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ON MULTIFRACTALITY, SCHWARZIAN DERIVATIVE AND ASYMPTOTIC VARIANCE OF WHOLE-PLANE SLE

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INTRODUCTION

In 1916 the German mathematician Ludwig Bieberbach firstly stated his famous conjecture, so-called Bieberbach's conjecture, which provide a necessary condition on Taylor coefficients of univalent functions on the unit disk. The conjecture says that: If f belongs to the class $\mathcal{S}$, i.e, a univalent function on the unit disk $\mathbb{D}$ and has the Taylor series $\sum_{n\geq 0} a_n z^n$ with the normalization $a_0 = 0, a_1 = 1$ then

$$|a_n| \leq n.$$

Bieberbach proved the conjecture in the case of $n = 2$ in the same year. This conjecture was then partly proved by many mathematicians ($n = 3, 4, 5$) before the French mathematician Louis de Branges finished its journey with a complete proof in 1985. Today Bieberbach's conjecture is a classical theorem of complexe analysis. For approximately 70 years of Bieberbach's theorem's history, beside its intrinsic value, the efforts to prove it have contributed important results and new theories to mathematics. One of these contributions is the Loewner's theory introduced by Karl Loewner in 1923 to solve Bieberbach's probleme for $n = 3$. Indeed, de Branges has used Loewner's theory as a crucial argument in his famous proof. The original idea of Loewner is to imbed the univalent function into a special flow governed by a nice vector field for which good estimates of the coefficients are available and then recover estimates for the initial function.

Loewner introduced a representation of the subset of $\mathcal{S}$ which consists of functions whose image are slit domains in $\mathbb{C}$. This subset is dense in $\mathcal{S}$ in the topology of locally uniform convergence. He showed that a function in this subset can be embedded in to a family of functions $(f_t(z))_{t\geq 0} \subset \mathcal{S}$

$$\begin{aligned} f_t(z) &= e^t z + \sum_{n=2}^{\infty} a_n z^n, \\ f_0(z) &= f(z). \end{aligned}$$

Moreover, the family $(f_t(z))_{t\geq 0}$ satisfies the equation

$$\frac{\partial}{\partial t} f_t(z) = z \frac{\partial}{\partial z} f_t(z) \frac{\lambda(t) + z}{\lambda(t) - z}, \tag{1}$$

where $\lambda : [0, \infty) \to \partial\mathbb{D}$ is a continuous function on the unit circle. With the sole information that $|\lambda(t)| = 1, \forall t$, he could prove that $|a_3| \leq 3|a_1|$.

Interestingly, the derivation of Eq. (1) above is only half of the story. There is indeed a converse: given any continuous function $\lambda : [0, +\infty[\to \mathbb{C}$ with $|\lambda(t)| = 1$

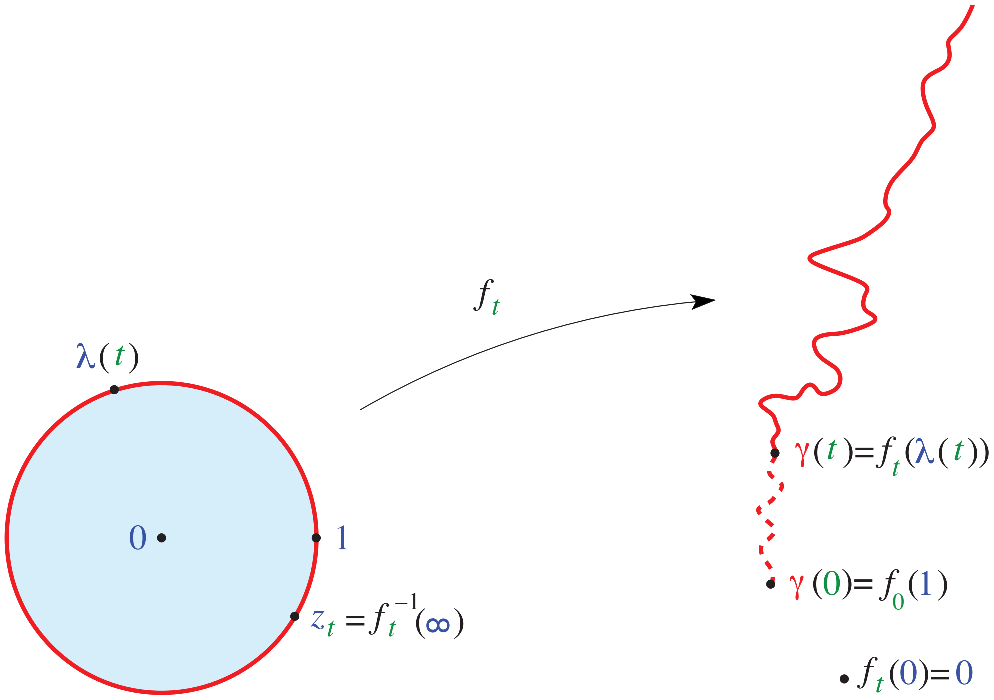


Figure 1 – *Loewner map* $z \mapsto f_t(z)$ *from* $\mathbb{D}$ *to the slit domain* $\Omega_t = \mathbb{C}\backslash\gamma([t,\infty))$ *(here slit by a single curve* $\gamma([t,\infty))$ *for* $\mathrm{SLE}_{\kappa\le 4}$*). One has* $f_t(0) = 0, \forall t \ge 0$*. At* $t = 0$*, the driving function* $\lambda(0) = 1$*, so that the image of* $z = 1$ *is at the tip* $\gamma(0) = f_0(1)$ *of the curve. Source: [5]*

for $t \ge 0$, the the Loewner equation (1), supplemented by the boundary ("initial") condition, $\lim_{t\to+\infty} f_t(e^{-t}z) = z$, has a solution $(t,z) \mapsto f_t(z)$, such that $(f_t(z))_{t\ge 0}$ is a chain of Riemann maps onto simply connected domains (Ω_t) that are increasing with t.

In 1999, in his work on the planar uniform spanning tree (UST) and the planar loop-erased random walk (LERW) probabilistic processes, Schramm [15] introduced into the Loewner equation the *random* driving function,

$$\lambda(t) := e^{i\sqrt{\kappa}B_t}, \tag{2}$$

where B_t is standard one dimensional Brownian motion and κ a non-negative parameter, thereby making Eq. (1) a stochastic PDE, and creating the celebrated *Schramm-Loewner Evolution* SLE_κ. Schramm-Loewner Evolution were proved to be the scaling limit of some stochastic processes in plane and conjectured to describe the scaling limit of various other processes. Because of its importance, *Schramm-Loewner Evolution* have become one of the most interested objects in statistical mechanic. A generalization of *Schramm-Loewner Evolution*, so-called *Lévy-Loewner Evolution*, is also concurrently investigated. This is a stochastic growth process defined to be a solution of the Loewner equation (1) with the driving function $\lambda(t) = e^{iL_t}$, where L_t is a Lévy process.

The above introduction is aimed at giving an overview about *Schramm-Loewner*

Evolution which is the object of research in this thesis. In the rest of this introduction section, we present our particular problems as well as delineate the contents and structures of the chapters of the thesis. The thesis includes four chapters that respectively deal with the generalization of moments of SLE maps, the generalized integral mean spectrum of SLE and the generalized universal spectrum, the Schwarzian derivative of SLE, McMullen's question on the relation between Minkovski dimension and asymptotic variance for SLE map.

The starting motivation of Chapter 2 is to revisite the Bieberbach's conjecture, namely, the Milin's conjecture in the framework of Schramm-Loewner Evolution. For this purpose, we study the logarithmic coefficients of whole-plane SLE_κ and the generalizations thereof, which are obtained by introducing *generalized moments* for the whole-plane SLE map, $\mathbb{E}(|f'(z)|^p/|f(z)|^q)$, for $(p,q) \in \mathbb{R}^2$. We generalize the properties obtained in Ref. [4], [5], [12], [13] to the mixed moments, along integrability curves in the moment plane $(p,q) \in \mathbb{R}^2$ depending continuously on κ, by extending the so-called Beliaev–Smirnov equation to this case. The generalization of this integrability property to the m-fold transform of f is also given.

In chapter 3, with results obtained from chapter 2 we proceed an analysis of the generalized spectrum of SLE maps. We define a generalized integral means spectrum, $\beta(p,q;\kappa)$, corresponding to the singular behavior of the mixed moments above. The average generalized spectrum of whole-plane SLE takes four possible forms, separated by five phase transition lines in the moment plane $\mathbb{R}^2$. The manifold identities so obtained in the (p,q)-plane encompass all previous results. Rigorous and non-rigorous analyises will be presented to check this manifold of spectrum.

Chapter 4 deals with the expectation of Schwarzian derivative of SLE map. Let f be a holomorphic function, its Schwarzian derivative is defined by

$$(Sf)(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Schwarzian derivative plays an important role in the theory of univalent function and conformal mapping. It is known that $|(Sf)(z)| \leq \frac{6}{(1-|z|^2)^2}$ whenever f is univalent in D. As a consequence of Nehari's criterion of univalence, we also have a sufficient condition on Schwarzian derivative for a meromorphic function in $\mathbb{D}$ to be univalent, that is

$$|(Sf)(z)| \leq \frac{2}{(1-|z|^2)^2}.$$

In their paper [4] (see also [5]), the authors posed for the first time the problem of considering Schwarzian derivative of SLE. They calculated $\mathbb{E}(|(Sf)(0)|^2)$ for f being the whole-plane SLE map and showed that $\mathbb{E}(|(Sf)(0)|^2) \leq 4$ is reached for $\kappa \geq 8$. They then asked for the interpretative meaning of this fact. In chapter 3 we give an approach to the problem of considering expectations of Schwarzian derivative of SLE and its second moment throughout differential equations. We start with a generalization of the main results in the first chapter to a more general moment function depending on parameters. The point is that for some particular restrictions of this generalized moment function we have quantities that are related to Schwarzian derivatie and its second moment by limitations. We are thus able to perform equations satisfied by these quantities from those of the moment function and obtain an exact formula of expectation of Schwarzian derivative of SLE.

The next problem considered in this thesis is the question asked by McMullen on the the relation between Minkovski dimension and asymptotic variance concerning a family of conformal mappings. In 2008, when working on the Weil-Peterson metric, McMullen [14] used the thermodynamic formalism to connect dynamical features of some special holomorphic families of conformal maps

$$\varphi_t : \mathbb{D}^* \to \mathbb{C}, \quad \varphi_0(z) = z, \quad \text{where } \mathbb{D}^* = \{z : |z| > 1\}.$$

One of those is the relation between the infinitesimal growth of the Hausdorff dimension of the Jordan curves $\varphi_t(\mathbb{S}_1)$ and the asymptotic variance of $v = \frac{d\varphi_t}{dt}\Big|_{t=0}$

$$2\frac{d^2}{dt^2}\Big|_{t=0} \text{H.dim}\varphi_t(\mathbb{S}_1) = \sigma^2(v'). \tag{3}$$

Here σ^2 is the McMullen's asymptotic variance of a Bloch function given by

$$\sigma^2(b) = \limsup_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi} |b(re^{i\theta})|d\theta, \quad .$$

McMullen also asked: Under what conditions of holomorphic families of conformal mappings the identity (3) is true?

To answer this question for inside of the disc, M.Zinsmeister and THN.Le [9] considered the mappings

$$\phi_t(z) = \int_0^z e^{tb(u)}du \quad b \in \mathcal{B} : \text{set of Bloch functions.} \tag{4}$$

and used a probability argument to describe a relatively large class of functions in $\mathcal{B}$ for which (φ_t) defined by (4) satisfies (3), where the Hausdorff dimension is replaced by the Minkovski dimension. In fact, they dealt with the following identity

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{M. dim } \phi_t(\mathbb{S}^1) = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=1} |b(z)|^2|dz|, \tag{5}$$

and the equivalent equality

$$\lim_{p\to 0} \frac{2\beta(p,\phi)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=1} |b(z)|^2|dz|, \tag{6}$$

where $\beta(p,\phi)$ is the integral mean spectrum of the function ϕ defined by $\phi'(z) = \exp(b(z))$. In the same article they also constructed a Bloch function b and showed that (5) doesn't holds for the family (φ_t) (4) corresponding to this b.

In chapter 4 we show that (6) is true in a sense of expectation for SLE_2. Indeed, we obtain the explicit expression of $\mathbb{E}(|\log f'(z)|^2)$ where $f = f_0$ is the interior whole-plane SLE_2 map at time 0 and the equality

$$\lim_{p\to 0} \frac{2\bar{\beta}(p)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|}\int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2)d\theta, \tag{7}$$

where $\bar{\beta}(p)$ is the average integral mean spectrum of the interior whole-plane SLE_2 map. To do that we firstly use the martingale argument to get an equation satisfied by t$\mathbb{E}(|\log f'(z)|^2)$. Note that this equation is performed for a arbitrary κ. We then solve the equation in the case of $\kappa = 2$ by inductive method applied for Taylor coefficients of the solution.

The results presented in this thesis are based on joint works of the author with others. Namely, Chap 2 and Chapter 3 are from joint works with B.Duplantier, TB.Le and M.Zinsmeister. Chapter 4 is from a joint work with M.Zinsmeister. Chapter 5 is from a joint work with TB.Le and M.Zinsmeister.

Chapter 1

STOCHASTIC LOEWNER EVOLUTION

In this chapter, we will provide mathematical backgrounds and introduce essential notions of Loewner's theory. In particular, we focus on various variants of *Loewner equation* which encode properties of growing (or shrinking) domains generated by Jordan arcs. We then introduce the setting of O. Schramm to establish the *Schamm–Loewner evolution*, a stochastic version of *Loewner evolution.* Some properties of Stochastic–Loewner evolutions which are necessary for the next chapters will also be given.

1.1 Simply connected domains

An arc in a metric space X is a continuous mapping $\gamma : [a, b] \subset R \to X$. If $\gamma(a) = \gamma(b)$ then the arc is called *closed.* Two arcs γ_1, γ_2 defined on the same interval $[a, b]$ are said to be *homotopic* if there exists $\Gamma : [a, b] \times [0, 1] \to X$ continuous such that

$$\forall s \in [a, b], \Gamma(s, 0) = \gamma_1, \Gamma(s, 1) = \gamma_2.$$

Definition 1.1.1. *The space X is called simply connected if it is connected and every closed arc $\gamma : [a, b] \to X$ is homotopic to a constant arc $\gamma_0 : [a, b] \to \gamma(a)$.*

Intuitively, within that space, every closed arc cannot continuously degenerate to a single point. When X is a planar domain we have the following equivalent characterizations of simply connected domains:

Theorem 1.1.1. *For a connected open subset Ω of $\mathbb{C}$ the followings are equivalent:*

(i) Ω is simply connected,

(ii) $\overline{\mathbb{C}} \setminus \Omega$ is connected,

(iii) For any closed arc γ whose image lies in Ω and any $z \notin \Omega$, $Ind(z, \gamma) = 0$.

Here, $Ind(z, \gamma)$ is the *index of z with respect to γ* which stands for the variation of the argument (measured in number of turns) of $\gamma(t) - z$ along $[a, b]$. When γ is piecewise C^1 this quantity is also equal to

$$\frac{1}{i2\pi} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} ds = \frac{1}{i2\pi} \int_\gamma \frac{1}{\zeta - z} d\zeta.$$

Theorem 1.1.2. *(Riemann). Let Ω is a simply connected proper subdomain of $\mathbb{C}$ and $w \in \Omega$. Then there exists a unique biholomorphic map $g : \Omega \to \mathbb{D}$ such that $g(w) = 0, g'(w) > 0$.*

An equivalent statement is that there exists a unique holomorphic bijection $f : \mathbb{D} \to \Omega$ sending 0 to $z_0 \in \Omega$ and $f'(0) > 0$. This specific map is called the *Riemann map* of Ω for z_0.

We have also a slight different version of Riemann mapping theorem which states for domains containing ∞.

Definition 1.1.2. *A set $K \in \mathbb{C}$ is called CCF-set or CCF-compact if it is compact, connected with connected complement, containing 0 but not reduced to this point.*

Definition 1.1.3. *The complement in $\overline{\mathbb{C}}$ of a CCF-set is called a CCF-domain.*

In order to state a Riemann mapping theorem for these domains we define the holomorphicity at ∞ for a mapping fixing ∞, using the structure at ∞.

Definition 1.1.4. *If $\Omega = \overline{\mathbb{C}} \backslash K$, where K is a $CCF-compact$, and $f : \Omega \to \overline{\mathbb{C}} \backslash \{0\}$ is a mapping fixing ∞, we say that f is holomorphihc at ∞ if the mapping*

$$\tilde{f}(z) = \frac{1}{f(1/z)}$$

is holomorphic at 0.

The limit $\lim_{z\to\infty} \frac{f(z)}{z}$ exists and equals to $\frac{1}{\tilde{f}'(0)}$ whenever f is holomorphihc at ∞. We denote this limit by $f'(\infty)$.

By using the reference CCF-domain $\Delta = \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$, we now introduce another version of Riemann mapping theorem:

Theorem 1.1.3. *If K is a CCF-compact then there exists a unique holomorphic bijection $f : \Delta \to \Omega$ such that $f(\infty) = \infty$ and $f'(\infty) > 0$.*

We also say that f is the *Riemann map* of Ω. The quantity $f'(\infty)$ is called the *logarithmic capacity* of K and is denoted by *cap*(K). The following property justifies the denomination:

Proposition 1.1.1. *The quantity cap(K) is increasing in the sense that if $K_1 \subsetneq K_2$ are two distint CCF-compact then cap$(K_1) <$ cap(K_2).*

1.2 Caratheodory convergence theorem

Definition 1.2.1. *Let $(U_n)_{n\geq 0}$ is a sequence of a open sets in $\mathbb{C}$ containing 0. Let V_n is the connected component of the interior of $\cap_{k\geq n} U_k$ containing 0. The kernel of the sequence is defined to be the union of the $V_n's$, provided it is non-empty; otherwise it is defined to be $\{0\}$.*

Since the definition, the kernel is either a connected open set containing 0 or the one point set $\{0\}$.

For the case of domain containing ∞, the definition of the kernel is similar.

Definition 1.2.2. *The kernel of a sequence* $(\Omega_n)_{n\geq 0}$ *of CCF-domains is the union of all domains* $U \subset \overline{\mathbb{C}}$ *such that* $\infty \in U$ *and* $U \subset \Omega_n$ *for* n *large enough. If no such domain exists we say that the kernel is* ∞*.*

Definition 1.2.3. *A sequence of open sets in* $\mathbb{C}$ *(or CCF-domains) is said to converge to a kernel if each subsequence has the same kernel.*

We now state the Caratheodory convergence theorem:

Theorem 1.2.1. *(Caratheodory convergence theorem). Let* (f_n) *be a sequence of holomorphic univalent functions on the unit disk* $\mathbb{D}$*, normalized so that* $f_n(0) = 0$ *and* $f_n'(0) > 0$*. Then* f_n *converges uniformly on compacta in* $\mathbb{D}$ *to a function* f *if and only if* $U_n = f_n(\mathbb{D})$ *converges to its kernel and this kernel is not* $\mathbb{C}$*.*
If the kernel is $\{0\}$*, then* $f = 0$*.*
Otherwise the kernel is a connected open set U*,* f *is univalent on* $\mathbb{D}$ *and* $f(\mathbb{D}) = U$*.*

There is another version of this theorem for the case of domains containing ∞:

Theorem 1.2.2. *Let* (Ω_n) *be a sequence of CCF-domains and* (f_n) *the corresponding sequence of Riemann maps. The the sequence* (f_n) *is uniformly convergent on compact sunsets of* Δ *if and only if* Ω_n *converges in sense of Caratheodory to a kernel distinct from* $\overline{\mathbb{C}} \setminus \{0\}$*. If* Ω_n *converges and* Ω *denotes its kernel then*

(i) If $\Omega = \infty$ *then* $f_n \to \infty$ *uniformly on compact subsets of* Δ*.*

(ii) If $\Omega = \overline{\mathbb{C}} \setminus \{0\}$ *then* $f_n \to 0$ *uniformly on compact subsets of* $\Delta \setminus \{\infty\}$*.*

(iii) Otherwise, f_n *converges to* f*, the Riemann mapping of* Ω*.*

1.3 Loewner equation

1.3.1 Radial Loewner equation

In this section we will introduce two versions of a Loewner evolution variant, so-called *radial Loewner evolution.* The first concerns biholomorphic mappings from the unit disk onto its subdomains whereas the second concerns biholomorphic mappings from the outside of the unit disk onto its subdomains. We prefer firstly deal with the exterior case because it is the historical one considered by Loewner.

1.3.1.1 Exterior radial evolution

The central objects of Loewner's theory are special families of univalent functions, called *Loewner chain.* The definition of Loewner chains is based on the notion of *subordination.*

Definition 1.3.1. *Let* $f, g : \Delta \to \overline{\mathbb{C}}$ *be two holomorphic functions with* $f(\infty) = \infty, g(\infty) = \infty$*. We say* f *is subordinate to* g *(denoted* $f \prec g$*) if and only if there exists* $\varphi : \Delta \to \Delta$ *holomorphic and fixing* ∞ *such that*

$$f(z) = g(\varphi(z))$$

for all $z \in \Delta$*.*

We now define a Loewner chain:

Definition 1.3.2. *The family* $(f_t)_{t\geq 0}$ *of holomorphic univalent mappings from* Δ *to* $\overline{\mathbb{C}}$ *is called a Loewner chain if:*

(i) $f_t(z) = e^t z + b_0(t) + \frac{b_1(t)}{z} + ...,$

(ii) $f_t \prec f_s$ *if* $0 \leq s \leq t.$

Let $\gamma : [0, +\infty) \to \Delta$ is a Jordan arc in Δ which is starting from a point $\gamma(0)$ on the unit circle and growing to ∞. We denote $K_t := \overline{\mathbb{D}} \cup \gamma([0,t])$ and $\Omega_t := \overline{\mathbb{C}} \setminus K_t$. $(\Omega_t)_{t\geq 0}$ is then a decreasing family of CCF-subdomains of Δ. The Riemann mapping theorem implies that for each CCF-domain Ω_t, there is a biholomorphic map $f_t : \Delta \to \Omega_t$ such that $f_t(\infty) = \infty$ and $f_t'(\infty) > 0$. We may write $f_t(z) = c(t)z + b_0(t) + \frac{b_1(t)}{z} + ...,\ \ c(t) > 0$ where $c(t)$ is the logarithmic capacity of K_t. Because $(K_t)_{t\geq 0}$ is a strictly increasing family of CCF-sets, the function $c(t)$ is strictly decreasing in t.

It is also easy to see that the family $(\Omega_t)_{t\geq 0}$ is continuous in the sense of Caratheodory convergence. By mean of Caratheodory convergence theorem, the family $(f_t)_{t\geq 0}$ is continuous in t for the topology of uniform convergence on compact sets in Δ. In particular, the function $t \mapsto c(t)$ is continuous.

We now make the following assumptions: $\lim_{t\to+\infty} c(t) = +\infty$ and $c(0) = 1$. Frequently we will assume that $\Omega_0 = \Delta$. With these conditions, one may perform a time-change and assume that $c(t) = e^t$.
We also have $f_t \prec f_s$ for $0 \leq s < t$ with the holomorphic function φ in Definition 1.3.1 is now $f_s^{-1} \circ f_t$. The family $(f_t)_{t\geq 0}$ is thus a Loewner chain.

Loewner proved that a Loewner chain is not only continuous but also absolutely continuous in time, therefore almost everywhere differentiable. He also proved that a Loewner chain can be described by a partial differential equation, in particular,

Theorem 1.3.1. *(Loewner 1923). Let* $f_t(z)$ *defined as above, then there exists a continuous function* $\lambda : [0, +\infty) \to \partial\mathbb{D}$ *such that almost everywhere in* t *we have for all* $z \in \Delta$,

$$\frac{\partial f_t(z)}{\partial t} = -z \frac{\partial f_t(z)}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}, \quad f_0(z) = z, \tag{1.1}$$

In fact, $\lambda(t) = f_t^{-1}(\gamma(t))$ for $t \in [0, +\infty)$. It is noticed that one can extend f_t^{-1} by continuity to $\gamma(t)$, so that $f_t^{-1}(\gamma(t))$ is well-defined.
The equation (1.1) is called *Loewner equation*. We also say that $\lambda(t)$ is the *driving function* of the Loewner chain $(f_t)_{t\geq 0}$.

Let $g_t : \Omega_t \to \Delta$ is the inverse function of f_t, then $g_t(z)$ is the unique solution of the equation

$$\frac{\partial g_t(z)}{\partial t} = g_t(z) \frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}, \quad g_0(z) = z. \tag{1.2}$$

The Loewner's theory contains another important fact on the Loewner equation: Theorem 1.3.1 has a converse. If $\lambda : [0, +\infty) \to \partial\mathbb{D}$ is a continuous function then there is a unique Loewner chain $(f_t)_{t\geq 0}$ such that it is a solution of Eq. (1.1). This Loewner chain corresponds to a decreasing family of CCF-domains $\Omega_t = \overline{\mathbb{C}} \setminus K_t$ in

Δ. However, the CCF-sets K_t, in this case, do not have to be $\overline{\mathbb{D}} \cup \gamma([0,t])$ for some Jordan arc γ.

We now introduce a notion which is the case of the stochastic version of Loewner chains considered in the next chapters:

Definition 1.3.3. *Let $\gamma : [0,+\infty) \to \Delta$ be a continuous curve such that $|\gamma(0)| = 1$. We say that the process $(\Omega)_{t\geq 0}$ is generated by γ if for $t \geq 0$ we have that Ω_t is the unbounded component of the complement of $\overline{\mathbb{D}} \cup \gamma([0,t])$.*

1.3.1.2 Interior radial Loewner evolution

Definition 1.3.4. *The unique solution of the following differential equation is called an interior radial Loewner evolution*

$$\frac{\partial f_t(z)}{\partial t} = -z\frac{\partial f_t(z)}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z}, \quad f_0(z) = z, \tag{1.3}$$

where $z \in \mathbb{D}$ and $\lambda : [0,+\infty) \to \partial\mathbb{D}$ is a continuous function.

The family $(f_t)_{t\geq 0}$ includes the Riemann mappings of domains Ω_t such that $(\Omega_t)_{t\geq 0}$ is a continuously decreasing family of subdomains containing 0 of the unit disk. Every function $f_t(z)$ may be written as

$$f_t(z) = e^{-t}z + a_2(t)z^2 + a_3(t)z^3 \ldots.$$

Let $g_t : \Omega_t \to \mathbb{D}$ is the inverse function of f_t, then $g_t(z)$ is the unique solution of the equation

$$\frac{\partial g_t(z)}{\partial t} = g_t(z)\frac{\lambda(t)+g_t(z)}{\lambda(t)-g_t(z)}, \quad g_0(z) = z. \tag{1.4}$$

We have also the notion of processes generated by a curve

Definition 1.3.5. *Let $\gamma : [0,+\infty) \to \mathbb{D}$ be a continuous curve such that $|\gamma(0)| = 1$. We say that the process $(\Omega)_{t\geq 0}$ is generated by γ if for $t \geq 0$ we have that Ω_t is the component containing 0 of $\mathbb{D} \setminus \gamma([0,t])$.*

1.3.2 Whole-plane Loewner equation

We introduce another variant of Loewner evolutions. There are also two versions corresponding to the inside and the outside of the unit disk.

1.3.2.1 Interior whole-plane Loewner evolution

Definition 1.3.6. *Let f and g be holomorphic univalent functions on the unit disk $\mathbb{D}$ with $f(0) = 0, g(0) = 0$. We say f is subordinate to g (denoted $f \prec g$) if and only if there is a univalent mapping $\varphi : \mathbb{D} \to \mathbb{D}$ fixing 0 such that*

$$f(z) = g(\varphi(z))$$

for all $|z| < 1$.

Definition 1.3.7. *The family $(f_t)_{t\geq 0}$ of holomorphic univalent mappings from $\mathbb{D}$ to $\mathbb{C}$ is called a Loewner chain if:*

(i) $f_t(z) = e^t z + a_2(t)z^2 + a_3(t)z^3 + \ldots,$

(ii) $f_s \prec f_t$ *if* $0 \leq s \leq t$.

Let $\gamma : [0, +\infty) \to \mathbb{C}$ is a Jordan arc joining a point $\gamma(0)$ on $\mathbb{C}$ to ∞ and not containing 0. If we denote by Ω_t the complement of $\gamma([t, +\infty))$ then $(\Omega_t)_{t\geq 0}$ is a increasing family of domains. The Riemann mapping theorem implies that for each domain Ω_t, there is a biholomorphic map $f_t : \mathbb{D} \to \Omega_t$ such that $f_t(0) = 0$ and $f_t'(0) > 0$. We may write $f_t(z) = c(t)z + a_2(t)z^2 + a_3(t)z^3 \ldots, c(t) > 0$.

Because the family $(\Omega_t)_{t\geq 0}$ is continuous in the sense of Caratheodory convergence, the family $(f_t)_{t\geq 0}$ is continuous in t for the topology of uniform convergence on compact subsets of $\mathbb{D}$. As a consequence, we have the continuity in t of $c(t)$. Moreover, for $0 \leq s < t$, by considering the function $f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}$ and using the *Swchwarz's lemma*, one may prove that $c(s) < c(t)$.
We now make the following assumptions: $c(0) = 1$ and $\lim_{t\to+\infty} c(t) = +\infty$. By a change of variable if necessary, we may assume that $c(t) = e^t$. This setting together with the fact that $f_s \prec f_t$ for $0 \leq s < t$ implies that $(f_t)_{t\geq 0}$ is a Loewner chain.

As in the radial case, the Loewner's theory show that $f_t(z)$ is absolutely continuous in t, in particular almost everywhere differentiable, and can be described by a differential equation.

Theorem 1.3.2. *Let $f_t(z)$ defined as above, then there exists a continuous function $\lambda : [0, +\infty) \to \partial\mathbb{D}$ such that almost everywhere in t we have for all $z \in \mathbb{D}$,*

$$\frac{\partial f_t(z)}{\partial t} = z \frac{\partial f_t(z)}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}. \tag{1.5}$$

The continuous function $\lambda(t)$ is determined by $\lambda(t) = f_t^{-1}(\gamma(t))$, the continuity extension of f_t^{-1} to the point $\gamma(t)$.
The equation (1.5) is called *Loewner equation.* We also say that $\lambda(t)$ is the *driving function* of the Loewner chain $(f_t)_{t\geq 0}$.

Inversely, let $\lambda : [0, +\infty) \to \partial\mathbb{D}$ be a continuous function, then there is a unique Loewner chain $(f_t)_{t\geq 0}$ which satisfies the equation (1.5). Moreover $(f_t)_{t\geq 0}$ includes the Riemann mappings of a increasing family $(\Omega_t)_{t\geq 0}$ of domains in $\mathbb{C}$ which continuously grows to the whole-plane. We also have the notion of a interior whole-plane Loewner evolution generated by a curve:

Definition 1.3.8. *Let $\gamma : [0, +\infty) \to \mathbb{C}$ be a continuous curve joining a point $\gamma(0)$ in $\mathbb{C}$ to ∞ and not containing 0. We say that the process $(\Omega)_{t\geq 0}$ is generated by γ if for $t \geq 0$ we have that Ω_t is the unbounded component of $\mathbb{C} \setminus \gamma([t, +\infty))$.*

1.3.2.2 Exterior whole-plane evolution

Definition 1.3.9. *The unique solution of the following differential equation is called an exterior whole-plane Loewner evolution*

$$\frac{\partial f_t(z)}{\partial t} = -z \frac{\partial f_t(z)}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}, \quad z \in \Delta, \tag{1.6}$$

where $\lambda : [0, +\infty) \to \partial\mathbb{D}$ *is a continuous function.*

The family $(f_t)_{t\geq 0}$ includes the Riemann mappings of CCF-domains Ω_t such that $(\Omega_t)_{t\geq 0}$ continuously grows to $\overline{\mathbb{C}} \setminus \{0\}$. Every function $f_t(z)$ may be written as

$$f_t(z) = e^t z + b_0(t) + \frac{b_1(t)}{z} + \dots.$$

The notion of processes generated by a curve in this case is defined in the frequent manner, i.e, domains of the evolution are the component containing ∞ of the complement of a Jordan arc.

1.3.3 Chordal Loewner equation

Let $\mathbb{H}$ be the upper half-plane $\{x + iy : y > 0\}$. Let $\gamma : [0, +\infty) \to \mathbb{H}$ is a Jordan arc in $\mathbb{H}$ which is starting from the origin and growing to ∞. We denote by Ω_t the complement of $\gamma[0, t]$ in $\mathbb{H}$. $(\Omega_t)_{t\geq 0}$ is then a decreasing family of subdomains of $\mathbb{H}$. With helps of Riemann mapping theorem and *Schwarz reflection principle*, one can prove that there exists a unique biholomorphic map $g_t : \Omega_t \to \mathbb{H}$ such that $\lim_{z\to\infty} g_t(z) - z = 0$. In this case, we say that g_t satisfies the *hydrodynamic normalization*. These normalized functions play a role as Riemann mappings in the radial case. Moreover, one can also write $g_t(z) = z + \frac{b_1(t)}{z} + \dots, \quad b_1(t) > 0$.
The Schwarz reflection principle extends each g_t to a Riemann mapping of a neighborhood of ∞. By mean of the Caratheodory convergence theorem, the family of the functions $f_t = g_t^{-1}$ is continuous for topology of uniform convergence on compact sets. As a consequence, $b_1(t)$ is continuous. Since the theory of half-plane capacity, $b_1(t)$ is also increasing.
We now make an assumption: $\lim_{t\to+\infty} b_1(t) = +\infty$. Changing variable if necessary, we may assume that $b_1(t) = 2t$.

The Loewner's theory shows that $g_t(z)$ is described by an ordinary differential equation

Theorem 1.3.3. *Let* $g_t(z)$ *defined as above, then there exists a continuous function* $\lambda : [0, +\infty) \to \mathbb{R}$ *such that almost everywhere in* t *we have for all* $z \in \Omega_t$,

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z. \tag{1.7}$$

As in the radial case, there also exists a converse: If $\lambda : [0, +\infty) \to \mathbb{R}$ is a continuous function, then the equation (1.7) has a unique solution $g_t(z)$. This solution is such that $(g_t)_{t\geq 0}$ includes the hydrodynamic normalized functions of a decreasing family of domains in $\mathbb{H}$.
The notion of processes generated by a curve is also defined in a similar way as previous cases.

1.4 Stochastic Loewner evolution

In 1999, in his work on the planar uniform spanning tree (UST) and the planar loop-erased random walk (LERW) probabilistic processes, O.Schramm [15] introduced into the Loewner equation a *random* driving function to create the celebrated

Schramm-Loewner Evolution SLE_κ. In the following definition, the stochastic setting is carried out for the radial Loewner equation in the unit disc.

Definition 1.4.1. *The radial Shramm–Loewner evolution or radial Stochastic–Loewner evolution(of parameter κ), denoted SLE_κ, in the unit disc is the solution of the stochastic PDE:*

$$\frac{\partial f_t(z)}{\partial t} = -z\frac{\partial f_t(z)}{\partial z}\frac{\lambda(t)+z}{\lambda(t)-z}, \quad f_0(z) = z, \tag{1.8}$$

where $\lambda(t) := e^{i\sqrt{\kappa}B_t}$ with B_t is the standard one dimensional Brownian motion and κ is a non-negative parameter.

In the above definition, the Brownian motion is a stochastic process characterized by the three fundamental properties:

i. Stationarity: if $0 \le s \le t$ then $B_t - B_s$ has the same law as B_{t-s}.

ii. Markov property: if $0 \le s \le t$ then $B_t - B_s$ is independent of B_{t-s}.

iii. Gaussianity: B_t has a normal distribution with mean 0 and variance t.

One can substitute the same random driving function $\lambda(t) := e^{i\sqrt{\kappa}B_t}$ into Eq. (1.1), Eq. (1.5), Eq. (1.6) to respectively define the exterior radial version as well as the versions of whole-plane variant of Schramm–Loewner evolution. In the chordal case, the corresponding driving function is $\lambda(t) = \sqrt{\kappa}B_t$.

Rhode and Schramm proved that the Schamm-Loewner evolution processes are almost surely generated by a curve for $\kappa \neq 8$. Moreover, there are phase transitions in parameter κ, namely, the curve is almost surely simple (does not intersect itself) for $\kappa \le 4$, the curve has double points for $4 < \kappa < 8$ and when $\kappa \ge 8$ the curve is a space-filling curve. These curves are also proved or conjectured to be the scaling limit of some two-dimensional lattice models in statistical mechanics, for instance, $\kappa = 2$ corresponds to the loop-erased random walk, SLE_κ with $\kappa = 8/3$ is conjectured to be the scaling limit of self-avoiding random walks, $\kappa = 4$ corresponds to the path of the harmonic explorer and contour lines of the Gaussian free field, SLE_κ with $\kappa = 6$ is the scaling limit of critical percolation on the triangular lattice....

As was remarked in [10] (Remark 4.18 and Section 4.3), if $U : (-\infty, +\infty) \to \mathbb{R}$ is a continuous function then (1.3) with $\lambda(t) = e^{iU(t)}$ can be solved for time variable $t \in (-\infty, +\infty)$, or equivalently, (1.4) can be solved for $t \in (-\infty, +\infty)$. In the equation (1.8) of the definition of interior radial SLE, one may consider the driving function $\lambda(t) = e^{i\sqrt{\kappa}B_t}$ with a two-side Brownian motion $B_t, -\infty < t < +\infty$. By following the same arguments, one can have an analogue of Lemma 1 in [2] (where the authors deal with the exterior radial SLE) for the interior version, i.e, the processes $f_t = g_t^{-1}$ and g_{-t} have the same law (up to conjugation by $e^{i\sqrt{k}B_t}$). We then redefine a radial SLE as

$$\tilde{f}_t(z) := g_{-t}(z) \overset{\text{(law)}}{=} e^{-i\sqrt{k}B_t} g_t^{-1}(e^{i\sqrt{k}B_t}), \quad t \in \mathbb{R}. \tag{1.9}$$

Then (conjugate, inverse) radial SLE process $\tilde{f}_t$ satisfies the ODE

$$\partial_t \tilde{f}_t(z) = \tilde{f}_t(z)\frac{\tilde{f}_t(z)+\lambda(t)}{\tilde{f}_t(z)-\lambda(t)}, \quad \tilde{f}_0(z) = z. \tag{1.10}$$

We have the following Markov property

Lemma 1.4.1.

$$\tilde{f}_t(z) = \lambda(s)\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s)). \tag{1.11}$$

A relation between the whole-plane SLE and the radial SLE is the following

Lemma 1.4.2. *The limit in law,* $\lim_{t\to+\infty} e^t \tilde{f}_t(z)$*, exists, and has the same law as the (time zero) interior whole-plane random map* $f_0(z)$*:*

$$\lim_{t\to+\infty} e^t \tilde{f}_t(z) \overset{\text{(law)}}{=} f_0(z).$$

Since SLE is a conformally invariant process, the exterior SLE$_\kappa$ map $\hat{f}_t$ and the interior SLE$_\kappa$ map f_t are related by the inversion

$$\hat{f}_t(z) \overset{\text{(law)}}{=} \frac{1}{f_t(\frac{1}{z})}, \quad z \in \Delta.$$

A generalization of *Schramm-Loewner Evolution*, so-called *Lévy-Loewner Evolution*, is also concurrently investigated. These are stochastic processes defined to be solutions of the Loewner equations with the driving function $\lambda(t) = e^{iL_t}$, where L_t is a *Lévy process*. Lévy processes generalize Brownian motions since they are assumed to satisfy only the two properties of Brownian motion: Stationary and Markov property. The essential difference with Brownian motions is that jumps are then allowed. The characteristic function of a Lévy process L_t has the form

$$\mathbb{E}(e^{i\xi L_t}) = e^{-t\eta(\xi)}, \tag{1.12}$$

where η (called the Lévy symbol) is a continuous complex function of $\xi \in \mathbb{R}$, satisfying (in addition to necessary Bchner-type conditions) $\eta(0) = 0$, and $\eta(-\xi) = \overline{\eta(\xi)}$. In the case of SLE$_\kappa$, the corresponding Lévy symbol is

$$\eta(\xi) = \frac{\kappa\xi^2}{2}.$$

We denote $\eta_k := \eta(k), k \in \mathbb{Z}$.

Chapter 2

GENERALIZED MOMENTS OF SLE

2.1 Logarithmic coefficients of SLE

2.1.1 Introduction

Let f be a univalent function in the class S then one can define the logathimic function

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

The coefficients γ_n, called *logarithmic coefficients* of $f(z)$, plays an important role in the theory of univalent functions as well as the geometric functions theory. A classical result on logarithmic coefficients is the *Milin's conjecture* which provides an inequality involving the these coefficients. This inequality possesses an special importance because it is a key link in the frame of de Brange's proof of the *Bieberbach's conjecture* on coefficients of univalent functions in the unit disk.
It is known that before de Branges, many methods were applied to attack the Bieberbach's conjecture but no one could completely prove it. Even so, the early approaches provided key ideas to the complete proof of de Branges. One of these contributions was made by Littlewood and Paley in 1932 when they showed that for an odd univalent function $h(z) = \sum_{n=1}^{\infty} c_n z^n$ in S, there exists a positive constant A independent to h and n such that

$$|c_n| \leq A.$$

They rather confidently conjectured that the universal bound A is 1 reached for the square root transform of the Koebe function. It is remarkable that if the conjecture of Littlewood and Paley is true then it immediately implies the Bieberbach's conjecture because the coefficients of a univalent function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is related to those of its square root transformation $z\sqrt{f(z^2)/z^2} = \sum_{k=1}^{\infty} c_{2k-1} z^{2k-1}$, which is an odd function, by

$$a_n = \sum_{k=1}^{n} c_{2(n-k)+1}.c_{2k-1}.$$

However, by using Loewner's method Fekete and Sgezö in 1933 could showed that $\max_{h \in S_{odd}} |c_5| = 1/2 + e^{-2/3} = 1.0134... > 1$ and thus disproved the *Littlewood-Paley conjecture.*

Although the original conjecture of Littlewood and Paley was wrong, their idea of estimating the coefficients of an univalent functions through those of its square root transform was very valuable. This idea was caught by Roberson and in 1936 he gave a weaker conjecture for coefficients c_n of odd functions in S

$$\sum_{k=1}^{n} |c_{2k-1}|^2 \leq n,$$

which also implies the Bieberbach's conjecture as a consequence of Cauchy-Schwarz inequality. In order to find the solution to Bieberbach's conjecture, the next step should then be attacking the *Roberson's conjecture.* For this step, principle contributions is due to the two soviet mathematicians Lebedev and Milin.

In their works on the relations between the Taylor series of a function and that of its logarithm, Lebedev and Milin proved that: If $\psi(z) = \sum_{k=0}^{\infty} \beta_k z^k$ with $\beta_0 = 1$ has positive radius of convergence, then the same is true for $\varphi(z) := \log \psi(z) = \sum_{k=1}^{\infty} \alpha_k z^k$, and the following coefficient relation is valid:

$$\frac{1}{n+1} \sum_{k=0}^{n} |\beta_k|^2 \leq \exp\left(\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) \left(k|\alpha_k|^2 - \frac{1}{k} \right) \right).$$

The equality holds for Koebe function and its rotated versions. This fact together with the fact that both Roberson's conjecture and *Lebedev-Milin inequality* state about upper bounds for means of quadratic moments suggested to Milin an approach to the Roberson's conjecture. We, for a univalent function $f \in S$, set $h(z) = z\sqrt{f(z^2)/z^2} = \sum_{k=1}^{\infty} c_{2k-1} z^{2k-1}$ and $\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$. From the definitions, one have

$$\log \frac{h(z)}{z} = \frac{1}{2} \log \frac{f(z^2)}{z^2} = \sum_{n=1}^{\infty} \gamma_n z^{2n}.$$

The Lebedev-Milin inequality then implies

$$\frac{1}{n+1} \sum_{k=1}^{n+1} |c_{2k-1}|^2 \leq \exp\left(\frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) \left(k|\gamma_k|^2 - \frac{1}{k} \right) \right).$$

Here if we make the following assumption

$$\sum_{k=1}^{n} (n+1-k) \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0,$$

then it is followed that

$$\frac{1}{n+1} \sum_{k=1}^{n+1} |c_{2k-1}|^2 \leq 1.$$

This is exactly the inequality conjectured by Roberson. The assumed essential inequality is called *Milin's conjecture.* The famous proof of Bieberbach's conjecture established by de Branges in 1984 has the above approach as its frame. He indeed proved the Milin's conjecture.

In the framework of the Schamm–Loewner evolution, the studying of logarithmic coefficients of SLE maps was started by [11]. The authors firstly realized that one can obtain an differential equation satisfied by the logarithmic function $\log \frac{f_t(z)}{z}$ of the SLE map $f_t(z)$ from the Loewner's equation (1). This equation then gives ordinary differential equations in time variable whose solutions are the logarithmic coefficients. By mean of the strong Markov property of Brownian process, they calculated the expectations of some first logarithmic coefficients as well as the expectations of their second moment. A theorem which states about explicit formulas for expected logarithmic coefficients of SLE maps and its proof were also eshtablised. In this section, we claim to continue the work [11], namely:

— In the first part of the section, we will give explicit formulas for the expected logarithmic coefficients of a Lévy–Loewner evolution. This result generalizes the main theorem in [11].

— In the rest, we will exactly formulate the expectations of the second moments of logarithmic coefficients of SLE_2 which are involved in a theorem and make a sketch for its proof. With this result we revisit the famous Milin's conjecture.

2.1.2 Expected logarithmic coefficients

We recall the calculations made in [11] to obtain the expectations of some first logarithmic coefficients of SLE_κ maps. The notation $f_t(z)$ will stand for the whole-plane SLE map at time t throughout this chapter except that sometimes, it will be used to denote LLE maps in parts concerning Lévy–Loewner evotution. If $h(z) := \log \frac{f_t(z)}{z}$ then the derivatives of $h(z)$ with respect to t and z are respectively

$$\dot{h} = \frac{\dot{f}_t}{f_t}, \qquad h' = \frac{f_t'}{f_t} - \frac{1}{z},$$

where $\dot{h}, h'$ are used to replace $\frac{\partial h}{\partial t}, \frac{\partial h}{\partial z}$ for reasons of concision. From Loewner equation (1), one gets an equation satisfied by h

$$\dot{h} = zh' \frac{\lambda(t) + z}{\lambda(t) - z} + \frac{\lambda(t) + z}{\lambda(t) - z}. \tag{2.1}$$

Assume that h has the Taylor series

$$\log \frac{f_t(z)}{z} = t + 2 \sum_{n \geq 1} \gamma_n z^n.$$

The logarithmic Loewner equation (2.1) then gives us differential recurrences of γ_n, namely,

$$\dot{\gamma}_1(t) - \gamma_1(t) = \overline{\lambda(t)}, \tag{2.2}$$

$$\dot{\gamma}_n(t) - n\gamma_n(t) = 2 \sum_{k=1}^{n-1} k\gamma_k(t) \overline{\lambda(t)}^{n-k} + \overline{\lambda(t)}^n, \quad n \geq 2. \tag{2.3}$$

These equations are equivalent to

$$\frac{\partial}{\partial t}(e^{-t}\gamma_1(t)) = e^{-t}\overline{\lambda(t)}, \tag{2.4}$$

$$\frac{\partial}{\partial t}(e^{-nt}\gamma_n(t)) = 2e^{-nt}\sum_{k=1}^{n-1} k\gamma_k(t)\overline{\lambda(t)}^{n-k} + e^{-nt}\overline{\lambda(t)}^{n}, \quad n \geq 2. \tag{2.5}$$

We know that $|\gamma_n(t)| \leq \sqrt{\frac{1}{k}(\sum_{k=1}^{n}\frac{1}{k}+\delta)}$ for all $n \geq 1$, where $\delta = 0.3118...$ is the *Milin constant*, as a consequence of the well-known *Milin's lemma* [6]. Therefore by integrating both sides of (2.4), (2.5) on $[0, +\infty)$ one arrive at

$$\gamma_1(t) = -e^t\int_t^\infty e^{-s}\overline{\lambda(s)}ds, \tag{2.6}$$

$$\gamma_n(t) = -e^{nt}\left[2\sum_{k=1}^{n-1} k\int_t^\infty e^{-ns}\gamma_k(s)\overline{\lambda(s)}^{n-k}ds + \int_t^\infty e^{-ns}\overline{\lambda(s)}^{n}ds\right], \quad n \geq 2. \tag{2.7}$$

Next, in order to obtain the expectation of a paticular $\gamma_n(t)$ one should notice that the integrals appearing in (2.6), (2.7) can be written as the linear combination of integrals of the form

$$\int_t^\infty ds_1 e^{-i\alpha_1\sqrt{\kappa}B_{s_1}-\beta_1 s_1}\int_{s_1}^\infty ds_2 e^{-i\alpha_2\sqrt{\kappa}B_{s_2}-\beta_2 s_2}...\int_{s_{k-1}}^\infty ds_k e^{-i\alpha_k\sqrt{\kappa}B_{s_k}-\beta_k s_k}. \tag{2.8}$$

By mean of the *Fubini's theorem*, these multiple integrals are integrals on the domains $\{t \leq s_1 \leq s_2 \leq ... \leq s_k < +\infty\}$ of the $k-$variable function

$$e^{-i\alpha_1\sqrt{\kappa}B_{s_1}-i\alpha_2\sqrt{\kappa}B_{s_2}...-i\alpha_k\sqrt{\kappa}B_{s_k}}e^{-\beta_1 s_1-\beta_2 s_2...-\beta_k s_k}.$$

One may write this function as

$$e^{-i\theta_1\sqrt{\kappa}(B_{s_1}-B_{s_2})}e^{-i\theta_2\sqrt{\kappa}(B_{s_2}-B_{s_3})}...e^{-i\theta_{n-1}\sqrt{\kappa}(B_{s_{n-1}}-B_{s_n})}e^{-i\theta_n\sqrt{\kappa}B_{s_n}}e^{-\beta_1 s_1-\beta_2 s_2...-\beta_k s_k}$$

and then respectively use the strong Markov property, the stationary property as well as the characteristic function of Brownian motion to obtain the expectation of the integrals (2.8) and therefore of $\gamma_n(t)$. For instance, one has, for $n = 1$,

$$\begin{aligned}
\mathbb{E}(\gamma_1(t)) &= \mathbb{E}\bigg(-e^t\int_t^\infty e^{-i\sqrt{\kappa}B_s-s}ds\bigg)\\
&= -e^t\int_t^\infty \mathbb{E}(e^{-i\sqrt{\kappa}B_s})e^{-s}ds\\
&= -e^t\int_t^\infty e^{-\frac{ks}{2}}e^{-s}ds = -\frac{2}{\kappa+2}e^{-\frac{\kappa t}{2}}.
\end{aligned}$$

In the case of $n = 2$, we can more clearly see the steps of the process mentioned above:

$$\begin{aligned}
\mathbb{E}(\gamma_2(t)) &= \mathbb{E}\bigg[-e^{2t}\bigg(2\int_t^\infty e^{-i\sqrt{\kappa}B_s-2s}\gamma_1(s)ds + \int_t^\infty e^{-i2\sqrt{\kappa}B_s-2s}ds\bigg)\bigg]\\
&= \mathbb{E}\bigg[-e^{2t}\bigg(-2\int_t^\infty e^{-i\sqrt{\kappa}B_{s_1}-s_1}ds_1\int_{s_1}^\infty e^{-i\sqrt{\kappa}B_{s_2}-s_2}ds_2 + \int_t^\infty e^{-i2\sqrt{\kappa}B_s-2s}ds\bigg)\bigg]\\
&= \mathbb{E}\bigg[-e^{2t}\bigg(-2\int_t^\infty\int_{s_1}^\infty e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})}e^{-2i\sqrt{\kappa}B_{s_2}}e^{-s_1-s_2}ds_2ds_1 + \int_t^\infty e^{-i2\sqrt{\kappa}B_s-2s}ds\bigg)\bigg].
\end{aligned}$$

Since the strong Markov property of Brownian motion, i.e, $B_{s_1} - B_{s_2}$ and B_{s_2} are independent for all $s_1 > s_2 \geq 0$, it is implied that $\mathbb{E}(e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})}e^{-2i\sqrt{\kappa}B_{s_2}}) = \mathbb{E}(e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})})\mathbb{E}(e^{-2i\sqrt{\kappa}B_{s_2}})$ and therefore

$$\begin{aligned}
&\mathbb{E}(\gamma_2(t))\\
&= -e^{2t}\Big(-2\int_t^\infty \int_{s_1}^\infty \mathbb{E}(e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})})\mathbb{E}(e^{-2i\sqrt{\kappa}B_{s_2}})e^{-s_1-s_2}ds_2ds_1 + \int_t^\infty \mathbb{E}(e^{-i2\sqrt{\kappa}B_s})e^{-2s}ds\Big)\\
&= -e^{2t}\Big(-2\int_t^\infty \int_{s_1}^\infty e^{-\frac{\kappa(s_1-s_2)}{2}}e^{-2\kappa s_2}e^{-s_1-s_2}ds_2ds_1 + \int_t^\infty \mathbb{E}(e^{-2\kappa s}e^{-2s}ds\Big)\\
&= -\frac{\kappa-2}{2(\kappa+1)(\kappa+2)}e^{-2\kappa t}.
\end{aligned}$$

We used the stationary of Brownian motion: for $B_{s_1} - B_{s_2}$ has the same law as $B_{s_1-s_2}$ for $s_1 \geq s_2 \geq 0$.
By the same method, one can also obtain

$$\mathbb{E}(\gamma_3(t)) = -\frac{(\kappa-2)(\kappa-1)}{2(\kappa+1)(\kappa+2)(\frac{9\kappa}{2}+3)}e^{-\frac{9\kappa t}{2}}.$$

In [11], the authors looked at the above specific cases and relized that

$$\begin{aligned}
\mathbb{E}(\gamma_1(t)) &= -\frac{\frac{\kappa}{2}-1}{\frac{\kappa}{2}+1}\frac{e^{-\frac{\kappa t}{2}}}{\frac{\kappa}{2}-1},\\
\mathbb{E}(\gamma_2(t)) &= -\frac{\frac{4\kappa}{2}-2}{\frac{4\kappa}{2}+2}\frac{\frac{\kappa}{2}-1}{\frac{\kappa}{2}+1}\frac{e^{-2\kappa t}}{\frac{4\kappa}{2}-2},\\
\mathbb{E}(\gamma_3(t) &= -\frac{\frac{9\kappa}{2}-3}{\frac{9\kappa}{2}+3}\frac{\frac{4\kappa}{2}-2}{\frac{4\kappa}{2}+2}\frac{\frac{\kappa}{2}-1}{\frac{\kappa}{2}+1}\frac{e^{-\frac{9\kappa t}{2}}}{\frac{9\kappa}{2}-3}
\end{aligned}$$

This fact led them to the following formulation for which they then also gave a proof:

Theorem 2.1.1. *(Le). Let $(f_t(z))_{t\geq 0}, z \in \mathbb{D}$, be the interior Schramm–Loewner whole-plane process driven by $\lambda(t) = e^{i\sqrt{\kappa}B_t}$ in Eq.* (1). *We write*

$$\log\frac{f_t(z)}{z} = t + 2\sum_{n\geq 1}\gamma_n(t)z^n.$$

Then

$$\mathbb{E}(\gamma_n(t)) = -\prod_{j=0}^{n-1}\frac{\frac{(n-j)^2\kappa}{2}-(n-j)}{\frac{(n-j)^2\kappa}{2}+(n-j)} \times \frac{e^{-\frac{n^2t}{2}}}{\frac{n^2\kappa}{2}-n}.$$

We now introduce a generalization of Theorem 2.1.1:

Theorem 2.1.2. *Let $(f_t)_{t\geq 0}$ be the interior Loewner whole-plane process driven by the Lévy process L_t with real Lévy symbol η and $\eta_j \geq 0$ for all positive integers j. We write*

$$\log\frac{f_t(z)}{z} = t + 2\sum_{n\geq 1}\gamma_n(t)z^n.$$

Then, the logarithmic function with stochastically rotated argument $\log \frac{f_t(e^{iL_t}z)}{e^{iL_t}z}$ *has the same law as* $\log \frac{f_0(z)}{z}$ *up to a difference* t*, i.e.,* $e^{niL_t}\gamma_n(t) \overset{\text{(law)}}{=} \gamma_n(0)$*. Setting* $\gamma_n = \gamma_n(0)$*, we have*

$$\mathbb{E}(\gamma_1) = -\frac{1}{\eta_1 + 1},$$

$$\mathbb{E}(\gamma_n) = -\frac{\prod_{k=1}^{n-1} \eta_k - k}{\prod_{k=1}^{n} \eta_k + k}, \quad n \geq 2.$$

The idea of our proof for this theorem originates in the proof of Theorem 3.1 in [5].

Proof. Recall that the Loewner equation for Lévy–Loewner maps $f_t(z)$ has the same type as that of SLE maps. The difference between them is just that in the Lévy setting, the driving function involves an Lévy process, namely $\lambda(t) = e^{-iL_t}$, instead of a Brownian motion. The equations (2.2) and (2.3) of logarithmic coefficients are thus obtained by the same arguments as in the SLE case.
For $n = 1$, since the reservation of the validity of (2.2) one also has that of (2.6). Recall that a Lévy process has a strong Markov property, which states that: $\forall s \geq t, L_s \overset{\text{(law)}}{=} L_t + \tilde{L}_{s-t}$, where $\tilde{L}_{s'}$ is an independent Lévy process, also started at $\tilde{L}_0 = 0$. The equation (2.6) is, by a change of variable $s \mapsto s + t$, rewritten as

$$\gamma_1(t) \overset{\text{(law)}}{=} -e^{-L_t} \int_0^\infty e^{-s} e^{-i\tilde{L}_s} ds \overset{\text{(law)}}{=} e^{-L_t}\gamma_1(0). \tag{2.9}$$

The expectation of $\gamma_1(0)$ can be obtained by setting $t = 0$ in (2.6) and making a short calculation of integral:

$$\begin{aligned}
\mathbb{E}(\gamma_1(0)) &= \mathbb{E}\Big(- e^t \int_0^\infty e^{-iL_s - s} ds \Big) \\
&= -e^t \int_0^\infty \mathbb{E}(e^{-iL_s}) e^{-s} ds \\
&= -e^t \int_0^\infty e^{-\eta_1 s} e^{-s} ds = -\frac{1}{\eta_1 + 1} e^{-\eta_1 t}.
\end{aligned}$$

In order to deal with the case of $n \geq 2$, we now put $u_n(t) := \gamma_n(t)e^{-nt}$ and $X_t := e^{-t-iL_t}$. From the equation (2.3), we have

$$\dot{u}_n(t) = 2\sum_{k=1}^{n-1} k X_t^{n-k} u_k(t) + X_t^n, \quad n \geq 2. \tag{2.10}$$

It is then implied that for $n \geq 3$,

$$\dot{u}_n(t) = X_t[\dot{u}_{n-1} + 2(n-1)u_{n-1}]. \tag{2.11}$$

On the other hand, as a consequence of the combination of (2.10) and the fact that $\dot{u}_1(t) = X_t$ which is implied by (2.2), the identity (2.11) also holds for $n = 2$.
Now, by considering $u_n(t)$ of the form

$$u_n(t) = -\int_t^{+\infty} ds X_s v_n(s), \tag{2.12}$$

we have $v_1(s) = 1$ and from (2.11),

$$v_n(t) = X_t v_{n-1}(t) - 2(n-1)\int_t^{+\infty} ds X_s v_{n-1}(s), \quad n \geq 2. \tag{2.13}$$

Let us define multiplicative and integral operators

$$\mathcal{X} v(t) := X_t v(t), \tag{2.14}$$

$$\mathcal{J} v(t) := -\int_t^{+\infty} ds X_s v(s). \tag{2.15}$$

Then (2.12) and (2.13) permit us to rewrite $u_n(t)$, for $n \geq 2$, as

$$u_n(t) = \mathcal{J} v_n(t) = \mathcal{J} \circ \prod_{k=1}^{n-1} \circ (\mathcal{X} + 2k\mathcal{J}) \circ \mathbb{1}, \tag{2.16}$$

where $\mathbb{1}$ is the constant function of value 1.
By making again use of the strong Markov property of Lévy process, we have that X_t satisfies the identity in law:

$$X_s \overset{\text{(law)}}{=} X_t \tilde{X}_{t-s}, \quad s \geq t, \tag{2.17}$$

where $\tilde{X}_{s'} := e^{-s' - \tilde{L}_{s'}}, s' \geq 0$, is an independent copy of that process, with $\tilde{X}_0 = 1$.
The operator $\mathcal{J}$ (2.15) can be rewritten as

$$\mathcal{J} v(t) \overset{\text{(law)}}{=} -X_t \int_0^{+\infty} ds \tilde{X}_s v(s+t) \tag{2.18}$$

$$= \mathcal{X} \circ \tilde{\mathcal{J}} v(t), \tag{2.19}$$

with $\tilde{\mathcal{J}} v(t) := -\int_0^{+\infty} ds \tilde{X}_s v(s+t)$. By iteration of the use of Markov property, Eq. (2.16) can be written as

$$u_n \overset{\text{(law)}}{=} \mathcal{J} \circ [\mathcal{X}(1 + 2(n-1)\tilde{\mathcal{J}}^{[n-1]})] \circ \ldots \circ [\mathcal{X}(1 + 2\tilde{\mathcal{J}}^{[1]})] \circ \mathbb{1} \tag{2.20}$$

$$\overset{\text{(law)}}{=} \mathcal{J} \circ \prod_{k=1}^{n-1} \circ [\mathcal{X}(1 + 2k\tilde{\mathcal{J}}^{[k]})] \circ \mathbb{1}, \tag{2.21}$$

where the integral operators $\tilde{\mathcal{J}}^{[k]}, k = 1, ..., n-1$, involve successive independent copies, $\tilde{X}_{s_k}^{[k]}, k = 1, ..., n-1$, of of the original exponential Lévy process X_s. We therefor arrive at the following explicit representation of the solution (2.16):

$$u_n(t) \overset{\text{(law)}}{=} -\int_t^{+\infty} ds X_s^n \prod_{k=1}^{n-1} \left(1 - 2k \int_0^{+\infty} ds_k (\tilde{X}_{s_k}^{[k]})^k\right). \tag{2.22}$$

Using again the identity in law (2.17) in (2.22), we arrive at

$$\begin{aligned} e^{inL_t}\gamma_n(t) &\overset{\text{(law)}}{=} -\int_0^{+\infty} ds\tilde{X}_s^n \prod_{k=1}^{n-1}\left(1-2k\int_0^{+\infty} ds_k(\tilde{X}_{s_k}^{[k]})^k\right) \\ &\overset{\text{(law)}}{=} \gamma_n(0). \end{aligned} \tag{2.23}$$

All factors in (2.23) involve successive independent copies of the Lévy process, and their expectations can now be taken independently. Recalling that the characteristic function of a Lévy process is $\mathbb{E}(e^{i\xi L_t}) = e^{-t\eta(\xi)}$, we then have $\mathbb{E}[(\tilde{X}_s)^k] = e^{-(\eta_k+k)s}$. Thus, for $n \geq 2$

$$\begin{aligned} \mathbb{E}[\gamma_n(0)] &= -\int_0^{+\infty} ds\mathbb{E}[\tilde{X}_s^n]\prod_{k-1}^{n-1}\left(1-2k\int_0^{+\infty} ds_k(\mathbb{E}[\tilde{X}_{s_k}^{[k]})^k]\right) \\ &= -\frac{1}{\eta_n+n}\prod_{k=1}^{n-1}\left(1-\frac{2k}{\eta_k+k}\right) \\ &= -\frac{\prod_{k=1}^{n-1}\eta_k-k}{\prod_{k=1}^{n}\eta_k+k}. \end{aligned} \tag{2.24}$$

□

Remark 2.1.1. *Because* $e^{niL_t}\gamma_n(t) \overset{\text{(law)}}{=} \gamma_n(0)$ *and* L_t *is independent to* L_0 *(therefore* e^{-niL_t} *is independent to* $\gamma_n(0)$ *for all* $n \geq 1$*), the expectation of* $\gamma_n(t)$ *can be obtained as*

$$\mathbb{E}(\gamma_1(t)) = \mathbb{E}(\gamma_1(0))\mathbb{E}(e^{-iL_t}) = -\frac{1}{\eta_1+1}e^{-t\eta_1}, \tag{2.25}$$

$$\mathbb{E}(\gamma_n(t)) = \mathbb{E}(\gamma_n(0))\mathbb{E}(e^{-niL_t}) = -\frac{\prod_{k=1}^{n-1}\eta_k-k}{\prod_{k=1}^{n}\eta_k+k}e^{-t\eta_n}, \quad n \geq 2. \tag{2.26}$$

The results in [11] turn out to be consequences by the Brownian setting $\eta_n = \frac{\kappa n^2}{2}$.

2.1.3 Moment of order 2

The arguments in the first part of section 2.1.2 does not only make one able to evaluate the expectations of a particular SLE logarithmic coefficient but also the expectation of its second moment. While a logarithmic coefficient is the linear combination of multiple integrals of the form (2.8), the second moment of this coefficient is a linear combination of products of such integrals with complex conjugate of others. Each of such products may of course be written as the sum of integrals (2.8).

Therefore the problem of evaluating the expectation of second moment of logarithmic coefficients turns back to that of calculating the expectations of integrals of the form (2.8), which one can do with given steps using the elementary properties of Brownian motion. By this way, the followings were obatined in [11]

$$\mathbb{E}(|\gamma_1(t)|^2) = \frac{2}{\kappa+2}, \mathbb{E}(|\gamma_2(t)|^2) = \frac{\kappa^2+16\kappa+12}{4(\kappa+1)(\kappa+2)(\kappa+6)}.$$

The independence of these expectations to t can be justified by Theorem 2.1.2 which states that $e^{ni\sqrt{\kappa}B_t}\gamma_n(t) \overset{\text{(law)}}{=} \gamma_n(0)$.

Follow the preceding section, we expect that a general formula can be found from specific cases. However, a essential difficulty appears in the calculation: A great number of integrals (2.8) to evaluate! If the linear representation of $\gamma_n(t)$ has k integrals of the form (2.8), its second moment contains k^2 products of these integrals with the complex conjugate of others. Moreover, if a such product is the product of an integral of a $k-$variable function with the conjugate of that of an $l-$variable function then this product is the sum of C^k_{k+l} integrals of the form (2.8). The total number of integrals becomes large and quickly increases in n. We thus need helps from computers. There is a computer programming, called *dynamic programming*[1], which allows us more quickly evaluate our expectations. This programming was used to solve a similar problem in [5]. Although we, with helps of computers, obtained values for some specific cases, we have not touched a general formula for all n and κ yet. It maybe impossible to find that formula. Interestingly, with $\kappa = 2$ we obtain remarkable values of expectations: for $n = 1...5$

$$\mathbb{E}(|\gamma_n(0)|^2) = \frac{1}{2n^2}. \tag{2.27}$$

We generally formulate this fact in the following theorem

Theorem 2.1.3. *Let $(f_t(z))_{t\geq 0}, z \in \mathbb{D}$, be the interior Schramm–Loewner whole-plane process driven by $\lambda(t) = e^{i\sqrt{\kappa}B_t}$ in Eq.* (1) *and let $f(z) := f_0(z)$, such that*

$$\log\frac{f(z)}{z} = 2\sum_{n\geq 1}\gamma_n z^n; \tag{2.28}$$

then, for $\kappa = 2$,

$$\mathbb{E}(|\gamma_n|^2) = \frac{1}{2n^2}, \quad \forall n \geq 1.$$

Proof. One firstly differentiates both sides of (2.28) to get

$$\frac{d}{dz}\log\frac{f(z)}{z} = \frac{f'(z)}{f(z)} - \frac{1}{z}. \tag{2.29}$$

This equation implies the Taylor series of the moment of order 2 $|zf'(z)/f(z)|^2$:

$$\left|z\frac{f'(z)}{f(z)}\right|^2 = 1 + 2\sum_{n\geq 1} n\gamma_n(z^n+\bar{z}^n) + \sum_{n\geq 1}\sum_{m\geq 1} nm\gamma_n\bar{\gamma}_m z^n\bar{z}^m. \tag{2.30}$$

1. We would like to thank Nguyen Thi Thuy Nga for her explanations on the dynamic programming.

Observe that the coefficient of $z^n\bar{z}^n$ in the right-hand side of (2.30) is $n^2|\gamma_n|^2$. One thus has the value of $\mathbb{E}(|\gamma_n|^2)$ for all $n \geq 1$ whenever he knows $\mathbb{E}(|zf'(z)/f(z)|^2)$. In the next section where generalized moments of SLE map are considered, we will show

Theorem 2.1.4. *Let f be the interior whole-plane* SLE_κ *map, in the same setting as in Theorem 2.1.3; then for $\kappa = 2$,*

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right) = \frac{(1-z)(1-\bar{z})}{1-z\bar{z}}.$$

Since Theorem 2.1.4, one has the series development

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right) = \frac{(1-z)(1-\bar{z})}{(1-z\bar{z})} = 1 - \sum_{n\geq 0} z^{n+1}\bar{z}^n - \sum_{n\geq 0} z^n\bar{z}^{n+1} + 2\sum_{n\geq 1} z^n\bar{z}^n. \quad (2.31)$$

A identification of the coefficients in $z^n\bar{z}^n$ of (2.30) and (2.31) finally leads us to

$$\mathbb{E}(|\gamma_n|^2) = \frac{1}{2n^2}, \quad n \geq 1.$$

□

Remark 2.1.2. *By identifying the other coefficients of* (2.30) *and* (2.31)*, one gets*

$$\mathbb{E}(\gamma_1) = -1/2, \quad \mathbb{E}(\gamma_n) = 0, \quad n \geq 2,$$
$$\mathbb{E}(\gamma_n\bar{\gamma}_{n+1}) = -\frac{1}{n(n+1)}, \quad \mathbb{E}(\gamma_n\bar{\gamma}_{n+k}) = 0, \quad n \geq 1, k \geq 2,$$

Remark 2.1.3. *Since Theorem 2.1.3, one have for* SLE_2

$$\mathbb{E}\left(\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right)\right) = -\frac{1}{2}\sum_{m=1}^{n}\sum_{k=1}^{m}\frac{1}{k} = -\frac{n+1}{2}\sum_{k=2}^{n+1}\frac{1}{k} < 0,$$

which confirms the validity "in expectation" of the Milin conjecture.

2.2 Moment problem and Martingale method

2.2.1 A rich algebraic structure of whole-plane SLE and Belyaev-Smirnov equations

In their work [4] of revisiting the Bieberbach conjecture in the framework of interior whole-plane SLE and LLE, the authors performed computations of $\mathbb{E}(|a_n|^2)$ for small n where a_n are the Taylor coefficients of the interior whole-plane SLE map at time 0

$$f(z) := f_0(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (2.32)$$

Since the obtained results, they then conjectured that: $\forall n \geq 2$

$$\mathbb{E}(|a_n|^2) = 1 \text{ for } \kappa = 6 \tag{2.33}$$
$$\mathbb{E}(|a_n|^2) = n \text{ for } \kappa = 2 \tag{2.34}$$

This conjecture was proved by I.Loutsenko in [12] (see also [13]). He used the Hasting's method [7] to derive a differential equation obeyed by the function $F(z,\bar{z}) = \mathbb{E}(f(z)^{\frac{p}{2}}\overline{f(z)}^{\frac{p}{2}})$ with parameter $p \in \mathbb{R}$, which he called the *correlation function.* Using $\partial := \partial_z, \bar{\partial} := \partial_{\bar{z}}$, this differential equation is written as

$$-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 F + \frac{z+1}{z-1} z\partial F + \frac{\bar{z}+1}{\bar{z}-1}z\bar{\partial}F - p\left[\frac{1}{(z-1)^2} + \frac{1}{(\bar{z}-1)^2} - 2\right]F = 0. \tag{2.35}$$

It is easy to see that if one can solve Eq. (2.35) in the case of $p = 2$, corresponding to $\mathbb{E}(|f(z)|^2)$, then explicit formulas of $\mathbb{E}(|a_n|^2)$ are obtained for all n by a identification of coefficients. Loutsenko considered the series form of the correlation function and transformed the differential equation into recursions of coefficients. He indeed showed that these recursions are solvable for $\kappa = 6, 2$ and arrived at (2.33),(2.34).

After their earlier draft [4], B.Duplantier, TTN.Nguyen, TPC.Nguyen and M. Zinsmeister [5] used a different method to obtain the same differential equation (2.35) as in [12]. This method is based on a martingale argument and uses the Markov property of the SLE map as well as Lemma 1.4.2 as important ingredients. Namely, the authors consider the martingale $\mathbb{E}(|\tilde{f}_t'(z)|^p|\mathcal{F}_s)$, where $\tilde{f}_t$ is the (conjugate, inverse) interior radial SLE$_\kappa$ map at time t defined by (1.9) and $\mathcal{F}_s$ denotes the σ–algebra generated by $\{B_\tau : \tau \leq s\}$. By using the Markov property, they showed that

$$\mathbb{E}(|\tilde{f}_t'(z)|^p|\mathcal{F}_s) = |\tilde{f}_s'(z)|^p \tilde{F}(z_s, t-s)$$

with $\tilde{F}(z,t) = \mathbb{E}(|\tilde{f}_t'(z)|^p), z_s = \tilde{f}_s(z)/\lambda(s)$. The fact that the ds drif term vanishes in the Itô derivative of a martingale made them able to obtain an equation obeyed by $\tilde{F}$. An application of Lemma 1.4.2 on this equation finally led to Eq. (2.35). It is also noted in [5] that the variables z and $\bar{z}$ can be considered as two independent complex variables in (2.35). Therefore, by setting $\bar{z} = 0$, one has an equation satisfied by $F(z,0) = \mathbb{E}[(f_0'(z))^{p/2}]$

$$\frac{\kappa}{2}(z\partial)^2 F(z,0) + \frac{z+1}{z-1}z\partial F(z,0) - p\left[\frac{1}{(z-1)^2} - 1\right]F(z,0) = 0. \tag{2.36}$$

The authors then considered special forms of solutions of (2.36), (2.35) and proved the following

Theorem 2.2.1. *(B.Duplantier, TTN.Nguyen, TPC.Nguyen, M.Zinsmeister). The whole-plane* SLE$_\kappa$ *map* $f_0(z)$ *has derivative moments*

$$\mathbb{E}[(f_0'(z))^{p/2}] = (1-z)^\alpha, \tag{2.37}$$
$$\mathbb{E}[|f_0'(z)|^p] = \frac{(1-z)^\alpha(1-\bar{z})^\alpha}{(1-z\bar{z})^\beta}, \tag{2.38}$$

for the special sets of exponents $p = \kappa\alpha(\alpha+1)/6 = (6+\kappa)(2+\kappa)/8\kappa$, *with* $\alpha = (6+\kappa)/2\kappa$ *and* $\beta = \kappa\alpha^2/2 = (6+\kappa)^2/8\kappa$.

Obviously, the equation (2.33) and (2.34) are consequences of this theorem. Note that Eq. (2.38) was also obtained by Loutsenko and Yermolayema in [13].

The above host of closed form results hints that the SLE_κ process, in its interior whole-plane version, has a rich algebraic structure. It is also notied that in order to derive these closed form results, differential equations obeyed by the moments $\mathbb{E}[(f_0'(z))^{p/2}]$ or $\mathbb{E}[|f_0'(z)|^p]$, called *Belyaev-Smirnov equations*, play an important role. The denomination comes from the fact that these equations (as well as the martingale method) were originally considered by D.Belyaev and S.Smirnov [2] to study the average integral means spectrum of the exterior whole-plane SLE_κ map. Since the Belyaev-Smirnov equations encode the expected moments of SLE_κ maps, they are not only useful for studying coefficient problems in the framework of SLE processes but also for studying other problems concerning moments of the derivative of SLE_κ maps, such as to determine integral mean spectra.

2.2.2 Martingale method

In this section, we will reintroduce the martingale method by using it to compute some expectations which are connected to the logarithmic coefficients of SLE_κ maps. In particular, we will prove Theorem 2.1.4, an important ingredient in the proof of Theorem 2.1.3.

2.2.2.1 A martingale computation

Proposition 2.2.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_2 map at time 0, in the same setting as in Theorem 2.1.3; we then have*

$$\mathbb{E}\left(z\frac{f'(z)}{f(z)}\right) = 1 - z.$$

A consequence of Proposition 2.2.1 is explicit formulas of the expected logarithmic coefficients of the SLE_2 map. Namely, one may multiply the both sides of (2.29) by z and then identifies the Taylor coefficients of their expectations to derive

Corollary 2.2.1. *Let $f(z)$ be the whole-plane SLE_κ map, in the same setting as in Theorem 2.1.3; then for $\kappa = 2$,*

$$\mathbb{E}(\gamma_n) = \begin{cases} -1/2, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Remark 2.2.1. *Corollary 2.2.1 confirms the validity of Theorem 2.1.1 and Theorem 2.1.2 in the case of SLE_2.*

We now give the proof of Proposition 2.2.1:

Proof. We firstly define

$$G(z) := \mathbb{E}\left(z\frac{f'(z)}{f(z)}\right). \tag{2.39}$$

Let us introduce the auxiliary, time-dependent, radial variant of the SLE one-point function $G(z)$ above,

$$\widetilde{G}(z,t) := \mathbb{E}\left(z\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)} \right), \tag{2.40}$$

where $\tilde{f}_t$ is a modified radial SLE map at time t defined by (1.9). Owing to Lemma 1.4.2, we have

$$\lim_{t\to+\infty} \widetilde{G}(z,t) = G(z). \tag{2.41}$$

We then use a martingale technique to obtain an equation satisfied by $\widetilde{G}(z,t)$. For $s \leq t$, define $\mathcal{M}_s := \mathbb{E}\left(\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}|\mathcal{F}_s\right)$, where $\mathcal{F}_s$ is the σ-algebra generated by $\{B_u,\, u \leq s\}$. $(\mathcal{M}_s)_{s\geq 0}$ is by construction a martingale. Because of the Markov property of SLE, we have

$$\begin{aligned}
\mathcal{M}_s = \mathbb{E}\bigg(\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}|\mathcal{F}_s\bigg) &= \mathbb{E}\bigg(\frac{\tilde{f}_s'(z)}{\lambda(s)}\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}|\mathcal{F}_s\bigg) \\
&= \frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\mathbb{E}\bigg(\frac{\tilde{f}_s(z)}{\lambda(s)}\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}|\mathcal{F}_s\bigg) \\
&= \frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\widetilde{G}(z_s,\tau),
\end{aligned}$$

where $z_s := \tilde{f}_s(z)/\lambda(s)$, and $\tau := t-s$.

We will need the derivatives of $\tilde{f}_s', \tilde{f}_s'$ and z_s. From Eq. (1.10), we have

$$\begin{aligned}
\partial_s\tilde{f}_s' = \partial_z\partial_s\tilde{f}_s = \partial_z\left(\tilde{f}_s\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right) \\
= \tilde{f}_s'\bigg(\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s-\lambda(s))^2}\bigg) = \tilde{f}_s'\bigg(1-\frac{2}{(1-z_s)^2}\bigg), \qquad (2.42)\\
\partial_s\tilde{f}_s = \tilde{f}_s\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} = \tilde{f}_s\frac{z_s+1}{z_s-1}, \qquad (2.43)\\
dz_s = z_s\left[\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\right]ds - iz_s\sqrt{\kappa}dB_s. \qquad (2.44)
\end{aligned}$$

Consider $\mathcal{M}_s$ as an Itô drift-diffusion process which is the composition of a twice differentiable scalar function and the vector $(\tilde{f}_s', \tilde{f}_s, z_s)$ of Itô processes, then the coefficient of the ds-drift term of the Itô derivative of $\mathcal{M}_s$ is obtained from the above as,

$$\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\left[-\frac{2z_s}{(1-z_s)^2}+z_s\left(\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\right)\partial_{z_s}-\partial_\tau-\frac{\kappa}{2}z_s^2\partial_{z_s}^2\right]\widetilde{G}(z_s,\tau), \tag{2.45}$$

and vanishes by the (local) martingale property. Because $\tilde{f}_s$ is univalent, $\tilde{f}_s'$ does not vanish in $\mathbb{D}$, therefore the bracket above vanishes.

Owing to the existence of the limit (2.41), we can now take the $\tau \to +\infty$ limit in the above, and obtain the ODE,

$$\begin{aligned}\mathcal{P}(\partial)[G(z)] &:= -\frac{2z}{(1-z)^2}G(z) + z\left(\frac{z+1}{z-1} - \frac{\kappa}{2}\right)G'(z) - \frac{\kappa}{2}z^2G''(z) \qquad (2.46)\\ &= \left[-\frac{2z}{(1-z)^2} + z\left(\frac{z+1}{z-1}\right)\partial_z - \frac{\kappa}{2}(z\partial_z)^2\right]G(z) = 0.\end{aligned}$$

We have exchanged the limit (2.41) and the derivatives $\partial_z, \partial_z^2, \partial_\tau$. It is justified by Lemma 1.4.2 and the dominated convergence theorem where the domination comes from Koebe distortion theorem.

Following Ref. [5], we now look for solutions to Eq. (2.46) of the form $\varphi_\alpha(z) := (1-z)^\alpha$. We have

$$\mathcal{P}(\partial)[\varphi_\alpha] = A(2,2,\alpha)\varphi_\alpha + B(2,\alpha)\varphi_{\alpha-1} + C(2,\alpha)\varphi_{\alpha-2},$$

where, in anticipation of the notation that will be introduced in Section 2.3 below,

$$\begin{aligned}A(2,2,\alpha) &:= \alpha - \frac{\kappa}{2}\alpha^2,\\ B(2,\alpha) &:= 2 - \left(3+\frac{\kappa}{2}\right)\alpha + \kappa\alpha^2,\\ C(2,\alpha) &:= -2 + \left(2+\frac{\kappa}{2}\right)\alpha - \frac{\kappa}{2}\alpha^2,\end{aligned}$$

with, identically, $A+B+C=0$. The linear independence of $\varphi_\alpha, \varphi_{\alpha-1}, \varphi_{\alpha-2}$ thus shows that $\mathcal{P}(\partial)[\varphi_\alpha] = 0$ is equivalent to $A=B=C=0$, which yields $\kappa = 2, \alpha = 1$, and $G(z) = 1-z$. $\square$

2.2.2.2 Proof of Theorem 2.1.4

As in section 2.2.2.1, we firstly introduce the function

$$G(z,\bar{z}) = \mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^2\right), \qquad (2.47)$$

and an auxiliary, time-dependent, radial variant of G

$$\tilde{G}(z,\bar{z}) = \mathbb{E}\left(\left|z\frac{\tilde{f}'(z)}{\tilde{f}(z)}\right|^2\right). \qquad (2.48)$$

From Lemma 1.4.2, the functions G and $\tilde{G}$ are related by

$$\lim_{t\to+\infty} \widetilde{G}(z,\bar{z},t) = G(z,\bar{z}). \qquad (2.49)$$

We now define $\mathcal{M}_s := \mathbb{E}\left(\left|\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}\right|^2 |\mathcal{F}_s\right)$, where $\mathcal{F}_s$ is the σ-algebra defined as in section 2.2.2.1. Then $(\mathcal{M}_s)_{s\geq 0}$ is a martingale. One can use the Markov property to represent $\mathcal{M}_s$ as

$$\begin{aligned}\mathcal{M}_s = \mathbb{E}\left(\left|\frac{\tilde{f}_t'(z)}{\tilde{f}_t(z)}\right|^2 |\mathcal{F}_s\right) &= \mathbb{E}\left(\left|\frac{\tilde{f}_s'(z)}{\lambda(s)}\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}\right|^2 |\mathcal{F}_s\right)\\ &= \left|\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\right|^2 \mathbb{E}\left(\left|\frac{\tilde{f}_s(z)}{\lambda(s)}\frac{\tilde{f}_{t-s}'(\tilde{f}_s(z)/\lambda(s))}{\tilde{f}_{t-s}(\tilde{f}_s(z)/\lambda(s))}\right|^2 |\mathcal{F}_s\right)\\ &= \left|\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\right|^2 \widetilde{G}(z_s, \bar{z}_s, \tau),\end{aligned}$$

where $z_s := \tilde{f}_s(z)/\lambda(s)$, and $\tau := t - s$.
We have that

$$\begin{aligned}\partial_s \log|\tilde{f}_s'| = \partial_s \Re(\log \tilde{f}_s') &= \Re\left[\frac{1}{\tilde{f}_s'}\partial_z\left(\tilde{f}_s\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right)\right]\\ &= \Re\left[\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s-\lambda(s))^2}\right]\\ &= 1 - \frac{1}{(1-z_s)^2} - \frac{1}{(1-\bar{z}_s)^2}, \qquad (2.50)\\ \partial_s \log|\tilde{f}_s| = \partial_s \Re(\log \tilde{f}_s) &= \Re\left(\frac{\partial_s \tilde{f}_s}{\tilde{f}_s}\right)\\ &= \Re\left(\frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right) = 1 + \frac{1}{z_s-1} + \frac{1}{\bar{z}_s-1}. \qquad (2.51)\end{aligned}$$

These equations imply that

$$\partial_s|\tilde{f}_s'| = |\tilde{f}_s'|\partial_s \log|\tilde{f}_s'| = |\tilde{f}_s'|\left(1 - \frac{1}{(1-z_s)^2} - \frac{1}{(1-\bar{z}_s)^2}\right), \qquad (2.52)$$

$$\partial_s|\tilde{f}_s| = |\tilde{f}_s|\partial_s \log|\tilde{f}_s| = |\tilde{f}_s|\left(1 + \frac{1}{z_s-1} + \frac{1}{\bar{z}_s-1}\right). \qquad (2.53)$$

Thank to Itô's lemma, we also have

$$dz_s = z_s\left[\frac{z_z+1}{z_s-1} - \frac{\kappa}{2}\right]ds - iz_s\sqrt{\kappa}dB_s, \qquad (2.54)$$

$$d\bar{z}_s = \bar{z}_s\left[\frac{\bar{z}_s+1}{\bar{z}_s-1} - \frac{\kappa}{2}\right]ds + i\bar{z}_s\sqrt{\kappa}dB_s. \qquad (2.55)$$

Consider $\mathcal{M}_s$ as an Itô drift-diffusion process which is the composition of a twice differentiable scalar function and the vector $(|\tilde{f}_s'|, |\tilde{f}_s|, z_s, \bar{z}_s)$, then the coefficient of the ds-drift term of the Itô derivative of $\mathcal{M}_s$ is obtained from (2.52),(2.53),(2.54), (2.55) as,

$$\begin{aligned}&\left|\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\right|^2\Bigg[-\frac{2}{(1-z_s)^2} - \frac{2}{(1-\bar{z}_s)^2} + \frac{2}{1-z_s} + \frac{2}{1-\bar{z}_s} + z_s\left(\frac{z_s+1}{z_s-1} - \frac{\kappa}{2}\right)\partial_{z_s}\\ &+ \bar{z}_s\left(\frac{\bar{z}_s+1}{\bar{z}_s-1} - \frac{\kappa}{2}\right)\partial_{\bar{z}_s} - \partial_\tau - \frac{\kappa}{2}z_s^2\partial_{z_s}^2 - \frac{\kappa}{2}\bar{z}_s^2\partial_{\bar{z}_s}^2 + \kappa z_s\bar{z}_s\partial_{z_s}\partial_{\bar{z}_s}\Bigg]\widetilde{G}(z_s, \bar{z}_s, \tau). \qquad (2.56)\end{aligned}$$

One can rewrite (2.56) as

$$\left|\frac{\tilde{f}_s'(z)}{\tilde{f}_s(z)}\right|^2\Bigg[-\frac{2}{(1-z_s)^2}-\frac{2}{(1-\bar{z}_s)^2}+\frac{2}{1-z_s}+\frac{2}{1-\bar{z}_s} \\ +z_s\frac{z_s+1}{z_s-1}\partial_{z_s}+\bar{z}_s\frac{\bar{z}_s+1}{\bar{z}_s-1}\partial_{\bar{z}_s}-\partial_\tau-\frac{\kappa}{2}(z_s\partial_{z_s}-\bar{z}_s\partial_{\bar{z}_s})^2\Bigg]\widetilde{G}(z_s,\bar{z}_s,\tau). \quad (2.57)$$

The (local) martingale property of the martingale $\mathcal{M}_s$ implies that the above quantity vanishes. Because $\tilde{f}_s$ is univalent, $\tilde{f}_s'$ does not vanish in $\mathbb{D}$, therefore the bracket above vanishes.

We now take the $\tau\to+\infty$ limit in the above and make use of the limit (2.49) to arrive at the PDE,

$$\mathcal{P}(D)[G(z,\bar{z})]=-\frac{\kappa}{2}(z\partial-\bar{z}\bar{\partial})^2G-\frac{1+z}{1-z}z\partial G-\frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G \\ +\left[-\frac{2}{(1-z)^2}-\frac{2}{(1-\bar{z})^2}+\frac{2}{1-z}+\frac{2}{1-\bar{z}}\right]G=0. \quad (2.58)$$

Following again Ref. [5], we look for solutions to Eq. (2.58) of the form $\varphi_\alpha(z)\overline{\varphi_\alpha(z)}P(z\bar{z})$ where $\varphi_\alpha(z):=(1-z)^\alpha$. We have

$$\mathcal{P}(\partial)[\varphi_\alpha\bar{\varphi}_\alpha P]=z\bar{z}\varphi_{\alpha-1}\bar{\varphi}_{\alpha-1}\left(\kappa\alpha^2P-2(1-z\bar{z})P'\right) \\ +(A(2,2,\alpha)\varphi_\alpha+B(2,\alpha)\varphi_{\alpha-1}+C(2,\alpha)\varphi_{\alpha-2})\bar{\varphi}_\alpha P \\ +\overline{A(2,2,\alpha)\varphi_\alpha+B(2,\alpha)\varphi_{\alpha-1}+C(2,\alpha)\varphi_{\alpha-2}}\varphi_\alpha P. \quad (2.59)$$

Since the consideration of the coefficients $A(2,2,\alpha),B(2,\alpha),C(2,\alpha)$ in section 2.2.2.1, when $\kappa=2,\alpha=1$ the two last lines of (2.59) vanish. We now consider the first line, in particular, the ODE

$$\kappa\alpha^2P(X)-2(1-X)P'(X)=0. \quad (2.60)$$

It is followed from the definition of G that $G(0,0)=1$, therefore $P(0)=1$. The ODE (2.60) with the initial condition $P(0)=1$ can be solved to obtain the unique solution

$$P(X)=\frac{1}{(1-X)^{\frac{\kappa\alpha^2}{2}}}. \quad (2.61)$$

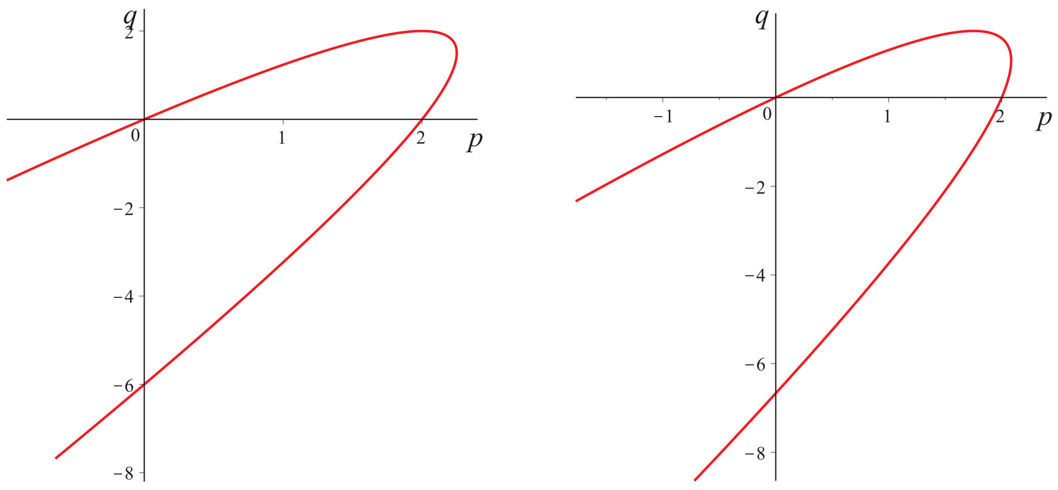
This fact together with the later fact on the vanishing of the two last lines of (2.59) leads us to the conclusion of Theorem 2.1.4.

2.3 SLE one-point function

Let us now turn to the natural generalization of Proposition 2.2.1.

Theorem 2.3.1. *Let $f(z)=f_0(z)$ be the interior whole-plane SLE_κ map at time zero, in the same setting as in Theorem 2.1.3. Consider the curve $\mathcal{R}$, defined parametrically by*

$$p=-\frac{\kappa}{2}\gamma^2+\left(2+\frac{\kappa}{2}\right)\gamma,\quad 2p-q=\left(1+\frac{\kappa}{2}\right)\gamma,\quad \gamma\in\mathbb{R}. \quad (2.62)$$



(a) $\kappa = 2$ (b) $\kappa = 6$

Figure 2.1 – *Integral curves $\mathcal{R}$ of Theorem 2.3.1, for $\kappa = 2$ and $\kappa = 6$. In addition to the origin, the $q = 0$ intersection point with the p-axis is at $p(\kappa) := (6+\kappa)(2+\kappa)/8\kappa$, with $p(2) = p(6) = 2$ [5, 12].*

On $\mathcal{R}$, the whole-plane SLE_κ *one-point function has the integrable form,*

$$\mathbb{E}\left(\frac{(f'(z))^{\frac{p}{2}}}{(f(z)/z)^{\frac{q}{2}}}\right) = (1-z)^{\gamma}.$$

Remark 2.3.1. *Eq. (2.62) describes a parabola in the (p, q) plane (see Fig. 2.1), which is given in Cartesian coordinates by*

$$2\kappa\left(\frac{2p-q}{2+\kappa}\right)^2 - (4+\kappa)\,\frac{2p-q}{2+\kappa} + p = 0, \tag{2.63}$$

with two branches,

$$\begin{aligned} &\gamma = \gamma_0^{\pm}(p) := \frac{1}{2\kappa}\left(4+\kappa \pm \sqrt{(4+\kappa)^2 - 8\kappa p}\right),\ \ p \leq \frac{(4+\kappa)^2}{8\kappa}, \\ &q = 2p - \left(1+\frac{\kappa}{2}\right)\gamma_0^{\pm}(p). \end{aligned} \tag{2.64}$$

or, equivalently,

$$2p = q + \frac{2+\kappa}{8\kappa}\left(6+\kappa \pm \sqrt{(6+\kappa)^2 - 16\kappa q}\right),\ \ q \leq \frac{(6+\kappa)^2}{16\kappa}. \tag{2.65}$$

Proof. Our aim is to derive an ODE satisfied by the whole-plane SLE one-point function,

$$G(z) := \mathbb{E}\left(z^{\frac{q}{2}}\frac{(f'(z))^{\frac{p}{2}}}{(f(z))^{\frac{q}{2}}}\right), \tag{2.66}$$

which, by construction, stays finite at the origin and such that $G(0) = 1$.

Let us introduce the shorthand notation,

$$X_t(z) := \frac{(\tilde{f}_t'(z))^{\frac{p}{2}}}{(\tilde{f}_t(z))^{\frac{q}{2}}}, \tag{2.67}$$

where $\tilde{f}_t$ is the conjugate, reversed radial SLE process in $\mathbb{D}$, as introduced by (1.9), and such that by Lemma 1.4.2, the limit, $\lim_{t\to+\infty} e^t \tilde{f}_t(z) \overset{\text{(law)}}{=} f_0(z)$, is the same in law as the whole-plane map at time zero. Applying the same method as in the previous section, we consider the time-dependent function

$$\widetilde{G}(z,t) := \mathbb{E}\left(z^{\frac{q}{2}} X_t(z)\right), \tag{2.68}$$

such that

$$\lim_{t\to+\infty} \exp\left(\frac{p-q}{2}t\right) \widetilde{G}(z,t) = G(z). \tag{2.69}$$

Consider now the martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$, defined by

$$\mathcal{M}_s = \mathbb{E}(X_t(z)|\mathcal{F}_s).$$

By the SLE Markov property we get, setting $z_s := \tilde{f}_s(z)/\lambda(s)$,

$$\mathcal{M}_s = X_s(z)\widetilde{G}(z_s,\tau), \quad \tau := t-s. \tag{2.70}$$

As before, the partial differential equation satisfied by $\widetilde{G}(z_s,\tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (2.70),

$$d\mathcal{M}_s = \widetilde{G}\, dX_s + X_s\, d\widetilde{G},$$

vanishes. The differential of X_s is simply computed from Eqs. (2.42) and (2.43) above as:

$$\begin{aligned} dX_s(z) &= X_s(z)F(z_s)ds, \\ F(z) &:= \frac{p}{2}\left[1-\frac{2}{(1-z)^2}\right] - \frac{q}{2}\left[1-\frac{2}{1-z}\right]. \end{aligned} \tag{2.71}$$

The Itô differential $d\widetilde{G}$ brings in the ds terms proportional to $\partial_{z_s}\widetilde{G}$, $\partial^2_{z_s}\widetilde{G}$, and $\partial_\tau\widetilde{G}$; therefore, in the PDE satisfied by $\widetilde{G}$, the latter terms are exactly the same as in the PDE (2.45). We therefore directly arrive at the vanishing condition of the overall drift term coefficient in $d\mathcal{M}_s$,

$$X_s(z)\left[F(z_s) + z_s\left(\frac{z_s+1}{z_s-1} - \frac{\kappa}{2}\right)\partial_z - \partial_\tau - \frac{\kappa}{2}z_s^2\partial_z^2\right]\widetilde{G}(z_s,\tau) = 0. \tag{2.72}$$

Since $X_s(z)$ does not vanish in $\mathbb{D}$, the bracket in (2.72) must identically vanish:

$$\left[F(z_s) + z_s\frac{z_s+1}{z_s-1}\partial_z - \partial_\tau - \frac{\kappa}{2}(z_s\partial_z)^2\right]\widetilde{G}(z_s,\tau) = 0, \tag{2.73}$$

where we used $z\partial_z + z^2\partial_z^2 = (z\partial_z)^2$.

To derive the ODE satisfied by $G(z)$ (2.66), we first recall its expression as the limit (2.68), which further implies

$$\lim_{\tau\to+\infty} \exp\left(\frac{p-q}{2}\tau\right)\partial_\tau \tilde{G}(z,\tau) = -\frac{p-q}{2}G(z).$$

Multiplying the PDE (2.72) satisfied by $\widetilde{G}$ by $\exp(\frac{p-q}{2}\tau)$ and letting $\tau \to +\infty$, we get

$$\begin{aligned}\mathcal{P}(\partial)[G(z)] &:= \left[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z + F(z) + \frac{p-q}{2}\right]G(z)\\ &= \left[-\frac{\kappa}{2}(z\partial_z)^2 - \frac{1+z}{1-z}z\partial_z - \frac{p}{(1-z)^2} + \frac{q}{1-z} + p - q\right]G(z) = 0.\end{aligned} \tag{2.74}$$

We now look specifically for solutions to (2.74), together with the boundary condition $G(0) = 1$, of the form $\varphi_\alpha(z) = (1-z)^\alpha$. This function satisfies the simple differential operator algebra [5]

$$\mathcal{P}(\partial)[\varphi_\alpha] = A(p,q,\alpha)\varphi_\alpha + B(q,\alpha)\varphi_{\alpha-1} + C(p,\alpha)\varphi_{\alpha-2}, \tag{2.75}$$

where

$$A(p,q,\alpha) := p - q + \alpha - \frac{\kappa}{2}\alpha^2, \tag{2.76}$$

$$B(q,\alpha) := q - \left(3 + \frac{\kappa}{2}\right)\alpha + \kappa\alpha^2, \tag{2.77}$$

$$C(p,\alpha) := -p + \left(2 + \frac{\kappa}{2}\right)\alpha - \frac{\kappa}{2}\alpha^2. \tag{2.78}$$

Obviously, one has $A+B+C=0$. Because $\varphi_\alpha, \varphi_{\varphi-1}, \varphi_{\alpha-2}$ are linearly independent, the condition $\mathcal{P}(\partial)[\varphi_\gamma] = 0$ is equivalent to the system $A = B = C = 0$, or equivalently $A = C = 0$, hence $C(p,\gamma) = 0$ and $A(p,q,\gamma) - C(p,\gamma) = 2p - q - (1+\kappa/2)\gamma = 0$. It yields precisely the parabola parametrization (2.62) given in Theorem 2.3.1, and has for solution (2.64). □

2.4 SLE two-point function

2.4.1 Beliaev–Smirnov type equations

In this section, we will determine the mixed moments of moduli, $\mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right)$, for (p,q) belonging to the same parabola $\mathcal{R}$ as in Theorem 2.3.1, and where $f = f_0$ is the (time zero) interior whole-plane SLE$_\kappa$ map.

In contradistinction to the method used in Refs. [2, 5] for writing a PDE obeyed by $\mathbb{E}(|f'(z)|^p)$, we shall use here a slightly different approach, building on the results obtained in Section 2.3. We shall study the SLE two-point function for $z_1, z_2 \in \mathbb{D}$,

$$G(z_1,\bar{z}_2) := \mathbb{E}\left(z_1^{\frac{q}{2}} \frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}} \overline{\left[z_2^{\frac{q}{2}} \frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}} \right]} \right). \tag{2.79}$$

As before, we define a time-dependent, auxiliary two-point function,

$$\begin{aligned}\widetilde{G}(z_1,\bar{z}_2,t) &:= \mathbb{E}\left(z_1^{\frac{q}{2}}\frac{(\tilde{f}_t'(z_1))^{\frac{p}{2}}}{(\tilde{f}_t(z_1))^{\frac{q}{2}}}\overline{\left[z_2^{\frac{q}{2}}\frac{(\tilde{f}_t'(z_2))^{\frac{p}{2}}}{(\tilde{f}_t(z_2))^{\frac{q}{2}}}\right]}\right)\\ &= \mathbb{E}\left(z_1^{\frac{q}{2}}X_t(z_1)\overline{z_2^{\frac{q}{2}}X_t(z_2)}\right),\end{aligned} \tag{2.80}$$

where as above $\tilde{f}_t$ is the reverse radial SLE$_\kappa$ process 1.9, and where we used the shorthand notation (2.67). This time, the two-point function (2.79) is the limit

$$\lim_{t\to+\infty} e^{(p-q)t}\widetilde{G}(z_1,\bar{z}_2,t) = G(z_1,\bar{z}_2). \tag{2.81}$$

Let us define the two-point martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$, with

$$\mathcal{M}_s := \mathbb{E}(X_t(z_1)\overline{X_t(z_2)}|\mathcal{F}_s).$$

By the Markov property of SLE,

$$\mathbb{E}\big(X_t(z_1)\overline{X_t(z_2)}|\mathcal{F}_s\big) = X_s(z_1)\overline{X_s(z_2)}\,\widetilde{G}(z_{1s},\bar{z}_{2s},\tau),\ \ \tau := t-s, \tag{2.82}$$

where

$$z_{1s} := \tilde{f}_s(z_1)/\lambda(s);\ \ \bar{z}_{2s} := \overline{\tilde{f}_s(z_2)/\lambda(s)} = \overline{\tilde{f}_s(z_2)}\lambda(s). \tag{2.83}$$

Their Itô differentials, dz_{1s} and $d\bar{z}_{2s}$, are as in (2.54), (2.55)

$$\begin{aligned}dz_{1s} &= z_{1s}\left[\frac{z_{1s}+1}{z_{1s}-1}-\frac{\kappa}{2}\right]ds - i\sqrt{\kappa}\,z_{1s}\,dB_s,\\ d\bar{z}_{2s} &= \bar{z}_{2s}\left[\frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1}-\frac{\kappa}{2}\right]ds + i\sqrt{\kappa}\,\bar{z}_{2s}\,dB_s.\end{aligned} \tag{2.84}$$

As before, the partial differential equation satisfied by $\widetilde{G}(z_{1s},z_{2s},\tau)$ is obtained by expressing the fact that the ds-drift term of the Itô differential of Eq. (2.82),

$$d\mathcal{M}_s = [dX_s(z_1)\overline{X_s(z_2)} + X_s(z_1)d\overline{X_s(z_2)}]\,\widetilde{G} + X_s(z_1)\overline{X_s(z_2)}\,d\widetilde{G}, \tag{2.85}$$

vanishes.

The differentials of X_s, $\overline{X_s}$ are as in Eq. (2.71) above:

$$\begin{aligned}dX_s(z_1) &= X_s(z_1)F(z_{1s})ds,\ \ d\overline{X_s(z_2)} = \overline{X_s(z_2)}F(\bar{z}_{2s})ds,\\ F(z) &:= \frac{p}{2}-\frac{q}{2}-\frac{p}{(1-z)^2}+\frac{q}{1-z}.\end{aligned} \tag{2.86}$$

We thus obtain the simple expression

$$d\mathcal{M}_s = X_s(z_1)\overline{X_s(z_2)}\left[[F(z_{1s})+F(\bar{z}_{2s})]\,\widetilde{G}\,ds + d\widetilde{G}\right], \tag{2.87}$$

and the vanishing of the ds-drift term in $d\mathcal{M}_s$ requires that of the drift term in the right-hand side bracket in (2.87), since $X_s(z)$ does not vanish in $\mathbb{D}$.

The Itô differential of $\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ can be obtained from Eqs. (2.84) and Itô calculus as

$$\begin{aligned} d\widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau) =& \partial_1 \widetilde{G}\, dz_{1s} + \bar{\partial}_2 \widetilde{G}\, d\bar{z}_{2s} - \partial_\tau \widetilde{G}\, ds \\ & - \frac{\kappa}{2} z_{1s}^2\, \partial_1^2 \widetilde{G}\, ds - \frac{\kappa}{2} \bar{z}_{2s}^2\, \bar{\partial}_2^2 \widetilde{G}\, ds + \kappa z_{1s}\, \bar{z}_{2s}\, \partial_1 \bar{\partial}_2 \widetilde{G}\, ds, \end{aligned} \tag{2.88}$$

where use was made of the shorthand notations, $\partial_1 := \partial_{z_1}$ and $\bar{\partial}_2 := \partial_{\bar{z}_2}$. We observe that the only coupling between the $z_{1s}, \bar{z}_{2s}$ variables arises in the last term of (2.88), the other terms simply resulting from the independent contributions of the z_{1s} and $\bar{z}_{2s}$ parts.

Using again the Itô differentials (2.84), we can rewrite (2.88) as

$$\begin{aligned} d\widetilde{G} =& -i\sqrt{\kappa}\left(z_{1s}\partial_1 - \bar{z}_{2s}\bar{\partial}_2\right) \widetilde{G}\, dB_s \\ & + \frac{z_{1s}+1}{z_{1s}-1} z_{1s}\partial_1 \widetilde{G}\, ds + \frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1} \bar{z}_{2s}\bar{\partial}_2 \widetilde{G}\, ds - \partial_\tau \widetilde{G}\, ds \\ & - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\,\bar{\partial}_2)^2 \widetilde{G}\, ds, \end{aligned} \tag{2.89}$$

where we used the obvious formal identity

$$(z_1\partial_1)^2 + (\bar{z}_2\,\bar{\partial}_2)^2 - 2z_1\partial_1\bar{z}_2\,\bar{\partial}_2 = (z_1\partial_1 - \bar{z}_2\,\bar{\partial}_2)^2. \tag{2.90}$$

At this stage, comparing the computations (2.87) and (2.89) above with those in the one-point martingale study in Section 2.3, it is clear that the PDE obeyed by $\widetilde{G} = \widetilde{G}(z_{1s}, \bar{z}_{2s}, \tau)$ is obtained as two duplicates of Eq. (2.73), completed as in (2.90) by the derivative coupling between variables z_{1s}, $\bar{z}_{2s}$:

$$\left[F(z_{1s}) + z_{1s}\frac{z_{1s}+1}{z_{1s}-1}\partial_1 + F(\bar{z}_{2s}) + \bar{z}_{2s}\frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1}\bar{\partial}_2 - \partial_\tau - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\bar{\partial}_2)^2\right]\widetilde{G} = 0. \tag{2.91}$$

The existence of the limit (2.81) further implies that of

$$\lim_{\tau\to\infty} e^{(p-q)\tau}\partial_\tau \tilde{G}(z_1, \bar{z}_2, \tau) = -(p-q)G(z_1, \bar{z}_2).$$

Multiplying the PDE (2.91) satisfied by $\widetilde{G}$ by $\exp((p-q)\tau)$ and letting $\tau \to +\infty$, then gives the expected PDE for $G(z_1, \bar{z}_2)$. It can be most compactly written in terms of the ODE (2.74) as

$$\left[\mathcal{P}(\partial_1) + \mathcal{P}(\bar{\partial}_2) + \kappa z_1\partial_1\bar{z}_2\bar{\partial}_2\right] G(z_1, \bar{z}_2) = 0, \tag{2.92}$$

and its fully explicit expression is

$$\begin{aligned} \mathcal{P}(D)[G(z_1, \bar{z}_2)] =& -\frac{\kappa}{2}(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)^2 G - \frac{1+z_1}{1-z_1} z_1\partial_1 G - \frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2 G \\ + \Bigg[-& \frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar{z}_2)^2} + \frac{q}{1-z_1} + \frac{q}{1-\bar{z}_2} + 2p - 2q\Bigg] G = 0. \end{aligned} \tag{2.93}$$

2.4.2 Moduli one-point function

Note that one can take the $z_1 = z_2 = z$ case in Definition (2.79) above, thereby obtaining the moduli one-point function,

$$G(z,\bar{z}) = \mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right). \tag{2.94}$$

Because of Eq. (2.93), it obeys the corresponding ODE,

$$\begin{aligned}\mathcal{P}(D)[G(z,\bar{z})] = &-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 G - \frac{1+z}{1-z} z\partial G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G \\ &+ \left[-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} + \frac{q}{1-z} + \frac{q}{1-\bar{z}} + 2p - 2q\right] G = 0,\end{aligned} \tag{2.95}$$

which is the generalization to $q \neq 0$ of the Beliaev–Smirnov equation (2.35) studied in Refs. [5] and [12].

2.4.3 Integrable case

Lemma 2.4.1. *The space of formal series $F(z_1, \bar{z}_2) = \sum_{k,\ell \in \mathbb{N}} a_{k,\ell} z_1^k \bar{z}_2^\ell$, with complex coefficients and that are solutions of the PDE* (2.93), *is one-dimensional.*

Proof. We assume that F is a solution to (2.93) with $F(0,0) = 0$; it suffices to prove that, necessarily, $F = 0$. We argue by contradiction: If not, consider the minimal (necessarily non constant) term $a_{k,l} z^k \bar{z}^\ell$ in the series of F, with $a_{k,\ell} \neq 0$ and $k + \ell$ minimal (and non vanishing). Then $\mathcal{P}(D)[F]$ (2.93) will have a minimal term, equal to $-a_{k,\ell}\left[\frac{\kappa}{2}(k-\ell)^2 + k + \ell\right] z_1^k \bar{z}_2^\ell$, which is non-zero, contradicting the fact that $\mathcal{P}(D)[F]$ vanishes. □

As a second step, following Ref. [5], let us consider the action of the operator $\mathcal{P}(D)$ of (2.93) on a function of the factorized form $\varphi(z_1)\varphi(\bar{z}_2)P(z_1, \bar{z}_2)$, which we write, in a shorthand notation, as $\varphi\bar{\varphi}P$. By Leibniz's rule, it is given by

$$\begin{aligned}\mathcal{P}(D)[\varphi\bar{\varphi}P] = &-\frac{\kappa}{2}\varphi\bar{\varphi}(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)^2 P - \kappa(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)(\varphi\bar{\varphi})(z_1\partial_1 - \bar{z}_2\bar{\partial}_2)P \\ &+ \kappa(z_1\partial_1\varphi)(\bar{z}_2\bar{\partial}_2\bar{\varphi})P - \varphi\bar{\varphi}\frac{1+z_1}{1-z_1}z_1\partial_1 P - \varphi\bar{\varphi}\frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2 P \\ &- \left[\frac{\kappa}{2}\bar{\varphi}(z_1\partial_1)^2\varphi + \frac{\kappa}{2}\varphi(\bar{z}_2\bar{\partial}_2)^2\bar{\varphi} + \bar{\varphi}\frac{1+z_1}{1-z_1}z_1\partial_1\varphi + \varphi\frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_2\bar{\varphi}\right] P \\ &+ \left[-\frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar{z}_2)^2} + \frac{q}{1-z_1} + \frac{q}{1-\bar{z}_2} + 2p - 2q\right]\varphi\bar{\varphi}P.\end{aligned}$$

Note that the operator $z_1\partial_1 - \bar{z}_2\bar{\partial}_2$ is antisymmetric with respect to $z_1, \bar{z}_2$; therefore, if we choose a symmetric function, $P(z_1, \bar{z}_2) = P(z_1\bar{z}_2)$, the first line of $\mathcal{P}(D)[\varphi\bar{\varphi}P]$ above identically vanishes.

One then looks for solutions to (2.93) of the particular form,

$$G(z_1, \bar{z}_2) = \varphi_\alpha(z_1)\varphi_\alpha(\bar{z}_2)P(z_1\bar{z}_2), \tag{2.96}$$

where, as before, $\varphi_\alpha(z) = (1-z)^\alpha$. The action of the differential operator then takes the simple form,

$$\begin{aligned}\mathcal{P}(D)[\varphi_\alpha\bar{\varphi}_\alpha P] =& z_1\bar{z}_2\varphi_{\alpha-1}\bar{\varphi}_{\alpha-1}\left(\kappa\alpha^2 P - 2(1-z_1\bar{z}_2)P'\right)\\ &+ \mathcal{P}(\partial_1)[\varphi_\alpha]\bar{\varphi}_\alpha P + \mathcal{P}(\partial_2)[\bar{\varphi}_\alpha]\varphi_\alpha P,\end{aligned}$$

where P' is the derivative of P with respect to $z_1\bar{z}_2$, and $\mathcal{P}(\partial)$ is the so-called boundary operator (2.74) [5].

The ODE, $\kappa\alpha^2 P(x) - 2(1-x)P'(x) = 0$ with $x = z_1\bar{z}_2$ and $P(0) = 1$, has for solution $P(z_1\bar{z}_2) = (1-z_1\bar{z}_2)^{-\kappa\alpha^2/2}$. It is then sufficient to pick for α the value $\gamma = \gamma_0^\pm(p)$ (2.64) such that $\mathcal{P}(\partial)[\varphi_\gamma] = 0$, as obtained in the proof of Theorem 2.3.1, to get a solution of the PDE, $\mathcal{P}(D)[\varphi_\gamma\bar{\varphi}_\gamma P] = 0$ (2.93). By uniqueness of the solution with $G(0,0) = 1$, it gives the explicit form of the SLE two-point function,

$$G(z_1,\bar{z}_2) = \varphi_\gamma(z_1)\varphi_\gamma(\bar{z}_2)(1-z_1\bar{z}_2)^{-\kappa\gamma^2/2}.$$

We thus get:

Theorem 2.4.1. *Let $f(z) = f_0(z)$ be the interior whole-plane* SLE_κ *map in the setting of Theorem 2.1.3; then, for (p,q) belonging to the parabola $\mathcal{R}$ defined in Theorem 2.3.1 by Eqs.* (2.62) *or* (2.63) *or* (2.64)*, and for any pair $(z_1,z_2) \in \mathbb{D}\times\mathbb{D}$,*

$$\mathbb{E}\left(z_1^{\frac{q}{2}}\frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}}\overline{\left[z_2^{\frac{q}{2}}\frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}}\right]}\right) = \frac{(1-z_1)^\gamma(1-\bar{z}_2)^\gamma}{(1-z_1\bar{z}_2)^\beta}, \quad \beta = \frac{\kappa}{2}\gamma^2.$$

Corollary 2.4.1. *In the same setting as in Theorem 2.4.1, we have for $z \in \mathbb{D}$,*

$$\mathbb{E}\left(|z|^q\frac{|f'(z)|^p}{|f(z)|^q}\right) = \frac{(1-z)^\gamma(1-\bar{z})^\gamma}{(1-z\bar{z})^\beta}, \quad \beta = \frac{\kappa}{2}\gamma^2,$$

for

$$\begin{aligned}\gamma &= \gamma_0^\pm(p) := \frac{1}{2\kappa}\left(4+\kappa\pm\sqrt{(4+\kappa)^2-8\kappa p}\right), \ p \le \frac{(4+\kappa)^2}{8\kappa},\\ q &= 2p - \left(1+\frac{\kappa}{2}\right)\gamma_0^\pm(p).\end{aligned}$$

Let us stress some particular cases of interest. First, the $p = 0$ case gives some integral means of f.

Corollary 2.4.2. *The interior whole-plane* SLE_κ *map has the integrable moments*

$$\mathbb{E}\left(\left[\frac{f(z_1)}{z_1}\right]^{\frac{(2+\kappa)(4+\kappa)}{4\kappa}}\left[\frac{\overline{f(z_2)}}{\bar{z}_2}\right]^{\frac{(2+\kappa)(4+\kappa)}{4\kappa}}\right) = \frac{(1-z_1)^{\frac{4+\kappa}{\kappa}}(1-\bar{z}_2)^{\frac{4+\kappa}{\kappa}}}{(1-z_1\bar{z}_2)^{\frac{(4+\kappa)^2}{2\kappa}}},$$

$$\mathbb{E}\left(\left|\frac{f(z)}{z}\right|^{\frac{(2+\kappa)(4+\kappa)}{2\kappa}}\right) = \frac{(1-z)^{\frac{4+\kappa}{\kappa}}(1-\bar{z})^{\frac{4+\kappa}{\kappa}}}{(1-z\bar{z})^{\frac{(4+\kappa)^2}{2\kappa}}}.$$

Second, taking $p = q$ yields the logarithmic integral means we started with:

Corollary 2.4.3. *The interior whole-plane* SLE$_\kappa$ *map* $f(z) = f_0(z)$ *has the integrable logarithmic moment*

$$\mathbb{E}\left(\left[z_1\frac{f'(z_1)}{f(z_1)}\right]^{\frac{2+\kappa}{2\kappa}}\left[\bar{z}_2\overline{\frac{f'(z_2)}{f(z_2)}}\right]^{\frac{2+\kappa}{2\kappa}}\right) = \frac{(1-z_1)^{\frac{2}{\kappa}}(1-\bar{z}_2)^{\frac{2}{\kappa}}}{(1-z_1\bar{z}_2)^{\frac{2}{\kappa}}},$$

$$\mathbb{E}\left(\left|z\frac{f'(z)}{f(z)}\right|^{\frac{2+\kappa}{\kappa}}\right) = \frac{(1-z)^{\frac{2}{\kappa}}(1-\bar{z})^{\frac{2}{\kappa}}}{(1-z\bar{z})^{\frac{2}{\kappa}}}.$$

Theorem 2.1.4 describes the $\kappa = 2$ case of the latter result.

2.5 Processes with $m-$fold symmetry

The results of Section 2.4 may be generalized to functions with m-fold symmetry, with m a positive integer, as was studied in [5]. For f in class $\mathcal{S}$, $f^{[m]}(z)$ is defined as being the holomorphic branch of $f(z^m)^{1/m}$ whose derivative is equal to 1 at 0. These are the functions in $\mathcal{S}$ whose Taylor series is of the form $f(z) = \sum_{k\geq 0} a_{mk+1}z^{mk+1}$. The $m = 2$ case corresponds to odd functions that play a crucial role in the theory of univalent functions.

One can also extend this definition to negative integers m, by considering then the m-fold transform of the outer whole-plane SLE as the conjugate by the inversion $z \mapsto 1/z$ of the $(-m)$-fold transform of the inner whole-plane SLE: $f^{[m]}(z) = 1/f^{[-m]}(1/z)$ for $m \in \mathbb{Z} \setminus \mathbb{N}$ and $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. The $m = -1$ case is of special interest: $f^{[-1]}$ maps the exterior of the unit disk onto the inverted image of $f(\mathbb{D})$, which is a domain with bounded boundary. Actually, for $f(z)$ the interior whole-plane SLE$_\kappa$ map considered in Ref. [5] and here, $f^{[-1]}(z^{-1})$ is precisely the exterior whole-plane SLE$_\kappa$ map introduced in Ref. [2].

The moments, $\mathbb{E}(|(f^{[m]})'(z)|^p)$ (for $m \in \mathbb{N} \setminus \{0\}$), as well as their associated integral means spectra were studied in Ref. [5]. Using Itô calculus, a PDE satisfied by these moments was derived for each value of m. The introduction of mixed (p, q) moments allows us to circumvent these calculations in a unified approach. To see this, notice that

$$(f^{[m]})'(z) = z^{m-1}f'(z^m)f(z^m)^{\frac{1}{m}-1}.$$

As a consequence,

$$\frac{|z|^q|(f^{[m]})'(z)|^p}{|f^{[m]}(z)|^q} = |z|^{q+p(m-1)}\frac{|f'(z^m)|^p}{|f(z^m)|^{p+\frac{q-p}{m}}},$$

so that we identically have

$$\mathbb{E}\left(|z|^q\frac{|(f^{[m]})'(z)|^p}{|f^{[m]}(z)|^q}\right) = G(z^m; p, q_m), \tag{2.97}$$

$$q_m = q_m(p, q) := p + \frac{q-p}{m}, \tag{2.98}$$

with the notation,

$$G(z;p,q) := G(z,\bar{z}) = \mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right), \tag{2.99}$$

where we have made explicit the dependence on the (p,q) parameters of the SLE moduli one-point function (2.94) introduced in Section 2.3. From Theorem 2.4.1, we immediately get the following.

Theorem 2.5.1. *Let $f^{[m]}$ be the m-fold whole-plane* SLE_κ *map, $m \in \mathbb{Z} \setminus \{0\}$, with $z \in \mathbb{D}$ for $m > 0$ and $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ for $m < 0$. Then,*

$$\mathbb{E}\left(|z|^q \frac{|(f^{[m]})'(z)|^p}{|f^{[m]}(z)|^q}\right) = \frac{(1-z^m)^\alpha(1-\bar{z}^m)^\alpha}{(1-(z\bar{z})^m)^{\frac{\kappa}{2}\alpha^2}},$$

for (p,q) belonging to the m-dependent parabola $\mathcal{R}^{[m]}$, given in parametric form by

$$p = \left(2+\frac{\kappa}{2}\right)\alpha - \frac{\kappa}{2}\alpha^2,\ \ q = \left(m+2+\frac{\kappa}{2}\right)\alpha - \frac{\kappa}{2}(m+1)\alpha^2,\ \ \alpha \in \mathbb{R}. \tag{2.100}$$

In Cartesian coordinates, an equivalent statement is

$$\alpha = \frac{(m+1)p-q}{m\left(1+\frac{\kappa}{2}\right)},$$

with

$$q = (m+1)p - m\frac{2+\kappa}{4\kappa}\left(4+\kappa \pm \sqrt{(4+\kappa)^2 - 8\kappa p}\right),\ \ p \leq \frac{(4+\kappa)^2}{8\kappa},$$

or,

$$p = \frac{q}{m+1} + \frac{m}{(m+1)^2}\frac{2+\kappa}{4\kappa}\left(2m+4+\kappa \pm \sqrt{(2m+4+\kappa)^2 - 8(m+1)\kappa q}\right),$$
$$q \leq \frac{(2m+4+\kappa)^2}{8(m+1)\kappa}.$$

As for logarithmic coefficients, first observe that trivially,

$$\log\frac{f^{[m]}(z)}{z} = \frac{1}{m}\log\frac{f(z^m)}{z^m}. \tag{2.101}$$

From this, and Theorem 2.1.3, we thus get

Corollary 2.5.1. *Let $f^{[m]}(z)$ be the m-fold whole-plane* SLE_2 *map and*

$$\log\frac{f^{[m]}(z)}{z} = 2\sum_{n\geq 1}\gamma_n^{[m]} z^n; \tag{2.102}$$

then

$$\mathbb{E}(|\gamma_n^{[m]}|^2) = \begin{cases} \frac{1}{2n^2} & n = mk,\quad k \geq 1 \\ 0 & \textit{otherwise.}\end{cases}$$

We can also see this result as a corollary of Theorem 2.5.1, which, for the logarithmic case $p = q$, and for any value of m, yields $p = q = 2$ for $\kappa = 2$ as the only integrable case.

Chapter 3

INTEGRAL MEAN SPECTRUM OF SLE$_\kappa$

3.1 Introduction

In this chapter we aim at generalizing to the setting of the present work the integral means spectrum analysis of Refs. [2], [5], [12], [13] concerning the whole-plane SLE. The original work by Beliaev–Smirnov [2] dealt with the exterior version, whereas Ref. [5] and this work concern the interior case. We thus look for the singular behavior of the integral,

$$\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right)|dz|, \tag{3.1}$$

for $r \to 1^-$, where f stands for the interior whole-plane SLE map (at time zero). The integral means spectrum $\beta(p,q)$ corresponding to this generalized moment integral is the exponent such that

$$\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right)|dz| \overset{(r\to 1^-)}{\asymp} (1-r)^{-\beta(p,q)}, \tag{3.2}$$

in the sense of the equivalence of the logarithms of both terms.

As mentioned in Section 1.4, it is interesting to remark that the map $\hat{f} := f^{[-1]}$,

$$\zeta \in \mathbb{C}\setminus\overline{\mathbb{D}} \mapsto f^{[-1]}(\zeta) := 1/f(1/\zeta),$$

is just the *exterior* whole-plane map from $\mathbb{C}\setminus\overline{\mathbb{D}}$ to the slit plane considered by Beliaev and Smirnov in Ref. [2]. We identically have for $0 < r < 1$:

$$\int_{r^{-1}\partial\mathbb{D}} \mathbb{E}\left(|\hat{f}'(\zeta)|^p\right)|d\zeta| = r^{2p-2}\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^{2p}}\right)|dz|. \tag{3.3}$$

We thus see that the standard integral mean of order $(p, q = 0)$ for the exterior whole-plane map studied in Ref. [2] *coincides* (up to an irrelevant power of r) with the (p,q) integral mean (3.1) for $q = 2p$, for the interior whole-plane map.

Remark 3.1.1. *Exterior-Interior Duality. More generally, we obviously have*

$$\int_{r^{-1}\partial\mathbb{D}} \mathbb{E}\left(\frac{|\hat{f}'(\zeta)|^p}{|\hat{f}(\zeta)|^{q'}}\right)|d\zeta| = r^{2p-2}\int_{r\partial\mathbb{D}} \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^{2p-q'}}\right)|dz|, \tag{3.4}$$

so that the (p,q') exterior integral means spectrum coincides with the (p,q) interior integral means spectrum for $q+q'=2p$. In particular, the $(p,0)$ interior derivative moments studied in Ref. [5] correspond to the $(p,2p)$ mixed moments of the Beliaev–Smirnov exterior map.

Hence the general setting introduced in this work unifies the integral means spectrum studies of Refs. [2] and [5] in a broader framework, that also covers the $p=q=q'$ logarithmic case, as well as the integral means of the map f (or $\hat{f}$) itself, in the $(0,q)$ (or $(0,-q)$) case.

3.1.1 Modified One-Point Function

Let us now consider the *modified* SLE moduli one-point function,

$$F(z,\bar{z}) := \frac{1}{|z|^q}G(z,\bar{z}) = \mathbb{E}\left(\frac{|f'(z)|^p}{|f(z)|^q}\right). \tag{3.5}$$

Because of Eq. (2.95), it obeys the modified PDE,

$$\begin{aligned}\mathcal{P}(D)[F(z,\bar{z})] = &-\frac{\kappa}{2}(z\partial-\bar{z}\bar{\partial})^2F - \frac{1+z}{1-z}z\partial F - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}F \\ &+\left[-\frac{p}{(1-z)^2}-\frac{p}{(1-\bar{z})^2}+2p-q\right]F(z,\bar{z})=0,\end{aligned} \tag{3.6}$$

which, of course, differs from Eq. (2.95). We can rewrite it as

$$\begin{aligned}\mathcal{P}(D)[F(z,\bar{z})] = &-\frac{\kappa}{2}(z\partial-\bar{z}\bar{\partial})^2F - \frac{1+z}{1-z}z\partial F - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}F \\ &-p\left[\frac{1}{(1-z)^2}+\frac{1}{(1-\bar{z})^2}+\sigma-1\right]F=0,\end{aligned} \tag{3.7}$$

in term of the important new parameter,

$$\sigma := q/p-1. \tag{3.8}$$

This PDE then exactly coincides with Eq. (106) in Ref. [5], where σ was meant to represent ±1, whereas here $\sigma\in\mathbb{R}$.

The value $\sigma=+1$ corresponds to the original Beliaev–Smirnov case, where the integral means spectrum successively involves three functions [2]:

$$\beta_{\mathrm{tip}}(p,\kappa) := -p-1+\frac{1}{4}(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}), \tag{3.9}$$

$$\text{for } p \le p_0'(\kappa) := -1-\frac{3\kappa}{8}; \tag{3.10}$$

$$\beta_0(p,\kappa) := -p+\frac{4+\kappa}{4\kappa}(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}), \tag{3.11}$$

$$\text{for } p_0'(\kappa) \le p \le p_0(\kappa);$$

$$\beta_{\mathrm{lin}}(p,\kappa) := p-\frac{(4+\kappa)^2}{16\kappa}, \tag{3.12}$$

$$\text{for } p \ge p_0(\kappa) := \frac{3(4+\kappa)^2}{32\kappa}. \tag{3.13}$$

As shown in Refs. [5, 12, 13] in the $\sigma = -1$ interior case, because of the unboundedness of the interior whole-plane SLE map, there exists a phase transition at $p = p^*(\kappa)$, with

$$\begin{aligned} p^*(\kappa) &:= \frac{1}{16\kappa}\left((4+\kappa)^2-4-2\sqrt{2(4+\kappa)^2+4}\right) \\ &= \frac{1}{32\kappa}\left(\sqrt{2(4+\kappa)^2+4}-6\right)\left(\sqrt{2(4+\kappa)^2+4}+2\right). \end{aligned} \tag{3.14}$$

The integral means spectrum is afterwards given by

$$\beta(p,\kappa) := 3p-\frac{1}{2}-\frac{1}{2}\sqrt{1+2\kappa p}, \text{ for } p \ge p^*(\kappa). \tag{3.15}$$

Since $p^*(\kappa) < p_0(\kappa)$ (3.13), this transition precedes and supersedes the transition from the bulk spectrum (3.11) towards the linear behavior (3.12).

The singularity analysis given in Ref. [5] led us to introduce the σ-dependent function

$$\beta_+^\sigma(p,\kappa) = (1-2\sigma)p-\frac{1}{2}\big(1+\sqrt{1-2\sigma\kappa p}\big). \tag{3.16}$$

For $\sigma = -1$, it recovers the integral means spectrum (3.15) above for the interior whole-plane SLE, while for $\sigma = +1$ it introduces a new spectrum,

$$\beta_+^{(+1)}(p,\kappa) = -p-\frac{1}{2}\big(1+\sqrt{1-2\kappa p}\big), \tag{3.17}$$

the relevance of which for the exterior whole-plane SLE case is analyzed in a joint work of D. Beliaev, B. Duplantier and M. Zinsmeister [1].

For general real values of σ (3.8), we can rewrite (3.16) as a function of (p,q,κ),

$$\beta_+^\sigma(p,\kappa) = \beta_1(p,q;\kappa) := 3p-2q-\frac{1}{2}-\frac{1}{2}\sqrt{1+2\kappa(p-q)}. \tag{3.18}$$

We claim that the spectrum generated by the integral means (3.1) in the general (p,q) case will involve the standard multifractal spectra (3.9), (3.11), (3.12), that are independent of q, and also the new (p,q)-dependent multifractal spectrum (3.18).

Phase transitions between these spectra will occur along lines drawn in the (p, q) plane.

Let us first describe the corresponding partition of the (p, q) plane into the respective domains of validity of the four spectra above. We thus need to determine the boundary curves where pairs (possibly triplets) of these spectra coincide, which are signaling the onset of the respective transitions.

3.1.2 Phase transition lines

The best way is perhaps to recall the analytical derivation of the various multifractal spectra as done in Ref. [5], which was based on the use of functions A (2.76), B (2.77) and C (2.78). It will be convenient to use the notation [5],

$$A^\sigma(p, \gamma) := -\frac{\kappa}{2}\gamma^2 + \gamma - \sigma p, \tag{3.19}$$

such that for $\sigma = q/p - 1$ (3.8),

$$A^\sigma(p, \gamma) = A(p, q; \gamma) = p - q + \gamma - \frac{\kappa}{2}\gamma^2, \tag{3.20}$$

as well as

$$B(q, \gamma) = q - \left(3 + \frac{\kappa}{2}\right)\gamma + \kappa\gamma^2, \tag{3.21}$$

$$C(p, \gamma) = -\frac{\kappa}{2}\gamma^2 + \left(2 + \frac{\kappa}{2}\right)\gamma - p, \tag{3.22}$$

$$\beta(p, \gamma) := \frac{\kappa}{2}\gamma^2 - C(p, \gamma) = \kappa\gamma^2 - \left(2 + \frac{\kappa}{2}\right)\gamma + p, \tag{3.23}$$

where the last function, $\beta(p, \gamma)$, is the so-called "spectrum function" [5]. Recall also that this function possesses an important duality property [5],

$$\beta(p, \gamma) = \beta(p, \gamma'), \;\; \gamma + \gamma' := \frac{2}{\kappa} + \frac{1}{2}. \tag{3.24}$$

Remark 3.1.2. *The B–S parameter γ_0, and bulk spectrum* (3.11) $\beta_0 := \beta(p, \gamma_0)$, *(corresponding to Eqs. (11) and (12) in Ref. [2]) are obtained from the equations (see Ref. [5]),*

$$C(p, \gamma_0) = 0; \quad \beta_0 = \beta(p, \gamma_0) = \kappa\gamma_0^2/2. \tag{3.25}$$

The two solutions to (3.25) *are $\gamma_0^\pm(p)$ as in Eq.* (2.64)*, where the lower branch $\gamma_0 := \gamma_0^-$ is the one selected for the bulk spectrum, $\beta_0(p) = \frac{1}{2}\kappa\gamma_0^-(p)^2$.*

This spectrum (3.11) *is defined only to the left of a vertical line in the (p, q) plane, as given by (see Fig. 3.1)*

$$\Delta_0 := \left\{p = \frac{(4 + \kappa)^2}{8\kappa}, q \in \mathbb{R}\right\}. \tag{3.26}$$

Remark 3.1.3. *The σ-dependent spectrum* (3.16) *is obtained from the equations*

$$A^\sigma(p,\gamma) = 0; \quad \beta(p,\gamma) = \kappa\gamma^2/2 - C(p,\gamma). \tag{3.27}$$

The solutions to Eq. (3.27) *are*

$$\gamma_\pm^\sigma(p) = \frac{1}{\kappa}\left(1 \pm \sqrt{1-2\sigma\kappa p}\right), \tag{3.28}$$

$$\beta_\pm^\sigma(p) = (1-2\sigma)p - \frac{\kappa}{2}\gamma_\pm^\sigma(p) = (1-2\sigma)p - \frac{1}{2}\left(1 \pm \sqrt{1-2\sigma\kappa p}\right). \tag{3.29}$$

The multifractal spectrum (3.16) *is then given by the upper branch $\beta_+^\sigma(p)$ [5]. Note also that this spectrum is defined only for $2\sigma\kappa p \le 1$, hence for points in the (p,q) plane below the oblique line (Fig. 3.1):*

$$\Delta_1 := \left\{(p,q) \in \mathbb{R}^2, q = p + \frac{1}{2\kappa}\right\}. \tag{3.30}$$

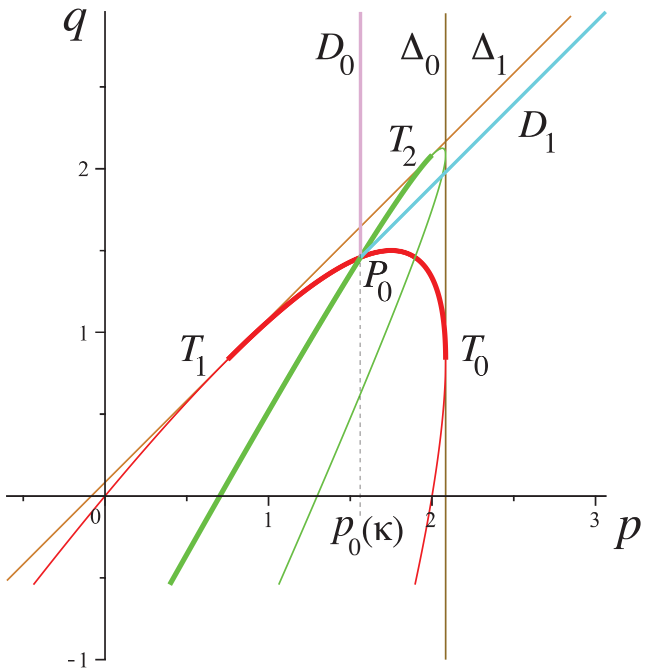


Figure 3.1 – *Red parabola $\mathcal{R}$* (3.32) *and green parabola $\mathcal{G}$* (3.37) *(for $\kappa = 6$). From the intersection point P_0* (3.39) *originate the two (half)-lines D_0* (3.41) *and D_1* (3.42)*. The bulk spectrum $\beta_0(p)$ and the generalized spectrum $\beta_1(p,q)$ coincide along the arc* (3.34) *of red parabola between its tangency points T_0 and T_1 with Δ_0 and Δ_1 (thick red line). They also coincide along the infinite left branch* (3.38) *of the green parabola, up to its tangency point T_2 to Δ_1 (thick green line). The $\beta_0(p)$ spectrum and the linear one $\beta_{\rm lin}(p)$ coincide along D_0, whereas $\beta_1(p,q)$ and $\beta_{\rm lin}(p)$ coincide along D_1.*

3.1.2.1 The Red Parabola

The parabola $\mathcal{R}$ of Theorems 2.3.1 and 2.4.1, which we shall hereafter call (and draw in) **red** (see Fig. 3.1), is given by the simultaneous conditions,

$$A^\sigma(p,\gamma) = A(p,q,\gamma) = 0,\ C(p,\gamma) = 0, \tag{3.31}$$

hence also $B(q,\gamma) = 0$, which recovers the parametric form (2.62)

$$\begin{aligned} p = p_{\mathcal{R}}(\gamma) &:= \left(2+\frac{\kappa}{2}\right)\gamma - \frac{\kappa}{2}\gamma^2, \\ q = q_{\mathcal{R}}(\gamma) &:= \left(3+\frac{\kappa}{2}\right)\gamma - \kappa\gamma^2,\ \gamma \in \mathbb{R}. \end{aligned} \tag{3.32}$$

By construction, the associated spectrum $\beta(p,\gamma)$ is therefore both of the B–S type, $\beta_0^\pm(p)$, and of the novel type, $\beta_\pm^\sigma(p)$. We successively have:

$$\gamma = \gamma_-^\sigma(p) = \gamma_0^-(p);\ \beta_-^\sigma(p) = \beta_0^-(p), \gamma \in (-\infty, 1/\kappa]\,, \tag{3.33}$$
$$\gamma = \gamma_+^\sigma(p) = \gamma_0^-(p);\ \beta_+^\sigma(p) = \beta_0^-(p), \gamma \in [1/\kappa, 2/\kappa + 1/2]\,, \tag{3.34}$$
$$\gamma = \gamma_+^\sigma(p) = \gamma_0^+(p);\ \beta_+^\sigma(p) = \beta_0^+(p), \gamma \in [2/\kappa + 1/2, +\infty)\,, \tag{3.35}$$

where the change of analytic branch from the first to the second line corresponds to a tangency at T_1 of the red parabola to the boundary line Δ_1, whereas the change from second to third corresponds to a tangency at T_0 to the vertical boundary line Δ_0. The interval where the multifractal spectra coincide, i.e., when $\beta_+^\sigma(p) = \beta_0^-(p)$, is thus given by line (3.34) in the equations above.

In Cartesian coordinates, the red parabola $\mathcal{R}$ (3.32) has for equation (2.63).

3.1.2.2 The Green Parabola

A second parabola in the (p,q) plane, hereafter called **green** (see Fig. 3.1) and denoted by $\mathcal{G}$, is such that the multifractal spectra $\beta_0^-(p)$ and $\beta_+^\sigma(p) = \beta(p,q;\kappa)$ coincide on part of it. We use the *duality* property (3.24) of the spectrum function [5], and set the simultaneous seed conditions,

$$\begin{aligned} &A^\sigma(p,\gamma') = A(p,q,\gamma') = 0,\ C(p,\gamma'') = 0, \\ &\gamma' + \gamma'' = 2/\kappa + 1/2, \end{aligned} \tag{3.36}$$

where γ' and γ'' are *dual* of each other and such that $\beta(p,\gamma') = \beta(p,\gamma'')$.

Eqs. (2.76) and (2.78) immediately give the parametric form for the green parabola,

$$\begin{aligned} p = p_{\mathcal{G}}(\gamma') &:= \frac{(4+\kappa)^2}{8\kappa} - \frac{\kappa}{2}\gamma'^2, \\ q = q_{\mathcal{G}}(\gamma') &:= \frac{(4+\kappa)^2}{8\kappa} + \gamma' - \kappa\gamma'^2,\ \gamma' \in \mathbb{R}. \end{aligned} \tag{3.37}$$

Along this locus, we successively have:

$$\begin{aligned} &\gamma' = \gamma_-^\sigma(p), \gamma'' = \gamma_0^+(p);\ \beta_-^\sigma(p) = \beta_0^+(p), \gamma' \in (-\infty, 0]\,, \\ &\gamma' = \gamma_-^\sigma(p), \gamma'' = \gamma_0^-(p);\ \beta_-^\sigma(p) = \beta_0^-(p), \gamma' \in \left[0, \kappa^{-1}\right], \\ &\gamma' = \gamma_+^\sigma(p), \gamma'' = \gamma_0^-(p);\ \beta_+^\sigma(p) = \beta_0^-(p), \gamma' \in \left[\kappa^{-1}, +\infty\right), \end{aligned} \tag{3.38}$$

where the changes of branches correspond to a tangency of the green parabola to Δ_0 followed by a tangency to Δ_1. The multifractal spectra coincide when $\beta_+^\sigma(p) = \beta_0^-(p)$, which corresponds to the third line (3.38) in the equations above, i.e., to the domain where $\gamma' \geq 1/\kappa$.

3.1.2.3 Quadruple point

The intersection of the red and green parabolas (3.32) and (3.37) can be found by combining the seed equations (3.31) and (3.36). We find either $\gamma = \gamma' = 1/\kappa + 1/4$, or $\gamma = 2/\kappa + 1/4, \gamma' = -1/4$, which lead to the two intersection points,

$$P_0 : p_0 = p_0(\kappa) = \frac{3(4+\kappa)^2}{32\kappa}, \qquad q_0 = \frac{(4+\kappa)(8+\kappa)}{16\kappa}, \tag{3.39}$$

$$P_1 : p_1 = \frac{(8+\kappa)(8+3\kappa)}{32\kappa}, \qquad q_0 = \frac{(4+\kappa)(8+\kappa)}{16\kappa}. \tag{3.40}$$

Note that these points have same ordinate, while the abscissa of the left-most one, P_0, is $p_0(\kappa)$ (3.13), where the integral means spectrum transits from the bulk form (3.11) to its linear form (3.12).

Through this intersection point P_0 further pass two important straight lines in the (p, q) plane.

Definition 3.1.1. *D_0 and D_1 are, respectively, the vertical line and the slope one line passing through point P_0, of equations*

$$D_0 := \{(p, q) : p = p_0\}, \tag{3.41}$$

$$D_1 := \left\{(p, q) : q - p = q_0 - p_0 = \frac{16 - \kappa^2}{32\kappa}\right\}. \tag{3.42}$$

A key property of D_1 is the following. The difference,

$$\beta_1(p, q; \kappa) - \beta_{\text{lin}}(p, \kappa) = \frac{1}{\kappa}\left(\frac{\kappa}{4} - \sqrt{1 + 2\kappa(p - q)}\right)^2, \tag{3.43}$$

is always positive, and vanishes only on line D_1, where

$$\forall (p, q) \in D_1, \;\; \beta_1(p, q; \kappa) = \beta_{\text{lin}}(p, \kappa) = p - \frac{(4+\kappa)^2}{16\kappa}. \tag{3.44}$$

3.1.2.4 The Blue Quartic

A third locus, the blue quartic $\mathcal{Q}$, will also play an important role, that is where the tip-spectrum, $\beta_{\text{tip}}(p; \kappa)$ (3.9), coincides with the novel spectrum, $\beta_+^\sigma(p) = \beta_1(p, q; \kappa)$. The tip spectrum is given by $\beta_{\text{tip}}(p; \kappa) = \beta(p, \gamma_0) - 2\gamma_0 - 1$, where γ_0 is solution to $C(p, \gamma_0) = 0$ and such that the tip contribution is positive, $2\gamma_0 + 1 \leq 0$ [2, 5]; this corresponds to the tip condition (3.10) [2]. In the (p, q) plane, this descibes the domain to the left of the straight line D_0' (Fig. 3.2), defined by

$$D_0' := \{(p, q) : p = p_0'(\kappa) = -1 - 3\kappa/8\}. \tag{3.45}$$

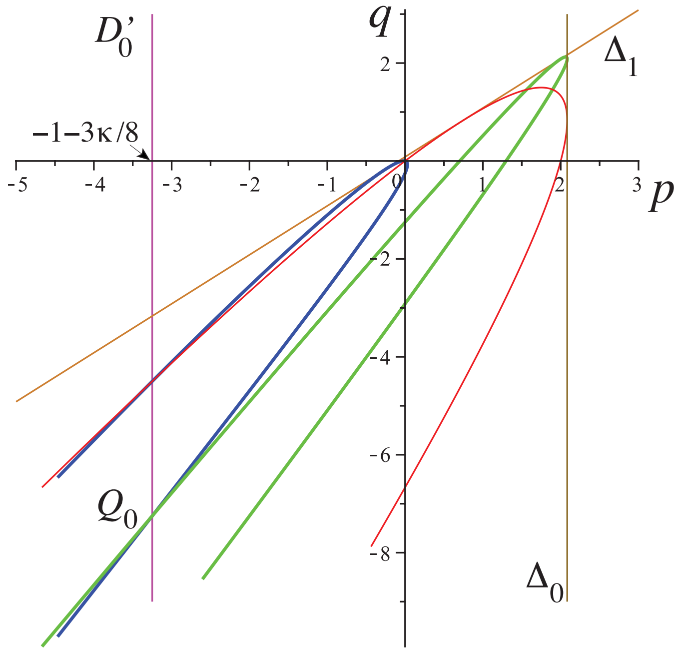


Figure 3.2 – *The blue quartic* (3.52) *for* $\kappa = 6$. *It intersects the green parabola at point* Q_0 (3.56) *and the red parabola at point* Q_1 (3.55) *(not marked), both of abscissa* $p_0'(\kappa) = -1 - 3\kappa/8$.

The generalized spectrum is given by $\beta_+^\sigma(p) = \beta(p, \gamma)$ where γ is solution to $A^\sigma(p, \gamma) = 0$. We therefore look for simultaneous solutions to the seed equations,

$$\begin{aligned} &\beta(p,\gamma) = \beta(p,\gamma_0) - 2\gamma_0 - 1, \;\; 2\gamma_0 + 1 \leq 0, \\ &A^\sigma(p,\gamma) = 0, \;\; C(p,\gamma_0) = 0. \end{aligned} \tag{3.46}$$

Using Eq. (3.20), we first find, as for the red and green parabolae,

$$q - p = \gamma - \frac{\kappa}{2}\gamma^2, \tag{3.47}$$

and from (3.23) and (3.22), by substitution in the above,

$$2p - q + \frac{1}{2} = \frac{\kappa}{4}(\gamma + \gamma_0), \tag{3.48}$$

$$\frac{4+\kappa}{2}\gamma - \kappa\gamma^2 - 1 = \frac{8+\kappa}{2}\gamma_0 - \kappa\gamma_0^2. \tag{3.49}$$

Solving for γ_0 in terms of γ gives

$$\gamma_0 = \gamma_0^\pm := \frac{8+\kappa}{4\kappa} \pm \frac{1}{2\kappa}\Delta^{\frac{1}{2}}(\gamma), \tag{3.50}$$

$$\Delta(\gamma) := 4\kappa^2\gamma^2 - 2\kappa(4+\kappa)\gamma + \frac{1}{4}(8+\kappa)^2 + 4\kappa, \tag{3.51}$$

with $\Delta(\gamma) > 0, \forall \gamma \in \mathbb{R}$. The tip relevance inequality in (3.46), $2\gamma_0 + 1 \leq 0$, implies the choice of the negative branch in (3.50): $\gamma_0 = \gamma_0^-$. We thus get the desired explicit parameterization of that branch of the quartic,

$$\begin{aligned} p = p_{\mathcal{Q}}(\gamma) &:= \frac{\kappa}{16} + \left(1 + \frac{\kappa}{4}\right)\gamma - \frac{\kappa}{2}\gamma^2 - \frac{1}{8}\Delta^{\frac{1}{2}}(\gamma), \\ q = q_{\mathcal{Q}}(\gamma) &:= p_{\mathcal{Q}}(\gamma) + \gamma - \frac{\kappa}{2}\gamma^2, \ \ \gamma \in \mathbb{R}. \end{aligned} \tag{3.52}$$

Remark 3.1.4. *Note that because of the very choice to parameterize the parabolae and the quartic by γ, such that A (3.20) vanishes, Eq. (3.47) holds for each of the pairs of parametric equations.*

We successively have along the branch (3.52) of the blue quartic:

$$\gamma = \gamma_-^\sigma(p);\ \beta_-^\sigma(p) = \beta_{\text{tip}}(p), \gamma \in (-\infty, 1/\kappa]\,,$$

$$\gamma = \gamma_+^\sigma(p);\ \beta_+^\sigma(p) = \beta_{\text{tip}}(p) < \beta_0^-(p), \gamma \in [1/\kappa, 1 + 2/\kappa]\,, \tag{3.53}$$

$$\gamma = \gamma_+^\sigma(p);\ \beta_+^\sigma(p) = \beta_{\text{tip}}(p) \geq \beta_0^-(p), \gamma \in [1 + 2/\kappa, +\infty)\,, \tag{3.54}$$

The intersection of the **blue** quartic (3.52) with the **red** parabola $\mathcal{R}$ (3.32) is located at

$$Q_1 : p_0' = -1 - \frac{3\kappa}{8}, q = -\frac{1}{2}(3 + \kappa);\ \ \gamma = \gamma_0 = -\frac{1}{2}, \tag{3.55}$$

followed by a second intersection at the origin, $p = q = 0$, for $\gamma = \frac{2}{\kappa}$ and $\gamma_0 = 0$.

The intersection of the **blue** quartic (3.52) with the **green** parabola $\mathcal{G}$ (3.37) is located at

$$Q_0 : p_0' = -1 - \frac{3\kappa}{8}, q_0' := -2 - \frac{7\kappa}{8};\ \ \gamma = \gamma' = 1 + \frac{2}{\kappa}, \gamma_0 = -\frac{1}{2}. \tag{3.56}$$

Notice that these two intersection points have same abscissae, $p_0'(\kappa)$ (3.10), where the transition for $\gamma_0 = -\frac{1}{2}$ from the bulk spectrum β_0 to the tip spectrum β_{tip} takes place. They are found by combining Eqs. (3.31) or Eqs. (3.36) with (3.46).

The tip spectrum and the generalized one coincide in both γ-intervals (3.53) and (3.54), which together parameterize the branch of the quartic located below its contact with Δ_1 (see Fig. 3.2). Because of the tip relevance condition (3.10), only the interval (3.54) describing the lower infinite branch of the quartic located to the *left* of Q_0 will matter for the integral means spectrum.

3.1.3 Whole-plane SLE$_\kappa$ generalized spectrum

3.1.3.1 Summary

Let us briefly summarize the results of Section 3.1.2. We know from Eq. (3.34) that the bulk spectrum $\beta_0(p)$ and the mixed spectrum $\beta_1(p, q)$ coincide along the finite sector of parabola $\mathcal{R}$ located between tangency points T_0 and T_1 (Fig. 3.1). From Eq. (3.38), we also know that they coincide along the infinite left branch of parabola $\mathcal{G}$ below the tangency point T_2 (Fig. 3.1).

The linear bulk spectrum $\beta_{\text{lin}}(p)$ coincides with $\beta_0(p)$ along line D_0 and supersedes the latter to the right of D_0 (Fig. 3.1). We know from (3.44) that $\beta_{\text{lin}}(p)$ and $\beta_1(p, q)$ coincide along the line D_1 (Fig. 3.1).

The tip spectrum $\beta_{\text{tip}}(p)$ coincides with $\beta_0(p)$ along line D_0', and supersedes it to the left of D_0'. We finally know from Eq. (3.54) that this tip spectrum $\beta_{\text{tip}}(p)$ coincides with $\beta_1(p,q)$ along the lower branch of the blue quartic located below point Q_0 (3.56) (Fig. 3.2).

The only possible scenario which thus emerges to construct the average generalized integral means spectrum by a continuous matching of the 4 different spectra along the phase transition lines described above, is the partition of the (p,q) plane in 4 different regions as indicated in Fig. 3.3:

— a part (I) to the left of D_0' and located above the blue quartic up to point Q_0, where the average integral means spectrum is $\beta_{\text{tip}}(p)$;

— an upper part (II) bounded by lines D_0', D_0, and located above the section of the green parabola between points Q_0 and P_0, where the spectrum is given by $\beta_0(p)$;

— an infinite wedge (III) of apex P_0 located between the upper half-lines D_0 and D_1, where the spectrum is given by $\beta_{\text{lin}}(p)$;

— a lower part (IV) whose boundary is the blue quartic up to point Q_0, followed by the arc of green parabola between points Q_0 and P_0, followed by the half-line D_1 above P_0 where the spectrum is $\beta_1(p,q)$.

The two wings T_1P_0 and P_0T_0 of the red parabola (Fig. 3.1), where we know from Theorem 2.4.1 that the average spectrum is given by $\beta_0(p) = \beta_1(p,q)$, can thus be seen as the respective extensions of region IV into II and of region II into IV.

This is summarized by the following theorem.

Theorem 3.1.1. *Separatrix curves for the generalized integral means spectrum of whole-plane* SLE_κ *are in the* (p,q) *plane (Fig. 3.3):*

— *(i) the vertical half-line* D_0 *above* $P_0 = (p_0, q_0)$ (3.39), *where* $p_0 = 3(4+\kappa)^2/32\kappa, q_0 = (4+\kappa)(8+\kappa)/16\kappa$;

— *(ii) the unit slope half-line* D_1 *originating at* P_0, *whose equation is* $q - p = (16-\kappa^2)/32\kappa$ *with* $p \geq p_0$;

— *(iii) the section of green parabola, with parametric coordinates* $\big(p_{\mathcal{G}}(\gamma), q_{\mathcal{G}}(\gamma)\big)$ (3.37) *for* $\gamma \in [1/4 + 1/\kappa, 1 + 2/\kappa]$, *between* P_0 *and* $Q_0 = (p_0', q_0')$ (3.56), *where* $p_0' = -1 - 3\kappa/8, q_0' = -2 - 7\kappa/8$;

— *(iv) the vertical half-line* D_0' *above point* Q_0;

— *(v) the branch of the blue quartic from* Q_0 *to* ∞, *with parametric coordinates* $\big(p_{\mathcal{Q}}(\gamma), q_{\mathcal{Q}}(\gamma)\big)$ (3.52) *for* $\gamma \in [1 + 2/\kappa, +\infty)$.

3.1.3.2 The B–S line

As mentioned above, the whole-plane SLE case studied by Beliaev and Smirnov corresponds to the $q = 2p$ line. Because of Eq. (2.63), it intersects the red parabola $\mathcal{R}$ only at $p = 0$. The green parabola $\mathcal{G}$ (3.37) has for Cartesian equation,

$$\frac{\kappa}{2}(2p-q)^2 - \frac{1}{8}(4+\kappa)^2\,(2p-q) + p + \frac{1}{128}(4+\kappa)^2(8+\kappa) = 0, \tag{3.57}$$

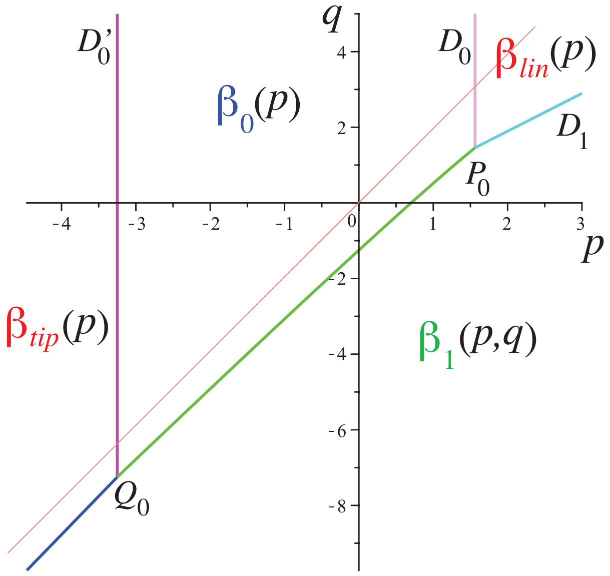


Figure 3.3 – *Respective domains of validity of integral means spectra* $\beta_{\text{tip}}(p)$, $\beta_0(p)$, $\beta_{\text{lin}}(p)$, *and* $\beta_1(p,q)$. *The thin straight line (coral)* $q = 2p$ *corresponds to the version of whole-plane SLE studied in Ref. [2]. It does not intersect the lower domain where* β_1 *holds.*

which shows that it intersects the B–S line at [1]

$$p = p_0''(\kappa) := -\frac{1}{128}(4+\kappa)^2(8+\kappa), \tag{3.58}$$

which is to the *left* of the tip transition line at $p_0'(\kappa) = -1 - \frac{3}{8}\kappa$ (3.10). The quartic $\mathcal{Q}$ (3.52) obeys

$$\left[\left(2p - q - \frac{\kappa}{16}\right)^2 - \frac{c}{4}\right]\left(2p - q - 1 - \frac{\kappa}{8}\right)(2p-q) = \frac{\kappa}{2}(p-q)\left(2p - q - \frac{1}{4} - \frac{\kappa}{8}\right)^2$$
$$c = c(\kappa) := \frac{1}{64}(8+\kappa)^2 + \frac{\kappa}{4}, \tag{3.59}$$

which immediately shows that the B–S line $q = 2p$ intersects $\mathcal{Q}$ only at the origin and stays above its lower branch.

The B–S line therefore does not intersect the segment of green parabola $\mathcal{G}$ between P_0 and Q_0, nor the quartic $\mathcal{Q}$ below Q_0 (Fig. 3.3). Thus the novel spectrum β_1 *does not* a priori appear in the version of whole-plane SLE considered in Ref. [2]. The B–S line nevertheless intersects $\mathcal{G}$ at p_0'' (3.58) to the left of Q_0, in a domain lying above the quartic and where the integral mean receives a non-vanishing contribution from the SLE tip. But if that integral mean is restricted to avoid a neighborhood of $z = 1$, whose image is the tip, only the bulk spectrum remains, and a phase transition will take place from β_0 to β_1 when the line $q = 2p$ crosses $\mathcal{G}$.

3.2 Integral mean spectrum on the red parabola

From Corollary 2.4.1, we have

Theorem 3.2.1. *Let* $f(z) = f_0(z)$ *be the interior whole-plane* SLE_κ *map in the setting of Theorem 2.1.3; then, for* (p, q) *belonging to the parabola* $\mathcal{R}$ *defined in Theorem 2.3.1 by Eqs.* (2.62) *or* (2.63) *or* (2.64),

$$\beta(p,q,\kappa) = \begin{cases} \frac{\kappa}{2}\gamma^2 - 2\gamma - 1 & \gamma \leq -\frac{1}{2} \\ \frac{\kappa}{2}\gamma^2 & \gamma > -\frac{1}{2}. \end{cases}$$

From (3.33),(3.34),(3.35), we can rewrite the spectrum on $\mathcal{R}$ as

$$\beta(p,q,\kappa) = \begin{cases} \beta_0^-(p) - 2\gamma - 1 & \gamma = \gamma_0^-(p) \leq -\frac{1}{2} \\ \beta_0^-(p) & -\frac{1}{2} < \gamma = \gamma_0^-(p) \leq \frac{1}{\kappa} + \frac{1}{4} \\ \beta_+^\sigma(p) & \gamma = \gamma_+^\sigma(p) > \frac{1}{\kappa} + \frac{1}{4}, \end{cases}$$

or in term of $\beta_{tip}, \beta_0, \beta_1$, as

$$\beta(p,q,\kappa) = \begin{cases} \beta_{tip}(p) & \gamma \leq -\frac{1}{2} \\ \beta_0(p) & -\frac{1}{2} < \gamma \leq \frac{1}{\kappa} + \frac{1}{4} \\ \beta_1(p,q) & \gamma > \frac{1}{\kappa} + \frac{1}{4}. \end{cases}$$

It is therefore yielded that the generalized spectrum β is the tip spectrum on the arc of the red parabola $\mathcal{R}$ lying on the left of the vertical line D_0', is the bulk spectrum on the arc of $\mathcal{R}$ between D_0' and the section of the green parabola $\mathcal{G}$ jointing P_0 and Q_0, is the β_1 spectrum on the arc of $\mathcal{R}$ lying on the right of that section of the green parabola $\mathcal{G}$.

3.3 Integral mean spectrum on a family of parabolas

The initial motivation of this section is to give precise values of the spectrum on a infinite family of parabolas in the $(p, q)-$plane, which has the parabola $\mathcal{R}$ as a member. The original idea of this section is from [12], [13]. It should be noticed that the analysis presented here is heuristic and some points are not rigorous, which should be more rigorously considered. We will obtain results which are consistent with the manifold of spectrum introduced in the Section 3.1

3.3.1 New parabolas

We recall that

$$G(z,\bar{z}) = \mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right) \tag{3.60}$$

is a solution of the ODE

$$\mathcal{P}(D)[G(z,\bar{z})] = -\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 G - \frac{1+z}{1-z} z\partial G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}G \\ + \left[-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} + \frac{q}{1-z} + \frac{q}{1-\bar{z}} + 2p - 2q \right] G = 0. \tag{3.61}$$

We now consider solutions of (3.61) of the form

$$G(z,\bar{z}) = (1-z)^\gamma (1-\bar{z})^\gamma \Theta(z,\bar{z}) =: \varphi_\gamma(z)\varphi_\gamma(\bar{z})\Theta(z,\bar{z}), \tag{3.62}$$

with γ satisfying

$$p = -\frac{\kappa}{2}\gamma^2 + (2 + \frac{\kappa}{2})\gamma, \tag{3.63}$$

which means $C(p,\gamma) = 0$, where $C(p,\gamma)$ is defined by (2.78). This condition can get rid of many singular terms in (3.61) (see also [5] and [13]).
By the Liebnitz's rule, the action of $\mathcal{P}(D)$ on $\varphi_\gamma(z)\varphi_\gamma(\bar{z})\Theta(z,\bar{z})$ is given by

$$\frac{\mathcal{P}(D)(\varphi_\gamma \bar{\varphi}_\gamma \Theta)}{\varphi_{\gamma-1}\bar{\varphi}_{\gamma-1}} = \\ -\frac{\kappa}{2}(1-z)(1-\bar{z})(z\partial - \bar{z}\bar{\partial})^2\Theta + (\kappa\gamma - 1)(z-\bar{z})(z\partial - \bar{z}\bar{\partial})\Theta - (1-z\bar{z})(z\partial + \bar{z}\bar{\partial})\Theta \\ + [(k\gamma^2 + 2A)z\bar{z} - (2A+B)(z+\bar{z}) + 2(A+B)]\Theta. \tag{3.64}$$

Here A, B are shorthand notations standing for $A(p,q,\gamma), B(q,\gamma)$ defined by (2.76), (2.77). Since $A + B + C = 0$ and $C(p,\gamma) = 0$, the last term in the coefficient of Θ vanishes. By plugging (2.76),(2.77),(2.78) into (3.64), one then arrives at

$$\frac{\mathcal{P}(D)(\varphi_\gamma \bar{\varphi}_\gamma \Theta)}{\varphi_{\gamma-1}\bar{\varphi}_{\gamma-1}} = -\frac{\kappa}{2}(1-z)(1-\bar{z})(z\partial - \bar{z}\bar{\partial})^2\Theta + (\kappa\gamma - 1)(z-\bar{z})(z\partial - \bar{z}\bar{\partial})\Theta \\ - (1-z\bar{z})(z\partial + \bar{z}\bar{\partial})\Theta + [Vz\bar{z} + U(z+\bar{z})]\Theta, \tag{3.65}$$

where

$$U = U(q,\gamma) := q - (3 + \frac{\kappa}{2})\gamma + \kappa\gamma^2 \tag{3.66}$$

$$V = V(q,\gamma) := -2q + (6+\kappa)\gamma - \kappa\gamma^2. \tag{3.67}$$

From above we see that $G(z,\bar{z}) = \varphi_\gamma(z)\varphi_\gamma(\bar{z})\Theta(z,\bar{z})$ is a solution of (3.61) if and only if $\Theta(z,\bar{z})$ obeys the equation

$$-\frac{\kappa}{2}(1-z)(1-\bar{z})(z\partial - \bar{z}\bar{\partial})^2\Theta + (\kappa\gamma - 1)(z-\bar{z})(z\partial - \bar{z}\bar{\partial})\Theta \\ - (1-z\bar{z})(z\partial + \bar{z}\bar{\partial})\Theta + [Vz\bar{z} + U(z+\bar{z})]\Theta = 0. \tag{3.68}$$

Assuming that $\Theta(z,\bar{z})$ has the series expansion as

$$\Theta(z,\bar{z}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \theta_{i,j}(\gamma,q,\kappa) z^{i-1}\bar{z}^{j-1}, \quad \theta_{1,1} = 1, \tag{3.69}$$

then the equation (3.68) give us the following recursions of $\theta_{i,j}$

$$\sum_{k=0}^{1}\sum_{l=0}^{1} C_{i,j}^{l,k}\theta_{i-l,j-k} = 0, \quad \theta_{1,1} = 1, \quad \theta_{i,j} = 0 \text{ for } i \leq 0 \text{ or } j \leq 0, \tag{3.70}$$

with

$$C_{i,j}^{0,0} = -\frac{\kappa}{2}(i-j)^2 - (i+j-2), \tag{3.71}$$
$$C_{i,j}^{0,1} = \frac{\kappa}{2}(i-j+1)^2 - (\kappa\gamma - 1)(i-j+1) + U, \quad C_{i,j}^{1,0} = C_{j,i}^{0,1}, \tag{3.72}$$
$$C_{i,j}^{1,1} = -\frac{\kappa}{2}(i-j)^2 + (i+j-4) + V. \tag{3.73}$$

These recursions imply that if one set

$$q = -\kappa\gamma^2 + \left(3 + \frac{1-2M}{2}\kappa\right)\gamma - \frac{M^2\kappa}{2} + M, \tag{3.74}$$

then $\theta_{i,j} = 0$ for $|i-j| > M$, or equivalently, the coefficient matrix $(\theta_{i,j})$ is $2M + 1-$diagonal. For instance, in the case of $M = 0$, Eq. (3.74) becomes the second equation in (2.62), which together with (3.63), from Corollary 2.4.1, imply

$$\Theta(z,\bar{z}) = \frac{1}{(1-z\bar{z})^{\frac{\kappa\gamma^2}{2}}}. \tag{3.75}$$

The coefficient matrix $(\theta_{i,j})$ is then diagonal, as implied by the above series analysis. By considering γ as a parameter, the points (p,q) determined by (3.63) and (3.74) perform a parabola in the $(p,q)-$plane for each M. We denote these parabolas by $\mathcal{P}_M, M = 0, 1, ...$ The parabola $\mathcal{R}$ defined in Theorem 2.3.1 by Eqs. (2.62) or (2.63) or (2.64) coincides with $\mathcal{P}_0$.

3.3.2 Integral mean spectrum as eigenvalues

In the previous section, we have showed that the $2M + 1-$truncations of the coefficient matrix $(\theta_{i,j})$ correspond to parabolas in the $(p,q)-$plane. We now show that on those parabolas, the integral mean spectrum $\beta(p,q)$ is determined by a eigenvalue of a tridiagonal matrix.

In the case of $2M + 1-$truncations, one can write $\Theta(z,\bar{z})$ as

$$\Theta(z,\bar{z}) = \sum_{n=-M}^{M} f_n(z\bar{z})z^n, \quad f_{-n}(\xi) = \xi^n f_n(\xi). \tag{3.76}$$

Because Θ is real, the coefficients $\theta_{i,j}$ are real and therefore f_n is real for $n = -M, ..., M$.

Substitute (3.76) into (3.68), one obtains

$$\sum_{n=-M}^{M}\Big[\Big(-\frac{\kappa}{2}n^2+n+V\Big)\xi f_n(\xi)+\Big(-\frac{\kappa}{2}n^2-n\Big)f_n(\xi)+2\xi(\xi-1)f_n'(\xi)\Big]z^n$$
$$+\sum_{n=-M}^{M}\Big(\frac{\kappa}{2}n^2+(\kappa\gamma-1)n+U\Big)f_n(\xi)z^{n+1}$$
$$+\sum_{n=-M}^{M}\Big(\frac{\kappa}{2}n^2-(\kappa\gamma-1)n+U\Big)\xi f_n(\xi)z^{n-1}=0. \tag{3.77}$$

By identifying "coefficients" of z^n for $n=-M+1,...,M-1$ in the both sides of (3.77), one arrives at

$$\xi A_{n+1}f_{n+1}+A_{-n+1}f_{n-1}+[B_n+(1-\xi)C_n]f_n+2\xi(\xi-1)f_n'=0, \tag{3.78}$$

where

$$A_n=\frac{\kappa}{2}n^2-(\kappa\gamma-1)n+U, \tag{3.79}$$
$$B_n=-\kappa n^2+V, \tag{3.80}$$
$$C_n=\frac{\kappa}{2}n^2-n-V. \tag{3.81}$$

Similarly, identifying "coefficients" of z^{-M} and z^M yields

$$\Big[\Big(-\frac{\kappa}{2}M^2-M+V\Big)\xi+\Big(-\frac{\kappa}{2}M^2+M\Big)\Big]f_{-M}$$
$$+\Big[\frac{\kappa}{2}(-M+1)^2-(\kappa\gamma-1)(-M+1)+U\Big]\xi f_{-M+1}+2\xi(\xi-1)f_{-M}'=0 \tag{3.82}$$

and

$$\Big[\Big(-\frac{\kappa}{2}M^2+M+V\Big)\xi+\Big(-\frac{\kappa}{2}M^2-M\Big)\Big]f_M$$
$$+\Big[\frac{\kappa}{2}(M-1)^2+(\kappa\gamma-1)(M-1)+U\Big]\xi f_{M-1}+2\xi(\xi-1)f_M'=0. \tag{3.83}$$

The functional vector $(f_n)_{n=-M..M}$ is a solution of the linear system of $2M+1$ first order ODEs (3.78), (3.82),(3.83). It is worth to note that if we have the additional condition $f_{-n}(\xi)=\xi^n f_n(\xi)$, which is satisfied by the "coefficients" f_n of Θ, then this system is reduced to $M+1$ ODEs corresponding to $n=0,...,M$. It is also noticed that from these $M+1$ ODEs of $f_0,f_1,...,f_M$ one can obtain a $M+1-$th order ODE for each f_n, which has polynomial coefficients and singularities at $1,0,\infty$.

We consider the first two cases of $M=0$ and $M=1$ as examples. When $M=0$ the linear system includes the only ODE

$$\kappa\gamma^2\xi f_0(\xi)+2\xi(\xi-1)f_0'(\xi)=0. \tag{3.84}$$

Note that f_0 coincides with the function P considered in the Section 2.4.3 and (3.84) is equivalent to the ODE obeyed by P in that section. These equations have the solution $f_0(\xi) = (1-\xi)^{-\kappa\gamma^2/2}$.

When $M = 1$, with the additional condition $f_{-n}(\xi) = \xi^n f_n(\xi)$ the linear system includes two ODEs

$$-4(\kappa\gamma - 1)f_1 + (\kappa\gamma^2 + 2\kappa\gamma + \kappa - 2)f_0 + 2(\xi - 1)f_0' = 0 \tag{3.85}$$

$$(-\kappa\gamma - \frac{\kappa}{2} + 1)f_0 + \left[\left(\frac{\kappa}{2} - 1 + \kappa\gamma^2 + 2\kappa\gamma\right)\xi - \frac{\kappa}{2} - 1\right]f_1 + 2\xi(\xi - 1)f_1' = 0. \tag{3.86}$$

We now study the asymptotic behavior of the $2M+1-$band solutions (3.76) near the unit circle, i.e. at $\xi \to 1$. Assume that f_n has the expansion at $\xi = 1$ as

$$f_n(\xi) = (1-\xi)^{-\tilde{\beta}_n} d_n + ..., d_n \neq 0 \quad n = -M, ..., M, \tag{3.87}$$

then by putting $\tilde{\beta} := \max_{n=-M..M} \tilde{\beta}_n$, one may write f_n as

$$f_n(\xi) = (1-\xi)^{-\tilde{\beta}}\psi_n + ..., \quad n = -M, ..., M. \tag{3.88}$$

These expansions are such that there is at least one coefficient $\psi_n \neq 0$.
By substituting (3.88) into the system (3.78), (3.82),(3.83), one obtains equations whose left hand sides are $O((1-\xi)^{-\tilde{\beta}})$. After dividing both sides of these equations by $(1-\xi)^{-\tilde{\beta}}$ and letting $\xi \to 1$, it yields

$$(-\kappa M^2 + V)\psi_{-M} + \left[\frac{\kappa}{2}(M-1)^2 + (\kappa\gamma - 1)(M-1) + U\right]\psi_{-M+1} = 2\tilde{\beta}\psi_{-M} \tag{3.89}$$

$$A_{n+1}\psi_{n+1} + A_{-n+1}\psi_{n-1} + B_n\psi_n = 2\tilde{\beta}\psi_n, \quad n = -M+1, ..., M-1 \tag{3.90}$$

$$\left[\frac{\kappa}{2}(M-1)^2 + (\kappa\gamma - 1)(M-1) + U\right]\psi_{M-1} + (-\kappa M^2 + V)\psi_M = 2\tilde{\beta}\psi_M. \tag{3.91}$$

The power $\tilde{\beta}$ and the vector $(\psi_n)_{n=-M..M}$ are thus respectively an eigenvalue and a corresponding eigenvector of the coefficient matrix (upto factor $\frac{1}{2}$) of the system (3.89),(3.90),(3.91) which is tridiagonal.

It is worth to note that the above tridiagonal matrix is symmetric with respect to n, therefore its eigenvectors are either symmetric or antisymmetric with respect to n. In the case of $2M+1-$truncations, because Θ is symmetric in $z, \bar{z}$, if ψ_n are related to Θ by (3.76) and (3.88) then $\psi_{-n} = \psi_n$. In this case, $(\psi_n)_{n=-M..M}$ is a symmetric (with respect to n) eigenvector of the tridiagonal matrix.

We introduce the function

$$\Psi(\varphi) := \sum_{n=-M}^{M} \psi_n e^{in\varphi}. \tag{3.92}$$

In the case of $2M+1-$truncations, the fact that ψ_n are real and $\psi_{-n} = \psi_n$ implies Ψ are real and even. Moreover, it follows from (3.88)

$$\Psi(\varphi) = \lim_{r\to 1^-} \frac{\Theta(re^{i\varphi}, re^{-i\varphi})}{(1-r^2)^{-\tilde{\beta}}}. \tag{3.93}$$

From (3.62) as well as the fact that $(1-z)^\gamma(1-\bar{z})^\gamma$ is no longer integrable when $\gamma < \frac{1}{2}$, we have

$$\beta(p,q,\kappa) = \begin{cases} \tilde{\beta} - 2\gamma - 1 & \gamma < -\frac{1}{2} \\ \tilde{\beta} & \gamma \geq -\frac{1}{2}, \end{cases} \tag{3.94}$$

provided that $\psi(\varphi)(0)$ does not vanish in the case of $\gamma < -\frac{1}{2}$. If $\psi(\varphi)(0) = 0$ when $\gamma < -\frac{1}{2}$ then $\beta-$spectrum is less than $\tilde{\beta} - 2\gamma - 1$, as we will see, this is the case. Therefore, the integral mean spectrum of whole-plane SLE_κ is determined by an eigenvalue of the tridiagonal matrix (3.89),(3.90),(3.91) .

3.3.3 Computation of eigenvalues

In this section, we will compute exact values of the eigenvalues of the tridiagonal matrix in the case of $2M+1-$truncations.

From the system (3.89),(3.90),(3.91), one can perform an equation for Ψ by multiplying both sides of the n-th equation by $e^{in\varphi}$ and then making the summation of these equation for n from $-M$ to M. Namely, one can obtain

$$\begin{aligned} &\frac{\kappa}{2}(-e^{-i\varphi} - e^{i\varphi} + 2)\partial^2_{\varphi\varphi}\Psi + i(\kappa\gamma - 1)(e^{-i\varphi} - e^{i\varphi})\partial_\varphi\Psi + [(e^{-i\varphi} + e^{i\varphi})U + V - 2\tilde{\beta}]\Psi \\ &+ \left(-\frac{\kappa}{2}M^2 - (\kappa\gamma - 1)M - U\right)e^{-i(M+1)\varphi}\psi_{-M} \\ &+ \left(-\frac{\kappa}{2}M^2 - (\kappa\gamma - 1)M - U\right)e^{i(M+1)\varphi}\psi_{M} = 0. \end{aligned} \tag{3.95}$$

One then uses the condition of $2M+1-$truncations (3.74) and the definition of U (3.66) to reduce the above equation to

$$\frac{\kappa}{2}(-e^{-i\varphi} - e^{i\varphi} + 2)\partial^2_{\varphi\varphi}\Psi + i(\kappa\gamma - 1)(e^{-i\varphi} - e^{i\varphi})\partial_\varphi\Psi + [(e^{-i\varphi} + e^{i\varphi})U + V - 2\tilde{\beta}]\Psi = 0. \tag{3.96}$$

This equation can be rewriten as

$$\kappa(1-\cos(\varphi))\partial^2_{\varphi\varphi}\Psi + 2(\kappa\gamma - 1)\sin(\varphi)\partial_\varphi\Psi + [2\cos(\varphi)U + V - 2\tilde{\beta}]\Psi = 0. \tag{3.97}$$

As was mentioned in the previous section, an eigenvector $(\psi_n)_{n=-M..M}$ of the system (3.89),(3.90),(3.91) is either symmetric or antisymmetric in n, i.e., either $\psi_{-n} = \psi_n$ or $\psi_{-n} = -\psi_n$. In the first case, Ψ is a polynomial of $\cos(\varphi)$. After a change of variable $x = \frac{1-\cos(\varphi)}{2}$, Ψ is a polynomial in x and Eq. (3.97) becomes

$$2\kappa x^2(1-x)\Psi'' + x[\kappa + 4(\kappa\gamma - 1) - 2(\kappa + 2(\kappa\gamma - 1))x]\Psi' + (2U + V - 2\tilde{\beta} - 4Ux)\Psi = 0, \tag{3.98}$$

where the notation Ψ denotes the same function as in (3.97) and Ψ', Ψ'' respectively stand for the first and the second derivatives of Ψ with respect to x.
In the second case, Ψ can be written as

$$\Psi(\varphi) = \mathrm{sign}(\sin(\varphi))\sqrt{\frac{1-\cos^2(\varphi)}{4}}\Psi_1(\varphi). \tag{3.99}$$

with Ψ_1 is a polynomial of $\cos(\varphi)$. From (3.97) and the fact that the derivative of the sign function is 0 expect at 0, the function $\Psi_2(\varphi) := \sqrt{\frac{1-\cos^2(\varphi)}{4}}\Psi_1(\varphi)$ obeys the equation (3.97). By changing again variable $x = \frac{1-\cos(\varphi)}{2}$, the function Ψ_2 is a solution of (3.98). Note that with this change of variable, Ψ_1 is a polynomial in x and $\Psi_2 = \sqrt{x(1-x)}\Psi_1$.

Conversely, if the equation (3.98) with a value of parameter $\tilde{\beta}$ has a solution Ψ which is a polynomial of x, then $\tilde{\beta}$ is a eigenvalue of the system (3.89),(3.90),(3.91) with a corresponding symmetric eigenvector. This eigenvector can be found from the solution Ψ by changing variable $x = \frac{1-\cos(\varphi)}{2}$ and the relation (3.92).
Also, if the equation (3.98) with a value of parameter $\tilde{\beta}$ has a solution of the form $\sqrt{x(1-x)}\Psi_1$ where Ψ_1 is a polynomial of x, then $\tilde{\beta}$ is a eigenvalue of the system (3.89),(3.90),(3.91) with a corresponding antisymmetric eigenvector. This eigenvector can be found from the solution Ψ by changing variable $x = \frac{1-\cos(\varphi)}{2}$ and the relations (3.99),(3.92).

Next, we will determine $2M+1$ different values of $\tilde{\beta}$ such that for each of them, there is a solution of (3.98) which is a polynomial of x or the product of a polynomial of x with $\sqrt{x(1-x)}$. Therefore we will determine all eigenvalues of the system (3.89),(3.90),(3.91).

Consider the solutions of (3.98) of the form

$$\Psi(x) = x^{\lambda/2}g(x). \tag{3.100}$$

By substituting (3.100) into (3.98), one obtains an equation obeyed by $g(x)$

$$\begin{aligned}&\kappa x^2(1-x)g'' + x[\kappa\lambda + A - (\kappa\lambda + B)x]g' \\ &\left[\kappa\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right) + \frac{\lambda}{2}A + U + \frac{V}{2} - \tilde{\beta} + \left(-\kappa\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right) - \frac{\lambda}{2}B - 2U\right)x\right]g = 0,\end{aligned} \tag{3.101}$$

where $P = \frac{\kappa}{2} + 2(\kappa\gamma - 1)$ and $Q = \kappa + 2(\kappa\gamma - 1)$.

If we set

$$\tilde{\beta} = \kappa\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right) + \frac{\lambda}{2}P + U + \frac{V}{2} =: \tilde{\beta}(\lambda) \tag{3.102}$$

then (3.101) becomes a hypergeometric equation

$$x(1-x)g'' + \left[\lambda + \frac{P}{\kappa} - \left(\lambda + \frac{Q}{\kappa}\right)x\right]g' + \left[-\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right) - \frac{\lambda}{2}\frac{Q}{\kappa} - \frac{2U}{\kappa}\right]g = 0. \tag{3.103}$$

One can rewrite this equation as

$$x(1-x)g''(x) + [c - (a+b+1)x]g'(x) - abg(x) = 0, \tag{3.104}$$

where

$$a = \frac{\lambda}{2} - M, \quad b = \frac{\lambda}{2} + 2\gamma - \frac{2}{\kappa} + M, \quad c = a + b + \frac{1}{2}. \tag{3.105}$$

Solutions of (3.104) has the general form

$$g = C_1 g_1 + C_2 g_2, \tag{3.106}$$

with

$$g_1(x) = {}_2F_1(a, b, a+b+\frac{1}{2}; x), \quad g_2(x) = \sqrt{1-x}\ {}_2F_1(a+\frac{1}{2}, b+\frac{1}{2}, \frac{3}{2}; 1-x). \quad (3.107)$$

Since $a = \frac{\lambda}{2} - M$, we see that if $\lambda = 2n$ for $n = 0, ..., M$ then g_1 is a polynomial of x, whereas if $\lambda = 2n+1$ for $n = 0, ..., M-1$ then g_2 is the product of $\sqrt{1-x}$ and a polynomial of x. Corresponding to these values of λ and these forms of g, by (3.100), the solution Ψ of (3.98) is a polynomial of x or a product of $\sqrt{x(1-x)}$ and a polynomial of x. Therefore, $\tilde{\beta}(2n), n = 0, ..., M$ and $\tilde{\beta}(2n+1), n = 0, ..., M-1$ are the eigenvalues of the tridiagonal matrix (3.89),(3.90),(3.91). Moreover, $\tilde{\beta}(2n), n = 0, ..., M$ correspond to symmetric eigenvectors, which is the case of the $2M+1-$truncation solution (3.76).

3.3.4 Phase transition of integral mean spectrum

In the previous section, we have proved that the eigenvector $\tilde{\beta}$ which determines $\beta-$integral mean spectrum by (3.94) takes value among $\tilde{\beta}(2n), n = 0, ..., M$

$$\tilde{\beta}(2n) = n\left[\kappa\left(n + 2\gamma - \frac{1}{2}\right) - 2\right] + \frac{k\gamma^2}{2}. \quad (3.108)$$

Because of the factor $x^{\frac{\lambda}{2}}$ in (3.100), where $x = \frac{1-\cos(\varphi)}{2}$ which vanishes as $z \to 1$, we have that if $\tilde{\beta} = \tilde{\beta}(2n)$ then

$$\beta(p, q, \kappa) = \begin{cases} \max\{\tilde{\beta}, \tilde{\beta} - 2\gamma - 1 - n\} & \gamma < -\frac{1}{2} \\ \tilde{\beta} & \gamma \geq -\frac{1}{2}. \end{cases} \quad (3.109)$$

Obviously, with $M = 0$, $\tilde{\beta}$ coincides with $\tilde{\beta}(0) = \frac{\kappa\gamma^2}{2}$. We thus have the $\beta-$spectrum in this case as in Theorem 3.2.1. However, when $M \geq 1$, $\tilde{\beta}$ does not have to coincide with a $\tilde{\beta}(2n)$ for all γ but may change continuously in γ from one to others of $\tilde{\beta}(2n), n = 0, ..., M$. Indeed, we will use the non-negativity and the continuity of $\beta-$spectrum to show that it is determined by the biggest among $\tilde{\beta}(2n)$ which is either $\tilde{\beta}(0)$ or $\tilde{\beta}(2M)$. It will be showed that there is a new phase transition of $\beta-$spectrum.

We firstly prove that there is a point γ_M such that for $\gamma > \gamma_M$, the eigenvalue $\tilde{\beta}(2M)$ is the biggest among $\tilde{\beta}(2n), n = -M, ..., M$ and it is strictly bigger than the others, whereas for $\gamma < \gamma_M$, the eigenvalue $\tilde{\beta}(0)$ is the biggest and strictly bigger than the others. We have

$$\tilde{\beta}(2n_2) - \tilde{\beta}(2n_1) = \frac{\kappa}{2}(n_2 - n_1)\left(n_1 + n_2 + 2\gamma - \frac{2}{\kappa} - \frac{1}{2}\right). \quad (3.110)$$

Let $n_2 = M, n_1 = 0$, one obtain the point γ_M for which $\tilde{\beta}(2M) = \tilde{\beta}(0)$ as

$$\gamma_M = -\frac{M}{2} + \frac{1}{\kappa} + \frac{1}{4}. \quad (3.111)$$

When $\gamma > \gamma_M$, from (3.110), one have for $n_2 = M$ and $0 \leq n_1 \leq M-1$

$$\tilde{\beta}(2M) - \tilde{\beta}(2n_1) > \frac{\kappa}{2}(M-n_1)n_1 \geq 0. \tag{3.112}$$

When $\gamma < \gamma_M$, also from (3.110), one have for $n_2 = 0$ and $1 \leq n_1 \leq M$

$$\tilde{\beta}(0) - \tilde{\beta}(2n_1) > \frac{-\kappa n_1}{2}(n_1 - M) \geq 0. \tag{3.113}$$

Therefore the biggest eigenvalue is either $\tilde{\beta}(0)$ or $\tilde{\beta}(2M)$ with a phase transition at γ_M.

Recall that our work in the present section and the previous section is based on the $2M+1-$truncation condition (3.74). We now represent this condition as

$$\kappa = \frac{2(M+3\gamma-q)}{M^2+2M\gamma+2\gamma^2-\gamma}, \tag{3.114}$$

and consider κ as a quantity depending on the two parameters γ, q. The condition $\kappa > 0$ is then equivalent to $\gamma > \frac{-M+q}{3}$. With this representation of the $2M+1-$truncation condition (3.74), one has

$$\tilde{\beta}(2n) = -\frac{2M^2n - M\gamma^2 - 2Mn^2 - 3\gamma^3 - 8\gamma^2 n + \gamma^2 q - 6\gamma n^2 + 4\gamma nq + 2n^2 q + Mn + \gamma n - qn}{M^2+2M\gamma+2\gamma^2-\gamma}, \tag{3.115}$$

therefore,

$$\tilde{\beta}(2n_2) - \tilde{\beta}(2n_1) = \frac{(n_2-n_1)(8\gamma^2 + (6n_1+6n_2-4q-1)\gamma - 2M^2 + 2Mn_1 + 2Mn_2 - 2qn_1 - 2qn_2 - M + q)}{M^2+2M\gamma+2\gamma^2-\gamma} \tag{3.116}$$

and

$$\gamma_M = -\frac{3}{8}M + \frac{q}{4} + \frac{1}{16} + \frac{1}{16}\sqrt{36M^2+16Mq+16q^2+20M-24q+1}. \tag{3.117}$$

The quantity under the square root symbol is always positive for $M \geq 1$ and $q \in \mathbb{R}$. In fact, for $n_2 = M, n_1 = 0$, (3.116) has two zeros, that are γ_M and another value γ'_M

$$\gamma'_M = -\frac{3}{8}M + \frac{q}{4} + \frac{1}{16} - \frac{1}{16}\sqrt{36M^2+16Mq+16q^2+20M-24q+1}. \tag{3.118}$$

However, the fact that $\gamma'_M < \frac{-M+q}{3} < \gamma_M$ for $M \geq 1, q \in \mathbb{R}$ implies that γ'_M is irrelevant. One still has the dominations of $\tilde{\beta}(0)$ and $\tilde{\beta}(2M)$ respectively on $(\frac{-M+q}{3}, \gamma_M)$ and $(\gamma_M, +\infty)$.

Our purpose is to determine the $\beta-$spectrum on the parabola $\mathcal{P}_M$ in the $(p,q)-$plane for all $\kappa > 0$, in other word, on a surface in a half space $\kappa > 0$ of the $(p,q,\kappa)-$

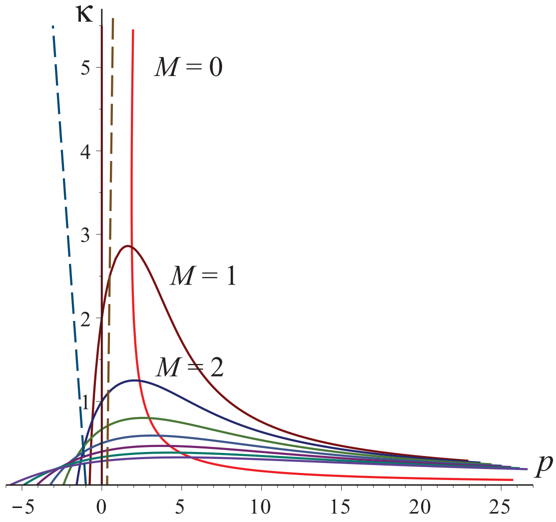


Figure 3.4 – *The curves defined by* (3.63), (3.114) *in the plane* $q = 0$. *The left dashed line,* $p = -1 - \frac{3\kappa}{8}$, *marks the phase transition at* $\gamma = -\frac{1}{2}$. *The right dashed curve marks the phase transition at* $\gamma = \gamma_M$. *Note that the case* $M = 0$ *corresponds to the union of the red curve and the haft line* $\{(0,\kappa) : \kappa > 0\}$.

space. To do that, we fix a real value of q and determine $\beta-$spectrum on the curve (3.63), (3.114) in the $(p,\kappa)-$plane. Figure 3.4 gives an example of these curves corresponding to $q = 0$.

Firstly, we consider the cases of $q \geq q_0 := M - \frac{3}{2}$, which is equivalent to $-\frac{1}{2} \leq \frac{-M+q}{3}$. Because the relevant value of the parameter γ is in $(\frac{-M+q}{3}, +\infty)$, from (3.94), the $\beta-$spectrum is determined by the eigenvalue $\tilde{\beta}$ by

$$\beta(p,\kappa) = \tilde{\beta}, \quad \text{for } \frac{-M+q}{3} < \gamma < +\infty. \tag{3.119}$$

It is important to note that the non-negativity and continuity of β in γ imply those of $\tilde{\beta}$. We will use these properties of $\tilde{\beta}$ as crucial arguments. The analysis in the previous sections has showed that $\tilde{\beta}$ can only take one value among $\tilde{\beta}(2n), n = 0, ..., M$ for each γ. By substituting $\gamma = \frac{-M+q}{3}$ into (3.115), one has

$$\tilde{\beta}(2n) = -2n, \quad n = 0, ..., M. \tag{3.120}$$

Since $\tilde{\beta}(2n)$ is a continuous function in γ for each n, $\tilde{\beta}(2n)$ is negative in a neighborhood of $\gamma = \frac{-M+q}{3}$ for $n = 1, ..., M$. Whereas, since $\tilde{\beta}(0)$ is increasing in a neighborhood of $\gamma = \frac{-M+q}{3}$, $\tilde{\beta}(0)$ is non-negative in a right neighborhood of that point. The non-negativity of $\tilde{\beta}$ implies that

$$\tilde{\beta} = \tilde{\beta}(0), \tag{3.121}$$

in a right neighborhood of $\gamma = \frac{-M+q}{3}$.

Because $\tilde{\beta}(0)$ is strictly bigger than every $\tilde{\beta}(2n)$ for $n = 1, ..., M$ in the interval $(\frac{-M+q}{3}, \gamma_M)$, it follows from the continuity of $\tilde{\beta}$ that $\tilde{\beta} = \tilde{\beta}(0)$ on $(\frac{-M+q}{3}, \gamma_M]$.

Next, we determine values of the eigenvalue $\tilde{\beta}$ (3.88) in a right neighborhood of γ_M. From (3.112) and (3.3.4), at $\gamma = \gamma_M$, one has

$$\tilde{\beta}(0) = \tilde{\beta}(2M) > \tilde{\beta}(2n), \quad n = 1, ..., M-1. \tag{3.122}$$

It is then implied that $\tilde{\beta}(0), \tilde{\beta}(2M) > \tilde{\beta}(2n)$ for $n = 1, ..., M-1$ in a neigborhood of γ_M. As a consequence of the continuity, $\tilde{\beta}$ can only take value among $\tilde{\beta}(0)$ and $\tilde{\beta}(2M)$ in the right part of that neighborhood with respect to γ_M. It is noticed that, because of the continuity of $\tilde{\beta}$ and the fact that $\tilde{\beta}(2M) > \tilde{\beta}(0)$ whenever $\gamma > \gamma_M$, $\tilde{\beta}$ identifies with either $\tilde{\beta}(0)$ or $\tilde{\beta}(2M)$ in all that right neighborhood. We will show that only $\tilde{\beta}(2M)$ is suitable for this identity. For this purpose, we consider the non-negativity of the function $g = g_1$ (3.100), (3.107) corresponding to the parameter $\tilde{\beta} = \tilde{\beta}(0)$ of the equation (3.101). As was mentioned in the previous sections, corresponding to the $2M+1-$truncations solution (3.76), (3.92), (3.100) , $g = g_1$ is non-negative and is a polynomial. We will indeed prove that there exists a right neighborhood of γ_M such that the corresponding polynomial $g = g_1$ (3.100), (3.107) of $\tilde{\beta}(0)$ is negative at $x = 1$. Let us set $\tilde{\beta} = \tilde{\beta}(0)$ in (3.101), then from (3.107),

$$g_1(1) = \frac{\Gamma(\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}, \tag{3.123}$$

where a, b are defined by (3.105). At $\gamma = \gamma_M$, one has

$$a + \frac{1}{2} = -M + \frac{1}{2}, \quad b + \frac{1}{2} = 1, \quad a + b + \frac{1}{2} = -M + 1. \tag{3.124}$$

One also has that $a + \frac{1}{2}, b + \frac{1}{2}, a + b + \frac{1}{2}$ are continuous functions in γ and $a + b + \frac{1}{2}$ is increasing in a neigborhood of γ_M. It follows that

$$a + \frac{1}{2} \overset{\gamma \to \gamma_M}{\longrightarrow} -M + \frac{1}{2}, \tag{3.125}$$

$$b + \frac{1}{2} \overset{\gamma \to \gamma_M}{\longrightarrow} 1, \tag{3.126}$$

$$a + b + \frac{1}{2} \overset{\gamma \to \gamma_M}{\longrightarrow} -M + 1 + 0^+ \tag{3.127}$$

Therefore, the sign properties of the Gamma function imply the existence of a enough small right neighborhood of γ_M such that $g_1(1)$ is negative.

We have showed that the eigenvalue $\tilde{\beta}$ cannot coincide with $\tilde{\beta}(0)$ in a right neigborhood of γ_M, therefore, it coincides with $\tilde{\beta}(2M)$ in a right neigborhood of γ_M. Because of the continuity of $\tilde{\beta}$ and $\tilde{\beta}(2M) > \tilde{\beta}(2n)$ for every $n = 0, ..., M-1$ on the right of γ_M, we have $\tilde{\beta} = \tilde{\beta}(2M)$ in $(\gamma_M, +\infty)$.

From the above analysis, we finally have, for $q \geq M - \frac{3}{2}$,

$$\beta(p, \kappa) = \tilde{\beta} = \begin{cases} \tilde{\beta}(0) & \frac{-M+q}{3} < \gamma < \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq \gamma_M, \end{cases} \tag{3.128}$$

where $\tilde{\beta}(0), \tilde{\beta}(2M)$ are defined by (3.115) and γ_M is defined by (3.117).

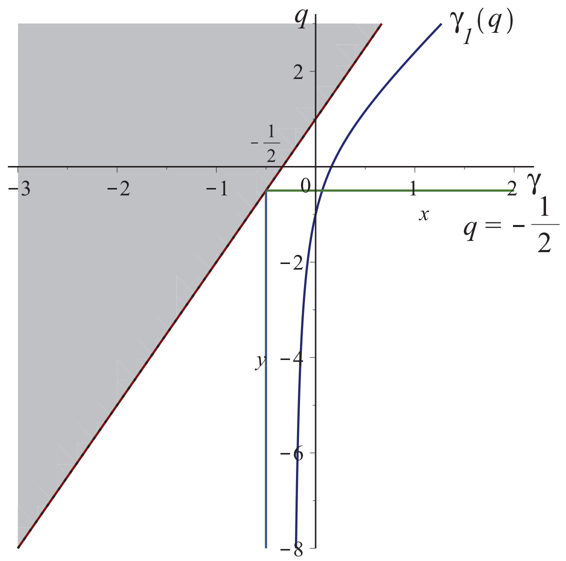


Figure 3.5 – *The (γ, p)-plane in the case of $M = 1$. The curve $\gamma_M(q)$ never intersects the vertical line $\gamma = -\frac{1}{2}$.*

We continue with studying the cases of $q < M - \frac{3}{2}$ which is equivalent to $\frac{-M+q}{3} < -\frac{1}{2}$. Here, it is more convenient to separately consider the cases $M = 1$ and $M \geq 2$.

In the case of $M = 1$, we always have $-\frac{1}{2} < \gamma_1 = \gamma_M$ for all $q \in \mathbb{R}$, in particular, in the $(\gamma, q)-$plane the curve $\gamma_1(q)$ does not intersect the vertical line $\gamma = -\frac{1}{2}$ and located to the right of that line and below the line $\gamma = \frac{-1+q}{3} = \frac{-M+q}{3}$ (see Figure 3.5). For a q such that $q < -\frac{1}{2} = M - \frac{3}{2}$, one has $\frac{-1+q}{3} < -\frac{1}{2} < \gamma_1$.

If $-\frac{1}{2} \leq \gamma \leq \gamma_1$ then the point (γ, q) can be jointed to a fixed point $T_0(\gamma_0, -\frac{1}{2})$ in the segment $\{(\gamma, -\frac{1}{2}) : -\frac{1}{2} < \gamma < \gamma_1(-\frac{1}{2})\}$ by a vertical segment. The continuity of the $\beta-$spectrum in q and the fact that $\tilde{\beta}(0) > \tilde{\beta}(2)$ in the region between the vertical line $\gamma = -\frac{1}{2}$ and the curve $\gamma_1(q)$ implies that $\beta = \tilde{\beta}(0)$ on the segment, in particular, at the point (γ, q).

If $\frac{-M+q}{3} = \frac{-1+q}{3} < \gamma < -\frac{1}{2}$ then the point (γ, q) can be jointed to the above fixed point $T_0(\gamma_0, -\frac{1}{2})$ by a polygon of two sides with the first side jointing (γ, q) and (γ_0, q), the second side jointing (γ_0, q) and $(\gamma_0, -\frac{1}{2})$. By the same reason as in the previous paragraph, the $\beta-$spectrum at (γ_0, q) is $\tilde{\beta}(0)$. The continuity of $\beta-$spectrum in γ implies that it coincides with $\tilde{\beta}(0)$ on the horizontal segment jointing (γ_0, q) and $(-\frac{1}{2}, q)$. On the horizontal segment from the point $(-\frac{1}{2}, q)$ to the point (γ, q), one has

$$\tilde{\beta}(0) - 2\gamma - 1 = \max\{\tilde{\beta}(0), \tilde{\beta}(0) - 2\gamma - 1\} \tag{3.129}$$

$$\tilde{\beta}(0) - 2\gamma - 1 > \max\{\tilde{\beta}(2), \tilde{\beta}(2) - 2\gamma - 1 - n\}. \tag{3.130}$$

From these facts and the continuity of the $\beta-$spectrum, it yields $\beta = \tilde{\beta}(0) - 2\gamma - 1$ on the segment jointing $(-\frac{1}{2}, q)$ and (γ, q), in particular, at the point (γ, q).

If $\gamma > \gamma_1(q)$ then the point (γ, q) can be jointed to a fixed point $T_0'(\gamma_0', -\frac{1}{2})$ on the half line $\{(\gamma, -\frac{1}{2}) : \gamma > \gamma_1(-\frac{1}{2})\}$ by a polygon of two sides. The first side is the vertical segment jointing T_0' and the point (γ_0', q). The second side is the horizontal segment from (γ_0', q) to (γ, q). The continuity of the $\beta-$ *spectrum* in q, γ and the fact that $\tilde{\beta}(2) > \tilde{\beta}(0)$ on the region located to the right of the curve $\gamma_1(q)$ imply that $\beta = \tilde{\beta}(2)$ on the polygon, therefore, at the point (γ, q).

By combining the two cases $q \geq M - \frac{3}{2}$ and $q < M - \frac{3}{2}$, we finally have for the case $M = 1$

$$\beta(p,\kappa) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \frac{-1+q}{3} < \gamma < -\frac{1}{2} \\ \tilde{\beta}(0) & -\frac{1}{2} \leq \gamma < \gamma_1 \\ \tilde{\beta}(2) & \gamma \geq \gamma_1. \end{cases} \tag{3.131}$$

We now turn to the case of $M \geq 2$. In this case, we have

$$-\frac{1}{2} < \gamma_M \quad \text{when } q > -\frac{8M-5}{2(2M-3)} \tag{3.132}$$

$$-\frac{1}{2} = \gamma_M \quad \text{when } q = -\frac{8M-5}{2(2M-3)} \tag{3.133}$$

$$-\frac{1}{2} > \gamma_M \quad \text{when } q < -\frac{8M-5}{2(2M-3)}. \tag{3.134}$$

We firstly consider $-\frac{8M-5}{2(2M-3)} \leq q < M - \frac{3}{2}$. One still has the validity of the arguments in the analysis for the case of $M = 1$ and $q < M - \frac{3}{2}$ for this case. Here, we note that $\tilde{\beta}(0) > \tilde{\beta}(2n)$ for $n = 1, ..., M$ in the region bounded by the three lines $q = M - \frac{3}{2}, q = -\frac{8M-5}{2(2M-3)}, \gamma = \frac{-M+q}{3}$ and the curve $\gamma_M(q)$ whereas $\tilde{\beta}(2M) > \tilde{\beta}(2n)$ for $n = 0, ..., M-1$ in the region located between the two lines $q = M - \frac{3}{2}, q = -\frac{8M-5}{2(2M-3)}$ and to the right of the curve $\gamma_M(q)$ in the $(\gamma, q)-$plane. The $\beta-$spectrum is then determined as

$$\beta(p,\kappa) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \frac{-M+q}{3} < \gamma < -\frac{1}{2} \\ \tilde{\beta}(0) & -\frac{1}{2} \leq \gamma < \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq \gamma_M. \end{cases} \tag{3.135}$$

Similarly, we can apply the same arguments for the case of $q = -\frac{8M-5}{2(2M-3)}$, or equivalently, $\frac{-M+q}{3} < -\frac{1}{2} = \gamma_M$. It then yields

$$\beta(p,\kappa) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \frac{-M+q}{3} < \gamma < -\frac{1}{2} = \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq -\frac{1}{2} = \gamma_M. \end{cases} \tag{3.136}$$

In the case of $q < -\frac{8M-5}{2(2M-3)}$, or equivalently, $\frac{-M+q}{3} < \gamma_M < -\frac{1}{2}$, the polygon argument can only be applied for $\frac{-M+q}{3} < \gamma \leq \gamma_M$ and $\gamma \geq -\frac{1}{2}$. Namely, we obtain

$$\beta(p,\kappa) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \frac{-M+q}{3} < \gamma < \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq -\frac{1}{2}. \end{cases} \tag{3.137}$$

At $\gamma = -\frac{1}{2}$, one has

$$\tilde{\beta}(2M) = \max\{\tilde{\beta}(2M), \tilde{\beta}(2M) - 2\gamma - 1 - M\} \tag{3.138}$$

$$\tilde{\beta}(2M) > \max\{\tilde{\beta}(2n), \tilde{\beta}(2n) - 2\gamma - 1 - n\}. \tag{3.139}$$

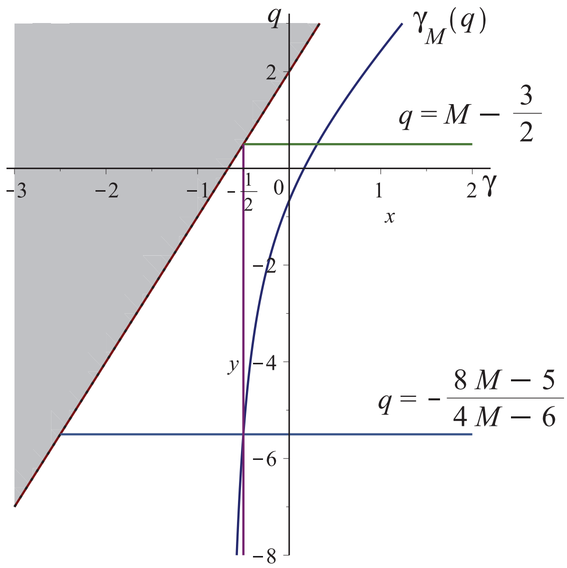


Figure 3.6 – *The (γ, p)-plane in the case of $M = 2$. The curve $\gamma_M(q)$ intersects the vertical line $\gamma = -\frac{1}{2}$ at $q = -\frac{8M-5}{2(2M-3)}$.*

Because $\tilde{\beta}(2n), n = 0, M$ are continuous functions in γ, we can have the $\beta-$spectrum in a neighborhood of $\gamma = -\frac{1}{2}$. Namely, in this neighborhood the β – *spectrum* coincides with $\beta(2M)$: there exists a $\epsilon > 0$ such that

$$\beta(p, \kappa) = \tilde{\beta}(2M), \quad -\frac{1}{2} - \epsilon < \gamma < -\frac{1}{2}. \tag{3.140}$$

Note that we are using the presentation (3.114) of the $2M + 1-$truncations condition (3.74) (and, correspondingly, also using (3.115), (3.117)). We now turn back to the original presentation (3.74) of this condition, where q is a quantity depending on the positive parameter κ, to rewrite the $\beta-$spectrum on the parabolas $\mathcal{P}_M$ as:
If $-\frac{1}{2} < -\frac{M}{2} + \frac{1}{\kappa} + \frac{1}{4} = \gamma_M$, then

$$\beta(p, q) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \gamma < -\frac{1}{2} \\ \tilde{\beta}(0) & -\frac{1}{2} \leq \gamma < \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq \gamma_M. \end{cases} \tag{3.141}$$

If $-\frac{1}{2} = -\frac{M}{2} + \frac{1}{\kappa} + \frac{1}{4} = \gamma_M$, then

$$\beta(p, q) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \gamma < -\frac{1}{2} = \gamma_M \\ \tilde{\beta}(2M) & \gamma \geq -\frac{1}{2} = \gamma_M. \end{cases} \tag{3.142}$$

If $-\frac{1}{2} > -\frac{M}{2} + \frac{1}{\kappa} + \frac{1}{4} = \gamma_M$, then the spectrum has not been completely found yet on all the parabola $\mathcal{P}_M$,

$$\beta(p, q) = \begin{cases} \tilde{\beta}(0) - 2\gamma - 1 & \gamma < \gamma_M \\ \tilde{\beta}(2M) & \gamma_M < -\frac{1}{2} - \epsilon < \gamma. \end{cases} \tag{3.143}$$

where ϵ is a positive constant. Here, $\tilde{\beta}(0)$ and $\tilde{\beta}(2M)$ are defined by (3.108).

Remark 3.3.1. *On the parabolas $\mathcal{P}_M$, we have the following identities*

— *If $\gamma < -\frac{1}{2}$ then $\tilde{\beta}(0) - 2\gamma - 1 = \beta_{tip}$.*

— *If $\gamma < \gamma_M$ then $\tilde{\beta}(0) = \beta_0$.*

— *If $\gamma > \gamma_M$ then $\tilde{\beta}(2M) = \beta_1$,*

where $\beta_{tip}, \beta_0, \beta_1$ are defined above. Therefore the results obtained in this work are consistant with the scheme of the generalized spectrum introduced in Section 3.1 .

Chapter 4

FOUR-POINT GENERALIZED FUNCTION AND SCHWARZIAN DERIVATIVE OF SLE

Let us recall the definition of Schwarzian derivative of a holomorphic function f, that is the quantity defined by

$$(Sf)(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

One of the most important properties of Schwarzian derivative is the invariance under möbius transformations, e.i., rational functions of the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. This property is a consequence of the fact that $(Sf)(z) = 0$ if and only if f is a möbius transformation which also indicates that Schwarzian derivative can be a tool to precisely measure the degree to which a function fails to be a möbius transformation.

In this chapter we continue using martingale techniques to perform equations satisfied by average Schwarzian derivative of SLE and its second moment, then obtain an explicit formula for the expectation of Schwarzian derivative. The main idea proceeds from the following observation: Let f is an univalent function in the unit disc, we introduce the function of two complexe variables

$$L(z, w) := \log\left(\frac{f(z) - f(w)}{z - w}\right), \tag{4.1}$$

it second mixed partial derivative is given by

$$\frac{\partial^2 L(z, w)}{\partial z \partial w} = \frac{f'(z)f'(w)}{(f(z) - f(w))^2} - \frac{1}{(z - w)^2}, \tag{4.2}$$

and the Schwarzian derivative of f is given by the formula

$$(Sf)(z) = 6\left.\frac{\partial^2 L(z, w)}{\partial z \partial w}\right|_{z=w} = 6 \lim_{w \to z} \frac{\partial^2 L(z, w)}{\partial z \partial w}. \tag{4.3}$$

We will firstly perform an equation of $\mathbb{E}\left(\frac{f'(z)f'(w)}{(f(z)-f(w))^2}-\frac{1}{z-w}\right)$ where f, as in above chapters, denotes the whole-plane SLE map at time 0. The martingale argument is again used for this step. Then we use the relation (4.3) in order to obtain an equation satisfied by $\mathbb{E}(Sf)$ (up to a constant factor). This equation is a non-homogeneous differential equation of order two with known initial conditions. Solving this equation gives us an exact formula of the expectation of Schwarzian derivative of SLE. The same argument is applied to obtain an equation for the expectation of the second moment of Schwarzian derivative.

When working on the first step of the above process, we consider the quantities

$$\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right) \quad \text{and} \quad \mathbb{E}\left(\frac{|z-w|^4}{|f(z)-f(w)|^4}|f'(z)|^2|f'(w)|^2\right). \tag{4.4}$$

It is remarkable that we can indeed go further to get a closed form of the following expectation

$$\mathbb{E}\left(\frac{(z_1-z_1')^{\frac{q}{2}}}{(f(z_1)-f(z_1'))^{\frac{q}{2}}}(f'(z_1))^{\frac{p}{2}}(f'(z_1'))^{\frac{p'}{2}}\overline{\frac{(z_2-z_2')^{\frac{q}{2}}}{(f(z_2)-f(z_2'))^{\frac{q}{2}}}(f'(z_2))^{\frac{p}{2}}(f'(z_2'))^{\frac{p'}{2}}}\right), \tag{4.5}$$

which generalizes Theorem 2.3.1 and Theorem 2.4.1 in Chapter 2. This result is presented in the first section of the present chapter.

4.1 Four-point generalized function

The first motivation of this section is to obtain explicit formula of the quantities (4.4). We are interested in these quantities because of their relations with some important quantities such as the Schwarzian derivative, Grunsky's coefficients and Grunsky's matrix of the whole-plane SLE$_\kappa$ map. Namely, from (4.2) and (4.3), they are related to the Schwarzian derivative of whole-plane SLE$_\kappa$ map f by

$$\mathbb{E}(S(f)(z)) = 6\lim_{w\to z}\frac{1}{(z-w)^2}\left[\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)-1\right], \tag{4.6}$$

$$\begin{aligned}\mathbb{E}(|S(f)(z)|^2) = 36\lim_{w\to z}\frac{1}{|z-w|^4}&\left[\mathbb{E}\left(\frac{|z-w|^4}{|f(z)-f(w)|^4}|f'(z)|^2|f'(w)|^2\right)+1\right.\\ &\left.-\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)-\overline{\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)}\right].\end{aligned} \tag{4.7}$$

Assume that the logarithmic function (4.1) has a series expansion as

$$\log\left(\frac{f(z)-f(w)}{z-w}\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{k,l}z^k w^l, \tag{4.8}$$

the $b_{k,l}$ is called the Grunsky's coefficient of f and the matrix $(\sqrt{kl}b_{k,l})$ is the corresponding Grunsky's matrix. From (4.2), the Grunsky's coefficients of the whole-plane SLE$_\kappa$ map f and the quantities (4.4) are related by

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\mathbb{E}(b_{k,l})klz^{k-1}w^{l-1} = \frac{1}{(z-w)^2}\left[\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)-1\right], \quad z,w\in\mathbb{D}. \tag{4.9}$$

To compute the quantities (4.4) we reuse the arguments which were applied to the proofs of Theorem 2.3.1 and Theorem 2.4.1. We indeed, with that method, give a more general result covering also Theorem 2.3.1 and Theorem 2.4.1, that is, Theorem 4.1.1.

4.1.1 SLE four-point function and Belyaev-Smirnov equation

The purpose of this section is to perform Belyaev-Smirnov type equations for the general moments (4.5). We will show that

Proposition 4.1.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ and*

$$H(z_1,\bar z_2,z_1',\bar z_2') = \mathbb{E}\left(\frac{(z_1-z_1')^{\frac{q}{2}}}{(f(z_1)-f(z_1'))^{\frac{q}{2}}}(f'(z_1))^{\frac{p}{2}}(f'(z_1'))^{\frac{p'}{2}}\,\overline{\frac{(z_2-z_2')^{\frac{q}{2}}}{(f(z_2)-f(z_2'))^{\frac{q}{2}}}}\,\overline{(f'(z_2))^{\frac{p}{2}}(f'(z_2'))^{\frac{p'}{2}}}\right). \tag{4.10}$$

Then H is a solution of the equation

$$\begin{aligned}
&-\frac{\kappa}{2}(z_1\partial z_1 - \bar z_2\bar\partial_{\bar z_2} + z_1'\partial z_1' - \bar z_2'\bar\partial_{\bar z_2'})^2 H\\
&-\frac{1+z_1}{1-z_1}z_1\partial z_1 H - \frac{1+\bar z_2}{1-\bar z_2}\bar z_2\bar\partial_{\bar z_2}H - \frac{1+z_1'}{1-z_1'}z_1'\partial_{z_1'}H - \frac{1+\bar z_2'}{1-\bar z_2'}\bar z_2'\bar\partial_{\bar z_2'}H\\
&+\Bigg[-\frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar z_2)^2} - \frac{p'}{(1-z_1')^2} - \frac{p'}{(1-\bar z_2')^2} + \frac{q}{(1-z_1)(1-z_1')} + \frac{q}{(1-\bar z_2)(1-\bar z_2')}\\
&+2(p+p') - 2q\Bigg]H = 0.
\end{aligned} \tag{4.11}$$

The following corollaries are obtained when we make specific settings on the variables z_1, z_1', z_2, z_2' in Proposition 4.1.1:

By taking limits $z_2 \to 0, z_2' \to 0$ in (4.37) and letting $z_1 = z, z_1' = \zeta$, it yields

Corollary 4.1.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ and*

$$F(z,\zeta) = \mathbb{E}\left(\frac{(z-\zeta)^{\frac{q}{2}}}{(f(z)-f(\zeta))^{\frac{q}{2}}}(f'(z))^{\frac{p}{2}}(f'(\zeta))^{\frac{p'}{2}}\right). \tag{4.12}$$

Then F is a solution of the equation

$$\begin{aligned}
&-\frac{\kappa}{2}(z\partial z + \zeta\partial\zeta)^2 F - \frac{1+z}{1-z}z\partial z F - \frac{1+\zeta}{1-\zeta}\zeta\partial\zeta F\\
&+\left[-\frac{p}{(1-z)^2} - \frac{p'}{(1-\zeta)^2} + \frac{q}{(1-z)(1-\zeta)} + p + p' - q\right]F = 0.
\end{aligned} \tag{4.13}$$

If we choose $z_1 = z_2 = z, z_1' = z_2' = \zeta$, it yields

Corollary 4.1.2. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ and*

$$G(z,\bar z,\zeta,\bar\zeta) = \mathbb{E}\left(\frac{|z-\zeta|^q}{|f(z)-f(\zeta)|^q}|f'(z)|^p|f'(\zeta)|^{p'}\right). \tag{4.14}$$

Then G is a solution of the equation

$$-\frac{\kappa}{2}(z\partial_z - \bar{z}\bar{\partial}_{\bar{z}} + \zeta\partial_\zeta - \bar{\zeta}\bar{\partial}_{\bar{\zeta}})^2 G - \frac{1+z}{1-z}z\partial_z G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}_{\bar{z}} G - \frac{1+\zeta}{1-\zeta}\zeta\partial_\zeta G - \frac{1+\bar{\zeta}}{1-\bar{\zeta}}\bar{\zeta}\bar{\partial}_{\bar{\zeta}} G$$
$$+\Bigg[-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} - \frac{p'}{(1-\zeta)^2} - \frac{p'}{(1-\bar{\zeta})^2} + \frac{q}{(1-z)(1-\zeta)} + \frac{q}{(1-\bar{z})(1-\bar{\zeta})}$$
$$+ 2(p+p') - 2q\Bigg] G = 0. \tag{4.15}$$

Let us now prove Proposition 4.1.1.

Proof. We firstly introduce a four-point auxiliary function

$$\tilde{H}(z_1, \bar{z}_2, z_1', \bar{z}_2', t) =$$
$$\mathbb{E}\Bigg(\frac{(z_1 - z_1')^{\frac{q}{2}}}{(\tilde{f}_t(z_1) - \tilde{f}_t(z_1'))^{\frac{q}{2}}}(\tilde{f}_t'(z_1))^{\frac{p}{2}}(\tilde{f}_t'(z_1'))^{\frac{p'}{2}}\overline{\frac{(z_2 - z_2')^{\frac{q}{2}}}{(\tilde{f}_t(z_2) - \tilde{f}_t(z_2'))^{\frac{q}{2}}}}\,\overline{(\tilde{f}_t'(z_2))^{\frac{p}{2}}(\tilde{f}_t'(z_2'))^{\frac{p'}{2}}}\Bigg),$$

where $\tilde{f}_t$ is the reverse radial SLE$_\kappa$ process (1.9). In the view point of Lemma 1.4.2, the function H is the limit in law

$$\lim_{\tau\to+\infty}(e^{p+p'-q}\tilde{H}(z_1, \bar{z}_2, z_1', \bar{z}_2', \tau)) \overset{\text{(law)}}{=} H(z_1, \bar{z}_2, z_1', \bar{z}_2'). \tag{4.16}$$

Let us now define, for $s \le t$, the conditional expectation

$$\mathcal{M}_s := \mathbb{E}\Bigg(\frac{(\tilde{f}_t'(z_1))^{\frac{p}{2}}(\tilde{f}_t'(z_1'))^{\frac{p'}{2}}}{(\tilde{f}_t(z_1) - \tilde{f}_t(z_1'))^{\frac{q}{2}}}\overline{\frac{(\tilde{f}_t'(z_2))^{\frac{p}{2}}(\tilde{f}_t'(z_2'))^{\frac{p'}{2}}}{(\tilde{f}_t(z_2) - \tilde{f}_t(z_2'))^{\frac{q}{2}}}}\Bigg|\mathcal{F}_s\Bigg), \tag{4.17}$$

where $\mathcal{F}_s$ is the σ-algebra generated by $\{B_u,\, u \le s\}$. Thank to Markov property of SLE, we can rewrite $\mathcal{M}_s$ as

$$\mathcal{M}_s = X_s(z_1, z_1')\overline{X_s(z_2, z_2')}\tilde{H}(z_{1s}, \bar{z}_{2s}, z_{1s}', \bar{z}_{2s}', \tau). \tag{4.18}$$

The quantities $X_s(z_i, z_i'), z_{is}, z_{is}', i = \overline{1,2}$ are defined by

$$X_s(z_i, z_i') := \frac{(\tilde{f}_s'(z_i))^{\frac{p}{2}}(\tilde{f}_s'(z_i'))^{\frac{p'}{2}}}{(\tilde{f}_s(z_i) - \tilde{f}_s(z_i'))^{\frac{q}{2}}},$$
$$z_{is} := \tilde{f}_s(z_i)/\lambda_s, \quad z_{is}' := \tilde{f}_s(z_i')/\lambda_s,$$

and $\tau := t - s$.

Note that, by its construction $(\mathcal{M}_s)_{s\ge 0}$ is a martingale and hence the ds-drift term in Itô derivative of $\mathcal{M}_s$ vanishes. We will use this fact to obtain an equation satisfied by $\tilde{H}$. From (4.18) we can represent the Itô derivative of $d\mathcal{M}_s$ in term of those of the factors in the right-hand side as

$$d\mathcal{M}_s = dX_s(z_1, z_1').\overline{X_s(z_2, z_2')}\tilde{H} + X_s(z_1, z_1')d\overline{X_s(z_2, z_2')}\tilde{H} + X_s(z_1, z_1')\overline{X_s(z_2, z_2')}d\tilde{H}. \tag{4.19}$$

On the other hand, from the fact that the Itô drift-diffusion process $X_s(z_i, z'_i)$ is the composition of a twice differentiable scalar function and the vector of Itô processes $(\tilde{f}'_s(z_i), \tilde{f}'_s(z'_i), \underline{\tilde{f}_s(z_i), \tilde{f}_s(z'_i)})$ and the equations (2.42), (2.43), the Itô differentials $dX_s(z_1, z'_1)$ and $d\overline{X_s(z_2, z'_2)}$ are obtained as

$$dX_s(z_1, z'_1) = X_s(z_1, z'_1)L(z_{1s}, z'_{1s})ds, \tag{4.20}$$

$$d\overline{X_s(z_2, z'_2)} = \overline{X_s(z_2, z'_{2s})}L(\bar{z}_{2s}, \bar{z}'_{2s})ds, \tag{4.21}$$

where

$$L(z, \zeta) := -\frac{p}{(1-z)^2} - \frac{p'}{(1-\zeta)^2} + \frac{q}{(1-z)(1-\zeta)} + \frac{p}{2} + \frac{p'}{2} - \frac{q}{2}.$$

In addition, by using Itô formula we also have

$$\begin{aligned} d\tilde{H} =& -\partial_\tau \tilde{H} ds + \partial_1 \tilde{H} dz_{1s} + \partial_2 \tilde{H} d\bar{z}_{2s} + \partial_3 \tilde{H} dz'_{1s} + \partial_4 \tilde{H} d\bar{z}'_{2s} \\ &+ \frac{\kappa}{2}(-\partial_1^2 - \partial_2^2 - \partial_3^2 - \partial_4^2 - 2\partial_1\partial_3 - 2\partial_2\partial_4 + 2\partial_1\partial_2 + 2\partial_1\partial_4 + 2\partial_2\partial_3 + 2\partial_3\partial_4)\tilde{H} ds, \end{aligned}$$

where $\partial_1, \partial_2, \partial_3, \partial_4$ respectively stand for $\partial_{z_{1s}}, \partial_{\bar{z}_{2s}}, \partial_{z'_{1s}}, \partial_{\bar{z}'_{2s}}$.

$$\begin{aligned} dz_{1s} &= z_{1s}\left[\frac{z_{1s}+1}{z_{1s}-1} - \frac{\kappa}{2}\right]ds - i\sqrt{\kappa}\, z_{1s}\, dB_s, \\ dz'_{1s} &= z'_{1s}\left[\frac{z'_{1s}+1}{z'_{1s}-1} - \frac{\kappa}{2}\right]ds - i\sqrt{\kappa}\, z'_{1s}\, dB_s, \\ d\bar{z}_{2s} &= \bar{z}_{2s}\left[\frac{\bar{z}_{2s}+1}{\bar{z}_{2s}-1} - \frac{\kappa}{2}\right]ds + i\sqrt{\kappa}\, \bar{z}_{2s}\, dB_s, \\ d\bar{z}'_{2s} &= \bar{z}'_{2s}\left[\frac{\bar{z}'_{2s}+1}{\bar{z}'_{2s}-1} - \frac{\kappa}{2}\right]ds + i\sqrt{\kappa}\, \bar{z}'_{2s}\, dB_s. \end{aligned}$$

Hence $d\mathcal{M}_s$ can be rewriten as the product of $X_s(z_1, z'_1).\overline{X_s(z_2, z'_2)}$ and the following

$$\begin{aligned} &\Bigg[L(z_{1s}, z'_{1s}) + L(\bar{z}_{2s}, \bar{z}'_{2s}) - \partial_\tau \tilde{H} - \frac{1+z_{1s}}{1-z_{1s}} z_{1s}\partial_1 \tilde{H} - \frac{1+\bar{z}_{2s}}{1-\bar{z}_{2s}} \bar{z}_{2s}\partial_2 \tilde{H} \\ &- \frac{1+z'_{1s}}{1-z'_{1s}} z'_{1s}\partial_3 \tilde{H} - \frac{1+\bar{z}'_{2s}}{1-\bar{z}'_{2s}} \bar{z}_{2s}\partial_4 \tilde{H} - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\partial_2 + z'_{1s}\partial_3 - \bar{z}'_{2s}\partial_4)^2 \tilde{H}\Bigg] ds \\ &- i\sqrt{\kappa}(z_{1s}\partial_1 - \bar{z}_{2s}\partial_2 + z'_{1s}\partial_3 - \bar{z}'_{2s}\partial_4)\tilde{H} dB_s. \end{aligned}$$

Because ds-term of $d\mathcal{M}_s$ vanishes as $(\mathcal{M}_s)_{s\geq 0}$ is a martingale, we get the equation

$$\begin{aligned} &L(z_{1s}, z'_{1s}) + L(\bar{z}_{2s}, \bar{z}'_{2s}) - \partial_\tau \tilde{H} - \frac{1+z_{1s}}{1-z_{1s}} z_{1s}\partial_1 \tilde{H} - \frac{1+\bar{z}_{2s}}{1-\bar{z}_{2s}} \bar{z}_{2s}\partial_2 \tilde{H} \\ &- \frac{1+z'_{1s}}{1-z'_{1s}} z'_{1s}\partial_3 \tilde{H} - \frac{1+\bar{z}'_{2s}}{1-\bar{z}'_{2s}} \bar{z}_{2s}\partial_4 \tilde{H} - \frac{\kappa}{2}(z_{1s}\partial_1 - \bar{z}_{2s}\partial_2 + z'_{1s}\partial_3 - \bar{z}'_{2s}\partial_4)^2 \tilde{H} = 0. \end{aligned}$$

Finally, we apply (4.16) to obtain the equation satisfied by H in Proposition 4.1.1.

□

4.1.2 Dual points and integrable case

From the equations obtained in the previous section, in this section, we give the explicit formula of general moments (4.5).

Theorem 4.1.1. *Let $f(z) = f_0(z)$ be the interior whole-plane* SLE_κ *map in the setting of Theorem 2.1.3; then, for dual pairs (p, q) and (p', q) both belonging to the parabola $\mathcal{R}$ defined in Theorem 2.3.1 by Eqs.* (2.62) *or* (2.63) *or* (2.64), *and for any quadruple $(z_1, z_2, z_1', z_2') \in \mathbb{D}^4$,*

$$\mathbb{E}\left(\frac{(z_1 - z_1')^{\frac{q}{2}}(f'(z_1))^{\frac{p}{2}}}{(f(z_1) - f(z_1'))^{\frac{q}{2}}}(f'(z_1'))^{\frac{p'}{2}}\overline{\left[(f'(z_2))^{\frac{p}{2}}\right]}\,\overline{\left[\frac{(z_2 - z_2')^{\frac{q}{2}}(f'(z_2'))^{\frac{p'}{2}}}{(f(z_2) - f(z_2'))^{\frac{q}{2}}}\right]}\right) \tag{4.22}$$
$$= \frac{(1 - z_1)^\gamma(1 - \bar{z}_2)^\gamma(1 - z_1')^{\gamma'}(1 - \bar{z}_2')^{\gamma'}}{(1 - z_1\bar{z}_2)^\beta(1 - z_1\bar{z}_2')^{\beta'}(1 - z_1'\bar{z}_2)^{\beta'}(1 - z_1'\bar{z}_2')^{\beta''}}, \quad \beta = \frac{\kappa}{2}\gamma^2, \beta' = \frac{\kappa}{2}\gamma\gamma', \beta'' = \frac{\kappa}{2}\gamma'^2.$$

Corollary 4.1.3. *In the same setting as in Theorem 4.1.1, we have for $z, \zeta \in \mathbb{D}$,*

$$\mathbb{E}\left(\frac{(z - \zeta)^{\frac{q}{2}}}{(f(z) - f(\zeta))^{\frac{q}{2}}}(f'(z))^{\frac{p}{2}}(f'(\zeta))^{\frac{p'}{2}}\right) = (1 - z)^\gamma(1 - \zeta)^{\gamma'}. \tag{4.23}$$

Corollary 4.1.4. *In the same setting as in Theorem 4.1.1, we have for $z, \zeta \in \mathbb{D}$,*

$$G(z, \bar{z}, \zeta, \bar{\zeta}) := \mathbb{E}\left(\frac{|z - \zeta|^q}{|f(z) - f(\zeta)|^q}|f'(z)|^p|f'(\zeta)|^{p'}\right) \tag{4.24}$$
$$= \frac{(1 - z)^\gamma(1 - \bar{z})^\gamma(1 - \zeta)^{\gamma'}(1 - \bar{\zeta})^{\gamma'}}{(1 - z\bar{z})^\beta(1 - z\bar{\zeta})^{\beta'}(1 - \zeta\bar{z})^{\beta'}(1 - \zeta\bar{\zeta})^{\beta''}}, \quad \beta = \frac{\kappa}{2}\gamma^2, \beta' = \frac{\kappa}{2}\gamma\gamma', \beta'' = \frac{\kappa}{2}\gamma'^2.$$

Although that Corollary 4.1.4 is a consequence of Theorem 4.1.1, we prefer to proceed firstly the proof of Corollary 4.1.4 for avoiding a redundancy of notations. The proof Theorem 4.1.1 will follow as an analogue.

Proof. We put

$$G(z, \bar{z}, \zeta, \bar{\zeta}) = \mathbb{E}\left(\frac{|z - \zeta|^q}{|f(z) - f(\zeta)|^q}|f'(z)|^p|f'(\zeta)|^{p'}\right).$$

Recall Corollary 4.1.2, G is a solution of the following equation

$$\begin{aligned}
&-\frac{\kappa}{2}(z\partial_z - \bar{z}\bar{\partial}_{\bar{z}} + \zeta\partial_\zeta - \bar{\zeta}\bar{\partial}_{\bar{\zeta}})^2 G \\
&-\frac{1+z}{1-z}z\partial_z G - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}_{\bar{z}} G - \frac{1+\zeta}{1-\zeta}\zeta\partial_\zeta G - \frac{1+\bar{\zeta}}{1-\bar{\zeta}}\bar{\zeta}\bar{\partial}_{\bar{\zeta}} G \\
&+\left[-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar{z})^2} - \frac{p'}{(1-\zeta)^2} - \frac{p'}{(1-\bar{\zeta})^2} + \frac{q}{(1-z)(1-\zeta)} + \frac{q}{(1-\bar{z})(1-\bar{\zeta})}\right. \\
&\left.+ 2(p + p') - 2q\right] G = 0.
\end{aligned} \tag{4.25}$$

Let us denote the differential operator in the the left hand side of (4.25) by $\mathcal{P}(D)$.

We now consider the action of $\mathcal{P}(D)$ on the function G of the form

$$G(z,\bar z,\zeta,\bar\zeta) = A(z,\bar z)B(\zeta,\bar\zeta)C(z\bar\zeta)D(\bar z\zeta). \tag{4.26}$$

Observe that

$$(z\partial_z - \bar z\bar\partial_{\bar z} + \zeta\partial_\zeta - \bar\zeta\bar\partial_{\bar\zeta})^2 = \\ (z\partial_z - \bar z\bar\partial_{\bar z})^2 + (\zeta\partial_\zeta - \bar\zeta\bar\partial_{\bar\zeta})^2 + 2(z\zeta\partial_{z\zeta} + \bar z\bar\zeta\partial_{\bar z\bar\zeta} - z\bar\zeta\partial_{z\bar\zeta} - \bar z\zeta\partial_{\bar z\zeta}).$$

The setting (4.26) together with Leibnitz's rule thus give us the development of $-\frac{\kappa}{2}(z\partial_z - \bar z\bar\partial_{\bar z} + \zeta\partial_\zeta - \bar\zeta\bar\partial_{\bar\zeta})^2 G$ as

$$-\frac{\kappa}{2}(z\partial_z - \bar z\bar\partial_{\bar z})^2 A.BCD - \frac{\kappa}{2}(\zeta\partial_\zeta - \bar\zeta\bar\partial_{\bar\zeta})^2 B.ACD \tag{4.27}$$
$$-\kappa(z\partial_z A.\zeta\partial_\zeta B + \bar z\bar\partial_{\bar z}A.\bar\zeta\bar\partial_{\bar\zeta}B - z\partial_z A.\bar\zeta\bar\partial_{\bar\zeta}B - \bar z\bar\partial_{\bar z}A.\zeta\partial_\zeta B)CD \tag{4.28}$$

By continuing to apply Leibnitz's rule to the other terms of $\mathcal{P}(D)(G)$ and finally using a summation, we then obtain

$$\begin{aligned}&\mathcal{P}(D)(ABCD) = \mathcal{P}_1(D)(A)BCD + \mathcal{P}_2(D)(B)ACD\\ &-\kappa(z\partial_z A.\zeta\partial_\zeta B + \bar z\bar\partial_{\bar z}A.\bar\zeta\bar\partial_{\bar\zeta}B - z\partial_z A.\bar\zeta\bar\partial_{\bar\zeta}B - \bar z\bar\partial_{\bar z}A.\zeta\partial_\zeta B)CD\\ &-\frac{2z\bar\zeta(1-z\bar\zeta)}{(1-z)(1-\bar\zeta)}ABC'D - \frac{2\bar z\zeta(1-\bar z\zeta)}{(1-\bar z)(1-\zeta)}ABCD'\\ &+q\left[\frac{1}{(1-z)(1-\zeta)} + \frac{1}{(1-\bar z)(1-\bar\zeta)} - \frac{1}{1-z} - \frac{1}{1-\bar z} - \frac{1}{1-\zeta} - \frac{1}{1-\bar\zeta} + 2\right]ABCD\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}\mathcal{P}_1(D) := \Bigg[&-\frac{\kappa}{2}(z\partial_z - \bar z\bar\partial_{\bar z})^2 - \frac{1+z}{1-z}z\partial z - \frac{1+\bar z}{1-\bar z}\bar z\bar\partial_{\bar z}\\ &+\left(-\frac{p}{(1-z)^2} - \frac{p}{(1-\bar z)^2} + \frac{q}{1-z} + \frac{q}{1-\bar z}\right) + 2p - 2q\Bigg]\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\mathcal{P}_2(D) := \Bigg[&-\frac{\kappa}{2}(\zeta\partial_\zeta - \bar\zeta\bar\partial_{\bar\zeta})^2 - \frac{1+\zeta}{1-\zeta}z\partial\zeta - \frac{1+\bar\zeta}{1-\bar\zeta}\bar\zeta\bar\partial_{\bar\zeta}\\ &+\left(-\frac{p'}{(1-\zeta)^2} - \frac{p'}{(1-\bar\zeta)^2} + \frac{q}{1-\zeta} + \frac{q}{1-\bar\zeta}\right) + 2p' - 2q\Bigg].\end{aligned} \tag{4.31}$$

Let us now consider A and B of the forms

$$A(z,\bar z) = \frac{\varphi_\gamma(z)\varphi_\gamma(\bar z)}{(1-z\bar z)^\beta}, \tag{4.32}$$

$$B(\zeta,\bar\zeta) = \frac{\psi_{\gamma'}(\zeta)\psi_{\gamma'}(\bar\zeta)}{(1-\zeta\bar\zeta)^{\beta''}}, \tag{4.33}$$

where $\varphi_\gamma(z) = (1-z)^\gamma, \psi_{\gamma'}(\zeta) = (1-\zeta)^{\gamma'}$. Here the notations φ, ψ are used to distinguish considered variables. Namely, φ is corresponding to the variables $z, \bar{z}$ and ψ is corresponding to $\zeta, \bar{\zeta}$. In this setting, the three last lines of (4.29) is the product of $\frac{1}{(1-z\bar{z})^\beta}\frac{1}{(1-\zeta\bar{\zeta})^{\beta''}}$ and

$$\begin{aligned}
&z\bar{\zeta}\varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'-1}D[\kappa\gamma\gamma' C + 2(z\bar{\zeta}-1)C'] + \bar{z}\zeta\varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'-1}\bar{\psi}_{\gamma'}C[\kappa\gamma\gamma' D + 2(\bar{z}\zeta-1)D'] \\
&+ [q-\kappa\gamma\gamma'][\varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'-1}\bar{\psi}_{\gamma'} + \varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'}\bar{\psi}_{\gamma'-1} \\
&- \varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'-1}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'-1} + 2\varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'}]CD.
\end{aligned} \tag{4.34}$$

We then rewrite (4.29) as

$$\begin{aligned}
&\mathcal{P}(D)(ABCD) = \mathcal{P}_1(D)(A)BCD + \mathcal{P}_2(D)(B)ACD \\
&+ z\bar{\zeta}\varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'-1}D[\kappa\gamma\gamma' C + 2(z\bar{\zeta}-1)C'] + \bar{z}\zeta\varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'-1}\bar{\psi}_{\gamma'}C[\kappa\gamma\gamma' D + 2(\bar{z}\zeta-1)D'] \\
&+ [q-\kappa\gamma\gamma'][\varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'-1}\bar{\psi}_{\gamma'} + \varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'}\bar{\psi}_{\gamma'-1} \\
&- \varphi_{\gamma-1}\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_{\gamma-1}\psi_{\gamma'}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'-1}\bar{\psi}_{\gamma'} - \varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'-1} + 2\varphi_\gamma\bar{\varphi}_\gamma\psi_{\gamma'}\bar{\psi}_{\gamma'}]CD.
\end{aligned} \tag{4.35}$$

Note that if (γ, p, q), (γ', p', q) are dual points on the parabola $\mathcal{R}$ defined in Theorem 2.3.1 by Eqs. (2.62) and A, B are respectively of the forms (4.32), (4.33) where $\beta = \frac{\kappa\gamma^2}{2}$, $\beta'' = \frac{\kappa\gamma'^2}{2}$, then

$$P_1(D)(A) = 0, \quad \mathcal{P}_2(D)(B) = 0 \quad \text{and } q - \kappa\gamma\gamma' = 0.$$

The first two identities are immediately implied from Theorem 2.4.1, whereas the third one is a consequence of the fact that γ, γ' are solutions of Eq. (2.62).

On the other hand the ODE $\kappa\gamma\gamma' P + 2(X-1)P' = 0$ with the initial condition $P(0) = 1$ has the solution

$$P(X) = \frac{1}{(1-X)^{\beta'}} \quad \text{where} \quad \beta' = \frac{\kappa\gamma\gamma'}{2}. \tag{4.36}$$

These facts imply that for

$$G(z, \bar{z}, \zeta, \bar{\zeta}) = \frac{\varphi_\gamma(z)\varphi_\gamma(\bar{z})}{(1-z\bar{z})^\beta}\frac{\psi_{\gamma'}(\zeta)\psi_{\gamma'}(\bar{\zeta})}{(1-\zeta\bar{\zeta})^{\beta''}}\frac{1}{(1-z\bar{\zeta})^{\beta'}}\frac{1}{(1-\bar{z}\zeta)^{\beta'}}$$

with $\gamma, p, \gamma', p', q, \beta, \beta'', \beta'$ as above, $\mathcal{P}(D)(G) = 0$, or equivalently, G is a solution of (4.25). □

Proof of Theorem 4.1.1

Let

$$H(z_1, \bar{z}_2, z_1', \bar{z}_2') = \mathbb{E}\left(\frac{(z_1 - z_1')^{\frac{q}{2}}}{(f(z_1) - f(z_1'))^{\frac{q}{2}}}(f'(z_1))^{\frac{p}{2}}(f'(z_1'))^{\frac{p'}{2}}\,\overline{\frac{(z_2 - z_2')^{\frac{q}{2}}}{(f(z_2) - f(z_2'))^{\frac{q}{2}}}(f'(z_2))^{\frac{p}{2}}(f'(z_2'))^{\frac{p'}{2}}}\right).$$

Proposition 4.1.1 gives us the equation satisfied by H

$$\begin{aligned}
&-\frac{\kappa}{2}(z_1\partial z_1 - \bar{z}_2\bar{\partial}_{\bar{z}_2} + z_1'\partial z_1' - \bar{z}_2'\bar{\partial}_{\bar{z}_2'})^2 H \\
&-\frac{1+z_1}{1-z_1}z_1\partial z_1 H - \frac{1+\bar{z}_2}{1-\bar{z}_2}\bar{z}_2\bar{\partial}_{\bar{z}_2}H - \frac{1+z_1'}{1-z_1'}z_1'\partial_{z_1'}H - \frac{1+\bar{z}_2'}{1-\bar{z}_2'}\bar{z}_2'\bar{\partial}_{\bar{z}_2'}H \\
&+\Bigg[-\frac{p}{(1-z_1)^2} - \frac{p}{(1-\bar{z}_2)^2} - \frac{p'}{(1-z_1')^2} - \frac{p'}{(1-\bar{z}_2')^2} + \frac{q}{(1-z_1)(1-z_1')} + \frac{q}{(1-\bar{z}_2)(1-\bar{z}_2')} \\
&+2(p+p') - 2q\Bigg]H = 0.
\end{aligned} \tag{4.37}$$

Obviously, Eq. (4.37) and Eq. (4.25) are symbolically almost the same except that G is replaced by H and $z, \bar{z}, \zeta, \bar{\zeta}$ are respectively replaced by $z_1, \bar{z}_2, z_1', \bar{z}_2'$. Another noticeable point is that the variables $z, \bar{z}, \zeta, \bar{\zeta}$ are considered as independent variables in the proof of Corollary 4.1.4. Hence the next steps of the proof are completely a analogue of the proof of Corollary 4.1.4 but with new settings of functions and variables.

4.2 Schwarzian of SLE

In this section, we present the following results:

- An explicit formula of the expectation of the Schwarzian derivative of the SLE_κ map.
- An PDE obeyed by the expected second moduli moment of the Schwarzian derivative of the SLE_κ map with that we recover results obtained in [4] (see also [5]) on this object.

4.2.1 Expectation of Schwarzian

The main result of this section is the following

Theorem 4.2.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ map at time 0 and $S(f)(z)$ is the Schwarzian derivative of $f(z)$, then*

$$\begin{aligned}
&\mathbb{E}((Sf)(z)) = \\
&-\frac{3(\kappa+2)}{2(3\kappa+8)}(-1)^{\frac{1}{2}\frac{\sqrt{\kappa^2-24\kappa+16}-\kappa-4}{\kappa}}(z-1)^{-\frac{1}{2}\frac{\sqrt{\kappa^2-24\kappa+16}-\kappa-4}{\kappa}}{}_2F_1\left[\begin{matrix}-\frac{1}{2}\frac{\sqrt{\kappa^2-24\kappa+16}-\kappa}{\kappa}, -\frac{1}{2}\frac{\sqrt{\kappa^2-24\kappa+16}-\kappa-4}{\kappa} \\ \frac{\kappa+2}{\kappa}\end{matrix}; z\right. \\
&+\frac{3}{2(3\kappa+8)}\frac{(\kappa+2)z^2-(2\kappa+12)z+\kappa+2}{(z-1)^2}.
\end{aligned} \tag{4.38}$$

The proof of Theorem 4.2.1 is two-fold:

First, we use the relation (4.3) and Corollary 4.1.1 to find a ODE obeyed by the expected Schwarzian derivative (up to a constant factor) of the SLE_κ map $f(z) := f_0(z)$.

Second, we provide initial conditions to the above ODE and solve it to obtain an explicit formula of $\mathbb{E}(f(z))$.

4.2.1.1 Equation of expected Schwarzian

An equation obeyed by the Schwarzian of the whole-plane SLE$_\kappa$ map is derived as is stated in the following proposition

Proposition 4.2.1. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE$_\kappa$ and $(Sf)(z)$ is its Schwarzian derivative. Then $\mathbb{E}((Sf)(z))$ is a solution of the equation*

$$-\frac{\kappa}{2}(z\partial_z)^2 S - \left(\frac{1+z}{1-z} + 2\kappa\right) z\partial_z S - 2\left(\frac{2}{(1-z)^2} + k - 1\right) S - \frac{12}{(1-z)^4} = 0. \quad (4.39)$$

Proof. We put

$$F(z,\zeta) := \mathbb{E}\left((z-\zeta)^2 \frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2}\right). \quad (4.40)$$

In Corollary 4.1.1, by setting $p = p' = 2, q = 4$, we have that F is a solution of the equation

$$\begin{aligned} &-\frac{\kappa}{2}(z\partial_z + \zeta\partial_\zeta)^2 F - \frac{1+z}{1-z} z\partial_z F - \frac{1+\zeta}{1-\zeta}\zeta\partial_\zeta F \\ &+ \left[-\frac{2}{(1-z)^2} - \frac{2}{(1-\zeta)^2} + \frac{4}{(1-z)(1-\zeta)}\right] F = 0. \end{aligned} \quad (4.41)$$

Let us define a two-point auxiliary function

$$F_{aux}(z,\zeta) := \mathbb{E}\left(\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right), \quad (4.42)$$

then

$$F(z,\zeta) = (z-\zeta)^2 F_{aux}(z,\zeta) + 1. \quad (4.43)$$

After taking (4.43) into Eq. (4.41), we obtain an equation satisfied by F_{aux} as

$$\begin{aligned} -\frac{\kappa}{2}(z\partial_z + \zeta\partial_\zeta)^2 F_{aux} - \left(\frac{1+z}{1-z} + 2\kappa\right) z\partial_z F_{aux} - \left(\frac{1+\zeta}{1-\zeta} + 2\kappa\right)\zeta\partial_\zeta F_{aux} \\ - 2\left(\frac{2}{(1-z)(1-\zeta)} + k - 1\right) F_{aux} - \frac{2}{(1-z)^2(1-\zeta)^2} = 0. \end{aligned} \quad (4.44)$$

One has the following relation between the Schwarzian derivative of f and the function F_{aux}

$$(Sf)(z) = 6 \lim_{\zeta\to z}\left(\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right) = 6\lim_{\zeta\to z} F_{aux}(z,\zeta). \quad (4.45)$$

Because of this relation, by taking $\zeta \to z$ in (4.44), we arrive at an equation for $(Sf)(z)$ as

$$-\frac{\kappa}{2}(z\partial_z)^2 S - \left(\frac{1+z}{1-z} + 2\kappa\right) z\partial_z S - 2\left(\frac{2}{(1-z)^2} + k - 1\right) S - \frac{12}{(1-z)^4} = 0.$$

□

4.2.1.2 Initial conditions

The equation (4.39) is an ODE of order two which can be solved to get a unique solution with given initial conditions. We now proceed calculations of the expectations $\mathbb{E}((Sf)(0))$ and $\mathbb{E}((Sf)'(0))$ and then provide initial conditions to Eq. (4.39), namely, we will prove the following

Proposition 4.2.2.

$$\mathbb{E}((Sf)(0)) = \frac{-6}{\kappa+1}, \quad \mathbb{E}((Sf)'(0)) = -\frac{32\kappa}{(3\kappa+2)(\kappa+1)}. \tag{4.46}$$

Proof. Assume that $f(z)$ has a Taylor series

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... = z + \sum_{n=2}^{\infty} a_n z^n.$$

We then have

$$(Sf)(0) = \frac{f'''(0)}{f'(0)} - \frac{3}{2}\left(\frac{f''(0)}{f'(0)}\right)^2 = 6(a_3 - a_2^2), \tag{4.47}$$

$$(Sf)'(0) = \frac{f^{(4)}(0)}{f'(0)} - 4\frac{f''(0)f'''(0)}{(f'(0))^2} + 3\left(\frac{f''(0)}{f'(0)}\right)^3 = 24(a_4 - 2a_2a_3 + a_2^3). \tag{4.48}$$

Passing to expectations leads to

$$\mathbb{E}((Sf)(0)) = 6\mathbb{E}(a_3 - a_2^2), \tag{4.49}$$

$$\mathbb{E}((Sf)'(0)) = 24\mathbb{E}(a_4 - 2a_2a_3 + a_2^3). \tag{4.50}$$

Let us recall that the expectation of SLE map $f(z)$ was obtained in [5]. The authors also established a more general result for an LLE, concretely,

$$\mathbb{E}(a_n) = \prod_{k=0}^{n-2} \frac{\eta_k - k - 2}{\eta_{k+1} + k + 1}, \quad n \geq 3, \tag{4.51}$$

where η is the Lévy symbol and $\eta_k = \eta(k), k \in \mathbb{N}$. We thus, in our setting, have

$$\mathbb{E}(a_3) = -\frac{\kappa - 6}{(\kappa+1)(\kappa+2)}, \tag{4.52}$$

$$\mathbb{E}(a_4) = -\frac{4}{3}\frac{(\kappa-6)(\kappa-2)}{(\kappa+1)(\kappa+2)(3\kappa+2)}. \tag{4.53}$$

To arrive at the values of (4.49), (4.50), we now calculate $\mathbb{E}(a_2^2), \mathbb{E}(a_2a_3)$ and $\mathbb{E}(a_2^3)$. From Loewner equation (1), we have the following differential equations

$$\dot{a}_2 - a_2 = 2\bar{\lambda}, \tag{4.54}$$

$$\dot{a}_3 - 2a_3 = 4a_2\bar{\lambda} + 2\bar{\lambda}^2. \tag{4.55}$$

The first differential equation (4.54) (together with the uniform bound, $\forall t \geq 0, |a_2(t)| \leq C_2 < +\infty$) yields

$$a_2(t) = -2 \int_t^{+\infty} e^{-s} \bar{\lambda}(s) ds, \tag{4.56}$$

therefore

$$a_2(0) = -2 \int_0^{+\infty} e^{-s} \bar{\lambda}(s) ds. \tag{4.57}$$

In the same way, the second one (4.55) leads to

$$a_3(0) = -4 \int_0^{+\infty} e^{-2s} a_2(s) \bar{\lambda}(s) ds - 2 \int_0^{+\infty} e^{-2s} \bar{\lambda}^2 ds. \tag{4.58}$$

The first integral involves a_2 which has the integral form (4.56). The formula for $a_3(0)$ then reduces to

$$\begin{aligned} a_3(0) &= 4 \Big(\int_0^{+\infty} e^{-s} \bar{\lambda}(s) ds \Big)^2 - 2 \int_0^{+\infty} e^{-2s} \bar{\lambda}^2 ds \\ &= a_2^2 - 2 \int_0^{+\infty} e^{-2s} \bar{\lambda}^2 ds. \end{aligned} \tag{4.59}$$

From (4.57), we have

$$\begin{aligned} a_2^2 &= 4 \int_0^{\infty} \int_0^{\infty} e^{-s_1 - s_2} \bar{\lambda}(s_1) \bar{\lambda}(s_2) ds_1 ds_2 \\ &= 4 \int_0^{\infty} \int_0^{\infty} e^{-s_1 - s_2} e^{-i\sqrt{\kappa}(B_{s_1} + B_{s_2})} ds_1 ds_2 \\ &= 8 \int_0^{\infty} \int_0^{s_1} e^{-s_1 - s_2} e^{-i\sqrt{\kappa}(B_{s_1} - B_{s_2})} e^{-2i\sqrt{\kappa} B_{s_2}} ds_1 ds_2. \end{aligned} \tag{4.60}$$

Thank to Fubini's theorem and the characteristic function of Brownian motion, we then arrive at

$$\begin{aligned} \mathbb{E}(a_2^2) &= 8 \int_0^{\infty} \int_0^{s_1} e^{-s_1 - s_2} \mathbb{E}(e^{-i\sqrt{\kappa}(B_{s_1} - B_{s_2})}) \mathbb{E}(e^{-2i\sqrt{\kappa} B_{s_2}}) ds_1 ds_2 \\ &= 8 \int_0^{\infty} \int_0^{s_1} e^{-(1+\frac{\kappa}{2})s_1} e^{-(1+\frac{3}{2}\kappa)s_2} ds_1 ds_2 \\ &= \frac{8}{(\kappa+2)(\kappa+1)}. \end{aligned} \tag{4.61}$$

As we now have the formulas (4.52), (4.61), plugging them into (4.49) yeilds

$$\mathbb{E}((Sf)(0)) = 6\mathbb{E}(a_3 - a_2^2) = \frac{-6}{\kappa+1}. \tag{4.62}$$

Next, let us consider a_2^3 and a_2a_3. The identity (4.57) implies that

$$\begin{aligned} a_2^3 &= -8\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-s_1-s_2-s_3}\bar{\lambda}(s_1)\bar{\lambda}(s_2)\bar{\lambda}(s_3)ds_1ds_2ds_3 \\ &= -8\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-s_1-s_2-s_3}e^{-i\sqrt{\kappa}(B_{s_1}+B_{s_2}+B_{s_3})}ds_1ds_2ds_3 \\ &= -48\int_0^{+\infty}\int_0^{s_1}\int_0^{s_2} e^{-s_1-s_2-s_3}e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})}e^{-2i\sqrt{\kappa}(B_{s_2}-B_{s_3})}e^{-3i\sqrt{\kappa}B_{s_3})}ds_1ds_2ds_3, \end{aligned} \tag{4.63}$$

whereas (4.57) together with (4.59) implies

$$a_2a_3 = a_2^3 + I, \tag{4.64}$$

where

$$\begin{aligned} I &= 4\int_0^{+\infty}\int_0^{+\infty} e^{-s_1-2s_2}\bar{\lambda}(s_1)\bar{\lambda}^2(s_2)ds_1ds_2 \\ &= 4\int_0^{+\infty}\int_0^{+\infty} e^{-s_1-2s_2}e^{-i\sqrt{\kappa}(B_{s_1}+2B_{s_2})}ds_1ds_2. \end{aligned} \tag{4.65}$$

Corresponding to the partition of the quadrant $\mathbb{R}^+\times\mathbb{R}^+ = \Big\{(s_1,s_2) : 0 \le s_i < +\infty\Big\}$ into $\Big\{(s_1,s_2)\in\mathbb{R}^+\times\mathbb{R}^+, s_1 > s_2\Big\}$ and $\Big\{(s_1,s_2)\in\mathbb{R}^+\times\mathbb{R}^+, s_1 < s_2\Big\}$, the quantity I can be rewriten as the sum of I_1 and I_2:

$$I_1 = \int_0^{+\infty}\int_0^{s_1} e^{-s_1-2s_2}e^{-i\sqrt{\kappa}(B_{s_1}-B_{s_2})}e^{-3i\sqrt{\kappa}B_{s_2}}ds_2ds_1 \tag{4.66}$$

$$I_2 = \int_0^{+\infty}\int_0^{s_2} e^{-s_1-2s_2}e^{-2i\sqrt{\kappa}(B_{s_2}-B_{s_1})}e^{-3i\sqrt{\kappa}B_{s_1}}ds_1ds_2. \tag{4.67}$$

Apply again Fubini's theorem to (4.63) and make use of the characteristic function of Brownian motion, we obtain

$$\begin{aligned} \mathbb{E}(a_2^3) &= -48\int_0^{+\infty}\int_0^{s_1}\int_0^{s_2} e^{-(1+\frac{\kappa}{2})s_1}e^{-(1+\frac{3}{2}\kappa)s_2}e^{-(1+\frac{5}{2}\kappa)s_3}ds_3ds_2ds_1 \\ &= \frac{-32}{(3\kappa+2)(\kappa+2)(\kappa+1)}. \end{aligned} \tag{4.68}$$

Similarly, from (4.66) and (4.67) we also have

$$\mathbb{E}(I_1) = \frac{4}{3}\frac{1}{(\kappa+2)(3\kappa+2)}, \tag{4.69}$$

$$\mathbb{E}(I_2) = \frac{1}{3}\frac{1}{(\kappa+1)(3\kappa+2)}. \tag{4.70}$$

Therefore

$$\mathbb{E}(a_2a_3) = \frac{-32}{(3\kappa+2)(\kappa+2)(\kappa+1)} + \frac{4}{3}\frac{5\kappa+6}{(3\kappa+2)(\kappa+1)(\kappa+2)}. \tag{4.71}$$

The equations (4.53), (4.68) and (4.71) permit us to obtain

$$\mathbb{E}((Sf)'(0)) = 24\mathbb{E}(a_4 - 2a_2a_3 + a_2^3) = -\frac{32\kappa}{(3\kappa+2)(\kappa+1)}. \tag{4.72}$$

□

By solving the ODE (4.39) with the two initial conditions (4.46), we obtain a unique solution (4.38) which gives a explicit formula of the expected Schwarzian derivative of the whole-plane SLE_κ map.

4.2.2 Second moduli moment of Schwarzian

The same idea as in the previous section can be applied to obtain an equation satisfied by $\mathbb{E}(|(Sf)(z)|^2)$. In this section, we will prove the following

Theorem 4.2.2. *Let $f(z) = f_0(z)$ be the interior whole-plane SLE_κ map at time 0 and $(Sf)(z)$ is its Schwarzian derivative. Then $\mathbb{E}(|(Sf)(z)|^2)$ is a solution of the equation*

$$\begin{aligned}&-\frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2M - \frac{1+z}{1-z}z\partial_z M - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\partial_{\bar{z}}M\\ &-4\left(\frac{1}{(1-z)^2} + \frac{1}{(1-\bar{z})^2} - 1\right)M - \frac{12\bar{S}}{(1-z)^4} - \frac{12S}{(1-\bar{z})^4} = 0,\end{aligned} \tag{4.73}$$

where $S(z) = \mathbb{E}((Sf)(z))$ is the expected Schwarzian derivative of f.

Proof. Define the auxiliary function

$$\begin{aligned}G_{aux}(z,\bar{z},\zeta,\bar{\zeta}) &= \left|\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right|^2\\ &= \frac{|f'(z)|^2|f'(\zeta)|^2}{|f(z)-f(\zeta)|^4} + \frac{1}{|z-\zeta|^4} - \frac{1}{(z-\zeta)^2}\overline{\left(\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2}\right)} - \frac{1}{(\bar{z}-\bar{\zeta})^2}\frac{f'(z)f'(\zeta)}{(f(z)-f(\zeta))^2},\end{aligned} \tag{4.74}$$

then, from (4.3) we have

$$|(Sf)(z)|^2 = 36\lim_{\zeta\to z,\bar{\zeta}\to\bar{z}} G_{aux}(z,\bar{z},\zeta,\bar{\zeta}). \tag{4.75}$$

We consider the function

$$G(z,\bar{z},\zeta,\bar{\zeta}) = \mathbb{E}\left(\frac{|z-\zeta|^4}{|f(z)-f(\zeta)|^4}|f'(z)|^2|f'(\zeta)|^2\right). \tag{4.76}$$

From Corollary 4.1.2, G is a solution of the equation

$$\begin{aligned}&-\frac{\kappa}{2}(z\partial z - \bar{z}\bar{\partial}_{\bar{z}} + \zeta\partial_\zeta - \bar{\zeta}\bar{\partial}_{\bar{\zeta}})^2G - \frac{1+z}{1-z}z\partial zG - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}_{\bar{z}}G - \frac{1+\zeta}{1-\zeta}\zeta\partial\zeta G - \frac{1+\bar{\zeta}}{1-\bar{\zeta}}\bar{\zeta}\bar{\partial}_{\bar{\zeta}}G\\ &+\left[-\frac{2}{(1-z)^2} - \frac{2}{(1-\bar{z})^2} - \frac{2}{(1-\zeta)^2} - \frac{2}{(1-\bar{\zeta})^2} + \frac{4}{(1-z)(1-\zeta)} + \frac{4}{(1-\bar{z})(1-\bar{\zeta})}\right]G = 0.\end{aligned} \tag{4.77}$$

Because G and G_{aux} are related by

$$G = |z-\zeta|^4 G_{aux} + F + \bar{F} - 1, \tag{4.78}$$

where F is defined by (4.40), from Eq. (4.77) we can obtain an equation satisfied by G_{aux}. By plugging (4.78) into (4.77), we arrive at an equation satisfied by G_{aux}

$$\begin{aligned}
&-\frac{\kappa}{2}(z\partial z - \bar{z}\bar{\partial}_{\bar{z}} + \zeta\partial_\zeta - \bar{\zeta}\bar{\partial}_{\bar{\zeta}})^2 G_{aux} \\
&-\frac{1+z}{1-z} z\partial z G_{aux} - \frac{1+\bar{z}}{1-\bar{z}}\bar{z}\bar{\partial}_{\bar{z}} G_{aux} - \frac{1+\zeta}{1-\zeta}\zeta\partial\zeta G_{aux} - \frac{1+\bar{\zeta}}{1-\bar{\zeta}}\bar{\zeta}\bar{\partial}_{\bar{\zeta}} G_{aux} \\
&+2\Bigg[\frac{2(z\bar{z}\zeta\bar{\zeta} - z\bar{z}\zeta - z\bar{z}\bar{\zeta} - z\zeta\bar{\zeta} - \bar{z}\zeta\bar{\zeta} + z\bar{z} + \zeta\bar{\zeta} + z\bar{\zeta} + \bar{z}\zeta - 1)}{(1-z)(1-\bar{z})(1-\zeta)(1-\bar{\zeta})} \\
&-\left(\frac{1}{1-z} - \frac{1}{1-\zeta}\right)^2 - \left(\frac{1}{1-\bar{z}} - \frac{1}{1-\bar{\zeta}}\right)^2\Bigg] G_{aux} \\
&-\frac{2\overline{F_{aux}}}{(1-z)^2(1-\zeta)^2} - \frac{2F_{aux}}{(1-\bar{z})^2)(1-\bar{\zeta})^2} = 0,
\end{aligned} \tag{4.79}$$

where the function F_{aux} is defined by (4.42). Finally, letting $\zeta \to z, \bar{\zeta} \to \bar{z}$ in (4.79) and using (4.75) as well as (4.45) lead us to the equation (4.73) obeyed by $M(z,\bar{z})$ in Theorem 4.2.2. □

Remark 4.2.1. *In Eq.* (4.73), *if we set* $z = \bar{z} = 0$, *then it yields*

$$M(0,0) = -3(S(0) + \overline{S(0)}).$$

The first equality in (4.46) *implies*

$$\mathbb{E}(|(Sf)(0)|^2) = M(0,0) = \frac{36}{\kappa+1}.$$

Hence Theorem 4.2.2 recovers the result obtained in [4] on the Schwarzian derivative of the SLE_κ *map.*

Chapter 5

ASYMPTOTIC VARIANCE OF SLE

Let $(\phi_t), t \in U$, a general analytic family of conformal maps on the unit disc with $\phi_0 = id$ and $\phi_t(0) = 0, \forall t \in U$ where U is a neighborhood of 0. Then one may write

$$\phi_t(z) = \int_0^z e^{\log \phi_t'(u)} du, z \in \mathbb{D}$$

and

$$\frac{\partial}{\partial t}\phi_t(z) = \int_0^z \frac{\partial}{\partial t}(\log \phi_t'(u)) e^{\log \phi_t'(u)} du.$$

The function $b(z) = V'(z) = \frac{\partial}{\partial t}(\log \phi_t'(z))|_{t=0}$ belongs to the Bloch space $\mathcal{B}$ which is defined by

$$\mathcal{B} = \{\text{b holomorphic in } \mathbb{D}, \sup_{\mathbb{D}}(1-|z|^2)|b'(z)| < \infty\}.$$

In [14], McMullen asked under what conditions on the family (ϕ_t) it is true that

$$2\frac{d^2}{dt^2}\text{H.dim}\phi_t(\partial\mathbb{D})\bigg|_{t=0} = \sigma^2(b), \tag{5.1}$$

where $\text{H.dim}\phi_t(\partial\mathbb{D})$ is the Hausdorff dimension of $\phi_t(\partial\mathbb{D})$ and $\sigma^2(b)$ is the McMullen's asymptotic variance of a Bloch function b given by

$$\sigma^2(b) = \limsup_{r\to 1^-} \frac{1}{2\pi|\log(1-r)|}\int_0^{2\pi} |b(re^{i\theta})| d\theta.$$

Conversely, if b is a Bloch function, then the family of functions

$$\phi_t(z) = \int_0^z e^{tb(u)} du \tag{5.2}$$

is an analytic family. There exists a neighborhood U of 0 such that if $t \in U$ then ϕ_t is a conformal map with quasiconformal extension.

In [9], by using a probability argument M.Zinsmeister and THN.Le described a relatively large class of functions in $\mathcal{B}$ for which (φ_t) defined by (5.2) satisfies (5.1), where the Hausdorff dimension is replaced by the Minkovski dimension

$$\frac{d^2}{dt^2}\bigg|_{t=0} \text{M. dim } \phi_t(\partial\mathbb{D}) = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=1} |b(z)|^2 |dz|. \tag{5.3}$$

Namely, they proved that

$$\lim_{p\to 0} \frac{2\beta(p,\phi)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_{|z|=1} |b(z)|^2 |dz|, \tag{5.4}$$

which is, in this setting of ϕ_t, equivalent to (5.3). Here $\beta(p,\phi)$ is the integral means spectrum of ϕ, $\phi'(z) = \exp(b(z))$, defined by

$$\beta(p,\phi) = \limsup_{r\to 1^-} \frac{\log(\int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta)}{|\log(1-r)|}, p \in \mathbb{R}. \tag{5.5}$$

The starting motivation of this chapter is to prove (5.4) in expectation for the interior whole plane SLE_2 map. Concretely, we will prove the following

Theorem 5.0.1. *Let $f := f_0$ be the interior whole-plane* SLE_κ *map at time 0 and $\bar\beta(p)$ be the average integral means spectrum of f defined by*

$$\bar\beta(p) = \limsup_{r\to 1^-} \frac{\log(\int_0^{2\pi} \mathbb{E}(|f'(re^{i\theta})|^p) d\theta)}{|\log(1-r)|}, p \in \mathbb{R},$$

then, for $\kappa = 2$,

$$\lim_{p\to 0} \frac{2\bar\beta(p)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2) d\theta. \tag{5.6}$$

The average integral means spectrum of the interior whole-plane SLE_κ was obtained in [5]. Therefore, to prove Theorem 5.0.1, we have to determined the right hand side of (5.6). For this purpose, we will proceed the two following steps:

— Using martingale techniques to derive an equation satisfied by $\mathbb{E}(|\log f'(z)|^2)$ for SLE_2 (as we will see, one can derive that for all $\kappa > 0$).

— Solving the equation obtained in the first step for $\kappa = 2$ by a consideration of its series form to get an explicit expression of $\mathbb{E}(|\log f'(z)|^2)$. This expression verifies the relation (5.6).

5.1 Logarithmic expectation of SLE

In this section we present results concerning the logarithmic expectation $F(z) := \mathbb{E}(\log f'(z))$ including a differential equation satisfied by F and the exact formula of the derivative of F (hence F). These results will be useful to find the equation of the second logarithmic moment $\mathbb{E}(|\log f'(z)|^2)$ and to find it solutions with that we deal in the two next sections.

Theorem 5.1.1. *Let $f := f_0$ be the interior whole-plane* SLE_κ *map at time 0 and $F(z) = \mathbb{E}(\log f'(z))$, then F satisfies the equation*

$$-\frac{\kappa}{2} z^2 \partial_{zz}^2 F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big) = 0, \tag{5.7}$$

or the equivalent equation,

$$-\frac{\kappa}{2}z\partial_{zz}^2F+\Big(\frac{z+1}{z-1}-\frac{\kappa}{2}\Big)\partial_zF+\frac{2(z-2)}{(z-1)^2}=0. \tag{5.8}$$

It follows that

$$\partial_z(z)F=\mathbb{E}\Big(\frac{f''(z)}{f'(z)}\Big)=\frac{4}{\kappa}\frac{(1-z)^{\frac{\kappa}{4}}}{z^{\frac{2}{\kappa}+1}}\int_0^z\frac{u^{\frac{2}{\kappa}}(u-2)}{(1-u)^{\frac{4}{\kappa}+2}}du. \tag{5.9}$$

Proof. Let us firstly introduce the time-dependent, auxiliary function

$$\tilde{F}(z,t):=\mathbb{E}(\log\tilde{f}_t) \tag{5.10}$$

where $\tilde{f}_t$ is the reverse radial SLE$_\kappa$ process 1.9. As a consequence of lemma 1.4.2, the function F is the limit in law

$$\lim_{t\to+\infty}(t+\tilde{F}(z,t))\overset{\text{(law)}}{=}F(z). \tag{5.11}$$

We now consider the conditional expectation

$$\mathcal{M}_s:=\mathbb{E}(\log\tilde{f}'_t|\mathcal{F}_s). \tag{5.12}$$

By Markov property of SLE

$$\log\tilde{f}'_t(z)=\log\tilde{f}'_s(z)+\log\tilde{f}'_{t-s}(z_s),\qquad z_s:=\frac{\tilde{f}_s(z)}{\lambda_s}. \tag{5.13}$$

Thus

$$\mathcal{M}_s=\log\tilde{f}'_s(z)+\tilde{F}(\tau,z_s), \tag{5.14}$$

where $\tau:=t-s$ et $\tilde{F}(t,z):=\mathbb{E}(\log\tilde{f}'_t)$.
We know that $(\mathcal{M}_s)_{t\geq s\geq 0}$ is a martingale. This fact implies that the ds-drift term of the Itô derivative of $\mathcal{M}_s$ vanishes, that permits us to obtain an equation satisfied by $\tilde{F}$. We now calculate that ds-drift term.

By regarding $\mathcal{M}_s$ as a stochastic process governed by the two processes $\log\tilde{f}'_s(z)$ and z_s, the Itô derivative of $\mathcal{M}_s$ is determined by

$$d\mathcal{M}_s=d\log\tilde{f}'_s(z)-\partial_\tau\tilde{F}ds+\partial_{z_s}\tilde{F}dz_s+\frac{1}{2}\partial^2_{z_sz_s}\tilde{F}dz_sdz_s. \tag{5.15}$$

Since the Itô differentials $d\log\tilde{f}'$ and dz_s are written in term of ds and dB_s as

$$d\log\tilde{f}'_s(z)=\Big(1-\frac{2}{(z_s-1)^2}\Big)ds \tag{5.16}$$

$$dz_s=z_s\Big(\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\Big)ds-i\sqrt{\kappa}z_sdB_s, \tag{5.17}$$

we obtain the coefficient of the drift term of $d\mathcal{M}_s$

$$\tilde{\mathcal{P}}_{sing}(D)(\tilde{F}):=1-\frac{2}{(z_s-1)^2}-\tilde{F}_\tau+\Big(\frac{z_s+1}{z_s-1}-\frac{\kappa}{2}\Big)z_s\partial_{z_s}\tilde{F}-\frac{\kappa}{2}z_s^2\partial^2_{z_sz_s}\tilde{F}. \tag{5.18}$$

The vanishing of this quantity give us the equation

$$\tilde{\mathcal{P}}_{sing}(D)(\tilde{F}) = 1 - \frac{2}{(z_s-1)^2} - \tilde{F}_\tau + \Big(\frac{z_s+1}{z_s-1} - \frac{\kappa}{2}\Big) z_s \partial_{z_s} \tilde{F} - \frac{\kappa}{2} z_s^2 \partial^2_{z_s z_s} \tilde{F} = 0. \quad (5.19)$$

Finally we use (5.11) to find the equation satisfied by F

$$2\Big(1 - \frac{1}{(z-1)^2}\Big) + \Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big) z \partial_z F - \frac{\kappa}{2} z^2 \partial_{zz} F = 0. \quad (5.20)$$

After eliminating the factor z, it yields an equivalent equation

$$-\frac{\kappa}{2} z \partial^2_{zz} F + \Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big) \partial_z F + \frac{2(z-2)}{(z-1)^2} = 0. \quad (5.21)$$

This equation is an ODE of order one of $\partial_z F$ with the initial condition $\partial_z F(0) = \frac{-8}{2+\kappa}$, obtained by substituting $z = 0$ into (5.21). One may use the integrating factor method to solve this equation and get the expression (5.9) of $\partial_z F$. □

From Theorem 5.1.1, simple formulas can be obtained for particular cases. For example, in the case of $\kappa = 2$,

$$\partial_z F(z) = -\frac{4}{3} + \frac{2}{3}\frac{1}{z-1}, \quad (5.22)$$

for $\kappa = 1$,

$$\partial_z F(z) = \frac{7}{15} z - \frac{28}{15} + \frac{4}{5}\frac{1}{z-1}, \quad (5.23)$$

for $\kappa = 4$, by putting $w := \sqrt{z}$

$$\partial_z F(z) = \frac{w^2-1}{8w^3}\Big(\frac{10w - 6w^3}{(w^2-1)^2} + 5\log\frac{1-w}{1+w}\Big). \quad (5.24)$$

The exact formula of F can also be obtained for certain values of κ by integrating both sides of (5.9). For instance, with $\kappa = 2$ we have

$$F(z) = -\frac{4}{3} z + \frac{2}{3}\log(1-z), \quad (5.25)$$

with $\kappa = 1$

$$F(z) = \frac{7}{30} z^2 - \frac{28}{15} z + \frac{4}{5}\log(1-z). \quad (5.26)$$

Remark 5.1.1. *If $\kappa = \frac{2}{n}$ then*

$$F(z) = P_n(z) + \frac{2n}{2n+1}\log(1-z), \quad (5.27)$$

where P_n is a polynomial of degree n.

Remark 5.1.2. *For general κ*

$$\partial_z F(z) \sim \frac{4}{4+\kappa}\frac{1}{z-1}, z \to 1 \quad (5.28)$$

so

$$F(z) \sim \frac{4}{4+\kappa}\log(1-z), z \to 1. \quad (5.29)$$

5.2 Second logarithmic moment

In order to obtain the value of the asymptotic variance we need the integral means on circles $\{|z| = r\}$ of the second moduli logarithmic moment $\mathbb{E}(|\log f'|^2)$. For this motivation, in this section we continue following the martingale argument to find an equation satisfied by $G(z) := \mathbb{E}(|\log f'|^2)$. Next, we consider G of a special form

$$G(z, \bar{z}) = F(z)\overline{F(z)} + R(z, \bar{z}) \tag{5.30}$$

and show that R is the solution of a differential equation, intuitively, which may be easier to deal with than one satisfied by G.

We now consider the martingale $(\mathcal{N}_s)_{t\geq s\geq 0}$ defined by

$$\mathcal{N}_s := \mathbb{E}(|\log \tilde{f}_t'|^2|\mathcal{F}_s). \tag{5.31}$$

Recall that the martingale argument is based on the fact that the ds-drift term in Itô derivative of a martingale vanishes. As in the preceding sections, we firstly calculate the ds term of $\mathcal{N}_s$ to find an equation of the auxiliary function $\tilde{G}$ which is defined as following

$$\tilde{G}(t, z, \bar{z}) := \mathbb{E}(|\log \tilde{f}_t'|^2), \tag{5.32}$$

where $\tilde{f}_t$ is the reverse radial SLE_κ process (1.9).

We rewrite $\mathcal{N}_s$ by using the Markov property of SLE

$$\mathcal{N}_s = |\log \tilde{f}_s'(z)|^2 + \tilde{G}(\tau, z_s, \bar{z}_s) + \log \tilde{f}_s'(z)\overline{\tilde{F}(\tau, z_s)} + \overline{\log \tilde{f}_s'(z)}\tilde{F}(\tau, z_s). \tag{5.33}$$

It is noted that $\tilde{F}$ is defined in the preceding section by (5.10).

For reasons of concision, we hereafter use the acronym dsCoeff for the phrase "coefficient of ds in Itô derivative". Thank to the linearity of Itô derivative, one can perform the ds term of $d\mathcal{N}_s$ as the sum of those of the three terms in the right hand side of (5.33). Since the coefficients of ds in $d|\log \tilde{f}_s'(z)|^2$, $d\log \tilde{f}_s'(z)\overline{\tilde{F}(\tau, z_s)}$ and $d\overline{\log \tilde{f}_s'(z)}\tilde{F}(\tau, z_s)$ are respectively

$$\begin{aligned}
&\partial_s \log \tilde{f}_t'(z)\overline{\log \tilde{f}_t'(z)} + \log \tilde{f}_t'(z)\partial_s\overline{\log \tilde{f}_t'(z)}\\
&\partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \log \tilde{f}_t'(z).ds\text{Coeff of } \overline{\tilde{F}(\tau, z_s)}\\
&\partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s) + \overline{\log \tilde{f}_t'(z)}.ds\text{Coeff of } \tilde{F}(\tau, z_s),
\end{aligned}$$

the ds-coefficient of $d\mathcal{N}_s$ is obtained as

$$\begin{aligned}
&ds\text{Coeff of } \tilde{G}(\tau, z_s, \bar{z}_s) + \partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s)\\
&+ \overline{\log \tilde{f}_t'(z)}(\partial_s \log \tilde{f}_t'(z) + ds\text{Coeff of } \tilde{F}(\tau, z_s))\\
&+ \log \tilde{f}_t'(z)(\partial_s\overline{\log \tilde{f}_t'(z)} + ds\text{Coeff of } \overline{\tilde{F}(\tau, z_s)}).
\end{aligned} \tag{5.34}$$

Note that

$$\begin{aligned}
&\partial_s \log \tilde{f}_t'(z) + ds\text{Coeff of } \tilde{F}(\tau, z_s)\\
\text{and } &\partial_s\overline{\log \tilde{f}_t'(z)} + ds\text{Coeff of } \overline{\tilde{F}(\tau, z_s)})
\end{aligned}$$

vanish because the first is the coefficient of the ds-drift term in Itô derivative of the martingale $(\mathcal{M}_s)_{t\geq s\geq 0}$ defined by (5.12) while the second is the complex conjugate of the first one. The ds-drift term coefficient of $d\mathcal{N}_s$ is thus reduced as

$$ds\text{Coeff of } \tilde{G}(\tau, z_s, \bar{z}_s) + \partial_s \log \tilde{f}_t'(z)\overline{\tilde{F}(\tau, z_s)} + \partial_s\overline{\log \tilde{f}_t'(z)}\tilde{F}(\tau, z_s). \tag{5.35}$$

We now expand the terms appear in (5.35). Namely, we have

$$\begin{aligned}\partial_s \log \tilde{f}_s' &= \frac{\partial_z\left[\tilde{f}_s \frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)}\right]}{\tilde{f}_s'} = \frac{\tilde{f}_s+\lambda(s)}{\tilde{f}_s-\lambda(s)} - \frac{2\lambda(s)\tilde{f}_s}{(\tilde{f}_s-\lambda(s))^2}\\ &= 1 - \frac{2}{(1-z_s)^2},\\ \partial_s\overline{\log \tilde{f}_s'} &= 1 - \frac{2}{(1-\bar{z}_s)^2}.\end{aligned}$$

In addition, by applying again Itô formula to $\tilde{G}(\tau, z_s, \bar{z}_s)$, we also get

$$ds\text{Coeff of } \tilde{G}(\tau, z_s, \bar{z}_s) = \frac{z_s+1}{z_s-1} z_s\partial_{z_s}\widetilde{G} + \frac{\bar{z}_s+1}{\bar{z}_s-1}\bar{z}_s\bar{\partial}_{\bar{z}_s}\widetilde{G} - \partial_\tau\widetilde{G} - \frac{\kappa}{2}(z_s\partial_{z_s} - \bar{z}_s\bar{\partial}_{\bar{z}_s})^2\widetilde{G}. \tag{5.36}$$

Let us denote the differential operator in the right hand side of (5.36) by $\mathcal{P}_{prin}(D)$. The quantity (5.35) then equals to

$$\mathcal{P}_{prin}(D)(\tilde{G}) + \Big(1 - \frac{2}{(1-z_s)^2}\Big)\overline{\tilde{F}(\tau, z_s)} + \Big(1 - \frac{2}{(1-\bar{z}_s)^2}\Big)\tilde{F}(\tau, z_s). \tag{5.37}$$

We recall that this quantity vanishes since $\mathcal{N}_s$ is a martingale. One thus has an equation satisfied by $\tilde{G}$

$$\mathcal{P}_{prin}(D)(\tilde{G}) + \Big(1 - \frac{2}{(1-z_s)^2}\Big)\overline{\tilde{F}(\tau, z_s)} + \Big(1 - \frac{2}{(1-\bar{z}_s)^2}\Big)\tilde{F}(\tau, z_s) = 0. \tag{5.38}$$

We continue by putting (5.38) into the limitation as τ tends to $+\infty$ and using Lemma 1.4.2 to get an equation of G. Before doing that, let us rewrite $|\log \tilde{f}_\tau'|^2$ as following

$$|\log \tilde{f}_\tau'|^2 = |\log e^\tau \tilde{f}_\tau'|^2 - \tau(\log \tilde{f}_\tau' + \overline{\log \tilde{f}_\tau'}) - \tau^2. \tag{5.39}$$

This identity implies that

$$\partial_{z_s}\tilde{G} = \partial_{z_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial_{z_s}\tilde{F}(\tau, z_s) \tag{5.40}$$

$$\partial_{\bar{z}_s}\tilde{G} = \partial_{\bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial_{\bar{z}_s}\overline{\tilde{F}(\tau, z_s)} \tag{5.41}$$

$$\partial^2_{z_s z_s}\tilde{G} = \partial^2_{z_s z_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial^2_{z_s z_s}\tilde{F}(\tau, z_s) \tag{5.42}$$

$$\partial^2_{\bar{z}_s\bar{z}_s}\tilde{G} = \partial^2_{\bar{z}_s\bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) - \tau\partial^2_{\bar{z}_s\bar{z}_s}\overline{\tilde{F}(\tau, z_s)} \tag{5.43}$$

$$\partial^2_{z_s\bar{z}_s}\tilde{G} = \partial^2_{z_s\bar{z}_s}\mathbb{E}(|\log e^\tau \tilde{f}_\tau'|^2) \tag{5.44}$$

$$\partial_\tau\tilde{G} = \partial_\tau\mathbb{E}(|\log e^\tau \tilde{f}_t'|^2) - \mathbb{E}(\log e^\tau \tilde{f}_\tau') - \tau\partial_\tau\tilde{F}(\tau, z_s) - \overline{\mathbb{E}(\log e^\tau \tilde{f}_\tau')} - \tau\partial_\tau\overline{\tilde{F}(\tau, z_s)}. \tag{5.45}$$

In (5.38) by replacing the terms that appear in the left-hand side of the above identities by their corresponding right-hand side terms, we arrive at

$$\mathcal{P}_{prin}(D)(\mathbb{E}(|\log e^\tau \tilde{f}'_\tau|^2)) + 2\Big(1 - \frac{1}{(1-\bar{z}_s)^2}\Big)\mathbb{E}(\log e^\tau \tilde{f}'_\tau) + 2\Big(1 - \frac{1}{(1-z_s)^2}\Big)\overline{\mathbb{E}(\log e^\tau \tilde{f}'_\tau)}$$
$$- \tau\tilde{\mathcal{P}}_{sing}(D)(\tilde{F})(\tau, z_s) - \tau\overline{\tilde{\mathcal{P}}_{sing}(D)(\tilde{F})(\tau, z_s)} = 0. \tag{5.46}$$

Note that $\tilde{\mathcal{P}}_{sing}(\tilde{F})(\tau, z_s)$ is the ds-term coefficient in $\mathrm{d}\mathcal{M}_s$, therefore vanishes. One can thus get rid of the second line of (5.46) and obtains

$$\mathcal{P}_{prin}(D)(\mathbb{E}(|\log e^\tau \tilde{f}'_\tau|^2) + 2\Big(1 - \frac{1}{(1-\bar{z}_s)^2}\Big)\mathbb{E}(\log e^\tau \tilde{f}'_\tau)$$
$$+ 2\Big(1 - \frac{1}{(1-z_s)^2}\Big)\overline{\mathbb{E}(\log e^\tau \tilde{f}'_\tau)} = 0. \tag{5.47}$$

Finally, the Lemma 1.4.2 is used to derive an equation satisfied by G.

Proposition 5.2.1. *Let $f := f_0$ be the interior whole-plane* SLE_κ *map at time 0 and $G(z,\bar{z}) = \mathbb{E}(|\log f'|^2)$, then G satisfies the equation*

$$\mathcal{P}_{prin}(D)(G) + 2\Big(1 - \frac{1}{(\bar{z}-1)^2}\Big)F(z) + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\overline{F(z)} = 0 \tag{5.48}$$

where

$$\mathcal{P}_{prin}(D) = \frac{z+1}{z-1}z\partial_z + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\partial_{\bar{z}} - \frac{\kappa}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})^2.$$

We now consider function G of the form

$$G(z,\bar{z}) = F(z)\overline{F(z)} + R(z,\bar{z}), \tag{5.49}$$

where F is the solution of Eq. (5.7) and $\overline{F(z)}$ is its complex conjugate. By replacing $G(z,\bar{z})$ by $F(z)\overline{F(z)} + R(z,\bar{z})$, the left hand side of (5.48) becomes

$$\mathcal{P}_{prin}(D)(R) + \kappa z\bar{z}\partial_z F\partial_{\bar{z}}\bar{F}$$
$$+ \left[-\frac{\kappa}{2}z^2\partial^2_{zz}F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\right]\bar{F}$$
$$+ \overline{\left[-\frac{\kappa}{2}z^2\partial^2_{zz}F + z\Big(\frac{z+1}{z-1} - \frac{\kappa}{2}\Big)\partial_z F + 2\Big(1 - \frac{1}{(z-1)^2}\Big)\right]}F. \tag{5.50}$$

Since Eq. (5.1.1), the two last lines vanish. The quantity (5.50) is thus simplified as

$$\mathcal{P}_{prin}(D)(R) + \kappa z\bar{z}\partial_z F\partial_{\bar{z}}\bar{F}. \tag{5.51}$$

That leads us to the following

Proposition 5.2.2.

$$\mathcal{P}_{prin}(D)(R) = -\kappa z\bar{z}\partial_z F\partial_{\bar{z}}\overline{F(z)}. \tag{5.52}$$

5.3 Asymptotic variance of SLE_2

The function R is analytic in the bi-disk $\mathbb{D}\times\mathbb{D}$, we thus write it in a series form

$$R(z,\bar{z}) = \sum_{n\geq 0} a_{n,m} z^n \bar{z}^m. \tag{5.53}$$

Since the normalization of SLE map f, $a_{0,0}=0$.

The equation (5.52) give us recurrent relations of the coefficients $a_{n,m}$. In the case of $\kappa=2$, these recurrent relations make us able to obtain easily an explicit formula of R and thus of G. Let us set $\kappa=2$ and put the series form (5.53) of R into Eq. (5.52) then after identifying the two sides of the equation, we obtain

$$a_{1,0} = a_{0,1} = a_{2,0} = a_{0,2} = 0, \tag{5.54}$$

$$a_{1,1} = 4, a_{2,2} = \frac{14}{9}, \tag{5.55}$$

$$\begin{aligned} a_{n,m} = {} & \frac{1}{(n-m)^2+n+m}\Bigg[\Bigg((n-m-1)^2 - n + m + 1\Bigg)a_{n-1,m} \\ & + \Bigg((n-m+1)^2 + n - m + 1\Bigg)a_{n,m-1} + \Bigg(-(n-m)^2 + n + m - 2\Bigg)a_{n-1,m-1}\Bigg]. \end{aligned} \tag{5.56}$$

By using the inductive method, we can prove that

$$a_{1,1} = 4 \tag{5.57}$$

$$a_{n,n} = \frac{4}{3}\left(\frac{4}{n^2} + \frac{1}{3n}\right) \qquad \forall n \geq 2 \tag{5.58}$$

$$a_{n,n+1} = a_{n+1,n} = \frac{-8}{3n(n+1)} \qquad \forall n \geq 1 \tag{5.59}$$

$$a_{n,m} = 0 \quad \text{otherwise.} \tag{5.60}$$

These identities are equivalent to

$$R(z,\bar{z}) = \frac{4}{3}\left[-\frac{4}{3}z\bar{z} + \frac{2(z+\bar{z})}{z\bar{z}}\int_0^{z\bar{z}} \log(1-u)du - 4\int_0^{z\bar{z}} \frac{\log(1-u)}{u}du - \frac{1}{3}\log(1-z\bar{z})\right], \tag{5.61}$$

and hence

$$\begin{aligned} G(z,\bar{z}) = {} & \left(-\frac{4}{3}z + \frac{2}{3}\log(1-z)\right)\left(-\frac{4}{3}\bar{z} + \frac{2}{3}\log(1-\bar{z})\right) \\ & + \frac{4}{3}\left[-\frac{4}{3}z\bar{z} + \frac{2(z+\bar{z})}{z\bar{z}}\int_0^{z\bar{z}} \log(1-u)du - 4\int_0^{z\bar{z}} \frac{\log(1-u)}{u}du - \frac{1}{3}\log(1-z\bar{z})\right]. \end{aligned} \tag{5.62}$$

The equation (5.62) directly implies the main result of this section

Theorem 5.3.1. *Let $f := f_0$ be the interior whole-plane* SLE_2 *map at time 0 then*

$$\lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|}\int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2)d\theta = \frac{2}{9}. \tag{5.63}$$

Proof. By using the Maclaurin expansion of the logarithmic function $\log(1-z)$

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1,$$

one rewrite the function $G(z, \bar{z})$ (5.62) as

$$\begin{aligned} G(z,\bar{z}) =& \frac{8}{9}\left(z \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n} + \bar{z} \sum_{n=1}^{\infty} \frac{z^n}{n} \right) + \frac{4}{9} \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{n=1}^{\infty} \frac{\bar{z}^n}{n} \\ & - \frac{8}{3}(z+\bar{z}) \sum_{n=1}^{\infty} \frac{(z\bar{z})^n}{n(n+1)} + \frac{16}{3} \sum_{n=1}^{\infty} \frac{(z\bar{z})^n}{n^2} - \frac{4}{9} \log(1 - z\bar{z}). \end{aligned} \tag{5.64}$$

The Plancherel's theorem then yields

$$\frac{1}{2\pi} \int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2) d\theta = \frac{16}{9} r^2 + \frac{52}{9} \sum_{n=1}^{\infty} \frac{r^{2n}}{n^2} - \frac{4}{9} \log(1 - r^2), \quad r < 1. \tag{5.65}$$

It follows that

$$\begin{aligned} \lim_{r \to 1^-} \frac{1}{4\pi |\log(1-r)|} \int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2) d\theta &= \lim_{r \to 1^-} \frac{\frac{16}{9} r^2 + \frac{52}{9} \sum_{n=1}^{\infty} \frac{r^{2n}}{n^2} - \frac{4}{9} \log(1-r^2)}{-2\log(1-r)} \\ &= \frac{2}{9}. \end{aligned} \tag{5.66}$$

□

Return to Theorem 5.0.1, let us recall that the average integral mean spectrum of the interior whole-plane SLE$_\kappa$ was obtained in [5] as

$$\bar{\beta}(p) = \begin{cases} \beta_{tip}(p,\kappa) & p < p_0'(\kappa) \\ \beta_0(p,\kappa) & p_0'(\kappa) \le p < p^*(\kappa) \\ \beta_1(p,\kappa) & p^*(\kappa) \le p, \end{cases} \tag{5.67}$$

where

$$\beta_{tip}(p,\kappa) = -p - 1 + \frac{1}{4}(4 + \kappa - \sqrt{(4+\kappa)^2 - 8\kappa p}), \tag{5.68}$$

$$\beta_0(p,\kappa) = -p + \frac{4+\kappa}{4\kappa}(4 + \kappa - \sqrt{(4+\kappa)^2 - 8\kappa p}), \tag{5.69}$$

$$\beta_1(p,\kappa) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p} \tag{5.70}$$

and

$$p_0'(\kappa) = -1 - \frac{3\kappa}{8}, \tag{5.71}$$

$$p^*(\kappa) = \frac{1}{32\kappa}\left(\sqrt{2(4+\kappa)^2 + 4} - 6\right)\left(\sqrt{2(4+\kappa)^2 + 4} + 2\right). \tag{5.72}$$

It follows that the development of $\bar{\beta}$ at $p = 0$ is

$$\bar{\beta}(p) = \frac{2\kappa}{(4+\kappa)^2} p^2 + o(p^2). \tag{5.73}$$

In particular, with $\kappa = 2$, we have that

$$\lim_{p \to 0} \frac{2\bar{\beta}(p)}{p^2} = \frac{2}{9}. \tag{5.74}$$

This fact together with Theorem 5.3.1 conclude the proof of Theorem 5.0.1.

We believe that the relation (5.6) is true for SLE_κ for all $\kappa > 0$. However, we have not arrived at a proof of this general result yet.

Chapter 6

CONCLUSIONS AND PERSPECTIVES

6.1 Conclusions

In this thesis, we studied the following problems:

Firstly, we continued the work started in [11] on consideration the logarithmic coefficients of the interior whole-plane SLE maps. In that work, the authors had given an explicit formula for the expected logarithmic coefficients and proceeded calculations of some first moduli second moments of these coefficients. Here, we mimicked the proof given in [11] to generalize the explicit formula of the expected logarithmic coefficients for the interior whole-plane LLE maps. We also proved a general result for SLE_2 by giving an explicit formula to the moduli second moments of all logarithmic coefficients. With this result, we revisited the Milin's conjecture in the case of SLE_2, namely, we arrived at the conclusion of the conjecture in a sense of expectation. It is noticed that in order to have an explicit formula to the moduli second moments of all logarithmic coefficients of the SLE_2 map, we calculated the expectation $\mathbb{E}\left(|z|^2\left|\frac{f'(z)}{f(z)}\right|^2\right)$.

Secondly, we generalize the results on moments of the interior whole-plane SLE map in [4], [5]. By using a martingale method and considering the so-called Beliaev-Smirnov equations, we yielded a closed form of the expected general moments $\mathbb{E}\left(z^{\frac{q}{2}}\frac{(f'(z))^{\frac{q}{2}})}{(f(z))^{\frac{q}{2}})}\right)$, $\mathbb{E}\left(|z|^q\frac{|f'(z)|^p}{|f(z)|^q}\right)$ and $\mathbb{E}\left(z_1^{\frac{q}{2}}\frac{(f'(z_1))^{\frac{p}{2}}}{(f(z_1))^{\frac{q}{2}}}\overline{\left[z_2^{\frac{q}{2}}\frac{(f'(z_2))^{\frac{p}{2}}}{(f(z_2))^{\frac{q}{2}}}\right]}\right)$ for (p,q) belonging to a parabola $\mathcal{R}$ in the real $(p,q)-$plane. This result is then extent to a more general type of mixed moment

$$\mathbb{E}\left(\frac{(z_1-z_1')^{\frac{q}{2}}(f'(z_1))^{\frac{p}{2}}}{(f(z_1)-f(z_1'))^{\frac{q}{2}}}(f'(z_1'))^{\frac{p'}{2}}\overline{\left[(f'(z_2))^{\frac{p}{2}}\right]}\,\overline{\left[\frac{(z_2-z_2')^{\frac{q}{2}}(f'(z_2'))^{\frac{p'}{2}}}{(f(z_2)-f(z_2'))^{\frac{q}{2}}}\right]}\right),$$

namely, for a pair $(p,q)\in\mathcal{R}$, the Beliaev-Smirnov type equations and closed forms of the above expected moment are derivied.

We introduced the generalized integral means spectrum, $\beta(p,q;\kappa)$, corresponding to the singular behavior of the mixed moments above. This generalized integral means spectrum includes the standard integral means spectrum of interior whole-plane SLE and the standard integral means spectrum of exterior whole-plane SLE as specific cases when one respectively sets $q = 0$ and $q = 2p$. These standard spectrum were explicitly calculated in term of the spectrum $\beta_{tip}(p), \beta_0(p), \beta_1(p,q), \beta_{lin}(p)$ introduced in [1],[2],[5]. In this thesis, we proposed a manifold of the spectrum in the moment $(p,q)-$plane , in which the average generalized spectrum of whole-plane SLE take four possible above forms, separated by five phase transition lines. We checked conjecture for (p,q) on the above parabola $\mathcal{R}$ by the obtained closed form of $\mathbb{E}\left(|z|^q \frac{|f'(z)|^p}{|f(z)|^q}\right)$ and applied the method introduced in [13] to check the conjecture for an infinite family of parabolas in the $(p,q)-$plane.

We also calculated explicitly the expectation of the Schwarzian derivative of the interior whole-plane SLE map by considering the Beaev-Smirnov type equation obeyed by $\mathbb{E}\left((z-w)^2 \frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)$ and a relation between the function in the expectation and the Schwarzian derivative of $f(z)$. The same argument was applied to obtain an equation satisfied by the moduli second moment of the Schwarzian derivative. This equation allows us to recover a result obtained in [4], that is the value of the moduli second moment of the Schwarzian derivative of the interior whole-plane SLE map at the origin.

Finally, we made use of the martingale method to reduce an explicit formula of $\mathbb{E}(|\log f'(z)|^2)$ and then proved the relation

$$\lim_{p\to 0} \frac{2\bar{\beta}(p)}{p^2} = \lim_{r\to 1^-} \frac{1}{4\pi|\log(1-r)|} \int_0^{2\pi} \mathbb{E}(|\log f'(re^i\theta)|^2)d\theta,$$

where $f(z)$ is the interior whole-plane SLE_2 map at time 0 and β is the corresponding average integral means spectrum. This relation is an analogue of that considered in [9] concerning a question raised by McMullen in [14].

6.2 Perspectives

The investigation of moments or logarithmic coefficients of the interior whole-plane SLE map began with computer calculations of Taylor coefficients [4] or logarithmic coefficients [11] of small indexes. The exciting results obtained by these calculations have motivated the analytic studies. However, as was mentioned in Section 2.1.3, the quantity of computations increases quickly as the index increases. This fact requires that the algorithm of computing programmings should be improved, we then may expect interesting explicit values of coefficients of higher indexes which motivate more analytic studies.

In this thesis, the problem of studying moments of the interior whole-plane SLE maps is considered for real values of moments. The extension of results obtained here and the applications the method used for the real moments case to the complex moments case will be an interesting and considerable problem.

In Chapter 3, the conjecture on the average integral means spectrum was proposed for all pairs (p,q) in the plane $\mathbb{R}^2$. That conjecture was checked for subsets of the moment $(p, q)-$plane, namely, for the parabola $\mathcal{R}$ and for an infinite family of parabolas by rigorous and non-rigorous methods. In [3], the authors also rigorously proved that the conjecture is true for a "large" part of the moment plane. The work of making rigorous some obtained results and proving the conjecture for the uncovered parts of the moment plane should be continued. As the integral means spectrum and the generalized integral means spectrum may be defined with complex arguments (corresponding to complex order of moments), a considerable future work is to study these spectrum in a complex setting of arguments.

It is showed that the expected moduli second moment of the Schwarzian derivative of the interior whole-plane SLE map satisfies a PDE. This fact suggests an approach to the study of that Schwarzian derivative. To exploite the PDE encoding the the expected moduli second moment of the Schwarzian derivative of the interior whole-plane SLE map will give more informations about the Schwarzian derivative and the SLE and the SLE map itself. It is noticed that the above PDE was obtained from another PDE obeyed by a mixed moment and a relation between that mixed moment and the Schwarzian derivative. As we have a host of PDEs satisfied by mixed moments (Propositions 4.1.1, Corollary 4.1.1, Corollary 4.1.2), if a quantity is related to a mixed moments obeying one of these equations then one may expect results on that quantity by exploiting the PDE obeyed by the corresponding mixed moments, for instance, one may investigate the Grunsky coefficients by considering the PDE satisfied by $\mathbb{E}\left((z-w)^2\frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)$.

The equation

$$\lim_{p\to 0}\frac{2\bar{\beta}(p)}{p^2} = \lim_{r\to 1^-}\frac{1}{4\pi|\log(1-r)|}\int_0^{2\pi}\mathbb{E}(|\log f'(re^i\theta)|^2)d\theta,$$

is obtained for f being the interior whole-plane SLE_2 map. To arrive at this result, we have solved the equation (5.52) (therefore solved Eq (5.48)) for $\kappa = 2$. For a general positive κ, it may be impossible to have an explicit formula of the solution of (5.52) (or (5.48)) as in the case of $\kappa = 2$, however, an asymptotic behavior of the solution as $r \to 1$ is sufficient to prove or disprove the above relation. We believe that the relation is true for all $\kappa > 0$ and to prove that should be an interesting future work.

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Xuan Hieu HO

SUR LA MUTIFRACTALITÉ, LA DÉRIVÉE SCHWARZIENE ET LA VARIANCE ASYMPTOTIQUE DE WHOLE-PLANE SLE

Résumé :

Soit f une instance du whole-plane SLE_κ: on sait que pour certaines valeurs de κ, p les moments dérivés $\mathbb{E}(|f'(z)|^p)$ peuvent être écrits sous une forme fermée, étude qui a permis de mettre au jour une nouvelle phase du spectre des moyennes intégrales. Le but de cette thèse est une étude des moments généralisés $\frac{|f'(z)|^p}{|f(z)|^q}$: cette étude permet de confirmer la structure algébrique riche du whole-plane SLE. On montre que les formes fermées des moments mixtes $\mathbb{E}\big(\frac{|f'(z)|^p}{|f(z)|^q}\big)$ apparaissent sur une famille dénombrable de paraboles du plan (p, q), en étendant les équations de Beliaev-Smirnov à ce cas. Nous introduisons également le spectre généralisé $\beta(p, q; \kappa)$, correspondant au comportement asymptotiques des moyennes intégrales mixtes. Le spectre généralisé moyen du whole-plane SLE prend quatre formes possibles, séparés par cinq séparatrices dans $\mathbb{R}^2$. Nous proposons également une approche semblable pour la dérivée Schwarziene $S(f)(z)$ de l'application de SLE. Les calculs sur les équations de Beliaev-Smirnov d'une certaine générale forme de moment mène à une formulation explicite de $\mathbb{E}(S(f)(z))$. Nous étudions finalement la variance asymptotique de McMullen et démontrons une relation entre la croissance infinitésimale du spectre de la moyenne intégrale et la variance asymptotique pour SLE_2.

Mots clés : Whole-plane SLE, moment logarithmique, equation de Beliaev-Smirnov, généralisé spectre de la moyène intégrale, dérivée Schwarziene, variance asymptotique de McMullen.

ON MULTIFRACTALITY, SCHWARZIAN DERIVATIVE AND ASYMPTOTIC VARIANCE OF WHOLE-PLANE SLE

Abstract :

Let f an instance of the whole-plane SLE_κ conformal map from the unit disk $\mathbb{D}$ to the slit plane: We know that for certain values of κ, p the derivative moments $\mathbb{E}(|f'(z)|^p)$ can be written in a closed form, study that has updated a new phase of the integral means spectrum. The goal of this thesis is a study on generalized moments $\frac{|f'(z)|^p}{|f(z)|^q}$: This study permit confirm the rich algebraic structure of the whole-plane version of SLE. It will be showed that closed forms of the mixed moments $\mathbb{E}\big(\frac{|f'(z)|^p}{|f(z)|^q}\big)$ can be obtained on a countable family of parabolas in the moment plane (p, q), by extending the so-called Beliaev–Smirnov equation to this case. We also introduce the generalized integral means spectrum, $\beta(p, q; \kappa)$, corresponding to the singular behavior of the mixed moments. The average generalized spectrum of whole-plane SLE takes four possible forms, separated by five phase transition lines in $\mathbb{R}^2$. We also propose a similar approach for the Schwarzian derivative $S(f)(z)$ of SLE maps. Computations on the Beliaev–Smirnov equation of a certain general form of moment lead to an explicit formula of $\mathbb{E}(S(f)(z))$. We finally study the McMullen asymptotic variance and prove a relation between the infinitesimal growth of the integral mean spectrum and the asymptotic variance in an expectation sense for SLE_2.

Keywords : Whole-plane SLE, logarithmic moment, Beliaev-Smirnov equation, generalized integral means spectrum, Schwarzian derivative, McMullen's asymptotic variance.

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