Positivity and qualitative properties of solutions of fourth-order elliptic equations

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Abstract

This thesis concerns the study of semilinear fourth-order elliptic boundary value problems and, in particular, deals with qualitative properties of the solutions. Such problems arise in various fields, from Plate Theory to Conformal Geometry and, compared to their second-order counterparts, they present intrinsic difficulties, mainly due to the lack of the maximum principle.

It is well-known that in presence of Dirichlet boundary conditions the positivity preserving property does not hold in general. On the other hand, if we consider Navier boundary conditions, it is easy to infer positivity by decoupling the fourth-order problem into a system of second-order problems. Therefore, it is interesting to study this question in the case of *Steklov* boundary conditions, which are an intermediate situation between the former and the latter. These conditions naturally appear in the study of the minimizers of the *Kirchhoff-Love functional*, which represents the energy of a hinged and loaded thin plate in dependence of a parameter σ, usually referred to as the Poisson ratio.

In the first part of the thesis we find sufficient assumptions on the domain to obtain the *positivity* of the minimizer of the Kirchhoff-Love functional in the physical relevant context regarding the parameter σ, extending a previous result by Parini and Stylianou in [75]. Then, for such domains, we study a generalized version of the functional, which corresponds to a *semilinear* Steklov boundary value problem. In particular, using variational techniques, we investigate existence and positivity of the ground states, as well as their asymptotic behaviour for the relevant values of σ.

The second part of the work is focused on the special context of the *critical dimension* for the biharmonic operator, namely $\mathbb{R}^4$. Indeed, the well-known Trudinger-Moser inequality, generalized to polyharmonic operators by Adams, allows to consider exponential nonlinearities. In this setting, we establish *uniform a-priori bounds* for solutions of a fourth-order semilinear problem endowed either with Dirichlet or Navier boundary conditions, with a rather general positive and subcritical nonlinearity. Our results complete the picture for the a-priori bounds issue in the polyharmonic context, and complement the works of Oswald [73] and Soranzo [85], which considered the same problem in the subcritical dimensions, namely $\mathbb{R}^N$ with $N \geq 5$. Our argument combines some uniform estimates near the boundary introduced by de Figuereido, Lions and Nussbaum in [29] and a blow-up analysis in the spirit of Robert and Wei, [81]. Finally, using Krasnosel'skii degree theory, we infer the existence of a positive solution for these problems. Our results apply in the case of the ball and, under an additional assumption on the solutions, extend to general smooth domains.

Résumé

Cette thèse concerne l'étude de certains problèmes elliptiques sémilinéaries d'ordre 4 et, notamment, des propriétés qualitatives des solutions. Ces problèmes apparaîssent dans nombreux domaines, par example dans la Théorie des Plaques et dans la Géometrie Conforme et, comparés aux leur homologues du deuxième ordre, ils présentent des difficultés intrinsèques, surtout liées à l'absence d'un principe de maximum.

Il est bien connu que la propriété de préservation de la positivité n'est pas valable en présence des conditions au bord de Dirichlet, en général. De l'autre côté, si on considère les conditions au bord de Navier, on peut la montrer aisément, en découplant le problème au quatrième ordre en un système d'ordre deux. Pour cette raison, il est intéressant d'étudier le cas des conditions au bord de *Steklov*, qui se posent entre les deux susmentionnées. Ces conditions apparaîssent naturellement dans l'étude des minimiseurs de la *fonctionnelle de Kirchhoff-Love*, qui représente l'énergie d'une plaque encastrée soumise à l'action d'une force extérieure, en dépendence d'un paramètre σ, qui prend le nom de rapport de Poisson.

Dans la première partie de la thèse on trouve des conditions suffisantes sur le domaine telles que les minimiseurs de la fonctionnelle de Kirchhoff-Love soient *positifs* dans le cadre physiquement significatif par rapport au paramètre σ, en généralisant un résultat de Parini et Stylianou de [75]. En plus, pour ces domaines, on étudie une version généralisée de la fonctionnelle qui correspond à un problème *semilinéaire* avec conditions de Steklov. En particulier, en utilisant des techniques variationnelles, on examine l'existence et la positivité des états fondamentaux, ainsi que leur comportement asymptotique pour les valeurs significatives de σ.

La deuxième partie de la thèse est consacrée au cadre special de la dimension critique pour l'opérateur biharmonique, notamment $\mathbb{R}^4$. En fait, la célèbre inégalité de Trudinger-Moser, généralisée par Adams aux opérateurs polyharmoniques, permet de considérer des nonlinéarités exponentielles. Dans ce contexte, on établie des *estimations uniformes a-priori* pour les solutions des problèmes semilinéaires d'ordre 4 avec conditions au bord de Dirichlet ou Navier et avec une nonlinéarité positive et souscritique assez générale. Ces résultats complètent le tableau pour la question des estimations a-priori dans le contexte polyharmonique et ils s'ajoutent aux travaux de Oswald [73] et de Soranzo [85], qui considéraient le même problème dans les dimensions souscritiques, c'est-à-dire dans $\mathbb{R}^N$ avec $N \geq 5$. Nos arguments combinent des estimations uniformes proche du bord qui ont été introduites par de Figuereido, Lions et Nussbaum dans [29] et une méthode de blow-up dans l'esprit de Robert et Wei, [81]. Enfin, en utilisant la théorie du degré de Krasnosel'skii, on obtient l'existence d'une solution positive pour ces problèmes. Nos résultats s'appliquent dans le cas de la boule et, sous une condition supplémentaire sur les solutions, ils s'ètèndent aux domaines réguliers bornés.

Sommario

Principale argomento di questa tesi è lo studio di problemi ellittici semilineari di ordine quattro ed in particolare delle proprietà qualitative delle soluzioni. Problemi di questo tipo hanno origine in diversi ambiti, ad esempio nella Teoria delle Piastre nella Geometria Conforme, e, rispetto agli analoghi del second'ordine, presentano intrinseche difficoltà, principalmente dovute all'assenza del principio di massimo.

È noto che in presenza di condizioni al bordo di tipo Dirichlet in generale non si ha la proprietà di preservazione della positività; d'altro canto, se si considerano le condizioni al bordo di tipo Navier, è agevole dedurre la positività scomponendo il problema del quart'ordine in un sistema di problemi di ordine due. Pertanto è interessante studiare la medesima questione considerando le condizioni al bordo di tipo *Steklov*, che si pongono come un caso intermedio tra le due precedenti. Queste ultime sorgono in maniera naturale nello studio dei minimi del *funzionale di Kirchhoff-Love,* il quale rappresenta, in dipendenza da un parametro σ, detto rapporto di Poisson, l'energia di una piastra incastrata sottoposta all'azione di una forza esterna.

Nella prima parte della tesi si ottengono condizioni sufficienti sul dominio per stabilire la positività dei minimi del funzionale di Kirchhoff-Love nel contesto fisicamente rilevante per il parametro σ, estendendo un precedente risultato di Parini e Stylianou in [75]. In seguito, per tali dominii viene studiata una generalizzazione del funzionale che corrisponde ad un problema *semilineare* con condizioni di Steklov. In particolare, tramite tecniche variazionali, si studiano esistenza e positività dei ground states, oltre al loro comportamento asintotico per i valori significativi di σ.

La seconda parte della tesi è dedicata al caso speciale della *dimensione critica* per l'operatore biarmonico, ossia $\mathbb{R}^4$. Infatti la celebre disuguaglianza di Trudinger-Moser, generalizzata ad operatori poliarmonici da Adams, consente di considerare nonlinearità esponenziali. In questo contesto, otteniamo stime uniformi a priori per le soluzioni di problemi semilineari di ordine 4, sia in presenza di condizioni al bordo di Dirichlet che di Navier, e per nonlinearità positive e sottocritiche di carattere piuttosto generale. I nostri risultati completano il quadro per tale questione nel contesto poliarmonico e complementano i lavori di Oswald [73] e Soranzo [85], i quali considerarono il medesimo problema nelle dimensioni sottocritiche, ossia in $\mathbb{R}^N$ con $N \geq 5$. La nostra analisi si fonda principalmente su alcune stime uniformi in un intorno del bordo, prendendo spunto dalle corrispettive di de Figuereido, Lions e Nussbaum in [29], e su una analisi di tipo blow-up nello spirito di Robert e Wei, [81]. Infine, utilizzando la teoria del grado topologico di Krasnosel'skii, viene dedotta l'esistenza di una soluzione positiva per tali problemi. I nostri risultati si applicano al caso della bolla e, imponendo una condizione ulteriore sulle soluzioni, si estendono a domìni limitati regolari.

Contents

Chapter 1

Introduction

In the last decades, fourth order PDEs (and, more in general, higher-order PDEs) have become an important research field. Arising from Physics or Differential Geometry, they are very challenging from an analytical point of view. One of the main well-known reasons is that, in general, for these problems the maximum principle does not hold, and consequently many familiar techniques from second-order equations do not extend to this context. Therefore, the development of several new methods for their investigation turned out to be necessary.
In this thesis, the central focus is on semilinear fourth-order boundary value problems of the kind

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ B(x,u,D^\alpha u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, namely an open and connected subset of $\mathbb{R}^N$, $N \geq 2$, the bilaplace operator Δ^2 is defined as $f \mapsto \Delta^2 f := -\Delta(-\Delta f)$, and $B(x,u,D^\alpha u)$ are compatible boundary conditions involving also the derivatives of u. Moreover, in general, h will be a subcritical nonlinearity. We are mainly interested in proving existence/nonexistence and positivity of (weak) solutions of (1.1), as well as a priori estimates.

1.1 The positivity preserving property

The first case one usually studies when dealing with problems of the form (1.1), is the one in which h does not depend on the unknown u, that is, the *linear* equation

$$\Delta^2 u = h(x) \qquad \text{in } \Omega. \tag{1.2}$$

This problem arises in the theory of plates: in this setting, $\Omega \subset \mathbb{R}^2$ models a thin plate and u represents the displacement from the unloaded horizontal position, when subjected to an external force h. The first natural question that one may wonder is the following: if the direction of the force is the same at every point, does the whole plate bend in a coherent way or, conversely, are there some areas in which it moves in the opposite direction? Expressing the question in mathematical terms: if $h \geq 0$, is it true that $u \geq 0$ in Ω? When this holds, we say that the problem (1.2) satisfies the *Positivity Preserving Property* (shortly denoted by *PPP*).

A large part of the literature tried to give a complete description of this phenomenon. It turns out that the boundary conditions prescribed on $\partial\Omega$ have a strong impact to the behaviour of solutions. As a first choice, one may impose *Navier boundary conditions*, namely

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega. \tag{1.3}$$

As a consequence, (1.2)-(1.3) splits into a system of second-order Dirichlet problems by setting $v := -\Delta u$ and the standard maximum principle applies.
If instead one considers the second natural choice to complement equation (1.2), that is, *Dirichlet boundary conditions*

$$u = u_n = 0 \qquad \text{on } \partial\Omega, \tag{1.4}$$

where u_n stands for the outer normal derivative on $\partial\Omega$, the situation is completely different. Indeed, in this setting it is still an open question to characterize the class of domains for which the PPP holds. So far, we know that for balls this property is ensured, as the classical work of Boggio [12] shows; on the other hand, contrary to the general expectation and to the first conjectures by Hadamard and Boggio himself, it is not true that convexity and/or smoothness of the boundary are sufficient conditions. There are several counterexamples in this direction: we refer to [91] for a short survey or to the monograph [40] for details. Quite recently, some authors, among which Dall'Acqua, Grunau, Robert and Sweers, have found some classes of domains for which the PPP holds, namely, small smooth deformations of the ball and of the limaçon, the latter in dimension 2 (see [25, 47]).

In the study of the thin plate model, a third kind of boundary conditions naturally comes out, named after *Steklov* for their first appearance in [88]:

$$u = \Delta u - a(x)u_n = 0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

where, in general, a is a continuous function on $\partial\Omega$. Indeed, the elastic energy of a thin loaded plate is modeled by the *Kirchhoff-Love functional*

$$I_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega fu. \tag{1.6}$$

Here, f is the density of the force applied to the plate and σ is usually referred to as the Poisson ratio, a parameter which measures the transverse expansion (resp. contraction) of the material, according to its positive (resp. negative) sign, when subjected to an external compressing force. More precisely, it is defined by $\sigma := \frac{\lambda}{2(\lambda+\mu)}$, where the Lamé constants λ, μ, depend on the material, and usually there holds $\lambda \geq 0$ and $\mu > 0$, so that $0 \leq \sigma \leq \frac{1}{2}$. Although there exist some exotic materials with negative Poisson ratio (see [55]), it is always true that $\sigma > -1$.

If now we assume that the plate is *hinged* on its boundary, namely fixed ($u = 0$ on $\partial\Omega$) but, unlike the clamped case, we do not prescribe $u_n = 0$ on $\partial\Omega$, then the natural context to settle the problem is the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$. Minimizing the energy I_σ in this space gives the Euler equation

$$\int_\Omega \Big(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) - fv\Big)dxdy = 0$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Assuming more regularity of the minimizer u, that is, $u \in H^4(\Omega)$ and then integrating by parts, one finds that u is the solution of the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where κ stands for the *signed curvature* of $\partial\Omega$ (positive on strictly convex parts). Notice that the second term in the energy functional I_σ has no influence on the equation: its contribution is in the second boundary condition, which comes out when integrating by parts. These Steklov boundary conditions may thus be considered as an intermediate situation between the Navier (when $\sigma = 1$) and the Dirichlet boundary conditions (seen as the limit case as $\sigma \to +\infty$). For further details on the physical model we refer to [97] as well as [22, p.250], while on the derivation of (1.7) to [40, 42, 16, 92].

The question whether the PPP holds for the general Steklov problem (1.2)-(1.5) has been addressed in [42] and a complete description of the possible scenarios has been found: nonexistence, existence and positivity, existence without necessarily positivity. However, these results for the solutions of (1.2)-(1.5) do not apply immediately to the case of the minimizers of the Kirchhoff-Love functional I_σ. Indeed, the two problems are equivalent only in the case of a smooth boundary. Parini and Stylianou in [75] investigated directly this problem, finding that *if Ω is convex and its boundary is sufficiently smooth, namely of class $C^{2,1}$*, then one can apply the results in [42] and thus obtain that any minimizer of I_σ is positive, once a positive source $f \in L^2(\Omega)$ is applied. The key point in their work consists of showing that under these assumptions the functional I_σ may be rewritten in a more convenient way, namely

$$I_\sigma(u) = \int_\Omega \frac{|\Delta u|^2}{2} - \frac{(1-\sigma)}{2} \int_{\partial\Omega} \kappa u_n^2 - \int_\Omega f u. \tag{1.8}$$

However, the high regularity which the authors assumed on $\partial\Omega$ was needed only for technical reasons. In the first part of this thesis, we manage to relax the assumptions on the boundary and obtain the same positivity statement:

Theorem 1.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with boundary of class $C^{1,1}$. Suppose $\sigma \in (-1, 1]$ and $0 \leq f \in L^2(\Omega)$. Then the minimizer of the Kirchhoff-Love functional I_σ is positive in Ω.*

The proof is achieved in the same spirit of [75] by means of a different density argument, which allows to have a less regular boundary. Notice that these are the least hypothesis we can consider to have a well-defined curvature $\kappa \in L^\infty(\Omega)$ in (1.7). We believe that the result remains true assuming only for instance Lipschitz regularity on $\partial\Omega$ and, thus, considering directly the functional (1.6), but a practicable way is still not clear.

The second main topic of the thesis is the study of semilinear problems of the form (1.1). Again, the choice of the boundary conditions plays a big role. In general, fourth-order problem like (1.1) are often endowed with Dirichlet or Navier boundary conditions. To have an overview of the main results available in the literature, the best reference in this field is the book [40], where several eigenvalue and semilinear problems are collected. In this thesis, we investigate two semilinear problems which arise in connection with two different questions in the field. The first one is the direct prosecution of the previous analysis of the Steklov boundary value problem (1.7) considering now a nonlinearity $h = h(x, u)$, typically of power growth in the second variable. The main interest is to prove the *positivity of least-energy solutions* in dependence of the boundary parameter σ, by means of variational methods, and Chapter 2 is dedicated to this investigation. The second part of the thesis, contained in Chapter 3, is devoted to the study of a semilinear Dirichlet problem in $\mathbb{R}^4$, the critical dimension for the fourth-order Sobolev embeddings. This means that by the Adams extension of the well-known Trudinger-Moser inequality, exponential nonlinearities are allowed. Our main interest is to prove *uniform a-priori estimates* for weak solutions, a crucial step in order to infer the existence by topological methods.

Finally, we let at the end of the thesis a brief chapter where we point out some open problems which arose during the present study, as well as an Appendix containing some known results, which are useful in our analysis.

Before entering into the details of Chapters 2 and 3, we give here a brief overview of those problems and sketch our principal results.

1.2 A semilinear problem associated to the Kirchhoff-Love functional

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $F : \Omega \times \mathbb{R} \to \mathbb{R}^+$. We study the following generalization of the Kirchhoff-Love functional I_σ:

$$J_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega F(x,u)dx,$$

where now we let σ be any parameter in $\mathbb{R}$. In case we restrict σ to the physical meaningful interval, the functional J_σ represents the elastic energy of a thin plate with a fixed boundary and subjected to a density load which is related to the displacement of the plate itself. The reader may consider F as a model for an elastic force.

Our interest is focused on the least-energy critical points of J_σ, usually referred to as *ground states*. In fact, these are the most interesting critical points from a physical point of view and, moreover, a large number of variational techniques apply. As mentioned before, in presence of a smooth boundary and via integration by parts, one sees that ground states of J_σ correspond to least-energy solutions of the problem

$$\begin{cases} \Delta^2 u = f(x,u) & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $f(x,s) := \partial_s F(x,s)$.

In the literature, semilinear Steklov boundary value problems of kind (1.9) have begun to appear quite recently in a series of papers by Gazzola, Berchio, Weth and many co-authors. They considered the associated eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - d u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

highlighting typical features which make it really different from the usual second-order eigenvalue problems, especially when dealing with isoperimetric issues (see [38, 16, 6]). Moreover some subcritical and critical problems are addressed in [8, 9, 41, 7], in the case of general Steklov boundary conditions (1.5) with a constant $a \in \mathbb{R}$.

Our aim, instead, is to provide a rather complete description of the ground states of the semilinear Kirchhoff-Love functional J_σ, with particular attention to their existence and positivity. Setting $f(x,u) = g(x)|u|^{p-1}u$ with $p \in (0,1) \cup (1,+\infty)$ and $0 < g \in L^1(\Omega)$, first we prove a nonexistence result for large negative values of σ (namely, for $\sigma \leq \sigma^*(\Omega) \leq -1$) and existence for the complementary interval $\sigma > \sigma^*$ by some variational arguments involving the *Nehari manifold*. Indeed, it is well-known that the ground state solutions correspond to the minima on the Nehari set

$$\mathcal{N} := \{u \in H^2(\Omega) \cap H^1_0(\Omega) \setminus \{0\} \mid J'_\sigma(u)[u] = 0\},$$

where J'_σ denotes the Fréchet derivative of the functional J_σ. This characterization will be also used in the proof of their positivity. To this aim, we use different techniques according to the value of σ and again, as the linear case, the convexity of the domain is necessary in our arguments. Indeed, assuming $\partial\Omega$ of class $C^{1,1}$ and exploiting the equivalent form of J_σ, if Ω is convex, then the curvature κ is positive and the second term in (1.8) has a sign. Thus, a distinction between the cases $\sigma \in (\sigma^*, 1)$ and $\sigma > 1$ comes naturally. In the first one, which contains the physical relevant interval, we deduce positivity comparing the value of J_σ on a ground state with that on a multiple of its superharmonic function. Then, an asymptotic analysis to the Navier problem yields an extension the positivity result in a small right neighborhood of $\sigma = 1$. Finally, induced by a comparison with the respective Dirichlet problem, which is positivity preserving in some special domains, we are led to a stronger positivity result for ground states of J_σ by means of Moreau's dual cone decomposition.

We may summarize the main results of Chapter 2 as follows:

Theorem 1.2.1 (Existence, Positivity). *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and let $f(x,s) = g(x)|s|^{p-1}s$, with $p \in (0,1) \cup (1,+\infty)$ and $g \in L^1(\Omega)$, $g > 0$ a.e. in Ω. Then there exist $\sigma^* \leq -1$ and $\sigma_1 > 1$ (depending on Ω and the latter possibly infinite) such that the functional J_σ has no positive critical points if $\sigma \leq \sigma^*$, while it admits (at least) a positive ground state if $\sigma \in (\sigma^*, \sigma_1)$.*

Theorem 1.2.2 (Asymptotic behaviour). *Under the previous assumptions for Ω and f, let $(u_k)_k$ be a sequence of ground states for the respective sequence of functionals $(J_{\sigma_k})_k$. Up to a subsequence,*

i) if $\sigma_k \searrow \sigma^*$, then $u_k \to 0$ in $H^2(\Omega)$ in the case $p > 1$, while $u_k \to +\infty$ in $L^\infty(\Omega)$ if $p \in (0,1)$;

ii) if $\sigma_k \to 1$, then $u_k \to \overline{u}$ in $W^{2,q}(\Omega)$ for every $q \geq 1$, where $\overline{u}$ is a ground state for the Navier problem;

iii) if $\sigma_k \to +\infty$, then $u_k \to U$ in $H^2(\Omega)$, where U is a ground state for the Dirichlet problem.

Notice that Theorem 1.2.1 might also be seen as an extension to the semilinear setting of the main positivity results established by Gazzola and Sweers [42, Theorem 4.1] for the linear case.

Two brief sections at the end of Chapter 2 complement our analysis. In the first one we obtain further results for positive radial solutions when Ω is a ball in $\mathbb{R}^2$, while in the second we investigate what happens if we assume that Ω is not convex. We show that the aforementioned analysis still holds in those cases for which the PPP holds for the corresponding Dirichlet problems. This will also show that convexity is not a necessary hypothesis for positivity of the minimizer of the linear Kirchhoff-Love functional I_σ.

1.3 A class of semilinear problems with exponential nonlinearities

If in the first topic of the thesis the main role was played by the boundary conditions, the central actors of its second part, contained in Chapter 3, are the *exponential nonlinearities*. There are several reasons to study boundary value problems with such nonlinearities, either coming from physical models, or from the Conformal Geometry, or more from pure analytical questions.

The Gel'fand problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \subset \mathbb{R}^N,\ N \geq 2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

arises in the study of the steady states of the parabolic problem

$$\begin{cases} v_t = \Delta v + \lambda(1 - \varepsilon v)^m e^{\frac{v}{1+\varepsilon v}} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

in the approximation regime $\varepsilon << 1$. Usually, (1.11) is known as the *solid fuel ignition model* and it is derived as a model of the thermal reaction process in a combustible, non deformable material of constant density during the ignition period. Here, λ is known as the Frank-Kamenetskii parameter, v is an adimensional temperature and $\frac{1}{\varepsilon}$ the activation energy.

The problem (1.10) arises also in the context of astrophysical models of stellar structures. The total pressure of a gaseous star is given by the Stefan-Boltzmann law as a sum of the kinetic and radiation pressure

$$P = \frac{k}{\mu H}\rho T + \frac{a}{3}T^4,$$

where $\rho = \rho(r)$ is the density distribution inside the star, supposed to be a radial quantity, $T = T(r)$ is the temperature, and k, μ, H are physical constant, respectively the Boltzmann constant, the mean molecular weight and the mass of the proton. In the case of isothermal conditions, then we may write $P = \bar{k}\rho + \bar{D}$, with $\bar{k}, \bar{D}$ constants. Being interested in the density distribution $\rho(r)$, and recalling the equilibrium law for a gaseous star

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho,$$

where the mass $M(r)$ enclosed inside a spherical surface of radius r is given by

$$M(r) = \int_0^r 4\pi t^2 \rho(t)dt,$$

we end up with

$$-4\pi G\rho = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \bar{k}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 d(\log\rho)}{dr}\right).$$

The change of variable $\psi(r) := \log(\rho(r))$ gives formally then

$$-\Delta\psi = \frac{4\pi G}{\bar{k}}e^{\psi}.$$

For further details of this model, we refer to [18, p.155].

A second important motivation to the study of semilinear problems with exponential growth emerges in the context of Differential Geometry. Let (M, g_0) be a compact Riemannian manifold of dimension $N = 2$ and $K : M \to \mathbb{R}$ be a smooth function. The interest is to find a metric g conformal to g_0 (that is, which differs from g_0 only by a smooth positive factor) such that K is the scalar curvature of the new metric g. Setting $g = e^{2u}g_0$, the problem is reduced to find solutions of

$$\Delta_{g_0}u + Ke^{2u} = K_0,$$

where Δ_{g_0} denotes the Beltrami-Laplace operator of (M, g_0) and K_0 is the scalar curvature of g_0. In the planar case, that is, M being a domain in $\mathbb{R}^2$ endowed with the Euclidean metric, then $K_0 = 0$ and we retrieve once again equation (1.10). This problem has been generalized to higher-dimensional Riemannian manifolds (M, g_0) by the introduction of the *Paneitz operator* in [74], for $N = 4$ defined as

$$P_g : f \in C^\infty(M) \mapsto \Delta_g^2 f + \mathrm{div}(\tfrac{2}{3}R_g g - 2Ric_g)df,$$

where R_g and Ric_g denote respectively the scalar and the Ricci curvatures of g. Analogously to the 2-dimensional context, if we choose a conformally invariant metric $g = e^{4u}g_0$, then the function u satisfies

$$P_g u + 2Q_0 = 2Qe^{4u}, \tag{1.12}$$

where Q_0 is the original Q-curvature of (M, g_0) and Q is the prescribed Q-curvature in the new metric g. The Q-curvature has been introduced by Branson and Ørsted in [10]. Moreover, in the special case of M being a domain of $\mathbb{R}^4$ and g_0 the Euclidean metric, the equation (1.12) reduces to

$$\Delta^2 u = 6e^{4u}. \tag{1.13}$$

Both equations (1.10) and (1.13) have been widely investigated in the context of Conformal Geometry: for the Euclidean case, among others, we refer to [19, 60, 99, 65] for important characterizations of solutions with finite energy, as well as [13, 59, 2, 80] for concentration-compactness issues.

The dimension 4 for the biharmonic operator and, more generally, the dimension $2m$ for the polyharmonic operator $(-\Delta)^m$ are in some sense peculiar. Indeed, it is well known that the classical Sobolev embedding $H_0^m(\Omega) \hookrightarrow L^p(\Omega)$, where $\Omega \subset \mathbb{R}^{2m}$ is a smooth bounded domain, holds for any $p \geq 1$ but fails for $p = +\infty$. Consequently, one may ask which is the maximal growth function $g : \mathbb{R} \to \mathbb{R}^+$ such that

$$\int_\Omega g(u)dx < +\infty \quad \text{for all } u \in H_0^m(\Omega).$$

When $m = 1$ and $N = 2$ the answer has been given independently by Pohožaev in 1965 and Trudinger in 1967 and then refined by in 1979, showing that g is exponential. More precisely, Moser obtained

$$\sup_{\|\nabla u\|_N = 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} dx \quad \begin{cases} \leq C|\Omega|, & \text{if } \alpha \leq \alpha_N, \\ = +\infty & \text{if } \alpha > \alpha_N. \end{cases} \tag{1.14}$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, denoting by ω_n the volume of the unit ball in dimension n. Later on, in 1988, Adams generalized (1.14) to the polyharmonic context

$$\sup_{\|\nabla^m u\|_{\frac{N}{m}} \leq 1} \int_\Omega e^{\beta|u|^{\frac{N}{N-m}}} dx \quad \begin{cases} \leq C|\Omega|, & \text{if } \beta \leq \beta_{N,m}, \\ = +\infty & \text{if } \beta > \beta_{N,m}, \end{cases} \tag{1.15}$$

where $\beta_{N,m}$ can be given explicitly in terms of Gamma functions. We also point out a further generalization by Tarsi in [94] when $H^m(\Omega)$ is endowed with Navier boundary conditions. These inequalities allowed to study problems like (1.10) and, more generally, of type

$$\begin{cases} (-\Delta)^m u = h(u) & \text{in } \Omega \subset \mathbb{R}^{2m}, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $B(u)$ stands for either Dirichlet or Navier boundary conditions, and the growth of $t \mapsto h(t)$ is controlled by an exponential map $t \mapsto e^{t^2}$.

In this second part of the thesis we study the issue of finding *uniform a-priori bounds* for positive weak solutions of the problems

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

when Ω is a smooth bounded domain in $\mathbb{R}^4$ and h has in general an exponential growth in the second variable. In other words, we look for a constant C, depending on the domain and on the nonlinearity, such that for any solution of each problem (1.16) it holds $\|u\|_{L^\infty(\Omega)} \le C$.

A priori bounds have attracted much attention since the work of Brezis and Turner [14] in 1977. Indeed, beside their own interest, they often play an important role in the analysis of the existence of positive weak solutions by means of Liouville's theorems or topological methods. If we go back to the second-order Dirichlet problems in dimension $N \ge 3$, and hence with a power-type nonlinearity, the question has been addressed in the seminal paper by Gidas and Spruck [43] where the authors first developed the blow-up technique to treat subcritical nonlinearities satisfying a sort of separation of variables at infinity, namely

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^p} = a(x) \in C(\overline{\Omega}), \qquad 1 < p < 2^* - 1 := \frac{N+2}{N-2}, \tag{1.17}$$

uniformly in $x \in \overline{\Omega}$. Almost at the same time, de Figueiredo, Lions and Nussbaum obtained in [29] a similar result using a different approach based on the moving planes method. In the critical dimension 2, in virtue of the Trudinger-Moser inequality, the aforementioned results may suggest that an a-priori bound can be established up to the critical nonlinearity $t \mapsto e^{t^2}$. Nevertheless, this turns out to be false. Indeed, as the seminal paper by Brezis and Merle [13] shows, a-priori estimates may be found *only* if the growth is exponential or less: they provide examples of unbounded solutions of the problem

$$\begin{cases} -\Delta u = V(x)e^{u^\alpha} & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.18}$$

with $\alpha \in (1,2)$ and $0 \le V \in C(\overline{\Omega})$. On the other hand, they established that if the potential V is positive and bounded and $\alpha = 1$, solutions of (1.18) are locally uniformly bounded inside Ω, assuming an L^1 control on the right-hand side. This, together with some estimates near the boundary in [29], proves the global statement, at least for convex domains. See also [20] for similar issues.

In the higher-order framework, the usual distinction regarding the boundary conditions is again essential: indeed, the second problem in (1.16) can be studied as a system of second-order Dirichlet problems, while the first in (1.16) has to face the lack of the maximum principle. Therefore, the polyharmonic generalization of the uniform estimates of Gidas-Spruck and de Figueiredo-Lions-Nussbaum has been carried out in [73, 85] when $\Omega \subset \mathbb{R}^{2m}$ is a ball. An acute extension of these works appeared in [77, 78], where the authors managed to settle the problem in any smooth domain of $\mathbb{R}^{2m}$, by proving a Liouville's type result in the half-space.

Concerning the polyharmonic case with an exponential nonlinearity, only few results are known and they only deal with the special nonlinearity $h(x,u) = V(x)e^u$, with $V \ge 0$. Regarding the Dirichlet problem, the analysis in [13] has been extended to the fourth-order case by Adimurthi, Robert and Struwe in [2] and to the general polyharmonic context by Martinazzi in [66]; uniform bounds for the Navier problem have been established by Lin and Wei, see [61, Corollary 2.3]. A parallel field of research is devoted to the study of the *mean field equation*

$$(-\Delta)^m u = \rho \frac{V(x)e^u}{\int_\Omega V(x)e^u dx} \quad \text{in } \Omega \subset \mathbb{R}^{2m},$$

especially dealing with concentration-compactness issues with respect to the parameter ρ. We refer to [70, 63] for the second-order problem and to [98, 81] for the fourth-order generalization, as well as to the references therein.

It is worth to mention that in [58] Lorca, Ubilla and Ruf establish uniform a-priori bounds in presence of exponential nonlinearities for a different operator, namely the N-laplacian in its critical dimension N; the nonlinearities involved there are either growing less than e^{t^α} for some $\alpha \in (0,1)$, or behaving like e^t. The authors use Orlicz spaces techniques to cover the first alternative and some arguments inspired by Brezis and Merle for the second case. Although in his Ph.D. thesis [76, Chapter 6] Passalacqua improved their result, allowing a larger class of nonlinearities, the gap between the growths e^{t^α} and e^t was not completely filled: for instance, the growth $f(t) = e^t(1+t)^{-\alpha}$ with $\alpha > 1$ was not allowed. Covering the remaining cases seems not attainable with those techniques. Let us also mention in passing that a similar gap occurs also when dealing with coupled elliptic systems in critical dimension, see [30].

Here, instead, we consider directly problems (1.16) with a general positive nonlinearity h which is assumed superlinear and subcritical or critical in the sense of Brezis-Merle; our main results may be roughly summarized in the following Theorems 1.3.1 (for the Dirichlet case) and 1.3.2 (regarding the Navier case), while the precise statements will be given in Chapter 3. In particular we shall mainly focus on the first, due to the lack of all second-order tools based on maximum principles. This is the main motivation for which we are restricting to the case of the ball. Nonetheless, we shall show that the result still applies to any smooth bounded domain Ω provided a control on the energy of solutions is assumed (Theorem 3.4.1), or a good boundary behaviour of the Green function holds (cf. Remark 20).

Theorem 1.3.1. *Let $\mathcal{B} \subset \mathbb{R}^4$ be a ball and h be a positive nonlinearity such that*

$$\lim_{t\to+\infty} \frac{h(x,t)}{f(t)} = a(x) \in C(\bar{\mathcal{B}})$$

uniformly in $\bar{\mathcal{B}}$ and let $f \in C^1(\mathbb{R}, \mathbb{R}^+)$ be increasing, superlinear and satisfy

$$\lim_{t\to+\infty} \frac{f'(t)}{f(t)} \in [0, +\infty). \tag{1.19}$$

Then, there exists a positive constant $C = C(h)$ such that for any weak solution u of the Dirichlet problem

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \mathcal{B}, \\ u = u_n = 0 & \text{on } \partial\mathcal{B}, \end{cases} \tag{1.20}$$

there holds $\|u\|_{L^\infty(\mathcal{B})} \leq C$.

On the other hand, the same result in presence of Navier boundary conditions applies for any smooth bounded *convex* domain.

Theorem 1.3.2. *Let $\Omega \subset \mathbb{R}^4$ be a smooth convex domain and h be as in Theorem 1.3.1. Then, there exists a positive constant $\bar{C} = \bar{C}(h, \Omega)$ such that for any weak solution u of the Navier problem*

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.21}$$

there holds $\|u\|_{L^\infty(\Omega)} \le \bar{C}$.

Let us remark that assumption (1.19) means that f is "controlled" by the map $t \mapsto e^{\gamma t}$ for some $\gamma > 0$ for large values of t. We also point out that our results are sharp, in the sense that if the function f behaves like $t \mapsto e^{t^\alpha}$ for some $\alpha \in (1, 2)$, then we provide examples of unbounded solutions of both (1.20) and (1.21).

The main argument which proves Theorems 1.3.1 and 1.3.2 may be sketched as follows. Firstly we obtain uniform boundary estimates for any solution of the problems (1.20)-(1.21). This leads to a uniform L^1-estimate of the right-hand side $h(x, u)$. Here, the main obstructions concerning the Dirichlet problem are mainly due to the lack of the maximum principle and of good Green function estimates near the boundary. Then, assuming by contradiction the existence of an unbounded sequence of solutions, we apply a blow-up strategy inspired by [81] leading to a problem in the whole $\mathbb{R}^4$. A contradiction is found by means of a Pohožaev identity.

An application of the Krasnosel'skii genus theory permits to infer from Theorems 1.3.1-1.3.2 the existence of a positive solution for the problems (1.20)-(1.21):

Theorem 1.3.3. *If in addition to the assumptions of Theorem 1.3.1 and with the notation therein, suppose also that there holds*

$$\limsup_{t \to 0^+} \frac{h(x,t)}{t} < \tilde{\lambda}_1 \quad \text{uniformly in } \overline{\mathcal{B}}, \tag{1.22}$$

where $\tilde{\lambda}_1$ is the first eigenvalue of Δ^2 with the Dirichlet boundary conditions. Then problem (1.20) admits a positive solution.

A similar statement holds also for the Navier problem (1.21). We remark that the additional assumption (1.22) is not only a matter of technicality but in some cases it is also necessary for the existence.

Finally, we conclude Chapter 3 providing some generalizations of Theorem 1.3.1: to a larger class of domains, as briefly mentioned before, under some additional assumptions on solutions in the spirit of Brezis-Merle, and to the respective polyharmonic problems.

1.4 Notation

Throughout the whole thesis, we use the following notation.

Let $N \ge 2$. We say $\Omega \subset \mathbb{R}^N$ is a *domain* when it is open and connected; moreover, Ω has a boundary of class $C^{k,\alpha}$ with $\alpha \in (0, 1)$ (resp. $C^{k,1}$) when $\partial\Omega$ can be described in local coordinates by a C^k function with α-Hölder (resp. Lipschitz) continuous k-th derivatives. Moreover, $d_\Omega(x)$ denotes the distance of $x \in \Omega$ from the boundary $\partial\Omega$.

The topological dual of a normed space X is denoted by X^*. Moreover, dx denotes the standard Lebesgue metric and $d\sigma$ the Hausdorff measure on the boundary. When the context is clear, they are is usually omitted.

We denote by $\nabla^k u$ the tensor of the derivatives of u of order k and $|\nabla^k u|$ its euclidean norm, namely

$$|\nabla^k u|^2 := \sum_{|\alpha|=k} |D^\alpha u|^2,$$

where the sum is on all multi-indices with length k. In particular $\nabla^2 u$ stands for the Hessian matrix of u. The derivatives may be denoted also by subscripts (u_x, u_{xy}, ...), and u_n and u_τ are the normal and the tangential derivative of u, n and τ being respectively the unit exterior normal and the unit tangent vector.

In the sequel, C indicates a generic constant, whose value may vary from line to line, and also within the same line.

Finally, we recall the definition of Sobolev spaces and their embeddings into Lebesgue and classical spaces. Let Ω be a Lebesgue-measurable subset of $\mathbb{R}^N$ and let $p \in (0, +\infty]$ and $k \in \mathbb{N}$: we define

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \,|\, \exists \text{ weak derivative } D^\alpha f \in L^p(\Omega),\ \forall |\alpha| \le k\},$$

α being a multi-index, endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha f\|_p^p \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{|\alpha|\le k}(\|D^\alpha f\|_\infty) & \text{if } p = \infty, \end{cases}$$

where $\|\cdot\|_p$ is denoting the usual norm in $L^p(\Omega)$.
The Hilbert spaces $W^{k,2}(\Omega)$ are usually denoted by $H^k(\Omega)$ and, moreover, we define $H_0^k(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^k(\Omega)}}$, namely the completion of the space of smooth function which are compactly supported in Ω in the norm of $H^k(\Omega)$. Equivalently, H_0^k may be seen as the subspace of $H^k(\Omega)$ of functions which have zero trace on the boundary. Finally, we recall that the space $W^{s,p}$ with $s \in (0,1)$ denotes the fractional Sobolev space endowed with the Gagliardo seminorm. We refer to [32] for a survey about the main properties of these spaces.

Lemma 1.4.1 (Sobolev embeddings, see for instance Theorem 4.12 [1])**.** *Let $\Omega \subset \mathbb{R}^N$ satisfy the strong Lipschitz condition*[1] *and let $j \ge 0$, $m \ge 1$ be integers and $1 \le p < +\infty$.*

- *If either $mp > N$ or $m = N$ and $p = 1$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \le q \le +\infty$;*
- *If $mp = N$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \le q < +\infty$;*
- *If $mp < N$, then $W^{m+j,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ for $p \le q \le p^* := \frac{Np}{N-mp}$;*
- *If $mp > N > (m-1)p$, then $W^{m+j,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega})$ for all $0 < \lambda < m - \frac{N}{p}$;*
- *If $N = (m-1)p$, then $W^{m+j,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega})$ for all $0 < \lambda < 1$.*

Moreover, if $|\Omega| < +\infty$, then the embeddings into Lebesgue or Sobolev spaces hold also for $1 \le q \le p$.

[1] For a definition of the *Strong Local Lipschitz Condition*, see [1, §4.9]; if Ω is bounded, it simply reduces to the condition of a locally Lipschitz boundary.

Chapter 2

Fourth-order problems related to the Kirchhoff-Love functional.

In this first part of the thesis, we study a generalization of the Kirchhoff-Love functional, namely

$$J_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega F(x,u)dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, the parameter σ lies in $\mathbb{R}$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}^+$ has a power growth in the second variable. Since we suppose the plate to be hinged, the appropriate setting in order to look for critical points of J_σ is the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$.

Firstly, we study the *linear* functional, namely when $F(x,u) = f(x)u$, looking for sufficient conditions which yield positivity for the (unique) minimizer in the physical relevant context, that is, when $\sigma \in (-1,1]$. Section 2.1 is devoted to this analysis, which extends a previous result by Parini and Stylianou in [75].

In Section 2.2 we investigate the functional J_σ with a general nonlinearity $F(x,u)$. We focus on ground states, which are the least-energy critical points, and we distinguish between subquadratic and superquadratic growth of F. In the first case, existence and positivity of ground states basically follow along the same lines as for the linear case; on the other hand, in the second case, more sophisticated variational techniques are involved, namely the method of the Nehari manifold. We start our analysis in Subsection 2.2.1 in the same setting for σ as we did for the linear functional I_σ, that is $\sigma \in (-1,1]$. Then Subsections 2.2.2-2.2.6 are devoted to the study of existence and positivity outside the interval $(-1,1]$, where different techniques are required to deduce positivity, in particular when $\sigma > 1$, as the second term of the functional changes sign. To this aim, but also as an independent goal, we also investigate the asymptotic behaviour of ground states solutions for the extremal values of σ, as well as for the Navier and Dirichlet cases (Subsections 2.2.3-2.2.5). Finally, further results for radial solutions and for positivity in nonconvex domains are given in Subsections 2.2.7 and 2.2.8.

This chapter, with the exception of Subsection 2.2.8, is an adaptation of the paper [82].

2.1 The linear Kirchhoff-Love functional

Let Ω be a bounded domain in $\mathbb{R}^2$, the parameter σ lie in the physical meaningful interval $(-1, 1]$ and consider the functional J_σ with $F(x, u) = f(x)u$. With this special choice, we retrieve the standard Kirchhoff-Love functional

$$I_\sigma := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega f(x)udx,$$

which represents the energy of the thin hinged plate Ω under the action of the vertical external force of density f. We recall that, formally, via integration by parts one finds that the minima of J_σ are solutions of the linear fourth-order Steklov problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The main interest, therefore, is to analyse whether the positivity preserving property holds for such solutions, when dealing with a nonnegative source term f. A first positive answer to this problem can be found in [75], where the authors proved the following:

Theorem 2.1.1 ([75], Theorem 3.1). *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with a boundary of class $C^{2,1}$. Assume $\sigma \in (-1, 1]$ and $f \in L^2(\Omega)$. Then the minimizer u_σ of I_σ is the unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ of (2.1). If, moreover, $f \geq 0$ and $f \not\equiv 0$, then $u_\sigma > 0$ in Ω.*

As briefly mentioned in the Introduction, the strategy of the proof is to show that the second term appearing in the functional I_σ can be equivalently rewritten as a boundary term depending on the curvature and, thus, it does not influence the equation inside the domain.

What stands out in the statement of Theorem 2.1.1 are the geometric assumptions on the domain: the *convexity* and the *regularity of the boundary*. It is known that if we "reject" both of them, e.g. considering an L-shaped domain, peculiar phenomena occur near the re-entrant corner (see [72]). Nevertheless, we may wonder to what extent this result holds true when assuming either less regularity on the boundary or when dealing with nonconvex (but sufficiently smooth) domains. A remark which supports the idea that those assumptions may be refined is that, in the proof of Theorem 2.1.1, the $C^{2,1}$ regularity of $\partial\Omega$ is necessary only because the authors take advantage of the density of $H^3(\Omega) \cap H_0^1(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$. The feeling was that this seemed to be only a technical requirement and this actually turned out to be true. Indeed, by a different density argument, we prove that the PPP holds relaxing the regularity of the boundary:

Theorem 2.1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with boundary of class $C^{1,1}$. Suppose $\sigma \in (-1, 1]$ and $f \in L^1(\Omega)$. If $f \geq 0$ and $f \not\equiv 0$, then the minimizer of the Kirchhoff-Love functional I_σ is positive in Ω.*

After a first subsection in which we gather some preliminary results about equivalence of norms in $H^2(\Omega) \cap H_0^1(\Omega)$, the whole section is devoted to the proof of Theorem 2.1.2. Again, the key point will be the equivalent formulation of the second term in I_σ as in [75], but our proof relies on a subtler density argument due to Stylianou, [89, Theorem 2.2.4].

The second issue about the convexity assumption is left at the end of the chapter, see Subsection 2.2.8. We anticipate here that we will show a class of nonconvex (but very smooth) domains in which the PPP still holds.

2.1.1 Equivalence of norms in $H^2(\Omega) \cap H_0^1(\Omega)$

We begin recalling some known useful facts about equivalence of norms in the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$.

Definition 2.1.1. We say that $\Omega \subset \mathbb{R}^N$ satisfies a *uniform external ball condition* if there exists $R > 0$ such that for all $x \in \partial\Omega$ there exists a ball B_R of radius R such that $x \in \partial B_R$ and $B_R \subset \mathbb{R}^N \setminus \overline{\Omega}$.

Lemma 2.1.3. *Let $\Omega \subset \mathbb{R}^N$ bounded with a Lipschitz boundary. Then $\||\nabla^2 \cdot |\|_2$ and $\| \cdot \|_{H^2(\Omega)}$ are equivalent norms on $H^2(\Omega) \cap H_0^1(\Omega)$. If in addition Ω satisfies a uniform external ball condition, then also $\|\Delta \cdot \|_2$ defines an equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. We refer to [71, Corollary 5.4] for a proof of the first statement, while the second is due to Adolfsson, see [3]. □

Throughout the whole chapter, $C_0 = C_0(\Omega)$ and $C_A = C_A(\Omega)$ denote the smallest positive constants such that

$$\|u\|^2_{H^2(\Omega)} := \|u\|_2^2 + \||\nabla u|\|_2^2 + \||\nabla^2 u|\|_2^2 \leq C_0 \||\nabla^2 u|\|_2^2 \tag{2.2}$$

and

$$\|u\|^2_{H^2(\Omega)} \leq C_A \|\Delta u\|_2^2 \tag{2.3}$$

for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

The next result states that the first two terms of I_σ together are interpretable as a square of an equivalent norm in our space $H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 2.1.4. *Let $\Omega \subset \mathbb{R}^N$ bounded with a Lipschitz boundary and $\sigma \in (-1, 1)$. Then*

$$\|u\|_{H_\sigma(\Omega)} := \left(\int_\Omega (\Delta u)^2 - 2(1-\sigma) \int_\Omega det(\nabla^2 u) \right)^{\frac{1}{2}} \tag{2.4}$$

defines a norm on $H^2(\Omega) \cap H_0^1(\Omega)$ equivalent to the standard one.

Proof. Firstly

$$\begin{aligned} \|u\|^2_{H_\sigma(\Omega)} &= \int_\Omega u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2\sigma(u_{xx}u_{yy} - u_{xy}^2) \\ &\leq \||\nabla^2 u|\|_2^2 + 2|\sigma| \left(\frac{u_{xx}^2 + u_{yy}^2}{2} + u_{xy}^2 \right) = (1 + |\sigma|) \||\nabla^2 u|\|_2^2. \end{aligned}$$

Moreover, one has

$$
\begin{aligned}
\|u\|^2_{H_\sigma(\Omega)} &= \int_\Omega u_{xx}^2 + u_{yy}^2 + 2(1-\sigma)u_{xy}^2 + 2\sigma u_{xx}u_{yy} \\
&\geq \int_\Omega u_{xx}^2 + u_{yy}^2 + 2(1-\sigma)u_{xy}^2 - |\sigma|(u_{xx}^2 + u_{yy}^2) \geq (1-|\sigma|)\||\nabla^2 u|\|_2^2.
\end{aligned}
\tag{2.5}
$$

The proof is completed by applying Lemma 2.1.3 and noticing that the map

$$(u,v)_{H_\sigma} \mapsto \int_\Omega \Delta u \Delta v - (1-\sigma)\int_\Omega u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$$

defines a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$ for every $\sigma \in (-1,1)$ by (2.5). □

2.1.2 A crucial identity

A rather standard technique which is often used to prove positivity of least-energy solutions in the context of fourth-order problems is the method of the superharmonic function. Roughly speaking, by means of I_σ, we compare our minimizer u with the function $\tilde{u}$ defined as the unique element of $H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta \tilde{u} = |\Delta u| & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to see by maximum principle that $\tilde{u} > |u|$, so we can compare the respective first and third term of I_σ. Nevertheless, we have no information about the behaviour of the determinant of the Hessian matrix of $\tilde{u}$ with respect to the same term of u. The strategy applied by Parini and Stylianou to overcome this problem when dealing with smooth and convex domains, was to rewrite it in an appropriate way and transform it into a boundary term. Here, we want to obtain the same result also for a less smooth domain.

Theorem 2.1.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then, for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$:*

$$\int_\Omega det(\nabla^2 u) = \frac{1}{2}\int_{\partial\Omega} \kappa u_n^2. \tag{F}$$

We split the proof in several lemmas, the starting point being a careful integration by parts inferred by Parini and Stylianou. For any $u \in H^2(\Omega)$, set

$$K(u) := \int_\Omega det(\nabla^2 u)dx. \tag{2.6}$$

Then, for any $C^\infty(\overline{\Omega})$, we have

$$\langle K'(u), \varphi\rangle := \int_\Omega (\varphi_{xx}u_{yy} + \varphi_{yy}u_{xx} - 2\varphi_{xy}u_{xy}).$$

It is thus clear that we can study K' instead of K as $\langle K'(v), v\rangle = 2K(v)$ follows from the definitions.

Lemma 2.1.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then for all $v \in H^2(\Omega)$ and $\varphi \in C^\infty(\overline{\Omega})$ there holds*

$$\langle K'(v), \varphi \rangle = \int_{\partial\Omega} (\kappa \varphi_n v_n + \varphi_{\tau\tau} v_n - \varphi_{\tau n} v_\tau). \tag{F_{PS}}$$

Hence, for any $v \in C^\infty(\overline{\Omega})$:

$$K(v) = \frac{1}{2}\langle K'(v), v\rangle = \frac{1}{2}\int_{\partial\Omega} (\kappa v_n^2 - (v_{n\tau} + v_{\tau n}) v_\tau). \tag{$\mathrm{F}_{PS}2$}$$

Proof. The identity (F_{PS}) can be found in [75, Lemma 2.5]. We give here the proof for sake of completeness. Denote by n_i, τ_i, $i = 1, 2$ the i-th coordinate of the unit vectors n, τ. Integrating by parts, one obtains

$$\begin{aligned}\int_\Omega \varphi_{xy} v_{xy} &= \int_{\partial\Omega} (\varphi_{xy} v_x n_2 - \varphi_{xyy} v n_1) + \int_\Omega \varphi_{xxyy} v \\ &= \int_{\partial\Omega} (\varphi_{xy} v_y n_1 - \varphi_{xxy} v n_2) + \int_\Omega \varphi_{xxyy} v\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \varphi_{xx} v_{yy} &= \int_{\partial\Omega} (\varphi_{xx} v_y n_2 - \varphi_{xxy} v n_2) + \int_\Omega \varphi_{xxyy} v, \\ \int_\Omega \varphi_{yy} v_{xx} &= \int_{\partial\Omega} (\varphi_{yy} v_x n_1 - \varphi_{xyy} v n_1) + \int_\Omega \varphi_{xxyy} v.\end{aligned}$$

Notice that if $\partial\Omega \in C^{1,1}$, κ is well-defined in $L^\infty(\partial\Omega)$ and the following relation holds (see, for instance, [87, Chapter 4]):

$$\Delta\varphi = \varphi_{nn} + \varphi_{\tau\tau} + \kappa\varphi_n.$$

Moreover, recalling the decompositions $\varphi_x = n_1\varphi_n + \tau_1\varphi_\tau$ and $\varphi_y = n_2\varphi_n + \tau_2\varphi_\tau$, we have

$$\begin{aligned}\langle K'(v), \varphi\rangle &= \int_{\partial\Omega} (\varphi_{xx} v_y n_2 + \varphi_{yy} v_x n_1 - \varphi_{xy} v_x n_2 - \varphi_{xy} v_y n_1) \\ &= \int_{\partial\Omega} \Delta\varphi v_n - \int_{\partial\Omega} (\varphi_{xx} v_x n_1 + \varphi_{yy} v_y n_2 + \varphi_{xy} v_x n_2 + \varphi_{xy} v_y n_1) \\ &= \int_{\partial\Omega} \Delta\varphi v_n - \int_{\partial\Omega} ((\varphi_x)_n v_x + (\varphi_y)_n v_y) \\ &= \int_{\partial\Omega} (\Delta\varphi v_n - (n_1 v_x + n_2 v_y)\varphi_{nn} - (\tau_1 v_x + \tau_2 v_y)\varphi_{\tau n}) \\ &= \int_{\partial\Omega} (\varphi_{\tau\tau} v_n - \varphi_{\tau n} v_\tau + \kappa\varphi_n v_n).\end{aligned}$$

Let now $\varphi = v \in C^\infty(\overline{\Omega})$. Since $\partial\Omega$ is a closed curve and by the definition of the tangential derivative (i.e. as $\frac{d}{ds}u(\gamma(s))$, where γ is the parametrization of the curve $\partial\Omega$ in the arch parameter s), then

$$\int_{\partial\Omega} (v_{n\tau} v_\tau + v_n v_{\tau\tau}) = \int_{\partial\Omega} (v_n v_\tau)_\tau = 0,$$

and ($\mathrm{F}_{PS}2$) follows. □

The remaining strategy for the proof of Theorem 2.1.5 consists of two steps: using $(\mathrm{F}_{PS}2)$, we firstly prove that (F) holds also for every $v \in C_0^{1,1}(\overline{\Omega}) := \{u \in C^{1,1}(\overline{\Omega}) \mid u_{|\partial\Omega} = 0\}$; then, by a density result, we transfer (F) from $C_0^{1,1}(\overline{\Omega})$ to $H^2(\Omega) \cap H_0^1(\Omega)$. We will make use of the following Lemma, which makes a well-known result more precise:

Lemma 2.1.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1 and $u \in C^{1,1}(\overline{\Omega})$. Then there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $u_k \to u$ in $H^2(\Omega)$ and $\|u_k\|_{W^{2,\infty}(\Omega)} \leq C\|u\|_{W^{2,\infty}(\Omega)}$ for some positive constant C.*

Proof. First of all notice that $C^{1,1}(\overline{\Omega})$ can be equivalently seen as $W^{2,\infty}(\Omega)$, which is a subset of $H^2(\Omega)$ since Ω is a bounded domain; moreover the fact that $C^\infty(\overline{\Omega})$ is dense in $H^2(\Omega)$ in $H^2(\Omega)$ norm if $\partial\Omega$ is of class C^1 is a standard fact (see [36, section 5.3.3, Theorem 3]), so the only statement to be verified is the $W^{2,\infty}(\Omega)$ estimate. Since the main tool in the proof of the $H^2(\Omega)$ convergence is the local approximation, which is achieved by mollification, we only have to prove that the same inequality holds there. So, let $v \in L^\infty(\Omega)$, $\varepsilon > 0$ and consider

$$v_\varepsilon(x) := (\eta_\varepsilon * v)(x) = \int_{B_\varepsilon(0)} \eta_\varepsilon(y)v(x-y)dy,$$

where η_ε is the standard mollifier in $\mathbb{R}^N$, that is $\eta_\varepsilon := \varepsilon^{-n}\eta(\frac{x}{\varepsilon})$ and

$$\eta(x) = \tilde{C}e^{\frac{1}{|x|^2-1}}\chi_{B_1(0)}(x),$$

where $\tilde{C} > 0$ such that $\int_{B_1(0)} \eta(z)dz = 1$. So v_ε is well-defined in $\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$, we have $v_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and $\int_{B_\varepsilon(0)} \eta_\varepsilon(z)dz = 1$ holds.
We claim that $\|v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \|v\|_{L^\infty(\Omega)}$. Indeed,

$$\|v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \sup_{x\in\Omega_\varepsilon} \int_{B_\varepsilon(0)} |\eta_\varepsilon(z)||v(x-z)|dz \leq \|v\|_{L^\infty(\Omega)} \int_{B_\varepsilon(0)} |\eta_\varepsilon(z)|dz = \|v\|_{L^\infty(\Omega)}.$$

The same inequality holds also for derivatives of v, because for any admissible multiindex α we have $D^\alpha(v_\varepsilon) = (D^\alpha(v))_\varepsilon$ (see [44, Lemma 7.3]). At this point, following the aforementioned proof of [36], it is easy to derive the desired result. □

Proposition 2.1.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then, for all $u \in C_0^{1,1}(\overline{\Omega})$:*

$$\int_\Omega det(\nabla^2 u) = \frac{1}{2}\int_{\partial\Omega} \kappa u_n^2.$$

Proof. Applying Lemma 2.1.7, let $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ be a sequence converging to u in $H^2(\Omega)$, whose norms in $W^{2,\infty}$ are controlled by the $W^{2,\infty}$ norm of u. By $(\mathrm{F}_{PS}2)$, we know that

$$K(u_k) = \frac{1}{2}\int_{\partial\Omega} [\kappa(u_k)_n^2 - ((u_k)_{n\tau} + (u_k)_{\tau n})(u_k)_\tau].$$

By the convergence in $H^2(\Omega)$, using the definition (2.6) of K, one clearly has $K(u_k) \to K(u)$; moreover, since $\kappa \in L^\infty(\partial\Omega)$ and using the trace theorem, one can deduce also

$$\int_{\partial\Omega} \kappa (u_k)_n^2 \to \int_{\partial\Omega} \kappa u_n^2.$$

Finally we have to consider the terms in which tangential derivatives are involved. Similarly to the normal derivative, one has $(u_k)_\tau \to u_\tau$ in $L^2(\partial\Omega)$, so $(u_k)_\tau \to 0$ in $L^2(\partial\Omega)$, since $u_{|\partial\Omega} = 0$. Furthermore,

$$(u_k)_{n\tau} = \nabla(u_k)_n \cdot \tau = \nabla(\nabla u_k \cdot n) \cdot \tau = (\nabla^2 u_k \cdot n + \nabla u_k \cdot \nabla n) \cdot \tau$$

and (see [87, Chapter 4])

$$(u_k)_{\tau n} = \sum_{i,j=1}^{2} \frac{\partial^2 u_k}{\partial x_i \partial x_j} \tau_i n_j$$

and one can infer that $(u_k)_{n\tau}$ and $(u_k)_{\tau n}$ are uniformly bounded in $L^2(\partial\Omega)$. In fact, since $(u_k)_k \subset C^\infty(\overline{\Omega})$ and using Lemma 2.1.7:

$$\begin{aligned}
\|(u_k)_{n\tau}\|_{L^2(\partial\Omega)} &\leq |\partial\Omega|^{1/2} \|(u_k)_{n\tau}\|_{L^\infty(\partial\Omega)} \\
&\leq |\partial\Omega|^{1/2} (\||\nabla^2 u_k \cdot n|\|_{L^\infty(\partial\Omega)} + \||\nabla u_k \cdot \nabla n|\|_{L^\infty(\partial\Omega)}) \\
&\leq 2|\partial\Omega|^{1/2} \|n\|_{W^{1,\infty}(\partial\Omega)} \|u_k\|_{W^{2,\infty}(\Omega)} \leq C(\Omega) \|u\|_{W^{2,\infty}(\Omega)}
\end{aligned}$$

and similarly for $(u_k)_{\tau n}$. Consequently,

$$\int_{\partial\Omega} \Big((u_k)_{n\tau} + (u_k)_{\tau n} \Big) (u_k)_\tau \to 0.$$

□

In order to extend (F) to the space $H^2(\Omega) \cap H^1_0(\Omega)$, we need a density result (Lemma 2.1.9 below) which is taken from [89, Theorem 2.2.4] and which can be adapted to our context: in fact, it concerns C^2 functions and diffeomorphisms but, with a little care, one can obtain the same result also in the class $C^{1,1}$.

Definition 2.1.2. ([1], §3.40, p.77) Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^N$ onto a domain $G \subset \mathbb{R}^N$ having inverse $\Psi := \Phi^{-1}$. We say that Φ is a $C^{1,1}$ diffeomorphism if, writing $\Phi = (\Phi_1, ..., \Phi_N)$ and $\Psi = (\Psi_1, ..., \Psi_N)$, then $\Phi_i \in C^{1,1}(\overline{\Omega})$ and $\Psi_i \in C^{1,1}(\overline{G})$ for every $i \in \{1, ..., N\}$.

Lemma 2.1.9. *Let $\Omega \subset \mathbb{R}^N$ be bounded and open such that for every $x \in \partial\Omega$ there exists a $j \in \{0, ..., N-1\}$, $\varepsilon > 0$ and a $C^{1,1}$-diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$, such that the following hold:*

- $\Phi(x) = 0$;
- $\Phi(B_\varepsilon(x) \cap \Omega) \subset S_j := \{x = (x_1, ..., x_N) \in \Omega \,|\, x_i > 0 \,, \forall i > j\}$;
- $\Phi(B_\varepsilon(x) \cap \partial\Omega) \subset \partial S_j$.

Then:

$$\overline{C_0^{1,1}(\overline{\Omega})}^{\|\cdot\|_{H^2(\Omega)}} = H^2(\Omega) \cap H_0^1(\Omega).$$

Proof of Theorem 2.1.5. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$; since the assumptions on the boundary are clearly fulfilled if $\partial\Omega$ is of class $C^{1,1}$, applying Lemma 2.1.9 we get an approximating sequence $(u_k)_{k\in\mathbb{N}} \subset C_0^{1,1}(\overline{\Omega})$ converging in $H^2(\Omega)$ to u. With the same steps as in the proof of Proposition 2.1.8, by the $H^2(\Omega)$ convergence, we have both $K(u_k) \to K(u)$ and $\int_{\partial\Omega} \kappa(u_k)_n^2 \to \int_{\partial\Omega} \kappa u_n^2$ and one concludes by the uniqueness of the limit. □

2.1.3 Existence and positivity for the minimizer of I_σ

Assuming that $\partial\Omega$ is of class $C^{1,1}$, Theorem 2.1.5 enables us to rewrite the functional I_σ in a more convenient way, namely

$$I_\sigma(u) = \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2 - \int_\Omega f(x)u.$$

for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$. With this formulation, now we are able to establish the positivity of the minimizer of the functional I_σ in convex domains with boundary of class $C^{1,1}$ if the density function $f(x)$ is nonnegative. We will make use of the method of the superharmonic function, which goes back to [96], see also the works [8, 42]. Its core is contained in the following lemma.

Definition 2.1.3. We say that u is *superharmonic* in Ω when $-\Delta u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$; u is *strictly superharmonic* when we have in addition that $-\Delta u \not\equiv 0$.

Notice that, by the strong maximum principle, a superharmonic function is either constant or strictly superharmonic and thus positive in Ω.

Lemma 2.1.10. *Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain; fix $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and define $\tilde{u}$ as the unique solution in $H_0^1(\Omega)$ of the Poisson problem:*

$$\begin{cases} -\Delta\tilde{u} = |\Delta u| & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Then $\tilde{u} \in H^2(\Omega)$ and either $\tilde{u} > |u|$ in Ω and $\tilde{u}_n^2 \geq u_n^2$ on $\partial\Omega$ or $\tilde{u} = u$ in Ω.

Proof. Since Ω is convex by assumption, it satisfies in particular a uniform external ball condition and thus we infer $\tilde{u} \in H^2(\Omega)$ by Lemma 2.1.3. Suppose $\tilde{u} \not\equiv u$. Since in particular $-\Delta\tilde{u} \geq \Delta u$ holds, by the maximum principle for strong solutions (see for instance [44, Theorem 9.6]), one has $\tilde{u} > -u$ in Ω and so $\tilde{u}_n \leq -u_n$. Similarly, $-\Delta\tilde{u} \geq -\Delta u$, implies also $\tilde{u} > u$ and $\tilde{u}_n \leq u_n$. Combining them, the result is proved. □

We have now all elements to finally prove Theorem 2.1.2.

Proof of Theorem 2.1.2. The existence of a unique minimizer of I_σ in $H^2(\Omega)\cap H^1_0(\Omega)$ is an application of Lax-Milgram theorem since, by Lemma 2.1.4,

$$I_\sigma(u) = \frac{1}{2}\|u\|^2_{H_\sigma} - \int_\Omega f(x)u,$$

with $f \in L^1(\Omega) \subset (H^2(\Omega) \cap H^1_0(\Omega))^*$. Moreover, by convexity, $\kappa \geq 0$ a.e. on $\partial\Omega$. From u, define its superharmonic function $\tilde{u}$ as in Lemma 2.1.10. Supposing $\tilde{u} \not\equiv u$, by that result we infer

$$\begin{aligned} I_\sigma(\tilde{u}) &= \int_\Omega \frac{(\Delta\tilde{u})^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa\tilde{u}_n^2 - \int_\Omega f(x)\tilde{u} \\ &< \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2 - \int_\Omega f(x)u = I_\sigma(u), \end{aligned}$$

which is a contradiction. Hence, necessarily $\tilde{u}$ coincides with u, so $-\Delta u = -\Delta\tilde{u} = |\Delta u| \geq 0$. As $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$, we deduce $u > 0$ in Ω. □

Remark 1. Notice that the convexity and the regularity assumptions were only needed to infer positivity, while, to prove existence, one only needs the hypothesis of Lemma 2.1.4, namely a Lipschitz boundary (together with the outer uniform ball condition if $\sigma = 1$, see Lemma 2.1.3).

2.2 A generalized Kirchhoff-Love functional

Let us now consider the following generalization of the Kirchhoff-Love functional, namely $J_\sigma : H^2(\Omega) \cap H^1_0(\Omega) \to \mathbb{R}$ defined as

$$J_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega F(x,u)dx.$$

Hereafter, we assume that Ω is a bounded domain in $\mathbb{R}^2$. Concerning the nonlinearity, the functional J_σ is well-defined once we impose $F(\cdot, s) \in L^1(\Omega)$ and $F(x,\cdot) \in C^1(\mathbb{R})$ (and thus there exists $f(x,\cdot)$ continuous such that $F(x,s) = \int_0^s f(x,t)dt$) and a power-type growth control on F, namely the existence of $a, b \in L^1(\Omega)$ such that $|F(x,s)| \leq a(x) + b(x)|s|^q$ for some $q > 0$. With these assumptions on F, it is a standard fact to prove that J_σ is a C^1 functional with Fréchet derivative

$$J'_\sigma(u)[v] = \int_\Omega \Delta u\Delta v - (1-\sigma)\int_\Omega (u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}) - \int_\Omega f(x,u)v.$$

Our main goal is to investigate existence and positivity of ground states of J_σ in dependence on the value of the parameter σ, which will play an important role in the whole analysis. We will show that existence holds for any σ over a threshold parameter $\sigma^* \leq -1$ and positivity will last up to a second parameter $\sigma_1 > 1$, possibly infinite. Moreover, we investigate the asymptotic behaviour of ground states near the relevant special cases for σ, that are σ^*, 1 (the Navier case) and ∞ (the Dirichlet case). Our main results can be summarized as in Theorems 1.2.1-1.2.2 given in the Introduction.

As briefly mentioned therein, if the boundary is smooth enough ($\partial\Omega$ of class $C^{4,\alpha}$ for $\alpha > 0$), standard elliptic regularity results apply and one can integrate by parts the Euler-Lagrange equation from J_σ to see that critical points satisfy the semilinear boundary value problem

$$\begin{cases} \Delta^2 u = f(x,u) & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

On the other hand, assuming only that the boundary is of class $C^{1,1}$, the signed curvature is well-defined in $L^\infty(\Omega)$ and we can have a weak formulation of problem (2.8). More precisely, in this case, by *weak solution* of (2.8) here we mean a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfies

$$\int_\Omega \Delta u \Delta\varphi - (1-\sigma)\int_{\partial\Omega} \kappa u_n \varphi_n = \int_\Omega f(x,u)\varphi \qquad \forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.9}$$

Consequently, we can equivalently say "ground states of J_σ" or "ground state solutions for (2.8)". For a proof of the equivalence of the two problems, we refer to [42].

Since the geometry of the functional plays an important role, from now on we have to distinguish between the sublinear case, that is, when the density f has at most a slow linear growth in the real variable (as specified in the following), and the superlinear case, the opposite one. In fact, we will see that in the first case J_σ behaves similarly to the linear Kirchhoff-Love functional I_σ since it is coercive and ground states are global minima, while, in the second case, J_σ has a mountain pass geometry and the ground states are saddle points.

We relegate at the end of this subsection a brief comment about linear growths, for instance $f(x,u) = \lambda g(x)u$, since (2.8) becomes an eigenvalue problem and can be investigated with standard techniques.

2.2.1 Existence and positivity for $\sigma \in (-1,1]$

Sublinear case

Proposition 2.2.1. *Let Ω be a bounded domain with Lipschitz boundary and $\sigma \in (-1,1)$. Let $p \in (0,2)$ and suppose*

$$|F(x,s)| \le d(x) + c(x)|s|^p + \frac{1}{2}(1-|\sigma|)C_0^{-1}s^2, \tag{H}$$

where $c, d \in L^1(\Omega)$. Then the functional J_σ is weakly lower semi-continuous and coercive, hence there exists a global minimizer of J_σ in $H^2(\Omega) \cap H_0^1(\Omega)$. The same conclusion holds if $\sigma = 1$, provided Ω satisfies also a uniform outer ball condition.

Proof. Let $(u_k)_{k\in\mathbb{N}} \subset H^2(\Omega) \cap H_0^1(\Omega) \ni u$ be such that $u_k \rightharpoonup u$ weakly in $H^2(\Omega)$; since it is bounded in $H^2(\Omega)$ and consequently in $L^\infty(\Omega)$, one has

$$|F(x,u_k)| \le d(x) + c(x)M^p + \frac{1}{2}(1-|\sigma|)C_0^{-1}M^2,$$

for some $M > 0$, which is integrable over Ω. Moreover, by the compactness of the embedding $H^2(\Omega) \hookrightarrow L^p(\Omega)$, there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that $u_{k_j} \to u$ in $L^p(\Omega)$ for a suitable $p \geq 1$, so $F(x, u_{k_j}(x)) \to F(x, u(x))$ a.e. in Ω by continuity of $F(x, \cdot)$. Hence, by the dominated convergence theorem, we have $\int_\Omega F(x, u_{k_j}) \to \int_\Omega F(x, u)$. This, together with the weakly lower semicontinuity of the norm, implies the same property for J_σ. If $\sigma \in (-1, 1)$, by (2.2):

$$\begin{aligned}J_\sigma(u) &\geq \frac{1}{2}(1-|\sigma|)\||\nabla^2 u|\|_2^2 - \|d\|_1 - C^p\|c\|_1\|u\|_{H^2(\Omega)}^p - \frac{1}{2}(1-|\sigma|)C_0^{-1}\|u\|_2^2 \\ &\geq \frac{1}{2}(1-|\sigma|)C_0^{-1}\||\nabla^2 u|\|_2^2 - \|c\|_1 C^p C_0^{\frac{p}{2}}\||\nabla^2 u|\|_2^p - \|d\|_1;\end{aligned}$$

by Lemma 2.1.3, we deduce that $J_\sigma(u) \to +\infty$ as $\|u\|_{H^2(\Omega)} \to +\infty$, since $p \in (0, 2)$. Easier computations provide a similar estimate to conclude the proof also in the case $\sigma = 1$. $\square$

As for the linear functional I_σ, these global minimizer are positive in Ω:

Proposition 2.2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and $\sigma \in (-1, 1]$. In addition to the assumption (H), suppose also that $f \geq 0$ and is positive on a subset of positive measure. If $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a nontrivial minimizer of J_σ, then u is strictly superharmonic in Ω, and thus positive.*

Proof. The strategy is the same as in Theorem 2.1.2: define the superharmonic function $\tilde{u}$ of u and suppose they do not coincide. Then, by Lemma 2.1.10, we have $\tilde{u} > |u|$ and $\tilde{u}_n^2 \geq u_n^2$. Recalling that $\kappa \geq 0$ by convexity, we get

$$\begin{aligned}J_\sigma(\tilde{u}) &= \int_\Omega \frac{(\Delta\tilde{u})^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa\tilde{u}_n^2 - \int_\Omega F(x, \tilde{u}) \\ &\leq \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2 - \int_\Omega F(x, \tilde{u}) \\ &< \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2 - \int_\Omega F(x, u)dx = J_\sigma(u),\end{aligned}$$

which is a contradiction, since u is a minimizer of J_σ. Notice that the last inequality holds since $\frac{\partial F(x,s)}{\partial s}\big|_{s=t} = f(x, t) \geq 0$. The conclusion follows an in the proof of Theorem 2.1.2. $\square$

Remark 2 (A model case). A simple nonlinearity which satisfies assumption (H) is

$$F(x, u) = g(x)|u|^{p+1} + d(x)u$$

where $p \in (0, 1)$ and $d, g \in L^1(\Omega)$. Notice that if we apply Propositions 2.2.1-2.2.2 with $g = 0$, we retrieve the results of Section 2.1 about the linear Kirchhoff-Love functional I_σ.

Remark 3. It is clear that, when $f(x, 0) \neq 0$, by Proposition 2.2.1, we always find a *nontrivial* global minimizer, which is positive by Proposition 2.2.2. For homogeneous nonlinearities this is not true in general, but still holds for our model $f(x, s) =$

$g(x)|s|^{p-1}s$. Indeed, let u be a global minimum of J_σ and test the relation $J'_\sigma(u) = 0$ with u itself: we get

$$\|u\|^2_{H_\sigma} - \int_\Omega g(x)|u|^{p+1}dx = 0.$$

This implies, since $p \in (0, 1)$,

$$J_\sigma(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{H_\sigma} < 0,$$

so it is clear that in the minimization process we do not fall on the null function. The same argument holds for more general nonlinearities $f(x, u)$, provided

$$\frac{f(x,u)u}{2} - F(x, u) < 0 \qquad \text{for all } u \in H^2(\Omega) \cap H^1_0(\Omega).$$

For instance, $f(x, s) = g(x)|s|^{p-1}s + h(x)|s|^{q-1}s$, for $g, h > 0$, $p, q \in (0, 1)$.

Superlinear case

The strategy applied in Remark 3, that is, to test the derivative of a functional evaluated in a function with the function itself and from this obtain further information, constitutes the base idea of the *Nehari manifold* method. This technique has been successfully applied to several different problems, and we refer to [93] for a detailed description of the method. We will make use of this strategy to infer the existence of (nontrivial) ground states of J_σ in the context of a mountain-pass geometry due to the superquadratic character of the third term of J_σ. To this aim, the structure of the problem being more involved than the sublinear case, we focus on the nonlinearity

$$f(x, u) = g(x)|u|^{p-1}u, \quad \text{where } 0 < g \in L^1(\Omega) \text{ and } p > 1. \tag{2.10}$$

Indeed, the functional

$$J_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u) - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1}$$

is not coercive anymore: in fact, fixing any $u \in H^2(\Omega) \cap H^1_0(\Omega) \setminus \{0\}$, we have $J_\sigma(tu) \to -\infty$ as $t \to +\infty$.

We define the *Nehari manifold* of J_σ as the set

$$\mathcal{N}_\sigma := \{u \in (H^2(\Omega) \cap H^1_0(\Omega)) \setminus \{0\} \mid J'_\sigma(u)[u] = 0\},$$

which clearly contains all nontrivial critical points of J_σ. Notice that $u \in \mathcal{N}_\sigma$ if and only if

$$\int_\Omega (\Delta u)^2 - 2(1-\sigma)\int_\Omega det(\nabla^2 u) = \int_\Omega g(x)|u|^{p+1},$$

so one has two equivalent formulations for J_σ restricted on $\mathcal{N}_\sigma$:

$$J_{\sigma|_{\mathcal{N}_\sigma}}(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega g(x)|u|^{p+1} = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2_{H_\sigma}, \tag{2.11}$$

which implies $J_{\sigma|\mathcal{N}_\sigma}(u) > 0$ for every $u \in \mathcal{N}_\sigma$.
Concerning the existence, we only need that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain (if $\sigma = 1$, Ω should satisfy also a uniform outer ball condition). Our arguments take some inspiration from [17, 46]. After some preliminary results which describe the geometry of $\mathcal{N}_\sigma$, we show that in the manifold the infimum of J_σ is attained and then, using a deformation lemma, we prove it is a critical point for J_σ in $H^2(\Omega) \cap H_0^1(\Omega)$.
A crucial step is to study what happens on the half-lines of $H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 2.2.3. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$ and the half-line r_u be defined as $r_u := \{tu \,|\, t > 0\}$. The intersection between r_u and $\mathcal{N}_\sigma$ consists of a unique point $t^*(u)u$, where*

$$t^*(u) := \left(\frac{\int_\Omega (\Delta u)^2 - 2(1-\sigma) \int_\Omega det(\nabla^2 u)}{\int_\Omega g(x)|u|^{p+1}} \right)^{\frac{1}{p-1}}. \tag{2.12}$$

Moreover $J_\sigma(t^*(u)u) = \max_{t>0} J_\sigma(tu)$.

Proof. For $t > 0$ and a fixed $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$, then $tu \in \mathcal{N}_\sigma$ if and only if

$$t^2 \left[\int_\Omega (\Delta u)^2 - 2(1-\sigma) \int_\Omega det(\nabla^2 u) \right] = t^{p+1} \int_\Omega g(x)|u|^{p+1},$$

from which we deduce $t = t^*(u)$. Moreover, define

$$\eta(t) := J_\sigma(tu) = \frac{t^2}{2} \left[\int_\Omega (\Delta u)^2 - 2(1-\sigma) \int_\Omega det(\nabla^2 u) \right] - \frac{t^{p+1}}{p+1} \int_\Omega g(x)|u|^{p+1}.$$

If we look for $\bar{t} > 0$ such that $\eta'(\bar{t}) = 0$, we find again that $\bar{t} = t^*(u)$ and, since $\eta'(t)(t - t^*(u)) < 0$ for $t \neq t^*(u)$, we have that $t^*(u)u$ is the unique global maximum on the half-line r_u. $\square$

Lemma 2.2.4. *The Nehari manifold is bounded away from 0, i.e. $0 \notin \overline{\mathcal{N}_\sigma}$.*

Proof. Suppose first that $\sigma \in (-1, 1)$ and let $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$. By Lemmas 2.1.4 and 2.2.3 and the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, the following chain of inequalities holds:

$$\begin{aligned} (1 + |\sigma|)\|t^*(u)u\|^2_{H^2(\Omega)} &\geq \|t^*(u)u\|^2_{H_\sigma(\Omega)} \\ &= (t^*(u))^{p+1} \int_\Omega g(x)|u|^{p+1} \\ &\geq (C_0^{-1}(1-|\sigma|))^{\frac{p+1}{p-1}} \frac{\|u\|^{\frac{2(p+1)}{p-1}}_{H^2(\Omega)}}{(\int_\Omega g(x)|u|^{p+1})^{\frac{2}{p-1}}} \\ &\geq C(\Omega, p, \sigma) \frac{\|u\|^{\frac{2(p+1)}{p-1}}_{H^2(\Omega)}}{(\|g\|_1 \|u\|^{p+1}_{H^2(\Omega)})^{\frac{2}{p-1}}} = \frac{C(\Omega, p, \sigma)}{\|g\|_1^{\frac{2}{p-1}}}. \end{aligned}$$

If $\sigma = 1$, one can deduce the same result using the equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$ given by $\|\Delta \cdot\|_2$. In both cases, there exists a uniform bound from below for the $H^2(\Omega)$ norm of the elements in the Nehari manifold and thus 0 cannot be a cluster point for $\mathcal{N}_\sigma$. $\square$

Proposition 2.2.5. *There exists $u \in \mathcal{N}_\sigma$ such that $J_\sigma(u) = \inf_{v \in \mathcal{N}_\sigma} J_\sigma(v) =: c$*

Proof. As already noticed, $c \geq 0$, since J_σ attains positive values on $\mathcal{N}_\sigma$. Let now $(u_k)_{k\in\mathbb{N}} \subset \mathcal{N}_\sigma$ be a minimizing sequence for J_σ: we claim that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^2(\Omega)$. In fact, if $\sigma \in (-1, 1)$, there exists a constant $C > 0$ such that, for every $k \in \mathbb{N}$,

$$C \geq J_\sigma(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|^2_{H_\sigma(\Omega)} \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)(1 - |\sigma|)C_0^{-1}\|u_k\|^2_{H^2(\Omega)},$$

while (2.3) provides the right estimate in the case $\sigma = 1$. Hence, there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset \mathcal{N}_\sigma$ and $u \in H^2(\Omega) \cap H^1_0(\Omega) \setminus \{0\}$ such that $u_{k_j} \rightharpoonup u$ weakly in $H^2(\Omega)$ (and so weakly in $(H^2(\Omega) \cap H^1_0(\Omega), \|\cdot\|_{H\sigma})$ by Lemma 2.1.4) and strongly in $L^\infty(\Omega)$ by compact embedding. Consider now $t^* = t^*(u)$ such that $t^*u \in \mathcal{N}_\sigma$: by weak semicontinuity of the norm

$$\begin{aligned}
c &= \inf_{v\in\mathcal{N}_\sigma} J_\sigma(v) \\
&\leq J(t^*u) = (t^*)^2\left[\int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u)\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \\
&\leq \liminf_{j\to+\infty}\left((t^*)^2\left[\int_\Omega \frac{(\Delta u_{k_j})^2}{2} - (1-\sigma)\int_\Omega det(\nabla^2 u_{k_j})\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|u_{k_j}|^{p+1}}{p+1}\right) \\
&= \liminf_{j\to+\infty} J_\sigma(t^*u_{k_j}) \leq \liminf_{j\to+\infty} J_\sigma(u_{k_j}) = c
\end{aligned} \tag{2.13}$$

where the last inequality holds because the supremum of J_σ in each half-line $\{tu_{k_j} \mid t > 0\}$ is achieved exactly in u_{k_j} by Lemma 2.2.3. Hence, the infimum of J_σ on $\mathcal{N}_\sigma$ is attained on t^*u. ☐

In the proof of Proposition 2.2.5 something weird happened: we took a minimizing sequence, which converges to an element u and we proved that there exists $\alpha = t^*(u) \in \mathbb{R}$ such that αu is the minimum point of our functional J_σ. One expects that the minimum is u itself and not a dilation of it. Indeed, one may show that $t^* = 1$. In fact, with the same notation as in that proof, from (2.13) we deduce $J_\sigma(u_{k_j}) \to c = J_\sigma(t^*u)$ by construction and $t^*u \in \mathcal{N}_\sigma$, so

$$J_\sigma(u_{k_j}) \to \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega g(x)|t^*u|^{p+1}.$$

Moreover, we took the sequence to be in the Nehari manifold itself, so $J_\sigma(u_{k_j}) = (\frac{1}{2} - \frac{1}{p+1})\int_\Omega g(x)|u_{k_j}|^{p+1}$, and we have that $u_{k_j} \to u$ strongly in $L^\infty(\Omega)$, thus

$$J_\sigma(u_{k_j}) \to \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega g(x)|u|^{p+1}.$$

By the uniqueness of the limit, we must have $t^* = 1$, that is $u \in \mathcal{N}_\sigma$.

Theorem 2.2.6. *The minimum u of J_σ in $\mathcal{N}_\sigma$ is a critical point for J_σ in $H^2(\Omega) \cap H^1_0(\Omega)$.*

Proof. Suppose by contradiction that u is not a critical point. Since the functional is C^1, there exists a ball centered in u and $\varepsilon > 0$ such that, for all $v \in B$,

$$c - \varepsilon \leq J_\sigma(v) \leq c + \varepsilon,$$

$$\|J'_\sigma(v)\|_{(H^2(\Omega)\cap H_0^1(\Omega))^*} \geq \tfrac{\varepsilon}{2},$$

where $c = J_\sigma(u) = \inf\limits_{v\in\mathcal{N}_\sigma} J_\sigma(v)$. Notice that on the half-line r_u, the point u is the global maximum, so $J_\sigma(v) < c$ for each $v \in B \cap r_u$, $v \neq u$.
If we set $a = c - \varepsilon$, $b = c + \varepsilon$, $\delta = 8$, $S = \overline{B_r(u)}$ and $S_0 = \overline{H^2(\Omega) \cap H_0^1(\Omega) \setminus B'}$, where $r > 0$ such that $B_r(u) \subset\subset B' \subset\subset B$, applying [39, Proposition 5.1.25], there exists a locally Lipschitz homotopy of homeomorphisms Γ_t on $H^2(\Omega) \cap H_0^1(\Omega)$ such that:

(i) $t \mapsto J_\sigma(\Gamma(t, v))$ is decreasing in $B_r(u)$ and, in general, non-increasing;

(ii) $J_\sigma(\Gamma(t, v)) = v$ for $v \in S_0$ and $t \in [0, 1]$, and so also for all $v \in \partial B$.

From (i) we deduce that $J_\sigma(\Gamma(t, v)) < c$ for every $v \in B \cap r_u$ and $t \neq 0$. Moreover, define the map: $\psi : B \cap r_u \to \mathbb{R}$ such that

$$\psi(v) := J'_\sigma(\Gamma(1, v))[\Gamma(1, v)]$$

and consider $v \in \partial B \cap r_u$, so there exists $\alpha \neq 1$ such that $v = \alpha u$: we know from (ii) that $\Gamma(1, v) = v$ and, by Lemma 2.2.3, $J'_\sigma(\alpha u)[\alpha u] > 0$ if $\alpha \in (0, 1)$ and $J'_\sigma(\alpha u)[\alpha u] < 0$ if $\alpha \in (1, +\infty)$, so $\psi(v)(v-u) < 0$ on $\partial B \cap r_u$. As a result, since one can think at ψ as a continuous map from $[x_1, x_2] \to \mathbb{R}$, where x_1 and x_2 correspond to the intersections between the half line r_u and the ball B, and since $\psi(x_1) > 0$ and $\psi(x_2) < 0$, there exists a zero of ψ in (x_1, x_2), i.e. there exists $\bar{v} \in B \cap r_u$ such that $J'_\sigma(\Gamma(1, \bar{v}))[\Gamma(1, \bar{v})] = 0$.
Setting $w := \Gamma(1, \bar{v})$, we have $w \in \mathcal{N}_\sigma$ and $J_\sigma(w) = J_\sigma(\Gamma(1, \bar{v})) < c = \inf\limits_{v\in\mathcal{N}_\sigma} J_\sigma(v)$, a contradiction. ☐

Remark 4. So far, we proved the existence of a ground state for J_σ. Actually, one can say more about the existence of general critical points by means of the Krasnosel'skii genus theory (see [5, Section 10.2]). In fact, since our framework is subcritical, it is quite standard to prove the Palais-Smale condition for J_σ by compact embedding of $H^2(\Omega)$ in every Lebesgue space. Moreover, our functional is C^1, even and bounded from below on the unit sphere of $H^2(\Omega) \cap H_0^1(\Omega)$: indeed, if $\|u\|_{H_\sigma(\Omega)} = 1$, then $\|u\|_\infty < C$ for some $C > 0$, so

$$J_\sigma(u) = \frac{1}{2} - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \geq \frac{1}{2} - \frac{C^{p+1}\|g\|_1}{p+1} > -\infty.$$

Hence, by [5, Proposition 10.8], one can ensure the existence of an *infinite number of couples of critical points*. The same argument may also be applied for the general sublinear case, provided $F(x, s) = F(x, -s)$ for every $s \in \mathbb{R}$.

Once the existence of ground states is achieved by Theorem 2.2.6, their positivity basically follows as in Proposition 2.2.2, imposing the same conditions on Ω. Nevertheless, this time the standard argument is not sufficient to have a contradiction since we do not know whether $\tilde{u} \in \mathcal{N}_\sigma$, but we can rely on the characterization of those points due to Lemma 2.2.3.

Proposition 2.2.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and $\sigma \in (-1, 1]$. Moreover, suppose $f(x, u) = g(x)|u|^{p-1}u$, where $p > 1$ and $g \in L^1(\Omega)$ positive a.e. in Ω. Then the ground states of the functional J_σ are positive in Ω.*

Proof. Suppose, by contradiction, that there exists $u \in \mathcal{N}_\sigma$ such that $J_\sigma(u) = \inf\{J_\sigma(v) \,|\, v \in \mathcal{N}_\sigma\}$ and u is not positive. In the same spirit of the proof of Proposition 2.2.2, consider the superharmonic function $\tilde{u}$ associated to u and suppose they are not the same. By Lemma 2.2.3 there exists $t^* := t^*(\tilde{u}) \in \mathbb{R}^+$ such that $t^*\tilde{u} \in \mathcal{N}_\sigma$. Then,

$$\begin{aligned}
J_\sigma(t^*\tilde{u}) &= (t^*)^2\left[\int_\Omega \frac{(\Delta\tilde{u})^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa\tilde{u}_n^2\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|\tilde{u}|^{p+1}}{p+1} \\
&< (t^*)^2\left[\int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \\
&= J_\sigma(t^*u) \le J_\sigma(u),
\end{aligned}$$

which is a contradiction. Notice that the last inequality holds since, by Lemma 2.2.3, J_σ restricted to every half-line attains its maximum on the Nehari manifold. Thus necessarily $\tilde{u}$ coincides with u, which implies that u is strictly superharmonic and thus positive. □

A-priori bounds in the sublinear case

In our sublinear model case $f(x, s) = g(x)|s|^{p-1}s$, $p \in (0, 1)$, something more may be deduced: in fact, Lemma 2.2.3 still applies and, with the same steps as in the proof of Lemma 2.2.4, (reversing the inequalities since now $p - 1 < 0$), one ends up with

$$\|u\|_{H^2(\Omega)} \le \left(\frac{\|g\|_1 C(\Omega)}{(1-|\sigma|)C_0^{-1}}\right)^{\frac{1}{1-p}} \qquad \text{for all } u \in \mathcal{N}_\sigma.$$

As a result, we can state the following:

Proposition 2.2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and let $g \in L^1(\Omega)$ be positive a.e. in Ω. For every $\sigma \in (-1, 1)$ fixed, all critical points of J_σ with $f(x, s) = g(x)|s|^{p-1}s$ and $p \in (0, 1)$ are uniformly bounded in $H^2(\Omega)$.*

Notice that by continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one may also infer an a-priori L^∞ bound for all critical points of J_σ. The estimate becomes also uniform with respect to σ if we restrict $\sigma \in I \Subset (-1, 1)$.

An eigenvalue problem

In the whole chapter we focus on the nonresonant case $p \neq 1$. Let us briefly discuss here what happens if we consider the eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda g(x) u & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

when $0 < g \in L^1(\Omega)$ and, as usual, $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with $C^{1,1}$ boundary. By the scaling invariance of (2.14), we can consider the minimization in $H^2(\Omega) \cap H_0^1(\Omega)$ of the following Rayleigh quotient:

$$R_\sigma(v) = \frac{\|v\|_{H_\sigma}^2}{\int_\Omega g(x) v^2}$$

and apply the standard technique, showing that the first eigenvalue λ_1 is simple and the first eigenfunction is strictly of one sign in Ω. For further details, we refer to [8, Theorem 4].

Moreover, supposing $g \in L^\infty(\Omega)$, by standard theory (see for instance [39, Chapter 6]) there exists a countable sequence of positive eigenvalues for problem (2.14) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty$ and that the respective eigenfunctions form an orthonormal basis of $H^2(\Omega) \cap H_0^1(\Omega)$ with respect to the scalar product $(\cdot,\cdot)_{H_\sigma}$. Indeed, if we consider the solution operator of the linear problem $K_\sigma : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ such that $f \mapsto K_\sigma f := u$ where u is the unique critical point of the functional I_σ, it is easy to infer that K_σ is linear, compact from L^2 to itself, self-adjoint and positive (that is, $(K_\sigma f, f) \geq 0$ for every $f \in L^2(\Omega)$).

Finally, one can also deduce an estimate from below for λ_1. In fact, if we consider $f(x,s) = \lambda g(x) s$ with $\lambda < (1-|\sigma|) C_0^{-1} \|g\|_\infty^{-1}$, it is possible to show that J_σ is strictly convex, so the global minimizer found in Proposition 2.2.1 is unique. Since $u = 0$ is clearly a critical point of J_σ, then there are no other nontrivial critical points. Consequently, $\lambda_1 \geq (1-|\sigma|) C_0^{-1} \|g\|_\infty^{-1}$.

2.2.2 Beyond the physical bounds: $\sigma \leq -1$

So far, we studied the existence of critical points of the functional J_σ and we established positivity of the ground states, always under the assumptions of $\sigma \in (-1, 1]$ and (regarding the positivity) regularity and convexity of Ω. The role of the lower bound $\sigma > -1$ is in particular explained by Lemma 2.1.4, as it was necessary for showing that $\|\cdot\|_{H_\sigma}$ is an equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$. On the other hand, the upper bound $\sigma \leq 1$, together with the convexity assumption, was striking in the proof of the positivity, since it allows the boundary term of J_σ to have a positive fixed sign.

In this section, we study what happens to the ground states of J_σ if we let the parameter be in the whole $\mathbb{R}$, in particular concerning the relationship between the existence and the lower bound $\sigma > -1$ and between the positivity and the upper bound $\sigma \leq 1$. To this aim, we divide the subject into two subsections, one for each problem, addressing the latter in Subsection 2.2.6. In both, we always assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain of class $C^{1,1}$ so that Theorem 2.1.5 holds. Moreover, as it seems more interesting from a mathematical point of view, we mainly focus on the superlinear case $f(x,u) = g(x)|u|^{p-1}u$ with $p > 1$, pointing out, if needed, the necessary adaptation for the sublinear power $p \in (0,1)$.

A Steklov eigenvalue problem

Let us begin by recalling some known facts about the *Steklov eigenvalue problem* (for this topic, we refer to [42] or, for the case $\kappa = 1$, [8, 15]):

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = d\kappa u_n & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

We define a *Steklov eigenvalue* to be a real value d such that (2.15) admits a non-trivial weak solution, named *Steklov eigenfunction*, i.e. $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $u \neq 0$, such that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega \Delta u \Delta \varphi - d \int_{\partial\Omega} \kappa u_n \varphi_n = 0. \tag{2.16}$$

First of all, d must be positive. In fact, if u is a Steklov eigenfunction, taking $u = \varphi$ in (2.16):

$$d \int_{\partial\Omega} \kappa (u_n)^2 = \int_\Omega (\Delta u)^2 > 0,$$

since $\|\Delta \cdot\|_2$ is a norm in $H^2(\Omega) \cap H_0^1(\Omega)$. As $\kappa \geq 0$, we have both $d > 0$ and $\int_{\partial\Omega} \kappa u_n^2 > 0$. As a complementary result, in order to show nontrivial solutions of (2.15), without loss of generality, we can restrict to the subset

$$\mathcal{H} = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \,\middle|\, \int_{\partial\Omega} \kappa (u_n)^2 \neq 0 \right\}.$$

Definition 2.2.1. We denote by $\tilde{\delta}_1(\Omega)$ the *first Steklov eigenvalue*:

$$\tilde{\delta}_1(\Omega) := \inf_{\mathcal{H}\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\int_{\partial\Omega} \kappa u_n^2}.$$

Proposition 2.2.9. *The first Steklov eigenvalue is attained, positive and there exists a unique (up to a multiplicative constant) corresponding Steklov eigenfunction, which is positive in Ω.*

Proof. We refer to [42, Lemma 4.4], just noticing that the continuity of the curvature assumed therein is not necessary to obtain this result. □

A nonexistence and an existence result

From Proposition 2.2.9, it is easy to deduce a nonexistence result for positive solutions if σ is negative enough:

Proposition 2.2.10. *If $\sigma \leq \sigma^* := 1 - \tilde{\delta}_1(\Omega)$, there is no nonnegative nontrivial solution for the Steklov problem* (2.8).

Proof. Let u be a nonnegative solution for (2.8) and $\Phi_1 > 0$ be the first Steklov eigenfunction; we use Φ_1 as a test function in (2.9):

$$\int_\Omega \Delta u \Delta \Phi_1 - (1-\sigma) \int_{\partial\Omega} \kappa u_n (\Phi_1)_n = \int_\Omega g(x) u^p \Phi_1.$$

Then, interpreting u this time as a test function in (2.16), we have

$$\int_\Omega \Delta u \Delta \Phi_1 = \tilde{\delta}_1(\Omega) \int_{\partial\Omega} \kappa (\Phi_1)_n u_n.$$

Combining the two equalities,

$$(\tilde{\delta}_1(\Omega) - (1-\sigma)) \int_{\partial\Omega} \kappa (\Phi_1)_n u_n = \int_\Omega g(x) u^p \Phi_1 > 0.$$

Again by positivity of u and Φ_1, we have $u_n \leq 0$ and $(\Phi_1)_n \leq 0$. Therefore, since $\kappa \geq 0$, we infer $\tilde{\delta}_1(\Omega) - 1 + \sigma > 0$. □

Remark 5. We already proved that our problem (2.8) admits positive solutions whenever $\sigma \in (-1, 1]$ with the same assumptions on Ω. Hence, we infer that, $\tilde{\delta}_1(\Omega) \geq 2$ and we have equality if $\Omega = B_1(0)$ (see [8, Proposition 12]). This result was already proved for C^2 bounded convex domains of $\mathbb{R}^2$ by Parini and Stylianou in [75, Remark 3.3], using Fichera's duality principle.

The next step is to investigate what happens if $\sigma \in (\sigma^*, -1]$ in case this interval is nonempty. We will show that the existence and the positivity results found for $\sigma \in (-1, 1]$ can be extended for this case. In fact, the only restriction we have to overcome, is the fact that here Lemma 2.1.4 is not the right way to prove that the first two terms in the functional J_σ define indeed a norm on $H^2(\Omega) \cap H^1_0(\Omega)$.

Lemma 2.2.11. *For every $\sigma > \sigma^*$, the map*

$$u \mapsto \left[\int_\Omega (\Delta u)^2 - (1-\sigma) \int_{\partial\Omega} \kappa (u_n)^2 \right]^{\frac{1}{2}} = \|u\|_{H_\sigma}$$

is a norm in $H^2(\Omega) \cap H^1_0(\Omega)$ equivalent to the standard norm.

Proof. By the definition of $\tilde{\delta}_1(\Omega)$ as an inf, we have $\|\Delta u\|_2^2 \geq \tilde{\delta}_1(\Omega) \int_{\partial\Omega} \kappa u_n^2$ for each $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and so, if $d > 0$ (which corresponds to $\sigma < 1$),

$$\int_\Omega (\Delta u)^2 \geq \int_\Omega (\Delta u)^2 - d \int_{\partial\Omega} \kappa u_n^2 \geq \left(1 - \frac{d}{\tilde{\delta}_1(\Omega)}\right) \int_\Omega (\Delta u)^2. \tag{2.17}$$

On the other hand, if $d < 0$ (so that $\sigma > 1$),

$$\int_\Omega (\Delta u)^2 \leq \int_\Omega (\Delta u)^2 + |d| \int_{\partial\Omega} \kappa u_n^2 \leq \left(1 + \frac{|d|}{\tilde{\delta}_1(\Omega)}\right) \int_\Omega (\Delta u)^2.$$

As a result, we have to impose that $d < \tilde{\delta}_1(\Omega)$ to have the positivity of the constant in the first estimate, while no restriction occurs in the second. The proof is completed noticing that the map

$$(u,v)_{H_\sigma} \mapsto \int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} \kappa u_n v_n$$

defines a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$ by inequality (2.17) for all $d < \tilde{\delta}_1(\Omega)$. $\square$

Proposition 2.2.12. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with boundary $C^{1,1}$ and suppose $\sigma \in (\sigma^*, -1]$; then the functional J_σ admits a positive ground state.*

Proof. It is sufficient to notice that Lemma 2.2.4 holds for these values of σ if we replace Lemma 2.1.4 by Lemma 2.2.11, while all the other propositions that led to the existence and the positivity of ground states are not affected by this change. $\square$

Remark 6. It is clear that, once Lemma 2.2.11 is established, the *existence* of a ground state of J_σ is ensured for *any* $\sigma > \sigma^*$. The issue of its positivity for the remaining part of this interval, that is $\sigma > 1$, will be addressed in Section 2.2.6.

Remark 7. (Sublinear Case) Both Propositions 2.2.10 and 2.2.12 hold in the case of a function $f(x,u)$ which verifies the assumption (H) (modifying in a suitable way the constant in front of the quadratic term) and $f \geq 0$, $f \not\equiv 0$.

2.2.3 Asymptotic analysis for ground states of J_σ as $\sigma \to \sigma^*$

Having now the existence of positive ground state solutions for $\sigma \in (\sigma^*, 1]$ and having shown that there are no positive solutions if $\sigma \leq \sigma^*$, a natural question that arises is to determine the behaviour as $\sigma_k \searrow \sigma^*$ of a sequence $(u_k)_{k\in\mathbb{N}}$, each of them being a ground state for the respective functional J_{σ_k}. We will find an antipodal result for $f(x,u) = g(x)|u|^{p-1}u$ as $p \in (1,+\infty)$ or $p \in (0,1)$.
The following proof is an adaptation of [9, Theorem 1], which covers the critical case $f(x,u) = |u|^{2^*-2}u$, when the dimension is $N \geq 5$. Moreover, the authors considered a slightly different notion of solution, that is, the minimizers of the Rayleigh quotient associated to the boundary value problem:

$$R_\sigma(u) := \frac{\|\Delta u\|_2^2 - (1-\sigma)\int_{\partial\Omega} \kappa u_n^2}{(\int_\Omega g(x)|u|^{p+1})^{\frac{2}{p+1}}}.$$

However, it is a standard fact to prove that every ground state of J_σ is also a minimizer of R_σ, while the converse is also true, up to a multiplication by a constant.

Theorem 2.2.13. *Let Ω be as in Proposition 2.2.12 and $\sigma_k \searrow \sigma^*$ as $k \to +\infty$. If $p \in (0,1)$, then $\|u_k\|_\infty \to +\infty$, while, if $p > 1$, then $\|u_k\|_{H^2(\Omega)} \to 0$.*

Proof. Let $p > 0$, $p \neq 1$; by the remark above, each ground state u_k is such that

$$R_{\sigma_k}(u_k) = \inf_{0 \neq u \in H^2(\Omega)\cap H_0^1(\Omega)} R_{\sigma_k}(u) := \Sigma_{\sigma_k} \geq 0.$$

By Proposition 2.2.9, there exists a positive first Steklov eigenfunction Φ_1; since we have $\|\Delta\Phi_1\|_2^2 = (1-\sigma^*)\int_{\partial\Omega}\kappa(\Phi_1)_n^2$, we have

$$0 \leq \Sigma_{\sigma_k} \leq R_{\sigma_k}(\Phi_1) = (\sigma_k - \sigma^*)\frac{\int_{\partial\Omega}\kappa(\Phi_1)_n^2}{\left(\int_\Omega g(x)|\Phi_1|^{p+1}\right)^{\frac{2}{p+1}}} \to 0$$

as $k \to +\infty$. Moreover, since u_k is a ground state for J_{σ_k}, $\|\Delta u_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega}\kappa(u_k)_n^2 = \int_\Omega g(x)|u_k|^{p+1}$ and, since $R_{\sigma_k}(u_k) = \Sigma_{\sigma_k}$, we deduce

$$\left(\int_\Omega g(x)|u_k|^{p+1}\right)^{\frac{p-1}{p+1}} = \Sigma_{\sigma_k} \to 0.$$

Hence, if $p > 1$, then $\int_\Omega g(x)|u_k|^{p+1} \to 0$; otherwise, if $p \in (0,1)$, then $\int_\Omega g(x)|u_k|^{p+1} \to +\infty$, which implies, by the Hölder inequality as $g \in L^1(\Omega)$, that $\|u_k\|_\infty \to +\infty$.
We have now to prove that, if $p > 1$, this convergence to 0 is actually in the natural norm $H^2(\Omega)$. By Lemma 2.2.11, $\|\cdot\|_{H_{\sigma_k}}$ is a norm in $H^2(\Omega) \cap H_0^1(\Omega)$ for every k, so we are able to decompose in that norm the Hilbert space as $H^2(\Omega) \cap H_0^1(\Omega) = span(\Phi_1) \oplus [span(\Phi_1)]^\perp$. Thus, for every k there exist a unique $\alpha_k \in \mathbb{R}$ and $\psi_k \in [span(\Phi_1)]^\perp$ such that $u_k = \alpha_k\Phi_1 + \psi_k$. Hence, for k large enough,

$$\begin{aligned} o(1) \geq \int_\Omega g(x)|u_k|^{p+1} &= \|\Delta u_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega}\kappa(u_k)_n^2 = (u_k, u_k)_{H_{\sigma_k}} \\ &= \alpha_k^2(\Phi_1, \Phi_1)_{H_{\sigma_k}} + (\psi_k, \psi_k)_{H_{\sigma_k}}. \end{aligned} \tag{2.18}$$

First of all,

$$(\Phi_1, \Phi_1)_{H_{\sigma_k}} = \|\Delta\Phi_1\|_2^2 - (1-\sigma_k)\int_{\partial\Omega}\kappa(\Phi_1)_n^2 = (\sigma_k - \sigma^*)\int_{\partial\Omega}\kappa(\Phi_1)_n^2. \tag{2.19}$$

Moreover, denoting by $\tilde{\delta_2}(\Omega)$ the second eigenvalue of the Steklov problem, i.e.

$$\tilde{\delta_2}(\Omega) = \inf_{span(\Phi_1)^\perp\setminus\{0\}} \frac{\|\Delta v\|_2^2}{\int_{\partial\Omega}\kappa v_n^2},$$

and defining $\sigma^{**} := 1 - \tilde{\delta_2}(\Omega)$, we get

$$\|\Delta\psi_k\|_2^2 \geq (1-\sigma^{**})\int_{\partial\Omega}\kappa(\psi_k)_n^2,$$

from which

$$\begin{aligned} (\psi_k, \psi_k)_{H_{\sigma_k}} &= \|\Delta\psi_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega}\kappa(\psi_k)_n^2 \\ &\geq \|\Delta\psi_k\|_2^2 - \frac{1-\sigma_k}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2 = \frac{\sigma_k - \sigma^{**}}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2. \end{aligned} \tag{2.20}$$

As a result, combining (2.18) with (2.19) and (2.20), we get:

$$o(1) \geq \int_\Omega g(x)|u_k|^{p+1} = \alpha_k^2(\sigma_k - \sigma^*)\int_{\partial\Omega}\kappa(\Phi_1)_n^2 + \frac{\sigma_k - \sigma^{**}}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2.$$

Since by Proposition 2.2.9 the first Steklov eigenfunction is simple, we have $\sigma^{**} < \sigma^*$ and, recalling that $\sigma_k > \sigma^*$ by assumption, necessarily $\|\Delta\psi_k\|_2 \to 0$. Hence,

$$\begin{aligned}\int_\Omega g(x)|\alpha_k\Phi_1|^{p+1} &\le \int_\Omega g(x)[|u_k| + |\psi_k|]^{p+1} \le 2^p \int_\Omega g(x)[|u_k|^{p+1} + |\psi_k|^{p+1}] \\ &\le 2^p \int_\Omega g(x)|u_k|^{p+1} + C^{p+1}(\Omega)\|g\|_1\|\psi_k\|_{H^2(\Omega)} \to 0.\end{aligned}$$

As a result, $\alpha_k \to 0$ and we finally obtain

$$\|u_k\|_{H^2(\Omega)} \le |\alpha_k|\|\Phi_1\|_{H^2(\Omega)} + \|\psi_k\|_{H^2(\Omega)} \to 0.$$

□

If we read carefully the proof of Theorem 2.2.13, we notice that the fact that each u_k is a ground state for J_σ was necessary only to deduce that $\int_\Omega g(x)|u_k|^{p+1} \to 0$, while to prove the convergence to 0 in $H^2(\Omega)$ norm it was only sufficient that each u_k is a critical point (actually, an element of the Nehari manifold $\mathcal{N}_{\sigma_k}$, since the only step of the proof involved is (2.18)). Therefore, we can state the following result, which will be useful when looking at the radial case in Section 2.2.7:

Lemma 2.2.14. *Let $(u_k)_k$ be a sequence of critical points of J_{σ_k} in the superlinear case, such that $\int_\Omega g(x)|u_k|^{p+1} \to 0$ as $\sigma_k \searrow \sigma^*$. Then $\|u_k\|_{H^2(\Omega)} \to 0$.*

2.2.4 Asymptotic behaviour of ground states of J_σ as $\sigma \to 1$

As mentioned at the beginning of Subsection 2.2.2, we shall investigate the behaviour of the ground states of J_σ when $\sigma > 1$. If, on one hand, the extension of the existence result is straightforward since $\|\cdot\|_{H_\sigma(\Omega)}$ is still a norm on $H^2(\Omega) \cap H^1_0(\Omega)$ by Lemma 2.2.11 (see Remark 6), on the other hand, the extension of positivity is more involved. A way to prove it at least in a right neighborhood of $\sigma = 1$ is by an argument which relies on the convergence of the ground states of the Steklov problem to the respective of the Navier problem. In this section we present this asymptotic approach, while we leave the proofs of the positivity in Subsection 2.2.6.

Again, we assume hereafter that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with $C^{1,1}$ boundary, and that the nonlinearity satisfies (2.10). Moreover, $(u_k)_{k\in\mathbb{N}}$ will always denote a sequence of ground states solutions of the Steklov problems

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u - (1-\sigma_k)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.21}$$

for a sequence $(\sigma_k)_{k\in\mathbb{N}}$ converging to 1. In order to underline the peculiarity of the problem when $\sigma = 1$, we set $J_{NAV} := J_1$, whose critical points are the weak solution of the following Navier problem:

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

Finally, $\overline{u}$ will always denote a ground state of J_{NAV}. Our first purpose is to prove the convergence $u_k \to \overline{u}$ in the natural norm, i.e. in $H^2(\Omega)$, as $\sigma_k \to 1$, without any distinction about the sign of $1-\sigma_k$. Then, we will promote it to a stronger norm.

First, we show that it is enough to prove a weak convergence.

Lemma 2.2.15. *Let $(u_k)_{k\in\mathbb{N}}$ and $\overline{u}$ be as specified above. If $u_k \rightharpoonup \overline{u}$ weakly in $H^2(\Omega)$ as $\sigma_k \to 1$, then (up to a subsequence) $u_k \to \overline{u}$ strongly in $H^2(\Omega)$.*

Proof. As $u_k \rightharpoonup \overline{u}$ weakly in $H^2(\Omega)$, there exists $M > 0$ such that $\|u_k\|^2_{H^2(\Omega)} \leq M$. Moreover, u_k is a solution of (2.21) for each $k \in \mathbb{N}$ and $\overline{u}$ of the Navier problem (2.22), thus, for every test function $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$:

$$\int_\Omega \Delta u_k \Delta\varphi - (1-\sigma_k)\int_{\partial\Omega} \kappa (u_k)_n \varphi_n = \int_\Omega g(x)|u_k|^{p-1}u_k\varphi, \tag{2.23}$$

$$\int_\Omega \Delta\overline{u}\Delta\varphi = \int_\Omega g(x)|\overline{u}|^{p-1}\overline{u}\varphi.$$

Hence

$$C_A^{-1}\|u_k-\overline{u}\|^2_{H^2(\Omega)} \leq \|\Delta u_k - \Delta\overline{u}\|_2^2 = \int_\Omega \Delta u_k \Delta(u_k-\overline{u}) - \int_\Omega \Delta\overline{u}\Delta(u_k-\overline{u})$$
$$= (1-\sigma_k)\int_{\partial\Omega}\kappa(u_k)_n(u_k-\overline{u})_n + \left[\int_\Omega g(x)|u_k|^{p-1}u_k(u_k-\overline{u}) - \int_\Omega g(x)|\overline{u}|^{p-1}\overline{u}(u_k-\overline{u})\right].$$

For the first term:

$$\left|(1-\sigma_k)\int_{\partial\Omega}\kappa(u_k)_n(u_k-\overline{u})_n\right| \leq |1-\sigma_k|C_T^2\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{H^2(\Omega)}\|u_k-\overline{u}\|_{H^2(\Omega)}$$
$$\leq |1-\sigma_k|C_T^2\|\kappa\|_{L^\infty(\partial\Omega)}M(M+\|\overline{u}\|_{H^2(\Omega)}) \to 0,$$

where C_T is the constant in the trace theorem. Concerning the second, it is enough to invoke the dominated convergence theorem as we have pointwise convergence and since

$$\left|g(x)(|u_k|^{p-1}u_k - |\overline{u}|^{p-1}\overline{u})(u_k-\overline{u})\right| \leq |g(x)|[C(\Omega)^pM^p + |\overline{u}|^p][C(\Omega)M+\overline{u}] \in L^1(\Omega),$$

where $C(\Omega)$ is the constant in the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. □

Remark 8. This result holds not only for ground states, but for generic solutions, i.e. if $(u_k)_{k\in\mathbb{N}}$ is a sequence of weak solutions of the Steklov problem (2.21) and $\overline{u}$ a weak solution of the Navier problem (2.22) and we know that $u_k \rightharpoonup \overline{u}$ weakly in $H^2(\Omega)$, then, up to a subsequence, it converges strongly too.

A crucial observation is that the Nehari manifolds are nested with respect to the parameter σ:

Lemma 2.2.16. *Let $\sigma_1 < \sigma_2$ and fix $u \in H^2(\Omega)\cap H^1_0(\Omega)\setminus\{0\}$. Then*

$$t^*_{\sigma_1}(u) \leq t^*_{\sigma_2}(u).$$

Proof. Indeed, $-(1-\sigma_1) < -(1-\sigma_2)$ and so, by (2.12),

$$t^*_{\sigma_1}(u)^{p-1} = \frac{\int_\Omega (\Delta u)^2 - (1-\sigma_1)\int_{\partial\Omega} \kappa u_n^2}{\int_\Omega g(x)|u|^{p+1}} \leq \frac{\int_\Omega (\Delta u)^2 - (1-\sigma_2)\int_{\partial\Omega} \kappa u_n^2}{\int_\Omega g(x)|u|^{p+1}} = t^*_{\sigma_2}(u)^{p-1}.$$

□

Notice that if $u \in H^2_0(\Omega)$ then one has the equality; if we suppose moreover that $\kappa > 0$ a.e., we deduce also the converse.

Proposition 2.2.17. *The sequence of ground states $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^2(\Omega)$.*

Proof. Set k_{max} such that $\sigma_{k_{max}} = \max\{(\sigma_k)_{k\in\mathbb{N}}, 1\}$ and so $u_{k_{max}}$ is a ground state for $J_{\sigma_{k_{max}}}$ (with the convention that if $\sigma_{k_{max}}$=1, then $u_{k_{max}}$ is a ground state for J_{NAV}). Defining $w_k := t^*_{\sigma_k}(u_{k_{max}})u_{k_{max}} \in \mathcal{N}_{\sigma_k}$, that is, the "projection" of $u_{k_{max}}$ on the Nehari manifold $\mathcal{N}_{\sigma_k}$ along its half-line, one has

$$\int_\Omega g(x)|u_k|^{p+1} \leq \int_\Omega g(x)|w_k|^{p+1} \leq \int_\Omega g(x)|u_{k_{max}}|^{p+1}. \tag{2.24}$$

Indeed, the first inequality comes from the fact that u_k is a ground state of J_{σ_k}, which has the equivalent formulation (2.11); the second is obtained by Lemma 2.2.16 since

$$\begin{aligned}\int_\Omega g(x)|w_k|^{p+1} &= (t^*_{\sigma_k}(u_{k_{max}}))^{p+1}\int_\Omega g(x)|u_{k_{max}}|^{p+1}\\ &\leq (t^*_{\sigma_{k_{max}}}(u_{k_{max}}))^{p+1}\int_\Omega g(x)|u_{k_{max}}|^{p+1} = \int_\Omega g(x)|u_{k_{max}}|^{p+1}.\end{aligned}$$

Furthermore, for any $\sigma > 0$ (and here we can assume it without loss of generality),

$$\int_\Omega (\Delta u)^2 - (1-\sigma)\int_{\partial\Omega} \kappa u_n^2 \geq \min\{\sigma, 1\} C_A(\Omega)\|u\|^2_{H^2(\Omega)}. \tag{2.25}$$

In fact, if $\sigma \in [1, +\infty)$ the proof is straightforward since $-(1-\sigma) \geq 0$, otherwise, if $\sigma \in (0, 1)$:

$$\begin{aligned}&\int_\Omega (\Delta u)^2 - (1-\sigma)\int_{\partial\Omega} \kappa u_n^2 = \int_\Omega (\Delta u)^2 + 2(1-\sigma)\int_\Omega (-det(\nabla^2 u))\\ &= \int_\Omega \left[u_{xx}^2 + u_{yy}^2 + 2\sigma u_{xx}u_{yy} + 2(1-\sigma)u_{xy}^2\right] \geq \sigma\int_\Omega (\Delta u)^2 + 2(1-\sigma)\int_\Omega u_{xy}^2\\ &\geq \sigma\int_\Omega (\Delta u)^2 \geq \sigma C_A^{-1}(\Omega)\|u\|^2_{H^2(\Omega)}.\end{aligned}$$

Combining (2.24) with (2.25), we get:

$$\|u_k\|^2_{H^2(\Omega)} \leq \frac{C_A(\Omega)}{\min\{\sigma_k, 1\}}\int_\Omega g(x)|u_k|^{p+1} \leq \frac{C_A(\Omega)}{\min\{\sigma_k, 1\}}\int_\Omega g(x)|u_{k_{max}}|^{p+1},$$

which is the estimate we needed. □

As a direct consequence of Proposition 2.2.17, the sequence $(u_k)_{k\in\mathbb{N}}$, up to a subsequence, is weakly convergent to some $u_\infty \in H^2(\Omega)\cap H_0^1(\Omega)$ with strong convergence in $L^\infty(\Omega)$. It is also easy to see that u_∞ is a weak solution of the Navier problem (2.22): it is enough to apply to (2.23) the weak convergence in $H^2(\Omega)$, the strong convergence in $L^2(\partial\Omega)$ of the normal derivatives and the dominated convergence theorem. As a consequence, by Proposition 2.2.15, the convergence $u_k \to u_\infty$ is strong in $H^2(\Omega)$.

Theorem 2.2.18. *Let $\sigma_k \to 1$ and $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with boundary of class $C^{1,1}$. Then the sequence $(u_k)_{k\in\mathbb{N}}$ of ground state solutions for the Steklov problems (2.21) admits a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ which converges in $H^2(\Omega)$ to u_∞, which is a ground state for the Navier problem (2.22), and thus strictly superharmonic.*

Proof. Clearly, as u_∞ is weak solution of (2.22), we have $J_{NAV}(u_\infty) \geq \inf_{\mathcal{N}_{NAV}} J_{NAV}$. Now we have to prove the converse inequality. Firstly, we have $J_{NAV}(u_\infty) \leq \liminf_{k\to+\infty} J_{\sigma_k}(u_k)$. Indeed,

$$\begin{aligned}\liminf_{k\to+\infty} J_{\sigma_k}(u_k) &= \liminf_{k\to+\infty}\int_\Omega \frac{(\Delta u_k)^2}{2} - \lim_{k\to+\infty}\frac{1-\sigma_k}{2}\int_{\partial\Omega}\kappa(u_k)_n^2 - \lim_{k\to+\infty}\int_\Omega \frac{g(x)|u_k|^{p+1}}{p+1}\\ &\geq \int_\Omega \frac{(\Delta u_\infty)^2}{2} - \int_\Omega \frac{g(x)|u_\infty|^{p+1}}{p+1} = J_{NAV}(u_\infty),\end{aligned}$$

having used the compactness of the map $\partial_n : H^2(\Omega) \to L^2(\partial\Omega)$ and the dominated convergence theorem. Moreover, if we suppose $\sigma_k < 1$ for k large enough, by Lemma 2.2.16 (with a similar argument to that in (2.24)), for all $k \in \mathbb{N}$ we have

$$J_{\sigma_k}(u_k) = \left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega g(x)|u_k|^{p+1} \leq \left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega g(x)|u_\infty|^{p+1} = J_{NAV}(u_\infty), \tag{2.26}$$

so in this case we are done. If otherwise $\sigma_k > 1$ for a infinite number of indices,(2.26) does not hold. In this case, without loss of generality, we can assume that $\sigma_k \searrow 1$. By the existence theorems in Subsection 2.2, we know that there exists a ground state $\overline{u} \in H^2(\Omega)\cap H_0^1(\Omega)$ for J_{NAV} and we define $\overline{u}_k := t^*_{\sigma_k}(\overline{u})\overline{u}$ to be the "projection" on the Nehari manifold $\mathcal{N}_{\sigma_k}$. Then $\|\overline{u}_k - \overline{u}\|_{H^2(\Omega)} = |1-t^*_{\sigma_k}(\overline{u})|\|\overline{u}\|_{H^2(\Omega)}$ with

$$1-(t^*_{\sigma_k}(\overline{u}))^{p-1} \overset{[\overline{u}\in\mathcal{N}_{NAV}]}{=} (t^*_{NAV}(\overline{u}))^{p-1} - (t^*_{\sigma_k}(\overline{u}))^{p-1} = 2(1-\sigma_k)\frac{\int_\Omega det(\nabla^2\overline{u})}{\int_\Omega g(x)|\overline{u}|^{p+1}} \to 0,$$

so $\overline{u}_k \to \overline{u}$ in $H^2(\Omega)$, which implies

$$\int_\Omega g(x)|\overline{u}_k|^{p+1} \to \int_\Omega g(x)|\overline{u}|^{p+1}. \tag{2.27}$$

Nevertheless, since u_k is a ground state of J_{σ_k},

$$\int_\Omega g(x)|\overline{u}_k|^{p+1} \overset{[\overline{u}_k\in\mathcal{N}_{\sigma_k}]}{=} (\tfrac{1}{2}-\tfrac{1}{p+1})J_{\sigma_k}(\overline{u}_k) \geq (\tfrac{1}{2}-\tfrac{1}{p+1})J_{\sigma_k}(u_k) \overset{[u_k\in\mathcal{N}_{\sigma_k}]}{=} \int_\Omega g(x)|u_k|^{p+1}; \tag{2.28}$$

furthermore, since we assumed $\sigma_k > 1$ and by Lemma 2.2.16,

$$\begin{aligned}\int_\Omega g(x)|u_k|^{p+1} &\geq \int_\Omega g(x)|t^*_{NAV}(u_k)u_k|^{p+1} = (\tfrac{1}{2} - \tfrac{1}{p+1})J_{NAV}(t^*_{NAV}(u_k)u_k) \\ &\geq (\tfrac{1}{2} - \tfrac{1}{p+1})J_{NAV}(\overline{u}) = \int_\Omega g(x)|\overline{u}|^{p+1}.\end{aligned} \tag{2.29}$$

Combining (2.27), (2.28) and (2.29), we find

$$\int_\Omega g(x)|u_k|^{p+1} \to \int_\Omega g(x)|\overline{u}|^{p+1}, \tag{2.30}$$

from which $J_{\sigma_k}(u_k) \to J_{NAV}(\overline{u})$, which completes our equality.
To conclude, notice that we have already obtained in the proof of Proposition 2.2.7 that ground states of the Navier problem (2.22) are strictly superharmonic. □

Remark 9. The same analysis may be adapted also for the sublinear case $p \in (0, 1)$, paying attention to some minor changes: for instance, Lemma 2.2.16 holds with the reverse inequality, but this compensates for the fact that this time the coefficient $\frac{1}{2} - \frac{1}{p}$ in the equivalent formulation of J_σ is negative.

In order to apply this convergence argument to the study of the positivity of ground states for a right neighborhood of $\sigma = 1$, we need a stronger result, namely, we need to upgrade our convergence to a stronger norm. Indeed, we need also a uniform control on the normal derivative on the boundary, which is not given by the simple H^2 norm.

A first step is to investigate, for a fixed $\sigma > \sigma^*$, the regularity of the solutions of (2.21) and (2.22) with just a slightly more regular boundary (actually, we have to impose that $\partial\Omega$ is of class C^2). This will be obtained by means of the following lemma by Gazzola, Grunau and Sweers, which follows from a result by Agmon, Douglis and Nirenberg [4, Theorem 15.3', p.707]:

Lemma 2.2.19 ([40], Corollary 2.23). *Let $q > 1$ and take an integer $m \geq 4$. Assume that $\partial\Omega \in C^m$ and $a \in C^{m-2}$, then there exists $C = C(m, q, a, \Omega) > 0$ such that*

$$\|u\|_{W^{m,q}(\Omega)} \leq C\left(\|u\|_q + \|\Delta^2 u\|_{W^{m-4,q}(\Omega)} + \|u\|_{W^{m-\frac{1}{q},q}(\partial\Omega)} + \|\Delta u - a u_n\|_{W^{m-2-\frac{1}{q},q}(\partial\Omega)} \right),$$

for every $u \in W^{m,q}(\Omega)$. The same statement holds for any $m \geq 2$ provided the norms on the right-hand side are suitably interpreted.

Hence we have to define $\Delta^2 u$ as a distribution in $W^{-2,q}(\Omega)$, i.e. acting on functions in $W^{2,q'}_0(\Omega)$. Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$ be a weak solution of (2.8); we define the linear functional over $H^2(\Omega)$:

$$\Delta^2 u : H^2(\Omega) \ni \varphi \mapsto < \Delta^2 u, \varphi > := \int_\Omega \Delta u \Delta \varphi$$

which is well-defined and continuous. If we let

$$u^p_g : \varphi \mapsto < u^p_g, \varphi > := \int_\Omega g(x)|u|^{p-1}u\varphi,$$

it is clearly well-defined and continuous on $W_0^{2,q'}(\Omega)$ and, by the weak formulation of the PDE, on the subset $H_0^2(\Omega)$ it acts identically as $\Delta^2 u$. As a result, we define

$$\Delta^2 u : W_0^{2,q'}(\Omega) \ni \varphi \mapsto < \Delta^2 u, \varphi > := \int_\Omega g(x)|u|^{p-1}u\varphi. \tag{2.31}$$

Proposition 2.2.20. *If $\partial\Omega \in C^2$, for every $\sigma > \sigma^*$ the weak solutions of Steklov and Navier problems* (2.21) *and* (2.22) *lie in $W^{2,q}(\Omega)$ for every $q > 2$.*

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a weak solution of (2.8). Applying Lemma 2.2.19 with $m = 2$ and $a = (1-\sigma)\kappa \in \mathcal{C}^0(\partial\Omega)$ ($a = 0$ for the Navier case), we find:

$$\|u\|_{W^{2,q}(\Omega)} \le C(q,\sigma,\Omega)\bigg(\|u\|_q + \|\Delta^2 u\|_{W^{-2,q}(\Omega)}\bigg),$$

which is well-defined in view of (2.31). Since

$$\|\Delta^2 u\|_{W^{-2,q}(\Omega)} = \sup_{0\neq\varphi\in W_0^{2,q'}(\Omega)} \frac{\left|\int_\Omega g(x)|u|^{p-1}u\varphi\right|}{\|\varphi\|_{W_0^{2,q'}(\Omega)}} \le C(p,q,\Omega)\|g\|_1\|u\|^p_{H^2(\Omega)}, \tag{2.32}$$

we finally deduce from (2.32) that

$$\|u\|_{W^{2,q}(\Omega)} \le C(q,\sigma,\Omega)\bigg(\|u\|_q + C(p,q,\Omega)\|g\|_1\|u\|^p_{H^2(\Omega)}\bigg) < +\infty.$$

□

We stress that we did not use either the fact that u is a ground state solution, or its positivity: the above result holds true for every weak solution of Steklov and Navier problems.

Proposition 2.2.21. *Let Ω be of class C^2 and $(u_k)_{k\in\mathbb{N}}$ be a sequence of weak solutions for the Steklov problems* (2.21) *converging in $H^2(\Omega)$ to $\overline{u}$, a weak solution for the Navier problem* (2.22). *Then the convergence is in $W^{2,q}(\Omega)$ for every $q \ge 2$.*

Proof. Let $q \ge 2$ and apply the regularity estimate of Lemma (2.2.19) to $u_k - \overline{u}$ with $m = 2$, $a = 0$:

$$\|u_k-\overline{u}\|_{W^{2,q}(\Omega)} \le C(q,\Omega)\big(\|u_k-\overline{u}\|_q+\|\Delta^2 u_k-\Delta^2\overline{u}\|_{W^{-2,q}(\Omega)}+|1-\sigma_k|\|\kappa(u_k)_n\|_{W^{-\frac{1}{q},q}(\partial\Omega)}\big), \tag{2.33}$$

using that on $\partial\Omega$ we have $\Delta(u_k-\overline{u}) - a(u_k-\overline{u})_n = \Delta u_k - \Delta\overline{u} = (1-\sigma_k)\kappa(u_k)_n$. By (2.31) and the dominated convergence theorem:

$$\|\Delta^2 u_k - \Delta^2\overline{u}\|_{W^{-2,q}(\Omega)} = \sup_{0\neq\varphi\in W_0^{2,q'}(\Omega)} \frac{\left|\int_\Omega g(x)|u_k|^{p-1}u_k\varphi - \int_\Omega g(x)|\overline{u}|^{p-1}\overline{u}\varphi\right|}{\|\varphi\|_{W_0^{2,q'}(\Omega)}} \to 0,$$

similarly to (2.32). We need now to prove that $(\kappa(u_k)_n)_{k\in\mathbb{N}}$ is bounded in $W^{-\frac{1}{q},q}(\partial\Omega)$. Notice that if we provide a uniform bound in $L^q(\partial\Omega)$, then we are done. In

fact $W^{-\frac{1}{q},q}(\partial\Omega) := \big(W^{\frac{1}{q},q'}(\partial\Omega)\big)^*$ and $W^{\frac{1}{q},q'}(\partial\Omega) \hookrightarrow L^{q'}(\partial\Omega)$, so we directly infer $W^{-\frac{1}{q},q}(\partial\Omega) \hookleftarrow L^q(\partial\Omega)$.
Moreover, it is known that, with our assumptions on $\partial\Omega$, the normal trace of functions in $W^{s,p}(\Omega)$ lies in $L^p(\partial\Omega)$, provided $s > 1+\frac{1}{p}$ (for this and some further sharper results, see [64, Theorem 2]). Hence,

$$\begin{aligned}\|\kappa(u_k)_n\|_{W^{-\frac{1}{q},q}(\partial\Omega)} &\le C(q,\Omega)\|\kappa(u_k)_n\|_{L^q(\partial\Omega)} \le C(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|(u_k)_n\|_{L^q(\partial\Omega)} \\ &\le C(q,\Omega,s)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{W^{s,q}(\Omega)},\end{aligned} \tag{2.34}$$

for some $s > 1+\frac{1}{q}$. Thus, we need to find an appropriate fractional Sobolev space in which $H^2(\Omega)$ is embedded. We claim that $H^2(\Omega) \hookrightarrow W^{1+3/2q,q}(\Omega)$. Actually, it is enough to prove that $H^1(\Omega) := W^{1,2}(\Omega) \hookrightarrow W^{3/2q,q}(\Omega)$ by the definition of $W^{s,p}(\Omega)$ for $s > 1$. So, let $u \in W^{1,2}(\Omega)$; by the Stein total extension theorem [1, Theorem 5.24] there exists $U \in W^{1,2}(\mathbb{R}^2)$ such that $U_{|\Omega} = u$ a.e. and $\|U\|_{W^{1,2}(\mathbb{R}^2)} \le C\|u\|_{W^{1,2}(\Omega)}$ for some positive constant independent of u. Applying an interpolation result (see Theorem A.0.1 in the Appendix) to U with $\theta = \frac{3}{2q}$ and the Sobolev embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{4q-6}(\mathbb{R}^2)$ since $4q-6 \ge 2$:

$$\|U\|_{W^{3/2q,q}(\mathbb{R}^2)} \le C\|U\|^{\frac{3}{2q}}_{W^{1,2}(\mathbb{R}^2)}\|U\|^{1-\frac{3}{2q}}_{L^{4q-6}(\mathbb{R}^2)} \le C_1\|U\|_{W^{1,2}(\mathbb{R}^2)}.$$

Hence,

$$\|u\|_{W^{3/2q,q}(\Omega)} = \|U\|_{W^{3/2q,q}(\Omega)} \le \|U\|_{W^{3/2q,q}(\mathbb{R}^2)} \le C_1\|U\|_{W^{1,2}(\mathbb{R}^2)} \le C_2\|u\|_{W^{1,2}(\Omega)}.$$

As a result, noticing that $s = 1+\frac{3}{2q} > 1+\frac{1}{q}$, we can continue (2.34), obtaining:

$$\|\kappa(u_k)_n\|_{W^{-\frac{1}{q},q}(\partial\Omega)} \le C(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{W^{1+3/2q,q}(\Omega)} \le \tilde{C}(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{H^2(\Omega)},$$

which is uniformly bounded in k. Combining estimate (2.33) with the ones above for the second and the third terms of (2.33), we finally end up with the strong convergence in $W^{2,q}(\Omega)$. □

2.2.5 The Dirichlet problem and an asymptotic analysis as $\sigma \to +\infty$

There is another relevant case in which an interesting asymptotic analysis may be performed: the Dirichlet problem

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.35}$$

seen as the limit problem as $\sigma \to \infty$. We do not expect to deduce any positivity argument for sufficiently large values of σ from a possible convergence result, since the normal derivative on the boundary of the limiting problem vanishes. However, we can hope to infer from it a further evidence that the positivity might hold for all values of $\sigma > \sigma^*$, at least when dealing with some special domains (see the subsequent Subsection 2.2.6). To this aim, we prove the following:

Theorem 2.2.22. *Let $\sigma_k \to +\infty$ and Ω be a bounded convex domain in $\mathbb{R}^2$ with boundary of class $C^{1,1}$. Assume also that the curvature κ is positive a.e on $\partial\Omega$. Then the sequence $(u_k)_{k\in\mathbb{N}}$ of ground states of $(J_{\sigma_k})_{k\in\mathbb{N}}$ admits a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ convergent in $H^2(\Omega)$ to U, which is a ground state for the Dirichlet problem* (2.35).

The argument is similar to that we used in Subsection 2.2.4 for the convergence to the Navier problem, but now we have to pay attention to the fact that in this case the two functional spaces are different ($H^2(\Omega) \cap H_0^1(\Omega)$ for the Steklov problem and $H_0^2(\Omega)$ for the Dirichlet). We are not giving here the details of the proof of the existence of ground states of (2.35), as it can be obtained as for the Steklov framework by the Nehari method of Section 2.2. In what follows, as usual, u_k will always denote a ground state for J_{σ_k} and U a ground state for $J_{DIR} : H_0^2(\Omega) \to \mathbb{R}$ defined as

$$J_{DIR}(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \frac{1}{p+1}\int_\Omega g(x)|u|^{p+1},$$

whose critical points are weak solutions of (2.35). Moreover, similarly to the Steklov case, we define the Nehari manifold for J_{DIR}:

$$\mathcal{N}_{DIR} := \{u \in H_0^2(\Omega) \setminus \{0\} \mid J'_{DIR}(u)[u] = 0\}.$$

First of all, notice that for each σ

$$J_\sigma|_{H_0^2(\Omega)} = J_{DIR}, \tag{2.36}$$

so $\mathcal{N}_\sigma$ restricted to the subspace $H_0^2(\Omega)$ coincides with $\mathcal{N}_{DIR}$.

Proof of Theorem 2.2.22. We follow the same steps as in Subsection 2.2.4 to deduce Theorem 2.2.18. Firstly, we prove that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^2(\Omega)$. Indeed, fix $W \in H_0^2(\Omega)$ a ground state for the Dirichlet problem (2.35), then

$$\begin{aligned} \|\Delta u_k\|_2^2 &\le \int_\Omega (\Delta u_k)^2 - (1-\sigma_k)\int_{\partial\Omega} \kappa (u_k)_n^2 = \int_\Omega g(x)|u_k|^{p+1} \\ &= \inf_{v\in\mathcal{N}_{\sigma_k}} \int_\Omega g(x)|v|^{p+1} \le \inf_{v\in\mathcal{N}_{\sigma_k}\cap H_0^2(\Omega)} \int_\Omega g(x)|v|^{p+1} = \int_\Omega g(x)|W|^{p+1}. \end{aligned} \tag{2.37}$$

Hence, there exists $U \in H^2(\Omega) \cap H_0^1(\Omega)$ such that, up to a subsequence, $u_k \rightharpoonup U$ weakly in $H^2(\Omega)$. Moreover, (2.37) implies that

$$0 \le (\sigma_k - 1)\int_{\partial\Omega} \kappa (u_k)_n^2 \le \int_\Omega g(x)|u_k|^{p+1} \le C(\Omega,p)\|g\|_1 \|u_k\|_{H^2(\Omega)}^{p+1} \le D(\Omega,p,g)$$

and, taking into account that $\sigma_k \to +\infty$, we deduce that

$$\int_{\partial\Omega} \kappa (u_k)_n^2 \to 0.$$

By the compactness of the map $\partial_n : H^2(\Omega) \to L^2(\partial\Omega)$, we have also that

$$\int_{\partial\Omega} \kappa (u_k)_n^2 \to \int_{\partial\Omega} \kappa U_n^2.$$

Hence, combining the two and recalling that we assumed $\kappa > 0$ on $\partial\Omega$, we deduce that $U_n \equiv 0$ on $\partial\Omega$ and thus $U \in H_0^2(\Omega)$.
Finally, testing the weak formulation of problem (2.21) with $\varphi \in H_0^2(\Omega)$ and passing to the limit as $k \to +\infty$, we get

$$\int_\Omega \Delta U \Delta \varphi = \int_\Omega g(x)|U|^{p-1}U\varphi,$$

so U is a solution of the Dirichlet problem (2.35) and, similarly to Lemma 2.2.15, we can prove that the convergence is strong in $H^2(\Omega)$. It remains to prove that U is actually a ground state for J_{DIR}. Let $W \in H_0^2(\Omega)$ be a ground state solution of J_{DIR}. Then, by (2.36):

$$m = J_{DIR}(W) = J_{\sigma_k}(t^*_{\sigma_k}(W)W) \geq \inf_{\mathcal{N}_{\sigma_k} \cap H_0^2(\Omega)} J_{\sigma_k} \geq \inf_{\mathcal{N}_{\sigma_k}} J_{\sigma_k} = J_{\sigma_k}(u_k),$$

hence we deduce that $m \geq \liminf_{k\to+\infty} J_{\sigma_k}(u_k)$. Moreover, by strong convergence,

$$J_{DIR}(U) = (\tfrac{1}{2} - \tfrac{1}{p+1}) \int_\Omega g(x)|U|^{p+1} = \lim_{k\to+\infty} (\tfrac{1}{2} - \tfrac{1}{p+1}) \int_\Omega g(x)|u_k|^{p+1} = \lim_{k\to+\infty} J_{\sigma_k}(u_k).$$

Finally, since U is a solution of the Dirichlet problem (2.35), we have $U \in \mathcal{N}_{DIR}$, so:

$$m \leq J_{DIR}(U) \leq \liminf_{k\to+\infty} J_{\sigma_k}(u_k) \leq m.$$

□

2.2.6 Beyond the physical bounds: $\sigma > 1$

This subsection is devoted to establish the positivity of ground states for $\sigma > 1$. We will provide two different proofs (which will produce two slightly different results): the first is relied on the convergence analysis given in Subsection 2.2.4, while the second is based on the method of dual cones, connecting our semilinear problem with the linear one. Again, for convenience, we assume the exponent of the nonlinearity (2.10) to be $p > 1$.

First method for positivity: a convergence argument

Let us start by noticing that, by Morrey's embeddings, the convergence in $W^{2,q}(\Omega)$ for every $q \geq 2$ of Proposition 2.2.21 implies the convergence in $C^{1,\alpha}(\overline{\Omega})$ for every $\alpha < 1$, so in particular in $C^1(\overline{\Omega})$. This will be the main ingredient in the next proof.

Proposition 2.2.23. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain of class C^2 and $(u_k)_{k\in\mathbb{N}}$ be a sequence of ground states for the functional J_{σ_k} with $\sigma_k \searrow 1$. Then there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ and $j_0 \in \mathbb{N}$ such that $u_{k_j} > 0$ in Ω for every $j \geq j_0$.*

Proof. By Propositions 2.2.20 and 2.2.21 and by the previous observation, we know that, up to a subsequence, $u_k \to \overline{u}$ in $C^1(\overline{\Omega})$ for some $\overline{u}$, a ground state for J_{NAV}.

Since Ω has a C^2 boundary, the interior sphere condition holds and one can extend the outer normal vector n in a small neighborhood $\omega_0 \subset \Omega$ of $\partial\Omega$ and thus define here $\overline{u}_n := \nabla\overline{u} \cdot n$ (see [87, Chapter 4]). Moreover, since $\overline{u}$ is strictly superharmonic, the normal derivative $\overline{u}_n$ is negative on $\partial\Omega$ and, by compactness of $\partial\Omega$ and continuity of $\overline{u}_n$, there exists $\alpha > 0$ such that

$$\overline{u}_{n|\partial\Omega} \leq -\alpha < 0.$$

Hence, again by continuity, there exists a second neighborhood $\omega \subset \omega_0$ of $\partial\Omega$ such that

$$\overline{u}_{n|\omega} \leq -\frac{2}{3}\alpha < 0.$$

Take now $\varepsilon_1 = \frac{\alpha}{3}$: by the $C^1(\overline{\Omega})$ convergence, there exists $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$ and $x \in \omega$:

$$\begin{aligned}|(u_k)_n(x)| &\geq |\overline{u}_n(x)| - |(u_k)_n(x) - \overline{u}_n(x)| \\ &> \frac{2\alpha}{3} - |||n|||_{L^\infty(\omega)}|||\nabla u_k - \nabla\overline{u}|||_{L^\infty(\Omega)} > \frac{2\alpha}{3} - \varepsilon_1 > \frac{\alpha}{3}.\end{aligned}$$

By the interior sphere condition, the map $\omega \to \partial\Omega$, $x \mapsto x_0$ such that $d(x, x_0) = \inf\{d(x,y)|y \in \partial\Omega\}$ is well defined and the vector $x - x_0$ has the same direction as $n(x)$ and $n(x_0)$. Hence by Lagrange Theorem and recalling that $u_{k|\partial\Omega} = 0$, for $x \in \omega$:

$$|u_k(x)| = |u_k(x) - u_k(x_0)| \geq \min_{y\in[x_0,x]} |(u_k)_n(y)||x - x_0| > \frac{\alpha}{3}|x - x_0| > 0. \quad (2.38)$$

Moreover, notice that by compactness of $\Omega_0 := \Omega \setminus \omega$ we have

$$\overline{u}_{|\Omega_0} \geq \min_{\Omega_0} \overline{u} := m > 0$$

and so by the uniform convergence it is easy to deduce that, for k large enough, $u_k(x) > \frac{m}{2}$ for every $x \in \Omega_0$. The result follows by combining this with (2.38). □

Theorem 2.2.24. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain of class C^2; then there exists $\sigma_1 > 1$ such that for every $\sigma \in (1, \sigma_1)$ the ground states of J_σ are positive in Ω.*

Proof. By contradiction, suppose that such σ_1 does not exist. Hence we would be able to find a sequence $(\sigma_k) \searrow 1$ such that for each of them there exists a ground state u_k for J_{σ_k} which is not positive. This would contradict Proposition 2.2.23. □

Remark 10. As we are dealing with continuous functions, since $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we are interested in the strict positivity *everywhere in* Ω and not only a.e. in Ω. Theorem 2.2.24 gives a positive answer for this question: in fact, as $\overline{u} \in H^2(\Omega) = W^{2,N}(\Omega)$ is strictly superharmonic, by the strong maximum principle for strong solutions [44, Theorem 9.6], we deduce that it cannot achieve its minimum on the interior of Ω, thus $\overline{u}(x) > 0$ *for every* $x \in \Omega$. By the C^1 convergence we deduce the same strict inequality for u_σ, with $\sigma \in (1, \sigma_1)$.

Second method for positivity: a dual cone decomposition

The positivity result of Theorem 2.2.24 leaves us a bit unsatisfied, for two main reasons. On one hand, we have no estimates which quantify "how large" this right neighborhood of $\sigma = 1$ is and, moreover, if we think of the Dirichlet problem (2.35), seen as the limit case as $\sigma \to +\infty$, then there exists at least one case in which we know that its ground states are positive, namely when Ω is a ball in $\mathbb{R}^2$ (see [37] for the case $g \equiv 1$, but the arguments therein hold also in our general situation). Thus, we expect to be able to completely extend the positivity for such domains. We point out that proving positivity of ground states of (2.35) is quite a hard subject, since it strongly relies on the geometry of the domain, even in the linear case, namely $f(x, u) = f(x)$, as recalled in the Introduction.
Roughly speaking, this technique allows to split any function u which belongs to an Hilbert space, into a sum of a positive term and (in the best case) a negative one, both belonging to the same Hilbert space. In practice, this replaces the decomposition $u = u^+ - u^-$ which is not available in the context of high-order Sobolev spaces. This method will allows to obtain from a supposed sign-changing ground state solution u, a function w of one sign and in the same space with a strictly lower energy level, leading to a contradiction.

At the end of this subsection, one may also compare the resulting analysis with the respective one for the linear problem with the same boundary conditions, due to Gazzola and Sweers in [42].

Definition 2.2.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$ and fix $\sigma \in \mathbb{R}$. The linear Steklov boundary problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.39}$$

is *positivity preserving in* Ω if there exists a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \geq 0$ implies $u \geq 0$, and this holds for each $f \in L^2(\Omega)$. We shorten this by saying "Ω is a [PPP$_\sigma$] domain for (2.39)".

Our aim is to apply the dual cone decomposition (see Theorem A.0.2 in the Appendix) to the Hilbert space $(H^2(\Omega) \cap H_0^1(\Omega); \|\cdot\|_{H_\sigma})$, where $\|\cdot\|_{H_\sigma}$ is the norm (2.4), and $K := \{v \in H \,|\, v \geq 0\}$, the cone of nonnegative functions, looking for a decomposition of each element into a positive and a negative part. Hence, we need a characterization of the dual cone K^*:

Lemma 2.2.25. *If Ω is a [PPP$_\sigma$] domain for* (2.39) *for a fixed $\sigma \in \mathbb{R}$, then $K^* \subseteq \{w \in H \,|\, w < 0 \text{ a.e.}\} \cup \{0\}$.*

Proof. We adapt here the proof of [40, Proposition 3.6]. Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ and let $v_\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique weak solution of the linear problem

$$\begin{cases} \Delta^2 v_\varphi = \varphi & \text{in } \Omega \\ v_\varphi = \Delta v_\varphi - (1-\sigma)\kappa (v_\varphi)_n = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, for every test function $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$(v_\varphi, w)_{H_\sigma} := \int_\Omega \Delta v_\varphi \Delta w - (1-\sigma) \int_{\partial\Omega} \kappa (v_\varphi)_n w_n = \int_\Omega \varphi w.$$

Hence, suppose $w = u \in K^*$: as Ω is a [PPP$_\sigma$] domain and $\varphi \geq 0$, we deduce that $v_\varphi \geq 0$, so $v_\varphi \in K$ and thus $(v_\varphi, u)_{H_\sigma} \leq 0$. As a result, we obtain that $\int_\Omega \varphi u \leq 0$ holds for every $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, implying that $u \leq 0$ a.e. in Ω.
Moreover, let us suppose that the null-set of u, namely $N := \{x \in \Omega \,|\, u(x) = 0\}$, has positive measure, consider $\psi := \chi_N \neq 0$ and let v_0 be the unique solution of the linear Navier problem:

$$\begin{cases} \Delta^2 v_0 = \psi & \text{in } \Omega, \\ v_0 = \Delta v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.40}$$

Then v_0 is strictly superharmonic by the maximum principle, thus $v_0 > 0$ and, by the Hopf Lemma, $(v_0)_n < 0$. As a result, for any function $v \in H^2(\Omega) \cap H_0^1(\Omega)$ one can produce two positive constants α, β such that $v + \alpha v_0 \geq 0$ and $v - \beta v_0 \leq 0$. Moreover we claim that $(u, v_0)_{H_\sigma} \geq 0$. In fact, as v_0 is the weak solution of (2.40) and by definition of ψ:

$$\int_\Omega \Delta u \Delta v_0 = \int_\Omega u\psi = \int_N u = 0.$$

Thus, since $\sigma > 1$, $\kappa \geq 0$, $u_n \leq 0$ as $u \geq 0$, and $(v_0)_n < 0$:

$$(u, v_0)_{H_\sigma} := \int_\Omega \Delta u \Delta v_0 - (1-\sigma) \int_{\partial\Omega} \kappa u_n (v_0)_n \geq 0.$$

As a result, recalling that $u \in K^*$, $v + \alpha v_0 \in K$ and $v - \beta v_0 \in (-K)$, we have the chain of inequalities:

$$\begin{aligned} 0 &\geq (u, v + \alpha v_0)_{H_\sigma} = (u, v)_{H_\sigma} + \alpha (u, v_0)_{H_\sigma} \geq (u, v)_{H_\sigma} \\ &\geq (u, v)_{H_\sigma} - \beta (u, v_0)_{H_\sigma} = (u, v - \beta v_0)_{H_\sigma} \geq 0, \end{aligned}$$

which implies $(u, v)_{H_\sigma} = 0$ and holds for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence this is true also for v defined as the unique solution of the Steklov problem:

$$\begin{cases} \Delta^2 v = u & \text{in } \Omega, \\ v = \Delta v - (1-\sigma)\kappa v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

and, using u as a test function, we deduce that

$$0 = (u, v)_{H_\sigma} = \int_\Omega u^2 = \|u\|_2^2,$$

which implies $u = 0$ a.e. □

Proposition 2.2.26. *Let $\sigma > 1$ and suppose Ω is a [PPP$_\sigma$] domain for* (2.39). *Then the ground states of J_σ are either positive a.e. or negative a.e.*

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a ground state and suppose by contradiction that u is sign-changing. Denoting as before the cone of nonnegative functions by K, by Theorem A.0.2 and Lemma 2.2.25 there exists a unique couple $(u_1, u_2) \in K \times K^*$ such that $u = u_1 + u_2$, $u_1 \geq 0$ and $u_2 < 0$, and such that $(u_1, u_2)_{H_\sigma} = 0$. Moreover, u is supposed to change sign, so $u_1 \neq 0$. Defining $w := u_1 - u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have $w > |u|$. Indeed,

$$\begin{aligned} w &= u_1 - u_2 > u_1 + u_2 = u, \\ w &= u_1 - u_2 > -u_1 - u_2 = -u. \end{aligned}$$

Consequently, $\int_\Omega g(x)|w|^{p+1} > \int_\Omega g(x)|u|^{p+1}$ and, since the decomposition is orthogonal under that norm, $\|w\|_{H_\sigma}^2 = \|u_1\|_{H_\sigma}^2 + \|u_2\|_{H_\sigma}^2 = \|u\|_{H_\sigma}^2$. Moreover, by Lemma 2.2.3, there exists $t^* := t^*(w) \in (0, +\infty)$ such that $w^* := t^*(w)w \in \mathcal{N}_\sigma$. Therefore,

$$\begin{aligned} J_\sigma(w^*) &= \frac{(t^*)^2}{2}\|w\|_{H_\sigma}^2 - \frac{(t^*)^{p+1}}{p+1}\int_\Omega g(x)|w|^{p+1} \\ &< \frac{(t^*)^2}{2}\|u\|_{H_\sigma}^2 - \frac{(t^*)^{p+1}}{p+1}\int_\Omega g(x)|u|^{p+1} = J_\sigma(t^*(w)u) \leq J_\sigma(u), \end{aligned}$$

since u is the maximum of J_σ on the half-line $\{tu \,|\, t \in (0, +\infty)\}$ by Lemma 2.2.3. This is again a contradiction, since u was the infimum of J_σ on the Nehari manifold $\mathcal{N}_\sigma$. Hence, there must hold $u \geq 0$.
Finally, as u is a critical point of J_σ, we have for each a positive test function $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$:

$$(u, \varphi)_{H_\sigma} = \int_\Omega \Delta u \Delta \varphi - (1 - \sigma)\int_{\partial\Omega} \kappa u_n \varphi_n = \int_\Omega g(x) u^p \varphi \geq 0,$$

which implies $-u \in K^*$. Applying Lemma 2.2.25, we get $-u < 0$, that is, $u > 0$. $\square$

As a consequence, the problem of proving positivity of ground state is led back to a problem of positivity preserving for the linear problem, which was already investigated by Gazzola and Sweers.
In the sequel, $f \gneq 0$ means $f(x) \geq 0$ for all x and $f \neq 0$.

Lemma 2.2.27 ([42], Theorem 4.1)**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with $\partial\Omega$ of class C^2 and let $0 \lneq \beta \in C(\partial\Omega)$. Then there exist $\delta_{1,\beta} = \delta_{1,\beta}(\Omega) \in (0, +\infty)$ and $\delta_{c,\beta} = \delta_{c,\beta}(\Omega) \in [-\infty, 0)$ such that, if $\alpha \in C(\partial\Omega)$ and $\delta_{c,\beta}\beta < \alpha \lneq \delta_{1,\beta}\beta$ and we consider the following linear problem*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - \alpha u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.41}$$

then $0 \lneq f \in L^2(\Omega)$ implies $u > 0$ in Ω.

Theorem 2.2.28. *Let $\sigma > 1$ and $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class C^2. There exists $\tilde{\delta}_c(\Omega) \in (1, +\infty]$ such that if $\sigma \in (1, \tilde{\delta}_c(\Omega))$, the ground states of the functional J_σ are either positive a.e. or negative a.e.*

Proof. Choosing $\beta = \kappa$ in Lemma 2.2.27, we infer the existence of $\delta_{c,\kappa}(\Omega) \in [-\infty, 0)$ such that if $(1-\sigma)\kappa \geq \delta_{c,\kappa}(\Omega)\kappa$, then the positivity preserving for problem (2.39) holds in Ω. Hence, defining $\tilde{\delta}_c(\Omega) := 1 + |\delta_{c,\kappa}(\Omega)|$, we can apply Proposition 2.2.26, provided $\sigma < \tilde{\delta}_c(\Omega)$. □

Remark 11. Again, up to some easy modifications in the proofs, both the convergence in Theorem 2.2.22 and the positivity result in Theorem 2.2.28 hold also in the sublinear case $p \in (0, 1)$.

Comparing Theorems 2.2.28 and 2.2.24, one may argue that we have nothing more than what we already knew: in both we obtain the existence of $\sigma_1 = \sigma_1(\Omega) > 1$ such that for all $\sigma \in (1, \sigma_1)$ the ground state solutions of problem (2.8) are positive. Nevertheless, in Theorem 2.2.28 we get further precise information about how the interval of positivity depends on the domain, relating it strongly with the positivity preserving property. This fact is striking in the case of the disc and allows us to finally answer the question which we asked at the beginning of the present analysis.

Corollary 2.2.29. *Let $\mathcal{B} \subset \mathbb{R}^2$ be a disc and let $\sigma > 1$. Then the ground states of the functional J_σ are either positive a.e. or negative a.e.*

Proof. It is enough to notice that here $\kappa = 1$ and applying [42, Corollary 2.9] one can deduce $\delta_{c,\kappa}(\mathcal{B}) = -\infty$, which implies $\tilde{\delta}_c(\mathcal{B}) = +\infty$. □

The ball is not the unique case in which we have $\tilde{\delta}_c(\Omega) = +\infty$. In the sequel, we denote by $G_\Omega : \overline{\Omega} \times \overline{\Omega} \setminus \{(x, x) : x \in \overline{\Omega}\} \to \mathbb{R}$ the Green function of the Dirichlet problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.42}$$

Lemma 2.2.30 ([42], Theorem 2.6). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with $\partial\Omega \in C^{4,\gamma}$ and let $0 \lneq \beta \in C(\partial\Omega)$. If*

$$G_\Omega(x, y) \geq c d_\Omega(x)^2 d_\Omega(y)^2, \tag{2.43}$$

then, for all $\alpha \in C(\partial\Omega)$ with $\alpha \lneq \delta_{1,\beta}\beta$ and $0 \lneq f \in C(\overline{\Omega})$, the weak solution of (2.41) satisfies $u > 0$ in Ω.

Therefore, any sufficiently smooth bounded domain such that (2.43) holds is a domain for which $\tilde{\delta}_c(\Omega) = +\infty$. Indeed, the fact we have a smaller class of $f \geq 0$ such that (2.41) has a positive solution has no consequences on the proof of positivity for the semilinear problem, since we only need it for $f \in C_c^\infty(\Omega)$. Nonetheless, the estimate from below (2.43) is a stronger condition than the positivity preserving property itself, thus it is far from being easy to infer. The unique examples available in the literature are provided by Dall'Acqua and Sweers and concern a special class of domains, referred to as *limaçons*, and their smooth deformations.

Definition 2.2.3. Let $a \in [0, \frac{1}{2}]$. The *limaçon* of parameter a is defined as the set:

$$\Omega_a := \{(\rho\cos(\varphi), \rho\sin(\varphi)) \in \mathbb{R}^2 \,|\, 0 \leq \rho < 1 + 2a\cos(\varphi)\}.$$

For $0 \leq a \leq \frac{1}{2}$, the curve $\rho = 1 + 2a\cos(\varphi)$ is non self-intersecting. Special values of the parameter a are the following:

- $a = 0$: Ω_0 is the unit disc;
- $a = \frac{1}{4}$: Ω_a is convex if and only if $a \in [0, \frac{1}{4}]$;
- $a = \frac{1}{2}$: $\Omega_{\frac{1}{2}}$ is the cardioid.

Definition 2.2.4. Let $\varepsilon > 0$, $\gamma \in (0,1)$, $k \in \mathbb{N}$ and Ω, Ω' be two domains in $\mathbb{R}^N$. We say that Ω is *ε-close to Ω' in $C^{k,\gamma}$-sense* if there exists a $C^{k,\gamma}$ mapping $g : \overline{\Omega'} \to \overline{\Omega}$ such that $g(\overline{\Omega'}) = \overline{\Omega}$ and $\|g - Id\|_{C^{k,\gamma}(\overline{\Omega'})} \leq \varepsilon$.

The following result has been established by Dall'Acqua in [23, Theorem 5.3.2] (see also [25], Theorem 3.1.3 and Remark 3.4.3).

Lemma 2.2.31. *Let $\bar{a} \in (\frac{1}{4}, \frac{\sqrt{6}}{6})$ and $\gamma \in (0,1)$. Then there exist $\varepsilon_0 > 0$ and $c_1, c_2 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and $a \in [0, \bar{a}]$ the following holds: if Ω is ε-close in $C^{2,\gamma}$-sense to the limaçon Ω_a, then the Green function G_Ω of* (2.42) *satisfies*

$$0 < c_1 D_\Omega(x,y) \leq G_\Omega(x,y) \leq c_2 D_\Omega(x,y) \quad \textit{for every } x, y \in \Omega, \tag{2.44}$$

where

$$D_\Omega(x,y) := d_\Omega(x) d_\Omega(y) \min\left\{1, \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right\}.$$

Thanks to Lemma 2.2.31, we can improve Corollary 2.2.29, enlarging the class of domains for which a "full positivity" result applies for the ground states of J_σ:

Proposition 2.2.32. *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain of class $C^{4,\alpha}$ which is ε-close in $C^{2,\gamma}$-sense to a limaçon as in Lemma 2.2.31. Then the ground states of J_σ are positive, provided $\sigma > \sigma^*(\Omega)$.*

Proof. The result is proved by applying Lemmas 2.2.31, 2.2.30, 2.2.27 and Proposition 2.2.26, noticing that the estimate from below (2.43) easily follows from (2.44). □

One should finally notice that here the positivity found by the dual cones method is up to a subset of the domain with zero Lebesgue measure, so *almost everywhere* in Ω. This is the price we have to pay to extend the positivity beyond the parameter σ_1 found in Theorem 2.2.24 (cf. Remark 10).

2.2.7 Radial case

This section is devoted to some further investigation when the domain is a disc in $\mathbb{R}^2$ and the function g is radial. In particular, we want to analyse existence, positivity and some qualitative properties of radially symmetric solutions. Moreover, we establish the counterpart of the convergence results of Subsections 2.2.3-2.2.5

for general radial positive solutions.

For simplicity, we focus on the problem

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \mathcal{B}, \\ u = \Delta u - (1-\sigma)u_n = 0 & \text{on } \partial\mathcal{B}, \end{cases} \tag{2.45}$$

where $\mathcal{B} := B_1(0) \subset \mathbb{R}^2$, $g = g(|x|)$ lies in $L^1(\mathcal{B})$ and it is strictly positive inside $\mathcal{B}$. Moreover, we let the parameter $\sigma \in \mathbb{R}$ and $p \in (0,1) \cup (1,+\infty)$ to cover both the sublinear and the superlinear case. Notice that the curvature does not appear in the mixed boundary condition since $\kappa(\mathcal{B}) \equiv 1$.

Positive radially decreasing solutions and uniform bounds

First of all, by Proposition 2.2.10, our analysis concerns only the range $\sigma > -1$: in fact, if $\Omega = \mathcal{B}$, one has $\sigma^* = -1$, since the first Steklov eigenvalue $\tilde{\delta}_1(\mathcal{B}) = 2$ (see [8, Proposition 12]). Retracing exactly the same steps of Subsection 2.2, it is quite easy to obtain the existence of a positive radial solution. In fact, confining ourselves to the closed subspace of radial functions

$$H_{rad}(\mathcal{B}) := \{u \in H^2(\mathcal{B}) \cap H_0^1(\mathcal{B}) \,|\, u(x) = u(|x|),\ \forall x \in \mathcal{B}\} = Fix_{O(2)}(H^2(\mathcal{B}) \cap H_0^1(\mathcal{B})),$$

we deduce the existence of a critical point of J_σ restricted to $H_{rad}(\mathcal{B})$. Then it is enough to notice that J_σ is invariant under the action of $O(2)$ and to apply the Principle of Symmetric Criticality due to Palais (see Theorem A.0.3 in the Appendix), retrieving that these points are critical for J_σ also with respect to the whole space.
Finally, if we restrict to the interval $(-1,1]$, the positivity of such critical points is proved as in Propositions 2.2.2 and 2.2.7, realizing that the superharmonic function of a radially symmetric function is radial too (see (2.7)). On the other hand, if $\sigma > 1$, one can apply the dual cone decomposition to the Hilbert space $H_{rad}(\mathcal{B})$ and argue as in Lemma 2.2.25 and Proposition 2.2.26, taking into account that $\mathcal{B}$ is a [PPP$_\sigma$] domain for every $\sigma > -1$. Summarizing, we have shown the following:

Proposition 2.2.33. *Let $p \in (0,1) \cup (1,+\infty)$, $g = g(|x|) \in L^1(\mathcal{B})$, $g > 0$. If $\sigma \le -1$, there is no positive nonnegative nontrivial solution for (2.45), while, if $\sigma > -1$, there exists at least a positive radial solution, which is strictly superharmonic whenever $\sigma \in (-1,1]$.*

Let us now focus on qualitative properties of radial positive solutions of (2.45). The first result concerns the radial behaviour, while the second the uniform boundedness in $L^\infty(\mathcal{B})$. Before proving these results, one should notice that such solutions are strong, namely in $W^{4,q}(\mathcal{B})$, provided $g \in L^q(\mathcal{B})$ for some $q > 2$ and also classical assuming in addition that $g \in W^{1,q}(\mathcal{B})$ for some $q > 2$. This is a straightforward application of Lemma 2.2.19 combined with Morrey's embeddings.

Lemma 2.2.34. *Let $B_R(0)$ be the ball of radius R in $\mathbb{R}^2$ centered in 0, $q > 2$ and $\tilde{h} \in W^{2,q}(B_R(0))$ be radial. Defining $h : [0, R] \to \mathbb{R}$ to be its restriction to the radial variable, for all $t \in [0, R]$ the following equality holds:*

$$th'(t) = \int_0^t s\Delta h(s)ds.$$

Proof. If h is of class C^2, it comes directly from integration by parts and from the radial representation of the laplacian as

$$\Delta\tilde{h}(x) = h''(|x|) + \frac{1}{|x|}h'(|x|).$$

Otherwise, let $(\tilde{f}_k)_{k\in\mathbb{N}} \subset C^\infty(\bar{B}_R(0))$ be such that $\tilde{f}_k \to \tilde{h}$ in $W^{2,q}(B_R(0))$, so in $C^1(\bar{B}_R(0))$. Since $\tilde{h}$ is radial, we claim that it is possible to choose each $\tilde{f}_k$ to be radial and we denote its restriction to the radial variable as f_k. If so, for every $k \in \mathbb{N}$ we have:

$$tf_k'(t) = \int_0^t s\Delta f_k(s)ds.$$

As a result, as $k \to +\infty$:

$$\left|\int_0^t s(\Delta f_k(s) - \Delta h(s))ds\right| = \frac{1}{2\pi}\|\Delta\tilde{f}_k - \Delta\tilde{h}\|_{L^1(B_t(0))} \leq C(q)\|\tilde{f}_k - \tilde{h}\|_{W^{2,q}(B_R(0))} \to 0.$$

The result is proved by the convergence in $C^1(\bar{B}_R(0))$ and the uniqueness of the limit. Now we have to justify our previous claim. Since $\tilde{h} \in W^{2,q}(B_R(0))$, we have

$$\sum_{i,\alpha}\int_{B_R(0)}\left|\frac{\partial^\alpha\tilde{h}}{\partial i^\alpha}(x,y)\right|^q dxdy < +\infty,$$

where $i \in \{x, y\}$ and α is a multi-index of length $0 \leq |\alpha| \leq 2$. Since each $\frac{\partial^\alpha\tilde{h}}{\partial i^\alpha}$ is radial, this is equivalent to saying that $h \in W^{2,q}([0, R], r)$, that is the weighted Sobolev space with weight r. Hence, by [54, Theorem 7.4] ($M = \{0\}$, $\varepsilon = 1$ in notation therein), there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset C^\infty([0, R])$ such that $f_k \to h$ in $W^{2,q}([0, R], r)$, that is

$$\sum_{i,\alpha}\int_0^R r\left|\frac{\partial^\alpha h}{\partial i^\alpha}(r) - f_k(r)\right|^q dr \to 0.$$

Hence, defining $F_k(x) := f_k(|x|)$, each $F_k \in C^\infty(\bar{B}_R(0))$, is radial and

$$\begin{aligned}\|\tilde{h} - F_k\|_{W^{2,q}(B_R(0))} &= \sum_{i,\alpha}\int_{B_R(0)}\left|\frac{\partial^\alpha\tilde{h}}{\partial i^\alpha}(x,y) - F_k(x,y)\right|^q dxdy \\ &= 2\pi\sum_{i,\alpha}\int_0^R r\left|\frac{\partial^\alpha h}{\partial i^\alpha}(r) - f_k(r)\right|^q dr \to 0,\end{aligned}$$

and the claim is proved. □

Proposition 2.2.35 (Radial Decay). *Assume $g \in L^q(\mathcal{B})$ for some $q > 2$, g is radial and $g > 0$. and let $u \not\equiv 0$ be a nonnegative radial solution of* (2.45) *with $\sigma \in (-1, 1]$ and $p \in (0, 1) \cup (1, +\infty)$. Then u is strictly radially decreasing, thus $u > 0$ in $\mathcal{B}$.*

Proof. By the assumption on g, we infer that u is a strong solution, thus $w := \Delta u \in W^{2,q}(\mathcal{B})$. Since $\Delta w = \Delta^2 u = g(|x|)u^p \geq 0$ in $[0, 1]$, applying Lemma 2.2.34, we have $w' > 0$ in $(0, 1]$. Hence Δu is strictly increasing in $(0, 1]$. Moreover, since u is nonnegative and $u(1) = 0$, we have $u'(1) \leq 0$; hence, using the second boundary condition, $\Delta u(1) = (1 - \sigma)u'(1) \leq 0$. Since Δu is strictly increasing in $(0, 1]$, we deduce that $\Delta u < 0$ in $[0, 1)$, and finally, applying again Lemma 2.2.34, $u' < 0$ in $(0, 1]$. □

In the next result we find a uniform upper bound for positive radial solutions of (2.45), which may be seen as a superlinear counterpart of Proposition 2.2.8. We will make use of a blow up method which goes back to the work [43] of Gidas and Spruck, and which was adapted to the polyharmonic case by Reichel and Weth in [77, 78]. Briefly, our argument is the following: supposing the existence of a sequence of positive radial solutions with diverging L^∞ norm, we rescale each of them in order to have another sequence of functions with the same L^∞ norm, satisfying the same equation in nested domains which tend to occupy the whole $\mathbb{R}^2$. Then we show that, up to a subsequence, it converges uniformly on compact subsets to a continuous nonnegative but nontrivial function. This turns out to be a solution of the same equation on $\mathbb{R}^2$ and to contradict a Liouville's-type result by Wei and Xu (see Theorem A.0.5 in the Appendix, with $N = 2$ and $m = 2$).

Proposition 2.2.36. *Let $\sigma \in (-1, 1]$ and $g \in L^q(\mathcal{B})$ for some $q > 2$, radial and $g > 0$. Suppose also that g is continuous in 0. Then, there exists $C > 0$ independent of σ such that $\|u\|_\infty \leq C$ for every u radial positive solution of* (2.45).

Proof. By contradiction, suppose there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of radial positive solutions such that $\|v_k\|_\infty \nearrow +\infty$. According to Proposition 2.2.35, each v_k is radially decreasing, so $v_k(0) = \|v_k\|_\infty \nearrow +\infty$. For each $k \geq 1$, define

$$u_k(x) = \lambda_k^{\frac{4}{p-1}} v_k(\lambda_k x),$$

where $\lambda_k \in \mathbb{R}^+$ are such that $\lambda_k^{\frac{4}{p-1}} = 1/v_k(0)$. With this choice, each u_k satisfies

$$\begin{cases} \Delta^2 u_k = g(|\lambda_k x|)u_k^p & \text{in } B_{\frac{1}{\lambda_k}}(0), \\ u_k = \Delta u_k - (1-\sigma)\lambda_k (u_k)_n = 0 & \text{on } \partial B_{\frac{1}{\lambda_k}}(0), \end{cases}$$

is in $W^{4,q}\big(B_{\frac{1}{\lambda_k}}(0)\big)$, radially decreasing and

$$\|u_k\|_{L^\infty(B_{\frac{1}{\lambda_k}}(0))} = u_k(0) = \lambda_k^{\frac{4}{p-1}} v_k(0) = 1. \tag{2.46}$$

We claim that the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded on compact sets of $\mathbb{R}^2$ in $W^{4,q}$ norm. In fact, let $K \subset \mathbb{R}^2$ be compact, then there exists $\rho > 0$ such that

$B_\rho(0) \supset K$ and, for k large enough, each u_k is well defined in K since $B_{\frac{1}{\lambda_k}}(0) \supset B_{2\rho}(0)$ definitively. For such k, by (2.46) and applying the local estimate of Lemma A.0.4 in the Appendix with $\Omega = B_{2\rho}(0)$, $m = N = 2$ and $\delta = 1/2$, we get

$$\begin{aligned} \|u_k\|_{W^{4,q}(K)} \leq \|u_k\|_{W^{4,q}(B_\rho(0))} &\leq \tfrac{C(\rho,q)}{(1/2^4)}(\|\Delta^2 u_k\|_{L^q(B_{2\rho}(0))} + \|u_k\|_{L^q(B_{2\rho}(0))}) \\ &\leq 16C(\rho,q)(\|g(|\lambda_k \cdot|)\|_{L^q(B_{2\rho}(0))} + |B_{2\rho}(0)|^{\frac{1}{q}}). \end{aligned} \tag{2.47}$$

Moreover, fixing $\varepsilon > 0$ and supposing k large enough,

$$\|g(|\lambda_k \cdot|)\|_{L^q(B_{2\rho}(0))} = (4\pi\rho^2)^{\frac{1}{q}} \left(\frac{1}{|B_{2\rho\lambda_k}(0)|} \int_{B_{2\rho\lambda_k}(0)} |g(y)|^q dy \right)^{\frac{1}{q}} \leq (4\pi\rho^2)^{\frac{1}{q}} g(0) + \varepsilon \tag{2.48}$$

where the last inequality follows from the Lebesgue differentiation theorem. Hence, combining (2.47) with (2.48), we infer $\|u_k\|_{W^{4,q}(K)} \leq C(p,q,K,g)$, so uniformly in k. Incidentally, notice that this constant does not depend on σ. Hence we find $u \in W^{4,q}(K)$ such that, up to subsequences, $u_k \to u$ in $C^3(K)$, where $u \in C^3(\mathbb{R}^2)$, $u \geq 0$ and $u(0) = 1$ by (2.46) and satisfying

$$\Delta^2 u = g(0)u^p \quad \text{in } \mathbb{R}^2.$$

so, by a bootstrap method, we deduce that u is also a classical solution. Finally, setting for all $x \in \mathbb{R}^2$ $w(x) := u(bx)$ with $b := g(0)^{-1/4}$, one has w is a nonnegative solution of

$$\Delta^2 w = w^p \quad \text{in } \mathbb{R}^2,$$

with $w(0) = u(0) = 1$, which contradicts Theorem A.0.5. $\square$

Convergence results

We want to investigate what happens to radial solutions at the endpoints of the interval $(-1, 1]$ in which σ lies. More precisely, our aim is to examine if any result similar to Theorems 2.2.13 and 2.2.18 can be found assuming $(u_k)_{k\in\mathbb{N}}$ to be a sequence of positive radial solutions of (2.45) with $\sigma = \sigma_k$ but without imposing any "minimizing" requirement. Unless otherwise stated, we assume $g \equiv 1$ and $p > 1$.

Let us start with the behaviour for $\sigma_k \to 1$, where the main ideas are taken from the same result for ground states. Notice that we know everything for the Navier problem in the ball: in fact, Dalmasso proved in [26] that there exists a unique positive solution, which is radially symmetric and radially decreasing thanks to a result by Troy, [95].

Proposition 2.2.37. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of positive radial solutions of (2.45) with $\sigma_k \nearrow 1$. Then $u_k \to \overline{u}$ in $H^2(\mathcal{B})$, where $\overline{u}$ is the unique positive solution of the Navier problem.*

Proof. We firstly claim that such a sequence is bounded in $H^2(\mathcal{B})$. Indeed, by Proposition 2.2.36:

$$\|u_k\|^2_{H^2(\mathcal{B})} \leq C_0\|\Delta u_k\|_2^2 \leq C_0\left(1 - \frac{1-\sigma_k}{2}\right)^{-1}\|u_k\|^2_{H_{\sigma_k}} = \frac{2C_0}{1+\sigma_k}\|u_k\|^{p+1}_{p+1} \leq 2\pi C_0 C^{p+1}.$$

Hence, we can extract a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that there exists $\overline{v} \in H^2(\mathcal{B}) \cap H_0^1(\mathcal{B})$ such that $u_{k_j} \rightharpoonup \overline{v}$ weakly in $H^2(\mathcal{B})$. By Proposition 2.2.15, together with Remark 8, one can infer that this subsequence is actually strongly convergent in $H^2(\mathcal{B})$ and then that $\overline{v}$ is a weak solution of the Navier problem (thus classical by regularity theory). Moreover, since the convergence is pointwise, we immediately deduce that $\overline{v}$ is nonnegative, radially symmetric and radially non-increasing. Furthermore, by Proposition 2.2.35, $\overline{v}$ is actually strictly decreasing and positive in $\mathcal{B}$, so it coincides with the unique positive solution $\overline{u}$ of the Navier problem. By the uniqueness of the limit and applying Urysohn subsequence principle, we retrieve the convergence of the whole sequence $(u_k)_{k\in\mathbb{N}}$ from which we started. □

Let us now investigate the case $\sigma_k \searrow -1$. As already noticed in Lemma 2.2.14, it is enough to understand the behaviour of the $L^{p+1}(\mathcal{B})$ norm of a sequence of solutions to infer the convergence in the $H^2(\mathcal{B})$ norm. Since the proof of Theorem 2.2.13 strongly relies on the fact that it deals with ground states, we need a different technique. The first step is a Pohožaev-type identity by Mitidieri in [68]: it will allow to prove an inequality involving $L^p(\mathcal{B})$ and $L^{p+1}(\mathcal{B})$ norms which, combined with the uniform bound of Proposition 2.2.36, will lead us to the convergence result.

Lemma 2.2.38 ([68], Proposition 2.2)**.** *Let Ω be a smooth domain and $u \in C^4(\overline{\Omega})$. The following identity holds:*

$$\int_\Omega (\Delta^2 u) x\cdot\nabla u - \frac{N}{2}\int_\Omega (\Delta u)^2 - (N-2)\int_\Omega \nabla\Delta u\cdot\nabla u$$
$$= -\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2 x\cdot n + \int_{\partial\Omega}\Big((\Delta u)_n(x\cdot\nabla u) + u_n(x\cdot\nabla\Delta u) - \nabla\Delta u\cdot\nabla u(x\cdot n)\Big).$$

Corollary 2.2.39. *Suppose u is a positive solution for problem* (2.45) *with $g\equiv 1$, then the following identity holds:*

$$\int_{\partial B_R}\left((\Delta u)_n + (1-\sigma)\left(1-\frac{1-\sigma}{2}\right)u_n\right)u_n = -\left(1+\frac{2}{p+1}\right)\int_{B_R} u^{p+1}. \tag{2.49}$$

Proof. By similar computations as in [9, Section 6], from Lemma 2.2.38 one infers:

$$\left(\frac{N-4}{2}-\frac{N}{p+1}\right)\int_\Omega u^{p+1} = \int_{\partial\Omega}\left(x\cdot\nabla\Delta u+\frac{N}{2}(1-\sigma)\kappa u_n-\frac{1}{2}(1-\sigma)^2\kappa^2 u_n(x\cdot n)\right)u_n.$$

If $N=2$ and $\Omega=\mathcal{B}$, we have $x=n$ and $\kappa=1$, so $x\cdot\nabla\Delta u=(\Delta u)_n$ and (2.49) follows. □

The next result follows from some ideas of Berchio and Gazzola contained in [7, Proposition 4].

Lemma 2.2.40. *Let $\sigma\in(-1,1)$ and u be a positive radial solution of problem* (2.45) *with $g\equiv 1$. Then the following estimate holds:*

$$\|u\|_{p+1}^{p+1} \geq \frac{3}{64}\left(1-\frac{3}{64}(1-\sigma)\right)\frac{1}{\pi(1+\sigma)}\frac{p+1}{p+3}\|\Delta^2 u\|_1^2. \tag{2.50}$$

Proof. By radial symmetry, (2.49) reduces to

$$2(\Delta u)'(1)u'(1) + (1-\sigma)(1+\sigma)(u'(1))^2 = -\frac{p+3}{p+1}\frac{1}{\pi}\int_{\mathcal{B}} u^{p+1}, \tag{2.51}$$

and by the divergence theorem we have

$$u'(1) = \frac{1}{2\pi}\int_{\mathcal{B}} \Delta u \quad \text{and} \quad (\Delta u)'(1) = \frac{1}{2\pi}\int_{\mathcal{B}} \Delta^2 u. \tag{2.52}$$

Moreover, let w be the first Steklov eigenfunction $w(x) = \frac{1}{4}(1-|x|^2)$, which verifies $\Delta w = -1$ in B, $w_{n|\partial B} = w'(1) = -1/2$, $w_{|\partial B} = w(1) = 0$. Then integrating by parts, using the properties of w and the boundary conditions:

$$\begin{aligned} -\int_{\mathcal{B}} \Delta u &= \int_{\mathcal{B}} \Delta w \Delta u = \int_{\mathcal{B}} w\Delta^2 u + \int_{\partial\mathcal{B}} w_n \Delta u = \int_{\mathcal{B}} w\Delta^2 u - \frac{1}{2}\int_{\partial\mathcal{B}} \Delta u = \\ &= \int_{\mathcal{B}} w\Delta^2 u - \frac{1-\sigma}{2}\int_{\partial\mathcal{B}} u_n = \int_{\partial\mathcal{B}} w\Delta^2 u - \frac{1-\sigma}{2}\int_{\mathcal{B}} \Delta u, \end{aligned}$$

where the last equality comes from the divergence theorem. Therefore we get

$$\int_{\mathcal{B}} w\Delta^2 u = -\frac{1+\sigma}{2}\int_{\mathcal{B}} \Delta u. \tag{2.53}$$

and hence, by (2.51), (2.52) and (2.53),

$$\left(\int_{\mathcal{B}} \Delta^2 u - (1-\sigma)\int_{\mathcal{B}} w\Delta^2 u\right)\int_{\mathcal{B}} w\Delta^2 u = \frac{p+3}{p+1}(1+\sigma)\pi\int_{\mathcal{B}} u^{p+1}. \tag{2.54}$$

Noticing that $0 \le w \le 1/4$, we claim that

$$\frac{3}{64}\int_{\mathcal{B}} \Delta^2 u \le \int_{\mathcal{B}} w\Delta^2 u \le \frac{1}{4}\int_{\mathcal{B}} \Delta^2 u. \tag{2.55}$$

Indeed, w and u radially decreasing and so is $\Delta^2 u$, hence

$$\begin{aligned} \int_{\mathcal{B}} \Delta^2 u &= \int_{\mathcal{B}\setminus B_{\frac{1}{2}}(0)} \Delta^2 u + \int_{B_{\frac{1}{2}}(0)} \Delta^2 u = |\mathcal{B}\setminus B_{\frac{1}{2}}(0)|\Delta^2 u(\tfrac{1}{2}) \\ &\le \frac{1}{w(\frac{1}{2})}\left(1 + \frac{|\mathcal{B}\setminus B_{\frac{1}{2}}(0)|}{|B_{\frac{1}{2}}(0)|}\right)\int_{\mathcal{B}} w\Delta^2 u = \frac{64}{3}\int_{\mathcal{B}} w\Delta^2 u, \end{aligned}$$

which yields the first inequality in (2.55), while the second is straightforward.

Hence, defining now $d := (1-\sigma)$, $s := \int_{\mathcal{B}} w\Delta^2 u$ and $A := \int_{\mathcal{B}} \Delta^2 u$, the left-hand side of (2.54) becomes

$$As - ds^2, \quad \text{with } s \in [\tfrac{3}{64}A, \tfrac{1}{4}A].$$

Since $d > 0$, we know $\psi : s \mapsto As - ds^2$ is a concave function, so it attains its minimum on the extremal values of the interval: in this case, with $0 < d < 2$, one has

$$\psi(s) \ge \tfrac{3}{64}(1 - \tfrac{3}{64}d)A^2.$$

Combining this with (2.54), one finds the desired estimate (2.50). $\square$

Theorem 2.2.41. *Let $\sigma_k \searrow -1$ and $(u_k)_{k\in\mathbb{N}}$ be a sequence of positive radial functions, each of them solution of the problem* (2.45) *with $g \equiv 1$ and $\sigma = \sigma_k$. Then, $u_k \to 0$ in $H^2(\mathcal{B})$.*

Proof. By Lemma 2.2.14, it is enough to prove the convergence in $L^{p+1}(\mathcal{B})$ norm. Since every solution of (2.45) is smooth, we have $\|\Delta^2 u_k\|_1 = \|u_k\|_p^p$. Moreover, by the uniform L^∞ estimate found in Proposition 2.2.36, we know that there exists a constant $C > 0$ not depending on σ_k, such that

$$\|u_k\|_{p+1}^{p+1} \leq \|u_k\|_\infty^{p+1}|\mathcal{B}| \leq \pi C^{p+1}.$$

As a result, using the estimate provided by Lemma 2.2.40, one has

$$\frac{1+\sigma_k}{1-\frac{3}{64}(1-\sigma_k)} \geq \frac{p+1}{p+3}\frac{3}{64\pi^2 C^{p+1}}\|u_k\|_p^{2p},$$

so, letting $\sigma_k \to -1$ we deduce $\|u_k\|_p \to 0$. This, together with the $L^\infty(\mathcal{B})$ estimate of Proposition 2.2.36, gives us the convergence in $L^{p+1}(\mathcal{B})$ and so the desired result. □

2.2.8 Positivity in nonconvex domains

So far, the hypothesis of convexity for our domain was always indispensable in order to prove existence and positivity. Indeed, we needed it to establish that

$$\|u\|_{H_\sigma} = \left[\int_\Omega (\Delta u)^2 - (1-\sigma)\int_{\partial\Omega}\kappa u_n^2\right]^{\frac{1}{2}}$$

is an equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$ provided $\partial\Omega$ of class $C^{1,1}$, see Lemma 2.2.11. Then, we needed that the linear Steklov problem (2.39) was positivity preserving, and this was deduced by applying Lemma 2.2.27 with $\beta = \kappa$, which has to be a positive function on $\partial\Omega$.
On the other hand, we know that the Navier problem is positivity preserving for any domain, using a simple application of the second-order maximum principle and, moreover, we already saw in Lemma 2.2.31 examples of nonconvex domains for which the positivity preserving property holds true, regarding the linear Dirichlet problem (2.42), together with the useful Green function estimate (2.43). It is thus natural to conjecture that at least for those domains the ground states of J_σ are positive. This section answers affirmatively, extending successfully the method of the dual cones presented in Subsection 2.2.6.

Throughout this subsection, regarding our domains, we assume:

(H$_{PPP}$) $\Omega \subset \mathbb{R}^2$ is a bounded domain of class $C^{4,\alpha}$ which is ε-close in $C^{2,\gamma}$-sense to a limaçon Ω_a, with $a \in [0,\bar{a}]$ and $\varepsilon \in [0,\varepsilon_0]$, $\bar{a}$ and ε_0 being fixed by Lemma 2.2.31.

Moreover, we focus on the superlinear nonlinearity $f(x,u) = g(x)|u|^{p-1}u$ with $p > 1$, the case $p \in (0,1)$ being similar.

The case $\sigma < 1$

A first natural choice is to consider a different boundary function β, namely $\beta = |\kappa|$. In this case, in order to apply Theorem 2.2.27, we have to compare it with our boundary function $\alpha = (1-\sigma)\kappa$. By that result, we infer the existence of two parameters $\delta_{1,|\kappa|}(\Omega) > 0$ and $\delta_{c,|\kappa|}(\Omega) \in [-\infty, 0)$ such that if $\delta_{c,|\kappa|}|\kappa| < (1-\sigma)\kappa \lneq \delta_{1,|\kappa|}|\kappa|$, where $\alpha := (1-\sigma)\kappa$, one has the positivity preserving property for the linear Steklov problem (2.39). If we assume (H_{PPP}), then Lemmas 2.2.31 and 2.2.30 show that $\delta_{c,|\kappa|}|\kappa| = -\infty$. Hence, we only have to check

$$(1-\sigma)\kappa(x) \lneq \delta_{1,|\kappa|}|\kappa|(x) \quad \text{for every } x \in \partial\Omega :$$

- if x belongs to a convex part of the boundary, then $\kappa(x) \geq 0$ and so we retrieve the condition $\sigma > 1 - \delta_{1,|\kappa|}$;
- otherwise, if $\kappa(x) < 0$, we have $\kappa(x) = -|\kappa|(x)$ and so we find an upper bound: $\sigma < 1 + \delta_{1,|\kappa|}$, which is always satisfied if $\sigma < 1$.

We have just proved the following result:

Proposition 2.2.42. *Let $\Omega \subset \mathbb{R}^2$ satisfy condition (H_{PPP}) and $\sigma \in (1-\delta_{1,|\kappa|}, 1]$. Then the linear Steklov problem* (2.39) *is positivity preserving.*

Remark 12. It is clear that if Ω is also convex, we retrieve the lower bound $\sigma > \sigma^*$.

Theorem 2.2.43. *Let $\Omega \subset \mathbb{R}^2$ satisfy (H_{PPP}), $0 < g \in L^1(\Omega)$ and $\sigma \in (1-\delta_{1,|\kappa|}, 1]$. Then the ground states of J_σ are positive a.e. in Ω.*

Proof. The only fact which has to be proved is that for such domains and values of σ, the map $\|\cdot\|_{H_\sigma}$ is still an equivalent norm on $H^2(\Omega) \cap H^1_0(\Omega)$. On one hand,

$$\begin{aligned} \|u\|^2_{H_\sigma} &= \|\Delta u\|_2^2 + (1-\sigma)\int_{\partial\Omega}(-\kappa)u_n^2 \leq \|\Delta u\|_2^2 + (1-\sigma)\int_{\partial\Omega}|\kappa|u_n^2 \\ &\leq \|\Delta u\|_2^2 + (1-\sigma)\frac{\|\Delta u\|_2^2}{\delta_{1,|\kappa|}} = \left[1 + \frac{1-\sigma}{\delta_{1,|\kappa|}}\right]\|\Delta u\|_2^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|u\|^2_{H_\sigma} &= \|\Delta u\|_2^2 + (1-\sigma)\int_{\partial\Omega}(-\kappa)u_n^2 \geq \|\Delta u\|_2^2 - (1-\sigma)\int_{\partial\Omega}|\kappa|u_n^2 \\ &\geq \|\Delta u\|_2^2 - (\sigma-1)\frac{\|\Delta u\|_2^2}{\delta_{1,|\kappa|}} = \left[1 - \frac{1-\sigma}{\delta_{1,|\kappa|}}\right]\|\Delta u\|_2^2. \end{aligned}$$

At this point, we can repeat the proof of Theorem 2.2.28. Indeed, Ω is a [PPP_σ] domain by Proposition 2.2.42, so the dual cone of the positive cone is composed by negative functions and we may apply the dual cones decomposition as in Proposition 2.2.26 to obtain the positivity result. □

The case $\sigma > 1$

For these values of σ, the strategy applied so far produces an artificial upper bound, namely $\sigma < 1 + \delta_{1,|\kappa|}$. This is particularly unsatisfying since the Dirichlet problem is positivity preserving in domains for which the condition (H_{PPP}) holds and thus we expect to retrieve a result comparable with Theorem 2.2.28, that is, without any upper bound for σ. The main difficulty was due to the strained comparison between κ and $|\kappa|$ to infer the positivity preserving property for the linear Steklov problem. So, let us rewrite our functional as

$$\begin{aligned} J_\sigma(u) &= \left[\frac{1}{2}\int_\Omega (\Delta u)^2 - \frac{1-\sigma}{2}\int_{\partial\Omega} |\kappa| u_n^2\right] + \frac{1-\sigma}{2}\int_{\partial\Omega} [|\kappa| - \kappa] u_n^2 - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1} = \\ &= \frac{1}{2} N_\sigma(u)^2 + (1-\sigma)\int_{\partial\Omega} \kappa^- u_n^2 - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1}, \end{aligned}$$

where $\kappa^- := \max\{0, -\kappa\} = \frac{1}{2}(|\kappa| - \kappa)$ is the negative part of the curvature and the map N_σ is defined on $H^2(\Omega) \cap H_0^1(\Omega)$ as

$$N_\sigma(u) := \left[\|\Delta u\|_2^2 - (1-\sigma)\int_{\partial\Omega} |\kappa| u_n^2\right]^{\frac{1}{2}}$$

This reminds to our semilinear problem (2.8) with a second boundary condition

$$\Delta u = (1-\sigma)|\kappa| u_n.$$

Lemma 2.2.44. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class C^2 and assume $\sigma > 1$. Then the map N_σ is a norm on $H^2(\Omega) \cap H_0^1(\Omega)$, equivalent to the standard one. Moreover, set $K_{N_\sigma} := \{u \in H^2(\Omega) \cap H_0^1(\Omega) \,|\, u \geq 0\}$. If $\sigma \in (1, 1 + |\delta_{c,|\kappa|}|)$, then $K_{N_\sigma}^* \subseteq \{w \in H \,|\, w < 0 \text{ a.e.}\} \cup \{0\}$.*

Proof. Firstly, $N_\sigma(\cdot)$ is indeed a norm since

$$(u, v)_{N_\sigma} := \int_\Omega \Delta u \Delta v - (1-\sigma)\int_{\partial\Omega} |\kappa| u_n v_n$$

is a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, recalling the definition of $\delta_{1,|\kappa|}$, we have

$$\|\Delta u\|_2^2 \leq \|\Delta u\|_2^2 + (\sigma - 1)\int_{\partial\Omega} |\kappa| u_n^2 \leq \|\Delta u\|_2^2 + (\sigma-1)\frac{\|\Delta u\|_2^2}{\delta_{1,|\kappa|}} = \left[1 + \frac{\sigma-1}{\delta_{1,|\kappa|}}\right]\|\Delta u\|_2^2.$$

Let now $\varphi \in C_c^\infty(\Omega)$ be nonnegative and let $v_\varphi \in H_0^1(\Omega)$ be the unique weak solution of the linear problem

$$\begin{cases} \Delta^2 v_\varphi = \varphi & \text{in } \Omega, \\ v_\varphi = \Delta v_\varphi - (1-\sigma)|\kappa|(v_\varphi)_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.56}$$

Then $v_\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ by regularity assumptions on $\partial\Omega$, and $v_\varphi \geq 0$ since we assumed the condition [PPP_σ], so $v_\varphi \in K_{N_\sigma}$. Hence, for each $u \in K_{N_\sigma}^*$,

$$0 \geq (v_\varphi, u)_{N_\sigma} = \int_\Omega \Delta u \Delta v_\varphi - (1-\sigma)\int_{\partial\Omega} |\kappa| u_n (v_\varphi)_n = \int_\Omega u\varphi,$$

which implies $u \leq 0$ a.e. in Ω. Moreover, with similar steps as in Lemma 2.2.25, we deduce also $u < 0$ a.e. In conclusion, our problem reduces again to the investigation of the positivity preserving property for the linear Steklov problem (2.56). Thanks to the new boundary condition involving $|\kappa|$, an application of Lemma 2.2.27 with $\alpha = (1-\sigma)|\kappa|$ and $\beta = |\kappa|$ shows that the condition [PPP$_\sigma$] holds whenever $\sigma \in (1-\delta_{1,|\kappa|}, 1+|\delta_{c,|\kappa|}|)$. $\square$

We have finally obtained the nonconvex extension of Proposition 2.2.32.

Theorem 2.2.45. *Let $\Omega \subset \mathbb{R}^2$ satisfy the condition (H_{PPP}) and assume $p > 1$, $0 < g \in L^1(\Omega)$ and $\sigma > 1$. Then the ground states of J_σ are positive a.e. in Ω.*

Proof. Suppose $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a sign-changing ground state. By dual cone decomposition, one may write $u = u_1 + u_2$ with $u_1 \in K_{N_\sigma}$, $0 \not\equiv u_2 \in K^*_{N_\sigma}$ and $(u_1, u_2)_{N_\sigma} = 0$. Defining $w = u_1 - u_2 > 0$, we have:

$$w > |u|, \qquad w_n^2 \geq u_n^2, \qquad N_\sigma(w)^2 = N_\sigma(u)^2.$$

For $\sigma \in (1-\delta_{1,|\kappa|}, 1]$ the result is already achieved in Theorem 2.2.43, so suppose here $\sigma > 1$. By Lemma 2.2.3, there exists a unique $t^*(w) > 0$ such that $t^*(w)w \in \mathcal{N}_\sigma$ and thus

$$\begin{aligned}
J_\sigma(t^*(w)w) &= t^*(w)^2 \left[\frac{1}{2}N_\sigma(w)^2 + (\sigma-1)\int_{\partial\Omega} \kappa^-(-w_n^2)\right] - t^*(w)^{p+1}\int_\Omega \frac{g(x)|w|^{p+1}}{p+1} \\
&< t^*(w)^2 \left[\frac{1}{2}N_\sigma(u)^2 + (\sigma-1)\int_{\partial\Omega} \kappa^-(-u_n^2)\right] - t^*(w)^{p+1}\int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \\
&= J_\sigma(t^*(w)u) \leq J_\sigma(u).
\end{aligned}$$

Since u is a minimum of J_σ in $\mathcal{N}_\sigma$, being a ground state, we have our contradiction. In both cases, this implies $u_2 = 0$ and consequently that $u = u_1 \geq 0$. With the same argument as in Proposition 2.2.26, we conclude the strict positivity of u in Ω. $\square$

Remark 13. The same statement holds by letting $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class C^2 and for $\sigma \in [1, 1+|\delta_{c,|\kappa|}|)$.

Remark 14. Theorem 2.2.45 gives a partial answer to a question which was posed in our work [82].

Chapter 3

A-priori bounds for fourth-order problems in critical dimension

In this second part of the thesis we establish uniform a-priori bounds for solutions of the semilinear problems

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ B(u, D^\alpha u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^4$, $B(u, D^\alpha u) = 0$ stands for homogeneous Dirichlet ($u = u_n = 0$) or Navier ($u = \Delta u = 0$) boundary conditions, and $h : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a positive superlinear and subcritical function in the sense of the Trudinger-Moser-Adams inequality (1.15). Typically, we consider h with an exponential behaviour in the second variable. In other words, our goal is to find a constant C, depending on the domain and on the nonlinearity, such that for any solution of each problem (1.16) there holds $\|u\|_{L^\infty(\Omega)} \leq C$.

Our analysis is mainly inspired by the work of [29] concerning the control of the behaviour near the boundary and of [81] regarding the blow-up technique to obtain the estimates in the interior of our domain. The results we get, extend and in some cases complete the analysis of the uniform a-priori bounds in the context of fourth-order boundary value problems in the literature.

This chapter is an adaptation of the forthcoming paper [83] and it is organized as follows. In Section 3.1 we introduce the definitions and we give the precise statements of the main results and some details about the class of the nonlinearities we consider. Section 3.2 and Section 3.3 are devoted to the analysis of the problem endowed with Dirichlet boundary conditions, the former regarding the behaviour of solutions near the boundary, and the latter in the interior, by a blow-up technique. Some generalizations of the results achieved are considered in Section 3.4. The Navier problem can be studied with a similar analysis and it is briefly addressed in Section 3.5. Finally, in Section 3.6 we deduce an existence result from our a-priori estimates and in Section 3.7 we present a counterexample which shows that the class of growth considered is in some sense sharp.

3.1 Definitions and main results

Throughout this chapter, unless otherwise stated, we focus on *Dirichlet* boundary conditions, confining the investigation of the Navier problem in Section 3.5.

Let $\Omega \subset \mathbb{R}^4$ be a smooth domain, that is, of class $C^{4,\alpha}$ for some $\alpha \in (0,1)$ and consider the Dirichlet boundary value problem

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Definition 3.1.1. We say $u \in H_0^2(\Omega)$ is a *weak solution* of (3.1) if for every $\varphi \in H_0^2(\Omega)$, we have:

$$\int_\Omega \Delta u \Delta \varphi = \int_\Omega h(x,u)\varphi. \tag{3.2}$$

The nonlinearity $h : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is always assumed to satisfy the conditions:

H1) $h \in L^\infty(\Omega \times [0,\tau])$ for all $\tau \in \mathbb{R}^+$;

H2) there exist functions $f \in C^1([0,+\infty))$ satisfying assumption (A) below and $0 < a \in L^\infty(\Omega) \cap C(\Omega)$ such that

$$\lim_{t\to+\infty} \frac{h(x,t)}{f(t)} = a(x) \qquad \text{uniformly in } \overline{\Omega}.$$

Definition 3.1.2. A function $f \in C^1([0,+\infty))$ satisfies assumption (A) if

A1) $f > 0$ and $f'(t) \geq 0$ for any $t > M$ for some $M \in \mathbb{R}$;

A2) f is superlinear at ∞, that is, $\lim\limits_{t\to+\infty} \frac{f(t)}{t} = +\infty$;

A3) there exists $\lim\limits_{t\to+\infty} \frac{f'(t)}{f(t)} \in [0,+\infty)$.

Our assumptions on the nonlinearity h may be seen as a generalization of those of Reichel and Weth in [77], which are the fourth-order counterpart of (1.17). Indeed, we are prescribing a sort of separation of variables at ∞ and a growth for the real variable which follows a "special" profile f. Nevertheless, at a first sight, assumption (*A3*) on this function f may be not completely clear. The next proposition provides then a characterization of such f, showing that (*A3*) is equivalent to require a control from above by a suitable exponential function. This means, also, that our analysis is not restricted to a precise profile at ∞ as in [85, 77, 76], but we include (almost) any growth in t which is controlled from above by $e^{\gamma t}$ for some $\gamma > 0$. Notice also that the function a might vanish on $\partial\Omega$: this will make our preliminary estimates a little more technical.

Proposition 3.1.1. *Let f satisfy assumption (A1) of Definition 3.1.2.*
a) Let $\gamma > 0$ and suppose the existence of

$$\lim_{t\to+\infty} \frac{e^{\gamma t}}{f(t)} \in (0,+\infty] \quad \textit{and} \quad \lim_{t\to+\infty} \left(\frac{e^{\gamma t}}{f(t)}\right)' \geq 0. \tag{3.3}$$

Then $\lim_{t\to+\infty} \dfrac{f'(t)}{f(t)} \in [0,\gamma]$.

b) Suppose $\lim_{t\to+\infty} \dfrac{f'(t)}{f(t)} := \tilde{\gamma} \in [0,+\infty)$. *Then there exists* $\gamma > 0$ *such that* (3.3) *holds.*

Proof. a) It is clear from the assumption (*A1*) that $\liminf_{t\to+\infty} \frac{f'(t)}{f(t)} \geq 0$. Moreover,

$$\left(\frac{e^{\gamma t}}{f(t)}\right)' = \frac{e^{\gamma t}}{f(t)}\left(\gamma - \frac{f'(t)}{f(t)}\right), \tag{3.4}$$

thus

$$0 \leq \lim_{t\to+\infty} \frac{f(t)}{e^{\gamma t}} \lim_{t\to+\infty} \left(\frac{e^{\gamma t}}{f(t)}\right)' = \lim_{t\to+\infty} \left(\gamma - \frac{f'(t)}{f(t)}\right),$$

which clearly implies our claim.

b) From our hypothesis, there exists $M > 0$ such that $f'(t) \leq (\tilde{\gamma}+1)f(t)$ for any $t \geq M$. Defining $\gamma := \tilde{\gamma}+1$ and integrating on $[M,t]$, we get $f(t) \leq f(M) + \gamma\int_M^t f(s)ds$ and, applying Gronwall Lemma, $f(t) \leq f(M)e^{-\gamma M}e^{\gamma t}$ which in turn implies $\frac{e^{\gamma t}}{f(t)} \geq C(M)$ for any $t \geq M$. This means, in particular, $\liminf_{t\to+\infty} \frac{e^{\gamma t}}{f(t)} > 0$. Moreover (3.4) implies $\frac{e^{\gamma t}}{f(t)}$ is nondecreasing, so there exists the limit as $t \to +\infty$ of $\frac{e^{\gamma t}}{f(t)}$ and it is positive. Again by (3.4) we have the existence of the limit as $t \to +\infty$ of $\left(\frac{e^{\gamma t}}{f(t)}\right)' \geq 0$. □

Remark 15. From (*H1*)-(*H2*) it follows that for each $\varepsilon > 0$ there exists a constant $d_\varepsilon \geq 0$ such that

$$(1-\varepsilon)a(x)f(t) - d_\varepsilon \leq h(x,t) \leq (1+\varepsilon)a(x)f(t) + d_\varepsilon \quad \text{for all } t \geq 0,\ x \in \Omega. \tag{3.5}$$

Indeed, (*H2*) implies the existence of $\tau(\varepsilon)$ such that (3.5) holds with $d_\varepsilon = 0$ for $t \geq \tau_\varepsilon$. Moreover, by (*H1*),

$$\sup_{x\in\Omega} \max_{t\in[0,\tau_\varepsilon]} h(x,t) \leq C(\tau_\varepsilon).$$

Therefore, $h(x,t) \leq (1+\varepsilon)a(x)f(t) + C(\tau_\varepsilon)$ for all $t \geq 0$. Finally, since

$$h(x,t) \geq \begin{cases} 0 & \text{if } t \in [0,\tau_\varepsilon], \\ (1-\varepsilon)a(x)f(t) & \text{if } t \geq \tau_\varepsilon, \end{cases}$$

and defining $d_\varepsilon := \max\{C(\tau_\varepsilon), (1-\varepsilon)\|a\|_\infty \max_{t\in[0,\tau_\varepsilon]} f(t)\}$, then (3.5) follows.

It will be useful in the sequel to distinguish among the admissible growths, in dependence of "how far" they are from being exponential, according to the following definition:

Definition 3.1.3. Let h satisfy assumptions (*H1*)-(*H2*) and f be as in (*H2*). We say that h is *subcritical* if

$$\lim_{t\to+\infty} \frac{f'(t)}{f(t)} = 0. \tag{3.6}$$

On the other hand, we say that h is *critical* if

$$\lim_{t\to+\infty} \frac{f'(t)}{f(t)} \in (0,+\infty). \tag{3.7}$$

Roughly speaking, the behaviour of the nonlinearities of *subcritical* type largely differs from the exponential map. An example is $h(x,t) = a(x)\log^{\theta}(t+1)t^p e^{t^{\alpha}}$ with $\theta, p \geq 0$, $\alpha \in [0,1)$ and $a \in C(\overline{\Omega})$. The class of *critical* functions gathers instead maps which are quite close to e^t with respect to the second variable, not affecting too much its exponential behaviour; model nonlinearities for this case are $h(x,t) = a(x)\frac{e^{\gamma t}}{(t+1)^q}$ with $q, \gamma > 0$.

The results we find for subcritical and critical nonlinearities, which we present here, are very similar. Nevertheless, we have to consider this distinction because, after a preliminary common analysis, the limiting equation found by means of a blow-up method will be linear in the first case and nonlinear in the second, thus the critical case will be more involved.

The main results of our analysis may be summarized as follows:

Theorem 3.1.2. *Let $\mathcal{B}$ be a ball in $\mathbb{R}^4$ and h be a subcritical nonlinearity satisfying assumptions (H1)-(H2) and, moreover, one of the following:*

i) $a(\cdot) \geq a_0 > 0$ *and* $h(\cdot,0) \in L^{\infty}(\mathcal{B})$;

ii) $\lim\limits_{t\to+\infty} \frac{f(t)}{t^{\alpha}} = +\infty$, *for some* $\alpha > 1$.

Then there exists $C > 0$ such that $\|u\|_{L^{\infty}(\mathcal{B})} \leq C$ for all weak solutions u of (3.1).

Let us now define

$$F(t) := \int_0^t f(s)ds \qquad \text{and} \qquad H(x,t) := \int_0^t h(x,s)ds. \tag{3.8}$$

Notice that, by de l'Hôpital's Theorem and assumption (*H2*),

$$\lim_{t\to+\infty} \frac{H(x,t)}{F(t)} = \lim_{t\to+\infty} \frac{h(x,t)}{f(t)} = a(x) \tag{3.9}$$

uniformly with respect to $x \in \overline{\Omega}$.

Theorem 3.1.3. *Let $\mathcal{B}$ be a ball in $\mathbb{R}^4$ and h be a critical nonlinearity satisfying assumptions (H1)-(H2). Suppose moreover that there exist functions $0 \leq B \in L^{\infty}(\mathcal{B})$, $0 \leq D \in L^1(\mathcal{B})$ such that*

$$|\nabla_x H(x,t)| \leq B(x)F(t) + D(x), \quad \textit{for any} \quad t \geq 0,\, x \in \mathcal{B}. \tag{H3}$$

Then there exists $C > 0$ such that $\|u\|_{C^4(\bar{\mathcal{B}})} \leq C$ for all weak solutions u of (3.1).

Before going into the details of the proofs, let us make some remarks about these two results:

1. It is easy to infer that in the critical case the condition (*ii*) of Theorem 3.1.2 is automatically satisfied, therefore it has not to be required as an assumption as therein.

2. The additional assumption (H3), when applied to a model nonlinearity $h(x,t) = a(x)f(t)$, is nothing but a uniform control on ∇a, and is thus equivalent to the request $a \in W^{1,\infty}(\Omega)$. Roughly speaking, this condition will allow us to control H and $\nabla_x H$ in a similar way.

3. As mentioned before, the assumptions on f gather all superlinear profiles which can be controlled by a map $t \to e^{\gamma t}$ for some $\gamma > 0$. This bound from above on the growth of f reveals to be *sharp*. Indeed, in the spirit of Brezis and Merle, in Section 3.7 we provide a counterexample for a class of nonlinearities which, although being subcritical, do not satisfy assumption (*A3*).

As mentioned in the Introduction, the main motivation to study a-priori bounds for semilinear elliptic problems is to infer the existence of solutions. The subsequent Theorem 3.1.4 is in fact obtained applying the Krasnosel'skii topological degree theory by means of Theorems 3.1.2 and 3.1.3. We remark that the existence of solutions for semilinear Dirichlet problems like (3.1) can be obtained also up to the critical growth (in the sense of Adams, cf. (1.15)) $t \mapsto e^{t^2}$ by variational methods. We refer to [56, 57] for this subject for general Dirichlet polyharmonic problems with subcritical and critical exponential nonlinearities.

To fix the notation, here and in the sequel, we denote by $\tilde{\lambda}_1(\Omega)$ (resp. $\lambda_1(\Omega)$) the first eigenvalue of Δ^2 (resp. $-\Delta$) in the domain Ω subjected to Dirichlet boundary conditions, omitting the domain whenever it is clear from the context.

Theorem 3.1.4. *Let $\mathcal{B}$ be a ball in $\mathbb{R}^4$ and $h : \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumptions of Theorems 3.1.2 or 3.1.3 and, in addition, $h(\cdot, t) \in C(\bar{\mathcal{B}})$ (resp. $C^{0,\gamma}(\bar{\mathcal{B}})$ for some $\gamma \in (0,1)$) for any $t \geq 0$ and*

$$\limsup_{t \to 0^+} \frac{h(x,t)}{t} < \tilde{\lambda}_1 \quad \text{uniformly in } x \in \mathcal{B}. \tag{3.10}$$

Then, problem (3.1) admits a positive strong (resp. classical) solution.

Remark 16. The assumption $\Omega = \mathcal{B}$ is not an intrinsic restriction but only a consequence of the hypothesis of Theorems (3.1.2)-(3.1.3).

Remark 17. The assumptions of Theorem 3.1.4 are, for instance, satisfied by $h(x,t) = a(x)t^p e^{t^\alpha}$ for any $\alpha \in [0,1)$, $p > 1$ regarding the subcritical context, or $h(x,t) = a(x)t^p \frac{e^{\theta t}}{(t+1)^\gamma}$ for any $\gamma \geq 0$, $p > 1$ and $\theta > 0$ for the critical framework.

Before entering into the details of the arguments, let us fix some notation.

Let $f, g : \Omega \to \mathbb{R}^+$: we say that $f \preceq g$ if there exists a constant $c > 0$ such that $f(t) \leq c g(t)$ for all $t \in \Omega$, and we write $f \simeq g$ if both $f \preceq g$ and $g \preceq f$ hold. Finally, we define $f \wedge g := \min\{f, g\}$.

3.2 A-priori estimates near the boundary and on the right-hand side

The main result of this section provides an uniform bound for solutions of (3.1) in a neighborhood of $\partial\Omega$ and a uniform L^1 control on the right-hand side of the equation.

Proposition 3.2.1. *Let $\mathcal{B}$ the unit ball in $\mathbb{R}^4$. Let h verify assumptions (H1)-(H2) and suppose one of the following holds:*

i) $a(\cdot) \geq a_0 > 0$ and $h(\cdot, 0) \in L^\infty(\mathcal{B})$;

ii) $\lim\limits_{t\to+\infty} \frac{f(t)}{t^\alpha} = +\infty$, for some $\alpha > 1$.

Then there exist $C_1, \Lambda > 0$ and a small neighborhood $\omega \subset \mathcal{B}$ of $\partial\mathcal{B}$ such that

$$\|u\|_{L^\infty(\omega)} \leq C_1 \qquad \textit{and} \qquad \int_{\mathcal{B}} h(x,u)dx \leq \Lambda$$

for all (positive) weak solutions u of (3.1)*.*

In other words, Proposition 3.2.1 prevents the boundary blow-up and, further, yields a uniform L^1 bound on the right-hand side of our problem (3.1), which will be essential for the blow-up technique in Section 3.3. The proof is inspired by some arguments which first appeared in [29] applied to second order elliptic problems, and which are a very flexible tool, since they adapt to many other contexts, for instance $N-$Laplacian problems ([58]) or elliptic systems ([30]). Notice that all these problems do not have to deal with the lack of maximum principle, as for the fourth-order context. At the present, as recalled in the Introduction, the most general setting we can consider to be sure that the positivity preserving property holds for fourth-order Dirichlet boundary value problems is to work in small deformations of the ball (see [47]). Nevertheless, although most of our results hold for this kind of domains, in the subsequent Lemma 3.2.6 we need a precise behaviour of the Green function of the operator Δ^2 subjected to Dirichlet boundary conditions near the boundary, so we have to further restrict to the case of a ball.

In the following, we will often make use of the Green function $G_{\Delta^2,\Omega}$ associated to the biharmonic operator with Dirichlet boundary conditions. We recall that $G_{\Delta^2,\Omega} : \overline{\Omega} \times \overline{\Omega} \setminus \{(x,x) : x \in \overline{\Omega}\} \to \mathbb{R}$ is defined as the unique function such that for any $g \in L^2(\Omega)$,

$$u(x) := \int_\Omega G_{\Delta^2,\Omega}(x,y)\, g(y)\, dy$$

is the unique solution in $H_0^2(\Omega)$ of the equation $\Delta^2 u = g$ in Ω. From now on, we will use the shorter notation G_Ω to underline the dependence on the domain, since the operator and the boundary conditions are fixed.

We collect here some results about pointwise estimates of G_Ω and of its derivatives, as they will be needed in the sequel. They go back to the work of Krasovskiĭ

[53] later on refined by Dall'Acqua and Sweers in [24]. However, very smooth boundaries are required therein. This formulation which assumes less regularity can be deduced combining [47, Theorem 4] and [48, Theorem 2].

Lemma 3.2.2. *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain of class $C^{4,\gamma}$ for some $\gamma \in (0,1)$. There exists a positive constant C depending on the domain, such that for all $x, y \in \Omega$, $x \neq y$, there holds*

$$|G_\Omega(x,y)| \leq C \log\left(2 + \frac{1}{|x-y|}\right), \tag{3.11}$$

$$|\nabla^i G_\Omega(x,y)| \leq \frac{C}{|x-y|^i}, \quad \text{for any } i \geq 1.$$

For further and sharper results in this direction, we refer to [49] and to the monograph [40]. In the special case of $\Omega = \mathcal{B}$, thanks to the explicit Boggio's formula, also the estimate from below of $G_\mathcal{B}$ may be obtained. Therefore, the following sharp two-sided estimate holds:

Lemma 3.2.3 ([40], Theorem 4.6)**.** *In $\overline{B} \times \overline{B}$ we have*

$$G_\mathcal{B}(x,y) \simeq \log\left(1 + \left(\frac{d_\mathcal{B}(x) d_\mathcal{B}(y)}{|x-y|^2}\right)^2\right). \tag{3.12}$$

In the whole section, we suppose h as in Proposition 3.2.1 and we denote by $\tilde{\varphi}_1$ the first eigenfunction of Δ^2 in Ω subjected to Dirichlet boundary conditions. Notice that $\tilde{\varphi}_1 > 0$ if Ω is positivity preserving (see [40, Theorem 3.7] and remarks below).

We split the proof of Proposition 3.2.1 in some steps. First, we obtain a local uniform estimate for the right-hand side of equation (3.1). Here the assumptions (*i*)-(*ii*) of Theorem 3.1.2 will play a significant role. If the domain is a ball, then the proof follows rather easily by pointwise Green function estimates from below which are available in this context. Refining the approach of [29], we also give a proof which does not rely on these estimates for general positivity preserving domains, provided some assumptions on the function $a(\cdot)$ are imposed according to the behaviour of f. Indeed, unlike similar arguments in [20, 30, 31], we are able to include in our analysis also functions $a(\cdot)$ which may vanish on the boundary $\partial\Omega$. The boundary estimate and the L^1 bound for the right-hand side are then proved in Lemmas 3.2.7 and 3.2.8, once the key Lemma 3.2.6 is obtained.

Lemma 3.2.4 (Local a-priori L^1 estimate in the ball)**.** *Suppose that condition (ii) of Proposition 3.2.1 holds, there exists a constant $C > 0$ such that*

$$\int_\mathcal{B} h(x,u)\tilde{\varphi}_1 = \tilde{\lambda}_1 \int_\mathcal{B} u\tilde{\varphi}_1 \leq C, \tag{3.13}$$

for all weak solutions u of (3.1).

Proof. We follow here some ideas of [33], see also [86]. Let c_1, c_2 be positive constants such that

$$c_1 d_{\mathcal{B}}^2(x) \le \tilde{\varphi}_1(x) \le c_2 d_{\mathcal{B}}^2(x)$$

for all $x \in \mathcal{B}$, see [21, Lemma 3]. By the representation formula and the pointwise estimate of the Green function (see [50], Proposition 2.3 and Remark 3):

$$G_{\mathcal{B}}(x,y) \ge C \log\left(2 + \frac{d_{\mathcal{B}}(y)}{|x-y|}\right)\left(1 \wedge \frac{d_{\mathcal{B}}^2(x) d_{\mathcal{B}}^2(y)}{|x-y|^4}\right),$$

for all solutions u of (3.1) there holds

$$\begin{aligned} u(x) &\ge C \int_{\mathcal{B}} \log\left(2 + \frac{d_{\mathcal{B}}(y)}{|x-y|}\right)\left(1 \wedge \frac{d_{\mathcal{B}}^2(x) d_{\mathcal{B}}^2(y)}{|x-y|^4}\right) h(y, u(y)) dy \\ &\ge C d_{\mathcal{B}}^2(x) \int_{\mathcal{B}} d_{\mathcal{B}}^2(y) h(y, u(y)) dy \ge C c_2^{-2} \tilde{\varphi}_1^2(x) \int_{\mathcal{B}} \tilde{\varphi}_1^2(y) h(y,u) dy. \end{aligned} \tag{3.14}$$

Moreover, by (3.5) with $\varepsilon = \frac{1}{2}$ and condition (ii), we have

$$\begin{aligned} \int_{\mathcal{B}} h(x,u)\tilde{\varphi}_1(x)dx &\ge \frac{1}{2}\int_{\mathcal{B}} a(x) f(u) \tilde{\varphi}_1(x) dx - d \int_{\mathcal{B}} \tilde{\varphi}_1(x) dx \\ &\ge \int_{\mathcal{B}} a(x)(C u^{\gamma}(x) - D)\tilde{\varphi}_1(x) dx - d \int_{\mathcal{B}} \tilde{\varphi}_1(x) dx \end{aligned}$$

where C and D are suitable positive constant. Therefore, by (3.14),

$$\begin{aligned} &\int_{\mathcal{B}} h(x,u)\tilde{\varphi}_1(x)dx + C(\|a\|_1, \gamma) \ge C \int_{\mathcal{B}} a(x) u^{\gamma}(x) \tilde{\varphi}_1(x) dx \\ &\ge C \left(\int_{\mathcal{B}} a(x) \tilde{\varphi}_1^{1+\gamma}(x) dx \right) \left(\int_{\mathcal{B}} h(x,u) \tilde{\varphi}_1(x) dx \right)^{\gamma}. \end{aligned}$$

Since $\gamma > 1$ and all constants are positive, then $\int_{\Omega} h(x,u)\tilde{\varphi}_1 \le C$. Finally, the equality in (3.13) is proven simply by testing (3.2) with $\varphi = \tilde{\varphi}_1 > 0$:

$$\int_{\Omega} h(x,u)\tilde{\varphi}_1 = \int_{\Omega} \Delta u \Delta \tilde{\varphi}_1 = \tilde{\lambda}_1 \int_{\Omega} u \tilde{\varphi}_1.$$

□

The next result is an extension of the previous local L^1 estimate also for *general positivity preserving domains*, provided the map $a(\cdot)$ satisfies some additional assumptions according to the growth of f. We remark that if the growth of $h(x,\cdot)$ is exponential these additional assumptions on a are mild.

Lemma 3.2.5 (Local a-priori L^1 estimate in general positivity preserving domains)**.** *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain where the positivity preserving property holds. Let h verify assumptions (H1)-(H2) and suppose one of the following:*

i) $\lim\limits_{t\to+\infty} \frac{f(t)}{t} = +\infty$, *$a(\cdot) \ge a_0 > 0$ and $h(\cdot, 0) \in L^1(\Omega)$;*

ii) $\lim_{t\to+\infty} \frac{f(t)}{t^\alpha} = +\infty$, *for* $\alpha > 1$, $a(\cdot)^{-\frac{1}{\alpha-1}} \in L^1(\Omega)$ *and* $h(\cdot,0) \in L^\infty(\Omega)$*;*

iii) $\lim_{t\to+\infty} \frac{f(t)}{e^{t^\beta}} = +\infty$, *for* $\beta \in (0,1]$, $\log(a(\cdot)) \in L^{1/\beta}(\Omega)$ *and* $h(\cdot,0) \in L^1(\Omega)$.

Then there exists a constant $C > 0$ *such that* (3.13) *holds for all weak solutions* u *of* (3.1).

Proof. As in the previous proof, testing (3.2) with $\varphi = \tilde\varphi_1 > 0$ we get

$$\int_\Omega h(x,u)\tilde\varphi_1 = \tilde\lambda_1 \int_\Omega u\tilde\varphi_1.$$

Since $t \mapsto h(\cdot,t)$ is superlinear for large t by assumption (*A2*), we know that for every $M > 0$ and $x \in \Omega$ there exists $t_0(M,x) > 0$ such that $h(x,t) \geq Mt + h(x,0)$ for all $t \geq t_0(M,x)$. More precisely, fixed $x \in \Omega$, we may choose t_0 as the last point of intersection between the graphs of $h(x,t)$ and of $g(t) = Mt + h(x,0)$. Setting $A := \{x \in \Omega \,|\, u(x) \geq t_0(x)\}$, we have

$$\begin{aligned}\int_\Omega u\tilde\varphi_1 &= \int_A u\tilde\varphi_1 + \int_{\Omega\setminus A} u\tilde\varphi_1 \leq \frac{1}{M}\int_\Omega h(x,u)\tilde\varphi_1 - \frac{1}{M}\int_A h(x,0)\tilde\varphi_1 + \int_{\Omega\setminus A} t_0(x)\tilde\varphi_1\\ &\leq \frac{\tilde\lambda_1}{M}\int_\Omega u\tilde\varphi_1 + \|\tilde\varphi_1\|_\infty\Big(\frac{1}{M}\|h(\cdot,0)\|_1 + \int_\Omega t_0(x)dx\Big).\end{aligned}$$

If we choose $M = 2\tilde\lambda_1$, then we deduce

$$\int_\Omega u\tilde\varphi_1 \leq 2\|\tilde\varphi_1\|_\infty\Big(\frac{1}{2\tilde\lambda_1}\|h(\cdot,0)\|_1 + \int_\Omega t_0(x)dx\Big),$$

which is now independent on u. However, we have to be sure that this is a finite quantity, so we have to link somehow t_0 to the integrability of a. Hence, we split the proof in three cases according to our possible assumptions:

Suppose (i) holds: by (3.5),

$$\frac{1}{2}a(x)f(t_0(x)) - d \leq h(x,t_0(x)) = Mt_0(x) + h(x,0) \tag{3.15}$$

that is, since $a(x) \geq a_0 > 0$,

$$f(t_0(x)) \leq \frac{2M}{a_0}t_0(x) + \frac{2}{a_0}h(x,0).$$

Since f is superlinear and $h(\cdot,0) \in L^1(\Omega)$, then $\int_\Omega t_0$ is bounded.

Suppose (ii) holds: then there exists a constant $N \geq 0$ such that $f(t) \geq t^\alpha - N$ for all $t \geq 0$. Hence, by (3.5) with the choice $\varepsilon = \frac{1}{2}$, by definition of t_0 and since $a(\cdot), h(\cdot,0) \in L^\infty(\Omega)$,

$$\frac{1}{2}a(x)t_0(x)^\alpha \leq Mt_0(x) + h(x,0) + d + \frac{1}{2}a(x)N \leq Mt_0(x) + C \tag{3.16}$$

for some $C > 0$. Recall the Young inequality, which may be stated in the following form: let $a, b > 0$, then for any $\varepsilon > 0$, one has

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q/p}}, \tag{3.17}$$

where $p > 1$, $q = \frac{p}{p-1}$. Apply (3.17) with $a = t_0(x)$, $p = \alpha$, $b = M$ and choose $\varepsilon = \varepsilon(x) = \frac{\alpha}{4}a(x)$. Then from (3.16) we get

$$t_0(x) \leq \frac{C}{a(x)^{\frac{1}{\alpha}}}\left(1+\frac{1}{a(x)^{\frac{1}{\alpha-1}}}\right)^{\frac{1}{\alpha}} \leq \frac{C}{a(x)^{\frac{1}{\alpha}}}\left(1+\frac{1}{a(x)^{\frac{1}{\alpha(\alpha-1)}}}\right) = C\left(\frac{1}{a(x)^{\frac{1}{\alpha}}}+\frac{1}{a(x)^{\frac{1}{\alpha-1}}}\right).$$

Consequently, since we know that $a \in L^\infty(\Omega)$, then once we impose that $a(\cdot)^{-\frac{1}{\alpha-1}} \in L^1(\Omega)$, we infer $t_0 \in L^1(\Omega)$, our goal.

Suppose (iii) holds: there exists clearly a constant $C_0 > 0$ such that $f(t) \geq t^2 - C_0$ for any $t \geq 0$, so, $t \leq \sqrt{f(t)} + \sqrt{C_0}$. From (3.15), set $\tau := \sqrt{f(t_0(x))}$; we obtain the inequality

$$\tfrac{1}{2}a(x)\tau^2 - M\tau - [d + h(x,0)] \leq 0. \tag{3.18}$$

Recalling $a \in L^\infty(\Omega)$, from (3.18) we find

$$f(t_0(x)) \leq C\frac{(1+h(x,0))}{a(x)^2}.$$

Using now our assumption on f in *(iii)*, we obtain

$$t_0(x) \leq C[1 + \log^{\frac{1}{\beta}}(1+h(x,0)) + |\log^{\frac{1}{\beta}}(a(x))|],$$

which is integrable on Ω by using our assumption on a and $h(x,0)$. ☐

Remark 18. It is easy to show that the same conclusion of Lemma 3.2.5 holds once instead of *(ii)* one considers the following condition:
(ii)' $\lim_{t\to+\infty} \frac{f(t)}{t^\alpha} = +\infty$, for some $\alpha > 1$, $a(\cdot)^{-\frac{1}{\alpha-1}} \in L^1(\Omega)$ and $h(\cdot,0)a(\cdot)^{-1} \in L^{\frac{1}{\alpha}}(\Omega)$.

Remark 19 (Example). The class of functions considered for a in the condition *(iii)* of Lemma 3.2.5 contains also functions which vanish on $\partial\Omega$ with order more than polynomial. For instance, if $\Omega = \mathcal{B}$ is the unit ball in $\mathbb{R}^4$, taking

$$a(x) = e^{-\frac{1}{(1-|x|)^\beta}},$$

then $\log(a(\cdot)) \in L^{1/\alpha}(B)$ for all $\beta < \alpha$. Indeed,

$$\int_B \left(\frac{1}{(1-|x|)^\beta}\right)^{1/\alpha} = C_4\int_0^1 \frac{r^3}{(1-r)^{\beta/\alpha}} < +\infty \quad \Leftrightarrow \quad \beta < \alpha.$$

Lemma 3.2.6. *Let u be a solution of* (3.1)*. There exist $\delta > 0$ and $\varepsilon > 0$ such that $\nabla u(x)\cdot\theta \leq 0$ for every $x \in \mathcal{B}_\varepsilon := \{x \in \mathcal{B} \,|\, d(x,\partial\mathcal{B}) < \varepsilon\}$ and for every direction θ such that $|\theta - n(x)| < \delta$.*

Proof. By Lemma 3.2.3, we know that the Green function is positive and vanishes on $\partial\mathcal{B}$ precisely of order 2 (see also [42]). This means that for fixed $y \in \mathcal{B}$ and $x \to \partial\mathcal{B}$ one has:

$$\frac{1}{C(y)}(1-|x|)^2 \leq G_{\mathcal{B}}(x,y) \leq C(y)(1-|x|)^2, \qquad C(y) > 0. \tag{3.19}$$

Since on the right and on the left we have the same behaviour, (3.19) implies that

$$\frac{\partial}{\partial|x|}G_{\mathcal{B}}(x,y) \simeq -2C(y)(1-|x|).$$

Following [29], the statement of the lemma is equivalent to prove that if $x_0 \in \partial\mathcal{B}$ with $(x_0)_1 = \langle n(x), n(x_0)\rangle > 0$, there exists $\delta > 0$ such that in $\mathcal{B} \cap \{|x-x_0| < \delta\}$, $\frac{\partial u}{\partial x_1} < 0$. Defining $h(x) = (1-|x|)^2$, we have $\nabla h(x) = -2(1-|x|)\left(\frac{x_1}{|x|}; \frac{x_2}{|x|}\right)$ and so

$$\frac{\partial u}{\partial x_1}(x) = \int_{\mathcal{B}} \frac{\partial}{\partial x_1} G_{\mathcal{B}}(x,y)h(y,u(y))dy \leq -2\int_{\mathcal{B}}(1-|x|)\frac{x_1}{|x|}C(y)h(y,u(y))dy < 0$$

for each $x \in \mathcal{B} \cap \{|x-x_0| < \delta\}$ for $\delta > 0$ small enough. □

Remark 20. As pointed out at the beginning of this section, this is the only step in which we do require that the domain is a ball, as we need the two-sided boundary estimate (3.19). Actually, one may replace the assumption $\Omega = \mathcal{B}$ with Ω *positivity preserving domain such that*

$$G_\Omega(x,y) \geq c d_\Omega(x)^2 d_\Omega(y)^2 \tag{3.20}$$

in $\Omega \times \Omega$. See also Open Problem 5 in Section 4.

Roughly speaking, Lemma 3.2.6 shows that near $\partial\mathcal{B}$ all solutions are uniformly decreasing in some outwards directions. This enables us to relate the behaviour of a solution close to the boundary with its local properties. The outcome is exactly Proposition 3.2.1, whose proof is contained in the following two lemmas.

Lemma 3.2.7. *There exists a neighborhood* ω *of* $\partial\mathcal{B}$ *and* $C_1 > 0$ *such that* $\|u\|_{L^\infty(\omega)} \leq C_1$ *for all weak solutions* u *of* (3.1).

Proof. By Lemma 3.2.6, arguing as in [29], one may infer that for every $x \in \omega_r := \{x \in B \,|\, d(x,\partial\mathcal{B}) < r\}$ there exists a set I_x and a constant $\gamma > 0$ independent of x such that $|I_x| \geq \gamma$, $I_x \subseteq \mathcal{B} \setminus \omega_{\frac{r}{2}}$ and $u(y) \geq u(x)$ for all $y \in I_x$. Taking $x \in \omega_r$, by positivity of $\tilde{\varphi}_1$ and either by Lemma 3.2.4 if we assume (ii) or Lemma 3.2.5 assuming instead (i):

$$C \geq \int_{\mathcal{B}} h(x,u)\tilde{\varphi}_1 dx \geq \tilde{\lambda}_1 \int_{\mathcal{B}\setminus\omega_{\frac{r}{2}}} u\tilde{\varphi}_1 \geq \tilde{\lambda}_1 \min_{\mathcal{B}\setminus\omega_{\frac{r}{2}}} \tilde{\varphi}_1 \int_{I_x} u(y)dy \geq c(\mathcal{B})\gamma u(x),$$

which implies the uniform L^∞ boundedness of u in ω_r. □

Lemma 3.2.8. *There exists a constant* $\Lambda > 0$ *such that* $\int_{\mathcal{B}} h(x,u)dx \leq \Lambda$ *for all weak solutions* u *of* (3.1).

Proof. By (3.5), Lemma 3.2.7, the positivity of $\tilde{\varphi}_1$ and Lemma 3.2.5 or 3.2.4,

$$\begin{aligned}\int_{\mathcal{B}} h(x,u)dx &= \int_{\omega} h(x,u)dx + \int_{\mathcal{B}\setminus\omega} h(x,u)dx \\ &\le 2f(C_1)\|a\|_{L^1(\mathcal{B})} + d|\mathcal{B}| + \frac{1}{m(\omega)}\int_{\mathcal{B}\setminus\omega} h(x,u)\tilde{\varphi}_1 dx \\ &\le 2f(C_1)\|a\|_{L^1(\mathcal{B})} + d|\mathcal{B}| + \frac{1}{m(\omega)}\int_{\mathcal{B}} h(x,u)\tilde{\varphi}_1 dx \le \Lambda(\mathcal{B},h)\end{aligned}$$

having defined $m(\omega) := \min_{\mathcal{B}\setminus\omega}\tilde{\varphi}_1 > 0$. □

3.3 Uniform bounds inside the domain

By now, we know that solutions of problem (3.1) in the ball $\mathcal{B}$ are bounded near the boundary, but they might become arbitrarily large around a point inside the domain. This is the situation we want to exclude via a blow-up argument in order to complete the proofs of Theorems 3.1.2 and 3.1.3. The beginning of the argument is the same for both of them and follows the approach of [81]: we define a sequence of rescaled functions on some expanding domains, which turns out to be locally compact in a Hölder space, and we find a limit profile v satisfying an equation in $\mathbb{R}^4$. In the subcritical framework it will be quite easy to find a contradiction as the limiting equation is linear, while in the critical case some further investigation will be needed.

Although the analysis that we present here concerns the problem (3.1) in $\mathcal{B}$, in this section we use the notation Ω to indicate the ball. This will be explained in Section 3.4.1, where we show that the same argument can be easily applied also for general smooth domains.

Let us start supposing by contradiction that there is a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions of problem (3.1) and of maximum points $(x_k)_{k\in\mathbb{N}} \subset \Omega$ such that

$$u_k(x_k) = \|u_k\|_{L^\infty(\Omega)} =: M_k \nearrow +\infty. \tag{3.21}$$

Since Ω is bounded, necessarily the points x_k accumulate to a limit point which has positive distance from the boundary, by Proposition 3.2.1. So, up to a subsequence, $x_k \to x_\infty \in \Omega$. Moreover, we define the rescaled functions $v_k : \Omega_k \to \mathbb{R}$ as

$$v_k(x) := u_k(x_k + \mu_k x) - M_k, \tag{3.22}$$

where the scaling is

$$\mu_k := \frac{1}{(f(M_k))^{1/4}} \to 0 \quad \text{as } k \to +\infty \tag{3.23}$$

and the expanding domains are $\Omega_k := \frac{\Omega - x_k}{\mu_k}$. Notice that $x_\infty \in \Omega$ implies $\Omega_k \nearrow \mathbb{R}^4$. Let us compute what is $|\Delta^2 v_k|$:

$$\begin{aligned}|\Delta^2 v_k(x)| = \mu_k^4|(\Delta^2 u_k)(x_k+\mu_k x)| &= \frac{h(x_k+\mu_k x, u_k(x_k+\mu_k x))}{f(M_k)} \\ &\le (1+\varepsilon)a(x_k+\mu_k x)\frac{f(u_k(x_k+\mu_k x))}{f(M_k)} + \frac{d_\varepsilon}{M_k},\end{aligned} \tag{3.24}$$

by (3.5), so it is uniformly bounded. Moreover, we obtain the following result:

Lemma 3.3.1. *Let $x \in B_R(0)$. There holds $|\nabla^i v_k(x)| \le C(R)$ for any $i \in \{0,1,2,3\}$.*

Proof. By the representation formula for derivatives and Lemma 3.2.2,

$$
\begin{aligned}
|\nabla^i v_k(x)| &= |\mu_k^i \nabla^i u_k(x_k+\mu_k x)| = \mu_k^i \left| \int_\Omega \nabla_x^i G_\Omega(x_k+\mu_k x, y) h(y, u_k(y)) dy \right| \\
&\le C\mu_k^i \int_{\Omega \setminus B_{2R\mu_k}(x_k)} \frac{h(y,u_k(y))}{|x_k+\mu_k x - y|^i} dy + C\mu_k^i \int_{B_{2R\mu_k}(x_k)} \frac{h(y,u_k(y))}{|x_k+\mu_k x - y|^i} dy.
\end{aligned}
$$

In $\Omega \setminus B_{2R\mu_k}(x_k)$ there holds $|x_k+\mu_k x - y| \ge |y-x_k| - \mu_k|x| \ge 2R\mu_k - R\mu_k = R\mu_k$, while in $B_{2R\mu_k}(x_k)$ we have $f(u_k(y)) \le f(M_k) = \mu_k^{-4}$ (this follows from $0 \le f'(t) \to +\infty$). Hence, by Proposition 3.2.1 and (3.5) with $\varepsilon = \frac{1}{2}$,

$$
\begin{aligned}
|\nabla^i v_k(x)| &\le CR^{-i}\Lambda + C\mu_k^i \int_{B_{2R\mu_k}(x_k)} \frac{2a(y)f(u_k(y)) + d}{|x_k+\mu_k x - y|^i} dy \\
&\le CR^{-i}\Lambda + C(2\|a\|_\infty + d\mu_k^4)\mu_k^{i-4} \int_{B_{2R\mu_k}(x_k)} \frac{1}{|x_k+\mu_k x - y|^i} dy.
\end{aligned}
\tag{3.25}
$$

Using the change of variable $y = \mu_k z + x_k$, the last integral becomes

$$
\int_{B_{2R\mu_k}(x_k)} \frac{1}{|x_k+\mu_k x - y|^i} dy = \int_{B_{2R}(0)} \frac{1}{\mu_k^i |x-z|^i} \mu_k^4 dz = \mu_k^{4-i} \int_{B_{2R}(0)} \frac{1}{|z-x|^i} dz.
$$

Inserting it into (3.25), we obtain

$$
|\nabla^i v_k(x)| \le CR^{-i}\Lambda + (2\|a\|_\infty + d\mu_k^4) C \int_0^{2R} \rho^{3-i} d\rho,
$$

which is finite for $i \in \{1,2,3\}$ since $\mu_k \to 0$ as $k \to +\infty$.
If we finally take $i=0$ and $x \in B_R(0)$,

$$
|v_k(x)| = |v_k(x) - v_k(0)| \le \sup_{B_R(0)} |\nabla v_k||x| \le C(R).
$$

□

Summarizing, we are able to control both $\Delta^2 v_k$ in (3.24) and, by Lemma 3.3.1, $\nabla^i v_k$ locally in $\mathbb{R}^4$. Hence, the local boundedness of the sequence $(v_k)_{k\in\mathbb{N}}$ is achieved by Lemma A.0.4, see the Appendix. This means that $(v_k)_{k\in\mathbb{N}}$ is bounded in $W^{4,p}_{loc}(\mathbb{R}^4)$ so, by compact embedding, there exists $v \in C^3(\mathbb{R}^4)$ such that $v_k \to v$ in $C^{3,\gamma}_{loc}(\mathbb{R}^4)$ for any $\gamma \in (0,1)$, satisfying $v \le 0$ and $v(0) = 0$. Looking for the equation satisfied by v in $\mathbb{R}^4$, one may rewrite (3.24) and obtain

$$
\Delta^2 v_k(x) \le (1+\varepsilon) a(x_k+\mu_k x) e^{\log(f(u_k(x_k+\mu_k x))) - \log(f(M_k))} + \tfrac{d_\varepsilon}{M_k}. \tag{3.26}
$$

Taking the first-order Taylor expansion of $\log \circ f$ around M_k, one finds

$$
\log(f(u_k(x_k+\mu_k x))) = \log(f(M_k)) + \frac{f'(z_k(x))}{f(z_k(x))}(u_k(x_k+\mu_k x) - M_k),
$$

where $z_k(x) := M_k + \theta(u_k(x_k+\mu_k x) - M_k) = M_k + \theta v_k(x)$, $\theta \in (0,1)$. Hence, (3.26) becomes

$$\Delta^2 v_k(x) \leq (1+\varepsilon)a(x_k+\mu_k x)e^{\frac{f'(z_k(x))}{f(z_k(x))}v_k(x)} + \tfrac{d_\varepsilon}{M_k}. \tag{3.27}$$

Analogously, the following lower bound holds:

$$\Delta^2 v_k(x) \geq (1-\varepsilon)a(x_k+\mu_k x)e^{\frac{f'(z_k(x))}{f(z_k(x))}v_k(x)} - \tfrac{d_\varepsilon}{M_k}. \tag{3.28}$$

Since $v_k \to v$ uniformly on compact sets and $M_k \to +\infty$, then $z_k(x) \to +\infty$ uniformly on compact sets, so the behaviour at ∞ of the term $\frac{f'(z_k(x))}{f(z_k(x))}$ is determined, roughly speaking, on how far is the nonlinearity f from being critical. As a result, we split our analysis according to whether f is subcritical or critical in the sense of Definition 3.1.3.

3.3.1 The subcritical case

In this framework, besides our standard assumptions on h, we further assume

$$\lim_{t\to+\infty} \frac{f'(t)}{f(t)} = 0$$

according to Definition 3.1.3. Therefore, taking the limit as $k \to +\infty$ in (3.27) and (3.28), we find

$$(1-\varepsilon)a(x_\infty) \leq \Delta^2 v \leq (1+\varepsilon)a(x_\infty) \qquad \text{in } \mathbb{R}^4,$$

which implies, by the arbitrariness of $\varepsilon > 0$:

$$\Delta^2 v = a(x_\infty) \qquad \text{in } \mathbb{R}^4.$$

Incidentally notice that $a(x_\infty) \neq 0$ since $x_\infty \in \Omega$ and here $a > 0$. Nevertheless, since this equation is satisfied by the limit profile v, we can quite easily deduce a contradiction. Indeed, we have the following chain of inequalities, which is a consequence of the Taylor expansion already used, Fatou Lemma, (3.5) with $\varepsilon = \frac{1}{2}$ and finally Proposition 3.2.1:

$$\begin{aligned}
+\infty = \int_{\mathbb{R}^4} a(x_\infty) &= \int_{\mathbb{R}^4} \lim_{k\to+\infty} a(x_k+\mu_k x)e^{\frac{f'(z_k(x))}{f(z_k(x))}v_k(x)}\chi_{\Omega_k}(x)dx \\
&= \int_{\mathbb{R}^4} \lim_{k\to+\infty} a(x_k+\mu_k x)e^{\log(f(u_k(x_k+\mu_k x)))-\log(f(M_k))}\chi_{\Omega_k}(x)dx \\
&\leq 2\liminf_{k\to+\infty} \int_{\Omega_k} \frac{[\frac{1}{2}a(x_k+\mu_k x)f(u_k(x_k+\mu_k x)) - d] + d}{f(M_k)}dx \\
&\leq 2\liminf_{k\to+\infty} \left[\int_{\Omega_k} \frac{h(x_k+\mu_k x, u_k(x_k+\mu_k x))}{f(M_k)}dx + d\int_{\Omega_k} \frac{dx}{f(M_k)}\right] \\
&\overset{[y=x_k+\mu_k x]}{=} 2\liminf_{k\to+\infty} \left[\int_\Omega h(y,u_k(y))dy + d|\Omega|\right] \leq 2[\Lambda + d|\Omega|],
\end{aligned}$$

where the last inequality is due to Proposition 3.2.1. This contradiction proves Theorem 3.1.2.

3.3.2 The critical case

Let us come back to inequalities (3.27) and (3.28), recalling that in this case we are assuming

$$\lim_{t\to+\infty}\frac{f'(t)}{f(t)} =: \beta \in (0,+\infty).$$

This time, once we take the limit as $k \to +\infty$ in both of them, and again recalling that ε is arbitrary, we find

$$\Delta^2 v = a(x_\infty)e^{\beta v} \qquad \text{in } \mathbb{R}^4, \tag{*}$$

where $a(x_\infty) > 0$ as before. Hence, the chain of inequalities used in Section 3.3.1 does not yield to a contradiction, so we have to bring further our investigation: we will see that the key point will be the characterization of the limit profile of the rescaled functions and a Pohozaev-type identity. Nevertheless, with exactly the same computations, one may infer that the limit profile v has finite energy:

Lemma 3.3.2. $\int_{\mathbb{R}^4} e^{\beta v} < +\infty$.

Proof. As in the proof of Theorem 3.1.2:

$$\begin{aligned} a(x_\infty)\int_{\mathbb{R}^4} e^{\beta v} &= \int_{\mathbb{R}^4}\lim_{k\to+\infty} a(x_k+\mu_k x)e^{\frac{f'(z_k(x))}{f(z_k(x))}v_k(x)}\chi_{\Omega_k}(x)dx \\ &\le 2\liminf_{k\to+\infty}\left[\int_\Omega h(y,u_k(y))dy + d|\Omega|\right] \le 2[\Lambda + d|\Omega|]. \end{aligned}$$

$\square$

Since v solves equation (*), in order to characterize it precisely, we need some further information about its growth at ∞.

Lemma 3.3.3 ([67], Lemma 4)**.** *For all $i = 1, 2, 3$ and $p \in [1, \frac{4}{i})$, there exists a constant $C(i, p) > 0$ such that $\|\nabla^i u_k\|^p_{L^p(B_r(x_0))} \le Cr^{4-ip}$ for any $B_r(x_0) \subset \Omega$.*

Proof. By the Green representation formula and Lemma 3.2.2 we have

$$|\nabla^i u_k(x)| \le \int_\Omega |\nabla^i_x G\Omega(x,y)|h(y,u_k(y))dy \le C\int_\Omega \frac{1}{|x-y|^i}h(y,u_k(y))dy.$$

Thus, for any $\varphi \in C^\infty_c(B_r(x_0))$ and p' being the conjugate exponent of p, we have

$$\begin{aligned} \int_{B_r(x_0)} |\nabla^i u_k(x)||\varphi(x)|dx &\le \int_{B_r(x_0)}\left(\int_\Omega |\nabla^i_x G_\Omega(x,y)|h(y,u_k(y))dy\right)|\varphi(x)|dx \\ &\le C\int_\Omega\left(h(y,u_k(y))\int_{B_r(x_0)}|x-y|^{-i}|\varphi(x)|dx\right)dy \\ &\le C\int_\Omega h(y,u_k(y))\||x-y|^{-i}\|_{L^p(B_r(x_0))}\|\varphi\|_{L^{p'}(B_r(x_0))}dy \\ &\le \Lambda r^{4-ip}\|\varphi\|_{L^{p'}(B_r(x_0))}, \end{aligned}$$

using Proposition 3.2.1 and the boundedness of Ω. By duality, this yields our claim.

$\square$

Lemma 3.3.4. $v(x) = o(|x|^2)$ *as* $|x| \to +\infty$.

Proof. Firstly, by (3.22) and Lemma 3.3.3 with $i = 2$ and $p = 1$, there holds

$$\begin{aligned}\int_{B_R(0)} |\Delta v_k| = \mu_k^2 \int_{B_R(0)} |\Delta u_k(x_k + \mu_k x)| dx = \mu_k^{-2} \int_{B_{R\mu_k}(x_k)} |\Delta u_k| \\ \leq \mu_k^{-2} C(R\mu_k)^{4-2} = CR^2.\end{aligned} \tag{3.29}$$

Suppose now by contradiction that $v(x) = o(|x|^2)$ as $|x| \to +\infty$ does not hold. By Lin [60] we would infer that there exists $b > 0$ such that $-\Delta v(x) \geq b$ for every $x \in \mathbb{R}^4$. This, combined with (3.29) and Fatou's Lemma, would imply

$$CbR^4 \leq \int_{B_R(0)} |\Delta v| \leq \liminf_{k\to+\infty} \int_{B_R(0)} |\Delta v_k| \leq CR^2,$$

which contradicts the arbitrariness of $R > 0$. This proves our claim. □

We can now apply a Liouville-type result by Martinazzi (see Theorem A.0.6 in the Appendix) and determine explicitly v:

Lemma 3.3.5. $v(x) = -c_1 \log(1 + c_2|x|^2)$ *for some* $c_i = c_i(\beta, a(x_\infty))$, $i = 1, 2$. *Moreover,*

$$\lim_{R\to+\infty} \liminf_{k\to+\infty} \int_{B_{R\mu_k}(x_k)} h(y, u_k(y)) dy \geq \theta > 0. \tag{3.30}$$

Proof. Parameters in the explicit formula of v may be found by standard computations from Theorem A.0.6 with $x_0 = 0$ (in the notation therein) since $v(0) = 0$. Moreover, arguing as in Section 3.3.1, we have

$$\begin{aligned}0 < 2\theta := a(x_\infty) \int_{\mathbb{R}^4} e^{\beta v} &= \lim_{R\to+\infty} \int_{B_R(0)} \lim_{k\to+\infty} a(x_k + \mu_k x) e^{\left(\frac{f'(z_k(x))}{f(z_k(x))}\right) v_k(x)} dx \\ &\leq \lim_{R\to+\infty} \liminf_{k\to+\infty} \int_{B_R(0)} a(x_k + \mu_k x) e^{\left(\frac{f'(z_k(x))}{f(z_k(x))}\right) v_k(x)} dx \\ &= 2 \lim_{R\to+\infty} \liminf_{k\to+\infty} \left[\int_{B_R(0)} \frac{\frac{1}{2} a(x_k + \mu_k x) f(u_k(x_k + \mu_k x) - d)}{f(M_k)} dx + \frac{d}{f(M_k)} |B_R(0)| \right] \\ &\leq 2 \lim_{R\to+\infty} \liminf_{k\to+\infty} \int_{B_{R\mu_k}(x_k)} h(y, u_k(y)) dy.\end{aligned}$$

□

So far, we have investigated the behaviour of each u_k around one maximum point x_k. This is indeed what happens for each sequence of blow-up points, as stated in the next result:

Lemma 3.3.6. *There are* $N > 0$ *and converging sequences* $x_{k,i} \to x^{(i)}$, $1 \leq i \leq N$, *with* $\lim_{k\to+\infty} u_k(x_{k,i}) = +\infty$ *such that, setting*

$$v_{k,i}(x) := u_k(x_{k,i} + \mu_{k,i} x) - u_k(x_{k,i}), \qquad \mu_{k,i} := (f(u_k(x_{k,i})))^{-1/4},$$

we have

(i) $\lim_{k\to+\infty} \frac{|x_{k,i}-x_{k,j}|}{\mu_{k,i}} = +\infty$ *for* $1 \le i \ne j \le N$*;*

(ii) $v_{k,i} \to v$ *in* $C^{3,\gamma}_{loc}(\mathbb{R}^4)$*, for* $1 \le i \le N$*, where* v *is defined in Lemma 3.3.5 and estimate* (3.30) *still holds;*

(iii) $\inf_{1\le i\le N} |x - x_{k,i}|^4 h(x, u_k(x)) \le C$ *for every* $x \in \Omega$*;*

(iv) $\inf_{1\le i\le N} |x - x_{k,i}|^j |\nabla^j u_k(x)| \le C$ *for every* $x \in \Omega$ *and* $1 \le j \le 4$*.*

Proof. The proof mainly follows the arguments of [81, Claims 5-7] (see also [67, Lemmas 7-8]) with some modifications as in our Lemmas 3.3.1-3.3.5.
We say that the *property* $\mathcal{H}_p$ holds if there exist p sequences of blow-up points, that is, $(x_{k,i})_{i=1}^p \subset \Omega$ such that *(i-ii)* hold. Notice that Lemmas 3.3.1-3.3.5 imply that $\mathcal{H}_1$ holds. We first show the following alternative:
Claim 1: *Supposing property* $\mathcal{H}_p$ *holds for some* $p \in \mathbb{N}\setminus\{0\}$*, then either* $\mathcal{H}_{p+1}$ *holds or there exists a constant* C *independent of* k *such that*

$$\inf_{1\le i\le p} |x - x_{k,i}|^4 h(x, u_k(x)) \le C \qquad \text{for any } x \in \Omega. \tag{3.31}$$

Let $w_k(x) := \inf_{1\le i\le p} |x - x_{k,i}|^4 h(x, u_k(x))$ and suppose the uniform bound (3.31) does not hold, i.e. $\|w_k\|_\infty \to +\infty$ and denote by $(y_k)_k \subset \Omega$ the maximum points. Our aim is to prove that such points are the ones that, together with $(x_{k,i})_{i=1}^p$, verify property $\mathcal{H}_{p+1}$. Define $\gamma_k := f(u_k(y_k))^{-1/4}$ and let

$$\tilde{u}_k(x) := u_k(y_k + \gamma_k x) - u_k(y_k).$$

Firstly, $\frac{|y_k - x_{k,i}|}{\gamma_k} \to +\infty$ for any i. Indeed, by (3.5)

$$\begin{aligned} +\infty \leftarrow w_k(y_k) &\le \inf_{i=1,\cdots,p} |x - x_{k,i}|^4 (2\|a\|_\infty f(u_k(y_k)) + d) \\ &\le 2\|a\|_\infty \inf_{1\le i\le p} \left(\frac{|y_k - x_{k,i}|}{\gamma_k}\right)^4 + d(diam(\Omega))^4. \end{aligned} \tag{3.32}$$

Hence, *(i)* is proved once we show also $\frac{|y_k - x_{k,i}|}{\mu_{k,i}} \to +\infty$ for any i. Suppose by contradiction that $y_k - x_{k,i} = O(\mu_{k,i})$, that is, $y_k = x_{k,i} + \mu_{k,i}\theta_{k,i}$ for some $|\theta_{k,i}| < C$. Notice that $(y_k)_k$, being a blow-up sequence for $(w_k)_k$, is a blow-up sequence for $(u_k)_k$ too. Indeed, with the same computations as in (3.32), one infers $f(u_k(y_k)) \to \infty$, which implies $u_k(y_k) \to \infty$. Hence,

$$\begin{aligned} w_k(y_k) &= |y_k - x_{k,i}|^4 h(y_k, u_k(y_k)) = \mu_{k,i}^4 |\theta_{k,i}|^4 h(y_k, u_k(y_k)) \\ &\le \mu_{k,i}^4 |\theta_{k,i}|^4 (2a(y_k) f(u_k(y_k)) + d) \\ &= |\theta_{k,i}|^4 (2a(x_{k,i} + \mu_{k,i}\theta_{k,i}) e^{\log(f(u_k(x_{k,i}+\mu_{k,i}\theta_{k,i}))) - \log(f(u_k(x_{k,i})))} + \tfrac{d}{f(u_k(x_{k,i}))}). \end{aligned}$$

Since $\log(f(u_k(x_{k,i}+\mu_{k,i}\theta_{k,i}))) = \log(f(u_k(x_{k,i}))) + \frac{f'(z_{k,i}(\theta_{k,i}))}{f(z_{k,i}(\theta_{k,i}))} v_{k,i}(\theta_{k,i})$, where $z_{k,i}(\theta_{k,i}) := u_k(x_{k,i}) + t(u_k(x_{k,i} + \mu_{k,i}\theta_{k,i}) - u_k(x_{k,i})) = u_k(x_{k,i}) + t v_{k,i}(\theta_{k,i})$, for some $t \in (0,1)$, then

$$\begin{aligned} &= |\theta_{k,i}|^4 \left(2a(x_{k,i} + \mu_{k,i}\theta_{k,i}) e^{\frac{f'(z_{k,i}(\theta_{k,i}))}{f(z_{k,i}(\theta_{k,i}))} v_{k,i}(\theta_{k,i})} + \frac{d}{f(u_k(x_{k,i}))} \right) \\ &\to 2|\theta_{\infty,i}|^4 a(x_{\infty,i}) e^{\beta v(\theta_{\infty,i})}, \end{aligned} \tag{3.33}$$

In fact, $u_k(x_{k,i}) \to +\infty$, while $v_{k,i}(\theta_{k,i})$ stays bounded by the same computations as in Lemma 3.3.1 ($\theta_{k,i}$ being bounded), which yields $z_{k,i} \to +\infty$ too and (3.33) follows by assumption (*A3*). Moreover, we have an explicit profile of v by Lemma 3.3.5, so we finally have

$$\limsup_{k\to+\infty} w_k(y_k) \leq 2|\theta_{\infty,i}|^4 \frac{a(x_{\infty,i})}{(1+c_2|\theta_{\infty,i}|^2)^{c_1\beta}} < \infty,$$

which contradicts $w_k(y_k) \to +\infty$. Hence *(i)* is proved.

In order to prove *(ii)*, we have to show that the sequence $(\tilde{u}_k)_k$ is uniformly bounded. Nevertheless, in contrast of what happened for the blow-up sequence $(v_k)_k$ defined in (3.22), for which $0 \leq \frac{f(u_k(x_k+\mu_k x))}{f(M_k)} \leq 1$ (x_k being as in (3.21)), here this term may be unbounded, and so $\Delta^2\tilde{u}_k$, losing the boundedness required by Lemma A.0.4. We now show that this cannot happen. To this aim, let $R>0$ and $\varepsilon \in (0,1)$ be fixed and $x \in B_R(0)$. We have $w_k(y_k+\gamma_k x) \leq w_k(y_k)$ which, rewritten, is

$$\frac{h(y_k+\gamma_k x, u_k(y_k+\gamma_k x))}{h(y_k, u_k(y_k))} \leq \left(\frac{\inf_{1\leq i\leq p}|y_k - x_{k,i}|}{\inf_{1\leq i\leq p}|y_k+\gamma_k x - x_{k,i}|}\right)^4.$$

Since $\frac{|y_k-x_{k,i}|}{\gamma_k} \to +\infty$, there exists $\bar{k}(R,\varepsilon) > 0$ such that for any $k \geq \bar{k}$ we have $\gamma_k R \leq \varepsilon|y_k - x_{k,i}|$. Hence, $|y_k+\gamma_k x - x_{k,i}| \geq |y_k - x_{k,i}| - \gamma_k R \geq (1-\varepsilon)|y_k - x_{k,i}|$ and thus,

$$\frac{h(y_k+\gamma_k x, u_k(y_k+\gamma_k x))}{h(y_k, u_k(y_k))} \leq \frac{1}{(1-\varepsilon)^4}. \tag{3.34}$$

Therefore, by (3.34) and (3.5), we have

$$\begin{aligned}
\frac{1}{(1-\varepsilon)^4} &\geq \frac{\frac{1}{2}a(y_k+\gamma_k x)f(u_k(y_k+\gamma_k x)) - d}{2a(y_k)f(u_k(y_k))) + d} \\
&\geq \frac{\frac{1}{4}a(y_\infty)f(u_k(y_k+\gamma_k x))}{2\|a\|_\infty f(u_k(y_k))) + d} - \frac{d}{2\|a\|_\infty f(u_k(y_k)) + d} \\
&\geq \frac{1}{b}\frac{f(u_k(y_k+\gamma_k x))}{f(u_k(y_k))} - C,
\end{aligned}$$

for some $b, C > 0$, since $f(u_k(y_k)) \to +\infty$. This means $\frac{f(u_k(y_k+\gamma_k x)))}{f(u_k(y_k))} \leq \tilde{C}$ and, in turn, together with (3.24), that $(\Delta^2\tilde{u}_k)_k$ is uniformly bounded. This leads to the compactness of $(\tilde{u}_k)_k$ in $W^{4,q}(\Omega)$ for any $q \geq 1$ and with the same arguments as in Lemmas 3.3.1-3.3.5, we obtain *(ii)*. This completes the proof of Claim 1.

Claim 2: *There exists $N \geq 1$ such that both property $\mathcal{H}_N$ and* (3.31) *hold.* Suppose by contradiction that the property $\mathcal{H}_p$ holds for any $p \in \mathbb{N}$ fixed. Hence, by (*i-ii*) for any $k \in \mathbb{N}$ one can find p points $(x_{k,i})_{i=1}^p$ and disjoint balls $B_{R\mu_{k,i}}(x_{k,i})$ such that (3.30) holds. Thus,

$$\begin{aligned}
\Lambda \geq \int_\Omega h(x,u_k(x))dx &\geq \int_{\bigcup_{i=1}^p B_{R\mu_{k,i}}(x_{k,i})} h(x,u_k(x))dx \\
&= \sum_{i=1}^p \int_{B_{R\mu_{k,i}}(x_{k,i})} h(x,u_k(x))dx \geq p\theta,
\end{aligned}$$

an upper bound for p, a contradiction.

Finally, we have to prove *(iv)*. By the Green representation formula and the estimates of Lemma 3.2.2, we have

$$|\nabla^j u_k(x)| \le C \int_\Omega |x-y|^{-j} h(y, u_k(y)) dy.$$

For any $k \in \mathbb{N}$ and $i \in \{1, \cdots, N\}$, define $\Omega_{k,i} := \{x \in \Omega \,|\, |x - x_{k,i}| = \inf_{1\le i\le N} |x - x_{k,i}|\}$, the set containing all points in Ω which are nearer to the blow-up point $x_{k,i}$, and moreover $B_{k,i} := B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})$. Decompose $\Omega = \cup_i((\Omega_{k,i} \cap B_{k,i}) \cup (\Omega_{k,i} \setminus B_{k,i}))$ and, consequently,

$$|\nabla^j u_k(x)| \le C \sum_{i=1}^N \left(\int_{\Omega_{k,i}\cap B_{k,i}} \frac{h(y, u_k(y))}{|x-y|^j} dy + \int_{\Omega_{k,i}\setminus B_{k,i}} \frac{h(y, u_k(y))}{|x-y|^j} dy \right). \tag{3.35}$$

Notice that $|x - y| \ge \frac{1}{2}|x - x_{k,i}|$ for any $y \in \Omega_{k,i} \cap B_{k,i}$, while on $\Omega_{k,i} \setminus B_{k,i}$ we apply inequality *(iii)*, obtaining from (3.35) and by Proposition 3.2.1,

$$|\nabla^j u_k(x)| \le \frac{2^j C \Lambda N}{|x - x_{k,i}|^j} + C \sum_{i=1}^N \int_{\Omega_{k,i}\setminus B_{k,i}} \frac{dy}{|x-y|^j |y - x_{k,i}|^4}. \tag{3.36}$$

To estimate the second term, we decompose $\Omega_{k,i} \setminus B_{k,i} = \Omega^{(1)}_{k,i} \cup \Omega^{(2)}_{k,i}$, where

$$\Omega^{(1)}_{k,i} := (\Omega_{k,i} \setminus B_{k,i}) \cap B_{2|x-x_{k,i}|}(x) \quad \text{and} \quad \Omega^{(2)}_{k,i} := (\Omega_{k,i} \setminus B_{k,i}) \setminus B_{2|x-x_{k,i}|}(x).$$

Then, $|y - x_{k,i}| \ge \frac{1}{2}|x - x_{k,i}|$ for any $y \in \Omega^{(1)}_{k,i}$ since it gathers points outside $B_{k,i}$; moreover, $\Omega^{(1)}_{k,i} \subset B_{2|x-x_{k,i}|}(x)$, so

$$\begin{aligned} \int_{\Omega^{(1)}_{k,i}} \frac{dy}{|x-y|^j |y-x_{k,i}|^4} &\le \frac{C}{|x-x_{k,i}|^4} \int_{B_{2|x-x_{k,i}|}(x)} \frac{dy}{|x-y|^j} \\ &= \frac{C}{|x-x_{k,i}|^4} \int_0^{2|x-x_{k,i}|} \rho^{3-j} d\rho \le \frac{C}{|x-x_{k,i}|^j}. \end{aligned} \tag{3.37}$$

On the other hand, for any $y \in \Omega^{(2)}_{k,i}$, there holds $|y - x| \le |y - x_{k,i}| + |x - x_{k,i}| \le \frac{3}{2}|y - x_{k,i}|$, thus

$$\begin{aligned} \int_{\Omega^{(2)}_{k,i}} \frac{dy}{|x-y|^j |y-x_{k,i}|^4} &\le C \int_{(B_{\frac{|x-x_{k,i}|}{2}}(x))^c} \frac{dy}{|y-x_{k,i}|^{4+j}} \\ &\le C \int_{\frac{|x-x_{k,i}|}{2}}^{+\infty} \rho^{-j} d\rho \le \frac{C}{|x-x_{k,i}|^j}. \end{aligned} \tag{3.38}$$

(iv) is finally obtained by (3.36), (3.37) and (3.38). □

Denote by S the set of blow-up points, that is, $S := \{y \,|\, \exists y_k \to y, u_k(y_k) \to +\infty\}$. Lemma 3.3.6 has two important consequences. First, S coincides with the set $\{x^{(i)}, 1 \le i \le N\}$, and therefore, S is finite. In fact, suppose by contradiction that there exist $\bar{x} \notin (x^{(i)})_{i=1}^N$ and a sequence $\bar{x}_k \to \bar{x}$ such that $u_k(\bar{x}_k) \to +\infty$. Since $N < \infty$, one has $\inf_{k,i} |\bar{x}_k - x^{(i)}| \geq \bar{d} > 0$. Notice also that $d(\bar{x}, \partial\Omega) \geq \eta > 0$ by Proposition 3.2.1, so $a(\bar{x}_k) \geq a_0 > 0$. Hence, by *(iii)* of Lemma 3.3.6 and (3.5) with $\varepsilon = \frac{1}{2}$, we get

$$\frac{C}{\bar{d}} \geq h(\bar{x}_k, u_k(\bar{x}_k)) \geq \frac{1}{2} a_0 f(u_k(\bar{x}_k)) - d,$$

which in turn implies $u_k(\bar{x}_k) \leq C$ by the superlinearity of f.

Moreover, we also deduce a local boundedness of $(u_k)_k$ outside S:

$$\|u_k\|_{W^{4,\infty}_{loc}(\overline{\Omega}\setminus S)} \leq C. \tag{3.39}$$

Indeed, let $K \subset\subset \overline{\Omega} \setminus S$ and $r > 0$ such that $K \cap B_r(x^{(i)}) = \emptyset$ for each $x^{(i)} \in S$. Firstly, $|u_k| \leq C$ since there are no blow-up points in K; moreover, $\inf_{1\leq i\leq N} |x - x_{k,i}| \geq \frac{r}{2}$ by construction, so by *(iv)* we infer $|\nabla^j u_k(x)| \leq C(r)$ for any $j \in \{1, \cdots, 4\}$.

In order to conclude the proof of Theorem 3.1.3, we need a Pohožaev identity which can be found in [68] (see also [81, Lemma 2.2]). For the sake of completeness, we sketch its proof.

Lemma 3.3.7. *Let $u \in H^4(\Omega)$ be a strong solution of $\Delta^2 u = h(x, u)$ in Ω. Then, for any $y \in \mathbb{R}^4$, we have*

$$4\int_\Omega H(x,u)dx + \int_\Omega \langle x - y, \nabla_x H(x,u)\rangle dx = \int_{\partial\Omega} \langle x - y, n(x)\rangle H(x,u) d\sigma + b(y,u),$$

where H is defined in (3.8) and b collects all remaining boundary terms:

$$\begin{aligned} b(y,u) := &\frac{1}{2}\int_{\partial\Omega} (\Delta u)^2 \langle x - y, n(x)\rangle d\sigma - 2\int_{\partial\Omega} u_n \Delta u d\sigma - \int_{\partial\Omega} (\Delta u)_n \langle x - y, \nabla u\rangle d\sigma \\ &- \int_{\partial\Omega} u_n \langle x - y, \nabla(\Delta u)\rangle d\sigma + \int_{\partial\Omega} \langle \nabla(\Delta u), \nabla u\rangle \langle x - y, n(x)\rangle d\sigma \end{aligned}$$

Sketch of the Proof. The following identities may be proved with standard and tedious computations:

$$\begin{aligned} &\mathrm{div}[(x - y, \nabla\Delta u)\nabla u + (x - y, \nabla u)\nabla\Delta u - (\nabla u, \nabla\Delta u)(x - y)] \\ &= (x - y, \nabla\Delta u)\Delta u + (x - y, \nabla u)\Delta^2 u - 2(\nabla u, \nabla\Delta u); \end{aligned} \tag{3.40}$$

$$\mathrm{div}\left[\tfrac{1}{2}(\Delta u)^2 (x - y) - 2\Delta u \nabla u\right] = \Delta u(\nabla\Delta u, x - y) - 2(\nabla\Delta u); \tag{3.41}$$

$$\begin{aligned} \mathrm{div}[(x - y)H(x,u)] &= 4H(x,u) + (x - y, \nabla_x H(x,u)) + (x - y, \nabla_t H(x,u)) \\ &= 4H(x,u) + (x - y, \nabla_x H(x,u)) + (x - y, \nabla u)h(x,u). \end{aligned} \tag{3.42}$$

Applying the Divergence Theorem to (3.40) one has

$$\begin{aligned} &\int_{\partial\Omega} (\Delta u)_n \langle x - y, \nabla u\rangle dx + \int_{\partial\Omega} u_n \langle x - y, \nabla(\Delta u)\rangle dx - \int_{\partial\Omega} \langle \nabla(\Delta u), \nabla u\rangle \langle x - y, n(x)\rangle dx \\ &= \int_\Omega (x - y, \nabla\Delta u)\Delta u + \int_\Omega (x - y, \nabla u)\Delta^2 u - 2\int_\Omega (\nabla u, \nabla\Delta u) \\ &= \int_\Omega (x - y, \nabla\Delta u)\Delta u - 2\int_\Omega (\nabla u, \nabla\Delta u) + \int_\Omega (x - y, \nabla u)h(x,u). \end{aligned}$$

Once we apply (3.41) to the first two terms and (3.42) to the last one and in both the Divergence Theorem again, the proof is completed. □

Proof of Theorem 3.1.3. Let $x_0 \in S$ and $r > 0$ be sufficiently small such that $B_r(x_0) \cap S = \{x_0\}$ and apply the identity of Lemma 3.3.7 in $B_r(x_0)$ with $y = x_0$ and $u = u_k$ which belongs to $H^4(\Omega)$ by elliptic regularity:

$$
\begin{aligned}
&4\int_{B_r(x_0)} H(x,u_k)dx + \int_{B_r(x_0)} \langle x-y, \nabla_x H(x,u_k)\rangle dx \\
&= \int_{\partial B_r(x_0)} \langle x-x_0, n(x)\rangle H(x,u_k)d\sigma + \frac{1}{2}\int_{\partial B_r(x_0)} (\Delta u_k)^2 \langle x-x_0, n(x)\rangle d\sigma \\
&- 2\int_{\partial B_r(x_0)} (u_k)_n \Delta u_k d\sigma - \int_{\partial B_r(x_0)} (\Delta u_k)_n \langle x-x_0, \nabla u_k\rangle d\sigma \\
&- \int_{\partial B_r(x_0)} (u_k)_n \langle x-x_0, \nabla(\Delta u_k)\rangle d\sigma + \int_{\partial B_r(x_0)} \langle \nabla(\Delta u_k), \nabla u_k\rangle \langle x-x_0, n(x)\rangle d\sigma.
\end{aligned}
$$

Since F, defined in (3.8), is continuous in $\mathbb{R}^+$ and $\partial B_r(x_0) \subset\subset \overline{\Omega} \setminus S$, (3.39) and (3.9) imply

$$
\int_{\partial B_r(x_0)} \langle x-x_0, n(x)\rangle H(x,u_k)d\sigma \le 2r\int_{\partial B_r(x_0)} (a(x)F(u_k)+d)d\sigma + d|\partial B_r(x_0)| = o(r).
$$

Hence, applying (3.39) on the boundary terms in the right-hand side,

$$
4\int_{B_r(x_0)} H(x,u_k)dx + \int_{B_r(x_0)} \langle x-y, \nabla_x H(x,u_k)\rangle dx = o(r). \tag{3.43}
$$

We want to bound from below the left-hand side of (3.43) with a positive constant independent of r, to get the contradiction. Let $0 < \varepsilon < \frac{1}{2}$ arbitrary; (3.9) and the assumption (H3) imply

$$
\begin{aligned}
&4\int_{B_r(x_0)} H(x,u_k)dx + \int_{B_r(x_0)} \langle x-y, \nabla_x H(x,u_k)\rangle dx \\
&\ge 4(1-\varepsilon)\int_{B_r(x_0)} a(x)F(u_k)dx - 4d_\varepsilon |B_r(x_0)| - \int_{B_r(x_0)} rB(x)F(u_k) - r\int_{B_r(x_0)} D(x) \\
&\ge \left(4(1-\varepsilon)\frac{a(x_0)}{2} - r\|B\|_\infty\right)\int_{B_r(x_0)} F(u_k)dx - d_\varepsilon C r^4 - r\int_{B_r(x_0)} D(x),
\end{aligned}
$$

when $r > 0$ is so small that $a(x) > \frac{a(x_0)}{2}$ for any $x \in B_r(x_0)$. Supposing further, up to a smaller value of r that $r\|B\|_\infty \le \frac{a(x_0)}{2}$, we find

$$
4\int_{B_r(x_0)} H(x,u_k)dx + \int_{B_r(x_0)} \langle x-y, \nabla_x H(x,u_k)\rangle dx + o(r) \ge \frac{a(x_0)}{2}\int_{B_r(x_0)} F(u_k)dx. \tag{3.44}
$$

We now claim there exists $m \ge 1$ such that

$$
F(t) \ge \frac{1}{m} f(t) \qquad \text{for any } t \ge 0. \tag{3.45}
$$

Indeed, let us define

$$m := \max\left\{1, \max_{t\geq 0}\frac{f'(t)}{f(t)}\right\}.$$

Then $1 \leq m < +\infty$ since $0 \leq f \in C^1([0,+\infty))$ by $(A1)$, together with the limit assumption $(A3)$. Hence

$$\left(F - \frac{1}{m}f\right)'(t) = f(t) - \frac{1}{m}f'(t) = f(t)\left(1 - \frac{1}{m}\frac{f'(t)}{f(t)}\right) \geq 0,$$

$$\left(F - \frac{1}{m}f\right)(0) = f(0)\left(1 - \frac{1}{m}\right) \geq 0,$$

which together imply (3.45). Finally, we further estimate from below the left-hand side of (3.44), by (3.45) and Lemma 3.3.5, obtaining

$$\begin{aligned}
\frac{a(x_0)}{2}\int_{B_r(x_0)} F(u_k) &\geq \frac{a(x_0)}{2m}\int_{B_r(x_0)} f(u_k) \\
&\geq \frac{a(x_0)}{4m\|a\|_\infty}\left[\int_{B_r(x_0)} (2a(x)f(u_k) + d) - d|B_r(x_0)|\right] \\
&\geq \frac{a(x_0)}{4m\|a\|_\infty}\left[\int_{B_r(x_0)} h(x, u_k(x))dx - d|B_r(x_0)|\right] \\
&\geq \frac{a(x_0)}{4m\|a\|_\infty}\frac{\theta}{4} + o(r),
\end{aligned}$$

This contradicts (3.43). As a consequence, we deduce $S = \emptyset$ and thus the boundedness of $(u_k)_{k\in\mathbb{N}}$ in $C^4(\overline{\Omega})$. $\square$

3.4 Some extensions of Theorems 3.1.2 and 3.1.3

The uniform a-priori estimates obtained in Theorems 3.1.2-3.1.3 deal with a large range of nonlinearities (the basic requirements being positivity, superlinearity and a control at ∞ by an exponential map), but they apply uniquely for the case of the ball. In this section we present a sufficient condition in the spirit of Brezis-Merle [13, Open problem 2] which enables us to generalize our results to smooth domains. Finally, we show that the present analysis can be similarly carried out also in the polyharmonic framework.

3.4.1 Extension to general smooth domains

The restrictions to the extension of Theorems 3.1.2 and 3.1.3 to more general domains are mainly two. Firstly, unless the domain is positivity preserving, we cannot guarantee that solutions of (3.1), as well as the first eigenfunction $\tilde{\varphi}_1$, are positive, and all estimates of Section 3.2 rely on this fact. Secondly, the two-sided estimate (3.12) is available only for balls, so we are able to conclude the argument for the boundary estimate in Lemma 3.2.7 only in this case. Nevertheless, once we have $\int_\Omega h(x,u) \leq C$ available, a careful reading of Sections 3.3.1 and 3.3.2 shows that the blow-up argument applies for *any* domain, provided the rescaled domains expand to cover all $\mathbb{R}^4$. The following result shows indeed that, once the uniform bound on the right-hand side is satisfied, the uniform bound can be obtained.

Theorem 3.4.1. *Let $\Omega \subset \mathbb{R}^4$ be a bounded $C^{4,\gamma}$ smooth domain, $\gamma \in (0,1)$ and h satisfy (H1)-(H2) with $0 < a_0 \leq a(\cdot) \in C(\overline{\Omega})$. Suppose one of the following conditions holds:*

1) h is subcritical as specified in (3.6);

2) h is critical as specified in (3.7) and (H3) holds in Ω.

Furthermore, suppose there exists $\Lambda > 0$ such that

$$\int_\Omega h(x,u)dx \leq \Lambda$$

for all weak solutions of (3.1). Then there exists $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for all weak solutions of (3.1). In addition, if (2) holds, solutions are also bounded in $C^4(\overline{\Omega})$.

Notice that, since the blow-up points may concentrate to $x_\infty \in \partial\Omega$, we have to impose strict positivity on the coefficient a to be sure that $a(x_\infty) > 0$. In both cases, the argument is essentially the one contained in Sections 3.3.1 and 3.3.2, up to some preliminary verifications.

Proof. By contradiction, suppose the existence of points $(x_k)_{k\in\mathbb{N}} \subset \Omega$ and solutions $(u_k)_{k\in\mathbb{N}} \subset H_0^2(\Omega)$ such that $M_k := \|u_k\|_{L^\infty(\Omega)} = u_k(x_k) \nearrow +\infty$. We define Ω_k and μ_k as in (3.23) as well as the rescaled functions v_k as in (3.22). Since Ω is bounded, the maximum points x_k accumulate to some $x_\infty \in \overline{\Omega}$. We claim that, in any case, $\lim_{k\to+\infty} \frac{d(x_k,\partial\Omega)}{\mu_k} = +\infty$, so that $\Omega_k \nearrow \mathbb{R}^4$. Indeed, suppose by contradiction that $d(x_k, \partial\Omega) = O(\mu_k)$, that is, up to an affine transformation, $\Omega_k \to (-\infty, 0) \times \mathbb{R}^3$. Letting $R > 0$ and $x \in B_R(0) \cap \overline{\Omega}_k$, with the same computations of Lemma 3.3.1, we infer $|\nabla^i v_k| \leq C(R)$ for any $x \in B_R(0) \cap \overline{\Omega}_k$. Choosing $x \in B_R(0) \cap \partial\Omega_k$, so that $v_k(x) = -M_k$, we would get

$$M_k = |v_k(x)| = |v_k(x) - v_k(0)| \leq CR,$$

a contradiction. For the subcritical case *(1)*, now it is enough to repeat the same compactness arguments as well as the contradiction provided in Section 3.3.1, to infer the a-priori bound. On the other hand, if we suppose *(2)*, we have to take care a little bit more the fact that x_∞ may belong to $\partial\Omega$. In particular, once we find that the limit function $v \in C^\infty(\mathbb{R}^4)$ satisfies $\Delta^2 v = a(x_\infty)e^{\beta v}$ in $\mathbb{R}^4$ and the analogues of Lemmas 3.3.2-3.3.6, the argument via the Pohožaev identity is a bit different, since it might happen that $x_\infty \in \partial\Omega$. In this case, we have to consider the identity of Lemma 3.3.7 integrated in $B_r(x_0) \cap \Omega$ with $x_0 \in S \cap \partial\Omega$ (we recall that S is the set of blow-up points), so two kinds of boundary terms appear: the ones relative to $\Omega \cap \partial B_r(x_0)$ and the others to $\partial\Omega \cap B_r(x_0)$. All terms of the second kind vanish by the Dirichlet boundary conditions except for the term $\int_{\partial\Omega\cap B_r(x_0)} (\Delta u_k)^2 \langle x - y, n(x)\rangle d\sigma$. Therefore, we have to choose in a clever way a sequence of points $(y_k)_{k\in\mathbb{N}}$ so that also this term vanishes. Following [81], we define $y_k := x_0 + \rho_{k,r} n(x_0)$, where

$$\rho_{k,r} := \frac{\int_{\partial\Omega\cap B_r(x_0)} (\Delta u_k)^2 \langle x - x_0, n(x)\rangle dx}{\int_{\partial\Omega\cap B_r(x_0)} (\Delta u_k)^2 \langle n(x_0), n(x)\rangle dx}$$

and, up to a smaller value, we may choose r so that $\frac{1}{2} \leq \langle n(x_0), n(x)\rangle \leq 1$ for all $x \in \overline{B}_r(x_0) \cap \Omega$. With these choices, we have $|\rho_{k.r}| \leq 2r$ and

$$\int_{\partial\Omega\cap B_r(x_0)} (\Delta u_k)^2 \langle x - y_k, n(x)\rangle dx = 0.$$

Applying now the identity of Lemma 3.3.7 in $\Omega \cap B_r(x_0)$ with $y = y_k$ and $u = u_k$, one retrieves

$$4\int_{\Omega\cap B_r(x_0)} H(x, u_k)dx + \int_{\Omega\cap B_r(x_0)} \langle x - y, \nabla_x H(x, u_k)\rangle dx = o(r).$$

and the conclusion follows from the same argument as in the Proof of Theorem 3.1.3. $\square$

3.4.2 Extension to the polyharmonic case

By now, we considered our problem (3.1) in dimension 4, the critical dimension for the fourth-order Sobolev inequalities. One may further ask if the same results hold true, once we consider the related Dirichlet problem for the *polyharmonic* operator:

$$\begin{cases} (-\Delta)^m u = h(x, u) & \text{in } \Omega, \\ u = \partial_n u = \cdots = \partial_n^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.46}$$

where $\Omega \subset \mathbb{R}^{2m}$ is a bounded smooth domain (that is, of class $C^{2m,\gamma}$, for some $\gamma \in (0, 1)$), $m \geq 2$. Indeed, $2m$ the critical dimension in the sense of the Trudinger-Moser-Adams inequality. Moreover, we assume h verifies (*H1*)-(*H2*). Herein, we mean by *weak solution* of (3.46) a function $u \in H_0^m(\Omega)$ such that

$$\int_\Omega \nabla^m u \nabla^m \varphi = \int_\Omega h(x, u)\varphi,$$

for every $\varphi \in H_0^m(\Omega)$, with the convention

$$\nabla^m := \begin{cases} \Delta^{m/2}, & m \text{ odd}, \\ \nabla\Delta^{(m-1)/2}, & m \text{ even}. \end{cases}$$

Theorem 3.4.2. *Let $\mathcal{B}$ be a ball in $\mathbb{R}^{2m}$ and h be a nonlinearity satisfying assumptions (H1)-(H2). Suppose moreover that one of the following holds:*

I) h is subcritical as specified in (3.6) *and either (i) or (ii) of Proposition 3.2.1 holds;*

II) h is critical as specified in (3.7) *and* (H3) *holds.*

Then there exists $C > 0$ such that $\|u\|_{L^\infty(\mathcal{B})} \le C$ for all weak solutions of (3.46)*. In addition, if (II) holds, solutions are also bounded in $C^{2m}(\overline{\mathcal{B}})$.*

Proof. An analogue of Proposition 3.2.1 in dimension $2m$ holds, following the same arguments of Section 3.2 with only evident changes. In particular, Lemmas 3.2.4, 3.2.5 and 3.2.6 hold by considering $\lambda_1^{(m)}$ and $\varphi_1^{(m)}$ as respectively the first eigenvalue and the first eigenfunction in Ω of the operator $(-\Delta)^m$ subjected to Dirichlet boundary conditions and since we have the same behaviour of the Green function, that is, it vanishes near the boundary precisely of order m (see [40, Theorem 4.6]):

$$G_{(-\Delta)^m,\mathcal{B}}(x,y) \simeq \log\left(1 + \left(\frac{d_{\mathcal{B}}(x)d_{\mathcal{B}}(y)}{|x-y|^2}\right)^m\right).$$

Then, Lemmas 3.2.7 and 3.2.8 easily follow, since they rely on properties that do not depend on the differential operator. Let us focus now on the blow-up argument. A careful reading of Section 3.3 may convince the reader that all statements are still valid for problem (3.46) once we adapt the scaling as

$$\mu_k^{(m)} := \frac{1}{(f(M_k))^{\frac{1}{2m}}}.$$

In particular, Lemma 3.3.1 holds for any $i \in \{0, 1, \cdots, 2m-1\}$. Hence, again by the local compactness guaranteed by Lemma A.0.4, we find $v \in C^{2m-1}(\mathbb{R}^{2m})$ satisfying

$$(-\Delta)^m v = a(x_\infty)e^{\beta v} \quad \text{in } \mathbb{R}^{2m}. \tag{$*_m$}$$

In the subcritical case *(I)*, one has $\beta = 0$, an the contradiction is found exactly as in Section 3.3.1. On the other hand, in the critical framework *(II)*, that is when $\beta \in (0, +\infty)$, the characterization of entire solutions of equation ($*_m$) again follows by Theorem A.0.6. Consequently, repeating the same steps as in Section 3.3.2, one has the result. It is only worth to mention how the Pohožaev identity modifies to fit in this context: for any $u \in C^{2m}(\overline{\Omega})$ solution of $(-\Delta)^m u = h(x,u)$ in $\Omega \subset \mathbb{R}^{2m}$ and for any $y \in \mathbb{R}^{2m}$, there holds:

$$\begin{aligned}
&2m\int_\Omega H(x,u)dx + \int_\Omega \langle x-y, \nabla_x H(x,u)\rangle dx \\
&= \int_{\partial\Omega} \langle x-y, n(x)\rangle H(x,u)d\sigma + \frac{1}{2}\int_{\partial\Omega} (\nabla^m u)^2 \langle x-y, n(x)\rangle d\sigma \\
&+ \sum_{j=0}^{m-1} (-1)^{m+j} \int_{\partial\Omega} \langle n(x), \nabla^j((x-y)\nabla u)\nabla^{2m-1-j}u(x)\rangle d\sigma.
\end{aligned}$$

□

At this point, it is straightforward to combine the arguments presented in Section 3.4.1 with the necessary modifications mentioned in the proof of Theorem 3.4.2 to obtain the following generalization of Theorem 3.4.1:

Corollary 3.4.3. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded $C^{2m,\gamma}$ smooth domain, $\gamma \in (0,1)$ and h be a nonlinearity satisfying assumptions (H1)-(H2) with $0 < a_0 \leq a(\cdot) \in C(\overline{\Omega})$. Suppose one of the conditions (I)-(II) is satisfied and that there exists $\Lambda > 0$ such that $\int_\Omega h(x,u)dx \leq \Lambda$ for all weak solutions of* (3.46)*. Then there exists $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for all weak solutions of* (3.46)*. In addition, if (II) holds, solutions are also bounded in $C^{2m}(\overline{\Omega})$.*

3.5 The Navier boundary conditions

So far, we studied the fourth-order problem (3.1) endowed with Dirichlet boundary conditions. In this section, we show that a similar analysis can be provided also when considering *Navier boundary conditions*:

$$\begin{cases} \Delta^2 u = h(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.47}$$

when $\Omega \subset \mathbb{R}^4$ is a bounded *convex* and smooth domain. As well-known, in this context this fourth-order problem can be seen as a system of coupled second-order problems:

$$\begin{cases} -\Delta u = w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta w = h(x,u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.48}$$

and thus the maximum principle holds. Hence, we are able to prove the a-priori uniform bounds as we did in the ball for the Dirichlet case.

A-priori bounds for Navier problems (3.47) and, in general, for systems of second-order equations have been investigated in several papers, most of the literature deals with power-type nonlinearities in dimension $N \geq 5$ (see [85, 52, 84] and references therein) or in $N = 4$ for the special case $h(x,t) = e^t$ ([61, 98, 62]). General exponential nonlinearities are the subject of a series of papers by de Figueiredo, do Ó and Ruf ([30, 31]), dealing with elliptic systems in dimension 2.

Recall that $u \in H^2(\Omega) \cap H^1_0(\Omega)$ is a *weak solution* of (3.47) whenever

$$\int_\Omega \Delta u \Delta \varphi = \int_\Omega h(x,u)\varphi \tag{3.49}$$

for any test function $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$. Our main results are the following.

Theorem 3.5.1. *Let Ω be a bounded and convex $C^{2,\gamma}$ domain in $\mathbb{R}^4$ and h be a subcritical nonlinearity satisfying (H1)-(H2) and those specified in Proposition 3.5.4. Then there exists $C > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq C$ for all weak solutions u of* (3.47)*.*

Theorem 3.5.2. *Let Ω be a bounded and convex $C^{2,\gamma}$ domain in $\mathbb{R}^4$ and h be a critical nonlinearity satisfying (H1)-(H2) and those specified in Proposition 3.5.4. Suppose moreover that there exist functions $0 \leq B \in L^\infty(\Omega)$, $0 \leq D \in L^1(\Omega)$ such that* (H3) *holds. Then there exists $C > 0$ such that $\|u\|_{C^4(\overline{\Omega})} \leq C$ for all weak solutions u of* (3.47)*.*

This section is devoted to the proof of Theorems 3.5.1-3.5.2, following the strategy used in Sections 3.2-3.3.

As for Dirichlet boundary conditions, the first step in order to get the a-priori uniform estimates on solutions is to obtain a uniform control near the boundary and of the right-hand side of (3.47). In Section 3.2, the key point was the behaviour of the Green function near the boundary of the ball. If we also assume the convexity of our domain Ω, its counterpart for Navier boundary conditions is the moving planes technique. Indeed, denoting $\Omega_r := \{x \in \Omega \,|\, d(x, \partial\Omega) < r\}$, we have the following:

Lemma 3.5.3. *Assume $\Omega \subset \mathbb{R}^4$ is a bounded $C^{2,\gamma}$ convex domain and that there exist $\bar{r}, \bar{\delta} > 0$ such that $h(\cdot, t) \in C^1(\Omega_{\bar{r}})$ for all $t \geq 0$ and $\nabla_x h(x,t) \cdot \theta \leq 0$ for all $x \in \Omega_{\bar{r}}$, $t \geq 0$ and unit vectors θ such that $|\theta - n(x)| \leq \bar{\delta}$. Then, there exist $r, \delta > 0$ such that $\nabla u(x) \cdot \theta \leq 0$ for all $x \in \Omega_r$ and $|\theta - n(x)| \leq \delta$, for any solution u of* (3.47).

Proof. The proof is an adaptation of [30, Lemma 3.2]. Each point $x \in \mathbb{R}^4$ will be denoted by $x = (x_1, \hat{x}_1)$ to isolate the first component, $\hat{x}_1$ standing for (x_2, x_3, x_4). We can assume without loss of generality, that $\Omega \subset \mathbb{R}^4_+ := \{x = (x_1, \hat{x}_1) \in \mathbb{R}^4 \,|\, x_1 > 0\}$ and that $0 \in \partial\Omega$. Moreover, let us define the cap $\Sigma_\lambda := \{x \in \Omega \,|\, x_1 < \lambda\}$, the reflected cap $\Sigma_\lambda := \{(2\lambda - x_1, \hat{x}_1) \,|\, (x_1, \hat{x}_1) \in \Sigma_\lambda\}$ and the segment dividing the two $T_\lambda := \{x \in \Omega \,|\, x_1 = \lambda\}$. By convexity, there exists $\bar{\lambda}(\bar{r}) > 0$ such that $\Sigma_\lambda \cup \Sigma'_\lambda \subset \Omega_{\bar{r}}$ for any $\lambda \in (0, \bar{\lambda})$. For such λ, define

$$w_\lambda(x) := u(2\lambda - x_1, \hat{x}_1) - u(x).$$

Therefore,

$$\begin{aligned}
\Delta^2 w_\lambda(x) &= \Delta^2 u(2\lambda - x_1, \hat{x}_1) - \Delta^2 u(x) \\
&= h((2\lambda - x_1, \hat{x}_1), u(2\lambda - x_1, \hat{x}_1)) - h((x_1, \hat{x}_1), u(x_1, \hat{x}_1)) \\
&\geq h((x_1, \hat{x}_1), u(2\lambda - x_1, \hat{x}_1)) - h((x_1, \hat{x}_1), u(x_1, \hat{x}_1))
\end{aligned}$$

since our assumption with $\theta = -x_1$ reads as $\frac{\partial h}{\partial x_1}(x,t) \geq 0$ for any $x \in \Omega_{\bar{r}}$. Using the mean value theorem,

$$\Delta^2 w_\lambda(x) \geq \frac{\partial h}{\partial t}((x_1, \hat{x}_1), \eta(x_1, \hat{x}_1))[u(2\lambda - x_1, \hat{x}_1) - u(x)] = c(x) w_\lambda(x),$$

where $\eta(x_1, \hat{x}_1)$ is a real number between $u(2\lambda - x_1, \hat{x}_1)$ and $u(x)$. For λ sufficiently small and positive, Σ_λ has small measure. Thus, the maximum principle for cooperative elliptic systems in small domains (see [28]) implies $w_\lambda \geq 0$, which means that near the boundary $\partial\Omega$ the function u is increasing along the direction x_1. The general statement follows by a compactness argument. $\square$

With this result available, we can thus recover the analogous of Proposition 3.2.1, whose proof is only sketched, being very similar:

Proposition 3.5.4. *Let Ω and h be as in Lemma 3.5.3. Suppose also that one of the conditions (i)-(ii) of Proposition 3.2.1 holds. Then there exist $r, C_1, \Lambda > 0$ such that*

$$\|u\|_{L^\infty(\Omega_r)} \le C_1 \qquad \textit{and} \qquad \int_\Omega h(x,u)dx \le \Lambda \tag{3.50}$$

for all (positive) weak solutions u of (3.47)*, where $\Omega_r := \{x \in \Omega \,|\, d(x, \partial\Omega) < r\}$.*

Sketch of the Proof. Recall that λ_1 and φ_1 are respectively the first eigenvalue and the first eigenfunction of $(-\Delta)$ with Dirichlet boundary conditions in Ω. Choosing $\varphi = \varphi_1$ in (3.49), we get

$$\int_\Omega h(x,u)\varphi_1 = \int_\Omega \Delta u \Delta\varphi_1 = -\lambda_1 \int_\Omega \Delta u \varphi_1 = \lambda_1 \int_\Omega \nabla u \nabla \varphi_1 = \lambda_1^2 \int_\Omega u\varphi_1.$$

Exploiting the superlinearity of h, with the same steps as in Lemma 3.2.5 and with the notation therein, we infer

$$\int_\Omega u\varphi_1 \le 2\|\varphi_1\|_\infty \left(\frac{1}{2\lambda_1^2} \|h(\cdot, 0)\|_1 + \int_\Omega t_0(x)dx \right).$$

By conditions (i)-(iii) of Lemma 3.2.5, we can bound the term involving t_0, obtaining

$$\int_\Omega h(x,u)\varphi_1 dx \le C \tag{3.51}$$

for some positive constant C independent of u. This uniform estimate, together with Lemma 3.5.3 allow to retrace Lemmas 3.2.7 -3.2.8 and get the desired bounds (3.50). On the other hand, the local L^1 bound (3.51) may be also obtained retracing the proof of Lemma 3.2.4, where the Green function estimates therein are replaced by the analogues for the Navier boundary conditions (see [40, Proposition 4.13] and the subsequent Proposition 3.5.5). □

Let now exclude internal blow-up. The main ingredients of the arguments for the Dirichlet case contained in Section 3.3 are the uniform estimates of Proposition 3.2.1 and the estimates on the Green function and of its derivatives provided by Lemma 3.2.2: once we have these two, Theorems 3.1.2 and 3.1.3 follow at once, as Section 3.3 shows. Therefore, by Proposition 3.5.4, we just have to find the counterpart of Lemma 3.2.2 for the Navier boundary conditions. A reference for the estimates for the Green function $G_{NAV}(x,y)$ is [51]; nevertheless, although we believe they already exist in the literature, we were not able to find any reference for the estimates of its derivatives. Hence, we prove here the following:

Proposition 3.5.5. *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain of class $C^{1,1}$ and let $G_{NAV}(x,y)$ be the Green function in Ω of the biharmonic operator subjected to Navier boundary conditions. There exists $C > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have that*

$$|G_{NAV}(x,y)| \le C \log\left(1 + \frac{d_\Omega(x) d_\Omega(y)}{|x-y|^2}\right), \tag{3.52}$$

$$|\nabla_x G_{NAV}(x,y)| \le \frac{C}{|x-y|}.$$

Proof. In view of the decomposition (3.48), G_{NAV} can be described as an iterated Green function for the Laplace operator with Dirichlet boundary conditions, denoted by $G_{-\Delta,\Omega}$. Indeed, if u is the solution of $\Delta^2 u = f$ with $u = \Delta u = 0$, then

$$\begin{aligned}\nabla^i u(x) &= \int_\Omega \nabla_x^i G_{-\Delta,\Omega}(x,y)v(y)dy \\ &= \int_\Omega \nabla_x^i G_{-\Delta,\Omega}(x,y)\Big(\int_\Omega G_{-\Delta,\Omega}(y,z)f(z)dz\Big)dy \\ &= \int_\Omega \Big(\int_\Omega \nabla_x^i G_{-\Delta,\Omega}(x,y)G_{-\Delta,\Omega}(y,z)dy\Big)f(z)dz,\end{aligned}$$

which yields

$$\nabla_x^i G_{NAV}(x,z) = \int_\Omega \nabla_x^i G_{-\Delta,\Omega}(x,y)G_{-\Delta,\Omega}(y,z)dy. \tag{3.53}$$

For a proof of (3.52) (that is, when $i = 0$), we refer to [40, Proposition 4.13] as well as for its generalizations in [51, Theorem 1.2]. Let now $i = 1$ and recall the estimates for $G_{-\Delta,\Omega}$ in $\mathbb{R}^4$, which go back to the work of Widman [100] (cf. [90] for this formulation):

$$|G_{-\Delta,\Omega}(x,y)| \preceq |x-y|^{-2}\Big(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\Big), \tag{3.54}$$

$$|\nabla_x G_{-\Delta,\Omega}(x,y)| \preceq |x-y|^{-3}\Big(1 \wedge \frac{d(y)}{|x-y|}\Big). \tag{3.55}$$

In the sequel, we strictly follow [51]. Fix $x, z \in \Omega$, define $\mathcal{O}_x := B_{\frac{2}{3}|x-z|}(x) \cap \Omega$ and similarly $\mathcal{O}_z := B_{\frac{2}{3}|x-z|}(z) \cap \Omega$ and let $\mathcal{R} := \Omega \setminus (\mathcal{O}_x \cup \mathcal{O}_z)$. By (3.53), (3.54) and (3.55),

$$|\nabla_x^i G_{NAV}(x,z)| \preceq \int_\Omega \frac{\big(1 \wedge \frac{d(y)}{|x-y|}\big)\big(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\big)}{|x-y|^3|y-z|^2}dy =: \int_\Omega Q(x,y,z)dy.$$

On $\mathcal{O}_x$ we have $|y-z| \sim |x-z|$. Indeed, $\frac{1}{3}|x-z| \le |y-z| \le \frac{5}{3}|x-z|$. Hence,

$$\begin{aligned}\int_{\mathcal{O}_x} Q(x,y,z)dy &\preceq \int_{\mathcal{O}_x} |x-y|^{-3}|y-z|^{-2}dy \preceq |x-z|^{-2}\int_{\mathcal{O}_x}|x-y|^{-3}dy \\ &\preceq |x-z|^{-2}\int_0^{\frac{2}{3}|x-z|} d\rho \preceq \frac{1}{|x-z|}.\end{aligned} \tag{3.56}$$

Analogously, on $\mathcal{O}_z$ there holds $|x-y| \sim |x-z|$ and thus

$$\begin{aligned}\int_{\mathcal{O}_z} Q(x,y,z)dy &\preceq \int_{\mathcal{O}_z} |x-y|^{-3}|y-z|^{-2}dy \preceq |x-z|^{-3}\int_{\mathcal{O}_z}|y-z|^{-2}dy \\ &\preceq |x-z|^{-3}\int_0^{\frac{2}{3}|x-z|} \rho d\rho \preceq \frac{1}{|x-z|}.\end{aligned} \tag{3.57}$$

On $\mathcal{R}$ there holds $|y-z| \sim |x-y|$. In fact, on one hand $|y-z| \ge \frac{2}{3}|x-y|$, so $|x-y| \le |x-z| + |z-y| \le \frac{5}{2}|y-z|$, and, on the other, $|y-x| \ge \frac{2}{3}|x-z|$, implying

$|y-z| \leq |x-y|$. Furthermore, the following relation holds (see [50, Lemma 3.2] as well as [40, Lemma 4.5]):

$$\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right) \sim \left(1 \wedge \frac{d(y)}{|y-z|}\right)\left(1 \wedge \frac{d(z)}{|y-z|}\right) \leq 1 \wedge \frac{d(z)}{|y-z|}.$$

As a result, we get

$$\begin{aligned}\int_{\mathcal{R}} Q(x,y,z)dy &\preceq \int_{\mathcal{R}} |x-y|^{-5}\left(1 \wedge \frac{d(y)}{|x-y|}\right)\left(1 \wedge \frac{d(z)}{|y-z|}\right) \\ &\preceq \int_{\mathcal{O}_x^c} |x-y|^{-5}\left(1 \wedge \frac{1}{|x-y|}\right)\left(1 \wedge \frac{d(z)}{|x-y|}\right).\end{aligned} \tag{3.58}$$

To estimate (3.58) we distinguish two cases depending on the reciprocal distance of x and z compared with their distance from the boundary. We denote by D a sufficiently large radius so that $\mathcal{R} \subset B_D(x) \setminus \mathcal{O}_x$.

Case $|x-z|^2 \leq d(z)$. If so, one has $\frac{2}{3}|x-z| \leq \sqrt{d(z)}$ and thus we continue (3.58) as follows:

$$\begin{aligned}\int_{\mathcal{R}} Q(x,y,z)dy &\preceq \int_{\frac{2}{3}|x-z|}^{D} \rho^{-5}\left(1 \wedge \frac{1}{\rho}\right)\left(1 \wedge \frac{d(z)}{\rho}\right)\rho^3 d\rho \\ &\preceq \int_{\frac{2}{3}|x-z|}^{\sqrt{d(z)}} \rho^{-2}d\rho + d(z)\int_{\sqrt{d(z)}}^{D} \rho^{-4}d\rho \\ &\preceq \frac{1}{|x-z|} + \frac{1}{d(z)} \preceq \frac{1}{|x-z|}.\end{aligned} \tag{3.59}$$

Case $|x-z|^2 > d(z)$. Then,

$$\int_{\mathcal{R}} Q(x,y,z)dy \preceq d(z)\int_{\frac{2}{3}|x-z|}^{D} \rho^{-4}d\rho \preceq \frac{d(z)}{|x-z|^3} < \frac{1}{|x-z|}. \tag{3.60}$$

The result is obtained summing up estimates (3.56), (3.57), (3.59) and (3.60). □

The estimate on $\nabla_x G_{NAV}$ provided by Proposition 3.5.5 allows to recover most of the arguments of Section 3.3, in particular the compactness of the sequence of the rescaled functions $(v_k)_k$, leading then to the contradictory argument which proves Theorem 3.5.1. On the other hand, to deal with critical nonlinearities, we need also the estimates for the higher-order derivatives. However, they easily follow by means of the decomposition into coupled systems (3.48) and the estimates on the Green functions for the Laplacian. Indeed, we have

$$w(x) = \int_\Omega G_{-\Delta,\Omega}(x,y)h(y,u(y))dy \quad \text{and} \quad \nabla w(x) = \int_\Omega \nabla_x G_{-\Delta,\Omega}(x,y)h(y,u(y))dy.$$

Thus, by (3.54) and (3.55), we infer

$$|w(x)| \leq C\int_\Omega \frac{1}{|x-y|^2}h(y,u(y))dy \quad \text{and} \quad |\nabla w(x)| \leq C\int_\Omega \frac{1}{|x-y|^3}h(y,u(y))dy,$$

which is enough for our purposes recalling that $|w| = |\Delta u|$. This, indeed, allow us to retrace Lemmas 3.3.3 and 3.3.6 and prove Theorem 3.5.2.

Remark 21. It should be now clear that an analogue statement to Theorem 3.4.1 holds also when dealing with Navier boundary conditions and domains which are not necessarily convex.

3.6 Existence results

The a-priori bound for solutions of (3.1) obtained in the previous sections is essential to apply topological methods and infer the existence of positive solutions. We follow the standard approach which has been widely applied in the literature (see, for instance, [29, 85, 31]), and which relies on a well-known result firstly due to Krasnosel'skii, which may be equivalently stated in the following way (see [27, Theorem 3.1] and the subsequent results):

Lemma 3.6.1. *Let X be a Banach space and $K \subset X$ a cone, which induces a partial order in X defined as follows: $x \leq y$ if and only if $y - x \in K$. Moreover, let $\Phi : K \to K$ be a compact map with $\Phi(0) = 0$ and suppose there exist $0 < r < R$ and $\tau > 0$ such that:*

1. *there exists a bounded linear operator $A : X \to X$ such that $A(K) \subset K$ with spectral radius $r(A) < 1$ and such that $\Phi(x) \leq Ax$ for all $x \in K$ with $\|x\| = r$;*
2. *there exists $\Psi : K \times [0, +\infty) \to K$ such that*
   - *(a) $\Psi(x, 0) = \Phi(x)$;*
   - *(b) $\Psi(x, t) \neq x$ for all $t \geq 0$ and $\|x\| = R$;*
   - *(c) $\Psi(x, t) \neq x$ for all $t \geq \tau$ and $\|x\| \leq R$.*

Then Φ has a fixed point $\overline{x} \in K$ with $r < \|\overline{x}\| < R$.

Proof of Theorem 3.1.4. Let $X = C(\overline{\mathcal{B}})$ and $K := \{f \in C(\overline{\mathcal{B}}) \,|\, f \geq 0\}$ be the closed cone of nonnegative functions, which induces the standard pointwise order on $C(\overline{\mathcal{B}})$. Moreover, let $T := (\Delta^2)^{-1}$ be the inverse of the bilaplace operator with Dirichlet boundary conditions, that is, $Tg = w$ if and only if w solves

$$\begin{cases} \Delta^2 w = g & \text{in } \mathcal{B}, \\ w = w_n = 0 & \text{on } \partial\mathcal{B}. \end{cases}$$

Then $T : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$ is a linear compact and positive operator. Defining now $\Phi := T \circ h(x, \cdot)$, then $\Phi(K) \subseteq K$ by maximum principle and Φ is a bounded compact operator, by the continuity of h.
By our assumptions on h, there exist $\alpha \in (0, \tilde{\lambda}_1)$ and $t_0 > 0$ such that $h(x, t) \leq \alpha t$ for any $x \in \mathcal{B}$, $t \in (0, t_0)$. Hence, defining $A := \alpha T$, then:

$$\Phi(u) = T(h(x, u)) \leq T(\alpha u) = \alpha T(u) = A(u).$$

Moreover $A(0) = 0$ by definition, $A(K) \subset K$ and

$$r(A) = \alpha \max\{\lambda \,|\, \lambda \text{ is an eigenvalue of } A^{-1}\} = \frac{\alpha}{\tilde{\lambda}_1} < 1,$$

so condition *1* of Lemma 3.6.1 is verified.

Let us now define $\Psi(u,t) := T(h(x,u+t))$, for any $t \geq 0$. It is clear that $\Psi(u,0) = \Phi(u)$. We are thus led to study the following family of problems:

$$\begin{cases} \Delta^2 u = h(x, u+t) & \text{in } \mathcal{B}, \\ u = u_n = 0 & \text{on } \partial\mathcal{B}. \end{cases} \tag{P$_t$}$$

With the same steps as in the proof of Lemma 3.2.5, we can get

$$\int_{\mathcal{B}} h(x, u+t)\tilde{\varphi}_1 dx \leq C, \tag{3.61}$$

where the constant C is independent of t; this in turns implies

$$\int_{\mathcal{B}} a(x)(u+t)\tilde{\varphi}_1 dx \leq C.$$

Since u, a and $\tilde{\varphi}_1$ are positive, we necessarily find that t is bounded. This means that there are no (positive) solutions of problem (P$_t$) when $t > \tilde{T}$ for some $\tilde{T} > 0$ which depends on Ω and h, so condition (*2c*) is fulfilled. Hence, we can restrict to $t \in [0, \tilde{T}]$ and prove the uniform a-priori bound for these values of the parameter t. In fact, in view of (3.61), firstly one can repeat the same steps of the proofs in Sections 3.2 and produce the same uniform estimates of Proposition 3.2.1. In particular, we get

$$\int_{\mathcal{B}} h(x, u+t) dx \leq \Lambda(h, \tilde{T}),$$

uniformly with respect to $t \in [0, \tilde{T}]$. This is sufficient to guarantee that the contradictory argument of Section 3.3 can be retraced for solutions of problem (P$_t$) with only minor adaptations, finding an a-priori bound which depends only on h and $\tilde{T}$. Rephrased, this means that there are no solutions of problem (P$_t$) for *any* $t \in [0, \tilde{T}]$ with $\|u\| \geq R$ for some $R = R(\tilde{T}) > 0$ (and thus also for *any* $t \geq 0$). Hence, condition (*2b*) is verified.

Since all assumptions of Lemma 3.6.1 are then ensured, the existence of a positive solution of problem (3.1) follows. Regularity of such solutions is given by the standard elliptic arguments. □

Remark 22. It should be clear from Theorem 3.4.2 that all the arguments used in the proof of Theorem 3.1.4 apply also for the polyharmonic context and a higher-order analogue of Theorem 3.1.4 holds.

We point out that the assumption (3.10) which was imposed to obtain condition *1* of Lemma 3.6.1, is also necessary in same cases. Indeed, if one considers $h(x,s) = \lambda s f(s)$, with $f(s) \geq 1$ for any $s \geq 0$ (for instance, $f(s) = e^{s^\alpha}$ with $\alpha \in (0,1]$), then a simple calculation shows that if $\lambda > \tilde{\lambda}_1$, we have no (positive) solutions for problem (3.1) in any domain. Indeed, by integration by parts, one gets

$$\tilde{\lambda}_1 \int_\Omega u\tilde{\varphi}_1 = \int_\Omega \Delta u \Delta\tilde{\varphi}_1 = \lambda \int_\Omega u f(u)\tilde{\varphi}_1 \geq \lambda \int_\Omega u\tilde{\varphi}_1,$$

which implies $\lambda \leq \tilde{\lambda}_1$.

3.7 A counterexample

In Theorems 3.1.2-3.1.3 and in their subsequent generalizations, the maximal growth allowed for the nonlinearity is of kind $t \mapsto e^{\gamma t}$ for some $\gamma > 0$. On the other hand, when dealing with polynomial nonlinearities in higher dimensions, we know e.g. from [77] that the analogue a-priori bounds can be reached until the critical threshold. One is thus induced to think that this is a matter of technicality and one may improve our result until the Trudinger-Moser-Adams critical growth $t \mapsto e^{t^2}$. This section shows that our result is instead sharp. Indeed, we present an example of a problem of type (3.1) with a growth of kind $t \mapsto e^{t^\alpha}$ with $\alpha \in (1, 2)$ which admits unbounded solutions.

Let $f(t) = e^{t^\alpha}$ with $\alpha \in (1, 2)$ and fix $1 < \gamma < 2 - \frac{1}{\alpha}$. Notice that f satisfies assumptions (*A1*) and (*A2*) but not (*A3*). In $\mathcal{B} \subset \mathbb{R}^4$ consider the function

$$u(x) = u(|x|) = u(r) := |\log(r^4 \log^\gamma(\tfrac{e}{r}))|^{\frac{1}{\alpha}}.$$

It is easy to see that $e^{u^\alpha} \in L^1(\mathcal{B})$. Defining now $a(x) := (\Delta^2 u)e^{-u^\alpha}$, then u satisfies

$$\begin{cases} \Delta^2 u = a(x)e^{u^\alpha} & \text{in } \mathcal{B}, \\ u = 0 & \text{on } \partial\mathcal{B}. \end{cases}$$

With similar computations as in [13, Example 2], one verifies that a behaves near 0 like $|\log(r)|^{\frac{1}{\alpha}-2}$ and therefore $a \in L^\infty(B_\varepsilon(0))$ for a suitable $\varepsilon \in (0, 1)$. On the other hand, $u \notin L^\infty(\mathcal{B})$ since $u \sim |\log(r)|^{\frac{1}{\alpha}}$ near the origin. However, $u_n(x) \neq 0$ on $\partial\mathcal{B}$. Hence, let us introduce $\eta \in C_c^\infty(\mathcal{B})$ be such that $\eta(x) = \eta(r) \geq 0$ and decreasing in the radial variable, $\eta \equiv 1$ if $r \in [0, \frac{1}{2}]$ and $\eta \equiv 0$ if $r \in [\frac{3}{4}, 1]$. Moreover, denote by $\rho \in (0, \frac{3}{4}]$ the radius such that $B_\rho(0) = \{x \in \Omega \,|\, \eta(x) \neq 0\} \subset\subset \mathcal{B}$. If we now set $w := u\eta$, then w satisfies

$$\begin{cases} \Delta^2 w = \overline{a}(x)e^{w^\alpha} & \text{in } B_\rho(0), \\ w = w_n = 0 & \text{on } \partial B_\rho(0), \end{cases}$$

with $\overline{a}(x) := (\Delta^2 w)e^{-w^\alpha}$. Moreover, we have:

1. $e^{w^\alpha} \in L^1(B_\rho(0))$ as $w \leq u$ in $B_\rho(0)$;
2. $\overline{a}$ is positive and continuous in $B_\rho(0)$;
3. $\overline{a} \in L^\infty(B_\rho(0))$: in fact, firstly, away from a neighborhood of 0 and of $\partial\mathcal{B}$, all derivatives of u are bounded and η has compact support in $\mathcal{B}$; furthermore we have $\overline{a} \equiv a$ in $B_{\frac{1}{2}}(0)$ by construction;
4. $w \notin L^\infty(B_\rho(0))$ since $w \equiv u$ in $B_{\frac{1}{2}}(0)$.

This example shows that the assumption $\lim_{t\to+\infty} \frac{f'(t)}{f(t)} \in [0, +\infty)$ is indeed sharp.

Remark 23. This counterexample also works when dealing with Navier boundary conditions, since by construction $\nabla^i w = 0$ on $\partial B_\rho(0)$ for any $i \geq 0$.

Chapter 4

Open problems and perspectives

In this last chapter we collect some possible extensions and unsolved questions.

Concerning Chapter 2...

- *If Ω is a ball, are the ground states of J_σ radially symmetric?*

In fact, we deduced the existence of ground states and of radial solutions which are indeed ground states among all possible radial solutions; both of them are positive and have the same behaviour when $\sigma \to -1$ and $\sigma \to 1$. But no standard techniques such as the Talenti symmetrization principle seem to apply (except for the Navier case) to prove that these classes of functions are indeed the same.

- *Are the radial positive solutions radially decreasing if $\sigma > 1$?*

Indeed, the radial decay property proved in Proposition 2.2.35 does not apply in this setting and, by now, we cannot extend Proposition 2.2.36 for these values of σ.

Moreover, in the spirit of [26] and [37]:

- *Can we say something about the uniqueness of (at least) the positive radially symmetric ground state of J_σ, for some values of σ?*

Finally, all the techniques developed from Section 2.2 strongly relied on the assumptions we made on the boundary, that is $\partial\Omega$ of class $C^{1,1}$ in order to have $\kappa \in L^\infty(\partial\Omega)$. In particular, Theorem 2.1.5 allowed us to rewrite in an appropriate way our functional.

- *May we deduce the positivity of ground states of J_σ for domains with less regularity on the boundary?*

In the particular case of a convex polygon $\mathcal{P}$, it is known that ground states of J_σ are positive for every σ: in fact, the superharmonic method applies easily once we have $\int_{\mathcal{P}} det(\nabla^2 u) = 0$ thanks to a result by Grisvard [45, Lemma 2.2.2]. We believe that positivity for ground states of J_σ still holds imposing, for instance, only Lipschitz regularity for $\partial\Omega$.

Concerning Chapter 3...

We proved the a-priori bound for solutions of problem (3.1) when the domain is a ball. Nevertheless, as specified in Section 3.4.1, the blow-up technique may be applied independently of our domain, provided we already have the uniform estimate $\int_\Omega h(x,u) \leq C$. Moreover, the arguments presented in Section 3.2 hold for any (regular) bounded positivity preserving domain in $\mathbb{R}^4$, *except for* Lemma 3.2.6.

- *Can we extend Lemma 3.2.6 (at least) for small deformations of the ball?*

Indeed, these are the only explicit class of domains which are known to satisfy the positivity preserving property when dealing with Dirichlet boundary conditions in dimension $N \geq 3$ (see Grunau and Robert, [47]). It would be sufficient to prove that the Green function inequality (3.20) holds for such domains (cf. Remark 20).

Finally:

- *May we extend the present investigation also for other problems in critical dimension, for instance involving the N-laplacian operator in bounded domains of $\mathbb{R}^N$?*

This will lead a generalization of the results in [58, 76]. Notice that in that context most of the tools we used are available, since the analysis near the boundary has been already achieved in [58], and the quasilinear Liouville's equation has been recently studied by Morlando and Esposito in [35, 34]. However, we have to replace all arguments which rely on the Green function estimates.

Appendix A

Some useful classical results

In this appendix, we only list some important results used several times during the proofs presented in this work.

Interpolation in Fractional Sobolev Spaces

Theorem A.0.1 ([13], Corollary 2). *For $0 \le s_1 < s_2 < +\infty$, $1 < p_1, p_2 < +\infty$, for every s, p such that $s = \theta s_1 + (1-\theta)s_2$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, we have*

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \le C\|f\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^N)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^N)}.$$

The Dual Cone Decomposition

Often, this decomposition is named after Moreau since his work [69].

Definition A.0.1. Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and $K \subset H$ be a nonempty closed convex cone. Its *dual cone* K^* is defined as

$$K^* := \{w \in H \,|\, (w,v)_H \le 0,\ \forall v \in K\}.$$

Theorem A.0.2 ([40], Theorem 3.4). *Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and K and K^* as before. Then for every $u \in H$, there exists a unique couple $(u_1, u_2) \in K \times K^*$ such that $u = u_1 + u_2$ and $(u_1, u_2)_H = 0$.*

The Principle of Symmetric Criticality

We follow the exposition in [101, Theorem 1.28].

Definition A.0.2. The action of a topological group G on a normed space X is a continuous map

$$G \times X \to X : [g,u] \mapsto gu$$

such that

$$1 \cdot u = u, \qquad (gh)u = g(hu), \qquad u \mapsto gu \text{ is linear.}$$

The action is isometric if $\|gu\| = \|u\|$; the space of invariant points is defined by

$$Fix(G) := \{u \in X \,|\, gu = u, \forall g \in G\}.$$

A set $A \subset X$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G$. A map $f : X \to X$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Theorem A.0.3 (Principle of Symmetric Criticality, Palais, 1979). *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\varphi \in C^1(X;\mathbb{R})$ is invariant and if u is a critical point of φ restricted to $Fix(G)$, then u is a critical point of φ.*

A local regularity result and Liouville's theorems for higher-order equations

The following local regularity estimate is a particular case of a more general result by Reichel and Weth:

Lemma A.0.4. *([77], Corollary 6) Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $m \in \mathbb{N}$, $h \in L^p(\Omega)$ for some $p \in (1, +\infty)$ and suppose $u \in W^{2m,p}(\Omega)$ satisfies*

$$(-\Delta)^m u = h \quad in \ \Omega.$$

Then there exists a constant $C = C(R, N, p, m)$ such that for any $\delta \in (0, 1)$,

$$\|u\|_{W^{2m,p}(B_{\delta R}(0))} \leq \frac{C}{(1-\delta)^{2m}}(\|h\|_{L^p(B_R(0))} + \|u\|_{L^p(B_R(0))}).$$

Finally, we collect some Liouville's theorems which are an important tool in our blow-up arguments:

Theorem A.0.5. *([99], Theorem 1.4) Let $m \in \mathbb{N}$ and assume that $p > 1$ if $N \leq 2m$ and $1 < p \leq \frac{N+2m}{N-2m}$ if $N > 2m$. If u is a classical nonnegative solution of*

$$(-\Delta)^m u = u^p \quad in \ \mathbb{R}^N,$$

then $u \equiv 0$.

Theorem A.0.6. *([65], Theorem 2) Suppose u is a solution of*

$$\begin{cases} (-\Delta)^m u = (N-1)! e^{Nu} \ \ on \ \mathbb{R}^N, \ N = 2m, \\ \int_{\mathbb{R}^N} e^{Nu} < +\infty, \end{cases}$$

such that $u(x) = o(|x|^2)$ as $|x| \to \infty$. Then $u(x)$ is symmetric with respect to some point $x_0 \in \mathbb{R}^N$ and there exists some $\lambda > 0$ so that

$$u(x) = \log\left(\frac{2\lambda}{\lambda^2 + |x-x_0|^2}\right) \quad for \ all \ x \in \mathbb{R}^N.$$

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