Stabilization of a class of nonlinear systems with passivity properties

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Contents

Notations & Acronyms

Mathematical Notation

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}^n$	Linear space of dimension n.
$\mathbb{R}_{\geq 0}$	Field of nonnegative real numbers.
x_i	Denotes the i-th element of the vector x.
$\|x\|^2$	Square of the Euclidean norm, *i.e.*, $\|x\|^2 := x^\top x$
$\|x\|_S^2$	The weighted square Euclidean norm, i.e., $\|x\|_S^2 := x^\top S x$.
$\mathbf{0}_n$	Column vector of zeros of dimension n.
e_i	Denotes the i-th Euclidean basis vector of $\mathbb{R}^n$.
A_i	Denotes the $i-th$ column of the matrix A.
A_{ij}	Denotes the $ij-th$ element of the matrix A.
$\texttt{sym}\{A\}$	The symmetric part of the square matrix A.
$\texttt{skew}\{A\}$	Returns the skew-symmetric part of the square matrix A.
I_n	The identity matrix of size $n \times n$.
$\mathbf{0}_{n\times s}$	Matrix of zeros of dimension $n \times s$.
F_*	For the distinguished element $x_* \in \mathbb{R}^n$ and any mapping $F : \mathbb{R}^n \to \mathbb{R}^s$, we define the constant matrix $F_* := F(x_*)$.
$\mathrm{diag}\{\cdot\}$	Diagonal matrix of the input arguments.
$\det\{\cdot\}$	Determinant of the A matrix.
$g^\dagger$	Left pseudo-inverse of the matrix g, *i.e.*, $g^\dagger := (g^\top g)^{-1} g^\top$.
$g^\perp$	Left full-rank annihilator of the matrix g, *i.e.*, $g^\perp g = 0$.
$(\cdot)^{-1}$	Inverse operator.
$(\cdot)^\top$	Transpose operator.
$\frac{d}{dt}(\cdot) = \dot{(\cdot)}$	Total time derivative.
Φ', Φ''	For mappings of scalar argument $\Phi : \mathbb{R} \to \mathbb{R}^s$ denote, respectively, first and second order differentiation.
$\nabla H(x)$	For $H : \mathbb{R}^n \to \mathbb{R}$, it refers to the gradient operator of a function, i.e., $\nabla H(x) := \left(\frac{\partial H(x)}{\partial x}\right)^\top$.
$\nabla^2 H(x)$	For $H : \mathbb{R}^n \to \mathbb{R}$, it refers to the Hessian operator of a function, i.e., $\nabla^2 H(x) := \left(\frac{\partial^2 H(x)}{\partial x^2}\right)^\top$.
$\nabla \mathcal{C}(x)$	For $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla \mathcal{C}(x) := [\nabla \mathcal{C}_1(x), \ldots, \nabla \mathcal{C}_m(x)]$.

Unless indicated otherwise, all vectors are column vectors. All mappings are assumed sufficiently smooth. To simplify the expressions, the arguments of all mappings are explicitly written only the first time that the mapping is defined.

When it is clear from the context, the subindex in ∇ is omitted.

Acronyms

PBC	Passivity-Based Control
EB	Energy Balancing
SPBC	Standard Passivity-Based Control
AS	Asymptotically Stable
GAS	Global Asymptotically Stable
PI	Proportional and Integral
PID	Proportional-Integral-Derivative
PDE	Partial Differential Equations
PH	Port-Hamiltonian
IOHD	Input-Output Hamiltonian systems with Dissipation
CbI	Control by Interconnection
IDA	Interconnection and Damping Assignment
LTI	Linear Time Invariant
EL	Euler Lagrange
PFL	Partial Feedback Linearization
AMM	Assumed Modes Method
PWM	Pulse-Width Modulation
DAC	Digital to Analog Converter
Cu-Be	Copper Beryllium

Synthèse en français

Le concept d'énergie joue un rôle fondamental dans la modélisation, l'analyse et le contrôle des systèmes physiques. Ce concept est bien connu dans la science et l'ingénierie, et est une pierre angulaire dans la poursuite d'un cadre unifié pour la modélisation appropriée des phénomènes du monde réel. Dans le cadre basé sur l'énergie, les systèmes physiques sont considérés comme l'interconnexion des éléments de stockage, de dissipation et de routage d'énergie. Dans ce cadre, les systèmes dynamiques sont traités comme des dispositifs de transformation d'énergie, ce qui permet d'analyser des systèmes non linéaires complexes comme l'interconnexion de sous-systèmes plus simples.

La passivité est une propriété physique qui a son origine dans la théorie des circuits où un réseau électrique est passif si tous ses éléments sont passifs, c'est-à-dire si ses éléments ne peuvent pas fournir de l'énergie au monde extérieur. Dans la théorie du contrôle, un système dynamique –dans sa représentation d'état– est appelé passif si le flux d'énergie d'entrée est supérieur ou égal à la différence entre les énergies stockées initiale et finale. Ce dernier peut être représenté mathématiquement en termes d'énergie et de puissance, respectivement, par les inégalités suivantes:

$$\begin{aligned} H(x(t_1)) - H(x(t_0)) &\leq \int_{t_0}^{t_1} y^\top(t)u(t)\mathrm{d}t \\ \dot{H} &\leq y^\top u, \end{aligned}$$

où x est l'état du système, u, y sont l'entrée et la sortie, respectivement, et dont le produit a des unités de puissance. $H(x)$ représente l'énergie stockée et l'intervalle de temps satisfait la condition $t_1 \geq t_0$.

L'énergie d'un système détermine son comportement, de ce fait, intuitivement, la propriété de passivité est liée à la stabilité du système. Cette déduction est en général vraie. En effet, en tenant compte de certaines conditions techniques et de la définition de la stabilité considérée, un système passif est stable. Pour les systèmes passifs, un des résultats fondamentaux est que l'interconnexion de deux systèmes passifs –en passant par un sous-système d'interconnexion adéquat– produit à nouveau un système passif. Ce dernier résultat joue un rôle déterminant dans la conception et l'analyse des systèmes passifs. Par ailleurs, c'est important de souligner que, bien que la passivité soit en principe une propriété physique, dans la théorie de la passivité, il n'est pas nécessaire que la fonction d'énergie du système corresponde à la représentation d'une énergie physique réelle, cela agrandit l'ensemble des systèmes pour lesquels la théorie de la passivité fournit une procédure méthodique d'analyse et conception de contrôle.

Dans le cadre énergétique, il s'avère qu'un moyen naturel de contrôler un système physique est de modifier sa fonction d'énergie, cette procédure est connue sous le nom de façonnage de l'énergie. Contrairement à certaines techniques classiques qui tentent d'imposer un comportement souhaité à l'aide d'un système d'annulation des non-linéarités et des contrôleurs de gains élevés, l'approche énergétique tente d'exploiter la structure et les propriétés physiques (quand c'est le cas) du système. L'idée de base de la commande basée sur la passivité (PBC) est de concevoir un contrôleur qui accomplit la tâche de contrôle, par exemple la stabilisation, en rendant le système passif par rapport à la fonction d'énergie souhaitée et en injectant un amortissement. Les différentes techniques PBC peuvent être classées dans deux classes principales:

- Les techniques qui choisissent *a priori* la structure des fonctions d'énergie à assigner et après conçoivent la loi de commande qui rend la fonction d'énergie souhaitée non-croissante.

- Les techniques où la structure en boucle fermée est choisie, puis la famille des fonctions d'énergie possibles se caractérise par une équation différentielle partielle (PDE).

Malgré son attrait pratique et théorique, les techniques PBC sont limitées par certaines contraintes, telles que l'amortissement présent dans le système à contrôler ou la complexité des PDEs à être résolues. L'objectif principal de cette thèse est d'élargir l'applicabilité des techniques PBC, pour ce faire, nous explorons l'utilisation de différentes sorties passives dans la conception de la loi de commande. En particulier, nous nous concentrons sur le contrôle par l'interconnexion (CbI), l'équilibrage énergétique (EB) et les régulateurs proportionnel-intégral-dérivé (PID).

Les résultats présentés dans cette thèse sont principalement basés sur les travaux [7, 13, 42, 45, 60] dans lesquels plusieurs techniques PBC, telles que CbI, EB et PID-PBC, sont étudiées sous différents angles. Par conséquent, les PBC formulés dans ce travail étendent les méthodologies qui y sont rapportées. Les principales contributions de cette thèse sont énumérées ci-dessous.

- Sur la base des sorties passives étudiées dans [42, 60], nous proposons un paramétrage qui caractérise les sorties passives pour les systèmes Hamiltoniens à ports (PH).

- Un cadre unifié de l'utilisation des différentes sorties passives dans deux techniques PBC bien connues, c'est-à-dire CbI et EB. Par la suite, nous fournissons une interprétation physique des régulateurs. Enfin, nous comparons les résultats obtenus dans les deux approches.

- La proposition d'une nouvelle méthodologie dans laquelle la fonction d'énergie en boucle fermée est façonnée sans résoudre des PDEs et les lois de commande résultantes ont la structure d'un PID de la sortie passive. Nous présentons le résultat en deux parties: tout d'abord, nous construisons un PI qui garantit que le système global a un équilibre stable au point souhaité. Ensuite, nous ajoutons le terme dérivatif. Nous analysons le scénario du régulateur PID à l'aide des différentes sorties passives précédemment caractérisées.

- L'application d'un schéma PID-PBC, récemment proposé dans [13], à un système mécanique complexe, à savoir un pendule inversé ultra flexible, représenté sous la forme d'un modèle contraint EL. La conception du régulateur, la preuve de stabilité, ainsi que les simulations et les résultats expérimentaux sont présentés pour montrer l'applicabilité de cette technique aux systèmes physiques.

Présentation de la thèse

La thèse est organisée de la façon suivante:

Une introduction prolongée en anglais est présentée dans le Chapitre 1.

En Chapitre 2 nous revoyons le théorème de Hill-Moylan [22] pour établir la propriété de cyclo-passivité des systèmes PH. De plus, nous proposons un paramétrage des sorties cyclo-passives pour ce type de systèmes. Ce paramétrage est donné en termes de paramètres de la plante et de deux mappages libres. Ensuite, la sortie cyclo-passive proposée est comparée à celles rapportées dans [7, 14, 36, 43, 41, 42, 58, 60] et il est montré que le nouveau paramétrage est adapté pour représenter chacune des sorties cyclo-passives précédemment rapportées dans la littérature. Enfin, nous présentons la génération de nouvelles sorties cyclo-passives en utilisant une représentation alternative du système PH comme cela est fait dans [42].

Dans le Chapitre 3 nous revenons sur les résultats en CbI et EB présentés en [7, 42, 60]. Dans ce chapitre, nous présentons les résultats en CbI et EB en utilisant la paramétrisation de la sortie cyclo-passive proposée dans le Chapitre 2. En plus, l'approche EB est dotée d'une interprétation physique et est liée à CbI, où il est montré que sous certaines circonstances, l'ensemble des PDEs à être résoudre pour CbI coïncide avec l'ensemble des PDEs à résolu pour EB. Ces PDEs sont résumées dans deux tableaux qui fournissent un cadre unifié pour les deux approches. À la fin du chapitre, nous présentons deux exemples pour illustrer l'application de CbI et EB pour la stabilisation d'un équilibre désiré.

En Chapitre 4 nous présentons une procédure constructive pour stabiliser les systèmes PH sans qu'il soit nécessaire de résoudre des PDEs. La méthodologie est, en principe, développée en considérant la sortie de façonnage de la puissance (y_{PS}) comme variable de port, ce résultat a été présenté dans [4]. Dans cette procédure le façonnage de l'énergie s'effectue via un retour d'état négatif. Par ailleurs, la méthodologie est comparée aux techniques d'assignation d'interconnexion et amortissement (IDA) et EB, en établissant une relation entre eux. Ensuite, nous explorons l'utilisation de la sortie passive plus générale pour la conception du PI ainsi que la relaxation de la condition d'intégrabilité. Enfin, nous présentons plusieurs exemples pour illustrer la méthode.

Le Chapitre 5 est un complément du Chapitre 4. Nous proposons deux alternatives pour la conception des régulateurs PID. Tout d'abord un régulateur PID de la sortie naturelle dont conditions pour avoir un terme dérivatif calculable sont étudiées. Le deuxième régulateur PID est construit en deux étapes: le retour négatif d'un terme intégral de la sortie passive de degré relatif zéro et après l'addition des termes proportionnel et dérivatif de la sortie naturelle du nouveau système PH. Enfin, nous présentons plusieurs exemples pour illustrer la méthode.

Le Chapitre 6 est consacré à l'application du régulateur PID présenté en [13]. Dans ce chapitre nous étudions le système du pendule inversé ultra flexible, lequel est représenté sous la forme d'un modèle contraint EL. La première étape de la conception du PID est de réduire le modèle contraint à un modèle purement différentiel. Par la suite, la conception du contrôle et la preuve de stabilité en boucle fermée sont effectuées comme dans [13], où l'ensemble des gains est choisi par l'analyse du système linéarisé en boucle fermée. Enfin, des simulations et des résultats expérimentaux sont présentés pour corroborér le résultat théorique.

Finalement, au Chapitre 7 nous présentons les conclusions et des travaux futurs dans la même ligne de recherche que les résultats exposés précédemment.

Chapter 1

Introduction

The control of physical systems has been studied and applied for several centuries for many purposes. Notably, in the last few decades we have been witnesses of a huge development in control theory, particularly, by the end of the 80s a complete theory for linear systems was developed, in [3] a brief history of the Automatic Control is presented. The tools of analysis and the techniques of control synthesis for general linear systems have been applied in many practical situations that fit this theory. Nevertheless, the different control tasks, *e.g.*, regulation or tracking of trajectories are far from be synthesized for nonlinear systems. As a matter of fact, new technological developments and the incursion in sciences as economics or biology, have brought the necessity of a more complex analysis and control theory. In spite of the extensive study of nonlinear systems, the existing techniques for its stabilization are available only for special classes of them. The material reported in this work is aimed at revisiting and trying to develop theory for a well-defined class of nonlinear systems to be controlled, which covers a broad spectrum of physical systems.

1.1 An Energy-Based Framework for Physical Systems

For physical systems, the concept of *energy* plays a fundamental role in science and engineering practice, establishing the *lingua franca* among different physical domains, *e.g.*, mechanical, electrical, thermal, hydraulic, etc. Therefore, the concept of energy is a cornerstone in the pursuit of a unified framework for appropriately modeling the real world phenomena. In this approach of modeling and analysis, the physical systems are regarded as the interconnection of three type of ideal elements:

- *Energy-storing elements.* All the components that store energy, *e.g.*, ideal inductor, ideal spring, ideal reservoir, etc. A system is dynamic if has storing elements.

- *Energy-dissipating elements.* All the components that dissipate energy, *e.g.*, ideal electrical resistor, ideal friction, ideal heat resistor, etc. This kind of components transform the energy in an irreversible way.

- *Energy-routing elements*. All the components that only redirect the power flow in the overall system, including ideal constrains, transformers and gyrators, *e.g.*, ideal electrical transformer, ideal turbine, ideal gear-box, etc. In contrast with the dissipative elements, the energy transformation in this case is reversible.

Whilst, the interaction between the system and its environment is represented by an external port, *e.g.*, sources, actuators, sensors, etc. An exhaustive discussion on this topic may be found in [14, 59, 61].

In this energy-based framework the dynamical systems are considered as energy-transforming devices. The latter has several advantages in the analysis and control of nonlinear systems, one of them is that the analysis of complex dynamical systems can be carried out via their decomposition into simpler sub-systems that, upon interconnection, add up their energies to determine the full original system's behavior. Another advantage is that the energy will provide fundamental information for the control design, and in some cases, a physical interpretation of the controller. All these features will be discussed with more detail in later sections of this Chapter.

1.2 Passivity and Energy-Shaping

Passivity is a physical property that has its origin in circuit theory where an electrical network is passive if all its elements are passive, that is, if its elements cannot deliver energy to the outside world, for further details of passivity in circuit theory we refer the reader to [10]. In 1972 the seminal work [62] provided a general theory for dissipative systems including the passivity concept as a particular case of dissipativeness, this theory was further developed in [21, 22, 63] and recently revisited in [32]. In control theory, a dynamical system –in its state-space representation– is called passive if the input flow of energy is greater than or equal to the difference between the initial and the final stored energies[1]. The latter is mathematically represented by the following inequality

$$H(x(t_1)) - H(x(t_0)) \leq \int_{t_0}^{t_1} y^\top(t)u(t)\mathrm{d}t \tag{1.1}$$

where x is the state of the system, u, y are the input and the output, respectively, and whose product has power units, H represents the stored energy and the time interval satisfies that $t_1 \geq t_0$.

It has been mentioned that the energy of a system determines its behavior, thence, intuitively the passivity property is related to the stability of the system. This deduction is in general true, in fact, taking into account some technical conditions and the definition of stability under consideration, a passive system is stable. A major result for passive systems is that the interconnection of two passive systems –via an adequate interconnection subsystem– yields into a passive system, in particular, the negative (and positive in some cases) feedback interconnection of two passive systems is again passive, for further details see

[1]In [32] these systems are called *internally passive* systems. It should be point out that there exist different definitions of passive systems depending on several factors, as for example, their representation. Nonetheless, an equivalence (or at least a connection) between the different definitions can be established. For further details see [62, 32]

chapters 3 and 4 of [58]. The latter result is instrumental for control design and analysis of passive systems. It is important to underscore that, while the passivity is in principle a physical property, in passivity theory is not required that the system's energy (storage) function corresponds to real physical energy, this enlarges the set of systems for which the passivity theory provides a methodic procedure of analysis and control design.

In the energy-based framework, it turns out that a natural way to control a physical system is to modify its energy function, this procedure is known as *energy-shaping*. In contrast with some classical techniques that try to impose a desired behavior trough nonlinearities cancellation and high gains controllers, the energy-shaping approach try to exploits the structure and the physical properties (when is the case) of the system. The aforementioned procedure has its roots in the pioneering works of Takegaki and Arimoto [55] and Jonckheere [25] published in 1981. The term Passivity-Based Control (PBC) was coined in [40] where it was observed that passivity theory provides a suitable mathematical framework to formalize the energy-shaping technique. The basic idea of PBC is to design a controller that accomplishes the control task, *e.g.*, stabilization, by rendering the system passive with respect to a desired energy function and injecting damping. Since the publication of [40] more than two decades ago, PBC has attracted the attention of many researchers and currently counts among the most popular controller design techniques. There are many variations of the basic PBC idea, and we refer the interested reader to [14, 38, 52, 58] for further details and a list of references.

The different PBC techniques can be classified in two main classes:

- The techniques that *a priori* select the structure of the energy functions to be assigned and then design the controller that renders the desired energy function non-increasing. Within this class we find the proportional integral derivative (PID) PBC and the energy-balancing (EB) controllers.
- The techniques where the closed-loop structure is chosen and then the family of possible energy functions is characterized by a partial differential equation (PDE). In this approach we find the interconnection and damping assignment (IDA) and the control by interconnection (CbI).

Despite of the practical and theoretical appealing that offers a methodical and, in some cases, intuitive procedure as PBC, there are different constraints and limitations that arise during the control design. This issues depend on the PBC technique and the control objective, *e.g.*, the called dissipation obstacle in CbI, the necessity of solutions for the PDEs in IDA, the imposition of a closed-loop structure that reduces the set of stabilizable plants (see [37] for an example), etc. Several works has developed alternatives to overcome this limitations and some of them will be discussed in the following chapters of this document.

1.3 Port-Hamiltonian Systems

The major part of this thesis work is devoted to the analysis and control of dynamical systems that admit a particular representation, called port-Hamiltonian (PH) model. The PH models are endowed with a special geometric structure where it is underscored the importance of the energy function, the interconnection pattern and the dissipation of the system, which are the essential ingredients of PBC. Henceforth, we refer a dynamical system that can be represented by a PH model as a PH system. As shown in [14, 58, 59], the PH systems encompass a very large class of physical nonlinear systems, furthermore, this approach is particularly useful for a systematic mathematical treatment of multi-physics systems, that is, systems containing subsystems from different physical domains. Moreover, the PH systems are not limited to physical systems, in fact, an advantage of this approach is its extension to physical system models with virtual system components, which may or not may mimick a physical dynamics. A clear example of the latter is a controller, programmed into a microprocessor, interconnected to a physical system.

There are many representations of PH systems, in this thesis work we are particularly interested in the called input-state-output form, where the state is assumed finite dimensional and the port variables are the input and output vectors. This representation of PH systems satisfies the called cyclo-passivity[2] inequality, given by

$$\dot{H} \leq y^\top u, \tag{1.2}$$

where H is the energy function and u, y are the input and output vector, respectively. It is clear that, integrating the equation above from t_0 to t_1 we obtain the inequality described in (1.1). Notice that, while u usually denotes the control input, y is not fixed in the inequality (1.2). Hence, all the possible signals y that satisfy (1.2) for an input u are called cyclo-passive outputs.

The present work is focused on the application of PBC to PH systems and, in Chapter 6, to Euler-Lagrange (EL) systems. Particularly, we are interested in constructive methods of control design that, exploiting the different cyclo-passive outputs, overcome the constraints of the PBC techniques enlarging the applicability of this control approach.

1.4 Thesis Overview and Contributions

The results presented in this thesis are mainly based on the works [7, 13, 42, 45, 60]. Therefore, the PBCs here formulated extend the methodologies reported therein. A brief introduction of this line of research as well as the main contributions of the thesis is described in the following.

In [42] a CbI is the interconnection, via a lossless subsystem, of the PH system to be stabilized with a dynamic controller, which is another PH system. Hence, a straightforward application of the Passivity Theorem [11] shows that the closed-loop system is still cyclo-passive and its storage function is described by the addition of the energy functions of the plant and the controller. To assign to the closed-loop storage function a desired shape it is necessary to relate the states of the plant and the controller, this is done via the generation of invariant sets, defined by the so-called Casimir functions. In [43] it has been shown that in its simplest formulation CbI is hampered by the so-called dissipation obstacle which, roughly speaking, means that the damped coordinates cannot be shaped. In [42], with the aim of extending the applicability of CbI, in

[2] The difference between passivity and cyclo-passivity will be briefly discussed in the next chapter.

particular to overcome the aforementioned dissipation obstacle, two extensions in the methodology are proposed: first the use of the so-called power shaping output, instead of the natural output, in the CbI design. Second, the use of a state-modulated interconnection scheme.

On the other hand, in [60] CbI is studied from a geometrical point of view. Where, starting from the concept of *Dirac structure*, a deep analysis of the geometrical properties of PH systems leads to the proposition of a more general cyclo-passive output. Furthermore, it is shown that the use of the more general output is suitable to overcome the dissipation obstacle. To shape the overall energy function it is, again, necessary the generation of Casimir functions to relate the states of the plant with the states of the controller.

Following the results reported in [42, 60], in the current thesis we propose an alternative parameterization of the general cyclo-passive output, which depends on parameters of the plant and two free mappings. An advantage of this parameterization is that it avoids the sign condition over the so-called generalized damping matrix imposed on the output proposed in [60]. Moreover, it is proved that the other cyclo-passive outputs reported in the literature can be expressed in terms of the before mentioned parameterization. Therefore, the CbI and EB-PBC approaches are studied and compared using this parameterized cyclo-passive output. Another contribution in this line of research is a unified framework, given in form of tables, where the PDEs to be solved, in these approaches, for all the cyclo-passive outputs are provided and compared.

In [13], a PID controller without solving PDEs has been proposed for a class of mechanical systems, which are represented as EL systems. In contrast with CbI, this PID controller is a static-feedback that does not preserves the original structure of the plant. Indeed, in this case the controller only pays attention to the energy shaping, ensuring that the overall energy function satisfies the cyclo-passive inequality and furthermore it qualifies as Lyapunov function to prove stability of a desired equilibrium point. This novel PID controller is constructed in two steps: first a partial feedback linearization (PFL) [54] applied that transforms the system into Spong's normal form—if this system is still EL, two new passive outputs are immediately identified. Second, a classical PID around a suitable combination of these passive outputs completes the design.

Following the results on EB-PBC presented in [7] and the PID controller reported in [13], we present a PI-like controller for a class of PH systems. The starting point in the design of this controller is the well-known power shaping output [36]. Thus, we shape the energy of the overall system adding a suitable function of the first integral of the before mentioned cyclo-passive output. The proposed design allow us to assign the equilibrium of the closed-loop system without solving PDEs. Moreover, it is shown that, under some conditions, the desired equilibrium is stable in the sense of Lyapunov. As an additional contribution, we extend the PI-like controller in two ways, first, we propose an input-output change that enlarges the set of stabilizable plants by this controller. Second, we construct the PI around the parameterization of the general cyclo-passive output. As a final result, in this line of research, we proposed two PID-PBC based on the cyclo-passive outputs of the PH system. Similar to the PI case, we state the conditions under which these controllers assign and stabilize the desired equilibrium of the overall system.

A final contribution of this thesis is the application of the PID-PBC reported in [13] to an ultra flexible inverted pendulum whose model is borrowed from

[45]. In this case, the PID-PBC cannot be directly applied due to a constraint present in the EL model. Therefore, a first step towards the stabilization is the elimination of this constraint. Once a suitable model is obtained, the application of the methodology is straightforward and the tuning gains of the PID are selected based on the poles of the linearized closed-loop system and the level curves of the potential energy of the overall system. Finally, simulations and experimental results are reported corroborating the theoretical ones.

1.5 Outline of the Thesis

The current thesis is organized as follows.

In Chapter 2, we revisit the Hill-Moylan's Theorem [22] to establish the cyclo-passivity property of PH systems. Moreover, we propose a parameterization of the cyclo-passive outputs for PH systems. This parameterization is given in terms of parameters of the plant and two free mappings. The proposed cyclo-passive output is compared with the ones reported in [7, 14, 36, 43, 41, 42, 58, 60] and it is shown that the new parameterization is suitable to represent each one of the cyclo-passive outputs previously reported. Finally, we present the generation of new cyclo-passive outputs using an alternative representation of the PH system as is done in [42].

In Chapter 3, we revisit the results on CbI and EB reported in [7, 42, 60]. In this chapter we present the results on CbI and EB using the parameterization of the cyclo-passive output proposed in Chapter 2. Additionally, the EB approach is endowed with a physical interpretation and it is related to the CbI, where it is shown that under some circumstances the set of PDEs to be solved for CbI coincides with the set of PDEs to be solved for EB. These PDEs are summarized in two tables which provide an unified framework for both approaches. At the end of the chapter we present two examples which illustrates the application of CbI and EB for stabilization of a desired equilibrium. The main results of this chapter has been partially reported in [33].

In Chapter 4, we present a constructive procedure to stabilize PH systems without the necessity of solving PDEs. This procedure has been reported in [4], where the energy shaping is carried out through the negative feedback of the first integral of the energy shaping output. We state the conditions to design the PI-like controller in such way that stabilizes the desired equilibrium in closed-loop. Moreover, the methodology is compared with IDA-PBC and EB-PBC establishing a relation between them. We also present two extensions of the results reported in [4]: first, the modification the external port of the PH system to relax the integrability condition present in the design of the PI-like controller. Second, the use of the parameterized cyclo-passive output proposed in Chapter 2, instead of the power shaping output, to design the PI-like stabilizer. The last part of the chapter consists in four examples which illustrates the applicability of the technique, its limitations and the enlargement of the set of stabilizable PH systems via the aforementioned extensions.

Chapter 5 is a complement of Chapter 4. In this chapter we propose two alternatives to design a PID-like controller based on the results reported in the previous chapter. The first controller is based on the natural output, hence the necessary conditions to have a *computable* derivative term on the natural output are studied. In this case, a constraint similar to the *dissipation obstacle* in CbI arises. The second PID-like controller is constructed in two steps: first, based on the PI-like controller of Chapter 4 we propose an integral-like term on the power shaping output. Then, we add the proportional and derivative terms on the natural output of the new PH system. In the last part of the chapter we present two examples which illustrates the applicability of both PID controllers and how they enlarge the class of stabilizable plants with respect to the controller reported in the previous chapter.

In contrast with the previous chapters, in Chapter 6 the representation of the system under study is the EL one. This chapter is entirely devoted to the application of the PID controller reported in [13] to the ultra flexible inverted pendulum whose constrained EL model has been reported in [45]. Due to the complex nature of the energy function of the system, PBC techniques which rely on the solution of PDEs are not suitable for control design purposes. Furthermore, an extra step is needed to apply the PID controller proposed in [13], that is, the projection of the system on the manifold defined by the constraint. Once the system is represented in a suitable, form the PID controller is designed as in the aforementioned reference. Thus, the set of gains are selected via the analysis of the linearized closed-loop system and corroborated through simulations. In the last section of the chapter some experimental results are reported. This chapter has been reported in [17].

Finally, to wrap up the current thesis, in Chapter 7 we present some concluding remarks and future work in the same line of research as the results exposed in the previous chapters.

Chapter 2

Passive outputs of PH systems

The objective of this chapter is to provide a theoretical framework for PH systems and their passive properties. Towards this end, we start recalling the Hill-Moylan's Theorem and the so-called input-state-output representation of PH systems. The main result of this chapter is the fully characterization of the passive outputs for the aforementioned class of systems, considering the Hamiltonian as storage function. Moreover, it is shown that the passive outputs reported in the literature—see [7, 14, 36, 43, 41, 42, 58, 60]—can be described by this parameterization. Additionally, we present the parameterization of the new passive outputs generated by alternative storage functions. The passive outputs presented in this chapter will be instrumental to establish the results of the following chapters.

2.1 Preliminaries

Consider a general nonlinear system of the form

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) + j(x)u \end{array} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output, with $n \geq m$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is the input matrix, which is full rank[1], $j : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$.

2.1.1 Passive systems

Definition 2.1.1 *System* (2.1) *is said to be cyclo-passive if there exist a differentiable function $H : \mathbb{R}^n \to \mathbb{R}$ (called storage function) that satisfies the power balance inequality*

$$\dot{H} \leq y^\top u \tag{2.2}$$

when evaluated along the trajectories of (2.1). *Additionally, if H is bounded from below, then* (2.1) *is a passive system.*

[1]In the sequel we will consider the under-actuated case, where $n > m$ and rank $\{g\} = m$.

Given the definition above, it is clear that every passive system is cyclo-passive, nonetheless, the converse is not true. In terms of energy, the difference between passive systems and cyclo-passive systems is elucidated as follows: while a passive system is not allowed to generate energy for any trajectory, for cyclo-passive systems, this behavior must hold only for closed trajectories, in other words, the cyclo-passivity inequality (2.2) is required to be satisfied only for the subset of inputs that return the state to its initial value. A physical example of a cyclo-passive, but not passive, system is a circuit built from typical resistors and negative capacitors or inductors, which can store negative energy. For further details on the difference between cyclo-passive systems and passive systems we refer the reader to [32, 58].

The Hill-Moylan's Theorem [22], establishes the conditions that a nonlinear system, represented by (2.1), must satisfy to be called (cyclo-)passive. For the sake of completeness and clarity we recall the aforementioned theorem below.

Theorem 2.1.1 *The system* (2.1) *is cyclo-passive with storage function* $H(x)$ *if and only if, for some* $q \in \mathbb{N}$, *there exist functions* $l : \mathbb{R}^n \to \mathbb{R}^q$ *and* $w : \mathbb{R}^n \to \mathbb{R}^{q \times m}$ *such that*

$$\begin{array}{rcl} -|l(x)|^2 & = & (\nabla H(x))^\top f(x) \\ h(x) & = & g^T(x)\nabla H(x) + 2w^\top(x)l(x) \\ |w(x)|^2 & = & \mathtt{sym}\{j(x)\} \end{array} \tag{2.3}$$

Proof: This proof is a particular case of the one presented for Theorem 9 in Chapter 5 of [32].

To establish the *necessity*, we, first, replace (2.1) in the inequality (2.2) yielding[2]

$$(\nabla H)^\top (f + gu) \leq (h + ju)^\top u.$$

The latter inequality is equivalent to

$$\begin{array}{rcl} 0 & \leq & -(\nabla H)^\top f + (h - g^\top \nabla H)^\top u + u^\top j^\top u \\ & = & \begin{bmatrix} 1 & u^\top \end{bmatrix} \begin{bmatrix} -(\nabla H)^\top f & \frac{1}{2}(h - g^\top \nabla H)^\top \\ \frac{1}{2}(h - g^\top \nabla H) & j \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix}. \end{array} \tag{2.4}$$

Notice that the inequality above must hold for all x and u. Furthermore, its right-hand term has only linear and quadratic functions in u, hence without loss of generality, it can be expressed as a quadratic form in the variable u. That is,

$$|l(x) + w(x)u|^2 = l^\top(x)l(x) + 2w^\top(x)l(x) + w^\top(x)w(x) \tag{2.5}$$

where the functions $l(x)$ and $w(x)$ are non-unique. Now, equating the coefficients of u in the right-hand terms of (2.4) and (2.5) we obtain the conditions given in (2.3). For *sufficiency*, we simply replace the equations (2.3) into the time derivative of H to get

$$\begin{array}{rcl} \dot{H} & = & (\nabla H)^\top (f + gu) \\ & = & -l^\top l + h^\top u - 2l^\top wu \\ & = & -l^\top l - 2l^\top wu - u^\top j^\top u + y^\top u \\ & = & -|l(x) + w(x)u|^2 + y^\top u \leq y^\top u. \end{array}$$

□

[2]To simplify notation, throughout the remaining of some proofs the argument x is omitted from all mappings.

2.2 Passivity of PH systems

We now focus on PH systems whose dynamics is described by the standard input-state-output representation[3] [14, 58]

$$\begin{array}{rcl} \dot{x} & = & (\mathcal{J}(x) - \mathcal{R}(x))\, \nabla H(x) + g(x)u \\ y_{(\cdot)} & = & h(x) + j(x)u, \end{array} \tag{2.6}$$

where $H(x)$ is the energy (storage) function of the system, namely the Hamiltonian, which we assume is bounded from below [4], $\mathcal{J}, \mathcal{R} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with $\mathcal{J}(x) = -\mathcal{J}^\top(x)$ and $\mathcal{R}(x) = \mathcal{R}^\top(x) \geq 0$, are the interconnection and damping matrices, respectively. To simplify the notation in the sequel we define the matrix $F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$,

$$F(x) := \mathcal{J}(x) - \mathcal{R}(x).$$

To streamline the characterization of *all* passive outputs for the PH system (2.6), we introduce a (non-unique) factorization of the dissipation matrix of the form

$$\mathcal{R}(x) = \phi^\top(x)\phi(x), \tag{2.7}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^{q \times n}$, with $q \in \mathbb{N}$ satisfying $q \geq \operatorname{rank}\{\mathcal{R}(x)\}$. We recall the basic linear algebra fact that $\mathcal{R}(x) \geq 0$ if and only if such a factor exists [23].

Proposition 2.2.1 *Consider the PH system* (2.6). *The following statements are equivalent.*

(S1) The mapping $u \mapsto y$ is passive with storage function $H(x)$.

(S2) For any factorization of the dissipation matrix $\mathcal{R}(x)$ of the form (2.7) *the mappings $h(x)$ and $j(x)$ can be expressed as*

$$\begin{array}{rcl} h(x) & = & \left(g(x) + 2\phi^\top(x)w(x)\right)^\top \nabla H(x) \\ j(x) & = & w^\top(x)w(x) + D(x), \end{array} \tag{2.8}$$

for some mappings $w : \mathbb{R}^n \to \mathbb{R}^{q \times m}$ and $D : \mathbb{R}^n \to \mathbb{R}^{m \times m}$, with $D(x)$ skew-symmetric.

Proof: From Theorem 2.1.1, see also [58, 32], it follows that the system (2.6) is passive if and only if

$$\left[\begin{array}{cc} 2(\nabla H)^\top \mathcal{R} \nabla H & (g^\top \nabla H - h)^\top \\ g^\top \nabla H - h & -(j + j^\top) \end{array} \right] \leq 0.$$

To prove that (S2) implies (S1) replace (2.7) and the definitions of $h(x)$ and $j(x)$, given in (2.8), to get

$$2 \left[\begin{array}{cc} -|\phi \nabla H|^2 & -(\nabla H)^\top \phi^\top w \\ -w^\top \phi \nabla H & -|w|^2 \end{array} \right] \leq 0,$$

which is always satisfied.

[3]In the PH framework we use the symbol $y_{(\cdot)}$ to distinguish the general passive output from the so-called natural output which is often denoted as y.

[4]This assumption is made to simplify the presentation, since in this case we deal with passivity of the PH system instead of cyclo-passivity.

The proof that (S1) implies (S2) proceeds as follows [52]. Assume $u \mapsto y$ is passive with storage function $H(x)$ and define the mapping $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$

$$d(x,u) := -\dot{H} + u^\top (h(x) + j(x)u) \geq 0. \tag{2.9}$$

Evaluating $\dot{H}$ along the trajectories of (2.6) and using (2.7) we get

$$d = (\nabla H)^\top \phi^\top \phi \nabla H + u^\top (h - g^\top \nabla H) + \frac{1}{2} u^\top (j + j^\top) u.$$

Because $d(x,u)$ is quadratic in u and nonnegative for all u and x, there exist a (non-unique) matrix valued function $w(x)$ such that

$$d = (\nabla H)^\top \phi^\top \phi \nabla H + 2 w^\top \phi \nabla H + u^\top w^\top w u.$$

The proof that $h(x)$ and $j(x)$ take the form (2.8) is established equating the terms in u and invoking the skew-symmetry of $D(x)$.

□

2.3 Particular cases of passive outputs

In this subsection we prove that all passive outputs of the PH system (2.6) reported in the literature are particular cases, or alternative representations, of the output (2.8).

Proposition 2.3.1 *Consider the output $y_{(\cdot)}$, given in the second equation of* (2.6)*, and its parameterization* (2.8)*. The following implications hold true.*

- *Natural output [43, 58]:*

$$\left.\begin{array}{l} w(x) = 0 \\ D(x) = 0 \end{array}\right\} \Rightarrow \; y_{(\cdot)} = y := g^\top(x) \nabla H(x). \tag{2.10}$$

- *Power-shaping output of [36] with $F(x)$ full rank:*

$$\begin{array}{c} \left.\begin{array}{rcl} w(x) &=& \phi(x) F^{-1}(x) g(x) \\ D(x) &=& -g^\top(x) F^{-\top}(x) \mathcal{J}(x) F^{-1}(x) g(x) \end{array}\right\} \\ \Rightarrow y_{(\cdot)} = y_{\mathtt{PS}} = -g^\top(x) F^{-\top}(x) \dot{x}. \end{array} \tag{2.11}$$

- *The alternative output of [14, 41, 60] with generalized damping matrix verifying*

$$\mathcal{Z}(x) := \begin{bmatrix} \mathcal{R}(x) & T(x) \\ T^\top(x) & S(x) \end{bmatrix} \geq 0 : \tag{2.12}$$

$$\begin{array}{c} \left.\begin{array}{rcl} S(x) &=& w^\top(x) w(x) \\ T(x) &=& \phi^\top(x) w(x) \end{array}\right\} \\ \Rightarrow y_{(\cdot)} = y_{\mathtt{VV}} := (g(x) + 2T(x))^\top \nabla H(x) + (S(x) + D(x)) u. \end{array} \tag{2.13}$$

- *Power-shaping output of [36] with $F(x)$ not full rank but verifying*

$$\begin{aligned} F^\top(x)(F^\dagger(x))^\top(x)F(x) &= F(x) && (2.14)\\ span\{g(x)\} &\subseteq span\{F(x)\}: && (2.15)\end{aligned}$$

$$\left.\begin{aligned} w(x) = \phi(x)F^\dagger(x)g(x)\\ D(x) = -g^\top(x)(F^\dagger(x))^\top(x)\mathcal{J}(x)F^\dagger(x)g(x)\end{aligned}\right\}$$
$$\Rightarrow y_{(\cdot)} = y_{\mathtt{PS}} = -g^\top(x)(F^\dagger(x))^\top\dot{x}. \tag{2.16}$$

Proof:
The proofs of (2.10) and (2.13) follow via direct replacement of the definitions of $w(x)$ and $D(x)$ in (2.8). For the latter notice that the generalized damping matrix takes the form

$$\mathcal{Z}(x) = \begin{bmatrix} \phi^\top(x)\phi(x) & \phi^\top(x)w(x)\\ w^\top(x)\phi(x) & w^\top(x)w(x)\end{bmatrix}$$

which clearly satisfies the condition (2.12). Furthermore, since $q \geq \text{rank }\{\mathcal{R}(x)\}$ and $w(x)$ is free, taking the integer q large enough it is possible to construct any matrix $T(x)$ such that

$$\text{rank }\{T(x)\} \leq \max\{\text{rank }\{\mathcal{R}(x)\}, \text{rank }\{S(x)\}\}.$$

To prove (2.11) replace the definitions of $w(x)$ and $D(x)$ in (2.8) to get

$$\begin{aligned} y_{(\cdot)} &= (g + 2\phi^\top\phi F^{-1}g)^\top\nabla H + g^\top F^{-\top}(\phi^\top\phi - \mathcal{J})F^{-1}gu\\ &= (g + 2\mathcal{R}F^{-1}g)^\top\nabla H + g^\top F^{-\top}(\mathcal{R} - \mathcal{J})F^{-1}gu\\ &= ((I + 2\mathcal{R}F^{-1})g)^\top\nabla H - g^\top F^{-\top}FF^{-1}gu\\ &= ((F + 2\mathcal{R})F^{-1}g)^\top\nabla H - g^\top F^{-\top}gu\\ &= ((\mathcal{J} + \mathcal{R})F^{-1}g)^\top\nabla H - g^\top F^{-\top}gu\\ &= -(F^\top F^{-1}g)^\top\nabla H - g^\top F^{-\top}gu\\ &= -g^\top F^{-\top}F\nabla H - g^\top F^{-\top}gu\\ &= -g^\top F^{-\top}(F\nabla H + gu)\\ &= -g^\top F^{-\top}\dot{x}.\end{aligned}$$

Finally, we proceed to establish (2.16). Towards this end we recall Lemmata A.2 and A.3 given in Appendix A—see also [7, 48]—that state that (2.14) is equivalent to the existence of a mapping $Z : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ solution of the equation

$$F^\top ZF = -F, \tag{2.17}$$

and that (2.15) and (2.17) imply that

$$F^\top Zg = -g, \tag{2.18}$$

respectively. Now, defining Z as

$$Z := -(F^\dagger)^\top FF^\dagger \tag{2.19}$$

and with the choice of w given in (2.16) it is easy to verify that

$$\begin{aligned}
(g+2\phi^\top w)^\top &= (g+2\phi^\top \phi F^\dagger g)^\top \\
&= (-F^\top Z g + 2\phi^\top \phi F^\dagger g)^\top \\
&= (F^\top (F^\dagger)^\top F F^\dagger g + 2\mathcal{R} F^\dagger g)^\top \\
&= (F F^\dagger g + 2\mathcal{R} F^\dagger g)^\top \\
&= ((F+2\mathcal{R}) F^\dagger g)^\top \\
&= ((\mathcal{J}+\mathcal{R}) F^\dagger g)^\top \\
&= (-F^\top F^\dagger g)^\top \\
&= -g^\top (F^\dagger)^\top F. \qquad (2.20)
\end{aligned}$$

where we have used (2.18) in the second equation, (2.19) in the third equation and (2.14) in the fourth and ninth equation. Then, using the w and D given in (2.16) we get

$$\begin{aligned}
w^\top w + D &= g^\top (F^\dagger)^\top \phi^\top \phi F^\dagger g - g^\top (F^\dagger)^\top \mathcal{J} F^\dagger g \\
&= g^\top (F^\dagger)^\top \mathcal{R} F^\dagger g - g^\top (F^\dagger)^\top \mathcal{J} F^\dagger g \\
&= -g^\top (F^\dagger)^\top (\mathcal{J}-\mathcal{R}) F^\dagger g \\
&= -g^\top (F^\dagger)^\top F F^\dagger g \\
&= g^\top (F^\dagger)^\top F^\top Z g \\
&= -g^\top (F^\dagger)^\top g \qquad (2.21)
\end{aligned}$$

where we have used (2.14) in the fifth equation, (2.18) and (2.19) in the last equation. Replacing (2.20) and (2.21) in (2.8) and (2.6) we obtain

$$y_{(\cdot)} = -g^\top(x)(F^\dagger(x))^\top \dot{x},$$

which completes the proof.

□

2.4 Generating new passive outputs

It turns out that a PH system has an infinite number of representations, and consequently of storage functions $H_s(x)$ such that $\dot{H}_s \leq 0$. Indeed, every pair $(F_s(x), H_s(x))$ that satisfies

$$F_s(x)\nabla H_s(x) = F(x)\nabla H(x), \qquad (2.22)$$

with $F_s(x)$ verifying $\mathtt{sym}\{F_s(x)\} \leq 0$, is an alternative representation of the system (2.6). Thus, the first equation of (2.6) can be expressed as

$$\dot{x} = F_s(x)\nabla H_s(x) + g(x)u. \qquad (2.23)$$

Although the dynamics (2.6) and (2.23) are identical, the passive outputs for the latter are not the same as the ones defined as $y_{(\cdot)}$ in (2.6). The new passive outputs $y_{(\cdot)s}$, correspondent to the storage functions $H_s(x)$, are characterized in the proposition below.

Proposition 2.4.1 *Let*

$$\begin{array}{rcl} \mathcal{J}_s(x) & = & \mathtt{skew}\{F_s(x)\} \\ \mathcal{R}_s(x) & = & -\mathtt{sym}\{F_s(x)\} = \phi_s(x)\phi_s^\top(x), \end{array} \tag{2.24}$$

where $\phi_s : \mathbb{R}^n \to \mathbb{R}^{q\times n}$*, with* $q \in \mathbb{N}$ *satisfying* $q \geq$ *rank* $\{\mathcal{R}_s(x)\}$*. Consider the system* (2.23)*, then the inequality*

$$\dot{H}_s \leq y_{(\cdot)s}^\top u \tag{2.25}$$

holds for any function $y_{(\cdot)s}$ *that can be expressed as*

$$y_{\mathtt{wD}s} = \left(g(x) + 2\phi_s^\top(x)w_s(x)\right)^\top \nabla H_s(x) + (w_s^\top(x)w_s(x) + D_s(x))u, \tag{2.26}$$

for some mappings $w_s : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ *and* $D_s : \mathbb{R}^n \to \mathbb{R}^{m\times m}$*, with* $D_s(x)$ *skew-symmetric.*

Proof: The proof follows from Proposition 2.2.1.

□

The following proposition—see also [42]—provides a constructive procedure to identify alternative storage functions.

Proposition 2.4.2 *For all full rank matrices* F_s *solution of the PDE*

$$\nabla(F_s^{-1}F\nabla H) = \left(\nabla(F_s^{-1}F\nabla H)\right)^\top, \tag{2.27}$$

with F_s *verifying* (2.24)*, there exists a storage function* H_s *such that* (2.22) *holds.*

Proof: The Poincare's Lemma[5] states that (2.27) is necessary and sufficient for the existence of H_s such that

$$\nabla H_s = F_s^{-1}F\nabla H \tag{2.28}$$

which is equivalent to (2.22).

□

In order to make a distinction between all possible passive outputs with respect to H, in the sequel we will denote the general output given in (2.6) and its parameterization (2.8) as $y_{\mathtt{wD}}$. Additionally, for passive outputs with respect to H_s we add a subindex s to the respective output, that is, $y_{\mathtt{wD}s}$ denotes the output described in (2.26), the power shaping output is given by

$$\begin{array}{rcl} y_{\mathtt{PS}s} & := & -g^\top(x)(F_s^\dagger(x))^\top \dot{x} \;\; \text{or} \\ & = & -g^\top(x)F_s^{-\top}(x)\dot{x}, \end{array} \tag{2.29}$$

and the natural output is defined as

$$y_s := g^\top(x)\nabla H_s(x). \tag{2.30}$$

[5] See Lemma A.1 in Appendix A.

Chapter 3

CbI of PH systems

We consider in this chapter the PBC techniques of CbI and EB-PBC. These techniques have been previously studied in [9, 43, 42, 35] to which we refer the reader for additional information. In particular, in [42] the relationships between CbI and EB-PBC of PH systems are thoroughly explored. In that paper a CbI is the interconnection through a loss-less subsystem of the PH system to be controlled and a *dynamic* controller, which is another PH system with its own state variables and energy function. Since passivity is invariant with respect to lossless interconnections the overall system is still passive with new energy function the sum of the energy functions of the plant and the controller.[1] On the other hand, EB-PBC in [42] is viewed as a particular *static* state-feedback $\hat{u}_c(x)$ that satisfies the power balance $\dot{H}_a = -\hat{u}_c^\top y$, for some function $H_a(x)$. Clearly, setting $u = \hat{u}_c(x) + v$, with v an external signal, yields $\dot{H} + \dot{H}_a \leq v^\top y$, ensuring passivity of the closed-loop system with new energy function $H(x) + H_a(x)$. The fact that the closed-loop energy function is the sum of the systems and controller energies motivates the qualifier "energy-balancing".

In this chapter we propose to view EB-PBC as a particular instance of CbI where the controller is a *regulated source* and the interconnection is the standard negative feedback. This simple notational modification permits to put in a unified framework both controller design techniques. At a more fundamental level, viewing EB-PBC as interconnected subsystems is consistent with the behavioral framework [46], which rightfully claims that the classical input-to-output assignment perspective is unsuitable to deal, at an appropriately general level, with the basic tenets of systems theory.

The main objective of this chapter is to explore the advantages of using the different passive outputs—reported in the previous chapter—in CbI and EB PBC. Unfortunately, we prove that the use of this general passive output *does not enlarge*, with respect to the existing results, the class of systems for which CbI is applicable.

Another contribution of the current chapter is that the derivation of the PDEs to be solved is rather straightforward and should be contrasted with the more complicated one reported in [42]—where the PDEs are obtained via the selection of the desired dissipation.

[1] To assign to the overall energy function a desired shape, it is necessary to "relate" the states of the plant and the controller via the generation of invariant sets—a key step that is discussed later in the paper.

3.1 Standard CbI

Consider the PH system Σ, given by (2.6) with $y_{(\cdot)} = y$. The starting point for CbI is the power-balance equation

$$\dot{H} = -|\nabla H|^2_{\mathcal{R}} + u^\top y. \tag{3.1}$$

Using the fact that $\mathcal{R} \geq 0$ we obtain the bound

$$\dot{H} \leq u^\top y, \tag{3.2}$$

that we refer as cyclo-passivity inequality.

We consider the simplest PH controller consisting of m integrators

$$\Sigma_c : \begin{cases} \dot{x}_c &= u_c \\ y_c &= \nabla H_c(x_c), \end{cases} \tag{3.3}$$

where $x_c, u_c, y_c \in \mathbb{R}^m$ and the controller Hamiltonian $H_c : \mathbb{R}^m \to \mathbb{R}$ is to be defined. Clearly, the controller is lossless, that is, it satisfies

$$\dot{H}_c = 0. \tag{3.4}$$

The regulator, Σ_c, and the plant Σ, are coupled via the standard interconnection subsystem, corresponding to the usual negative feedback interconnection, defined as

$$\Sigma_I : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right. . \tag{3.5}$$

The Figure 3.1 represents the simplest formulation of the CbI approach.

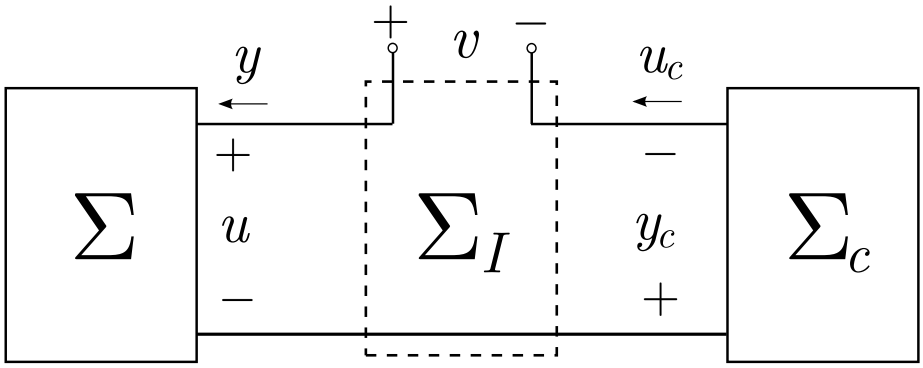

Figure 3.1: Standard CbI scheme with negative feedback interconnection (3.5).

Adding up (3.2) and (3.4), and using (3.5), we get

$$\dot{H} + \dot{H}_c \leq v^\top y. \tag{3.6}$$

Hence, the interconnected system is also passive with port variables (v, y) and new energy function the sum of the energy functions of the plant and the controller. The objective of CbI is to assign to the overall energy function a desired shape—in the simplest case of equilibrium stabilization to assign a minimum at the desired point. Hence, it is necessary to "relate" the states of the plant and the controller looking for conserved quantities (dynamical invariants) of the overall system. If such quantities can be found we can generate Lyapunov function candidates combining the conserved quantities and the energy function. To mathematically formalize the discussion above we need the following definitions.

Definition 3.1.1 *The set of assignable equilibria of PH systems, described in* (2.6)*, is given by*

$$\mathcal{E} := \left\{ x \in \mathbb{R}^n \mid g^{\perp}(x)F(x)\nabla H(x) = 0 \right\}. \tag{3.7}$$

Definition 3.1.2 *Consider the PH system* Σ *given in* (2.6)*, with* $y_{(\cdot)} = y$*, interconnected with the PH controller* Σ_c *defined in (3.3) via the standard power–preserving interconnection* (3.5)*. The function* $\mathcal{C}(x) - x_c$*, where* $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$*, is a Casimir function of the interconnected system if and only if* $\dot{\mathcal{C}} - \dot{x}_c = 0$ for all H *and* H_c*. That is, if and only if,* $\mathcal{C}$ *is a solution of the PDEs*

$$\left[\begin{array}{c|c} (\nabla\mathcal{C})^\top & -I_m \end{array}\right] \left[\begin{array}{cc} F & -g \\ g^\top & 0 \end{array}\right] = 0. \tag{3.8}$$

The following proposition, whose proof follows immediately from (3.6) and Definition 3.1.2, provides the standard formulation of CbI for equilibrium stabilization.[2]

Proposition 3.1.1 *Consider the system* (2.6)*, with* $y_{(\cdot)} = y$*, interconnected with the controller* (3.3) *via* (3.5)*. Assume,* $\mathcal{C}(x) - x_c$ *is a Casimir function of the interconnected system. Then, for all* $\Phi : \mathbb{R}^m \to \mathbb{R}$*, the function*

$$W(x, x_c) := H(x) + H_c(x_c) + \Phi(\mathcal{C}(x) - x_c) \tag{3.9}$$

satisfies

$$\dot{W} \leq v^\top y.$$

Moreover, if $(x_*, x_{c_*}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$ *is an equilibrium of the interconnected system with* $v = 0$ *and*

$$(x_*, x_{c_*}) = \arg\min W(x, x_c),$$

and it is isolated, then (x_*, x_{c_*}) *is stable in the sense of Lyapunov with Lyapunov function* W*.*

3.1.1 Extending the applicability of CbI

In [42], see also [7], it is shown that a necessary condition for the solvability of the PDEs (3.8) is

$$\mathcal{R}\nabla\mathcal{C} = 0, \tag{3.10}$$

which implies that

$$\mathcal{R}\nabla_x\Phi = 0$$

whose consequence is that the coordinates where dissipation is present cannot be "shaped", this limitation is known as the *dissipation obstacle* [43].[3] It is

[2]See [14, 42] for further details.

[3]See [30] for some new insight into the implications of the dissipation obstacle in CbI.

also possible to express this phenomenon in terms of the energy provided to the plant by the controller system, as states the proposition below.

Proposition 3.1.2 *Let $x_* \in \mathcal{E}$ be the equilibrium of the PH system* (2.6) *to be stabilized via CbI, and u, y the corresponding input and output. If the PDE* (3.8) *admits a solution then $u_*^\top y_* = 0$.*

Proof: From (3.10), we have that $F^\top \nabla \mathcal{C} = g$ is equivalent to

$$F\nabla\mathcal{C} = -g. \tag{3.11}$$

On the other hand, at the equilibrium point $\dot{x} = 0$, then we have the following chain of equalities

$$\begin{aligned}
F_*(\nabla H)_* + g_* u_* &= 0\\
F_*(\nabla H)_* - F_*(\nabla\mathcal{C})_* u_* &= 0\\
F_*((\nabla H)_* - (\nabla\mathcal{C})_* u_*) &= 0\\
\Rightarrow ((\nabla H)_* - (\nabla\mathcal{C})_* u_*)^\top F_*((\nabla H)_* - (\nabla\mathcal{C})_* u_*) &= 0\\
\Leftrightarrow ((\nabla H)_* - (\nabla\mathcal{C})_* u_*)^\top \mathcal{R}_*((\nabla H)_* - (\nabla\mathcal{C})_* u_*) &= 0.
\end{aligned}$$

where we used (3.11) to obtain the second equality. Replacing the condition $\mathcal{R}_*(\nabla\mathcal{C})_* = 0$ in the latter expression we get

$$(\nabla H)_*^\top \mathcal{R}_*(\nabla H)_* = 0. \tag{3.12}$$

Whilst the power balance equation at the equilibrium is given by

$$-(\nabla H)_*^\top \mathcal{R}_*(\nabla H)_* + u_*^\top y_* = 0$$

replacing (3.12) in the equation above

$$u_*^\top y_* = 0.$$

□

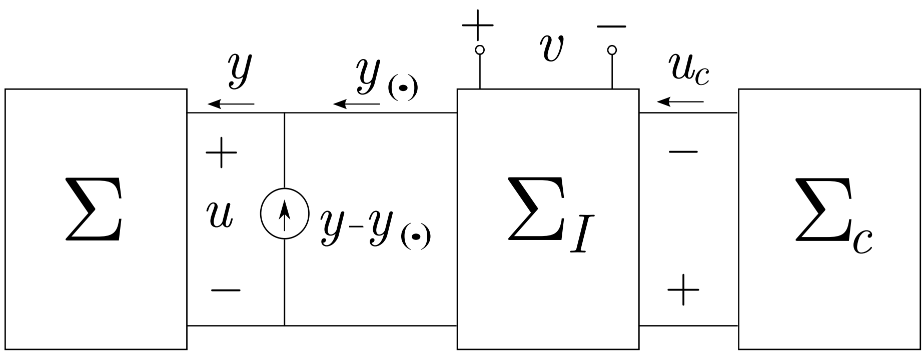

Figure 3.2: Electrical circuit analog of the CbI scheme with external port variables $(v, y_{(\cdot)})$.

To extend the domain of applicability of CbI—in particular, to overcome the dissipation obstacle—it is necessary to modify the port variable y. Therefore, in the following section we formulate the CbI approach considering the different passive outputs defined in the previous chapter. The modification of the port variable is schematically represented by the addition of a current source as shown in Figure 3.2.

To further extend the realm of application of CbI, in Section 5 of [42] the simple negative feedback Σ_I is replaced by a *state-modulated* power-preserving interconnection[4] of the form

$$\Sigma_I^{\mathtt{SM}} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha^\top(x) & 0 \end{bmatrix} \begin{bmatrix} y_{(\cdot)} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}, \right. \tag{3.13}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is chosen by the designer. The aforementioned reference presents the results of CbI using (3.13) and the passive outputs y, $y_{\mathtt{PS}}$. In this chapter we will only revisit these results for comparison purposes.

3.2 EB-PBC as a CbI with regulated sources

In Section 3.1 we assumed that the controller is another dynamical system. Actually, it is possible to use the framework of CbI when the controller is a *static, state-regulated* source and Σ_I is given by (3.5), as shown in Figure 3.3.[5] Even though the source does not contain energy storing elements it is clear that it injects energy into the system. The key point here is to make this energy a *function of the state of the plant.* Indeed, replacing in

$$\dot{H} \leq u^\top y_{(\cdot)}$$

the control law

$$u = \hat{u}_c(x) + v, \tag{3.14}$$

with $\hat{u}_c : \mathbb{R}^n \to \mathbb{R}^m$ to be defined, we get

$$\dot{H} \leq \hat{u}_c^\top y_{(\cdot)} + v^\top y_{(\cdot)}.$$

Therefore, if the power balance equation

$$\dot{H}_a = -\hat{u}_c^\top y_{(\cdot)} \tag{3.15}$$

holds for some energy function $H_a : \mathbb{R}^n \to \mathbb{R}$—that depends on the state of the plant x—we get

$$\dot{H} + \dot{H}_a \leq v^\top y_{(\cdot)}. \tag{3.16}$$

Hence, the interconnected system is passive with new storage function

$$H_d(x) := H(x) + H_a(x).$$

Moreover, if $x_* \in \mathbb{R}^n$ is an equilibrium of the interconnected system with $v = 0$ and

$$x_* = \arg\min\{H_d(x)\},$$

[4]See [58] for further details about this interconnection.

[5]The choice of a *voltage* source is done for clarity of presentation. All further developments can be carried out selecting a *current* source instead. This underscores the fact that in CbI there is no need to *a priori* impose an input–output causality relation.

and is isolated, it is stable with Lyapunov function H_d.

Following the terminology used in the literature, *e.g.,* [7, 42, 43, 44], we refer to this version of CbI as EB-PBC. We use the symbol H_a, instead of H_c to underscore that H_a is an "added" energy, function of the plant state x, while the latter is a *bona fide* energy function depending on the state x_c of the energy storing elements of the controller.

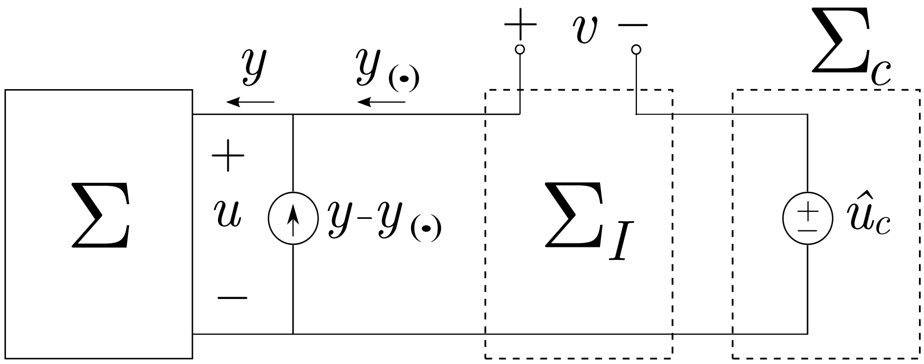

Figure 3.3: Electrical circuit analog of the EB-PBC scheme with $u = \hat{u}_c(x) + v$ and external port variables $(v, y_{(\cdot)})$.

In contrast with the Standard CbI of Section 3.1, in EB-PBC it is not necessary to look for conserved quantities. Indeed, in this case, the added energy is already a function of the plant states and the overall energy can be shaped with a suitable selection of H_a. It is clear that (3.15) defines a PDE in the unknown function H_a—parameterized by the free function $\hat{u}_c$. It is remarkable that in spite of their fundamental difference the PDEs that need to be solved coincide, under certain circumstances, in both approaches—as shown in [42] and corroborated below.

3.3 CbI with the different passive outputs

As discussed in Section 3.1, see Proposition 3.1.1, the stability analysis in CbI proceeds by adding to the systems total energy $H + H_c$ a *cross–term* of the coordinates of the plant and the controller. This cross–term is an arbitrary function, *i.e.*, Φ, of the Casimir functions, which are defined solving some PDEs. The PDEs that must be solved for CbI with y are given in (3.8). In this section we identify the PDEs that need to be solved for the application of CbI with the passive outputs $y_{\mathtt{PS}}$ and $y_{\mathtt{wD}}$.

3.3.1 CbI with the power shaping output

Although the PDEs for CbI with $y_{\mathtt{PS}}$ and $y_{\mathtt{PS}s}$ are given in Proposition 6 of [42] the simple propositions below give a clearer characterization of them.

Proposition 3.3.1 *Consider the PH system Σ, with $y_{(\cdot)} = y_{\mathtt{PS}}$, interconnected with the PH controller Σ_c defined in (3.3) via the standard power–preserving* *interconnection* (3.5). *Assume F is full rank*[6] *and verifies*

$$\nabla(F^{-1}g) = \left(\nabla(F^{-1}g)\right)^{\top}. \tag{3.17}$$

(i) *The solutions $\mathcal{C}$ of the PDE*

$$\nabla\mathcal{C} = -F^{-1}g, \tag{3.18}$$

define Casimir functions for the interconnected system.

(ii) *For all $\Phi : \mathbb{R}^m \to \mathbb{R}$, the function* (3.9) *satisfies*

$$\dot{W} \le v^{\top} y_{\mathsf{PS}}.$$

(iii) *If $(x_*, x_{c_*}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$ is an equilibrium of the interconnected system with $v = 0$ and it is an isolated minimum of W, then it is stable in the sense of Lyapunov with Lyapunov function W.*

Proof: *In Proposition 6 of [42] it is shown that the PDEs in this case are $F\nabla\mathcal{C} = -g$. Noting that F is full rank it follows from Poincare's Lemma that a necessary and sufficient condition for solvability of this PDEs is* (3.17). *This completes the proof.*

□

Proposition 3.3.2 *Consider the PH system* (2.23), (2.29) *interconnected with the PH controller Σ_c defined in (3.3) via the standard power-preserving interconnection* (3.5). *Consider matrices F_s verifying* (2.24), (2.27) *and the additional integrability condition*

$$\nabla(F_s^{-1}g) = \left(\nabla(F_s^{-1}g)\right)^{\top}. \tag{3.19}$$

(i) *The solutions $\mathcal{C}$ of the PDE*

$$\nabla\mathcal{C} = -F_s^{-1}g,$$

define Casimir functions for the interconnected system.

(iii) *For all $\Phi : \mathbb{R}^m \to \mathbb{R}$, the function*

$$W_s(x, x_c) := H_s(x) + H_c(x_c) + \Phi(\mathcal{C}(x) - x_c). \tag{3.20}$$

satisfies

$$\dot{W}_s \le v^{\top} y_{\mathsf{PS}s}.$$

(iv) *If $(x_*, x_{c_*}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$ is an equilibrium of the interconnected system with $v = 0$ and it is an isolated minimum of W_s, then it is stable in the sense of Lyapunov with Lyapunov function W_s.*

Proof: *The proof is identical to the proof of Proposition 3.3.1 using F_s instead of F.*

□

[6] This assumption can be relaxed by conditions (2.14), (2.15) and using $F^{\dagger}$ instead of F^{-1}.

3.3.2 CbI with the general passive output

The PDEs for CbI with $y_{\mathtt{wD}}$ and $y_{\mathtt{wD}s}$ are, respectively, described in the following propositions.

Proposition 3.3.3 *Consider the PH system Σ, with $y_{(\cdot)} = y_{\mathtt{wD}}$, interconnected with the PH controller Σ_c defined in (3.3) via the standard power-preserving interconnection* (3.5)*.*

(i) *A* necessary *condition for the existence of mappings $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathcal{C}(x) - x_c$ are Casimir functions of the interconnected system is that the parameters w and D are chosen as*

$$\begin{array}{rcl} w & = & -\phi\nabla\mathcal{C} \\ D & = & -\nabla\mathcal{C}^\top \mathcal{J}\nabla\mathcal{C}, \end{array} \tag{3.21}$$

with $\mathcal{C}$ the solutions of the PDEs

$$F\nabla\mathcal{C} = -g. \tag{3.22}$$

(ii) *For all $\Phi : \mathbb{R}^m \to \mathbb{R}$, the function* (3.9) *satisfies*

$$\dot{W} \leq v^\top y_{\mathtt{wD}}.$$

(iii) *If $(x_*, x_{c*}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$ is an equilibrium of the interconnected system with $v = 0$ and it is an isolated minimum of W, then it is stable in the sense of Lyapunov with Lyapunov function W.*

Proof: The dynamics of the interconnected system is

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} F & -g \\ (g + 2\phi^\top w)^\top & -(w^\top w + D) \end{bmatrix} \begin{bmatrix} \nabla H \\ \nabla H_c \end{bmatrix} + \begin{bmatrix} g \\ w^\top w + D \end{bmatrix} v.$$

Now, $\dot{\mathcal{C}} - \dot{x}_c = 0$ for all H, H_c and v, if and only if

$$\left[\begin{array}{c|c} (\nabla\mathcal{C})^\top & -I_m \end{array} \right] \begin{bmatrix} F & -g \\ (g + 2\phi^\top w)^\top & -(w^\top w + D) \end{bmatrix} = 0, \tag{3.23}$$

which can be equivalently written as

$$(\nabla\mathcal{C})^\top F = (g + 2\phi^\top w)^\top \tag{3.24}$$
$$(\nabla\mathcal{C})^\top g = w^\top w + D \tag{3.25}$$

Replacing g from (3.24) into (3.25) yields

$$(\nabla\mathcal{C})^\top F^\top \nabla\mathcal{C} - 2(\nabla\mathcal{C})^\top \phi^\top w = w^\top w + D.$$

The symmetric part of the equation above is

$$w^\top w = -(\nabla\mathcal{C})^\top \mathcal{R}\nabla\mathcal{C} + (\nabla\mathcal{C})^\top \phi^\top w + w^\top \phi\nabla\mathcal{C}, \tag{3.26}$$

while, the skew-symmetric part is

$$D = -\left((\nabla \mathcal{C})^\top \mathcal{J} \nabla \mathcal{C} + (\nabla \mathcal{C})^\top \phi^\top w - w^\top \phi \nabla \mathcal{C}\right). \tag{3.27}$$

Now, using the factorization of $\mathcal{R}$ given in (2.7) we can write (3.26) as

$$(w + \phi \nabla \mathcal{C})^\top (w + \phi \nabla \mathcal{C}) \quad = \quad 0,$$

which is satisfied if and only if

$$w = -\phi \nabla \mathcal{C}. \tag{3.28}$$

Replacing (3.28) in (3.27) yields

$$\begin{aligned} D &= -\left((\nabla \mathcal{C})^\top \mathcal{J} \nabla \mathcal{C} + (\nabla \mathcal{C})^\top \phi^\top \phi \nabla \mathcal{C} - (\nabla \mathcal{C})^\top \phi^\top \phi \nabla \mathcal{C}\right) \\ &= -(\nabla \mathcal{C})^\top \mathcal{J} \nabla \mathcal{C}. \end{aligned}$$

The proof is completed replacing (3.28) in (3.24) that yields (3.22).

□

Proposition 3.3.4 *Consider the PH system* (2.23), (2.26) *interconnected with the PH controller* Σ_c *defined in (3.3) via the standard power-preserving interconnection* (3.5).

(i) *A* necessary *condition for the existence of mappings* $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$ *such that* $\mathcal{C}(x) - x_c$ *are Casimir functions of the interconnected system is that the parameters* w_s *and* D_s *are chosen as zero or*

$$\begin{aligned} w_s &= -\phi_s \nabla \mathcal{C} \\ D_s &= -\nabla \mathcal{C}^\top \mathcal{J}_s \nabla \mathcal{C}, \end{aligned} \tag{3.29}$$

with $\mathcal{C}$ *the solutions of the PDEs*

$$F_s \nabla \mathcal{C} = -g. \tag{3.30}$$

(ii) *For all* $\Phi : \mathbb{R}^m \to \mathbb{R}$*, the function* (3.20) *satisfies*

$$\dot{W}_s \leq v^\top y_{\mathtt{wDs}}.$$

(iii) *If* $(x_*, x_{c_*}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$ *is an equilibrium of the interconnected system with* $v = 0$ *and it is an isolated minimum of* W_s *then it is stable in the sense of Lyapunov with Lyapunov function* W_s*.*

Proof: The proof is identical to the proof of Proposition 3.3.3 using F_s instead of F.

□

Remark 3.3.1 *Some straightforward computations show that replacing the interconnection subsystem* (3.5) *by* (3.13)*, the PDEs* (3.22) *and* (3.30) *become*

$$\begin{aligned} F \nabla \mathcal{C} &= -g\alpha \\ F_s \nabla \mathcal{C} &= -g\alpha. \end{aligned}$$

3.4 EB-PBC with the general passive output

In this section we consider EB-PBC using the outputs $y_{\mathtt{wD}}$ and $y_{\mathtt{wD}s}$. The assumption below is needed in this subsection.

Assumption 3.4.1 *Assume there is a mapping (not necessarily unique) $L : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with rank m, such that*

$$g(x) = -F(x)L(x). \tag{3.31}$$

Proposition 3.4.1 *Under Assumption 3.4.1, fix w and D in (2.8) as*

$$\begin{aligned} w &= -\phi L \\ D &= -L^\top \mathcal{J} L. \end{aligned} \tag{3.32}$$

The control $u = \hat{u}_c(x) + v$ with

$$\hat{u}_c(x) = -L^\dagger(x)\nabla H_a(x), \tag{3.33}$$

where $H_a : \mathbb{R}^n \to \mathbb{R}$ is a solution of the PDE

$$L^\perp \nabla H_a = 0, \tag{3.34}$$

ensures

$$\hat{u}_c^\top y_{\mathtt{wD}} = -\dot{H}_a.$$

Moreover, the closed-loop system satisfies

$$\dot{H} + \dot{H}_a = -|\nabla H - Lu|^2_{\mathcal{R}} + v^\top y_{\mathtt{wD}}.$$

Proof: *Replacing (3.32) in (2.8) we get*

$$\begin{aligned} y_{\mathtt{wD}} &= (g - 2\mathcal{R}L)^\top \nabla H + L^\top(\mathcal{R} - \mathcal{J})Lu \\ &= (-FL - 2\mathcal{R}L)^\top \nabla H + L^\top(\mathcal{R} - \mathcal{J})Lu \\ &= \left((-\mathcal{J} + \mathcal{R} - 2\mathcal{R})L\right)^\top \nabla H + L^\top(\mathcal{R} - \mathcal{J})Lu \\ &= (F^\top L)^\top \nabla H - L^\top F L u \\ &= L^\top F \nabla H + L^\top g u \\ &= L^\top \dot{x} \end{aligned}$$

where, for the second and fifth identity we used (3.31). Then

$$\hat{u}_c^\top y_{\mathtt{wD}} = \hat{u}_c^\top L^\top \dot{x}. \tag{3.35}$$

Now, from Lemma A.4 we have that, (3.33) and (3.34) are equivalent to

$$L\hat{u}_c = -\nabla H_a.$$

Replacing the latter equation in (3.35), we get

$$\hat{u}_c^\top y_{\mathtt{wD}} = -(\nabla H_a)^\top \dot{x} = -\dot{H}_a. \tag{3.36}$$

Note that, for $y_{\mathtt{wD}}$ the power-balance equation takes the form

$$\dot{H} = u^\top y_{\mathtt{wD}} - |\phi\nabla H + wu|^2.$$

The proof is completed replacing (3.36) and (3.32) in the expression above.

□

Assumption 3.4.2 Assume there is a mapping (not necessarily unique) $L_s : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with rank m, such that

$$g(x) = -F_s(x)L_s(x), \tag{3.37}$$

where F_s verifies (2.24) and (2.27).

Proposition 3.4.2 *Under Assumption 3.4.2, fix w_s and D_s in* (2.26) *as*

$$\begin{aligned} w_s &= -\phi_s L_s \\ D_s &= -L_s^\top \mathcal{J}_s L_s. \end{aligned} \tag{3.38}$$

The control $u = \hat{u}_c(x) + v$ with

$$\hat{u}_c(x) = -L_s^\dagger(x)\nabla H_a(x), \tag{3.39}$$

where $H_a : \mathbb{R}^n \to \mathbb{R}$ is a solution of the PDE

$$L_s^\perp \nabla H_a = 0, \tag{3.40}$$

ensures

$$\hat{u}_c^\top y_{\mathtt{wDs}} = -\dot{H}_a.$$

Moreover, the closed-loop system satisfies

$$\dot{H}_s + \dot{H}_a = -|\nabla H_s - L_s u|^2_{\mathcal{R}_s} + v^\top y_{\mathtt{wDs}}.$$

Proof: *The proof is identical to the proof of Proposition 3.4.1 using F_s, H_s instead of F, H.*

□

Remark 3.4.1 *Propositions 3.4.1 and 3.4.2 exhibit that, using the corresponding passive output, the solutions of the PDEs in CbI are also suitable solutions of the PDEs that arise in the EB-PBC approach. Nevertheless, we stress the fact that the set of mappings L such that $L^\top \dot{x} = y_{\mathtt{wD}}$ is not fully characterized by* (3.31)*. Indeed, L must be a solution of*

$$\begin{bmatrix} (\nabla H(x))^\top F^\top(x) \\ g^\top(x) \end{bmatrix} L(x) = \begin{bmatrix} (\nabla H(x))^\top (g(x) + 2\phi^\top(x)w(x)) \\ w^\top(x)w(x) + D(x) \end{bmatrix}. \tag{3.41}$$

A similar analysis can be carried out for L_s. This difference between the Casimir functions and the integrals of the passive output will be further discussed in the next chapter.

3.4.1 Comparing CbI, EB and IDA

We find it convenient to recall the PDEs that must be solved and the conditions for the various PBCs studied in this chapter. In order to do this we present Tables 3.1 and 3.2, where a greater rank number implies a bigger set of solutions for the corresponding PDE.

The relationship between the various CbIs is obvious. Also, since F full rank implies that L in (3.31) is unique and full rank, the PDEs of EB-PBC with $y_{\mathtt{wD}}$ become in this case

$$g^\perp F \nabla H_a = 0, \tag{3.42}$$

which coincide with the PDEs of CbI with $y_{\mathtt{PS}}$ (or $y_{\mathtt{wD}}$) with $\Sigma_I^{\mathtt{SM}}$. The equivalence of the PDEs of this two methods when F is *not full rank* is less obvious but can be established with Lemma A.5, given in Appendix A. The proof of the latter is presented for the case $m = 1$ but it can be extended *verbatim* to the general case evaluating (3.31) column-by-column.

Unlike the other EB-PBCs studied in this chapter, IDA-PBC does not proceed from the creation of new passive outputs, hence this methodology is not suitable for a CbI implementation. There are several reasons that motivate us to include IDA-PBC[7] in Table 3.1. The most important one is that it is the most powerful PBC technique available to date, in the sense of being applicable to the largest class of PH systems. A second reason is that it has been widely adopted in many practical applications, including mechanical, electromechanical, power electronic and power systems.

Rank	EB	$\det\{F\} \neq 0$	$H_s = H$	PDEs
2	y	$\times$	$\checkmark$	$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla H_a = 0$
4	y_s	$\times$	$\times$	$\begin{bmatrix} g^\perp F_s \\ g^\top \end{bmatrix} \nabla H_a = 0$
6	$y_{\mathtt{PS}}$	$\checkmark$	$\checkmark$	$g^\perp F \nabla H_a = 0$
8	$y_{\mathtt{PS}s}$	$\times$	$\times$	$g^\perp F_s \nabla H_a = 0$
6	$y_{\mathtt{wD}}$	$\times$	$\checkmark$	$FL = -g,\ L^\perp \nabla H_a = 0$
8	$y_{\mathtt{wD}s}$	$\times$	$\times$	$F_s L_s = -g,\ L_s^\perp \nabla H_a = 0$
6	Basic IDA	$\times$	$\checkmark$	$g^\perp F \nabla H_a = 0$
9	IDA	$\times$	$\checkmark$	$g^\perp F_d \nabla H_a = g^\perp (F - F_d) \nabla H$

Table 3.1: PDEs to be solved in EB-PBC.

[7]See [35, 43] and Subsection 4.4.2 in Chapter 4

Rank	CbI	$\det\{F\} \neq 0$	$H_s = H$	$\Sigma_{\mathtt{I}}^{\mathtt{SM}}$	PDEs
1	y	$\times$	✓	$\times$	$\begin{bmatrix} F \\ g^\top \end{bmatrix} \nabla\mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$
2	$y + \mathtt{SM}$	$\times$	✓	✓	$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla\mathcal{C} = 0$
3	y_s	$\times$	$\times$	$\times$	$\begin{bmatrix} F_s \\ g^\top \end{bmatrix} \nabla\mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$
4	$y_s + \mathtt{SM}$	$\times$	$\times$	✓	$\begin{bmatrix} g^\perp F_s \\ g^\top \end{bmatrix} \nabla\mathcal{C} = 0$
5	$y_{\mathtt{PS}}$	✓	✓	$\times$	$\nabla\mathcal{C} = -F^{-1}g$
6	$y_{\mathtt{PS}} + \mathtt{SM}$	✓	✓	✓	$g^\perp F \nabla\mathcal{C} = 0$
7	$y_{\mathtt{PS}s}$	$\times$	$\times$	$\times$	$\nabla\mathcal{C} = -F_s^{-1}g$
8	$y_{\mathtt{PS}s} + \mathtt{SM}$	$\times$	✓	✓	$g^\perp F_s \nabla\mathcal{C} = 0$
5	$y_{\mathtt{wD}}$	$\times$	✓	$\times$	$\nabla\mathcal{C} = -F^{-1}g$
6	$y_{\mathtt{wD}} + \mathtt{SM}$	$\times$	✓	✓	$g^\perp F \nabla\mathcal{C} = 0$
7	$y_{\mathtt{wD}s}$	$\times$	$\times$	$\times$	$\nabla\mathcal{C} = -F_s^{-1}g$
8	$y_{\mathtt{wD}s} + \mathtt{SM}$	$\times$	$\times$	✓	$g^\perp F_s \nabla\mathcal{C} = 0$

Table 3.2: PDEs to be solved in CbI.

3.5 Examples

In this section we present two examples that illustrate the main results given in this chapter. In the first example we show the effects of the dissipation obstacle and the manner in which the modification of the output variable helps to overcome it. In the second example we illustrate the practicality of the generation of new passive outputs presented in the previous chapter—see Section 2.4.

3.5.1 RLC circuit

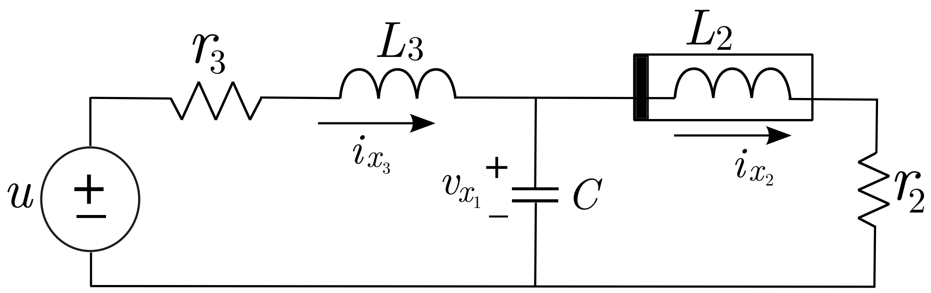

Figure 3.4: Electrical circuit

Consider the RLC circuit given in Figure 3.4 composed by two linear resistors r_2, r_3, a linear capacitor C, a linear inductor L_3 and a nonlinear inductor L_2 whose constitutive equation is given by

$$i_{x_2} = a \tanh x_2.$$

This circuit is described by the PH model

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -r_2 & 0 \\ -1 & 0 & -r_3 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{3.43}$$

with $r_2 > 0, r_3 > 0$ and

$$H = \frac{1}{2C} x_1^2 + a \ln \cosh x_2 + \frac{1}{2L_3} x_3^2, \tag{3.44}$$

where the constant parameters a, L_3, C are positive. The control objective is to stabilize the voltage $V_{r_2} = r_2 a \tanh x_2$ in a constant value, hence the desired equilibrium is given by

$$x_* = (C r_2 a \tanh x_{2*}, x_{2*}, L_3 a \tanh x_{2*}).$$

The gradient and the Hessian of H are given by

$$\nabla H = \begin{bmatrix} \frac{1}{C} x_1 \\ a \tanh x_2 \\ \frac{1}{L_3} x_3 \end{bmatrix}, \quad \nabla^2 H = \mathrm{diag}\{\tfrac{1}{C}, \tfrac{a}{\cosh^2 x_2}, \tfrac{1}{L_3}\}.$$

Whence it is clear that $(\nabla H)_* \neq \mathbf{0}_3$.

CbI

Consider the system (3.43)-(3.44) with $y_{(\cdot)} = y$. Then, the PDE (3.8) takes the form

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & -r_2 & 0 \\ 1 & 0 & -r_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{C}}{\partial x_1} \\ \frac{\partial \mathcal{C}}{\partial x_2} \\ \frac{\partial \mathcal{C}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that $\nabla \mathcal{C} = \mathbf{0}_3$. Therefore we are not able to shape any state with this selection of output variable. Below we propose a solution to overcome the dissipation obstacle present in this example.

Proposition 3.5.1 *Consider the system* (3.43)-(3.44)*, with* $y_{(\cdot)} = y_{\mathtt{PS}}$*, interconnected with* (3.3) *via* (3.5)*. The following holds true.*

(i) *The function*

$$C = \frac{1}{r_2 + r_3} \left(r_2 \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \right), \tag{3.45}$$

where $\tilde{x}_i = x_i - x_{i*}$*, defines a Casimir function for the system.*

(ii) *The point* $(x_*, (r_2 + r_3)^2 a \tanh x_{2*})$ *is a stable equilibrium of the closed-loop system with Lyapunov function* W_s*, defined in* (3.20)*, where*

$$H_c = \tfrac{1}{2(r_2+r_3)^2} x_c^2, \quad \Phi = \tfrac{1}{2(r_2+r_3)^2} \left(r_2 \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - x_c \right)^2. \tag{3.46}$$

Proof: To establish the proof note that, the PDE (3.18) takes the form

$$\begin{aligned} \nabla \mathcal{C} &= -\tfrac{1}{r_2+r_3} \begin{bmatrix} -r_2 r_3 & r_3 & -r_2 \\ -r_3 & -1 & -1 \\ r_2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \tfrac{1}{r_2+r_3} \begin{bmatrix} r_2 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Thus, (3.45) is a solution of the PDE above.

Now, from (3.44) and (3.46) we have

$$\begin{aligned} \nabla W &= \begin{bmatrix} \frac{1}{C} x_1 \\ a \tanh x_2 \\ \frac{1}{L_3} x_3 \\ \frac{1}{(r_2+r_3)^2} x_c \end{bmatrix} + \tfrac{1}{(r_2+r_3)^2} \left(r_2 \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - x_c \right) \begin{bmatrix} r_2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \\ \nabla^2 W &= \begin{bmatrix} \frac{1}{C} & 0 & 0 & 0 \\ 0 & \frac{a}{\cosh^2 x_2} & 0 & 0 \\ 0 & 0 & \frac{1}{L_3} & 0 \\ 0 & 0 & 0 & \frac{1}{(r_2+r_3)^2} \end{bmatrix} + \tfrac{1}{(r_2+r_3)^2} \begin{bmatrix} r_2^2 & r_2 & r_2 & -r_2 \\ r_2 & 1 & 1 & -1 \\ r_2 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

Hence, evaluating both expressions above at $(x, x_c) = (x_*, (r_2 + r_3)^2 a \tanh x_{2*})$ and noting that $\frac{a}{\cosh^2 x_{2*}} > 0$, we get

$$(\nabla W)_* = \mathbf{0}_4, \quad (\nabla^2 W)_* > 0. \tag{3.47}$$

This completes the proof.

□

The Figure 3.5 represents the closed-loop system as an electrical circuit, with

$$\begin{aligned} y - y_{\mathsf{PS}} &= -2\mathcal{R}\nabla H + u \\ &= -2\left(r_2 a \tanh x_2 + \tfrac{r_3}{L_3} x_3\right) + \tfrac{1}{C_c} x_c \end{aligned}$$

where

$$C_c := (r_2 + r_3)^2.$$

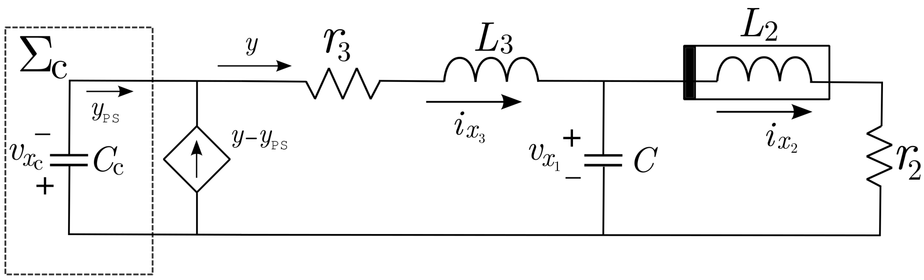

Figure 3.5: Electrical circuit resultant of the CbI

EB-PBC

In the following proposition we present an EB-PBC which solves the stabilization problem of the RLC circuit.

Proposition 3.5.2 *Consider the system* (3.43)-(3.44) *in closed-loop with the controller* $u = \hat{u}_c$*, where*

$$\hat{u}_c = -\frac{1}{r_2 + r_3}\left(r_2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - (r_2 + r_3)^2 a \tanh x_{2*}\right) \tag{3.48}$$

with $\tilde{x}_i = x_i - x_{i*}$*. The equilibrium point* $(Cr_2 a \tanh x_{2*}, x_{2*}, L_3 a \tanh x_{2*})$ *is asymptotically stable with Lyapunov function*

$$H_d = H + H_a,$$

where

$$H_a = \frac{1}{2(r_2 + r_3)^2}(r_2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - (r_2 + r_3)^2 a \tanh x_{2*})^2. \tag{3.49}$$

Proof: To establish the proof, first, notice that the equation (3.31) takes the form

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & -r_2 & 0 \\ -1 & 0 & -r_3 \end{bmatrix} L = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \iff L = \frac{1}{r_2 + r_3}\begin{bmatrix} r_2 \\ 1 \\ 1 \end{bmatrix}.$$

The expression above, in combination with (3.49), yields

$$\begin{aligned} L^{\dagger}\nabla H_a &= \tfrac{1}{r_2+r_3}\left(r_2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - (r_2+r_3)^2 a\tanh x_{2*}\right) \\ &= -\hat{u}_c, \end{aligned}$$

where we used (3.48) to obtain the second equality. Thus, from Proposition 3.4.1 it follows that

$$\dot{H}_d = -|\nabla H - L\hat{u}_c|^2_{\mathcal{R}}. \tag{3.50}$$

Moreover, the gradient and the Hessian of H_d are given by

$$\begin{aligned} \nabla H_d &= \begin{bmatrix} \frac{1}{C}x_1 \\ a\tanh x_2 \\ \frac{1}{L_3}x_3 \end{bmatrix} \\ &\quad + \tfrac{1}{(r_2+r_3)^2}\left(r_2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 - (r_2+r_3)^2 a\tanh x_{2*}\right)\begin{bmatrix} r_2 \\ 1 \\ 1 \end{bmatrix} \\ \nabla^2 H_d &= \tfrac{1}{(r_2+r_3)^2}\begin{bmatrix} \frac{(r_2+r_3)^2}{C} + r_2^2 & r_2 & r_2 \\ r_2 & 1 + \frac{a(r_2+r_3)^2}{\cosh^2 x_2} & 1 \\ r_2 & 1 & 1 + \frac{(r_2+r_3)^2}{L_3} \end{bmatrix}. \end{aligned}$$

Therefore, evaluating both expressions above at x_* we have

$$(\nabla H_d)_* = \mathbf{0}_3, \quad (\nabla^2 H_d)_*.$$

The latter ensures the stability of the equilibrium.

To prove the convergence of the trajectories, note that from (3.50)

$$\begin{aligned} \dot{H}_d = 0 \quad &\Rightarrow \quad \mathcal{R}\left(\nabla H - L\hat{u}_c\right) = 0 \\ &\Rightarrow \quad \left\{ \begin{array}{rcl} L_3 a\tanh x_2 &=& x_3 \\ \mathcal{R}F^{-1}\dot{x} &=& 0. \end{array} \right. \end{aligned} \tag{3.51}$$

Combining both expressions in (3.51), we have

$$\left.\begin{array}{rcl} \dot{x}_1 &=& 0 \\ \dot{x}_2 &=& -\dot{x}_3 \end{array}\right\} \Rightarrow \ddot{x}_1 = \frac{a}{\cosh^2 x_2}\dot{x}_2 - \frac{1}{L_3}\dot{x}_3 = \left(\frac{a}{\cosh^2 x_2} + \frac{1}{L_3}\right)\dot{x}_2 = 0$$

which implies

$$\dot{x}_2 = 0 \Rightarrow \left\{ \begin{array}{rcl} \dot{x}_3 &=& 0 \\ x_1 &=& Cr_2 a\tanh x_2 \end{array} \right. \Rightarrow x_2 = x_{2*}.$$

Hence, from (3.51) and the expression above we conclude that $x = x_*$. The proof is completed invoking the Barbashin-Krasovskii's Theorem.[8]

□

Simulations are carried out with the following parameters: $C = 2\mu F$, $r_2 = 1.2k\Omega$, $r_3 = 550\Omega$, $L_3 = 50mH$ and $a = 30mA$. The desired voltage in r_2 is $7V$, which is obtained with a constant charge $x_{2*} = 197mH$. The Figure 3.6 shows the simulation results for the closed-loop system under initial conditions $x_0 = \mathbf{0}_3$.

[8]See [26].

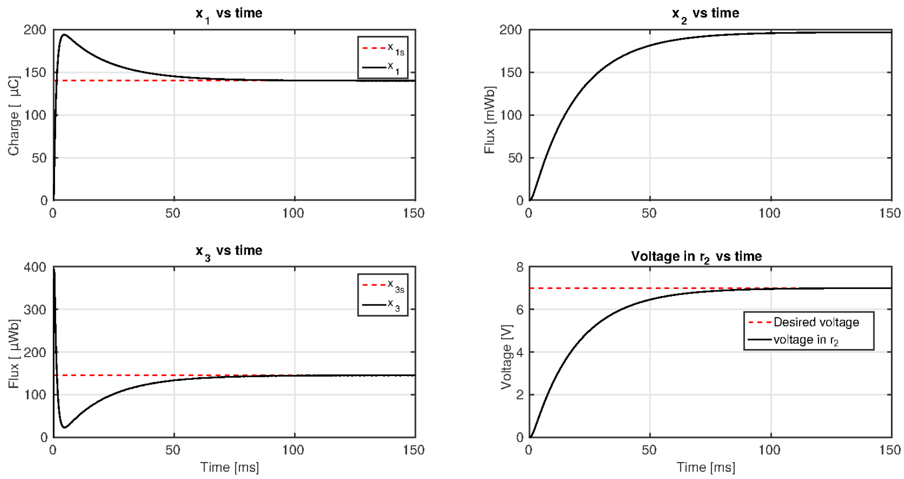

Figure 3.6: Simulated response of the regulated RLC circuit.

3.5.2 Generation of new passive outputs

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{3.52}$$

where

$$H = \cos x_2 + 1 + a x_1, \tag{3.53}$$

with the constant a different from zero and $y_{(\cdot)} = y_{\mathtt{wD}}$, where w and D are defined in (3.21). The control objective is to stabilize the equilibrium point $x_* = (x_{1*}, 0)$, which clearly satisfies $x_* \in \mathcal{E}$. The gradient and the Hessian of the storage function are given by

$$\nabla H = \begin{bmatrix} a \\ -\sin x_2 \end{bmatrix}, \quad \nabla^2 H = \begin{bmatrix} 0 & 0 \\ 0 & -\cos x_2 \end{bmatrix}. \tag{3.54}$$

Notice that

$$(\nabla H)_* = \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad (\nabla^2 H)_* = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence, both states need to be shaped to have a minimum at x_*.

On the other hand, the Casimir functions for this system must satisfy

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{C}}{\partial x_1} \\ \frac{\partial \mathcal{C}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and thus

$$\nabla \mathcal{C} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{3.55}$$

From the PDE above it is clear that the second state cannot be shaped for any passive output with respect to H, and in consequence, the plant is not stabilizable by CbI considering H as storage function.

Proposition 3.5.3 *Consider the system* (3.52)-(3.53). *The following holds true.*

(i) *An alternative representation for the system is given by*

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \nabla H_s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{3.56}$$

where

$$H_s = -\cos x_2 + 1 - a(x_1 + x_2). \tag{3.57}$$

(ii) *Consider the system* (3.56)-(3.57) *interconnected with* (3.3) *via* (3.5). *The point* $(x_*, 0)$ *is a stable equilibrium of the closed-loop system with Lyapunov function* W_s*, defined in* (3.20)*, where*

$$H_c = \tfrac{1}{2}(x_c + a)^2, \quad \Phi = \tfrac{1}{2}(x_1 + x_2 - x_c - x_{1*} + a)^2. \tag{3.58}$$

Proof: To establish the proof, first, note that

$$F\nabla H = \begin{bmatrix} \sin x_2 \\ a \end{bmatrix}.$$

Moreover, from (3.56) we have

$$\begin{aligned} F_s \nabla H_s &= \begin{bmatrix} \sin x_2 \\ a \end{bmatrix} \\ &= F\nabla H. \end{aligned}$$

Now, consider $y_{(\cdot)s} = y_{\mathtt{wD}s}$ with w_s and D_s selected as in (3.29). Then, the PDE (3.30) takes the form

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{C}}{\partial x_1} \\ \frac{\partial \mathcal{C}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

which implies

$$\nabla \mathcal{C} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, the function

$$\mathcal{C} = x_1 + x_2 - x_{1*} + a$$

defines a Casimir function for the system (3.56)-(3.57).

From (3.57) and (3.58), we have

$$\nabla W_s = \begin{bmatrix} -a \\ -a + \sin x_2 \\ x_c + a \end{bmatrix} + (x_1 + x_2 - x_{1*} + a - x_c) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Whence, $(\nabla W_s)_* = \mathbf{0}_3$. Moreover

$$\nabla^2 W_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Hence

$$(\nabla^2 W_s)_* = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} > 0.$$

The latter implies that $\arg\min\{W_s\} = (x_*, 0)$. This completes the proof.

$\square$

Chapter 4

Shaping the energy without solving PDEs

An energy shaping controller for *mechanical systems* that does not require the solution of PDEs has been recently proposed in [13]. In this chapter we pursue this research line considering the more general case of PH systems [58]. The starting point of the design is the well-known power shaping output [36], which defines a passive output for the PH system with storage function its energy function. As is well–known a PI controller around this output preserves the passivity of the closed-loop. In [4] it is shown that, if the power shaping output is "integrable", the integral action of the PI is passive with storage function defined as a quadratic term of the "integral" of the power shaping output, which depends on the plant state. In this way we can generate a *new storage function* for the closed-loop constructed as the sum of this function and the original energy function of the PH system. Adding a suitably chosen constant to the control makes this function *positive definite*, which then qualifies as a Lyapunov function for the closed–loop system. The condition imposed on the power shaping output boils down to a classical integrability condition of some computable vector fields, hence it can be readily verified.

In this chapter we extend the results of [4] constructing a PI controller around the general output $y_{\mathtt{wD}}$ and providing alternatives to overcome the integrability condition over the passive output. Another contribution is a comparison between the resultant PI with the Casimir function studied in the previous chapter, from which it is shown that the set of stabilizable plants with the present methodology is larger than the set stabilizable via Casimir generation.

4.1 Preliminaries

The following assumptions identify the class of PH systems for which the proposed control strategy is applicable.

Assumption 4.1.1 *The matrix $F(x)$, in* (2.6)*, is full rank or verifies* (2.14) *and* (2.15)[1]*.*

[1] The methodology is presented for F full rank, nevertheless, the results can be extended considering $F^{\dagger}$ instead of F^{-1}.

Assumption 4.1.2 The vector fields $F^{-1}(x)g_i(x)$, with $g_i(x),\ i = 1, \ldots, m$, the columns of the matrix $g(x)$, are *gradient vector fields.* That is,

$$\nabla\left(F^{-1}(x)g_i(x)\right) = \left\{\nabla\left(F^{-1}(x)g_i(x)\right)\right\}^\top. \tag{4.1}$$

As shown in Chapter 2—see also [36, 42]—if Assumption 4.1.1 holds, then $y_{\mathtt{PS}}$ is a passive output for the PH system (2.6) with storage function $H(x)$. More precisely, the following dissipation inequality holds

$$\dot{H} \leq u^\top y_{\mathtt{PS}}. \tag{4.2}$$

On the other hand, recalling Poincare's Lemma it is easy to see that Assumption 4.1.2 ensures the existence of a mapping $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\nabla\gamma(x) = -F^{-1}(x)g(x). \tag{4.3}$$

Furthermore,

$$\dot{\gamma} = (\nabla\gamma(x))^\top \dot{x} = y_{\mathtt{PS}}. \tag{4.4}$$

4.2 Energy Shaping

In this section we define a static state-feedback such that the system (2.6), with $y(\cdot) = y_{\mathtt{PS}}$, in closed–loop with this control preserves passivity of the mapping $v \mapsto y_{\mathtt{PS}}$ but with a suitably *modified storage function.*

Proposition 4.2.1 Suppose Assumptions 4.1.1 and 4.1.2 hold. Define the mapping $u_{\mathtt{PS}} : \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{aligned} u_{\mathtt{PS}}(x) \quad := \quad & \left(I - K_P g^\top(x) F^{-\top}(x) g(x)\right)^{-1} \Big\{ K_P g^\top(x) F^{-\top}(x) F(x) \nabla H(x) \\ & - K_I(\gamma(x) + \kappa) \Big\} \end{aligned} \tag{4.5}$$

where γ satisfies (4.3)[2] and $\kappa \in \mathbb{R}^m$ and $K_P, K_I \in \mathbb{R}^{m\times m}$, $K_I, K_P > 0$, are *free parameters.* The system (2.6) in closed–loop with the control $u = u_{\mathtt{PS}}(x) + v$ defines a passive mapping $v \mapsto y_{\mathtt{PS}}$ with storage function

$$H_d(x) = H(x) + \frac{1}{2}|\gamma(x) + \kappa|^2_{K_I}. \tag{4.6}$$

Proof: To establish the proof, first, notice that from (2.11) and (4.4) the control (4.5) reduces to

$$u_{\mathtt{PS}}(x) = -K_I(\gamma + \kappa) - K_P y_{\mathtt{PS}}. \tag{4.7}$$

Therefore, differentiating (4.6) we get

$$\begin{aligned} \dot{H}_d \quad &= \quad \dot{H} + y_{\mathtt{PS}}^\top K_I(\gamma + \kappa) \\ &\leq \quad y_{\mathtt{PS}}^\top (u + K_I(\gamma + \kappa)) \\ &= \quad y_{\mathtt{PS}}^\top (v - K_P y_{\mathtt{PS}}) \leq y_{\mathtt{PS}}^\top v, \end{aligned}$$

where we used (4.4) in the first equality, (3.1) in the first inequality, (4.7) for the second equality, respectively, and $K_P > 0$ for the last inequality.

□

[2] Notice that the existence of $\gamma(x)$ is ensured by Assumption 4.1.2 and it can be computed via direct integration.

Remark 4.2.1 Strictly speaking, a PI controller on the passive output $y_{\mathtt{PS}}$ has the form

$$u(t) = -K_P y_{\mathtt{PS}}(t) - K_I \int_0^t y_{\mathtt{PS}}(\tau) d\tau. \tag{4.8}$$

Hence, at first glance it might seem appealing to propose γ in (4.4) as

$$\gamma(x(t)) = \int y_{\mathtt{PS}}(t) dt.$$

However, replacing the latter in (4.8) yields

$$u = -K_P y_{\mathtt{PS}}(t) - K_I(\gamma(x(t)) + K_I(\gamma(x(0)),$$

which clearly depends on the initial conditions of the plant. This makes this approach fragile and impractical to its implementation.

In contrast with the PI controller (4.8), the control law $u_{\mathtt{PS}}$ of Proposition 4.2.1 is independent of the initial conditions of the plant but still has a PI-like architecture since

$$\frac{d}{dt}(\gamma + \kappa) = y_{\mathtt{PS}}.$$

Despite the technical differences, in the sequel we refer to the control law $u_{\mathtt{PS}}$ as PI-PBC.

Remark 4.2.2 The condition of integrability of the vector fields $F^{-1}(x)g_i(x)$ appears also in the context of CbI of PH systems, as a necessary and sufficient condition for existence of Casimir functions, see Proposition 3.3.1 in Chapter 3.

4.3 Stabilization

From Proposition 4.2.1 it is clear that if the new storage function $H_d(x)$ is positive definite (with respect to the desired equilibrium x_*) it qualifies as a *bona fide* Lyapunov function for the closed-loop system (with $v = 0$) that ensures stability of x_*. This fact is stated in the proposition below where we also give an easily verifiable condition to check positivity of $H_d(x)$.

Proposition 4.3.1 *Consider the system* (2.6)*, verifying Assumptions 4.1.1 and 4.1.2, in closed-loop with the control* $u = u_{\mathtt{PS}}(x)$*, where* $u_{\mathtt{PS}}(x)$ *is given by* (4.5)*. Fix*

$$\kappa := K_I^{-1} g_*^\dagger F_* (\nabla H)_* - \gamma_*. \tag{4.9}$$

If $x_* \in \mathcal{E}$ *and*

$$(\nabla^2 H_d)_* > 0 \tag{4.10}$$

with $H_d(x)$ *defined in* (4.6)*, then* x_* *is* stable *(in the sense of Lyapunov) with Lyapunov function* $H_d(x)$*. It is* asymptotically *stable if* $y_{\mathtt{PS}}$*, defined in* (2.11)*, is a detectable output, that is, if the following implication is true*

$$y_{\mathtt{PS}}(t) \equiv 0 \Longrightarrow \lim_{t\to\infty} x(t) = x_*.$$

Proof: First, we will prove that x_* is an equilibrium point of the closed-loop system. From (2.11) we have that $y_{\mathtt{PS}_*} = 0$, hence (4.7)—at the equilibrium—becomes $u_{\mathtt{BC}_*} = -K_I(\gamma_* + \kappa)$. The proof is completed noting that the choice of κ given in (4.9), together with the fact that $x_* \in \mathcal{E}$, guarantees that

$$F_*(\nabla H)_* - g_* \left(K_I(\gamma_* + \kappa) \right) = 0. \tag{4.11}$$

To prove the stability claim we recall that from Proposition 4.2.1 and $v = 0$ we have that $\dot{H}_d \leq -K_P |y_{\mathtt{PS}}|^2 \leq 0$. Hence, invoking classical Lyapunov Theory [26], it suffices to prove that $H_d(x)$ is a positive definite function. From (4.11) we get

$$(\nabla H)_* = F_*^{-1} g_* K_I(\gamma_* + \kappa). \tag{4.12}$$

Computing the gradient of $H_d(x)$ at the equilibrium yields

$$\begin{aligned} (\nabla H_d)_* &= (\nabla H)_* + (\nabla \gamma)_* K_I(\gamma_* + \kappa) \\ &= (\nabla H)_* - F_*^{-1} g_* K_I(\gamma_* + \kappa) = 0, \end{aligned}$$

where the second and third identities are obtained replacing (4.3) and (4.12), respectively. This ensures that x_* is a critical point of $H_d(x)$. The proof is completed recalling that (4.10) is a sufficient condition for x_* to be an isolated minimum of $H_d(x)$.

□

Remark 4.3.1 *With the aim of preserving a PI-like architecture of the controller, a particular structure is imposed on H_d, that is, the open-loop energy plus a quadratic term in $\gamma + \kappa$. Nonetheless, a more general energy function H_d can be designed, namely*

$$H_d = H + \Phi(\gamma),$$

where $(\nabla \Phi)_* = g_*^\dagger F_*(\nabla H)_*$. See Proposition 4.5.2 for further details.

4.4 Relation with Classical PBCs

In this section we discuss the relationship between the new controller and the classical PBC techniques of EB and IDA.[3]

4.4.1 EB-PBC

The basic idea of EB-PBC (with the output $y_{\mathtt{PS}}$) is to look for a state feedback $u_{\mathtt{EB}} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\dot{H}_a = -u_{\mathtt{EB}}^\top y_{\mathtt{PS}},$$

for some "added" energy function $H_a : \mathbb{R}^n \to \mathbb{R}$. In this case, setting $u = u_{\mathtt{EB}}(x)$ transforms the passivity inequality (4.2) into

$$\dot{H} + \dot{H}_a \leq 0,$$

and if $H(x) + H_a(x)$ is positive definite the closed-loop system will have a stable equilibrium at x_*. The following proposition states that, for a suitable choice of the tuning gains, the new controller is an EB-PBC.

[3]The interested reader is referred to [35, 42, 43] for further details on EB-PBC and IDA-PBC.

Proposition 4.4.1 Consider the PH system (2.6) verifying Assumptions 4.1.1 and 4.1.2. Fix $K_P = 0$ in $u_{\mathtt{PS}}(x)$. Then, the control $u = u_{\mathtt{PS}}(x)$ is an EB-PBC with added energy function

$$H_a(x) := \frac{1}{2}|\gamma(x) + \kappa|^2_{K_I}. \tag{4.13}$$

Proof: For $K_P = 0$ the mapping $u_{\mathtt{PS}}(x)$, given in (4.7), reduces to

$$u_{\mathtt{PS}}(x) = -K_I\left(\gamma(x) + \kappa\right). \tag{4.14}$$

On the other hand, from (4.4) and (4.13) we have

$$\dot{H}_a = y_{\mathtt{PS}}^\top K_I(\gamma + \kappa) = -y_{\mathtt{PS}}^\top u_{\mathtt{PS}},$$

completing the proof.

□

4.4.2 IDA-PBC

In IDA-PBC we fix the desired interconnection and damping matrices, hence, fix the matrix $F_d : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $\mathtt{sym}\{F_d(x)\} \leq 0$, and look for a control $u = u_{\mathtt{IDA}}(x)$ such that the closed-loop has the form

$$\dot{x} = F_d(x)\nabla H_d(x);$$

for some energy function $H_d : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, which has a minimum at the desired equilibrium. It is easy to show that the assignable energy functions $H_d(x)$ are characterized by the solutions of the following PDE

$$g^\perp(x)\Big\{F_d(x)\nabla H_d(x) - F(x)\nabla H(x)\Big\} = 0, \tag{4.15}$$

and the control is uniquely defined as

$$u_{\mathtt{IDA}}(x) := g^\dagger(x)\Big\{F_d(x)\nabla H_d(x) - F(x)\nabla H(x)\Big\}. \tag{4.16}$$

The proposition below establishes the relation between IDA-PBC and the controller of Proposition 4.2.1.

Proposition 4.4.2 Consider the PH system (2.6) verifying Assumptions 4.1.1 and 4.1.2. Fix $K_P = 0$ in $u_{\mathtt{PS}}(x)$ and select the desired interconnection and damping matrices as

$$F_d(x) = F(x). \tag{4.17}$$

Then, the energy function $H_d(x)$ defined in (4.6) and the control $u = u_{\mathtt{PS}}(x)$ given in (4.5) satisfy the IDA-PBC equations (4.15) and (4.16), respectively.

Proof: Replacing the gradient of $H_d(x)$, given by

$$\nabla H_d = \nabla H - F^{-1}gK_I(\gamma + \kappa),$$

in the PDE (4.15) we get

$$g^\perp\Big\{F\left(\nabla H - F^{-1}gK_I(\gamma+\kappa)\right) - F\nabla H\Big\} = g^\perp gK_I(\gamma+\kappa) = 0$$

On the other hand, the control law (4.5) is given by (4.14), which satisfies (4.16) since, using (4.17),

$$\begin{aligned} u_{\mathtt{IDA}} &= g^{\dagger}\Big\{F\left(\nabla H - F^{-1}gK_I(\gamma+\kappa)\right) - F\nabla H\Big\} \\ &= -g^{\dagger}gK_I(\gamma+\kappa) = u_{\mathtt{PS}}. \end{aligned}$$

□

Remark 4.4.1 *It is well-known that the IDA-PBC*[4] *preserves the PH structure in closed-loop. Hence, from Proposition 4.4.2 it is clear that, fixing $K_P = 0$ in* (4.7)*, the closed-loop system is a PH system.*

4.5 Extensions of the PI

In this section we provide two extensions of the controller of Proposition 4.2.1. First, we explore an alternative to relax Assumption (3.4.2) by modifying the input-output port of the system. In the second part of this section we analyze the use of $y_{\mathtt{wD}}$ instead of $y_{\mathtt{PS}}$ to construct the PI controller and the differences between Casimir functions and the first integrals of the passive output.

4.5.1 Change of coordinates

In this subsection we prove that the class of PH systems for which (4.3) is solvable can be enlarged via an *input* change of coordinates. For ease of presentation we restrict ourselves to the case of full rank $F(x)$, nonetheless, the extension to the non-full rank case is straightforward. In this case, the passive output of interest is the power shaping output [36] given in (2.11). We recall that the key question for energy shaping with the methodology proposed in previous sections is the existence of a mapping $\gamma(x)$ such that (4.4) holds. Now, introducing an input change of coordinates

$$u = M(x)\bar{u}, \tag{4.18}$$

with $M : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ *full rank*, the power balance becomes

$$\dot{H} \le \bar{u}^\top \bar{y}_{\mathtt{PS}},$$

with the new input $\bar{u}$ and the new output

$$\bar{y}_{\mathtt{PS}} := -M^\top(x)g^\top(x)F^{-\top}(x)\dot{x}. \tag{4.19}$$

It is clear then that imposing the integrability conditions to the new vector fields $F^{-1}(x)g(x)M_i(x)$ guarantees the existence of a mapping $\gamma(x)$ such that

$$\dot{\gamma} = \bar{y}_{\mathtt{PS}}.$$

The proposition below states under which conditions there exists a full rank matrix $M(x)$ such that the required integrability conditions are satisfied.

Proposition 4.5.1 *Define a mapping $\Lambda : \mathbb{R}^n \to \mathbb{R}^{n\times(n-m)}$ verifying*

[4] See [43, 35]

(C1) rank $\{\Lambda(x)\} = n - m$

(C2) and

$$g^\top(x)F^{-\top}(x)\Lambda(x) = 0. \tag{4.20}$$

There exists a full rank matrix $M : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ *such that*

$$F^{-1}(x)g(x)M(x) = \nabla\gamma(x), \tag{4.21}$$

where $\gamma : \mathbb{R}^n \to \mathbb{R}^m$*, if and only if the distribution*

$$\Delta := \mathit{span}\{\Lambda(x)\} \tag{4.22}$$

is involutive, that is, if the Lie brackets of vector fields in Δ *remain in* Δ*.*

Proof: The proof proceeds as follows. First of all notice that, since $M(x)$ is full rank, we have that

$$\ker\left\{M^\top(x)g^\top(x)F^{-\top}(x)\right\} = \ker\left\{g^\top(x)F^{-\top}(x)\right\}.$$

Moreover, in view of (4.20),

$$\Delta^\perp := \operatorname{span}\left\{M^\top(x)g^\top(x)F^{-\top}(x)\right\}$$

is a co-distribution of Δ. The proof is completed invoking Frobenius Theorem [53], the fact that a distribution Δ is completely integrable if and only if there exist $\gamma_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, ..., m$, such that its co-distribution is given by

$$\Delta^\perp = \operatorname{span}\left\{(\nabla\gamma_1(x))^\top, ..., (\nabla\gamma_m(x))^\top\right\},$$

and defining $\gamma(x) := \operatorname{col}(\gamma_1(x), \dots, \gamma_m(x))$.

□

Remark 4.5.1 *The matrix* $M(x)$ *modifies the input-output port of the PH system* (2.6) *and thus the negative feedback scheme. This is equivalent to changing the interconnection subsystem* (3.5) *for* (3.13) *in CbI—see Subsection 3.1.1 in Chapter 3.*

4.5.2 First integrals

The selection $y_{(\cdot)} = y_{\mathtt{PS}}$ is appealing for three main reasons: the integrability condition is straightforward to verify, the energy shaping is done without the solution of PDEs and, in addition—as it is shown in Proposition 4.3.1—, an appropriate selection of κ ensures that x_* is a critical point of the closed-loop energy function. In spite of these advantages, the set of stabilizable PH systems using the PI approach presented in previous sections can be enlarged with the selection of $y_{\mathtt{wD}}$ as output variable.

The proposition below establishes a more general construction of the PI controller introduced in Propositions 4.2.1 and 4.3.1.

Proposition 4.5.2 *Consider the PH system* (2.6)*,* (2.8)*. Assume there exist mappings* $w(x)$ *and* $D(x)$ *such that the PDE*

$$\begin{bmatrix}(\nabla H(x))^\top F^\top(x)\\ g^\top(x)\end{bmatrix}\nabla\gamma(x) = \begin{bmatrix}(\nabla H(x))^\top(g(x) + 2\phi^\top(x)w(x))\\ w^\top(x)w(x) - D(x)\end{bmatrix} \tag{4.23}$$

admits a solution $\gamma : \mathbb{R}^n \to \mathbb{R}^m$*. Then, the following statements hold true.*

(i) *The system* (2.6), (2.8) *in closed-loop with the controller*

$$\begin{aligned} u &= v + \left\{I + K_P(w^\top(x)w(x) + D(x))\right\}^{-1}\Big\{ -\nabla\Phi \\ &\quad -K_P\left(g(x) + 2\phi^\top(x)w(x)\right)^\top \nabla H(x)\Big\} \end{aligned} \tag{4.24}$$

defines a passive mapping $v \mapsto y_{\mathtt{wD}}$ *with storage function*

$$H_d = H + \Phi(\gamma) \tag{4.25}$$

where $\Phi : \mathbb{R}^m \to \mathbb{R}$ *is to be defined.*

(ii) *Fix* $v = 0$ *in* (4.24)*. If the equilibrium* $x_* \in \mathcal{E}$ *satisfies*

$$\arg\min\{H_d\} = x_*,$$

and it is isolated, then is stable in the sense of Lyapunov with Lyapunov function H_d*.*

(ii) *The point is asymptotically stable if (ii) holds and*

$$\dot{H}_d \equiv 0 \Longleftrightarrow x = x_*. \tag{4.26}$$

Proof: Notice that the existence of γ solution of (4.23) implies that

$$\dot{\gamma} = y_{\mathtt{wD}}. \tag{4.27}$$

On the other hand, from (2.8) and (4.24) the control law reduces to

$$u = -\nabla\Phi - K_P y_{\mathtt{ywD}}. \tag{4.28}$$

Therefore the time derivative of H_d is given by

$$\begin{aligned} \dot{H}_d &\leq y_{\mathtt{wD}}^\top u + \dot{x}^\top \nabla\gamma\nabla\Phi \\ &= y_{\mathtt{wD}}^\top (u + \nabla\Phi) \\ &= -|y_{\mathtt{wD}}|^2_{K_P} + y_{\mathtt{wD}}^\top v \leq y_{\mathtt{wD}}^\top v \end{aligned}$$

where we used (4.27) and (4.28) to obtain the first and the second equality, respectively. The rest of the proof is completed invoking classical Lyapunov Theory and the Barbashin-Krasovskii's Theorem.

□

Discussion on first integrals vs Casimir functions

In Proposition 4.5.2 the energy shaping is carried out by the generation of the passive output's *first integrals*. Comparing the PDEs (3.23) and (4.23), we notice the absence of the term $\nabla H(x)$ in the first set of equations. This absence forces a particular selection of the parameters w and D given in (3.21). As a result of this specific selection, the passive output $y_{\mathtt{wD}}$ does not enlarge the set of PH systems for which CbI is applicable as was proved in Chapter 3. Nonetheless, this constraint in the choice of the parameters w, D is not present in the generation of first integrals or EB approach.[5] Clearly, fixing $\mathcal{C} = \gamma(x)$,

[5]See Remark 3.4.1 in Chapter 2.

any solution $\mathcal{C}$ of (3.23) is also a solution of (4.23), with the set of solutions of (3.23) being *strictly contained* in the set of solutions of (4.23). Indeed, the set of functions γ that solve (4.23) is larger than the set of solutions of (3.23) due to the presence of the term $\nabla H(x)$. An example that illustrates this point is given in Section 4.6.

On the other hand, Proposition 4.4.1 establishes a relationship between the PI controller (4.7) and the EB-PBC. Moreover, fixing $L = \nabla\gamma$, the equation (4.23) is clearly the same as (3.41), this corroborates the relation between both approaches.

Derivations similar to the ones done in Proposition 4.5.2 are reported in Section 7.1 of [58] where, following the construction of [28], new passive outputs—called "alternate" in [58]—are used for CbI. There is a relation also with input-output Hamiltonian systems with dissipation (IOHD) studied in [57], for which the integrability condition (4.1) is implicitly assumed. See these references for further details.

4.6 Examples

In this section we apply the proposed controller to a physical system and investigate, with the example of LTI systems, some of the limitations of the method.

4.6.1 Micro electro-mechanical optical switch

Consider the PH representation of the optical switch system[6]

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix} \nabla H(x) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u. \tag{4.29}$$

The energy function is

$$H(x) = \frac{1}{2m}x_2^2 + \frac{1}{2}a_1x_1^2 + \frac{1}{4}a_2x_1^4 + \frac{1}{2c_1(x_1+c_0)}x_3^2,$$

where x_1, x_2 are, respectively, the mass of the comb driver actuator and its momentum, x_3 denotes the charge in the capacitor, u is the voltage applied on the electrodes, $a_1 > 0, a_2 > 0$ are the spring constants, $b > 0, r > 0$ are resistive elements, $c_0 > 0, c_1 > 0$ are constants that determine the capacitance function and, finally, $m > 0$ denotes the mass of the actuator. It is important to underscore the physical constraint $x_1 > 0$. See [5] for further details on the model.

The assignable equilibria for this system is

$$(x_{2_*}, x_{3_*}) = \left(0, (c_0 + x_{1_*})\sqrt{2c_1x_{1_*}(a_1 + a_2x_{1_*}^2)}\right) \tag{4.30}$$

and the goal is to stabilize at the desire constant position $x_{1_*} > 0$.

Clearly, F is full rank. Also, some simple calculations using (2.11) prove that $y_{\mathtt{PS}} = r\dot{x}_3$, therefore $\gamma(x) = rx_3$. Hence, Assumptions 4.1.1 and 4.1.2 hold.

[6] See [5].

It only remains to show that the conditions of Proposition 4.3.1 hold. Some simple computations yield

$$(\nabla^2 H_d)_* = \begin{bmatrix} a_1 + 3a_2 x_{1_*}^2 + d_1^2 d_2 & 0 & -d_1 d_2 \\ 0 & \frac{1}{m} & 0 \\ -d_1 d_2 & 0 & d_2 + r^2 K_I \end{bmatrix},$$

with

$$\begin{array}{rcl} d_1 & := & \sqrt{2c_1 x_{1_*}(a_1 + a_2 x_{1_*}^2)} \\ d_2 & := & \frac{1}{c_1(c_0 + x_{1_*})}. \end{array} \tag{4.31}$$

Hence, for all $K_I > 0$ the condition (4.10) holds and x_* is a stable equilibrium for the closed-loop.

To prove asymptotic stability, first, note that from (4.9) we get

$$\kappa = -\frac{1}{K_I} \frac{x_{3_*}}{r c_1 (c_0 + x_{1_*})} - r x_{3_*}. \tag{4.32}$$

Second, in the residual set where $\dot{H}_d \equiv 0$, we have

$$\dot{H}_d = \dot{x}^\top F^{-1} \dot{x} - |y_{\mathsf{PS}}|_{K_P}^2 = 0 \iff \left\{ \begin{array}{rcl} \dot{x}_1 & = & 0 \\ \dot{x}_3 & = & 0. \end{array} \right.$$

Furthermore, the following chain of implications hold true

$$\begin{array}{rcl} \dot{x}_1 = 0 & \iff & x_2 = 0 \\ & \implies & \dot{x}_2 = 0 \\ & \implies & x_3 = (x_1 + c_0)\sqrt{2c_1 x_1 (a_1 + a_2 x_1^2)}. \end{array} \tag{4.33}$$

Hence, since $\dot{x}_3 = 0$, we have

$$\begin{array}{rcl} \frac{x_3}{c_1(x_1+c_0)} + K_I (r x_3 + \kappa) & = & 0 \\ \iff \frac{\sqrt{2c_1 x_1 (a_1 + a_2 x_1^2)}}{c_1(x_1+c_0)} - \frac{\sqrt{2c_1 x_{1*} (a_1 + a_2 x_{1*}^2)}}{c_1(x_{1*}+c_0)} + K_I r (x_3 - x_{3*}) & = & 0 \\ \implies x_1 & = & x_{1*}, \end{array}$$

where we used (4.33) and (4.32). Moreover, the analysis above implies that $x = x_*$ and thus

$$\dot{H}_d \equiv 0 \iff x = x_*.$$

Simulations

Simulation results are presented in Figure 4.1. Based on the results reported in [5], the system parameters were chosen as $c_0 = 15 \times 10^{-6}$, $c_1 = 35.6 \times 10^{-9}$, $m = 2.35 \times 10^{-9}$, $a_1 = 0.46$, $a_2 = 0.0973$, $b = 5.5 \times 10^{-7}$ and $r = 100$. Figure 4.1 shows the closed-loop system response with initial conditions $x_0 = \mathbf{0}_3$ and the gain selection $K_P = 1000$, $K_I = 5 \times 10^{-3}$. The control objective is to stabilize x_1 at $x_{1*} = 7 \times 10^{-5}$. Thus, from (4.30), $x_{2*} = 0$ and $x_{3*} = 1.2870 \times 10^{-10}$. As it can be noticed from the plot, $x_1 > 0$, which agrees with the physical constraint.

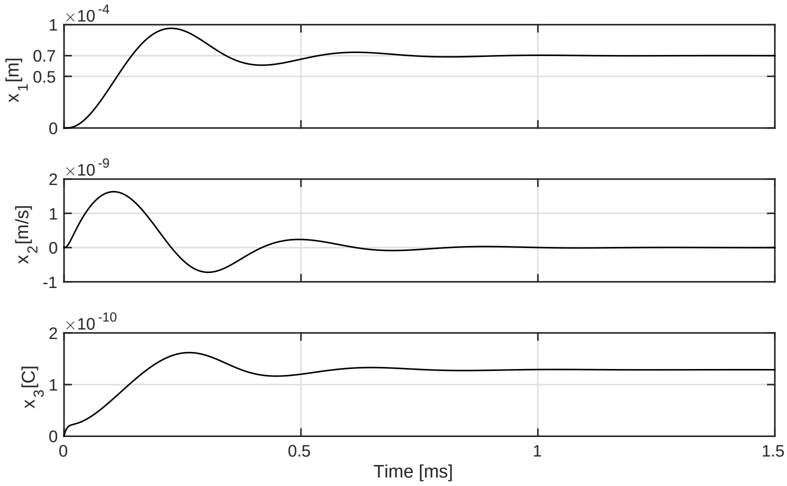

Figure 4.1: Simulation results of system (4.29) in closed-loop with the PI-PBC.

4.6.2 LTI systems: Controllability is not enough

In the important paper [47] it was shown that IDA-PBC for LTI systems is a *universal stabilizer*, in the sense that it is applicable to all stabilizable systems. On the other hand, it was shown in [37] that stabilizability is *not enough* for IDA-PBC of mechanical system. Indeed, in Proposition 4.1 of [37] it is shown that if the system has *uncontrollable modes*, an additional condition of the pole location, which is stronger than stabilizability, must be imposed for stabilization with IDA-PBC.

The difference between these two cases is that, while for general IDA-PBC there is no constraint on the structure of the desired energy function, for mechanical systems a *particular structure* is imposed to it. Since in the methodology proposed in this paper there is also a constraint on the desired energy function, namely (4.6), it is expected that a condition stronger than stabilizability should be imposed for the method to apply. Actually, we will prove that unlike IDA-PBC for mechanical systems even controllability is not enough for the proposed method to work.

Now, recall that for LTI systems the energy function is of the form $H(x) = \frac{1}{2}x^\top Qx$, the matrices F and g are constant and, without loss of generality, we can take $x_* = 0$. Therefore, the control (4.5) becomes a simple linear, state-feedback of the form

$$u_{\mathtt{PS}}(x) = K_L x \tag{4.34}$$

with

$$K_L := \left(I_m - K_P g^\top F^{-\top} g\right)^{-1}\left(K_P g^\top F^{-\top} F Q + K_I g^\top F^{-\top}\right).$$

Notice that for linear systems, with $x_* = 0$, the constant vector κ given in (4.9) is equal to zero. To prove the aforementioned conjecture we will construct an LTI, controllable PH system for which the controller (4.34) yields an unstable closed-loop system *for all* values of the tuning gains K_P, K_I. It is important to note that the Lyapunov stability test used in Proposition 4.3.1 is sufficient, but not necessary—even for LTI systems. Therefore, instability must be proved checking directly the closed-loop system matrix. Also, the sign constraints imposed to the tuning gains, which are required to ensure positivity of the shaped energy function, need not be imposed in the LTI case where, as indicated above, a stability analysis—other than Lyapunov—will be carried out.

Consider the following *controllable*, LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{4.35}$$

with $a > 0$. Some simple calculations show that it admits a PH representation

$$\dot{x} = FQx + gu, \tag{4.36}$$

with

$$F := \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, Q := \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}, g := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.37}$$

where F is full rank and satisfies $\mathtt{sym}\{F\} \leq 0$. We remark that Assumption 4.1.2 is always satisfied for single input LTI systems.

Proposition 4.6.1 *Consider the LTI, PH system* (4.36), (4.37)*, in closed-loop with the controller* (4.34)*. For all positive values of the controller gains K_P and K_I the closed-loop system is unstable.*

Proof: From (4.37), the gain K_L is given by

$$K_L = \begin{bmatrix} K_I & K_P \end{bmatrix}.$$

Therefore,

$$u_{\mathtt{PS}}(x) = K_I x_1 + K_P x_2.$$

Hence, the closed-loop system takes the form

$$\dot{x} = A_{cl} x,$$

where

$$A_{cl} := \begin{bmatrix} 0 & 1 \\ a + K_I & 1 + K_P \end{bmatrix}.$$

The eigenvalues of the closed-loop system matrix are

$$\lambda_{1,2}\{A_{cl}\} = \frac{K_P + 1}{2} \pm \frac{\sqrt{(K_P + 1)^2 + 4(a + K_I)}}{2}.$$

Note that for any positive gains K_P, K_I the closed-loop system has at least one eigenvalue with positive real part. This completes the proof.

□

4.6.3 First integrals example

Consider the PH system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \nabla H + \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} u \tag{4.38}$$

with

$$H(x) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}x_3^2. \tag{4.39}$$

The control objective is to stabilize the point $x_* = (0, 0, x_{3*})$, with $x_{3*} < 0$.

Computing the gradient and the Hessian of H, we have

$$\nabla H = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{bmatrix}, \quad \nabla^2 H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.40}$$

whence it is clear that $(\nabla H)_* \neq 0$ and $\nabla^2 H$ is not full rank.

Proposition 4.6.2 *Consider the system* (4.38) *with output variable* $y_{\mathtt{wD}}$. *The following holds true.*

(i) *There are no Casimir functions* $\mathcal{C}(x)$ *solution of the PDE*

$$F\nabla\mathcal{C} = -g. \tag{4.41}$$

(ii) *The function*

$$\gamma(x) := \frac{1}{2}x_1^2 + x_3 \tag{4.42}$$

satisfies

$$\dot{\gamma} = y_{\mathtt{wD}}$$

with

$$w = \begin{bmatrix} x_1 \\ 0 \\ -1 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.43}$$

Proof: Casimirs. Given F and g in (4.38), the PDE (4.41) takes the form

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{C}}{\partial x_1} \\ \frac{\partial \mathcal{C}}{\partial x_2} \\ \frac{\partial \mathcal{C}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -x_1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{\partial \mathcal{C}}{\partial x_2} \\ -\frac{\partial \mathcal{C}}{\partial x_1} \\ -\frac{\partial \mathcal{C}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -x_1 \\ 0 \\ -1 \end{bmatrix}$$

The equation above implies that

$$\nabla\mathcal{C} = \begin{bmatrix} 0 \\ -x_1 \\ 1 \end{bmatrix}$$

and thus

$$\nabla^2 \mathcal{C} \neq (\nabla^2 \mathcal{C})^\top.$$

This part of the proof is completed invoking Poincare's Lemma.
First integrals. Replace (4.40) and (4.43) in (2.8), then

$$\begin{aligned} y_{\mathtt{wD}} &= \begin{bmatrix} x_1 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{bmatrix} + (x_1^2 + 1)u \\ &= x_1(x_1 + x_2) - x_3 + (x_1^2 + 1)u \end{aligned} \tag{4.44}$$

On the other hand,

$$\begin{aligned} \dot{\gamma} &= \begin{bmatrix} x_1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \\ &= x_1 \dot{x}_1 + \dot{x}_3 \\ &= x_1(x_1 + x_2 + x_1 u) - x_3 + u \\ &= y_{\mathtt{wD}}, \end{aligned}$$

where we used (4.44) to obtain the last equality.

□

Remark 4.6.1 *Considering as output variable the power shaping output, it is not possible to find a function γ such that $\dot{\gamma} = y_{\mathtt{PS}}$. This is easy to verify since*

$$\nabla \gamma = -F^{-1} g = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \\ 1 \end{bmatrix}$$

which is clearly not integrable. Moreover, still not integrable for any $M(x)$ in the input change of coordinates (4.18)*.*

Proposition 4.6.3 *Consider the system* (4.38) *in closed-loop with the controller*

$$u = -\frac{1}{K_P w^\top w + 1} \left\{ K_I(\gamma + \kappa) + K_P (g + 2\phi^\top w)^\top \nabla H \right\} \tag{4.45}$$

with constant gains $K_P, K_I > 0$ and

$$\kappa := -x_{3*} - \frac{x_{3*}}{K_I}. \tag{4.46}$$

The equilibrium x_* is asymptotically stable, in the sense of Lyapunov, with Lyapunov function

$$H_d = H + \frac{1}{2} K_I (\gamma + \kappa)^2.$$

Proof: Note that the control law (4.45) can be rewritten as

$$u = -K_I(\gamma + \kappa) - K_P y_{\mathtt{wD}}. \tag{4.47}$$

On the other hand, the time derivative of H_d is given by

$$\begin{aligned} \dot{H}_d &= -|\phi \nabla H + wu|^2 + y_{\mathtt{wD}} u + y_{\mathtt{wD}} K_I(\gamma + \kappa) \\ &= -|\phi \nabla H + wu|^2 + y_{\mathtt{wD}}(u + K_I(\gamma + \kappa)) \\ &= -|\phi \nabla H + wu|^2 - K_P |y_{\mathtt{wD}}|^2, \end{aligned}$$

where we used (4.47) to obtain the last equality. Moreover

$$
\begin{aligned}
\nabla H_d &= \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{bmatrix} + K_I(\gamma + \kappa) \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} \\
\nabla^2 H_d &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + K_I \begin{bmatrix} x_1^2 & 0 & x_1 \\ 0 & 0 & 0 \\ x_1 & 0 & 1 \end{bmatrix} + K_I(\gamma + \kappa) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Evaluating both expressions above at x_*, we get

$$
\begin{aligned}
(\nabla H_d)_* &= \begin{bmatrix} 0 \\ 0 \\ x_{3*} \end{bmatrix} - x_{3*} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
(\nabla^2 H_d)_* &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + K_I \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - x_{3*} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0.
\end{aligned}
$$

The stability of the point is proven invoking Lyapunov Theory.

Now, to establish the convergence to the equilibrium, note that $\dot{H}_d = 0$ if and only if

$$
\begin{aligned}
\phi \nabla H + wu &= 0 && (4.48) \\
y_{\mathtt{wD}} &= 0. && (4.49)
\end{aligned}
$$

From (4.49) we have

$$
u = -K_I \left(\frac{1}{2} x_1^2 + x_3 + \kappa \right).
$$

Replacing the latter in (4.48) we get

$$
\begin{bmatrix} -x_1 K_I(\frac{1}{2}x_1^2 + x_3 + \kappa) \\ 0 \\ x_3 + K_I(\frac{1}{2}x_1^2 + x_3 + \kappa) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

The latter implies that

$$
\begin{aligned}
x_1 \left(\frac{1}{2} x_1^2 + x_3 + \kappa \right) &= 0 && (4.50) \\
-\frac{K_I}{1 + K_I} \left(\frac{1}{2} x_1^2 + \kappa \right) &= x_3. && (4.51)
\end{aligned}
$$

Replacing (4.46) and (4.51) in (4.50) we obtain

$$
\frac{1}{K_I + 1} x_1 \left(\frac{1}{2} x_1^2 - x_{3*} - \frac{1}{K_I} x_{3*} \right) = 0. \tag{4.52}
$$

The equation above only has solution for $x_1 = 0$, replacing the latter in (4.51) we get

$$
x_3 = \frac{K_I}{1 + K_I} \left(x_{3*} + \frac{1}{K_I} x_{3*} \right) \iff x_3 = x_{3*}.
$$

Now, notice that

$$x_1 = 0 \Rightarrow \dot{x}_1 = 0 \Rightarrow x_2 = 0. \tag{4.53}$$

Hence, we conclude that

$$\dot{H}_d = 0 \Longleftrightarrow x = x_*. \tag{4.54}$$

The proof is completed recalling the Barbashin-Krasovskii's Theorem.

□

Simulations

Figure 4.2 shows the simulation results of the closed-loop system for $x_{3*} = -4$, initial conditions at $x_0 = (4, -2, 2)$ and choosing the gains as $K_I = 3$, $K_P = 1$.

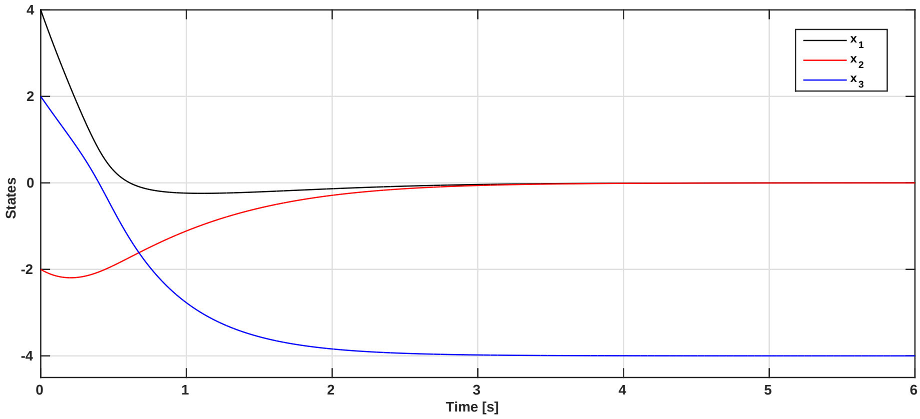

Figure 4.2: Simulation results of system (4.38) in closed-loop with (4.45).

From the Figure 4.2 can be noticed that the control objective is achieved, that is, the states of the closed-loop system converge to the desired value.

4.6.4 Change of coordinates for energy shaping

Consider the PH system

$$\dot{x} = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \nabla H(x) + \begin{bmatrix} -x_3 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \tag{4.55}$$

with

$$\begin{aligned} H(x) &= \tfrac{1}{2}|x|^2 \\ \mathcal{E} &= \left\{ x \in \mathbb{R}^3 \mid x_2 - x_3 - x_1(x_3 + 1) = 0 \right\}. \end{aligned}$$

The control objective is to stabilize the equilibrium $x_* = (1, 3, 1)$, which belongs to $\mathcal{E}$. Note that Assumption 4.1.1 is satisfied, then the first step towards the control design is to verify that Assumption 4.1.2 holds. Therefore, we compute

$$-F^{-1}g(x) = \begin{bmatrix} 1 & 0 \\ x_3 & 1 \\ -1 & 1 \end{bmatrix}.$$

Using Poincare's Lemma we can prove that the vector fields of $F^{-1}g_i(x)$ are not integrable. Thus, the PID-PBC design of Propositions 4.2.1 and 4.3.1 is not applicable. We investigate now the possibility of extending it with the input change of coordinates (4.18), as proposed in Subsection 4.5.1.

A full-rank basis for the kernel of $g^\top(x)F^{-\top}(x)$ is given by

$$\Lambda(x) = \begin{bmatrix} x_3 + 1 \\ -1 \\ 1 \end{bmatrix},$$

whose span defines an involutive distribution and thus Proposition 4.5.1 ensures the *existence* of the required full-rank mapping $M : \mathbb{R}^3 \to \mathbb{R}^{2\times 2}$ that defines input change of coordinates.

To compute $M(x)$ we invoke again Poincare's Lemma and solve the PDEs

$$\nabla(F^{-1}g(x)m_i(x)) = \left[\nabla(F^{-1}g(x)m_i(x))\right]^\top, \; i = 1, 2,$$

where $m_i : \mathbb{R}^3 \to \mathbb{R}^2$ are the columns of $M(x)$. A simple solution to these PDEs is given by

$$M(x) = \begin{bmatrix} 1 & 0 \\ x_2 & 1 \end{bmatrix}. \tag{4.56}$$

Moreover

$$-F^{-1}g(x)M(x) = \begin{bmatrix} 1 & 0 \\ x_2 + x_3 & 1 \\ -1 + x_2 & 1 \end{bmatrix}.$$

Integrating the columns of (4.6.4) we get the mapping

$$\gamma(x) = \begin{bmatrix} x_1 + x_2 x_3 - x_3 + \frac{1}{2}x_2^2 \\ x_2 + x_3 \end{bmatrix}, \tag{4.57}$$

that satisfies $\bar{y}_{\mathtt{PS}} = \dot{\gamma}$.

In the proposition below we proceed with the design of the PI-PBC described in (4.5) with the new input $\bar{u}$, defined in (4.18), and the new output $\bar{y}$, given by (4.19).

Proposition 4.6.4 *Consider the PH system* (4.55) *in closed-loop with the controller $u = M(x)\bar{u}$, with $M(x)$ defined in* (4.56)*, and*

$$\bar{u} = \left\{I_2 - K_P M^\top g^\top F^{-\top} g M\right\}^{-1} \left\{-K_I(\gamma + \kappa) + K_P M^\top g^\top F^{-\top} F \nabla H\right\} \tag{4.58}$$

where $K_P = diag\{k_p, k_p\}, K_I = diag\{k_i, k_i\}$ with $k_p, k_i > 0$, γ is given in (4.57) *and*

$$\kappa = \begin{bmatrix} -\frac{1}{k_i} - 7.5 \\ \frac{1}{k_i} - 4 \end{bmatrix}. \tag{4.59}$$

Then, $x_* = (1, 3, 1)$ is a locally asymptotically stable equilibrium of the closed-loop system with Lyapunov function H_d defined in (4.6).

Proof: To establish the proof, first, note that the control law (4.58) can be rewritten as

$$\bar{u}_{\mathtt{PS}} = -K_P \bar{y}_{\mathtt{PS}} - K_I(\gamma + \kappa). \tag{4.60}$$

Moreover

$$\begin{aligned} \dot{H}_d &= \dot{x}^\top F^{-1}\dot{x} + \bar{y}_{\mathsf{PS}}^\top\left(\bar{u} + K_I(\gamma+\kappa)\right) \\ &= -\dot{x}_3^2 - |\bar{y}_{\mathsf{PS}}|_{K_P}^2 \leq 0. \end{aligned} \tag{4.61}$$

On the other hand,

$$\begin{aligned} (\nabla H_d)_* &= (\nabla H)_* - F^{-1} g_* M_* K_I\left(\gamma_* + \kappa\right) \\ &= \mathbf{0}_3, \end{aligned}$$

where we used the value of κ given in (4.59). Furthermore,

$$(\nabla^2 H_d)_* = \begin{bmatrix} k_i+1 & 4k_i & 2k_i \\ 4k_i & 17k_i & 9k_i-1 \\ 2k_i & 9k_i-1 & 5k_i+1 \end{bmatrix}$$

whose Schur complement analysis shows that is positive definite for any $k_i > 0.03$.

To prove asymptotic stability, note that from (4.61) we have

$$\dot{H}_d \equiv 0 \Longleftrightarrow \left\{ \begin{array}{ccc} \dot{x}_3 & = & 0 \\ \bar{y}_{\mathsf{PS}} & = & 0 \end{array} \right\} \Longrightarrow \dot{x} = \mathbf{0}_3.$$

Furthermore, the expression above implies that

$$\begin{aligned} F\nabla H - gMK_I(\gamma+\kappa) &= \mathbf{0}_3 \\ F\nabla H_d &= \mathbf{0}_3. \end{aligned}$$

Thus, since F is full rank, we have

$$\nabla H_d = \mathbf{0}_3. \tag{4.62}$$

The rest of the proof relies in the fact that x_* is an isolated minimum of H_d. Moreover, for a neighborhood of x_*, the only solution of (4.62) is x_*.

□

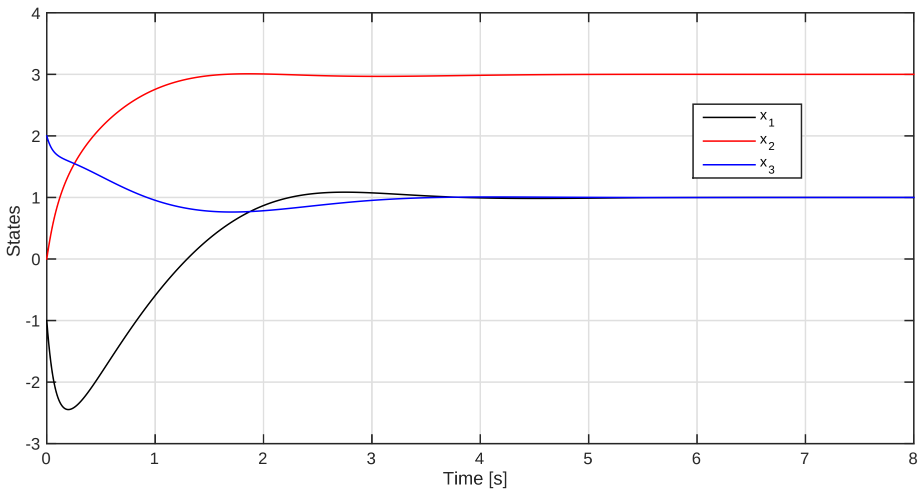

Figure 4.3: Simulation results of the system (4.55) in closed-loop with (4.18).

Simulations

Figure 4.3 shows the response of the closed-loop system with initial conditions $x_0 = (-1, 0, 2)$ and choosing the PI-PBC gains as $K_I = \text{diag}\{1.5, 1.5\}$, $K_P = \text{diag}\{2, 2\}$. The simulation results confirm the convergence of the trajectories to the desired equilibrium.

Chapter 5

PID controller

Proportional-Integral-Derivative (PID) controllers overwhelmingly dominate engineering applications where the control objective is to regulate some signal around a desired value. Commissioning of PIDs reduces to the suitable selection of the controller gains, which is a difficult task for wide ranging operating systems, where the validity of a linearized approximation is limited. Although gain scheduling, auto tuning and adaptation provide some help to overcome this problem, they suffer from well documented drawbacks that include being time consuming and fragility of the design [1]. In contrast with this scenario in PID-PBC, where the PID is wrapped around a passive output, the gain tuning step is trivialized, as convergence of the output to zero and $\mathcal{L}_2$-stability of the closed-loop system is guaranteed *for all* positive gains—among which the designer selects those that ensure best *transient performance*.

However, it is often the case that the signal to be regulated is not a passive output or its reference output is nonzero. Another scenario of practical interest is when the control objective is to drive the full system *state* to a desired constant value. A classical example is under-actuated mechanical systems, whose passive outputs are the actuated velocities, but in most applications the objective is to drive *all positions* to some desired constant values. To address these problems two approaches have been adopted in the literature, first, to identify passive systems for which the PID controller on the original passive outputs assigns the equilibrium and preserves the passivity but with a *new storage function* that has a minimum at the desired equilibrium, which then qualifies as a Lyapunov function for the latter. The identification of these systems boils down to imposing some integrability conditions that allows us to express the integral term of the PID as a function of the systems state. Second, to give conditions under which the *incremental model* of the system is also passive [16, 19, 24], property called "shifted passivity" in [58]. In this case, adding the PID around the incremental variables ensures, not only that the incremental output goes to zero, but also that the desired equilibrium is assigned to the closed-loop. The first approach has been pursued in [13, 50] for mechanical systems and in the last chapter we studied the construction of a PI for general PH systems. PID-PBCs have been designed following the second line of research in [8, 19, 51] for power converters, in [31] for photovoltaic systems and in [6] for general RLC circuits. The addition of integral actions has also been proposed to robustify PBCs, *vis-à-vis* external disturbances, in [12, 16, 39, 49].

Surprisingly, in the controllers developed in Chapter 4 the derivative term is absent. This is directly related to the relative degree of the passive outputs used for its construction. Motivated by the wide range of applicability of PID controllers, this chapter is devoted to the construction of PID stabilizers based on the passive outputs reported in Chapter 2. Towards this end, we propose two different schemes of PID-PBC and we show that the addition of the derivative term enlarges the class of stabilizable plants with respect to the controllers reported in Chapter 4.

5.1 PID controller

In this section we provide the necessary conditions for the construction of a PID controller based on the passive output of the PH system (2.6). Therefore, we restrict our attention to PID controllers of the form

$$u = -K_P y_{(\cdot)} - K_I(\gamma(x) + \kappa) - K_D \dot{y}_{(\cdot)} \tag{5.1}$$

where $\gamma : \mathbb{R}^m \to \mathbb{R}$ satisfies

$$\dot{\gamma} = y_{(\cdot)},$$

$\kappa \in \mathbb{R}^m$ is constant and the symmetric positive constant matrices $K_P, K_I, K_D \in \mathbb{R}^{m\times m}$ are the PID tuning gains.

5.1.1 Preliminaries

The first step towards the formulation of the PID controller is to ensure that the control law (5.1) can be computed without differentiation nor singularities that may arise due to the presence of the derivative term $\dot{y}_{(\cdot)}$. Clearly, the derivative term can be added only when the output $y_{(\cdot)}$ has relative degree equal to one, that is, when $w(x) = 0$ and $D(x) = 0$, hence $y_{(\cdot)}$ is the natural output defined in (2.10).

The following assumptions identify the class of PH systems for which the proposed control strategy is applicable.

Assumption 5.1.1 *There exists a function $\gamma : \mathbb{R}^m \to \mathbb{R}$ that solves the following PDE*

$$\begin{bmatrix} g^\top(x) \\ (\nabla H(x))^\top F^\top(x) \end{bmatrix} \nabla\gamma(x) = \begin{bmatrix} \mathbf{0}_{m\times m} \\ (\nabla H(x))^\top g(x) \end{bmatrix}. \tag{5.2}$$

Assumption 5.1.2 *The mapping $K : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, defined as*

$$K(x) := I_m + K_D(\nabla y)^\top g(x),$$

is full rank.

Proposition 5.1.1 *Fix $y_{(\cdot)} = y$ and suppose Assumption 5.1.1 and 5.1.2 holds. The control law* (5.1) *takes the form*

$$u = -K^{-1}(x)\left\{K_P y + K_I(\gamma(x) + \kappa) + K_D(\nabla y)^\top F(x)\nabla H(x)\right\}, \tag{5.3}$$

with γ solution of (5.2).

Proof: The proof is straightforward from (2.10) and Assumption 5.1.2. □

Before closing this subsection we note that in [58] PID control is viewed from a different perspective. Namely, assuming that $\dot{y}$ is *computable*, it is shown that the closed-loop system can be represented as a PH system with *algebraic constraints*. However, leaving aside the complexity of computing $\dot{y}$, the stability analysis of this kind of systems remains an essentially open question.

5.1.2 $\mathcal{L}_2$-stability analysis

PID controllers define input strictly passive mappings [58]. Thus, the Passivity Theorem [11, 58] allows to immediately conclude output strict passivity—hence, $\mathcal{L}_2$-stability—of the closed-loop system. The proposition below establishes this result for the PH system (2.6) in closed-loop with the PID-PBC (5.3).

Proposition 5.1.2 *Consider the PH system* (2.6) *in closed-loop with the PID-PBC* (5.3) *with an external signal* $d(x)$. *The operator* $d \mapsto y$ *is* $\mathcal{L}_2$*-stable. More precisely, there exists* $\eta \in \mathbb{R}$ *such that*

$$\int_0^t |y(s)|^2 ds \leq \frac{1}{\lambda_{\min}\{K_P\}} \int_0^t |d(s)|^2 ds + \eta, \ \forall t \geq 0.$$

Proof: Proposition 2.2.1 ensures passivity of the mapping $\Sigma : u \mapsto y$ defined by the PH system. On the other hand, output strict passivity of the mapping $\Sigma_c : y \mapsto (-u)$, defined by the PID-PBC, is proved noting that

$$u = -K_P y - K_I(\gamma(x) + \kappa) - K_D \dot{y}. \tag{5.4}$$

Therefore

$$\begin{array}{rcl} y^\top(-u) & = & y^\top K_P y + y^\top K_I(\gamma + \kappa) + y^\top K_D \dot{y} \\ & \geq & \lambda_{\min}(K_P)|y|^2 + \dot{\gamma}^\top K_I(\gamma + \kappa) + y^\top K_D \dot{y}. \end{array}$$

Integrating the expression above we get

$$\int_0^t y(s)(-u(s))ds \geq \lambda_{\min}(K_P) \int_0^t |y(s)|^2 ds - |\gamma(0)|^2_{K_I} - |y(0)|^2_{K_D}, \ \forall t \geq 0.$$

The rest of proof follows directly from the Passivity Theorem [11]. □

5.1.3 Lyapunov Stability Analysis

A first step of the stability analysis is to ensure that x_* is an *equilibrium* of the closed-loop system. In contrast with the PI controller proposed in Chapter 4, apart from the integrability condition, the plant to be stabilized needs to satisfy an additional assumption to ensure that the PID controller (5.3) can assign the desired equilibrium of the closed-loop system. This assumption is formulated below.

Assumption 5.1.3 *Let* $x_* \in \mathcal{E}$. *The PH system* (2.6) *evaluated at* x_* *verifies*

$$\mathcal{R}_* (\nabla H)_* = \mathbf{0}_n. \tag{5.5}$$

The proposition below states that a necessary condition to assign the equilibrium of the closed-loop system is that Assumption 5.1.3 holds. This clearly stymies the applicability of the PID-PBC.

Proposition 5.1.3 *Let x_* an equilibrium of the open-loop system. Consider the PH system* (2.6), (2.10) *in closed-loop with the PID-PBC* (5.3). *Hence, x_* is an equilibrium of the closed-loop system only if Assumption* (5.1.3) *is satisfied.*

Proof: Note that Assumption 5.1.1 implies that

$$\begin{aligned}\dot{\gamma}(x) &= (\nabla\gamma)^\top \dot{x} \\ &= g^\top(x)\nabla H(x) = y.\end{aligned}$$

Therefore, $y_* = 0$. Moreover, since $x_* \in \mathcal{E}$, the following chain of implications hold true.

$$\begin{aligned} F_*(\nabla H)_* + g_* u_* &= 0 \\ \Longrightarrow (\nabla H)_*^\top F_*(\nabla H)_* + (\nabla H)_*^\top g_* u_* &= 0 \\ \Longrightarrow (\nabla H)_*^\top \mathcal{R}_*(\nabla H)_* &= 0 \\ \Longleftrightarrow \mathcal{R}_*(\nabla H)_* &= 0. \end{aligned}$$

This completes the proof.

□

Remark 5.1.1 *Assumption 5.1.3 is analogous to the dissipation obstacle present in CbI. In other words, dissipation cannot be present on the coordinates to be shaped.*

The proposition below establishes conditions for which the point x_* is a stable equilibrium, in the sense of Lyapunov, of the closed-loop system.

Proposition 5.1.4 *Consider the PH system* (2.6), (2.10) *in closed-loop with the PID-PBC* (5.3). *Suppose that Assumptions 5.1.1-5.1.3 hold. Let the closed-loop energy function*

$$H_d = H + \frac{1}{2}|\gamma + \kappa|_{K_I}^2 + \frac{1}{2}|y|_{K_D}^2. \tag{5.6}$$

(i) If $x_* \in \mathcal{E}$ and

$$\arg\min H_d(x) = x_*, \tag{5.7}$$

and it is isolated. Then, the closed-loop system has a stable *equilibrium at x_*, with Lyapunov function H_d.*

(ii) The equilibrium is asymptotically *stable if*

$$\dot{H}_d \equiv 0 \Longleftrightarrow x = x_*. \tag{5.8}$$

Proof: To establish the proof we first show that H_d is non-increasing along the trajectories of the closed-loop system. Towards this end we compute

$$\begin{aligned}\dot{H}_d &\leq y^\top u + y^\top K_I(\gamma + \kappa) + y^\top K_D \dot{y} \\ &= -|y|_{K_P}^2 \leq 0,\end{aligned}$$

where we used (5.4) to obtain the equality.

Now, note that (5.7) implies that H_d is positive definite with respect to x_*. The proof is completed invoking Lyapunov Theory and the Barbashin-Krasovskii's Theorem.

□

5.2 Alternative PID controller

In this section we propose an alternative PID controller based on the passive outputs of the PH system (2.6). The starting point of this new PID-PBC is the key property underscore in the Remark 4.4.1, that is, the integral action in the controller $u_{\mathtt{PS}}$ of Propositions 4.2.1 preserves the PH structure of the closed-loop system. Hence, the new PH system is passive and the desired equilibrium is already a critical point of its storage function. Furthermore, a new static-feedback will add a proportional and derivative term of the new PH system's output completing the PID-PBC design.

5.2.1 Preliminaries

In this subsection we define some mappings and assumptions which are instrumental in the design of the PID-PBC scheme proposed in this section.

Towards the construction of the new controller, suppose Assumptions 4.1.1 and 4.1.2 hold. Then, define the mappings $H_a : \mathbb{R}^n \to \mathbb{R}$ and $y_a : \mathbb{R}^n \to \mathbb{R}^m$ as follows

$$\begin{aligned} H_a(x) &:= H(x) + \frac{1}{2}|\gamma + \kappa|^2_{K_I}, \qquad (5.9) \\ y_a &:= g^\top(x)\nabla H_a(x), \qquad (5.10) \end{aligned}$$

where γ verifies (4.3).

As in Section 5.1, the following assumption is necessary to ensure that the PID-PBC can be implemented without singularities due to the presence of the derivative term.

Assumption 5.2.1 *The mapping $K_{\mathtt{PSN}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, defined as*

$$K_{\mathtt{PSN}}(x) := I_m + K_D(\nabla y_a)^\top g(x),$$

is full rank.

If Assumption 5.2.1 holds, then it is possible to define the mapping $v_{\mathtt{PS}}; \mathbb{R}^n \to \mathbb{R}^m$ as

$$v_{\mathtt{PS}}(x) := -K_{\mathtt{PSN}}^{-1}\left\{K_P y_a + K_D(\nabla y_a)^\top F(x)\nabla H_a(x)\right\}. \qquad (5.11)$$

5.2.2 Energy shaping

In this subsection we define a new PID controller around $y_{\mathtt{PS}}$ and y_a such that the closed-loop system preserves passivity properties.

Proposition 5.2.1 *Suppose Assumptions 4.1.1, 4.1.2 and 5.1.3 hold. Define the mapping $u_{\mathtt{PSN}} : \mathbb{R}^n \to \mathbb{R}^m$*

$$u_{\mathtt{PSN}}(x) := -K_I(\gamma(x) + \kappa) + v_{\mathtt{PS}}, \qquad (5.12)$$

where γ verifies (4.3)[1]*, $\kappa \in \mathbb{R}^m$ is constant and the symmetric positive constant matrices $K_P, K_I, K_D \in \mathbb{R}^{m\times m}$ are the PID tuning gains. The system* (2.6) *in*

[1]The result is presented for F full rank, nonetheless, some simple calculations show that the same result holds for F not full rank, using $F^\dagger$ instead of F^{-1}.

closed-loop with the control $u = u_{\mathtt{PSN}} + v$ defines a passive mapping $v \mapsto y_a$ with storage function

$$H_d = H(x) + \frac{1}{2}|\gamma + \kappa|^2_{K_I} + \frac{1}{2}|y_a|^2_{K_D}. \tag{5.13}$$

Proof: To establish the proof, first, note that from (5.9) and (5.10) the expression (5.11) takes the form

$$v_{\mathtt{PS}} = -K_P y_a - K_D \dot{y}_a. \tag{5.14}$$

On the other hand, the derivative of H_a, defined in (5.9), along the trajectories of the closed-loop system verifies

$$\begin{aligned} \dot{H}_a &= (\nabla H_a)^\top \dot{x} \\ &= (\nabla H_a)^\top (F\nabla H + gu) \\ &= (\nabla H_a)^\top F \nabla H_a + (\nabla H_a)^\top g(v_{\mathtt{PS}} + v) \\ &= -|\nabla H_a|^2_{\mathcal{R}} + y_a^\top (v_{\mathtt{PS}} + v), \end{aligned} \tag{5.15}$$

with y_a defined as in (5.10). Furthermore,

$$\begin{aligned} \dot{H}_d &= \dot{H}_a + y_a^\top K_D \dot{y}_a \\ &= -|\nabla H_a|^2_{\mathcal{R}} + y_a^\top (v_{\mathtt{PS}} + v) + y_a^\top K_D \dot{y}_a \\ &= -|\nabla H_a|^2_{\mathcal{R}} - |y_a|^2_{K_P} + y_a^\top v \le y_a^\top v, \end{aligned}$$

where we used (5.15) and (5.14) to obtain the second and the last equality, respectively.

□

Remark 5.2.1 *In contrast with the PID-PBC on the natural output proposed in Section 5.1, the controller $u_{\mathtt{PSN}}$ is not constructed around a passive output but is consists in integral[2], proportional and derivative terms of two different mappings. This can be understood as a control law composed by two loops, where the first one is an integral term of $y_{\mathtt{PS}}$ and the second loop adds the proportional and derivative terms of y_a.*

5.2.3 Stabilization

In this subsection we present the main result of stabilization of PH systems with the PID-PBC presented in Proposition 5.2.1.

The proposition below establishes the conditions for which the new storage function $H_d(x)$ is positive definite (with respect to the desired equilibrium x_*) and thus qualifies as a Lyapunov function for the closed-loop system (with $v = 0$).

Proposition 5.2.2 *Consider the system* (2.6)*, verifying Assumptions 4.1.1, 4.1.2 and 5.1.3, in closed-loop with the control $u = u_{\mathtt{PSN}}$, defined* (5.12)*. Select κ as in* (4.9)*.*

[2] See Remark 4.2.1.

If $x_* \in \mathcal{E}$ and

$$(\nabla^2 H_d)_* > 0 \tag{5.16}$$

with $H_d(x)$ defined in (5.13)*, then x_* is* stable *(in the sense of Lyapunov) with Lyapunov function $H_d(x)$. It is* asymptotically *stable if the following implication is true*

$$\dot{H}_d \equiv 0 \Longleftrightarrow x = x_*.$$

Proof: From Proposition 4.3.1 it follows that

$$(\nabla H_a)_* = \mathbf{0}_n.$$

Hence, $y_{a*} = 0$. Furthermore,

$$(\nabla H_d)_* = (\nabla H_a)_* + (\nabla y_a)_* K_D y_{a*} = \mathbf{0}_n.$$

Hence, (5.16) in combination with the expression above imply that H_d is positive definite with respect to x_*. The proof is completed invoking Lyapunov Theory and the Barbashin-Krasovskii's Theorem.

□

Remark 5.2.2 *We stress the fact that, in contrast with the PID-PBC proposed in Section 5.1, the PID controller* (5.12)-(5.11)*—under Assumptions 4.1.1 and 4.1.2—ensures that the point $x_* \in \mathcal{E}$ can be assigned as an equilibrium of the closed-loop system even if Assumption 5.1.3 does not hold.*

5.3 Examples

In this section we present two examples for which the PI controller presented in Chapter 4 is not suitable but can be stabilized via the PID-PBC proposed in this chapter. The first example is the LTI studied in the previous chapter and its general version. The second example shows

5.3.1 LTI continued

In this subsection we show that the LTI system studied in Subsection 4.6.2 in Chapter 4 can be stabilized with the PID-PBC proposed in Section 5.1. Moreover, this result can be extended to the *controllability canonical* form of LTI systems of dimension 2.

Proposition 5.3.1 *Consider the system* (4.36)*,* (4.37) *in closed-loop with*

$$u = \frac{(K_I + aK_D)}{1 - K_D} x1 + \frac{(K_P + K_D)}{1 - K_D} x_2, \tag{5.17}$$

with $K_D \neq 1$. There exists a set of positive gains K_I, K_P, K_D such that the origin is a stable equilibrium of the closed-loop system.

Proof: The first step to establish the proof is to show that Assumptions 5.1.1-5.1.3 are satisfied. Towards this end, note that $y = -x_2$, whence we propose $\gamma = -x_1$ as a solution to (5.2). Moreover,

$$(\nabla y)^\top g = -1$$

hence, Assumption 5.1.2 holds for any $K_D \neq 0$. On the other hand, since $x_* = \mathbf{0}_2$, Assumption 5.1.3 is satisfied and $\kappa = 0$.

Now, some simple calculations show that the PID-PBC given in (5.3) takes the form (5.17). Moreover the closed-loop system is given by

$$\dot{x} = A_{cl} x$$

where

$$A_{cl} := \begin{bmatrix} 0 & 1 \\ \frac{K_I + a}{1 - K_D} & \frac{K_P + 1}{1 - K_D} \end{bmatrix}.$$

The eigenvalues of the closed-loop system matrix are

$$\lambda_{1,2}\{A_{cl}\} = \frac{K_P + 1}{2(1 - K_D)} \pm \frac{\sqrt{(K_P + 1)^2 + 4(a + K_I)(1 - K_D)}}{2(1 - K_D)}.$$

The proof is completed noting that for $K_D > 1$ and $(K_I + a) \geq 0$, the eigenvalues of A_{cl} have negative real part.

□

The Proposition below establishes that any controllable LTI system of dimension 2 can be stabilized by the PID-PBC proposed in Section 5.1.

Proposition 5.3.2 *Consider a LTI system described by*

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{5.18}$$

Then, for any constant parameters a_1, a_2, there exists a set of positive gains K_I, K_P, K_D for the PID-PBC (5.3) *such that the origin is a stable equilibrium of the closed-loop system.*

Proof: To establish the proof, first, note that the system (5.18) admits a PH representation[3]

$$\begin{aligned} \dot{x} &= FQx + gu, \\ F &:= \begin{bmatrix} 0 & -sign(a_2) \\ sign(a_2) & -|a_2| \end{bmatrix} \\ Q &:= \mathrm{diag}\{sign(a_2)a_1, -sign(a_2)\} \\ g &:= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{5.19}$$

Now, we proceed to corroborate that Assumptions 5.1.1–5.1.3 are satisfied. In order to do that note that $y = -sign(a_2)x_2$, thus we propose $\gamma = -sign(a_2)x_1$ as a solution of the PDE (5.2), which implies that Assumptions 5.1.1 and 5.1.3 are satisfied (with $\kappa = 0$). Furthermore,

$$(\nabla y)^\top g = -sign(a_2),$$

hence the selection $K_D \neq 1$ ensures that Assumption 5.1.2 holds.

[3]We will consider $sign(0) = 1$.

Finally, to prove stability of the closed-loop system note that the control law (5.3) takes the form

$$u = \frac{sign(a_2)(K_I + a_1 K_D)}{1 - sign(a_2)K_D} x_1 + \frac{sign(a_2)(K_P + a_2 K_D)}{1 - sign(a_2)K_D} x_2.$$

Moreover, the closed-loop system takes the form

$$\dot{x} = A_{cl} x,$$

where

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ \frac{a_1 + sign(a_2)K_I}{1 - sign(a_2)K_D} & \frac{a_2 + sign(a_2)K_P}{1 - sign(a_2)K_D} \end{bmatrix}$$

The eigenvalues of A_{cl} are given by

$$\lambda_{1,2}\{A_{cl}\} = \begin{cases} \frac{a_2 + K_P}{2(1-K_D)} \pm \frac{\sqrt{(a_2+K_P)^2 + 4(a_1+K_I)(1-K_D)}}{2(1-K_D)} & \text{if} \quad a_2 \geq 0, \\ \frac{a_2 - K_P}{2(1+K_D)} \pm \frac{\sqrt{(a_2-K_P)^2 + 4(a_1-K_I)(1+K_D)}}{2(1+K_D)} & \text{if} \quad a_2 < 0. \end{cases}$$

The proof is completed noting that

$$\left.\begin{array}{r} K_D > 1, (a_1 + K_I) \geq 0 \quad \text{for} \quad a_2 \geq 0, \\ (a_1 - K_I) < 0 \quad \text{for} \quad a_2 < 0 \end{array}\right\} \implies \Re\{\lambda_{1,2}\{A_{cl}\}\} < 0.$$

□

5.3.2 Alternative PID-PBC

Consider the PH system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \nabla H + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \tag{5.20}$$

with

$$H = \frac{1}{2}(x_1 + x_2)^2 + a x_1 + 1 + \cos x_3, \tag{5.21}$$

where the constant parameter a is different from zero and $x_3 \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. Define $\mathcal{X} := \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, then the equilibria set of this system is given by

$$\mathcal{E} := \left\{ x \in \mathbb{R}^2 \times \mathcal{X} \mid \sin x_3 = 0 \right\}. \tag{5.22}$$

The control objective is to stabilize the constant point $x_* = (x_{1*}, x_{2*}, 0)$, with x_{1*} independent from x_{2*}. Note that the PID-PBC (5.3) is not suitable since

$$\mathcal{R}(\nabla H)_* = \begin{bmatrix} 0 \\ x_{1*} + x_{2*} \\ 0 \end{bmatrix}. \tag{5.23}$$

Moreover, the latter expression is equal to $\mathbf{0}_3$ if and only if $x_{1*} = -x_{2*}$, which is not necessarily true.

On the other hand, the matrix F is full rank and

$$-F^{-1}g = \nabla\gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which admits a solution linear in x_1 and x_2. The latter implies that Assumptions 4.1.1 and 4.1.2 are satisfied. Furthermore, we propose

$$\gamma = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}. \tag{5.24}$$

Now, selecting $K_I = \text{diag}\{k_{i1}, k_{i2}\}$ with $k_{i1}, k_{i2} > 0$, the gradient and the Hessian of H_a, defined in (5.9), are given by

$$\begin{aligned} \nabla H_a &= \begin{bmatrix} x_1 + x_2 + a + k_{i2}(x_1 + \kappa_2) \\ x_1 + x_2 + k_{i1}(x_2 + \kappa_1) \\ -\sin x_3 \end{bmatrix} \\ \nabla^2 H_a &= \begin{bmatrix} 1 + k_{i2} & 1 & 0 \\ 1 & 1 + k_{i1} & 0 \\ 0 & 0 & -\cos x_3 \end{bmatrix}. \end{aligned} \tag{5.25}$$

Therefore, evaluating (5.25) at x_* we obtain $(\nabla H_a)_* = \mathbf{0}_3$ for

$$\kappa = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} -x_{2*} - \frac{x_{1*} + x_{2*}}{k_{i1}} \\ -x_{1*} - \frac{a + x_{1*} + x_{2*}}{k_{i2}} \end{bmatrix}. \tag{5.26}$$

Note that the Hessian of H_a has no definite sign at x_*, and in consequence, the PI proposed in Chapter 4 cannot ensure the stability in closed-loop of the equilibrium. Therefore, in order to add a derivative term that helps to shape the energy of the overall system, we compute

$$\begin{aligned} y_a &= \begin{bmatrix} x_1 + x_2 + k_{i1}(x_2 + \kappa_1) \\ \sin x_3 \end{bmatrix} \\ \nabla y_a &= \begin{bmatrix} 1 & 0 \\ 1 + k_{i1} & 0 \\ 0 & -\cos x_3 \end{bmatrix}. \end{aligned} \tag{5.27}$$

In the proposition below we design the PID controller proposed in Propositions 5.2.1 and 5.2.2.

Proposition 5.3.3 *Consider the system* (5.20)–(5.21) *in closed-loop with the PID controller defined in* (5.12)–(5.11)*, where* γ, κ *are chosen as in* (5.24) *and* (5.26)*, respectively,* H_a, y_a *are described by* (5.25) *and* (5.27)*, respectively; and the gain matrices are selected as*

$$K_I = diag\{k_{i1}, k_{i2}\}, \quad K_P = diag\{k_{p1}, k_{p2}\}, \quad K_D = \begin{bmatrix} k_{d1} & k_{d2} \\ k_{d2} & k_{d3} \end{bmatrix},$$

with $k_{i1}, k_{i2}, k_{p1}, k_{p2}, k_{d1}, k_{d3} > 0$ *and* $k_{d1}k_{d3} > k_{d2}$*. Then, the point* $x_* = (x_{1*}, x_{2*}, 0)$ *is an asymptotically stable equilibrium of the closed-loop system with Lyapunov function* H_d *defined as in* (5.13)*.*

Proof: It has been shown that Assumptions 4.1.1 and 4.1.2 hold. Therefore, to construct the PID-PBC it remains to prove that Assumption 5.2.1 is satisfied. Towards this end, we compute

$$\begin{aligned} K_{\mathtt{PSN}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} k_{d1} & k_{d2} \\ k_{d2} & k_{d3} \end{bmatrix} \begin{bmatrix} 1 & 1+k_{i1} & 0 \\ 0 & 0 & -\cos x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+k_{d1}(1+k_{i1}) & -k_{d2}\cos x_3 \\ k_{d2}(1+k_{i1}) & 1-k_{d3}\cos x_3 \end{bmatrix}, \end{aligned} \tag{5.28}$$

whose determinant is given by

$$\det\{K_{\mathtt{PSN}}\} = 1 + k_{d1}(1+k_{i1}) + \left\{(k_{d2}^2 - k_{d1}k_{d3})(1+k_{i1}) - k_{d3}\right\}\cos x_3.$$

Hence, an appropriate selection of the tuning gains ensures that the mapping $K_{\mathtt{PSN}}$ is invertible and thus the Assumption 5.2.1 holds.

On the other hand, the Hessian of the overall system is given by

$$\nabla^2 H_d = \begin{bmatrix} k_{d1}+1+k_{i2} & k_{d1}(1+k_{i1})+1 & -k_{d2}\cos x_3 \\ k_{d1}(1+k_{i1})+1 & k_{d1}(1+k_{i1})^2+1+k_{i1} & -k_{d2}(1+k_{i1})\cos x_3 \\ -k_{d2}\cos x_3 & -k_{d2}(1+k_{i1})\cos x_3 & k_{d3}\cos^2 x_3 - \cos x_3 \end{bmatrix}.$$

Therefore, evaluating the latter at x_* we have

$$(\nabla^2 H_d)_* = \begin{bmatrix} k_{d1}+1+k_{i2} & k_{d1}(1+k_{i1})+1 & -k_{d2} \\ k_{d1}(1+k_{i1})+1 & k_{d1}(1+k_{i1})^2+1+k_{i1} & -k_{d2}(1+k_{i1}) \\ -k_{d2} & -k_{d2}(1+k_{i1}) & k_{d3}-1 \end{bmatrix}.$$

Moreover, a Schur complement analysis shows that $(\nabla^2 H_d)_* > 0$ for $k_{d3} > 1$. Hence the stability of the point x_* follows from Propositions 5.2.1 and 5.2.2.

To prove the convergence of the trajectories to the desired equilibrium, note that

$$\dot{H}_d \equiv 0 \Longleftrightarrow \left\{ \begin{array}{rcl} y_a &=& 0 \\ \dot{x}_1 &=& 0 \\ \dot{x}_2 &=& 0. \end{array} \right.$$

Furthermore, the latter implies

$$\begin{aligned} x_3 &= 0 \\ \dot{x}_3 &= 0 \\ x_1 + x_2 + k_{i1}(x_2+\kappa_1) &= 0 \qquad (5.29) \\ x_1 + x_2 + a + k_{i2}(x_1+\kappa_2) &= 0. \qquad (5.30) \end{aligned}$$

Hence, combining (5.29) and (5.30) we obtain

$$\begin{aligned} k_{i1}(x_2+\kappa_1) &= a + k_{i2}(x_1+\kappa_2) \\ \Longleftrightarrow k_{i1}(x_2 - x_{2*}) &= k_{i2}(x_1 - x_{1*}) \end{aligned} \tag{5.31}$$

where we used (5.26). Moreover, replacing the latter in (5.29) we get

$$\begin{aligned} x_1 - x_{1*} + \tfrac{k_{i2}}{k_{i1}}(x_1 - x_{1*}) + k_{i2}(x_1 - x_{1*}) &= 0 \\ \Longleftrightarrow (k_{i1} + k_{i2} + k_{i1}k_{i2})(x_1 - x_{1*}) &= 0 \\ \Longleftrightarrow x_1 &= x_{1*}. \end{aligned}$$

Therefore, substituting the latter in (5.31), we have that $x_2 = x_{2*}$ and thus we conclude that $\dot{H}_d \equiv 0 \Leftrightarrow x = x_*$. This completes the proof.

□

Simulations

Figure 5.1 shows the simulation results of the closed-loop system with the constant parameter $a = 2$ and the desired equilibrium at $x_* = (2, 1, 0)$. The simulations are carried out for initial conditions $x_0 = (0, -1, \frac{\pi}{3})$ and selecting the tuning gains as

$$K_I = \mathrm{diag}\{3, 2\}, \quad K_P = \mathrm{diag}\{5, 6\}, \quad K_D = \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}.$$

From the figure it is clear that the states converge to the desired equilibrium. This corroborates the result of Proposition 5.3.3.

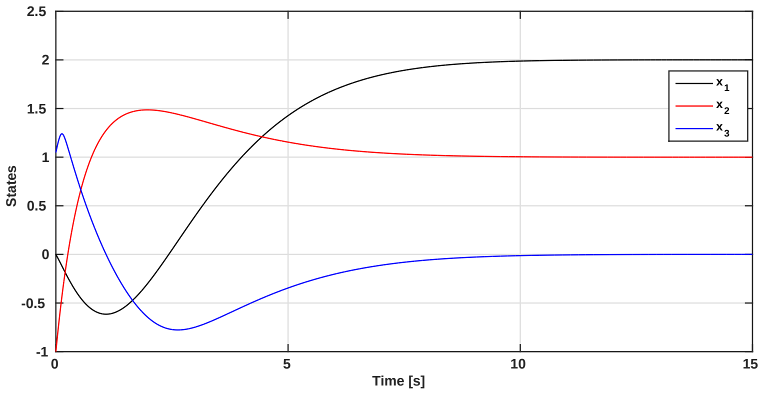

Figure 5.1: Simulation results of system (5.20) in closed-loop with (5.12).

Chapter 6

Flexible inverted pendulum

The problem of stabilization of under-actuated mechanical systems, both in the domain of ordinary and partial differential equations, has been widely addressed by several control researchers in recent years. In the domain of flexible mechanisms and robots, flexibility in the links is the main source of under actuation. If the deformations due to flexibility are small it is possible to use an *unconstrained* Lagrange formulation and invoke the Assumed Modes Method (AMM) [29] to obtain a simple, finite-dimensional model—see [15] for a recent literature review. This modeling procedure, however, is inapplicable for systems with large deformations, for which a *constrained* EL formulation is required. This approach has been adopted in [45] to derive an accurate model for a single ultra-flexible link fixed to a cart. Potential energy change owing to ultra-large deformations in the presence of gravity is considered in [45] using the constant length of the beam as a holonomic constraint. For a survey on recent control techniques for this class of systems see [45, 56, 2].

The objective of this Chapter is to design an energy shaping controller with *guaranteed stability properties* for the model of a single ultra-flexible link fixed to a cart reported in [45]. As is well known [34] the application of energy shaping controllers is stymied by the need to solve PDEs that identify the mechanical structure (Lagrangian or Hamiltonian) that is assigned to the closed-loop. To propose a truly constructive energy shaping scheme, that does not require the solution of PDEs, it was recently proposed in [13] to relax the constraint of preservation in closed-loop of the EL structure. The design in [13] proceeds in two steps, first, we apply a partial feedback linearization (PFL) [54] that transforms the system into Spong's normal form—if this system is still EL, two new passive outputs are immediately identified. Second, a classical PID around a suitable combination of these passive outputs completes the design.

In this chapter it is shown that this technique, developed for standard EL systems in [13], is also applicable to the constrained EL system at hand. This extension is far from obvious, because the (lower order) dynamics that results from the projection of the system on the manifold defined by the constraint *is not* an EL system. In spite of this fact it is shown that, because of the workless nature of the forces introduced by the constraints, it is still possible to identify the two new passive outputs to which the PID is applied.

6.1 System dynamics and problem formulation

In [45] a dynamic model that accurately describes the behavior of the single ultra-flexible link fixed to a cart depicted in Fig. 6.1 is reported. The main feature of this model, which distinguishes it from other models, is that to take into account large deformations of the link its length is assumed *constant*—giving rise to a holonomic constraint. The model is rigorously developed using a constrained EL formalism, combined with a standard application of the AMM, and its validity is experimentally corroborated. In this section we present this model, first, in its constrained EL form and then in a reduced form—obtained via the elimination of the constrained equations.

6.1.1 Constrained EL model

The model reported in [45] admits a constrained EL representation of the form

$$\begin{aligned} D(q)\ddot{q} + C(q,\dot{q})\dot{q} + B(q) + R\dot{q} &= e_3\tau + \lambda A(q) \\ \Gamma(q) &= 0, \end{aligned} \tag{6.1}$$

where $q = \mathrm{col}(\theta, x_e, z) \in \mathbb{D} \times \mathbb{R}_{\geq 0} \times \mathbb{R}$ are the generalized coordinates, $R \geq 0$ is a matrix of damping coefficients. $D > 0$ is the inertia matrix, $C\dot{q}$ are the Coriolis and centrifugal forces, B is a conservative force vector due to potential energy, τ is the control vector, λA is a vector of virtual forces due to the holonomic constraint, with λ the Lagrange multiplier, and Γ is the (constant length) constraint function given by

$$\Gamma(q) := \int_0^{x_e} \sqrt{1 + [\theta\phi'(x)]^2}\mathrm{d}x - L, \tag{6.2}$$

with $L > 0$ the length of the link and ϕ the mode shape function of the AMM [29] reported in [27], that is,

$$\phi(x) = \cosh\left(\frac{\eta x}{L}\right) - \cos\left(\frac{\eta x}{L}\right) + \gamma\left[\sin\left(\frac{\eta x}{L}\right) - \sinh\left(\frac{\eta x}{L}\right)\right],$$

where η and γ are constants defined in Table 6.1. The analysis made in [45] considers only one mode where the deflection $\alpha(\theta, x)$ is given by

$$\alpha(x, \theta) = \phi(x)\theta.$$

The different terms entering into the System (6.1) are defined as

$$\begin{aligned} D(q) &:= \begin{bmatrix} D_1(x_e) & 0 & D_2(x_e) \\ 0 & D_3 & 0 \\ D_2(x_e) & 0 & D_4 \end{bmatrix}, \\ A(q) &:= \nabla\Gamma(q) = \begin{bmatrix} A_1(\theta, x_e) \\ A_2(\theta, x_e) \\ 0 \end{bmatrix}, \\ R &:= \mathrm{diag}\{R_1, 0, R_3\}, \\ C(q,\dot{q}) &:= \begin{bmatrix} \frac{1}{2}C_1(x_e)\dot{x}_e & \delta(x_e, \dot{\theta}, \dot{z}) & \frac{1}{2}C_2(x_e)\dot{x}_e \\ -\delta(x_e, \dot{\theta}, \dot{z}) & 0 & -\frac{1}{2}C_2(x_e)\dot{\theta} \\ \frac{1}{2}C_2(x_e)\dot{x}_e & \frac{1}{2}C_2(x_e)\dot{\theta} & 0 \end{bmatrix}, \end{aligned}$$

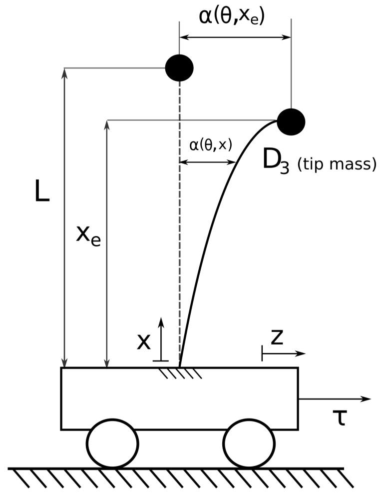

Figure 6.1: Single ultra-flexible link with base excitation

with

$$\delta(x_e, \dot{\theta}, \dot{z}) := \frac{1}{2} C_1(x_e)\dot{\theta} + \frac{1}{2} C_2(x_e)\dot{z},$$

and

$$B(q) := \nabla V(q) = \begin{bmatrix} B_1(\theta, x_e) \\ B_2(\theta, x_e) \\ 0 \end{bmatrix} \tag{6.3}$$

where V is the potential energy of the system given by

$$V(q) = \frac{1}{2} EI \int_0^{x_e} \frac{[\theta\phi''(x)]^2}{\left\{1 + [\theta\phi'(x)]^2\right\}^3} \mathrm{d}x - D_3 g(L - x_e),$$

E, I, D_3, R_1, R_3 are constant parameters and the remaining functions are defined as follows

$$\begin{aligned}
A_1(\theta, x_e) &:= \int_0^{x_e} \frac{\theta[\phi'(x)]^2}{\sqrt{1 + [\theta\phi'(x)]^2}} \mathrm{d}x, \\
A_2(\theta, x_e) &:= \sqrt{1 + [\theta\phi'(x_e)]^2}. \\
B_1(\theta, x_e) &:= EI \int_0^{x_e} \frac{\theta[\phi''(x)]^2 \left\{1 - 2[\theta\phi'(x)]^2\right\}}{\left\{1 + [\theta\phi'(x)]^2\right\}^4} \mathrm{d}x, \\
B_2(\theta, x_e) &:= \frac{1}{2} \frac{EI[\theta\phi''(x_e)]^2}{\left\{1 + [\theta\phi'(x_2)]^2\right\}^3} + D_3 g. \\
C_1(x_e) &:= 2D_3\phi(x_e)\phi'(x_e), \quad C_2(x_e) := D_3\phi'(x_e). \\
D_1(x_e) &:= \rho\mathcal{A}_0 \int_0^L [\phi(x)]^2 \mathrm{d}x + D_3[\phi(x_e)]^2,
\end{aligned}$$

$$\begin{aligned} D_2(x_e) &:= D_3\phi(x_e) + \rho\mathcal{A}_0 \int_0^L \phi(x)\mathrm{d}x, \\ D_4 &:= D_3 + M_c + \rho\mathcal{A}_0 L. \end{aligned}$$

Problem formulation: Given the system (6.1) find a control input τ that places the beam at its vertical position with the cart stopped at the zero position, *i.e.*, that renders the point $q_* := (0, L, 0)$ a (locally) asymptotically stable equilibrium.

Remark 6.1.1 In [45] the model (6.1) is obtained applying EL equations to the constrained Lagrangian

$$\mathcal{L}(q, \dot{q}, \lambda) = T(\dot{q}, q) - V(q) + \lambda\Gamma(q)$$

where λ is a Lagrange multiplier and T is the kinetic energy of the system given by

$$T(\dot{q}, q) = \frac{1}{2}\dot{q}^\top D(q)\dot{q}.$$

Remark 6.1.2 It should be noted that the well-known [40] skew-symmetry property

$$\dot{D}(q) = C(q, \dot{q}) + C^\top(q, \dot{q}), \tag{6.4}$$

is satisfied. Nevertheless, this important property is of no use for controller design in the present context.

Remark 6.1.3 In [45] the analysis of the open-loop equilibria of (6.1) is carried out. In particular, it is proven that the open-loop equilibrium set is given by

$$\mathcal{E} := \{(\theta, x_e, z) \in \mathbb{D} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \mid A_1 B_2 - A_2 B_1 = 0\}, \tag{6.5}$$

where the arguments θ, x_e are omitted in A_i, B_i, with $i = 1, 2$. Furthermore, and not surprisingly, it is shown that the desired equilibrium $q_* \in \mathcal{E}$ and is unstable.

6.1.2 Reduced purely differential model

In this subsection we apply the standard constraint differentiation procedure [20] to transform the algebro-differential equations (6.1) to a purely differential form of reduced order.

Proposition 6.1.1 The system dynamics (6.1) is equivalent to

$$\begin{aligned} D_\theta(\theta)\ddot{\theta} + D_z(\theta)\ddot{z} + C_\theta(\theta)\dot{\theta}^2 + R_1\dot{\theta} + B_\theta(\theta) &= 0 \\ D_z(\theta)\ddot{\theta} + D_4\ddot{z} + C_z(\theta)\dot{\theta}^2 + R_3\dot{z} &= \tau \end{aligned} \tag{6.6}$$

with the functions $D_\theta, C_\theta, B_\theta, D_z$ and C_z given in (6.9).

Proof: *Differentiating the constraint equation (6.2), we get*

$$\begin{aligned} A_1(\theta, x_e)\dot{\theta} + A_2(\theta, x_e)\dot{x}_e &= 0 \\ A_1(\theta, x_e)\ddot{\theta} + A_2(\theta, x_e)\ddot{x}_e + A_3(\theta, x_e)\dot{\theta}\dot{x}_e + A_4(\theta, x_e)\dot{x}_e^2 & \\ +A_5(\theta, x_e)\dot{\theta}^2 &= 0, \end{aligned} \tag{6.7}$$

where

$$
\begin{aligned}
A_3(\theta, x_e) &:= \frac{2\theta[\phi'(x_e)]^2}{\sqrt{1+[\theta\phi'(x_e)]^2}}, \\
A_4(\theta, x_e) &:= \frac{\theta^2\phi'(x_e)\phi''(x_e)}{\sqrt{1+[\theta\phi'(x_e)]^2}}, \\
A_5(\theta, x_e) &:= \int_0^{x_e} \frac{[\phi'(x)]^2}{\left\{1+[\theta\phi'(x)]^2\right\}^{\frac{3}{2}}} \mathrm{d}x.
\end{aligned}
$$

Now, the partial derivative with respect to x_e *of the constraint* (6.2)*, given by* $A_2(\theta, x_e)$*, is clearly bounded away from zero. Thus, invoking the Implicit Function Theorem [26] we can guarantee the existence of a function* $\hat{x}_e(\theta)$ *such that*

$$
\Gamma(\theta, \hat{x}_e(\theta)) = 0.
$$

Equivalently, it is possible to express x_e *in terms of* θ*, that is*

$$
x_e = \hat{x}_e(\theta). \tag{6.8}
$$

Replacing (6.7) *in* (6.1) *it is possible to eliminate the Lagrange multiplier* λ*—as done in [45]. Moreover, using* (6.8)*, we can eliminate the coordinate* x_e *to reduce the order of the system. After some lengthy, but straightforward calculations, this leads to the equations* (6.6) *with the definitions*

$$
\begin{aligned}
D_\theta(\theta) &:= D_1(\hat{x}_e(\theta)) + D_3\frac{A_1^2(\theta, \hat{x}_e(\theta))}{A_2^2(\theta, \hat{x}_e(\theta))} \\
C_\theta(\theta) &:= D_3\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2^2(\theta, \hat{x}_e(\theta))}\zeta - \frac{1}{2}C_1(\hat{x}_e(\theta))\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2(\theta, \hat{x}_e(\theta))} \\
B_\theta(\theta) &:= B_1(\theta, \hat{x}_e(\theta)) - B_2(\theta, \hat{x}_e(\theta))\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2(\theta, \hat{x}_e(\theta))} \\
D_z(\theta) &:= D_2(\hat{x}_e(\theta)) \\
C_z(\theta) &:= -C_2(\hat{x}_e(\theta))\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2(\theta, \hat{x}_e(\theta))},
\end{aligned} \tag{6.9}
$$

where

$$
\zeta = A_5(\theta, \hat{x}_e(\theta)) + A_4(\theta, \hat{x}_e(\theta))\frac{A_1^2(\theta, \hat{x}_e(\theta))}{A_2^2(\theta, \hat{x}_e(\theta))} - A_3(\theta, \hat{x}_e(\theta))\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2(\theta, \hat{x}_e(\theta))}.
$$

$\square$

Remark 6.1.4 The first equation in (6.7) can be rewritten as follows

$$
A^\top(q)\dot{q} = 0. \tag{6.10}
$$

Consequently, differentiating the total energy of (6.1)—given by $H(q, \dot{q}) := T(q, \dot{q}) + V(q)$—and using the skew-symmetry property (6.4) yields the usual power balance equation

$$
\dot{H} = -\dot{q}^\top R\dot{q} + \dot{q}_3\tau.
$$

This means that the virtual forces introduced in the equations due to constrained Lagrange formulation are workless, that is, they are not responsible for addition or removal of energy from the system. This key property is used later to identify the passive outputs used for the design of the energy shaping controller.

Remark 6.1.5 The admissible initial conditions for the reduced system (6.6) are restricted to the set

$$\{(\theta, z) \in \mathbb{D} \times \mathbb{R} \mid \Gamma(\theta, \hat{x}_e(\theta)) = 0\},$$

where, clearly, the system evolves.

6.2 Energy Shaping Control

As explained in the introduction the energy shaping control of [13] proceeds in three steps: a partial feedback linearization, identification of two passive outputs and the addition of a PID loop around a suitable combination of these outputs. These steps are applied to the system (6.6) in the following subsections.

6.2.1 Partial feedback linearization

The lemma below describes a first static state–feedback that performs the PFL of the system (6.6).

Lemma 6.2.1 Consider the system (6.6) in closed–loop with the control

$$\tau = R_3\dot{z} + \left(C_z - \frac{D_z}{D_\theta}C_\theta\right)\dot{\theta}^2 - \frac{D_z}{D_\theta}R_1\dot{\theta} - \frac{D_z}{D_\theta}B_\theta + \left(D_4 - \frac{D_z^2}{D_\theta}\right)u. \tag{6.11}$$

Then, the system can be written in Spong's normal form

$$\begin{aligned} D_\theta(\theta)\ddot{\theta} + C_\theta(\theta)\dot{\theta}^2 + R_1\dot{\theta} + B_\theta(\theta) &= G_\theta(\theta)u \\ \ddot{z} &= u, \end{aligned} \tag{6.12}$$

where

$$G_\theta(\theta) := -D_z(\theta).$$

Proof: *The proof proceeds rewriting the first equation of* (6.6) *as follows*

$$\ddot{\theta} = -\frac{1}{D_\theta}\left(D_z\ddot{z} + C_\theta\dot{\theta}^2 + R_1\dot{\theta} + B_\theta\right). \tag{6.13}$$

Now, replacing the latter expression in the second equation of (6.6)*, we get*

$$-\frac{D_z}{D_\theta}\left(D_z\ddot{z} + C_\theta\dot{\theta}^2 + R_1\dot{\theta} + B_\theta\right) + D_4\ddot{z} + C_z\dot{\theta}^2 + R_3\dot{z} = \tau.$$

Substituting the control law (6.11) *in the equation above we obtain the second equality of* (6.12)*. The first equation results, replacing $\ddot{z} = u$ in the first equation of* (6.6)*.* □

6.2.2 Identification of the passive outputs

In the following lemma the new cyclo-passive maps for the system (6.12) are identified.

Lemma 6.2.2 Consider the system (6.12). The signals

$$\begin{aligned} y_a &:= \dot{z} \\ y_u &:= G_\theta(\theta)\dot{\theta}, \end{aligned}$$

define cyclo–passive maps $u \mapsto y_a$ and $u \mapsto y_u$ with storage functions

$$\begin{aligned} H_a(z) &= \frac{1}{2}\dot{z}^2 && (6.14) \\ H_u(\theta) &= \frac{1}{2}D_\theta(\theta)\dot{\theta}^2 + V_\theta(\theta), && (6.15) \end{aligned}$$

respectively, where

$$V_\theta(\theta) := V(\theta, \hat{x}_e(\theta)).$$

More precisely, the time derivative of the functions H_a and H_u along the solutions of (6.12) satisfy the dissipation inequalities

$$\dot{H}_a \leq u y_a, \quad \dot{H}_u \leq u y_u. \tag{6.16}$$

Proof: *First, notice that*

$$\begin{aligned} \dot{V}_\theta &= \frac{\partial V}{\partial \theta}\dot{\theta} + \frac{\partial V}{\partial \hat{x}_e}\dot{\hat{x}}_e \\ &= B_1(\theta, x_e)\dot{\theta} + B_2(\theta, x_e)\dot{\hat{x}}_e \\ &= B_1(\theta, \hat{x}_e(\theta))\dot{\theta} + B_2(\theta, \hat{x}_e(\theta))\dot{x}_e \\ &= \left[B_1(\theta, \hat{x}_e(\theta)) - B_2(\theta, \hat{x}_e(\theta))\frac{A_1(\theta, \hat{x}_e(\theta))}{A_2(\theta, \hat{x}_e(\theta))}\right]\dot{\theta} \\ &= B_\theta(\theta)\dot{\theta}, \end{aligned} \tag{6.17}$$

where we have used (6.3) *for the second identity,* (6.8) *for the third one and the first equation in* (6.7) *for the fourth one.*

Now, we will prove that

$$\dot{D}_\theta = 2C_\theta\dot{\theta}. \tag{6.18}$$

Indeed, computing the time derivative of D_θ we get

$$\begin{aligned} \dot{D}_\theta &= \dot{D}_1 + 2D_3\frac{A_1}{A_2^2}\dot{A}_1 - 2D_3\frac{A_1^2}{A_2^3}\dot{A}_2 \\ &= \frac{\partial D_1}{\partial \hat{x}_e}\dot{\hat{x}}_e + 2D_3\frac{A_1}{A_2^2}\left(\frac{\partial A_1}{\partial \theta}\dot{\theta} + \frac{\partial A_1}{\partial \hat{x}_e}\dot{\hat{x}}_e\right) - 2D_3\frac{A_1^2}{A_2^3}\left(\frac{\partial A_2}{\partial \theta}\dot{\theta} + \frac{\partial A_2}{\partial \hat{x}_e}\dot{\hat{x}}_e\right) \\ &= C_1\dot{x}_e + 2D_3\frac{A_1}{A_2^2}\left[A_5\dot{\theta} + \frac{1}{2}A_3\dot{x}_e - \frac{A_1}{A_2}\left(\frac{1}{2}A_3\dot{\theta} + A_4\dot{x}_e\right)\right] \\ &= \left[-C_1\frac{A_1}{A_2} + 2D_3\frac{A_1}{A_2^2}\zeta\right]\dot{\theta} \\ &= 2C_\theta\dot{\theta}, \end{aligned}$$

where we used the first equation of (6.7) *to eliminate* $\dot{x}_e$ *in the fourth identity, the arguments* $\theta, \hat{x}_e(\theta)$ *are omitted.*

The time derivative of (6.15) *along the system trajectories is*

$$\begin{aligned} \dot{H}_u &= D_\theta \dot{\theta}\ddot{\theta} + \frac{1}{2}\dot{D}_\theta \dot{\theta}^2 + \dot{V}_\theta \\ &= \left(G_\theta u - C_\theta \dot{\theta}^2 - R_1 \dot{\theta} + \frac{1}{2}\dot{D}_\theta \dot{\theta} \right) \dot{\theta} \\ &= -R_1 \dot{\theta}^2 + G_\theta \dot{\theta} u \le u y_u. \end{aligned}$$

where (6.13) *and* (6.17) *were used in the second equality while the third one was obtained invoking* (6.18).

On the other hand, the time derivative of (6.14) *along the system trajectories is*

$$\dot{H}_a = \dot{z}\ddot{z} = u y_a.$$

This completes the proof. □

6.2.3 PID controller

Similarly to [13] the controller design is completed adding a PID around a suitably weighted sum of the two cyclo-passive outputs (y_a and y_u) identified in Lemma 6.2.2. More precisely, the controller implements the relationship

$$k_e u = -\left(K_P \tilde{y} + K_I \int_0^t \tilde{y}(s) ds + K_D \dot{\tilde{y}} \right), \tag{6.19}$$

where

$$\tilde{y} := k_a y_a + k_u y_u \tag{6.20}$$

with $k_e, k_a, k_u \in \mathbb{R}$ and $K_P, K_I, K_D \in \mathbb{R}_{\ge 0}$ the PID gains. As explained in [13], and illustrated below, these gains are selected to shape the energy function.

To implement the controller (6.19) *without differentiation* the term $\dot{\tilde{y}}$ is replaced by its evaluation along the system dynamics (6.12). Since the system is relative degree one this brings along some terms depending on u that are moved to the left hand side of (6.19). Some lengthy, but straightforward, calculations show that (6.19) is equivalent to

$$K(\theta) u = -\left(K_P \tilde{y} + K_I \int_0^t \tilde{y}(s) ds \right) - K_D k_u S(\theta, \dot{\theta}),$$

where we defined the functions

$$\begin{aligned} S(\theta, \dot{\theta}) &:= \dot{G}_\theta \dot{\theta} - \frac{G_\theta}{D_\theta} \left[C_\theta \dot{\theta}^2 + R_1 \dot{\theta} + B_\theta \right] \\ K(\theta) &:= k_e + K_D \left[k_a + k_u \frac{G_\theta^2(\theta)}{D_\theta(\theta)} \right]. \end{aligned}$$

Clearly, a sufficient condition for the controller to be implementable is that the function K is bounded away from zero, that is,

$$|K| \ge \delta > 0. \tag{6.21}$$

To analyze the stability of the system (6.12) in closed–loop with the PID (6.19), (6.20) we propose the function

$$W(t,\tilde{y},\theta,\dot{\theta},\dot{z}) := k_e[k_a H_a(\dot{z}) + k_u H_u(\theta,\dot{\theta})] + \frac{K_I}{2}\left(\int_0^t \tilde{y}(s)ds\right)^2 + \frac{K_D}{2}\tilde{y}^2,$$

and make the reasonable assumption that the friction forces acting on the beam are negligible, hence set $R_1 = 0$. The derivative of W yields

$$\begin{aligned}\dot{W} &= k_e(k_a\dot{H}_a + k_u\dot{H}_u) + K_I\tilde{y}\int_0^t \tilde{y}(s)ds + \frac{K_D}{2}\tilde{y}\dot{\tilde{y}} \\ &= k_e\tilde{y}u + K_I\tilde{y}\int_0^t \tilde{y}(s)ds + \frac{K_D}{2}\tilde{y}\dot{\tilde{y}} \\ &= -K_P\tilde{y}^2,\end{aligned}$$

where we used the dissipation inequalities (6.16)—that under the assumption $R_1 = 0$ become equalities—to get the second identity and replaced (6.19) to find the last one.

The final step in our stability analysis is to show that the function W can be expressed as a positive definite (with respect to the desired equilibrium) function of the state $(\theta, z, \dot{\theta}, \dot{z})$ of the system (6.12). Notice that for this reduced system the desired equilibrium is simply the *origin*.

To express W as a function of the state we only need to deal with the integral term. For, we define the function

$$V_N(\theta) := -D_3\int_0^\theta \phi(\hat{x}_e(s))\mathrm{d}s - \left(\rho\mathcal{A}_0\int_0^L \phi(x)\mathrm{d}x\right)\theta,$$

whose time derivative is given by

$$\begin{aligned}\dot{V}_N &= -D_3\phi(\hat{x}_e(\theta))\dot{\theta} - \left(\rho\mathcal{A}_0\int_0^L \phi(x)\mathrm{d}x\right)\dot{\theta} \\ &= -D_z\dot{\theta} \\ &= G_\theta\dot{\theta} \\ &= y_u. \end{aligned} \tag{6.22}$$

Consequently

$$\int_0^t \tilde{y}(s)ds = k_a z(t) + k_u V_N(\theta(t)) + c,$$

where $c \in \mathbb{R}$ is an integration constant. Using the latter and the definitions of H_a, H_u and $\tilde{y}$ we can prove that, up to an additive constant,

$$W(t,\tilde{y},\theta,\dot{\theta},\dot{z}) = \frac{1}{2}\begin{bmatrix}\dot{\theta}\\ \dot{z}\end{bmatrix}^\top D_d(\theta)\begin{bmatrix}\dot{\theta}\\ \dot{z}\end{bmatrix} + V_d(\theta,z) =: H_d(\theta,z,\dot{\theta},\dot{z}) \tag{6.23}$$

where we defined

$$D_d(\theta) := \begin{bmatrix}k_e k_u D_\theta(\theta) + k_u^2 K_D G_\theta^2(\theta) & k_a k_u K_D G_\theta(\theta) \\ k_a k_u K_D G_\theta(\theta) & k_e k_a + k_a^2 K_D\end{bmatrix} \tag{6.24}$$

$$V_d(\theta,z) := k_e k_u V_\theta(\theta) + \frac{1}{2}K_I\left[k_a z + k_u V_N(\theta)\right]^2. \tag{6.25}$$

Remark 6.2.1 Without the assumption that $R_1 = 0$ a term $-k_e k_u R_1 \dot{\theta}^2$ appears in $\dot{W}$. As will be shown below, see also [13] and Remark 6.2.2, to make the upward position a minimum of the total energy function H_d it is necessary to flip the potential energy of the pendulum, which is done selecting $k_e k_u < 0$, making positive the dissipation term. The deleterious effect of dissipation in energy shaping methods is well known and has been reported in various references [64, 18].

6.2.4 Main stability result

The proposition below, which essentially gives conditions on the controller gains to ensure H_d is positive definite, is the main result of this section.

Proposition 6.2.1 Consider the system (6.1) in closed–loop with the controller (6.11) where the outer–loop control u is given by the PID (6.19) with

$$\tilde{y} = k_a \dot{z} + k_u G_\theta(\theta)\dot{\theta}. \tag{6.26}$$

Set the constant gains k_e, k_a and the PID gains K_P, K_I and K_D to *arbitrary positive* numbers while k_u is negative and, for some small $\epsilon > 0$, satisfies

$$k_u \leq -\kappa \left(k_a + \frac{k_e}{K_D} \right) - \epsilon, \tag{6.27}$$

where κ is a positive constant verifying

$$\kappa \geq \frac{D_\theta(\theta)}{G_\theta^2(\theta)}. \tag{6.28}$$

(i) The origin of the reduced dynamics (6.6), which corresponds to the desired equilibrium $q_* = (0, L, 0)$ of (6.1), is *stable* with Lyapunov function H_d given in (6.23).

(ii) It is *asymptotically* stable if the signal $\tilde{y}$ defined in (6.26) is detectable with respect to (6.12).

Proof: *In Lemma 6.2.1 it has been shown that the dynamics of the system* (6.1) *in closed-loop with the controller* (6.11) *is described by* (6.12)*. Therefore, given the derivations above, it only remains to prove that the non–increasing function H_d, defined in* (6.23)*, is positive definite. This will be established proving that, under the conditions of the proposition $D_d > 0$ and V_d has an isolated minimum at the origin.*

To prove the first claim notice from (6.24) *that the* $(2,2)$ *term of D_d is positive. Hence it only remains to show that its determinant is also positive. Now,*

$$\begin{aligned} \det\{D_d\} &= k_e k_u D_\theta \left(k_e k_a + k_a^2 K_D \right) + k_e k_u^2 k_a K_D G_\theta^2 \\ &= k_e k_u \left[D_\theta \left(k_e k_a + k_a^2 K_D \right) + k_u k_a K_D G_\theta^2 \right] \\ &= k_e k_u k_a K_D G_\theta^2 \left[\frac{D_\theta}{G_\theta^2} \left(k_a + \frac{k_e}{K_D} \right) + k_u \right]. \end{aligned}$$

Since $k_e k_u < 0$*, the term outside the brackets is negative. Furthermore, if* (6.27) *and* (6.28) *are satisfied, the term inside the brackets is also negative, yielding* $\det\{D_d\} > 0$.

We proceed now to prove that the condition (6.21)*, which ensures realizability of the control* (6.19)*, is satisfied. This is established noticing that*

$$\frac{D_\theta(\theta)}{G_\theta^2(\theta)} K_D K(\theta) = \frac{D_\theta(\theta)}{G_\theta^2(\theta)} \left(k_a + \frac{k_e}{K_D} \right) + k_u.$$

Hence, invoking (6.27)*, we have that* (6.21) *holds.*

To establish the proof of the second claim we compute the gradient of V_d *as*

$$\begin{aligned} \nabla V_d &= \begin{bmatrix} k_e k_u \nabla V_\theta + K_I k_u \nabla V_N \left(k_a z + k_u V_N \right) \\ K_I k_a \left(k_a z + k_u V_N \right) \end{bmatrix} \\ &= \begin{bmatrix} k_e k_u B_\theta - K_I k_u D_z \left(k_a z + k_u V_N \right) \\ K_I k_a \left(k_a z + k_u V_N \right) \end{bmatrix}. \end{aligned}$$

Using the fact that $B_\theta(0) = 0$ *and* $V_N(0) = 0$ *we conclude that* $\nabla V_d(0) = col(0,0)$*. Now, the Hessian of* V_d *is given by*

$$\begin{aligned} \nabla^2 V_d &= \begin{bmatrix} k_e k_u \nabla^2 V_\theta + K_I k_u \nabla^2 (k_a z + k_u V_N) & -K_I k_u k_a D_z \\ -K_I k_u k_a D_z & K_I k_a^2 \end{bmatrix} \\ &= \begin{bmatrix} \nu(\theta, z) & -K_I k_u k_a D_z \\ -K_I k_u k_a D_z & K_I k_a^2 \end{bmatrix}, \end{aligned}$$

where we defined the function

$$\nu(\theta, z) := k_e k_u \nabla^2 V_\theta + K_I k_u^2 D_z^2 - K_I k_u \nabla D_z \left(k_a z + k_u V_N \right).$$

Evaluating it at the origin yields

$$\nabla^2 V_d(0) = \begin{bmatrix} \nu(0) & -K_I k_u k_a D_z(0) \\ -K_I k_u k_a D_z(0) & K_I k_a^2 \end{bmatrix}, \tag{6.29}$$

where

$$\nu(0) = k_e k_u \nabla^2 V_\theta(0) + K_I k_u^2 D_z^2(0).$$

Now,

$$\nabla^2 V_\theta(0) = EI \int_0^{\hat{x}_e(0)} [\phi''(x)]^2 \mathrm{d}x - D_3 g \int_0^{\hat{x}_e(0)} [\phi'(x)]^2 \mathrm{d}x,$$

which can be shown to be negative [45]. Since $k_e k_u < 0$ *the* $(1,1)$ *term of* $\nabla^2 V_d(0)$ *is positive. Moreover,*

$$\det\{\nabla^2 V_d(0)\} = k_e k_u \nabla^2 V_\theta(0) K_I k_a^2,$$

which is also positive, ensuring$\nabla^2 V_d(0) > 0$.

The previous analysis ensures that the origin is an isolated minimum for the function V_d *as claimed above. The proof is completed invoking classical Lyapunov theory [26].* □

Remark 6.2.2 Notice that the condition $\nabla^2 V_\theta(0) < 0$ is consistent with the well known fact that the upward pendulum position is unstable in open–loop. Similarly to the rigid case [13] the maximum of the open–loop potential energy is transformed into a minimum in closed–loop multiplying V_θ by the negative number $k_e k_u$—see (6.25).

Remark 6.2.3 The critical condition (6.27) is satisfied in a neighborhood of the origin replacing C by

$$\frac{D_3\phi^2(L) + \rho\mathcal{A}_0 \int_0^L \phi^2(x)dx}{\left[D_3\phi(L) + \rho\mathcal{A}_0 \int_0^L \phi(x)dx\right]^2} = \frac{D_\theta(0)}{G_\theta^2(0)}.$$

Remark 6.2.4 The term $k_a z + k_u V_N(\theta)$ in (6.25) is a new potential energy corresponding to a virtual spring attached to the cart—thereby enabling to stabilize the cart position.

Remark 6.2.5 The choice of the free gains of Proposition 6.2.1 is given just as an illustration. From the proof it is clear that, depending on the particular problem, other (possibly less conservative) choices are available.

6.3 Simulation Results

Table 6.1: System parameters

Parameter	**Symbol**	**Value**	**Units**
Pendulum cross section area	$\mathcal{A}_o$	8×10^{-6}	m^2
Young's modulus	E	9×10^{10}	$\frac{N}{m^2}$
Gravitational acceleration	g	9.81	$\frac{m}{seg^2}$
Moment of inertia	I	1.066×10^{-13}	$kg \cdot m^2$
Pendulum length	L	0.305	m
Tip mass	M	2.75×10^{-2}	kg
Cart mass	M_c	0.1	kg
Function of the system natural frequency	η	1.1741	–
Dimensionless constant	γ	0.9049	–
Pendulum density	ρ	8400	$\frac{kg}{m^3}$
Viscous friction at the pendulum base	R_1	9.86×10^{-4}	$\frac{kg}{seg}$
Viscous friction between the rail and the cart	R_3	7.69	$\frac{kg}{seg}$

In this section we assess the performance of the proposed controller via Matlab® simulations choosing different sets of gains and different initial conditions. We simulated the system (6.12) in closed–loop with the PID controller (6.19), (6.26) with the parameters given in Table 6.1.

We have chosen three different initial conditions, given in Table 6.2, corresponding to radically different scenarios of the system. Namely, an arbitrary point (ICs1), one of the stable open-loop equilibria (ICs2) and an initial condition with the cart far from the origin and the tip mass located at the unstable open–loop equilibrium (ICs3).

For the selection of suitable gains for the controller, we fixed the gain $k_e = 1$ and linearized the closed–loop system. We based our criterion to choose the gains, always satisfying (6.21) and (6.27), and observing the eigenvalues of the closed–loop matrix of the linearized system around the desired equilibrium point. Particular attention has been paid to the eigenvalue closest to the imaginary axis, which is directly related to the rate of convergence of the cart position. Three sets of gains were selected and they are given in Table 6.3. For the Set 1 the real part of the slowest pole was -0.58, -0.75 for the Set 2 and -1.33 for Set 3.

Table 6.2: Initial conditions

	θ $[m]$	z $[m]$	$\dot{\theta}$ $[m/s]$	$\dot{z}$ $[m/s]$
ICs 1	-0.08	-0.1	0	0
ICs 2	0.134	0	0	0
ICs 3	0	-0.15	0	0

Table 6.3: Gains sets

	k_e	k_a	k_u	K_D	K_P	K_I
Set 1	1	0.5	-50.77	1.47	1.94	0.35
Set 2	1	1	-61.37	1.28	1.92	0.52
Set 3	1	1	-43.04	2.18	3.66	1.35

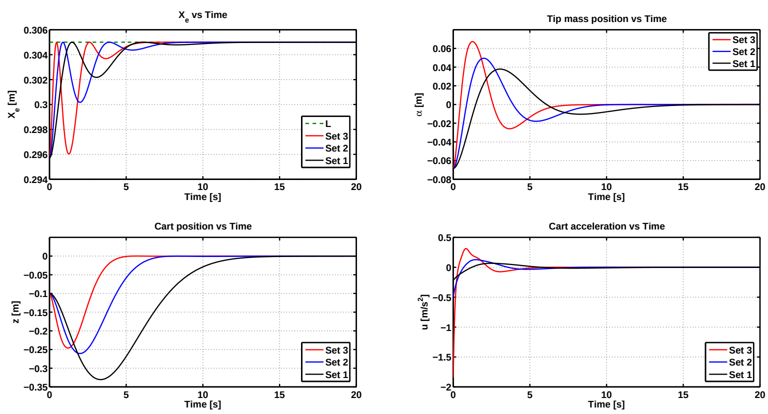

Figure 6.2: Simulation results for ICs 1

Simulation results of the energy shaping control are shown in Figures 6.2–6.4, where the variation of the cart position and control input acceleration is

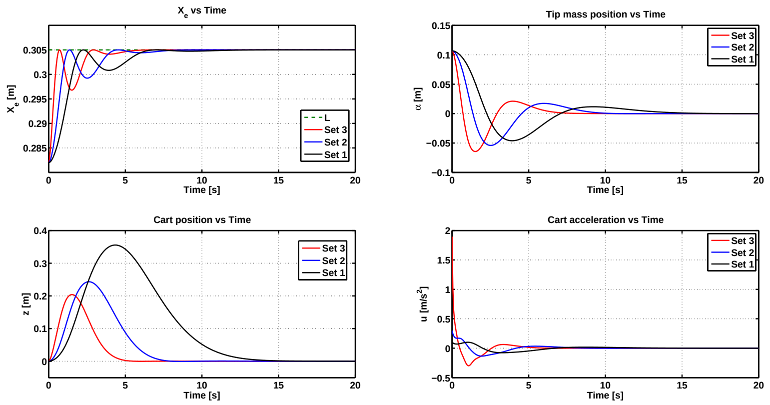

Figure 6.3: Simulation results for ICs 2

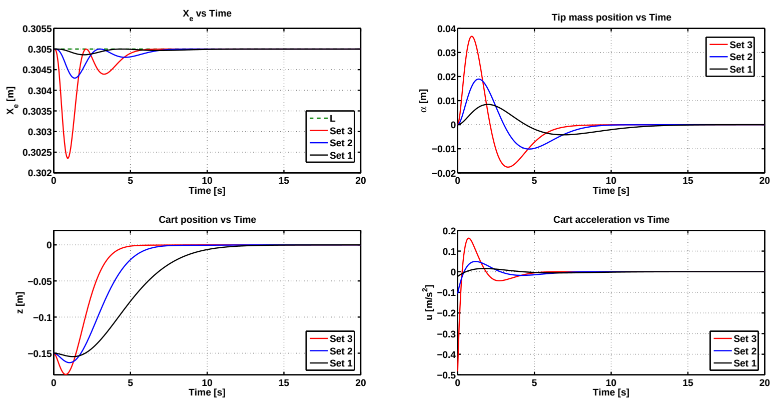

Figure 6.4: Simulation results for ICs 3

observed to be within practical limits, hence the control objective of simultaneous stabilization of cart position while suppressing the cantilever vibrations is achieved.

To evaluate the effect of the gains on the estimate of the domain of attraction of the closed-loop systems provided by the Lyapunov function H_d we show in Figure 6.5 the level curves of the desired potential energy V_d for each set of gains. As expected, there is a trade off between convergence rate and the size of the domain of attraction—as the slowest closed–loop pole of the linearized system moves farther to the left the closed sub level sets shrink.

6.4 Experimental Results

Experiments were also carried out to assess the performance of the proposed controller. The physical description of the experimental setup is provided in the Appendix B. The partial feedback linearization was replaced by the stan-

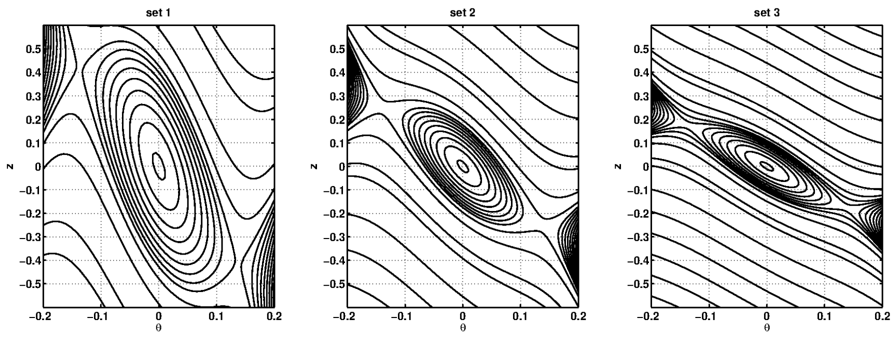

Figure 6.5: Level curves of the desired potential energy $V_d(\theta, z)$ for the different sets of gains

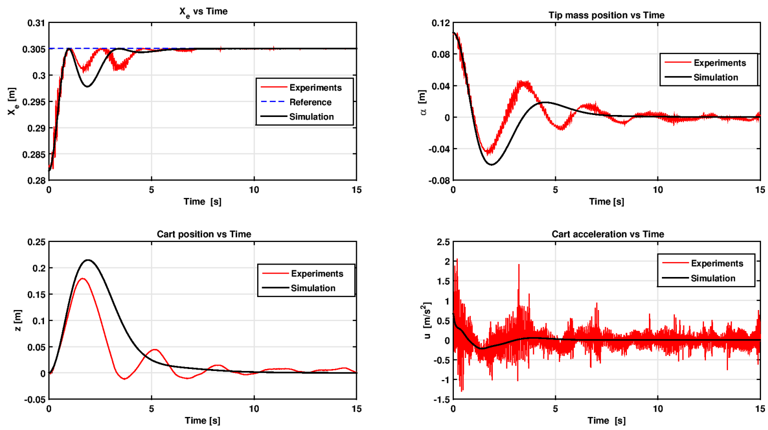

Figure 6.6: Comparison of simulation and experimental results for ICs 2

dard procedure of obtaining the desired trajectories for z via the integration of the cart acceleration, which is numerically reconstructed. It was observed that the gains used in the simulation do stabilize the physical pendulum but with a reduced domain of attraction, that is, placing the pendulum closer to the upward position. It is not surprising that the domain of attraction predicted by the model is reduced in the practical application. To show a comparison of the simulation and the experiment starting from the same initial conditions and using the same controller it was decided to select another set of gains. Figure 6.6 presents the comparison of simulation and experiment for the set of initial conditions ICs 2 and the set of gains: $k_e = 1$, $k_a = 1$, $k_u = -47.5$, $K_D = 1.9$, $K_P = 3$ and $K_I = 0.9$. The results demonstrate that the control task is achieved in a similar time although the trajectory in the experiments shows more oscillations with high frequency components of the vibrations of the beam. These oscillations are not captured by the simulation model that, as explained in Section 6.1.1, retains only the first deflection mode. However, as shown in the plots, these high frequency vibrations degrade the transient performance but do not induce instability.

A video of the simulations and experiments can be watched at

`https://youtu.be/aG53XaQPP3c`.

Chapter 7

Conclusions & Future work

In this chapter we summarize the main results presented in the thesis. Additionally, we propose some points to be addressed as future work in the same line of research of the problems studied in previous chapters.

7.1 Concluding remarks

Below, we present some concluding remarks of this thesis.

- All the passive outputs reported in the literature can be represented by the parameterization given in (2.8).
- EB-PBC can be understood as CbI with regulated sources as is exposed in Chapter 3.
- The PDEs to be solved in CbI using $y_{\mathtt{PS}}$ and $y_{\mathtt{wD}}$ are the same when F is full rank.
- The solutions of the PDEs to be solved in CbI are also solutions of the PDEs to be solved in EB-PBC.
- If the passive output $y_{\mathtt{PS}}$ is integrable, then it is possible to design a PI-PBC based on $y_{\mathtt{PS}}$ such that:
  - The energy shaping is carried without the necessity of solving PDEs.
  - The PI-PBC assigns the desired equilibrium to the overall system with an appropriate selection of κ.
  - The closed-loop system has a stable equilibrium point at the desired equilibrium if the conditions given in Proposition 4.3.1 hold.
- The integral action of the PI-PBC preserves the Hamiltonian structure of the closed-loop system.
- The PI-PBC is a particular case of IDA-PBC.
- EB-PBC using $y_{\mathtt{PS}}$ is a particular case of the PI-PBC.
- An input change of coordinates enlarges the class of systems that are stabilizable with the PI-PBC. This is shown in Chapter 4 Subsection 4.5.1.

- Another extension of the PI-PBC, proposed in Chapter 4 Section 4.2, is to use the first integrals of the more general passive output $y_{\mathtt{wD}}$.
- In LTI systems, controllability is not a sufficient condition to might be stabilized by the PI-PBC.
- A PID-PBC based on the natural output can be constructed if Assumptions 5.1.1 and 5.1.2 are satisfied.
- The PID-PBC based on the natural output is hampered by the dissipation obstacle.
- All the controllable LTI system of dimension 2 can be stabilized by the PID-PBC.
- An alternative PID-PBC based on two different passive outputs can be designed as is exposed in Proposition 5.2.1. In this case, the equilibrium assignment is not stymied by the dissipation obstacle.
- In general, the derivative term in both PID-PBC designs destroys the Hamiltonian structure for the closed-loop system.
- The applicability of the PID-PBC proposed in [13] can be extended to EL system with constraints as is done in Chapter 6. Where the control objective is achieved using a simplified model of the ultra flexible inverted pendulum.

7.2 Future work

The results reported in the previous chapters have motivated the following future work.

- To study the CbI approach proposing as controller an IOHD system. Then, compare the results with the controllers reported in Chapter 3.
- To establish a comparison between IDA-PBC and the PID-PBCs reported in Chapter 5.
- To design a PID-PBC similar to the one constructed in Proposition 5.2.1, using $y_{\mathtt{wD}}$ instead of $y_{\mathtt{PS}}$.
- To look for a more general result for LTI systems in closed-loop with the PID-PBC presented in Chapter 5.
- To generalize the result for constrained EL systems in closed-loop with the PID-PBC presented in Chapter 6.
- To look for alternatives in the design of the PID-PBC of Chapter 6 such that the PFL is not needed.

Appendices

Appendix A

Lemmata

Lemma A.1 (Poincare's Lemma) *Given* $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathcal{C}^1$. *There exists* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *such that* $\nabla\varphi = f$ *if and only if* $\nabla f = (\nabla f)^\top$.

Lemma A.2 *The equation*

$$F^\top Z F = -F \tag{A.1}$$

with unknown $Z : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, *is consistent (i.e., at least one such* Z *exists) if and only if* (2.14) *is satisfied.*

Proof: Equation (A.1) is a particular case of the linear matrix equation

$$AXB = C, \tag{A.2}$$

where X is unknown. According to Theorem 2.3.2 of [48] the equation above is consistent if and only if

$$AA^\dagger C B^\dagger B = C. \tag{A.3}$$

By matching the terms in (A.1) and (A.2) we get

$$A = F^\top, \quad X = Z, \quad B = F, \quad \text{and} \quad C = -F.$$

Replacing these in (A.3) we obtain

$$-F^\top (F^\dagger)^\top F F^\dagger F = -F \Longleftrightarrow F^\top (F^\dagger)^\top F = F,$$

where we used the definition of generalized inverse.

□

Lemma A.3 *Equations* (2.15) *and* (A.1) *imply that*

$$F^\top Z = -g. \tag{A.4}$$

Proof: Equation (2.15) implies the existence of a mapping $\beta : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that

$$g = F\beta.$$

On the other hand, equation (A.1) implies that

$$F^\top Z F = -F\beta$$

for any β. Combining the last two equations yields (A.4).

□

Lemma A.4 *Let $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $m < n$ with rank $g = m$. For any $b \in \mathbb{R}^n$, $a \in \mathbb{R}^m$*

$$b + ga = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} g^\perp b &=& 0 \\ a &=& -(g^\top g)^{-1} g^\top b. \end{array} \right.$$

Proof: Since rank $\{g^\perp\} = n - m$, the n-dimensional matrix $\begin{bmatrix} g^\perp \\ g^\dagger \end{bmatrix}$ is full rank. Hence

$$b + ga = 0 \quad \Leftrightarrow \quad \begin{bmatrix} g^\perp \\ g^\dagger \end{bmatrix} (b + ga) = 0.$$

Moreover, the right-hand term of the equivalence above takes the form

$$\begin{array}{rcl} g^\perp b &=& 0 \\ g^\dagger b + a &=& 0. \end{array}$$

□

Lemma A.5 Consider the mappings $L : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ verifying

$$\begin{array}{rcl} F + F^\top &\leq& 0 \\ FL &=& -g. \end{array}$$

The sets

$$\begin{array}{rcl} \mathcal{S}_1 &:=& \left\{ z \in \mathbb{R}^n \mid L^\perp z = 0 \right\} \\ \mathcal{S}_2 &:=& \left\{ z \in \mathbb{R}^n \mid g^\perp F z = 0 \right\}, \end{array}$$

verify $\mathcal{S}_1 = \mathcal{S}_2$.

Proof: *The chain of implications below proves that $z \in \mathcal{S}_1 \Rightarrow z \in \mathcal{S}_2$.*

$$\begin{array}{rcl} z \in \mathcal{S}_1 &\Rightarrow& \exists \alpha \in \mathbb{R} \text{ such that } z = L\alpha \\ &\Rightarrow& Fz = FL\alpha \\ &\Rightarrow& Fz = -g\alpha \\ &\Rightarrow& g^\perp F z = 0 \\ &\Leftrightarrow& z \in \mathcal{S}_2. \end{array}$$

The opposite direction, that is, $z \in \mathcal{S}_2 \Rightarrow z \in \mathcal{S}_1$, is established by contradiction.

$$\begin{array}{rcl} z \notin \mathcal{S}_1 &\Rightarrow& z \neq L\beta, \ \forall \beta \in \mathbb{R} \\ &\Rightarrow& Fz \neq FL\beta \\ &\Rightarrow& Fz \neq -g\beta \\ &\Rightarrow& g^\perp F z \neq 0 \\ &\Leftrightarrow& z \notin \mathcal{S}_2. \end{array}$$

□

Appendix B

Experimental Implementation

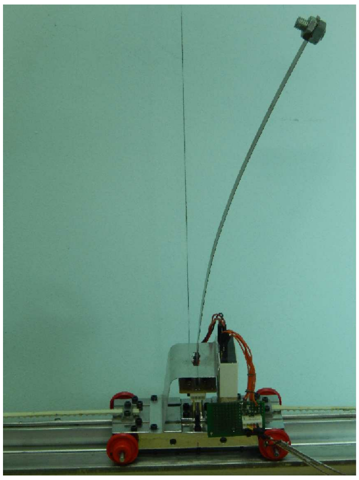

Figure B.1: Inverted flexible pendulum.

Figure B.1 shows the picture of the setup used for experimental implementation. A fatigue resistant Cu-Be alloy material is used for fabrication of the beam. Cart is guided by a rail and driven through a toothed belt driven by a motor (Maxon Motor AG: 236670). An encoder reads the position z of the motor and hence the cart. An H-bridge amplifier (Nex Robotics Hercules 36V,15A) is used to drive the motor. Strain gauges (TML Tokyo Sokki Kenkyujo Co.: FLA-5-11) in full bridge configuration along with an amplifier (DataQ Instruments 5B38-02) are used for feedback θ. The derivatives $\dot{\theta}$ and $\dot{z}$ are computed numerically using a digital derivative filter. Interfacing of the motor, strain amplifier, and encoder is done with data acquisition system ds 1104 from dSPACE GmbH via PWM, DAC, and encoder interfaces. Careful horizontal leveling of the cart and rail, and meticulous adjustment of the beam and the center of gravity of the tip mass is carried out to make sure that the unstable equilibrium is perfectly vertical and other equilibria are symmetric about the unstable equilibrium position.

Several nonlinear terms in the control law (6.19) are integral function of θ with length constraint giving $\dot{x}_e$ as limit of integration and thus are computationally demanding to evaluate in real time. Hence a look up table arrangement is used for evaluation of these terms in real time. Appropriate signal conditioning is used to balance detrimental effects of noise on one side and filter delay on the other.

Bibliography

[1] K. J. Astrom and T. Hagglund. The Future of PID Control. *Control Engineering Practice*, 9(11):1163–1175, 2001.

[2] E. Bayo. A Finite–Element Approach to Control the End–Point Motion of a Single–Link Flexible Robot. *Journal of Robotic systems*, 4(1):63–75, 1987.

[3] S. Bennett. A Brief History of Automatic Control. *IEEE Control Systems*, 16(3):17–25, 1996.

[4] P. Borja, R. Cisneros, and R. Ortega. A Constructive Procedure for Energy Shaping of Port–Hamiltonian Systems. *Automatica*, 72:230–234, 2016.

[5] B. Borovic, C. Hong, A.Q. Liu, L. Xie, and F.L. Lewis. Control of a MEMS Optical Switch. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 3, pages 3039–3044, Dec 2004.

[6] F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco. Proportional Plus Integral Control for Set Point Regulation of a Class of Nonlinear RLC Circuits. *Circuits, Systems and Signal Processing*, 28(4):609–615, 2009.

[7] F. Castaños and R. Ortega. Energy-Balancing Passivity-Based Control is Equivalent to Dissipation and Output Invariance. *Systems and Control Letters*, 58(8):553–560, 2009.

[8] R. Cisneros, M. Pirro, G. Bergna, R. Ortega, G. Ippoliti, and M. Molinas. Global Tracking Passivity–Based PI Control of Bilinear Systems and Its Application to the Boost and Modular Multilevel Converters. *Control Engineering Practice*, 43:109–119, 2015.

[9] M. Dalsmo and A. van der Schaft. On Representations and Integrability of Mathematical Structures in Energy–Conserving Physical Systems. *SIAM J. Opt. and Control*, 37(1):354–369, 1999.

[10] C.A. Desoer and E.S. Kuh. *Basic Circuit Theory*. McGraw-Hill Book Company, 1969.

[11] C.A. Desoer and M. Vidyasagar. *Feedback Systems: Input–Output Properties*. Academic Press, New York, 1975.

[12] A. Donaire and S. Junco. On the Addition of Integral Action to Port–Controlled Hamiltonian Systems. *Automatica*, 45(8):1910–1916, 2009.

[13] A. Donaire, R. Mehra, R. Ortega, S. Satpute, J.G. Romero, F. Kazi, and N.M. Singh. Shaping the Energy of Mechanical Systems Without Solving Partial Differential Equations. *Automatic Control, IEEE Transactions on*, 6(8):1051–1056, 2016.

[14] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx. *Modeling and Control of Complex Physical Systems: The Port–Hamiltonian Approach*. Springer Science & Business Media, 2009.

[15] S.K. Dwivedy and P. Eberhard. Dynamic Analysis of Flexible Manipulators, A Literature Review. *Mechanism and machine theory*, 41(7):749–777, 2006.

[16] J. Ferguson, R.H. Middleton, and A. Donaire. Disturbance Rejection Via Control by Interconnection of Port–Hamiltonian Systems. In *54th IEEE Conference on Decision and Control*, pages 507–512, Dec 2015.

[17] P. S. Gandhi, P. Borja, and R. Ortega. Energy shaping control of an inverted flexible pendulum fixed to a cart. *Control Engineering Practice*, 56:27–36, 2016.

[18] F. Gómez-Estern and A. van der Schaft. Physical Damping in IDA–PBC Controlled Underactuated Mechanical Systems. *European Journal of Control*, 10(5):451–468, 2004.

[19] M. Hernandez-Gomez, R. Ortega, F. Lamnabhi-Lagarrigue, and G. Escobar. Adaptive PI Stabilization of Switched Power Converters. *IEEE Transactions Control Systems Technology*, 18(3):688–698, 2010.

[20] D. Hill and I. Mareels. Stability Theory for Differential/Algebraic Systems With Application to Power Systems. In *Robust Control of Linear Systems and Nonlinear Control*, pages 437–445. Springer, 1990.

[21] D. Hill and P. Moylan. The Stability of Nonlinear Dissipative Systems. *IEEE Transactions on Automatic Control*, 21(5):708–711, 1976.

[22] D. Hill and P. Moylan. Dissipative Dynamical Systems: Basic Input–Output and State Properties. *Journal of the Franklin Institute*, 309(5):327–357, 1980.

[23] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[24] R. Jayawardhana, B. amd Ortega, E. Garcia-Canseco, and F. Castaños. Passivity of Nonlinear Incremental Systems: Application to PI Stabilization of Nonlinear RLC Circuits. *Systems and Control Letters*, 56(9):618–622, 2007.

[25] E. Jonckheere. Lagrangian Theory of Large Scale Systems. In *Proc. European Conf. Circuit Th. and Design*, pages 626–629, 1981.

[26] H. Khalil. *Nonlinear Systems*. Prentice-Hall, New Jersey, third edition, 2002.

[27] P.A.A. Laura, J.L. Pombo, and E.A. Susemihl. A Note on the Vibrations of a Clamped-Free Beam with a Mass at the Free End. *Journal of Sound and Vibration*, 37(2):161–168, 1974.

[28] B. Maschke, R. Ortega, and A. van der Schaft. Energy–Based Lyapunov Functions for Forced Hamiltonian Systems With Dissipation. *IEEE Transactions Automatic Control*, 20(8):1498–1502, 2000.

[29] L. Meirovitch. *Elements of Vibration Analysis.* McGraw-Hill, 1975.

[30] Z. Meng, R. Ortega, D. Jeltsema, and H. Su. Further Deleterious Effects of the Dissipation Obstacle on Control by Interconnetction of Port–Hamiltonian Systems. *Automatica*, to appear.

[31] C. Meza, D. Biel, D. Jeltsema, and J.M.A. Scherpen. Lyapunov–Based Control Scheme for Single–Phase Grid–Connected PV Central Inverters. *IEEE Control Systems Magazine*, 20(2):520–529, 2012.

[32] P. Moylan. *Dissipative Systems and Stability.* Springer–Verlag, 2013.

[33] R. Ortega and P. Borja. New Results on Control by Interconnection and Energy-Balancing Passivity-based control of Port-Hamiltonian Systems. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 2346–2351, Dec 2014.

[34] R. Ortega, A. Donaire, and J.G. Romero. *Passivity–Based Control of Mechanical Systems.* Springer Verlag, 2015.

[35] R. Ortega and E. García-Canseco. Interconnection and Damping Assignment Passivity-Based Control: A Survey. *European Journal of Control*, 10(5):432 – 450, 2004.

[36] R. Ortega, D. Jeltsema, and J.M.A. Scherpen. Power shaping: A New Paradigm for Stabilization of Nonlinear RLC Circuits. *Automatic Control, IEEE Transactions on*, 48(10):1762–1767, Oct 2003.

[37] R. Ortega, Z. Liu, and H. Su. Control via Interconnection and Damping Assignment of Linear Time-Invariant Systems: A Tutorial. *International Journal of Control*, 85(5):603–611, 2012.

[38] R. Ortega, A. Loria, P. Nicklasson, and H. Sira-Ramirez. *Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications.* Communications and Control Engineering. Springer Verlag, London, 1998.

[39] R. Ortega and J.G. Romero. Robust Integral Control of Port–Hamiltonian Systems: The Case of Non-Passive Outputs With Unmatched Disturbances. *Systems and Control Letters*, 61(1):11–17, 2012.

[40] R. Ortega and M.W. Spong. Adaptive Motion Control of Rigid Robots: A Tutorial. *Automatica*, 25(6):877–888, 1989.

[41] R. Ortega, M.W. Spong, F. Gomez-Estern, and G. Blankenstein. Stabilization of a Class of Underactuated Mechanical Systems Via Interconnection and Damping Assignment. *Automatic Control, IEEE Transactions on*, 47(8):1218–1233, Aug 2002.

[42] R. Ortega, A. van der Schaft, F. Castaños, and A. Astolfi. Control by Interconnection and Standard Passivity-Based Control of Port-Hamiltonian Systems. *Automatic Control, IEEE Transactions on*, 53(11):2527–2542, 2008.

[43] R. Ortega, A. van der Schaft, I. Mareels, and B. Maschke. Putting Energy Back in Control. *Control Systems Magazine, IEEE*, 21(2):18–33, Apr 2001.

[44] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar. Energy–Shaping of Port-Controlled Hamiltonian Systems by Interconnection. In *Decision and Control (CDC), 1999 IEEE 38th Annual Conference on*, volume 2, pages 1646–1651, 1999.

[45] O. Patil and P.S. Gandhi. On the Dynamics and Multiple Equilibria of An Inverted Flexible Pendulum With Tip Mass on A Cart. *Journal of Dynamic Systems, Measurement, and Control*, 136(4):041017, 2014.

[46] J. W. Polderman and J. C. Willems. *Introduction to Mathematical System Theory: A Behavioral Approach*. Springer–Verlag, 1998.

[47] S. Prajna, A. van der Schaft, and G. Meinsma. An LMI Approach to Stabilization of Linear Port-Controlled Hamiltonian Systems. *Systems & Control Letters*, 45(5):371 – 385, 2002.

[48] C. Rao and S. Mitra. *Generalized Inverse of Matrices and Its Applications*. Wiley, New York, 1971.

[49] J.G. Romero, A. Donaire, and R. Ortega. Robust Energy Shaping Control of Mechanical Systems. *Systems and Control Letters*, 62(9):770–780, 2013.

[50] J.G. Romero, R. Ortega, and A. Donaire. Robust Energy Shaping Control of Mechanical Systems Without Solving PDEs and Extension to Constant Speed Tracking. *IEEE Transactions Automatic Control*, 61(11):3551–3556, 2016.

[51] S. Sanders and G. Verghese. Lyapunov–Based Control for Switched Power Converters. *IEEE Transactions Power Electronics*, 7(1):17–24, 1992.

[52] R. Sepulchre, M. Jankovic, and P. Kokotovic. *Constructive Nonlinear Control*. Springer, New York, 1997.

[53] M. Spivak. *A Comprehensive Introduction to Differential Geometry*. Perish, Inc, third edition, 1999.

[54] M.W. Spong. Underactuated Mechanical Systems. In *Control problems in robotics and automation*, pages 135–150. Springer, 1998.

[55] M. Takegaki and S. Arimoto. A New Feedback Method for Dynamic Control of Manipulators. *ASME. J. Dyn. Sys., Meas., Control.*, 103(2):119–125, June 1981.

[56] D.E. Torfs, R. Vuerinckx, J. Swevers, and J. Schoukens. Comparison of Two Feed–Forward Design Methods Aiming at Accurate Trajectory Tracking of the End Point of a Flexible Robot Arm. *Control Systems Technology, IEEE Transactions on*, 6(1):2–14, 1998.

[57] A. van der Schaft. Interconnections of Input–Output Hamiltonian Systems With Dissipation. In *55th IEEE Conference on Decision and Control*, pages 4686–4691, Dec 2016.

[58] A. van der Schaft. *L_2–Gain and Passivity Techniques in Nonlinear Control.* Springer, Berlin, third edition, 2016.

[59] A. van der Schaft and D. Jeltsema. Port–Hamiltonian Systems Theory: An Introductory Overview. *Foundations and Trends in Systems and Control*, 1(2–3):173–378, 2014.

[60] A. Venkatraman and A. van der Schaft. Energy Shaping of Port-Hamiltonian Systems by Using Alternate Passive Input-Output Pairs. *European Journal of Control*, 16(6):665 – 677, 2010.

[61] P.E. Wellstead. *Introduction to Physical System Modelling.* Academic Press London, 1979.

[62] J.C. Willems. Dissipative Dynamical Systems Part I: General Theory. *Arch Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[63] J.C. Willems. Dissipative Dynamical Systems. *European Journal of Control*, 13(2-3):134–151, 2007.

[64] C. Woolsey, C.K. Reddy, A.M. Bloch, D.E. Chang, N.E. Leonard, and J.E. Marsden. Controlled Lagrangian Systems with Gyroscopic Forcing and Dissipation. *European Journal of Control*, 10(5):478–496, 2004.

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Stabilisation d'une classe de systèmes non linéaires avec propriétés de passivité.

Mots clefs : Systèmes non linéaires, systèmes hamiltoniens à ports, régulateurs PID, énergie, passivité, function de stockage.

Résumé : Dans cette thèse, nous abordons le problème de la stabilisation des systèmes non linéaires. En particulier, nous nous concentrons sur les modèles où l'énergie joue un rôle fondamental. Ce cadre énergétique est adapté pour capturer les phénomènes de plusieurs domaines physiques tels que les systèmes mécaniques, les systèmes électriques, les systèmes hydrauliques, etc. Le point de départ des contrôleurs proposés sont les concepts de système passif, des sorties passives et des fonctions d'énergie (ou stockage). Dans ce travail, nous étudions deux classes de systèmes dynamiques, à savoir les Hamiltoniens à ports (PH) et les Euler-Lagrange (EL), qui conviennent pour représenter de nombreux processus physiques. Une première étape vers la construction des contrôleurs est de montrer la passivité des systèmes PH et la caractérisation de leurs sorties passives. Par la suite, nous explorons l'utilisation des différentes sorties passives dans deux techniques bien connues de contrôle par passivité (PBC), c'est-à-dire le contrôle par interconnexion (CbI) et l'équilibrage énergétique (EB), et nous comparons les résultats obtenus dans les deux approches. De plus, nous proposons une nouvelle méthodologie dans laquelle la loi de commande est composée d'un terme proportionnel (P), un terme intégral (I) et, éventuellement, un terme dérivatif (D) de la sortie passive. Dans cette stratégie, l'énergie du système en boucle fermée est façonnée sans qu'il soit nécessaire de résoudre des équations différentielles partielles (PDE). Nous analysons le scénario du contrôleur PID à l'aide des différentes sorties passives précédemment caractérisées. Enfin, nous appliquons un schéma PID-PBC récemment proposé dans la littérature à un système mécanique complexe, à savoir un pendule inversé ultra flexible, représenté sous la forme d'un modèle contraint EL. La conception du contrôleur, la preuve de la stabilité, ainsi que les simulations et les résultats expérimentaux sont présentés pour montrer l'applicabilité de cette technique aux systèmes physiques.

Title : Stabilization of a class of nonlinear systems with passivity properties.

Keywords : Nonlinear systems, port-Hamiltonian systems, PID regulators, energy, passivity, storage function.

Abstract : In this thesis we address the problem of stabilization of nonlinear systems. In particular, we focus on models where the energy plays a fundamental role. This energy-based framework is suitable to capture the phenomena of several physical domains, such as mechanical systems, electrical systems, hydraulic systems, etc. The starting point in the proposed controllers are the concepts of passive system, passive outputs and energy (storage) functions. In this work we study two classes of dynamical systems, namely port-Hamiltonian (PH) and Euler-Lagrange (EL), which are suitable to represent many physical processes. A first step towards the controller design is to show the passivity of the PH systems and the characterization of their passive outputs. Thereafter, we explore the use of the different passive outputs in two well-known passivity-based control (PBC) techniques, that is control by interconnection (CbI) and energy balancing (EB), and we compare the obtained results in both approaches. In addition, we propose a novel methodology in which the controller consists in a proportional (P), an integral (I) and, possibly, a derivative (D) term of the passive output. In this approach the energy of the closed-loop system is shaped without the necessity of solving partial differential equations (PDEs). We analyze the scenario of the PID controller using the different passive outputs previously characterized. Finally, we apply a PID-PBC scheme recently proposed in the literature to a complex mechanical system, namely an ultra flexible inverted pendulum, which is represented as a constrained EL model. The controller design, the stability proof, as well as simulations and experimental results are presented to show the applicability of this technique to physical systems.

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