defined by

$$\begin{cases} b_{11} &= \dfrac{\lambda\left(\lambda(t_1-t_2)+b(t_1+t_2)\right)\left(e^{t_1}+e^{-t_1}+e^{t_2}+e^{-t_2}\right)}{4t_1t_2(t_1-t_2)}, \\ b_{12} &= \dfrac{\lambda b(e^{t_1}-e^{-t_1}+e^{t_2}-e^{-t_2})+(\lambda^2+t_1t_2)(e^{t_1}-e^{-t_1}-e^{t_2}+e^{-t_2})}{4t_1t_2(t_1-t_2)}, \\ b_{13} &= \dfrac{i}{4}\left(e^{t_1}+e^{-t_1}-e^{t_2}-e^{-t_2}\right), \\ b_{14} &= \dfrac{i}{4t_1t_2}\left(t_2\left(e^{t_1}-e^{-t_1}\right)-t_1\left(e^{t_2}-e^{-t_2}\right)\right), \\ b_{21} &= \dfrac{-\lambda}{4t_1t_2(t_1-t_2)}\left((\lambda-b)(t_1t_2+\lambda^2+\lambda b)\left(e^{t_1}-e^{-t_1}\right)-(\lambda+b)(t_1t_2+\lambda^2-\lambda b)(e^{t_2}-e^{-t_2})\right), \\ b_{23} &= \dfrac{-i\lambda}{4t_1t_2(t_1-t_2)}\left((\lambda-b)(t_1t_2+\lambda^2+\lambda b)(e^{t_1}-e^{-t_1})+(\lambda+b)(t_1t_2+\lambda^2-\lambda b)(e^{t_2}-e^{-t_2})\right), \end{cases}$$

where $t_1 = \sqrt{b\lambda - \lambda^2}$ and $t_2 = \sqrt{-b\lambda - \lambda^2}$. Performing advanced calculation for the exponential of matrix B and $-B$, we obtain the following matrix

$$e^B = \begin{pmatrix} A_1 & \frac{1}{\lambda}A_2 - \frac{b}{2\lambda^2}\left(A_4 + \frac{b}{4}A_1\right) & iA_3 & \frac{-i}{\lambda}A_4 - \frac{ib}{2\lambda^2}\left(\frac{b}{4}A_3 - A_2\right) \\ \frac{b}{2}\left(\frac{b}{4}A_1 - A_4\right) & A_1 & \frac{ib}{2}\left(A_2 + \frac{b}{4}A_3\right) & iA_3 \\ -iA_3 & \frac{i}{\lambda}A_4 + \frac{ib}{2\lambda^2}\left(\frac{b}{4}A_3 - A_2\right) & A_1 & \frac{1}{\lambda}A_2 - \frac{b}{2\lambda^2}\left(A_4 + \frac{b}{4}A_1\right) \\ -\frac{ib}{2}\left(A_2 + \frac{b}{4}A_3\right) & -iA_3 & \frac{b}{2}\left(\frac{b}{4}A_1 - A_4\right) & A_1 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\lambda A_2 & 0 & i\lambda A_4 & 0 \\ 0 & 0 & 0 & 0 \\ i\lambda A_4 & 0 & -\lambda A_2 & 0 \end{pmatrix} + \begin{pmatrix} o(1) & O\left(\frac{1}{\lambda^3}\right) & o(1) & O\left(\frac{1}{\lambda^3}\right) \\ o(1) & o(1) & o(1) & o(1) \\ o(1) & O\left(\frac{1}{\lambda^3}\right) & o(1) & O\left(\frac{1}{\lambda^3}\right) \\ o(1) & o(1) & o(1) & o(1) \end{pmatrix},$$

and the expression of e^{-B} is given by

$$e^{-B}=\begin{pmatrix} A_1 & -\frac{1}{\lambda}A_2+\frac{b}{2\lambda^2}\left(A_4+\frac{b}{4}A_1\right) & iA_3 & \frac{i}{\lambda}A_4+\frac{ib}{2\lambda^2}\left(\frac{b}{4}A_3+A_2\right)\\ -\frac{b}{2}\left(\frac{b}{4}A_1-A_4\right) & A_1 & -\frac{ib}{2}\left(A_2+\frac{b}{4}A_3\right) & iA_3\\ iA_3 & \frac{i}{\lambda}A_4+\frac{ib}{2\lambda^2}\left(\frac{b}{4}A_3-A_2\right) & A_1 & -\frac{1}{\lambda}A_2+\frac{b}{2\lambda^2}\left(A_4+\frac{b}{4}A_1\right)\\ -\frac{ib}{2}\left(A_2+\frac{b}{4}A_3\right) & iA_3 & -\frac{b}{2}\left(\frac{b}{4}A_1-A_4\right) & A_1 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 & 0\\ \lambda A_2 & 0 & -i\lambda A_4 & 0\\ 0 & 0 & 0 & 0\\ i\lambda A_4 & 0 & +\lambda A_2 & 0\end{pmatrix}+\begin{pmatrix} o(1) & O\left(\frac{1}{\lambda^3}\right) & o(1) & O\left(\frac{1}{\lambda^3}\right)\\ o(1) & o(1) & o(1) & o(1)\\ o(1) & O\left(\frac{1}{\lambda^3}\right) & o(1) & O\left(\frac{1}{\lambda^3}\right)\\ o(1) & o(1) & o(1) & o(1)\end{pmatrix},$$

where,

$$A_1=\cos(\lambda)\cos\left(\frac{b}{2}\right),\quad A_2=\sin(\lambda)\cos\left(\frac{b}{2}\right),\quad A_3=\sin(\lambda)\sin\left(\frac{b}{2}\right)\ \text{ and }\ A_4=\cos(\lambda)\sin\left(\frac{b}{2}\right).$$

On the another hand, using Lemma 3.4.2, we get

$$Y(1)=\left(\frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}},\frac{o(1)}{\lambda^{\frac{\ell}{2}}},0,y_x(1)\right)^{\top}. \tag{3.4.29}$$

Using the expression of e^B and e^{-B}, we get

$$e^{-B}\int_0^1 e^{B(1-z)}F(z)dz=\left(\frac{o(1)}{\lambda^{\ell}},\frac{o(1)}{\lambda^{\ell-1}},\frac{o(1)}{\lambda^{\ell}},\frac{o(1)}{\lambda^{\ell-1}}\right)^{\top}. \tag{3.4.30}$$

Using equations (3.4.28)-(3.4.30), we get

$$\frac{1}{\lambda}A_2y_x(1)-\frac{b}{2\lambda^2}\left(A_4+\frac{b}{4}A_1\right)y_x(1) = \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}, \tag{3.4.31}$$

$$\frac{1}{\lambda}A_4y_x(1)+\frac{ib}{2\lambda^2}\left(\frac{b}{4}A_3+A_2\right)y_x(1) = \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}. \tag{3.4.32}$$

Case 1 : Suppose that $b=k\pi$. Assume that $b=(2s+1)\pi$, then $\cos\left(\frac{b}{2}\right)=0$ then $A_1=A_2=0$, using this fact and multiplying equation (3.4.32) by λ^2, we get

$$A_4y_x(1)=\frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-2}}. \tag{3.4.33}$$

Inserting equation (3.4.33) in equation (3.4.32), we get

$$\frac{1}{\lambda^2}A_3 y_x(1) = \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}}. \tag{3.4.34}$$

Multiplying equation (3.4.34) by λ^2, we get

$$A_3 y_x(1) = \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-3}}. \tag{3.4.35}$$

Consequently, squaring equations (3.4.33) and (3.4.35), tacking the sum and using the fact $A_3^2 + A_4^2 = \sin^2\left(\frac{b}{2}\right)$, we get

$$|y_x(1)|^2 = \frac{o(1)}{\lambda^{\ell+2\alpha-6}}.$$

The same proof for $b = 2s\pi$.

Case 2 : Suppose that $b \neq k\pi$, then from equations (3.4.31) and (3.4.32), we get

$$\begin{aligned} \frac{1}{\lambda}A_4 y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}, && (3.4.36)\\ \frac{1}{\lambda}A_2 y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}}. && (3.4.37)\\ & && (3.4.38) \end{aligned}$$

Multiplying equations (3.4.36) and (3.4.37) by λ, we get

$$\begin{aligned} A_4 y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}}, && (3.4.39)\\ A_2 y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}}, && (3.4.40) \end{aligned}$$

Using the fact $b \neq k\pi$, then $\cos(\frac{b}{2}) \neq 0$ and $\sin\left(\frac{b}{2}\right) \neq 0$, then from equations (3.4.39) and (3.4.40), we get

$$\begin{aligned} \cos(\lambda) y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}}, && (3.4.41)\\ \sin(\lambda) y_x(1) &= \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}},. && (3.4.42) \end{aligned}$$

Squared equations (3.4.41) and (3.4.42), we get

$$|y_x(1)|^2 = \frac{o(1)}{\lambda^{\ell+2\alpha-2}}. \tag{3.4.43}$$

The proof has been completed.

Proof of the Theorem 3.4.1 :

Case 1 : If $b \in \pi\mathbb{Z}$, take $\ell = 6 - 2\alpha$ then

$$|y_x(1)|^2 = o(1). \tag{3.4.44}$$

Take $h = x$ in equation (3.4.18) and using equation (3.4.44), we get

$$-\int_0^1 h'|\lambda u|^2 dx - \int_0^1 h'|u_x|^2 dx - \int_0^1 h'|\lambda y|^2 dx - \int_0^1 h'|y_x|^2 dx = o(1). \tag{3.4.45}$$

In the another hand, we have

$$\int_{-\infty}^{+\infty} |\omega|^2 d\xi \leq \frac{1}{\eta} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\omega|^2 d\xi = \frac{o(1)}{\lambda^{6-2\alpha}}. \tag{3.4.46}$$

Using equations (3.4.9), (3.4.45) and (3.4.46), we get $\|U\|_{\mathcal{H}} = o(1)$ which contradicts (3.4.3). This implies that

$$\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - \mathcal{A}) \|_{\mathcal{L}(\mathcal{H})} = O(\lambda^{6-2\alpha}).$$

Case 2 : For $b \notin \pi\mathbb{Z}$, take $\ell = 2 - 2\alpha$ in equation (3.4.23), then we get

$$|y_x(1)|^2 = o(1). \tag{3.4.47}$$

Choose $h = x$ in equation (3.4.18) and using equation (3.4.47), we get

$$-\int_0^1 h'|\lambda u|^2 dx - \int_0^1 h'|u_x|^2 dx - \int_0^1 h'|\lambda y|^2 dx - \int_0^1 h'|y_x|^2 dx = o(1). \tag{3.4.48}$$

In the another hand, we have

$$\int_{-\infty}^{+\infty} |\omega|^2 d\xi \leq \frac{1}{\eta} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\omega|^2 d\xi = \frac{o(1)}{\lambda^{\ell}}. \tag{3.4.49}$$

Using equations (3.4.9) and (3.4.48), we get $\|U\|_{\mathcal{H}} = o(1)$ which contradicts (3.4.3). This implies that

$$\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - \mathcal{A}) \|_{\mathcal{L}(\mathcal{H})} = O(\lambda^{2-2\alpha}).$$

3.4.2 Polynomial Stability in the general case i.e. $a \neq 1$

In this subsection, assume $a \neq 1$, $\eta > 0$ and b satisfy the condition (C). We study the asymptotic behavior of system (3.2.5)-(3.2.10). Our main result is the following theorem

Theorem 3.4.5. *Assume that $\eta > 0$, $a \neq 1$ and b satisfies condition* (C). *If ($a \in \mathbb{Q}$ and b small enough) or $\sqrt{a} \in \mathbb{Q}$, then, for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$*

independent of U_0 such that the energy of the strong solution U of (3.2.11) satisfies the following estimation

$$E(t) \leq \frac{C_2}{t^{\frac{1}{3-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}, \qquad \forall t > 0. \tag{3.4.50}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

Proof: Following Borichev and Tomilov, a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space verify

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{$H1$}$$

and

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{|\lambda|^{\ell}} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty. \tag{$H2$}$$

Condition $(H1)$ was already proved, we will prove $(H2)$ using an argument of contradiction. Suppose that $(H2)$ is false, then there exist a real sequence (λ_n) and a sequence $U^n = (u^n, v^n, y^n, z^n, \omega^n) \in D(\mathcal{A})$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \quad \|U^n\| = \|(u^n, v^n, y^n, z^n, \omega^n)\| = 1, \tag{3.4.51}$$

$$\lambda_n^{\ell}(i\lambda_n I - \mathcal{A})U^n = (f_1^n, g_1^n, f_2^n, g_2^n, f_3^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}. \tag{3.4.52}$$

Detailing equation (3.4.52), we get

$$\begin{aligned}
i\lambda_n u^n - v^n &= \frac{f_1^n}{\lambda_n^{\ell}} \longrightarrow 0 \quad \text{in} \quad H_L^1(0,1), && (3.4.53)\\
i\lambda_n v^n - u^n_{xx} + bz_n &= \frac{g_1^n}{\lambda_n^{\ell}} \longrightarrow 0 \quad \text{in} \quad L^2(0,1), && (3.4.54)\\
i\lambda_n y^n - z^n &= \frac{f_2^n}{\lambda_n^{\ell}} \longrightarrow 0 \quad \text{in} \quad H_0^1(0,1), && (3.4.55)\\
i\lambda_n z^n - ay^n_{xx} - bv^n &= \frac{g_2^n}{\lambda_n^{\ell}} \longrightarrow 0 \quad \text{in} \quad L^2(0,1), && (3.4.56)\\
i\lambda_n \omega^n + (\xi^2+\eta)\omega^n - v^n(1)\mu(\xi) &= \frac{f_3^n(\xi)}{\lambda_n^{\ell}} \longrightarrow 0 \quad \text{in} \quad L^2(-\infty,+\infty). && (3.4.57)
\end{aligned}$$

For the simplicity, we dropped the index n. Eliminate v and z in equations (3.4.53) and (3.4.55) by (3.4.54) and (3.4.56), we obtain the reduced system

$$\begin{aligned}
\lambda^2 u + u_{xx} - i\lambda b y &= -\frac{g_1 + i\lambda f_1 + bf_2}{\lambda^{\ell}}, && (3.4.58)\\
\lambda^2 y + ay_{xx} + i\lambda b u &= -\frac{g_2 + i\lambda f_2 - bf_1}{\lambda^{\ell}},. && (3.4.59)
\end{aligned}$$

Let $Y=(u,u_x,y,y_x)$, then system (3.4.58) and (3.4.59), could be written as

$$Y_x = BY + F. \tag{3.4.60}$$

Where

$$B=\begin{pmatrix} 0 & 1 & 0 & 0\\ -\lambda^2 & 0 & i\lambda b & 0\\ 0 & 0 & 0 & 1\\ \frac{-i\lambda b}{a} & 0 & \frac{-\lambda^2}{a} & 0\end{pmatrix},\quad F=\begin{pmatrix} 0\\ -\frac{g_1+i\lambda f_1+f_2}{\lambda^\ell}\\ 0\\ -\frac{g_2+i\lambda g_1-bf_1}{\lambda^\ell}\end{pmatrix}\quad \text{and}\quad Y_0=\begin{pmatrix}0\\ u_x(0)\\ 0\\ y_x(0)\end{pmatrix}.$$

The solution of equation (3.4.60) at 1 is given by

$$Y(1)=e^{B}Y_0+\int_0^1 e^{B(x-z)}F(z)dz, \tag{3.4.61}$$

from equation (3.4.61), we obtain

$$e^{-B}Y(1)=Y_0+e^{-B}\int_0^1 e^{B(1-z)}F(z)dz, \tag{3.4.62}$$

where

$$e^{B}=\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14}\\ b_{21} & b_{11} & b_{23} & b_{13}\\ -\frac{b_{13}}{a} & -\frac{b_{14}}{a} & b_{33} & b_{34}\\ -\frac{b_{23}}{a} & -\frac{b_{13}}{a} & b_{43} & b_{33}\end{pmatrix}\quad \text{and}\quad e^{-B}=\begin{pmatrix} b_{11} & -b_{12} & b_{13} & -b_{14}\\ -b_{21} & b_{11} & -b_{23} & b_{13}\\ -\frac{b_{13}}{a} & \frac{b_{14}}{a} & b_{33} & -b_{34}\\ \frac{b_{23}}{a} & -\frac{b_{13}}{a} & -b_{43} & b_{33},\end{pmatrix}$$

and

$$\left\{\begin{array}{lll}
b_{11} & = & \dfrac{((a-1)\lambda+\Delta)(e^{t_1}+e^{-t_1})+((1-a)\lambda+\Delta)(e^{t_2}+e^{-t_2})}{4\Delta},\\
b_{12} & = & \dfrac{a\sqrt{2}\left(((a-1)\lambda-\Delta)t_1\left(e^{-t_2}-e^{t_2}\right)+((a-1)\lambda+\Delta)t_2\left(e^{t_1}-e^{-t_1}\right)\right)}{4t_1t_2\Delta},\\
b_{13} & = & \dfrac{iab}{2\Delta}\left(e^{t_2}+e^{-t_2}-e^{t1}-e^{-t_1}\right),\\
b_{14} & = & \dfrac{-ia^2b\sqrt{2}}{2t_1t_2\Delta}\left(t_2\left(e^{t_1}-e^{-t_1}\right)-t_1\left(e^{t_2}-e^{-t_2}\right)\right),\\
b_{21} & = & \dfrac{-a\lambda\sqrt{2}\left(((a-1)\lambda^2+\lambda\Delta+2b^2)\,t_2\left(e^{t_1}-e^{-t_1}\right)+((a-1)\lambda^2+\lambda\Delta-2b^2)\,t_1\left(e^{t_2}-e^{-t_2}\right)\right)}{4t_1t_2\Delta},\\
b_{23} & = & \dfrac{iab\lambda\sqrt{2}\left(((a+1)\lambda+\Delta)\,t_2\left(e^{t_1}-e^{-t_1}\right)\right)+\left((\Delta-(a+1)\lambda)\,t_1\left(e^{t_2}-e^{-t_2}\right)\right)}{4t_1t_2\Delta},\\
b_{33} & = & \dfrac{((1-a)\lambda+\Delta)\left(e^{t_1}+e^{-t_1}\right)+((a-1)\lambda+\Delta)\left(e^{t_2}+e^{-t_2}\right)}{4\Delta},\\
b_{34} & = & \dfrac{a\sqrt{2}\left(((a-1)\lambda+\Delta)\,t_1\left(e^{t_2}-e^{-t_2}\right)+((1-a)\lambda+\Delta)\,t_2\left(e^{t_1}-e^{-t_1}\right)\right)}{4t_1t_2\Delta},\\
b_{43} & = & \dfrac{\lambda\sqrt{2}\left(((1-a)\lambda^2-\lambda\Delta+2ab^2)\,t_1\left(e^{t_2}-e^{-t_2}\right)+((a-1)\lambda^2-\lambda\Delta-2ab^2)\,t_2\left(e^{t_1}-e^{-t_1}\right)\right)}{4t_1t_2\Delta},
\end{array}\right.$$

where

$$t_1=\frac{\sqrt{-2a\lambda\left((a+1)\lambda+\Delta\right)}}{2},\quad t_2=\frac{\sqrt{2a\lambda\left(\Delta-(a+1)\lambda\right)}}{2}\quad \text{and}\quad \Delta=\sqrt{(a-1)^2\lambda^2+4ab^2}.$$

Performing advanced calculation for the exponential of matrix B and $-B$, we obtain the following matrix

$$e^B=\begin{pmatrix}
\cos(\lambda) & 0 & 0 & 0\\
-\lambda\sin(\lambda)-\frac{b^2}{2(a-1)}\cos(\lambda) & \cos(\lambda) & \frac{ib}{(a-1)}\left(a\sin(\lambda)-\sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right) & 0\\
0 & 0 & \cos\left(\frac{\lambda}{\sqrt{a}}\right) & 0\\
\frac{-ib}{a(a-1)}\left(a\sin(\lambda)-\sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right) & 0 & -\frac{\lambda}{\sqrt{a}}\sin\left(\frac{\lambda}{\sqrt{a}}\right)+\frac{b^2}{2(a-1)}\cos\left(\frac{\lambda}{\sqrt{a}}\right) & \cos\left(\frac{\lambda}{\sqrt{a}}\right)
\end{pmatrix}$$

$$+(o_{ij}),$$

where $o_{ij}=\frac{O(1)}{\lambda}$. In particular, we have

$$o_{14}=\frac{iab}{(a-1)\lambda^2}\left(\sin(\lambda)+\sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right)+\frac{iab^3}{2(a-1)^2\lambda^3}\left(\cos(\lambda)-a\cos\left(\frac{\lambda}{\sqrt{a}}\right)\right)+\frac{O(1)}{\lambda^4},$$

$$
\begin{aligned}
o_{31} &= -\frac{ib}{(a-1)\lambda}\left(\cos\left(\frac{\lambda}{\sqrt{a}} - \cos(\lambda)\right)\right) + \frac{O(1)}{\lambda^2},\\
o_{32} &= -\frac{ib}{(a-1)\lambda^2}\left(\sin(\lambda) + \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right) + \frac{O(1)}{\lambda^3},\\
o_{34} &= \frac{\sqrt{a}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{a}}\right) - \frac{ab^2}{2(a-1)\lambda^2}\cos\left(\frac{\lambda}{\sqrt{a}}\right) + \frac{O(1)}{\lambda^3}.
\end{aligned}
$$

Our aim is to show that $y_x(1) = o(1)$, suppose that $y_x(1) = 1$. Using the expression of e^B, e^{-B}, F and Lemma 3.4.2 we get

$$
e^{-B}\int_0^1 e^{B(1-z)}F(z)dz = \left(\frac{o(1)}{\lambda^\ell}, \frac{o(1)}{\lambda^{\ell-1}}, \frac{o(1)}{\lambda^\ell}, \frac{o(1)}{\lambda^{\ell-1}}\right)^\top \tag{3.4.63}
$$

and

$$
e^{-B}Y(1) = \left(\tilde{e}_{14} + \frac{o(1)}{\lambda^{\frac{\ell}{2}}}, \tilde{e}_{24}y_x(1) + \frac{o(1)}{\lambda^{\frac{\ell}{2}-1}}, \tilde{e}_{34}y_x(1) + \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}, \tilde{e}_{44} + \frac{o(1)}{\lambda^{\frac{\ell}{2}+1}}\right)^\top. \tag{3.4.64}
$$

Inserting equations (3.4.63) and (3.4.64) in (3.4.62) and using the expression of o_{14} and o_{34}, we get

$$
\begin{aligned}
\frac{iab}{(a-1)\lambda^2}\left(\sin(\lambda) + \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right) + \frac{iab^3}{2(a-1)^2\lambda}\left(\cos(\lambda) - a\cos\left(\frac{\lambda}{\sqrt{a}}\right)\right)&\\
+\frac{O(1)}{\lambda^4} + \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}} + \frac{o(1)}{\lambda^\ell} = 0&,
\end{aligned} \tag{3.4.65}
$$

and

$$
-\frac{\sqrt{a}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{a}}\right) + \frac{ab^2}{2(a-1)\lambda^2}\cos\left(\frac{\lambda}{\sqrt{a}}\right) + \frac{O(1)}{\lambda^3} + \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha}} + \frac{o(1)}{\lambda^\ell} = 0 \tag{3.4.66}
$$

Multiplying equations (3.4.65) and (3.4.66) by λ^2 and $-\frac{\lambda}{\sqrt{a}}$ respectively, we get

$$
\sin\left(\lambda + \frac{b^2}{2(a-1)\lambda}\right) = \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-2}} + \frac{o(1)}{\lambda^{\ell-2}}, \tag{3.4.67}
$$

$$
\sin\left(\frac{\lambda}{\sqrt{a}} - \frac{b^2\sqrt{a}}{2(a-1)\lambda}\right) = \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^{\frac{\ell}{2}+\alpha-1}} + \frac{o(1)}{\lambda^{\ell-1}}. \tag{3.4.68}
$$

It follows from equations (3.4.67)-(3.4.68), there exists $n, m \in \mathbb{Z}$ such that

$$\lambda = n\pi - \frac{b^2}{2(a-1)\lambda} + \frac{O(1)}{\lambda} + \frac{o(1)}{\lambda^{\min\left(\frac{\ell}{2}+\alpha-2,\ell-2\right)}} \tag{3.4.69}$$

$$\frac{\lambda}{\sqrt{a}} = m\pi + \frac{\sqrt{a}b^2}{2(a-1)\lambda} + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^{\min\left(\frac{\ell}{2}+\alpha-1,\ell-1\right)}} \tag{3.4.70}$$

Using the fact that λ is big enough i.e. $\lambda \sim \pi n \sim \pi\sqrt{a}m$, then by tacking the squares of equations (3.4.69) and (3.4.70), we get respectively

$$\lambda^2 = n^2\pi^2 - \frac{b^2}{a-1} + \frac{O(1)}{\lambda} + \frac{o(1)}{\lambda^{\min\left(\frac{\ell}{2}+\alpha-3,\ell-3\right)}}, \tag{3.4.71}$$

$$\lambda^2 = am^2\pi^2 + \frac{ab^2}{a-1} + \frac{O(1)}{\lambda} + \frac{o(1)}{\lambda^{\min\left(\frac{\ell}{2}+\alpha-2,\ell-2\right)}}. \tag{3.4.72}$$

Combining equations (3.4.70)-(3.4.71), we get

$$n^2\pi^2 - am^2\pi^2 = b^2\left(\frac{a+1}{a-1}\right) + \frac{O(1)}{\lambda} + \frac{o(1)}{\lambda^{\min\left(\frac{\ell}{2}+\alpha-3,\ell-3\right)}}. \tag{3.4.73}$$

Take $\ell = 6 - 2\alpha$ in equation (3.4.73), we get

$$n^2\pi^2 - am^2\pi^2 = b^2\left(\frac{a+1}{a-1}\right) + \frac{O(1)}{\lambda} + o(1). \tag{3.4.74}$$

We distinguish three case. Indeed,

Case 1 : Assume that there exists $p_0, q_0 \in \mathbb{Z}$ such that $a = \frac{p_0^2}{q_0^2} = \frac{n^2}{m^2}$. Then from equation (3.4.74), we get the following contradiction

$$0 = b^2\left(\frac{a+1}{a-1}\right) + \frac{O(1)}{\lambda} + o(1).$$

Case 2 : If $a = \frac{p_0^2}{q_0^2} \neq \frac{n^2}{m^2}$. Then from equation (3.4.74), we get

$$n^2 - \frac{p_0^2}{q_0^2}m^2 = \frac{b^2}{\pi^2}\left(\frac{a+1}{a-1}\right) + o(1) + \frac{O(1)}{\lambda}.$$

Equivalently, we obtain

$$\frac{nq_0 - p_0m}{q_0} = \frac{b^2}{\pi^2}\left(\frac{a+1}{a-1}\right)\frac{q_0}{n_0q_0 + p_0m} + \frac{o(1)}{\lambda} + \frac{O(1)}{\lambda^2}. \tag{3.4.75}$$

Them, we get the following contradiction

$$\frac{1}{q_0} \leq \frac{O(1)}{\lambda} + o(1).$$

Consequently, from **Case 1** and **Case 2**, the system is polynomially stable for $\sqrt{a} \in \mathbb{Q}$.
Case 3 : Assume that there exists $p_0, q_0 \in \mathbb{Z}$ such that $a = \frac{p_0}{q_0}$ and $a \neq \frac{p^2}{q^2}$ for all $p, q \in \mathbb{Z}$. Then from equation (3.4.74), we have

$$\left| \frac{q_0 n^2 - p_0 m^2}{q_0} \right| \leq \frac{b^2}{\pi^2} \left(\frac{a+1}{a-1} \right) + \frac{O(1)}{\lambda} + o(1). \tag{3.4.76}$$

Since b is small enough, we can assume that

$$b^2 \leq \frac{\pi^2 (a-1)}{2q_0(a+1)}. \tag{3.4.77}$$

Consequently, using equations (3.4.76) and (3.4.77), we get the following contradiction

$$\frac{1}{2q_0} \leq \frac{1}{q_0} - \frac{b^2(1 + a\sqrt{a})}{\pi(a-1)} \leq \frac{O(1)}{\lambda} + o(1). \tag{3.4.78}$$

Finally, the system is polynomially stable for $a \in \mathbb{Q}$ and b small enough.

3.5 Conclusion

We have studied the influence of the coefficients on the indirect stabilisation ba a fractional derivative control in the sense of Caputo of order $\alpha \in (0,1)$ and $\eta \geq 0$ of a system of wave equation coupled via the velocity terms. If the wave speeds are equal ($a = 1$), $\eta > 0$ and if the coupling parameter $b = k\pi$ (resp.$b \neq k\pi$), $k \in \mathbb{Z}$ and it is outside a discrete set of exceptional values, a non-uniform stability is expected. Then, using a frequency domain approach combining with a multiplier method, we have proved a polynomial energy decay rate of type $\frac{1}{t^{\frac{1}{1-\alpha}}}$ (resp. $\frac{1}{t^{\frac{1}{3-\alpha}}}$). In the general case, when $a \neq 1$ a non uniform stability is expected. Finally, if $\sqrt{a}$ is a rational number or (a is a rational number and b is small enough) and if b is outside another discrete set of exceptional values, using a frequency domain approach, we proved a polynomial energy decay rate of type $\frac{1}{t^{\frac{1}{3-\alpha}}}$. But, it is interesting to remark that both energy decay in Theorem 3.4.1 (resp. Theorem 3.4.5) approach $\frac{1}{t}$ and (resp. $\frac{1}{\sqrt{t}}$) as $\alpha \to 1$ which is the energy decay given in [48].
We conjecturer that the remaining cases could be analyzed in the same way with a slower polynomial decay rate.

CHAPITRE *4*

ON THE STABILITY OF MULTIDIMENSIONAL COUPLED WAVE EQUATIONS WITH ONE BOUNDARY FRACTIONAL DAMPING

4.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with a Lipschitz boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 open subsets of Γ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ_1 is non empty. We consider the multidimensional coupled wave equations

$$\begin{aligned} u_{tt} - \Delta u + b y_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (4.1.1)\\ y_{tt} - a\Delta y - b u_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, && (4.1.2)\\ u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, && (4.1.3)\\ y &= 0, \quad \text{on} \quad \Gamma \times \mathbb{R}^+, && (4.1.4)\\ \frac{\partial u}{\partial \nu} + \gamma \partial_t^{\alpha,\eta} u &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, && (4.1.5) \end{aligned}$$

where ν is the unit outward normal vector along the boundary Γ_1, γ is a positive constant involved in the boundary control, $a > 0$ and $b \in \mathbb{R}_*$. The notation $\partial_t^{\alpha,\eta}$ stands the generalized Caputo's fractional derivative see [20] of order α with respect to the time variable and is defined by

$$\partial_t^{\alpha,\eta}\omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s)ds, \quad 0 < \alpha < 1, \; \eta \geq 0.$$

The system (4.1.1)-(4.1.5) is considered with initial conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{where} \quad x \in \Omega, && (4.1.6)\\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad \text{where} \quad x \in \Omega, && (4.1.7) \end{aligned}$$

The fractional derivative operator of order α, $0<\alpha<1$, is defined by

$$[D^\alpha f](t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{df}{d\tau}(\tau)d\tau. \tag{4.1.8}$$

The fractional differentiation is inverse operation of fractional integration that is defined by

$$[I^\alpha f](t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau, \quad 0<\alpha<1. \tag{4.1.9}$$

From equations (4.1.8), (4.1.9), clearly

$$[D^\alpha f] = I^{1-\alpha}Df. \tag{4.1.10}$$

Now, we present marginally distinctive forms of (4.1.8) and (4.1.9). These exponentially modified fractional integro-differential operators an will be denoted by us follows

$$[D^{\alpha,\eta} f](t) = \int_0^1 \frac{(t-\tau)^{-\alpha}e^{-\eta(t-\tau)}}{\Gamma(1-\alpha)} \frac{df}{d\tau}(\tau)d\tau \tag{4.1.11}$$

and

$$[I^{\alpha,\eta} f](t) = \int_0^t \frac{(t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}}{\Gamma(\alpha)} f(\tau)d\tau. \tag{4.1.12}$$

Note that the two operators D^α and $D^{\alpha,\eta}$ differ just by their Kernels. $D^{\alpha,\eta}$ is merely Caputo's fractional derivative operator, expect for its exponential factor. Thus, similar to identity (4.1.10), we do have

$$[D^{\alpha,\eta} f] = I^{1-\alpha,\eta}Df. \tag{4.1.13}$$

The order of our derivatives is between 0 and 1.

The boundary fractional damping of the type $\partial_t^{\alpha,\eta}u$ where $0<\alpha<1$, $\eta\geq 0$ arising from the material property has been used in several applications such as in physical, chemical, biological, ecological phenomena. For more details we refer the readers to [46], [47], [10], [11], [12] and [44]. In theoretical point of view, fractional derivatives involves singular and non-integrable kernels ($t^{-\alpha}$, $0<\alpha<1$). This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.

There are a few number of publications concerning the stabilization of distributed system with fractional damping. In [46], B. Mbodje considered a $1-d$ wave equation with boundary fractional damping acting on a part of the boundary of the domain :

$$\left\{\begin{array}{rcll} \partial_t^2 u(x,t)-\partial_x^2 u(x,t) & = & 0, & 0<x<1, \quad t>0,\\ u(0,t) & = & 0. & \\ \partial_x u(1,t) & = & -\gamma\partial_t^{\alpha,\eta}u(1,t), & 0<\alpha<1, \quad \eta\geq 0,\\ u(x,0) & = & u_0(x), & \\ \partial_t u(x,0) & = & v_0(x), & \end{array}\right. \tag{4.1.14}$$

Firstly, he proved that system (4.1.14) is not uniformly stable, on other words its energy has no exponential decay rate. However, using LaSalle's invariance principle, he proved that system (4.1.14) is strongly stable for usual initial data. Secondly, he established a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. In [3], we considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain :

$$\left\{\begin{array}{lcll} u_{tt}-\Delta u & = & 0, & \text{in} \quad \Omega\times\mathbb{R}^+,\\ u & = & 0, & \text{on} \quad \Gamma_0\times\mathbb{R}^+,\\ \dfrac{\partial u}{\partial\nu}+\gamma\partial_t^{\alpha,\eta} & = & 0, & \text{on} \quad \Gamma_1\times\mathbb{R}^+,\\ u(x,0) & = & u_0(x) & \text{in} \quad \Omega,\\ u_t(x,0) & = & u_1(x), & \text{in} \quad \Omega. \end{array}\right. \tag{4.1.15}$$

Firstly, combining a general criteria of Arendt and Batty with Holmgren's theorem we showed the strong stability of system(4.1.15) in the absence of the compactness of the resolvent and without any additional geometric conditions. Next, we show that our system is not uniformly stable in general, since it is the case of the interval. Hence, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method. Indeed, by assuming that the boundary control region satisfy the Geometric Control Condition **(GCC)** and by using the exponential decay of the wave equation with a standard damping

$$\partial_\nu u(x,t)+u_t(x,t)=0, \quad \text{on} \quad \gamma_1\times\mathbb{R}_+^*$$

we established a polynomial energy decay rate for smooth solutions, which depends on the order of the fractional derivative. In [23], Zhang and Dai considered the multidimensional wave equation with boundary source term and fractional dissipation defined by

$$\begin{cases} u_{tt}-\Delta u &=& 0, & x\in\Omega & t>0,\\ u &=& 0, & x\in\Gamma_0 & t>0,\\ \dfrac{\partial u}{\partial\mu}+\partial_t^{\alpha}u &=& |u|^{m-1}u, & x\in\Gamma_1 & t>0,\\ u(x,0) &=& u_0, & x\in\Omega, & \\ u_t(x,0) &=& u_1(x), & x\in\Omega & \end{cases} \tag{4.1.16}$$

where $m>1$. They proved by Fourrier transforms and the Hardy-Littelwood-Sobolev inequality the exponential stability for sufficiently large initial data.

In [2], Benaissa and al. considered the Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type defined by

$$\begin{cases} \varphi_{tt}(x,t)+\varphi_{xxxx}(x,t) &=& 0, & \text{in} &]0,L[\times]0,+\infty[,\\ \varphi(0,t)=\varphi_x(0,t) &=& 0, & \text{in} &]0,+\infty[,\\ \varphi_{xx}(L,t) &=& 0, & \text{in} &]0,+\infty,\\ \varphi_{xxx}(L,t) &=& \gamma\partial_t^{\alpha,\eta}\varphi(L,t), & \text{in} &]0,+\infty[\end{cases} \tag{4.1.17}$$

where $0<\alpha<1$, $\eta\geq 0$ and $\gamma>0$. They proved, under the condition $\eta=0$, by a spectral analysis, the non uniform stability. On the other hand, for $\eta>0$, they also proved that the energy of system (4.1.17) decay as time goes to infinity as $\frac{1}{t^{\frac{1}{1-\alpha}}}$.

In [4] see also ([5]-[6]), Alabau-Boussouira studied the boundary indirect stabilization of a system of two level second order evolution equations coupled through the zero order terms. The lack of uniform stability as proved in the case where the ratio of the wave propagation speeds of the two equation is equal to $\frac{1}{k^2}$ with k being an integer and Ω os a cubic domain in $\mathbb{R}^3$, or by a compact perturbation argument and a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ is obtained by a general integral inequality in the case where the wave propagates at the same speed and Ω is a star-shaped domain in $\mathbb{R}^N$. These results are very interesting but not optimal.

In [8], Ammari and Mehrenberger, gave a characterization of the stability of a system of two evolution equations coupling through the velocity terms subject one bounded viscous feedbacks.

In [41] Liu and Rao, considered a system of two coupled wave equations with one boundary damping described by

$$\begin{cases} u_{tt}-a\Delta u+\alpha y &=& 0, & \text{in} & \Omega\times\mathbb{R}^+,\\ y_{tt}-\Delta y+\alpha u &=& 0, & \text{in} & \Omega\times\mathbb{R}^+,\\ a\partial_\nu u+\gamma u+u_t &=& 0, & \text{on} & \Gamma_1\times\mathbb{R}^+,\\ u &=& 0, & \text{on} & \Gamma_0\times\mathbb{R}^+,\\ y &=& 0, & \text{on} & \Gamma\times\mathbb{R}^+ \end{cases} \tag{4.1.18}$$

They established that, under some arithmetic condition on the ratio of the wave propagation speeds of the two equations, a polynomial energy decay rate for smooth initial data on a 1-dimensional domain. Furthermore, under the equality speed wave propagation, they proved that the energy of the system (4.1.18) decays at the rate $\frac{1}{t}$ for smooth initial data on a $N-$dimensional domain Ω with usual geometrical condition.

In [7] Ammar-Khodja and Bader studied the simultaneous boundary stabilization of a system of two wave equations coupling through the velocity terms described by by

$$\begin{cases} u_{tt} - u_{xx} + b(x)y_t & = 0, \quad \text{in} \quad (0,1) \times (0,+\infty), \\ y_{tt} - ay_{xx} - b(x)u_t & = 0, \quad \text{in} \quad (0,1) \times (0,+\infty), \\ y_t(0,t) - \alpha\left(y_x(0,t) + u_t(0,t)\right) & = 0, \quad \text{in} \quad (0,+\infty), \\ u_x(0,t) - \alpha y_t(0,t) & = 0, \quad \text{in} \quad (0,+\infty), \\ u(1,t) = y(1,t) & = 0, \quad \text{in} \quad (0,+\infty) \end{cases} \tag{4.1.19}$$

where a and α are two constants strictly positives and $b \in \mathcal{C}^0([0,1])$. They proved, in the general case, when $a \neq 1$, the system (4.1.19) is uniformly stable if and only if it is strongly stable and there exists $p, q \in \mathbb{Z}$ such that $a = \frac{(2p+1)^2}{q^2}$. Moreover, under the equal speed wave propagation condition i.e. $(a = 1)$, They proved that, system (4.1.19) is uniformly stable if and only if it is strongly stable and the coupling parameter $b(x)$ verifies that $\int_0^1 b(x)dx \neq \frac{(2k+1)\pi}{2}$ for any $k \in \mathbb{Z}$. Note that, system (4.1.19) is damped by two related boundary controls.
In [48] and [61], Najdi and Wehbe considered a $1-d$ coupled wave equations on its indirect boundary stabilization defined by :

$$\begin{cases} u_{tt} - u_{xx} + by_t & = 0, \quad \text{in} \quad (0,1) \times \mathbb{R}_+^*, \\ y_{tt} - ay_{xx} - bu_t & = 0, \quad \text{in} \quad (0,1) \times \mathbb{R}_+^*, \\ y_x(0,t) - y_t(0,t) & = 0, \quad \text{in} \quad \mathbb{R}_+^*, \\ u(1,t) = y(1,t) = u(0,t) & = 0, \quad \text{in} \quad \mathbb{R}_+^*, \end{cases} \tag{4.1.20}$$

where $a > 0$ and $b \in \mathbb{R}^*$ are constants. Firstly, they proved that system (4.1.20) is strongly stable if and only if the coupling parameter b is outside a discrete set S of exceptional values. Next, for $b \notin S$, they proved that the energy decay rate of system (4.1.20) is greatly influenced by the nature of the coupling parameter b (an additional condition on b) and by the arithmetic property of the ratio of the wave propagation speeds a.
In [59], Toufayli considered a multidimensional system of coupled wave equations to on boun-

dary feedbacks described by

$$\begin{cases} u_{tt} - \Delta u + b y_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ y_{tt} - a\Delta y - b u_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \partial_\nu y - y_t &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ u &= 0, \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ y &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+. \end{cases} \tag{4.1.21}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain of class C^2, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, ν is the unit normal vector to Γ_1 pointing toward the exterior of Ω, $a > 0$ and $b \in \mathbb{R}^*$ are constants. Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, she established an exponential energy decay rate.
The polynomial energy decay rate occurs in many control problems where the open-loop systems are strongly stable, but not exponentially stable (see [30]). We quote [35], [36] for wave equations with local internal or boundary damping, [14] and [40]for abstract system, [56], [62] for systems of coupled wave-heat equations.

This chapter is organized as follows : In Subsection 4.2.1, we reformulate the system (4.1.1)-(4.1.7) into an augmented model system by coupling the wave equation with a suitable equation and we prove the well-posedness of our system by semigroup approach. In the subsection 2.2.2, Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, using a general criteria of Arendt and Batty theorem, we show that the strong stability of our system for in the absence of the compactness of the resolvent and under the multiplier control condition noted by **(MGC)**. In Section 4.3, under the equal speed wave propagation and the coupling parameter b verify another condition, we look for a polynomial decay rate for smooth initial data for our system by applying a frequency domain approach combining with a multiplier method. Indeed, by assuming that the boundary control region satisfy the Multiplier Geometric Control Condition **(MGC)**, we establish a polynomial energy decay for smooth solution of type $\frac{1}{t^{\frac{1}{1-\alpha}}}$.

4.2 Well-Posedness and Strong Stability

In this section, we will study the strong stability of system (4.1.1)-(4.1.7) in the absence of the compactness of the resolvent and by the (MGC) condition defined in Definition **??**. First, we will study the existence, uniqueness and regularity of the system of our system.

4.2.1 Augmented model and well-Posedness

Firstly, we reformulate system (4.1.1)-(4.1.7). For this aim, we use Theorem 1.6.1, system (4.1.1)-(4.1.7) may be recast into the following model :

$$\begin{aligned} u_{tt} - \Delta u + b y_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, & (4.2.1)\\ y_{tt} - a\Delta y - b u_t &= 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, & (4.2.2)\\ \partial_t \omega(\xi,t) + \left(|\xi|^2 + \eta\right)\omega(\xi,t) - \mu(\xi)\partial_t u(x,t) &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+,\ \xi \in \mathbb{R}^d, & (4.2.3)\\ u &= 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, & (4.2.4)\\ y &= 0, \quad \text{on} \quad \Gamma \times \mathbb{R}^+, & (4.2.5)\\ \frac{\partial u}{\partial \nu} + \gamma\kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi,t)d\xi &= 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+ & (4.2.6) \end{aligned}$$

where γ is a positive constant, $\eta \geq 0$ and $\kappa = \frac{2\sin(\alpha\pi)\Gamma\left(\frac{d}{2}+1\right)}{d\pi^{\frac{d}{2}+1}}$. Finally, system (4.2.1)-(4.2.6) is considered with the following initial conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad &\text{where} \quad x \in \Omega, & (4.2.7)\\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) \quad &\text{where} \quad x \in \Omega, & (4.2.8)\\ \omega(\xi,0) = 0 \qquad &\text{where} \quad \xi \in \mathbb{R}^d. & (4.2.9) \end{aligned}$$

Our main interest is the existence, uniqueness and regularity of the solution of this system. We define the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}^d), \tag{4.2.10}$$

equipped with following inner product

$$\left\langle U, \widetilde{U} \right\rangle_{\mathcal{H}} = \int_\Omega \left(v\bar{\tilde{v}} + \nabla u \nabla \bar{\tilde{u}} + z\bar{\tilde{z}} + a\nabla y \nabla \bar{\tilde{y}} \right) dx + \tilde{\gamma}\kappa \int_{\mathbb{R}^d} \omega(\xi)\bar{\tilde{\omega}}(\xi)d\xi$$

where $\tilde{\gamma} = \gamma\,|\Gamma_1|$ and $U = (u,v,y,z,\omega), \widetilde{U} = (\tilde{u},\tilde{v},\tilde{y},\tilde{z},\tilde{\omega}) \in \mathcal{H}$ and $H^1_{\Gamma_0}(\Omega)$ is given by

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega), \quad u = 0 \ \text{ on } \ \Gamma_0 \right\}.$$

The energy of the solution of system (4.2.1)-(4.2.9) is defined by :

$$E(t) = \frac{1}{2}\left\|(u,u_t,y,y_t,\omega)\right\|^2_{\mathcal{H}}. \tag{4.2.11}$$

For smooth solution, a direct computation gives

$$E'(t) = -\tilde{\gamma}\kappa \int_{\mathbb{R}^d} \left(|\xi|^2 + \eta\right) |\omega(\xi,t)|^2 d\xi. \tag{4.2.12}$$

Then, system (4.2.1)-(4.2.9) is dissipative in the sense that its energy is a non-increasing function of the time variable t. Now, we define the linear unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u,v,y,z,\omega)^\top \in \mathcal{H};\ \Delta u \in L^2(\Omega), \quad y \in H^2(\Omega) \cap H_0^1(\Omega), \\ v \in H^1_{\Gamma_0}(\Omega), \quad z \in H_0^1(\Omega), \quad -(|\xi|^2+\eta)\omega + v|_{\Gamma_1}\mu(\xi) \in L^2(\mathbb{R}^d), \\ \dfrac{\partial u}{\partial \nu} + \gamma\kappa \displaystyle\int_{\mathbb{R}^d} \mu(\xi)\omega(\xi)d\xi = 0 \ \text{ on } \ \Gamma_1, \quad |\xi|\omega \in L^2(\mathbb{R}^d) \end{array} \right\} \tag{4.2.13}$$

and

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - bz \\ z \\ a\Delta y + bv \\ -(|\xi|^2+\eta)\omega + v|_{\Gamma_1}\mu(\xi). \end{pmatrix}$$

By denoting $v = u_t$ and $z = y_t$ and $U_0 = (u_0, v_0, y_0, z_0, \omega_0)^\top$, system (4.2.1)-(4.2.9) can be written as an abstract linear evolution equation on the space $\mathcal{H}$

$$\left\{ \begin{array}{l} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{array} \right. \tag{4.2.14}$$

Proposition 4.2.1. *For $\eta \geq 0$, the unbounded linear operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof: For all $U = (u,v,y,z,\omega)^\top \in D(\mathcal{A})$, we have

$$\Re\left(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}\right) = -\tilde{\gamma}\kappa \int_{\mathbb{R}^d} (|\xi|^2+\eta)|\omega(\xi,t)|^2 d\xi \leq 0. \tag{4.2.15}$$

which implies that $\mathcal{A}$ is dissipative in the sense that its energy is a non-increasing function of the time variable t. Now, let $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, we prove the existence of $U = (u,v,y,z,\omega)^\top \in D(\mathcal{A})$, unique solution of the equation

$$(I - \mathcal{A})\, U = F. \tag{4.2.16}$$

Equivalently, we have the following system

$$
\begin{aligned}
u-v &= f_1, && (4.2.17)\\
v-\Delta u+bz &= f_2, && (4.2.18)\\
y-z &= f_3, && (4.2.19)\\
z-(a\Delta y+bv) &= f_4, && (4.2.20)\\
\omega+(|\xi|^2+\eta)\omega-v|_{\Gamma_1}\mu(\xi) &= f_5(\xi). && (4.2.21)
\end{aligned}
$$

By equation (4.2.21), we get

$$
\omega(\xi)=\frac{f_5(\xi)+v|_{\Gamma_1}\mu(\xi)}{|\xi|^2+\eta+1}. \tag{4.2.22}
$$

From equations (4.2.17) and (4.2.19), we have

$$
v=u-f_1 \quad \text{and} \quad z=y-f_3. \tag{4.2.23}
$$

Using equation (4.2.23) in equations (4.2.18) and (4.2.20), we get the following problem

$$
\begin{aligned}
u-\Delta u+by &= f_1+f_2+bf_3, && (4.2.24)\\
y-a\Delta y-bu &= f_3+f_4-bf_1 && (4.2.25)
\end{aligned}
$$

with the boundary conditions

$$
u=0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial u}{\partial \nu}=-\gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi,t)d\xi \quad \text{on} \quad \Gamma_1, \quad y=0 \quad \text{on} \quad \Gamma. \tag{4.2.26}
$$

Using equations (4.2.22) and (4.2.23), we get

$$
\omega(\xi)=\frac{f_5(\xi)}{|\xi|^2+\eta+1}+\frac{u|_{\Gamma_1}\mu(\xi)}{|\xi|^2+\eta+1}-\frac{f_1\mu(\xi)}{|\xi|^2+\eta+1}. \tag{4.2.27}
$$

Let $\phi=(\varphi_1,\varphi_2)\in H^1_{\Gamma_0}(\Omega)\times H^1_0(\Omega)$ test function. Multiply equations (4.2.24) and (4.2.25) by φ_1 and φ_2 respectively, we obtain

$$
\begin{aligned}
\int_\Omega u\varphi_1dx+\int_\Omega\nabla u\nabla\varphi_1dx-\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\varphi_1d\Gamma+b\int_\Omega y\varphi_1dx &= \int_\Omega F_1\varphi_1dx. && (4.2.28)\\
\int_\Omega y\varphi_2dx+a\int_\Omega\nabla y\nabla\varphi_2dx-b\int_\Omega u\varphi_2dx &= \int_\Omega F_2\varphi_2dx. && (4.2.29)
\end{aligned}
$$

where $F_1=f_1+f_2+bf_3$ and $F_2=f_3+f_4-bf_1$. Using equations (4.2.6) and (4.2.27), we get

$$
-\int_{\Gamma_1}\frac{\partial u}{\partial\nu}\varphi_1d\Gamma=M_1\int_{\Gamma_1}\varphi_1d\Gamma+M_2\int_{\Gamma_1}u\varphi_1d\Gamma-M_2\int_{\Gamma_1}f_1\varphi_1d\Gamma. \tag{4.2.30}
$$

where,

$$M = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu(\xi) f_5(\xi)}{|\xi|^2 + \eta + 1} d\xi, \quad \text{and} \quad N = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{|\xi|^2 + \eta + 1} d\xi. \tag{4.2.31}$$

using the fact $f_5(\xi) \in L^2(\mathbb{R}^d)$ and for $\alpha \in]0,1[$ it easy to check that $M, N < +\infty$. Adding equations (4.2.28) and (4.2.29), we obtain

$$a((u,y),(\varphi_1,\varphi_2)) = L(\varphi_1,\varphi_2), \quad \forall(\varphi_1,\varphi_2) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega) \tag{4.2.32}$$

where

$$\begin{aligned} a((u,y),(\varphi_1,\varphi_2)) &= \int_\Omega u\varphi_1 dx \int_\Omega \nabla u \nabla \varphi_1 dx + \int_\omega y\varphi_2 dx + a\int_\Omega \nabla y \nabla \varphi_2 dx \\ &\quad + M_2 \int_{\Gamma_1} u\varphi_1 d\Gamma + b\int_\Omega y\varphi_1 dx - b\int_\Omega u\varphi_2 dx. \end{aligned} \tag{4.2.33}$$

and

$$L(\varphi_1,\varphi_2) = \int_\Omega F_1\varphi_1 dx + \int_\Omega F_2\varphi_2 dx - M_1\int_{\Gamma_1} \varphi_1 d\Gamma + M_2 \int_{\Gamma_1} f_1\varphi_1 d\Gamma. \tag{4.2.34}$$

Using Lax-Milgram, we deduce that their exists $(u,y) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$ unique solution of the variationnel problem (4.2.32), using the fact $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and the regularity of the Laplaciana we deduce that $(u,v,y,z,\omega) \in D(\mathcal{A})$.

From proposition 4.2.1, we have the operator $\mathcal{A}$ is maximal on $\mathcal{H}$ and consequently, generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ following Lummer-Philipps theorem (see [42] and [51]). Then, the solution of the evolution equation (4.2.14) admits the following representationL

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (4.2.14). Hence, semi-group theory allows to show the next existence and uniqueness results :

Theorem 4.2.2. *For any $U_0 \in \mathcal{H}$, problem (4.2.14) admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

4.2.2 Strong Stability

In this subsection, we study the strong stability of system (4.2.1)-(4.2.9) in the sense that its energy converges to zero when t goes to infinity for all initial data in $\mathcal{H}$ under the condition where $a = 1$ and b small enough. It easy to see that the resolvent of $\mathcal{A}$ is not compact, then the classical methods such as Lasalle's invariance principle or the spectrum decomposition theory on Benchimol are not applicable in this case. We use then a general criteria of Arendt-Batty, following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements. We will prove the following stability result

Theorem 4.2.3. *Suppose that $\eta \geq 0$, $a = 1$ and b small enough, then the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, i.e, for all $U_0 \in \mathcal{H}$, we have*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 4.2.3, we need the following lemmas.

Lemma 4.2.4. *Assume that $\eta \geq 0$, $a = 1$ and b small enough. Then, for all $\lambda \in \mathbb{R}$, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}$$

.

Proof: Let $U \in D(\mathcal{A})$ and let $\lambda \in \mathbb{R}$, such that

$$\mathcal{A}U = i\lambda U. \tag{4.2.35}$$

Equivalently, we have

$$\begin{aligned} v &= i\lambda u, && (4.2.36)\\ \Delta u - bz &= i\lambda v, && (4.2.37)\\ z &= i\lambda y, && (4.2.38)\\ \Delta y + bv &= i\lambda z, && (4.2.39)\\ -\left(|\xi|^2 + \eta\right)\omega + v|_{\Gamma_1}\mu(\xi) &= i\lambda\omega. && (4.2.40) \end{aligned}$$

Next, a straightforward computation gives

$$\Re\left(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}\right) = -\gamma\kappa \int_{\mathbb{R}^d} \left(|\xi|^2 + \eta\right)|\omega|^2 d\xi. \tag{4.2.41}$$

Then using equation (4.2.35) and (4.2.41) we deduce that

$$\omega = 0 \quad \text{a.e. in} \quad \mathbb{R}^d. \tag{4.2.42}$$

It follows, from (4.2.13) and (4.2.40), that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad v = 0 \quad \text{on} \quad \Gamma_1. \tag{4.2.43}$$

If $\lambda = 0$, then $v = z = 0$, then we obtain the following systems

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega \times \mathbb{R}^+, \\ u &= 0 \text{ on } \Gamma_0 \times \mathbb{R}^+, \\ \dfrac{\partial u}{\partial \nu} &= 0 \text{ on } \Gamma_1 \times \mathbb{R}^+, \end{cases} \tag{4.2.44}$$

and

$$\begin{cases} \Delta y &= 0 \text{ in } \Omega \times \mathbb{R}^+, \\ y &= 0 \text{ on } \Gamma \times \mathbb{R}^+. \end{cases} \tag{4.2.45}$$

It clear that problem (4.2.44) and (4.2.45) have $u = 0$ and $y = 0$ unique solution respectively, then $U = 0$, Which contradict the hypothesis $U \neq 0$.
If $\lambda \neq 0$, using equation (4.2.43) and (4.2.36), we get

$$u = 0 \quad \text{on} \quad \Gamma_1. \tag{4.2.46}$$

Eliminating v and z in equations (4.2.36) and (4.2.38) in equations (4.2.37) and (4.2.39), we obtain the following system

$$\lambda^2 u + \Delta u - i\lambda b y = 0 \quad \text{in} \quad \Omega, \tag{4.2.47}$$

$$\lambda^2 y + \Delta y + i\lambda b u = 0 \quad \text{in} \quad \Omega, \tag{4.2.48}$$

$$u = 0 \quad \text{on} \quad \Gamma \tag{4.2.49}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1, \tag{4.2.50}$$

$$y = 0 \quad \text{on} \quad \Gamma. \tag{4.2.51}$$

We divide the proof into several steps.
Step 1. Multiplying equations (4.2.47) and (4.2.48) by $\bar{y}$ and $\bar{u}$ respectively, using green formula and the boundary conditions, we get

$$\int_\Omega \lambda^2 u\bar{y}dx - \int_\Omega \nabla u \cdot \nabla \bar{y}dx - i\int_\Omega \lambda b|y|^2 dx = 0, \tag{4.2.52}$$

$$\int_\Omega \lambda^2 y\bar{u}dx - \int_\Omega \nabla y \cdot \nabla \bar{u}dx + i\int_\Omega \lambda b|u|^2 dx = 0. \tag{4.2.53}$$

Adding equations (4.2.52) and (4.2.53) and tacking the imaginary part, we obtain

$$\int_\Omega |u|^2 dx = \int_\Omega |y|^2 dx. \tag{4.2.54}$$

Step 2. Multiplying equation (4.2.47) by $\bar{u}$, using green formula and the boundary conditions, we get

$$\int_\Omega \lambda^2 |u|^2 dx - \int_\Omega |\nabla u|^2 dx = -\Re\left(-i\int_\Omega \lambda b y \bar{u} dx\right). \tag{4.2.55}$$

Step 3. Multiplying equation (4.2.47) by $2(m\cdot\nabla\bar{u})$, we get

$$2\int_\Omega \lambda^2 u(m\cdot\nabla\bar{u})dx + 2\int_\Omega \Delta u(m\cdot\nabla\bar{u})dx = 2i\int_\Omega \lambda b y(m\cdot\nabla\bar{u})dx. \tag{4.2.56}$$

$U\in D(\mathcal{A})$, then the regularity is sufficiently for applying an integration on the second integral in the left hand said in equation (4.2.56). Then we obtain

$$2\int_\Omega \Delta u(m\cdot\nabla\bar{u})dx = -2\int_\Omega \nabla u\cdot\nabla\left(m\cdot\nabla\bar{u}\right)dx + 2\int_\Gamma \frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma. \tag{4.2.57}$$

Using the green formula, we get

$$-2\Re\left(\int_\Omega \nabla u\cdot\nabla(m\cdot\nabla\bar{u})dx\right) = (N-2)\int_\Omega |\nabla u|^2 dx - \int_\Gamma (m\cdot\nu)|\nabla u|^2 dx. \tag{4.2.58}$$

Inserting equation (4.2.58) in equation (4.2.57) and using equations (4.2.49) and (4.2.50), we get

$$2\Re\left(\int_\Omega \Delta u(m\cdot\nabla\bar{u})dx\right) = (N-2)\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma. \tag{4.2.59}$$

In the another hand, it easy to see that

$$2\int_\Omega \lambda^2 u(m\cdot\nabla\bar{u})dx = -d\lambda^2\int_\Omega |u|^2 dx. \tag{4.2.60}$$

Inserting equations (4.2.59) and (4.2.60) in equation (4.2.56), we get

$$d\lambda^2\int_\Omega |u|^2 dx + (2-d)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma = -2\Re\left(\int_\Omega i\lambda b y(m\cdot\nabla\bar{u})\right)dx. \tag{4.2.61}$$

Multiplying equations (4.2.55) by $1-N$, and tacking the sum of this equation and (4.2.61), we get

$$\lambda^2\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx - \int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma = \Im\left(\lambda b\int_\Omega y((d-1)\bar{u} + 2(m\cdot\nabla\bar{u}))dx\right). \tag{4.2.62}$$

Using Cauchy-Shwartz inequality in the right hand side equation (4.2.62), then for $\varepsilon>0$, we get

$$\lambda^2\int_\Omega |u|^2dx+\int_\Omega|\nabla u|^2dx-\int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma\leq (d-1)|\lambda||b|\left(\int_\Omega\frac{|y|^2}{2\varepsilon}dx\right)^{\frac{1}{2}}\left(\int_\Omega 2\varepsilon|u|^2dx\right)^{\frac{1}{2}}$$
$$+2\|m\|_\infty|\lambda||b|\left(\int_\Omega\frac{|y|^2}{\varepsilon}dx\right)^{\frac{1}{2}}\left(\int_\Omega\varepsilon|\nabla u|^2dx\right)^{\frac{1}{2}}. \quad (4.2.63)$$

Using Young and Poincaré inequality, we get

$$\lambda^2\int_\Omega |u|^2dx+\int_\Omega|\nabla u|^2dx-\int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2\leq\left(\frac{(d-1)^2|\lambda|^2|b|^2}{4\varepsilon}+\frac{\|m\|_\infty|\lambda|^2|b|^2}{\varepsilon}\right)\int_\Omega|y|^2dx$$
$$+\varepsilon(1+C)\int_\Omega|\nabla u|^2dx \quad (4.2.64)$$

where $C=\frac{1}{\alpha}$ and α is the smallest eigen value of $-\Delta$ in $H_0^1(\Omega)$. Now, using the geometric condition (**??**) and equation (4.2.54), we get

$$\lambda^2\int_\Omega |u|^2dx+\int_\Omega|\nabla u|^2dx\leq\lambda^2b^2\left(\frac{(d-1)^2}{4\varepsilon}+\frac{\|m\|_\infty^2}{\varepsilon}\right)\int_\Omega|y|^2dx+\varepsilon(1+C)\int_\Omega|\nabla u|^2dx. \quad (4.2.65)$$

Tacking $\varepsilon=\frac{1}{1+C}$ in equation (4.2.65) and using equation (4.2.54), we get

$$\int_\Omega|u|^2dx\leq b^2(1+C)\left(\frac{(d-1)^2}{4}+\|m\|_\infty^2\right)\int_\Omega|u|^2dx. \quad (4.2.66)$$

Using the fact b is small enough, we obtain

$$\int_\Omega|u|^2dx=0. \quad (4.2.67)$$

such that b satisfy

$$1-b^2(1+C)\left(\frac{(d-1)^2}{4}+\|m\|_\infty^2\right)>0. \quad (4.2.68)$$

Finally, combining equations (4.2.54) and (4.2.67), we deduce that

$$u=y=0, \quad (4.2.69)$$

using equation (4.2.36), (4.2.38) and equation (4.2.69), we obtain

$$v=z=0. \quad (4.2.70)$$

Consequently, using equation (4.2.41), (4.2.69) and (4.2.70), we obtain $U=0$, which contradict the hypothesis $U \neq 0$. The proof has been completed.

Lemma 4.2.5. *Assume that $\eta = 0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0 \in \sigma(\mathcal{A})$.*

Proof: First, let $\varphi_k \in H^1_{\Gamma_0}(\Omega)$ and $\psi_k \in H^1_0(\Omega)$ be an eigenfunctions respectively of the following problems

$$\begin{cases} -\Delta\varphi_k & = & \mu_k^2\varphi_k & \text{in} & \Omega, \\ \varphi_k & = & 0, & \text{on} & \Gamma_0, \\ \frac{\partial\varphi_k}{\partial\nu} & = & 0 & \text{on} & \Gamma_1 \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\psi_k & = & 0 & \text{in} & \Omega, \\ \varphi_k & = & 0, & \text{on} & \Gamma_0, \\ \frac{\partial\varphi_k}{\partial\nu} & = & 0 & \text{on} & \Gamma_1. \end{cases} \tag{4.2.71}$$

such that

$$\|\varphi_k\|_{H^1_{\Gamma_0}(\Omega)} = \int_\Omega |\nabla\varphi_k|^2 dx.$$

Next, define the vector $F = (\varphi_k, 0, 0, 0, 0) \in \mathcal{H}$. Assume that there exists $U = (u, v, y, z, \omega) \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = F.$$

It follows that

$$\begin{cases} v = -\varphi_k, & \text{in} & \Omega \\ z = 0 & \text{in} & \Omega, \\ -|\xi|^2\omega + \mu(\xi)v = 0 & \text{on} & \Gamma_1, \end{cases} \tag{4.2.72}$$

and

$$\begin{cases} \Delta u & = & 0, & \text{in} & \Omega, \\ \Delta y & = & 0, & \text{in} & \Omega, \\ u & = & 0, & \text{on} & \Gamma_0, \\ \dfrac{\partial u}{\partial\nu} + \gamma\kappa\displaystyle\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi)d\xi & = & 0, & \text{on} & \Gamma_1, \\ y & = & 0, & \text{on} & \Gamma. \end{cases} \tag{4.2.73}$$

From (4.2.72), we deduce that $\omega(\xi) = |\xi|^{\frac{2\alpha-d-4}{2}}\varphi_k|_{\Gamma_1}$. We easily check that, for $\alpha \in]0,1[$, the function $\omega(\xi) \notin L^2(\mathbb{R}^d)$. So, the assumption of the existence of U is false and consequently the operator $-\mathcal{A}$ is not invertible.

Lemma 4.2.6. *Assume that ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0, \lambda \in \mathbb{R}^*$). Then, for any $f \in L^2(\Omega)$,* *the following problem*

$$\begin{cases} \lambda^2 u + \Delta u - i\lambda b y & = h \quad in \quad \Omega, \\ \lambda^2 y + \Delta y + i\lambda b u & = g \quad in \quad \Omega, \\ u & = 0 \quad on \quad \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right) u & = 0 \quad on \quad \Gamma_1 \\ y & = 0 \quad on \quad \Gamma. \end{cases} \tag{4.2.74}$$

where

$$c_1 = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu^2(\xi)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi \quad and \quad c_2 = \gamma\kappa \int_{\mathbb{R}^d} \frac{\mu^2(\xi)\left(|\xi|^2 + \eta\right)}{\lambda^2 + (|\xi|^2 + \eta)^2} d\xi, \tag{4.2.75}$$

admits a unique solution $(u, y) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$.

Proof: First, it easy to check that, if ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), then, for $\alpha \in]0, 1[$ the coefficients $c_1(\lambda, \eta)$ and $c_2(\lambda, \eta)$ are well defined. Moreover, if $\eta > 0$ and $\lambda = 0$, then, using Lax-Milligram's theorem we deduce that system (4.2.74) admits a unique solution $(u, y) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$. Now, we assume that $\eta \geq 0$ and $\lambda' in \mathbb{R}^*$ and let us consider the following problem

$$\begin{cases} -\Delta u + i\lambda b y & = h \quad \text{in} \quad \Omega, \\ -\Delta y - i\lambda b u & = g \quad \text{in} \quad \Omega, \\ u & = 0 \quad \text{on} \quad \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right) u & = 0 \quad \text{on} \quad \Gamma_1 \\ y & = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{4.2.76}$$

Next, we give a variational formation of (4.2.76). For this aim, find $(u, y) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$, such that

$$a\left((u, y), (\varphi, \psi)\right) = L(\varphi, \psi) \quad \forall\, (\varphi, \psi) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega). \tag{4.2.77}$$

where

$$\begin{aligned} a\left((u, y), (\varphi, \psi)\right) &= \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla y \nabla \psi dx + \lambda^2 c_1 \int_{\Gamma_1} u\varphi d\Gamma \\ &\quad + i\lambda c_2 \int_{\Gamma_1} u\varphi d\Gamma + i\lambda b \left(\int_\Omega (y\varphi - u\psi)\, dx\right) \end{aligned}$$

and

$$L(\varphi, \psi) = \int_\Omega (h\varphi + g\psi)\, dx.$$

It is clear that a is bilinear, continuous and coercive on the space $\left(H^1_{\Gamma_0}(\Omega)\, H^1_0(\Omega)\right)^2$. Finally, by Lax-Milligram's theorem, the variational problem (4.2.77) admits a unique solution $(u,y) \in H^1_{\Gamma_0}(\Omega)\, H^1_0 \times (\Omega)$. We defined the operator $\mathcal{L}$, by

$$\mathcal{L}U = \begin{pmatrix} -\Delta u + i\lambda by \\ -\Delta y - i\lambda bu \end{pmatrix}, \quad \text{where } \; U = (u,y)^\top \tag{4.2.78}$$

Then $\mathcal{L}$ is an isomorphism from $H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$ into $H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$. Using the compactness embedding from $L^2(\Omega) \times L^2(\Omega)$ into $\left(H^1_{\Gamma_0}(\Omega)\right)' \times H^{-1}(\Omega)$ and $H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$, we deduce that $\mathcal{L}^{-1}$ is compact from $L^2(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times L^2(\Omega)$. Then applying $\mathcal{L}^{-1}$ to (4.2.74), we get

$$\left(\lambda^2 \mathcal{L}^{-1} - I\right) U = \mathcal{L}^{-1} f. \tag{4.2.79}$$

The same computation in Lemma 4.2.4 shows $\ker\left(\lambda^2 \mathcal{L}^{-1} - I\right) = \{0\}$ for b small enough. Then following Fredholm's alternative, the equation (4.2.79) admits a unique solution.

Lemma 4.2.7. *If $\lambda > 0$, for all $\lambda \in \mathbb{R}$, we have*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

while if $\eta = 0$ for all $\lambda \in \mathbb{R}^*$, we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof: We give the proof in the case $\eta > 0$, the proof of the second statement is fully similar. Let $\lambda \in \mathbb{R}$ and $F = (f_1, f_2, f_3, f_4, f_5)^\top \in \mathcal{H}$, then we look for $U = (u, v, y, z, \omega)^\top \in D(\mathcal{A})$ solution of

$$(i\lambda - \mathcal{A})U = F. \tag{4.2.80}$$

Equivalently, we have

$$\begin{cases} i\lambda u - v & = & f_1, & \text{in} & \Omega, \\ i\lambda v - \Delta u + bz & = & f_2, & \text{in} & \Omega, \\ i\lambda y - \Delta z & = & f_3, & \text{in} & \Omega, \\ i\lambda z - \Delta y - bv & = & f_4, & \text{in} & \Omega, \\ i\lambda \omega + (|\xi|^2 + \eta)\omega - v\mu(\xi) & = & f_5, & \text{on} & \Gamma_1 \end{cases}$$

as before, by eliminating v, z and ω from the above system and using the fact that

$$\partial_\nu u + \gamma\kappa \int_{\mathbb{R}^d} \mu(\xi)\omega(\xi)d\xi = 0 \quad \text{on} \quad \Gamma_1,$$

we get the following system :

$$\begin{cases} \lambda^2 u + \Delta u - i\lambda b y & = & -f_2 - i\lambda f_1 - b f_3 & \text{in} & \Omega, \\ \lambda^2 y + \Delta y + i\lambda b u & = & -f_4 - i\lambda f_3 + b f_1 & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \Gamma_0, \\ y & = & 0 & \text{on} & \Gamma, \\ \dfrac{\partial u}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right) u & = & -i\lambda c_1 f_1 + c_2 f_1 + I^1_{f_5} + I^2_{f_5} & \text{on} & \Gamma_1. \end{cases} \tag{4.2.81}$$

where c_1, c_2 is defined in equation (4.2.75) and $I^2_{f_5}, I^2_{f_5}$ are given by

$$I^1_{f_5}(\lambda,\eta) = i\lambda\gamma\kappa \int_{\mathbb{R}^d} \frac{f_5(\xi)\mu(\xi)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi \quad \text{and} \quad I^2_{f_5}(\lambda,\eta) = -\gamma\kappa \int_{\mathbb{R}^d} \frac{f_5(\xi)\mu(\xi)(|\xi|^2+\eta)}{\lambda^2 + (|\xi|^2+\eta)^2} d\xi.$$

It easy to check that, for $f_5 \in L^2(\mathbb{R}^d)$ and $\alpha \in]0,1[$, the integrals $I^1_{f_5}$ and $I^2_{f_5}$ are will defined. First, let $(\varphi_{f_5}, \psi_{f_5}) \in H_{\Gamma_0} \times H^1_0(\Omega)$ be defined by

$$\begin{cases} -\Delta \varphi_{f_5} + i\lambda b \psi_{f_5} & = & 0 & \text{in} & \Omega \\ -\Delta \psi_{f_5} - i\lambda b \psi_{f_5} & = & 0 & \text{in} & \Omega \\ \varphi_{f_5} & = & 0 & \text{on} & \Gamma_0, \\ \psi_{f_5} & = & 0 & \text{on} & \Gamma, \\ \dfrac{\partial \varphi_{f_5}}{\partial \nu} & = & I^1_{f_5} + I^2_{f_5} & \text{on} & \Gamma_1. \end{cases} \tag{4.2.82}$$

Then setting $\tilde{u} = u + \varphi_{f_5}$ and $\tilde{y} = y + \psi_{f_5}$ in (4.2.82), then we get

$$\begin{cases} \lambda^2 \tilde{u} + \Delta \tilde{u} - i\lambda b \tilde{y} & = & \lambda^2 \varphi_{f_5} - f_2 - i\lambda f_1 - b f_3 & \text{in} & \Omega, \\ \lambda^2 \tilde{y} + \Delta \tilde{y} + i\lambda b \tilde{u} & = & \lambda^2 \psi_{f_5} - f_4 - i\lambda f_3 + b f_1 & \text{in} & \Omega, \\ \tilde{u} & = & 0 & \text{on} & \Gamma_0, \\ \tilde{y} & = & 0 & \text{on} & \Gamma, \\ \dfrac{\partial \tilde{u}}{\partial \nu} + \left(\lambda^2 c_1 + i\lambda c_2\right) \tilde{u} & = & -i\lambda c_1 f_1 + c_2 f_1 + (\lambda^2 c_1 + i\lambda c_2)\, \varphi_{f_5} & \text{on} & \Gamma_1. \end{cases} \tag{4.2.83}$$

Next, let $\theta \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)$ such that

$$\theta = 0, \quad \frac{\partial \theta}{\partial \nu} = -i\lambda c_1 f_1 + c_2 f_1 + \left(\lambda^2 c_1 + i\lambda c_2\right) \varphi_{f_5} \quad \text{on} \quad '\Gamma_1. \tag{4.2.84}$$

Then setting $\chi = \tilde{u} - \theta$, we get

$$\begin{cases} \lambda^2\chi + \Delta\chi - i\lambda b\tilde{y} & = \lambda^2\varphi_{f_5} - \lambda^2\theta - \Delta\theta - f_2 - i\lambda f_1 - bf_3 & \text{in} & \Omega, \\ \lambda^2\tilde{y} + \Delta\tilde{y} + i\lambda b\chi & = \lambda^2\psi_{f_5} - i\lambda b\theta - f_4 - i\lambda f_3 + bf_1 & \text{in} & \Omega, \\ \chi & = 0 & \text{on} & \Gamma_0, \\ \dfrac{\partial\chi}{\partial\nu} + \left(\lambda^2 c_1 + i\lambda c_2\right)\chi & = 0 & \text{on} & \Gamma_1 \end{cases} \tag{4.2.85}$$

Using Lemma 4.2.6 problem (4.2.85) has a unique solution $\chi \in H^1_{\Gamma_0}(\Omega)$ and therefore problem (4.2.81) has a unique solution $(u, y) \in H^1_{\Gamma_0}(\Omega) \times H^1_0(\Omega)$. By defining $v = i\lambda u - f_1$, $z = i\lambda u - f_3$ in Ω and

$$\omega = \frac{f_5(\xi)}{i\lambda + |\xi|^2 + \eta} + \frac{i\lambda u_{\Gamma_1}\mu(\xi)}{i\lambda + |\xi|^2 + \eta} - \frac{f_1|_{\Gamma_1}\mu(\xi)}{i\lambda + |\xi|^2 + \eta}$$

we deduce that $U = (u, v, y, z, \omega)$ belongs to $D(\mathcal{A})$ and is solution of (4.2.80). This complete the proof.

Proof of Theorem 4.2.3. Following a general criteria of Arendt-Batty see ..., the C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, from Lemma 4.2.4 we directly deduce that $\mathcal{A}$ has non pure imaginary eigenvalues. Next, using Lemmas 4.2.5 and 4.2.7, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. The proof is thus completed.

4.3 Polynomial Stability

This section is devoted to the study of the polynomial stability of system (4.2.1)-(4.2.9) in the case $\eta > 0$, $a = 1, b$ small enough and under the **(MGC)** condition defined in Definition 2.3.3. For the purpose, we will use a frequency domain approach, namely we will use Theorem 2.4 of [18] (see also [14, 15, 40]) that we partially recall.

Theorem 4.3.1. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\ell > 0$ the following conditions are equivalent*

$$\|(is - A)^{-1}\| = O(|s|^\ell), \quad s \to \infty, \tag{4.3.1}$$

$$\|T(t)A^{-1}\| = O(t^{-1/\ell}), \quad t \to \infty. \tag{4.3.2}$$

As the condition $i\mathbb{R} \subset \rho(\mathcal{A})$ was already checked in Theorem 4.2.3, it remains to prove that condition (4.3.1) holds. This is made with the help of a multiplier method under the (**MGC**) condition defined in Definition 2.3.3. We define the two opens sets by

$$\begin{cases} \Gamma_0^\varepsilon &= \left\{x \in \Omega; \quad \inf_{y\in\Gamma_0} |x-y| \le \varepsilon\right\}, \\ \Gamma_1^\varepsilon &= \left\{x \in \Omega; \quad \inf_{y\in\Gamma_1} |x-y| \le \varepsilon\right\} \end{cases}$$

and we define the function θ by

$$\begin{cases} \theta \equiv 0 & \text{on} \quad \Gamma_0, \\ \theta \equiv 1 & \text{on} \quad \Gamma_1, \\ \theta(x) \in [0,1]. \end{cases} \tag{4.3.3}$$

Since $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, we can choose $\varepsilon > 0$ small enough such that

$$\Gamma_0 \subset \Gamma_0^\varepsilon \quad \text{and} \quad \Gamma_1 \subset \Gamma_1^\varepsilon.$$

Next, we present the main result of this section

Theorem 4.3.2. *Assume that $a = 1$, $\eta > 0$ and b small enough. Then, for all initial data $U_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of U_0, such that the energy of the strong solution U of (4.2.14), satisfies the following estimation*

$$E(t,U) \le C \frac{1}{t^{\frac{1}{1-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.3.4}$$

In particular, for $U_0 \in \mathcal{H}$, the energy converges to zero at t goes to infinity.

As announced in Theorem 4.1, by tacking $\ell = 2-2\alpha$, the polynomial energy decay (4.3.4) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{$H1$}$$

and

$$\sup_{|\lambda|\in\mathbb{R}} \frac{1}{|\lambda|^\ell} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty \tag{$H2$}$$

are satisfied. Condition $(H1)$ is already proved in Theorem 4.2.3. We will prove condition $(H2)$ using an argument of contradiction. For this purpose, suppose $(H2)$ is false, then there exists a real sequence (λ_n), with $|\lambda_n| \to +\infty$ and a sequence $(U^n) \subset D(\mathcal{A})$, verifying the following conditions

$$\|U^n\|_{\mathcal{H}} = \|(u^n, v^n, y^n, z^n, \omega^n)\|_{\mathcal{H}} = 1 \tag{4.3.5}$$

and

$$\lambda_n^\ell (i\lambda_n - \mathcal{A})U^n = (f_1^n, f_2^n, f_3^n, f_4^n, f_5^n) \to 0 \quad \text{in} \quad \mathcal{H}. \tag{4.3.6}$$

For the simplicity, we drop the index n. Detailing equation (4.3.6), we get

$$\begin{aligned} i\lambda u - v &= \frac{f_1}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad H^1_{\Gamma_0}(\Omega), && (4.3.7)\\ i\lambda v - \Delta u + bz &= \frac{f_2}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\Omega), && (4.3.8)\\ i\lambda y - z &= \frac{f_3}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad H^1_0(\Omega), && (4.3.9)\\ i\lambda z - \Delta y - bv &= \frac{f_4}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\Omega), && (4.3.10)\\ i\lambda\omega + (|\xi|^2+\eta)\omega - v|_{\Gamma_1}\mu(\xi) &= \frac{f_5}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d). && (4.3.11) \end{aligned}$$

Note U is uniformly bounded in $\mathcal{H}$. Then, taking the inner product of (4.3.6) with U in $\mathcal{H}$, we get

$$-\gamma\kappa\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2 d\xi = \Re\left((i\lambda I - \mathcal{A})U, U\right)_{\mathcal{H}} = \frac{o(1)}{\lambda^\ell}. \tag{4.3.12}$$

As below, by eliminating v and z from the above system, we get

$$\begin{aligned} \lambda^2 u + \Delta u - i\lambda b y &= -\frac{f_2}{\lambda^\ell} - \frac{i\lambda f_1}{\lambda^\ell} - \frac{b f_3}{\lambda^\ell}, && (4.3.13)\\ \lambda^2 y + \Delta y + i\lambda b u &= -\frac{f_4}{\lambda^\ell} - \frac{i\lambda f_3}{\lambda^\ell} + \frac{b f_1}{\lambda^\ell}. && (4.3.14) \end{aligned}$$

Lemma 4.3.3. *Assume that $\eta > 0$. Then the solution $(u, v, y, z, \omega) \in D(\mathcal{A})$ of (4.3.7)-(4.3.11) satisfies the following asymptotic behavior estimation*

$$\begin{aligned} \|u\|_{L^2(\Omega)} &= \frac{O(1)}{\lambda}, && (4.3.15)\\ \|y\|_{L^2(\Omega)} &= \frac{O(1)}{\lambda}, && (4.3.16)\\ \|\partial_\nu u\|_{L^2(\Gamma_1)} &= \frac{o(1)}{\lambda^{1-\alpha}}, && (4.3.17)\\ \|u\|_{L^2(\Gamma_1)} &= \frac{o(1)}{\lambda}. && (4.3.18) \end{aligned}$$

Proof: Using equations (4.3.6), (4.3.7) and (4.3.9), we deduce directly the estimations (4.3.15)-(4.3.16). Now, from the boundary condition

$$\partial_\nu u + \gamma\kappa\int_{\mathbb{R}^d}\mu(\xi)\omega(\xi)d\xi = 0 \quad \text{on} \quad \Gamma_1$$

we get

$$|\partial_\nu u| \leq \gamma\kappa\left(\int_{\mathbb{R}^d}\frac{\mu^2(\xi)}{|\xi|^2+\eta}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2 d\xi\right)^{\frac{1}{2}}. \tag{4.3.19}$$

Then, combining equation (4.3.12) and equations (4.3.19), we obtain the desired equation (4.3.17). Finally multiplying equation (4.3.11) by $(i\lambda+|\xi|^2+\eta)^{-1}\mu(\xi)$, integrating over $\mathbb{R}^d$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$A_1|v|_{\Gamma 1}| \leq A_2\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)|\omega|^2 d\xi\right)^{\frac{1}{2}}+\frac{1}{|\lambda|^{\ell}}\left(\int_{\mathbb{R}^d}|f_3(\xi)|^2 d\xi\right)^{\frac{1}{2}}, \tag{4.3.20}$$

where,

$$A_1=\int_{\mathbb{R}^d}\frac{|\mu(\xi)|^2}{(|\lambda|+|\xi|^2+\eta)}d\xi,\quad A_2=\left(\int_{\mathbb{R}^d}\frac{|\mu(\xi)|^2}{|\xi|^2+\eta}d\xi\right)^{\frac{1}{2}}\quad\text{and}\quad A_3=\left(\int_{\mathbb{R}^d}\frac{|\mu(\xi)|^2}{(|\lambda|+|\xi|^2+\eta)^2}d\xi\right)^{\frac{1}{2}}.$$

Using lemma 1.6.2 , we obtain

$$A_1=c(|\lambda|+\eta)^{\alpha-1}\quad\text{and}\quad A_3=\tilde{c}(|\lambda|+\eta)^{\frac{\alpha}{2}-1} \tag{4.3.21}$$

where c and $\tilde{c}$ are two positive constants. Inserting equation (4.3.12) and (4.3.21) in equation (4.3.20) and using the fact that $\ell=2-2\alpha$, we get

$$\|v\|_{L^2(\Gamma_1)}=o(1). \tag{4.3.22}$$

It follows, from (4.3.7), that equation (4.3.18) holds. The proof has been completed.

Lemma 4.3.4. *Assume that $\eta>0$. Then, the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of (4.3.7)-(4.3.11) satisfies the following asymptotic behavior estimation*

$$\|\partial_\nu y\|_{L^2(\Gamma_1)}=O(1)+\frac{o(1)}{\lambda^{1-\alpha}}. \tag{4.3.23}$$

Proof: Multiplying equation (4.3.14) by $2\theta(m\cdot\nabla\bar{y})$, we obtain

$$\begin{aligned}2\int_\Omega\lambda^2 y\theta(m\cdot\nabla\bar{y})dx+2\int_\Omega\Delta y\theta(m\cdot\nabla\bar{y})dx+2i\int_\Omega\lambda bu\theta(m\cdot\nabla\bar{y})dx=\\ -2\int_\Omega\theta\left(\frac{f_4}{\lambda^\ell}+\frac{i\lambda f_3}{\lambda^\ell}-\frac{bf_1}{\lambda^\ell}\right)(m\cdot\nabla\bar{y})dx.\end{aligned} \tag{4.3.24}$$

Firstly, using the facts that ∇y is bounded in $L^2(\Omega)$, $\|f_1\|_{H^1_{\Gamma_0}(\Omega)}=o(1)$ and $\|f_4\|_{L^2(\Omega)}=o(1)$, we get

$$-2\int_\Omega\theta\left(\frac{f_4}{\lambda^\ell}-b\frac{f_1}{\lambda^\ell}\right)(m\cdot\nabla\bar{y})\,dx=\frac{o(1)}{\lambda^\ell}. \tag{4.3.25}$$

On the other hand, using Green formula and the fact that $y=0$ on Γ for the second term of the right hand side of equation (4.3.24), we get

$$-2\int_{\Omega}\frac{i\lambda\theta f_3}{\lambda^{\ell}}(m\cdot\nabla\bar{y})dx = 2\int_{\Omega} i\frac{\lambda\bar{y}\cdot\nabla(\theta m f_3)}{\lambda^{\ell}}. \tag{4.3.26}$$

So, using equation (4.3.16) and the fact that $\|f_3\|_{H_0^1(\Omega)} = o(1)$, we obtain, from (4.3.26) that

$$-2\int_{\Omega}\frac{i\lambda\theta f_3}{\lambda^{\ell}}(m\cdot\nabla\bar{y})dx = \frac{o(1)}{\lambda^{\ell}}. \tag{4.3.27}$$

Secondly, using Green formula and the fact $y = 0$ on Γ, for the first term of the left hand side of equation (4.3.24), we get

$$2\int_{\Omega}\lambda^2 y\theta\left(m\cdot\nabla\bar{y}\right)dx = -\int_{\Omega}\left(d\theta + (m\cdot\nabla\theta)\right)|\lambda y|^2 dx. \tag{4.3.28}$$

Next, using the Green formula, for the second term of he left hand side of equation (4.3.24), we get

$$2\Re\left(\int_{\Omega}\Delta y\theta(m\cdot\nabla\bar{y})dx\right) = -2\Re\left(\int_{\Omega}\nabla y\cdot\nabla\left(\theta(m\cdot\nabla\bar{y})\right)dx\right) + 2\int_{\Gamma}\theta(\partial_{\nu}y)\left(m\cdot\nabla\bar{y}\right)d\Gamma. \tag{4.3.29}$$

Furthermore, using Green formula for the first term on right hand side of equation (4.3.29), we get

$$\left\{\begin{array}{rcl} -2\Re\left(\displaystyle\int_{\Omega}\nabla y\cdot\nabla\left(\theta(m\cdot\nabla\bar{y})\right)dx\right) &=& \displaystyle\int_{\Omega}(m\cdot\nabla\theta)|\nabla y|^2 dx + (d-2)\int_{\Omega}\theta|\nabla y|^2 dx \\ && \displaystyle -\int_{\Gamma}\theta(m\cdot\nu)|\nabla y|^2 d\Gamma \\ && \displaystyle -2\Re\left(\int_{\Omega}(\nabla y\cdot\nabla\theta)(m\cdot\nabla\bar{y})dx\right). \end{array}\right. \tag{4.3.30}$$

Then, combining equation (4.3.29), (4.3.30) and using the fact that $y = \frac{\partial y}{\partial \tau} = 0$ on Γ , we get

$$\left\{\begin{array}{rcl} 2\Re\left(\displaystyle\int_{\Omega}\Delta y\theta(m\cdot\nabla\bar{y})dx\right) &=& \displaystyle -2\Re\left(\int_{\Omega}(\nabla y\cdot\nabla\theta)(m\cdot\nabla\bar{y})dx\right) \\ && \displaystyle -(2-d)\int_{\Omega}\theta|\nabla y|^2 dx + \int_{\Gamma_1}\theta(m\cdot\nu)\left|\partial_{\nu}y\right|^2 d\Gamma_1 \\ && \displaystyle +\int_{\Omega}(m\cdot\nabla\theta)|\nabla y|^2 dx. \end{array}\right. \tag{4.3.31}$$

So, inserting equations (4.3.25), (4.3.27), (4.3.28) and (4.3.31) in equation (4.3.24), we obtain

$$\left\{\begin{array}{lcl}\displaystyle\int_{\Gamma_1}\theta(m\cdot\nu)\left|\partial_\nu y\right|^2 d\Gamma &=& \displaystyle\int_\Omega\left(d\theta+(m\cdot\nabla\theta)\right)|\lambda y|^2dx-(d-2)\int_\Omega\theta|\nabla y|^2dx\\ && \displaystyle+2\Re\left(\int_\Omega(\nabla y\cdot\nabla\theta)(m\cdot\nabla\bar{y})dx\right)-\int_\Omega(m\cdot\nabla\theta)|\nabla y|^2dx\\ && \displaystyle-2\Re\left(i\lambda\int_\Omega bu\theta(m\cdot\nabla\bar{y})dx\right)+\frac{o(1)}{\lambda^\ell}.\end{array}\right. \tag{4.3.32}$$

Finally, using equation (4.3.16), the fact that ∇y is bounded in $L^2(\Omega)$, $\theta=1$ on Γ_1 and $\ell=2-2\alpha$, we obtain the desired equation (4.3.23). The proof is thus complete.

Lemma 4.3.5. *Assume that $\eta>0$. Then the solution $(u,v,y,z,\omega)\in D(\mathcal{A})$ of (4.3.7)-(4.3.11) satisfies the following asymptotic behavior estimation*

$$\int_\Omega|\lambda u|^2dx-\int_\Omega|\lambda y|^2dx=o(1). \tag{4.3.33}$$

Proof:[proof] Multiplying equations (4.3.13) and (4.3.14) by $\lambda\bar{y}$ and $\lambda\bar{u}$ respectively, using Green formula, we obtain

$$\int_\Omega\lambda^3u\bar{y}dx-\lambda\int_\Omega\nabla u\nabla\bar{y}dx-ib\int_\Omega|\lambda y|^2dx=-\int_\Omega\left(\frac{f_2}{\lambda^\ell}+i\frac{\lambda f_1}{\lambda^\ell}+\frac{bf_3}{\lambda^\ell}\right)\lambda\bar{y}dx \tag{4.3.34}$$

and

$$\begin{array}{r}\displaystyle\int_\Omega\lambda^3y\bar{u}dx-\lambda\int_\Omega\nabla y\nabla\bar{u}dx+\lambda\int_{\Gamma_1}(\partial_\nu y)\bar{u}d\Gamma_1+ib\int_\Omega|\lambda u|^2dx=\\ \displaystyle-\int_\Omega\left(\frac{f_4}{\lambda^\ell}+\frac{i\lambda f_3}{\lambda^\ell}-\frac{bf_1}{\lambda^\ell}\right)\lambda\bar{u}dx.\end{array} \tag{4.3.35}$$

Firstly, using equations (4.3.15), (4.3.16) and the facts that $\|f_1\|_{H^1_{\Gamma_0}(\Omega)}=o(1)$, $\|f_2\|_{L^2(\Omega)}=o(1)$, $\|f_3\|_{H^1_0(\Omega)}=o(1)$ and $\|f_4\|_{L^2(\Omega)}=o(1)$, we get

$$\int_\Omega\left(\frac{f_2}{\lambda^\ell}+\frac{bf_3}{\lambda^\ell}\right)\lambda\bar{y}dx = \frac{o(1)}{\lambda^\ell}, \tag{4.3.36}$$

$$\int_\Omega\left(\frac{f_4}{\lambda^\ell}-\frac{bf_1}{\lambda^\ell}\right)\lambda\bar{u}dx = \frac{o(1)}{\lambda^\ell}, \tag{4.3.37}$$

Using equations Lemma 4.3.3 and (4.3.23), we get

$$\lambda\int_{\Gamma_1}\frac{\partial y}{\partial\nu}\bar{u}d\Gamma_1=o(1). \tag{4.3.38}$$

Next, multiplying equations (4.3.13) and (4.3.14) respectively by $\bar{f}_3$ and $\bar{f}_1$ and integrating, we get

$$\int_\Omega \lambda^2 u\bar{f}_3 - \int_\Omega \nabla u \nabla \bar{f}_3 dx = i\lambda b \int_\Omega y\bar{f}_3 dx = -\int_\Omega \left(\frac{f_2}{\lambda^\ell} + i\frac{\lambda f_1}{\lambda^\ell} + \frac{bf_3}{\lambda^\ell}\right)\bar{f}_3 dx \tag{4.3.39}$$

and

$$\begin{aligned}\int_\Omega \lambda^2 y\bar{f}_1 dx - \int_\Omega \nabla y \nabla \bar{f}_1 dx + \int_{\Gamma_1} (\partial_\nu y)\bar{f}_1 d\Gamma_1 + i\lambda b \int_\Omega u\bar{f}_1 dx \\ = -\int_\Omega \left(\frac{f_4}{\lambda^\ell} + i\frac{\lambda f_3}{\lambda^\ell} - \frac{bf_1}{\lambda^\ell}\right)\bar{f}_1 dx.\end{aligned} \tag{4.3.40}$$

Using equations (4.3.15), (4.3.16) and (4.3.23), and the facts that $\|f_1\|_{H^1_{\Gamma_0}(\Omega)} = o(1)$, $\|f_2\|_{L^2(\Omega)} = o(1)$, $\|f_3\|_{H^1_0(\Omega)} = o(1)$, $\|f_4\|_{L^2(\Omega)} = o(1)$ and ∇u, ∇y are bounded in $L^2(\Omega)$, we obtain, from (4.3.39)-(4.3.40)

$$\int_\Omega \lambda^2 u\bar{f}_3 dx = -i\int_\Omega \frac{\lambda f_1 \bar{f}_3}{\lambda^\ell} dx + o(1), \tag{4.3.41}$$

$$\int_\Omega \lambda^2 y\bar{f}_1 dx = -i\int_\Omega \frac{\lambda f_3 \bar{f}_1}{\lambda^\ell} dx + o(1).. \tag{4.3.42}$$

Now, combining equation (4.3.34), (4.3.36) and (4.3.42) we get

$$\int_\Omega \lambda^3 u\bar{y} dx - \lambda \int_\Omega \nabla u \nabla \bar{y} dx - ib\int_\Omega |\lambda y|^2 dx = \int_\Omega \frac{\lambda \bar{f}_3 f_1}{\lambda^{2\ell}} dx + \frac{o(1)}{\lambda^\ell}. \tag{4.3.43}$$

On the other hand, combining equation (4.3.35), (4.3.37), (4.3.38) and (4.3.41) we obtain

$$\int_\Omega \lambda^3 y\bar{u} dx - \lambda \int_\Omega \nabla y \nabla \bar{u} dx + ib\int_\Omega |\lambda u|^2 dx = \int_\Omega \frac{\lambda \bar{f}_1 f_3}{\lambda^{2\ell}} dx + o(1). \tag{4.3.44}$$

Finally, adding equations (4.3.43)-(4.3.44), use the fact that $\ell = 2 - 2\alpha$ and tacking the imaginary part, we get the desired equation (4.3.33). The proof is thus complete.

Lemma 4.3.6. *Assume that $\eta > 0$. Then the solution $(u, v, y, z, \omega) \in D(\mathcal{A})$ of (4.3.7)-(4.3.11) satisfies the following asymptotic behavior estimation*

$$\int_\Omega |\lambda u|^2 dx - \int_\Omega |\nabla u|^2 dx = o(1) \quad \textit{and} \quad \int_\Omega |\lambda y|^2 dx - \int_\Omega |\nabla y|^2 dx = o(1). \tag{4.3.45}$$

Proof:[proof] Multiplying equation (4.3.13) by $\bar{u}$, using Green formula, we get

$$\int_\Omega |\lambda u|^2 dx - \int_\Omega |\nabla u|^2 dx + \int_{\Gamma_1} \frac{\partial u}{\partial \nu}\bar{u} d\Gamma_1 - ib\int_\Omega \lambda y\bar{u} dx = \frac{o(1)}{\lambda^\ell}. \tag{4.3.46}$$

Multiplying equation (4.3.14) by $\bar{y}$, using green formula, we get

$$\int_\Omega |\lambda y|^2 dx - \int_\Omega |\nabla y|^2 dx + ib\int_\Omega \lambda u \bar{y} dx = \frac{o(1)}{\lambda^\ell} \tag{4.3.47}$$

Using Lemma 4.3.3 in equation (4.3.46), we get the desired (4.3.45). The proof is thus complete.

Lemma 4.3.7. *Assume that* $\eta > 0$. *If* $|b| \leq \frac{1}{\|m\|_\infty}$ *then the solution* $(u, v, y, z, \omega) \in D(\mathcal{A})$ *of* (4.3.7)-(4.3.11) *satisfies the following asymptotic behavior estimation*

$$\int_\Omega |\lambda u|^2 dx = o(1). \tag{4.3.48}$$

Proof: Multiplying equation (4.3.13) by $2(m \cdot \nabla \bar{u})$, we get

$$\begin{aligned} 2\int_\Omega \lambda^2 u(m\cdot\nabla\bar{u})dx + 2\int_\Omega \Delta u(m\cdot\nabla\bar{u})dx - 2i\lambda b\int_\Omega by(m\cdot\nabla\bar{u})dx = \\ -2\int_\Omega\left(\frac{f_2}{\lambda^\ell} + \frac{i\lambda f_1}{\lambda^\ell} + \frac{bf_3}{\lambda^\ell}\right)(m\cdot\nabla\bar{u})dx. \end{aligned} \tag{4.3.49}$$

Firstly, using the fact that $\|f_2\|_{L^2(\Omega)} = o(1)$, $\|f_3\|_{H_0^1(\Omega)} = o(1)$, $\nabla\bar{u}$ is bounded in $L^2(\Omega)$, and the fact that $\ell = 2-2\alpha$, we get

$$-2\int_\Omega\left(\frac{f_2}{\lambda^\ell} + \frac{bf_3}{\lambda^\ell}\right)(m\cdot\nabla\bar{u})dx = \frac{o(1)}{\lambda^{2-2\alpha}}. \tag{4.3.50}$$

On the other hand, we say that

$$2i\lambda\int_\Omega \frac{f_1}{\lambda^\ell}(m\cdot\nabla\bar{u})dx = -2\frac{i\lambda}{\lambda^\ell}\int_\Omega \bar{u}\cdot\nabla(f_1 m)dx + \frac{2i\lambda}{\lambda^\ell}\int_{\Gamma_1}(mf\cdot\nu)\bar{u}d\Gamma_1. \tag{4.3.51}$$

So, using equation (4.3.15), (4.3.18), $\|f_1\|_{H^1_{\Gamma_0}(\Omega)} = o(1)$ and the fact that $\ell = 2-2\alpha$, we obtain, from (4.3.51) that

$$2i\lambda\int_\Omega \frac{f_1}{\lambda^\ell}(m\cdot\nabla\bar{u})dx = \frac{o(1)}{\lambda^{2-2\alpha}}. \tag{4.3.52}$$

Secondly, using integration by parts, we get

$$2\int_\Omega \lambda^2 u(m\cdot\nabla\bar{u})dx = -d\int_\Omega |\lambda u|^2 dx + \lambda^2\int_{\Gamma_1}(m\cdot\nu)|u|^2 d\Gamma_1. \tag{4.3.53}$$

So, using equation (4.3.18) in equation (4.3.53) and the fact that $\ell = 2-2\alpha$ we obtain

$$\lambda^2\int_{\Gamma_1}(m\cdot\nu)|u|^2 d\Gamma_1 = o(1). \tag{4.3.54}$$

Combining equation (4.3.53) and (4.3.54), we get

$$2\int_{\Omega}\lambda^2 u(m\cdot\nabla u)dx=-d\int_{\Omega}|\lambda u|^2dx+o(1). \tag{4.3.55}$$

Next, using Green formula, we get

$$2\int_{\Omega}\Delta u(m\cdot\nabla\bar{u})dx=(d-2)\int_{\Omega}|\nabla u|^2dx+2\int_{\Gamma}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma-\int_{\Gamma}(m\cdot\nu)|\nabla u|^2d\Gamma. \tag{4.3.56}$$

So, inserting equation (4.3.25), (4.3.27), (4.3.55), (4.3.56) in equation (4.3.49) and tacking the reel part, we get

$$\begin{aligned}-d\int_{\Omega}|\lambda u|^2dx+(d-2)\int_{\Omega}|\nabla u|^2dx+2\Re\left(\int_{\Gamma}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)-\int_{\Gamma}(m\cdot\nu)|\nabla u|^2d\Gamma&\\ +2\lambda b\Re\left(-i\int_{\Omega}y(m\cdot\nabla\bar{u})dx\right)=o(1).&\end{aligned} \tag{4.3.57}$$

Using the fact $\frac{\partial u}{\partial\tau}=0$ on Γ_0, we get

$$\begin{aligned}-2\Re\left(\int_{\Gamma}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)+\int_{\Gamma}(m\cdot\nu)|\nabla u|^2d\Gamma=-\int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma&\\ -2\Re\left(\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)+\int_{\Gamma_1}(m\cdot\nu)|\nabla u|^2d\Gamma&\end{aligned} \tag{4.3.58}$$

Let $\varepsilon>0$, so by Young inequality, we get

$$2\Re\left(\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)\leq\frac{\|m\|_{\infty}^2}{\varepsilon}\int_{\Gamma_1}\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma_1+\varepsilon\int_{\Gamma_1}|\nabla u|^2d\Gamma. \tag{4.3.59}$$

Using equation (4.3.17) in (4.3.59) and the fact that $\ell=2-2\alpha$, we get

$$2\Re\left(\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)\leq\varepsilon\int_{\Gamma_1}|\nabla u|^2d\Gamma+\frac{o(1)}{\lambda^{2-2\alpha}}. \tag{4.3.60}$$

Inserting equations (4.3.60) in equation (4.3.58), we get

$$\begin{aligned}-2\Re\left(\int_{\Gamma}\frac{\partial u}{\partial\nu}(m\cdot\nabla\bar{u})d\Gamma\right)+\int_{\Gamma}(m\cdot\nu)|\nabla u|^2d\Gamma\geq-\int_{\Gamma_0}(m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2d\Gamma&\\ -\varepsilon\int_{\Gamma_1}|\nabla u|^2d\Gamma+\int_{\Gamma_1}(m\cdot\nu)|\nabla u|^2d\Gamma+\frac{o(1)}{\lambda^{2-2\alpha}}.&\end{aligned} \tag{4.3.61}$$

Using the (**MGC**) condition, in equation (4.3.61), we get

$$-2\Re\left(\int_{\Gamma}\frac{\partial u}{\partial \nu}(m\cdot\nabla\bar{u})d\Gamma\right)+\int_{\Gamma}(m\cdot\nu)|\nabla u|^2 d\Gamma \geq \frac{o(1)}{\lambda^{2-2\alpha}}+(\delta-\varepsilon)\int_{\Gamma_1}|\nabla u|^2 d\Gamma. \tag{4.3.62}$$

Tacking $\varepsilon<\delta$, then we get

$$-2\Re\left(\int_{\Gamma}\frac{\partial u}{\partial \nu}(m\cdot\nabla\bar{u})d\Gamma\right)+\int_{\Gamma}(m\cdot\nu)|\nabla u|^2 d\Gamma \geq \frac{o(1)}{\lambda^{2-2\alpha}}. \tag{4.3.63}$$

Inserting equation (4.3.63) in equation (4.3.57), we get

$$d\int_{\Omega}|\lambda u|^2dx+(2-d)\int_{\Omega}|\nabla u|^2dx\leq 2\lambda b\Re\left(-i\int_{\Omega}y(m\cdot\nabla\bar{u})dx\right)+o(1). \tag{4.3.64}$$

Multiplying equation (4.3.45) by $1-d$ and combining with equation (4.3.64), we get

$$\int_{\Omega}|\lambda u|^2dx+\int_{\Omega}|\nabla u|^2dx\leq 2\Re\left(-i\int_{\Omega}b\lambda y(m\cdot\nabla\bar{u})dx\right)+o(1). \tag{4.3.65}$$

Using Young inequality, we get

$$2\Re\left(-i\int_{\Omega}b\lambda y(m\cdot\nabla\bar{u})dx\right)\leq \|m\|_{\infty}^2 b^2\int_{\Omega}|\lambda y|^2dx+\int_{\Omega}|\nabla u|^2dx. \tag{4.3.66}$$

Inserting equation (4.3.66) in equation (4.3.65), we get

$$\int_{\Omega}|\lambda u|^2dx+\int_{\Omega}|\nabla u|^2dx\leq \|m\|_{\infty}^2 b^2\int_{\Omega}|\lambda y|^2dx+\int_{\Omega}|\nabla u|^2dx+o(1). \tag{4.3.67}$$

Using Lemma 4.3.5, we get

$$\int_{\Omega}(1-\|m\|_{\infty}^2 b^2)|\lambda u|^2dx\leq o(1). \tag{4.3.68}$$

Finally, using the fact that $|b|<\frac{1}{\|m\|_{\infty}}$ in equation (4.3.68), we get the desired equation (4.3.48). Thus the proof is complete.

Proof of Theorem 4.3.1. Using (4.3.33), (4.3.45) and (4.3.48), we get

$$\int_{\Omega}|\nabla u|^2dx=o(1),\quad \int_{\Omega}|\lambda y|^2dx=o(1),\quad \text{and}\quad \int_{\Omega}|\nabla y|^2dx=o(1). \tag{4.3.69}$$

It follows, from (4.3.12), (4.3.48) and (4.3.69), that $\|U\|_{\mathcal{H}}=o(1)$ which a contradiction with (4.3.5). Consequently condition $(H2)$ holds and the energy of smooth solution of system (4.2.1)-(4.2.9) decays polynomial to zero as t goes to infinity where $a=1$ and b small enough and verifying $|b|<\frac{1}{\|m\|_{\infty}}$. Finally, using the density of the domains $D(\mathcal{A})$ in $\mathcal{H}$, we can easily prove that the energy of weak solution of system (4.2.1)-(4.2.9) decays to zero as t goes to infinity.

The proof has been completed.

CHAPITRE 5

INDIRECT STABILITY OF A SYSTEM OF STRONGLY COUPLED WAVE EQUATONS WITH LOCAL KELVIN-VOIGHT DAMPING

5.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipchitz boundary Γ. We consider the following two wave equations coupled with a viscoelastic damping around the boundary Γ :

$$\begin{cases} \rho_1(x)u_{tt} - \operatorname{div}(a_1(x)\nabla u + b(x)\nabla u_t) + \alpha y_t = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \rho_2(x)y_{tt} - \operatorname{div}(a_2(x)\nabla y) - \alpha u_t = 0 & \text{in} \quad \Omega \times \mathbb{R}^+, \end{cases} \tag{5.1.1}$$

with the following initial conditions :

$$u(\cdot,0) = u_0\left(\cdot\right), \; y(\cdot,0) = y_0\left(\cdot\right), \; u_t(\cdot,0) = u_1\left(\cdot\right), \; y_t(\cdot,0) = y_1\left(\cdot\right) \quad \text{in } \Omega, \tag{5.1.2}$$

and the following boundary conditions :

$$u\left(x,t\right) = y\left(x,t\right) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+. \tag{5.1.3}$$

The functions $\rho_1, \rho_2, a_1, a_2, b \in L^\infty(\Omega)$ such that

$$\rho_1(x) \geq \rho_1, \; \rho_2(x) \geq \rho_2, \; a_1(x) \geq a_1, \; a_2(x) \geq a_2$$

and α is a real constant number.
The local viscoelastic damping is a natural phenomena of bodies which have one part made of viscoelastic material, and the other is made of elastic material. There are a few number of publications concerning the wave equation with local viscoelastic damping. In [39], Liu and Rao

studied the stability of a wave equations with local viscoelastic damping distributed around the boundary of the domain. They proved that the energy of the system goes exponentially to zero for all usual initial data.
K. Liu and Z. Liu in [38], considered the longitudinal and transversal vibrations of the Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. They proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is not exponentially stable. In [48], they consider a system of wave equations which are weakly coupled and partially damped by one locally distributed Kelvin-Voigt damping. The first equation is effectively damped, the second equation is indirectly damped through the coupling parameter. Firstly, using a unique continuation result based on a Carleman estimate, they show that the system is strongly stable for all usual initial data. Secondly, using a spectral approach, we show that the system is not uniformly exponentially stable. Then, it is natural to expect a polynomial energy decay rate. For this aim, using a frequency domain approach combined with piece wise multiplier method, we establish a polynomial energy decay rate.
This chapter is organized as follows. First, in section 2, we show the well-posedness of the system and using a general criteria of Arendt-Batty see [9] and a mild continuation of Hormander Theorem 8.3.1 (see [28]), we show the strong stability of the system in the absence of the compactness of the resolvent. Next, in section 3, using a spectrum approach, we prove the non-uniform stability of the system. Finally, in section 4, we establish an optimal polynomial energy decay rate as $\frac{1}{t}$ for smooth functions by a frequency domain approach combined with a piece ways multiplier method.

5.2 Well-Posedness and Strong Stability

5.2.1 Well posedness of the problem

In this part, using a semigroup approach, we establish well-posedness result for the system Kelvin Voight with viscoelastic damping $\operatorname{div}(b(x)\nabla u_t)$ be applied around the boundary Γ.
Now, the energy of system (5.1.1)-(5.1.3) is given by

$$E(t)=\frac{1}{2}\int_\Omega\left(\rho_1(x)|u_t|^2+\rho_2(x)|y_t|^2+a_1(x)|\nabla u|^2+a_2(x)|\nabla y|^2\right)dx. \tag{5.2.1}$$

Then a straightforward computation gives

$$E'(t)=-\int_\Omega b(x)|\nabla u_t|^2dx\le 0.$$

Thus, the system (5.1.1)-(5.1.3) is dissipative in the sense that its energy is non increasing with respect to the time t. For any $\gamma > 0$ we define the $\gamma-$neighborhood O_γ of the boundary Γ as follows

$$O_\gamma = \{x \in \Omega \ : \quad |x-y| \leq \gamma, \quad y \in \Gamma\}. \tag{5.2.2}$$

More precisely, we assume that

$$\begin{cases} \rho_1(x) \geq \rho_1 > 0, \ \rho_2(x) \geq \rho_2 > 0, \ a_1(x) \geq a_1 > 0, \ a_2(x) \geq a_2 > 0 \quad \text{for all } x \in \Omega, \\ \\ b(x) \geq b_0 > 0 \quad \text{for all } x \in O_\gamma. \end{cases} \tag{H}$$

Next, we define the Hilbert space

$$\mathcal{H} = \left(H_0^1(\Omega) \times L^2(\Omega)\right)^2$$

endowed with the inner product

$$\left\langle U, \tilde{U} \right\rangle = \int_\Omega \left(a_1 \nabla u \cdot \nabla \bar{\tilde{u}} + a_2 \nabla y \cdot \nabla \bar{\tilde{y}} + \rho_1 v \bar{\tilde{v}} + \rho_2 z \bar{\tilde{z}}\right) dx,$$

for all $U = (u, v, y, z)^\top \in \mathcal{H}$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})^\top \in \mathcal{H}$. Finally, we define the unbounded linear operator $\mathcal{A}$ by

$$\begin{aligned} D(\mathcal{A}) = \Big\{ &(u, v, y, z) \in \mathcal{H} : \ \operatorname{div}(a_1(x)\nabla u + b(x)\nabla v) \in L^2(\Omega), \\ &\qquad \operatorname{div}(a_2(x)\nabla y) \in L^2(\Omega) \quad \text{and} \quad v, z \in H_0^1(\Omega) \Big\}, \end{aligned} \tag{5.2.3}$$

and

$$\mathcal{A}\begin{pmatrix} u \\ v \\ y \\ z \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{1}{\rho_1(x)}\left(\operatorname{div}(a_1(x)\nabla u + b(x)\nabla v) - \alpha z\right) \\ z \\ \dfrac{1}{\rho_2(x)}\left(\operatorname{div}(a_2(x)\nabla y) + \alpha v\right) \end{pmatrix}.$$

If $U = (u, u_t, y, y_t)^\top$ is a regular solution of system (5.1.1)-(5.1.3), then we rewrite this system as the following evolution equation

$$\begin{cases} U_t = \mathcal{A}U \\ U(0) = U_0 \end{cases} \tag{5.2.4}$$

where $U_0 = (u_0, u_1, y_0, y_1)^\top$.

Proposition 5.2.1. *Under hypothesis* (H)*, the unbounded linear operator* $\mathcal{A}$ *is m-dissipative in the energy space* $\mathcal{H}$*.*

Proof: For all $U=(u,v,y,z)^\top \in D(\mathcal{A})$, we have

$$\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}} = -\int_\Omega b(x)|\nabla v|^2 dx \leq 0, \tag{5.2.5}$$

which implies that $\mathcal{A}$ is dissipative. Now, let $F=(f_1,f_2,f_3,f_4)^\top \in \mathcal{H}$, we prove the existence of

$$U=(u,v,y,z)^\top \in D(\mathcal{A})$$

unique solution of the equation

$$-\mathcal{A}U=F$$

Equivalently, we have the following system

$$\begin{aligned} -v &= f_1 && (5.2.6)\\ -\operatorname{div}(a_1\nabla u + b\nabla v) + \alpha z &= \rho_1 f_2 && (5.2.7)\\ -z &= f_3 && (5.2.8)\\ -\operatorname{div}(a_2(x)\nabla y) - \alpha v &= \rho_2 f_4 && (5.2.9)\end{aligned}$$

Inserting (5.2.6), (5.2.8) in (5.2.7) and (5.2.9), we get

$$\begin{aligned} -\operatorname{div}(a_1\nabla u - b\nabla f_1) &= \rho_1 f_2 + \alpha f_3 && (5.2.10)\\ -\operatorname{div}(a_2\nabla y) &= \rho_2 f_4 - \alpha f_1 && (5.2.11)\end{aligned}$$

Multiplying (5.2.10) and (5.2.11) by $\bar{\varphi}$ and $\bar{\psi}$ respectively, and integrating their sum, we get

$$\begin{aligned}\int_\Omega \left(a_1\nabla u\cdot\nabla\bar{\varphi} + a_2\nabla y\cdot\nabla\bar{\psi}\right)dx &= \int_\Omega \left(\left(\rho_1 f_2 + \alpha f_3\right)\bar{\varphi} + \left(b\nabla f_1\cdot\nabla\bar{\varphi}\right)\right)dx\\ &\quad + \int_\Omega \left(\rho_2 f_4 - \alpha f_1\right)\bar{\psi}dx,\ \forall\ (\varphi,\psi)\in H_0^1(\Omega)\times H_0^1(\Omega).\end{aligned} \tag{5.2.12}$$

Using Lax-Milgram Theorem (see [51]), we deduce that (5.2.12) admits a unique solution (u,y) in $H_0^1(\Omega)\times H_0^1(\Omega)$. Clearly this solution satisfies (5.2.10)-(5.2.11) by choosing appropriated test functions. Thus, using (5.2.6)-(5.2.8) and classical regularity arguments, we conclude that $-\mathcal{A}U=F$ admits a unique solution $U\in D\left(\mathcal{A}\right)$ and $0\in D\left(\mathcal{A}\right)$. Since $D\left(\mathcal{A}\right)$ is dense in $\mathcal{H}$, then, by the resolvent identity, for small $\lambda>0$ we have $R\left(\lambda I-\mathcal{A}\right)=\mathcal{H}$ (see Theorem 1.2.4 in [42]) and $\mathcal{A}$ is m-dissipative in $\mathcal{H}$. The proof is thus complete.

Thanks to Lumer-Phillips Theorem (see [42] and [51]), we deduce that $\mathcal{A}$ generates a C_0-semigroup of contraction $e^{t\mathcal{A}}$ in $\mathcal{H}$ and therefore problem (5.1.1)-(5.1.3) is well-posed. Then we have the following result :

Theorem 5.2.2. *For any $U_0 \in \mathcal{H}$, problem* (5.2.4) *admits a unique weak solution*

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

5.2.2 Strong Stability

In this part, we use a general criteria of Arendt-Batty in [9] to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the system (5.1.1)-(5.1.3) in the absence of the compactness of the resolvent of $\mathcal{A}$. Throughout this part, we assume that

$$a_1, a_2 \in C^{0,1}(\overline{\Omega}).$$

Theorem 5.2.3. *Under hypothesis* (H)*, the C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of* (5.2.4) *satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 5.2.3, we need the following two lemmas.

Lemma 5.2.4. *Under hypothesis* (H)*, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \qquad \forall\ \lambda \in \mathbb{R}.$$

Proof: From Proposition 5.2.1, $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. Suppose that there exists a real number $\lambda \neq 0$ and $U = (u, v, y, z)^\top \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{5.2.13}$$

From (5.2.5) and (5.2.13), we have

$$0 = \Re\left(i\lambda\|U\|^2_{\mathcal{H}}\right) = \Re(\langle\mathcal{A}U, U\rangle_{\mathcal{H}}) = -\int_\Omega b(x)|\nabla v|^2 dx. \tag{5.2.14}$$

This together with the condition (H) and Poincaré's inequality implies that

$$b(x)\nabla v = 0 \text{ in } \Omega \ \text{ and } \ v = 0 \text{ in } O_\gamma. \tag{5.2.15}$$

Inserting (5.2.15) into (5.2.13), we get the following system

$$v = i\lambda u \quad \text{in} \quad \Omega, \tag{5.2.16}$$

$$\operatorname{div}(a_1 \nabla u) - \alpha z = i\lambda \rho_1 v \quad \text{in} \quad \Omega, \tag{5.2.17}$$

$$z = i\lambda y \quad \text{in} \quad \Omega, \tag{5.2.18}$$

$$\operatorname{div}(a_2 \nabla y) + \alpha v = i\lambda \rho_2 z \quad \text{in} \quad \Omega. \tag{5.2.19}$$

Using (5.2.15) and (5.2.16), we get

$$u \equiv 0 \ \text{ on } \ O_\gamma. \tag{5.2.20}$$

Inserting (5.2.18) in (5.2.17), we get

$$\operatorname{div}(a_1 \nabla u) - \alpha i \lambda y = i\rho_1 \lambda v. \tag{5.2.21}$$

From (5.2.20) and (5.2.21), we have

$$y \equiv 0 \quad \text{on} \quad O_\gamma. \tag{5.2.22}$$

Using (5.2.19) and (5.2.22), we get

$$z \equiv 0 \ \text{ on } \ O_\gamma. \tag{5.2.23}$$

Moreover, from (5.2.15), (5.2.20), (5.2.22) and (5.2.23), we get

$$U \equiv 0 \ \text{ on } \ O_\gamma. \tag{5.2.24}$$

Next, inserting (5.2.16), (5.2.18) in (5.2.17) and (5.2.19), we get the following system

$$\begin{cases} \mathcal{A}_1 = \rho_1 \lambda^2 u + \operatorname{div}(a_1 \nabla u) - \alpha z = 0 \quad \text{in} \quad \Omega, \\ \mathcal{A}_2 = \rho_2 \lambda^2 y + \operatorname{div}(a_2 \nabla y) + \alpha v = 0 \quad \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad u = y = 0 \quad in \quad O_\gamma. \end{cases} \tag{5.2.25}$$

Using the fact that $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, we deduce that the solution (u, y) of system (5.2.25) belong to $H_c^2(\Omega) \times H_c^2(\Omega)$, where

$$H_c^2(\Omega) = \left\{ u \ : \ u \text{ in } H^2(\Omega) \text{ and with compact support} \right\}.$$

From Theorem 8.3.1 see [28], there exist $C > 0$ and $\tau_0 >> 1$, such that for all $\tau > \tau_0$, we have

$$\tau^3 \int_\Omega e^{2\tau\varphi} |f|^2 dx + \tau \int_\Omega e^{2\tau\varphi} |\nabla f|^2 dx \leq C \int_\Omega e^{2\tau\varphi} |\Delta f|^2 dx \qquad \forall f \in C_0^\infty(\Omega), \tag{5.2.26}$$

where $\varphi(x) = \frac{|x-x_0|^2}{2}$ for a fixed $x_0 \notin \overline{\Omega}$. By density argument we extend equation (5.2.26) into space $H_c^2(\Omega)$. Now, by taking $f = u$ and $f = y$ in equation (5.2.26), we get

$$\tau^3 \int_\Omega e^{2\tau\varphi}|u|^2 dx + \tau \int_\Omega e^{2\tau\varphi}|\nabla u|^2 dx \leq C_1 \int_\Omega e^{2\tau\varphi}|\Delta u|^2 dx, \tag{5.2.27}$$

$$\tau^3 \int_\Omega e^{2\tau\varphi}|y|^2 dx + \tau \int_\Omega e^{2\tau\varphi}|\nabla y|^2 dx \leq C_2 \int_\Omega e^{2\tau\varphi}|\Delta y|^2 dx. \tag{5.2.28}$$

Adding (5.2.27) and (5.2.28), we get

$$\begin{aligned}\tau^3 \int_\Omega e^{2\tau\varphi}(|u|^2 + |y|^2) dx + \tau \int_\Omega e^{2\tau\varphi}(|\nabla u|^2 + |\nabla y|^2) dx \\ \leq 2C \int_\Omega e^{2\tau\varphi}(|a\Delta u|^2 + |a\Delta y|^2) dx.\end{aligned} \tag{5.2.29}$$

From (5.2.25) and (5.2.29), we get

$$\begin{aligned}(\tau^3 - C_3) \int_\Omega e^{2\tau\varphi}(|u|^2 + |y|^2) dx + (\tau - C_4) \int_\Omega e^{2\tau\varphi}(|\nabla u|^2 + |\nabla y|^2) dx \\ \leq C_5 \int_\Omega e^{2\tau\varphi}(|\mathcal{A}_1 u|^2 + |\mathcal{A}_2 y|^2) dx,\end{aligned} \tag{5.2.30}$$

where $C_1,\ C_2$ and C_3 are positive constants. Finally, take τ such that $\tau^3 - C_1 \geq \frac{1}{2}$ and $\tau - C_2 \geq \frac{1}{2}$, we deduce that $u = 0$ and $y = 0$ in Ω. The proof is thus complete.

Lemma 5.2.5. *Under hypothesis* (H), $i\lambda I - \mathcal{A}$ *is surjective for all* $\lambda \in \mathbb{R}$.

Proof: Since $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^\star$. For any

$$F = (f_1, f_2, f_3, f_4)^\top, \quad \lambda \in \mathbb{R}^\star,$$

we prove the existence of

$$U = (u, v, y, z)^\top \in D(\mathcal{A})$$

solution for the following equation

$$(i\lambda I - \mathcal{A})U = F$$

Equivalently, we have

$$\begin{aligned} i\lambda u - v &= f_1, && (5.2.31)\\ i\lambda v - \frac{1}{\rho_1}\left(\operatorname{div}(a_1\nabla u + b\nabla v) - \alpha z\right) &= f_2, && (5.2.32)\\ i\lambda y - z &= f_3, && (5.2.33)\\ i\lambda z - \frac{1}{\rho_2}\left(\operatorname{div}(a_2\nabla y) + \alpha v\right) &= f_4. && (5.2.34)\end{aligned}$$

From (5.2.31) and (5.2.33), we have

$$v = i\lambda u - f_1 \text{ and } \quad z = i\lambda y - f_3. \tag{5.2.35}$$

Inserting (5.2.35) in (5.2.32) and (5.2.34), we get the following system

$$\left\{\begin{aligned} -\lambda^2 u - \frac{1}{\rho_1}\operatorname{div}\left((a_1 + i\lambda b)\nabla u\right) + i\frac{\alpha\lambda}{\rho_1}y &= h_1,\\ -\lambda^2 y - \frac{1}{\rho_2}\operatorname{div}(a_2\nabla y) - i\frac{\alpha\lambda}{\rho_2}u &= h_2,\end{aligned}\right. \tag{5.2.36}$$

where

$$\left\{\begin{aligned} h_1 &= i\lambda f_1 + f_2 - \frac{1}{\rho_1}\operatorname{div}(b\nabla f_1) + \frac{\alpha}{\rho_1}f_3,\\ h_2 &= -\frac{\alpha}{\rho_2}f_1 + i\lambda f_3 + f_4.\end{aligned}\right. \tag{5.2.37}$$

Define the operators

$$\mathcal{L}\mathcal{U} = \begin{pmatrix} -\frac{1}{\rho_1}\operatorname{div}\left((a_1 + i\lambda b)\nabla u\right) + i\frac{\alpha\lambda}{\rho_1}y\\ -\frac{1}{\rho_2}\operatorname{div}(a_2\nabla y) - i\frac{\alpha\lambda}{\rho_2}u \end{pmatrix}, \quad \forall\, \mathcal{U} = (u,y)^\top \in \left(H_0^1(\Omega)\right)^2.$$

Using Lax-Milgram theorem, it is easy to show that $\mathcal{L}$ is an isomorphism from $(H_0^1(\Omega))^2$ onto $(H_0^1(\Omega))^2$. Let $\mathcal{U} = (u,y)^\top$ and $h = (h_1,h_2)^\top$, then we transform system (5.2.36) into the following form

$$\mathcal{U} - \lambda^2\mathcal{L}^{-1}\mathcal{U} = \mathcal{L}^{-1}F. \tag{5.2.38}$$

Using the compactness embeddings from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and from $H_0^1(\Omega)$ into $L^2(\Omega)$ we deduce that the operator $\mathcal{L}^{-1}$ is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Consequently, by Fredholm alternative, proving the existence of $\mathcal{U}$ solution of (5.2.38) reduces to proving the injectivity of the operator $Id - \lambda^2\mathcal{L}^{-1}$. Indeed, if $\widetilde{\mathcal{U}} = (\tilde{u},\tilde{y})^\top \in \ker\left(Id - \mathcal{L}^{-1}\right)$, then we have $\lambda^2\widetilde{U} - \mathcal{L}\widetilde{U} = 0$.

It follows that

$$\left\{\begin{array}{rcl} -\lambda^2\tilde{u} - \dfrac{1}{\rho_1}\operatorname{div}\left((a_1+i\lambda b)\nabla\tilde{u}\right) + i\dfrac{\alpha\lambda}{\rho_1}\tilde{y} &=& 0, \\ -\lambda^2\tilde{y} - \dfrac{1}{\rho_2}\operatorname{div}(a_2\nabla\tilde{y}) - i\dfrac{\alpha\lambda}{\rho_2}\tilde{u} &=& 0. \end{array}\right. \tag{5.2.39}$$

Now, it is easy to see that if $(\tilde{u},\tilde{y})$ is a solution of (5.2.39) then the vector $\hat{U}=(\tilde{u},i\tilde{u},\tilde{y},i\tilde{y})$ belongs to $D(\mathcal{A})$ and we have $i\lambda\hat{U}-\mathcal{A}\hat{U}=0$. Therefore, by lemma 5.2.4, we get $\hat{U}$ and so $\ker(Id-\lambda^2\mathcal{L})=\{0\}$. Thanks to Fredholm alternative, the equation (5.2.38) admits a unique solution $\mathcal{U}=(u,y)\in\left(H_0^1(\Omega)\right)^2$. Thus using (5.2.35) and a classical regularity arguments, we conclude that $(i\lambda I-\mathcal{A})\,U=F$ admits a unique solution $U\in D(\mathcal{A})$. The proof is thus complete.

Proof of Theorem 5.2.3. Following a general criteria of Arendt-Batty in [9], the C_0-semigroup $e^{t\mathcal{A}}$ of contractions is strongly stable if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A})\cap i\mathbb{R}$ is countable. By Lemma 5.2.4, the operator $\mathcal{A}$ has no pure imaginary eigenvalues and by Lemma 5.2.5, $R(i\lambda-\mathcal{A})=\mathcal{H}$ for all $\lambda\in\mathbb{R}$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A})\cap i\mathbb{R}=\emptyset$. The proof is thus complete.

Remark 5.2.6. *We mention [53] for a direct approach of the strong stability of Kirchhoff plates in the absence of compactness of the resolvent.*

5.3 Non Uniform Stability

In this section, our goal is to show that the system (5.1.1)-(5.1.3) is not exponentially stable. Throughout, this part, we assume that

$$a_1,a_2,\rho_1,\rho_2\in\mathbb{R}^+\quad\text{and}\quad b\in\mathbb{R}_*^+ \tag{H'}$$

Theorem 5.3.1. *Under hypothesis* (H')*, the system* (5.1.1)-(5.1.3) *is not uniformly stable in the energy space* $\mathcal{H}$.

For the proof of Theorem 5.3.1, we aim to show that an infinite of eigenvalues of $\mathcal{A}$ approach the imaginary axis. First, we determine the characteristic equation satisfied by the eigenvalues of $\mathcal{A}$. For the aim, let $\lambda\in\mathbb{C}$ be an eigenvalue of $\mathcal{A}$ and let $U=(u,v,y,z)^\top\in D(\mathcal{A})$ be an associated eigenvector such that $\|U\|_{\mathcal{H}} = 1$. Then

$$v = \lambda u, \tag{5.3.1}$$

$$\operatorname{div}(a_1 \nabla u + b \nabla v) - \alpha z = \lambda \rho_1 v, \tag{5.3.2}$$

$$z = \lambda y, \tag{5.3.3}$$

$$\operatorname{div}(a_2 \nabla y) + \alpha v = \lambda \rho_2 z. \tag{5.3.4}$$

Inserting (5.3.1), (5.3.3) in (5.3.2) and (5.3.4), we get

$$\rho_1 \lambda^2 u - (a_1 + \lambda b)\Delta u + \alpha \lambda y = 0, \tag{5.3.5}$$

$$\rho_2 \lambda^2 y - a_2 \Delta y - \alpha \lambda u = 0. \tag{5.3.6}$$

Lemma 5.3.2. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$. Then $\Re(\lambda)$ is bounded.*

Proof: Multiplying (5.3.5) and (5.3.6) by $\bar{u}$ and $\bar{y}$ respectively, and integrating their sum, then take the real part of the resulting equation, we get

$$\Re(\lambda)\, b\|\nabla u\|^2 + \rho_1\|\lambda u\|^2 + a_1\|\nabla u\|^2 + \rho_2\|\lambda y\|^2 + a_2\|\nabla y\|^2 = 0. \tag{5.3.7}$$

Since $\|U\|_{\mathcal{H}} = 1$, then we have $\rho_1\|\lambda u\|^2 + \rho_2\|\lambda y\|^2 + a\|\nabla u\|^2 + a\|\nabla y\|^2$ and $\|\nabla u\|^2$ are bounded. Consequentially there exists constant $\kappa > 0$, such that

$$-\kappa \leq \Re(\lambda) < 0.$$

The proof is thus complete.

Proposition 5.3.3. *There exists $k_0 \in \mathbb{N}^\star$ sufficiently large such that*

$$\sigma(\mathcal{A}) \supset \sigma_0 \cup \sigma_1,$$

where

$$\sigma_0 = \{\lambda_k\}_{k \in J}, \quad \sigma_1 = \{\lambda_k\}_{|k| \geq k_0}, \quad \sigma_0 \cap \sigma_1 = \emptyset.$$

Moreover, J is a finite set, and

$$\lambda_k = i\sqrt{\frac{a}{\rho_2}}\mu_k - \frac{\alpha^2}{2b\rho_2\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right). \tag{5.3.8}$$

Proof: From (5.3.6), we have

$$u = \frac{1}{\alpha\lambda}\left[\rho_2\lambda^2 y - a_2\Delta y\right]. \tag{5.3.9}$$

Inserting (5.3.9) in (5.3.5), we get the following equation

$$a_2(a_1+\lambda b)\Delta^2 y - \left[(a_2\rho_1+a_1\rho_2)\lambda^2+\rho_2 b\lambda^3\right]\Delta y + \lambda^2\left(\rho_1\rho_2\lambda^2+\alpha^2\right)y = 0. \tag{5.3.10}$$

Now, let φ_k be an normalized eigenvector of the following problem

$$\begin{cases} -\Delta\varphi_k = \mu_k^2\varphi_k & \text{in} \quad \Omega, \\ \varphi_k = 0 & \text{on} \quad \Gamma. \end{cases} \tag{5.3.11}$$

Then by taking $y=\varphi_k$ in (5.3.10), we deduce the following characteristic equation

$$P(\lambda) = \rho_1\rho_2\lambda^4 + \rho_2 b\mu_k^2\lambda^3 + \left[(a_2\rho_1+a_1\rho_2)\mu_k^2+\alpha^2\right]\lambda^2 + a_2 b\mu_k^4\lambda + a_1a_2\mu_k^4. \tag{5.3.12}$$

Next, taking $\xi = \frac{\lambda}{\mu_k}$ and $\zeta_k = \frac{1}{\mu_k}$ in (5.3.12), we get

$$\rho_2 b\xi^3 + a_2 b\xi + a_1a_2\xi_k + \alpha^2\xi^2\xi_k^3 + (a_2\rho_1+a_1\rho_2)\,\xi^2\xi_k + \rho_1\rho_2\xi^4\xi_k = 0. \tag{5.3.13}$$

Since $\frac{\lambda}{\mu_k}$ is bounded and $\xi_k \to 0$, then thanks to Rouché's Theorem, there exists k_0 large enough such that for all $|k| \geq k_0$ the large roots of the polynomial P are close to the roots of the polynomial

$$P_0(\xi) = \rho_2 b\xi^3 + a_2 b\xi.$$

Moreover, the large roots of P_0 satisfy the following asymptotic

$$\xi = i\sqrt{\frac{a_2}{\rho_2}}. \tag{5.3.14}$$

Then from equation (5.3.14) and Rouché's Theorem, we get the large roots of P satisfy the asymptotic equations

$$\xi = i\sqrt{\frac{a_2}{\rho_2}} + \varepsilon_k \quad \text{where} \quad \lim_{|k|\to+\infty}\varepsilon_k = 0. \tag{5.3.15}$$

Inserting equation (5.3.15) in equation (5.3.13), we get

$$\varepsilon_k = o\left(\frac{1}{\mu_k}\right) \quad \text{and} \quad \lambda_k = i\sqrt{\frac{a_2}{\rho_2}}\mu_k + \tilde{\varepsilon}_k, \quad \text{where} \quad \lim_{|k|\to+\infty}\tilde{\varepsilon}_k = 0. \tag{5.3.16}$$

Multiplying equation (5.3.12) $\frac{1}{\mu_k^4}$, we get

$$\frac{\rho_1\rho_2}{\mu_k^4}\lambda^4 + \frac{\rho_2 b}{\mu_k^2}\lambda^3 + \frac{(a_2\rho_1+a_1\rho_2)}{\mu_k^2}\lambda^2 + \frac{\alpha^2}{\mu_k^2}\lambda^2 + a_2 b\lambda + a_1a_2 = 0. \tag{5.3.17}$$

Inserting equation (5.3.16) in equation (5.3.17), we get

$$\tilde{\varepsilon}_k = -\frac{\alpha^2}{2b\rho_2\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right). \tag{5.3.18}$$

The proof thus is complete.

Proof of Theorem 5.3.1. From Proposition 5.3.3 the large eigenvalues in (5.3.8) approach the imaginary axis and therefore the system (5.1.1)-(5.1.3) is not uniformly stable in the energy space $\mathcal{H}$.

5.4 Polynomial Stability

In this section we prove that the system (5.1.1)-(5.1.2) is polynomially stable in the energy space $\mathcal{H}$. Throughout, this part, we assume that

$$a_1, a_2, \rho_1, \rho_2, b \in C^{1,1}(\overline{\Omega}). \tag{C1}$$

Also, we assume that the following supplementary conditions.
There exists a function $q \in C^1(\Omega, \mathbb{R}^N)$ and $0 < \alpha < \beta < \gamma$, such that

$$\partial_j q_k = \partial_k q_j, \quad \operatorname{div}(a_1 a_2 \rho_2 q),\ \operatorname{div}(a_1 a_2 \rho_1 q) \in C^{0,1}(\Omega_\beta) \quad \text{and} \quad q = 0 \quad \text{on} \quad O_\alpha, \tag{C2}$$

There exists a constant $\sigma_1 > 0$, such that

$$2a_2\partial_j(a_{1k}q_k) + a_1(q_k\partial_j a_2 + q_j\partial_k a_2) + \left[a_1\left(\frac{a_2}{\rho_2}q\nabla\rho_2 - q\nabla a_2\right)\right] I \geq \sigma_1 I, \quad \forall x \in \Omega_\beta. \tag{C3}$$

There exists a constant $\sigma_2 > 0$, such that

$$2a_1\partial_j(a_{2k}q_k) + a_2(q_k\partial_j a_1 + q_j\partial_k a_1) + \left[a_2\left(\frac{a_1}{\rho_1}q\nabla\rho_1 - q\nabla a_1\right)\right] I \geq \sigma_2 I, \quad \forall x \in \Omega_\beta. \tag{C4}$$

There exists a constant $M > 0$ such that for all $v \in H_0^1(\Omega)$, we have

$$|(q\cdot\nabla v)\,\nabla b - (q\cdot\nabla b)\nabla v| \leq M\sqrt{b}\,|\nabla v|, \quad \forall x \in \Omega_\beta. \tag{C5}$$

Theorem 5.4.1. *Assume that conditions* (H), (C1) − (C5) *are satisfied. Then for all initial data* $U_0 \in D(\mathcal{A})$*, there exists a constant* $C > 0$ *independent of* U_0*, such that the energy of the* *strong solution U of (5.2.4) satisfying the following estimation :*

$$E(t,U) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t>0. \tag{5.4.1}$$

In particular, for $U_0 \in \mathcal{H}$, the energy $E(t,U)$ converges to zero as t goes to infinity.

Following Borichev and Tomilov, a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ verify (5.4.1) if

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{H1}$$

and

$$\frac{1}{\lambda^2}\|(i\lambda-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty. \tag{H2}$$

We Know that condition $(H1)$ is verified. Our goal now is to prove that condition $(H2)$ is satisfied. To this aim, we proceed by a contradiction argument. Suppose that $(H2)$ does not hold, then there exist a sequence $(\lambda_n)_n \in \mathbb{R}$ and a sequence $(U_n) \subset D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = \|(u_n,v_n,y_n,z_n)\|_{\mathcal{H}} = 1 \tag{5.4.2}$$

and

$$\lambda_n^2(i\lambda-\mathcal{A})U = (f_1,g_1,f_2,g_2) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}, \tag{5.4.3}$$

are satisfied. For simplicity, We drop the index n . By detailing equation $(H2)$, we get the following system

$$i\lambda u - v = \frac{f_1}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad H_0^1(\Omega), \tag{5.4.4}$$

$$i\lambda\rho_1 v - (\mathrm{div}(a_1\nabla u + b\nabla v) - \alpha z) = \frac{\rho_1 g_1}{\lambda^2} \longrightarrow 0 \quad \text{in} \quad L^2(\Omega), \tag{5.4.5}$$

$$i\lambda y - z = \frac{f_2}{\lambda^\ell} \longrightarrow 0 \quad \text{in} \quad H_0^1(\Omega), \tag{5.4.6}$$

$$i\lambda\rho_2 z - (\mathrm{div}(a_2\nabla y) + \alpha v) = \frac{\rho_2 g_2}{\lambda^2} \longrightarrow 0 \quad \text{in} \quad L^2(\Omega). \tag{5.4.7}$$

Lemma 5.4.2. *Assume that conditions* (H), (C1) − (C5) *holds. Then the solution $(u,v,y,z) \in D(\mathcal{A})$ of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\|\nabla v\|_{L^2(O_\gamma)} = \frac{o(1)}{\lambda} \quad \textit{and} \quad \|v\|_{L^2(O_\gamma)} = \frac{o(1)}{\lambda}.$$

Proof: Multiply in $\mathcal{H}$ equation $(H2)$ by the uniformaly bounded sequence $U=(u,v,y,z)$, we get

$$\int_\Omega b(x)|\nabla v|^2 dx = -\Re\left(\langle(i\lambda I-\mathcal{A})U,U\rangle_{\mathcal{H}}\right) = \frac{o(1)}{\lambda^2}.$$

It follows that

$$\|b(x)\nabla v\|_{L^2(\Omega)} = \frac{o(1)}{\lambda}. \tag{5.4.8}$$

Using (C) and Poincaré inequality in equation (5.4.8), we get the second estimation desired.

Lemma 5.4.3. *Assume that conditions* (H), (C1) − (C5) *holds. Then, the solution* $(u, v, y, z) \in D(\mathcal{A})$ *of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \quad \|u\|_{L^2(O_\gamma)} = \frac{o(1)}{\lambda^2}, \text{ and } \|\nabla u\|_{L^2(O_\gamma)} = \frac{o(1)}{\lambda^2}.$$

Proof: First, using equations (5.4.4), (5.4.6) and the fact $\|U\|_{\mathcal{H}} = 1$, we get the first and the second estimation. Second using equation (5.4.4) and Lemma 5.4.2, we get the third and the fourth estimation. The proof has been completed.

Lemma 5.4.4. *Assume that conditions* (H), (C1) − (C5) *holds. Then, the solution* $(u, v, y, z) \in D(\mathcal{A})$ *of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\int_{O_\gamma} |z|^2 \, dx = o(1) \quad \text{and} \quad \int_{O_\gamma} |\nabla y|^2 dx = o(1). \tag{5.4.9}$$

Proof: We define the cut-off function $\eta \in C^1(\overline{\Omega})$ by

$$\eta = \begin{cases} 1 & \text{on} & O_{\gamma-\varepsilon}, \\ 0 & \text{on} & \Omega_\gamma, \\ 0 \leq \eta \leq 1 & \text{otherwith.} \end{cases}$$

Multiplying (5.4.5) by $\eta\bar{z}$ in $L^2(\Omega)$. Then using the fact that z is uniformly bounded in $L^2(\Omega)$ and g_1 converges to zero in $L^2(\Omega)$, we get

$$i\int_\Omega \lambda\eta v\bar{z}dx + \int_\Omega (a_1\nabla u + b\nabla v)\cdot\nabla\left(\frac{\eta\bar{z}}{\rho_1}\right)dx + \alpha\int_\Omega \frac{\eta}{\rho_1}|z|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{5.4.10}$$

Multiplying (5.4.7) by $\eta\bar{v}$ in $L^2(\Omega)$. Then using the fact that v is uniformly bounded in $L^2(\Omega)$ and g_2 converges to zero in $L^2(\Omega)$, we get

$$i\int_\Omega \lambda\eta z\bar{v}dx + \int_\Omega a_2\nabla y\cdot\nabla\left(\frac{\eta\bar{v}}{\rho_2}\right)dx - \alpha\int_\Omega \frac{\eta}{\rho_2}|v|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{5.4.11}$$

Adding equations (5.4.10)-(5.4.11), taking real part of the resulting equation and using Lemma 5.4.2, we get

$$\Re\left(\int_\Omega (a_1\nabla u + b\nabla v)\cdot\nabla\left(\frac{\eta\bar{z}}{\rho_1}\right)dx\right) + \alpha\int_\Omega \frac{\eta}{\rho_1}|z|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{5.4.12}$$

Using Lemmas 5.4.2, 5.4.3, and the fact that $\frac{\nabla z}{\lambda}$ and z are uniformly bounded in $L^2(\Omega)$, we get

$$\Re\left(\int_\Omega (a_1\nabla u + b\nabla v)\cdot\nabla\left(\frac{\eta\bar{z}}{\rho_1}\right)dx\right) = o(1). \tag{5.4.13}$$

From (5.4.12) and (5.4.13), we get

$$\int_\Omega \frac{\eta}{\rho_1}|z|^2dx = o(1). \tag{5.4.14}$$

Now, using (5.4.14), hypothesis (H) and the fact that $\eta = 1$ on O_γ, we get the first estimation of (5.4.9). Next, Multiplying (5.4.7) by $\eta\bar{y}$ in $L^2(\Omega)$. Then using the fact that λy is uniformly bounded in $L^2(\Omega)$ and g_2 converges to zero in $L^2(\Omega)$, we get

$$i\int_\Omega \eta\lambda z\bar{y}dx + \int_\Omega a_2\nabla y\cdot\nabla\left(\frac{\eta\bar{y}}{\rho_2}\right)dx - \alpha\int_\Omega \frac{\eta}{\rho_2}v\bar{y}dx = o\left(\frac{1}{\lambda^2}\right). \tag{5.4.15}$$

Using Lemmas 5.4.2, 5.4.3, and (5.4.14)-(5.4.15), we get

$$\int_\Omega a_2\nabla y\cdot\nabla\left(\frac{\eta\bar{y}}{\rho_2}\right)dx = o(1). \tag{5.4.16}$$

Using (5.4.16), the definition of η and the fact that $\nabla y,\ \lambda y$ are uniformly bounded in $L^2(\Omega)$, we get

$$\int_\Omega \frac{a_2\eta}{\rho_2}|\nabla y|^2dx = o(1). \tag{5.4.17}$$

Finally, using (5.4.17), condition (H) and the fact that $\eta = 1$ on O_γ, we get the second estimation of (5.4.9). Thus the proof is complete.

Corollary 5.4.5. *Assume that conditions* $(\mathrm{H}), (\mathrm{C1})-(\mathrm{C5})$ *holds. Then, from Lemmas 5.4.3-5.4.4 the solution* $(u,v,y,z)\in D(\mathcal{A})$ *of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\|U\|_{\mathcal{H}} = o(1) \quad \textit{on} \quad O_\gamma. \tag{5.4.18}$$

Now, we need to show that

$$\|U\|_{\mathcal{H}} = o(1) \quad \text{on} \quad \Omega_\beta.$$

Lemma 5.4.6. *Assume that conditions* $(\mathrm{H}), (\mathrm{C1})-(\mathrm{C6})$ *holds. Then, the solution* $(u,v,y,z)\in D(\mathcal{A})$ *of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\int_\Omega \frac{a_2}{2}\left(2a_2\partial_j(a_{1k}q_k) + a_1q_k\partial_ja_2 + a_1q_j\partial_ka_2\right)\partial_jy\partial_k\bar{y}dx + \int_\Omega \frac{a_1a_2}{2}\left[\frac{a_2}{\rho_2}q\cdot\nabla\rho_2 - q\cdot\nabla a_2\right]|\nabla y|^2dx \\ - \Re\left(i\alpha\lambda\int_\Omega a_1a_2uq\cdot\nabla\bar{y}dx\right) = o(1).$$

Proof: Define the following multiplier

$$N = a_2 \nabla y. \tag{5.4.19}$$

Eliminate v and z in equation (5.4.7) by (5.4.4) and (5.4.6), we get

$$-\lambda^2 \rho_2 y - \operatorname{div}(N) - i\alpha\lambda u = \frac{o(1)}{\lambda}. \tag{5.4.20}$$

Multiply equation (5.4.20) by $q \cdot a_1 \bar{N}$, we get

$$-\lambda^2 \int_\Omega \rho_2 y q \cdot a_1 \bar{N} dx - \int_\Omega \operatorname{div}(N) q \cdot a_1 \bar{N} dx - i\alpha \int_\Omega \lambda u q \cdot a_1 \bar{N} dx = \frac{o(1)}{\lambda}. \tag{5.4.21}$$

Using Green formula, we get

$$-\lambda^2 \Re\left(\int_\Omega \rho_2 y q \cdot a_1 \bar{N} dx\right) = -\lambda^2 \Re\left(\int_\Omega a_1 a_2 \rho_2 y q \cdot \nabla \bar{y} dx\right) = \frac{1}{2}\int_\Omega \operatorname{div}(a_1 a_2 \rho_2 q)|\lambda y|^2 dx. \tag{5.4.22}$$

Now, let $h \in C^{0,1}(\bar{\Omega})$. Multiply equation (5.4.20) by $h\bar{y}$, using Lemma 5.4.3, we get

$$-\int_\Omega h|\lambda y|^2 dx + \int_\Omega \frac{a_2}{\rho_2} h |\nabla y|^2 dx + \int_\Omega \frac{a_2}{\rho_2} \bar{y} \nabla h \cdot \nabla y dx = o(1). \tag{5.4.23}$$

Take $h = \operatorname{div}(a_1 a_2 \rho_2 q)$, using the fact ∇y is uniformly bounded in $L^2(\Omega)$ and Lemma 5.4.3 in equation (5.4.23), we get

$$\int_\Omega \operatorname{div}(a_1 a_2 \rho_2 q)|\lambda y|^2 dx = \int_\Omega \frac{a_2}{\rho_2} \operatorname{div}(a_1 a_2 \rho_2 q)|\nabla y|^2 dx + o(1). \tag{5.4.24}$$

From equations (5.4.22) and (5.4.24), we obtain

$$-\lambda^2 \Re\left(\int_\Omega \rho_2 y q \cdot a_1 \bar{N} dx\right) = \int_\Omega \frac{a_2}{2\rho_2} \operatorname{div}(a_1 a_2 \rho_2 q)|\nabla y|^2 dx + o(1). \tag{5.4.25}$$

In the another hand, we have

$$\begin{aligned}
-\Re \int_\Omega \operatorname{div}(N) q \cdot a_1 \bar{N} dx &= -\Re \int_\Omega \partial_j N_j q_k a_{1k} \bar{N}_k dx \\
&= \Re \int_\Omega (N_j \partial_j (a_{1k} q_k) \bar{N}_k + N_j q_k a_{1k} \partial_j \bar{N}_k) dx \\
&= \Re \int_\Omega (N_j \partial_j (a_{1k} q_k) \bar{N}_k + N_j q_k a_{1k} \partial_k \bar{N}_j) dx \\
&+ \Re \int_\Omega N_j q_k a_{1k} (\partial_j \bar{N}_k - \partial_k \bar{N}_j)) dx.
\end{aligned} \tag{5.4.26}$$

Using Green formula, we get

$$
\begin{aligned}
-\Re\left(\int_\Omega \operatorname{div}(N)q\cdot a_1\bar{N}dx\right) = {} & \Re\left(\int_\Omega \left(N_j\partial_j(a_{1k}q_k)\bar{N}_k - \frac{1}{2}\operatorname{div}(a_1q)|N|^2\right)dx\right)\\
& + \Re\left(\int_\Omega N\cdot[(a_1q\cdot\nabla\bar{y})\nabla a_2 - (a_1q\cdot\nabla a_2)\nabla\bar{y}]dx\right).
\end{aligned}
\tag{5.4.27}
$$

Inserting equations (5.4.25) and (5.4.27) in (5.4.21), we get

$$
\begin{aligned}
\int_\Omega \frac{a_2}{2\rho_2}\operatorname{div}(a_1a_2\rho_2q)|\nabla y|^2dx + \Re\Bigg(\int_\Omega & \left(N_j\partial_j(a_{1k}q_k)\bar{N}_k - \frac{1}{2}\operatorname{div}(a_1q)|N|^2\right)dx\\
& + \int_\Omega N\cdot[(a_1q\cdot\nabla\bar{y})\nabla a_2 - (a_1q\cdot\nabla a_2)\nabla\bar{y}]dx\Bigg)\\
& -\Re\left(i\alpha\lambda\int_\Omega a_1a_2uq\cdot\nabla\bar{y}dx\right) = o(1).
\end{aligned}
\tag{5.4.28}
$$

It follows that

$$
\begin{aligned}
\int_\Omega \frac{a_2}{2\rho_2}\operatorname{div}(a_1a_2\rho_2q)|\nabla y|^2dx + \Re\Bigg(\int_\Omega & a_2^2\left(\partial_jy\partial_j(a_{1k}q_k)\partial_k\bar{y} - \frac{1}{2}\operatorname{div}(a_1q)|\nabla y|^2\right)dx\\
& + \int_\Omega a_1a_2\nabla y\cdot[(q\cdot\nabla\bar{y})\nabla a_2 - (q\cdot\nabla a_2)\nabla\bar{y}]dx\Bigg)\\
& -\Re\left(i\alpha\lambda\int_\Omega a_1a_2uq\cdot\nabla\bar{y}dx\right) = o(1).
\end{aligned}
\tag{5.4.29}
$$

A direct calculation, gives

$$
\begin{aligned}
\int_\Omega \frac{a_2}{2\rho_2}\operatorname{div}(a_1a_2\rho_2q)|\nabla y|^2dx + \Re\left(\int_\Omega a_2^2\left(\partial_jy\partial_j(a_{1k}q_k)\partial_k\bar{y} - \frac{1}{2}\operatorname{div}(a_1q)|\nabla y|^2\right)dx\right)&\\
+\Re\left(\int_\Omega \left[\frac{a_1a_2}{2}(q_k\partial_ja_2 + q_j\partial_ka_2)\partial_jy\partial_k\bar{y} - a_1a_2q\cdot\nabla a_2|\nabla y|^2\right]dx\right)&\\
-\Re\left(i\alpha\lambda\int_\Omega a_1a_2uq\cdot\nabla\bar{y}dx\right) = o(1).&
\end{aligned}
\tag{5.4.30}
$$

This implies that

$$
\begin{aligned}
\int_\Omega \frac{a_2}{2}\Big[(2a_2\partial_j(a_{1k}q_k) + a_1q_k\partial_ja_2 + a_1q_j\partial_ka_2)\partial_jy\partial_k\bar{y}\Big]dx&\\
+\int_\Omega \frac{a_1a_2}{2}\left(\frac{a_2}{\rho_2}q\cdot\nabla\rho_2 - q\cdot\nabla a_2\right)|\nabla y|^2dx&\\
-\Re\left(i\alpha\lambda\int_\Omega a_1a_2uq\cdot\nabla\bar{y}dx\right) = o(1).&
\end{aligned}
\tag{5.4.31}
$$

The proof has been completed.

Lemma 5.4.7. *Assume that conditions* (H), (C1) − (C5) *holds. Then, the solution* $(u, v, y, z) \in$

$D(\mathcal{A})$ of equations (5.4.4)-(5.4.7) satisfying the following estimations

$$\int_\Omega \frac{a_1}{2}\left(2a_1\partial_j(a_{2k}q_k)+a_2q_k\partial_j a_2+a_2q_j\partial_k a_2\right)\partial_j u\partial_k\bar{u}dx+\int_\Omega \frac{a_1a_2}{2}\left[\frac{a_1}{\rho_1}q\cdot\nabla\rho_1-q\cdot\nabla a_1\right]|\nabla y|^2dx \\ -\Re\left(i\alpha\lambda\int_\Omega a_1a_2yq\cdot\nabla\bar{u}dx\right)=o(1).$$

Proof: Define the following multiplier

$$M=a_1\nabla u+b\nabla v. \tag{5.4.32}$$

From equations (5.4.4), (5.4.6) and (5.4.5) we have

$$-\lambda^2\rho_1u-\operatorname{div}(a_1\nabla u+b\nabla v)+i\alpha\lambda y=\frac{o(1)}{\lambda}. \tag{5.4.33}$$

Multiplying equation (5.4.32) by $q\cdot a_2\bar{M}$, we obtain

$$-\lambda^2\int_\Omega \rho_1uq\cdot a_2\bar{M}dx-\int_\Omega \operatorname{div}(a_1\nabla u+b\nabla v)q\cdot a_2\bar{M}dx+i\alpha\int_\Omega yq\cdot a_2\bar{M}dx=\frac{o(1)}{\lambda}. \tag{5.4.34}$$

Using Green formula, we get

$$\begin{aligned}-\lambda^2\Re\left(\int_\Omega \rho_1uq\cdot a_2\bar{M}dx\right)&=-\lambda^2\Re\left(\int_\Omega a_2\rho_1uq\cdot(a_1\nabla u+b\nabla v)dx\right)\\&=\frac{1}{2}\int_\Omega \operatorname{div}(a_1a_2\rho_1q)|\lambda u|^2dx+o(1).\end{aligned} \tag{5.4.35}$$

Let $\hat{h}\in C^{0,1}(\bar{\Omega})$. Multiplying equations (5.4.4) and (5.4.5) by $i\lambda\hat{h}u$ and $\hat{h}u$ respectively and using Lemma 5.4.3- 5.4.4, we get

$$-\int_\Omega \hat{h}|\lambda u|^2dx-i\lambda\int_\Omega hv\bar{u}dx \;=\; \frac{o(1)}{\lambda^2},, \tag{5.4.36}$$

$$i\lambda\int_\Omega \hat{h}v\bar{u}dx+\int_\Omega \frac{a_1}{\rho_1}\hat{h}|\nabla u|^2dx \;=\; o(1),. \tag{5.4.37}$$

Adding equations (5.4.36), (5.4.37) and take the reel part, we get

$$-\int_\Omega \hat{h}|\lambda u|^2dx+\int_\Omega \frac{a_1}{\rho_1}\hat{h}|\nabla u|^2dx=o(1). \tag{5.4.38}$$

Using equation (5.4.35), (5.4.38) and take $\hat{h}=\operatorname{div}(a_1a_2\rho_1q)$, we get

$$-\lambda^2\Re\left(\int_\Omega \rho_1uq\cdot a_2\bar{M}dx\right)=\int_\Omega \frac{a_2}{2\rho_2}\operatorname{div}(a_1a_2\rho_1q)|\nabla u|^2dx+o(1). \tag{5.4.39}$$

Furthermore, we have

$$\begin{aligned}
-\Re\int_\Omega \operatorname{div}(M)q\cdot a_2\bar{M}dx &= -\Re\int_\Omega \partial_j M_j q_k a_{2k}\bar{M}_k dx\\
&= \Re\int_\Omega (M_j\partial_j(a_{2k}q_k)\bar{M}_k + M_j q_k a_{2k}\partial_j\bar{M}_k)dx\\
&= \Re\int_\Omega (M_j\partial_j(a_{2k}q_k)\bar{M}_k + M_j q_k a_{2k}\partial_k\bar{M}_j)dx\\
&+ \Re\int_\Omega M_j a_{2k} q_k(\partial_j\bar{M}_k - \partial_k\bar{M}_j))dx.
\end{aligned} \tag{5.4.40}$$

Using Green formula in equation (5.4.40), we get

$$\begin{aligned}
-\Re\left(\int_\Omega \operatorname{div}(M)q\cdot a_2\bar{M}dx\right) &= \Re\left(\int_\Omega M_j\partial_j(a_{2k}q_k)\bar{M}_k - \frac{1}{2}|M|^2\operatorname{div}(a_2 q)dx\right)\\
&+ \Re\left(\int_\Omega M\cdot\left[(a_2 q\cdot\nabla\bar{u})\nabla a_1 - (a_2 q\cdot\nabla a_1)\nabla\bar{u}\right]dx\right)\\
&+ \Re\left(\int_\Omega a_2 M\cdot\left[(q\cdot\nabla\bar{v})\nabla b - (q\cdot\nabla b)\nabla\bar{v}\right]dx\right).
\end{aligned} \tag{5.4.41}$$

Using Condition $(C5)$, then from equation (5.4.41) we get

$$\begin{aligned}
\int_\Omega \frac{a_1}{2\rho_1}\operatorname{div}(a_1a_2\rho_1 q)|\nabla u|^2 dx + \Re\left(\int_\Omega M_j\partial_j(a_2k)\bar{M}_k - \frac{1}{2}|M|^2\operatorname{div}(a_2 q)dx\right)&\\
+ \Re\left(\int_\Omega M\cdot\left[(a_2 q\cdot\nabla\bar{u})\nabla a_1 - (a_2 q\cdot\nabla a_1)\nabla\bar{u}\right]dx\right)&\\
+ \Re\left(i\alpha\lambda\int_\Omega a_1a_2 y q\cdot\nabla\bar{u}dx\right) = o(1).&
\end{aligned} \tag{5.4.42}$$

A direct calculation gives,

$$\begin{aligned}
\int_\Omega \frac{a_1}{2\rho_1}\operatorname{div}(a_1a_2\rho_1 q)|\nabla u|^2 dx + \Re\left(\int_\Omega a_1^2\left(\partial_j u\partial_j(a_{2k}q_k)\partial_k\bar{u} - \frac{1}{2}|M|^2\operatorname{div}(a_2 q)\right)dx\right)&\\
+\Re\left(\int_\Omega \frac{a_1a_2}{2}(q_k\partial_j a_1 + q_j\partial_k a_1)\partial_j u\partial_k\bar{u} - a_1a_2 q\cdot\nabla a_1|\nabla u|^2 dx\right)&\\
+\Re\left(i\alpha\lambda\int_\Omega a_1a_2 y q\cdot\nabla\bar{u}dx\right) = o(1).&
\end{aligned} \tag{5.4.43}$$

This implies that

$$\begin{aligned}
\int_\Omega \frac{a_1}{2}(2a_1\partial_j(a_{2k}q_k) + a_2 q_k\partial_j a_1 + a_2 q_j\partial_k a_1)\partial_j u\partial_k\bar{u}dx&\\
+\int_\Omega \frac{a_1a_2}{2}\left[\frac{a_1}{\rho_1}q\cdot\nabla\rho_1 - q\cdot\nabla a_1\right]|\nabla u|^2 dx&\\
+\Re\left(i\alpha\lambda\int_\Omega a_1a_2 y q\cdot\nabla\bar{u}dx\right) = o(1).&
\end{aligned} \tag{5.4.44}$$

The proof has been completed.

Lemma 5.4.8. *Assume that conditions* $(\mathrm{H}),(\mathrm{C1})-(\mathrm{C5})$ *holds. Then, the solution* $(u,v,y,z)\in D(\mathcal{A})$ *of equations* (5.4.4)-(5.4.7) *satisfying the following estimations*

$$\|U\|_{\mathcal{H}}=o(1) \quad on \quad \Omega_\beta$$

.

Proof: Adding equations (5.4.31) and (5.4.44), it easy to see that

$$\int_\Omega A_1\partial_j y\partial_k\bar{y}dx+\int_\Omega A_2|\nabla y|^2dx+\int_\Omega A_3\partial_j u\partial_k\bar{u}dx+\int_\Omega A_4|\nabla u|^2dx=o(1), \tag{5.4.45}$$

where

$$\begin{cases} A_1=\dfrac{a_2}{2}((2a_2\partial_j(a_{1k}q_k)+a_1q_k\partial_ja_2+a_1q_j\partial_ka_2)),\\ A_2=\dfrac{a_1a_2}{2}\left(\dfrac{a_2}{\rho_2}q\cdot\nabla\rho_2-q\cdot\nabla a_2\right),\\ A_3=\dfrac{a_1}{2}(2a_1\partial_j(a_{2k}q_k)+a_2q_k\partial_ja_1+a_2q_j\partial_ka_1),\\ A_4=\dfrac{a_1a_2}{2}\left[\dfrac{a_1}{\rho_1}q\cdot\nabla\rho_1-q\cdot\nabla a_1\right]. \end{cases}$$

Using condition (C_3) and (C_4) in equation (5.4.45), then we get

$$\|\nabla u\|_{L^2(\Omega_\beta)}=o(1) \quad \text{and} \quad \|\nabla y\|_{L^2(\Omega_\beta)}=o(1). \tag{5.4.46}$$

Consequently, from equations (5.4.24), (5.4.38) and (5.4.46), we obtain

$$\|U\|_{\mathcal{H}}=o(1) \quad \text{on} \quad \Omega_\beta.$$

The proof has been completed.

Proof of Theorem 5.4.1 Using The fact $\|U\|_{\mathcal{H}}=o(1)$ on O_γ and Lemma 5.4.8, we get $\|U\|_{\mathcal{H}}=o\,(1)$ over Ω which contradicts (H2). This implies that

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda Id-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=O\left(\lambda^2\right).$$

The result follows from [18].

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Université Libanaise

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Thèse de Doctorat

Mohammad AKIL

Quelques problèmes de stabilisation directe et indirecte d'équations d'ondes par des contrôles de type fractionnaire frontière ou de type Kelvin-Voight localisé

Résumé

Cette thèse est consacrée à l'étude de la stabilisation directe et indirecte de différents systèmes d'équations d'ondes avec un contrôle frontière de type fractionnaire ou un contrôle local viscoélastique de type Kelvin-Voight. Nous considérons, d'abords, la stabilisation de l'équation d'ondes multidimensionnel avec un contrôle frontière fractionnaire au sens de Caputo. Sous des conditions géométriques optimales, nous établissons un taux de décroissance polynomial de l'énergie de système. Ensuite, nous nous intéressons à l'étude de la stabilisation d'un système de deux équations d'ondes couplées via les termes de vitesses, dont une seulement est amortie avec contrôle frontière de type fractionnaire au sens de Caputo. Nous montrons différents résultats de stabilités dans le cas 1-d et N-d. Finalement, nous étudions la stabilité d'un système de deux équations d'ondes couplées avec un seul amortissement viscoélastique localement distribué de type Kelvin-Voight.

Abstract

This thesis is devoted to study the stabilization of the system of waves equations with one boundary fractional damping acting on apart of the boundary of the domain and the stabilization of a system of waves equations with locally viscoelastic damping of Kelvin-Voight type. First, we study the stability of the multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. Second, we study the stability of the system of coupled one-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Next, we study the stability of the system of coupled multi-dimensional wave equation with one fractional damping acting on a part of the boundary of the domain. Finally, we study the stability of the multidimensional waves equations with locally viscoelastic damping of Kelvin-Voight is applied for one equation around the boundary of the domain.

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