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Stability and Stabilization of Networked Systems

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To Antonio Loría
and Elena Panteley
with thanks.

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Contents

Aperçu de la thèse

Ce mémoire présente le travail accompli au cours des trois dernières années sur la coordination des systèmes multi-agents et, en particulier, sur le contrôle en formation des véhicules non-holonomes. En générale, résoudre un problème de coordination distribuée pour un système multi-agent consiste à synthétiser l'entrée de commande pour chaque agent afin de permettre à certaines grandeurs d'intérêt dans le groupe de systèmes de réaliser une tâche commune, par exemple, former une certaine posture géométrique, suivre un leader commun, ou bien décrire un comportement commun en régime permanent (synchronisation).

Selon la procédure de conception des lois de commandes, deux types d'approches se distinguent, les approches *centralisées* et les approches *distribuées*. Dans le premier cas, chaque système reçoit une information globale qui consiste en le comportement de référence qu'il est sensé produire en régime permanent. Dans ce cas, le problème de coordination entre les différents systèmes est réduit à la commande en poursuite de chaque système séparément vers son comportement de référence. Dans l'approche distribuée, l'entrée de commande de chaque agent est conçue en utilisant uniquement des informations locales qui proviennent d'un certain groupe d'agents appelé le groupe de voisins. L'interaction entre les agents se caractérise, des lors, par un graphe de communication.

Les solutions distribuées aux problèmes de coordination des systèmes multiagents ont été largement étudiées en automatique, nous citons par exemple:

[43], [93], [22], [79] and [106], la dernière référence est un état de l'art sur le sujet.

Deux axes principaux de recherche sont identifiés dans le contexte de la coordination distribuée des systèmes multi-agents. Le premier apparaît lorsqu'on considère la commande distribuée en présence de contraintes sur le processus de communication entre les agents, ce qui inclut le cas où le transfert d'informations est unidirectionnel [102], [90], variable dans le temps, on parle dans ce cas de graph temps-variant [104], [73], ou affecté par des retards de transmission ou des échantillonnages [1]. Le second cas survient lorsqu'on considère la dynamique individuelle des agents, par exemple, le cas général des systèmes linéaires [54], les systèmes non-linéaires identiques [40], ou les systèmes non-linéaires hétérogènes [98], [120].

Le problème général de la coordination distribuée du mouvement d'un groupe d'agents mobiles a été aussi largement étudié dans le domaine de l'ingénierie automatique au cours des dernières décennies. Un tel intérêt est dû à l'importance d'une telle coordination dans de nombreuses applications, nous citons par exemple le cas des robots mobiles [29], des véhicules aériens sans pilote [32], des véhicules sous-marins autonomes [13], satellites [57], aéronefs et engins spatiaux [103], etc.

Parmi les problèmes les plus importants en coordination distribuée, deux catégories de problèmes se distinguent:

Problème de consensus sans leader. Dans ce cas, l'objectif est de parvenir à un arrangement entre les coordonnées des agents et de les faire converger asymptotiquement vers une posture commune. Les agents peuvent échanger uniquement des informations avec un certain nombre de voisins. Le problème de consensus sans leader a été étudié, par exemple, pour le cas des systèmes linéaires de premier ordre et de second ordre [73], [56], [109], et aussi pour le cas de certaines classes de systèmes non-linéaires [90, 92, 120]. Dans certaines applications, l'arrangement entre les états des systèmes diffère légèrement du consensus classique, dans le sens où, au lieu de faire converger les états vers une valeur commune, les agents devraient former une posture géométrique qui pourrait être constante ou variable dans le temps. Ce type de problème est souvent appelé problème de formation à base de consensus. Il convient de souligner qu'un changement de coordonnées est souvent adopté afin de permettre la transformation du problème de formation en un problème de consensus [29].

Problème de consensus avec leader. L'objectif, dans ce cas, est de parvenir à un arrangement entre les agents tout en poursuivant une trajectoire commune générée par un agent leader. Comme dans le cas précédent, seule l'information concernant les postures des agents voisins (qui peuvent inclure le leader) est accessible pour chaque agent. L'interaction entre les agents, incluant le leader, se caractérise par un graphe d'interconnexion augmenté. Le plus souvent, le comportement du système leader a une grande influence à la fois sur la conception des lois de commande et aussi sur l'analyse de la boucle fermée.

Dans ce document, les deux problèmes décrits ci-dessus sont étudiés dans le cas où les agents sont des robots mobiles non-holonomes. La commande du robot mobile non-holonome a été un domaine de recherche très actif en automatique non-lineaire au cours des deux dernières décennies, voir par exemple [49] pour un état de l'art sur la commande de ce type de systèmes; En règle générale, la commande d'un robot mobile non-holonome consiste à résoudre l'un des trois problèmes suivants:

Le problème général de poursuite. Il consiste à définir un robot virtuel qui génère une trajectoire de référence que le robot commandé est doit poursuivre. En général, les vitesses du robot leader sont des fonctions variables dans le temps, ainsi le système en boucle fermée est le plus souvent non-linéaire et temps-variant – voir les chapitres 2-3.

Le problème de stabilisation. Il consiste à stabiliser les trajectoires du robot vers une posture de consigne constant. Ce problème est pertinent en raison de la contrainte non-holonome qui empêche la résolution du problème en utilisant des lois de rétroaction lisses et autonomes [15]. Le problème de stabilisation peut être reformulé en un problème de leader-suiveur en introduisant un leader dont les vitesses sont égales à zéro.

Le problème de poursuite-stabilisation simultanés. Il consiste à concevoir un contrôleur unifié qui résout le problème de leader-suiveur pour le cas général des vitesses du leader — voir Chapitre 2 pour une discussion plus détaillée.

L'extension naturelle du **problème de stabilisation** d'un véhicules non-holonomes au cas multi-agent est le problème de **consensus sans leader** qui est étudié dans le chapitre 4 sous l'hypothèse d'un graphe bidirectionnel connecté et d'une communication affectée par un retard variant dans le temps et borné. Le problème de **leader-suiveur** pour un groupe de robots mobiles a également été considéré dans cette thèse. Selon les vitesses du leader, Les chapitres 2 et 3 étudient les trois problèmes suivants, sous l'hypothèse d'un graphe constant ayant une topologie particulière qui est celle de l'arbre générateur dirigé.

1)- Problème de poursuite leader-suiveur. Dans ce cas, on résout le problème de consensus leader-suiveur en supposant que les vitesses du leader décrivent une function général variante dans le temps, de sorte que la norme de ses vitesses est un signal a excitation permanente – voir Définition A.6.

2)- Problème de rendez-vous robuste leader-suiveur. Dans ce cas, les vitesses du leader convergent vers zéro.

3)- Problème de poursuite-rendez-vous simultanés. Dans ce cas, on propose un contrôleur unifié qui résout le problème de consensus leader-suiveur pour toutes les configurations possibles des vitesses du leader.

Notre approche consiste à transformer chacun des problèmes cités précédemment en un problème de stabilisation d'un ensemble invariant. Nos outils d'analyse reposent principalement sur la construction de fonctions de Lyapunov et de Lyapunov-Krasovskii strictes pour des systèmes non-linéaires variant dans le temps et/ou retardés. Ces fonctions sont, par la suite, utilisées pour établir des résultats de stabilité uniforme et de robustesse pour le cas des robots mobiles.

Le premier chapitre de ce manuscrit présente des résultats techniques sur la stabilité des systèmes linéaires temps-variant inspirées du livre [72]. Notamment, on présente les méthodes essentielles pour la construction de fonctions de Lyapunov strictes. Ces méthodes sont employées dans tous les chapitres qui suivent pour la conception des lois de commande et pour analyse de stabilité de la boucle fermée pour le cas des robots mobiles en formation distribuée.

Introduction and Contributions

We present in this memoir the work accomplished in the last three years on multi-agent coordination and in particular, on formation control of non-holonomic vehicles. Generally speaking, solving multi-agent coordination problem consists on designing the control input for each agent in order to allow certain quantities of interest in the group of systems to realize a common task, for example, reaching a certain geometric pattern, following a common leader agent or describing a common steady state behavior.

Depending on the control design procedure, we distinguish the *centralized* and the *distributed* approaches. In the first approach each system receives a global information which consists of its reference behavior. In this case, the multi-agent coordination problem is reduced to the stabilization of each system separately toward its reference behavior. In the distributed approach the control input for each agent is designed using only local knowledge that is received from some agents called neighbors. The interaction between the agents is characterized by a communication graph.

Distributed solution to multi-agent coordination, consensus or synchronization problems have been extensively studied in the control literature, we cite for example:

[43], [93], [22], [79] and [106], where the last reference is a survey on this topic.

Two principle research axes can be identified in the context of distributed multi-agent coordination. The first one appears when considering distributed control in the presence of communication constraints between the agents, which include the case when the transfer of information is unidirectional [102], [90], unreliable links with time-varying graph topology [104], [73], delayed or sampled transfer of information [1] to name few. The second one arises when considering individual dynamics of the agents, for example, general linear systems [54], nonlinear homogeneous systems [40], or heterogeneous nonlinear systems [98], [120].

The general problem of distributed coordinated motion of mobile agents has been extensively studied in control engineering during the last decades. Such an interest is caused by importance of such a coordination in many different engineering applications, we cite here mobile robots [29], unmanned air vehicles [32], autonomous underwater vehicles [13], satellites [57], aircraft and spacecraft [103], etc.

Among existing approaches to the coordination task we mention here the following two problems:

Leaderless consensus problem. In this case the objective is to reach an agreement between the agents and in particular coordinates to make them converge asymptotically to a common value. In this case, agents can exchange information only with their neighbors. The leaderless consensus problem of multiple dynamical systems has been extensively studied, for example, linear systems, including first, second order and general linear systems are considered in [56,73,109], and different classes of nonlinear systems are considered in [90,92,120].

In some applications, an agreement between the systems is slightly different from the classical consensus, in the sense that instead of common value, the agents should follow some geometric pattern that can be constant or time varying. This type of problem is often referred to as leaderless consensus problem. It should be underlined here that an appropriate change of coordinates allows to transform the formation task into consensus one [29].

Leader-follower consensus problem. In this case the objective is to reach an agreement between the agents defined by a common trajectory generated by a leader agent. As in the previous case only the information of the neighboring agents (and may be the leader), is accessible to the agents. The interaction between the agents, including the leader, is characterized by an augmented graph of interconnections. Usually, the behavior of the leader system has a great influence both on the control design and on the closed-loop analysis.

In this document we study the two above described problems in the case where the agents are modeled as a nonholonomic mobile robots. The control of nonholonomic mobile robot has been an active research field in the control community during the last two decades see for example [49] for a survey on the control of nonholonomic vehicles; generally speaking, controlling a nonholonomic mobile robot consists of solving one of the following three problems.

The general leader-follower problem. It consists in defining a virtual robot that generates a reference trajectory to be followed by the controlled robot. In general, the velocities of the virtual robot are time varying functions, as a result the closed-loop system is usually nonlinear and time varying–see Chapters 2-3.

The stabilization problem. It consists in stabilization of the robot trajectories to a constant set point. This problem is relevant because of the nonholonomic restriction that enables the use of any smooth autonomous feedback law [15]. The stabilization problem can be recast as a leader-follower problem by introducing a leader, whose velocities are equal to zero.

The simultaneous tracking-stabilization problem. It consists in the design of a unified controller that solves the leader-follower problem both in the case where the leader's velocities are either general time varying functions or equal to zero — see Chapter 2 for more detailed discussion.

The natural extension of the **stabilization problem** for nonholonomic vehicles to the multi-agent case is the **leaderless consensus** problem which we study in Chapter 4 under assumptions of a general bidirectional graph and time varying communication delays. The **leader-follower** problem for a multiple nonholonomic mobile robots has also been considered in this thesis. Depending on the leader's velocities, Chapters 2 and 3 study the three following problems, respectively, under a particular constant communication graph topology that is a directed spanning tree.

1)- Leader-follower tracking problem. In this case, we solve the leader-follower consensus problem under the assumption that the leader vehicle describes a general time varying path, such that, the norm of its velocities is persistently exciting,–see Definition A.6.

2)- Leader-follower robust agreement problem. In this case, we solve the leader-follower consensus problem when the leader's velocities converge to zero.

3)- Simultaneous tracking-agreement problem. In this case, we design a unified controller that solves the leader-follower consensus problem for all possible configurations of the leader's velocities.

Our approach consists in transforming each one of the problems cited above into a stabilization problem of an invariant set. Our analysis tools are based, mainly, on the construction of strict Lyapunov functions and strict Lyapunov-Krasovskii functionals for nonlinear time varying and/or delayed systems. These functions are then used to establish stability and robustness results in the area of mobile robot control.

The first chapter of this manuscript presents our basic technical results of stability for time varying linear systems. Notably, we present therein the essential methods for the construction of the strict Lyapunov functions. These methods we employ in all the subsequent chapters in the control design and the analysis of mobile robots. The Lyapunov functions that we employ follow ideas proposed in [72]. However, the constructions that we present for the specific case-studies of time-varying systems in Chapter 1, and for mobile robots, in the subsequent chapters, are original. Moreover, to the best of our knowledge, for the problems of formation control for autonomous vehicles, we are the first to provide strict Lyapunov functions.

Our contributions are described in further detail below.

Contributions of the thesis

We briefly summarize the main results of this thesis, chapter by chapter, and cite related publications. References correspond to the list of publications presented in p. 18.

- Chapter 1: We present some results on stability of persistently excited linear time-varying systems with particular structures. Such systems appear in diverse problems, which include the analysis of model-reference adaptive systems, persistently-excited observers, consensus of systems interconnected through time-varying links and systems with time-varying input gain. The originality of our statements lies in the fact that we provide smooth strict Lyapunov functions hence, our proofs are constructive and direct. Moreover, we establish uniform global exponential stability with explicit stability and decay estimates.

  This chapter formed the subject of the following publications on: [(iv),(iii)].

- Chapter 2: We present controllers for leader-follower formation *tracking* and *robust agreement* control problems for a group of autonomous non-holonomic vehicles. We consider general models composed of a velocity kinematics and a generic force-balance equations. We assume that, each robot has a unique leader and only the swarm leader robot knows the reference trajectory, but each robot may have one or several followers. That is, the graph topology is a spanning tree. For the *tracking* case, we establish uniform global asymptotic stability of the closed-loop system under the assumption that the virtual vehicle velocities are persistently exciting. The analysis relies on the construction of a strict Lyapunov function for the position tracking error dynamics and a recursive argument for cascaded systems. For the *robust agreement* case, we control the group of robots that follow trajectories with a vanishing reference velocities. The control design is based on a δ-persistently exciting controller (for the kinematics model) that is robust to decaying perturbations. We construct strict Lyapunov functions to guarantee integral input-to-state stability and small input-to state stability of the closed-loop system at the kinematic level. At the same time we design a dynamic level controller that ensures asymptotic convergence of the formation trajectories even in the case when the inertia parameters are unknown.

  These results were originally presented in the following publications with A. Loría and E. Panteley: [(viii), (i), (xii), (v), (xiii)].

- Chapter 3: We solve the leader-follower *simultaneous tracking-stabilization control* problem for a force-controlled nonholonomic mobile robots, assuming that the leader's velocities are either integrable *(parking problem)* or Persistently Exciting *(tracking problem)*. We introduce a simple control law that allows to extend the idea of control design proposed in [119] to a more general class of controllers and, then, to more general scenarios of the leader's velocities. In particular we assume that the leader's velocities are either converging to zero or persistently exciting. This permits to solve the leader-follower simultaneous tracking-agreement problem for a group of force-controlled nonholonomic mobile robots, under a spanning tree communication topology rooted at the virtual leader. We introduce a simple decentralized control law and establish, for each agent, convergence to zero of the tracking errors relatively to its neighbor.

Stability proofs that we present are based on the construction of strict Lyapunov functions for classes of nonlinear time-varying systems and robustness analysis tools such as iISS the strong iISS notions.

Publications related to the material presented in this chapter are in preparation with A. Loría and E. Panteley: [(xviii), (xix)].

- Chapter 4: We present a novel decentralized consensus-based formation controllers for swarms of nonholonomic vehicles both for the kinematic and the dynamic models. We solve the leaderless consensus problem with a desired orientations (*partial consensus case*), and the leaderless consensus problem in both positions and orientations (*full consensus case*). Moreover, we consider a case where that the system interconnections are affected by time-varying delays. The network is modeled as an undirected, static and connected graph. The controllers that we propose are a smooth time-varying δ-persistently exciting controllers of the PD and PID type. The stability analysis is carried out using a novel strict Lyapunov function for both cases.

  The material of this chapter was prepared in collaboration with E. Nuño-Ortega, A. Bautista-Castillo, From University of Guadalajara, A. Loría and E. Panteley [(ix), (xv), (xvi)].

For clarity of exposition we have decided to present in this thesis only our results on formation control of mobile robots and related topics. Thus, some of our results, cited below,were excluded from the manuscript, some of them are either published or under review and the others are in preparation:

- The papers [(vi), (xvii)] are a joint works with E. Panteley and A. Loria on the synchronization of heterogeneous oscillators using singular perturbation approach.

- The papers [(xi), (xiv)] are joint work with D. Belleter, C. Paliotta, and K. Y. Pettersen, from NTNU Trondheim, where we studied local and global path following problems for underactuated marine vessels in the presence of unknown ocean currents using an observer based approach.

- The publication [(ii)] is a joint work with N. R. Chowdhury, S. Sukumar, from IIT Bombay, and A. Loría where we studied consensus problem under time-varying bidirectional graph containing a persistently exciting spanning tree.

List of publications

The following is an exhaustive list of publications written during the past three years, that are either published, accepted for publication, or still under review. It contains but is not restricted to the contents of this document.

Journal papers

i/ M. Maghenem, A. Loría, and E. Panteley, "Formation-tracking control of autonomous vehicles under relaxed persistency of excitation conditions," *IEEE Trans. on Contr. Syst. Techn.*, 2016. Provisionally accepted.

ii/ N. R. Chowdhury, S. Sukumar, M. Maghenem, and A. Loría, "On the estimation of algebraic connectivity in graphs with persistently exciting interconnections," *Int. J. of Contr.*, 2016. Pre-published online.
DOI: http://dx.doi.org/10.1080/00207179.2016.1272006.

iii/ M. Maghenem and A. Loría, "Strict Lyapunov functions for time-varying systems with persistency of excitation," *Automatica*, vol. 78, pp. 274–279, 2017. Pre-published online. DOI: 10.1016/j.automatica.2016.12.029.

iv/ M. Maghenem and A. Loría, "Lyapunov functions for persistently-excited cascaded time-varying systems: application in consensus analysis," *IEEE Trans. Automat. Control*, 2016. Pre-published online. DOI: 10.1109/TAC.2016.2610099.

Conference papers

v/ M. Maghenem, A. Loría, and E. Panteley, "A robust δ-persistently exciting controller for formation-agreement stabilization of multiple mobile robots," in *Proc. IEEE American Control Conference*, (Seatle, WA), 2017. To appear.

vi/ M. Maghenem, E. Panteley, and A. Loría, "Synchronization of networked Andronov-Hopf oscillators using singular perturbation approach," in *Proc. 55th IEEE Conf. Decision and Control*, (Las Vegas, NV, USA), 2016. To appear.

vii/ M. Maghenem, A. Loría, and E. Panteley, "A strict yapunovfunction for nonholonomic systems under persistently-exciting controllers," in *IFAC NOLCOS 2016*, (Monterey, CA, USA), 2016. To appear.

viii/ M. Maghenem, A. Loría, and E. Panteley, "Lyapunov-based formation-tracking control of nonholonomic systems under persistency of excitation," in *IFAC NOLCOS 2016*, (Monterey, CA, USA), 2016. To appear.

ix/ M. Maghenem, A. Bautista-Castillo, E. Nuño, A. Loría, and E. Panteley, "Consensus-based formation control of nonholonomic robots using a strict Lyapunov function," in *IFAC World Congress*, 2017. To appear.

x/ M. Maghenem, A. Loría, and E. Panteley, "Global tracking-stabilization control of mobile robots with parametric uncertainty," in *IFAC World Congress*, 2017. To appear.

xi/ M. Maghenem, D. Belleter, C. Paliotta, and K. Y. Pettersen, "Observer Based Path Following for Underactuated Marine Vessels in the Presence of Ocean Currents: A Local Approach," in *IFAC World Congress*, 2017. To appear. See also: arXiv preprint arXiv:1704.00573.

Technical reports

Papers under review

xii/ M. Maghenem, A. Loría, and E. Panteley, "A cascades approach to formation-tracking stabilization of force-controlled autonomous vehicles," *IEEE Trans. on Automatic Control*, 2016. In review. See also: https://hal.archives-ouvertes.fr/hal-01364791/document.

xiii/ M. Maghenem, A. Loría, and E. Panteley, “A robust δ-persistently exciting controller for leader-follower tracking-agreement of multiple vehicles,” *Eurpean J. of Control*, 2016. In review.

Papers in preparation

xiv/ M. Maghenem, D. Belleter, C. Paliotta, and K. Y. Pettersen, “ Observer Based Path Following for Underactuated Marine Vessels in the Presence of Ocean Currents: A Global Approach”.

xv/ M. Maghenem, A. Bautista-Castillo, E. Nuño, A. Loría, and E. Panteley, “Full and partial consensus-based Formation Control of nonholonomic Robots using a Strict Lyapunov function”.

xvi/ M. Maghenem, A. Bautista-Castillo, E. Nuño, A. Loría, and E. Panteley, “ Strict Lyapunov-Krasovskii functional for leaderless consensus problem of non-holonomic Robots under general time-varying delay ”.

xvii/ M. Maghenem, E. Panteley, and A. Loría, “ Singular-Perturbations-Based Analysis of Synchronization in Heterogeneous Networks ”.

xviii/ M. Maghenem, A. Loría, and E. Panteley, “A universal controller for tracking and stabilization control of nonholonomic vehicles ”.

xix/ M. Maghenem, A. Loría, and E. Panteley, “ A universal controller for leader-follower formation tracking and agreement control of nonholonomic vehicles ”.

Notations

Notations

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}_{\geq 0}$	Field of positive real numbers.
$\mathbb{R}^n$	Linear space of real vectors of dimension n.
$\mathbb{R}^{n\times m}$	Ring of matrices of size $n \times m$.
x_i	The i-th element of the vector x.
I_n	The identity matrix of size $n \times n$.
$\mathbf{1}$	Column vector of ones of dimension n.
diag$(\cdot)$	Diagonal matrix of the input arguments.
col$(\cdot)$	Column vector of the input arguments.
$\bar{x}$	The diagonal matrix representation of x, i.e., $\bar{x} = \text{diag}(x_i)$.
$\lvert x\rvert$	The Euclidean norm of x.
$\lvert x\rvert_\infty$	For a time varying vector $x(t)$ denote, $\sup_{t\geq 0}\{x(t)\}$.
$\lvert x\rvert_{\mathcal{A}}$	For a set $\mathcal{A} \subset \mathbb{R}^n$ denote, $\min_{y\in\mathcal{A}} \lvert x - y\rvert$.
$A^\top$	The transpose matrix to A.
$A^\perp$	The orthogonal matrix to A, i.e., $A^\top A^\perp = 0$.
$\lvert A\rvert$	For a matrix A denote, induced Euclidean norm of A.
$\lvert M(t)\rvert_\infty$	For a time varying matrix $M(t)$ denote, $\sup_{t\geq 0}\{\lvert M(t)\rvert\}$.
$\otimes$	The Kronecker product.
$\dot{f},\ \ddot{f}$	For function of scalar argument $f : \mathbb{R} \to \mathbb{R}^s$ denote, respectively, first and second order differentiation.
$\mathcal{K}$	Class of positive continuous and strictly increasing functions, $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $f(0) = 0$.
$\mathcal{K}_\infty$	Class of functions $f \in \mathcal{K}$, with $f(\infty) = \infty$.
$\mathcal{L}$	Class of positive continuous and strictly decreasing functions, $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $f(\infty) = 0$.
$\mathcal{KL}$	Class of positive and continuous functions $f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $f(\cdot, y) \in \mathcal{K}^\infty$ and $f(x, \cdot) \in \mathcal{L}$.

$\mathcal{L}_p$	The space of $p(>0)$ integrable functions, $f:\mathbb{R}_{\geq 0}\to\mathbb{R}^n \in \mathcal{L}_p$ $\Rightarrow \left[\int_0^\infty \|f(s)\|^p\, ds\right]^{\frac{1}{p}} < \infty$.
x_t	For $x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, denote the functional $x_t(\theta) := x_t(t+\theta)$, for all $\theta\in[-T,0]$.
$C[-T,0]$	The space of functions which are continuous on $[-T,0]$.
$\|x_t\|_{\mathcal{A}}$	For a functional $x_t \in C[-T,0]$ denote, $\max_{\theta\in[-T,0]} \|x(t+\theta)\|_{\mathcal{A}}$.
$W[-T,0]$	The space of functions which are absolutely continuous on $[-T,0]$, and have square integrable first order derivatives.
$\\|x_t\\|_{\mathcal{A}}$	For a functional $x_t\in W[-T,0]$ denote, $\max_{\theta\in[-T,0]} \|x(t+\theta)\|_{\mathcal{A}} + [\int_{-T}^0 \|\dot{x}(t+s)\|^2\, ds]^{1/2}$.
$\mathcal{L}_2[-T,0]$	The space of square integrable functions on $[-T,0]$.

For a symmetric positive semi-definite matrix $L\in\mathbb{R}^{n\times n}$, we define

$\lambda_M(L)$	The maximum eigenvalue of L.
$\lambda_m(L)$	The minimum eigenvalue of L.
$\lambda_i(L)$	The ith eigenvalue of L greater then $\lambda_m(L)$.

Acronyms

a.e.	Almost Everywhere
PE	Persistently Exciting
US	Uniformly Stable
UAS	Uniformly Asymptotically Stable
UES	Uniformly Exponentially Stable
UGAS	Uniformly Globally Asymptotically Stable
UGES	Uniformly Globally Exponentially Stable
ISS	Input-to-State Stability
iISS	integral Input-to-State Stability
PD	Proportional and Derivative
PI	Proportional and Integral
PID	Proportional, Integral and Derivative
SLF	Strict Lyapunov Function
SLKF	Strict Lyapunov Krasovskii Functional

Chapter 1

Strict Lyapunov functions for time-varying systems with persistency of excitation

Establishing uniform asymptotic stability of the origin for time-varying systems is a difficult task in general, even for linear systems. For instance, for the latter, eigenvalue analysis is generally inconclusive, even for boundedness of the solutions. Much of the control literature in which time-varying systems appear, relies on generic methods of proof that are based on "signal chasing" arguments such as Barbălat's lemma, properties of functions in $\mathcal{L}_p$ spaces, *etc*. In general, finding a strict Lyapunov function (that is, which is positive definite, radially unbounded and with negative definite derivative) is an extremely challenging problem.

The notion of persistency of excitation, which was originally introduced in the context of systems identification [11], is known to be necessary and sufficient for uniform exponential stability of certain linear time-varying systems [82]. Early proofs of such statement rely on concepts such as uniform complete observability [83], output injection arguments [7] and other (rather intricate) methods tailored specifically for linear systems [41].

In so-called model-reference adaptive control [86], persistency of excitation plays a fundamental role as a necessary and sufficient condition for uniform global asymptotic stability. For functions that depend on the state and time, however, persistency of excitation must be redefined and the stability anaysis demands a special treatment. For instance, on occasions it appears convenient to analyse nonlinear time-varying systems as linear time-varying [47, p. 659]. Such method of analysis renders possible the extension of stability tools devoted to linear time-varying systems with persistency of excitation, to the realm of nonlinear systems [63]. Nevertheless, as it is showed in the latter reference, it is fundamental to take special care in imposing a uniform variant of persistency of excitation, independent of the initial conditions.

More recently, new notions of peristency of excitation tailored to establish uniform attractivity for nonlinear time-varying systems, were introduced in [50, 51, 66, 99]. In the first two, links between persistence of excitation and detectability are established. In the latter two, necessary and sufficient conditions for uniform global asymptotic stability of generic nonlinear time-varying systems are given.

Beyond stability analysis, persistency of excitation plays a fundamental role in control design, as for instance, in systems in which the control input is multiplied by a time-varying function –see [61]. Such is the case of certain systems in aerospace engineering applications –see *e.g.*, [113], [4], and [69].

Persistency of excitation appears naturally in control design when there is a structural impediment to use autonomous smooth feedback, as in the case of chain-form systems [64, 108]. In [108], under a change of coordinates and a preliminary feedback, the closed-loop system is transformed into a so-called skew-symmetric system, roughly of the form $\dot{x} = Ax + Bu$ with $u \in \mathbb{R}$ where $A \in \mathbb{R}^{n\times n}$ is diagonal with only one element different from zero and $B \in \mathbb{R}^{n\times n}$ is skew-symmetric. Then, following the design rationale from [108], in [64] uniform global asymptotic stability was established for the closed-loop systems using controllers with persistency of excitation. Other control applications include the stabilization of parameterized time-varying systems [116] and the analysis and design of observers for bilinear systems [14, 123].

As we shall show here, persistency of excitation also provides a naturally relaxed condition for the solution to the so-called *consensus* problem [105] for systems with time-varying interconnections. In this scenario, stating conditions of persistency of excitation on the communication channels is particularly useful to characterize the "minimal reliability" of the channels [115]. In much of the existing literature, however, the study of consensus under time-varying communication links makes use of trajectory based approaches by means of a *non differentiable* Lyapunov functions to establish the contraction of trajectories. See for instance the seminal work of Moreau [79] in which the communication signals take a arbitrarily positive values. Similar problems are treated, for example, in [37] and [38] under relatively relaxed conditions on communication signals and on the graph topologies.

Furthermore, on top of stability and stabilisation one must also recognize the question of *performance.* Specifically, to determine explicit exponential estimates that relate the property of persistency of excitation to the overshoot and convergence rates. For the so-called "gradient" systems explicit bounds were independently provided in [16] and [63]. For more complex cases, such as that of model-reference adaptive control systems see [58]. It is to be noted, however, that the methods of proof in these references are rather intricate since they do not rely on the construction of strict Lyapunov functions.

In this first chapter we present the technical basis for the presentation of our contributions in the subsequent chapters. We broach several case-studies of stability analysis of time-varying systems:

- cascaded systems [97],
- consensus under spanning tree topology and time-varying communication links [105],
- model-reference adaptive control [88],
- stabilization of non-holonomic systems [108],
- systems with time-varying input gain [19, 61].

For all these case-studies we establish statements of uniform global exponential stability via Lyapunov's direct method. For each of these we give concrete examples in which our results are useful. From a technical viewpoint, the design of our Lyapunov functions is mostly inspired by [74] but we also use the results in [76] and [72], mainly for the strictification of Lyapunov functions with a non-positive persistently-exciting bounds on the time-derivatives.

Each of the case studies broached here is representative of a wide research area hence, we do not develop them in depth. In the subsequent chapters we present part of the work we accomplished in the period of this thesis (36 months). For clarity of exposition we chose to focus on problems of stabilization and formation control (consensus) of autonomous vehicles.

1.1 Case-study: a comparison positive system

We start with a simple statement that, in addition to setting the basis for our results, is interesting in its own right. Consider the differential equation

$$\dot{v} = -q(t)v, \qquad v \in \mathbb{R} \tag{1.1}$$

where $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Invoking standard results on adaptive control – see *e.g.*, [41], one may conclude that the origin is UGES if and only if $\sqrt{q}$ is continuous and persistently exciting, see Definition A.6, that is, if there exist T, $\mu > 0$ such that

$$\int_t^{t+T} q(s)ds > \mu \quad \forall\, t \geq 0. \tag{1.2}$$

Here, we establish the same result, by providing a strict Lyapunov function. The construction method for this and all other Lyapunov functions in this memoir is inspired from [72, 75]. It relies on a functional that is defined upon a locally Lipschitz bounded persistently exciting function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with bounded first derivative (a.e.), that is, we assume that there exists a constant $\bar{\psi} > 0$, such that

$$\max\left\{|\psi(t)|_\infty,\ |\dot{\psi}(t)|_\infty\right\} \leq \bar{\psi} \qquad \text{a.e.} \tag{1.3}$$

Then, we define the functional $\Upsilon : (\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \to \mathbb{R}$, such that, for all $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\Upsilon_\psi(t) := 1 + 2\bar{\psi}T - \frac{2}{T}\int_t^{t+T}\int_t^m \psi(s)ds\, dm \tag{1.4}$$

and, for further development, we remark that this function is lower and upper bounded, in particular,

$$1 \leq \Upsilon_\psi(t) < \bar{\Upsilon}_\psi := 1 + 2\bar{\psi}T. \tag{1.5}$$

Furthermore, after the fundamental theorem of calculus, the derivative of this function has the form

$$\dot{\Upsilon}_\psi(t) = -\frac{2}{T}\int_t^{t+T} \psi(s)ds + 2\psi(t) \tag{1.6}$$

then, using persistency of excitation of the signal $\sqrt{\psi(t)}$, we can upperbound the derivative of Υ_ψ as

$$\dot{\Upsilon}_\psi(t) \leq -\frac{2\mu}{T} + 2\psi(t). \tag{1.7}$$

Remark 1.1. *The function*

$$p(t) := -\frac{2}{T}\int_t^{t+T}\int_t^m \psi(s)ds\, dm \tag{1.8}$$

was first introduced in [74] under the equivalent form

$$p(t) = \int_t^{t+T} (s - t - T)\, q(s)ds, \tag{1.9}$$

which is obtained by simple change of the order of integration.

The following statement presents a strict Lyapunov function which establishes this, otherwise well-known, result –*cf.* [60].

Lemma 1.1. *Let* $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *be essentially bounded and let inequality* (1.2) *hold. Under these conditions, for the system* (1.1), *the function* $W : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, *defined by*

$$W(t, v) \;=\; \frac{1}{2}\Upsilon_q(t)v^2 \tag{1.10}$$

is a strict Lyapunov function and the origin $\{v = 0\}$ *is uniformly globally exponentially stable.*

Proof. Let $\bar{q}$ be such that $|q(t)| \leq \bar{q}$ for all $t \geq t_0$ and define $p_M := \bar{q}T$. defining the function $p(t)$ using (1.8), we obtain that $q(t) \geq 0$, $-p_M \leq p(t) \leq 0$, and $|p(t)| \leq p_M$ for all $t \geq 0$ hence, $W(t, v)$ can be bounded as

$$\frac{1}{2}v^2 \leq W(t,v) \leq \big[\frac{1}{2} + \bar{q}T\big]v^2. \tag{1.11}$$

The derivative of W along the trajectories of (1.1) yields

$$\dot{W}(t,v) = -\Big[q(t)\Upsilon_q(t) - \frac{\dot{\Upsilon}_p}{2}\Big]v^2,$$

then using (1.5) and (1.6) we obtain

$$\dot{W} \leq -\left[\frac{1}{T}\int_t^{t+T} q(s)ds\right]v^2 \qquad \forall t \geq 0 \tag{1.12}$$

and, in view of (1.2), we obtain that for all $t \geq t_0$ and $v \in \mathbb{R}$

$$\dot{W}(t,v) \leq -\frac{\mu}{T}v^2. \tag{1.13}$$

Finally, using (1.11), we also have

$$\dot{W}(t,v) \leq -\frac{2\mu}{(1+2\bar{q}T)T}W(t,v) \tag{1.14}$$

which, by integrating along the trajectories, yields

$$|v(t)| \leq \sqrt{1+2\bar{q}T}\,|v(t_\circ)|\exp\Big[-\frac{\mu(t-t_\circ)}{(1+2\bar{q}T)T}\Big] \quad \forall t \geq t_0. \tag{1.15}$$

□□□

Remark 1.2. *The requirement that* $q(t) \geq 0$ *is not necessary —see [60, Lemma 1], that is, under an extra condition on the excitation parameters* (T, μ)*, one can establish UGES of* (1.1) *under* (1.2) *without requiring* $q(t)$ *to be positive. For example, one possible way to derive the extra condition on the parameters* (T, μ) *is to decompose* $q(t)$ *as* $q(t) := q_1(t) + q_2(t)$ *where* $q_1(t) \geq 0$ *verifies* (1.9) *and* $q_2(t)$ *is bounded, then using the Lyapunov function provided in* (1.10)*, in which we replace* $q(t)$ *by* $q_1(t)$*, one can easily derive a sufficient condition, that relies the excitation parameters* (T, μ) *to the upper bounds of both* $q_1(t)$ *and* $q_2(t)$*, such that the time-derivative of* (1.10)*, along trajectories of* (1.1)*, is negative definite.*

The simplicity of Lemma 1.1 should not eclipse its utility in stability analysis. For instance, along with the comparison lemma [47, Lemma 2.5], it may be used to establish uniform global asymptotic stability, with guaranteed convergence rates, for certain nonlinear time-varying systems. To see this, consider the equation

$$\dot{z} = f(t, z) \tag{1.16}$$

and let $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be positive definite, proper and decrescent, that is, assume that there exist α_1, $\alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|z|) \leq V(t, z) \leq \alpha_2(|z|). \tag{1.17}$$

Assume, further, that there exists a globally Lipschitz continuous function $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, satisfying (1.2),

$$\dot{V}(t, z) \leq -q(t)V(t, z). \tag{1.18}$$

Then, let us define $v(t) := V(t, z(t))$, so that $\dot{v}(t) \leq -q(t)v(t)$ for all $t \geq 0$. In view of the monotonicity properties of V and the comparison lemma, Lemma 1.1 directly establishes UGAS of the origin, $\{z = 0\}$, with an explicit decay estimate. Indeed, from (1.15), (1.17) and the comparison Lemma, we obtain

$$|z(t)| \leq \alpha_1^{-1}\Big(k_v \alpha_2(|z_\circ|) e^{-\lambda_v (t - t_\circ)}\Big) \tag{1.19a}$$

$$\lambda_v := \frac{\mu}{k_v^2 T}, \qquad k_v := \sqrt{1 + 2\bar{q}T}. \tag{1.19b}$$

Example: Nonlinear observer design

To illustrate further the utility of Lemma 1.1, consider the problem of designing an observer for a bilinear system

$$\dot{x} \ = \ A(u, y)x + B(u, y) \tag{1.20a}$$

$$y \ = \ C(u, y)x. \tag{1.20b}$$

Since the system is linear in the unmeasured variable, we may proceed with a "Luenberger-like" design. To that end, let $\hat{x}$ denote the state estimate and let us define its dynamics through the equation

$$\dot{\hat{x}} = A(u, y)\hat{x} + B(u, y) - L(u, y)C(u, y)[\hat{x} - x] \tag{1.21}$$

where the observer gain, L, is to be designed in order to ensure that the origin of the estimation-errors system is UGES.

Proposition 1.1. *Consider the system* (1.20) *and the observer* (1.21). *Let L be continuous, and let u, y be such that there exist a continuously-differentiable function $P : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a continuous function $q_m : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and positive constants p_m, p_M, μ and T such that:*

(i) defining $\mathcal{A}(t) := A(u(t), y(t)) - L(u(t), y(t))C(u(t), y(t))$ and $Q(t) := -\dot{P}(t) - P(t)\mathcal{A}(t) - \mathcal{A}(t)^\top P(t)$, we have

$$Q(t) \geq q_m(t)I \geq 0 \quad \forall\, t \geq 0;$$

(ii) $\sqrt{q_m}$ is persistently exciting uniformly in $y(t)$ and $u(t)$ i.e., it satisfies (1.2) *with μ and T independent of the initial conditions*[1]*;*

(iii) the matrix $P(t)$ is uniformly positive definite and bounded, i.e.,

$$p_m I \leq P(t) \leq p_M I.$$

Then, the estimation errors $z(t)$ satisfy the bound

$$|z(t)| \leq k_v \sqrt{\frac{p_M}{p_m}} |z_\circ| e^{-\lambda_v (t - t_\circ)} \tag{1.22}$$

where k_v and λ_v are defined in (1.19b). □

Proof. Let the estimation errors be defined as $z := \hat{x} - x$ hence,

$$\dot{z} = \mathcal{A}(t)z. \tag{1.23}$$

Then, consider the function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $V(t, z) := z^\top P(t) z$. This function satisfies (1.17) with $\alpha_1(s) := p_m s^2$ and $\alpha_2(s) := p_M s^2$. Moreover, defining $q(t) := \frac{q_m(t)}{p_M}$, a direct computation shows that the time-derivative of V along the trajectories of (1.23) satisfies (1.18). Therefore, by Lemma 1.1, we see that

$$\mathcal{W}(t, z) := \frac{1}{2} \Upsilon_q(t) [z^\top P(t) z]^2$$

is a Lyapunov function for the estimation error dynamics (1.23) and, moreover, (1.19a) holds which, in this case, is equivalent to (1.22). □□□

The statement of Proposition 1.1 generalizes some results that rely on a uniform

[1]See [63] for details.

complete *observability* condition, *e.g.*, the choice:

$$\dot{P} = -\varepsilon P - \left[A(u,y)^\top P + PA(u,y)\right] + 2C^\top C \tag{1.24a}$$

$$L := P^{-1}C^\top, \qquad P(t_\circ) \geq p_m I, \tag{1.24b}$$

commonly used in observer design for bilinear systems –*cf.* [14], guarantees that $P(t)$, hence $Q(t) := \varepsilon P(t)$, is positive definite and bounded, for all $t \geq T$. The persistency of excitation condition on Q, imposed in Proposition 1.1, is less restrictive than positivity; moreover, the gain $L(t)$ as defined in (1.24b) may reach very high values [14]. Yet, the advantage of this choice is that it leads directly to an exponential-convergence estimate and provides a strict Lyapunov function for the estimation error-system. That is, this construction naturally lends itself for output-feedback high-gain designs, notably for systems with Lipschitz non-linearities –see *e.g.*, [2]. On the other hand, for such type of systems, notably chaotic oscillators, the main result in [68] provides an observer of the type of (1.21), under the less restrictive persistency of excitation condition on $Q(t)$. Thus, the statement of Proposition 1.1 covers all the previously mentioned results by providing an explicit stability bound under the weaker condition of persistency of excitation.

1.2 Case-study: Cascaded systems

We extend the result in Lemma 1.1 by establishing a statement of stability for linear cascaded systems that are persistently excited. We broach two case-studies: first, that of a chain of single integrators and, second, a more general case of multivariable systems.

1.2.1 Chain of single integrators

For the sake of illustration, let us start with the 2nd-order system:

$$\dot{x}_1 = -a_1(t)x_1 + a_{12}(t)x_2 \tag{1.25a}$$

$$\dot{x}_2 = -a_2(t)x_2 \tag{1.25b}$$

under the assumption that a_1, a_2 and a_{12} are continuous, uniformly bounded functions, and a_1, a_2 non negative having persistently exciting square root.

For this system, exponential stability of the origin $\{x_1 = x_2 = 0\}$ may be assessed following a direct cascades argument. Indeed, this follows, *e.g.*, from the results in [97] observing that, by Lemma 1.1, the respective origins of

$$\dot{x}_1 = -a_1(t)x_1 \qquad \dot{x}_2 = -a_2(t)x_2 \tag{1.26}$$

are UGES and $a_2(t)$ is bounded hence, the solutions $x_1(t)$ of equation (1.25a) are uniformly globally bounded. The statement also follows from the fact that (1.25a) is ISS with Lyapunov function $W(t, x_1)$ defined by (1.10) and input x_2. However, even though the cascades argument is straightforward for the case of two interconnected systems, the argument is hard to extend to cascades of $n > 2$ time-varying systems,

$$\Sigma_n' : \begin{cases} \dot{x}_1 = & -\, a_1(t)x_1 + a_{12}(t)x_2 \\ \dot{x}_2 = & -\, a_2(t)x_2 + a_{23}(t)x_3 \\ & \vdots \\ \dot{x}_{n-1} = & -\, a_{n-1}(t)x_{n-1} + a_{(n-1)n}(t)x_n \\ \dot{x}_n = & -\, a_n(t)x_n, \end{cases} \tag{1.27}$$

relying purely on converse Lyapunov theorems. Our next statement removes this difficulty by providing a strict Lyapunov function.

Theorem 1.1. *Consider the system* (1.27) *under the following hypotheses:*

Assumption 1.1. *(Non-negativity): $a_i(t) \geq 0$ for all $i \leq n$ and all $t \geq 0$.*

Assumption 1.2. *(Boundedness): There exists $\bar{a} > 0$ such that*

$$\max\left\{ |a_i(t)|, \left|a_{(i-1)i}(t)\right| \right\} \leq \bar{a}$$

for all $t \geq 0$ and all $i \leq n$.

Assumption 1.3. *(Persistency of Excitation): There exist μ, $T > 0$ such that*

$$\int_t^{t+T} a_i(s)ds > \mu \quad \forall i \leq n, \; \forall\, t \geq 0. \tag{1.28}$$

Then, defining $\beta_1 := 1$ and, for each $i \leq n$,

$$\beta_i := \frac{4\beta_{i-1}T^2}{\mu^2}\left[\,(1 + 2\bar{a}T)\,\bar{a}\right]^2, \quad \forall i \geq 2, \tag{1.29}$$

the function $V_n : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined as

$$V_n(t, x) := x^\top P(t) x \tag{1.30}$$

with

$$P(t) := \frac{1}{2} diag\left(\Upsilon_{a_i}(t)\beta_i\right),$$

is a strict Lyapunov function, and

$$\dot{V}_n(t,x) \le -\frac{\mu}{2T} x^T diag\left(\beta_i\right) x. \tag{1.31}$$

Consequently, the origin is uniformly globally exponentially stable.

The proof is reported in Appendix B.1.

Remark 1.3. *From the previous theorem it also follows that the trajectories of the system* (1.27) *satisfy*

$$|x(t)|^2 \le \alpha_M |x_\circ|^2 e^{-(\mu/2T\alpha_M)(t-t_\circ)} \qquad \forall t \ge t_\circ$$

where $\alpha_M := 1 + (2T + \beta_n)\bar{a}$*. To see this, we observe that the Lyapunov function* $V_n(t,x)$ *satisfies (since* $\beta_n > \beta_{n-1} > \cdots \beta_1 = 0$*)*

$$(1/2)\alpha_M |x|^2 \ge V_n(t,x) \ge (1/2)|x|^2.$$

Example: consensus under spanning tree

To illustrate the utility of the case studied in Theorem 1.1, we consider a classical tracking consensus problem concerning n agents interconnected in a spanning-tree topology with time-varying interconnection gains. In this case, each agent communicates only with two neighbors. Even though here we consider that each agent communicates always with the same neighbours, in general, this does not need to be the case –*cf.* [6]. We limit our case-study to this topology because in concrete cases of formation control, or follow-the-leader tracking control for that matter, using such communication topology excludes communication redundancy. This idea is pursued in Chapters 2-4 for the case of multiple nonholonomic mobile robots, where we are interested to solve the leader-follower problem under different configurations of the leader's velocities.

From a strictly theoretical viewpoint, however, our stability statement *per se* in this section is covered by, *e.g.*, [80]. On the other hand, as far as we know, we provide for the first time a *strict smooth* Lyapunov function which, in turn, allows to establish input-to-state stability (ISS) of the closed-loop system – see Appendix A.4.

Thus, let us consider n dynamical systems defined by

$$\dot{z}_i = f_i(t, z_i) + u_i, \qquad z_i \in \mathbb{R}^m, \; i \le n \tag{1.32}$$

which are required to follow a reference trajectory $z^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ generated by an exogenous system $\dot{z}^* := f^*(t, z^*)$. We assume that only the controller for the last (n-th) agent has access to the reference trajectory. Then, the ith agent receives information from the $i+1$st, thereby establishing a spanning-tree topology, albeit through unreliable channels.

To recast this consensus-tracking problem into a stabilization one we introduce the error system with state variables $x_i := z_i - z_{i+1}$ for all $i \leq n$, with $z_{n+1} := z^*$ and $f_{n+1} := f^*$. That is,

$$\dot{x}_i = f_i(t, x_i + z_{i+1}(t)) - f_{i+1}(t, z_{i+1}(t)) + u_i - u_{i+1} \tag{1.33a}$$

$$\dot{x}_n = f_n(t, x_n + z^*(t)) - f^*(t, z^*(t)) + u_n. \tag{1.33b}$$

With this change of coordinates, the consensus problem boils down to stabilizing the origin $\{x = 0\}$, with $x := [x_1, \cdots, x_n]^\top$, for the non-autonomous system (1.33). To do so, we use the control inputs

$$u_i := -\gamma a_i(t)[z_i - z_{i+1}] + w_i, \qquad a_i(t) \geq 0, \quad \forall\, t \geq 0 \tag{1.34}$$

where the functions a_i are assumed to be bounded and persistently exciting, $\gamma > 0$ is the interconnection strength, and w_i denote "additional" inputs to be defined. Then, the closed-loop system is

$$\dot{x}_i = -\gamma a_i(t)x_i + \gamma a_{i+1}(t)x_{i+1} + \psi_i(t, x_i) + v_i \tag{1.35a}$$

$$\dot{x}_n = -\gamma a_n(t)x_n + \psi_n(t, x_n) + v_n \tag{1.35b}$$

with $v_i := w_i - w_{i+1}$ and

$$\psi_i(t, x_i, z_{i+1}) := f_i(t, x_i + z_{i+1}(t)) - f_{i+1}(t, z_{i+1}(t)), \quad i \leq n. \tag{1.36}$$

Note that the system (1.35) may be regarded as a "perturbed" version of (1.27) hence, the following statement, which implies robust consensus-tracking of (1.32), follows as a corollary of Theorem 1.1.

Lemma 1.2. *Consider the system* (1.35) *under assumptions 1.1–1.3. For each $i \leq n$, let v_i be measurable functions, let $\psi_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m$ be such that there exist continuously-differentiable class $\mathcal{K}_\infty$ functions L_i such that*

$$|\psi_i(t, x_i)| \leq L_i(|x_i|). \tag{1.37}$$

Let R_i be such that for all $x_i \in B_{R_i}$, where $B_{R_i} := \{x_i \in \mathbb{R} : |x_i| \le R_i\}$,

$$\Big|\frac{\partial L_i}{\partial s}(|x_i|)\Big| \le \ell_i$$

and the interconnection strength γ is such that

$$\frac{\mu\gamma}{2T} > 2\ell_i\big[1 + \bar{a}2T\big].$$

Then, the system (1.35) *is input-to-state-stable from the input $v := [v_1, \cdots, v_n]^\top$, for all initial conditions $t_\circ \ge 0$ and $x_{i\circ} \in \mathbb{R}^n$ which produce complete trajectories satisfying $x_i(t, t_\circ, x_{i\circ}) \in B_{R_i}$.*

Sketch of proof:

Following the proof of Theorem 1.1 the Lyapunov function V_n defined in (1.30) satisfies

$$\dot{V}_n(t, x) \le -\sum_{i=1}^{n}\left[\frac{\mu\gamma}{2T} - \ell_i\big[1 + \bar{a}2T\big]\right]x_i^2 + \big[1 + \bar{a}2T\big]x_i v_i \tag{1.38}$$

for all $x_i \in B_{R_i}$. Then, we see that inequality $|v_i| \le \ell_i|x_i|$ imply that

$$\dot{V}_n(t, x) \le -\sum_{i=1}^{n}\left[\frac{\mu\gamma}{2T} - 2\ell_i\big[1 + \bar{a}2T\big]\right]x_i^2. \tag{1.39}$$

Then, it follows that the function V_n defined in (1.30) is an ISS Lyapunov function for all $x_i \in B_{R_i}$ and each $i \le n$. Hence, the system is input-to-state stable for all initial conditions $t_\circ \ge 0$, $x_{i\circ} \in B_{R_i}$ generating complete trajectories that satisfy $|x_i(t, t_\circ, x_{i\circ})| \le R_i$ for all $t \ge t_\circ \ge 0$ and all $i \le n$. ■

1.2.2 Multivariable cascaded linear time-varying systems

Let us consider now, the cascade of multivariable linear-time-varying persistently-excited systems

$$\begin{aligned}
\dot{x}_1 &= A_1(t)x_1 + B_1(t)x_2 \\
&\vdots \\
\dot{x}_{n-1} &= A_{n-1}(t)x_{n-1} + B_{n-1}(t)x_n \\
\dot{x}_n &= A_n(t)x_n, \qquad x_i \in \mathbb{R}^m,
\end{aligned} \tag{1.40}$$

where $B(t)$ and $A(t) : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n\times n}$ are continuously differentiable, and the following hypotheses hold:

Assumption 1.4. *(Boundedness) There exists $\bar{B} > 0$ such that $|B_i|_\infty \le \bar{B}$.*

Assumption 1.5. *(Lyapunov Equation) There exist positive definite matrices $P_i(t)$, positive semi-definite matrices $Q_i(t)$, positive constants P_{iM}, P_{im} and time-varying function $q_{im} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that; for all $t \geq 0$,*

$$P_{im} I_n \leq P_i(t) \leq P_{iM} I_n \tag{1.41}$$

$$0 \leq q_{im}(t) I_n \leq Q_i(t) \tag{1.42}$$

$$\dot{P}_i + A_i^\top P_i + P_i A_i = -Q_i(t) \tag{1.43}$$

Assumption 1.6. *(Persistency of excitation) There exists a positive constants μ, T, such that:*

$$\int_t^{t+T} q_{im}(s) ds > \mu \qquad \forall t \geq 0. \tag{1.44}$$

This type of systems generalizes that of the single chain of integrators presented previously. We have the following.

Theorem 1.2. *Under assumptions 1.4, 1.5 and 1.6 there exists a quadratic strict differentiable Lyapunov function for* (1.40).

Proof. For each $i \leq n$, let us define $V_i(t, x) = x_i^\top P_i(t) x_i$. The derivative of each V_i along the trajectories of (1.40), satisfies

$$\begin{aligned}
\dot{V}_1 \leq & - x_1^\top Q_1(t) x_1 + 2 x_1^\top P_1 B_1(t) x_2 \\
& \vdots \\
\dot{V}_{n-1} \leq & - x_{n-1}^\top Q_{n-1}(t) x_{n-1} + 2 x_{n-1}^\top P_{n-1} B_{n-1}(t) x_n \\
\dot{V}_n \leq & - x_n^\top Q_n(t) x_n.
\end{aligned} \tag{1.45}$$

Then, consider the modified Lyapunov function $W_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{nm} \to \mathbb{R}_{\geq 0}$ defined by $W_i(t, x) = \left(\Upsilon_{q_{im}}(t) + 2P_{iM}\right) V_i(t, x)$. Using (1.6) and (1.5) we obtain

$$\begin{aligned}
\dot{W}_1 \leq & - \frac{2\mu}{T} x_1^\top P_1 x_1 + 2 q_{1m}(t) x_1^\top P_1 x_1 - 2 P_{1M} x_1^\top Q_1(t) x_1 + 2\left(\Upsilon_{q_{1m}}(t) + 2P_{1M}\right) x_1^\top P_1 B_1 x_2 \\
& \vdots \\
\dot{W}_{n-1} \leq & - \frac{2\mu}{T} x_{n-1}^\top P_{n-1} x_{n-1} + 2 q_{n-1m}(t) x_{n-1}^\top P_{n-1} x_{n-1} - 2 P_{n-1M} x_{n-1}^\top Q_{n-1}(t) x_{n-1} \\
& + 2\left(\Upsilon_{q_{n-1m}}(t) + 2P_{n-1M}\right) x_{n-1}^\top P_{n-1} B_{n-1} x_n \\
\dot{W}_n \leq & - \frac{2\mu}{T} x_1^\top P_1 x_1 + 2 q_{1m}(t) x_1^\top P_1 x_1 - 2 P_{1M} x_1^\top Q_1(t) x_1
\end{aligned}$$

We define $\phi_i(t) := \Upsilon_{q_{im}}(t) + 2P_{iM}$, a nonsingular matrices $\nu_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ such that $P_i(t) = \nu_i(t)^\top \nu_i(t)$ and $M_i(t) = \phi_i(t) \nu_i(t) B_i \nu_{i+1}(t)^{-1}$. Then using Assumption 1.6, from

(1.45) we obtain that derivatives of the functions $W_i(t,x)$ can be bounded as

$$\begin{aligned}
\dot{W}_1 \leq& -\frac{2\mu}{T}\left|\nu_1 x_1\right|^2 + 2(\nu_1 x_1)^\top M_1(t)\nu_2(t)x_2\\
&\vdots\\
\dot{W}_{n-1} \leq& -\frac{2\mu}{T}\left|\nu_{n-1}^\top x_{n-1}\right|^2 + 2(\nu_{n-1}x_{n-1})^\top M_{n-1}(t)\nu_n(t)x_n\\
\dot{W}_n \leq& -\frac{2\mu}{T}\left|\nu_n^\top x_n\right|^2.
\end{aligned} \tag{1.46}$$

Using the inequality

$$2(\nu_i x_i)^\top M_i \nu_{i+1}x_{i+1} \leq \frac{\mu}{T}\left|\nu_i x_i\right|^2 + \frac{T}{\mu}\left|M_i\right|_\infty^2 \left|\nu_{i+1}x_{i+1}\right|^2$$

to estimate cross terms in (1.46), we obtain the following bounds for the derivatives

$$\begin{aligned}
\dot{W}_1 \leq& -\frac{\mu}{T}\left|\nu_1 x_1\right|^2 + \frac{T}{\mu}\left|M_1\right|_\infty^2\left|\nu_2 x_2\right|^2\\
&\vdots\\
\dot{W}_{n-1} \leq& -\frac{\mu}{T}\left|\nu_{n-1}^\top x_{n-1}\right|^2 + \frac{T}{\mu}\left|M_{n-1}\right|_\infty^2\left|\nu_n x_n\right|^2\\
\dot{W}_n \leq& -\frac{\mu}{T}\left|\nu_n^\top x_n\right|^2.
\end{aligned} \tag{1.47}$$

Finally, the strict Lyapunov function for the system (1.40) is given by

$$\mathcal{W}(t,x) = \sum_{i=1}^{n}\beta_i W_i(t,x_i)$$

where $\beta_1 = 1$ and $\beta_{i+1} = \frac{2T^2}{\mu^2}\left|M_i\right|_\infty^2 \beta_i$, while its derivative satisfies the inequality

$$\dot{\mathcal{W}}(t,x) = -\frac{\mu}{T}\left|\nu_1 x_1\right|^2 - \sum_{i=2}^{n}\frac{T}{\mu}\beta_{i-1}\left|M_{i-1}\right|_\infty\left|\nu_i x_i\right|^2.$$

□□□

Example: master-slave synchronization

In order to illustrate the use of Theorem 1.2, let us consider the following case-study of consensus-tracking control of Lagrangian systems,

$$D_i(q_i)\ddot{q}_i + C_i(q_i,\dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \qquad \tau_i,\ q_i \in \mathbb{R}^p. \tag{1.48}$$

The functions D_i, C_i and g_i are, respectively, the inertia matrix, the Coriolis matrix and the potential forces vector. The control torques are denoted by τ_i.

We consider the problem of tracking and mutual synchronization —see [89] in which all systems are required to follow a common exogenous trajectory $t \mapsto q^*$. Now, we assume that the systems are interconnected in a spanning-tree topology through unreliable links hence, on intervals of time the nodes may be isolated.

First, to each system we apply the preliminary linearizing feedback (this is possible because D is full rank) $\tau_i = D_i(q_i)u_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i)$ so that the equation of each node becomes $\ddot{q}_i = u_i$.

Then, emulating the unreliability of the communication channel by a square-pulse function $a : \mathbb{R}_{\geq 0} \to \{0, \bar{a}\}$ the control input becomes

$$u_i = a(t)[-k_1(q_i - q_{i+1}) - k_2(\dot{q}_i - \dot{q}_{i+1}) + \ddot{q}_{i+1}]$$

that is, the control is active only when $a(t) = \bar{a} > 0$.

Now, for each $i \leq n$, define $x_i := [q_i^\top \; \dot{q}_i^\top]^\top - [q_{i+1}^\top \; \dot{q}_{i+1}^\top]^\top$. We see that the error dynamics, in closed-loop, takes the form

$$\dot{x}_i = A_i(t)x_i + B_i(t)x_{i+1} + v_i(t), \quad i \leq n-1$$

where the perturbation v_i, which stems from the fact the "feedforward" term $\ddot{q}_{i+1}$ in u_i is not available all the time, is defined as $v_i(t) := [a(t)-1][\ddot{q}_{i+1}(t)-\ddot{q}_{i+2}(t)]$. Furthermore,

$$A_i(t) := \begin{bmatrix} 0 & 1 \\ -a(t)k_1 & -a(t)k_2 \end{bmatrix}, \quad B_i(t) = \begin{bmatrix} 0 \\ a(t) \end{bmatrix}$$

and, for $i = n$ we have $\dot{x}_n = A_n(t)x_n + v_n$ with $v_n(t) := [a(t) - 1]\ddot{q}^*(t)$.

By Theorem 1.2, for $v_i \equiv 0$, the origin is uniformly exponentially stable and admits a strict smooth Lyapunov function provided that Assumptions 1.4–1.6 hold. To verify these assumptions, we follow the second construction in [61] for double integrators with time-varying persistently-exciting input gain, $\ddot{x} = \alpha(t)u$, and define

$$a(t) := \frac{\alpha(t)}{\alpha(t) + \varepsilon}, \quad \varepsilon \in (0,1). \tag{1.49}$$

In the current example we used $k_1 = k_2 = 1$ for all agents but, in general, different gains may be used.

Following the reasoning proposed in [61], we decompose the matrices A_i $(i = \{1, ..., n-1\})$ as follows

$$A_i(t) = A_{i0} + \frac{\epsilon}{\epsilon + \alpha(t)} A_{i1}, \tag{1.50}$$

where

$$A_{i0} := \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \; A_{i1} := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

We choose $\alpha(t)$ as a positive periodic pulse function of period $T = 40$s, with a duty cycle of 70% and $\varepsilon = 0.01$. Hence, $a(t) \approx \alpha(t)$ is also positive and $\sqrt{a(t)}$ is persistently exciting –see the bottom plot in Figure 1.1, thus the assumptions 1.1–1.3 hold.

The "nominal" dynamics $\dot{x}_i = A_i(t)x_i$ has been studied in [61, Proposition 2]. In our case, if we take $Q_i := 0.16255\, I_2$, then there exists $P_i \in \mathbb{R}^2$ constant positive definite such that

$$A_{i0}^T P_i + P_i A_{i0} = -Q_i,$$

and

$$A_i^T(t) P_i + P_i A_i(t) = -q_{im}(t) I_2.$$

with $q_{im}(t) \geq 0$ and $q_{im}(t) \approx a(t)$. So we can see that Assumptions 1.5 and 1.6 hold for this particular choice of $\alpha(t)$. Using Theorem 1.2, with $v_i(t) \equiv 0$, we conclude UGES hence, formation tracking control of (1.48). Input-to-state stability with respect to the disturbance v_i also may be concluded. Simulation results are presented in Figure 1.1, for the case when all systems follow the reference trajectory $q^*(t) = \sin(t)$. The steady-state error depicted in the zoomed portion of the figure illustrates the ISS statement.

1.3 Case-study: spiraling systems

In this second part, we address the question of stability for linear time-varying systems of the general form

$$\dot{x} = [\mathcal{A}_o(t) + \mathcal{A}_s(t)]x, \quad x \in \mathbb{R}^n \tag{1.51}$$

where $\mathcal{A}_o$ and $\mathcal{A}_s$ are bounded differentiable mappings $\mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$. The model (1.51) has two essential constituting parts: the so-called *oscillating* drift $\mathcal{A}_o(t)x$ and the *steering* drift $\mathcal{A}_s(t)x$. In words, it is assumed that under the action of the former, the trajectories of (1.51) tend to oscillate while under the action of the latter, there exists a vanishing output $y := C(t)^\top x$. Under a detectability argument, provided by persistency of excitation, the trajectories tend to the origin while describing attenuated oscillations. Hence the name of *spiraling systems*.

To characterize the steering and oscillating properties of the system's dynamics we"' introduce the following assumption

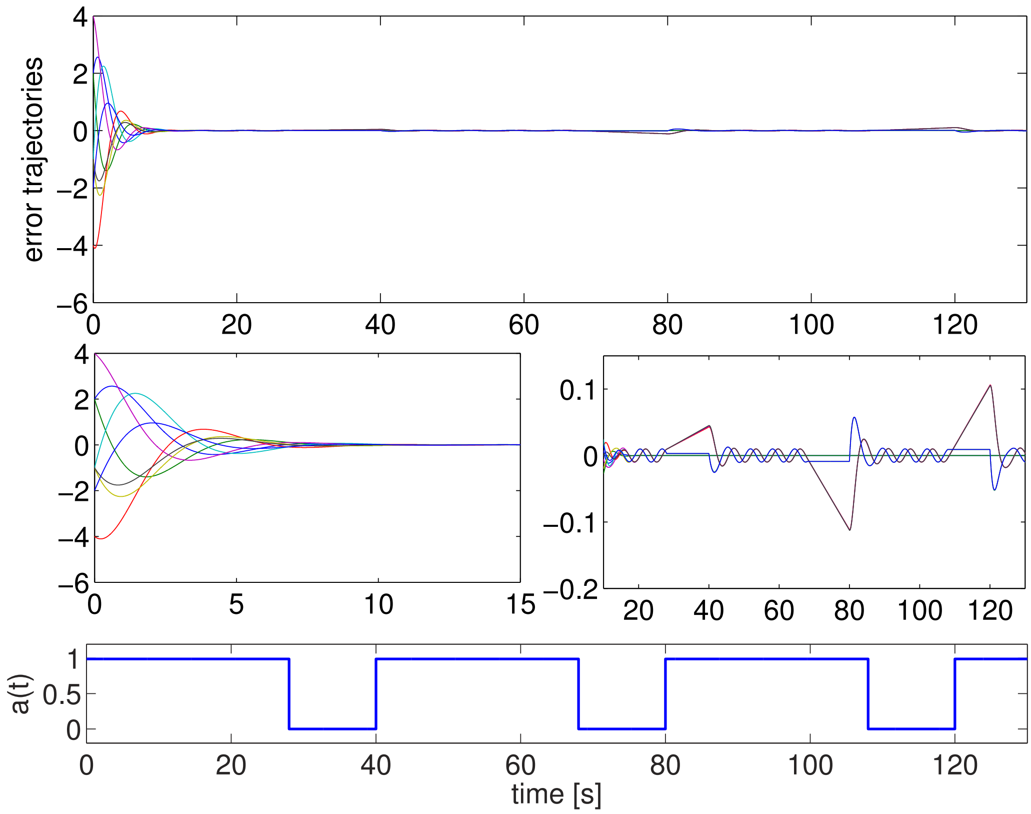

Figure 1.1: Mutual synchronization of four Lagrangian systems

Assumption 1.7. *1) There exist two bounded smooth functions P_s and Q_s taking values from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^{n\times n}$ such that, for all $t \geq 0$, $P_s(t)$ is symmetric positive definite, $Q_s(t)$ is symmetric positive semi-definite, $Q_s(t) \not\equiv 0$, and*

$$\mathcal{A}_s(t)^\top P_s(t) + P_s(t)\mathcal{A}_s(t) + \dot{P}_s(t) = -Q_s(t). \tag{1.52}$$

2) There exists a smooth bounded function $P_o : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ such that, for all $t \geq 0$, $P_o(t)$ is symmetric positive definite and

$$\mathcal{A}_o(t)^\top P_o(t) + P_o(t)\mathcal{A}_o(t) + \dot{P}_o(t) = 0. \tag{1.53}$$

Assumption 1.7 is fairly relaxed so it is not sufficient for exponential stability. The following counter-example illustrates this fact, while our results establish further conditions that correlate $\mathcal{A}_s$ and $\mathcal{A}_o$ to ensure exponential stability.

Example 1. Let a and b be piece-wise constant periodic functions taking non-negative values and persistently exciting, and let

$$\mathcal{A}_s(t) := \begin{bmatrix} -a(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_o(t) := \begin{bmatrix} 0 & -b(t) \\ b(t) & 0 \end{bmatrix}.$$

For each integer $n \geq 0$, let $J_n := (\pi/\bar{b})[2n+1,\, 2n+2]$. Then, for all $t \in J_n$, let $a(t) := \bar{a}$ and $b(t) := 0$ while for all $t \notin J_n$ we have $a(t) := 0$ and $b(t) := \bar{b}$. Hence, $a(t)b(t) \equiv 0$ and the trajectories generated by (1.51) satisfy the dynamics of a system whose dynamics switches between Σ_a and Σ_b, defined as

$$\Sigma_a \quad : \quad \begin{cases} \dot{x}_1(t) = -\,\bar{b}x_2(t) \\ \dot{x}_2(t) = \bar{b}x_1(t), \end{cases} \qquad \forall\, t \notin J_n$$

$$\Sigma_b \quad : \quad \begin{cases} \dot{x}_1(t) = -\,\bar{a}x_1(t) \\ \dot{x}_2(t) = 0 \end{cases} \qquad \forall\, t \in J_n.$$

This system satisfies Assumption 1.7 with $P_o = P_s = I_2$ and $Q_s(t) = \begin{bmatrix} a(t) & 0 \\ 0 & 0 \end{bmatrix}$ and yet, the analytic computation of its solutions, from the initial condition $(x_{1\circ}, x_{2\circ}) = (0, -1)$ shows that they do not converge. Indeed, for all $t \in [0, \pi/\bar{b}]$ the mode Σ_a is active, which yields to $(x_1(t), x_2(t)) = (\sin(\bar{b}t), -\cos(\bar{b}t))$ for all $t \in [0, \pi/\bar{b}]$. At $t_0 := \pi/\bar{b}$ the system switches to the mode Σ_b with the initial condition $(x_1(t_0), x_2(t_0)) = (0, 1)$, the trajectories remain constants for all $t \in J_0$. by induction we can see that for all $t_n := (\pi/\bar{b})(2n+1)$ (t_n the initial time of each sequence J_n) we have $(x_1(t_n), x_2(t_n)) = (0, \pm 1)$ and the trajectories remain constants along all the interval J_n. □

Thus, additional assumptions, relating properties of the matrices P_o and P_s, should be imposed to ensure exponential stability of the origin. Below we present two results that address two case studies of spiraling systems and we present some technical results that cover the state of the art in this topic.

1.3.1 Case-study: "adaptive control" systems

First, let us consider the case where matrix $\mathcal{A}_s(t)$ is constant, while the matrix $\mathcal{A}_o(t)$ is skew-symmetric. This type of systems appears in the analysis of adaptive control systems, for example, we recover the class of systems studied in [82]. If, in particular, $\mathcal{A}_o \equiv 0$ and $\mathcal{A}_s(t)$ is negative semidefinite we recover the systems studied in [83]. A particular case of the latter are "gradient-type" adaptive systems, defined as $\dot{x} = -\phi(t)\phi(t)^\top x$, for which it is well known that persistency of excitation of ϕ is necessary and sufficient for uniform exponential stability [83]. There are various distinct proofs of this fact in the literature –see *e.g.* [16,63]; as far as we know, the first strict Lyapunov function was provided recently in [20].

More generally, the system (1.51) also includes the familiar equation [41, 47, 88]

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\theta}} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{A}_s} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -B(t) \\ C(t) & 0 \end{bmatrix}}_{\mathcal{A}_o} \begin{bmatrix} e \\ \tilde{\theta} \end{bmatrix} \tag{1.54}$$

for which there exists $P = P^\top > 0$ such that $A^\top P + PA < 0$ (*i.e.*, A is Hurwitz) and $C(t) := B(t)^\top P$. In this case, Assumption 1.7 holds with $P_o = P_s := \text{diag}(P, I)$. For such systems, which appear in the context of model-reference adaptive control (in which case e represents tracking errors and $\tilde{\theta}$ estimation errors), it is well known that if in addition $B(t)$ is bounded with a bounded derivative, and $B(t)$ is also persistently exciting, the origin is uniformly exponentially stable.

Stability analysis for this adaptive control schame can be found in numerous textbooks and research monographs, see for instance [41, 47, 88]. However, the first strict Lyapunov function for model-reference adaptive control systems was provided only recently in [75] —see also [74]. More precisely, in this reference vectors B and C coincide, i.e. $B(t) = C(t)$ and depend both on time and the state, and $A := A(x_1)$ satisfies $x_1^\top A(x_1)x_1 \geq c|x_1|^2$ for some $c > 0$. Our first result (Theorem 1.3) provides a strict Lyapunov function for the case in which $\mathcal{A}_s$ in (1.54) is time-varying and satisfies Lyapunov equation (1.52) hence, we relax the uniform-positivity condition on A imposed [75]. The method consists in constructing a strict Lyapunov function starting from a non strict one that satisfies $\dot{V}(t, x) \leq -q(t)V(t, x)$ —*cf.* Section 1.1.

In particular, consider the system

$$\begin{cases} \dot{x}_1 = -A(t)x_1 - B(t)^\top x_2, & x_1 \in \mathbb{R}^n \\ \dot{x}_2 = C(t)x_1, & x_2 \in \mathbb{R}^m \end{cases} \tag{1.55}$$

where matrices $A(t)$ and $B(t)$ are uniformly bounded and have uniformly bounded derivatives (a.e.).

The following result not only ensures exponential stability of this system but also gives a strict Lyapunov function.

Theorem 1.3. *For the system* (1.55) *assume that* $B \in C^1$ *and there exists a positive definite matrix function* $P \in C^1$ *and positive semi-definite bounded matrix function* $Q \in C^1$*, such* *that*

$$P_m I \leq P(t) \leq P_M I \tag{1.56}$$
$$\dot{P}(t) - A(t)^\top P(t) - P(t)A(t) = -Q(t)$$
$$C(t) = B(t)P(t).$$

In addition, assume that the function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *defined by*

$$\psi(t) := \lambda_m(Q(t))\sqrt{\lambda_m(B(t)B(t)^\top)},$$

where λ_m *denotes the smallest eigenvalue, is persistently exciting and satisfies* (1.3). *Then, the null solution of* (1.55) *is uniformly exponentially stable and the system admits the strict Lyapunov function*

$$V(t,x) \ = \lambda_m^2(Q(t))x_1 B(t)^\top x_2 + \frac{1}{2}\Big[\Upsilon_{\psi^2}(t) + \alpha\Big]\left[x_1^\top P(t)x_1 + |x_2|^2\right]$$

with

$$\begin{aligned}\alpha \ \geq \ & (2T/\mu)\lambda_m^3(Q)|\dot{B}|_\infty^2 + (8T/\mu)\lambda_m(Q)\dot{\lambda}_m^2(Q)\,|B|_\infty^2 + (2T/\mu)\lambda_m^3(Q)|A^\top B^\top|_\infty^2 \\ & + \lambda_m^2(Q)|B^\top|_\infty(1+1/P_m) + 2\lambda_m(Q)\lambda_m(BB^\top)P_M + 2\lambda_m(Q)\left|B^\top C\right|_\infty. \end{aligned} \tag{1.57}$$

Indeed, we have

$$\dot{V}(t,x) \leq -(\mu/4T)\big[x_1^\top P x_1 + |x_2|^2\big].$$

Proof. In view of (1.5), the boundedness of B, Q, and P, as well as (1.57), V is positive definite and radially unbounded. Indeed,

$$\begin{aligned} V(t,x) \ & \geq \ \frac{(\alpha+1)}{2}\left[\lambda_m(P)\,|x_1|^2 + |x_2|^2\right] - \frac{1}{2}\lambda_m^2(Q(t))\,|B(t)|_\infty\left[|x_1|^2 + |x_2|^2\right] \\ & \geq \ \frac{1}{2}\left[\lambda_m(P)\,|x_1|^2 + |x_2|^2\right] \end{aligned} \tag{1.58}$$

and

$$V(t,x) \leq \left[\bar{\Upsilon}_{\psi^2} + \alpha\right]\left[\lambda_m(P)\,|x_1|^2 + |x_2|^2\right] + \lambda_m^2(Q(t))\,|B|_\infty\left[|x_1|^2 + |x_2|^2\right] \tag{1.59}$$

we conclude using (1.57) that there exist $\eta_1, \eta_2 > 0$ such that

$$\eta_1\,|x|^2 \leq V(t,x) \leq \eta_2\,|x|^2\,, \quad x = [x_1^\top\, x_2^\top]^\top.$$

The time-derivative of V along trajectories of (1.55) yields

$$\begin{aligned}
\dot{V}(t,x) &= 2\lambda_m(Q)\dot{\lambda}_m(Q)x_1^\top B^\top x_2 - \lambda_m^2(Q)x_1^\top A^\top B^\top x_2 - \lambda_m^2(Q)x_2^\top BB^\top x_2 \\
&\quad +\lambda_m^2(Q)x_1^\top B^\top Cx_1 + \lambda_m^2(Q)x_1^\top \dot{B}^\top x_2 - \frac{\alpha}{2}x_1^\top Qx_1 \\
&\quad -\frac{\mu}{T}\left[x_1^\top Px_1 + |x_2|^2\right] + \psi^2\left[x_1^\top Px_1 + |x_2|^2\right] \\
&\leq -\frac{\alpha}{2}\lambda_m(Q)\,|x_1|^2 + \lambda_m^2(Q)\lambda_m(BB^\top)x_1^\top Px_1 + \lambda_m^2(Q)x_1^\top B^\top Cx_1 \\
&\quad -\lambda_m^2(Q)x_1^\top A^\top B^\top x_2 + 2\lambda_m(Q)\dot{\lambda}_m(Q)x_1^\top B^\top x_2 + \lambda_m^2(Q)x_1^\top \dot{B}^\top x_2 \\
&\quad -\frac{\mu}{T}\,|x_2|^2 - \frac{\mu}{T}x_1^\top Px_1. \qquad (1.60)
\end{aligned}$$

Then, we use the inequalities

$$\lambda_m^2(Q)x_1^\top \dot{B}^\top x_2 \leq \frac{\epsilon}{2}\lambda_m^4(Q)|\dot{B}|_\infty^2\,|x_1|^2 + \frac{1}{2\epsilon}\,|x_2|^2\,,$$

$$2\lambda_m(Q)\dot{\lambda}_m(Q)x_1^\top B^\top x_2 \leq 2\epsilon\lambda_m^2(Q)\dot{\lambda}_m^2(Q)\,|B|_\infty^2\,|x_1|^2 + \frac{1}{2\epsilon}\,|x_2|^2\,,$$

$$\lambda_m^2(Q)x_1^\top A^\top B^\top x_2 \leq \frac{\epsilon}{2}\lambda_m^4(Q)\left|A^\top B^\top\right|_\infty^2|x_1|^2 + \frac{1}{2\epsilon}\,|x_2|^2\,,$$

which hold for any $\epsilon > 0$. Hence, setting $\epsilon = 2T/\mu$ and in view of (1.57), it follows that $\dot{V} \leq -(\mu/4T)|x_2|^2 - (\mu/T)x_1^\top Px_1$. □□□

In the particular case of planar systems, i.e. x_1, $x_2 \in \mathbb{R}$, Theorem 1.3 reduces to the following statement, which plays a key role in robustness analysis of the closed-loop systems considered in the next chapters.

Corollary 1.1. *Consider the system*

$$\begin{cases} \dot{x}_1 = -\,a(t)x_1 - b(t)x_2, & a(t) \geq 0 \\ \dot{x}_2 = b(t)x_1, \end{cases} \qquad (1.61)$$

Then, provided that a and b satisfy (1.3) *with $\bar{a}$ and $\bar{b}$ respectively. Assume, in addition, that $\psi := ab$ is persistently exciting. Then, the function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, defined as*

$$V(t,x) = a(t)^2 b(t)x_1x_2 + \frac{1}{2}\Big[\Upsilon_{a^2b^2}(t) + \alpha\Big][x_1^2 + x_2^2]$$

with

$$\alpha \geq 2\bar{a}\bar{b}^2 \left[1 + \frac{\bar{a}^2 T}{4\mu}(3+\bar{a})^2\right] \tag{1.62}$$

satisfies

$$\dot{V}(t,x) \leq -(\mu/2T)\left[x_1^2 + x_2^2\right].$$

Example: Master-slave synchronization

In order to illustrate the utility of Theorem 1.3, we consider a simple example treating the master-slave synchronization problem for two harmonic oscillators, the slave system $\dot{z} = A(t)z + Bu$ and the master system $\dot{z}^* = A(t)z^*$, where

$$A(t) := \begin{bmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

That is, both oscillators spin at the same variable frequency $\omega(t)$, but out of phase. Then, the problem consists in ensuring that $z(t) \to z^*(t)$ exponentially fast under the assumption that the oscillators are linked through an unreliable channel.

To solve this problem the control law is designed so that the closed-loop system has the structure given by equation (1.51): a steering drift and an oscillating drift. The latter is natural to the harmonic oscillators while the former may be added through the simple static output feedback $u = -a(t)[z_1 - z_1^*]$. Indeed, note that the closed-loop system has exactly the form (1.61) with $x_1 := z_1 - z_1^*$, $x_2 := z_2 - z_2^*$ and $b(t) := \omega(t)$. We conclude that phase-lock synchronization is achieved provided that a and ω are bounded, have bounded derivatives, and their product is persistently exciting.

Remark 1.4. *Note that the closed-loop system in this case is similar to that in Example 1 with $b(t) = \omega(t)$. Hence, we conclude that persistency of excitation of $a(t)$, which ensures the steering of x_1 to zero, and that of $\omega(t)$, which contributes to propagate the stabilization effect of $a(t)$, does not suffice* alone *to ensure the attractivity of the origin. For the stabilization effect to be properly propagated from one coordinate to another it is required that the product of $a(t)\omega(t)$ is persistently exciting.*

1.3.2 Case-study: "skew-symmetric" systems

The case study addressed in this final section is motivated by stabilization problems where non-autonomous feedback are imposed by the control problem. These include: leader-follower tracking control [53], stabilization of non-holonomic systems [64, 108], stabilization of systems with time-varying input gain [61, 113].

We consider a particular class of systems defined by the following ordinary differential equation

$$\dot{x} = -b(t)^2 BB^\top x + a(t)Ax \tag{1.63}$$

where matrix $A \in \mathbb{R}^{n\times n}$ is neutrally stable, and matrix $A_o(t) = a(t)A$ satisfies (1.53), that is, there exists matrix $P_o \in \mathbb{R}^{n\times n}$, constant positive definite, such that

$$A_o(t)^T P_o + P_o A_o(t) = 0,$$

the pair (A, B) is controllable, and both $a(t)$ and $b(t)$ are scalar functions defined on $\mathbb{R}_{\geq 0}$, such that the product $a(t)^2 b(t)$ is persistently exciting. It is easy to see that under the imposed conditions, Assumption 1.7 holds with $\mathcal{A}_s(t) := -b(t)^2 BB^\top$ and $P_s = I_n$.

Equation (1.63) intersects with the class of systems studied in [82] and covers the class of systems studied in [19], where uniform global exponential stability is ensured for the particular case that $a(t) \equiv 1$ and A is skew-symmetric. More significantly, in the latter reference the proof is trajectory-based whereas here, we give a strict Lyapunov function (Theorem 1.4).

Notice that the so-called skew-symmetric systems [108] represent a particular case of the system (1.63). Indeed, the seminal work [108] on stabilization of nonholonomic systems in chain form:

$$\begin{aligned}
\dot{z}_1 &= u_2 \\
\dot{z}_i &= u_1 z_{i-1} \\
\dot{z}_m &= u_1,
\end{aligned}$$

where $z_i \in \mathbb{R}$, shows that using a suitable smooth global change of coordinates $z \mapsto x$ and a preliminary feedback $u_2(t, z)$ in the new coordinates, the system may be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{m-1} \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 u_1 & \cdots & 0 \\ u_1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -k_{m-1}u_1 \\ 0 & \cdots & u_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{bmatrix} \tag{1.64}$$

$$\dot{x}_m = u_1. \tag{1.65}$$

The term *skew-symmetric* was introduced in [108] motivated by the fact that the matrix in (1.64) may be written as the sum of $\mathcal{A}_s$:=diag$[-k_1\, 0 \cdots 0]$, which satisfies (1.52) with $P_s = I_m$, and a neutrally stable ("skew-symmetric") matrix $\mathcal{A}_o$ that satisfies (1.53) with

P_o :=diag $[1,\ k_2,\ k_2k_3,\ \cdots,\ \prod_{i=2}^{m-1} k_i]$. Alternatively, (1.64) falls in the model (1.63) with $B = [1,\ 0 \cdots 0]$, $b \equiv \sqrt{k_1}$, and $a(t) := u_1(t, z(t))$. Now, following the rationale of [108] where non-uniform global asymptotic stability is proved, in [64] it is shown that under certain persistency of excitation assumption on the control input u_1, the origin of the system (1.64) is uniformly globally asymptotically stable. The proof is based on [63] and exploits the equation (1.64) as a linear-time-varying system obtained by replacing the nonlinear function $u_1(t, z)$ with a parametrized, by initial conditions, time signal $u_1(t, z(t))$ —see the discussion in the Introduction. However, such proof is very involved as it appeals to a recursive output-injection argument. Our approach allows to construct a strict Lyapunov function for this system and provide a direct proof.

Theorem 1.4. *Consider the system* (1.63). *Let us assume that the functions $a(\cdot)$, $b(\cdot)$ and their derivatives are bounded, i.e., there exists $\bar{a}$ and $\bar{b}$ such that* (1.3) *holds, and $\psi(t) := a(t)b(t)$ is persistently exciting. In addition, assume that the pair (A, B) is controllable and that there exist a constant positive definite matrix $P = P^\top \in \mathbb{R}^{n\times n}$, such that:*

$$p_m I_n \le P \le p_M I_n \tag{1.66a}$$

$$A^\top P + PA = 0, \tag{1.66b}$$

$$PBB^\top = BB^\top P := CC^\top. \tag{1.66c}$$

Define $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ as

$$V(t, x) \ := \ \frac{1}{2}\left[\gamma + \Upsilon_{a^4b^2}(t)\right] x^\top Px + b(t)^2 a(t)^3 x^\top PA \sum_{i=1}^{n} \beta_i \Gamma_i Px \tag{1.67}$$

where $\gamma := \gamma_1 + \gamma_2$,

$$\gamma_1 \ := \ \frac{T\bar{b}^6\bar{a}^6}{2\mu P_m}\left|\sum_{i=1}^{n}\beta_i C^\top \sum_{j=1}^{i}\left[A\Gamma_i P - \Gamma_i PA\right]\right|^2 \tag{1.68}$$

$$\gamma_2 \ := \ \frac{T\bar{a}^4}{\mu P_m}\sum_{i=1}^{n}\beta_i \left|M^\top PC\right|_\infty^2 + \bar{b}^2\bar{a}^3\left|P^{1/2}A\sum_{i=1}^{n}\beta_i\Gamma_i P^{1/2}\right|, \tag{1.69}$$

$$\Gamma_i \ := \ \sum_{j=1}^{i} A^{j-1}BB^\top A^{j-1\top}. \tag{1.70}$$

Under controlability of (A, B), the matrix $P^{1/2}A\Gamma_n A^\top P^{1/2}$ is non singular, and we take $\beta_n I \ge [P^{1/2}A\Gamma_n A^\top P^{1/2}]^{-1}$ and constants β_i are defined in reverse order, i.e., for each $i \in \{n -$

$1, \cdots, 1\}$,

$$\beta_i \geq \frac{2nT}{\mu P_m} \left| \left[PA^iB \right]^\top M \right|^2 \left[\sum_{k=i+1}^{n} \beta_k \right]^2 - \sum_{k=i+1}^{n-1} \beta_k \tag{1.71}$$

where $M = \left[2\dot{b}a + 3b\dot{a}\right]A + ba^2A^2$. *Then, the function* V *is a strict differentiable Lyapunov function for the system* (1.63) *and its origin is uniformly exponentially stable.*

The proof is reported in Appendix B.2.

Remark 1.5. *The strict Lyapunov function provided in Theorem 1.4 serves the corestone for the analysis approach proposed in Chapter 2. the construction of Lyapunov function for a skew-symmetric nonlinear time-varying systems in Proposition 2.1, and to establish some robustness results with respect to external perturbations.*

The following proposition extends Corollary 1.1 and, to some extent, Theorem 1.4 to a case of "skew-symmetric" systems, defined by (1.51), with

$$\mathcal{A}_s := \text{diag}(-a_1(t)^2 \;\; 0 \cdots 0), \tag{1.72a}$$

$$\mathcal{A}_o(t) := \begin{bmatrix} 0 & -a_2(t) & 0 & \cdots & 0 \\ a_2(t) & 0 & -a_3(t) & 0 & \vdots \\ 0 & a_3(t) & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -a_n(t) \\ 0 & \cdots & 0 & a_n(t) & 0 \end{bmatrix}. \tag{1.72b}$$

The following statement also generalizes [63, Theorem 2] and provides a direct proof for it, as opposed to the recursive output-injection argument used in this reference.

Proposition 1.2. *Let the functions* $a_i(t)$ *satisfy the bound* (1.3) *with* $\bar{a}$. *Assume in addition that the function* $\psi := \Pi_{i=1}^{n} a_i(t)$, *is persistently exciting, i.e., there exists* T *and* $\mu > 0$, *such that,*

$$\int_t^{t+T} \psi^2(s)ds \geq \mu > 0, \qquad \forall t \geq 0.$$

For each $i \in [2, n]$ *define* $\bar{x}_i := [x_1 \cdots x_i]^\top$ *and*

$$\Phi_i(t, \bar{x}_i) = \sum_{j=2}^{i} \left[a_j \prod_{\substack{k\,=\,2 \\ k\,\neq\,j}}^{i} a_k^2 \right] x_{j-1}x_j, \quad i \in [2, n].$$

Then, provided that α_i, *for* $i = n$ *down to* $i = 2$, *and* γ *satisfy the following:*

$$\alpha_n = 1, \qquad \alpha_{n-1} \;\; = \;\; \bar{a} + \frac{4n(n-1)^2\bar{a}^{2n}T}{\mu} \tag{1.73}$$

$$\alpha_i = \bar{a}^{n-i} + \frac{nT}{\mu}\left[\left[\sum_{j=i+1}^{n}\bar{a}^{2j}\alpha_j\right]\left[\sum_{j=i+1}^{n}\alpha_j\bar{a}^{2(j-i)}\right] + \left[\sum_{j=i+1}^{n}\alpha_j 2i\bar{a}^{(2j-1-i)}\right]^2\right] \quad (1.74)$$

$$\gamma \geq \frac{T}{\mu}(\bar{a}^2+1)\left[\left(\sum_{i=2}^{n}\alpha_i\bar{a}^{2i-1}\right)^2 + \left(\sum_{i=2}^{n}2i\alpha_i\bar{a}^{2i-1}\right)^2\right] + \sum_{i=2}^{n}\alpha_i\bar{a}^{2i-1}$$
$$+\frac{2T\bar{a}}{\mu}\left[\sum_{i=2}^{n}\alpha_i\bar{a}^{2i-1}\right]^2. \quad (1.75)$$

We have that the function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as

$$V(t,x) = \left[\frac{1}{2}\Upsilon_{\psi^2}(t) + \gamma\right]|x|^2 + a_1^2\left[\sum_{i=2}^{n}\alpha_i\Phi_i(t,\bar{x}_i)\right]$$

is a strict differentiable Lyapunov function for (1.51), (1.72), *and its derivative satisfies*

$$\dot{V}(t,x) \leq -\frac{\mu}{2T}|x|^2.$$

Consequently, the origin is uniformly globally exponentially stable. □

The proof of the latter statement is presented in Appendix B.3.

Example: Control of underactuated ships

To illustrate the utility of Theorem 1.4, we briefly consider the tracking control problem for underactuated ships that is solved in [53] under the assumption that the reference trajectories are persistently exciting. For the purpose of this chapter, we remark that the closed-loop system in this reference has the cascaded form

$$\dot{x}_1 = \left[\mathcal{A}_s + \mathcal{A}_o(t)\right]x_1 + G(t,x_1,x_2)x_2, \qquad x_1 \in \mathbb{R}^4 \quad (1.76)$$
$$\dot{x}_2 = Fx_2, \quad x_2 \in \mathbb{R}^2, \quad (1.77)$$

where $F \in \mathbb{R}^{2\times 2}$ is a Hurwitz constant matrix, $\mathcal{A}_s \in \mathbb{R}^{4\times 4}$ is a diagonal constant matrix with two negative elements and two zero elements, matrix $G(\cdot)$ has linear growth in x_1 and $\mathcal{A}_o(t)$ depends on the reference trajectories and satisfies the second part of Assumption 1.7 –see [53] for details. Following standard arguments for cascaded systems it is possible to establish uniform global asymptotic stability of the origin, provided that the same property holds for the nominal system $\dot{x}_1 = \left[\mathcal{A}_s(t) + \mathcal{A}_o(t)\right]x_1$. In [53] this is established under the assumption of persistency of excitation of the reference velocity along with uniform-complete-observability and output-injection arguments. Theorem 1.4 not only delivers a strict Lyapunov function to ensure exponential stability for the nominal x_1–dynamics but it also constitutes a fundamental step to carry on a robustness analysis with respect to unmodelled perturbations.

1.4 Conclusion

We have presented original strict Lyapunov functions for uniform exponential stability of linear time-varying systems with persistency of excitation that appear in a variety of problems including adaptive control systems, state estimation of bilinear systems, consensus with persistently-exciting interconnections, master-slave synchronization, *etc.* The utility of our theoretical findings is briefly demonstrated through concise but representative examples of meaningful control problems.

In the succeeding chapters we present a deeper analysis of a particular area: that of consensus and formation control of mobile robots, using controllers with persistency of excitation. Although many of our controllers are reminiscent of others that have appeared in the literature, our contributions lie in the establishment of strong properties such as uniform global asymptotic stability, (integral) intput-to-state stability and, most remarkably, in the construction of original Lyapunov functions for most of the control problems that we solve.

We believe that the construction of strict Lyapunov functions for *nonlinear* time-varying systems with structures as those investigated here may lead to a range of open problems in stability and control theory. Notably, the problem of establishing robustness properties (Input-to-output stability) is a well-motivated avenue of research for which our statements might be a starting point.

Chapter 2

Leader-follower formation control of nonholonomic vehicles

The landmark paper [46] introduced a follow-the-leader control approach for nonholonomic mobile robots which translates a robotics problem into a standard stabilization problem for time-varying systems. The approach consists in defining a virtual robot that generates a reference trajectory that is supposed to be followed by the controlled robot. In other words, the problem boils down to stabilizing the origin of the error dynamics between the reference and the actual robot's coordinates. This problem has been studied extensively in the literature; moreover, it naturally blends into the more general framework of leader-follower formation control. In this case, a swarm of robots is required to follow each other, thereby creating a “chain” of leaders and followers. From a graph theory view point, they compose what is known as a spanning tree.

Following the ideas from [46] and based on the technical tools illustrated in the previous chapter, we study the formation control problems in a variety of ways. Depending on the velocities of the virtual robot, that we shall denote by v_r (forward velocity) and ω_r (angular velocity), we distinguish the following:

Problem 2.1 (*Tracking*)**.** *It is assumed that the virtual reference vehicle describes a path with a time schedule that defines generic continuous reference functions v_r and ω_r — see Section 2.3.*

Problem 2.2 (*Stabilization*)**.** *It is assumed that the leader vehicle is static hence, $v_r \equiv \omega_r \equiv 0$.*

Problem 2.3 (*Parking*)**.** *It is assumed that the velocities of the virtual reference vehicle are "fastly" vanishing. Strictly speaking, the velocities (v_r, ω_r) are assumed to be integrable. This problem is considered in Chapter 3.*

Problem 2.4 (*Robust stabilization*)**.** *This is a generalization of the parking problem above. As in the previous case, velocities of the virtual vehicle are assumed to be vanishing. However, in contrast with parking problem, here we do not impose restriction on the speed of convergence of* (v_r, ω_r) *to zero, that is the assumption on the integrability of the leader's velocities is not imposed in this problem —see Section 2.5.*

The *robust stabilization* and the *parking* problems are particular scenarios of the general *tracking* problem, even if, technically speaking, their study is based on the study of *stabilization* problem.

Problem 2.5 (*Simultaneous tracking-stabilization*)**.** *In this case, it is required to design a universal controller which addresses both, the tracking and the parking problems —see Chapter 3.*

The generic leader-follower problem has been addressed in hundreds of articles since the early 1990s via a range of controllers and under distinct restrictions on the reference velocities. For example, in [107] the control design relies on the condition that at least one of the leader's velocities does not converge; in [95] simple linear time-varying controllers are given for which it is established that persistency of excitation of the reference angular velocity is necessary and sufficient for uniform exponential stabilization; in [21] where the translational leader's velocity is assumed to be greater than zero.

The *stabilization problem* has also been thoroughly studied; the motivation in the community, triggered by the famous Brockett's necessary condition which is not satisfied by non-holonomic systems. This implies that the system is not stabilizable via smooth static feedback. For example, in [10, 101] discontinuous controllers are provided, a time-varying continuous controllers are proposed in [84], and a smooth time-varying in [108] and in [64, 67]. In the latter, uniform global asymptotic stability is established.

In the case of the *parking* and the *robust stabilization problems* additional technical difficulties appear from the fact that reference velocities converge to zero hence, many of the schemes tailored for the generic tracking control problem fail in this case. Under the assumption that the reference trajectories converge fast enough (they are integrable) this problem was solved, for instance in —see [52], [27, 119].

It is important to stress that, most often, the constraints on the reference velocities impose a certain control design and, therefore, influence the statements that one can establish. Hence, it is clear that the simultaneous tracking stabilization control problem is the most challenging of all and, as far as we know, has only been treated in [27,52,85,119]

Furthermore, the problems previously described may also be posed to the scenario of *formation control*, in which a swarm of robots must advance in a coordinated manner, as a single robot. Hence, the problems above have their natural counterparts in the multi-agent framework. In order to solve the general formation tracking control problem for a multiple non-holonomic mobile robots, two main approaches exist in the literature, the virtual-structure and the leader-follower approach.

The virtual structure approach consists on defining a virtual formation moving along a desired path, and then controlling each robot to reach its corresponding position on the virtual structure [39]. This approach removes the hierarchy between agents in comparison with the leader-follower approach when the leader is not virtual, and allows some robustness of the formation. In [117] a virtual structure approach is adopted and a distributed coupling among agents is introduced in order to increase robustness of the formation.

The leader-follower approach has the advantage of allowing simpler controllers that are easily implementable. A comparison between the two methods is in [103]. In [118] a distributed virtual leader-follower formation tracking control problem is considered under a force-controlled model and parameter uncertainty. In [29] leader-follower formation tracking control problem is considered, for a general framework of nonholonomic systems in chained form, under the assumption of persistence of excitation on the rotational reference velocity, this solution has been extended in [30] to provide a *distributed* solution to the same problem.

In [28] the leader-follower formation tracking control problem is solved using a combination of the virtual structure approach in order to generate the reference trajectories for each agent, then an output feedback control law is designed in order to track each agent toward its reference trajectory. This work has been extended in [25], where the problem with collision avoidance is considered. Under the assumption that the robot is modeled as a point-mass (second-order integrators), time-varying formation configurations are considered in [114].

In this Chapter we solve leader-follower formation control problem under the configurations of the leader's velocities described in Problems 2.1 and 2.4. Some of our controllers are similar to what is proposed in the literature or inspired from it, but our technical hypotheses are relaxed. For instance, for the *tracking control* problem we assume that the sum of squares of the leader's velocities (v_r, ω_r) is persistently exciting. At the same time, we propose an original design for the *robust stabilization control* problem, in this scenario, no restrictions are imposed on the convergence rate to zero of the reference velocities, and still we obtain some strong robustness results.

In the case of formation control, we use a distributed approach and assume that the communications graph that contains both the leader and the followers, consists in a spanning tree. That is, each robot communicates only with one neighbouring "father" and transmits its coordinates to one or several neighbouring "children". While this may appear restrictive from a technical viewpoint, from the robotics viewpoint it has the clear advantage of reducing the number of sensors needed, the amount of processed data and is more natural.

To the best of our knowledge, some of our contributions were open questions. Such is the case, for instance, of the leader-follower *robust agreement control* problem, that is solved in this Chapter.

The *simultaneous tracking and robust agreement control* problem, which covers all scenarios, is another open problem that we have solved, but this is presented in Chapter 3.

In addition, most of our proofs rely on the construction of strict Lyapunov functions which, moreover, are used to establish statements of robustness in the (integral) input-to-state stability sense. All these are original contributions of this thesis.

2.1 Problem formulation

We start by introducing the dynamic model of a mobile robot, that we use here and in next chapter. That is, we consider force-controlled autonomous vehicles modelled by the equations

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{v} &= f_1(t,v,\omega,q) + g_1(t,v,\omega,q)u_1 \\ \dot{\omega} &= f_2(t,v,\omega,q) + g_2(t,v,\omega,q)u_2 \end{cases} \tag{2.2}$$

The variables v and ω denote the forward and angular velocities respectively, the first two elements of $q := [x\ y\ \theta]^\top$ correspond to the Cartesian coordinates of a point on the robot with respect to a fixed reference frame, and θ denotes the robot's orientation with respect to the same frame. The two control inputs are the torques u_1, u_2.

The Equations (2.1) correspond to the kinematic model while (2.2) correspond to the force-balance equations. The latter may take various forms, such as the Euler-Lagrange equations [35]; see also [26] in the context of mobile robots. In this memoir, we leave these equations undefined since our controllers are generic.

Generally speaking, the control strategy consists in designing virtual control laws at the kinematics level, i.e., considering v and ω as control inputs. Then, we design u_1 and u_2 to steer v and ω toward the ideal control laws v^* and ω^*. That is, if $v = v^*$ and $\omega = \omega^*$, the origin of the closed-loop system, for the kinematics equations is uniformly globally asymptotically stable. Moreover, for (2.1), we establish robustness statements in the sense of input-to-state stability hence, our statements are valid for *any* controller that guarantees the stabilization of the origin at the force level —Equations (2.2). Thus, except for the example provided in Section 2.2, we leave Equations (2.2) in generic form.

2.1.1 Single follower case

For clarity of exposition, we start by describing the most elementary scenario, that of leader-follower tracking control, as defined in [46]. Such problem consists in making the robot to follow a fictitious reference vehicle modeled by

$$\dot{x}_r = v_r \cos\theta_r \tag{2.3a}$$

$$\dot{y}_r = v_r \sin\theta_r \tag{2.3b}$$

$$\dot{\theta}_r = \omega_r, \tag{2.3c}$$

and which moves about with reference velocities $v_r(t)$ and $\omega_r(t)$.

More precisely, it is desired to steer the differences between the Cartesian coordinates to some values d_x, d_y, and to zero the orientation angles and the velocities of the two robots, that is, the quantities

$$p_\theta = \theta_r - \theta, \quad p_x = x_r - x - d_x, \quad p_y = y_r - y - d_y.$$

The distances d_x, d_y define the position of the robot with respect to the (virtual) leader. In general, these may be functions that depend on time and the state or may be assumed to be constant, depending on the desired path to be followed. In our study, we consider these distances to be defined as piece-wise constant functions –*cf.* [62].

Then, as it is customary, we transform the error coordinates $[p_\theta,\ p_x,\ p_y]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot, that is, we define

$$\begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_\theta \\ p_x \\ p_y \end{bmatrix}. \tag{2.4}$$

In these new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\begin{aligned} \dot{e}_\theta &= \omega_r(t) - \omega && (2.5a)\\ \dot{e}_x &= \omega e_y - v + v_r(t)\cos(e_\theta) && (2.5b)\\ \dot{e}_y &= -\omega e_x + v_r(t)\sin(e_\theta) && (2.5c) \end{aligned}$$

which is to be completed with Eqs (2.2).

Hence, the control problem reduces to steering the trajectories of (2.5) to zero via the inputs u_1 and u_2 in (2.2), i.e.,$\lim_{t\to\infty} e(t) = 0$. As we mentioned, a natural method consists in designing virtual control laws at the kinematic level, that is, w^* and v^*, and control inputs u_1 and u_2, depending on the latter, such that the origin $(e, \tilde{v}, \tilde{w}) = (0, 0, 0)$ with

$$\tilde{v} := v - v^*, \quad \tilde{\omega} := \omega - \omega^*, \quad e = [e_\theta \; e_x \; e_y]^\top, \tag{2.6}$$

is uniformly globally asymptotically stable.

2.1.2 Multiple followers case

The previous setting naturally extends to the case in which a swarm of n robots is required to follow a virtual leader, advancing in formation. This may be achieved in a variety of manners. Here, we assume that the ith robot follows a leader, indexed $i-1$, thereby forming a spanning-tree graph communication topology.

The geometry of the formation may be defined via the relative distances between any pair of leader-follower robots, d_{xi}, d_{yi} and it is independent of the communications graph (two robots may communicate independently of their relative positions). Then, the relative position error dynamics is given by a set of equations similar to (2.5), that is,

$$\begin{aligned} \dot{e}_{\theta i} &= \omega_{i-1}(t) - \omega_i && (2.7a)\\ \dot{e}_{xi} &= \omega_i e_{yi} - v_i + v_{i-1}(t)\cos(e_{\theta i}) && (2.7b)\\ \dot{e}_{yi} &= -\omega_i e_{xi} + v_{i-1}(t)\sin(e_{\theta i}) && (2.7c) \end{aligned}$$

For $i = 1$ we recover the error dynamics for the case of one robot following a virtual leader that is, by definition, $v_0 := v_r$ and $\omega_0 := \omega_r$. Then, we introduce the virtual controls (v_i^*, ω_i^*) depending on the type of problem under study, or more precisely, on the configuration of leader's velocities —see Problems 2.1-2.5 described on p. 52.

The velocities (v_i^*, ω_i^*) serve as references for the actual controls u_{1i} and u_{2i} in

$$\dot{v}_i = f_{1i}(t, v_i, \omega_i, e_i) + g_{1i}(t, v_i, \omega_i, e_i)u_{1i} \tag{2.8a}$$

$$\dot{\omega}_i = f_{2i}(t, v_i, \omega_i, e_i) + g_{2i}(t, v_i, \omega_i, e_i)u_{2i}, \qquad i \leq n \tag{2.8b}$$

whence, the velocity errors

$$\tilde{\omega}_i := \omega_i - \omega_i^*, \quad \tilde{v}_i := v_i - v_i^*.$$

As in the case of one follower, it is required to stabilize the origin of the closed-loop system. In particular, it is required that for all $i \leq n$,

$$\lim_{t\to\infty} e_i(t) = 0. \tag{2.9}$$

Remark 2.1. *Solving such a problem under a general directed graph remains an interesting open question. In Chapter 4, however, we solve the problem under general bi-directional graph and time-varying delay but only when the leader's velocities are equal to zero. In fact, due to the non-holonomic restriction a natural extension of the existing works on consensus problem of first and second order systems, [81, 102], to a multiple non-holonomic mobile robots case is not possible.*

2.2 Example of torque controller

Our contributions consists in controllers and stability proofs that concern the kinematics equation (2.7). We establish robustness statements with respect to converging (fastly) errors $\tilde{v}$ and $\tilde{\omega}$. In this section, we present an example of, an otherwise standard, control design at the force level. As we shall see, this is only one example of a force controller that may be used with our kinematics' controllers proposed in this and next chapters.

Consider the following model of wheeled mobile robots –*cf.* [25],

$$\dot{q}_i = J(q_i)\nu_i \tag{2.10a}$$

$$M\dot{\nu}_i + C(\dot{q}_i)\nu_i = \tau_i \tag{2.10b}$$

where τ_i is the torque control input; the variable $\nu_i := [\nu_{1i}\ \nu_{2i}]$ denotes the angular velocities of the two wheels, M is an inertia matrix (hence positive definite, symmetric),

C is the matrix of Coriolis forces (which is skew-symmetric), and

$$J(q_i) = \frac{r}{2} \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \\ 1/b & -1/b \end{bmatrix}$$

where r and b are positive constant parameters of the system. The relation between the wheels' velocities, ν_i, and the robot's velocities in the fixed frame, $\dot{q}_i$, is given by

$$\begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \frac{r}{2b} \begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \nu_{1i} \\ \nu_{2i} \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \nu_{1i} \\ \nu_{2i} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \tag{2.11}$$

which may be used in (2.10a) to obtain the model (2.1), (2.2) with

$$\begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} = \frac{r}{2b} \begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix} M^{-1}\tau_i$$

—see [25] for more details on this coordinate transformation.

Then, using (2.11), for any given virtual control inputs v_i^* and ω_i^*, we can compute $\nu_i^* := [\nu_{1i}^* \ \nu_{2i}^*]^\top$ and define the torque control input

$$\tau_i = M\dot{\nu}_i^* + C(J(q_i)\nu_i)\nu_i^* + D\nu_i^* - k_d\tilde{\nu}_i, \quad k_d > 0$$

where $\tilde{\nu}_i := \nu_i - \nu_i^*$. We see that the force error equations yields

$$M\dot{\tilde{\nu}}_i + \big[C(\dot{q}_i(t)) + D + k_d I\big]\tilde{\nu}_i = 0 \tag{2.12}$$

in which we have replaced $\dot{q}_i$ with the trajectories $\dot{q}_i(t)$ to regard this system as (linear) time-varying, with state $\tilde{\nu}_i$. Now, due to the skew-symmetry of $C(\cdot)$ the total derivative of

$$V(\tilde{\nu}_i) = \frac{1}{2}\tilde{\nu}_i^T M\tilde{\nu}_i,$$

along the trajectories of (2.12) yields

$$\dot{V}(\tilde{\nu}_i) \leq -k_d|\tilde{\nu}_i|^2. \tag{2.13}$$

Although this inequality holds independently of $\dot{q}_i(t)$, Eq. (2.12) is valid only on the interval of existence of $\dot{q}_i(t)$, denoted $[t_\circ, t^{\max})$, $t^{\max} \leq \infty$. Hence, so does (2.13) and, consequently,

$$|\tilde{\nu}_i(t)| \leq \kappa|\tilde{\nu}_i(t_\circ)|e^{-\lambda(t-t_\circ)} \qquad \forall\, t \in [t_\circ, t^{\max}) \tag{2.14}$$

for some κ and $\lambda > 0$. From (2.11) it is clear that a similar bound holds for $\eta_i(t) = [\tilde{v}_i(t)\; \tilde{\omega}_i(t)]$. In other words, the velocity errors tend exponentially to zero uniformly in the initial conditions and in the position error trajectories.

We assume that the inertia parameters and the constants contained in $C(\dot{q}_i)$ are unknown while r and b are considered to be known. Let $\hat{M}$ and $\hat{C}$ denote, respectively, the estimates of M and C. Furthermore, using,

$$\begin{bmatrix} \nu_{1i}^* \\ \nu_{2i}^* \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v_i^* \\ \omega_i^* \end{bmatrix}, \tag{2.15}$$

let us introduce the certainty-equivalence control law

$$\tau_i^* := \hat{M}\dot{\nu}_i^* + \hat{C}(\dot{q}_i)\nu_i^* - k_d\tilde{\nu}_i, \quad k_d > 0 \tag{2.16}$$

Then, let us define $\tilde{M} := \hat{M} - M$ and $\tilde{C} := \hat{C} - C$, so

$$\tau_i^* := M\dot{\nu}_i^* + C(\dot{q}_i)\nu_i^* - k_d\tilde{\nu}_i + \tilde{M}\dot{\nu}_i^* + \tilde{C}\nu_i^* \tag{2.17}$$

and, setting $\tau_i = \tau_i^*$ in (2.10b), we obtain the closed-loop equation

$$M\dot{\tilde{\nu}}_i + [C(\dot{q}_i) + k_d I]\tilde{\nu}_i = \Psi(\dot{q}_i, \dot{\nu}_i^*, \nu_i^*)^\top\tilde{\Theta}_i \tag{2.18}$$

where $\Theta_i \in \mathbb{R}^m$ is a vector of constant (unknown) lumped parameters in M and C, $\hat{\Theta}_i$ denotes the estimate of Θ_i, $\tilde{\Theta}_i := \hat{\Theta}_i - \Theta_i$ is the vector of estimation errors, and $\Psi : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{m\times 2}$ is a continuous known function. For this, we used the property that (2.10b) is linear in the constant lumped parameters. In addition, we use the passivity-based adaptation law –*cf.* [94],

$$\dot{\hat{\Theta}}_i = -\gamma\Psi(\dot{q}_i, \dot{\nu}_i^*, \nu_i^*)\tilde{\nu}_i, \qquad \gamma > 0. \tag{2.19}$$

Then, a direct computation shows that the total derivative of

$$V(\tilde{\nu}_i, \tilde{\Theta}_i) := \frac{1}{2}\big[|\tilde{\nu}_i|^2 + \frac{1}{\gamma}|\tilde{\Theta}_i|^2\big]$$

along the trajectories of (2.18), (2.19), yields

$$\dot{V}(\tilde{\nu}_i, \tilde{\Theta}_i) \leq -k_d|\tilde{\nu}_i|^2.$$

Integrating the latter to infinity we obtain that $\tilde{\nu} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\tilde{\Theta}_i \in \mathcal{L}_\infty$. It follows, *e.g.*, from [41, Lemma 3.2.5], that $\tilde{\nu}_i \to 0$ and, in view of (2.11),

$$\lim_{t\to\infty} |\tilde{v}_i(t)| + |\tilde{\omega}_i(t)| = 0. \tag{2.20}$$

As it may be appreciated, the property that the velocity tracking errors converge, *i.e.*, (2.20) is fairly weak. Nevertheless, it is established under the realistic conditions that the parameteres are unknown. Furthermore, the weakness of this property only makes the significance of our next statements stronger; we show that *all* our controllers are robust to the inputs $\tilde{v}$ and $\tilde{\omega} \to 0$. In a few cases, however, it is imposed that $\tilde{v} \in \mathcal{L}_2$ which is also established above.

2.3 Leader-follower tracking

We address now the *tracking control* goal as described in Problem 2.1 under the following relaxed assumption —*cf.* [34, 44, 45]

Assumption 2.1. *there exist positive numbers μ and T such that*

$$\int_t^{t+T} [\omega_r(s)^2 + v_r(s)^2] ds \geq \mu \quad \forall\, t \geq 0. \tag{2.21}$$

In [23], the authors proposed the controller

$$v^* \;:=\; v_r(t)\cos(e_\theta) + k_x e_x \tag{2.22a}$$

$$\omega^* \;:=\; \omega_r(t) + k_\theta e_\theta + v_r(t) k_y e_y \phi(e_\theta) \tag{2.22b}$$

where ϕ is the so-called 'sync' function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta} \tag{2.23}$$

and establish (non-uniform) convergence of the tracking errors under the assumption that the some of square of the leader's velocities converge to a non null value. In this chapter, for the same controller but under slightly relaxed conditions which is stated in term of persistency of excitation in Assumption 2.1, we establish uniform global asymptotic stability for the closed-loop system and for the first time, we provide a strict Lyapunov function.

The design of the controller (2.22), under Assumption 2.1, is motivated by the resulting structure of the error dynamics for the tracking errors, which is reminiscent of nonlinear adaptive control systems. Indeed, by setting $\omega = \omega^*$ and $v = v^*$, we obtain

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \underbrace{\begin{bmatrix} -k_\theta & 0 & -v_r(t)k_y\phi(e_\theta) \\ 0 & -k_x & \omega^*(t,e) \\ v_r(t)\phi(e_\theta) & -\omega^*(t,e) & 0 \end{bmatrix}}_{A_{v_r}(t,e)} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} \tag{2.24}$$

which has the structure of (1.54) except that, here, the "regressor" function $B(\cdot)$ depends on time and the state, as is generally the case in model-reference-adaptive control systems [48].

We obtain the crucial property that the trivial solution for this system is uniformly globally stable (it is uniformly stable and all solutions are uniformly globally bounded). To see this, note that the total derivative of $V_1 : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$, defined as

$$V_1(e) = \frac{1}{2}\left[e_x^2 + e_y^2 + \frac{1}{k_y}e_\theta^2\right] \tag{2.25}$$

corresponds to

$$\dot{V}_1(e) = -k_x e_x^2 - k_\theta e_\theta^2 \leq 0. \tag{2.26}$$

Furthermore, after [99], it may be concluded that the origin of this system is uniformly globally asymptotically stable provided that the vector $[-v_r(t)k_y\phi(e_\theta) \quad \omega^*(t,e)]$, subject to $e_\theta = 0$, is δ-persistently exciting with respect to e_y —see Appendix A.7. Roughly, this holds provided that this vector is persistently exciting for any $e_y \neq 0$; condition which, actually, reduces to (2.21). Thus, our first statement is the following.

Proposition 2.1 (Kinematic model)**.** *For the system* (2.24) *assume that Assumption 2.1 holds and there exist $\bar{\omega}_r$, $\bar{\dot{\omega}}_r$, $\bar{\nu}$, $\bar{\dot{\nu}} > 0$ such that*

$$|\omega_r|_\infty \leq \bar{\omega}_r, \quad |\dot{\omega}_r|_\infty \leq \bar{\dot{\omega}}_r, \quad |v_r|_\infty \leq \bar{v}_r, \quad |\dot{v}_r|_\infty \leq \bar{\dot{v}}_r. \tag{2.27}$$

Then, the origin is uniformly globally asymptotically stable and locally exponentially stable, for any positive values of the control gains k_x, k_y, and k_θ. Moreover, there exists a positive definite radially unbounded function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as the functional

$$V(t,e) := P_{[3]}(t,V_1)V_1(e) - \omega_r(t)e_x e_y + v_r(t)P_{[1]}(t,V_1)e_\theta e_y \tag{2.28}$$

where $P_{[k]} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function such that $P_{[k]}(\cdot, V_1)$ is uniformly bounded and $P_{[k]}(t, \cdot)$ is a polynomial of degree k with non-negative coefficients. In addition, $P_{[k]}(t, \cdot)$ has the property that yields the total derivative of V along the trajectories of (2.24) *satisfying*

$$\dot{V}(t, e) \leq -\frac{\mu}{T} V_1(e) - k_x e_x^2 - k_\theta e_\theta^2. \tag{2.29}$$

□

The contribution of Proposition 2.1 lies in its original proof which is based on Lyapunov's direct method and follows the method of construction proposed in Subsection 1.3.2 of Chapter 1. Next, we sketch the main proof steps that lead to the design of $V(t, e)$ in (2.28).

Sketch of the proof. Firstly, for any locally integrable function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\sup_{t \geq 0} |\varphi(t)| \leq \bar{\varphi}$, let us introduce

$$\Upsilon_\varphi(t) := 1 + 2\bar{\varphi}T - \frac{2}{T} \int_t^{t+T} \int_t^m \varphi(s) ds dm \tag{2.30}$$

–cf (1.4). Note that $\Upsilon_\varphi(t)$ has been introduced in (1.4) and satisfies:

$$\begin{gathered} \dot{\Upsilon}_\varphi(t) = -\frac{2}{T} \int_t^{t+T} \varphi(s) ds + 2\varphi(t), \\ 1 \leq \Upsilon_\varphi(t) < \bar{\Upsilon}_\varphi := 1 + 2\bar{\varphi}T \end{gathered} \tag{2.31}$$

In the sequel, we use this function with $\varphi = v_r^2 + \omega_r^2$. We also introduce several polynomial functions with positive coefficients, denoted by $\rho_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. These shall be defined as needed in a manner that the derivative of

$$\begin{aligned} V_2(t, e) \quad := \quad & \rho_1(V_1)V_1 + \left[\Upsilon_{v_r^2}(t) + \Upsilon_{\omega_r^2}(t)\right] V_1 - \omega_r(t) e_x e_y \\ & + v_r \rho_2(V_1) e_\theta e_y + \rho_3(V_1) V_1, \end{aligned} \tag{2.32}$$

with V_1 defined in (2.25), be negative definite. In addition, note that

$$V_2(t, e) \geq \frac{1}{2} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix}^\top \begin{bmatrix} \rho_3(V_1)/k_y & v_r \rho_2(V_1) & 0 \\ v_r \rho_2(V_1) & \rho_3(V_1) & -\omega_r \\ 0 & -\omega_r & \rho_3(V_1) \end{bmatrix} \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix}$$

so V_2 is positive definite and radially unbounded if the matrix in this inequality is positive semidefinite. The latter holds if ρ_3 satisfies

$$\rho_3(V_1) \geq 2\sqrt{k_y \bar{v}_r^2 \rho_2(V_1)^2 + \bar{\omega}_r^2}.$$

Finally, we introduce

$$V(t,e) = V_2(t,e) + V_1\rho_4(V_1) \tag{2.33}$$

which is also positive definite. We shall show that for an appropriate choice of the polynomials ρ_i, the total derivative of V along the trajectories of (2.24) yields

$$\dot{V}(t,e) \leq -\frac{\mu}{T}V_1(e) - k_x e_x^2 - k_\theta e_\theta^2, \quad \forall\, (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \tag{2.34}$$

To that end, we rewrite (2.24) in the output-injection form

$$\dot{e} \;=\; A^\circ_{v_r}(t,e)e + v_r[\phi(e_\theta) - 1]B^\circ(e_y)e \tag{2.35}$$

$$A^\circ_{v_r}(t,e) \;:=\; \begin{bmatrix} -k_\theta & 0 & -v_r k_y \\ 0 & -k_x & \varpi^\circ_{v_r} \\ v_r & -\varpi^\circ_{v_r} & 0 \end{bmatrix} \tag{2.36}$$

$$B^\circ(e) \;:=\; \begin{bmatrix} 0 & 0 & -k_y \\ 0 & 0 & k_y e_y \\ 1 & -k_y e_y & 0 \end{bmatrix} \tag{2.37}$$

$$\varpi^\circ_{v_r}(t,e) \;=\; \omega_r(t) + k_\theta e_\theta + v_r k_y e_y \tag{2.38}$$

This partition, which facilitates the analysis, is motivated by the fact that $v_r[\phi(e_\theta) - 1]B^\circ(e_y)e = 0$ if $e_\theta = 0$.

First, we establish that V_2 is a LF for $\dot{e} = A^\circ_{v_r}(t,e)e$. Then, we evaluate $\dot{V}$ including the output injection term $v_r[\phi(e_\theta) - 1]B^\circ(e_y)e$. See Appendix B.4 for a detailed development. ■

The value of having a strict Lyapunov function for (2.24) may not be overestimated. Notably, this allows to carry on with a robustness analysis vis-a-vis of the dynamics (2.2). For example, in order to solve the tracking control problem for (2.1), (2.2), using Proposition 2.2 below, it is only left to design u_1 and u_2 such that, given the references v^* and ω^*, the origin of the closed-loop dynamics

$$\dot{\tilde{v}} \;=\; f_{1cl}(t,\tilde{v},\tilde{\omega},e) \tag{2.39a}$$

$$\dot{\tilde{\omega}} \;=\; f_{2cl}(t,\tilde{v},\tilde{\omega},e) \tag{2.39b}$$

is globally exponentially stable uniformly in the initial conditions and in e. In Section 2.2 we presented an example of an effective force controller. However, in general, the design of the control inputs u_1 and u_2 depends on the problem setting and is beyond the scope of this thesis.

We rather emphasize that the overall error dynamics takes the convenient form

$$\begin{aligned} \dot{e} &= A_{v_r}(t,e)e + B(e)\eta, && (2.40a) \\ \dot{\eta} &= F_{cl}(t,\eta,e), \quad F_{cl} := [f_{1cl}\ f_{2cl}], && (2.40b) \end{aligned}$$

where

$$B(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}, \quad \eta := \begin{bmatrix} \tilde{v} \\ \tilde{\omega} \end{bmatrix}. \tag{2.41}$$

It is worth stressing that, based on the structure of the Lyapunov function in (2.28), one can also establish that the system in (2.40a) is integral-input-to-state stable with respect to η —see Definition A.4.

Proposition 2.2. *Consider the system* (2.40a) *with k_x, k_y, and k_θ arbitrary positive gains; assume, moreover, that the references satisfy Assumption 2.1 and* (2.27). *Then, the system* (2.40a) *is integral input-to-state stable with respect to the "input" η.* □

The proof of the last statement is reported in Appendix B.5.

Now, for the purpose of analysis, we replace the state e with the trajectories $e(t,\zeta(t_\circ))$ in (2.40b) so the closed-loop equations may be regarded as a cascaded nonlinear time-varying system with state $\zeta := [e^\top\ \eta^\top]^\top$. More precisely, in place of (2.40b) we write

$$\dot{\eta} = \tilde{F}_{cl}(t,\eta)$$

where $\tilde{F}_{cl}(t,\eta) = F_{cl}(t,\eta,e(t))$ –*cf.* [59]. Then, using arguments for cascaded systems from [96] we can establish the following proposition:

Proposition 2.3. *Consider the system* (2.40) *with initial conditions $(t_\circ,\zeta_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^5$. Assume that k_x, k_y, and k_θ are positive and that inequalities* (2.21) *and* (2.27) *hold. In addition, assume that the solutions are complete and the origin of* (2.40b) *is globally asymptotically stable, uniformly in the initial times $t_\circ \in \mathbb{R}_{\geq 0}$ and in the error trajectories $t \mapsto e$. Assume further that the trajectories $t \mapsto \eta$ are uniformly integrable, that is, there exists $\phi \in \mathcal{K}$ such that*

$$\int_{t_\circ}^{\infty} |\eta(\tau)| d\tau \leq \phi(|\zeta_\circ|) \quad \forall\, t \geq t_\circ \geq 0. \tag{2.42}$$

Then, the origin is uniformly globally asymptotically stable. □

Proof. From Proposition 2.1, the origin $\{e = 0\}$ is uniformly globally asymptotically stable for (2.24). By assumption the same property holds for (2.40b). Since, moreover, B is linear in e, the result follows from the main results in [96, Theorem 2]. □□□

Remark 2.2. *Technically, the function $\tilde{F}_{cl}$ is defined only on the interval of existence of $e(t)$, whence the assumption that the solutions exist on $[t_\circ, \infty)$. Nevertheless, this hypothesis may be dropped if we impose that $\eta \to 0$ uniformly in $e(t)$* only *on the interval of existence. This is considered in our main result later on —see Proposition 2.4.*

2.4 Leader-follower formation tracking control

We extend our previous results to the problem of multi-agent tracking control for a group of N robots modeled by (2.1) and (2.2). Similarly to the controller proposed previously, we define

$$v_i^* := v_{i-1}\cos(e_{\theta i}) + k_{xi}e_{xi} \tag{2.43}$$

$$\omega_i^* := \omega_{i-1} + k_{\theta i}e_{\theta_i} + v_{i-1}k_{yi}e_{yi}\phi(e_{\theta i}) \tag{2.44}$$

which serve as references for the actual controls u_{1i} and u_{2i} in (2.8). Next, we use the velocity errors

$$\tilde{\omega}_i := \omega_i - \omega_i^*, \quad \tilde{v}_i := v_i - v_i^*$$

and let us define $\Delta v_j := v_j - v_r$ and $\Delta\omega_j := \omega_j - \omega_r$ for all $j \leq n$ (by definition, $\Delta\omega_0 = \Delta v_0 = 0$). Then, we replace ω_i with $\tilde{\omega}_i + \omega_i^*$ and, respectively, v_i with $\tilde{v}_i + v_i^*$ in (2.7), and we use

$$v_i^* = [\Delta v_{i-1} + v_r]\cos(e_{\theta i}) + k_{xi}e_{xi} \tag{2.45}$$

$$\omega_i^* = \Delta\omega_{i-1} + \omega_r + k_{\theta i}e_{\theta_i} + [\Delta v_{i-1} + v_r]k_{yi}e_{yi}\phi(e_{\theta i}). \tag{2.46}$$

It follows that, for each pair of nodes, the error system takes the form

$$\dot{e}_i = A_{v_r}(t, e_i)e_i + G(t, e_i, \xi_i)e_i + B(e_i)\eta_i \tag{2.47}$$

–*cf.* (2.40a), where

$$\begin{aligned}
e_i &:= [e_{\theta_i}\, e_{x_i}\, e_{y_i}]^\top, \quad \eta_i := [\tilde{v}_i\, \tilde{\omega}_i]^\top \\
\xi_i &:= [\Delta\omega_{i-1}\; \Delta v_{i-1}]^\top \\
G &:= \begin{bmatrix} 0 & 0 & -k_y g_1 \\ 0 & 0 & g_2 \\ g_1 & -g_2 & 0 \end{bmatrix} \\
g_1 &:= \Delta v_{i-1} e_{y_i} \phi(e_{\theta_i}) \\
g_2 &:= \Delta\omega_{i-1} + k_y \Delta v_{i-1} e_{y_i} \phi(e_{\theta_i})
\end{aligned}$$

and B is defined in (2.41) —note that $G(t, e_i, 0) \equiv 0$. Thus, the overall closed-loop system has the convenient cascaded form (in reverse order):

$$\dot{e}_n = A_{v_r}(t, e_n)e_n + G(t, e_n, \xi_n)e_n + B(e_n)\eta_n \tag{2.48a}$$

$$\vdots$$

$$\dot{e}_2 = A_{v_r}(t, e_2)e_2 + G(t, e_2, \xi_2)e_2 + B(e_2)\eta_2 \tag{2.48b}$$

$$\dot{e}_1 = A_{v_r}(t, e_1)e_1 + B(e_1)\eta_1 \tag{2.48c}$$

and these closed-loop equations are complemented by the equations that stem from applying the actual control inputs in (2.8), that is,

$$\dot{\eta}_i = F_{i_{cl}}(t, \eta_i, e_i), \quad F_{i_{cl}} := [f_{i_{1cl}}\; f_{i_{2cl}}] \tag{2.49}$$

for all $i \leq n$.

To underline the good structural properties of the system (2.48)–(2.49) and to explain the rationale of our result, let us argue as follows. By assumption, the control inputs u_{1i} and u_{2i} are such that $\eta_i \to 0$, independently of the behaviour of e_i. Furthermore, we see from Equation (2.48c) that, as $\eta_1 \to 0$, we recover the system (2.24). Hence, using Proposition 2.1, we may conclude that $\eta_1 \to 0$ implies that $e_1 \to 0$. With this in mind, let us observe (2.48b). We have $\xi_2 := [\Delta\omega_1\; \Delta v_1]^\top$ where $\Delta\omega_1 = \omega_1 - \omega_r$ and $\Delta v_1 = v_1 - v_r$. On the other hand, by virtue of the control design, $e_1 = 0$ implies that $\omega_1^* = \omega_r$ and $v_1^* = v_r$, in which case we have $\Delta\omega_1 = \tilde{\omega}_1$ and $\Delta v_1 = \tilde{v}_1$. It follows that $e_1 \to 0$ and $\eta_1 \to 0$ imply that $\xi_2 \to 0$. In addition, as $\eta_2 \to 0$ (by the action of the controller at the force level), the terms $G(t, e_2, \xi_2)e_2 + B(e_2)\eta_2$ in (2.48b) vanish and (2.48b) becomes $\dot{e}_2 = A_{v_r}(t, e_2)e_2$. By Proposition 2.1 we conclude that e_2 also tends to zero. Carrying on by induction, we conclude that $e \to 0$.

Although intuitive, the previous arguments implicitly rely on the robustness of

$\dot{e}_i = A_{v_r}(t, e_i)e_i$ (*i.e.*, of the system (2.24)) with respect to the inputs η_i and ξ_i. More precisely, on the condition that the solutions exist on $[t_\circ, \infty)$ and, moreover, that they remain uniformly bounded during the transient. In the following statement, which is presented next, we relax these (technical) assumptions.

Proposition 2.4. *For each $i \leq n$, consider the system* (2.7), (2.8) *with control inputs u_{1i} and u_{2i} which are functions of $(t, v_i, \omega_i, e_i, v_i^*, \omega_i^*)$ and v_i^*, ω_i^* are defined in* (1.40) *and* (2.44) *respectively. Let conditions* (2.21) *and* (2.27) *hold. Let $\zeta_i := [e_i^\top \ \eta_i^\top]^\top$. In addition, assume that:*

Assumption 2.2. *for each i, there exists a function $\beta_i \in \mathcal{KL}$ such that, on the maximal interval of existence*[1] *of $t \mapsto e_i$,*

$$|\eta_i(t, t_\circ, \eta_{1\circ}, e_{i\circ})| \leq \beta(|\zeta_{i\circ}|, t - t_\circ) \tag{2.50}$$

and (2.42) *holds for some $\phi_i \in \mathcal{K}$.*

Then, $\{\zeta = 0\}$, where $\zeta := [\zeta_1^\top \cdots \zeta_n^\top]^\top$, is uniformly globally asymptotically stable. □

Assumption 2.2 means that $\eta_i(t)$ converge uniformly to zero while the trajectories $e_i(t)$ exist. In particular, if the system is forward complete 2.2 imposes uniform global asymptotic stability of (2.49). Even though this may be a strong hypothesis in a general context of nonlinear systems —see [59], it may be easily met in the case of formation tracking control, as we illustrate below.

Proof. The proof follows along the arguments developed below (2.49). For $i = 1$ the closed-loop dynamics, composed of (2.48c) and

$$\dot{\eta}_1 = F_{1_{cl}}(t, \eta_1, e_1(t)), \tag{2.51}$$

is defined on the interval of existence of $e_1(t)$, denoted $[t_\circ, t_{\max})$, and has a cascaded form. By assumption, η_1 satisfies the bound (2.50) for all $t \in [t_\circ, t_{\max})$ hence, on this interval,

$$\begin{aligned} \dot{V}_1(e_1(t)) &\leq \frac{\partial V_1}{\partial e_1}(e_1(t))B(e_1(t))\,|\eta_1(t)| = -\tilde{\omega}_1 e_{\theta 1}/k_{y1} - \tilde{v}_1 e_{x1} \\ &\leq c\sqrt{V_1(e_1(t))} \max_{[t_0, t_{max}]} \{|\eta_1(t)|\} \leq c' V_1(e_1(t)) + d \end{aligned} \tag{2.52}$$

where c is a positive number of innocuous value, $d > 0$ and $c' > \max_{[t_0, t_{max}]} \{|\eta_1(t)|\}$; both are independent of the initial time. Integrating on both sides of the latter from $t_\circ$

[1] If necessary, we consider the shortest maximal interval of existence among all the trajectories $e_i(t)$, with $i \leq n$.

to $t_{\max}$ we see that, by continuity of the solutions with respect to the initial conditions, this interval of integration may be stretched to infinity. By the definition of $V_1(e_1)$ we obtain that $e_1(t)$ exists on $[t_\circ, \infty)$ – cf. [47, page 74], [66, Proposition 1]. Moreover, since by definition $\Delta v_0 = \Delta\omega_0 = 0$, we conclude from (2.45) and (2.46), that v_1^* and ω_1^* exist along trajectories on $[t_\circ, \infty)$. It follows that the same property holds for $v_1(t)$ and $\omega_1(t)$ and, consequently, for $\xi_2(t)$ —recall that

$$\xi_2 := \begin{bmatrix} v_1 - v_r \\ \omega_1 - \omega_r \end{bmatrix}.$$

From forward completeness and condition 2.2 it follows, in turn, that $\eta_1 = 0$ is uniformly globally asymptotically stable for (2.51). Now we can apply a cascades argument for the system (2.48c), (2.51). Since B in (2.48c) is linear in e_1 and the origin of $\dot{e}_1 = A_{v_r}(t, e_1)e_1$ is uniformly globally asymptotically stable, the same property holds for the origin $(e_1, \eta_1) = (0, 0)$ —see [96, Theorem 2]. This means that there exists a class $\mathcal{KL}$ function β such that

$$|\zeta_1(t, t_\circ, \zeta_{1\circ})| \leq \beta(|\zeta_{1\circ}|, t - t_\circ) \quad \forall\, t \geq t_\circ \tag{2.53}$$

where we recall that $\zeta_i = [e_i^\top\ \eta_i^\top]$ for all $i \leq n$. In particular, $e_1(t)$, $\eta_1(t)$ and, consequently, $\xi_2(t)$, are uniformly globally bounded. To see this more clearly, we recall that, by definition, ξ_2 is a continuous function of the state ζ_1 and time and equals to zero if $\zeta_1 = 0$. Indeed, $\xi_2 = \psi(t, \zeta_1)$ where

$$\psi_1(t, \zeta_1) \;=\; \begin{bmatrix} \tilde{v}_1 + v_1^* - v_r \\ \tilde{\omega}_1 + \omega_1^* - \omega_r \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 + v_r(t)[\cos(e_{\theta 1}) - 1] + k_{x1}e_{x1} \\ \tilde{\omega}_1 + k_{\theta 1}e_{\theta_1} + v_r(t)k_{y1}e_{y1}\phi(e_{\theta 1}) \end{bmatrix} \tag{2.54}$$

Next, let $i = 2$ and consider the closed-loop equations:

$$\dot{e}_2 \;=\; A_{v_r}(t, e_2)e_2 + G(t, e_2, \psi_1(t, \zeta_1))e_2 + B(e_2)\eta_2 \tag{2.55a}$$

$$\dot{\zeta}_1 \;=\; F_{\zeta_1}(t, \zeta_1) \tag{2.55b}$$

$$\dot{\eta}_2 \;=\; F_{2_{cl}}(t, \eta_2, e_2(t)) \tag{2.55c}$$

Note that we replaced e_2 with $e_2(t)$ in (2.49) to obtain the "decoupled" dynamics equation (2.55c). Then, η_2 is regarded as a perturbation to the system

$$\dot{e}_2 \;=\; A_{v_r}(t, e_2)e_2 + G(t, e_2, \psi_1(t, \zeta_1))e_2 \tag{2.56a}$$

$$\dot{\zeta}_1 \;=\; F_{\zeta_1}(t, \zeta_1). \tag{2.56b}$$

which, in turn, is also in cascaded form. Now, in view of the structure of G, we have

$$\frac{\partial V_1}{\partial e_i} G(t, e_i, \xi_i) e_i = 0, \qquad \forall\, i \leq n \tag{2.57}$$

hence, the total derivative of V_1 along the trajectories of (2.55a) yields

$$\dot{V}_1(e_2(t)) \leq c\sqrt{V_1(e_2(t))}\, |\eta_2(t)|_{[t_0, t_{max}]} \leq c' V_1(e_2(t)) + d$$

with an appropriate redefinition of c and c' –*cf.* Ineq. (2.52). Completeness of $e_2(t)$, and therefore of $\eta_2(t)$, follows using similar arguments as for the case when $i = 1$. Consequently, by Assumption 2.2, the origin of (2.55c) is uniformly globally asymptotically stable.

To analyze the stability of the origin for (2.55) we invoke again [96, Theorem 2]. To that end, we only need to establish uniform global asymptotic stability for the system (2.56) (since B is linear and the origin of (2.55c) is uniformly globally asymptotically stable). For this, we invoke [97, Theorem 4] as follows: first, we remark that the respective origins of $\dot{e}_2 = A_{v_r}(t, e_2)e_2$ and (2.56b) are uniformly globally asymptotically stable. Second, note that condition A4 in [97, Theorem 4] is not needed here since we already established uniform forward completeness. Finally, [97, Ineq. (24)] holds trivially with $V = V_1$, in view of (2.57). We conclude that $(e_2, \zeta_1, \eta_2) = (0, 0, 0)$ is a uniformly globally asymptotically stable equilibrium of (2.55).

For $i = 3$ the closed-loop dynamics is

$$\dot{e}_3 = A_{v_r}(t, e_3)e_3 + G(t, e_3, \psi_2(t, \zeta_{12})e_3 + B(e_3)\eta_3 \tag{2.58a}$$

$$\dot{\zeta}_{12} =: F_{\zeta_{12}}(t, \zeta_{12}) \tag{2.58b}$$

$$\dot{\eta}_3 = F_{3_{cl}}(t, \eta_3, e_3(t)) \tag{2.58c}$$

where $\zeta_{12} := [\zeta_1^\top\ \zeta_2^\top]^\top$, $\zeta_2 := [e_2^\top\ \eta_2^\top]$, and

$$\psi_2(t, \zeta_{12}) := \begin{bmatrix} \tilde{v}_2 + [\xi_{21} + v_r(t)][\cos(e_{\theta 1}) + k_{x1}e_{x1} - v_r \\ \tilde{\omega}_2 + \xi_{22} + k_{\theta 1}e_{\theta_1} + v_r(t)k_{y1}e_{y1}\phi(e_{\theta 1}) \end{bmatrix}$$

which corresponds to ξ_3 —*cf.* (2.54). The previous arguments, as for the case $i = 2$, apply now to (2.58) so the result follows by induction. □□□

Remark 2.3. *An example of torque controller for* (2.2) *that guarantees the integrability of the vector* $[\tilde{v}_1,\ \tilde{\omega}_1, ...,\ \tilde{v}_N,\ \tilde{\omega}_N]$ *is presented in the first part, that is, when we assume that all the system parameters are known, we end up with equation* (2.14).

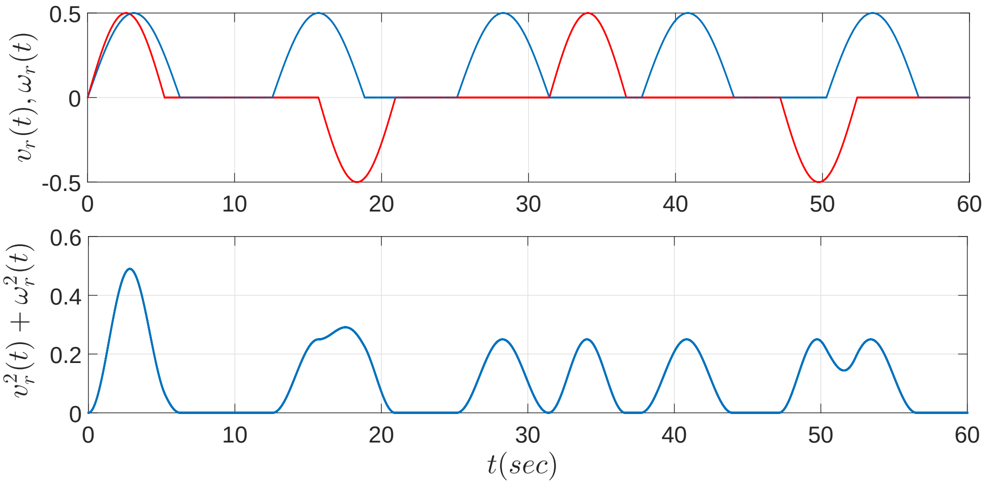

Figure 2.1: Reference velocities v_r and ω_r

2.4.1 Example

We consider a group of four mobile robots modeled as in (2.10a) and following a virtual leader (2.3). In this simulation, the desired formation shape of the four mobile robots is a diamond configuration that tracks the trajectory of the virtual leader. See Figure 2.7. We define the reference velocities v_r and ω_r in a way that there sum of squares is persistently exciting — see Figure 2.5. The physical parameters are taken from [34]:

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}, \quad C(\dot{q}_i) = \begin{bmatrix} 0 & c\omega \\ -c\omega & 0 \end{bmatrix},$$

with $m_1 = 0.6227$, $m_2 = -0.2577$, $c = 0.2025$, $r = 0.15$, and $b = 0.5$. The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1, 2, 4]$, $[x_2(0), y_2(0), \theta_2(0)] = [0, 2, 2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0, 5, 1]$ and $[x_4(0), y_4(0), \theta_4(0)] = [2, 2, 1]$; the control gains were set to $k_{x_i} = k_{y_i} = k_{\theta_i} = 1$. The formation shape with a certain desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0, 0]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [-1, 0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, -1/2]$ and $[d_{x_{3,4}}, d_{y_{3,4}}] = [0, 1]$. See Figure 2.3, The parameter $k_d = 15$. The results of the simulation are showed in Figures 2.2–2.3. In Figure 2.2, 2.4 we show the convergence of the tracking errors between the agent and its neighborhood, the control inputs and the parameter estimation errors.

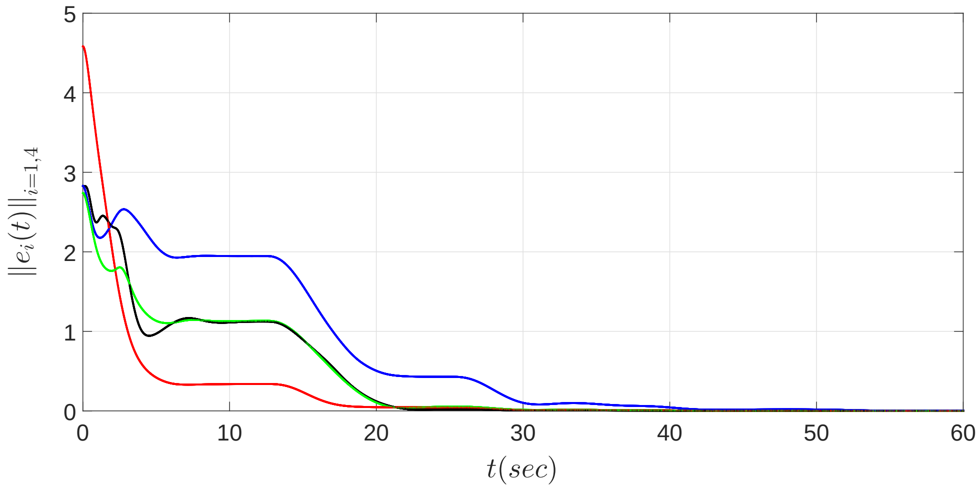

Figure 2.2: Exponential convergence of the relative errors (in norm) for each pair leader-follower

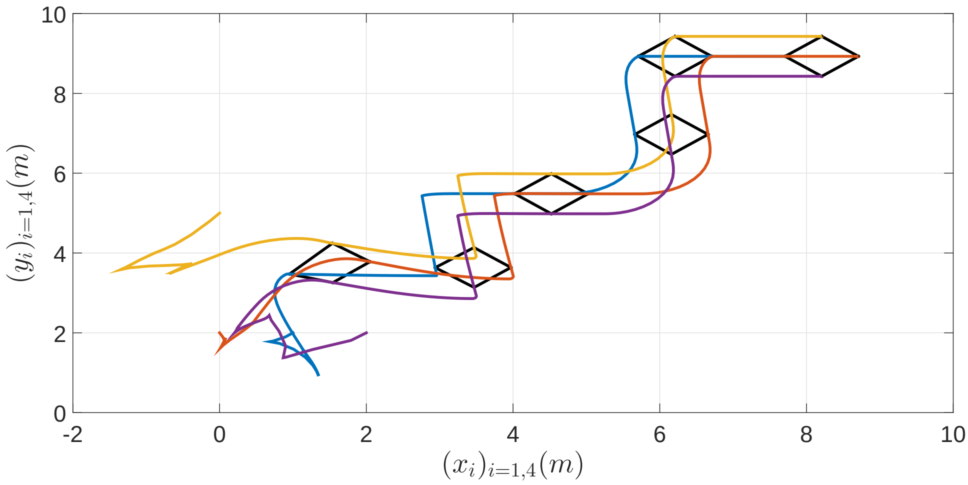

Figure 2.3: Illustration of the path-tracking in formation

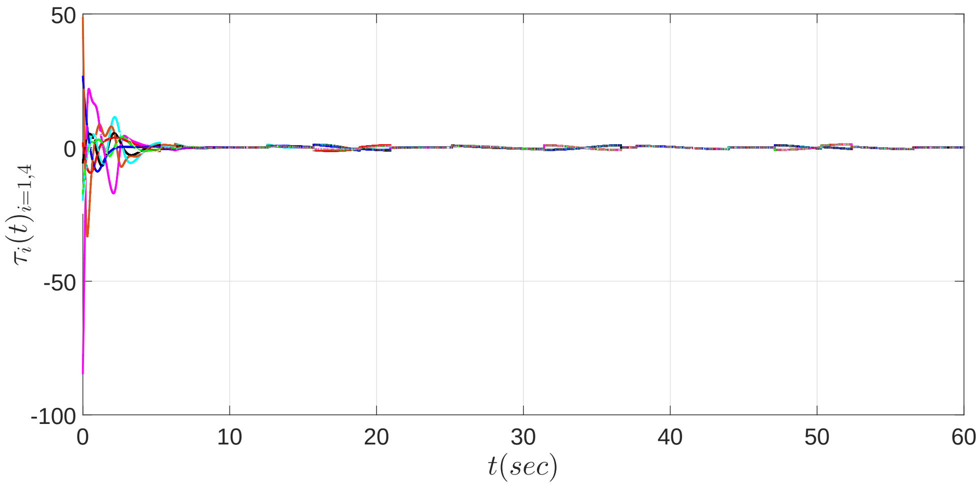

Figure 2.4: Illustration of the torque inputs for each agent

2.5 Leader-follower robust stabilization control

According to the *robust stabilization* control goal described in Problem 2.4, it is assumed that

$$\lim_{t\to\infty} |v_r(t)| + |\omega_r(t)| = 0 \tag{2.59}$$

hence, in particular, the persistency of excitation condition, crucial to solve tracking control problem, is violated.

Remark 2.4. *The latter restriction naturally excludes control methods based on conditions of persistency of excitation in Section 2.4 or, even more restrictive, that the references are always separated from zero –*cf. *[23, 26, 67].*

As we saw in Section 2.4, our control strategy consists in designing virtual control laws v^* and ω^* for the kinematics equations (2.5) and, then, using them as references for the dynamics equation (2.2). Our contribution resides in the fact that our kinematics controller is robust with respect to *any* controller at the force dynamics level. That is, we establish convergence of the tracking errors for any controller $[u_1, u_2]$ guaranteeing that $v \to v^*$ and $\omega \to \omega^*$, that is, the errors $\tilde{v} := v - v^*$ and $\tilde{\omega} = \omega - \omega^*$ verify

$$\lim_{t\to\infty} |\tilde{v}(t)| + |\tilde{\omega}(t)| = 0. \tag{2.60}$$

Consider the virtual control laws

$$v^* = k_x e_x + v_r(t)\cos e_\theta \tag{2.61a}$$

$$\omega^* = \omega_r(t) + k_\theta e_\theta + k_y \left[e_y^2 + e_x^2\right] p(t) \tag{2.61b}$$

under the standing assumption that $\dot{p}$ is persistently exciting —see Definition A.6, that is, let there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \dot{p}(s)^2 ds \geq \mu \quad \forall t \geq 0. \tag{2.62}$$

This type of controller is called δ-persistently exciting —see [64, 67, 119]. For instance, the term $\phi(t,x) := [e_y^2 + e_x^2]p(t)$, that appears in (2.61b), satisfies Definition A.7 with $x = [e_x,\ e_y]^\top$ and p being persistently exciting. The mechanism relies on the properties of $\phi(t,x)$ which, roughly speaking, is persistently exciting as long as the tracking errors are away from the origin.

For the controller (2.61), we establish strong integral input-to-state stability with respect to the reference trajectories v_r and ω_r, as well as the velocity tracking errors $\tilde{v} = v - v^*$ and $\tilde{\omega} = \omega - \omega^*$. In particular, the tracking errors converge to zero for *any* reference velocities satisfying (2.59) and any converging velocity errors, even slowly-converging.

Proposition 2.5. *Consider the system* (2.5) *with* $v = \tilde{v} + v^*$, $\omega = \tilde{\omega} + \omega^*$, *and* (2.61). *Let* k_x, k_θ, *and* $k_y > 0$ *and let* p *and* $\dot{p}$ *be bounded and persistently exciting. Then, the closed-loop system is strongly integral input-to-state stable with respect to* $\eta := [v_r\ \omega_r\ \tilde{v}\ \tilde{\omega}]^\top$. □

Proof. We start by writing the closed-loop system (2.5) with (2.61) in the form of a perturbed system, *i.e.*,

$$\dot{e} = A(t,e)e + B(e)\eta \tag{2.63}$$

where $e := [e_x\ e_y\ e_\theta]^\top$, η is a vanishing perturbation (due to (2.59) and (2.60)), and

$$A(t,e) \;:=\; \begin{bmatrix} -k_x & \psi(t,e) & 0 \\ -\psi(t,e) & 0 & 0 \\ -k_y p(t) e_x & -k_y p(t) e_y & -k_\theta \end{bmatrix},$$

$$B(e) \;=\; \begin{bmatrix} 0 & e_y & -1 & e_y \\ \sin(e_\theta) & -e_x & 0 & -e_x \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\psi(t,e) \;:=\; k_\theta e_\theta + k_y p(t) \left[e_y^2 + e_x^2\right].$$

Then, we carry out the analysis of stability for (2.63) in the following three steps:

1. we construct a strict Lyapunov function for the nominal system $\dot{e} = A(t,e)e$;

2. we use this Lyapunov function to establish the small input-to-state stability property with respect to the input η —see Appendix A.3 for the characterization of small input-to-state stability using Lyapunov functions;

3. we establish integral input-to-state-stability of (2.63) with respect to η —see Appendix A.4 for the characterization of small input-to-state stability using Lyapunov functions.

Remark 2.5. *Proving the last three items is equivalent, by definition, to establishing the strong input-to-state stability of* (2.63) *with respect to* η *—see Appendix A.5.*

Step 1. UGAS of the nominal system $\dot{e} = A(t,e)e$.

Let ϕ_m and $\phi_M > 0$ and consider the positive differentiable function $\phi : \mathbb{R}_{\geq 0} \to [\phi_m, \phi_M]$ satisfying

$$\dot{\phi} = -k_\theta \phi + k_y p(t). \tag{2.64}$$

Such function exists because p is bounded and persistently exciting [112]. Then, consider the new error coordinate

$$e_z = e_\theta + \phi(t) \left[e_y^2 + e_x^2\right],$$

which satisfies

$$\dot{e}_z = -k_\theta e_z - 2\phi k_x e_x^2. \tag{2.65}$$

Then, in the new coordinates, the nominal system becomes

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} -k_x & \dot{\phi}[e_y^2 + e_x^2] \\ -\dot{\phi}[e_y^2 + e_x^2] & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} + e_z \begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} \tag{2.66a}$$

$$\dot{e}_z = -k_\theta e_z - 2\phi k_x e_x^2 \tag{2.66b}$$

Now, since $\dot{p}$ is persistently exciting and ϕ satisfies the equation

$$\ddot{\phi} = -k_\theta \dot{\phi} + \dot{p} \tag{2.67}$$

we conclude that $\dot{\phi}$ is also persistently exciting [41, Lemma 4.8.3]. Based on these properties, Lemma 2.1, below, provides a strict differentiable Lyapunov function for (2.66).

Lemma 2.1 (set-point stabilization)**.** *Consider the following nonlinear time-varying system*

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} -k_x & \dot{\phi}[e_y^2 + e_x^2] \\ -\dot{\phi}[e_y^2 + e_x^2] & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} + e_z \begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} \tag{2.68a}$$

$$\dot{e}_z = -k_\theta e_z - 2\phi k_x e_x^2 \tag{2.68b}$$

let k_θ, $k_x > 0$, $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\dot{\phi}$ *be persistently exciting and let*

$$\max\left\{ |\phi|_\infty,\ |\dot{\phi}|_\infty,\ |\ddot{\phi}|_\infty \right\} \leq \bar{\phi}$$

where

$$|\phi|_\infty := \operatorname*{ess\,sup}_{t\geq 0} |\phi(t)|$$

Then, the system (2.68) *admits the following strict Lyapunov function*

$$V_3(t,e) := \gamma_1\big(V_1(e)\big)V_1(e) + V_2(t,e) + \gamma_2\big(V_1(e)\big)e_z^2 \tag{2.69}$$

where $V_1(e) := e_x^2 + e_y^2$.

$$V_2(t,e) \;\; := \;\; \gamma_3\big(V_1(e)\big)V_1(e) + \Upsilon_{\dot\phi(s)^2}(t)V_1(e)^3 - \dot\phi(t)V_1(e)e_xe_y \tag{2.70}$$

$$\Upsilon_{\dot\phi(s)^2}(t) \;\; := \;\; 1 + \bar\phi^2 T - \frac{1}{T}\int_t^{t+T}\int_t^m \dot\phi(s)^2 ds dm, \tag{2.71}$$

and $\gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *are positive polynomials of* V_1 *defined as*

$$\gamma_1(V_1) \;\; := \;\; \frac{\mu}{16Tk_x}V_1^2 + \frac{1}{2}\bar\phi V_1 + \frac{4k_x\bar\phi^2}{k_\theta}\gamma_2(V_1)V_1 + \frac{1}{2}, \tag{2.72}$$

$$\gamma_2(V_1) \;\; := \;\; \frac{8T\bar\phi^2 k_\theta}{\mu}V_1 + 1 \tag{2.73}$$

$$\gamma_3(V_1) \;\; := \;\; \frac{\bar\phi}{k_x}\left[2\bar\phi V_1^2 + \frac{1}{4}\big[3k_x + 1\big]V_1 + \frac{T\bar\phi}{\mu}\big[k_x^2 + 1\big]\right]. \tag{2.74}$$

and its derivative satisfies the inequality

$$\dot V_3(t,e) \leq -\frac{\mu}{8T}e_y^6 - k_\theta\gamma_2(V_1)e_z^2 - k_xe_x^2 - \frac{\mu}{4T}V_1^3. \tag{2.75}$$

□

The proof of Lemma 2.1 is presented in the Appendix B.6, the construction of V_3 is inspired by [72].

Step 2. Small ISS property.

We recall that a system $\dot x = f(t,x,\eta)$ is said to be "small ISS" if it is input-to-state stable for sufficiently small values of η. See the Appendix A.3 for precise definitions.

The proof of this property for the system (2.63) relies on the function V_3 constructed in Lemma 2.1 above; specifically on the order of growth in V_1. For the purpose of analysis we remark that V_3 in (2.69) can be written as

$$V_3(t,e) \equiv \mathcal{V}_3(t,e,V_1) \tag{2.76}$$

where

$$\mathcal{V}_3(t,e,V_1) \;:=\; \rho(t,V_1)V_1 - \dot{\phi}(t)V_1 e_x e_y + \gamma_2(V_1)e_z^2 \tag{2.77}$$

$$\rho(t,V_1) \;:=\; [\gamma_1(V_1)+\gamma_3(V_1)]V_1 + \Upsilon_{\dot{\phi}(s)^2}(t)V_1^3 \tag{2.78}$$

that is, $\rho : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function, uniformly bounded in t (since $|\Upsilon_{\dot{\phi}(s)^2}(t)| \leq 1 + \bar{\phi}^2 T$) and $\rho(t,\cdot)$ is a polynomial of degree 2 with strictly positive coefficients. In particular, since $\Upsilon_{\dot{\phi}(s)^2}(t) \geq 1$,

$$\frac{\partial \rho}{\partial V_1} \geq 0 \quad \forall\,(t,V_1) \in \mathbb{R}_{\geq 0}.\times \mathbb{R}_{\geq 0}$$

Now, by Lemma 2.1 the time-derivative of V_3 along the nominal system (2.66) satisfies (2.75) hence, the time-derivative of V_3 along the trajectories of (2.63) satisfies

$$\dot{V}_3 \leq -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 + \frac{\partial V_3}{\partial e}B(e)\eta. \tag{2.79}$$

Now, note that $B(e)\eta = K_1(\eta)e + K_2(\eta,e)$ where

$$K_1(\eta) := \begin{bmatrix} 0 & \omega_r+\tilde{\omega} & 0 \\ -(\omega_r+\tilde{\omega}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_2(\eta,e) = \begin{bmatrix} -\tilde{v} \\ v_r \sin e_\theta \\ -\tilde{\omega} \end{bmatrix},$$

so using the fact that

$$\frac{\partial V_1}{\partial e}K_1(\eta)e = 0,$$

we obtain

$$\begin{aligned}
\dot{V}_3 \;\leq\; & -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 - \dot{\phi}[\omega_r+\tilde{\omega}]V_1\left[e_y^2 - e_x^2\right] + \frac{\partial V_3}{\partial e}K_2(\eta,e) \\
\leq & -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 + \bar{\phi}\big[\,|\omega_r| + |\tilde{\omega}|\,\big]V_1^2 + \left|\frac{\partial V_3}{\partial e}\right||K_2| \\
& - k_x e_x^2 - \frac{\mu}{8T}e_y^6.
\end{aligned} \tag{2.80}$$

On the other hand, from (2.77) and (2.76) we obtain

$$\left|\frac{\partial V_3}{\partial e}\right| \leq 2\left[\frac{\partial \rho}{\partial V_1}V_1 + \rho(t,V_1) + \bar{\phi}V_1\right]\big[\,|e_y| + |e_x|\,\big] + 2\frac{\partial \gamma_2}{\partial V_1}\big[\,|e_y| + |e_x|\,\big]e_z^2$$
$$+ 4\gamma_2(V_1)\bar{\phi}\,|e_z|\,\big[\,|e_y| + |e_x|\,\big] + 2\gamma_2(V_1)|e_z|. \tag{2.81}$$

Next, let us introduce the positive polynomial of second degree

$$\gamma_4(V_1) := \frac{\partial \rho}{\partial V_1}V_1 + \rho(t,V_1) + \bar{\phi}V_1,$$

and the positive constant —see (2.73)

$$\alpha := \frac{\partial \gamma_2}{\partial V_1},$$

so that, using them in (2.81) and observing that $|K_2| \leq |\eta|$, we obtain

$$\begin{aligned}\dot{V}_3 \leq & -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 + 2\bar{\phi}|\eta|V_1^2 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 \\ & + 2\gamma_4(V_1)|\eta|\big[|e_y| + |e_x|\big] + 2\alpha|\eta|\big[|e_y| + |e_x|\big]e_z^2 \\ & + 4\gamma_2(V_1)\bar{\phi}|\eta||e_z|\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)|\eta||e_z|. \end{aligned} \tag{2.82}$$

Then, using the inequality $|e_z|\big[\,|e_y| + |e_x|\big] \leq e_z^2 + V_1/2$ in (2.82) we obtain

$$\begin{aligned}\dot{V}_3 \leq & -\frac{\mu}{4T}V_1^3 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 - \Big[\big[k_\theta - 4\bar{\phi}|\eta|\big]\gamma_2(V_1) - 2\alpha|\eta|\Big[|e_y| + |e_x|\Big] - |\eta|\Big]\,e_z^2 \\ & + 2\bar{\phi}|\eta|V_1^2 + 2\gamma_4(V_1)|\eta|\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)\bar{\phi}|\eta|V_1 + \gamma_2(V_1)^2|\eta| \\ \leq & -\Big[\frac{\mu}{4T}V_1^3 - \Phi_1(e_x,e_y)|\eta|\Big] - \Big[\frac{k_\theta}{2}\gamma_2(V_1) - \Phi_2(e_x,e_y)|\eta|\Big]e_z^2 \\ & - k_x e_x^2 - \frac{\mu}{8T}e_y^6 - \frac{k_\theta}{2}\gamma_2(V_1)e_z^2 \end{aligned} \tag{2.83}$$

where

$$\begin{aligned}\Phi_1 &:= 2\bar{\phi}V_1^2 + 2\gamma_4(V_1)\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)\bar{\phi}V_1 + \gamma_2(V_1)^2, \\ \Phi_2 &:= 4\bar{\phi}\gamma_2(V_1) + 2\alpha\big[|e_y| + |e_x|\big] + 1.\end{aligned}$$

Then, since $|e_y| + |e_x| \leq \sqrt{2V_1}$, $\gamma_2(V_1) = \mathcal{O}(V_1)$, and $\gamma_4(V_1) = \mathcal{O}(V_1^2)$ there exist positive constants a_i, with $i \in [0, 4]$, of innocuous values[1], such that

$$\begin{aligned}\Phi_1 &\leq [a_2 V_1^2 + a_1 V_1 + a_0][1 + a_4 V_1^{1/2}] &(2.84)\\ \Phi_2 &\leq a_1 V_1 + a_4 V_1^{1/2} + a_0. &(2.85)\end{aligned}$$

Furthermore, since $V_1^{1/2} \leq a_0 + a_1 V_1$ for all $a_0 \geq 1$, $a_1 \geq 1$, and $V_1 \geq 0$,

$$\begin{aligned}\Phi_1 &\leq a_3 V_1^3 + a_2 V_1^2 + a_1 V_1 + a_0 &(2.86)\\ \Phi_2 &\leq a_1 V_1 + a_0. &(2.87)\end{aligned}$$

Now, let $R > 0$ and assume that η satisfies the following bound

$$|\eta| \leq R \min\Big\{V_1(e)^3 + e_z^2,\ 1\Big\} \tag{2.88}$$

which, in particular, implies that $|\eta| \leq R$. We see that the factor of e_z^2 in (2.83) is non-positive for sufficiently small R. Now, in regards to the term involving Φ_1 in (2.83), note that in case that $V_1 \geq 1$, since $|\eta| \leq R$, we have $\Phi_1|\eta| \leq c_1 R V_1^3$, and $\Phi_2|\eta| \leq c_2 R V_1$ for some c_1, $c_2 > 0$. Otherwise, if $V_1 \leq 1$, then there exists c_3, $c_4 > 0$ such that $\Phi_{1,2} \leq c_{3,4}$ and, in view of (2.88),

$$\Phi_1|\eta| \leq c_3 R\,[V_1^3 + e_z^2], \qquad \Phi_2|\eta| \leq c_4 R\,[V_1 + 1] \tag{2.89}$$

We conclude that, for sufficiently small R, (2.83) and (2.88) imply that

$$\dot{V}_3(t,e) \leq -\frac{k_\theta}{4} e_z^2 - k_x e_x^2 - \frac{\mu}{8T} e_y^6$$

so the system is small-input-to-state stable.

Step 3. The iISS property.

The proof of Proposition 2.5 is finalized by establishing integral input-to-state stability of the system (2.63) with respect to η. To that end, consider the proper positive-definite Lyapunov function

$$W(t,e) = \ln\big(1 + V_3(t,e)\big) \tag{2.90}$$

[1] Conventionally, a_i (for any integer $i \geq 0$) denote positive coefficients of polynomials so, without loss of generality, we implicitly assume that they are redefined as needed, *e.g.*, $a_i := a_i a_j + a_i^2 - a_i \ldots$

and a positive definite function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(e) \geq \frac{1}{1 + V_3(t,e)} \left[k_x e_x^2 + \frac{\mu}{8T} e_y^6 + k_\theta e_z^2 \right] \tag{2.91}$$

Then, in view of (2.83), the time-derivative of W along the trajectories of (2.63) satisfies

$$\dot{W}(t,e) \quad \leq \quad -\alpha(e) + \frac{\Phi_1 + \Phi_2 e_z^2}{1 + V_3(t,e)} |\eta|. \tag{2.92}$$

From (2.72) and the fact that $V_2 \geq 0$ (see Lemma 2.1), there exist a_1, a_2, and $a_3 > 0$, such that

$$V_3(t,e) \geq a_3 V_1^3 + a_2 V_1^2 + a_1 V_1 + \gamma_2(V_1) e_z^2 \tag{2.93}$$

so, in view of (2.86), (2.87), and (2.73), the factor of $|\eta|$ in (2.92) is bounded that is, there exists $c > 0$ such that $\dot{W}(t,e) \leq -\alpha(e) + c|\eta|$, so the system (2.63) is integral input-to-state stable. □□□

2.6 Leader-follower robust agreement control

We extend now the statement of Proposition 2.5 to the problem of multi-agent robust agreement control for a group of N robots modeled by (2.1) and (2.2). Similarly to the controller proposed previously, we define

$$v_i^* \quad = \quad v_{i-1} \cos(e_{\theta_i}) + k_{xi} e_{x_i} \tag{2.94a}$$

$$\omega_i^* \quad = \quad \omega_{i-1} + k_{\theta_i} e_{\theta_i} + k_{y_i} p_i(t) \left[e_{y_i}^2 + e_{x_i}^2 \right] \tag{2.94b}$$

where $p_i : \mathbb{R}_{\geq 0} \to [p_{mi}, p_{Mi}]$, are bounded and smooth for all $i \leq N$ with bounded derivatives up to the second. Moreover, we assume that each p_i and its first derivative, $\dot{p}_i$, are persistently exciting.

Proposition 2.6. *Consider the network system composed by* (2.5) *for* $i = \{1, ..., N\}$, *let constants* k_{xi}, k_{yi}, $k_{\theta i} > 0$ *and let* p_i *and* $\dot{p}_i$ *be bounded and persistently exciting. Then, for the network system* (2.5), *tracking errors converge to zero, i.e.* (2.9) *holds for* $i = \{1, ..., N\}$ *when* $[v_r,\ \omega_r,\ \tilde{v}_1,\ \tilde{\omega}_1, ...,\ \tilde{v}_N,\ \tilde{\omega}_N]$ *converge to zero, where*

$$\tilde{v}_i = v_i - v_i^*, \qquad \tilde{\omega}_i = \omega_i - \omega_i^*.$$

□

Proof. To compact the notation, let us define

$$V_{1i}(e_i) := e_{x_i}^2 + e_{y_i}^2 \tag{2.95}$$

$$\psi_i(t, e_i) := k_{\theta i} e_{\theta_i} + k_{y_i} p_i(t) V_{1i}$$

so that, replacing

$$v_i = v_i^* + \tilde{v}_i, \quad \omega_i = \tilde{\omega}_i + \omega_i^*, \tag{2.96}$$

and (2.94) in (2.5) we obtain

$$\dot{e}_{x_i} = [\tilde{\omega}_i + \omega_{i-1} + \psi_i] e_{y_i} - \tilde{v}_i - k_{x_i} e_{x_i} \tag{2.97a}$$

$$\dot{e}_{y_i} = -[\tilde{\omega}_i + \omega_{i-1} + \psi_i] e_{x_i} + v_{i-1} \sin(e_{\theta_i}) \tag{2.97b}$$

$$\dot{e}_{\theta_i} = -\psi_i - \tilde{\omega}_i \tag{2.97c}$$

which has exactly the same structure as (2.63). Indeed, the equations (2.97) may be re-written in the compact form

$$\dot{e}_i = A_i(t, e_i) e_i + B(e_i) \eta_i \tag{2.98}$$

where $e_i := [e_{\theta_i}\ e_{x_i}\ e_{y_i}]^\top$,

$$\begin{aligned} A_i(t, e_i) &:= \begin{bmatrix} -k_{x_i} & \psi_i(t, e_i) & 0 \\ -\psi_i(t, e_i) & 0 & 0 \\ -k_{y_i} p_i(t) e_{x_i} & -k_{y_i} p_i(t) e_{y_i} & -k_{\theta_i} \end{bmatrix} \\ \eta_i &:= [v_{i-1}\ \omega_{i-1}\ \tilde{v}_i\ \tilde{\omega}_i]^\top. \end{aligned}$$

As η in Proposition 2.5, which contains v_r and ω_r, η_i may also be regarded as a vanishing perturbation. To see more clearly, we develop some expressions for v_{i-1} and ω_{i-1} to exhibit their dependence on v_r and ω_r. Using, recursively, (2.96) and (2.94a) we obtain

$$\begin{aligned}
v_{i-1} &= v_{i-2}\cos(e_{\theta_{i-1}}) + k_{x_{i-1}}e_{x_{i-1}} + \tilde{v}_{i-1} \\
&= \left[\tilde{v}_2 + v_{i-3}\cos(e_{\theta_{i-2}}) + k_{x_{i-2}}e_{x_{i-2}}\right]\cos(e_{\theta_{i-1}}) + \ k_{x_{i-1}}e_{x_{i-1}} + \tilde{v}_{i-1} \\
&= \tilde{v}_{i-1} + \tilde{v}_{i-2}\cos(e_{\theta_{i-1}}) + \ \tilde{v}_{i-3}\cos(e_{\theta_{i-2}})\cos(e_{\theta_{i-1}}) \\
&\quad + v_{i-4}\cos(e_{\theta_{i-3}})\cos(e_{\theta_{i-2}})\cos(e_{\theta_{i-1}}) \\
&\quad\quad + k_{x_{i-3}}e_{x_{i-3}}\cos(e_{\theta_{i-2}})\cos(e_{\theta_{i-1}}) \\
&\quad\quad + k_{x_{i-2}}e_{x_{i-2}}\cos(e_{\theta_{i-1}}) + k_{x_{i-1}}e_{x_{i-1}} \\
&\ \vdots \\
&= \sum_{j=1}^{i-1}\Big[\left[\tilde{v}_j + k_{x_j}e_{x_j}\right]\prod_{k=j+1}^{i-1}\cos(e_{\theta_k})\Big] + v_r\prod_{j=1}^{i-1}\cos(e_{\theta_j})
\end{aligned}$$

[2]while, from (2.94b),

$$\begin{aligned}
\omega_{i-1} &= \omega_{i-2} + k_{\theta_{i-1}}e_{\theta_{i-1}} + k_{y_{i-1}}p_{i-1}(t)V_{1i-1} + \tilde{\omega}_{i-1} \\
&= \omega_{i-3} + k_{\theta_{i-2}}e_{\theta_{i-2}} + k_{y_{i-2}}p_{i-2}(t)V_{1i-2} \\
&\quad + k_{\theta_{i-1}}e_{\theta_{i-1}} + k_{y_{i-1}}p_{i-1}(t)V_{1i-1} + \tilde{\omega}_{i-1} + \tilde{\omega}_{i-2} \\
&\ \vdots \\
&= \omega_r(t) + \sum_{j=1}^{i-1}\psi_j.
\end{aligned}$$

So we see that for each robot indexed by $i \le N$, v_{i-1} and ω_{i-1} depend on the tracking errors of all the followers, indexed up to $i-1$, including the reference vehicle. For $i = 1$, the system (2.98) corresponds to (2.63) hence, by Proposition 2.5, $e_1 \to 0$. For $i = 2$, $\eta_2 := [v_1,\ \omega_1,\ \tilde{v}_1,\ \tilde{\omega}_1]$ where

$$\begin{aligned}
v_1 &= \tilde{v}_1 + k_{x_1}e_{x_1} + v_r\cos(e_{\theta_1}) \\
\omega_1 &= \omega_r + k_{\theta 1}e_{\theta_1} + k_{y_1}p_1(t)V_{11}
\end{aligned}$$

hence, $\eta_2 \to 0$ and, by Proposition 2.5 we obtain that $e_2 \to 0$. The statement follows by induction. □□□

Remark 2.6. *An example of torque controller that guarantees convergence to zero of the vector* $[\tilde{v}_1,\ \tilde{\omega}_1, \ldots,\ \tilde{v}_N,\ \tilde{\omega}_N]$ *is presented in Subsection 2.2.*

[2]Conventionally, $\prod_{j=1}^{0}\cos(\cdot) = 1$.

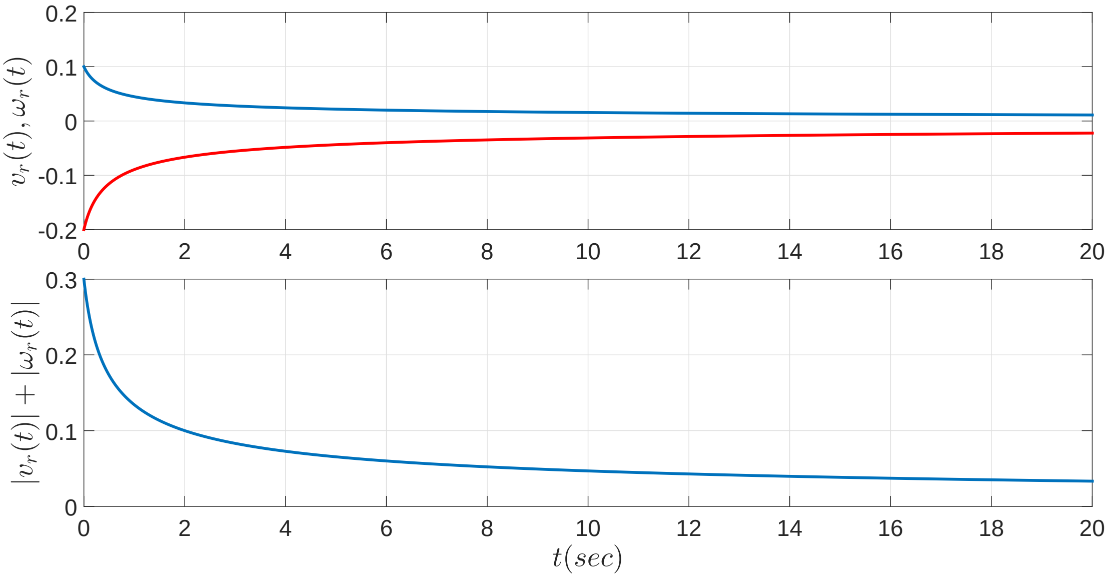

Figure 2.5: Reference velocities v_r and ω_r

2.6.1 Example

In this simulation, we consider a hexagonal desired formation shape for six mobile robots where one of them is a virtual leader. See Figure 2.7. We impose a slowly vanishing reference velocities (v_r, ω_r) (non integrable) — see Figure 2.5.

The physical parameters of the systems are presented in Subsection 2.4.1, while in this case we assume that the inertia parameters and the constants contained in $C(\dot{q}_i)$ are unknown, that is, we use in this case the adaptive torque controller in (2.16).

the desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0.5, -0.5]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [1, 0]$, $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, 1/2]$, $[d_{x_{3,4}}, d_{y_{3,4}}] = [0.5, -0.5]$ and $[d_{x_{4,5}}, d_{y_{4,5}}] = [1, 0]$. The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1, 3, 4]$, $[x_2(0), y_2(0), \theta_2(0)] = [0, 2, 2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0, 4, 1]$, $[x_4(0), y_4(0), \theta_4(0)] = [2, 2, 1]$ and $[x_5(0), y_5(0), \theta_5(0)] = [-2, 2, 1]$;

The control gains are set to $k_x = k_{x_i} = k_y = k_{y_i} = 2.5$ and $k_\theta = k_{\theta_i} = 1$ and the function $p(t) = 20\sin(t/8) + 1/4$, which has a persistently exciting time derivative. The parameters (γ, k_d) are taken equal to $(10^{-6}, 12)$. The parameters (γ, k_d) are taken equal to $(10^{-5}, 15)$, and $\hat{\Theta}(0) = (\hat{m}_1(0), \hat{m}_2(0), \hat{c}(0)) = (0, 0, 0)$.

In Figures 2.6, 2.8, and 2.9 we show the convergence of the tracking errors between each agent and its neighborhood, the control inputs and the parameter estimation errors, respectively.

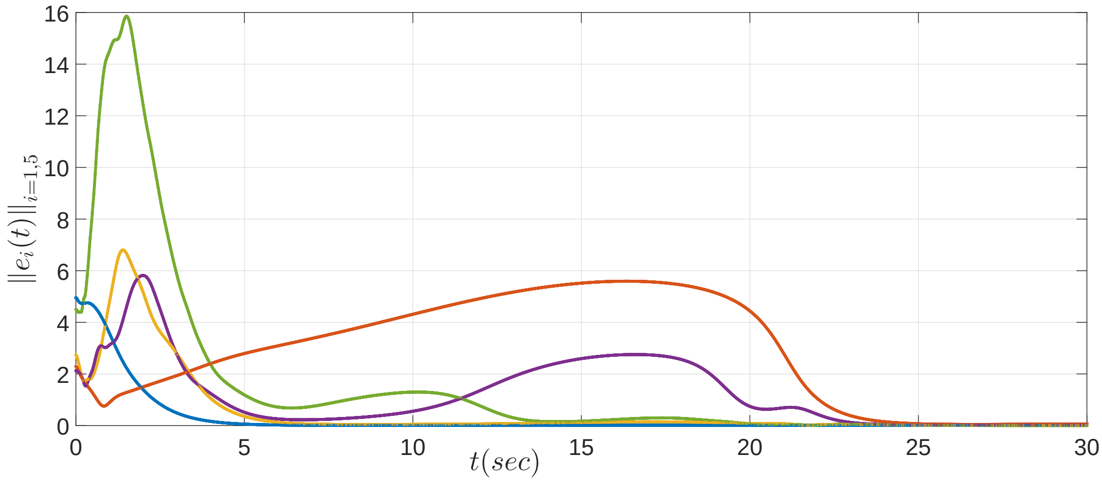

Figure 2.6: Exponential convergence of the relative errors (in norm) for each pair leader-follower

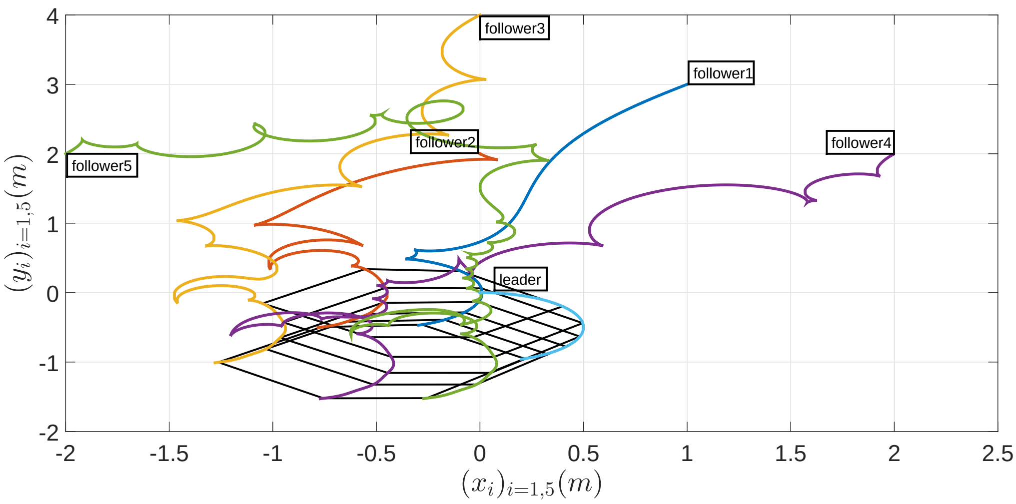

Figure 2.7: Illustration of the path-tracking in formation

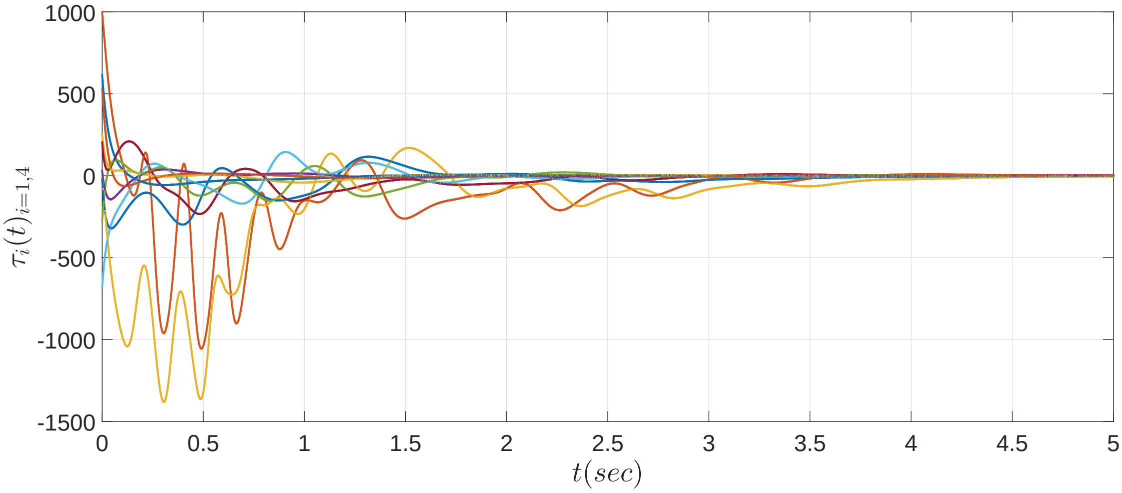

Figure 2.8: Illustration of the torque inputs for each agent

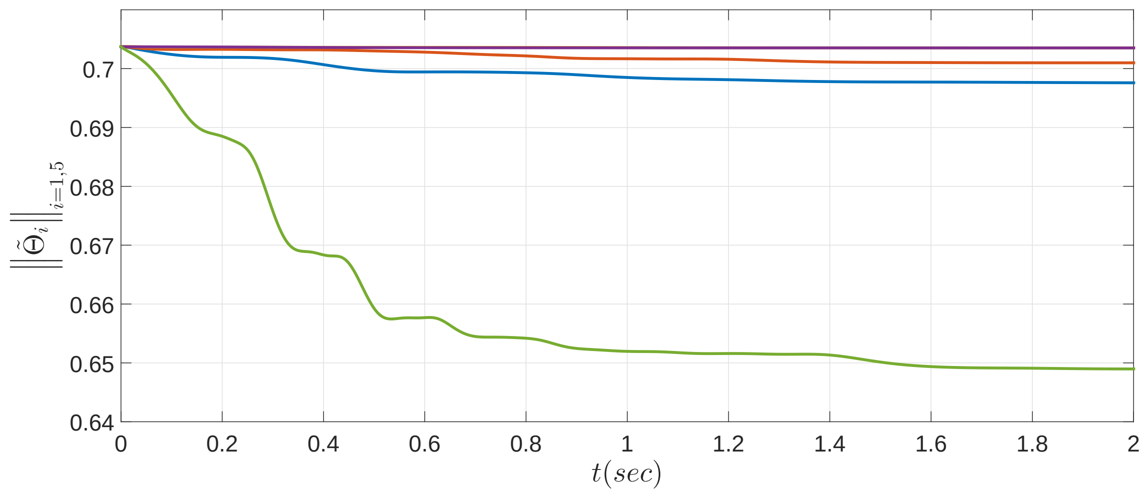

Figure 2.9: Illustration of the estimation parameter errors for each agent

2.7 Conclusion

In this chapter we considered the leader-follower control problem for single and multiple agent cases. We identified several control problems that impose distinct technical difficulties, depending on nature of the leader's velocities. First, we presented a formation-tracking controller for autonomous vehicles that ensures uniform global asymptotic stability of the closed-loop system, under the assumption that either the angular or the forward reference velocity is persistently exciting. Then, we considered the case where the leader's velocities converge to zero and presented a simple decentralized controller for leader-follower *robust agreement* problem. In both cases, we assumed that each robot has only one leader and may have one or more followers.

Moreover, a strict Lyapunov function is provided for the kinematic error dynamics. We decouple the problems at the velocity-kinematics and force-dynamics levels. Interestingly enough, our results apply to a range of controllers at the dynamic level. Thus, one can use a variety of control schemes for Lagrangian and Hamiltonian systems, including adaptive and output feedback control designs.

Chapter 3

Leader-follower simultaneous tracking-agreement control of nonholonomic vehicles

In the previous chapter we presented several problem formulations for the formation control of nonholonomic vehicles, and emphasized how different scenarios of the leader's velocities influence both the control design and the stability properties of the closed-loop system. Problem of unified controller that stabilizes the closed-loop globally asymptotically for different scenarios of the leader's velocities is a very challenging problem. Indeed, to the best of our knowledge the *simultaneous tracking-stabilization* problem for nonholonomic mobile robot has only been studied in [27, 52, 85, 119], where the goal is to design a unified velocity or torque controller for the follower robot in order to track the trajectories of the leader asymptotically under different scenarios of the leader's velocities. The possible scenarios include the case where the leader describes a general time-varying path *(tracking scenario)*, and stabilization scenario where the leader converge to a set point *(parking scenario)* or, in a more general case, where the leader's velocities converge to zero *robust stabilization scenario*.

In [52] a saturated time-varying velocity controller is proposed to track the leader's trajectories under different scenarios of the leader's velocities. In [85] a unified velocity controller is provided to solve the problem under all possible configurations of the leader's velocities using the concept of transverse functions. In [27] and [119], a unified torque controller is proposed in order to make the tracking error converging to the origin under a *tracking* and a *parking scenarios*. In [119], a nice idea has been used which consists of combining a tracking controller with stabilization controller via a weighted sum, the weight function depends on the leader's velocities and promotes each controller depending on the current scenario.

For the multi-agent formation case, the unified controller proposed in [85] has been extended to the leader-follower formation case in [118] under a general force controlled model of the mobile robot, while assuming that the leader's coordinates are accessible to all the network. Providing a distributed solution to the leader-follower *simultaneous tracking-agreement control* problem is an open question to the best of our knowledge.

In this chapter, we propose to extend the idea of control design established in [119] to a more general class of controllers, and thus to allow a more general scenarios of the leader's velocities as in [85]. Also our original proofs allow a straightforward extension to the leader-follower *simultaneous tracking-agreement* case under spanning communication graph topology. As in the previous chapter, our results are based on the construction of strict Lyapunov functions for a nonlinear time-varying systems [72], and robustness analysis tools such as the integral Input-to-state Stability [8, 9], and the Strong integral Input-to-state Stability [17, 18].

3.1 A larger class of controllers

The simultaneous tracking-stabilization control problem has been addressed in [26] and [119], where a unified control law is provided to guarantee the global attractivity of the origin of (2.5) under each one of the following scenarios:

S1: Tracking scenario. It is assumed that there exists T and $\mu > 0$ such that, for all $t \geq t_0$:

$$\int_t^{t+T} \big(|v_r(\tau)|^2 + |\omega_r(\tau)|^2\big) d\tau > \mu, \quad \forall t \geq t_0. \tag{3.1}$$

S2: Stabilization scenario. It is assumed that there exists $\beta > 0$ such that, for all $t \geq t_0$:

$$\int_{t_0}^{t} (|v_r(\tau)| + |\omega_r(\tau)|) d\tau < \beta, \quad \forall t \geq t_0. \tag{3.2}$$

Remark 3.1. *Obviously, the two scenarios cannot appear simultaneously, but the goal is to design a unified controller that guarantees the global attractivity of the origin of the closed-loop system* (2.5), *independently of the actual scenario of the leader's velocities.*

In the first part of this chapter we consider simultaneous tracking and parking problem and design a universal controller that achieves the trajectory tracking objectives

$$\lim_{t\to\infty} e(t) = 0 \tag{3.3}$$

and,

$$\lim_{t\to\infty} (\tilde{v}, \tilde{w}) = (0, 0) \tag{3.4}$$

under either of the two scenarios described above.

similar to Section 2.5, our contributions are the following:

- in the kinematic level, we propose control inputs v^* and ω^* that ensure uniform global asymptotic stability of the origin of (2.5);
- on the dynamic level, for the velocity error kinematics in closed-loop, we establish integral input-to-state stability with respect to the error velocities $[\tilde{v}, \tilde{\omega}]$;
- for *any* control inputs u_1 and u_2 ensuring that $\tilde{v} \to 0$ and $\tilde{\omega} \to 0$, we establish global attractivity of the origin provided that the error velocities $(\tilde{v}, \tilde{\omega})$ converge sufficiently fast (they are square integrable).

The control laws that ensure the properties above are:

$$v^* := v_r(t)\cos(e_\theta) + k_x e_x, \tag{3.5}$$

$$\omega^* := \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta) + \rho(t) k_y f(t, e_x, e_y) \tag{3.6}$$

where ϕ is the so-called *sync* function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta},$$

the weight function $\rho(t)$ is defined as

$$\rho(t) := \exp\left(-\int_0^t \left[\,|v_r(\tau)| + |\omega_r(\tau)|\,\right] d\tau\right), \tag{3.7}$$

and $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function defined such that the following technical assumptions hold.

Assumption 3.1. *There exist non-decreasing function $\sigma_1 : \mathbb{R}_\geq \to \mathbb{R}_{\geq 0}$ and a constant $\sigma_2 > 0$ such that*

$$\max\left\{|\frac{\partial f}{\partial t}|, \quad |\frac{\partial f}{\partial e_x}|, \quad |\frac{\partial f}{\partial e_y}|\right\} \leq \sigma_1(|e_x\, e_y|) \tag{3.8}$$

$$|f(t, e_x, e_y)| \leq \sigma_2 |e_x\, e_y|. \tag{3.9}$$

Assumption 3.2. *Let $f_\circ(t, e_y) := f(t, 0, e_y)$ then, $\frac{\partial f}{\partial t}(t, 0, e_y)$ is uniform $\delta-$persistently exciting with respect to e_y —see Definition A.7 or [65, Definition 3].*

Roughly speaking, the purpose of the function f is to excite the e_y–dynamics as long as $|e_y|$ is separated from zero.

The controller (3.6) which achieves both the tracking and the stabilization control goals, is a weighted sum of the tracking controller of [70],

$$\omega^*_{tra} := \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta),$$

and the stabilization controller that generalizes the ones proposed in [71,78,119],

$$\omega^*_{stab} := \omega_r + k_\theta e_\theta + k_y f(t, e_x, e_y).$$

The weight function $\rho(t)$ acts as a smoothly-switching supervisor promoting the application of either ω^*_{tra} or ω^*_{stab}, depending on the task scenario **S1** or **S2**. More precisely, from (3.7) we see that ρ satisfies

$$\dot{\rho} = -\left[\, |v_r(t)| + |\omega_r(t)| \,\right] \rho \tag{3.10}$$

and $\rho \to 0$ exponentially fast if (3.1) holds. Hence, the tracking scenario **S1** is promoted. If, instead, (3.2) holds, the reference velocities converge and $\rho(t) > \exp(-\beta)$. Hence, the action of the stabilization controller is favoured.

Remark 3.2. *The idea of such merging of two controllers for the scenarios* **S1** *and* **S2** *was initially introduced in [78]. The class of controllers satisfying Assumptions 3.1-3.2 covers those in [119]; in particular, the function f is not necessarily globally bounded and may depend only on e_y. A more significant contribution with respect to the literature is that we establish uniform global asymptotic stability for* (2.5) *in closed-loop with $(v, \omega) = (v^*, \omega^*)$; this is in contrast with [119] and [26] where it is proved that* (3.3) *holds. In addition, we establish integral ISS of* (2.5) *with respect to $[\tilde{v}, \tilde{\omega}]$.*

Proposition 3.1. *Consider the system* (2.5) *with $v = \tilde{v} + v^*$, $\omega = \tilde{\omega} + \omega^*$, and the virtual inputs* (3.5) *and* (3.6)*. Let k_x, k_θ, and $k_y > 0$.*

Assume that there exist $\bar{\omega}_r$, $\bar{\dot{\omega}}_r$, $\bar{v}_r$, $\bar{\dot{v}}_r > 0$ such that

$$|\omega_r|_\infty \leq \bar{\omega}_r, \quad |\dot{\omega}_r|_\infty \leq \bar{\dot{\omega}}_r, \quad |v_r|_\infty \leq \bar{v}_r, \quad |\dot{v}_r|_\infty \leq \bar{\dot{v}}_r. \tag{3.11}$$

Furthermore, let Assumptions 3.1-3.2 hold.

If either (3.1) *or* (3.2) *is satisfied, then the closed-loop system resulting from* (2.5), (2.6), (3.5), *and* (3.6) *has the following properties:*

(P1) *if* $\tilde{v} = \tilde{\omega} = 0$*, the origin* $\{e = 0\}$ *is uniformly globally asymptotically stable;*

(P2) *the closed-loop system is integral input-to-state stable with respect to* $\eta := [\tilde{v}\ \tilde{\omega}]^\top$*;*

(P3) *if* $\eta \to 0$ *and* $\eta \in \mathcal{L}_2$*, then* (3.3) *holds.* □

3.1.1 Proof of Proposition 3.1

For each scenario, **S1** and **S2** we establish uniform global asymptotic stability for the closed-loop kinematics equation (2.5) restricted to $\eta = 0$. Then, we establish the iISS with respect to η by showing that the closed-loop trajectories are bounded under the condition that η is square integrable —*cf.* [8].

Under Scenario S1

The proof of Proposition 3.1 under condition (3.1) is constructive, in particular, we provide a strict Lyapunov function for the closed loop system. To that end, we start by observing that the error system (2.5), (2.6), (3.5) and (3.6) has the form

$$\dot{e} = A_{v_r}(t, e)e + B_1(t, e)\rho(t) + B_2(e)\eta, \tag{3.12}$$

where

$$\begin{aligned} A_{v_r}(t, e) &:= \begin{bmatrix} -k_\theta & 0 & -v_r(t)k_y\phi(e_\theta) \\ 0 & -k_x & \omega^*(t, e) \\ v_r(t)\phi(e_\theta) & -\omega^*(t, e) & 0 \end{bmatrix}, \\ B_1(t, e) &:= \begin{bmatrix} -k_y f(t, e_x, e_y) \\ k_y f(t, e_x, e_y)e_y \\ -k_y f(t, e_x, e_y)e_x \end{bmatrix}, \quad B_2(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}. \end{aligned} \tag{3.13}$$

Writing the closed-loop dynamics as in (3.12) is convenient to stress that the "nominal" system $\dot{e} = A_{v_r}(t, e)e$ has a familiar structure encountered in model reference adaptive control, see Section 1.3.1. Moreover, defining

$$V_1(e) := \frac{1}{2}\left[e_x^2 + e_y^2 + \frac{1}{k_y}e_\theta^2\right], \tag{3.14}$$

we obtain, along the trajectories of $\dot{e} = A_{v_r}(t, e)e$,

$$\dot{V}_1(e) \leq -k_x e_x^2 - k_\theta e_\theta^2.$$

This is a fundamental first step in the construction of a strict Lyapunov function for the "perturbed" system (3.12).

To establish the proof in the case of scenario **S1**, we follow the steps $1-3$ below:

1) We build a strict Lyapunov function $V(t,e)$ for the nominal system $\dot{e} = A_{v_r}(t,e)e$. This establishes **P1**.

2) We construct strict Lyapunov function $W(t,e)$ for the perturbed system $\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho$.

3) We use $W(t,e)$ to prove integral ISS of (3.12) with respect to η (*i.e.*, **P2**) as well as the boundedness of the trajectories under the assumption that $\eta \in \mathcal{L}_2$. This and the assumption that $\eta \to 0$ implies (3.3), *i.e.*, **P3**.

Step 1. We establish UGAS for the nominal system

$$\dot{e} = A_{v_r}(t,e)e \tag{3.15}$$

via Lyapunov's direct method. After Proposition 2.1, there exists a positive definite radially unbounded function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as

$$\begin{aligned} V(t,e) \quad := \quad & P_{[3]}(t,V_1)V_1(e) - \omega_r(t)e_x e_y \\ & + v_r(t)P_{[1]}(t,V_1)e_\theta e_y, \end{aligned} \tag{3.16}$$

and such that

$$F_{[3]}(V_1) \leq V(t,e) \leq S_{[3]}(V_1), \tag{3.17}$$

where $V_1(e)$ is defined in (3.14), $F_{[3]}$, $S_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and $P_{[k]} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are smooth polynomials in $V_1(e)$ with strictly positive and bounded coefficients of degree 3 and k respectively. It is shown in Proposition 2.1 that the total derivative of $V(t,e)$ along the trajectories of (3.15) satisfies

$$\dot{V}(t,e) \leq -\frac{\mu}{T}V_1(e) - k_x e_x^2 - k_\theta e_\theta^2. \tag{3.18}$$

Hence uniform global asymptotic stability of the null solution of (3.15) follows.

Step 2. Now we construct a strict Lyapunov function for the system

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t). \tag{3.19}$$

To that end, we start by "reshaping" the function $V(t,e)$ defined in (3.16) to obtain a particular negative bound on its time-derivative. Let

$$Z(t,e) := Q_{[3]}(V_1)V_1(e) + V(t,e) \tag{3.20}$$

where $Q_{[3]}(V_1)$ is a third order polynomial with a strictly positive coefficients. Then, in view of (3.18), the total derivative of Z along the trajectories of (3.15) satisfies

$$\dot{Z}(t,e) \le -\frac{\mu}{T}V_1(e) - Q_{[3]}(V_1)\left[k_x e_x^2 + k_\theta e_\theta^2\right]. \tag{3.21}$$

Next, we recall that in view of (3.1) ρ is uniformly integrable hence, for any $\gamma > 0$, there exists $c > 0$ such that

$$G(t) := exp\left(-\gamma \int_0^t \rho(s)ds\right) \ \ge\ c > 0 \quad \forall t \ge 0. \tag{3.22}$$

Thus, since $Z(t,e)$ and $V(t,e)$ are positive definite radially unbounded —see (3.17) and (3.20), so is the function

$$W(t,e) \ \ := \ \ G(t)Z(t,e). \tag{3.23}$$

Indeed, we have

$$exp\left(-\gamma \int_0^\infty \rho(s)ds\right) Z(t,e) \le W(t,e) \le Z(t,e).$$

Now, the time-derivative of W along trajectories of (3.19) verifies

$$\begin{aligned} \dot{W}(t,e) \ &\le\ -Y(t,e) + \dot{G}(t)Z(t,e) + G(t)\frac{\partial\left(Q_{[3]}(V_1)V_1 + V\right)}{\partial e}B_1(t,e)\rho(t) \\ Y(t,e) \ &:=\ G(t)\Big[\frac{\mu}{T}V_1(e) + Q_{[3]}(V_1)\left[k_x e_x^2 + k_\theta e_\theta^2\right]\Big]. \end{aligned} \tag{3.24}$$

Note that, in view of (3.22), $Y(t,e)$ is positive definite. We proceed to show that the rest of the terms bounding $\dot{W}$ are negative semi-definite. To that end, we develop (dropping the arguments of $f(t,e_x,e_y)$)

$$\begin{aligned} \frac{\partial\left(Q_{[3]}(V_1)V_1 + V\right)}{\partial e}B_1(t,e) \ &=\ \frac{\partial\left(Q_{[3]}(V_1)V_1 + V\right)}{\partial V_1}\frac{\partial V_1}{\partial e}B_1(t,e) - \omega_r k_y f(\cdot)\left[e_x + e_y^2\right] \\ &\quad - v_r P_{[1]}(t,V_1)k_y f(\cdot)\left[e_\theta e_x + e_y\right] \end{aligned} \tag{3.25}$$

and we decompose $B_1(t,e)$ into

$$B_1(t,e) = \begin{bmatrix} -k_y f(\cdot) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k_y f(\cdot) \\ 0 & -k_y f(\cdot) & 0 \end{bmatrix} e.$$

Then, since

$$\frac{\partial V_1}{\partial e}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & k_y f(\cdot)\\ 0 & -k_y f(\cdot) & 0\end{bmatrix} e = 0,$$

it follows that

$$\frac{\partial V_1}{\partial e} B_1(t,e) = -\frac{\partial V_1}{\partial e_\theta} k_y f(\cdot) = -e_\theta f(\cdot).$$

Thus, using the latter equation, we obtain

$$\begin{aligned}\dot{W}(t,e) \leq & - Y(t,e) + \dot{G}(t)Z(t,e) - G(t)\rho(t)f(\cdot)\frac{\partial\left(Q_{[3]}(V_1)V_1 + V\right)}{\partial V_1} e_\theta \\ & + v_r f(\cdot)G(t)\rho(t)P_{[1]}(t,V_1)\left[-k_y e_\theta e_x - k_y e_y\right] \\ & + \omega_r G(t)\rho(t)f(\cdot)\left[-k_y e_x + k_y e_y^2\right]. \end{aligned} \tag{3.26}$$

In view of (3.9) and the boundedness of v_r and ω_r, there exists a polynomial $R_{[3]}(V_1)$ with non-negative coefficients, such that

$$\begin{aligned}R_{[3]}(V_1)V_1 \geq & - f(\cdot)\frac{\partial\left(Q_{[3]}(V_1)V_1 + V\right)}{\partial V_1} e_\theta + \omega_r f(\cdot)\left[-k_y e_x + k_y e_y^2\right] \\ & + v_r f(\cdot)P_{[1]}(t,V_1)\left[-k_y e_\theta e_x - k_y e_y\right]. \end{aligned} \tag{3.27}$$

Hence, since $V(t,e) \geq F_{[3]}(V_1)V_1$ —see (3.17), we obtain

$$\dot{W} \;\; \leq \;\; -Y(t,e) + \dot{G}(t)F_{[3]}(V_1)V_1 + G(t)\rho(t)R_{[3]}(V_1)V_1.$$

On the other hand, in view of (3.22), $\dot{G}(t) \leq -\gamma G(t)\rho(t)$ for any $\gamma > 0$ and the coefficients of $F_{[3]}(V_1)$ are strictly positive. Therefore, there exists $\gamma > 0$ such that

$$\gamma F_{[3]}(V_1) \geq R_{[3]}(V_1)$$

and, consequently, $\dot{W}(t,e) \leq -Y(t,e)$ for all $t \geq 0$ and all $e \in \mathbb{R}^3$. Uniform global asymptotic stability of the null solution of (3.19) follows.

Step 3. In order to establish iSS with respect to η and boundedness of the closed-loop trajectories subject to $\eta \in \mathcal{L}_2$, we proceed as in Proposition 2.2. Let

$$W_1(t,e) := \ln\left(1 + W(t,e)\right). \tag{3.28}$$

The derivative of W_1 along trajectories of (3.12) satisfies

$$\dot{W}_1 \quad \leq \quad -G_m \frac{\frac{\mu}{T} V_1(e) + Q_{[3]} \left[k_x e_x^2 + k_\theta e_\theta^2 \right]}{1 + W(t,e)} + \frac{\left| \frac{\partial W}{\partial e} B_2 \eta \right|}{1 + W(t,e)} \tag{3.29}$$

with $G_m := exp \left(-\gamma \int_0^\infty \rho(t) dt \right)$.

Next, we decompose $B_2(e)\eta$ introduced in (3.12) into

$$B_2(e)\eta := B_{21}(\eta) + B_{22}(\eta)e$$

where

$$B_{21}(\eta) := \begin{bmatrix} -\tilde{\omega} \\ -\tilde{v} \\ 0 \end{bmatrix}, \quad B_{22}(\eta) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\omega} \\ 0 & -\tilde{\omega} & 0 \end{bmatrix}.$$

Then, using the fact that $\frac{\partial V_1}{\partial e} B_{22}(\eta) e = 0$, defining

$$H(t,e) := Q_{[3]} + P_{[3]} + \frac{\partial Q_{[3]}}{\partial V_1} V_1 + \frac{\partial P_{[3]}}{\partial V_1} V_1 + \bar{v}_r \, |e_\theta| \, |e_y| \, \frac{\partial P_{[1]}}{\partial V_1},$$

and

$$\xi = \begin{bmatrix} \frac{e_\theta}{k_y} \\ e_x \end{bmatrix}, \tag{3.30}$$

we obtain

$$\begin{aligned} \left| \frac{\partial W}{\partial e} B_2 \eta \right| \leq \; & H(t,e)|\xi||\eta| + \bar{\omega}_r |e_y||\eta| + \bar{v}_r P_{[1]} |e_y||\eta| \\ & + \bar{\omega}_r V_1 |\eta| + \bar{v}_r P_{[1]} |e_\theta||e_x||\eta| \\ \leq \; & H(t,e) \left[\frac{1}{2\epsilon} |\xi|^2 + \frac{\epsilon}{2} |\eta|^2 \right] + \bar{\omega}_r \left[\frac{1}{2\epsilon} V_1 + \frac{\epsilon}{2} V_1 |\eta|^2 \right] \\ & + \bar{\omega}_r \left[\frac{1}{2\epsilon} V_1 + \frac{\epsilon}{2} |\eta|^2 \right] + \bar{v}_r \left[\frac{1}{2\epsilon} V_1 + \frac{\epsilon}{2} P_{[1]}^2 |\eta|^2 \right] \\ & + \bar{v}_r P_{[1]} \left[\frac{1}{2\epsilon} V_1 |e_\theta|^2 + \frac{\epsilon}{2} |\eta|^2 \right] \\ \leq \; & \left[H(t,e) + \bar{v}_r P_{[1]} k_y^2 V_1 \right] \frac{1}{2\epsilon} |\xi|^2 + \left[2\bar{\omega}_r + \bar{v}_r \right] \frac{1}{2\epsilon} V_1 \\ & + \frac{\epsilon}{2} |\eta|^2 \left[H(t,e) + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]} \right]. \end{aligned}$$

Next, we choose $\epsilon > 0$ such that

$$\frac{H(t,e) + \bar{v}_r P_{[1]} k_y^2 V_1}{\epsilon} |\xi|^2 \;\le\; G_m Q_{[3]} \left[k_x e_x^2 + k_\theta e_\theta^2 \right],$$
$$\frac{2\bar{\omega}_r + \bar{v}_r}{\epsilon} \;\le G_m \frac{\mu}{T}.$$

Such $\epsilon > 0$ exists because $Q_{[3]}$ is a third order polynomial of V_1 with strictly positive coefficients. So (3.29) becomes

$$\begin{aligned} \dot{W}_1 \;\; \le \;\; & -\frac{G_m}{2} \frac{\frac{\mu}{T} V_1(e) + Q_{[3]} \left[k_x e_x^2 + k_\theta e_\theta^2 \right]}{1 + W(t,e)} \\ & + \frac{D_{[3]}(V_1)}{1 + W(t,e)} \frac{\epsilon}{2} |\eta|^2 \end{aligned} \tag{3.31}$$

where $D_{[3]}(V_1)$ is a third order polynomial satisfying

$$H(t,e) + \bar{\omega}_r V_1 + \bar{\omega}_r + \bar{v}_r P_{[1]}^2 + \bar{v}_r P_{[1]} \le D_{[3]}.$$

From the positivity of V, (3.17), and the definition of W, we have

$$G_m Q_{[3]}(V_1) V_1 \le W_1(t,e) \le S_{[3]}(V_1) V_1 \tag{3.32}$$

hence,

$$\begin{aligned} \dot{W}_1 \;\; \le \;\; & -\frac{G_m}{2} \frac{\frac{\mu}{T} V_1(e) + Q_{[3]}(V_1) \left[k_x e_x^2 + k_\theta e_\theta^2 \right]}{1 + S_{[3]}(V_1)} \\ & + \frac{D_{[3]}(V_1)}{1 + G_m Q_{[3]}(V_1)} \frac{\epsilon}{2} |\eta|^2 . \end{aligned} \tag{3.33}$$

This implies the existence of a positive constant $c > 0$ and a positive definite function $\alpha(e)$ such that

$$\dot{W}_1 \le -\,\alpha(e) + c\, |\eta|^2 . \tag{3.34}$$

The result follows from Lemma A.4.

Under the scenario S2:

The proof of Proposition 3.1 under condition (3.2) relies on arguments for stability of cascaded systems as well as on tools tailored for systems with persistency of excitation.

We start by rewriting the closed-loop equations in a convenient form for the analysis under the conditions of Scenario S2. To that end, to compact the notation, let us introduce

$$\begin{aligned} f_\rho(t, e_x, e_y) &:= \rho(t) f(t, e_x, e_y) & (3.35) \\ \Phi(t, e_\theta, e_x, e_y) &= k_\theta e_\theta + k_y f_\rho(t, e_x, e_y) & (3.36) \end{aligned}$$

Then, the closed-loop equations become

$$\dot{e} = f_e(t, e) + g(t, e)\eta, \quad \eta = [\tilde{v}\ \tilde{\omega}]^\top, \tag{3.37}$$

where

$$\begin{aligned} f_e(t, e) &:= \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho - k_y v_r \phi(e_\theta) e_y \\ -k_x e_x + \Phi e_y + \big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_y \\ -\Phi e_x - \big[\omega_r + k_y v_r \phi(e_\theta) e_y\big] e_x + v_r \sin e_\theta \end{bmatrix}, \\ g(t, e) &:= \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}. \end{aligned}$$

Following the proof-lines of [97, Lemma 1] for cascaded systems, we establish the following for the system (3.37):

Claim 1. The solutions are uniformly globally bounded subject to $\eta \in \mathcal{L}_2$,

Claim 2. The origin of $\dot{e} = f_e(t, e)$ is uniformly globally asymptotically stable (*i.e.*, **P1**).

After [8] the last two claims together imply integral ISS with respect to η (*i.e.*, **P2**). Moreover, Claim 1 implies the convergence of the closed-loop trajectories to the origin provided that the input η tends to zero and is square integrable (*i.e.*, **P3**).

Proof of Claim 1. Let

$$W(e) := \ln(1 + V_1(e)), \qquad V_1(e) := \frac{1}{2}\left[\, e_x^2 + e_y^2 \,\right]. \tag{3.38}$$

The total derivative of V_1 along the trajectories of (3.37) yields

$$\dot{V}_1(e) \le -k_x e_x^2 + |e_x||\tilde{v}| + |v_r||\sin(e_\theta)||e_y| \tag{3.39}$$

hence,

$$\begin{aligned} \dot{W}(e) &\le \frac{1}{1 + V_1}\Big[-\frac{k_x}{2} e_x^2 + |v_r||e_y| + \frac{\tilde{v}^2}{2k_x} \Big] & (3.40) \\ &\le \frac{|e_y|}{1 + V_1}|v_r| + \frac{1}{2k_x[1 + V_1]}\tilde{v}^2. & (3.41) \end{aligned}$$

Integrating on both sides of (3.41) along the trajectories, from t_0 to t, and invoking the integrability of v_r and the square integrability of η we see that $W(e(t))$ is bounded for all $t \geq t_0$. Boundedness of $e_x(t)$ and $e_y(t)$ follows since W is positive definite and radially unbounded in (e_x, e_y).

Remark 3.3. *For further development, we also emphasize that proceeding as above from Inequality* (3.40) *we conclude that $e_x \in \mathcal{L}_2$, uniformly in the initial conditions.*

Next, we observe that the $\dot{e}_\theta$–equation in (3.37) corresponds to an exponentially stable system with bounded input $u(t) = -k_y v_r(t)\phi(e_\theta(t))e_y(t) - k_y f_\rho(t, e_x(t), e_y(t)) - \tilde{\omega}(t)$ hence, we also have $e_\theta \in \mathcal{L}_\infty$.

Proof of Claim 2.

Let $\eta = 0$ and, for further development, let us split the drift of the nominal system $\dot{e} = f_e(t, e)$ into the output injection form:

$$f_e(t, e) = F(t, e) + K(t, e) \tag{3.42}$$

where

$$K(t, e) := \begin{bmatrix} -k_y v_r \phi(e_\theta) e_y \\ \left[\omega_r + k_y v_r \phi(e_\theta) e_y\right] e_y \\ -\left[\omega_r + k_y v_r \phi(e_\theta) e_y\right] e_x + v_r \sin e_\theta \end{bmatrix} \tag{3.43}$$

and

$$F(t, e) := \begin{bmatrix} -k_\theta e_\theta - k_y f_\rho \\ -k_x e_x + \Phi e_y \\ -\Phi e_x \end{bmatrix}.$$

To establish UGAS for the origin of $\dot{e} = f_e(t, e)$ we invoke the output-injection lemma–see Appendix A.6. According to the latter, UGAS follows if:
a) there exist: an "output" y, non decreasing functions k_1, k_2, and β: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a class $\mathcal{K}_\infty$ function k, and a positive definite function γ such that, for all $t \geq 0$ and all $e \in \mathbb{R}^3$,

$$|K(t, e)| \leq k_1(|e|)k(|y|) \tag{3.44}$$

$$|y(t, e)| \leq k_2(|e|) \tag{3.45}$$

$$\int_\circ^\infty \gamma\big(|y(t)|\big)dt \leq \beta(|e(0)|); \tag{3.46}$$

b) the origin of $\dot{e} = f_e(t, e)$ is uniformly globally stable;
c) the origin of $\dot{e} = F(t, e)$ is UGAS.

Condition a. Using (3.43), a direct computation shows that there exists $c > 0$ such that

$$|K(t, e)| \leq c\left(|e|^2 + |e|\right) |\, [v_r\ \omega_r]\, |, \tag{3.47}$$

so (3.44) holds with $k_1(s) := c(s^2 + s)$, $k(s) := s$, and $y := [v_r\ \omega_r]$. Moreover, (3.45) and (3.46) hold with $\gamma(s) = s$, since $[v_r\ \omega_r] \in \mathcal{L}_1$, for a constant functions β and k_2 which, moreover, are independent of the initial state.

Condition b. Uniform global stability is tantamount to uniform stability and uniform global boundedness of the solutions —see [36]. The latter was established already for the closed-loop system under the action of the "perturbation" η hence, it holds all the more in this case, where $\eta = 0$.
In order to establish uniform stability, we use Lyapunov's direct method. Let $R > 0$ be arbitrary but fixed.
We claim that, for the system $\dot{e} = F(t, e)$, there exists a Lyapunov function candidate $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ and positive constants α_1, α_2, and α_3 such that

$$\alpha_1 |e|^2 \leq V(t,e) \leq \alpha_2 |e|^2 \qquad \forall t \geq 0,\ e \in \mathbb{R}^3 \tag{3.48}$$

$$\left|\tfrac{\partial V(t,e)}{\partial e}\right| \leq \alpha_3 |e| \qquad \forall t \geq 0,\ e \in \mathbb{R}^3 \tag{3.49}$$

$$\tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial e} F(t,e) \leq 0 \qquad \forall t \geq 0,\ e \in B_R. \tag{3.50}$$

Furthermore, from (3.47) it follows that, for all $t \geq 0$ and all $e \in B_R$,

$$|K(t,e)| \leq c(R+1)\big[|v_r| + |\omega_r|\big]|e|.$$

Then, evaluating the time derivative of V along the trajectories of (3.42), we obtain, for all $e \in \mathcal{B}_R$,

$$\begin{aligned} \dot{V}(t,e) &\leq \frac{\partial V(t,e)}{\partial e} K(t,e) \leq \alpha_3 c(R+1)\big[|v_r| + |\omega_r|\big]|e|^2 \\ &\leq \frac{\alpha_3 c(R+1)}{\alpha_1}\big[|v_r| + |\omega_r|\big] V(t,e). \end{aligned} \tag{3.51}$$

Defining $v(t) := V(t, e(t))$ and invoking the comparison lemma, we conclude, for all $e \in \mathcal{B}_R$, that

$$v(t) \leq \exp\left(\frac{c\alpha_3(R+1)}{\alpha_1} \int_{t_\circ}^{\infty} (|v_r(s)| + |\omega_r(s)|)\, ds\right) v(t_\circ)$$

and, in view of the integrability condition (3.2), we obtain

$$|e(t)|^2 \leq \frac{\alpha_2}{\alpha_1} \exp\left(\frac{\alpha_3 c(R+1)}{\alpha_1}\beta\right) |e(t_\circ)|^2$$

Thus, uniform stability of (3.42) follows.
It is left to construct a Lyapunov function candidate V for the system $\dot{e} = F(t,e)$, that

satisfies the conditions (3.48)-(3.50). To that end, consider the coordinates

$$e_z = e_\theta + g(t, e_y) \tag{3.52}$$

where $g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$g(t, e_y) := e^{-k_\theta(t-t_\circ)} g(t_\circ, e_y) + \int_{t_\circ}^{t} k_y e^{-k_\theta(t-s)} f(s, 0, e_y) ds$$

and, for further development we observe that

$$\frac{\partial g}{\partial t}(t, e_y) = -k_\theta g(t, e_y) + k_y f_\rho(t, 0, e_y). \tag{3.53}$$

Let $g(t_\circ, e_y)$ be such that $|g(t_\circ, e_y)| \leq |e_y|$ which implies, using Assumption 3.1, that

$$|g(t, e_y)| \leq (1 + k_y \sigma_2) |e_y| . \tag{3.54}$$

In the new coordinates, we obtain

$$\dot{e}_z = -k_\theta e_z - \frac{\partial g}{\partial e_y} \Phi e_x - k_y \tilde{f}(t, e_x, e_y)$$

where $\tilde{f}(t, e_x, e_y) := f_\rho(t, e_x, e_y) - \tilde{f}_\rho(t, 0, e_y)$. Then, Assumption 3.1 implies that for any $R > 0$ there exists a positive constant $c_R > 0$ such that

$$\max_{e \in B_R} \left\{ \sup_{t \geq 0} \left| \tilde{f}_\rho(t, e_x, e_y) \right|, \quad \sup_{t \geq 0} \left| \frac{\partial g}{\partial e_y} \Phi e_x \right| \right\} \leq c_R |e_x| .$$

Thus, consider the following Lyapunov function candidate

$$V(t, e) := \left[\frac{1}{2} \frac{c_R^2}{k_\theta k_x} + (1 + k_y \sigma_2)^2 \right] \left[e_x^2 + e_y^2 \right] + \frac{1}{2} e_z^2 \tag{3.55}$$

which trivially satisfies (3.49). Its total time derivative is

$$\begin{aligned} \dot{V}(t, e) &= -\frac{c_R^2}{k_\theta} e_x^2 - e_z \left[k_\theta e_z + \frac{\partial g}{\partial e_y} \Phi e_x + k_y \tilde{f}(t, e_x, e_y) \right] \\ &\leq -\frac{c_R^2}{k_\theta} e_x^2 - k_\theta e_z^2 - c_R |e_z||e_x| \leq 0, \quad \forall \in \mathcal{B}_R, \end{aligned} \tag{3.56}$$

so (3.50) holds. Using (3.54) and the inequalities

$$e_z^2 \geq e_\theta^2 - 2|e_\theta||g(t, e_y)| + |g(t, e_y)|^2 \geq \frac{1}{2} e_\theta^2 - (1 + k_y \sigma_2)^2 |e_y|^2.$$

$$e_z^2 \le e_\theta^2 + 2|e_\theta||g(t,e_y)| + |g(t,e_y)|^2 \le 2e_\theta^2 + 2(1+k_y\sigma_2)^2|e_y|^2,$$

we see that the following bounds on V follow

$$V(t,e) \ge \frac{1}{2}\frac{c_R^2}{k_\theta k_x}\left[e_x^2 + e_y^2\right] + \frac{1}{4}e_\theta^2$$

$$V(t,e) \le \left[\frac{1}{2}\frac{c_R^2}{k_\theta k_x} + 2(1+k_y\sigma_2)^2\right]\left[e_x^2 + e_y^2\right] + e_\theta^2.$$

Thus the inequalities in (3.48) also hold.

Condition c. Since the solutions are uniformly globally bounded, for any $r > 0$, there exists $R > 0$ such that $|e(t)| \le R := \{|e| \le R\}$ for all $t \ge t_\circ$, all $e_\circ \in B_r$, and all $t_\circ \ge 0$. It is only left to establish uniform global attractivity. To that end, we observe that the nominal $\dot{e} = F(t,e)$ has the form

$$\dot{e}_\theta = -k_\theta e_\theta - k_y f_\rho(t, e_x, e_y) \tag{3.57a}$$

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-k_x & \Phi_\theta(t,e_x,e_y)\\ -\Phi_\theta(t,e_x,e_y) & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} \tag{3.57b}$$

where, for each $e_\theta \in B_R$, we define the smooth *parameterised* function $\Phi_\theta : \mathbb{R}_{\ge 0}\times\mathbb{R}^2 \to \mathbb{R}$ as

$$\Phi_\theta(t,e_x,e_y) := \Phi(t,e_\theta,e_x,e_y).$$

Then, the system (3.57) may be regarded as a cascaded system —*cf.* [59]. Moreover, the system (3.57a) is input-to-state stable and the perturbation term $k_y f_\rho(t, e_x(t), e_y(t))$ is uniformly bounded. Therefore, in order to apply a statement for cascaded systems, we must establish that the origin of (3.57b) is globally asymptotically stable, uniformly in the initial conditions $(t_\circ, e_{x\circ}, e_{y\circ}) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^2$ and in the "parameter" $e_\theta \in B_R$. For this, we invoke [65, Theorem 3] as follows. Since $k_x > 0$ there is only left to show that $\Phi_\theta^\circ(t,e_y)e_y$, where

$$\Phi_\theta^\circ(t,e_y) := \Phi_\theta(t,0,e_y),$$

is uniformly δ-persistently exciting with respect to e_y, uniformly for any $\theta \in B_R$ —*cf.* Definition A.7, [65, Definition 3], [63]. Since Φ_θ° is smooth, it suffices to show that for any $|e_y| \ne 0$ and r, there exist T and μ such that

$$|e_y| \ne 0 \implies \int_t^{t+T}\left|\tilde{\Phi}_\theta^\circ(\tau,e_y)\right|d\tau \ge \mu \quad \forall\, t \ge 0 \tag{3.58}$$

—see [65, Lemma 1].

Remark 3.4. *In general, μ depends both on e_θ and on e_y, but since $e_\theta \in B_R$ and B_R is compact,* *by continuity, one can always choose the smallest qualifying μ, for each fixed e_y. Therefore, as in [65], μ may be chosen as a class $\mathcal{K}$ function dependent of $|e_y|$ only.*

Now, we show that (3.58) holds under Assumption 3.2. To that end, we remark that

$$\Phi_\theta^\circ(t, e_y) = k_\theta e_\theta + k_y \rho(t) f_\circ(t, e_y)$$

–*cf.* Eq. (3.36), satisfies

$$\dot{\Phi}_\theta^\circ \;\; = \;\; -k_\theta \Phi + k_y \dot{\rho} f_\circ + k_y \rho \frac{\partial f_\circ}{\partial t} - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

where we used $\dot{e}_\theta = -\Phi$ and $\dot{e}_y = \Phi e_x$. Therefore, defining

$$K_\Phi(t, e) \;\; := \;\; k_\theta[\Phi_\theta^\circ - \Phi] - k_y \rho \frac{\partial f_\circ}{\partial e_y} \Phi e_x$$

we obtain

$$\dot{\Phi}_\theta^\circ \;\; = \;\; -k_\theta \Phi_\theta^\circ - k_y \rho \frac{\partial f_\circ}{\partial t} + k_y \dot{\rho} f_\circ + K_\Phi(t, e).$$

The latter equation corresponds to that of a linear filter with state Φ_θ° and input

$$\Psi(t, e_y) := -k_y \rho(t) \frac{\partial f_\circ}{\partial t}(t, e_y) + k_y \dot{\rho}(t) f_\circ(t, e_y) + K_\Phi(t, e(t))$$

therefore, after [66, Property 4], Φ_θ° is uniformly δ-PE with respect to e_y, if so is Ψ. Now, from Assumption 3.2 and uniform global boundedness of the solutions, for any r there exists $c > 0$ such that

$$\left| k_y \dot{\rho}(t) f_\circ(t, e_y(t)) + K_\Phi(t, e(t)) \right| \leq c(r) \big[\, |e_x(t)| + |\dot{\rho}(t)| \,\big]$$

Therefore, uniform δ-PE with respect to e_y of Ψ follows from Assumption **A2** and the fact that $\dot{\rho}$ and e_x are uniformly square integrable. That $\dot{\rho} \in \mathcal{L}_2$, with a bound uniform in the initial times, follows from (3.10) because v_r, ω_r, and ρ are bounded and $|v_r| + |\omega_r|$ is uniformly integrable. That e_x is uniformly $\mathcal{L}_2$ follows from (3.40) —see Remark 3.3. This concludes the proof of UGAS for the nominal system $\dot{e} = f_e(t, e)$ hence, **Claim 2.** is proved.
This completes the proof of Proposition 3.1.

3.2 Control under relaxed conditions on the reference velocities

In [85], simultaneous stabilization and tracking problem has been addressed under a general tracking/stabilization scenario that includes all possible behavior of the leader's velocities (v_r, ω_r), by using the concept of transverse functions. In our case we propose to extend the idea of control design proposed in [119] and, moreover, we consider more general scenarios of leader's velocities as in [85], Advantage of our approach is that it allows a straightforward extension to the leader-follower formation case.

Consider the two following scenarios

S1: Tracking scenario. There exists T and μ such that, for all $t \geq t_0$:

$$\int_t^{t+T} \left(|v_r(\tau)|^2 + |\omega_r(\tau)|^2\right) d\tau > \mu > 0, \ \forall t \geq t_0. \tag{3.59}$$

S3: Robust stabilization scenario.

$$\lim_{t \to \infty} v_r(t) = 0 \tag{3.60a}$$

$$\lim_{t \to \infty} \omega_r(t) = 0 \tag{3.60b}$$

In this case, we propose the following family of virtual control laws at the kinematic level

$$v^* = v_r cos e_\theta + k_x e_x \tag{3.61}$$

$$\omega^* = \omega_r + k_\theta e_\theta + k_y e_y v_r \phi(e_\theta) + \rho(t) k_y p(t) \sqrt{e_x^2 + e_y^2}, \tag{3.62}$$

$$\rho(t) := \exp\left(-\int_{t_0}^{t} F\left(v_r(\tau), \omega_r(\tau)\right) d\tau\right) \tag{3.63}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a piecewise constant function that verifies the following

1. If **S3** holds, then,
$$\int_{t_0}^{t} F(v_r(\tau), \omega_r(\tau)) d\tau < \infty, \quad \forall t \geq 0$$

2. If **S1** holds, then, there exists T_1 and μ_1 such that

$$\int_t^{t+T_1} F\left(v_r(s), \omega_r(s)\right)^2 ds \geq \mu_1, \quad \forall t \geq 0.$$

Remark 3.5. *The definition of ρ in* (3.63) *covers that in* (3.7) *employed in the previous section.*

The following lemma establishes the existence of F by providing a candidate that satisfies the last two items

Lemma 3.1. *Let $\alpha(t) := \sqrt{v_r^2(t) + \omega_r^2(t)}$, where $v_r(t)$ and $\omega_r(t)$ are two scalar continuous functions. Assume that there exists $\bar{\alpha} > 0$ such that $|\alpha(t)|_\infty \leq \bar{\alpha}$. Then, the functional*

$$F(v_r, \omega_r) := K(\alpha) := \begin{cases} \alpha & \text{if } \alpha \in (0, \frac{\mu}{2T\bar{\alpha}}] \\ 0 & Otherwise \end{cases} \tag{3.64}$$

satisfies the following:

1. *$K(\alpha(t))$ is PE, if $\alpha(t)$ is PE.*

2. *$K(\alpha(t))$ is integrable, if $\lim_{t\to\infty} \alpha(t) = 0$.*

Proof. The proof of the second item is trivial, because $K(\alpha)$ is null after finite time T_f, and

$$\int_0^t K(\alpha(s))ds \leq \int_0^{T_f} K(\alpha(s))ds, \qquad \forall t \geq 0.$$

To prove the first item, we use [63, Lemma 2] which states that, if $\alpha(t)$ is PE $\left(\int_t^{t+T} \alpha(\tau)d\tau \geq \mu\right)$. Then for every $t \geq 0$ there exists a non null measure set

$$I_t := \{\tau \in [t, t+T] : |\alpha(\tau)| \geq a := \mu/(2T\bar{\alpha})\},$$

and

$$meas(I_t) \geq b := T\mu/(2T\bar{\alpha}^2 - \mu).$$

Using this lemma we obtain

$$\int_t^{t+T} K^2(\alpha(s))ds \geq \int_{I_t} K^2(\alpha(s))ds \geq \int_{I_t} a^2 ds \geq a^2 b > 0.$$

Hence is $K(\alpha(s))$ PE. □□□

Proposition 3.2. *Consider the system* (2.5) *with $v = \tilde{v} + v^*$, $\omega = \tilde{\omega} + \omega^*$, and the virtual inputs* (3.61) *and* (3.62). *Let k_x, k_θ, and $k_y > 0$; let p and $\dot{p}$ be bounded and persistently exciting, and assume that there exist $\bar{\omega}_r$, $\bar{\dot{\omega}}_r$, $\bar{v}_r$, $\bar{\dot{v}}_r > 0$ such that* (3.11) *holds. Then,*

1) *Under the condition of scenario **S1**, the closed-loop system is integral input-to-state stable with respect to* $\eta_1 := [\tilde{v}\ \tilde{\omega}]^\top$. *Moreover, if* η *is a converging square integrable function, then the closed-loop trajectories converge to the origin.*

2) *Under the condition of scenario **S3**, the closed-loop system is strongly integral input-to-state stable with respect to* $\eta_2 := [v_r\ \omega_r\ \tilde{v}\ \tilde{\omega}]^\top$.

□

3.2.1 Proof of Proposition 3.2

Under the scenario S1:

We decompose the closed-loop system as follows

$$\dot{e} = A_{v_r}(t,e)e + B_1(t,e)\rho(t) + B_2(e)\eta, \tag{3.65}$$

where

$$A_{v_r}(t,e) := \begin{bmatrix} -k_\theta & 0 & -v_r(t)k_y\phi(e_\theta) \\ 0 & -k_x & \omega^*(t,e) \\ v_r(t)\phi(e_\theta) & -\omega^*(t,e) & 0 \end{bmatrix}, \quad B_1(t,e) := \begin{bmatrix} -k_y p(t)\sqrt{e_y^2+e_x^2} \\ k_y p(t)\sqrt{e_y^2+e_x^2}e_y \\ -k_y p(t)\sqrt{e_y^2+e_x^2}e_x \end{bmatrix},$$
$$B_2(e) := \begin{bmatrix} 0 & -1 \\ -1 & e_y \\ 0 & -e_x \end{bmatrix}. \tag{3.66}$$

The proof under **S1**, follows exactly the same steps as in Proposition 3.1.

Under the scenario S3:

We start by rewriting the closed-loop system as

$$\dot{e} = A(t,e)e + B(e)\eta_2 \tag{3.67}$$

where

$$\begin{aligned} A(t,e) &:= \begin{bmatrix} -k_\theta & -k_y p_1(t)\frac{\sqrt{e_x^2+e_y^2}}{e_x} & -k_y p_1(t)\frac{\sqrt{e_x^2+e_y^2}}{e_y} \\ 0 & -k_x & \psi(t,e) \\ 0 & -\psi(t,e) & 0 \end{bmatrix}, \\ \psi(t,e) &:= k_\theta e_\theta + k_y p_1(t)\sqrt{e_y^2+e_x^2}. \\ B(e) &= \begin{bmatrix} -k_y e_y \phi(e_\theta) & 0 & 0 & -1 \\ k_y e_y^2 \phi(e_\theta) & e_y & -1 & e_y \\ \sin(e_\theta) - k_y e_x e_y \phi(e_\theta) & -e_x & 0 & -e_x \end{bmatrix} \end{aligned}$$

To establish the strong iISS property of the closed-loop system with respect to η_2, we follow the same proof steps as for Proposition 2.5. That is,

Step 1. we construct strict Lyapunov function for the closed-loop system when $\eta_2 = 0$;

Step 2. we establish the small ISS property of the closed-loop with respect to η_2;

Step 3. we establish the integral ISS property of the closed-loop with respect to η_2.

Remark 3.6. *To simplify the computations, we introduce*

$$p_1(t) := \rho(t)p(t), \tag{3.68}$$

where $\rho(t)$ is given in (3.63). *It is important to notice that under **S3**, $p_1(t)$ has the same properties as $p(t)$. That is, functions p_1 and $\dot{p}_1$ are bounded and, since $\dot{\rho}(t)$ converges to zero as $t \to \infty$, then $\dot{p}_1 (= \rho\dot{p} + p\dot{\rho})$ is PE if $\dot{p}$ is so –see Lemma A.9 in the Appendix.*

Step 1. UGAS of the nominal system $\dot{e} = A(t,e)e$.

Let ψ_m, ψ_M and $\psi_M \geq \psi_m > 0$ and consider a positive differentiable function $\psi : \mathbb{R}_{\geq 0} \to [\psi_m, \psi_M]$ satisfying

$$\dot{\psi} = -k_\theta \psi + k_y p_1(t). \tag{3.69}$$

Such a function exists because p_1 is bounded and persistently exciting [112]. Then, consider the new error coordinate

$$e_z = e_\theta + \psi(t)\sqrt{e_y^2 + e_x^2},$$

which satisfies

$$\dot{e}_z = -k_\theta e_z - \psi k_x \frac{e_x^2}{\sqrt{e_x^2 + e_y^2}}. \tag{3.70}$$

In the new coordinates, the nominal system becomes

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-k_x & \dot{\psi}\sqrt{e_y^2+e_x^2}\\ -\dot{\psi}\sqrt{e_y^2+e_x^2} & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} + e_z\begin{bmatrix}0 & k_\theta\\ -k_\theta & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} \tag{3.71a}$$

$$\dot{e}_z = -k_\theta e_z - \psi k_x \frac{e_x}{\sqrt{e_x^2+e_y^2}} e_x \tag{3.71b}$$

Remark 3.7. *Note that, by replacing e_z with the trajectories $e_z(t)$ the system* (3.71) *covers a cascaded form —see [59, 97]. Moreover, it is easy to show that e_x and e_y are bounded.*

Now, since $\dot{p}_1$ is persistently exciting (see Remark 3.6) and ψ satisfies the equation

$$\ddot{\psi} = -k_\theta\dot{\psi} + \dot{p}_1 \tag{3.72}$$

we can conclude that function $\dot{\psi}$ is also persistently exciting [41, Lemma 4.8.3]. Based on these properties, Lemma 3.2 below provides a strict differentiable Lyapunov function for the system (3.71).

Lemma 3.2 (set-point stabilization)**.** *Consider the following nonlinear time-varying system*

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-k_x & \dot{\psi}\sqrt{e_y^2+e_x^2}\\ -\dot{\psi}\sqrt{e_y^2+e_x^2} & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} + e_z\begin{bmatrix}0 & k_\theta\\ -k_\theta & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} \tag{3.73a}$$

$$\dot{e}_z = -k_\theta e_z - \psi k_x \frac{e_x^2}{\sqrt{e_x^2+e_y^2}} \tag{3.73b}$$

let k_θ, $k_x > 0$, $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\dot{\psi}$ be persistently exciting and let

$$\max\left\{\,|\psi|_\infty,\ |\dot{\psi}|_\infty,\ |\ddot{\psi}|_\infty\right\} \leq \bar{\psi}$$

where

$$|\psi|_\infty := \operatorname*{ess\,sup}_{t\geq 0} |\psi(t)|$$

Then, the system (3.73) *admits the following strict Lyapunov function:*

$$V_2(t,e) := P_1(V_1)V_1(e) + \Upsilon_{\dot{\psi}(s)^2}(t)V_1(e)^2 - \dot{\psi}(t)\sqrt{V_1}e_xe_y + Q_1(V_1)e_z^2 \tag{3.74}$$

where $V_1(e) := e_x^2 + e_y^2$,

$$\Upsilon_{\dot{\psi}(s)^2}(t) := 1 + \bar{\psi}^2 T - \frac{1}{T}\int_t^{t+T}\int_t^m \dot{\psi}(s)^2 ds dm \tag{3.75}$$

— cf (1.4)*, P_1 and $Q_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are first order polynomials of V_1 with positive coefficients*

defined as

$$
\begin{aligned}
P_1(V_1) &:= \frac{1}{k_x}\bar{\psi}^2 V_1 + \bar{\psi}\frac{V_1}{4} + \frac{k_x}{4k_\theta}\bar{\psi}^2 Q_1(V_1) + \bar{\psi}^2\frac{T(k_x+k_\theta)}{4\mu}V_1 + 1, &\quad (3.76)\\
Q_1(V_1) &:= \frac{T(k_x+k_\theta)}{\mu}\bar{\psi}^2 V_1 + 1. &\quad (3.77)
\end{aligned}
$$

with

$$
\dot{V}_2(t,e) \leq -\frac{1}{2}k_\theta Q_1(V_1)e_z^2 - \frac{\mu}{2T}V_1^2. \quad (3.78)
$$

The proof of Lemma 3.2 is presented in the Appendix B.7. **Step 2.** Small ISS property.

Similarly to the proof of Proposition 2.5 in the previous Chapter, the proof of the small ISS property, for the closed-loop system (3.67) with respect to η_2, relies on the function V_2 constructed in Lemma 3.2 above; specifically on its order of growth in V_1. For the purpose of analysis we recall that V_2 in (3.74) satisfies

$$
V_2(t,e) \geq P_1(V_1)V_1 - \dot{\psi}\sqrt{V_1}e_x e_y + Q_1(V_1)e_z^2, \quad (3.79)
$$

where $P_1(V_1)$ and $Q_1(V_1)$ are first order polynomials with respect to V_1 with positive coefficients and time-derivative of $V_2(t,e)$ along the nominal part verifies

$$
\dot{V}_2(t,e) = -\frac{\mu}{2T}V_1^2 - k_\theta\frac{Q_1(V_1)}{2}e_z^2 \quad (3.80)
$$

To establish the small ISS property of the closed-loop with respect to η, let us consider the time-derivative of V_2 along trajectories of (3.67)

$$
\dot{V}_2(\cdot) \leq -\frac{\mu}{2T}V_1^2 - k_\theta\frac{Q_1(V_1)}{2}e_z^2 + \frac{\partial V_2}{\partial e}B(e)\eta \quad (3.81)
$$

After decomposing $B(e)\eta$ as

$$
B(e)\eta := B_1(\eta,e)e + B_2(\eta,e), \quad (3.82)
$$

with

$$
B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (\tilde{\omega}+\omega_r+k_y v_r e_y\phi(e_\theta)) \\ 0 & -(\omega_r+\tilde{\omega}+k_y v_r e_y\phi(e_\theta)) & 0 \end{bmatrix},
$$

and

$$B_2 = \begin{bmatrix} -\tilde{\omega} - k_y v_r e_y \phi(e_\theta) \\ \tilde{v} \\ v_r \sin e_\theta \end{bmatrix}.$$

and using the fact that

$$\frac{\partial V_1}{\partial e} B_1 e = 0. \tag{3.83}$$

we obtain

$$\begin{aligned}
\dot{V}_2(\cdot) \leq & -\frac{\mu}{2T} V_1^2 - k_\theta \frac{Q_1(V_1)}{2} e_z^2 - 2\dot{\psi}(\omega_r + \tilde{\omega})\sqrt{V_1}\left[e_y^2 - e_x^2\right] - \\
& \dot{\psi}\sqrt{V_1} e_y v_r \left[e_y^2 - e_x^2\right] + \frac{\partial V_2}{\partial e} B_2 \\
\leq & -\frac{\mu}{2T} V_1^2 - k_\theta \frac{Q_1(V_1)}{2} e_z^2 + 2\bar{\psi}\left|\omega_r + \tilde{\omega}\right| \sqrt{V_1} V_1 \\
& + \bar{\psi}\sqrt{V_1}\left|e_y\right|\left|v_r\right| V_1 + \frac{\partial V_2}{\partial e} B_2.
\end{aligned} \tag{3.84}$$

Next, we upperbound the term $\frac{\partial V_2}{\partial e} B_2$

$$\begin{aligned}
\left|\frac{\partial V_2}{\partial e} B_2\right| \leq & \left(P_1(V_1) + \frac{\partial P_1}{\partial V_1} V_1 + 2\bar{\psi}\sqrt{V_1}\right) \left|[e_x, e_y]\right| \left|[\tilde{v},\ v_r]\right| \\
& + \left(\frac{\partial Q_1}{\partial V_1} e_z^2 \left|[e_x,\ e_y]\right| + Q_1(V_1)\left|\psi e_z\right|\right) \left|[\tilde{v},\ v_r]\right| \\
& + Q_1(V_1)\left|e_z\right|\left|e_y\right|\left|v_r\right| + Q_1(V_1)\left|e_z\right|\left|\tilde{\omega}\right|
\end{aligned} \tag{3.85}$$

to obtain the following bound on $\dot{V}_2$

$$\begin{aligned}
\dot{V}_2(\cdot) \leq & -\frac{\mu}{2T} V_1^2 - k_\theta \frac{Q_1(V_1)}{2} e_z^2 + 2\bar{\psi}\left|\omega_r + \tilde{\omega}\right| \sqrt{V_1} V_1 + \bar{\psi}\sqrt{V_1}\left|e_y\right|\left|v_r\right| V_1 \\
& + \left(P_1 + \frac{\partial P_1}{\partial V_1} V_1 + 2\bar{\psi}\sqrt{V_1}\right)\sqrt{V_1}\left|\eta\right| \\
& + \left(\frac{\partial Q_1}{\partial V_1} e_z^2 \sqrt{V_1} + Q_1(V_1)\bar{\psi}\left|e_z\right|\right)\left|\eta\right| \\
& + Q_1(V_1)\left|e_z\right|\left|e_y\right|\left|v_r\right| + Q_1(V_1)\left|e_z\right|\left|\tilde{\omega}\right|.
\end{aligned}$$

$$
\begin{aligned}
\leq & -\frac{\mu}{2T}V_1^2 - k_\theta \frac{Q_1(V_1)}{2}e_z^2 + 4\bar{\psi}\,|\eta|\,\sqrt{V_1}V_1 + \bar{\psi}\,|\eta|\,V_1^2 \\
& + \left(P_1 + \frac{\partial P_1}{\partial V_1}V_1 + 2\bar{\psi}\sqrt{V_1}\right)\sqrt{V_1}\,|\eta| \\
& + \left(\frac{\partial Q_1}{\partial V_1}V_1 e_z^2 + Q_1(V_1)\bar{\psi}^2 + Q_1(V_1)e_z^2\right)|\eta| \\
& + Q_1(V_1)(e_z^2 + V_1)\,|\eta| + Q_1(V_1)(e_z^2 + 1)\,|\eta|\,.
\end{aligned}
$$

$$
\begin{aligned}
\leq & - V_1^2\left[\frac{\mu}{2T} - \left(\bar{\psi} + Q_{11}\right)|\eta|\right] + 2P_{11}V_1\sqrt{V_1}\,|\eta| \\
& + 4\bar{\psi}\,|\eta|\,\sqrt{V_1}V_1 + 2\bar{\psi}\,|\eta|\,V_1 + P_{12}\sqrt{V_1}\,|\eta| \\
& + Q_1(V_1)\left[\bar{\psi}^2 + 1\right]|\eta| + Q_{12}V_1\,|\eta| - \frac{Q_1(V_1)}{2}e_z^2\left[\frac{k_\theta}{2} - 4\,|\eta|\right]. \qquad (3.86)
\end{aligned}
$$

where Q_{11}, Q_{12}, P_{11}, and P_{12} are positive constants, such that

$$
Q_1(V_1) := Q_{11}V_1 + Q_{12}, \qquad P_1(V_1) := P_{11}V_1 + P_{12}.
$$

So the small ISS property of (3.67) with respect to η follows by observing that the system is ISS with respect to η for all η satisfying the bound

$$
|\eta| < \min\left\{\frac{k_\theta}{8}, \frac{\mu}{2T\left(\bar{\psi} + Q_{11}\right)}\right\},
$$

Step 3. The iISS property.

The proof of Proposition 3.2 is finalized by establishing integral input-to-state stability of the system (3.67) with respect to η. To that end, we proceed similar to Proposition 3.1 and we consider the proper positive-definite Lyapunov function

$$
W_2(t,e) = \ln\left(1 + V_2(t,e)\right) \qquad (3.87)
$$

We can see that $W_2(t,e)$, is a proper Lyapunov function since so is V_2. Moreover, the total time-derivative of W_2 along trajectories of the closed-loop system yields

$$
\dot{W}_2 = \frac{\dot{V}_2}{1 + V_2} \leq -\frac{\frac{\mu}{2T}V_1^2 + k_\theta\frac{Q_1(V_1)}{2}e_z^2}{1 + V_2} + \frac{\partial V_2}{\partial e}B\eta/(1 + V_2) \qquad (3.88)
$$

From (3.76) and (3.77), we conclude that there exists a first order polynomial $g_1(V_1)$

with a strictly positive coefficients, such that

$$V_2(t,e) \geq g_1(V_1)\left(V_1 + e_z^2\right) \tag{3.89}$$

From this, it follows that there exists a class $\mathcal{K}$ function α such that:

$$\alpha(|e|) \leq \frac{\frac{\mu}{2T}V_1^2 + k_\theta \frac{g_1(V_1)}{2} e_z^2}{1 + V_2(t,e)}. \tag{3.90}$$

Then, using (3.82), (3.83) and the inequality (3.85), we can bound $\dot{W}_2(t,e)$ as

$$\begin{aligned}
\dot{W}_2 \leq & -\alpha(|e|) + \frac{2\bar{\psi}\,|\omega_r + \tilde{\omega}|\,\sqrt{V_1}V_1}{1 + g_1(V_1)V_1} + \frac{\bar{\psi}\sqrt{V_1}\,|e_y|\,|v_r|\,V_1}{1 + g_1(V_1)V_1} \\
& + \frac{\left(P_1(V_1) + \frac{\partial P_1}{\partial V_1}V_1 + 2\bar{\psi}\sqrt{V_1}\right)|[e_x,\ e_y]|\,|[\tilde{v},\ v_r]|}{1 + g_1(V_1)V_1} \\
& + \frac{\left(\frac{\partial Q_1}{\partial V_1}e_z^2\,|[e_x,\ e_y]| + Q_1(V_1)\,|\psi e_z|\right)|[\tilde{v},\ v_r]|}{1 + g_1(V_1)\left(e_z^2 + V_1\right)} \\
& + \frac{Q_1(V_1)\,|e_z|\,|e_y|\,|v_r| + Q_1(V_1)\,|e_z|\,|\tilde{\omega}|}{1 + g_1(V_1)(e_z^2 + V_1)}
\end{aligned}$$

$$\begin{aligned}
\leq & -\alpha(|e|) + \frac{2\bar{\psi}\,|\omega_r + \tilde{\omega}|\,\sqrt{V_1}V_1}{1 + g_1(V_1)V_1} + \frac{\bar{\psi}\,|v_r|\,V_1^2}{1 + g_1(V_1)V_1} \\
& + \frac{\left(P_1(V_1)\sqrt{V_1} + \frac{\partial P_1}{\partial V_1}V_1\sqrt{V_1} + 2\bar{\psi}V_1\right)|\eta|}{1 + g_1(V_1)V_1} \\
& + \frac{\left(\frac{\partial Q_1}{\partial V_1}e_z^2(V_1 + 1) + Q_1(V_1)\left(\psi^2 e_z^2 + 1\right)\right)|\eta|}{1 + g_1(V_1)\left(e_z^2 + V_1\right)} \\
& + \frac{Q_1(V_1)\left(e_z^2 + V_1\right)|v_r| + Q_1(V_1)\left(e_z^2 + 1\right)|\tilde{\omega}|}{1 + g_1(V_1)(e_z^2 + V_1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq -\alpha(|e|) + \frac{2\bar{\psi}\left|\omega_r + \tilde{\omega}\right|\sqrt{V_1}V_1}{1 + g_1(V_1)V_1} + \frac{\bar{\psi}\left|v_r\right|V_1^2}{1 + g_1(V_1)V_1} \\
&\quad + \frac{\left(P_1(V_1)\sqrt{V_1} + \frac{\partial P_1}{\partial V_1}V_1\sqrt{V_1} + 2\bar{\psi}V_1\right)|\eta|}{1 + g_1(V_1)V_1} \\
&\quad + \frac{|\eta|\,Q_1(V_1) + Q_1(V_1)V_1\left|v_r\right| + Q_1(V_1)\left|\tilde{\omega}\right|}{1 + g_1(V_1)V_1} \\
&\quad + \frac{|\eta|\,\frac{\partial Q_1}{\partial V_1}e_z^2 + |\eta|\,Q_1(V_1)\left(\psi^2 + 1\right)e_z^2}{1 + g_1(V_1)e_z^2} \\
&\quad + \frac{\left(\left|v_r\right| + \left|\tilde{\omega}\right|\right)Q_1(V_1)e_z^2}{1 + g_1(V_1)e_z^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\alpha(|e|) + |\eta|\,\frac{4\bar{\psi}\sqrt{V_1}V_1}{1 + g_1(V_1)V_1} + |\eta|\,\frac{\bar{\psi}V_1^2}{1 + g_1(V_1)V_1} \\
&\quad + |\eta|\,\frac{P_1(V_1)\sqrt{V_1} + \frac{\partial P_1}{\partial V_1}V_1\sqrt{V_1} + 2\bar{\psi}V_1}{1 + g_1(V_1)V_1} \\
&\quad + |\eta|\,\frac{Q_1(V_1) + Q_1(V_1)V_1 + Q_1(V_1)}{1 + g_1(V_1)V_1} \\
&\quad + |\eta|\,\frac{\frac{\partial Q_1}{\partial V_1}e_z^2 + Q_1(V_1)\left(\bar{\psi}^2 + 1\right)e_z^2}{1 + g_1(V_1)e_z^2} \\
&\quad + |\eta|\,\frac{2Q_1(V_1)e_z^2}{1 + g_1(V_1)e_z^2}.
\end{aligned}
$$

Since the functions $g_1(V_1)$, $P_1(V_1)$ and $Q_1(V_1)$ are first order polynomials with strictly positive coefficients, then all the fractionals in the last inequality are bounded, and therefore, there exists a constant $c > 0$ such that

$$
\dot{W}_3 \leq -\alpha(|e|) + c\,|\eta| \tag{3.91}
$$

hence the closed-loop system is iISS with respect to η. This complete the proof of Proposition 3.2.

If we compare the unified tracking/stabilization controllers proposed in Sections 3.1 and 3.2, it is easy to notice that the only difference is the more generic form for the function $\rho(t)$ that appears in the expression of ω^* — compare (3.7) and (3.63).

3.3 A leader-follower formation case

In this section we present extension of the unified controller design proposed in the previous section to the case of formation control.

To the best of our knowledge, unified controller for leader-follower simultaneous tracking stabilization formation problem is considered only in [118], extending the idea of control design proposed in [85] for individual vehicle. Controller proposed in this reference is a centralized one, indeed, accessibility of the leader's coordinates to all the agents in the network is required.

In this section we use controller from the previous section as a stumbling block for distributed controller design. Particular type of graph topology (spanning tree) and input to state stability properties of the closed-loop system allow a sequencing of the controller design for individual agents in the network and simplify drastically stability analysis of the networked system.

The controller of Proposition 3.2 is an important contribution, relative to that of Proposition 3.1. Indeed, the former guarantees small ISS property of the closed-loop system which renders (almost) direct the extension of our previous statements to the general case of formation control.

The unified controller proposed in [85] has been extended to the leader-follower formation case in [118] assuming the leader's coordinates to be accessible to all the network. In our case we relax the last assumption by considering a particular graph topology.

Similarly to (3.61)-(3.93) we introduce the virtual controls

$$v_i^* = v_{i-1} cos e_{\theta i} + k_{xi} e_{xi} \tag{3.92}$$

$$\omega_i^* = \omega_{i-1} + k_\theta e_{\theta i} + k_{yi} e_{yi} v_{i-1} \phi(e_{\theta i}) + \rho_{i-1}(t) k_{yi} p(t) \sqrt{e_{xi}^2 + e_{yi}^2} \tag{3.93}$$

where,

$$\rho_{i-1}(t) := \exp^{-\int_{t_0}^{t} F(v_{i-1}(\tau), \omega_{i-1}(\tau)) d\tau} \tag{3.94}$$

which at the dynamic level, serve as references for the actual controls u_{1i} and u_{2i} in

$$\dot{v}_i = f_{1i}(t, v_i, \omega_i, e_i) + g_{1i}(t, v_i, \omega_i, e_i) u_{1i} \tag{3.95a}$$

$$\dot{\omega}_i = f_{2i}(t, v_i, \omega_i, e_i) + g_{2i}(t, v_i, \omega_i, e_i) u_{2i}, \qquad i \le n. \tag{3.95b}$$

Proposition 3.3. *Consider the network system composed by* (2.7) *for* $i = \{1, ..., N\}$*, let constants* k_{xi}*,* k_{yi}*,* $k_{\theta i} > 0$ *and let* p_i *and* $\dot{p}_i$ *be bounded and persistently exciting, and assume that there exist* $\bar{\omega}_r$*,* $\bar{\dot{\omega}}_r$*,* $\bar{v}_r$*,* $\bar{\dot{v}}_r > 0$ *such that* (3.11) *holds. Then, for the network system* (2.7)*, the errors converge to zero, (i.e.* (2.9) *holds for* $i = \{1, ..., N\}$*), provided that the leader's velocities satisfies one of the scenarios **S1** and **S3**, and for all error velocities* $[\tilde{v}_1,\ \tilde{\omega}_1, ...,\ \tilde{v}_N,\ \tilde{\omega}_N]$ *square integrable and converging to zero.* □

Proof. **Under the scenario S1:**, we start by decomposing the closed-loop equation of each follower as

$$\dot{e}_i = A_{v_{i-1}}(t, e_i)e_i + B_{1i}(t, e_i)\rho_i(t) + B_{2i}(e_i)\eta_i, \tag{3.96}$$

where

$$A_{v_{i-1}}(t, e_i) := \begin{bmatrix} -k_{\theta i} & 0 & -v_{i-1}(t)k_{yi}\phi(e_{\theta i}) \\ 0 & -k_{xi} & \omega_i^*(t, e_i) \\ v_{i-1}(t)\phi(e_{\theta i}) & -\omega_i^*(t, e_i) & 0 \end{bmatrix}, \quad B_{1i}(t, e_i) := \begin{bmatrix} -k_{yi}p_i(t)e_{yi} \\ k_{yi}p_i(t)e_{yi}^2 \\ -k_{yi}p_i(t)e_{yi}e_{xi} \end{bmatrix},$$

$$B_{2i}(e_i) := \begin{bmatrix} 0 & -1 \\ -1 & e_{yi} \\ 0 & -e_{xi} \end{bmatrix}. \tag{3.97}$$

The proof under **S1** follows two steps.

First, we prove the forward completeness of the trajectories using the following Lyapunov function candidate

$$V_{1i}(t, e_i) := \frac{1}{2}\left[e_{xi}^2 + e_{yi}^2 + \frac{1}{k_{yi}}e_{\theta i}^2\right] \tag{3.98}$$

its time-derivative along trajectories of (3.96) satisfies

$$\dot{V}_{1i} := -k_{xi}e_{xi}^2 - \frac{k_{\theta i}}{k_{yi}}e_{\theta i}^2 - p_i\rho_i e_{\theta i}e_{yi} - \frac{1}{k_{yi}}e_{\theta i}\tilde{\omega}_i - e_{xi}\tilde{v}_i \tag{3.99}$$

Under the assumption on boundedness of signals p_i, ρ_i, $\tilde{\omega}_i$ and $\tilde{v}_i$, it is always possible to find two positive constants a_i and b_i, such that

$$\dot{V}_{1i} \leq a_i V_{1i} + b_i. \tag{3.100}$$

which implies the forward completeness of trajectories of the formation.

The second step, consists in repetitive use of Proposition 3.3, exploiting the cascaded structure of the system. Indeed, for the first follower the closed-loop is reduced to (3.65), which, under the scenario **S1**, is integral Input-to-State Stable with respect to the vector $\eta_1 := [\tilde{v}_1, \tilde{\omega}_1]$. As a result, using square-integrability of $\eta_1(t)$ and its convergence to zero, we obtain that errors $e_1(t)$ converge to zero. Consequently

$$\lim_{t\to\infty} v_1(t) = v_r(t), \qquad \lim_{t\to\infty} \omega_1(t) = \omega_r(t). \tag{3.101}$$

Moreover, there exists $\bar{c}_1 > 0$ such that

$$\max\{v_1, \dot{v}_1, \omega_1, \dot{\omega}_1\} \leq \bar{c}_1. \tag{3.102}$$

For $i = 2$ the closed-loop system (3.96) is equivalent to (3.65), if we replace v_r by v_1 and ω_r by ω_1. Using (3.101), (3.102) and Lemma A.9 from Appendix A.7, we conclude that there exists $t_1 > 0$ and $\mu_1 > 0$ such that for all $t \geq t_1$, we have

$$\int_t^{t+T} \left(v_1^2(s) + \omega_1^2(s)\right) ds \geq \mu_1, \quad \forall t \geq t_1.$$

As a result, Proposition 3.2 is applicable for all $t \geq t_1$. Having the forward completeness of trajectories, assuming the convergence and the square integrability of $\eta_2 := [\tilde{v}_2, \tilde{\omega}_2]$ we conclude that

$$\lim_{t\to\infty} |e_2(t)| = 0, \qquad \lim_{t\to\infty} v_2(t) = v_r(t), \qquad \lim_{t\to\infty} \omega_2(t) = \omega_r(t). \tag{3.103}$$

Moreover, there exists $\bar{c}_2 > 0$ such that

$$\max\{v_2, \dot{v}_2, \omega_2, \dot{\omega}_2\} \leq \bar{c}_2. \tag{3.104}$$

Repeating the same argument, we conclude the same properties for all the agents. Which proves the statement.

Under the scenario S3:, we decompose the closed-loop equation for each follower as follows

$$\dot{e}_i = A_i(t, e_i)e_i + B(e_i)\eta_{2i} \tag{3.105}$$

where $p_{1i}(t) = \rho_i(t)p_i(t)$, as used in (3.68), and

$$
\begin{aligned}
A_i(t, e_i) &:= \begin{bmatrix} -k_{\theta i} & -k_{yi}p_{1i}(t)\frac{\sqrt{e_{xi}^2+e_{yi}^2}}{e_{xi}} & -k_{yi}p_{1i}(t)\frac{\sqrt{e_{xi}^2+e_{yi}^2}}{e_{yi}} \\ 0 & -k_{xi} & \psi_i(t, e_i) \\ 0 & -\psi_i(t, e_i) & 0 \end{bmatrix}, \\
\psi_i(\cdot) &:= k_{\theta i}e_{\theta i} + k_{yi}p_{1i}(t)\sqrt{e_{yi}^2 + e_{xi}^2}, \\
B(e_i) &= \begin{bmatrix} -k_{yi}e_{yi}\phi(e_{\theta i}) & 0 & 0 & -1 \\ k_{yi}e_{yi}^2\phi(e_{\theta i}) & e_{yi} & -1 & e_{yi} \\ \sin(e_{\theta i}) - k_{yi}e_{xi}e_{yi}\phi(e_{\theta i}) & -e_{xi} & 0 & -e_{xi} \end{bmatrix}.
\end{aligned}
$$

The proof follows using Proposition 3.2 recursively.

For $i = 1$, the system (3.105) is reduced to (3.67) and Proposition 3.2 is applicable, and is strong iISS with respect to $\eta_{21} := [v_r, \omega_r, \tilde{v}_1, \tilde{\omega}_1]$. Consequently, when $\eta_{12} \to 0$, we have

$$e_1 \to 0, \quad v_1 \to 0, \quad \omega_1 \to 0.$$

Similarly, for $i = 2$, we have under convergence of $[v_1, \omega_1]$ the closed-loop (3.105) is strong iISS with respect to $\eta_{21} := [v_1, \omega_1, \tilde{v}_1, \tilde{\omega}_1]$. Consequently

$$e_2 \to 0, \quad v_2 \to 0, \quad \omega_2 \to 0.$$

□□□

Remark 3.8. *An example of torque controller for* (2.2) *that guarantees the square integrability of the vector* $[\tilde{v}_1,\ \tilde{\omega}_1, ...,\ \tilde{v}_N,\ \tilde{\omega}_N]$ *is presented in Subsection 2.2 of Chapter 2.*

3.4 Simulations

We consider a group of four mobile robots following a virtual leader, the desired formation shape is a diamond configuration that tracks the trajectory of the virtual leader. See Figures 3.8 or 3.4. In the first part of the simulations, we define the reference velocities v_r and ω_r in a way that they converge (slowly) to zero (robust stabilization scenario **S3**). In the second part (tracking scenario **S1**), the leader's velocities are designed such that their sum of square is persistently exciting–see Figures 3.1 and 3.2, respectively.

The physical parameters of the systems are introduced in Subsection 2.4.1, the inertia parameters and the constants contained in $C(\dot{q}_i)$ are supposed to be unknown. The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1, 3, 4]$,

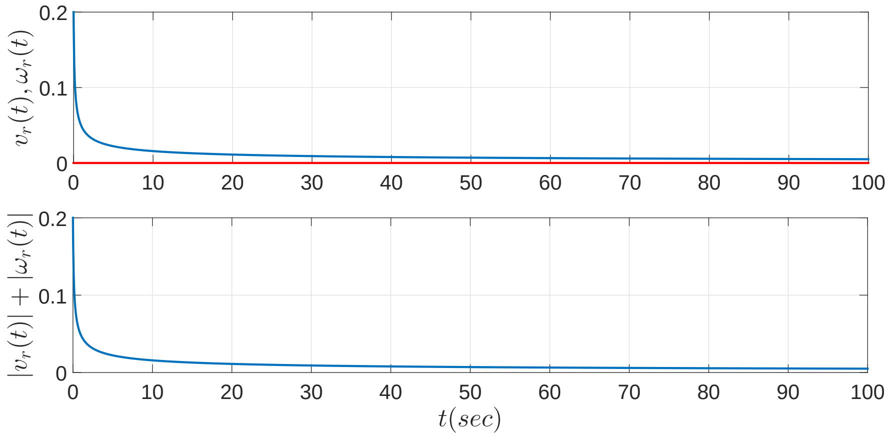

Figure 3.1: Reference velocities v_r and ω_r for the scenario **S3**

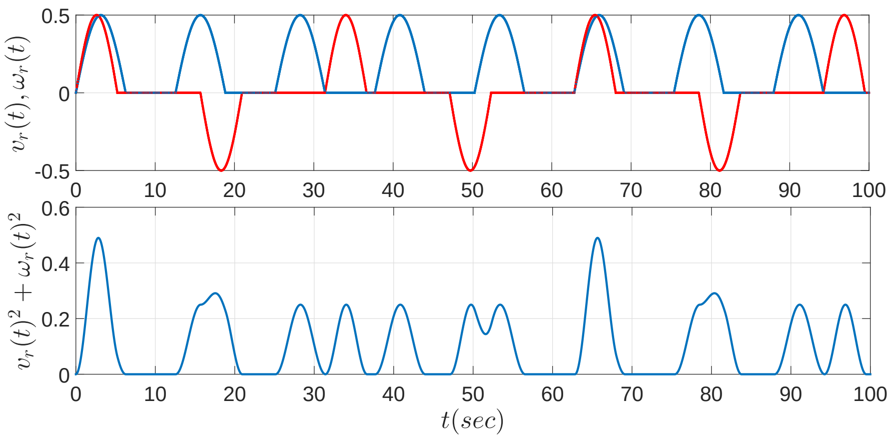

Figure 3.2: Reference velocities v_r and ω_r for the scenario **S1**

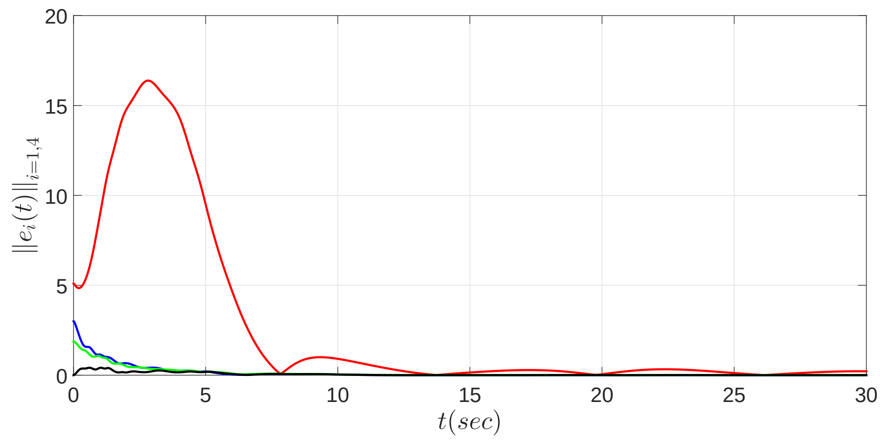

Figure 3.3: Relative errors (in norm) for each pair leader-follower under **S3**

$[x_2(0), y_2(0), \theta_2(0)] = [0, 2, 2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0, 4, 1]$ and $[x_4(0), y_4(0), \theta_4(0)] = [0, 3, 1]$; the control gains were set to $k_{x_i} = k_{y_i} = k_{\theta_i} = 1$ and the function $p(t) = 20\sin(0.5t)$, which has a persistently exciting time-derivative. The function F is designed as follows

$$F(a, b) := K(\sqrt{a^2 + b^2}) := \begin{Bmatrix} \sqrt{a^2 + b^2} & \forall \alpha \geq 0.1 \\ 0 & \text{Otherwise} \end{Bmatrix}.$$

The formation shape with a certain desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0, 0]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [-1, 0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, -1/2]$ and $[d_{x_{3,4}}, d_{y_{3,4}}] = [0, 1]$. See Figure 3.8, The parameters (γ, k_d) are taken equal to $(10^{-5}, 15)$, and $\hat{\Theta}(0) = (\hat{m}_1, \hat{m}_2, \hat{c}) = (0, 0, 0)$.

For the stabilization scenario **S3**, the results of the simulation are shown in Figures 3.3–3.4. In Figure 3.3, 3.5, and 3.6, we show the convergence of the tracking errors between the agent and its neighborhood, the control inputs and the parameter estimation errors, and in Figures 3.3–3.6. In Figure 3.7, 3.9, and 3.10 for the tracking scenario.

3.5 Conclusion

In this chapter we considered leader-follower simultaneous tracking and stabilization problem for nonholonomic vehicles. We proposed two kinematic level controllers that ensure uniform global asymptotic stability of the kinematic closed-loop system. On the dynamical level, the virtual kinematic level controllers serve as a reference for the controller design. For dynamical level we proved that any controller that ensures convergence to zero and square integrability of the velocity errors solves the leader-follower simultaneous tracking and stabilization problem. The extension of these

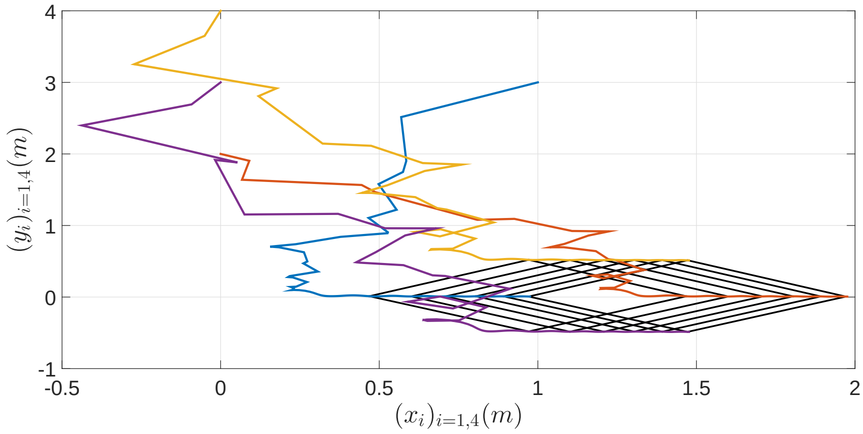

Figure 3.4: Illustration of the path-tracking in formation under **S3**

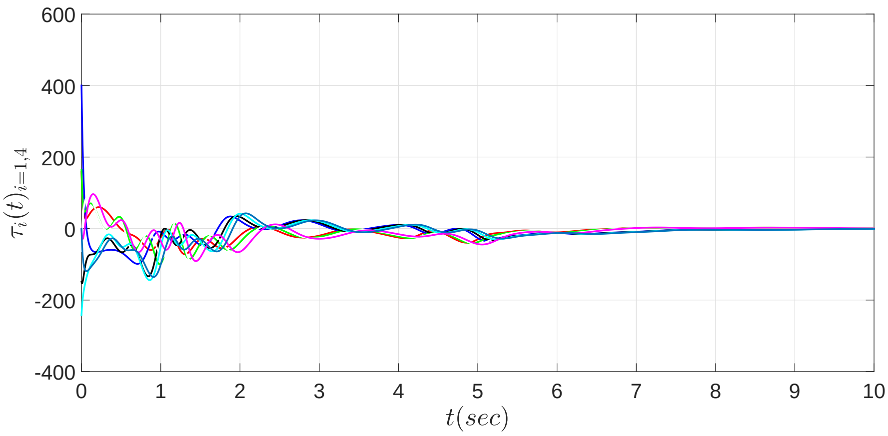

Figure 3.5: Illustration of the torque inputs for each agent under **S3**

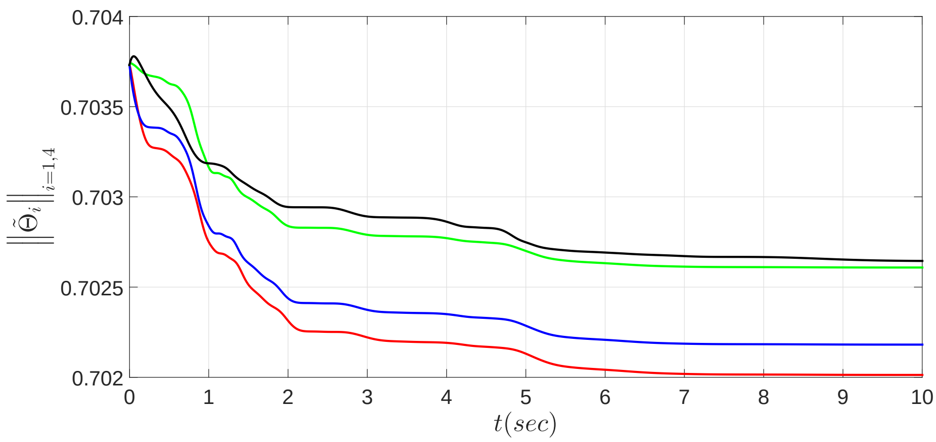

Figure 3.6: Illustration of the estimation parameter errors for each agent under **S3**

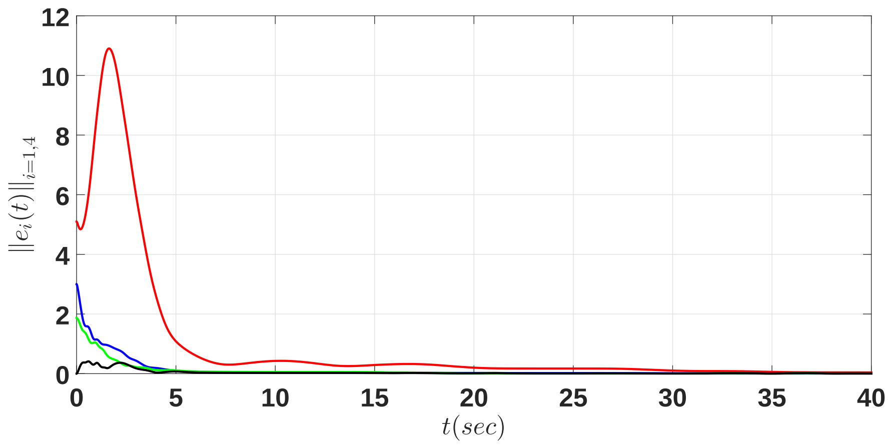

Figure 3.7: Relative errors (in norm) for each pair leader-follower under **S1**

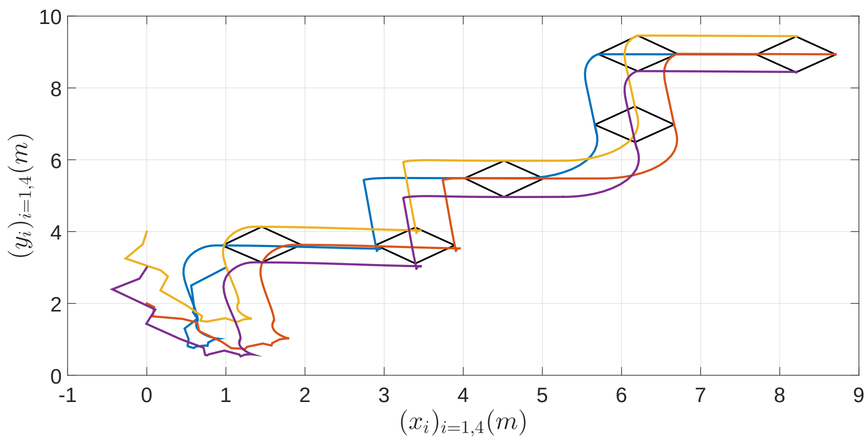

Figure 3.8: Illustration of the path-tracking in formation under **S1**

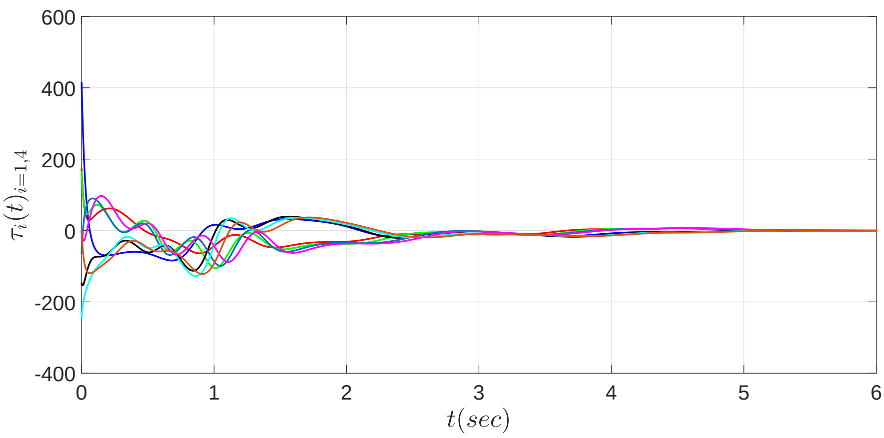

Figure 3.9: Illustration of the torque inputs for each agent under **S1**

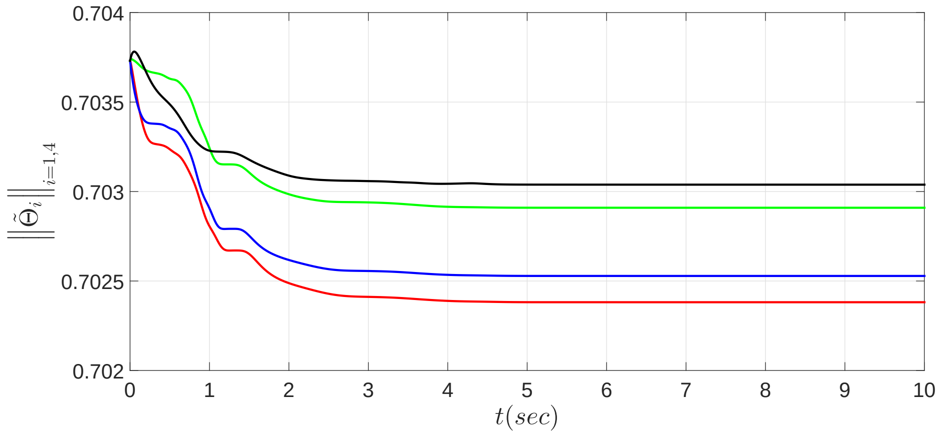

Figure 3.10: Illustration of the estimation parameter errors for each agent under **S1**

results to leader-follower simultaneous tracking and agreement formation problem, presented in Section 3.3, is based on the controller design from Section 3.2 and ensures asymptotic convergence of the formation errors under a spanning tree communication graph topology.

Chapter 4

Consensus-based formation control of nonholonomic robots under delayed interconnections

In previous chapters we addressed several problems on leader-follower formation control for swarms of mobile robots under two standing assumptions:(i) the communication is reliable (notably, without delay) and (ii) the communication topology is restricted to that of a spanning tree. On the other hand, the existence of a leader system that imposes particular behaviors to the formation imposes certain technical difficulties. In this chapter we restrict our attention to the leaderless consensus problem of multiple mobile robots, but under the assumption that the robots are interconnected in a general bidirectional graph and that the communications are affected by time-varying delays.

As we have mentioned in previous chapters, one of the main difficulties appearing in the formation control of nonholonomic systems is that the designed controller has to be either discontinuous or time-varying [15]. Different approaches have been proposed to deal with consensus-like control objectives. For instance, in [24] a decentralized feedback control is introduced that drives a system of multiple nonholonomic unicycles to a rendezvous point in terms of both position and orientation, the proposed control law is discontinuous and time-invariant. In [55] necessary and sufficient conditions for the feasibility of a class of position formations are presented. In [100] a distributed formation control law using a consensus-based approach is proposed to drive a group of agents to a desired geometric pattern. In [121] the position/orientation formation control problem for multiple nonholonomic agents using a time-varying controller that leads the agents to a given formation using only their orientation is addressed. To solve the consensus and formation-control problems, in [31] a cooperative control law that is robust to constant communication delays is presented. In [3] a distributed consensus control law is proposed for a network of nonholonomic agents in the presence of bounded disturbances with unknown dynamics in all inputs channels. For an undirected graph, in [100] a smooth time-varying controller is proposed; it is improved by adding in [12] a PD-like controller at the dynamical level. All these previous works, except for [12], solve the consensus problem without uniformity on the initial time, and they only consider the simplified case of vehicle kinematics.

In this chapter we solve two problems of consensus stabilization for nonholonomic vehicles interconnected through a bidirectional generic graph, under time-varying delays. In the first case, we assume that each robot adopts a particular orientation, i.e., consensus is pursued only in their Cartesian positions on the plane. In the second case, the robots are required to assume a common position and orientation. The solution is based on the design of $\delta-$PE controllers [64, 67]. We solve these problems under the assumption that the graph is static, connected and undirected, and that there exists a bounded time-varying delay in the interconnection.

As in previous chapters our proofs are constructive. Following [72], [74] and [33], we provide a novel strict Lyapunov-Krasovskii functionals (SLKF), to establish uniform global asymptotic stability of the consensus set. This is important to guarantee robustness with respect to bounded disturbances and to provide a method of gain tuning. To the best of our knowledge this is the first work that provides a SLKFs in this scenario.

4.1 Network model description

As it is customary in multi-agent consensus [90, 93], the complete dynamics of the systems is composed of two parts:
i) the dynamics of the nodes, which are described by a second order nonholonomic differential equations;
ii) the interconnection topology which is modeled using a Laplacian matrix [77].

4.1.1 Node dynamics

We recall the dynamical model of mobile robot given in (2.1) and (2.2), that is

$$\begin{cases} \dot{x}_i &= v_i \cos\theta_i \\ \dot{y}_i &= v_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{v}_i &= f_1(t, v_i, \omega_i, z_i) + g_1(t, v_i, \omega_i, z_i)u_{1i} \\ \dot{\omega}_i &= f_2(t, v_i, \omega_i, z_i) + g_2(t, v_i, \omega_i, z_i)u_{2i} \end{cases} \tag{4.2}$$

Assuming that $g_1(t, v_i, \omega_i, q_i)$ and $g_2(t, v_i, \omega_i, q_i)$ are invertible and using the complete knowledge of the system states and parameters, let

$$\begin{aligned} u_{1i} =& g_1(t, v_i, \omega_i, z_i)^{-} u_{vi} - f_1(t, v_i, \omega_i, z_i), \\ u_{2i} =& g_2(t, v_i, \omega_i, z_i)^{-} u_{\omega i} - f_2(t, v_i, \omega_i, z_i) \end{aligned}$$

so that we obtain the familiar second-order model

$$\begin{cases} \dot{x}_i &= v_i \cos\theta_i \\ \dot{y}_i &= v_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i, \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{v}_i &= u_{vi} \\ \dot{\omega}_i &= u_{\omega i}. \end{cases} \tag{4.4}$$

The consensus problem consists in making each vehicle achieve a certain position relative to an unknown barycenter. In addition, the vehicles may be required to adopt a common orientation or they may be allowed to adopt, each, a particular target orientation.

In a compact form, we consider the following model of N second order nonholonomic robots,

$$\dot{z} = \Phi(\theta)v \tag{4.5a}$$

$$\dot{v} = u_v \tag{4.5b}$$

$$\dot{\theta} = \omega \tag{4.5c}$$

$$\dot{\omega} = u_\omega \tag{4.5d}$$

where $z := [z_1^\top, ..., z_N^\top]^\top \in \mathbb{R}^{2N}$; $z_i := [x_i - \delta_{xi}, y_i - \delta_{yi}]^\top \in \mathbb{R}^2$ is the translational error of the global translational coordinates $[x_i, y_i] \in \mathbb{R}^2$, of the ith-robot, with respect to a constant vector $\delta_i := [\delta_{xi}, \delta_{yi}]^\top \in \mathbb{R}^2$; the global translational coordinates $[x_i, y_i]$ are expressed with respect to a fixed frame; the constant vector δ_i determines the desired position of the ith-robot relative to the barycenter of the formation z_c when $z_i = z_c$; $v := [v_1, ..., v_N]^\top \in \mathbb{R}^N$; v_i is the linear velocity, $\Phi(\theta) := \text{diag}[\phi(\theta_i)] \in \mathbb{R}^{2N\times N}$; $\phi(\theta_i) := [\cos(\theta_i), \sin(\theta_i)]^\top \in \mathbb{R}^2$; $\tilde{\theta} := \theta - \theta_d := [\theta_1 - \theta_{d1}, ..., \theta_N - \theta_{dN}]^\top \in \mathbb{R}^N$ is the orientation error of each robot; θ_d is a constant desired orientation; $\omega := [\omega_1, ..., \omega_N]^\top \in \mathbb{R}^N$; ω_i is the angular velocity; and $u_v := [u_{v1}, ..., u_{vN}]^\top \in \mathbb{R}^N$ and $u_\omega := [u_{\omega 1}, ..., u_{\omega N}]^\top \in \mathbb{R}^N$ are, respectively, the translational and the rotational control inputs.

Since θ_d is constant, the following two equations hold

$$\dot{\Phi}(\theta) = -\Phi(\theta)^{\perp}\bar{\omega}, \qquad \dot{\Phi}(\theta)^{\perp} = \Phi(\theta)\bar{\omega}, \tag{4.6}$$

where $\bar{\omega} = diag[\omega_i] \in \mathbb{R}^{N\times N}$, $\Phi(\theta)^{\perp} = diag[\phi(\theta_i)^{\perp}] \in \mathbb{R}^{2N\times N}$ and $\phi(\theta_i)^{\perp} = [\sin(\theta_i), -\cos(\theta_i)]^{\top}$.

The control objective is to steer each z_i toward a common position z_c, and each orientation θ_i toward a specified constant θ_{di} or to a common unknown orientation θ_c.

4.1.2 Interconnection Topology

The interconnection of the N agents is modeled using the Laplacian matrix $L := [\ell_{ij}] \in \mathbb{R}^{N\times N}$, whose elements are defined as

$$\ell_{ij} = \begin{cases} \sum\limits_{j\in\mathcal{N}_i} a_{ij} & i = j \\ -a_{ij} & i \neq j \end{cases} \tag{4.7}$$

where $\mathcal{N}_i$ is the set of agents transmitting information to the ith robot hence, $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise.

Similar to passivity-based (energy-shaping) synchronization [5, 91] and in order to ensure that the interconnection forces are generated by the gradient of a potential function, the following assumption is used in this chapter:

Assumption 4.1. *The interconnection graph is* undirected, static and connected.

Assumption 4.2. *The communication, from the $j-$th agent to the $i-$th agent, is subject to a variable time-delay $T_{ij}(t)$ with a known upper-bound T^*. Hence, it holds that*

$$0 \leq T_{ij}(t) \leq T^* < \infty. \tag{4.8}$$

Remark 4.1. *By construction, L has a zero row sum,* i.e., *$L1_N = 0$, where 1_N is a vector of N ones. Moreover, Assumption 4.1, ensures that L is symmetric, has a single zero-eigenvalue and the rest of the spectrum of L is positive. Thus, rank$(L) = N - 1$.*

4.2 Problem formulation

We solve the following two consensus problems: roughly speaking, in the first case the robots achieve consensus in relation to their location only; in the second case, they also achieve a common orientation, under a general time-varying delay.

Delayed Partial Consensus Problem. Consider a network of N nonholonomic robots satisfying (4.5). Design a decentralized controller verifying Assumptions 4.1-4.2 such that all robots positions converge, globally, uniformly, and asymptotically, to a given formation pattern with a desired given orientation $\theta_d \in \mathbb{R}^N$, *i.e.*, there exists $z_c \in \mathbb{R}^2$ such that

$$\lim_{t\to\infty} z(t) = 1_N \otimes z_c; \tag{4.9a}$$

$$\lim_{t\to\infty} \theta_i(t) = \theta_{di}, \tag{4.9b}$$

where $\theta_{di} \in \mathbb{R}$ is a given desired constant orientation for each robot, and z_c is the barycenter of the formation pattern.

Delayed Full Consensus Problem. Consider a network of N nonholonomic robots satisfying (4.5). Design a decentralized controller verifying Assumptions 4.1-4.2 such that all robots positions and orientation converge, globally, uniformly, and asymptotically, to a given formation pattern, *i.e.*, there exists $[z_c\, \theta_c] \in \mathbb{R}^3$ such that

$$\lim_{t\to\infty} z(t) = 1_N \otimes z_c; \tag{4.10a}$$

$$\lim_{t\to\infty} \theta_i(t) = \theta_c, \tag{4.10b}$$

where $[\theta_c\, z_c]$ are the barycenter of the formation pattern. As in previous chapters, we solve the afore-mentioned consensus problems by recasting them into classical stabilization problems (of the origin or of a set)[1]. To that end, we first need to introduce suitable error coordinates (e, s) such that if $(e, s) = 0$ then equivalently we have $z(t) = 1_N \otimes z_c$.

Let $i \leq n$,

$$e_i = \phi(\theta_i)^\top \sum_{j\in\mathcal{N}_i} a_{ij}(z_i - z_j),$$

$$s_i = \phi(\theta_i)^{\perp\top} \sum_{j\in\mathcal{N}_i} a_{ij}(z_i - z_j)$$

which, defining $e := [e_1 \dots e_N]$, $s := [s_1 \dots s_N]$, may be written in the equivalent vector form

$$e = \Phi(\theta)^\top L_2 z, \qquad s = \Phi(\theta)^{\perp\top} L_2 z. \tag{4.11}$$

On the other hand, in the presence of state delays, we introduce the delayed counterpart of (e_i, s_i), denoted by (e_{di}, s_{di}), as

$$e_{di} = \phi(\theta_i)^\top \sum_{j\in\mathcal{N}_i} a_{ij}(z_j(t - T_{ij}(t)) - z_i(t)),$$

[1]–see Appendix A.2 for basic definitions and characterizations of stability

$$s_{di} = \phi(\theta_i)^{\perp\top} \sum_{j\in\mathcal{N}_i} a_{ij}(z_j(t - T_{ij}(t)) - z_i(t)).$$

Correspondingly, in vector form we have

$$\begin{aligned} e_d =& \Phi(\theta)^\top L_2 z + \Phi(\theta)^\top \mathcal{A}(\dot{z}_t), \\ s_d =& \Phi(\theta)^{\perp\top} L_2 z + \Phi(\theta)^{\perp\top} \mathcal{A}(\dot{z}_t). \end{aligned} \tag{4.12}$$

where

$$\mathcal{A}(\dot{z}_t) = \begin{bmatrix} \sum_{j\in\mathcal{N}_1} a_{1j} \int_{t-T_{j1}(t)}^{t} \dot{z}_j(\delta) d\delta \\ \vdots \\ \sum_{j\in\mathcal{N}_N} a_{Nj} \int_{t-T_{jN}(t)}^{t} \dot{z}_j(\delta) d\delta \end{bmatrix} \tag{4.13}$$

and we recall that $L_2 = L \otimes I_2$.

Then, the control objective (4.9a) (or (4.10a)) is achieved if we prove that $(e_d, s_d, v) \to (0, 0, 0)$. In fact, having $v = 0$, implies that $\mathcal{A}(\dot{z}_t) = 0$ and $(e_d, s_d) = (e, s)$ then, after Lemma 4.1 below, we know that verifying the control objective (4.9a) (or (4.10a)) is equivalent to establishing that $(e, s) \to (0, 0)$.

Lemma 4.1. *Consider (e, s) given by (4.11), and assume that L satisfies Assumption 4.1. Then $L_2 z = 0 \Leftrightarrow (e, s) = (0, 0)$ and, moreover,*

$$\lambda_2(L) z^\top L_2 z \le |e|^2 + |s|^2 \le \lambda_N(L) z^\top L_2 z \tag{4.14}$$

where $\lambda_2(L)$ and $\lambda_N(L)$ are the second smallest and the largest eigenvalue of L, respectively.

Proof. Since the matrix $\begin{bmatrix} \Phi(\theta)^\top \\ \Phi(\theta)^{\perp\top} \end{bmatrix}$ is non singular. The first fact follows directly. For the second fact, we remark that $|e|^2 + |s|^2 = z^\top L_2^2 z = z^\top L_2^{\frac{1}{2}} L_2 L_2^{\frac{1}{2}} z$. Since $L_2^{\frac{1}{2}} z$ is orthogonal to the eigenspace associated to the zero eigenvalue of L_2, it holds that

$$\lambda_2(L) z^\top L_2^{\frac{1}{2}} L_2^{\frac{1}{2}} z \le z^\top L_2^{\frac{1}{2}} L_2 L_2^{\frac{1}{2}} z \le \lambda_N(L) z^\top L_2^{\frac{1}{2}} L_2^{\frac{1}{2}} z$$

so (4.14) follows. □□□

4.3 Control design and stability analysis

Before providing the control inputs for each case-study, we introduce the following useful functions.

First, we define the function $p : \mathbb{R}_{\ge 0} \to \mathbb{R}$ that satisfies the following assumption.

Assumption 4.3. *the function $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and its derivatives, up to the third, are bounded. Thus, there exists $b_p > 0$ such that*

$$\max\left\{|p|_\infty, |\dot{p}|_\infty, |\ddot{p}|_\infty, |p^{(3)}|_\infty\right\} \leq b_p.$$

Moreover, $\dot{p}(t)$ is persistently exciting, with excitation parameters (T, μ).

Next, we define the function $\bar{q} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{N\times N}$, as $\bar{q}(t) = \text{diag}(q_i(t))$ and which is related to $p(t)$ by the differential equation

$$\bar{q}^{(3)} + K_\alpha \ddot{\bar{q}} + K_I \dot{\bar{q}} = \dot{p} I_n, \tag{4.15}$$

where K_α and K_I are diagonal positive definite matrices. Also, we define the function $\bar{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{N\times N}$, as $\bar{f}(t) := \text{diag}(f_i(t))$ and which is related to $p(t)$ by the following differential equation

$$\ddot{\bar{f}} + K_{d\theta} \dot{\bar{f}} + K_{p\theta} \bar{f} = p(t) I_n, \tag{4.16}$$

in which, $K_{d\theta}$ and $K_{p\theta}$ are diagonal positive definite matrices.

If $\dot{p}(t)$ satisfies Assumption 4.3 then, after Lemma B.1, it follows that $\dot{f}_i$ and $\dot{q}_i$ are also persistently exciting [2] and so are the matrices $\dot{\bar{f}}(t) = \dot{\bar{f}}(t)$ and $\dot{\bar{q}}(t) = \dot{\bar{q}}(t)$ in the sense of Definition A.6. Furthermore, there exist $b_{\bar{f}} > 0$ and $b_{\bar{q}} > 0$ such that

$$\max\left\{\left|\bar{f}\right|_\infty, \left|\dot{\bar{f}}\right|_\infty, \left|\ddot{\bar{f}}\right|_\infty, \left|\bar{f}^{(3)}\right|_\infty\right\} \leq b_{\bar{f}}.$$

and

$$\max\left\{\left|\bar{q}\right|_\infty, \left|\dot{\bar{q}}\right|_\infty, \left|\ddot{\bar{q}}\right|_\infty, \left|\bar{q}^{(3)}\right|_\infty\right\} \leq b_{\bar{q}}.$$

Remark 4.2. *Lemma B.1 also provides an explicit estimation of the excitation parameters (T_f, μ_f) for $\dot{f}$, (T_q, μ_q) for $\dot{q}$, and the constants $b_{\bar{f}}$ and $b_{\bar{q}}$, which are used in the construction of the strict Lyapunov function.*

Finally, for a bounded function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$, with $|\psi|_\infty = b_\psi > 0$, we recall the function $\bar{\Upsilon}_{\psi^2} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{N\times N}$, as $\bar{\Upsilon}_{\psi^2}(t) := \text{diag}\left(\Upsilon_{\psi_i^2}(t)\right)$, with

$$\Upsilon_{\psi_i^2}(t) := 1 + 2b_\psi^2 T - \frac{2}{T}\int_t^{t+T}\int_t^m \psi_i(s)^2 ds\, dm \tag{4.17}$$

–cf equation (1.4) in Chapter 1.

Recall also that $\Upsilon_{\psi_i^2}(t)$ admits the following bounds $1 \leq \Upsilon_{\psi_i^2}(t) < b_{\Upsilon i} := 1 + 2b_\psi^2 T$

[2] This is reminiscent of the fact that the output of a stable proper minimum phase filter driven by a PE input is also PE –see [41, Lemma 4.8.3]

and, furthermore,

$$\dot{\Upsilon}_{\psi_i^2}(t) = -\frac{2}{T}\int_t^{t+T} \psi_i(s)^2 ds + 2\psi_i(t)^2. \tag{4.18}$$

Moreover, if ψ is persistently exciting, we obtain

$$\dot{\Upsilon}_{\psi_i^2}(t) \leq -\frac{2\mu}{T} + 2\psi_i(t)^2. \tag{4.19}$$

We are now ready to provide the translation and the rotation control laws (u_v, u_ω) to solve the partial and the full delayed and undelayed consensus problems.

4.3.1 Undelayed partial consensus problem

In the translational error coordinates (e, s), we employ a simple undelayed PD-like controller as it was originally proposed in [12], that is,

$$u_v = -K_{dt}v - K_{pt}e, \tag{4.20}$$

where K_{dt} and K_{pt} are diagonal positive definite matrices. For the rotational part, we propose the following controller

$$u_\omega = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s, e) \tag{4.21}$$

where $K_{d\theta}$ and $K_{p\theta}$ are diagonal positive definite matrices, and $\kappa(s, e)$ is defined as

$$\kappa(s, e) = \frac{1}{2}[s_1^2 + e_1^2, ..., s_N^2 + e_N^2]^\top \in \mathbb{R}^N. \tag{4.22}$$

The closed-loop system, which results from Equations (4.5), (4.11), (4.20), and (4.21), is

$$\dot{z} = \Phi(\theta)v \tag{4.23a}$$

$$\dot{v} = -K_{dt}v - K_{pt}e \tag{4.23b}$$

$$\dot{e} = -\bar{\omega}s + \Phi(\theta)^\top L_2 \Phi(\theta)v \tag{4.23c}$$

$$\dot{s} = \bar{\omega}e + \Phi(\theta)^{\perp\top} L_2 \Phi(\theta)v \tag{4.23d}$$

$$\dot{\tilde{\theta}} = \omega \tag{4.23e}$$

$$\dot{\omega} = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s, e). \tag{4.23f}$$

Thus, equations (4.23a)-(4.23b) determine the closed-loop dynamics for the translational dynamics while equations (4.23e)-(4.23f) determines the closed-loop dynamics of the rotational coordinates $(\tilde{\theta}, \omega)$. These can be viewed as a stable second order system with input $-p(t)\kappa(s,e)$ whose role is to excite the rotational velocity $\bar{\omega}$ when the errors (s,e) are different from zero.

We establish uniform global asymptotic stability of the origin of the system (4.23). Our proof is constructive as it relies on the construction of a strict Lyapunov function. To that end, we introduce the following change of coordinates:

$$e_\theta = \tilde{\theta} + \bar{f}(t)\kappa(s,e), \quad e_\omega = \omega + \dot{\bar{f}}(t)\kappa(s,e). \tag{4.24}$$

Next, let us define $X_t := [v^\top, e^\top, s^\top]^\top \in \mathbb{R}^{3N}$ and $X_r := [e_\theta^\top, e_\omega^\top]^\top \in \mathbb{R}^{2N}$ as the translational and the rotational components of the system, respectively. Additionally, let $\bar{e} = \mathrm{diag}(e_i)$, $\bar{s} = \mathrm{diag}(s_i)$, $\bar{e}_\omega = \mathrm{diag}(e_{\omega i})$ and $\bar{\kappa} = \mathrm{diag}(\kappa_i)$. Then using (4.16), we obtain

$$\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \dot{\bar{f}}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\dot{\bar{f}}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v \tag{4.25a}$$

$$\dot{X}_r = \begin{bmatrix} 0 & I_N \\ -K_{p\theta} & -K_{d\theta} \end{bmatrix} X_r + \begin{bmatrix} \bar{f} \\ \dot{\bar{f}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2 \right) \Phi v. \tag{4.25b}$$

We remark that in view of Lemma 4.1, $(X_t, X_r) = (0,0)$ is equivalent to $(v, z, \theta, \omega) = (0, 1_N \otimes z_c, \theta_d, 0)$, and the dynamics (4.23) is embedded in (4.25). Thus, solving the consensus problem is equivalent to proving uniform global asymptotic stability of the origin for (4.25).

Theorem 4.1. *Consider the system* (4.5) *in closed-loop with* (4.20) *and* (4.21)*. Assume that K_{dt}, K_{pt}, $K_{d\theta}$ and $K_{p\theta}$ are diagonal positive definite and Assumption 4.3 holds. Then, the origin $(X_t, \tilde{\theta}, \omega) = (0,0,0)$ is uniformally globally asymptotically stable.*

Proof. (Sketch) The proof is constructive; we provide a strict Lyapunov function. Only the main steps are given here, the complete proof is in Appendix B.9.

First, we observe that (4.25a) admits the following non-strict Lyapunov function

$$V(v,z) = v^\top K_{pt}^{-1} v + z^\top L_2 z. \tag{4.26}$$

Indeed, in view of (4.14), it is concluded that $V(v,z)$ is positive definite and radially unbounded with regards to $X_t = 0$, and using (4.14) we obtain

$$v^\top K_{pt}^{-1} v + \frac{1}{\lambda_N(L)} \left(e^T e + s^T s \right) \leq V(v,z) \leq v^\top K_{pt}^{-1} v + \frac{1}{\lambda_2(L)} \left(e^T e + s^T s \right).$$

Moreover, the time-derivative of V along the trajectories of (4.23) yields

$$\dot{V}(\theta, X_t) = -2v^\top K_{pt}^{-1} K_{dt} v. \tag{4.27}$$

The strict Lyapunov function for the closed-loop system (4.25) is

$$\Gamma(t, X_t, X_r) = W(t, X_t, V) + \rho_1(V) Z(X_r) + \rho_2(V) V \tag{4.28}$$

where

$$\begin{aligned} W &= \gamma(V)V + V\kappa^\top \bar{\Upsilon}_{\dot{f}^2}(t)\kappa + \alpha(V)e^\top v - c_1 V e^\top \bar{\dot{f}} s + c_1 b_f \lambda_N(L) V^2 \\ &\quad + \left(\lambda_N(L) + |K_{pt}|\right)\alpha(V)V, \end{aligned}$$

$$\rho_1(V) = \frac{2\sigma(V)}{c_2 \lambda_m(K_{d\theta})}\left[\alpha(V) + c_1 b_f V\right] + 1, \tag{4.29}$$

$$\sigma(V) = \max\left\{\frac{16Tc_1 b_f}{\mu}, \frac{4\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\alpha(V)V}{\gamma(V)}\right\},$$

$$\begin{aligned} \alpha(V) &= 4b_f^2 \lambda_N(L) V^2 \left|K_{pt}^{-1}\right| + 4c_1 b_f^2 \lambda_N(L)\left|K_{pt}^{-1}\right| V^2 + \frac{4c_1}{c_4}\left|\dot{f}^2\left(\Phi^{\perp\top} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-1}\right| V + \\ &\quad c_1^2 c_4 b_f^2 \left|K_{pt}^{-1}\right|, \end{aligned}$$

$$\begin{aligned} \gamma(V) &= 2c_4 V^2 \lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^\top L_2 \Phi\right|_\infty^2 + 2c_4 V^2 \lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp\top} L_2 \Phi\right|_\infty^2 && (4.30) \\ &\quad + \frac{\partial \alpha}{\partial V} V\left(|K_{pt}| + \lambda_N(L)\right) + \frac{c_4}{2} c_1 V + 2\alpha(V)\left|\Phi^\top L_2 \Phi\right|_\infty \left|K_{dt}^{-1}K_{pt}\right|_\infty && (4.31) \\ &\quad + \frac{c_4}{2}\left|K_{pt}K_{dt}^{-1}\right|\alpha^2(V) + \frac{c_4}{2}\alpha(V)\left|K_{dt}\right| + 2c_1 b_f \lambda_N(L) V + \frac{4}{c_4} V^2 \lambda_N(L)\left|K_{dt}^{-1}\right| && (4.32) \\ &\quad + \frac{c_4}{2} c_1^2 \left|K_{dt}^{-1}K_{pt}\right|\left|\dot{f}^2\left(\Phi^\top L_2 \Phi\right)^2\right|_\infty, && (4.33) \end{aligned}$$

$$\rho_2(V) = \rho_1(V)\rho_3(V)V, \quad \rho_3(V) = \frac{c_3 \lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|}{2}\left(\left|\Phi^\top L_2 \Phi\right|_\infty^2 + \left|\Phi^{\perp\top} L_2 \Phi\right|_\infty^2\right),$$

and the constants c_1, c_2, c_3 and c_4 are:

$$c_1 = 1 + \frac{\lambda_N(L)}{\max\left\{2, \frac{2T}{\mu}\left(1 + \frac{2N}{\lambda_2(L)}\right)\right\}}, \quad c_2 = \frac{2}{\lambda_m(K_{d\theta})} + \frac{\lambda_M(K_{d\theta}) + 1}{\lambda_m(K_{p\theta})} + 1, \tag{4.34}$$

$$c_3 = \max\left\{\frac{8\left(2c_2b_f+b_f\right)^2}{c_2\lambda_m(K_{d\theta})}, \frac{8\left(2c_2b_f\lambda_M(K_{p\theta})+b_f\right)^2}{\lambda_m(K_{p\theta})}\right\}, \quad c_4 = \max\left\{2, \frac{2T}{\mu}\left(2+\frac{8N}{\lambda_2(L)}\right)\right\}.$$

Since ρ_1 and ρ_2 are positive functions and radially unbounded, positive definiteness of Γ is ensured by the facts that $\Gamma(t,0,0)=0$, for all $t \geq 0$, and

$$W \geq \gamma(V)V,$$

$$\begin{aligned} W \leq\ & \gamma(V)V + V\kappa^\top(e,s)\bar{\Upsilon}_{\dot{f}^2}(t)\kappa(e,s) + 2c_1b_f\lambda_N(L)V^2 \\ & +2\left(\lambda_N(L)+|K_{pt}|\right)\alpha(V)V, \end{aligned}$$

$$Z \geq \min\left\{1, \lambda_m(K_{p\theta})\right\}\left(e_\theta^\top e_\theta + e_\omega^\top e_\omega\right),$$

$$Z \leq \max\left\{1+c_2, c_2\lambda_M(K_{p\theta})+1\right\}\left(e_\theta^\top e_\theta + e_\omega^\top e_\omega\right).$$

After some term chasing and long cumbersome manipulations we get

$$\dot{\Gamma} \leq -\frac{\mu}{4T}V^3 - \frac{\rho_1(V)}{8}\left[c_2e_\omega^\top K_{d\theta}e_\omega + e_\theta^\top K_{p\theta}e_\theta\right] - \frac{1}{4}\gamma(V)v^\top K_{dt}K_{pt}^{-1}v - \frac{1}{8}\alpha(V)e^\top K_{pt}e. \tag{4.35}$$

Therefore $\dot{\Gamma}$ is negative definite and Γ qualifies as a strict Lyapunov function for system (4.25). Global uniform asymptotic stability of the equilibrium $(X_t, X_r) = (0,0)$ is ensured and thus the consensus problem is solved. □□□

4.3.2 Delayed partial consensus problem

Using the delayed translational error coordinates (e_d, s_d), we employ the following delayed PD-like controller for the translational input

$$u_v = -K_{dt}v - K_{pt}e_d. \tag{4.36}$$

where K_{dt} and K_{pt} are diagonal positive definite matrices.
We introduce the rotational controller as

$$u_\omega = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s_d, e_d), \tag{4.37}$$

where $K_{d\theta}$ and $K_{p\theta}$ are diagonal positive definite matrices, $\tilde{\theta} = \theta - \theta_d$ and the function κ is defined in (4.22).

The closed-loop system resulting from the open loop equation (4.5) and the controllers (4.12), (4.36) and (4.37) is

$$\dot{z} = \Phi(\theta)v \tag{4.38a}$$
$$\dot{v} = -K_{dt}v - K_{pt}e_d \tag{4.38b}$$
$$\dot{e} = -\bar{\omega}s + \Phi(\theta)^\top L_2\Phi(\theta)v \tag{4.38c}$$
$$\dot{s} = \bar{\omega}e + \Phi(\theta)^{\perp\top} L_2\Phi(\theta)v \tag{4.38d}$$
$$\dot{\tilde{\theta}} = \omega \tag{4.38e}$$
$$\dot{\omega} = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s_d, e_d). \tag{4.38f}$$

The closed-loop equation (4.38) is similar to (4.23) in which we replaced in the vector field the errors (e, s) by their delayed version (e_d, s_d) introduced in (4.12). That is, we modify the Lyapunov function constructed for the system (4.23) into a strict Lyapunov-Krasovskii functional in order to establish uniform global asymptotic stability of the origin of the system (4.38). To that end, we rewrite $\kappa(e_d, s_d)$ as

$$\kappa(e_d, s_d) = \kappa(e, s) + \kappa_d(e, s, \theta, \dot{z}_t) \tag{4.39}$$

with,

$$\kappa_d(\cdot) = \frac{1}{2}\begin{bmatrix} \mathcal{A}_1^T(\dot{z}_t)\mathcal{A}_1(\dot{z}_t) + 2e_1\phi(\theta_1)^T\mathcal{A}_1(\dot{z}_t) + 2s_1\phi(\theta_1)^{\perp T}\mathcal{A}_1(\dot{z}_t) \\ \vdots \\ \mathcal{A}_N^T(\dot{z}_t)\mathcal{A}_N(\dot{z}_t) + 2e_N\phi(\theta_N)^T\mathcal{A}_N(\dot{z}_t) + 2s_N\phi(\theta_N)^{\perp T}\mathcal{A}_N(\dot{z}_t) \end{bmatrix} \tag{4.40}$$

where $\dot{z}_t$ denote the functional $\dot{z}_t(\theta) := \dot{z}(t+\theta)$, for all $\theta \in [-T, 0]$.

Let us use the same change of coordinates used in (4.24), that is

$$e_\theta = \tilde{\theta} + \bar{f}(t)\kappa(s, e), \quad e_\omega = \omega + \dot{\bar{f}}(t)\kappa(s, e) \tag{4.41}$$

where the matrix $\bar{f}$ verifies (4.16).
Next, having $X_t = [v^\top, e^\top, s^\top]^\top$, $X_r = [e_\theta^\top, e_\omega^\top]^\top$, $\bar{e} = \text{diag}(e_i)$, $\bar{s} = \text{diag}(s_i)$, $\bar{e}_\omega = \text{diag}(e_{\omega i})$, $\bar{\kappa} = \text{diag}(\kappa_i)$ and using (4.16), we get

$$\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \dot{\bar{f}}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\dot{\bar{f}}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v - \begin{bmatrix} K_{pt}\Phi(\theta)^T \\ 0 \\ 0 \end{bmatrix} \mathcal{A}(\dot{z}_t) \tag{4.42a}$$

$$\dot{X}_r = \begin{bmatrix} 0 & I_N \\ -K_{p\theta} & -K_{d\theta} \end{bmatrix} X_r + \begin{bmatrix} \bar{f} \\ \dot{\bar{f}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2 \right) \Phi v - \begin{bmatrix} 0 \\ p(t) \end{bmatrix} \kappa_d(e, s, \theta, \dot{z}_t). \quad \text{(4.42b)}$$

The next result establishes uniform global asymptotic stability of the origin $(X_t, X_r) = (0, 0)$ of the system (4.42) provided that the following assumption holds

Assumption 4.4. *The matrices K_{dt} and K_{pt} satisfy*

$$1 - \left(1 + N^2\bar{a}^2\right) T^* \lambda_M(K_{pt}K_{dt}^-) \geq 0. \quad \text{(4.43)}$$

Hence, we recover uniform global asymptotic stability of $(v, z, \theta, \omega) = (0, 1_N \otimes z_c, \theta_d, 0)$ in the original coordinates.

Theorem 4.2. *Consider the system* (4.5) *in closed loop with* (4.36) *and* (4.37). *Assume that: K_{dt}, K_{pt}, $K_{d\theta}$ and $K_{p\theta}$ are diagonal positive definite and Assumptions 4.1, 4.2, 4.3, 4.4 hold. Then, the origin of the closed-loop system in the original state space, i.e, $(e, s, v, \tilde{\theta}, \omega) = (0, 0, 0, 0, 0)$ is uniformally globally asymptotically stable.*

Proof. (Sketch) The proof is constructive; we provide a strict Lyapunov-Krasovskii functional. Only the main steps are given here, the complete proof is in Appendix B.10.

First, we observe that the translational part of the system admits the following non-strict Lyapunov-Krasovskii functional

$$V(v, z, \dot{z}_t) = v^\top K_{pt}^{-1} v + z^\top L_2 z + \int_{-T^*}^{0} \int_{t+\theta}^{t} \dot{z}(s)^T \dot{z}(s) ds d\theta, \quad \text{(4.44)}$$

where $T^* = \max_{i,j} \{T_{ij}\}$.

Indeed, in view of (4.14), and the following inequality

$$\int_{-T^*}^{0} \int_{t+\theta}^{t} \dot{z}(s)^T \dot{z}(s) ds d\theta \leq T^* \int_{t-T^*}^{t} |\dot{z}(s)|^2 \, ds,$$

it follows that the function V is positive definite and radially unbounded with respect to $X_t = 0$, that is, there exist two class $\mathcal{K}_\infty$ functions u and v, such that inequality (A.10) holds with respect to $X_t = 0$. Which implies that $V(v, z, \dot{z}_t)$ is Lyapunov-Krasovskii candidate with respect to $X_t = 0$. Moreover, the time-derivative of V along the trajectories of (4.38) is

$$\begin{aligned}
\dot{V} =& -2v^\top K_{pt}^{-1}K_{dt}v + 2v^T\Phi(\theta)^T\mathcal{A}(\dot{z}_t) + T^* v^T v - \int_{t-T^*}^{t} \dot{z}(s)^T\dot{z}(s)ds \\
\leq& -[2 - T^*\lambda_M(K_{pt}K_{dt}^{-})]v^\top K_{pt}^{-1}K_{dt}v - \frac{1}{\bar{a}^2N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T\dot{z}_i(s)ds \\
&+ 2v^T\Phi(\theta)^T\mathcal{A}(\dot{z}_t)
\end{aligned} \tag{4.45}$$

then, we apply Jensen's inequality

$$\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T\dot{z}_i(s)ds \leq -\frac{1}{T_{ij}^*}\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds \tag{4.46}$$

and, we use

$$|\mathcal{A}(\dot{z}_t)|^2 \leq N\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds \tag{4.47}$$

to obtain

$$\begin{aligned}
\dot{V} \leq& -[2 - T^*\lambda_M(K_{pt}K_{dt}^{-})]v^\top K_{pt}^{-1}K_{dt}v - \frac{1}{\bar{a}^2NT^*}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds \\
&+ \frac{N}{2\epsilon}\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds + \epsilon\lambda_M(K_{pt}K_{dt}^{-})v^T K_{pt}^{-1}K_{dt}v.
\end{aligned} \tag{4.48}$$

Taking $\epsilon = N^2\bar{a}T^*$ and the matrices K_{dt} and K_{pt} such that Assumption 4.4 is verified, we get

$$\dot{V} \leq -v^\top K_{pt}^{-1}K_{dt}v - \frac{1}{2\bar{a}^2NT^*}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds \tag{4.49}$$

The strict Lyapunov-Krasovskii functional for the closed-loop system (4.42) is

$$\Gamma(t, X_t, X_r, \dot{z}_t) = W(t, X_t, V, \dot{z}_t) + \rho_1(V)Z(X_r) + \rho_2(V)V \tag{4.50}$$

where

$$\begin{aligned}
W \;=\;& \gamma(V)V + V\kappa^\top\bar{\Upsilon}_{\dot{f}^2}(t)\kappa + \alpha(V)e^\top v - c_1Ve^\top\bar{\dot{f}}s + c_1b_f\lambda_N(L)V^2 \\
&+ (\lambda_N(L) + |K_{pt}|)\,\alpha(V)V,
\end{aligned}$$

$$Z = c_2\left(e_\omega^\top e_\omega + e_\theta^\top K_{p\theta} e_\theta\right) + e_\theta^\top e_\omega,$$

$$\rho_1(V) = \frac{2\sigma(V)}{c_2\lambda_m(K_{d\theta})}\left(\alpha(V) + c_1 b_f V\right) + 1 + V,$$

$$\sigma(V) = \max\left\{\frac{16Tc_1 b_f}{\mu}, \frac{4\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\alpha(V)V}{\gamma(V)}\right\},$$

$$\begin{aligned}\alpha(V) = {} & 4b_f^2\lambda_N(L)V^2\left|K_{pt}^{-1}\right| + 4c_1 b_f^2\lambda_N(L)\left|K_{pt}^{-1}\right|V^2 + \frac{4c_1}{c_4}\left|\dot{\bar{f}}^2\left(\Phi^{\perp\top}L_2\Phi\right)^2\right|_\infty\left|K_{dt}^{-1}\right|V \\ & + c_1^2 c_4 b_f^2\left|K_{pt}^{-1}\right|,\end{aligned}$$

$$\begin{aligned}\gamma(V) = {} & 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^\top L_2\Phi\right|_\infty^2 + 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp\top}L_2\Phi\right|_\infty^2 \\ & + \frac{\partial\alpha}{\partial V}V\left(|K_{pt}| + \lambda_N(L)\right) + \frac{c_4}{2}c_1V + 2\alpha(V)\left|\Phi^\top L_2\Phi\right|_\infty\left|K_{dt}^{-1}K_{pt}\right|_\infty \\ & + \frac{c_4}{2}\left|K_{pt}K_{dt}^{-1}\right|\alpha^2(V) + \frac{c_4}{2}\alpha(V)\left|K_{dt}\right| + 2c_1b_f\lambda_N(L)V + \frac{4}{c_4}V^2\lambda_N(L)\left|K_{dt}^{-1}\right| \\ & + \frac{c_4}{2}c_1^2\left|K_{dt}^{-1}K_{pt}\right|\left|\dot{\bar{f}}^2\left(\Phi^\top L_2\Phi\right)^2\right|_\infty + 8\bar{a}^2N^2T^*\lambda_M(K_{pt})\alpha(V),\end{aligned}$$

$$\rho_2(V) = b_p\rho_1(V)[1 + c_2]N^2\bar{a}^2T^*[\lambda_N(L)c_5V + c_6\rho_1(V)] + \rho_1(V)\rho_3(V)V,$$

$$\rho_3(V) = \frac{c_3\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|}{2}\left(\left|\Phi^\top L_2\Phi\right|_\infty^2 + \left|\Phi^{\perp\top}L_2\Phi\right|_\infty^2\right),$$

and the constants c_1, c_2, c_3, c_4, c_5 and c_6 are:

$$c_1 = 1 + \frac{\lambda_N(L)}{\max\left\{2, \frac{2T}{\mu}\left(1 + \frac{2N}{\lambda_2(L)}\right)\right\}}, \quad c_2 = \frac{2}{\lambda_m(K_{d\theta})} + \frac{\lambda_M(K_{d\theta}) + 1}{\lambda_m(K_{p\theta})} + 1,$$

$$c_3 = \max\left\{\frac{8\left(2c_2b_f + b_f\right)^2}{c_2\lambda_m(K_{d\theta})}, \frac{8\left(2c_2b_f\lambda_M(K_{p\theta}) + b_f\right)^2}{\lambda_m(K_{p\theta})}\right\},$$

$$c_4 = \max\left\{2, \frac{2T}{\mu}\left(2 + \frac{8N}{\lambda_2(L)}\right)\right\}, \quad c_5 = \frac{b_pN^2\bar{a}^2T^*[1 + c_2]}{\min\left\{1, \lambda_m(K_{d\theta})\right\}},$$

$$c_6 \;=\; 16 b_p[2\lambda_M(K_{d\theta}^{-}) + \lambda_M(K_{p\theta}^{-})].$$

Γ is Lyapunov-Krasovskii candidate with respect to the origin due to the fact that $\Gamma(t,0,0,0) = 0$, for all $t \geq 0$, V is Lyapunov-Krasovskii candidate with respect to $X_t = 0$, and the following inequalities

$$W \;\geq\; \gamma(V)V,$$

$$W \;\leq\; \gamma(V)V + V\kappa^\top(e,s)\bar{\Upsilon}_{\dot{f}^2}(t)\kappa(e,s) + 2c_1 b_f \lambda_N(L)V^2 + 2\left(\lambda_N(L) + |K_{pt}|\right)\alpha(V)V,$$

$$Z \;\geq\; \min\{1, \lambda_m(K_{p\theta})\}\left(e_\theta^\top e_\theta + e_\omega^\top e_\omega\right),$$

$$Z \;\leq\; \max\{1 + c_2, c_2\lambda_M(K_{p\theta}) + 1\}\left(e_\theta^\top e_\theta + e_\omega^\top e_\omega\right).$$

After some lengthy computations we obtain

$$\dot{\Gamma} \leq -\frac{\mu}{4T}V^3 - \frac{\rho_1(V)}{16}\left[c_2 e_\omega^\top K_{d\theta} e_\omega + e_\theta^\top K_{p\theta} e_\theta\right] - \frac{1}{4}\gamma(V)v^\top K_{dt} K_{pt}^{-1} v - \frac{1}{16}\alpha(V)e^\top K_{pt} e. \tag{4.51}$$

Therefore, $\dot{\Gamma}$ is negative definite and Γ qualifies as a strict Lyapunov-Krasovskii functional for the system (4.42). Global uniformly asymptotic stability of the equilibrium $(X_t, X_r) = (0,0)$ is ensured and thus the delayed partial consensus problem is solved. □□□

4.3.3 Undelayed full consensus problem

In this case-study, we employ the translational controller introduced in (4.20), that is

$$u_v = -K_{dt}v - K_{pt}e, \tag{4.52}$$

where by design, K_{dt} and K_{pt} are diagonal positive definite matrices.
The rotational controller u_ω is

$$u_\omega^f = -L\theta + L\bar{q}(t)\kappa(s,e) + \alpha, \tag{4.53a}$$

$$\dot{\alpha} = -K_\alpha \alpha - K_I \omega + \dot{p}\kappa(s,e), \tag{4.53b}$$

where K_α and K_I are diagonal positive definite matrices and $\kappa(s,e)$ is defined in (4.22).

We solve the full consensus problem by studying the closed-loop of (4.5) under the controllers (4.11), (4.20), and (4.53); we obtain

$$\dot{z} = \Phi(\theta)v \tag{4.54a}$$
$$\dot{v} = -K_{dt}v - K_{pt}e \tag{4.54b}$$
$$\dot{e} = -\bar{\omega}s + \Phi(\theta)^\top L_2 \Phi(\theta)v \tag{4.54c}$$
$$\dot{s} = \bar{\omega}e + \Phi(\theta)^{\perp\top} L_2 \Phi(\theta)v \tag{4.54d}$$
$$\dot{\theta} = \omega \tag{4.54e}$$
$$\dot{\omega} = -L\theta + \alpha - L\bar{q}(t)\kappa(s,e) \tag{4.54f}$$
$$\dot{\alpha} = -K_\alpha \alpha - K_I \omega - \dot{p}\kappa(s,e). \tag{4.54g}$$

The translational part (4.54b)-(4.54d) is the same as (4.23b)-(4.23d) in the undelayed partial consensus case, whereas, the rotation part in (4.54e)-(4.54g) has a PID-like structure instead of a PD-like structure as in (4.23e)-(4.23f).

We establish uniform global asymptotic stability of the invariant set

$$\mathcal{S} := \left\{ (v,e,s,\theta,\omega,\alpha) \in \mathbb{R}^{6N} \;:\; (v,e,s,L\theta,\omega,\alpha) = 0 \right\}. \tag{4.55}$$

The proof relies on the construction of a strict Lyapunov function. To that end, we introduce the following change of coordinates

$$e_\theta = \tilde{\theta} + \bar{q}(t)\kappa(s,e), \quad e_\omega = \omega + \dot{\bar{q}}(t)\kappa(s,e), \quad e_\alpha = \alpha + \ddot{\bar{q}}(t)\kappa(s,e). \tag{4.56}$$

Having $X_t = [v^\top, e^\top, s^\top]^\top$, we introduce the rotational component as $X_r := [e_\theta^\top, e_\omega^\top, e_\alpha^\top]^\top$, using $\bar{e} = \mathrm{diag}(e_i)$, $\bar{s} = \mathrm{diag}(s_i)$, $\bar{e}_\omega = \mathrm{diag}(e_{\omega i})$, $\bar{\kappa} = \mathrm{diag}(\kappa_i)$, and (4.15) we obtain

$$\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \dot{\bar{q}}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\dot{\bar{q}}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v \tag{4.57a}$$

$$\dot{X}_r = \begin{bmatrix} 0 & I_N & 0 \\ -L & 0 & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r + \begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \\ \ddot{\bar{q}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2 \right) \Phi v. \tag{4.57b}$$

Let us introduce the following set

$$\mathcal{D} := \left\{(X_t, X_r) \in \mathbb{R}^{6N} \ : \ (X_t, Le_\theta, e_\omega, e_\alpha) = 0\right\}. \tag{4.58}$$

Note that in view of Lemma 4.1, $(X_t, X_r) \in \mathcal{D}$ is equivalent to having $(v, z, \theta, \omega, \alpha)$ belonging to the set $\mathcal{S}$ introduced in (4.55). Thus, in the following, we analyze the stability of the set $\mathcal{D}$.

Theorem 4.3. *Consider the system* (4.5) *in closed-loop with* (4.20) *and* (4.53). *Assume that K_{dt}, K_{pt}, K_I and K_α are diagonal positive definite and Assumption 4.3 holds. Then, the set $\mathcal{D}$ of the closed-loop state space is uniformally globally asymptotically stable. Moreover, the proof is constructive; we provide a strict Lyapunov function.*

Proof. (Sketch) The complete proof is in Appendix B.11, here we include only the main steps.

First, the translational part of the system admits $V(v, z)$ as a non-strict Lyapunov function, using (4.26) and (4.27) and the fact that the translational part is the same in (4.57) and in (4.25).
The strict Lyapunov function for the closed-loop system (4.57) is

$$\Gamma(t, X_t, X_r) = W(t, X_t, V) + \rho_1(V)Z(X_r) + \rho_2(V)V \tag{4.59}$$

where

$$\begin{aligned} W &= \gamma(V)V + V\kappa^\top \bar{\Upsilon}_{\dot{q}^2}(t)\kappa + \alpha(V)e^\top v - c_1 V e^\top \bar{\dot{q}} s + c_1 b_q \lambda_N(L)V^2 \\ &\quad + (\lambda_N(L) + |K_{pt}|)\,\alpha(V)V, \\ Z &= c_2\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right) + c_5 e_\omega^T e_\alpha + e_\theta^T L e_\omega, \end{aligned}$$

$$\rho_1(V) = \frac{2\sigma(V)}{c_5\lambda_m(K_I)}\left(\alpha(V) + c_1 b_q V\right) + 1, \tag{4.60}$$

$$\sigma(V) = \max\left\{\frac{16Tc_1 b_q}{\mu}, \frac{4\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\alpha(V)V}{\gamma(V)}\right\}, \tag{4.61}$$

$$\begin{aligned} \alpha(V) &= 4b_q^2\lambda_N(L)V^2\left|K_{pt}^{-1}\right| + 4c_1 b_q^2 \lambda_N(L)\left|K_{pt}^{-1}\right|V^2 + \frac{4c_1}{c_4}\left|\bar{\dot{q}}^2\left(\Phi^{\perp\top}L_2\Phi\right)^2\right|_\infty \left|K_{dt}^{-1}\right| V \\ &\quad + c_1^2 c_4 b_q^2\left|K_{pt}^{-1}\right|, \end{aligned}$$

$$\begin{aligned}
\gamma(V) &= 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{q}^2}\Phi^\top L_2\Phi\right|_\infty^2 + 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{q}^2}\Phi^{\perp\top} L_2\Phi\right|_\infty^2 \\
&+\frac{\partial\alpha}{\partial V}V\left(|K_{pt}|+\lambda_N(L)\right)+\frac{c_4}{2}c_1V+2\alpha(V)\left|\Phi^\top L_2\Phi\right|_\infty\left|K_{dt}^{-1}K_{pt}\right|_\infty \\
&+\frac{c_4}{2}\left|K_{pt}K_{dt}^{-1}\right|\alpha^2(V)+\frac{c_4}{2}\alpha(V)\left|K_{dt}\right|+2c_1b_q\lambda_N(L)V+\frac{4}{c_4}V^2\lambda_N(L)\left|K_{dt}^{-1}\right| \\
&+\frac{c_4}{2}c_1^2\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\dot{q}}^2\left(\Phi^\top L_2\Phi\right)^2\right|_\infty,
\end{aligned}$$

$$\rho_2(V) = c_3\rho_3(V), \quad \rho_3(V) = V\rho_1(V)\lambda_n(L)\left[\left|\phi^TL\phi\right|_\infty+\left|\phi^{\perp T}L\phi\right|_\infty\right], \tag{4.62}$$

and the constants c_1, c_2, c_3 and c_4 are:

$$c_1 = 1+\frac{\lambda_N(L)}{\max\left\{2, \frac{2T}{\mu}\left(1+\frac{2N}{\lambda_2(L)}\right)\right\}},$$

$$\begin{aligned}
c_2 &= 4c_5\lambda(K_IK_\alpha)+4c_5\lambda_M(K_I^2K_\alpha)+4c_5^2\lambda_M(K_IK_\alpha^-) \\
&+ 4\lambda_M(K_IK_\alpha^-)+2\lambda_n(L)+4+2\lambda_M(K_I)c_5^2,
\end{aligned}$$

$$\begin{aligned}
c_3 &= 2b_q^2\Bigg[(2c_2+1)^2+\frac{\lambda_M(K_I^-)}{c_5}(2c_2+c_5+\lambda_n(L))^2+ \\
&\frac{\lambda_M(K_\alpha^-K_I)}{c_2}(2\lambda_M(K_I^-)+c_5)^2\Bigg]\left|K_{dt}^-K_{pt}\right|,
\end{aligned}$$

$$c_4 = \max\left\{2, \frac{2T}{\mu}\left(2+\frac{8N}{\lambda_2(L)}\right)\right\}, \quad c_5 = 4\lambda_n(L)\lambda_M(K_I^-).$$

Positive definiteness of Γ with respect to $\mathcal{D}$ is ensured, which means that $\Gamma(t, X_t, X_r) \geq 0$ and $\Gamma(t, X_t, X_r) = 0 \Leftrightarrow |[X_t\ X_r]|_{\mathcal{D}}$, for all $t \geq 0$), using the fact that ρ_1 and ρ_2 are positive radially unbounded functions and the following inequalities hold

$$W \geq \gamma(V)V,$$

$$W \leq \gamma(V)V + V\kappa^\top(e,s)\bar{\Upsilon}_{\dot{q}^2}(t)\kappa(e,s)+2c_1b_f\lambda_N(L)V^2+2\left(\lambda_N(L)+|K_{pt}|\right)\alpha(V)V,$$

and

$$\frac{c_2}{2}\left(e_\theta^TLe_\theta+e_\omega^Te_\omega+e_\alpha^TK_I^-e_\alpha\right) \leq Z \leq 2c_2\left(e_\theta^TLe_\theta+e_\omega^Te_\omega+e_\alpha^TK_I^-e_\alpha\right).$$

After some term chasing and some cumbersome manipulations we get

$$\begin{aligned}\dot{\Gamma} \leq & -\frac{\rho_1(V)}{8}\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_5 e_\omega^T K_I e_\omega + e_\theta^T L^2 e_\theta\right] \\ & -\frac{1}{4}\gamma(V) v^\top K_{dt} K_{pt}^{-1} v - \frac{1}{8}\alpha(V) e^\top K_{pt} e - \frac{\mu}{4T} V^3\end{aligned} \tag{4.63}$$

Therefore $\dot{\Gamma}$ is negative definite and Γ qualifies as a strict Lyapunov function for the system (4.57). Global uniformly asymptotic stability of the set $\mathcal{D}$ is ensured and thus the full consensus problem is solved. □□□

4.3.4 Delayed full consensus problem

For the translational controller, we employ the same delayed PD-like controller used in (4.36), that is,

$$u_v = -K_{dt} v - K_{pt} e_d. \tag{4.64}$$

where K_{dt} and K_{pt} are diagonal positive definite matrices.

For the rotational part, we introduce for each i

$$\begin{aligned} u_{\omega i} = & -K_{\omega i}\omega_i - K_{\omega i}\dot{q}_i \kappa_i(s_d, e_d) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\theta_i - \theta_j(t - T_{ij}(t))\right) + \alpha_i \\ & + \sum_{j\in\mathcal{N}_i} a_{ij}\left[q_i\kappa_i(s_d, e_d) - q_j(t - T_{ij}(t))\kappa_j(s_d(t - T_{ij}(t)), e_d(t - T_{ij}(t)))\right] \end{aligned} \tag{4.65a}$$

$$\dot{\alpha}_i = -K_{\alpha i}\alpha_i - K_{Ii}\omega_i + \dot{p}\kappa_i(s_d, e_d). \tag{4.65b}$$

with K_ω, K_α and K_I diagonal positive definite matrices.
Using the variable $e_\theta := \theta + \bar{q}\kappa(e, s)$, the control law (4.65) has the following form

$$u_\omega = -K_\omega \omega - K_\omega \dot{\bar{q}}\kappa(s_d, e_d) - (L e_\theta + \mathcal{A}(\dot{e}_{\theta t})) + \alpha - \tag{4.66a}$$

$$\begin{bmatrix} \vdots \\ \sum_{j\in\mathcal{N}_i} a_{ij} q_i\left[\kappa_i(s_d, e_d) - \kappa_i(s, e)\right] \\ \vdots \end{bmatrix} + \begin{bmatrix} \vdots \\ \sum_{j\in\mathcal{N}_i} a_{ij} q_j(t - T_{ij}) \begin{bmatrix} \kappa_j(s_d(t - T_{ij}), e_d(t - T_{ij})) - \\ \kappa_j(s(t - T_{ij}), e(t - T_{ij})) \end{bmatrix} \\ \vdots \end{bmatrix},$$

$$\dot{\alpha} = -K_\alpha \alpha - K_I \omega + \dot{p}\kappa(s_d, e_d), \tag{4.66b}$$

where

$$\mathcal{A}(\dot{e}_{\theta_t}) = \begin{bmatrix} \sum_{j\in\mathcal{N}_1} a_{1j} \int_{t-T_{j1}(t)}^{t} \dot{e}_{\theta j}(\delta)d\delta \\ \vdots \\ \sum_{j\in\mathcal{N}_N} a_{Nj} \int_{t-T_{jN}(t)}^{t} \dot{e}_{\theta j}(\delta)d\delta \end{bmatrix}, \tag{4.67}$$

$$Le_\theta + \mathcal{A}(\dot{e}_{\theta_t}) = \begin{bmatrix} \sum_{j\in\mathcal{N}_1} a_{1j}(e_{\theta 1} - e_{\theta j}(t - T_{j1})) \\ \vdots \\ \sum_{j\in\mathcal{N}_N} a_{Nj}(e_{\theta N} - e_{\theta j}(t - T_{jN})) \end{bmatrix}, \tag{4.68}$$

and the function κ defined in (4.22).
Using (4.39) and the matrix $D := \text{diag}\left[\sum_{j\in\mathcal{N}_i} a_{ij}\right]$, we obtain

$$u_\omega = -K_\omega\omega - K_\omega\bar{\dot{q}}\kappa(s_d, e_d) - (Le_\theta + \mathcal{A}(\dot{e}_{\theta t})) + \alpha - D\bar{q}\kappa_d(\cdot) + \mathcal{B}(t) \tag{4.69a}$$

$$\dot{\alpha} = -K_\alpha\alpha - K_I\omega + \dot{p}\kappa(s_d, e_d). \tag{4.69b}$$

where

$$\mathcal{B}(t) := \begin{bmatrix} \vdots \\ \sum_{j\in\mathcal{N}_i} a_{ij}q_j(t - T_{ij})\kappa_{dj}\left(s(t - T_{ij}), e(t - T_{ij}), \theta(t - T_{ij}), \dot{z}_{t-T_{ij}}\right) \\ \vdots \end{bmatrix},$$

In this part, we use the controllers (4.12), (4.36), and (4.65), in closed-loop with the system (4.5) to obtain

$$\dot{z} = \Phi(\theta)v \tag{4.70a}$$

$$\dot{v} = -K_{dt}v - K_{pt}e_d \tag{4.70b}$$

$$\dot{e} = -\bar{\omega}s + \Phi(\theta)^\top L_2\Phi(\theta)v \tag{4.70c}$$

$$\dot{s} = \bar{\omega}e + \Phi(\theta)^{\perp\top} L_2\Phi(\theta)v \tag{4.70d}$$

$$\dot{\theta} = \omega \tag{4.70e}$$

$$\dot{\omega} = -K_\omega\omega - Le_\theta + \alpha - \mathcal{A}(\dot{e}_{\theta t}) - K_\omega\bar{\dot{q}}\kappa(e_d, s_d) - D\bar{q}\kappa_d(t) + \mathcal{B}(t) \tag{4.70f}$$

$$\dot{\alpha} = -K_\alpha\alpha - K_I\omega - \dot{p}\kappa(s_d, e_d). \tag{4.70g}$$

The closed-loop equation (4.70) is similar to (4.54) in which we replaced the errors (e, s) by their delayed version (e_d, s_d) introduced in (4.12). In the next theorem, we propose to extend the strict Lyapunov function constructed for the the system (4.54) to a strict Lyapunov-Krasovskii functional in order to establish uniform global asymptotic stability of the set $\mathcal{S}$ introduced in (4.55) for the closed-loop system when $\mathcal{B}(\cdot) = 0$, then we use the output injection argument in Lemma A.7 to conclude the global uniform asymptotic stability of the global closed-loop system.

Remark 4.3. *The output injection argument in Lemma A.7 remains valid in the presence of uniformally bounded time delay at least when the unperturbed system admits a strict Lyapunov-Karasovskii functional.*

Let us recall the change of coordinates used in (4.56)

$$e_\theta = \tilde{\theta} + \bar{q}(t)\kappa(s,e), \quad e_\omega = \omega + \dot{\bar{q}}(t)\kappa(s,e), \quad e_\alpha = \alpha + \ddot{\bar{q}}(t)\kappa(s,e). \tag{4.71}$$

Having $X_t = [v^\top, e^\top, s^\top]^\top$, $X_r = [e_\theta^\top, e_\omega^\top, e_\alpha^\top]^\top$, $\bar{e} = \text{diag}(e_i)$, $\bar{s} = \text{diag}(s_i)$, $\bar{e}_\omega = \text{diag}(e_{\omega i})$, $\bar{\kappa} = \text{diag}(\kappa_i)$ and using (4.15), we obtain

$$\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \dot{\bar{q}}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\dot{\bar{q}}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v - \begin{bmatrix} K_{pt}\Phi(\theta)^T \\ 0 \\ 0 \end{bmatrix} \mathcal{A}(\dot{z}_t) \tag{4.72a}$$

$$\begin{aligned} \dot{X}_r = {} & \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \\ \ddot{\bar{q}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2\right) \Phi v \\ & - \begin{bmatrix} 0 \\ K_\omega \dot{\bar{q}} + D\bar{q} \\ \dot{p} \end{bmatrix} \kappa_d(e,s,\theta,\dot{z}_t) + \begin{bmatrix} 0 \\ \mathcal{B}(t) \\ 0 \end{bmatrix}. \end{aligned} \tag{4.72b}$$

Note that in view of Lemma 4.1, (X_t, X_r) belongs to the set $\mathcal{D}$, introduced in (4.58), is equivalent to having $(v, e, s, \theta, \omega, \alpha)$ belonging to the $\mathcal{S}$ introduced in (4.55). Thus, we propose to study the stability of the set $\mathcal{D}$, provided that the following extra assumption holds

Assumption 4.5.

$$\lambda_m(K_\omega) \geq \frac{2T^{*2}N^2\bar{a}^2}{c_2}\left[\frac{1}{6} + 2c_2\right],$$

with,

$$c_2 = 3\lambda_M(K_\omega) - 2\lambda_N(L) + \frac{3}{2}\lambda_M(K_I K_\alpha^-).$$

Theorem 4.4. *Consider the system* (4.5) *in closed-loop with* (4.36) *and* (4.65) *with* $\mathcal{B}(t) = 0$. *Assume that* K_{dt}*,* K_{pt}*,* K_ω*,* K_α *and* K_I *are diagonal positive definite and Assumptions 4.1, 4.2,* *4.3, 4.5 hold. Then, the set $\mathcal{D}$, introduced in* (4.58)*, of the closed-loop state space is uniformally globally asymptotically stable.*

Proof. (Sketch) The proof is constructive; we provide a strict Lyapunov-Krasovskii functional. The complete proof is in Appendix B.12, the main steps are the following

First, since the translational part of the system ((4.70b)-(4.70c)) is the same as in the partial delayed case ((4.38b)-(4.38c)) then, using (4.44) and (4.45), we conclude that it admits $V(v, z, \dot{z}_t)$ as a non-strict Lyapunov-Krasovskii functional with respect to $X_t = 0$.

The strict Lyapunov-Krasovskii functional for the closed-loop system (4.72) with $\mathcal{B}(t) = 0$ is

$$\Gamma(t, X_t, X_r, \dot{z}_t, \dot{e}_{\theta t}) = W(t, X_t, V, \dot{z}_t) + \rho_1(V) Z(X_r, \dot{e}_{\theta t}) + \rho_2(V) V \tag{4.73}$$

where

$$\begin{aligned} W &= \gamma(V)V + V\kappa^\top \bar{\Upsilon}_{\dot{q}^2}(t)\kappa + \alpha(V)e^\top v - c_1 V e^\top \bar{\dot{q}} s + c_1 b_q \lambda_N(L) V^2 \\ &\quad + (\lambda_N(L) + |K_{pt}|)\,\alpha(V)V, \end{aligned}$$

$$Z = c_2\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right) + e_\theta^T L e_\omega + c_5 \int_{-T^*}^{0}\int_{t+h}^{t} \dot{e}_\theta^T(s)\dot{e}_\theta(s)\,ds\,dh,$$

$$\rho_1(V) = \frac{2\sigma(V)}{c_2 \lambda_m(K_\omega)}\left(\alpha(V) + c_1 b_q V\right) + 1 + V,$$

$$\sigma(V) = \max\left\{\frac{16 T c_1 b_q}{\mu}, \frac{4\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\alpha(V)V}{\gamma(V)}\right\},$$

$$\begin{aligned} \alpha(V) &= 4b_q^2 \lambda_N(L) V^2 \left|K_{pt}^{-1}\right| + 4c_1 b_q^2 \lambda_N(L)\left|K_{pt}^{-1}\right| V^2 + \frac{4c_1}{c_4}\left|\bar{\dot{q}}^2\left(\Phi^{\perp\top} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-1}\right| V \\ &\quad + c_1^2 c_4 b_q^2 \left|K_{pt}^{-1}\right|, \end{aligned}$$

$$\begin{aligned}
\gamma(V) &= 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{q}^2}\Phi^\top L_2\Phi\right|_\infty^2 + 2c_4V^2\lambda_N(L)\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\Upsilon}_{\dot{q}^2}\Phi^{\perp\top}L_2\Phi\right|_\infty^2 \\
&+\frac{\partial\alpha}{\partial V}V\left(|K_{pt}|+\lambda_N(L)\right)+\frac{c_4}{2}c_1V+2\alpha(V)\left|\Phi^\top L_2\Phi\right|_\infty\left|K_{dt}^{-1}K_{pt}\right|_\infty \\
&+\frac{c_4}{2}\left|K_{pt}K_{dt}^{-1}\right|\alpha^2(V)+\frac{c_4}{2}\alpha(V)\left|K_{dt}\right|+2c_1b_q\lambda_N(L)V+\frac{4}{c_4}V^2\lambda_N(L)\left|K_{dt}^{-1}\right| \\
&+\frac{c_4}{2}c_1^2\left|K_{dt}^{-1}K_{pt}\right|\left|\bar{\dot{q}}^2\left(\Phi^\top L_2\Phi\right)^2\right|_\infty + 8\bar{a}^2N^2T^*\lambda_M(K_{pt})\alpha(V),
\end{aligned}$$

$$\begin{aligned}
\rho_2(V) &= c_3\rho_3(V)+\rho_1(V)b_qV\lambda_m(K_{pt}K_{dt}^-)\left(\lambda_N^2(L)+c_6c_5T^*\right)+ \\
&2\bar{a}^2N^2T^*\rho_1(V)\Bigg[\frac{c_2}{2}(b_q+b_p)c_7\rho_1(V)+c_6c_2b_q(1+\left(1+\lambda_M(K_\omega^-D)\right)^2\lambda_M(K_\omega))V\lambda_N(L)+ \\
&\frac{c_7}{2}\lambda_M(K_\omega)\left(1+\lambda_M(K_\omega^-D)\right)^2b_q\rho_1(V)+\frac{2\left(1+\lambda_M(K_\omega^-D)\right)^2\lambda_M(K_\omega)}{c_6}\lambda_N(L)Vb_q\Bigg],
\end{aligned}$$

$$\rho_3(V) = V\rho_1(V)\lambda_n(L)\left[\left|\phi^TL\phi\right|_\infty^2+\left|\phi^{\perp T}L\phi\right|_\infty^2\right],$$

and the constants $c_1 - c_7$ are:

$$c_1 = 1+\frac{\lambda_N(L)}{\max\left\{2,\frac{2T}{\mu}\left(1+\frac{2N}{\lambda_2(L)}\right)\right\}}, \quad c_2 = 3\lambda_M(K_\omega)+2\lambda_N(L)+\frac{3}{2}\lambda_M(K_IK_\alpha^-),$$

$$c_3 = b_q^2\left[(2c_2+1)^2+\lambda_M(K_\omega)(2c_2+\lambda_n(L))^2+\lambda_M(K_\alpha^-K_I)4\lambda_M(K_I^-)^2\right]\left|K_{dt}^-K_{pt}\right|,$$

$$c_4 = \max\left\{2,\frac{2T}{\mu}\left(2+\frac{8N}{\lambda_2(L)}\right)\right\}, \quad c_5 = \bar{a}^2N^2T^*\left[\frac{1}{6}+2c_2\right],$$

$$c_6 = 16b_q\left[\lambda_M(K_\omega)+\frac{c_5T^*}{c_2}+2+2\lambda_M(K_I^-K_\alpha)\right],$$

$$c_7 = 2N^2T^*\bar{a}^2\left[b_p\lambda_M^2(K_\omega)+\lambda_M(K_I^-)b_p+\lambda_M(K_\omega)\lambda_N(L)b_q\right]$$

Notice that the functional $Z(X_r,\dot{e}_{\theta t})$ is a strict Lyapunov-Krasovskii functional with respect to the set

$$\mathcal{D}_r := \left\{(Le_\theta,e_\omega,\alpha)\in\mathbb{R}^{3N} \;/\; Le_\theta=e_\omega=\alpha=0\right\},$$

under the assumption 4.5, for the following delayed system

$$\dot{X}_r = \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix}. \tag{4.74}$$

In deed, $Z(X_r, \dot{e}_{\theta t})$ is a Lyapunov-Krasovskii candidate with respect to the set $\mathcal{D}_r$ due to the following inequalities

$$\begin{aligned} Z &\geq \frac{c_2}{2}\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right), \\ Z &\leq 2c_2\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right) + 2c_5 T^* \int_{t-T^*}^{t} |\dot{e}_\theta(s)|^2\, ds \end{aligned}$$

then, one can easily find the two class $\mathcal{K}_\infty$ functions u and v such that inequality (A.10) holds with respect to the set $\mathcal{D}_r$.
The time-derivative of $Z(X_r, \dot{\theta}_t)$ along trajectories of (4.74) verifies

$$\begin{aligned} \dot{Z}(\cdot) = &- 2c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - e_\theta^T L^2 e_\theta - e_\theta^T L K_\omega e_\omega + e_\theta^T L e_\alpha + e_\omega^T L e_\omega \\ &- e_\theta^T L \mathcal{A}(\dot{e}_{\theta t}) - 2c_2 e_\omega^T \mathcal{A}(\dot{e}_{\theta t}) + c_5 T^* \dot{e}_\theta^T \dot{e}_\theta - c_5 \int_{t-T^*}^{t} \dot{e}_\theta^T(s)\dot{e}_\theta(s) ds. \end{aligned} \tag{4.75}$$

Using the fact that $K_\omega > I_N$ both with the following inequalities

$$e_\theta^T L e_\alpha \leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon}{2}\lambda_M(K_I K_\alpha^-) e_\alpha^T K_I^- K_\alpha e_\alpha$$

$$e_\omega^T L e_\omega \leq \lambda_N(L) e_\omega^T K_\omega e_\omega,$$

$$-2c_2 e_\omega^T \mathcal{A}(e_{\theta t}) \leq \epsilon_1 c_2 e_\omega^T K_\omega e_\omega + \frac{c_2}{\epsilon_1}\left|\mathcal{A}(e_{\theta t})\right|^2,$$

$$e_\theta^T L \mathcal{A}(e_{\theta t}) \leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon}{2}\left|\mathcal{A}(e_{\theta t})\right|^2,$$

$$e_\theta^T L K_\omega e_\omega \leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon \lambda_M(K_\omega)}{2},$$

$$|\mathcal{A}(e_{\theta t})|^2 \leq N \sum_{j=1}^{N}\sum_{i=1}^{N}\left(\int_{t-T_{ij}}^{t} \dot{e}_{\theta j}(s) ds\right)^2 a_{ij}^2,$$

$$\begin{aligned}\int_{t-T^*}^{t} \dot{e}_\theta^T(s)\dot{e}_\theta(s)ds &\geq \frac{1}{\bar{a}^2 N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{e}_{\theta j}^2(s)ds,\\ \int_{t-T_{ij}}^{t} a_{ij}^2\dot{e}_{\theta j}^2(s)ds &\geq \frac{1}{T^*}\left(\int_{t-T_{ij}}^{t} \dot{e}_{\theta j}(s)ds\right)^2 a_{ij}^2,\end{aligned}$$

taking $\epsilon_1 = \frac{1}{2}$, $\epsilon = 3$ and using Assumption 4.5 we obtain

$$\dot{Z}(\cdot) = -\frac{1}{2}c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{1}{2}e_\theta^T L^2 e_\theta. \tag{4.76}$$

Since V and Z are Lyapunov-Krasovskii candidates with respect to $X_t = 0$ and $\mathcal{D}_r$ respectively, then we conclude that Γ is so with respect to $\mathcal{D}$ using the following inequalities.

$$\begin{aligned} W &\geq \gamma(V)V,\\ W &\leq \gamma(V)V + V\kappa^\top(e,s)\bar{\Upsilon}_{\dot{q}^2}(t)\kappa(e,s) + 2c_1 b_f \lambda_N(L)V^2 + 2\left(\lambda_N(L) + |K_{pt}|\right)\alpha(V)V.\end{aligned}$$

Then, one can easily find the two class $\mathcal{K}_\infty$ functions u and v such that equation (A.10) holds with respect to the set $\mathcal{D}$ since it verifies $\mathcal{D} = \{X_t = 0\} \cap \mathcal{D}_r$.

After some term chasing and some cumbersome manipulations we get

$$\begin{aligned}\dot{\Gamma} \leq &-\frac{\rho_1(V)}{8}\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_2 e_\omega^T K_\omega e_\omega + e_\theta^T L^2 e_\theta\right]\\ &-\frac{1}{4}\gamma(V)v^\top K_{dt}K_{pt}^{-1}v - \frac{1}{16}\alpha(V)e^\top K_{pt}e - \frac{\mu}{4T}V^3.\end{aligned} \tag{4.77}$$

This implies that $\dot{\Gamma}$ is negative definite and Γ qualifies as a strict Lyapunov-Krasovskii functional for system (4.72). Global uniformly asymptotic stability of the set $\mathcal{D}$ is ensured and thus the full delayed consensus problem is solved. □□□

Corollary 4.1. *Assume that K_{dt}, K_{pt}, K_ω, K_α and K_I are diagonal positive definite matrices and let Assumptions 4.1, 4.2, 4.3, 4.5 hold. Then, for the closed-loop system* (4.72) *the set $\mathcal{D}$, introduced in* (4.58), *of the closed-loop state space is uniformally globally asymptotically stable.*

Proof. The proof of the corollary is a direct application of Lemma A.7. Indeed,

Item 1. the global closed-loop system (4.72) is uniformally globally stable, that is, using the non strict Lyapunov-Karasovskii functional V introduced in (4.44) with a time derivative along trajectories of (4.72a) given in (4.49), which concludes the uniform global stability of translational coordinates. The rotational part in (4.72b) is composed by the uniformally exponentially stable linear delayed system

$$\dot{X}_r = \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix}$$

which is ISS with respect to the bounded perturbation vector

$$\begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \\ \ddot{\bar{q}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2\right) \Phi v - \begin{bmatrix} 0 \\ K_\omega \dot{\bar{q}} + D\bar{q} \\ \dot{p} \end{bmatrix} \kappa_d(e, s, \theta, \dot{z}_t) + \begin{bmatrix} 0 \\ \mathcal{B}(t) \\ 0 \end{bmatrix}$$

that depends on the translational coordinates thus, the first item follows.

Item 2. the uniform global asymptotic stability of the unperturbed (system (4.72) with $\mathcal{B}(\cdot) = 0$) follows for theorem 4.4.

Item 3. the last condition to verify concerns the integrability of the vector $\mathcal{B}(\cdot)$, having

$$\begin{aligned} |\mathcal{B}(t)|^2 \leq & \sum_{i=1}^{N} \left(\sum_{j\in\mathcal{N}_i} a_{ij} q_j(t - T_{ij}) \kappa_{dj}(t - T_{ij}) \right)^2 \leq N \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} a_{ij}^2 q_j(t - T_{ij})^2 \kappa_{dj}(t - T_{ij})^2 \\ \leq & N\bar{a}^2 \, |q|_\infty^2 \sum_{j=1}^{N} \kappa_{dj}^2(t - T_j^*) \end{aligned} \tag{4.78}$$

where

$$T_j^*(t) := \underset{i}{\operatorname{argmax}} \left\{ \kappa_{dj}^2(t - T_{ij}(t)) \right\}.$$

Using (4.40) we obtain

$$\kappa_{dj}^2(t - T_j^*) \leq \left| g_j(t - T_j^*) \right| \mathcal{A}_j^T(\dot{z}_{t-T_j^*}) \mathcal{A}_j(\dot{z}_{t-T_j^*})$$

with

$$g_j(t) = \frac{1}{4} \left[\mathcal{A}_j(\dot{z}_t) + 2e_j\phi(\theta_j) + 2s_j\phi(\theta_j)^\perp \right] \left[\mathcal{A}_j^T(\dot{z}_t) + 2e_j\phi(\theta_j)^T + 2s_j\phi(\theta_j)^{\perp T} \right].$$

Using (4.13) we obtain the following inequality

$$\begin{aligned}\mathcal{A}_j^T(\dot{z}_{t-T_j^*})\mathcal{A}_j(\dot{z}_{t-T_j^*}) \leq & \left[\sum_{k\in\mathcal{N}_j} a_{kj}\int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta\right]^T \left[\sum_{k\in\mathcal{N}_j} a_{kj}\int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta\right] \\ \leq & N\sum_{k\in\mathcal{N}_j} a_{kj}^2 \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta) d\delta.\end{aligned}$$

From the last inequalities, we obtain

$$\begin{aligned}|\mathcal{B}(\cdot)|^2 \leq & N^2\bar{a}^2 |q|_\infty^2 \sum_{j=1}^{N} \left|g_j(t-T_j^*)\right| \sum_{k\in\mathcal{N}_j} a_{kj}^2 \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta) d\delta \\ \leq & N^2\bar{a}^2 |q|_\infty^2 \sup_j \left\{\left|g_j(t-T_j^*)\right|\right\} \sum_{j=1}^{N}\sum_{k\in\mathcal{N}_j} a_{kj}^2 \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta) d\delta.\end{aligned}$$

Having the translational part of the closed-loop system globally bounded using the non strict Lyapunov-Karasovskii candidate V introduced in (4.44), we conclude the global boundedness of the term $N^2\bar{a}^2 |q|_\infty^2 \sup_j \left\{\left|g_j(t-T_j^*)\right|\right\}$. Then, using the time derivative of V along the trajectories of the translational coordinates $X_t(t)$ in (4.49) we conclude the uniform integralbility of the term

$$\sum_{j=1}^{N}\sum_{k\in\mathcal{N}_j} a_{kj}^2 \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta)^T d\delta \int_{t-T_{kj}-T_j^*}^{t-T_j^*} \dot{z}_j(\delta) d\delta$$

since $T_j^* \leq T^*$, which conclude the proof of the corollary. □□□

Remark 4.4. *Two remarks are in order:*
i) for simplicity, and without losing generality, the function p is taken equal for all the agents;
ii) the function κ in (4.22) *may correspond to any class-$\mathcal{K}$ function with the following form $\kappa(s_d, e_d) = \frac{1}{2}[G(s_{d1}^2 + e_{d1}^2), ..., G(s_{dn}^2 + e_{dn}^2)]^\top$. The only condition on κ is that there exist two positive polynomials $P_1(\cdot)$ and $P_2(\cdot)$ such that:*

$$G(\cdot) \leq P_1(\cdot), \ and, \ \left|\frac{\partial G(\cdot)}{\partial(\cdot)}\right| \leq P_2(\cdot).$$

4.4 Conclusion

This chapter deals with the distributed formation control of multiple nonholonomic robots under a general time-varying delay. We report a novel decentralized consensus-based formation controllers that consider both, the kinematic and the dynamic model and a delayed exchanged information between the elements of the network, to uniformly and asymptotically solve the *partial* and the *full* consensus problems. The network is modeled as an undirected, static and connected graph. The controller has a smooth time-varying PD-like and PID-like scheme that is $\delta-$persistently exciting. Up to the authors' knowledge this is the first work that provides a strict Lyapunov function and a strict Lyapunov-Krasovskii functional, thereby guaranteeing uniform global asymptotic stability for the closed-loop system. Simulations, using a network with six agents, have been provided to illustrate our theoretical contributions.

Conclusions & Future Work

The following concluding remarks are in order.

In Chapter 1. we presented some technical results on uniform exponential stability of time-varying linear systems with particular structures that appear, for example, in the analysis of model-reference adaptive systems, persistently excited observers, consensus of systems interconnected through time-varying links and systems with time-varying input gain. Stability proofs we presented in this section are based on the explicit construction of strict Lyapunov functions. Such an approach allows not only to conclude stability and convergence properties of the system trajectories but also to give explicit decay estimates for the convergence rate.

In the subsequent chapters these stability results served as basis for the consensus and formation control of mobile robots using controllers with persistency of excitation.

In Chapters 2-3. we identified several control problems for swarms of mobile robots depending on nature of the leader's velocities, notably the leader-follower tracking, robust agreement and simultaneous tracking-agreement problems. In all three cases assuming a spanning tree communication graph topology, we considered two-stage controller design – first at the kinematic and then at the dynamic level. At the kinematics level, a nonlinear change of coordinates was used to transform the three problems into that of uniform global asymptotic stabilization of the origin. Stability analysis provided in these chapters relies on the extension of strict Lyapunov functions proposed in Chapter 1, cascaded systems design, notions of iISS and strong iISS and their characterization. In particular, we provided strict Lyapunov functions for the closed loop systems at the kinematic level and demonstrated that at the dynamic level one can use a variety of control schemes for Lagrangian and Hamiltonian systems that ensure square intergrability of velocity errors.

In Chapter 4. we restricted our attention to the leaderless consensus problem of multiple mobile robots, but under the assumption that the robots are interconnected in a general bidirectional graph and that the communications are affected by time-varying delays.

In particular, we considered 2 cases : in the first case, under the assumption that each robot adopts a particular orientation, i.e., consensus is pursued only in their Cartesian positions on the plane while in the second case, the robots required to assume both common position and orientation.

We proposed decentralized smooth time-varying PD-like and PID-like controllers that consider both, the kinematic and the dynamic models to uniformly and asymptotically solve the partial and the full consensus problems. Assuming that there are no delays in the communications, we designed new strict Lyapunov function that ensures uniform global asymptotic stability of the consensus set, these functions served the basis to construct strict Lyapunov-Krasovskii functional for the formation with delays.

Although many of controllers proposed in this thesis are reminiscent of others that have appeared in the literature, our contributions lie in the establishment of strong properties such as uniform global asymptotic stability, (integral) intput-to-state stability and, most remarkably, in the construction of original Lyapunov functions for most of the control problems that we addressed.

Chapter 1 is bound to present concrete case-studies of stability analysis for time-varying systems. Each of this case-studies, we believe, may serve as a departure point to different lines of research. In that light, the subsequent chapters are devoted to an in-depth study of one case-study: that of consensus and formation control of autonomous vehicles. Other concrete open questions include:

1. Design of strict differentiable Lyapunov function for the first order time varying consensus problem studied in Subsection 1.2.1 for the case of directed graphs.

2. Establish necessary and/or more relaxed sufficient conditions for uniform exponential stability of the spiraling systems (1.51).

3. Extension of the leader-follower tracking, robust-agreement and simultaneous tracking-agreement controllers to more general graph topologies and in the presence of time delay.

4. Extension of the results of Chapter 4 on partial and full consensus of mobile robots to the case of connected directed graph and to other types of moving agents.

Appendix A

Basic notions

A.1 Preliminaries

Our technical results are stability statements of nonlinear time-varying systems of the form

$$\dot{x} = f(t, x). \tag{A.1}$$

For simplicity, we assume that f is such that solutions exist and are unique.

A.2 Uniform Stability notions

Definition A.1. *Consider the time-varying dynamical system*

$$\dot{x} = f(t, x) \tag{A.2}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that the solutions of (A.2) *exist in finite time intervals for all initial condition $(t_0, x(t_0)) \in \mathbb{R} \times \mathbb{R}^n$ and admit an invariant set $\mathcal{A}$.*
The invariant set $\mathcal{A}$ is Uniformly Stable (US) if there exist $\alpha \in \mathcal{K}$ and $r > 0$, such that

$$|x(t, t_0, x(t_0))|_{\mathcal{A}} \leq \alpha(|x(t_0)|_{\mathcal{A}}) \qquad \forall t \geq t_0, \qquad \forall\, |x(t_0)|_{\mathcal{A}} \leq r. \tag{A.3}$$

The invariant set $\mathcal{A}$ is Uniformly Asymptotically Stable (UAS) if there exists $\beta \in \mathcal{KL}$, such that

$$|x(t, t_0, x(t_0))|_{\mathcal{A}} \leq \beta(t - t_0, |x(t_0)|_{\mathcal{A}}) \qquad \forall t \geq t_0, \qquad \forall\, |x(t_0)|_{\mathcal{A}} \leq r. \tag{A.4}$$

The invariant set $\mathcal{A}$ is Uniformly Exponentially Stable (UES) if there exists $\gamma_1,\ \gamma_2 > 0$, such that

$$|x(t,t_0,x(t_0))|_{\mathcal{A}} \le \gamma_1 \, |x(t_0)|_{\mathcal{A}} \, e^{(t-t_0)} \qquad \forall t \ge t_0, \qquad \forall\ |x(t_0)|_{\mathcal{A}} \le r. \tag{A.5}$$

The invariant set $\mathcal{A}$ is UGS, UGAS, UGES if equations (A.3), (A.4), (A.5) *hold, respectively, for all $r > 0$.*

Definition A.2. *Consider the delayed time-varying dynamical system*

$$\dot{x} = f(t, x_t), \quad t \ge t_0. \tag{A.6}$$

where $f : \mathbb{R}_{\ge 0} \times C[-T, 0] \to \mathbb{R}^n$ is continuous in both arguments and locally Lipschitz in the second argument. and admits an invariant set $\mathcal{A}$, that is,

$$|x(t_0 + \theta)|_{\mathcal{A}} = 0,\ \forall \theta \in [-T, 0] \Rightarrow f(t, x_{t_0}) = 0,\ \forall t \ge t_0.$$

The invariant set $\mathcal{A}$ is Uniformly Stable (US) if there exist $\alpha \in \mathcal{K}$ and $r > 0$, such that

$$|x(t,t_0,x_{t_0})|_{\mathcal{A}} \le \alpha(|x_{t_0}|_{\mathcal{A}}) \qquad \forall t \ge t_0, \qquad \forall\, |x_{t_0}|_{\mathcal{A}} \le r. \tag{A.7}$$

The invariant set $\mathcal{A}$ is Uniformly Asymptotically Stable (UAS) if there exists $\beta \in \mathcal{KL}$, such that

$$|x(t,t_0,x_{t_0})|_{\mathcal{A}} \le \beta(t - t_0, |x_{t_0}|_{\mathcal{A}}) \qquad \forall t \ge t_0, \qquad \forall\, |x_{t_0}|_{\mathcal{A}} \le r. \tag{A.8}$$

The invariant set $\mathcal{A}$ is Uniformly Exponentially Stable (UES) if there exists $\gamma_1,\ \gamma_2 > 0$, such that

$$|x(t,t_0,x_{t_0})|_{\mathcal{A}} \le \gamma_1 \, |x_{t_0}|_{\mathcal{A}} \, e^{(t-t_0)} \qquad \forall t \ge t_0, \qquad \forall\ |x_{t_0}|_{\mathcal{A}} \le r. \tag{A.9}$$

The invariant set $\mathcal{A}$ is UGS, UGAS, UGES if equations (A.3), (A.4), (A.5) *hold, respectively, for all $r > 0$.*

Lemma A.1 (Lyapunov characterization)**.** *Suppose $f : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}^n$; and that u; v; w : $\mathbb{R}_+ \to \mathbb{R}_{\ge 0}$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The invariant set $\mathcal{A}$ is uniformly stable if there exists a continuous functional $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}$, which is positive-definite radially unbounded with respect to $\mathcal{A}$, i.e.*

$$u(|x(t)|_{\mathcal{A}}) \le V(t, x) \le v(|x|_{\mathcal{A}}).$$

and such that its derivative along (A.6) *is non-positive in the sense that*

$$\dot{V}(t,x) \leq -w(|x(t)|_{\mathcal{A}}).$$

If $w(s) > 0$ for $s > 0$, then the invariant set $\mathcal{A}$ solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} = \infty$, then it is globally uniformly asymptotically stable.

Lemma A.2 (Lyapunov–Krasovskii characterization [33])**.** *Suppose $f : \mathbb{R}_{\geq 0}\times C[-T,0] \to \mathbb{R}^n$; and that u; v; $w : \mathbb{R}_+ \to \mathbb{R}_{\geq 0}$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The invariant set $\mathcal{A}$ is uniformly stable if there exists a continuous functional $V : \mathbb{R}_{\geq 0} \times W[-T,0] \times \mathcal{L}_2[-T,0] \to \mathbb{R}$, which is positive-definite radially unbounded with respect to $\mathcal{A}$, i.e.*

$$u(|x(t)|_{\mathcal{A}}) \leq V(t,x_t,\dot{x}_t) \leq v(\|x_t\|_{\mathcal{A}}). \tag{A.10}$$

and such that its derivative along (A.6) *is non-positive in the sense that*

$$\dot{V}(t,x_t,\dot{x}_t) \leq -w(|x(t)|_{\mathcal{A}}).$$

If $w(s) > 0$ for $s > 0$, then the invariant set $\mathcal{A}$ solution is uniformly asymptotically stable. If in addition $\lim_{s\to\infty} w(s) = \infty$, then it is globally uniformly asymptotically stable.

A.3 ISS and Lyapunov characterization

Definition A.3 (ISS [110])**.** *Consider the time-varying dynamical system*

$$\dot{x} = f(t,x,u) \tag{A.11}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is such that its solutions exist on the infinite time interval for all initial condition $(t_0, x(t_0)) \in \mathbb{R} \times \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.
The dynamical system (A.11) *is Input-to-State Stable (ISS), with respect to the input u, if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, and a class $\mathcal{K}^\infty$ function $\gamma(\cdot)$, such that:*

$$|x(t)| \leq \beta(|x(t_0)|, t-t_0) + \gamma\left(\sup_{t_0\leq s\leq\infty} |u(s)|\right) \tag{A.12}$$

Similarly, the dynamical system (A.11) *is small Input-to-State Stable (ISS), with respect to the input u, if there exists $r > 0$, such that equation* (A.12) *holds for $|u| \leq r$.*

Lemma A.3 ([47])**.** *Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov* *function such that:*

$$\underline{\alpha}\left(|x|\right) \leq V(t, X) \leq \overline{\alpha}\left(|x|\right) \tag{A.13}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W(x),\ \forall\, |x| \geq \rho\left(|u|\right) > 0 \tag{A.14}$$

Where $\underline{\alpha}$, $\overline{\alpha}$ are $\mathcal{K}^{\infty}$ functions, ρ a class $\mathcal{K}$ function, and W a continous PD function, which implies that the system $\dot{x} = f(t, x, u)$ is ISS with respect to the input u.
Similarly, if there exists $r > 0$ such that (A.14) *holds for $|u| \leq r$ then the system $\dot{x} = f(t, x, u)$ is small ISS with respect to the input u.*

A.4 integral ISS and Lyapunov characterization

Definition A.4 (Integral ISS [110]). *Consider the time-varying dynamical system*

$$\dot{x} = f(t, x, u) \tag{A.15}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is such that its solutions exist on the infinite time interval for all initial condition $(t_0, x(t_0)) \in \mathbb{R} \times \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.
The dynamical system (A.15) *is Integral Input-to-State Stable (iISS), with respect to the input u, if there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$, and a class $\mathcal{K}^{\infty}$ function $\gamma(\cdot)$, such that:*

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \int_{t_0}^{t} \gamma\left(|u(s)|\right) ds \tag{A.16}$$

Lemma A.4 ([42]). *Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that:*

$$\underline{\alpha}\left(|x|\right) \leq V(t, X) \leq \overline{\alpha}\left(|x|\right) \tag{A.17}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha_1(|x|) + \rho(|u|) \tag{A.18}$$

Where $\underline{\alpha}$, $\overline{\alpha}$ are $\mathcal{K}^{\infty}$ functions, α_1 a positive definite function called dissipation rate, and ρ a class $\mathcal{K}^{\infty}$ function, which implies that the system $\dot{x} = f(t, x, u)$ is integral ISS with respect to the input u.

A.5 Strong iISS

Definition A.5 (Strong iISS [17]). *Consider the time-varying dynamical system*

$$\dot{x} = f(t, x, u) \tag{A.19}$$

where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is such that its solutions exist on the infinite time interval for all initial condition $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.
The dynamical system (A.19) *is said to be strongly integral input-to-state stable (strongly iISS) with respect to u, if it is integral input-to-state stable (iISS) with respect to u, and input-to-state stable (ISS) with respect to sufficiently small values of u.*

Lemma A.5 ([18]). *Consider the following cascaded interconnected system:*

$$\dot{x}_1 = f_1(t, x_1, x_2, u_1) \tag{A.20}$$

$$\dot{x}_2 = f_2(t, x_2, u_2) \tag{A.21}$$

Where f_1 is strong iISS with respect to $[x_2\ u_1]$, and f_2 is so with respect to u_2, then the overall system is strong iISS with respect to $[u_1\ u_2]$.

Remark A.1. *The prove of the last lemma is provided in [18] for the autonomous case, but the prove may be directly extended to the non-autonomous case, because it uses a catalog of properties introduced in [111], [8] and [9], and the crucial part when establishing these properties uses the converse Lyapunov theorem for Asymptotic Stability which exists for the uniform asymptotic stability,see for example [122].*

A.6 Nonlinear output injection

Undelayed case

Lemma A.6 ([99]). *Consider the following system in the output injection form:*

$$\dot{x} = f(t, x) = F(t, x) + K(t, x) \tag{A.22}$$

The origin of (A.22) *is UGAS follows if:*

i. the origin of $\dot{x} = f(t, x)$ is uniformly globally stable;

ii. the origin of $\dot{x} = F(t, x)$ is UGAS;

iii. *there exist an "output" y, non decreasing functions k_1, k_2, β: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ function k, as well as a positive definite function γ such that*

$$|K(t,x)| \leq k_1(|x|)k(|y|) \tag{A.23}$$

$$|y(t,x)| \leq k_2(|x|) \tag{A.24}$$

$$\int_0^\infty \gamma\big(|y(t)|\big) \leq \beta(|x(0)|). \tag{A.25}$$

In the presence of time varying Delay

The following lemma is the extension of Lemma A.6 to the case of time varying delayed systems

Lemma A.7. *Consider the following delayed system in the output injection form:*

$$\dot{x} = f(t, x_t) = F(t, x_t) + K(t, x_t) \tag{A.26}$$

$f : \mathbb{R}_+ \times C[-T, 0] \to \mathbb{R}^n$ *uniformly bounded in t and sufficiently smooth. The origin of* (A.26) *is UGAS follows if:*

i. *the origin of $\dot{x} = f(t, x_t)$ is UGS;*

ii. *the origin of $\dot{x} = F(t, x_t)$ is UGAS and admits a strict differentiable Lyapunov-Karasovskii functional continuous functional $V : \mathbb{R}_+ \times W[-T, 0] \times \mathcal{L}_2[-T, 0] \to \mathbb{R}$ with the properties*

$$\underline{\alpha}(|x|) \leq V(t, x_t, \dot{x}_t) \leq \bar{\alpha}(\|x_t\|) \tag{A.27}$$

$$\dot{V}_F(t, x_t, \dot{x}_t) \leq -\alpha(|x|) \tag{A.28}$$

$$\left|\frac{\partial V(t,x_t,\dot{x}_t)}{\partial x}\right| \leq \alpha_\partial(\|x_t\|) \tag{A.29}$$

where the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \alpha_\partial \in \mathcal{K}_\infty$.

iii. *there exist an "output" $y : \mathbb{R}_+ \times C[-T, 0] \to \mathbb{R}^m$, non decreasing functions k_1, k_2, β: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ function k, as well as a positive definite function γ such that*

$$|K(t, x_t)| \leq k_1(|x_t|)k(|y|) \tag{A.30}$$

$$|y(t, x_t)| \leq k_2(|x_t|) \tag{A.31}$$

$$\int_{-T}^\infty \gamma\big(\,|y(t, x_t)|\,\big) \leq \beta(|x_t(0)|). \tag{A.32}$$

Proof. Since the system $\dot{x} = F(t, x_t)$ is UGAS, and admits a continuously differentiable Lyaunov-Karasovskii functional $V(t, x_t, \dot{x}_t)$.

Taking the derivative of $V(t, x)$ along the trajectories of A.26 we obtain

$$\dot{V}_{A.26}(t, x_t, \dot{x}_t) \leq -\alpha(|x|) + \alpha_\partial(\|x_t\|)k_1(|x_t|)k(|y|), \tag{A.33}$$

and since the system is UGS, there exist $\gamma_w \in \mathcal{K}_\infty$ such that

$$|x(t)| \leq \gamma_w(|x_{t_0}|), \ \forall t \geq t_0.$$

Moreover, $f(t, x_t)$ uniformly bounded in t and continuous in x_t with $f(t, 0) = 0$ then, there exist a class $\mathcal{K}$ function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|x_t\| \leq \kappa\left(|x_{t_0}|\right), \ \forall t \geq t_0 + T$$

and

$$\alpha_\partial(\|x_t\|)k_1(|x_t|) \leq \mu(|x_{t_0}|), \ \forall t \geq t_0 + T.$$

Let assume that

$$|x_{t_0}| \leq r, \qquad\qquad r > 0.$$

We claim that for a given positive definite function γ and $k \in \mathcal{K}_\infty$, for each $r, \ \nu > 0$ and $\Delta > 0$, there exists $\rho > 0$ such that

$$k(|y|) \leq \frac{\nu}{\mu(r)} + \rho\gamma(|y|) \qquad \forall \ |y| \leq \Delta \tag{A.34}$$

indeed, we can take

$$\rho := \sup_{s \in (0,\Delta]} \left\{ \frac{k(s) - \frac{\nu}{\mu(r)}}{\gamma(s)} \right\} = \max_{s \in \left[k^{-1}\left(\frac{\nu}{\mu(r)}\right), \Delta\right]} \left\{ \frac{k(s) - \frac{\nu}{\mu(r)}}{\gamma(s)} \right\}. \tag{A.35}$$

Let $\Delta := k_2 \circ \gamma_w(r)$.

Using all these definitions in (A.33) we obtain that for all $|y| \leq \Delta$ and any $\nu > 0$ there exists $\rho = \rho(r, l, \nu, \Delta)$ such that

$$\dot{V}_{A.26}(t, x_t, \dot{x}_t) \leq -\alpha(|x(t)|) + \mu(r)\left[\frac{\nu}{\mu(r)} + \rho\gamma(|y(t)|)\right] \tag{A.36}$$

$$= -\left[\alpha(|x(t)|) - \nu\right] + \mu(r)\rho\gamma(|y(t)|), \ \forall t \geq t_0 + T. \tag{A.37}$$

Let $\beta_{r\nu}(s) := \bar{\alpha}(\kappa(s)) + \mu(r)\rho\beta(s)$, integrating on both sides of the inequality above, from $t_\circ$ to ∞ and using A.27-A.29, we obtain that for any $\nu > 0$

$$\int_{t_\circ}^{\infty} [\alpha(|x(\tau)|) - \nu]\, d\tau = \int_{t_\circ}^{t_\circ+T} [\alpha(|x(\tau)|) - \nu]\, d\tau + \int_{t+T}^{\infty} [\alpha(|x(\tau)|) - \nu]\, d\tau \tag{A.38}$$

$$\leq T\alpha(\gamma_\omega(|x_{t_0}|)) + \bar{\alpha}(\kappa\,(|x_{t_0}|)) + \mu(r)\rho \int_{t_\circ}^{\infty} \gamma(|y(\tau)|) d\tau \tag{A.39}$$

$$\leq \beta_{r\nu}(|x_{t_0}|). \tag{A.40}$$

The proof is completed invoking integral Lemma A.8. □□□

Integral lemma in the presence of time varying Delay

The following lemma is the extension of [99, Lemma 2] to the time varying delayed systems

Lemma A.8. *Consider the following delayed system:*

$$\dot{x} = f(t, x_t) \tag{A.41}$$

$f : \mathbb{R}_+ \times C[-T, 0] \to \mathbb{R}^n$ *sufficiently smooth.*

The system (A.41) *is UGAS if it is UGS and there exists continuous positive definite function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *and for each* $r, \nu > 0$ *there exists* $\beta_{r\nu} > 0$*, such that for all* $(t_0, x_{t_0}) \in \mathbb{R}_+ \times B_r$*, all solutions* $x(\cdot, t_0, x_{t_0})$ *and all* $t \geq t_0$*,*

$$\int_{t_0}^{t} [\gamma(|x(\tau, t_0, x_{t_0})|) - \nu]\, d\tau \leq \beta_\nu(|x_{t_0}|)\,. \tag{A.42}$$

Proof. By assumption the system is UGS, thus we need to prove global uniform attractivity only.

From UGS it follows that there exists a class $\mathcal{K}_\infty$ function γ_w such that $|x(t)| \leq \gamma_w(|x_{t_0}|)$ for all $t \geq t_0$.

Fix r and ϵ such that $0 < \epsilon \leq r$ and let $\delta = \gamma_w^{-1}(\epsilon)$. Since the system is UGS, we only need to show that there exists $T^*(r, \epsilon) > 0$ such that for each t_0 and each $x_{t_0} \in B_r$ there exists a time $t' \in [t_0,\, t_0 + T^*]$, such that $|x(t', t_0, x_{t_0})| \leq \delta$. We proceed by *reductio ad absurdum.*

Let $\gamma_m(\delta, r) := \min_{s\in[\delta,\, \gamma_w(r)]} \gamma(s)$ and[1] assume that $|x(t, t_0, x_{t_0})| > \delta$ for all $t \in [t_0,\, t_0 + T^*]$

[1]Note that $\gamma_w(s) \geq s$ so $s \geq \gamma_w^{-1}(s)$. Therefore, $\gamma_w(r) \geq r \geq \epsilon \geq \gamma_w^{-1}(\epsilon)$, so the interval $[\delta,\, \gamma_w(r)]$ is

where $T^*(\epsilon, r) := \dfrac{1}{\nu}\bar{\beta}_\nu(r)$, where $\nu = \gamma_m(\delta)/2$ and $\bar{\beta}_\nu(r) = \sup\limits_{0\le s\le r} \beta_\nu(s)$. Then we find that

$$\int_{t_0}^{t_0+T^*} \gamma(\|x(\tau, t_0, x_{t_0})\|)d\tau > T^*\gamma_m(\delta) = 2\bar{\beta}_\nu(r)\,. \tag{A.43}$$

On the other hand, from (1.40) it follows that

$$\begin{aligned}\int_{t_0}^{t_0+T^*} \gamma(|x(\tau, t_0, x_{t_0})|)d\tau &\le \int_{t_0}^{t_0+T^*} [\gamma(|x(\tau, t_0, x_{t_0})|) - \nu]\, d\tau + \int_{t_0}^{t_0+T^*} \frac{1}{2}\gamma_m(\delta)d\tau \\ &\le \beta_\nu(|x_{t_0}|) + \bar{\beta}_\nu(r) \le 2\bar{\beta}_\nu(r)\end{aligned}$$

which contradicts A.43. Therefore, the origin is uniformly attractive. □□□

A.7 PE, δ-PE and Uniform δ-PE

Definition A.6 (Persistency of Excitation [87]). *A piecewise continuous and bounded function $\psi : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$ is said to be persistently exciting, with excitation parameters$-(T, \mu)$, if there exist $T, \mu > 0$ such that*

$$\int_t^{t+T} \psi(s)\psi(s)^T ds \ge \mu I_n \quad \forall\, t \ge 0. \tag{A.44}$$

Let $x \in \mathbb{R}^n$ be partitioned as $x^T := \mathrm{col}[x_1^T\ x_2^T]$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Define the column vector function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and the set $\mathcal{D}_1 := (\mathbb{R}^{n_1}\backslash \{0\}) \times \mathbb{R}^{n_2}$.

Definition A.7 (Uniform δ Persistency of Eexcitation [66]). *A function $\phi(\cdot, \cdot)$ where $t \to \phi(t, x)$ is locally integrable is said to be uniformly δ- persistently exciting (U$\delta-$PE) with respect to x_1 if for each $x \in \mathcal{D}_1$ there exist $\delta > 0$, $T > 0$ and $\mu > 0$ such that $\forall t \in \mathbb{R}_+$*

$$|z - x| \le \delta \Rightarrow \int_t^{t+T} |\phi(\tau, z)|\, d\tau \ge \mu \tag{A.45}$$

If $\phi(\cdot, \cdot)$ is U$\delta-$PE with respect to the whole state x we will simply say that "ϕ is U$\delta-$PE".

Consider the system

$$\dot{x} = f(t, x) \tag{A.46}$$

where $f : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that the solution to (A.46) is forward complete. Let $\phi(\cdot, x(\cdot, t_0, x_0))$ is locally integrable for each solution $x(\cdot, t_0, x_0)$.

nonempty.

Definition A.8 (Uniform δ Persistency of Excitation along trajectories [99]). *A function ϕ is called uniformly persistently exciting (U$\delta-$PE) with respect to x_1 (along trajectories of* (A.46)*) if for each r and $\delta > 0$ there exist constants $T(r,\delta) > 0$ and $\mu(r,\delta) > 0$, such that for all $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{B}_r$, all corresponding solutions satisfy*

$$\left\{ \min_{s\in[t,t+T]} |x_1(s)| \geq \delta \right\} \Rightarrow \left\{ \int_t^{t+T} \phi(\tau, x(\tau, t_0, x_0))^T \phi(\tau, x(\tau, t_0, x_0)) d\tau \right\}. \tag{A.47}$$

Remark A.2. *In general, for multivariable functions, the two properties, in Definitions. A.8 and A.7, are different. Neither one implies the other –see [66] however, for the type of functions of interest here, the following statement establishes a link between the two properties.*

Lemma A.9 ([87]). *If $u_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $u_1 \in \Omega_{(n,t_0,T)}$, and $u_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, with $u_2 \to 0$ as $t \to \infty$. Then $u_1 + u_2 \in \Omega_{(n,t_1,T)}$ for some $t_1 \geq t_0$.*
where

$$\Omega_{(n,t_0,T)} := \left\{ u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n / \int_t^{t+T} u^T(s) u(s) ds \geq \mu, \forall t \geq t_0 \right\}.$$

Appendix B

Proof of auxiliary results

B.1 Proof of theorem 1.1

The proof is constructed based upon that of Lemma 1.1. We show by recurrence that the Lyapunov function candidate V_n is positive definite, proper and its total derivative satisfies (1.31). Firstly, using (1.5), we conclude

$$\frac{1}{2}\text{diag}\left(\beta_i\right) \leq P(t) \leq \frac{1}{2}\text{diag}\left(\beta_i(1+2\bar{a}T)\right).$$

Next, notice that for $i \geq 1$ the system (1.27) corresponds to

$$\Sigma_i' : \dot{x}_i = -a_i(t)x_i$$

and

$$W_i(t, x_i) = \frac{1}{2}\Upsilon_{a_i}(t)x_i^2 \tag{B.1}$$

is a strict Lyapunov function for Σ_i'. The latter follows by mimicking the proof of Lemma 1.1 to obtain

$$\dot{W}_i(t, x_i) \leq -\frac{\mu}{T}x_i^2 \tag{B.2}$$

–*cf.* Eq. (1.13). For $n = 2$, the cascaded system Σ_2' corresponds to (1.25), for which we define the function $V_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ as

$$V_2(t, \bar{x}_{12}) = W_1(t, x_1) + \beta_2 W_2(t, x_2) \tag{B.3}$$

with $\bar{x}_{1j} := [x_1 \cdots x_j]^\top$ and, according to (1.29),

$$\beta_2 \geq \frac{4T^2}{\mu^2}\left[\bar{a}(1+2\bar{a}T)\right]^2. \tag{B.4}$$

Furthermore, using the bound $\bar{a} \geq \max\{a_i, a_{i+1}\} \geq 0$, following the proof-lines of Lemma 1.1, we see that the time-derivative of V_2 satisfies

$$\dot{V}_2(t, \bar{x}_{12}) \leq \dot{W}_1(t, x_1) - \beta_2 \frac{\mu}{T} x_2^2 \tag{B.5}$$

and, along the trajectories of (1.25a), $\dot{V}_1$ satisfies

$$\dot{W}_1(t, x_1) \leq -\frac{\mu}{T} x_1^2 + \Upsilon_{a_1}(t) x_1 a_{12}(t) x_2.$$

In turn, this implies that

$$\dot{V}_2(t, x) \leq -\frac{\mu}{2T}(x_1^2 + \frac{3}{2}\beta_2 x_2^2) + \phi_2(t, \bar{x}_{12}, \beta_2) \tag{B.6}$$

$$\phi_2(t, \bar{x}_{12}, \beta_2) := -\frac{\mu}{2T} x_1^2 - \beta_2 \frac{\mu}{4T} x_2^2 + \Upsilon_{a_1}(t) x_1 x_2 a_{12}(t). \tag{B.7}$$

Now, notice that $\phi_2 \leq 0$ if β_2 satisfies (B.4). To show this, we introduce

$$\epsilon := \frac{2T}{\mu} \tag{B.8}$$

and we use the triangle inequality

$$x_1 a_{12}(t) \Upsilon_{a_1}(t) x_2 \leq \frac{1}{2\epsilon} x_1^2 + \frac{1}{2}\epsilon \left(a_{12}(t)\Upsilon_{a_1}(t)\right)^2 x_2^2, \tag{B.9}$$

to obtain

$$\phi_2(t, \bar{x}_{12}, \beta_2) \leq -\frac{x_1^2}{2}\big[\frac{\mu}{T} - \frac{1}{\epsilon}\big] - x_2^2 \left[\beta_2 \frac{\mu}{4T} - \frac{1}{2}\epsilon \big[(1 + 2\bar{a}T)\bar{a}\big]^2\right].$$

From (B.8) and (B.4) it follows that $\phi_2 \leq 0$ hence, we conclude that

$$\dot{V}_2(t, \bar{x}_{12}) \leq -\frac{\mu}{2T} x_1^2 - \frac{3\mu}{4T}\beta_2 x_2^2. \tag{B.10}$$

Next, we proceed by induction. For any $j \in (2, n]$, let V_j be a strict Lyapunov function for Σ_j' –*cf.* (1.27), and let it be defined as

$$V_j(t, \bar{x}_{1j}) = V_{j-1}(t, \bar{x}_{1j-1}) + \beta_j W(t, x_j). \tag{B.11}$$

To evaluate its total time-derivative along the trajectories of Σ_j' we first see that

$$\dot{V}_{j-1}(t, \bar{x}_{1j-1}) \leq -\frac{\mu}{2T}\sum_{i=1}^{j-2} \beta_i x_i^2 - \frac{3\mu}{4T} x_{j-1}^2 + \frac{\partial V_{j-1}}{\partial x_{j-1}} a_{(j-1)j} x_j$$

and, in view of (B.11),

$$\frac{\partial V_{j-1}}{\partial x_{j-1}} = \left[\Upsilon_{a_{j-1}}(t)\beta_{j-1}\right]x_{j-1}. \tag{B.12}$$

Hence, it follows that

$$\dot{V}_j(t,\bar{x}_{1j}) \leq -\frac{\mu}{2T}\sum_{i=1}^{j-1}\beta_i x_i^2 - \frac{3\mu}{4T}x_j^2 + \phi_j(t,\bar{x}_{1j},\beta_j,\beta_{j-1})$$

where

$$\begin{aligned}\phi_j(\cdot) = & -\frac{\mu}{4T}\beta_{j-1}x_{j-1}^2 + \beta_{j-1}\Upsilon_{a_{j-1}}(t)a_{(j-1)j}(t)x_j x_{j-1} \\ & - \beta_j\frac{\mu}{4T}x_j^2.\end{aligned} \tag{B.13}$$

Now, in view of (B.7), the factor of $\beta_{j-1}\Upsilon_{a_{j-1}}(t)a_{(j-1)j}(t)x_j x_{j-1}$ is non-negative hence, applying the triangle inequality to the last two terms on the right-hand side of (B.13), we obtain that, for any $\epsilon > 0$,

$$\begin{aligned}\phi_j(\cdot) \leq & -\frac{\mu}{4T}\beta_{j-1}x_{j-1}^2 + \frac{\beta_{j-1}}{2\epsilon}x_{j-1}^2 + \beta_{j-1}\frac{\epsilon}{2}\left(\Upsilon_{a_{j-1}}(t)a_{(j-1)j}(t)\right)^2 x_j^2 \\ & - \beta_j\frac{\mu}{4T}x_j^2\end{aligned}$$

which, in turn, using (1.5), we btain

$$\phi_j(\cdot) \leq -\left[\frac{\mu}{4T} - \frac{1}{2\epsilon}\right]\beta_{j-1}x_{j-1}^2 + \left[\beta_{j-1}\frac{\epsilon}{2}\left(\left(1+2T\bar{a}\right)\bar{a}\right)^2 - \beta_j\frac{\mu}{4T}\right]x_j^2$$

for all $\epsilon \neq 0$. To render non-positive the factors of x_j^2 and x_{j-1}^2 above, we choose

$$\epsilon = \frac{2T}{\mu}.$$

Then, the factor of x_{j-1}^2 equals to zero if (B.8) holds, while the factor of $-x_j^2$ is non-negative if

$$\beta_j \geq \frac{4T^2}{\mu^2}\beta_{j-1}\left[\left(1+2\bar{a}T\right)\bar{a}\right]^2$$

for all $j \in (2,n]$ –*cf.* (1.29). It follows that $\phi_j \leq 0$ and, consequently,

$$\dot{V}_j(t,\bar{x}_{1j}) \leq -\frac{\mu}{2T}\sum_{i=1}^{j-1}x_i^2 - \frac{3\mu}{4T}x_j^2. \tag{B.14}$$

The latter holds for any integer $j \in [3,n]$ hence, together with (B.2) and (B.10), the inequality (1.31) follows. ∎

B.2 Proof of theorem 1.4

In view of (1.5), the boundedness of a and b, we have

$$V(t,x) \geq \frac{1}{2}\left[\gamma + \Upsilon_{a^4b^2}(t)\right] x^\top Px - \bar{b}^2\bar{a}^3 \left| P^{1/2} A \sum_{i=1}^{n} \beta_i \Gamma_i P^{1/2} \right| x^\top Px$$
$$V(t,x) \leq \frac{1}{2}\left[\gamma + \bar{\Upsilon}_{a^4b^2}(t)\right] x^\top Px + \bar{b}^2\bar{a}^3 \left| P^{1/2} A \sum_{i=1}^{n} \beta_i \Gamma_i P^{1/2} \right| x^\top Px$$

Using the bound $\gamma > \gamma_2$, the function V is positive definite and radially unbounded. Indeed, there exist η_1, $\eta_2 > 0$ such that

$$\eta_1 |x|^2 \leq V(t,x) \leq \eta_2 |x|^2 \tag{B.15}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$.

Next, we compute the total derivative of V along the trajectories of (1.63). We use the persistency of excitation of the product $\psi = a^2 b$, (1.7) and (1.66), and we reorganise some terms to obtain

$$\begin{aligned}
\dot{V} \quad \leq \quad & -\gamma b^2 x^\top CC^\top x - \frac{\mu}{T} x^\top Px - a^4 b^2 x^\top \left[\beta_n PA\Gamma_n A^\top P - P + \sum_{i=1}^{n-1} \beta_i PA\Gamma_i A^\top P\right] x \\
& -ba^2 \left[\left[2\dot{b}a + 3b\dot{a}\right][PAx]^\top + ba^2 [PA^2 x]^\top\right] \sum_{i=1}^{n} \beta_i \Gamma_i Px \\
& -b^4 a^3 [C^\top x]^\top \sum_{i=1}^{n} \beta_i C^\top \left[A\Gamma_i P - \Gamma_i PA\right] x.
\end{aligned} \tag{B.16}$$

To establish that $\dot{V}$ is negative definite we first note that, since the pair A, B is controllable by assumption, the matrix $\Phi_c \Phi_c^\top$ where Φ_c corresponds to Kalman's controllability matrix $\Phi_c := [B \; AB \; \cdots \; A^{n-1}B]$, is positive definite and satisfies $\Phi_c \Phi_c^\top = A\Gamma_n A^\top$. Hence, in view of the definition of β_n we have $-\beta_n x^\top PA\Gamma_n A^\top Px + x^\top Px \leq 0$. Therefore, the sum of the first two terms in the second line of (B.16) is non-positive. Next, note that the terms in the last line of (B.16) are bounded from above by

$$\gamma_1 b^2 |C^\top x|^2 + \frac{\mu P_m}{2T\bar{b}^2} |x|^2$$

where γ_1 is defined in (1.68). Hence, using $\Gamma_1 = BB^\top$ and $\Gamma_i = BB^\top + A\Gamma_{i-1}A$ for all

$i \geq 2$, as well as (1.66c), it follows that

$$\dot{V} \leq -ba^2 x^\top M^\top \left[P \sum_{i=1}^{n} \beta_i CC^\top + \sum_{i=2}^{n} \beta_i PA\Gamma_{i-1} A^\top P \right] x - \gamma_2 b^2 |C^\top x|^2 - \frac{\mu}{2T} x^\top Px$$
$$- b^2 a^4 x^\top \sum_{i=1}^{n-1} \beta_i PA\Gamma_i A^\top Px$$

where M is defined below (1.71). Next, observe that

$$\begin{aligned} \sum_{i=2}^{n} \beta_i PA\Gamma_{i-1} A^\top P &= \sum_{i=2}^{n} \beta_i \sum_{j=1}^{i-1} PA^j BB^\top A^{j\top} P = \sum_{j=1}^{n-1} PA^j BB^\top A^{j\top} P \sum_{i=j+1}^{n} \beta_i \\ \sum_{i=1}^{n-1} \beta_i PA\Gamma_i A^\top P &= \sum_{i=1}^{n-1} \beta_i \sum_{j=1}^{i} PA^j BB^\top A^{j\top} P = \sum_{j=1}^{n-1} PA^j BB^\top A^{j\top} P \sum_{i=j}^{n-1} \beta_i \end{aligned}$$

so, in view of (1.69), we obtain

$$\begin{aligned} \dot{V} &\leq -\frac{\mu}{4T} x^\top Px - b^2 a^4 x^\top \left[\sum_{j=1}^{n-1} PA^j BB^\top A^{j\top} P \sum_{i=j} \beta_i \right] x \\ &\quad -ba^2 x^\top M^\top \left[\sum_{j=1}^{n-1} PA^j BB^\top A^{j\top} P \sum_{i=j+1}^{n} \beta_i \right] x. \end{aligned} \tag{B.17}$$

Then, defining $Y_j := [PA^j B]^\top$, it follows that

$$\dot{V} \leq -\frac{\mu}{4T} x^\top Px - \sum_{j=1}^{n-1} \left[|ba^2 Y_j x|^2 \sum_{i=j}^{n-1} \beta_i + \left[Y_j Mx \right]^\top \left[ba^2 Y_j x \right] \sum_{i=j+1}^{n} \beta_i \right]. \tag{B.18}$$

Using the triangle inequality on the last term on the right hand side of (B.18), we see that for any $\epsilon_j \neq 0$,

$$\begin{aligned} \dot{V} &\leq -\frac{\mu}{4T} x^\top Px + \frac{1}{2} \sum_{j=1}^{n-1} \epsilon_j^2 |Y_j Mx|^2 \\ &\quad -\sum_{j=1}^{n-1} |ba^2 Y_j x|^2 \left[\left[\beta_j + \sum_{i=j+1}^{n-1} \beta_i \right] - \frac{1}{2\epsilon_j^2} \left[\sum_{i=j+1}^{n} \beta_i \right]^2 \right]. \end{aligned}$$

Now, on one hand, defining

$$\epsilon_j^2 = \frac{\mu P_m}{4nT \, |Y_j M|^2} \tag{B.19}$$

we obtain

$$-\frac{\mu}{8T}x^\top Px + \frac{1}{2}\sum_{j=1}^{n-1}\epsilon_j^2|Y_jMx|^2 \le 0. \tag{B.20}$$

On the other, in view of (1.71), we have

$$\beta_j + \sum_{i=j+1}^{n-1}\beta_i \ge \frac{1}{2\epsilon_j^2}\left(\sum_{i=j+1}^{n}\beta_i\right)^2 \ \ge\ 0.$$

Thus, we conclude that

$$\dot{V} \le -\frac{\mu}{8T}x^\top Px$$

which completes the proof. ■

B.3 Proof of Proposition 1.2

First, we remark that V is quadratic and having that a_i satisfy (1.3) then, using (1.75) and (1.5), we obtain

$$V(t,x) \ge [1/2+\gamma]\,|x|^2 - \sum_{i=2}^{n}\alpha_i\bar{a}^{2i-1}\,|x|^2 \ge 1/2\,|x|^2 \tag{B.21}$$

$$V(t,x) \le \left[\bar{\Upsilon}_{\psi^2}+\gamma\right]|x|^2 + \sum_{i=2}^{n}\alpha_i\bar{a}^{2i-1}\,|x|^2 \tag{B.22}$$

We conclude that V is quadratic positive definite and radially unbounded Lyapunov function candidate, that is, there exist c_1, $c_2 > 0$ such that

$$c_1|x|^2 \le V(t,x) \le c_2|x|^2.$$

Then, let introduce $\Pi_\ell(t) := \prod_{k=1}^{\ell}|a_k(t)|$, the time-derivative of the first term in V satisfies

$$\frac{d}{dt}\left\{\frac{1}{2}\sum_{i=1}^{n}\Upsilon_{\Pi_n^2}(t)x_i^2\right\} \le -\frac{\mu}{T}\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}\Pi_n^2(t)x_i^2 - a_1^2\Upsilon_{\Pi_n^2}(t)x_1^2. \tag{B.23}$$

On the other hand, the derivatives of Φ_i, satisfy the following.

Firstly, for $i=2$, we have

$$\frac{d}{dt}\{a_1^2\Phi_2\} \leq -\, a_1^2a_2^2x_2^2 + a_1^2\bar{a}^3\,|x_1x_2| + a_1^2\bar{a}^2\,|x_1x_3| + 4\,|a_1|\,\bar{a}^2\,|x_1x_2| + a_1^2\bar{a}^2x_1^2 \tag{B.24}$$

and, similarly, for each $i \in [3, n-1]$, we obtain

$$\begin{aligned}\frac{d}{dt}\{a_1^2\Phi_i\} \leq & 2i\sum_{j=1}^{i-1}\Pi_j\bar{a}^{(2i-1-j)}\,|x_jx_{j+1}| + \sum_{j=1}^{i-1}\Pi_j^2(t)\bar{a}^{2(i-j)}\,|x_jx_{j+2}| \\ & + a_1(t)^2\bar{a}^{2(i-1)}x_1^2 - \Pi_i^2(t)x_i^2 + a_1^2\bar{a}^{2i-1}\,|x_1x_2|\end{aligned} \tag{B.25}$$

while, for $i = n$,

$$\begin{aligned}\frac{d}{dt}\{a_1^2\Phi_n\} \leq & -\Pi_n^2(t)x_n^2 + a_1(t)^2\bar{a}^{2i-1}\,|x_1x_2| + \sum_{j=1}^{n-2}\Pi_j^2(t)\bar{a}^{2(n-j)}\,|x_jx_{j+2}| \\ & + a_1(t)^2\bar{a}^{2(n-1)}x_1^2 + 2n\sum_{j=1}^{n-1}\Pi_j(t)\bar{a}^{(2n-1-j)}\,|x_jx_{j+1}|.\end{aligned} \tag{B.26}$$

Thus, putting together (B.23) up to (B.26) we get

$$\begin{aligned}\dot{V}(t,x) \leq & -\frac{\mu}{2T}|x|^2 - \sum_{i=2}^{n}[\alpha_i - \bar{a}^{n-i}]\Pi_i^2(t)x_i^2 - \frac{\mu}{4T}|x|^2 + \sum_{i=3}^{n}\sum_{j=2}^{i-1}\Pi_j^2(t)\left[\bar{a}^{2(i-j)}\alpha_i\,|x_jx_{j+2}|\right] \\ & + \sum_{i=3}^{n}\sum_{j=2}^{i-1}\Pi_j(t)\left[\alpha_i 2i\bar{a}^{(2i-1-j)}\,|x_jx_{j+1}|\right] + \phi_1(t,x)\end{aligned}$$

where we defined

$$\begin{aligned}\phi_1(t,x) := & \sum_{i=2}^{n}\alpha_i\left[a_1(t)\bar{a}^{2i-1}\,|x_1x_2| + a_1^2\bar{a}^{2(i-1)}x_1^2 + a_1^2\bar{a}^{2(i-1)}\,|x_1x_3|\right. \\ & \left. +2i\,|a_1|\,\bar{a}^{2(i-1)}\,|x_1x_2|\right] - \gamma a_1(t)^2x_1^2 - \frac{\mu}{4T}|x|^2.\end{aligned} \tag{B.27}$$

Now, notice that all positive terms above are (bounded by) cross products of $|x_2|$ and $|x_3|$ with $|a_1x_1|$ hence, they may be upper-bounded by

$$\lambda[a_1x_1]^2 + \frac{1}{\lambda}[\epsilon_2x_2^2 + \epsilon_3x_3^2]$$

for appropriate values of ϵ_2, ϵ_3 and λ. It follows that for sufficiently small λ and choosing γ sufficiently large, as in (1.75), we have $\phi_1 \leq 0$. Therefore,

$$\dot{V}(t,x) \leq -\frac{\mu}{2T}|x|^2 + \phi_2(t,x)$$

where we defined

$$\begin{aligned}\phi_2(t,x) := &-\sum_{i=2}^{n}\left[\alpha_i - \bar{a}^{n-i}\right]\Pi_i(t)^2 x_i^2 - \frac{\mu}{4T}|x|^2 \\ &+\sum_{i=3}^{n}\sum_{j=2}^{i-1}\Pi_j(t)\left[\Pi_j(t)\bar{a}^{2(i-j)}\alpha_i\,|x_j x_{j+2}| + 2i\bar{a}^{(2i-1-j)}\,|x_j x_{j+1}|\right].\end{aligned}$$

It is left to prove that $\phi_2 \leq 0$. To that end, we start by changing the order of summation. Hence,

$$\begin{aligned}\phi_2(t,x) = &-\sum_{i=2}^{n}\left[\alpha_i - \bar{a}^{n-i}\right]\Pi_i(t)^2 x_i^2 - \frac{\mu}{4T}|x|^2 + \sum_{i=2}^{n-2}\Pi_i(t)^2\,|x_i x_{i+2}|\sum_{j=i+1}^{n}\bar{a}^{2(j-i)}\alpha_j \\ &+\sum_{i=2}^{n-1}\Pi_i(t)\,|x_i x_{i+1}|\sum_{j=i+1}^{n}\alpha_j 2i\bar{a}^{(2j-1-i)}\end{aligned}$$

which satisfies

$$\begin{aligned}\phi_2(t,x) \leq &-\frac{\mu}{4nT}x_n^2 + \sum_{i=2}^{n-2}\Bigg[-\left[\alpha_i - \bar{a}^{n-i}\right]\Pi_i^2 x_i^2 + \Pi_i^2\sum_{j=i+1}^{n}\bar{a}^{2(j-i)}\alpha_j - \frac{\mu}{4nT}\left[x_{i+1}^2 + x_{i+2}^2\right] \\ &+\Pi_i^2\sum_{j=i+1}^{n}\bar{a}^{2(j-i)}\alpha_j\,|x_i x_{i+2}| + \sum_{j=i+1}^{n}\alpha_j 2i\bar{a}^{(2j-1-i)}\Pi_i\,|x_i x_{i+1}|\Bigg] - \left[\alpha_{n-1} - \bar{a}\right]\Pi_{n-1}^2 x_{n-1}^2 \\ &+\left[\alpha_n 2(n-1)\bar{a}^n\right]\Pi_{n-1}|x_{n-1}x_n|\end{aligned}$$

Now, as for ϕ_1 in (B.27), we see that the cross terms in the first summation above may be upper-bounded using the triangle inequality. That is, we use

$$2|x_i x_{i+2}| \leq \epsilon_i x_i^2 + \frac{1}{\epsilon_i}x_{i+2}^2, \quad 2|x_i x_{i+1}| \leq \delta_i x_i^2 + \frac{1}{\delta_i}x_{i+1}^2,$$

which holds for any ϵ_i, $\delta_i > 0$. In particular, setting

$$\epsilon_i = \frac{2nT}{\mu}\sum_{j=i+1}^{n}\bar{a}^{2j}\alpha_j, \quad \delta_i = \frac{2nT}{\mu}\sum_{j=i+1}^{n}\left[\alpha_j 2i\bar{a}^{(2j-1-i)}\right]$$

we see that choosing $\alpha_n = 1$ and α_{n-1} according to (1.73), while

$$\alpha_i = \bar{a}^{n-i} + \frac{\mu}{4nT}\delta_i^2 + \frac{\epsilon_i}{2}\left[\sum_{j=i+1}^{n} \alpha_j \bar{a}^{2(j-i)}\right],$$

ϕ_2 is non-positive for all t and x –*cf.* (1.74).

B.4 Proof complement for Proposition 2.1

First, we show that the total derivative of V_2 along the trajectories of $\dot{e} = A^{\circ}_{v_r}(t,e)e$ is negative definite. Firstly, since ρ_1 is a polynomial that maps $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and V_1 satisfies (2.26),

$$\frac{d}{dt}\left\{\rho_1(V_1)V_1\right\} \leq -\rho_1(V_1)\big[k_x e_x^2 + k_\theta e_\theta^2\big]. \tag{B.28}$$

Next, we use (2.31), as well as $|e| \geq |e_y|$ and $\Upsilon_\varphi > 0$, to obtain

$$\begin{aligned}\frac{d}{dt}\left\{\big[\Upsilon_{v_r^2} + \Upsilon_{\omega_r^2}\big]V_1\right\} \leq &-\frac{2}{T}\left[\int_t^{t+T}\big[\omega_r(s)^2 + v_r(s)^2\big]ds\right]V_1 \\ &+ [\omega_r^2 + v_r^2]\big[e_x^2 + \frac{1}{k_y}e_\theta^2 + e_y^2\big]\end{aligned} \tag{B.29}$$

Then, using (2.36) and (2.38), we obtain

$$\begin{aligned}-\frac{d}{dt}\left\{\omega_r e_x e_y\right\} = &-\dot{\omega}_r e_x e_y - \omega_r\big[-k_x e_x e_y + \omega_r e_y^2 + k_\theta e_\theta e_y^2 \\ &+ k_y v_r e_y^3 - \omega_r e_x^2 - k_\theta e_\theta e_x^2 - k_y v_r e_y e_x^2 + v_r e_\theta e_x\big].\end{aligned} \tag{B.30}$$

Now, for the cross-terms we use the inequalities $2e_x e_y \leq \epsilon e_x^2 + (1/\epsilon)e_y^2$ and $2e_\theta e_y^2 \leq \epsilon V_1 e_\theta^2 + (1/\epsilon)e_y^2$, which hold for any $\epsilon > 0$, and we regroup some terms to obtain

$$\begin{aligned}-\frac{d}{dt}\left\{\omega_r e_x e_y\right\} \leq &-\omega_r\Big[-k_x e_x e_y + \omega_r e_y^2 + k_\theta e_\theta e_y^2 + k_y v_r e_y^3\Big] \\ &-\omega_r\Big[-\omega_r e_x^2 - k_\theta e_\theta e_x^2 - k_y v_r e_y e_x^2 + v_r e_\theta e_x\Big] - \dot{\omega}_r e_x e_y\end{aligned}$$

$$\begin{aligned}\leq &-\omega_r^2 e_y^2 + \bar{\omega}_r k_x\left(\frac{\epsilon}{2}e_x^2 + \frac{1}{2\epsilon}e_y^2\right) + \bar{\omega}_r k_\theta\left(\epsilon V_1 e_\theta^2 + \frac{1}{2\epsilon}e_y^2\right) \\ &+ \bar{\omega}_r \bar{v}_r k_y e_y^3 + \bar{\omega}_r^2 e_x^2 + \bar{\omega}_r\frac{k_\theta}{2}\left(e_\theta^2 + 2V_1 e_x^2\right) \\ &+ \bar{\omega}_r k_y \bar{v}_r\left(\frac{1}{2\epsilon}e_y^2 + \frac{\epsilon}{2}V_1 e_x^2\right) + \frac{\bar{\omega}_r \bar{v}_r}{2}\left(e_\theta^2 + e_x^2\right) + \bar{\dot{\omega}}_r\left(\frac{\epsilon}{2}e_x^2 + \frac{1}{2\epsilon}e_y^2\right)\end{aligned}$$

$$\begin{aligned}\leq &-\omega_r^2 e_y^2 + \bar{\omega}_r \bar{v}_r k_y e_y^3 + \frac{1}{2\epsilon}\Big[(k_x + k_\theta)\bar{\omega}_r + \bar{\omega}_r k_y \bar{v}_r + \bar{\dot{\omega}}_r\Big] e_y^2 \\ &+ \Big[\bar{\omega}_r k_x \frac{\epsilon}{2} + \bar{\omega}_r^2 + \bar{\omega}_r k_y \bar{v}_r \frac{\epsilon}{2} V_1 + \bar{\dot{\omega}}_r \frac{\epsilon}{2} + \frac{\bar{\omega}_r \bar{v}_r}{2} + \bar{\omega}_r k_\theta V_1\Big] e_x^2 \\ &+ \Big[\bar{\omega}_r k_\theta \epsilon V_1 + \bar{\omega}_r k_\theta + \frac{\bar{\omega}_r \bar{v}_r}{2}\Big] e_\theta^2 \end{aligned} \quad \text{(B.31)}$$

$$\begin{aligned}\leq &-\omega_r^2 e_y^2 + \frac{\epsilon}{2} v_r^2 V_1 e_y^2 + \rho_5(V_1) e_x^2 + \rho_6(V_1) e_\theta^2 \\ &+ \frac{1}{2\epsilon}\Big[\bar{\omega}_r^2 k_y^2 + (k_x + k_\theta)\bar{\omega}_r + \bar{\omega}_r k_y \bar{v}_r + \bar{\dot{\omega}}_r\Big] e_y^2 \end{aligned} \quad \text{(B.32)}$$

where ρ_5 and ρ_6 are first-order polynomials of V_1 defined as

$$\begin{aligned}\rho_5(V_1) &= \frac{\bar{\omega}_r}{2}\Big[\big(\epsilon k_y \bar{v}_r + 2k_\theta\big)V_1 + \big(k_x + \frac{\bar{\dot{\omega}}_r}{\bar{\omega}_r}\big)\epsilon + 2\bar{\omega}_r + \bar{v}_r\Big] \\ \rho_6(V_1) &= \bar{\omega}_r\Big[k_\theta(\epsilon V_1 + 1) + \frac{\bar{v}_r}{2}\Big].\end{aligned}$$

Next, we have

$$\begin{aligned}\frac{d}{dt}\left\{v_r \rho_2(V_1) e_\theta e_y\right\} = &-\rho_2(V_1) v_r^2 e_y^2 - v_r \rho_2(V_1)\big[k_\theta e_\theta e_y + \omega_r e_x e_\theta \\ &+ k_\theta e_\theta^2 e_x + k_y v_r e_y e_x e_\theta + v_r e_\theta^2\big] + \rho_2(V_1)\dot{v}_r e_\theta e_y \\ &- v_r \frac{\partial \rho_2}{\partial V_1} e_\theta e_y \big[k_x e_x^2 + k_\theta e_\theta^2\big].\end{aligned} \quad \text{(B.33)}$$

Hence, using again the triangle inequality to bound the cross-terms and regrouping them, we obtain

$$\begin{aligned}\frac{d}{dt}\left\{v_r \rho_2(V_1) e_\theta e_y\right\} \leq &-k_y v_r^2 \rho_2 e_y^2 - \rho_2 k_\theta v_r e_\theta e_y - \rho_2 v_r \big[\omega_r e_x e_\theta + k_\theta e_\theta^2 e_x + k_y v_r e_y e_x e_\theta\big] \\ &+ \rho_2 v_r^2 e_\theta^2 + \rho_2 \dot{v}_r e_\theta e_y - v_r \frac{\partial \rho_2(V_1)}{\partial V_1} e_\theta e_y \left(k_x e_x^2 + k_\theta e_\theta^2\right)\end{aligned}$$

$$
\begin{aligned}
\leq & -k_y v_r^2 \rho_2 e_y^2 + k_\theta \bar{v}_r \left(\frac{\epsilon}{2} \rho_2^2 e_\theta^2 + \frac{1}{2\epsilon} e_y^2 \right) + \rho_2 \bar{v}_r \frac{\bar{\omega}_r}{2} \left(e_x^2 + e_\theta^2 \right) \\
& + \rho_2 \bar{v}_r \frac{k_\theta}{2} \left(e_\theta^2 + V_1 e_x^2 \right) + \rho_2 k_y \bar{v}_r^2 \left(V_1 e_x^2 + \frac{e_\theta^2}{2} \right) + \bar{v}_r^2 \rho_2 e_\theta^2 + \\
& \dot{\bar{v}}_r \left(\frac{\epsilon}{2} \rho_2^2 e_\theta^2 + \frac{1}{2\epsilon} e_y^2 \right) + \bar{v}_r \left| \frac{\partial \rho_2(V_1)}{\partial V_1} \right| \max\{k_y, 1\} V_1 \left(k_x e_x^2 + k_\theta e_\theta^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
\leq & -k_y v_r^2 \rho_2 e_y^2 + \frac{1}{2\epsilon} \left(k_\theta \bar{v}_r + \dot{\bar{v}}_r \right) e_y^2 + \left[\rho_2 \bar{v}_r \frac{\bar{\omega}_r}{2} + k_\theta V_1 \rho_2 \bar{v}_r \right. \\
& \left. + \rho_2 k_y \bar{v}_r^2 V_1 + \bar{v}_r \left| \frac{\partial \rho_2(V_1)}{\partial V_1} \right| \max\{k_y, 1\} V_1 k_x \right] e_x^2 \\
& + \left[k_\theta \bar{v}_r \frac{\epsilon}{2} \rho_2^2 + \rho_2 \bar{v}_r \frac{\bar{\omega}_r}{2} + \rho_2 \bar{v}_r \frac{k_\theta}{2} + \rho_2 \frac{k_y}{2} \bar{v}_r^2 + \bar{v}_r^2 \rho_2 \right. \\
& \left. + \dot{\bar{v}}_r \frac{\epsilon}{2} \rho_2^2 + \bar{v}_r \left| \frac{\partial \rho_2(V_1)}{\partial V_1} \right| \max\{k_y, 1\} V_1 k_\theta \right] e_\theta^2
\end{aligned}
$$

$$
\leq -k_y v_r^2 \rho_2 e_y^2 + \rho_7(V_1) e_x^2 + \rho_8(V_1) e_\theta^2 + \frac{1}{2\epsilon} \left(k_\theta \bar{v}_r + \dot{\bar{v}}_r \right) e_y^2 \qquad \text{(B.34)}
$$

where ρ_7 and ρ_8 are second-order polynomials of V_1 satisfying

$$
\begin{aligned}
\rho_7(V_1) & \geq \rho_2 \bar{v}_r \left[\frac{\bar{\omega}_r}{2} + (k_\theta + k_y \bar{v}_r) V_1 \right] + \max\{k_y, 1\} k_x \bar{v}_r V_1 \left| \frac{\partial \rho_2}{\partial V_1} \right| \\
\rho_8(V_1) & \geq \frac{\bar{v}_r \rho_2(V_1)}{2} \left[\bar{\omega}_r + k_\theta(\epsilon \rho_2(V_1) + 1) + (k_y + 2) \bar{v}_r \right] \\
& \quad \dot{\bar{v}}_r \frac{\epsilon}{2} \rho_2(V_1)^2 + \bar{v}_r \left| \frac{\partial \rho_2}{\partial V_1} \right| \max\{k_y, 1\} k_\theta V_1.
\end{aligned}
$$

Now we put all the previous bounds together. Using (2.21) in (B.29), we obtain, in view of (B.31) and (B.34),

$$
\begin{aligned}
\frac{\partial V_2}{\partial t} + & \frac{\partial V_2}{\partial e} A_{v_r}^{\circ}(t, e) e \leq -\frac{2\mu}{T} V_1(e) - \left[k_y \rho_2(V_1) - 1 - \frac{\epsilon}{2} V_1 \right] v_r^2 e_y^2 \\
& + \frac{1}{2\epsilon} \left[\bar{\omega}_r \left[\bar{\omega}_r k_y^2 + k_x + k_\theta + k_y \bar{v}_r \right] + \dot{\bar{\omega}}_r + k_\theta \bar{v}_r + \dot{\bar{v}}_r \right] e_y^2 \\
& - e_x^2 \left[k_x \rho_1 - \rho_7 - \rho_5 - v_r^2 - \omega_r^2 \right] - e_\theta^2 \left[k_\theta \rho_1 - \rho_8 - \rho_6 - \frac{1}{k_y} \left(v_r^2 + \omega_r^2 \right) \right]. \qquad \text{(B.35)}
\end{aligned}
$$

Hence, defining

$$
\begin{aligned}
\epsilon &:= \frac{T}{\mu}\Big[\bar{\omega}_r\left[\bar{\omega}_r k_y^2 + k_x + k_\theta + k_y\bar{v}_r\right] + \dot{\bar{\omega}}_r + k_\theta\bar{v}_r + \dot{\bar{v}}_r\Big] \\
\rho_1(V_1) &:= 1 + \frac{1}{\min\{k_x, k_\theta\}}\Big[\rho_5 + \rho_6 + \rho_7 + \rho_8 + \big[1 + \frac{1}{k_y}\big]\left[\omega_r^2 + v_r^2\right]\Big]. \\
\rho_2(V_1) &:= \frac{1}{k_y}\left[1 + \frac{\epsilon}{2}V_1\right]
\end{aligned}
$$

we obtain

$$
\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial e}A^\circ_{v_r}(t,e)e \;\le\; -\frac{\mu}{T}V_1(e) - k_x e_x^2 - k_\theta e_\theta^2. \tag{B.36}
$$

That is, V_2 is a strong Lyapunov function for the nominal dynamics $\dot{e} = A^\circ_{v_r}(t,e)e$.

Next, we evaluate the total derivative of V along the trajectories of (2.35) (*i.e.*, including the output injection term). From (B.36) we obtain

$$
\begin{aligned}
\dot{V}_3(t,e) &\le \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial e}A^\circ_{v_r}(t,e)e + W(t,e) && \text{(B.37)} \\
W(t,e) &:= -k_\theta\rho_4(V_1)e_\theta^2 + v_r[\phi(e_\theta) - 1]\frac{\partial V_2}{\partial e}B^\circ(e_y)e && \text{(B.38)}
\end{aligned}
$$

for which we used (2.26), as well as the positivity of $\rho_4(V_1)$ and of $\partial\rho_4/\partial V_1$, to obtain

$$
\frac{d}{dt}\{V_1\rho_4(V_1)\} \;=\; \dot{V}_1\rho_4(V_1) + V_1\frac{\partial\rho_4}{\partial V_1}\dot{V}_1 \le -k_\theta\rho_4(V_1)e_\theta^2.
$$

We show that $W(t,e)$, defined in (B.38), is non-positive. To that end, note that

$$
[\phi(e_\theta) - 1] \le e_\theta^2 \tag{B.39}
$$

and, in view of the structure of B°, we have

$$
\frac{\partial V_1}{\partial e}B^\circ(e)e = 0
$$

hence,

$$
\frac{\partial V_2}{\partial e} = v_r\rho_2(V_1)[e_y\ 0\ e_\theta] - \omega_r[0\ e_y\ e_x]
$$

and, moreover,

$$
\begin{aligned}
\big|\,[e_y\ 0\ e_\theta]B^\circ(e)e\,\big| &= \big|-k_y e_y^2 + e_\theta^2 - k_y e_y e_x e_\theta\big| \le \big|e_\theta^2 - \frac{k_y}{2}e_y^2 + \frac{k_y}{2}e_x^2 e_\theta^2\big| \\
&\le 2k_yV_1 + 2k_y^2V_1^2 \\
\big|\,[0\ e_y\ e_x]B^\circ(e)e\,\big| &= \big|k_y e_y^3 + e_\theta e_x - k_y e_y e_x^2\big| \le 2k_yV_1^2 + \max\{k_y, 1\}V_1.
\end{aligned}
$$

Thus, $W(t,e) \leq 0$ if

$$\rho_4(V_1) \geq \frac{2\bar{v}_r \max\{k_y, 1\}}{k_\theta} \Big[\big[k_y \rho_2 \bar{v}_r + \bar{\omega}_r\big] V_1^2 + \big[\rho_2 \bar{v}_r + \bar{\omega}_r\big] V_1 \Big]$$

and (2.34) follows from (B.36) and (B.37).

B.5 Proof of Proposition 2.2

Consider the function $W : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined by

$$W(t,e) := \ln \big[1 + V(t,e)\big] \tag{B.40}$$

where $V : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is the continuously differentiable function defined in (2.28).

The total derivative of W along the trajectories of (2.47) yields

$$\dot{W}(t,e) \leq \frac{\dot{V}(t,e)}{1 + V(t,e)}$$

which, in virtue of (2.29) implies that

$$\dot{W}(t,e) \leq -\alpha(|e|) + \frac{\partial V}{\partial e} \frac{B(e)\eta}{1 + V(t,e)} \tag{B.41}$$

where

$$\alpha(|e|) = \frac{\mu}{2T} \frac{V_1(e)}{1 + V(t,e)}.$$

To establish the statement of the proposition we show that the second term on the right hand side of (B.41) is bounded from above by $\gamma|\eta|$ with $\gamma > 0$. For the sake of argument, remark that $V(t,e) = \mathcal{V}(t,e,V_1)$ where

$$\mathcal{V}(t,e,V_1) := P_{[3]}(t,V_1)V_1 - \omega_r(t) e_x e_y + v_r(t) P_{[1]}(t,V_1) e_\theta e_y$$

and, in addition, note that there exists a fourth-order polynomial $\tilde{P}_4(V_1)$ such that

$$\mathcal{V}(t,e,V_1) \geq \tilde{P}_4(V_1), \qquad \forall\, (t,e,V_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \times \mathbb{R}_{\geq 0}. \tag{B.42}$$

Furthermore,

$$\frac{\partial V}{\partial e} = \frac{\partial \mathcal{V}}{\partial V_1} \frac{\partial V_1}{\partial e} + \frac{\partial \mathcal{V}}{\partial e}$$

Therefore

$$\frac{\partial V}{\partial e}\left[B(e)\eta\right] = \frac{\partial \mathcal{V}}{\partial V_1}\frac{\partial V_1}{\partial e}B(e)\eta + \frac{\partial \mathcal{V}}{\partial e}\left[B(e)\eta\right]$$

Now, since $P_{[3]}$ is a polynomial of 3rd order, we have

$$\frac{\partial \mathcal{V}}{\partial V_1} = P'_{[3]}(V_1) + v_r(t)\frac{\partial P_{[1]}}{\partial V_1}e_\theta e_y$$

where $P'_{[3]} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the polynomial function of 3rd order defined as

$$P'_{[3]}(V_1) := \frac{\partial P_{[3]}}{\partial V_1}V_1 + P_{[3]}(V_1).$$

Then, since $P_{[1]}$ is a polynomial of 1st order and $e_\theta e_y \leq V_1(e)$, there exists $c > 0$ such that

$$\left|\frac{\partial \mathcal{V}}{\partial V_1}\right| \leq P'_{[3]}(V_1) + c\bar{v}_r V_1.$$

Furthermore, $B(e)$ is linear in e therefore, there exists $c > 0$ such that

$$\left|\frac{\partial V_1}{\partial e}B(e)\eta\right| \leq cV_1|\eta|$$

and, on the other hand,

$$\frac{\partial \mathcal{V}}{\partial e}^\top = \begin{bmatrix} v_r(t)P_{[1]}(t, V_1)e_y \\ -\omega_r(t)e_y \\ v_r(t)P_{[1]}(t, V_1)e_\theta - \omega_r(t)e_x \end{bmatrix} \tag{B.43}$$

Putting all these bounds together, we conclude that there exists a polynomial of fourth order $P'_4(V_1)$ such that

$$\left|\frac{\partial V}{\partial e}\left[B(e)\eta\right]\right| \leq P'_4(V_1)|\eta|.$$

and, therefore,

$$\dot{W}(t, e) \leq -\alpha(|e|) + c|\eta|$$

where

$$c := \limsup_{V_1 \geq 0} \frac{P'_4(V_1)}{1 + \tilde{P}_4(V_1)}$$

and the claim follows.

B.6 Proof of Lemma 2.1

First, we remark that V_2, hence V, is positive definite and radially unbounded. This follows since $\gamma_1(V_1)V_1 > (\bar{\phi}/2)V_1^2$ and

$$-\dot{\phi}(t)V_1(e)e_xe_y + \frac{\bar{\phi}}{2}V_1(e)^2 = \frac{V_1}{2}\begin{bmatrix} e_x \\ e_y \end{bmatrix}^\top \begin{bmatrix} \bar{\phi} & \dot{\phi} \\ \dot{\phi} & \bar{\phi} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} \geq 0.$$

Next, we proceed now to compute the total derivative of V. By the fundamental theorem of calculus, we have

$$\dot{\Upsilon}_{\dot{\phi}(s)^2}(t) = -\frac{1}{T}\int_t^{t+T} \dot{\phi}(s)^2 ds + \dot{\phi}(t)^2.$$

Now, let μ, $T > 0$ be generated by the assumption that $\dot{\phi}$ is persistently exciting. Then,

$$\dot{\Upsilon}_{\dot{\phi}(s)^2}(t) \leq -\frac{\mu}{T} + \dot{\phi}(t)^2.$$

Therefore, the time-derivative of V_2 along the trajectories of the system

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} -k_x & \dot{\phi}[e_y^2 + e_x^2] \\ -\dot{\phi}[e_y^2 + e_x^2] & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} \tag{B.44}$$

satisfies

$$\begin{aligned} \dot{V}_2 \;\; \leq \;\; & -\frac{\mu}{T}V_1^3 + \dot{\phi}^2 V_1^3 - \dot{\phi}^2 e_y^2 V_1^2 + k_x\dot{\phi}e_xe_yV_1 \\ & -2k_x\gamma_3(V_1)e_x^2 - \ddot{\phi}e_xe_yV_1 + 2\dot{\phi}e_yk_xe_x^3 + \dot{\phi}^2V_1^2e_x^2 \\ \leq \;\; & -\frac{\mu}{2T}V_1^3 - \frac{\mu}{2T}e_y^6 + \dot{\phi}^2\left(e_x^4 + 3e_x^2e_y^2 + 3e_y^4\right)e_x^2 \\ & +\dot{\phi}^2e_y^6 - \dot{\phi}^2e_y^2V_1^2 + k_x\dot{\phi}e_xe_y[e_x^2 + e_y^2] - 2k_x\gamma_3(V_1)e_x^2 \\ & -\ddot{\phi}e_xe_yV_1 + 2\dot{\phi}e_xe_yk_xe_x^2 + \dot{\phi}^2V_1^2e_x^2 \end{aligned}$$

Now, we use $V_1 = [e_x^2 + e_y^2]$ and the inequalities

$$\begin{array}{rcl}
\dot{\phi}^2 \left(e_x^4 + 3e_x^2 e_y^2 + 3e_y^4\right) e_x^2 & \leq & 3\bar{\phi}^2 V_1^2 e_x^2, \\
\dot{\phi}^2 e_y^6 - \dot{\phi}^2 e_y^2 V_1^2 & \leq & 0, \\
3\dot{\phi} e_x e_y k_x e_x^2 & \leq & \dfrac{3}{2} V_1 \bar{\phi} k_x e_x^2, \\
\dot{\phi} e_x e_y k_x e_y^2 & \leq & \dfrac{\bar{\phi}}{2}\big[\dfrac{1}{\epsilon} e_y^6 + \epsilon k_x^2 e_x^2\big], \\
-\ddot{\phi} e_x e_y [e_y^2 + e_x^2] & \leq & \dfrac{\bar{\phi}}{2}\big[\dfrac{1}{\epsilon} e_y^6 + \epsilon e_x^2 + e_x^2 V_1\big]
\end{array}$$

to obtain

$$\begin{array}{rcl}
\dot{V}_2 & \leq & -\dfrac{\mu}{2T} V_1^3 - \left[\dfrac{\mu}{2T} - \dfrac{\bar{\phi}}{\epsilon}\right] e_y^6 \\
& & - \left[2k_x \gamma_3(V_1) - 4\bar{\phi}^2 V_1^2 - \dfrac{3}{2}\bar{\phi} k_x V_1 - \dfrac{\epsilon\bar{\phi}}{2}[k_x^2 + 1] - \dfrac{\bar{\phi}}{2} V_1\right] e_x^2
\end{array}$$

so, setting $\epsilon = \frac{4T\bar{\phi}}{\mu}$ and $\gamma_3(V_1)$ as in (2.74), we obtain

$$\dot{V}_2 \leq -\frac{\mu}{2T} V_1^3 - \frac{\mu}{4T} e_y^6. \tag{B.45}$$

Next, we compute the total derivative of $V(t, \tilde{e})$ (recall that $\tilde{e} := [e_x\ e_y\ e_z]^\top$) in (2.69) along the trajectories of (2.66). Using (B.45), we obtain

$$\begin{array}{rcl}
\dot{V}_3 & \leq & -2\gamma_1(V_1) k_x e_x^2 - \dfrac{\mu}{2T} V_1^3 - \dfrac{\mu}{4T} e_y^6 \\
& & + \dfrac{\partial V_2}{\partial V_1} \dfrac{\partial V_1}{\partial [e_x\ e_y]^\top} \begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} e_z \\
& & - \dfrac{\partial(\dot{\phi} V_1 e_x e_y)}{\partial [e_x\ e_y]^\top} \begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} e_z \\
& & + 2\gamma_2(V_1) e_z [-k_\theta e_z - 2\phi k_x e_x^2].
\end{array}$$

However,

$$\frac{\partial V_1}{\partial [e_x\ e_y]^\top} \begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} = 0$$

hence,

$$\begin{array}{rcl}
\dot{V}_3 & \leq & -\dfrac{\mu}{4T} e_y^6 - k_\theta \dot{\phi} e_z [e_y^4 - e_x^4] - 2k_\theta \gamma_2(V_1) e_z^2 \\
& & -4\phi(t) k_x \gamma_2(V_1) e_z e_x^2 - 2\gamma_1(V_1) k_x e_x^2 - \dfrac{\mu}{2T} V_1^3.
\end{array}$$

Now, for any ϵ_1, $\epsilon_2 > 0$ we have

$$
\begin{aligned}
-k_\theta\dot{\phi}e_z e_y^4 &\leq \frac{1}{2\epsilon_1}\bar{\phi}k_\theta e_z^2 e_y^2 + \frac{\epsilon_1}{2}\bar{\phi}k_\theta e_y^6 \\
k_\theta\dot{\phi}e_z e_x^4 &\leq \frac{1}{2\epsilon_1}\bar{\phi}k_\theta e_z^2 V_1 + \frac{\epsilon_1}{2}\bar{\phi}k_\theta e_x^2 V_1^2 \\
-4\phi(t)k_x\gamma_2(V_1)e_z e_x^2 &\leq \frac{2\bar{\phi}}{\epsilon_2}k_x\gamma_2(V_1)e_z^2 + 2\epsilon_2\bar{\phi}k_x\gamma_2(V_1)V_1 e_x^2,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\dot{V}_3(t,e) \leq\; & -\frac{\mu}{4T}V_1^3 - \Big[\frac{\mu}{4T} - \frac{\epsilon_1}{2}\bar{\phi}k_\theta\Big]e_y^6 \\
& -\Big[2\gamma_1(V_1)k_x - \frac{\epsilon_1}{2}\bar{\phi}k_\theta V_1^2 - 2\epsilon_2 k_x\bar{\phi}\gamma_2(V_1)V_1\Big]e_x^2 \\
& -\Big[2k_\theta\gamma_2(V_1) - \big[\frac{k_\theta}{\epsilon_1}V_1 + \frac{2}{\epsilon_2}k_x\gamma_2(V_1)\big]\bar{\phi}\Big]e_z^2.
\end{aligned}
$$

So, setting

$$
\epsilon_1 := \frac{\mu}{4T\bar{\phi}k_\theta}, \quad \epsilon_2 := \frac{4k_x\bar{\phi}}{k_\theta}
$$

and using (2.72)–(2.74), (2.75) follows.

B.7 Proof of Lemma 3.2

The time-derivative of V_2 along trajectories of (3.73) satisfies the following inequalities

$$
\begin{aligned}
\dot{V}_2(t,e) \leq & -2P_1(V_1)k_x e_x^2 - \frac{\mu}{T}V_1^2 + \dot{\psi}^2 V_1^2 - \ddot{\psi}\sqrt{V_1}e_x e_y - \dot{\psi}\frac{e_x e_y}{\sqrt{V_1}}k_x e_x^2 + \dot{\psi}^2 V_1 e_x^2 \\
& - \dot{\psi} e_y^2 \sqrt{V_1}\left(-k_x e_x + \dot{\psi}\sqrt{V_1}\right) + \dot{\psi}k_\theta e_z \sqrt{V_1}\left(e_x^2 - e_y^2\right) \\
& + 2Q_1(V_1)e_z\left(-k_\theta e_z - \psi k_x \frac{e_x^2}{\sqrt{e_x^2+e_y^2}}\right) \\
\leq & -\left[2k_x P_1(V_1) - 2\dot{\psi}^2 V_1 - \dot{\psi}k_x\frac{\sqrt{V_1}}{2}\right]e_x^2 - \frac{\mu}{T}V_1^2 + k_x\dot{\psi}\sqrt{V_1}e_y^2 e_x + \dot{\psi}k_\theta V_1\sqrt{V_1}e_z - \\
& 2Q_1 k_\theta e_z^2 - 2Q_1(V_1)\psi k_x \frac{e_z e_x^2}{\sqrt{V_1}} \\
\leq & -\left[2k_x P_1(V_1) - 2\dot{\psi}^2 V_1 - \dot{\psi}k_x\frac{\sqrt{V_1}}{2}\right]e_x^2 - \frac{\mu}{T}V_1^2 + k_x\left|\dot{\psi}\right|V_1\left(\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}e_x^2\right) + \\
& \left|\dot{\psi}\right|k_\theta V_1\left(\frac{1}{2\epsilon}V_1 + \frac{\epsilon}{2}e_z^2\right) - 2Q_1 k_\theta e_z^2 + \frac{k_x}{\delta}Q_1(V_1)e_z^2 + \delta f^2 Q_1(V_1)k_x e_x^2 \\
\leq & -\left[2k_x P_1(V_1) - 2\dot{\psi}^2 V_1 - \dot{\psi}k_x\frac{\sqrt{V_1}}{2} - \delta\psi^2 Q_1(V_1)k_x - k_x\left|\dot{\psi}\right|\frac{\epsilon}{2}V_1\right]e_x^2 - \\
& \left[\frac{\mu}{T} - k_x\left|\dot{\psi}\right|\frac{1}{2\epsilon} - \frac{1}{2\epsilon}\left|\dot{\psi}\right|k_\theta\right]V_1^2 - \left[2Q_1(V_1)\left(k_\theta - \frac{k_x}{2\delta}\right) - \frac{\epsilon}{2}\left|\dot{\psi}\right|k_\theta V_1\right]e_z^2
\end{aligned}
\tag{B.46}
$$

We take: $\epsilon = \frac{T}{\mu}\left(k_x + k_\theta\right)\bar{\psi}$, and $\delta = \frac{k_x}{k_\theta}$, and:

$$
Q_1(V_1) \geq \epsilon\bar{\psi}V_1 + 1 \tag{B.47}
$$

and,

$$
P_1(V_1) \geq \frac{1}{k_x}\bar{\psi}^2 V_1 + \bar{\psi}\frac{\sqrt{V_1}}{4} + \frac{\delta}{4}\bar{\psi}^2 Q_1(V_1) + \bar{\psi}\frac{\epsilon}{4}V_1 \tag{B.48}
$$

We get finally,

$$
\dot{V}_2(t,X) \leq -\frac{\mu}{2T}V_1^2 - \frac{1}{2}Q_1(V_1)k_\theta e_z^2 \tag{B.49}
$$

B.8 New Filtration Lemma

The following lemmas extend a well-known filtration property of persistently exciting functions [41].

Lemma B.1. *Consider the two scalar second order systems:*

$$\ddot{f} + k_1\dot{f} + k_2 f = p(t) \tag{B.50}$$

and

$$q^{(3)} + k_1\ddot{q} + k_2\dot{q} = \dot{p}(t) \tag{B.51}$$

where $k_1, k_2 > 0$ *and* $p(t)$ *is a time-varying input such that* $\dot{p}(t)$ *is PE with excitation parameters*$-(T,\mu)$ *and there exists* $b_p > 0$ *such that* $\max\left\{p, \dot{p}, \ddot{p}, p^{(3)}\right\} \le b_p$. *Then* $\dot{f}(t)$ *and* $\dot{q}(t)$ *are both PE with excitation parameters*$-(T_f, \mu_f)$ *and*$-(T_q, \mu_q)$ *respectively, given by* $T_f = k_f T$,

$$\mu_f = \frac{\left(2(1+k_2^{-1})b_p r_f\right)^2}{b_p^2\left(1+k_1k_2^{-1}+k_2^{-1}\right)^2 T_f}, \tag{B.52}$$

and $k_f = \left[\frac{4\left(1+k_2^{-1}\right)b_p r_f}{\mu k_2^{-1}}\right] + 1$, $T_q = k_q T$,

$$\mu_q = \frac{(2b_p r_q(2+k_2))^2}{b_p^2(1+k_1+k_2)^2 T_q}, \tag{B.53}$$

and $k_q = \left[\frac{4}{\mu}b_p r_q(2+k_2)\right] + 1$.
Where

$$r_f^2 = \frac{(a+1)\dot{f}(0)^2 + 4(ak_2+1)\left(f(0)^2 + k_2^{-1}{b_p}^2\right) + \frac{b{b_p}^2}{c}}{\min\{1, k_2\}}, \tag{B.54}$$

$$r_q^2 = $$
$$2\frac{(a+1)\ddot{q}(0)^2 + 4(ak_2+1)\left(\dot{q}(0)^2 + k_2^{-2}b_p^2\right) + \frac{bb_p^2}{c} + k_2^{-2}b_p}{\min\{1, k_2\}},$$

$a = 2k_1^{-1} + k_1k_2^{-1} + k_2^{-1} + 1$, $b := 4k_2^{-1} + \frac{1}{ak_1k_2^2}$ *and* $c := \frac{1}{4}\frac{\min\{ak_1, k_2\}}{a+2+ak_2}$.
Furthermore,

$$\max\left\{f, \dot{f}, \ddot{f}, f^{(3)}\right\} \le b_f,$$

$$\max\left\{q, \dot{q}, \ddot{q}, q^{(3)}\right\} \le b_q,$$

with:

$$b_f = \left[2 + k_1 + k_2 + k_1^2 + k_1k_2 + k_2\right] r_f + [k_1 + k_2^{-} + 2]b_p. \tag{B.55}$$

$$b_q = \left(2 + k_1 + k_2 + \frac{2}{k_2}(k_1+1)\right) r_q + \left(1 + \frac{2}{k_2}\right) b_p + |q(0)|. \tag{B.56}$$

Proof. 1)- Consider the following linear change of coordinates for the first differential equation (B.50): $x = f - k_2^{-1}p(t)$, $y = \dot{f}$. Then $\dot{x} = y - k_2^{-1}\dot{p}(t)$ and $\dot{y} = -k_2 y - k_1 x$. First, note that the overall trajectories are bounded, i.e., there exists $r_f > 0$ that is a function of $(x(0), y(0), b_p)$, such that $|(x, y)| \leq r_f$, $\forall t \geq 0$. Consider now the following time-derivative

$$\begin{aligned} \frac{d}{dt}\left[-\dot{p}x - k_2^{-1}\ddot{p}y\right] =& [-\dot{p} + k_1 k_2^{-1}\ddot{p} - k_2^{-1}p^{(3)}]y + k_2^{-1}\dot{p}^2 \\ \geq& - b_p\left[1 + k_1 k_2^{-1} + k_2^{-1}\right]|y| + k_2^{-1}\dot{p}^2, \end{aligned}$$

then

$$\begin{aligned} & b_p\left[1 + k_1 k_2^{-1} + k_2^{-1}\right]\int_t^{t+k_f T}|y(s)|ds \geq \\ & \int_t^{t+k_f T}\frac{d}{ds}\left[\dot{p}(s)x(s) + k_2^{-1}\ddot{p}(s)y(s)\right]ds + \\ & k_2^{-1}\int_t^{t+k_f T}\dot{p}(s)^2 ds \geq -2\left(1 + k_2^{-1}\right)b_p r_f + k_2^{-1}k_f\mu \end{aligned}$$

where k_f is a positive integer and, to obtain the last inequality, we used the fact that trajectories are bounded and that $\dot{p}$ is PE with parameters (T, μ). Invoking the Cauchy-Schwartz inequality on $\int_t^{t+k_f T}|y(s)|ds$, we obtain

$$\begin{aligned} & b_p^2\left(1 + k_1 k_2^{-1} + k_2^{-1}\right)^2 k_f T\int_t^{t+k_f T}y(s)^2 ds \geq \\ & \left(k_2^{-1}k_f\mu - 2\left(1 + k_2^{-1}\right)b_p\right)^2 \end{aligned}$$

Finally, we get

$$\int_t^{t+k_f T}y(s)^2 ds \geq \frac{\left(k_2^{-1}k_f\mu - 2\left(1 + k_2^{-1}\right)b_p r_f\right)^2}{b_p^2\left(1 + K_1 k_2^{-1} + k_2^{-1}\right)^2 k_f T} \tag{B.57}$$

Taking $k_f = \left[\frac{4\left(1+k_2^{-1}\right)b_p r_f}{\mu k_2^{-1}}\right] + 1$, we find $T_f = k_f T$ and $\mu_f = \frac{\left(2(1+k_2^{-1})b_p r_f\right)^2}{b_p^2\left(1+k_1 k_2^{-1}+k_2^{-1}\right)^2 T_f}$, such that $\int\limits_t^{t+T_f} y(s)^2 ds \geq \mu_f$. 2)- Consider the second equation case using the notation $(x, y) = (\dot{q}, \ddot{q})$. First, note that the overall trajectories are bounded, i.e., there exists $r_q > 0$ that is a function of $(x(0), y(0), b_p)$, such that $\|(x, y)\| \leq r_q$, $\forall t \geq 0$. Consider now the following time-derivative

$$\begin{aligned} \frac{d}{dt}\left[\dot{p}y - (\ddot{p} - k_2\dot{p})x\right] =& -\left[\dot{p}k_1 + p^{(3)} - k_2\ddot{p}\right]x + \dot{p}^2 \\ \geq& - b_p\left[k_1 + 1 + k_2\right]|x| + \dot{p}^2, \end{aligned}$$

then

$$b_p\left[k_1+1+k_2\right]\int_t^{t+k_qT}|x(s)|ds \geq$$
$$\int_t^{t+k_qT}\frac{d}{ds}[\dot{p}y-(\ddot{p}-k_2\dot{p})x]ds+$$
$$\int_t^{t+k_qT}\dot{p}(s)^2ds \geq -2\left(2+k_2\right)b_pr+k_q\mu$$

where k_q is a positive integer and, to obtain the last inequality, we used the fact that trajectories are bounded and that $\dot{p}$ is PE with parameters (T,μ). Invoking the Cauchy-Schwartz inequality on $\int_t^{t+k_qT}|x(s)|ds$, we obtain

$$b_p^2\left(k_1+k_2+1\right)^2k_qT\int_t^{t+k_qT}x(s)^2ds \geq$$
$$\left(k_q\mu-2\left(2+k_2\right)b_pr_q\right)^2$$

Finally, we get

$$\int_t^{t+kT}x(s)^2ds \geq \frac{\left(2b_pr_q(2+k_2)\right)^2}{b_p^2(1+k_1+k_2)^2T_q} \tag{B.58}$$

Taking $k_q=\left[\frac{4}{\mu}b_pr_q(2+k_2)\right]+1$, we find $T_q=k_qT$ and $\mu_q=\frac{(2b_pr_q(2+k_2))^2}{b_p^2(1+k_1+k_2)^2T_q}$, such that $\int\limits_t^{t+T_q}x(s)^2ds \geq \mu_q$. In order to have an explicit estimation of (T_f,μ_f) and (T_q,μ_q) it only remains to estimate the upper bound of the trajectories r_f and r_q. For, let consider the first deferential equation (B.50) and let us define the Lyapunov function candidate $V(x,y)=a\left(y^2+k_2x^2\right)+xy$ with $x=f-k_2^-p(t)$, $y=\dot{f}$ and $a=2k_1^{-1}+k_1k_2^{-1}+k_2^{-1}+1$. $V(x,y)$ verifies the following bounds

$$\begin{aligned}\min\left\{1,k_2\right\}(y^2+x^2) \leq V(x,y) \leq \\ \max\left\{a+1,ak_2+1\right\}\left(x^2+y^2\right).\end{aligned} \tag{B.59}$$

$\dot{V}$, along the trajectories of the system, satisfies

$$\begin{aligned}\dot{V}(\cdot) \leq & -ak_1y^2+y^2-k_1yx-k_2x^2+2\dot{p}x+k_2^{-1}y\dot{p} \\ \leq & -\frac{a}{4}k_1y^2-\frac{1}{4}k_2x^2+\left[4k_2^{-1}+\frac{1}{ak_1k_2^2}\right]\dot{p}^2 \\ \leq & -cV+bb_p^2\end{aligned}$$

where $c:=\frac{1}{4}\frac{\min\{ak_1,k_2\}}{a+2+ak_2}$ and $b:=4k_2^{-1}+\frac{1}{ak_1k_2^2}$.

Since $x^2+y^2 \leq \frac{1}{\min\{1,k_2\}}V$, we can calculate the upper bound of the trajectories as

$$
\begin{aligned}
\|(x,y)\|^2 \leq & \frac{1}{\min\{1,k_2\}} \max\left\{V(0), \frac{bb_p^2}{c}\right\} \\
\leq & \frac{(a+1)\dot{f}(0)^2 + 4(ak_2+1)\left(f(0)^2 + k_2^{-2}b_p^2\right) + \frac{bb_p^2}{c}}{\min\{1,k_2\}} = r_f^2.
\end{aligned} \tag{B.60}
$$

To deduce the bound r_q using r_f we observe that the differential equations (B.50) is equivalent to (B.51) if we replace f by $\dot{q}$ and $\dot{f}$ by $\ddot{q}$, also under the assumption $\max\left\{p, \dot{p}, \ddot{p}, p^{(3)}\right\} \leq b_p$ we obtain,

$$
\begin{aligned}
&(\dot{q} - k_2 p)^2 + \ddot{q}^2 \leq \\
&\frac{(a+1)\ddot{q}(0)^2 + 4(ak_2+1)\left(\dot{q}(0)^2 + k_2^{-2}b_p^2\right) + \frac{bb_p^2}{c}}{\min\{1,k_2\}}.
\end{aligned} \tag{B.61}
$$

which implies,

$$
\begin{aligned}
&\dot{q}^2 + \ddot{q}^2 \leq \\
&2\frac{(a+1)\ddot{q}(0)^2 + 4(ak_2+1)\left(\dot{q}(0)^2 + k_2^{-2}b_p^2\right) + \frac{bb_p^2}{c} + k_2^{-2}b_p}{\min\{1,k_2\}} \\
&= r_q.
\end{aligned} \tag{B.62}
$$

Finally, from the system dynamics, (B.60) and (B.62), we can find that $f \leq r_f + k_2^{-}b_p$, $\dot{f} \leq r_f$, $\ddot{f} \leq (k_1+k_2)r_f + b_p$ and $f^{(3)} \leq (k_1^2 + k_1k_2 + k_2)r_f + (k_1+1)b_p$ so (B.55) follows. Also, $\dot{q} \leq r_q$, $\ddot{q} \leq r_q$, $q^{(3)} \leq k_1 r_q + k_2 r_q + b_p$ and $q \leq \frac{2}{k_2}b_p + \frac{2}{k_2}(k_1+1)r_q + |q(0)|$ so (B.56) follows. This concludes the proof. □□□

B.9 Proof of theorem 4.1

First, we remark that $W(\cdot)$ and $Z(\cdot)$ are positive definite radially unbounded and satisfy

$$
\begin{aligned}
W(\cdot) \geq & \gamma(V)V \\
W(\cdot) \leq & \gamma(V)V + V\kappa^T(e,s)\bar{\Upsilon}_{\dot{f}^2}(t)\kappa(e,s) + c_1 b_{\bar{f}} 2\lambda_N(L)V^2 + \\
& 2\left[\lambda_N(L) + |K_{pt}|\right]\alpha(V)V \\
Z(\cdot) \geq & \min\{1, \lambda_m(K_{p\theta})\}\left[e_\theta^T e_\theta + e_\omega^T e_\omega\right] \\
Z(\cdot) \leq & \max\{1 + c_2, c_2\lambda_M(K_{p\theta}) + 1\}\left[e_\theta^T e_\theta + e_\omega^T e_\omega\right].
\end{aligned} \tag{B.63}
$$

Next, we study the time derivative of each component in the Lyapunov function $\Gamma(\cdot)$ along the trajectories of the closed-loop system (4.25), that is,

$$\frac{d}{dt}\left(\gamma(V)V\right) \leq -\,\gamma(V)v^T K_{dt} K_{pt}^{-} v, \tag{B.64}$$

$$\frac{d}{dt}\left(\rho_2(V)V\right) \leq -\,\rho_2(V)v^T K_{dt} K_{pt}^{-} v, \tag{B.65}$$

$$\begin{aligned}\frac{d}{dt}\left(V\kappa^T\bar{\Upsilon}_{\dot{f}^2}(t)\kappa\right) &\leq -V\kappa^T\frac{1}{T}\int_t^{t+T}\dot{f}^2(s)ds\kappa + \\ &V\kappa^T\dot{f}^2(t)\kappa + 2V\kappa^T\bar{\Upsilon}_{\dot{f}^2}\left[\bar{e}\Phi^T L_2\Phi v + \bar{s}\Phi^{\perp T}L_2\Phi v\right].\end{aligned} \tag{B.66}$$

Since $\dot{f}(t)$ is persistently exciting, with parameters$-(\mu, T)$, then the following inequalities holds

$$\begin{aligned}\frac{d}{dt}\left(V\kappa^T\bar{\Upsilon}_{\dot{f}^2}(t)\kappa\right) &\leq -\frac{\mu}{T}V\kappa^T\kappa + V\kappa^T\dot{f}^2(t)\kappa + \\ &2V\kappa^T\bar{\Upsilon}_{\dot{f}^2}\left[\bar{e}\Phi^T L_2\Phi v + \bar{s}\Phi^{\perp T}L_2\Phi v\right] \\ &\leq -\frac{\mu}{T}V\kappa^T\kappa + V\kappa^T\dot{f}^2(t)\kappa + 2Ve^T\bar{\kappa}\bar{\Upsilon}_{\dot{f}^2}\Phi^T L_2\Phi v + \\ &2Vs^T\bar{\kappa}\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp T}L_2\Phi v\end{aligned} \tag{B.67}$$

where we used the facts that

$$\kappa^T\bar{\Upsilon}_{\dot{f}^2}\bar{e}\Phi^T L_2\Phi v = e^T\bar{\kappa}\bar{\Upsilon}_{\dot{f}^2}\Phi^T L_2\Phi v$$

and

$$\kappa^T\bar{\Upsilon}_{\dot{f}^2}\bar{s}\Phi^{\perp T}L_2\Phi v = s^T\bar{\kappa}\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp T}L_2\Phi v.$$

Moreover, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(V\kappa^T\bar{\Upsilon}_{\dot{f}^2}(t)\kappa\right) \leq -\frac{\mu}{T}V\kappa^T\kappa + V\kappa^T\dot{f}^2(t)\kappa + \\
&\quad V\left[\frac{1}{\epsilon}e^T\bar{\kappa}e + \epsilon v^T\Phi^T L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\bar{\kappa}\Phi^T L_2\Phi v\right] + \\
&\quad V\left[\frac{1}{\epsilon}s^T\bar{\kappa}s + \epsilon v^T\Phi^{\perp T} L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\bar{\kappa}\Phi^{\perp T} L_2\Phi v\right] \\
&\leq -\frac{\mu}{T}V\kappa^T\kappa + V\kappa^T\dot{f}^2(t)\kappa + V\frac{2}{\epsilon}\kappa^T\kappa + \\
&\quad V\epsilon v^T\Phi^T L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\kappa\Phi^T L_2\Phi v + \\
&\quad V\epsilon v^T\Phi^{\perp T} L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\kappa\Phi^{\perp T} L_2\Phi v \\
&\leq -\left[\frac{\mu}{T} - \frac{2}{\epsilon}\right]V\kappa^T\kappa + V\kappa^T\dot{f}^2(t)\kappa + \\
&\quad V^2\epsilon\lambda_N(L)\left(\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^T L_2\Phi\right|^2_\infty + \left|\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp T} L_2\Phi\right|^2_\infty\right) \times \\
&\quad \left|K^-_{dt}K_{pt}\right| v^T K_{dt}K^-_{pt}v
\end{aligned} \tag{B.68}
$$

where the following inequality

$$
\begin{aligned}
v^T\Phi^T L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\kappa\Phi^T L_2\Phi v \leq \lambda_N(L)V\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^T L_2\Phi\right|^2_\infty \left|K^-_{dt}K_{pt}\right| \times \\
v^T K_{dt}K^-_{pt}v,
\end{aligned} \tag{B.69}
$$

$$
\begin{aligned}
&v^T\Phi^{\perp T} L_2\Phi\bar{\Upsilon}^2_{\dot{f}^2}\kappa\Phi^{\perp T} L_2\Phi v \leq \lambda_N(L)V\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp T} L_2\Phi\right|^2_\infty \times \\
&\left|K^-_{dt}K_{pt}\right| v^T K_{dt}K^-_{pt}v,
\end{aligned} \tag{B.70}
$$

and

$$
s^T\bar{\kappa}s \leq 2\kappa^T\kappa
$$

are used in order to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(V\kappa^T\bar{\Upsilon}_{\dot{f}^2}(t)\kappa\right) \leq -\left[\frac{\mu}{T} - \frac{2}{\epsilon}\right]V\kappa^T\kappa + Vs^T\dot{f}^2(t)\bar{\kappa}s + \\
&b^2_{\bar{f}}\lambda_N(L)V^2\left|K^-_{pt}\right| e^T K_{pt}e + \\
&V^2\epsilon\lambda_N(L)\left(\left|\bar{\Upsilon}_{\dot{f}^2}\Phi^T L_2\Phi\right|^2_\infty + \left|\bar{\Upsilon}_{\dot{f}^2}\Phi^{\perp T} L_2\Phi\right|^2_\infty\right) \times \\
&\left|K^-_{dt}K_{pt}\right| v^T K_{dt}K^-_{pt}v
\end{aligned} \tag{B.71}
$$

where the following inequality is used

$$
V\kappa^T\dot{f}^2(t)\kappa \leq Vs^T\dot{f}^2(t)\bar{\kappa}s + b^2_{\bar{f}}\lambda_N(L)V^2\left|K^-_{pt}\right| e^T K_{pt}e.
$$

$$
\begin{aligned}
&\frac{d}{dt}\left(\alpha(V)e^{T}v\right) \leq -\left(\frac{\partial \alpha}{\partial V}\right) e^{T} v v^{T} K_{dt} K_{pt}^{-} v + \\
&\quad \alpha(V)\left[v^{T}\Phi^{T}L_{2}\Phi v + s^{T}\dot{f}\bar{\kappa}v - e^{T}K_{dt}v - e^{T}K_{pt}e - v^{T}\bar{e}_{\omega}s\right] \\
&\leq -\alpha(V)e^{T}K_{pt}e - \left(\frac{\partial \alpha}{\partial V}e^{T}v\right) v^{T}K_{dt}K_{pt}^{-}v + \\
&\quad \alpha(V)v^{T}\left(\Phi^{T}L_{2}\Phi K_{dt}^{-}K_{pt}\right)K_{dt}K_{pt}^{-}v + \\
&\quad \alpha(V)s^{T}\dot{f}\bar{\kappa}v - \alpha(V)e^{T}K_{pt}K_{dt}K_{pt}^{-}v - \alpha(V)v^{T}\bar{e}_{\omega}s \\
&\leq -\alpha(V)e^{T}K_{pt}e - \left(\frac{\partial \alpha}{\partial V}e^{T}v\right) v^{T}K_{dt}K_{pt}^{-}v + \\
&\quad \alpha(V)\left|\Phi^{T}L_{2}\Phi\right|_{\infty}\left|K_{dt}^{-}K_{pt}\right| v^{T}K_{dt}K_{pt}^{-}v + \\
&\quad \frac{1}{\epsilon}s^{T}\dot{f}^{2}\bar{\kappa}^{2}s + \frac{\epsilon}{4}\left|K_{pt}K_{dt}^{-}\right|\alpha^{2}(V)v^{T}K_{dt}K_{pt}^{-}v + \\
&\quad \frac{1}{\epsilon}\alpha(V)e^{T}K_{pt}e + \frac{\epsilon}{4}\alpha(V)\left|K_{dt}\right| v^{T}K_{dt}K_{pt}^{-}v - \alpha(V)v^{T}\bar{e}_{\omega}s
\end{aligned}
\tag{B.72}
$$

where the following inequalities are used

$$
v^{T}\Phi^{T}L_{2}\Phi v \leq \left|\Phi^{T}L_{2}\Phi\right|_{\infty}\left|K_{dt}^{-}K_{pt}\right| v^{T}K_{dt}K_{pt}^{-}v, \tag{B.73}
$$

$$
\alpha(V)s^{T}\dot{f}\bar{\kappa}v \leq \frac{1}{\epsilon}s^{T}\dot{f}^{2}\bar{\kappa}^{2}s + \frac{\epsilon}{4}\left|K_{pt}K_{dt}^{-}\right|\alpha^{2}(V)v^{T}K_{dt}K_{pt}^{-}v, \tag{B.74}
$$

and

$$
\alpha e^{T}K_{dt}v \leq \frac{1}{\epsilon}\alpha(V)e^{T}K_{pt}e + \frac{\epsilon}{4}\left|K_{dt}\right|\alpha(V)v^{T}K_{dt}K_{pt}^{-}v. \tag{B.75}
$$

Also, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\alpha(V)e^{T}v\right) \leq -\alpha(V)e^{T}K_{pt}e\left(1 - \frac{1}{\epsilon}\right) + \\
&\quad \frac{\lambda_{N}(L)}{\epsilon}V s^{T}\dot{f}^{2}\bar{\kappa}s - \alpha(V)v^{T}\bar{e}_{\omega}s + \\
&\quad \left[-\left(\frac{\partial \alpha}{\partial V}e^{T}v\right) + \alpha(V)\left|\Phi^{T}L_{2}\Phi\right|_{\infty}\left|K_{dt}^{-}K_{pt}\right| + \right. \\
&\quad \left.\frac{\epsilon}{4}\left|K_{pt}K_{dt}^{-}\right|\alpha^{2}(V) + \frac{\epsilon}{4}\alpha(V)\left|K_{dt}\right|\right] v^{T}K_{dt}K_{pt}^{-}v.
\end{aligned}
\tag{B.76}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(-c_1 V e^T \dot{f} s\right) \leq & -e^T \ddot{f} s V c_1 + c_1[e^T \dot{f} s] v^T K_{dt} K_{pt}^{-} v - \\
& c_1 V \left[v^T \Phi^T L_2 \Phi \dot{f} s + s^T \bar{\kappa} \dot{f}^2 s - e^T \dot{f}^2 \bar{\kappa} e + e^T \dot{f} \Phi^{\perp T} L_2 \Phi v - \right. \\
& \left. s^T \bar{e}_\omega \dot{f} s + e^T \dot{f} \bar{e}_\omega e \right] \\
\leq & - c_1 V s^T \dot{f}^2 \bar{\kappa} s + \frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 e^T e + \frac{1}{\epsilon} V^2 s^T s + \\
& \left(c_1 e^T \dot{f} s\right) v^T K_{dt} K_{pt}^{-} v + \\
& \frac{\epsilon}{4} c_1^2 \left|K_{dt}^{-} K_{pt}\right| \left|\dot{f}^2 \left(\Phi^T L_2 \Phi\right)^2\right|_\infty v^T K_{dt} K_{pt}^{-} v \\
& + \frac{1}{\epsilon} V^2 s^T s + b_{\bar{f}}^2 c_1 \lambda_N(L) V^2 e^T e + \\
& \frac{1}{\epsilon} c_1 \left|\dot{f}^2 \left(\Phi^{\perp T} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-}\right| V e^T K_{pt} e + \frac{\epsilon}{4} c_1 V v^T K_{dt} K_{pt}^{-} v + \\
& c_1 V s^T \dot{f} \bar{e}_\omega s - c_1 V e^T \dot{f} \bar{e}_\omega e
\end{aligned} \tag{B.77}
$$

where the following inequalities are used

$$
c_1 V e^T \ddot{f} s \leq \frac{\epsilon}{4} c_1^2 b_{\bar{f}}^2 e^T e + \frac{1}{\epsilon} V^2 s^T s, \tag{B.78}
$$

$$
\begin{aligned}
c_1 V v^T \Phi^T L_2 \Phi \dot{f} s \leq & \frac{\epsilon}{4} c_1^2 \left|\dot{f}^2 \left(\Phi^T L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-} K_{pt}\right| v^T K_{dt} K_{pt}^{-} v \\
& + \frac{1}{\epsilon} V^2 s^T s,
\end{aligned} \tag{B.79}
$$

and

$$
\begin{aligned}
c_1 V e^T \dot{f} \Phi^{\perp T} L \Phi v \leq & \frac{1}{\epsilon} c_1 V \left|\dot{f}^2 \left(\Phi^{\perp T} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-}\right| e^T K_{pt} e \\
& + \frac{\epsilon}{4} c_1 V v^T K_{dt} K_{pt}^{-} v.
\end{aligned} \tag{B.80}
$$

Then, we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(-c_1 V e^T \dot{f} s\right) \leq & -c_1 V s^T \dot{f}^2 \bar{\kappa} s + \left[c_1 e^T \dot{f} s + \right. \\
& \left. \frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 \left|K_{dt}^{-} K_{pt}\right| \left|\left(\Phi^T L_2 \Phi\right)^2\right|_\infty + \frac{\epsilon}{4} c_1 V \right] v^T K_{dt} K_{pt}^{-} v + \\
& \left[b_{\bar{f}}^2 c_1 \lambda_N(L) \left|K_{pt}^{-}\right| V^2 + \frac{1}{\epsilon} c_1 \left|\dot{f}^2 \left(\Phi^{\perp T} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-}\right| V \right. \\
& \left. + \frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 \left|K_{pt}^{-}\right| \right] e^T K_{pt} e + \\
& \frac{2}{\epsilon} V^2 s^T s + c_1 V s^T \dot{f} \bar{e}_\omega s - c_1 V e^T \dot{f} \bar{e}_\omega e.
\end{aligned} \tag{B.81}
$$

Next, we use the following inequality

$$V^2 s^T s \leq V^2 \lambda_N(L) \left|K_{dt}^{-}\right| v^T K_{pt}^{-} K_{dt} v + \frac{4n}{\lambda_2(L)} V \kappa^T \kappa \tag{B.82}$$

to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(-c_1 V e^T \dot{f} s\right) \leq -c_1 V s^T \dot{f}^2 \bar{\kappa} s + \Big[c_1 e^T \dot{f} s + \\
&\frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 \left|K_{dt}^{-} K_{pt}\right| \left|\left(\Phi^T L_2 \Phi\right)^2\right|_\infty + \frac{\epsilon}{4} c_1 V\Big] v^T K_{dt} K_{pt}^{-} v + \\
&\Big[b_{\bar{f}}^2 c_1 \lambda_N(L) \left|K_{pt}^{-}\right| V^2 + \frac{1}{\epsilon} c_1 \left|\dot{f}^2 \left(\Phi^{\perp T} L_2 \Phi\right)^2\right|_\infty \left|K_{dt}^{-}\right| V \\
&+ \frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 \left|K_{pt}^{-}\right|\Big] e^T K_{pt} e + \\
&\frac{2}{\epsilon} V^2 \lambda_N(L) \left|K_{dt}^{-}\right| v^T K_{pt}^{-} K_{dt} v + \frac{8n}{\lambda_2(L)\epsilon} V \kappa^T \kappa + \\
&c_1 V s^T \dot{f} \bar{e}_\omega s - c_1 V e^T \dot{f} \bar{e}_\omega e.
\end{aligned} \tag{B.83}$$

Using the previous inequalities, we are able to study the time derivative of $W(\cdot)$ along the trajectories of the closed-loop system (4.25), that is

$$\begin{aligned}
\dot{W}(\cdot) \leq &- \Big[\gamma(V) - V^2 \epsilon \lambda_N(L) \Big(\left|\bar{\Upsilon}_{\dot{f}^2} \Phi^T L_2 \Phi\right|_\infty^2 + \\
&\left|\bar{\Upsilon}_{\dot{f}^2} \Phi^{\perp T} L_2 \Phi\right|_\infty^2\Big) \left|K_{dt}^{-} K_{pt}\right| + \left(\frac{\partial \alpha}{\partial V} e^T v\right) - \\
&\alpha(V) \left|\Phi^T L_2 \Phi\right|_\infty \left|K_{dt}^{-} K_{pt}\right| - \frac{\epsilon}{4} \left|K_{pt}\right| \alpha^2(V) - \\
&\frac{\epsilon}{4} \alpha(V) \left|K_{dt}\right| - \left(c_1 e^T \dot{f} s\right) - \\
&\frac{\epsilon}{4} c_1^2 \left|K_{dt}^{-} K_{pt}\right| \left|\left(\Phi^T L_2 \Phi\right)^2 \dot{f}^2\right|_\infty - \frac{\epsilon}{4} c_1 V - \\
&\frac{2}{\epsilon} V^2 \lambda_N(L) \left|K_{dt}^{-}\right|\Big] v^T K_{dt} K_{pt}^{-} v \\
&- \Big[\alpha(V) \left(1 - \frac{1}{\epsilon}\right) - b_{\bar{f}}^2 \lambda_N(L) V^2 \left|K_{pt}^{-}\right| - \\
&b_{\bar{f}}^2 c_1 \lambda_N(L) \left|K_{pt}^{-}\right| V^2 - \frac{1}{\epsilon} c_1 \left|\dot{f}^2 \left(\Phi^{\perp T} L_2 \Phi\right)^2\right|_\infty \times \\
&\left|K_{dt}^{-}\right| V - \frac{\epsilon}{4} b_{\bar{f}}^2 c_1^2 \left|K_{pt}^{-}\right|\Big] e^T K_{pt} e \\
&- \left[c_1 - 1 - \frac{\lambda_N(L)}{\epsilon}\right] V s^T \dot{f}^2 \bar{\kappa} s - \left[\frac{\mu}{T} - \frac{2}{\epsilon} - \frac{8n}{\lambda_2(L)\epsilon}\right] V \kappa^T \kappa \\
&- \alpha(V) v^T \bar{e}_\omega s + c_1 V s^T \dot{f} \bar{e}_\omega s - c_1 V e^T \dot{f} \bar{e}_\omega e.
\end{aligned} \tag{B.84}$$

Taking $\epsilon = \max\left\{2, \frac{2T}{\mu}\left(2 + \frac{8n}{\lambda_2(L)}\right)\right\}$, $\gamma(V)$, $\alpha(V)$, and c_1 as in (4.30)–(4.34) respectively, we obtain

$$\begin{aligned}\dot{W}(\cdot) \leq &- \frac{1}{2}\gamma(V)v^T K_{dt} K_{pt}^{-} v - \frac{1}{4}\alpha(V)e^T K_{pt} e - \frac{\mu}{2T} V\kappa^T\kappa \\ &- \alpha(V)v^T \bar{e}_\omega s + c_1 V s^T \dot{f}\bar{e}_\omega s - \dot{f}c_1 V e^T \dot{f}\bar{e}_\omega e. \end{aligned} \tag{B.85}$$

Considering, now, the time derivative of the remaining part in the Lyapunov function $\Gamma(\cdot)$, that is,

$$\begin{aligned}&\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq \rho_1(V)\left[-c_2 e_\omega^T e_\omega + e_\omega^T e_\omega - e_\theta^T K_{p\theta} e_\theta - e_\theta^T e_\omega\right] + \\ &\quad c_2 2 e_\omega^T \dot{f}\rho_1(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad c_2 2 e_\theta^T K_{p\theta} f\rho_1(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad e_\omega^T f\rho_1(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad e_\theta^T \dot{f}\rho_1(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right]. \end{aligned} \tag{B.86}$$

Using the fact that $Z(X_r)$ is a strict Lyapunov function for the system

$$\dot{X}_r = \begin{bmatrix} 0 & I_n \\ -K_{p\theta} & -K_{d\theta}\end{bmatrix} X_r,$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq \rho_1(V)\left[-e_\omega^T\left[c_2 - I_n - \frac{\delta}{4}\right]e_\omega - e_\theta^T\left[K_{p\theta} - \frac{1}{\delta}\right]e_\theta\right] + \\ &\quad c_2 2 e_\omega^T \dot{f}\rho_1(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad c_2 2 e_\theta^T K_{p\theta} f\rho(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad e_\omega^T f\rho(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right] + \\ &\quad e_\theta^T \dot{f}\rho(V)\left[\bar{e}\Phi^T(\theta)L_2\Phi^T(\theta)v + \bar{s}\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right]. \end{aligned} \tag{B.87}$$

Taking $\delta = 2\frac{\lambda_M(K_{p\theta})}{\lambda_m(K_{p\theta})}$ and the parameter c_2 as in (4.34), we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\rho(V)Z(X_r)\right) \le \rho(V)&\left[-\frac{c_2}{2}e_\omega^T e_\omega - \frac{1}{2}e_\theta^T K_{p\theta}e_\theta\right] + \\
&\left(c_2 2b_{\bar f} + b_{\bar f}\right)\rho(V)\left[\left|e_\omega^T \bar e \Phi^T(\theta)L_2\Phi(\theta)v\right| + \right.\\
&\left.\left|e_\omega^T \bar s \Phi^{\perp T}(\theta)L_2\Phi(\theta)v\right|\right] + \\
&\left(c_2 2K_{p\theta}b_{\bar f} + b_{\bar f}\right)\rho(V)\left[\left|e_\theta^T \bar e \Phi^T(\theta)L\Phi(\theta)v\right| + \right.\\
&\left.\left|e_\theta^T \bar s \Phi^{\perp T}(\theta)L_2\Phi(\theta)v\right|\right]\\
\le &\rho(V)\left[-\frac{c_2}{2}e_\omega^T e_\omega - \frac{1}{2}e_\theta^T K_{p\theta}e_\theta\right] + \\
&\frac{\pi\rho}{2}\left|\bar e\Phi^T(\theta)L_2\Phi^T(\theta)v\right|^2 + \frac{\pi\rho}{2}\left|\bar s\Phi^{\perp T}(\theta)L_2\Phi^T(\theta)v\right|^2 + \\
&\left(c_2 2K_{p\theta}b_{\bar f} + b_{\bar f}\right)^2\rho(V)2\frac{e_\theta^T e_\theta}{\pi} + \left(c_2 2b_{\bar f} + b_{\bar f}\right)^2\rho(V)2\frac{e_\omega^T e_\omega}{\pi}.
\end{aligned} \tag{B.88}
$$

Next, taking $\pi = \max\left\{\frac{8\left(2c_2b_{\bar f}+b_{\bar f}\right)^2}{c_2\lambda_m(K_{d\theta})}, \frac{8\left(2c_2K_{p\theta}b_{\bar f}+b_{\bar f}\right)^2}{\lambda_m(K_{p\theta})}\right\}$, we get

$$
\begin{aligned}
\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \le -\rho_1(V)&\left[\frac{c_2}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta}e_\theta\right] + \\
&\frac{\pi\rho_1}{2}\left|\bar e\Phi^T(\theta)L_2\Phi(\theta)v\right|^2 + \frac{\pi\rho_1}{2}\left|\bar s\Phi^{\perp T}(\theta)L_2\Phi(\theta)v\right|^2.
\end{aligned} \tag{B.89}
$$

Finally, using the inequalities

$$
\left|\bar e\Phi^T L_2\Phi v\right|^2 \le \lambda_N(L)V\left|\Phi^T L_2\Phi\right|_\infty^2\left|K_{dt}^- K_{pt}\right| v^T K_{dt}K_{pt}^- v \tag{B.90}
$$

and

$$
\left|\bar s\Phi^{\perp T} L_2\Phi v\right|^2 \le \lambda_N(L)V\left|\Phi^{\perp T} L_2\Phi\right|_\infty^2\left|K_{dt}^- K_{pt}\right| v^T K_{dt}K_{pt}^- v \tag{B.91}
$$

we verify the following

$$
\begin{aligned}
\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \le -\rho_1(V)&\left[\frac{a}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta}e_\theta\right] + \\
&\frac{\pi}{2}\lambda_N(L)\left(\left|\Phi^T(\theta)L_2\Phi(\theta)\right|_\infty^2 + \right.\\
&\left.\left|\Phi^{\perp T}(\theta)L_2\Phi(\theta)\right|_\infty^2\right)\left|K_{dt}^- K_{pt}\right|\rho_1(V)Vv^T K_{dt}K_{pt}^- v\\
\le -&\rho_1(V)\left[\frac{c_2}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta}e_\theta\right] + \rho_2(V)v^T K_{dt}K_{pt}^- v.
\end{aligned} \tag{B.92}
$$

As a last step, we use the previous inequalities to show that the time derivative of the global Lyapunov function $\Gamma(\cdot)$ along trajectories of the system (4.25) is negative definite, that is,

$$\begin{aligned}\dot{\Gamma}(\cdot) \leq &- \rho_1(V)\left[\frac{c_2}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta} e_\theta\right] - \frac{1}{2}\gamma(V) v^T K_{dt} K_{pt}^{-} v \\ &- \frac{1}{4}\alpha(V) e^T K_{pt} e - \frac{\mu}{2T} V \kappa^T \kappa - \alpha(V) v^T \bar{e}_\omega s + \\ &c_1 V s^T \dot{f} \bar{e}_\omega s - c_1 V e^T \dot{f} \bar{e}_\omega e.\end{aligned} \tag{B.93}$$

Furthermore, using the following inequalities

$$\alpha(V) v^T \bar{e}_\omega s \leq \frac{\lambda_N(L)\left|K_{dt}^{-} K_{pt}\right|}{c_3}\alpha(V) V v^T K_{dt} K_{pt}^{-} v + \frac{c_3}{4}\alpha(V) e_\omega^T e_\omega$$

and

$$c_1 V e_\omega^T \dot{f}\left(\bar{s}s - \bar{e}e\right) \leq c_1 V b_{\bar{f}}\left[\frac{c_3}{4}e_\omega^T e_\omega + \frac{4}{c_3}\kappa^T \kappa\right],$$

we obtain

$$\begin{aligned}\dot{\Gamma}(\cdot) \leq &- \rho_1(V)\left[\frac{a}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta} e_\theta\right] - \frac{1}{2}\gamma(V) v^T K_{dt} K_{pt}^{-} v \\ &- \frac{1}{4}\alpha(V) e^T K_{pt} e - \frac{\mu}{2T} V \kappa^T \kappa + \\ &\frac{c_3}{4}\left(\alpha(V) + b_{\bar{f}} c_1 V\right) e_\omega^T e_\omega + \\ &\frac{\lambda_N(L)\left|K_{dt}^{-} K_{pt}\right|}{c_3}\alpha(V) V v^T K_{dt} K_{pt}^{-} v + \frac{4 c_1 b_{\bar{f}}}{c_3} V \kappa^T \kappa.\end{aligned} \tag{B.94}$$

Next, taking $c_3(V_1) = \max\left\{\frac{16 T c_1 b_{\bar{f}}}{\mu}, \frac{4\lambda_N(L)\left|K_{dt}^{-} K_{pt}\right|\alpha(V) V}{\gamma(V)}\right\}$, we obtain

$$\begin{aligned}\dot{\Gamma}(\cdot) \leq &- \rho_1(V)\left[\frac{c_2}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta} e_\theta\right] - \frac{1}{4}\gamma(V) v^T K_{dt} K_{pt}^{-} v \\ &- \frac{1}{8}\alpha(V) e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa + \\ &\frac{c_3}{4}\left(\alpha(V) + b_{\bar{f}} \beta V\right) e_\omega^T e_\omega.\end{aligned} \tag{B.95}$$

Finally, taking ρ_1 as in (4.29), we obtain

$$\begin{aligned}\dot{\Gamma}(\cdot) \leq &- \frac{\rho_1(V)}{2}\left[\frac{c_2}{4}e_\omega^T e_\omega + \frac{1}{4}e_\theta^T K_{p\theta} e_\theta\right] - \frac{1}{4}\gamma(V) v^T K_{dt} K_{pt}^{-} v \\ &- \frac{1}{8}\alpha(V) e^T K_{pt} e - \frac{\mu}{4T} V^3.\end{aligned} \tag{B.96}$$

Which completes the proof.

B.10 Proof of theorem 4.2

We start observing that the transnational part of the closed-loop system (4.42) admits the following non-strict Lyapunov-Krasovskii functional candidate

$$V(\theta, X_t, \dot{z}_t) = v^\top K_{pt}^{-1} v + z^\top L_2 z + \int_{-T^*}^{0} \int_{t+\theta}^{t} \dot{z}(s)^T \dot{z}(s) ds d\theta \tag{B.97}$$

where $T^* = \max_{i,j} \{T_{ij}\}$.
Indeed, in view of the following inequality

$$\int_{-T^*}^{0} \int_{t+\theta}^{t} \dot{z}(s)^T \dot{z}(s) ds d\theta \leq T^* \int_{t-T^*}^{t} |\dot{z}(s)|^2 ds$$

one can easily establish that $V(\theta, X_t, \dot{z}_t)$ is Lyapunov-Krasovskii functional candidate with respect to X_t. Moreover, the time derivative of $V(\cdot)$ along the trajectories of (4.42) is given by

$$\begin{aligned} \dot{V}(\cdot) =& - 2v^\top K_{pt}^{-1} K_{dt} v + 2v^T \Phi(\theta)^T \mathcal{A}(\dot{z}_t) + T^* v^T v - \int_{t-T^*}^{t} \dot{z}(s)^T \dot{z}(s) ds \\ \leq& - [2 - T^* \lambda_M (K_{pt} K_{dt}^{-})] v^\top K_{pt}^{-1} K_{dt} v - \frac{1}{\bar{a}^2 N} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T \dot{z}_i(s) ds + \\ & 2v^T \Phi(\theta)^T \mathcal{A}(\dot{z}_t). \end{aligned} \tag{B.98}$$

Applying Jensen's inequality, we obtain the following

$$\int_{t-T_{ij}}^{t} \dot{z}_i(s)^T \dot{z}_i(s) ds \leq -\frac{1}{T_{ij}^*} \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds. \tag{B.99}$$

Moreover, using the inequality

$$|\mathcal{A}(\dot{z}_t)|^2 \leq N \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds \tag{B.100}$$

we obtain

$$
\begin{aligned}
\dot{V}(\cdot) \leq & -[2 - T^* \lambda_M(K_{pt} K_{dt}^-)] v^\top K_{pt}^{-1} K_{dt} v - \\
& \frac{1}{\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds + \\
& \frac{N}{2\epsilon} \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds + \epsilon \lambda_M(K_{pt} K_{dt}^-) v^T K_{pt}^{-1} K_{dt} v. \quad \text{(B.101)}
\end{aligned}
$$

Next, taking $\epsilon = N^2 \bar{a} T^*$ and the matrices K_{dt} and K_{pt} such that Assumption 4.4 holds, we get

$$
\dot{V}(\cdot) \leq -v^\top K_{pt}^{-1} K_{dt} v - \frac{1}{2\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds \quad \text{(B.102)}
$$

From the previous section, the time derivative of $W(\cdot)$ along the trajectories of the following un-delayed system

$$
\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \bar{\dot{f}} \bar{\kappa} - \bar{e}_\omega \\ 0 & -\bar{\dot{f}} \bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v \quad \text{(B.103)}
$$

verifies the following upper bound

$$
\dot{W}(\cdot) \leq -\frac{1}{4} \gamma(V) v^T K_{dt} K_{pt}^- v - \frac{1}{8} \alpha(V) e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa + \frac{\rho_1(V)}{8} c_2 e_\omega^T K_{p\theta} e_\omega. \quad \text{(B.104)}
$$

If we consider the delayed case in (4.42a) we obtain

$$
\begin{aligned}
\dot{W}(\cdot) \leq & -\frac{1}{4} \gamma(V) v^T K_{dt} K_{pt}^- v - \frac{1}{8} \alpha(V) e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa + \frac{\rho_1(V)}{8} c_2 e_\omega^T K_{p\theta} e_\omega \\
& - \frac{\gamma(V)}{2\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds - \alpha(V) e^T K_{pt} \Phi(\theta)^T \mathcal{A}(\dot{z}_t) \\
& - \frac{\partial \alpha}{\partial V} e^T v \left(\frac{1}{2\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds \right). \quad \text{(B.105)}
\end{aligned}
$$

Using the inequalities

$$
-\alpha(V) e^T K_{pt} \Phi(\theta)^T \mathcal{A}(\dot{z}_t) \leq \frac{\alpha(V)}{16} e^T K_{pt} e + 4\alpha(V) \lambda_M(K_{pt}) |\mathcal{A}|^2
$$

and

$$|\mathcal{A}|^2 \leq N \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds$$

we obtain

$$\dot{W}(\cdot) \leq -\frac{1}{4}\gamma(V)v^T K_{dt} K_{pt}^{-} v - \frac{1}{16}\alpha(V)e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa + \frac{\rho_1(V)}{8} c_2 e_\omega^T K_{p\theta} e_\omega. \quad \text{(B.106)}$$

On the other hand, the time derivative $\rho_1(V)Z(X_r)$ along the following un-delayed system

$$\dot{X}_r = \begin{bmatrix} 0 & I_N \\ -K_{p\theta} & -K_{d\theta} \end{bmatrix} X_r + \begin{bmatrix} b_{\bar{f}} \\ \dot{\bar{f}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2 \right) \Phi v \quad \text{(B.107)}$$

satisfies the following upper bound

$$\begin{aligned} \frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\rho_1(V) \left[\frac{c_2}{4} e_\omega^T K_{d\theta} e_\omega + \frac{1}{4} e_\theta^T K_{p\theta} e_\theta \right] + \frac{1}{2}\rho_2(V) v^T K_{dt} K_{pt}^{-} v \\ & - Z(X_r)\dot{V}. \end{aligned} \quad \text{(B.108)}$$

If we consider the delayed case as in (4.42b), we obtain

$$\begin{aligned} \frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\rho_1(V) \left[\frac{c_2}{4} e_\omega^T K_{d\theta} e_\omega + \frac{1}{4} e_\theta^T K_{p\theta} e_\theta \right] \\ & + \frac{1}{2}\rho_2(V) v^T K_{dt} K_{pt}^{-} v + 2\rho_1(V) K_{pt} c_2 e_\omega^T \kappa_d + \rho_1(V) K_{pt} e_\theta^T \kappa_d \\ & - Z(X_r) \left(-\frac{1}{2\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds \right). \end{aligned} \quad \text{(B.109)}$$

Using the fact that

$$\kappa_d = \frac{1}{2} \begin{bmatrix} |\mathcal{A}_1|^2 \\ \vdots \\ |\mathcal{A}_N|^2 \end{bmatrix} + \bar{e}\Phi^T \mathcal{A}(\dot{z}_t) + \bar{s}\Phi^{\perp T} \mathcal{A}(\dot{z}_t)$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq &-\rho_1(V)\left[\frac{c_2}{4}e_\omega^T K_{d\theta}e_\omega + \frac{1}{4}e_\theta^T K_{p\theta}e_\theta\right] + \frac{1}{2}\rho_2(V)v^T K_{dt}K_{pt}^{-}v \\
&+ 2\rho_1(V)K_{pt}c_2 e_\omega^T \left[\frac{1}{2}\begin{bmatrix}|\mathcal{A}_1|^2\\ \vdots \\ |\mathcal{A}_N|^2\end{bmatrix} + \left(\bar{e}\Phi^T + \bar{s}\Phi^{\perp T}\right)\mathcal{A}(\dot{z}_t)\right] \\
&+ \rho_1(V)K_{pt}e_\theta^T \left[\frac{1}{2}\begin{bmatrix}|\mathcal{A}_1|^2\\ \vdots \\ |\mathcal{A}_N|^2\end{bmatrix} + \left(\bar{e}\Phi^T + \bar{s}\Phi^{\perp T}\right)\mathcal{A}(\dot{z}_t)\right] \\
&- Z(X_r)\left(\frac{1}{2\bar{a}^2 NT^*}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t}\dot{z}_i(s)ds\right).
\end{aligned} \tag{B.110}$$

Using the following inequalities

$$\lambda_M(K_{pt})c_2\rho_1(V)\,|e_\omega|\,|\mathcal{A}|^2 \leq \frac{\lambda_M(K_{pt})c_2}{2\epsilon_1}\,|e_\omega|^2\,|\mathcal{A}|^2 + \lambda_M(K_{pt})c_2\rho_1(V)^2\frac{\epsilon_1}{2}\,|\mathcal{A}|^2,$$

$$\frac{1}{2}\lambda_M(K_{pt})\rho_1(V)\,|e_\theta|\,|\mathcal{A}|^2 \leq \frac{\lambda_M(K_{pt})}{4\epsilon_1}\,|e_\theta|^2\,|\mathcal{A}|^2 + \lambda_M(K_{pt})\rho_1(V)^2\frac{\epsilon_1}{4}\,|\mathcal{A}|^2,$$

$$\begin{aligned}
2\rho_1(V)K_{pt}c_2 e_\omega^T\left(\bar{e}\Phi^T + \bar{s}\Phi^{\perp T}\right)\mathcal{A}(\dot{z}_t) \leq\, & 2\frac{\lambda_M(K_{pt})\rho_1(V)c_2}{\epsilon}\,|e_\omega|^2 \\
&+ \lambda_M(K_{pt})c_2\rho_1(V)\left(e^2+s^2\right)\epsilon\,|\mathcal{A}|^2,
\end{aligned}$$

and

$$\rho_1(V)K_{pt}e_\theta^T\left(\bar{e}\Phi^T + \bar{s}\Phi^{\perp T}\right)\mathcal{A}(\dot{z}_t) \leq \frac{\lambda_M(K_{pt})\rho_1(V)}{\epsilon}\,|e_\theta|^2 + \rho_1(V)\lambda_M(K_{pt})\left(e^2+s^2\right)\frac{\epsilon}{2}\,|\mathcal{A}|^2$$

we obtain

$$\begin{aligned}\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\rho_1(V)\left[\frac{c_2}{4}e_\omega^T K_{d\theta}e_\omega + \frac{1}{4}e_\theta^T K_{p\theta}e_\theta\right] + \frac{1}{2}\rho_2(V)v^T K_{dt}K_{pt}^{-}v \\ & + \frac{\lambda_M(K_{pt})\rho_1(V)}{\epsilon}\left(2c_2+1\right)\left(|e_\omega|^2 + |e_\theta|^2\right) \\ & + \frac{\lambda_M(K_{pt})}{2\epsilon_1}\left(c_2 + \frac{1}{2}\right)\left(|e_\omega|^2 + |e_\theta|^2\right)|\mathcal{A}|^2 \\ & + \lambda_M(K_{pt})\rho_1(V)\left[\left(c_2+\frac{1}{2}\right)\rho_1(V)\frac{\epsilon_1}{2} + \left(c_2+\frac{1}{2}\right)\lambda_N(L)V\epsilon\right]|\mathcal{A}|^2 \\ & - \frac{Z(X_r)}{2\bar{a}^2NT^*}\left(\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds\right). \end{aligned} \tag{B.111}$$

Taking $\epsilon_1 = c_5$, $\epsilon = c_6$, and using the inequalities

$$Z \geq \min\left\{1, \lambda_m(K_{p\theta})\right\}\left(e_\theta^\top e_\theta + e_\omega^\top e_\omega\right)$$

and

$$|\mathcal{A}|^2 \leq N\sum_{j=1}^{N}\sum_{i=1}^{N}a_{ij}^2\int_{t-T_{ij}}^{t}\dot{z}_i(s)^T ds\int_{t-T_{ij}}^{t}\dot{z}_i(s)ds$$

we obtain

$$\begin{aligned}\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\rho_1(V)\left[\frac{3c_2}{16}e_\omega^T K_{d\theta}e_\omega + \frac{3}{16}e_\theta^T K_{p\theta}e_\theta\right] + \frac{1}{2}\rho_2(V)v^T K_{dt}K_{pt}^{-}v \\ & + \lambda_M(K_{pt})\rho_1(V)\left[\left(c_2+\frac{1}{2}\right)\rho_1(V)\frac{\epsilon_1}{2} + \left(c_2+\frac{1}{2}\right)\lambda_N(L)V\epsilon\right]|\mathcal{A}|^2. \end{aligned} \tag{B.112}$$

Finally, we conclude

$$\begin{aligned}\dot{\Gamma}(\cdot) \leq & -\frac{1}{8}\gamma(V)v^T K_{dt}K_{pt}^{-}v - \frac{1}{16}\alpha(V)e^T K_{pt}e - \frac{\mu}{8T}V\kappa^T\kappa \\ & -\rho_1(V)\left[\frac{c_2}{16}e_\omega^T K_{d\theta}e_\omega + \frac{1}{16}e_\theta^T K_{p\theta}e_\theta\right]. \end{aligned} \tag{B.113}$$

Which completes the proof.

B.11 Proof of theorem 4.3

First, since $\rho_1(V)$ and $\rho_2(V)$, in (4.60) and (4.62) respectively, are strictly positive functions and radially unbounded, positive definiteness of Γ is ensured using the fact that

$\Gamma(t,0,0)=0$, for all $t\geq 0$,

$$W \;\;\geq\;\; \gamma(V)V,$$

$$W \;\;\leq\;\; \gamma(V)V+V\kappa^{\top}(e,s)\bar{\Upsilon}_{\dot{q}^2}(t)\kappa(e,s)+2c_1b_f\lambda_N(L)V^2+2\left(\lambda_N(L)+|K_{pt}|\right)\alpha(V)V,$$

and

$$\frac{c_2}{2}\left(e_\theta^T Le_\theta+e_\omega^T e_\omega+e_\alpha^T K_I^- e_\alpha\right) \;\;\leq Z\leq\;\; 2c_2\left(e_\theta^T Le_\theta+e_\omega^T e_\omega+e_\alpha^T K_I^- e_\alpha\right).$$

Furthermore, since the translational part of (4.57) is similar to the one presented in (4.57a), it becomes straightforward to conclude the following

$$\begin{aligned}\dot{W}(\cdot)\leq &-\frac{1}{2}\gamma(V)v^T K_{dt}K_{pt}^- v-\frac{1}{4}\alpha(V)e^T K_{pt}e-\frac{\mu}{2T}V\kappa^T\kappa\\ &-\alpha(V)v^T\bar{e}_\omega s+c_1Vs^T\dot{f}\bar{e}_\omega s-\dot{f}c_1Ve^T\dot{f}\bar{e}_\omega e.\end{aligned}\tag{B.114}$$

Then, using the following inequalities

$$\alpha(V)v^T\bar{e}_\omega s\leq\frac{\lambda_N(L)\left|K_{dt}^- K_{pt}\right|}{\sigma}\alpha(V)Vv^T K_{dt}K_{pt}^- v+\frac{\sigma}{4}\alpha(V)e_\omega^T e_\omega,$$

$$c_1Ve_\omega^T\dot{f}\left(\bar{s}s-\bar{e}e\right)\leq c_1Vb_{\bar{f}}\left[\frac{\sigma}{4}e_\omega^T e_\omega+\frac{4}{\sigma}\kappa^T\kappa\right],$$

and taking $\sigma=\max\left\{\frac{16Tc_1b_q}{\mu},\frac{4\lambda_N(L)\left|K_{dt}^- P\right|\alpha(V)V}{\gamma(V)}\right\}$, we obtain

$$\dot{W}(\cdot)\leq-\frac{1}{4}\gamma(V)v^T K_{dt}K_{pt}^- v-\frac{1}{8}\alpha(V)e^T K_{pt}e-\frac{\mu}{4T}V\kappa^T\kappa+\frac{\sigma}{4}\left(\alpha(V)+b_qc_1V\right)e_\omega^T e_\omega \tag{B.115}$$

and

$$\dot{W}(\cdot)\leq-\frac{1}{4}\gamma(V)v^T K_{dt}K_{pt}^- v-\frac{1}{8}\alpha(V)e^T K_{pt}e-\frac{\mu}{4T}V\kappa^T\kappa+\frac{\rho_1(V)}{8}c_5e_\omega^T K_I e_\omega. \tag{B.116}$$

Notice that the function $Z(X_r)$ is a strict Lyapunov function for the following system

$$\dot{X}_r=\begin{bmatrix}0 & I_N & 0\\ -L & 0 & I_N\\ 0 & -K_I & -K_\alpha\end{bmatrix}X_r, \tag{B.117}$$

it's time derivative along the trajectories of (B.117) satisfies

$$\dot{Z}(X_r) \leq -\frac{1}{2}\left[c_2\alpha^T K_I^- K_\alpha \alpha + c_2 e_\omega^T K_I e_\omega + e_\theta^T L^2 e_\theta\right] \tag{B.118}$$

and along the trajectories of (4.57) it satisfies

$$\begin{aligned}\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq &-\frac{1}{2}\rho_1(V)\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_5 e_\omega^T K_I e_\omega + e_\theta^T L^2 e_\theta\right]\\ &+\rho_1 e_\theta^T L\left[2c_2\bar{q} + \dot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v\\ &+\rho_1 e_\omega^T\left[2c_2\dot{\bar{q}} + Lb_{\bar{q}} + c_5\ddot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v\\ &+\rho_1 e_\alpha^T\left[2K_I^-\ddot{\bar{q}}c_2 + c_5\dot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v.\end{aligned} \tag{B.119}$$

Using the following inequalities

$$\begin{aligned}&e_\theta^T L\left[2c_2\bar{q} + \dot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v \leq \frac{1}{4}e_\theta^T L^2 e_\theta\\ &+2b_q^2\lambda_M(L)\left(2c_2+1\right)^2 V\left[\left|\Phi^T L\Phi\right|_\infty + \left|\Phi^{\perp T} L\Phi\right|_\infty\right]\left|K_{dt}^- K_{pt}\right| v^T K_{dt} K_{pt}^- v,\end{aligned} \tag{B.120}$$

$$\begin{aligned}&e_\omega^T\left[2c_2\dot{\bar{q}} + Lb_{\bar{q}} + c_5\ddot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v \leq \frac{1}{4}c_5 e_\omega^T K_I e_\omega\\ &+2b_q^2\lambda_M(L)\frac{\lambda_M(K_I^-)}{c_5}\left(2c_2 + c_5 + \lambda_N(L)\right)^2 V\\ &\times\left[\left|\Phi^T L\Phi\right|_\infty + \left|\Phi^{\perp T} L\Phi\right|_\infty\right]\left|K_{dt}^- K_{pt}\right| v^T K_{dt} K_{pt}^- v,\end{aligned} \tag{B.121}$$

and

$$\begin{aligned}\rho_1 e_\alpha^T\left[2K_I^-\ddot{\bar{q}}c_2 + c_5\dot{\bar{q}}\right]\left(\bar{e}\Phi^T L_2 + \bar{s}\Phi^{\perp T} L_2\right)\Phi v \leq &\frac{1}{4}c_2 e_\alpha^T K_I^- K_\alpha e_\alpha\\ &+2b_q^2\lambda_M(L)\frac{\lambda_M(K_\alpha^- K_I)}{c_2}\left(2c_2\lambda_M(K_I^-) + c_5\right)^2 V\\ &\times\left[\left|\Phi^T L\Phi\right|_\infty + \left|\Phi^{\perp T} L\Phi\right|_\infty\right]\left|K_{dt}^- K_{pt}\right| v^T K_{dt} K_{pt}^- v\end{aligned} \tag{B.122}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq &-\frac{1}{4}\rho_1(V)\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_5 e_\omega^T K_I e_\omega + e_\theta^T L^2 e_\theta\right]\\ &+2b_q^2\rho_1(V)\lambda_M(L)\Bigg[\left(2c_2+1\right)^2 + \frac{\lambda_M(K_I^-)}{c_5}\left(2c_2 + c_5 + \lambda_N(L)\right)^2\\ &+\frac{\lambda_M(K_\alpha^- K_I)}{c_2}\left(2c_2\lambda_M(K_I^-) + c_5\right)^2\Bigg] V\\ &\times\left[\left|\Phi^T L\Phi\right|_\infty + \left|\Phi^{\perp T} L\Phi\right|_\infty\right]\left|K_{dt}^- K_{pt}\right| v^T K_{dt} K_{pt}^- v,\end{aligned} \tag{B.123}$$

also

$$\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq -\frac{1}{4}\rho_1(V)\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_5 e_\omega^T K_\omega e_\omega + e_\theta^T L^2 e_\theta\right] \\ + \rho_2(V) v^T K_{dt} K_{pt}^- v. \tag{B.124}$$

Which yields to the following upper bound on the time derivative of the global Lyapunov function $\Gamma(\cdot)$ along the trajectories of (4.57)

$$\dot{\Gamma}(\cdot) \leq -\frac{1}{4}\gamma(V) v^T K_{dt} K_{pt}^- v - \frac{1}{8}\alpha(V) e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa \\ -\frac{1}{8}\rho_1(V)\left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_5 e_\omega^T K_I e_\omega + e_\theta^T L^2 e_\theta\right]. \tag{B.125}$$

Which completed the proof.

B.12 Proof of theorem 4.4

We start by invoking Section B.10, that is, we observe that the translational part of the closed-system (4.72), which is the same as in the partial delayed case in (4.42a), admits the following non-strict Lyapunov-Krasovskii functional candidate

$$V(\theta, X_t, \dot{z}_t) = v^\top K_{pt}^{-1} v + z^\top L_2 z + \int_{-T^*}^{0} \int_{t+\theta}^{t} \dot{z}(s)^T \dot{z}(s) ds d\theta. \tag{B.126}$$

where $T^* = \max_{i,j}\{T_{ij}\}$.
It's time derivative along the trajectories of the closed-loop system (4.72) satisfies the following upper bound

$$\dot{V}(\cdot) \leq -v^\top K_{pt}^{-1} K_{dt} v - \frac{1}{2\bar{a}^2 N T^*} \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds. \tag{B.127}$$

On the other hand, the functional $Z(X_r, \dot{e}_{\theta t})$ is a strict Lyapunov-Krasovskii functional, under Assumption 4.5, for the following delayed system

$$\dot{X}_r = \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix}. \tag{B.128}$$

Indeed, we have

$$\begin{aligned} Z &\geq \frac{c_2}{2}\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right), \\ Z &\leq 2c_2\left(e_\theta^T L e_\theta + e_\omega^T e_\omega + e_\alpha^T K_I^- e_\alpha\right) + 2c_5 T^* \int_{t-T^*}^{t} |\dot{e}_\theta(s)|^2 ds. \end{aligned}$$

and the time derivative of $Z(X_r, \dot{\theta}_t)$ along trajectories of (B.128) is given by

$$\begin{aligned} \dot{Z}(\cdot) = &- 2c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - e_\theta^T L^2 e_\theta - e_\theta^T L K_\omega e_\omega + e_\theta^T L e_\alpha + e_\omega^T L e_\omega \\ &- e_\theta^T L \mathcal{A}(\dot{e}_{\theta t}) - 2c_2 e_\omega^T \mathcal{A}(\dot{e}_{\theta t}) + c_5 T^* \dot{e}_\theta^T \dot{e}_\theta - c_5 \int_{t-T^*}^{t} \dot{e}_\theta^T(s)\dot{e}_\theta(s) ds. \end{aligned} \tag{B.129}$$

Using the fact that $K_\omega > I_n$ both with the following inequalities

$$\begin{aligned} e_\theta^T L e_\alpha &\leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon}{2}\lambda_M(K_I K_\alpha^-) e_\alpha^T K_I^- K_\alpha e_\alpha, \\ e_\omega^T L e_\omega &\leq \lambda_N(L) e_\omega^T K_\omega e_\omega, \\ -2c_2 e_\omega^T \mathcal{A}(e_{\theta t}) &\leq \epsilon_1 c_2 e_\omega^T K_\omega e_\omega + \frac{c_2}{\epsilon_1} |\mathcal{A}(e_{\theta t})|^2, \\ e_\theta^T L \mathcal{A}(e_{\theta t}) &\leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon}{2} |\mathcal{A}(e_{\theta t})|^2, \\ e_\theta^T L K_\omega e_\omega &\leq \frac{1}{2\epsilon} e_\theta^T L^2 e_\theta + \frac{\epsilon \lambda_M(K_\omega)}{2}, \\ |\mathcal{A}(e_{\theta t})|^2 &\leq N \sum_{j=1}^{N}\sum_{i=1}^{N}\left(\int_{t-T_{ij}}^{t} \dot{e}_{\theta j}(s) ds\right)^2 a_{ij}^2, \\ \int_{t-T^*}^{t} \dot{e}_\theta^T(s)\dot{e}_\theta(s) ds &\geq \frac{1}{\bar{a}^2 N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{e}_{\theta j}^2(s) ds, \\ \int_{t-T_{ij}}^{t} a_{ij}^2 \dot{e}_{\theta j}^2(s) ds &\geq \frac{1}{T^*}\left(\int_{t-T_{ij}}^{t} \dot{e}_{\theta j}(s) ds\right)^2 a_{ij}^2, \end{aligned}$$

taking $\epsilon_1 = \frac{1}{2}$, $\epsilon = 3$ and using Assumption 4.5, we obtain

$$\dot{Z}(\cdot) = -\frac{1}{2}c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{1}{2} e_\theta^T L^2 e_\theta. \tag{B.130}$$

Since V is Lyapunov Krasovskii functional candidate, $\Gamma(t, 0, 0, 0, 0) = 0$, for all $t \geq 0$, and the following inequalities hold

$$\begin{aligned} W &\geq \gamma(V)V, \\ W &\leq \gamma(V)V + V\kappa^\top(e, s)\bar{\Upsilon}_{\dot{q}^2}(t)\kappa(e, s) + 2c_1 b_f \lambda_N(L) V^2 + 2\left(\lambda_N(L) + |K_{pt}|\right)\alpha(V)V, \end{aligned}$$

then Γ is Lyapunov Krasovskii functional candidate for the closed-loop system.

Using the previous section, the time derivative of $W(\cdot)$ along the trajectories of the following un-delayed system

$$\dot{X}_t = \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \bar{\dot{f}}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\bar{\dot{f}}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v - \begin{bmatrix} K_{pt}\Phi(\theta)^T \\ 0 \\ 0 \end{bmatrix} \mathcal{A}(\dot{z}_t) \tag{B.131}$$

satisfies

$$\dot{W}(\cdot) \leq -\frac{1}{4}\gamma(V)v^T K_{dt} K_{pt}^{-} v - \frac{1}{16}\alpha(V)e^T K_{pt} e - \frac{\mu}{4T} V \kappa^T \kappa + \frac{\rho_1(V)}{8} c_2 e_\omega^T K_{p\theta} e_\omega. \tag{B.132}$$

Moreover, the time derivative $\rho_1(V)Z(X_r)$ along the following system

$$\dot{X}_r = \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix} \tag{B.133}$$

satisfies the following upper bound

$$\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq -\frac{\rho_1(V)}{2} c_2 \left[e_\alpha^T K_I^{-} K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{\rho_1(V)}{2} e_\theta^T L^2 e_\theta. \tag{B.134}$$

Now, if we consider the delayed system in (4.72b) when $\mathcal{B}(t) = 0$, which has the following form

$$\begin{aligned} \dot{X}_r = & \begin{bmatrix} 0 & I_N & 0 \\ -L & -K_\omega & I_N \\ 0 & -K_I & -K_\alpha \end{bmatrix} X_r - \begin{bmatrix} 0 \\ \mathcal{A}(\dot{e}_{\theta t}) \\ 0 \end{bmatrix} + \begin{bmatrix} b_{\bar{q}} \\ \dot{\bar{q}} \\ \ddot{\bar{q}} \end{bmatrix} \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2\right) \Phi v \\ & - \begin{bmatrix} 0 \\ K_\omega\left(\bar{q} + b_{\bar{q}} K_\omega^{-} D\right) \\ \dot{p} \end{bmatrix} \kappa_d(e, s, \theta, \dot{z}_t), \end{aligned} \tag{B.135}$$

$$M := \left(\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2\right)\Phi.$$

We obtain

$$\begin{aligned}
\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\frac{\rho_1(V)}{2}c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{\rho_1(V)}{2}e_\theta^T L^2 e_\theta \\
& - 2c_2\rho_1(V)\left[e_\omega^T K_\omega\left(\bar{\dot{q}} + b_{\bar{q}}K_\omega^- D\right)\kappa_d + \dot{p}e_\alpha^T K_I^-\kappa_d\right] \\
& - \rho_1(V)e_\theta^T L K_\omega\left(\bar{\dot{q}} + b_{\bar{q}}K_\omega^- D\right)\kappa_d \\
& + \rho_1(V)c_5 T^*\left[v^T M b_{\bar{q}}^2 M v + 2e_\omega^T b_{\bar{q}} M v\right] \\
& + \rho_1(V)e_\theta^T L\left(2c_2\bar{q} + \bar{\dot{q}}\right)Mv + \rho_1(V)e_\omega^T\left(c_2\bar{\dot{q}} + Lb_{\bar{q}}\right)Mv \\
& + \rho_1(V)e_\alpha^T\left(2c_2 K_I^-\bar{\dot{q}}\right)Mv \\
& - Z(X_r)\left(\frac{1}{2\bar{a}^2 NT^*}\sum_{j=1}^N\sum_{i=1}^N a_{ij}^2\int_{t-T_{ij}}^t \dot{z}_i(s)^T ds\int_{t-T_{ij}}^t \dot{z}_i(s)ds\right).
\end{aligned} \tag{B.136}$$

Using the following equality

$$\kappa_d = \frac{1}{2}\begin{bmatrix}|\mathcal{A}_1|^2\\ \vdots \\ |\mathcal{A}_N|^2\end{bmatrix} + \bar{e}\Phi^T\mathcal{A}(\dot{z}_t) + \bar{s}\Phi^{\perp T}\mathcal{A}(\dot{z}_t)$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\rho_1(V)Z(X_r)\right) \leq & -\frac{\rho_1(V)}{2}c_2\left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{\rho_1(V)}{2}e_\theta^T L^2 e_\theta \\
& - 2c_2\rho_1(V)\left[e_\omega^T K_\omega\left(\bar{\dot{q}} + b_{\bar{q}}K_\omega^- D\right) + \dot{p}e_\alpha^T K_I^-\right] \\
& * \left[\frac{1}{2}\begin{bmatrix}|\mathcal{A}_1|^2\\ \vdots \\ |\mathcal{A}_N|^2\end{bmatrix} + \bar{e}\Phi^T\mathcal{A}(\dot{z}_t) + \bar{s}\Phi^{\perp T}\mathcal{A}(\dot{z}_t)\right] \\
& - \rho_1(V)e_\theta^T L K_\omega\left(\bar{\dot{q}} + b_{\bar{q}}K_\omega^- D\right)\left[\frac{1}{2}\begin{bmatrix}|\mathcal{A}_1|^2\\ \vdots \\ |\mathcal{A}_N|^2\end{bmatrix} + \bar{e}\Phi^T\mathcal{A}(\dot{z}_t) + \bar{s}\Phi^{\perp T}\mathcal{A}(\dot{z}_t)\right] \\
& + \rho_1(V)c_5 T^*\left[v^T M b_{\bar{q}}^2 M v + 2e_\omega^T b_{\bar{q}} M v\right] \\
& + \rho_1(V)e_\theta^T L\left(2c_2\bar{q} + \bar{\dot{q}}\right)Mv + \rho_1(V)e_\omega^T\left(c_2\bar{\dot{q}} + Lb_{\bar{q}}\right)Mv \\
& + \rho_1(V)e_\alpha^T\left(2c_2 K_I^-\bar{\dot{q}}\right)Mv \\
& - Z(X_r)\left(\frac{1}{2\bar{a}^2 NT^*}\sum_{j=1}^N\sum_{i=1}^N a_{ij}^2\int_{t-T_{ij}}^t \dot{z}_i(s)^T ds\int_{t-T_{ij}}^t \dot{z}_i(s)ds\right).
\end{aligned} \tag{B.137}$$

Using the following inequalities

$$c_2\rho_1(V)b_q\left(1+\lambda_M(K_\omega^{-}D)\right)|K_\omega e_\omega|\,|\mathcal{A}|^2 \leq \frac{c_2 b_q}{2\epsilon_1}\lambda_M(K_\omega)^2\,|e_\omega|^2\,|\mathcal{A}|^2+\frac{c_2 b_q}{2}\left(1+\lambda_M(K_\omega^{-}D)\right)^2,$$

$$*\epsilon_1\,|\mathcal{A}|^2\,\rho_1(V)^2,$$

$$2c_2\rho_1(V)e_\omega^T K_\omega\left(\bar{\dot{q}}+b_{\bar{q}}K_\omega^{-}D\right)\left(\bar{e}\Phi^T+\bar{s}\Phi^{\perp T}\right)\mathcal{A}\leq 2\frac{c_2}{\epsilon}\rho_1(V)b_q e_\omega^T K_\omega e_\omega +$$

$$\epsilon c_2\left(1+\lambda_M(K_\omega^{-}D)\right)^2\rho_1 b_q\lambda_M(K_\omega)\left(e^2+s^2\right)|\mathcal{A}|^2,$$

$$c_2\rho_1(V)b_p\left|K_I^{-}e_\alpha\right||\mathcal{A}|^2\leq\frac{c_2 b_p}{2\epsilon_1}\left|K_I^{-}e_\alpha\right|^2|\mathcal{A}|^2+\frac{c_2 b_p\epsilon_1}{2}|\mathcal{A}|^2\rho_1^2(V),$$

$$2c_2\rho_1(V)e_\alpha^T K_I^{-}\dot{p}\left(\bar{e}\Phi^T+\bar{s}\Phi^{\perp T}\right)\mathcal{A}\leq 2\frac{c_2\lambda_M(K_\alpha^{-}K_I^{-})}{\epsilon}\rho_1(V)b_q e_\alpha^T K_\alpha K_I^{-}e_\alpha$$

$$+\epsilon c_2 b_q\rho_1(V)\left(e^2+s^2\right)|\mathcal{A}|^2,$$

$$\rho_1(V)\,|Le_\theta|\,\lambda_M(K_\omega)\left(1+\lambda_M(K_\omega^{-}D)\right)b_q\,|\mathcal{A}|^2\leq|\mathcal{A}|^2\,\lambda_M(K_\omega)b_q\frac{1}{2\epsilon_1}\,|Le_\theta|^2$$

$$+\frac{1}{2}\lambda_M(K_\omega)b_q\left(1+\lambda_M(K_\omega^{-}D)\right)^2|\mathcal{A}|^2\,\epsilon_1,$$

$$\rho_1(V)e_\theta^T LK_\omega\left(\bar{\dot{q}}+b_{\bar{q}}K_\omega^{-}D\right)\left(\bar{e}\phi^T+\bar{s}\phi^{\perp T}\right)\mathcal{A}\leq\lambda_M(K_\omega)\rho_1(V)\frac{b_q}{\epsilon}\left(e_\theta^T L^2 e_\theta\right)+$$

$$\frac{\lambda_M(K_\omega)}{2}\left(1+\lambda_M(K_\omega^{-}D)\right)^2\epsilon\left(e^2+s^2\right)b_q\,|\mathcal{A}|^2\,\rho_1(V),$$

$$\rho_1(V)e_\theta^T L\left(2c_2\bar{q}+\bar{\dot{q}}\right)Mv\leq\rho_1(V)\frac{b_q}{2\epsilon}\left|e_\theta^T L\right|^2+\frac{b_q\epsilon}{2}\rho_1(V)\left(2c_2+1\right)^2|M|^2\left|K_{dt}^{-}K_{pt}\right|v^T K_{dt}K_{pt}^{-}v,$$

$$\rho_1(V)e_\omega^T\left(c_2\bar{\dot{q}}+Lb_{\bar{q}}\right)Mv\leq\rho_1(V)\frac{c_2 b_q}{2\epsilon}e_\omega^T K_\omega e_\omega+\frac{b_q\epsilon}{2c_2}\rho_1(V)\lambda_M(K_\omega)\left(2c_2+\lambda_N(L)\right)^2$$

$$*\,|M|^2\left|K_{dt}^{-}K_{pt}\right|v^T K_{dt}K_{pt}^{-}v,$$

$$\rho_1(V)e_\alpha^T\left(2c_2K_I^{-}\bar{\ddot{q}}\right)Mv\leq\rho_1(V)\frac{c_2 b_q}{\epsilon}e_\alpha^T K_\alpha K_I^{-}e_\alpha+\epsilon\rho_1(V)\left(c_2\lambda_M(K_I^{-}K_\alpha^{-})b_q\right)$$

$$*\,|M|^2\left|K_{dt}^{-}K_{pt}\right|v^T K_{dt}K_{pt}^{-}v$$

$$c_5T^*\rho_1(V)v^T M^T b_{\bar{q}}^2 Mv\leq c_5T^*\rho_1(V)b_q^2\,|M|^2\left|K_{dt}K_{pt}^{-}\right|v^T K_{dt}K_{pt}^{-}v,$$

$$2c_5T^*\rho_1(V)e_\omega^T b_{\bar{q}}Mv\leq\frac{c_5T^*\rho_1(V)b_q}{\epsilon}e_\omega^T K_\omega e_\omega+\rho_1(V)\lambda_M(K_\omega^{-})\epsilon b_q c_5T^*\,|M|^2$$

$$*\left|K_{dt}^{-}K_{pt}\right|v^T K_{dt}K_{pt}^{-}v,$$

taking $\epsilon_1=c_7$, $\epsilon=c_6$ and using the inequalities

$$Z\geq\frac{c_2}{2}\left(e_\theta^T Le_\theta+e_\omega^T e_\omega+e_\alpha^T K_I^{-}e_\alpha\right),$$

$$|\mathcal{A}|^2 \leq N \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds$$

we obtain

$$\begin{aligned}\frac{d}{dt}\left(\rho_1(V) Z(X_r)\right) \leq & -\frac{\rho_1(V)}{4} c_2 \left[e_\alpha^T K_I^- K_\alpha e_\alpha + e_\omega^T K_\omega e_\omega\right] - \frac{\rho_1(V)}{4} e_\theta^T L^2 e_\theta \\ & + \frac{1}{2}\rho_2(V) v^t K_{dt} K_{pt}^- v + \frac{1}{2}\rho_2(V) \sum_{j=1}^{N} \sum_{i=1}^{N} a_{ij}^2 \int_{t-T_{ij}}^{t} \dot{z}_i(s)^T ds \int_{t-T_{ij}}^{t} \dot{z}_i(s) ds.\end{aligned} \tag{B.138}$$

Finally, we obtain

$$\begin{aligned}\dot{\Gamma} \leq & -\frac{\rho_1(V)}{8} \left[c_2 e_\alpha^T K_I^- K_\alpha e_\alpha + c_2 e_\omega^T K_\omega e_\omega + e_\theta^T L^2 e_\theta\right] \\ & - \frac{1}{4}\gamma(V) v^\top K_{dt} K_{pt}^{-1} v - \frac{1}{16}\alpha(V) e^\top K_{pt} e - \frac{\mu}{4T} V^3.\end{aligned} \tag{B.139}$$

Which completes the proof.

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ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Title : Stability and Stabilization of Networked Systems.

Keywords : Strict Lyapunov functions, nonholonomic mobile robots, consensus, leader-follower, multi-agent systems, Persistency of excitation.

Abstract : In this thesis, we propose a Lyapunov based approaches to address some distributed solutions to multi-agent coordination problems, more precisely, we consider a group of agents modeled as nonholonomic mobile robots, we provide a distributed control laws in order to solve the leader-follower and the leaderless consensus problems under different assumptions on the communication graph topology and on the leader's trajectories. The originality of this work relies on the closed-loop analysis approach, that is, it consists on transforming the last two problems into a global stabilization problem of an invariant set.
The stability analysis is mainly based on the construction of strict Lyapunov functions and strict Lyapunov-Krasovskii functionals for a classes of nonlinear time-varying and/or delayed systems.

Titre : Stabilité et Stabilisation des Systèmes en Réseaux.

Mots clefs : automatique, méthodes de Lyapunov, véhicules autonomes, robotique mobile, synchronisation.

Résumé : Dans cette thèse, des méthodes dites de Lyapunov sont proposées afin de résoudre des problèmes liés à la coordination distribuée des systèmes multiagent, plus précisément, un groupe de systèmes (agents) non-linéaires formés de robots mobiles non-holonomes est considéré. Pour ce groupe de systèmes, des lois de commande distribuée sont proposées dans le but de résoudre des problèmes de type leader-suiveur en formation et aussi des problèmes de type formation sans-leader par une approche de consensus, sous différentes hypothèses sur le graphe de communication et surtout sur les vitesses du leader. L'originalité de ce travail est dans l'approche proposée pour l'étude de stabilité de la boucle fermée, cette approche consiste à transformer les deux derniers problèmes en des problèmes de stabilisation globale asymptotique d'un ensemble invariant. L'analyse de stabilité est basée sur la construction de fonction de Lyapunov et de fonction de Lyapunov-Karasovskii strictes pour des classes de systèmes non-linéaires variant dans le temps présentant des retards bornés et variant dans le temps.

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