Exceptional Field Theory and Supergravity

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Théorie des Champs Exceptionnels et Supergravité

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Abstract

In this thesis, recent developments in Double Field Theory (DFT) and Exceptional Field Theory (EFT) are presented. They are reformulation of supergravity in which duality symmetries are made manifest before dimensional reduction. This is achieved through the definition of an extended spacetime that 'geometrises' the T-duality group $O(d,d)$ in DFT and exceptional U-duality groups in EFT. All functions on this extended space are subject to a covariant 'section constraint', whose solutions then restrict the coordinates dependency of the fields. There exist different solutions to the section constraint that correspond to different theories. In this sense, different theories are unified within the formalism of extended field theories. Moreover, extended field theories possess a powerful tool to study compactifications: the generalised Scherk-Schwarz ansatz. Here, we present several examples of the effectiveness of the generalised Scherk-Schwarz ansatz. In particular, we proved two conjectures regarding consistent truncations: the so-called Pauli reduction of the bosonic string on group manifolds and type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$. Another application is presented on the embedding of generalised type IIB within the $\mathrm{E}_{6(6)}$ EFT, which recently appeared in the study of integrable systems. Finally, we present the supersymmetric completion of the bosonic $\mathrm{E}_{8(8)}$ EFT.

This thesis is based on the following publications [1, 2, 3, 4, 5]. With my supervisor Henning Samtleben, we collaborated with Olaf Hohm on [1, 2], Christopher Pope on [3] and Marc Magro on [5].

Résumé

Dans cette thèse, nous présentons des avancements récents en Théorie des Champs Doubles (TCD) et Théories des Champs Exceptionnels (TCE). Ces théories ont la particularité d'être des reformulations de supergravité dans lesquelles les symétries de dualité sont explicites avant toute réduction dimensionnelle. Ces reformulations se basent sur la définition d'un espace-temps étendu qui géométrise le groupe de T-dualité en TCD et les groupes exceptionnels de U-dualité en TCE. Tous les champs de cet espace sont soumis à une contrainte de section qui restreint leur dépendance en coordonnées. Il existe plusieurs solutions à la contrainte de section, qui correspondent donc à des théories différentes. Dans ce sens, le formalisme des théories des champs étendues amène à une unification de ces théories. De plus, grâce à un outil spécifique aux théories des champs étendues, l'ansatz de Scherk-Schwarz généralisé, il est possible de réécrire les ansatz compliqué de type Kaluza-Klein en supergravité sous une forme élégante et compacte: un produit matriciel en dimensions supérieures. Ici, nous présentons plusieurs exemples de l'efficacité de l'ansatz de Scherk-Schwarz généralisé. En particulier, nous prouvons deux conjectures concernant les troncations cohérentes: la réduction dite "de Pauli" de la corde bosonique ainsi que la supergravité de type IIB sur $\mathrm{AdS}_5 \times \mathrm{S}^5$. La dernière application de cet ansatz concerne la théorie de type IIB généralisée, apparue récemment dans l'étude des système intégrables, et son plongement dans la TCE $\mathrm{E}_{6(6)}$. Enfin, nous présentons la complétion supersymétrique de la TCE $\mathrm{E}_{8(8)}$ bosonique.

Outline

In this thesis, we focus on recent developments in extended field theories. We start by reviewing the basics of Double Field Theory and Exceptional Field Theory which are covariant formulations of supergravity where dualities are now manifest symmetries. We then introduce the main asset of extended field theories, known as generalised Scherk-Schwarz ansatz (GSS). In this framework, one can show that a complicated reduction ansatz in the usual framework of Kaluza-Klein supergravity takes the very efficient form of a matrix product in higher dimensions. In this case, the consistency of the truncation is guaranteed providing the 'enlarged' twist matrices satisfy a set of differential equations.

Chapter 2 is dedicated to an application of the GSS in Double Field Theory. Within an $O(d,d)$ covariant formulation of the NS-NS sector of supergravity, we proved the consistency of the reduction of the $n+d$ dimensional bosonic string to n dimensions on any d-dimensional group manifold G, with the isometry group $G\times G$ as gauge group. This is know in the literature as a Pauli reduction, whose consistency was first conjectured in [6]. The proof relies on the construction of an explicit $SO(d,d)$ twist matrix in terms of the Killing vectors of the bi-invariant metric on G that solves the consistency conditions. From the twist matrix, it is then easy to read the full non-linear reduction ansatz for all fields, whose consistency is guaranteed by construction.

In chapter 3, we focus on the $\mathrm{E}_{6(6)}$ Exceptional Field Theory. Here, we present the IIB decomposition of all the fields of the theory by enforcing the $\mathrm{GL}(5)\times\mathrm{SL}(2)$ solution of the section constraint. We then move to the type IIB side, where we perform a 5+5 split à la Kaluza-Klein (but keeping the dependency on the internal coordinate) and the necessary field redefinitions such that the two theories could be compared. By matching the degrees of freedom on each side, we obtain the dictionary between the type IIB and EFT fields (whose dependency in the internal coordinates has been constrained by the solution of the section constraint as previously mentioned).

Two applications of the EFT/Type IIB dictionary and the generalised Scherk Schwarz ansatz are presented in chapter 4. The first application concerns the consistent truncation of type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ to the maximal SO(6) gauged supergravity. After a general analysis of the consistency conditions, we use explicit twist matrices together with the dictionary of chapter 3 to find the full set of IIB reduction formulas. Again, the generalised Scherk-Schwarz origin of the reduction guarantees its consistency and the conjecture formulated by Günaydin, Romans and Warner in [7] is proven. The remaining sections of this chapter regards another application of EFT to type IIB supergravity. In section (4.4), we present the recently found generalised type IIB field equations, in which a one-form replaces the gradient of the dilaton and is subject to a Bianchi-like identity. We then solve the Bianchi identities of generalised type IIB, which are deformed with respect to the usual ones. Finally, we show how the deformations of the field strengths can be obtained from a surprisingly simple Scherk-Schwarz ansatz upon picking a new solution of the section constraint.

The fifth chapter of this thesis regards the supersymmetric extension of the $E_{8(8)}$ EFT. After a review of the bosonic EFT, we introduce the different blocks of the generalised spin connection needed for the couplings to fermions. We establish 'uplifted' supersymmetry rules and show its algebra closes. We then give the supersymmetric lagrangian, whose full invariance under supersymmetry is proven in appendix D.

Chapter 1

Introduction

Our current understanding of fundamental physics involves four principal interactions: strong, weak, electromagnetic and gravitational. The first three are described by the Standard Model (SM) with an incredible precision. The key theoretical concept underlying the success of the Standard Model is gauge symmetry, which is the idea that symmetry transformations act independently at each point of spacetime rather than globally. For example, in the case of the Standard Model, the total gauge group is $SU(3)^{\text{strong}} \times (SU(2) \times U(1))^{\text{electroweak}}$. By Noether's theorem, this gauge symmetry gives rise to conserved charges which transform as Lorentz scalar (gauge charges) or Lorentz vector (energy-momentum from translational symmetry). It is only fair to ask if a conserved charge could also transform as a spinor, and what kind of symmetry this would generate ? The answer is local Supersymmetry or Supergravity.

The history of supersymmetry starts in 1971 with Golfand and Likhtman. They introduced 4 anti-commuting spinor generators in four dimensions to extend the Poincaré symmetry [8]. At the time, it was revolutionary because, in 1967, Coleman and Mandula had shown in their famous No-go theorem that the most general symmetry of the S-matrix is the direct product of Poincaré and some internal group. Otherwise, one can show that the S-matrix becomes trivial, which means that there are no interactions [9]. However, there was a loophole in this theorem. The proof was based on an implicit axiom: the symmetry generators were assumed to be bosonic. Since for supersymmetry the additional generators (supercharges) are spinors, it bypasses the no-go theorem. The consequence of these supercharges is that every particle of spin s has a superpartner or sparticle with same mass and a spin of $s \pm \frac{1}{2}$.

Particle	Spin	Sparticle	Spin
quark: q	$\frac{1}{2}$	squark : $\tilde{q}$	0
lepton: l	$\frac{1}{2}$	slepton : $\tilde{l}$	0
photon: γ	1	photino : $\tilde{\gamma}$	$\frac{1}{2}$
W boson	1	Wino: $\tilde{W}$	$\frac{1}{2}$
Z boson	1	Zino: $\tilde{Z}$	$\frac{1}{2}$

Table 1.1: Standard Model particles and their supersymmetric partners.

So why haven't we seen the superpartners already ? Indeed, if a selectron, the bosonic version of the electron, has the same mass m_e, it should be fairly easy to see it in experiments ! But not all symmetries are exact: a gauge symmetry may be 'spontaneously broken', which would explain the difference in masses of the superpartners. This is what produces the unification of the weak (massive Z and W bosons) and electromagnetic interactions (massless photon) into a $SU(2)_L \times U(1)^Y$ electroweak interaction in the Standard Model. The beauty of this symmetry lies in the unification of two very unalike interactions, different in both range and strength, yet with a common origin. This means that if supersymmetry were to exist, it must be an approximate symmetry, and must be broken at low energies.

In 1974, Wess and Zumino wrote down the first supersymmetric field theories in 4 dimensions [10]. This is often seen as the actual starting point of the systematic study of supersymmetry. At this time, they were several motivations for studying supersymmetry in itself:

- Supersymmetry provides natural candidates for Cold Dark Matter.
- Supersymmetric theories have softer ultraviolet divergences. For example, $\mathcal{N} = 4$ super Yang-Mills, the maximal supersymmetric extension of the Yang-Mills theory is finite. Also, in $D = 4$, $\mathcal{N} = 8$ maximal supergravity, several expected divergences are known to cancel. This led to speculations about whether this theory is free of any UV divergences at all ! [11].

To summarise, supersymmetric quantum field theories are easier to work with, and provide meaningful toy models.

We have seen that the rôle of gauge symmetry was preponderant in the development of the Standard Model. What about a gauged version of supersymmetry ? Would this be more interesting and powerful ? This was accomplished shortly after (1976) by Freedman, Ferrara and Van Nieuwenhuizen and independently by Deser and Zumino for $\mathcal{N} = 1$ supergravity [12, 13]. The latter used the first order formalism for General Relativity where the vielbein and the connection are treated as independent fields and were able to show that the sum of an Einstein term and a massless minimally coupled spin $\frac{3}{2}$ described by a Rarita-Schwinger action was invariant under local supersymmetry transformations. Reciprocally, since the algebra of supersymmetry field theories closes into translations, its gauged counterpart closes into diffeomorphisms and so is an extension of general relativity. It is then natural to refer to these theories as supergravity.

1.1 String theory, dualities and supergravity

The Standard Model, whose might has never been stronger since the discovery of the Higgs boson in 2012, cannot be the complete picture. On the theoretical side, the SM fails to describe gravitational interactions and provides no candidates for Cold Dark Matter particles ($\simeq 80\%$ of the mass of the Universe). These issues can be solved with string theory, thus providing unification of all known interactions.

In string theory, fundamental particles such as photons are not point particles anymore but can be seen as vibrational modes of open or closed strings. The size of the strings is typically set by the Planck length ($\simeq 10^{-33}$ cm) so at all measurable scales they appear as point particles but are in fact extended object and prevent the theory from having any UV divergences.

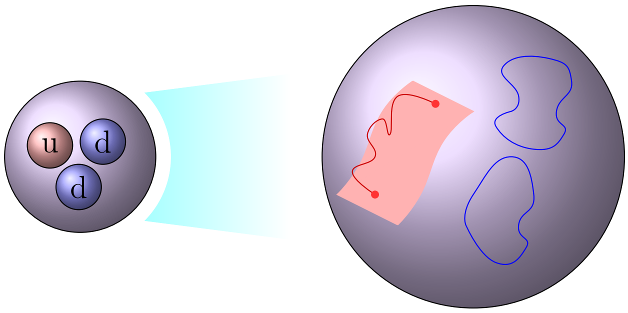

Figure 1.1: A neutron composed of a one up quark (red) and two down quark (blue) described by different string modes.

There are 5 different (and consistent) types of string theories related by dualities [14, 15, 16, 17] which once discovered, led to think that they were different limits of a same underlying theory called « M-theory » [18]. These dualities are called T and S-duality,

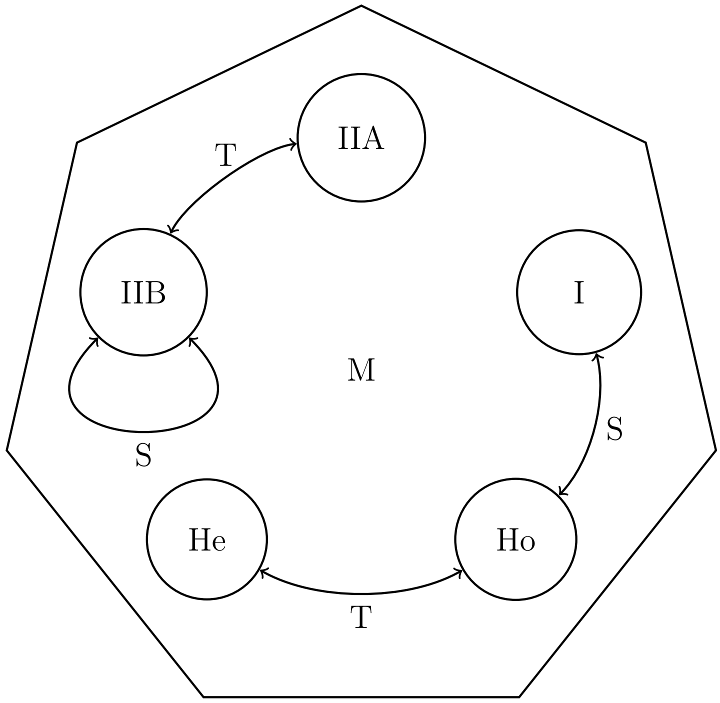

Figure 1.2: The different string theories are related by T and S dualities, and can be seen as different corners of the parameter space of M-theory. He, Ho respectively stands for $E_8 \times E_8$ and SO(32) heterotic string theories.

which mix into U-duality. In the simplest case, a T-duality is an equivalence between two theories with a compactified dimension, one with radius R, the other one with radius $\frac{\ell_s^2}{R}$, where ℓ_s is the string length scale. It relates type IIA and type IIB, as well as the two heterotic string theories. S-duality relates a theory with string coupling constant g_s with a theory with coupling constant $1/g_s$. It maps type IIB to itself (a particular case of the $SL(2,\mathbb{Z})$ symmetry of the theory) and type I to the SO(32) heterotic string theory. Since it relates a strongly interacting theory to a weakly interacting one, a regime where one can use perturbation theory, S-duality explains how three of the five original string theories behave at strong coupling. The remaining two exhibit an 11th dimension of size $g_s\ell_s$ at strong coupling, a circle in the case of type IIA and a line in the heterotic case. This the realm of M-theory, a true non-perturbative description of string theory that would unify the 5 superstring theories. While we have yet to find a complete formulation of M-theory, it is accurately described by the unique 11 dimensional supergravity at low energy [19].

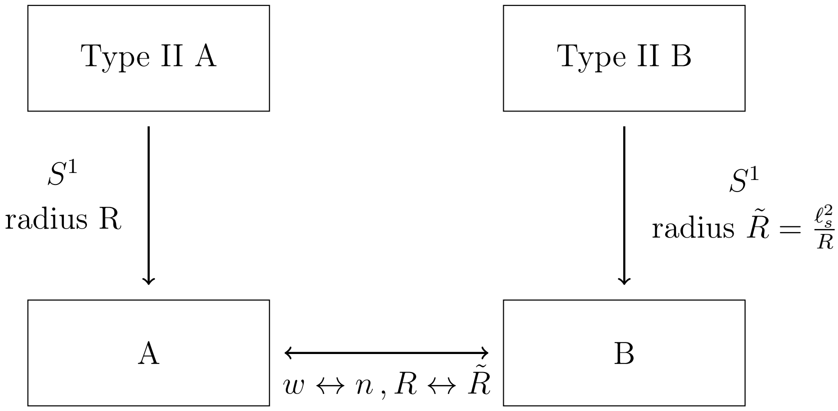

Figure 1.3: A simple example of T-duality. w and n are respectively the winding and momentum modes of the string in the compactified direction.

Since the discovery of 11 dimensional supergravity [20], several types of supergravity have been constructed in spacetime dimensions $D > 4$, and in particular two ten-dimensional supergravity theories called type IIA and type IIB. As their names hinted, they appear from Type IIA/B string theories as their low energy limit. We will mainly focus on the type IIB supergravity theory in this thesis. Recently, formulations that make the duality symmetries of string theory manifest in field theories have been developed:

- In the case of T-duality and its group $O(d, d)$, the duality covariant formulation is known as Double Field Theory (DFT) [21, 22, 23, 24]. It has already produced many interesting results [25, 26, 27] and has a fruitful interplay with generalised geometry [25, 28] in pure mathematics. Chapter 2 is dedicated to DFT and the proof of an old conjecture using the new tools offered by the extended formalism.
- Exceptional Field Theory (EFT) [29, 30, 31, 32, 33, 34, 35, 36] deals with the full U-duality groups and thus unifies different supergravity theories in one framework. In chapter 4, we will see two applications of the generalised Scherk-Schwarz ansatz in the context of the $\mathrm{E}_{6(6)}$ EFT.

One can also study the supersymmetric version of the extended field theories. It has been done in EFT in [37, 38] for the $\mathrm{E}_{6(6)}$, $\mathrm{E}_{7(7)}$ cases. In the last chapter of this thesis, we will be interested in the supersymmetric extension of the $E_{8(8)}$ EFT.

1.2 The bosonic string

Strings, as extended objects, sweep a two-dimensional worldsheet Σ in a n+d dimensional spacetime. This worldsheet is therefore parametrised by two parameters $\xi^\alpha = (\tau, \sigma)$, and in the case of the closed string $\sigma \sim \sigma + 2\pi$, is topologically a cylinder. The trajectory of the closed string in spacetime, a Riemannian manifold $(\mathcal{M}, G)$ of dimension D, may be

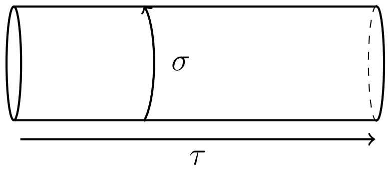

Figure 1.4: The worldsheet of a closed string

thought as the following map

$$X\colon \begin{cases} \Sigma \to \ \mathcal{M} \\ (\tau,\sigma) \mapsto \ X^\mu(\tau,\sigma) \end{cases}$$

In the simple case where the string propagates in a flat space-time, the simplest σ-model which can be built from this map is given by the Polyakov action

$$S_P \;=\; -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\ , \tag{1.2.1}$$

constructed in terms of the induced worldsheet metric $g_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ (the pull-back of the flat space-time metric to the worldsheet) and an independent dynamical worldsheet metric $\gamma_{\alpha\beta}$. The factor of α' in front is here for dimensional purposes, and has dimension -2 in $\bar{h} = c = 1$ units, i.e $\alpha' = \ell_s^2$ where ℓ_s is the string length scale. It is related to the tension of the string by

$$T \;=\; \frac{1}{2\pi\ell_s^2} = \frac{1}{2\pi\alpha'}\ . \tag{1.2.2}$$

Varying the action with respect to $\gamma_{\alpha\beta}$ and substituting gives

$$S \;=\; -T\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{\det\left(-g_{\alpha\beta}\right)}\ , \tag{1.2.3}$$

which is nothing else but the area sweeped by the string in the target space. In addition to the usual Poincaré invariance and reparametrisation invariance (worldsheet diffeomorphisms invariance), the action (1.2.1) exhibits an invariance under Weyl transformations

$$\gamma_{\alpha\beta} \;\to\; \tilde{\gamma}_{\alpha\beta} = \Omega(\tau,\sigma)\gamma_{\alpha\beta}\ . \tag{1.2.4}$$

One can define the 2d stress energy tensor

$$T_{\alpha\beta} \;=\; \frac{4\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}}\ , \tag{1.2.5}$$

and due to the Weyl symmetry, it is a traceless tensor, i.e. $T_{\alpha\beta}\gamma^{\alpha\beta} = 0$. The Weyl symmetry can be used to write the action in conformal gauge,

$$S \;=\; -\frac{T}{2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\partial^\alpha X\cdot\partial_\alpha X\ , \tag{1.2.6}$$

where we have introduced the scalar product $X \cdot X = X^\mu X^\nu \eta_{\mu\nu}$. Varying with respect to X gives a 2d wave equation,

$$\frac{\delta S}{\delta X_\mu} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = \partial_+\partial_- X^\mu = 0 \,, \tag{1.2.7}$$

where we have introduced the worldsheet light-cone coordinates

$$\xi^\pm = (\tau \pm \sigma) \,. \tag{1.2.8}$$

The most general solution is a superposition of waves moving to the left and to the right

$$X^\mu(\xi) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \,. \tag{1.2.9}$$

In the case of the closed string, imposing the closed string boundary conditions

$$X^\mu(\tau, 0) = X^\mu(\tau, 2\pi), \tag{1.2.10}$$

one gets the following mode decomposition [39]

$$X^\mu_L(\xi^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \xi^+ + i(\frac{\alpha'}{2})^{1/2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\xi^+} \,, \tag{1.2.11}$$

$$X^\mu_R(\xi^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \xi^- + i(\frac{\alpha'}{2})^{1/2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\xi^-} \,, \tag{1.2.12}$$

with x^μ and $p^\mu = \sqrt{2/\alpha'}\alpha^\mu_0$ the center of mass position and momentum. To keep the $X^\mu(\tau,\sigma)$ real, one also needs to impose $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$ and $\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*$. The general solutions (1.2.12) still have to satisfy the constraints coming from the tracelessness of the stress energy tensor (1.2.6). In light-cone gauge, the constraints are

$$(\partial_- X)^2 = (\partial_+ X)^2 = 0. \tag{1.2.13}$$

Expanding the constraints in terms of the Fourier modes

$$(\partial_- X)^2 = \alpha' \sum_n L_n e^{-in\xi^-} \,, \quad (\partial_+ X)^2 = \alpha' \sum_n \tilde{L}_n e^{-in\xi^+} \,, \tag{1.2.14}$$

one finds

$$\begin{aligned} L_n &= \sum_m \alpha_m \cdot \alpha_{n-m} \,, \\ \tilde{L}_n &= \sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} \,. \end{aligned} \tag{1.2.15}$$

In the classical theory, the vanishing of the energy momentum tensor thus translates to an infinite set of constraint

$$L_n = \tilde{L}_n = 0 \,, \quad \forall\, n \in \mathbb{Z} \,. \tag{1.2.16}$$

After canonical quantisation,

$$[\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\,\delta_{n+m}\,\eta^{\mu\nu}, \quad [x^\mu, p_\nu] = i\delta^\mu{}_\nu \ , \tag{1.2.17}$$

the α_m^μ and $\tilde{\alpha}_m^\mu$ commutation relations are those of the harmonic oscillator raising and lowering operator up to normalisation. As usual, one can now start building the Fock space of the theory by demanding the oscillator ground state of the string to be annihilated by the α_m,$\tilde{\alpha}_m$ for $m > 0$

$$\alpha_m^\mu |0;p\rangle = \tilde{\alpha}_m^\mu |0;p\rangle = 0 \, , \quad m > 0 \ , \tag{1.2.18}$$

for a string of center of mass momentum p^μ, the eigeinvalue of the momentum operator introduced in (1.2.17). Of course, this is not a positive definite Fock space as can be seen from the time components of the commutation relation. Only a subset of this space is physical. The states belonging to this subspace must obey a quantum analog of the constraints (1.2.16). Since the α and $\tilde{\alpha}$ are now operator, there is an ordering ambiguity in the Virasoro operators (1.2.15). They are now defined to be normal ordered

$$L_n \ \equiv \ \frac{1}{2}\sum_m : \alpha_m \cdot \alpha_{n-m} : \ = \ \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{+\infty} \alpha_m \cdot \alpha_{n-m} \ . \tag{1.2.19}$$

Due to the commutations relations (1.2.17), the normal ordering has only an importance for L_0 and $\tilde{L}_0$. Therefore, the quantum analog of the constraints (1.2.16) for a state $|\psi\rangle$ to be physical are

$$\begin{aligned} (L_0 - a)|\psi\rangle = 0 \ , \quad L_n|\psi\rangle = 0 \ , \ n > 0 \ , \\ (\tilde{L}_0 - a)|\psi\rangle = 0 \ , \quad \tilde{L}_n|\psi\rangle = 0 \ , \ n > 0 \end{aligned} \tag{1.2.20}$$

where a is an undetermined constant that takes care of the normal ordering ambiguity. Interestingly, the constraints involving L_0 and $\tilde{L}_0$ have a simple physical interpretation

$$0 \ = \ \frac{\alpha'}{4}p^2 + \sum_{n=1}^{+\infty} \alpha_n \cdot \alpha_{-n} - a \ , \quad 0 \ = \ \frac{\alpha'}{4}p^2 + \sum_{n=1}^{+\infty} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} - a \tag{1.2.21}$$

they tell us the effective mass of the string since in Minkovski space, one has $p_\mu p^\mu = -M^2$. In the following, we will work with the so called ‘critical’ string theory with space-time dimension $D = 26$ and zero-point energy of the oscillators $a = 1$. In this case, the spectrum is free of negative norm states, a necessary condition for a consistent causal and unitary theory. Adding and substracting the L_0 and $\tilde{L}_0$ constraint gives the ‘mass-shell ’ and the ‘level-matching’ conditions

$$M \ = \ \frac{2}{\alpha'}(N + \tilde{N} - 2) \ , \tag{1.2.22}$$

$$N \ = \ \tilde{N} \ , \tag{1.2.23}$$

with the left and right moving oscillation modes of the string $N = \sum_{n=1}^{+\infty} \alpha_n \cdot \alpha_{-n}$ and similarly for $\tilde{N}$. We can now look at the first excited state. In this case, $N = \tilde{N} = 1$ and thus $M = 0$ and this state can be written as

$$|\psi\rangle \;=\; H_{\mu\nu}\, \alpha_1^\mu \tilde{\alpha}_1^\nu \,|0;p\rangle \tag{1.2.24}$$

where we have acted on the oscillator vaccuum state of a single closed string of momemtum p^μ. As we are dealing with massless particles subject to the constraints (1.2.20), we have $p^\mu H_{\mu\nu} = 0$ such that H captures only transverse fluctuations of the string. Therefore, one decompose the polarisation H into representations of $SO(D)$ [40]

$$H_{\mu\nu} \;=\; h_{\mu\nu} \;+\; b_{\mu\nu} \;+\; \varphi\,, \tag{1.2.25}$$

with the first term being symmetric traceless, the second term antisymmetric and a final trace term. These string oscillations modes are identified to quanta of the following massless spacetime fields : the space-time metric $G_{\mu\nu}$, the B-field $B_{\mu\nu}$ and the dilaton ϕ. Up to now, we have seen that the quantisation of the closed bosonic string naturally gives the graviton and much more. Starting from the Polyakov action, it is useful to generalise it to an action describing string propagation in curved space-time, as well as taking into account the massless states of the closed string as part of the background.

It is well known that an electrically charged relativistic particle can be described by a one dimensional σ-model with the electromagnetic potential as background gauge field. In a similar way, the full bosonic string theory is described by a 2d σ-model with the Kalb-Ramond B-field as a background gauge field. Demanding the additional terms to be power-counting renormalizable and invariant under reparametrisation of the string worldsheet greatly restricts the kind of interaction that can be added to the action. The appropriate action is [41]

$$S = S_G + S_B + S_\phi \tag{1.2.26}$$

with

$$\begin{aligned}
S_G &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)\,,\\
S_B &= \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)\,,\\
S_\phi &= \frac{1}{4\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\gamma}\gamma^{\alpha\beta}R_{\alpha\beta}\,\phi(X)
\end{aligned} \tag{1.2.27}$$

The first term is the obvious generalisation of the Polyakov action to the curved spacetime case. The second term is the pullback of the spacetime 2-form B to the string worldsheet integrated over the string worldsheet ($\epsilon^{\alpha\beta}$ is a tensor density). The third term is the scalar curvature of the worldsheet multiplied by the dilaton, such that for the constant part of the dilaton (its VEV ϕ_0), it gives a contribution $\chi\,\phi_0$ where χ is the Euler characteristic of the worldsheet and the string coupling is given by $g_s = e^{\phi_0}$. While the first two terms are classically Weyl invariant, this is not the case for the last one (unless ϕ is constant). This may seem puzzling until one remembers that in general, the symmetries of the classical field theory are not always exact symmetries of the quantised field theory. Since α' play the role of the loop-expansion parameter in the action above and as the dilaton term is of higher order in α', one should compare only the classical effect of the last term to the quantum effect of the first two terms when checking for Weyl invariance. The breakdown of the Weyl invariance at the quantum level is governed by the trace anomaly of the stress-energy tensor [42]

$$\langle T^\alpha{}_\alpha\rangle = -\frac{1}{2\alpha'}\beta_{\mu\nu}(G)\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + \frac{1}{2\alpha'}\beta_{\mu\nu}(B)\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu + \frac{1}{2}\beta(\phi)R^{(2)} \ , \tag{1.2.28}$$

described in terms of the following β functionals

$$\beta_{\mu\nu}(G) \ = \ \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H^{\rho\sigma}{}_\mu H_{\nu\rho\sigma}\right) + \mathcal{O}(\alpha'^2) \tag{1.2.29}$$

$$\beta_{\mu\nu}(B) \ = \ \alpha'\left(-\frac{1}{2}\nabla^\rho H_{\mu\nu\rho} + \nabla^\rho\phi\, H_{\mu\nu\rho}\right) + \mathcal{O}(\alpha'^2) \tag{1.2.30}$$

$$\beta(\phi) \ = \ \alpha'\left(-\frac{1}{2}\nabla^\mu\nabla_\mu\phi + \nabla^\mu\phi\nabla_\mu\phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \mathcal{O}(\alpha'^2) \ . \tag{1.2.31}$$

To preserve Weyl invariance at the quantum level, one must impose $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\phi) = 0$. The vanishing of the beta functionals have a beautiful physical interpretation. They are the equation of motion of the so-called 'low energy effective action'

$$S \ = \ \frac{1}{2\kappa_0}\int d^{26}X\,\sqrt{|G|}\,e^{-2\phi}\Big(R + 4\,G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}\,H^{\mu\nu\rho}H_{\mu\nu\rho}\Big) \ , \tag{1.2.32}$$

where we have introduced the Ricci scalar for the spacetime metric $G_{\mu\nu}$ and the field strength of the B-field

$$H_{\mu\nu\rho} \ = \ 3\partial_{[\mu}B_{\nu\rho]} \ . \tag{1.2.33}$$

As we will show later, the low energy effective action above can be rewritten in a manifestly $O(d,d)$ invariant form within the formalism of Double Field Theory.

1.3 Compactification and T-duality

In section (1.1) of the introduction, we have seen the importance of dualities in string theory. In this section, we will develop further the discussion on T-duality. Since we seem to perceive only 4 dimensions and string theory predicts 11 dimensions, we need a mechanism to explain why we have yet to see the extra dimensions. Typically, one assumes they are curled up. We start with a simple example: the bosonic string with one dimension compactified on a circle of radius R (here the 25th). Spacetime is then $\mathbb{R}^{24,1} \times S^1$. This has the following effects: in the same way that a particle moving around this circle will have a quantised momentum in integer multiples of $1/R$, the momentum of the string in the 25^{th} direction is

$$p^{25} = \frac{n}{R}, \; n \in \mathbb{Z} \tag{1.3.1}$$

where the quantum number n is called the Kaluza-Klein momentum. However a string, as extended objects, can also wrap around the circle. The condition (1.2.10) may be relaxed to

$$X^\mu(\tau, 0) \;=\; X^\mu(\tau, 2\pi) + 2\pi R\, w, \tag{1.3.2}$$

Consequently, the mode decomposition is changed to

$$\begin{aligned} X_L^{25}(\xi^+) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p^{25}\xi^+ + \frac{1}{2}wR\xi^+ + \text{osc.} \equiv \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_L \xi^+ + \text{osc.}\,, &(1.3.3)\\ X_R^{25}(\xi^-) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p^{25}\xi^- - \frac{1}{2}wR\xi^- + \text{osc.} \equiv \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_R \xi^+ + \text{osc.}\,, &(1.3.4)\end{aligned}$$

where osc. stands for the same oscillator modes as in (1.2.12) and

$$p_L \;=\; \frac{n}{R} + \frac{wR}{\alpha'}\,, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}\,. \tag{1.3.5}$$

This change the mass-shell constraint (1.2.22) and level matching (1.2.23) to

$$\begin{aligned} M^2 &= \frac{n^2}{R^2} + (\frac{wR}{\alpha'})^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)\,, \\ N - \tilde{N} &= n\, w \end{aligned} \tag{1.3.6}$$

where we have defined $M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu$. We see two new terms in the mass-shell constraint. The first one is the contribution to the mass from the quantised momentum $(p^{25})^2 = n^2/R^2$. The second one tells us that since the string has tension $T = 1/(2\pi\alpha')$, it takes energy to stretch the string, accounted by the winding energy $E_w = 2\pi wRT = wR/\alpha'$. Now, under the exchange of the winding and momentum modes

$$w \leftrightarrow n \tag{1.3.7}$$

and the exchange of radii

$$R \leftrightarrow \tilde{R} = \alpha'/R\,, \tag{1.3.8}$$

the relations (1.3.6) are left invariant. This symmetry is called T-duality: it maps a theory compactified on a 'small' circle to a theory compactified on a 'large' circle, which is quite a departure from what we are used to with point particles. Under T-duality, we can see that p_L is left invariant while $p_R \to -p_R$. More generally, one can show this is the case for the left- and right-movers

$$X_L^{25} \to X_L^{25}\ , \qquad X_R^{25} \to -X_R^{25}\ , \tag{1.3.9}$$

meaning that the coordinate X^{25} is mapped to its dual

$$\tilde{X}^{25} = X_L^{25} - X_R^{25}\ , \tag{1.3.10}$$

with the dual coordinate satisfying

$$\partial_\alpha X^{25} = \epsilon_{\alpha\beta}\partial^\beta \tilde{X}^{25}\ . \tag{1.3.11}$$

This can be made explicit at the level of the σ-model assuming the background fields do not depend on the circular coordinate in standard Kaluza-Klein fashion (here X^{25} but we will call this direction $X^\bullet$ for generality) [43]

$$\begin{aligned} 4\pi\alpha' S \ &= \ \int \mathrm{d}^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}(-\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu} - 2V_\alpha\partial_\beta X^\mu G_{\mu\bullet} - G_{\bullet\bullet}V_\alpha V_\beta) \\ &\quad +\varepsilon^{\alpha\beta}(B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + 2B_{\bullet\mu}V_\alpha\partial_\beta X^\mu) \\ &\quad +\tilde{X}^\bullet\varepsilon^{\alpha\beta}\partial_\alpha V_\beta + \alpha'\sqrt{-\gamma}R^{(2)}\phi\ , \end{aligned} \tag{1.3.12}$$

where now $\mu = 0\ldots 24$ and we have introduced the Lagrange multiplier $\tilde{X}^\bullet$ whose variation gives

$$\varepsilon^{\alpha\beta}\partial_\alpha V_\beta = 0\ . \tag{1.3.13}$$

In the case where $V_\alpha = \partial_\alpha X^\bullet$, which solves the above equation, one recovers the usual action (1.2.26). What is more interesting is that the T-dual action can be found from the equations of motion of V_α and back substitution in the action (1.2.26)

$$\tilde{S} \ = \ S_{\tilde{G}} + S_{\tilde{B}} + S_{\tilde{\phi}} \tag{1.3.14}$$

with the dual fields given by the Buscher rules

$$\begin{aligned} \tilde{G}_{\bullet\bullet} \ &= \ \frac{1}{G_{\bullet\bullet}}\ , \quad \tilde{G}_{\bullet\mu} = \frac{B_{\bullet\mu}}{G_{\bullet\bullet}}\ , \quad \tilde{G}_{\mu\nu} = G_{\mu\nu} + \frac{B_{\bullet\mu}B_{\bullet\nu} - G_{\bullet\mu}G_{\bullet\nu}}{G_{\bullet\bullet}}\ , \\ \tilde{B}_{\bullet\mu} \ &= \ -\tilde{B}_{\mu\bullet} = \frac{G_{\bullet\mu}}{G_{\bullet\bullet}}\ , \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{G_{\bullet\mu}B_{\bullet\nu} - B_{\bullet\mu}G_{\bullet\nu}}{G_{\bullet\bullet}}\ , \end{aligned} \tag{1.3.15}$$

provided the dilaton shifts to $\tilde{\phi} = \phi - \frac{1}{2}\log(G_{\bullet\bullet})$ so to keep conformal invariance at one-loop. The Buscher rules can be written in terms of a factorised duality matrix

$$g \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbb{1} - e_\bullet & e_\bullet \\ e_\bullet & \mathbb{1} - e_\bullet \end{pmatrix}\ , \in \quad O(D,D) \tag{1.3.16}$$

($e_\bullet$ is a $D \times D$ matrix, non-zero only for its $\bullet\bullet$ component which is 1) acting on the background matrix $E_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + B_{\hat{\mu}\hat{\nu}}$, ($\hat{\mu} = \mu, \bullet$) by fractional linear transformation

$$\tilde{E} = (aE + b) \cdot (cE + d)^{-1} \tag{1.3.17}$$

with $\tilde{E}_{\hat{\mu}\hat{\nu}}$ being the dual background matrix. This makes appearent the link between the Buscher rules and the T-duality group. In the more general case where d dimensions are compactified on the torus T^d, the mass-shell constraint and level matching condition are most conveniently written in terms of a $2d$ vector $N^M = (w^m, n_m)$, $M = 1 \dots 2d, m = 1 \dots d$ transforming in the fundamental representation of $O(d, d, \mathbb{Z})$

$$\begin{aligned} M^2 &= N^M \mathcal{H}_{MN} N^N + \frac{2}{\alpha'}(N + \tilde{N} - 2) \,, \\ N - \tilde{N} &= N^M \eta_{MN} N^N \,, \end{aligned} \tag{1.3.18}$$

with $M^2 = -\sum_{\mu=0}^{25-d} p_\mu p^\mu$, the $O(d, d, \mathbb{R}) \equiv O(d, d)$ invariant matrix

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{1.3.19}$$

and the $2d \times 2d$ matrix

$$\mathcal{H}_{MN} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \in SO(d, d) \,. \tag{1.3.20}$$

The matrix $\mathcal{H}$ parametrizes the coset $\frac{O(d,d)}{O(d)\times O(d)}$ of dimension d^2,the moduli space of d-dimensional toroidal compactification of the bosonic string. As $\mathcal{H}$ and η are both element of $O(d, d)$,the constraints (1.3.6) are invariant under

$$\begin{pmatrix} n^m \\ w_m \end{pmatrix} \to \Lambda^{-1} \begin{pmatrix} n^m \\ w_m \end{pmatrix} \,, \quad \mathcal{H} \to \Lambda^T \mathcal{H} \Lambda \,, \tag{1.3.21}$$

where Λ is an element of $O(d, d, \mathbb{Z})$. This is known as the T-duality group.

1.4 Double Field Theory

In the previous section, we have seen that due to the extended nature of strings, the T-duality group $O(d, d, \mathbb{Z})$ emerges from a toroidal compactification of the bosonic string on T^d. The remnant of this symmetry at low energy is a global $O(d, d)$ symmetry (Maharana-Schwarz). Double Field Theory is a covariant reformulation of the NS-NS sector of supergravity which makes the T-duality group manifest before dimensional reduction. In its most common formulation, one introduces d dual coordinates

$$\mathbb{X}^M = \begin{pmatrix} x_\mu \\ \tilde{x}_\mu \end{pmatrix} \,. \tag{1.4.1}$$

$M = 1 \dots 2D,\ \mu = 1 \dots d$ so to treat the dual modes on an equal footing. Among the D coordinates, one could separate into n non-compact (external) coordinates and d compact coordinates as if one were to perform a dimensional reduction (on a torus for example). While the dual compact coordinates are related to usual winding modes of the closed bosonic string, their non-compact counterparts do not share this physical interpretation. This is taken care of in a $O(D,D)$ covariant manner by the section

$$\eta^{MN}\partial_M\partial_N \quad \equiv \quad 0\ , \quad \eta_{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ , \tag{1.4.2}$$

with η the $O(D,D)$ invariant metric which enforce that each fields or gauge parameters only depends on a 'physical' subset of the doubled coordinates. The constraint finds its origin in the level matching condition in closed string field theory, however it goes further: when one also enforces the constraint on products of fields and gauge parameters, one effectively reduces half of the coordinates. This is called the strong constraint and this is the one we will enforce in this thesis.

Let us now turn to the field content of the theory. As we have seen in the Buscher rules, T-duality mixes the metric with the B-field. Therefore, to have a manifestly duality invariant theory, one should combine them in a generalised metric $\mathcal{H}_{MN}$, a symmetric $2D \times 2D$ matrix

$$\mathcal{H}_{MN}(\mathbb{X}) = \begin{pmatrix} G_{\mu\nu} - B_{\mu\rho}G^{\rho\sigma}B_{\sigma\nu} & B_{\mu\rho}G^{\rho\nu} \\ -G^{\mu\rho}B_{\rho\nu} & G^{\mu\nu}\ . \end{pmatrix} \tag{1.4.3}$$

Together with the duality invariant dilaton $e^{-2d(\mathbb{X})} = e^{-2\phi}\sqrt{G}$, it captures the D^2+1 degrees of freedom of the massless state of the bosonic string. In this sense, DFT is reminiscent of the idea of Kaluza, where the 4-dimensional metric and the photon were unified in a higher dimensional theory. The resemblance with the original Kaluza-Klein idea does not stop here. The NS NS action

$$S \quad = \quad \frac{1}{2\kappa_0}\int d^{26}X\ \sqrt{|G|}\, e^{-2\phi}\Big(R + 4\, G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}\, H^{\mu\nu\rho}H_{\mu\nu\rho}\Big)\ , \tag{1.4.4}$$

is invariant under diffeomorphisms

$$\begin{aligned} \delta_\Lambda G_{\mu\nu} \quad &= \quad \mathcal{L}_\Lambda G_{\mu\nu} \\ \delta_\Lambda B_{\mu\nu} \quad &= \quad \mathcal{L}_\Lambda B_{\mu\nu} \\ \delta_\Lambda \phi \quad &= \quad \mathcal{L}_\Lambda \phi \end{aligned} \tag{1.4.5}$$

and gauge transformation of the B-field

$$\delta_{\tilde{\Lambda}} B_{\mu\nu} \quad = \quad 2\partial_{[\mu}\tilde{\Lambda}_{\nu]}\ . \tag{1.4.6}$$

In the formalism of DFT, both symmetries share a common geometric origin: they are encoded in the gauge transformations of the generalised metric and the $O(D,D)$ singlet dilaton with a generalised gauge parameter $\Lambda^M = (\Lambda^\mu, \tilde{\Lambda}_\mu)$

$$\begin{aligned}
\delta_\Lambda \mathcal{H}_{MN} &\equiv \mathbb{L}_\Lambda \mathcal{H}_{MN} = \Lambda^P \partial_P \mathcal{H}_{MN} + \mathbb{P}^P{}_M{}^K{}_L \partial_K \Lambda^L \mathcal{H}_{PN} + (M \leftrightarrow N) \ , && (1.4.7) \\
\delta_\Lambda e^{-2d} &\equiv \mathbb{L}_\xi e^{-2d} = \Lambda^M \partial_M e^{-2d} + e^{-2d} \partial_M \Lambda^M = \partial_M (\Lambda^M e^{-2d}) \ , && (1.4.8)
\end{aligned}$$

where we have introduced the generalised Lie derivative $\mathbb{L}$ whose action on a generalised vector of weight λ_V is given by

$$\begin{aligned}
\mathbb{L}_\Lambda V^M &= \Lambda^N \partial_N V^M - \mathbb{P}^M{}_P{}^K{}_L \partial_K \Lambda^L V^P + \lambda_V \partial_P \Lambda^P V^M \ , \\
\mathbb{P}^M{}_P{}^K{}_L &= \delta^M{}_L \delta^K{}_P - \eta^{MK} \eta_{PL} \ . && (1.4.9)
\end{aligned}$$

In the case where one enforces that no fields depend on the doubled coordinates $\tilde{x}_\mu$ (a solution of the section constraint (1.4.2)) one recovers the usual gauge transformations (1.4.5) and (1.4.6) of supergravity. In addition, the section constraint also ensures that the gauge algebra closes i.e

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}} \ , \qquad (1.4.10)$$

with the parameter Λ_{12}^M usually given by the Lie bracket in standard Riemannian geometry. Here, it is given by the C-bracket

$$\Lambda_{12}^M = [\Lambda_1, \Lambda_2]_C^M = 2\Lambda_{[1}^K \partial_K \Lambda_{2]}^M + \eta^{MK} \eta_{PQ} \Lambda_{[1}^P \partial_K \Lambda_{2]}^Q \ , \qquad (1.4.11)$$

which has non-vanishing Jacobiator. This unusual property has no physical consequence since one can show that the Jacobiator takes the following form

$$\begin{aligned}
J(\Lambda_1, \Lambda_2, \Lambda_3)^M &\equiv [[\Lambda_1, \Lambda_2]_C, \Lambda_3]_C^M + \text{ c.p.} \ , \\
&= \frac{1}{6} \eta^{MP} \partial_P N(\Lambda_1, \Lambda_2, \Lambda_3) \ , && (1.4.12)
\end{aligned}$$

where c.p. stands for cyclic permutations and $N(\Lambda_1, \Lambda_2, \Lambda_3)$ is a scalar known in the litterature as the Nijenhuis operator [44]. This is exactly of the form of a trivial parameter since for any generalised vector V, we have

$$\mathbb{L}_J V^M = 0 \ . \qquad (1.4.13)$$

Using this formulation, the NS-NS action above can be rewritten into a manifestly $O(D, D)$ covariant form

$$S = \int \mathrm{d}\mathbb{X}^{2D} e^{-2d} \mathcal{R}(\mathcal{H}, d) \ , \qquad (1.4.14)$$

close in spirit to the original GR action. The generalised Ricci scalar is given by [45]

$$\begin{aligned}
\mathcal{R} &= \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_L \mathcal{H}_{KN} - \partial_M \partial_N \mathcal{H}^{MN} \\
&\quad + 4\mathcal{H}^{MN} \partial_M \partial_N d - 4\mathcal{H}^{MN} \partial_M d \partial_N d + 4\partial_M \mathcal{H}^{MN} \partial_N d \ . && (1.4.15)
\end{aligned}$$

Alternatively, one could find the same result from a ground-up approach, by constructing the action from the invariance under generalised diffeomorphisms. In the next section, we will show it is possible to take into account not only the NS-NS sector of supergravity but the full bosonic sector of different supergravities, using the U-duality covariant formalism of exceptional field theories.

1.5 Exceptional Field Theory

Supergravity theories have a rather unique property: upon dimensional reduction, exceptional hidden symmetries emerge [46, 47]. When the unique 11D supergravity is compactified on a torus T^n, $n = 2 \ldots 8$, the hidden global symmetries of the lower dimensional theory span an $E_{n(n)} \equiv E_{n(n)}(\mathbb{R})$ Lie algebra. Naively, one would only expect $GL(n)$, a subgroup of $E_{n(n)}$, but not the whole $E_{n(n)}$ group. During the second superstring revolution, the groups $E_{n(n)}$ were interpreted as the low-energy remnant of the U-duality groups $E_{n(n)}(\mathbb{Z})$ of M-theory. However, until the construction of Exceptional Generalised Geometry (EGG) and exceptional field theories (EFT), a complete geometric interpretation of these groups was lacking.

The formulation of EFT is based on an extended spacetime that 'geometrises' the exceptional U-duality group. There are different EFT theories: although same in spirit, they differ in form as they are based on different exceptional groups. In the following, we choose to introduce the $\mathrm{E}_{6(6)}$ EFT as we will be using it for the main part of this thesis. In this EFT, all fields depend on $5 + 27$ coordinates (x^μ, Y^M), where $\mu, \nu = 0, \ldots, 4$, while lower and upper indices $M, N = 1, \ldots, 27$ label the (inequivalent) fundamental representations $\mathbf{27}$ and $\bar{\mathbf{27}}$ of $\mathrm{E}_{6(6)}$, respectively. All functions on this extended space are subject to a covariant 'section constraint' or 'strong constraint' that implies that locally the fields only live on a 'physical slice' of the extended space. In the present case this constraint can be written in terms of the invariant symmetric d-symbol d^{MNK} that $\mathrm{E}_{6(6)}$ admits as

$$d^{MNK} \partial_N \partial_K A \;=\; 0 \;, \qquad d^{MNK} \partial_N A \, \partial_K B \;=\; 0 \;, \tag{1.5.1}$$

for any arbitrary functions A, B on the extended space. In particular, this constraint holds for all fields and gauge parameters. It was shown in [33] that this constraint allows for two solutions. First, breaking $\mathrm{E}_{6(6)}$ to SL(6) the constraint is solved by fields depending on 6 internal coordinates, and we recover the spacetime of 11-dimensional supergravity. Second, breaking $\mathrm{E}_{6(6)}$ to SL(5) $\times$ SL(2) the constraint is solved by fields depending on 5 internal coordinates, and we recover the spacetime of type IIB supergravity. Indeed, upon picking one of these solutions one obtains a theory with the field content and symmetries of $D = 11$ or type IIB supergravity, respectively, but in a non-standard formulation. These formulations are obtained from the standard ones by splitting the coordinates and tensor fields a la Kaluza-Klein, however, without truncating the coordinate dependence, as pioneered by de Wit and Nicolai [48]. Nevertheless, in this way it is possible to describe both $D = 11$ and type IIB supergravity in one elegant framework.

As in DFT, one can obtain the action of EFT from the invariance under generalised diffeomorphisms (together with external diffeomorphisms in this case). The first step towards the action is therefore to define a gauge transformation with respect to an internal diffeomorphism parameter Λ^M for a vector V^M of weight λ_V

$$\delta V^M \;=\; \mathbb{L}_\Lambda V^M \;\equiv\; \Lambda^K \partial_K V^M - 6\,\mathbb{P}^M{}_N{}^K{}_L\,\partial_K \Lambda^L\, V^N + \lambda\,\partial_P \Lambda^P\, V^M \tag{1.5.2}$$

with the projector onto the adjoint representation in the tensor product $\mathbf{27} \otimes \bar{\mathbf{27}} = \mathbf{78} + \cdots$, which reads

$$\mathbb{P}^M{}_N{}^K{}_L \;\equiv\; (t_\alpha)_N{}^M (t^\alpha)_L{}^K \;=\; \frac{1}{18}\,\delta_N^M \delta_L^K + \frac{1}{6}\,\delta_N^K \delta_L^M - \frac{5}{3}\,d_{NLR} d^{MKR}\,. \tag{1.5.3}$$

We will refer to a tensor structure as transforming 'covariantly' iff its transformation is governed by the generalised Lie derivative (of some weight) and call such objects generalised tensors. Given the modified form of generalised Lie derivatives, as opposed to the conventional Lie derivatives, it is no longer clear that they are consistent. In particular, as in DFT, one should check that they satisfy an algebra, i.e., that they lead to gauge transformations that close.

An explicit computation then shows that the generalised Lie derivatives close according to

$$\big[\mathbb{L}_{\Lambda_1}, \mathbb{L}_{\Lambda_2}\big] \;=\; \mathbb{L}_{[\Lambda_1,\Lambda_2]_{\rm E}}\,, \tag{1.5.4}$$

where we have introduced the 'E-bracket'

$$\big[\Lambda_1, \Lambda_2\big]_{\rm E}^M \;\equiv\; 2\,\Lambda_{[1}^K \partial_K \Lambda_{2]}^M - 10\, d^{MNP}\, d_{KLP}\, \Lambda_{[1}^K \partial_N \Lambda_{2]}^L\,. \tag{1.5.5}$$

The first term of the bracket has the same form as the standard Lie bracket governing the algebra of standard diffeomorphisms. The second term explicitly involves the $\mathrm{E}_{6(6)}$ structure in form of the d-symbols, in a similar fashion to the $O(D,D)$ case. The E-bracket in EFT is the natural exceptional extension of the C-bracket introduced in the previous section. It also shares the non-vanishing Jacobiator property

$$J(\Lambda_1, \Lambda_2, \Lambda_3)^M = [[\Lambda_1, \Lambda_2]_E, \Lambda_3]_E^M + \text{c.p.} \neq 0 \tag{1.5.6}$$

but, as in DFT, the Jacobiator takes the form of a trivial parameter $J^M = d^{MNK} \partial_N \chi_K$. Therefore, the generalised Lie derivative with respect to this parameter vanishes and we have

$$[[\delta_{\Lambda_1}, \delta_{\Lambda_2}], \delta_{\Lambda_3}]_E + \text{c.p.}\,, \tag{1.5.7}$$

as expected. So far we have defined the generalised internal diffeomorphisms by generalised Lie derivatives. Since all fields are functions of internal and external coordinates Y^M and x^μ, respectively, we now need to set up a calculus that allows us to differentiate w.r.t. x^μ. Indeed, as for all fields and parameters, $\Lambda^M = \Lambda^M(x,Y)$ depends on the external x^μ and therefore the derivative ∂_μ of any tensor fields is not covariant in the above sense. In order to remedy this we introduce a gauge connection $\mathcal{A}_\mu{}^M$, of which we can think as taking values in the 'E-bracket algebra', and define the covariant derivatives

$$D_\mu \;\equiv\; \partial_\mu - \mathbb{L}_{\mathcal{A}_\mu} \; . \tag{1.5.8}$$

The covariant derivative of any generalised tensor then transforms covariantly provided the gauge vector transforms as $\delta_\Lambda \mathcal{A}_\mu{}^M = D_\mu \Lambda^M$, where we treat the gauge parameter Λ^M as a vector of weight $\lambda = \frac{1}{3}$. Next, we would like to define a field strength for $\mathcal{A}_\mu{}^M$. Naively, one would write the standard formula for the field strength or curvature of a gauge connection, but with the Lie bracket replaced by the E-bracket (1.5.5)

$$F_{\mu\nu}{}^M = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^M - [\mathcal{A}_\mu, \mathcal{A}_\nu]_E^M \; . \tag{1.5.9}$$

However, since the E-bracket does not satisfy the Jacobi identity the resulting object does not transform covariantly

$$\delta F_{\mu\nu}{}^M \;=\; 2D_{[\mu}\mathcal{A}_{\nu]}{}^M + 10 d^{MKR} d_{NLR} \partial_K (\mathcal{A}_\mu{}^N \delta \mathcal{A}_\nu{}^L) \; . \tag{1.5.10}$$

The failure of the covariance is due to the second term, which is of trivial form $d^{MNK}\partial_N \chi_K$. This suggest that we can repair it by introducing a two-form $\mathcal{B}_{\mu\nu\, M}$ with an appropriate gauge transformation and adding the term $d^{MNK}\,\partial_K \mathcal{B}_{\mu\nu\, N}$ to the field strength. This defines (the beginning of) the so-called tensor hierarchy, originally introduced in gauged supergravity [49, 50]. Using (1.5.5) we thus obtain the field strength

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^M \;=\;& 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^M - 2\,\mathcal{A}_{[\mu}{}^K \partial_K \mathcal{A}_{\nu]}{}^M + 10\, d^{MKR} d_{NLR}\, \mathcal{A}_{[\mu}{}^N\, \partial_K \mathcal{A}_{\nu]}{}^L \\ & + 10\, d^{MNK}\, \partial_K \mathcal{B}_{\mu\nu\, N} \; . \end{aligned} \tag{1.5.11}$$

Its general variation is given by

$$\delta \mathcal{F}_{\mu\nu}{}^M \;=\; 2D_{[\mu}\delta\mathcal{A}_{\mu]}{}^M + 10 d^{MNK}\partial_K \Delta \mathcal{B}_{\mu\nu\, N} \tag{1.5.12}$$

with

$$\Delta \mathcal{B}_{\mu\nu\, N} \;=\; \delta \mathcal{B}_{\mu\nu\, N} + d_{NKL}\mathcal{A}_{[\mu}{}^K \delta \mathcal{A}_{\nu]}{}^L \; . \tag{1.5.13}$$

One can show the field strength does transform covariantly i.e.

$$\delta \mathcal{F}_{\mu\nu}{}^M \;=\; \mathbb{L}_\Lambda \mathcal{F}_{\mu\nu}{}^M \; , \tag{1.5.14}$$

under the following gauge transformations of $\mathcal{A}$ and $\mathcal{B}$

$$\begin{aligned} \delta \mathcal{A}_\mu{}^M \;&=\; D_\mu \Lambda^M - 10 d^{MNK}\partial_K \Xi_{\mu N} \; , \\ \Delta \mathcal{B}_{\mu\nu\, N} \;&=\; 2D_{[\mu}\Xi_{\nu]M} + 10 d^{MNK}\Lambda^K \mathcal{F}_{\mu\nu}{}^L + \mathcal{O}_{\mu\nu M} \; , \end{aligned} \tag{1.5.15}$$

where we included the unspecified constrained term $\mathcal{O}_{\mu\nu\, M}$ that vanish under the projection

$$d^{MNK}\partial_K \mathcal{O}_{\mu\nu\, N} \;=\; 0\,. \tag{1.5.16}$$

Next, we would like to establish a Bianchi identity, but again, due to the failure of the E-bracket to satisfy the Jacobi identity, the naive identity does not hold, i.e., $\mathcal{DF} \neq 0$. This is also fixed by the presence of the 2-form in that the curl of the 2-form curvature gives the 3-form curvature of the 2-form. Specifically, we have the Bianchi identities

$$3\, D_{[\mu}\mathcal{F}_{\nu\rho]}{}^{M} \;=\; 10\, d^{MNK}\partial_K \mathcal{H}_{\mu\nu\rho\, N}\,. \tag{1.5.17}$$

The 3-form field strength $\mathcal{H}_{\mu\nu\rho\, M}$ is defined by this equations, up to terms that vanish under the projection with $d^{MNK}\partial_K$.

We close the introduction on EFT with the form of the action of the $\mathrm{E}_{6(6)}$ EFT. We have seen that the $\mathrm{E}_{6(6)}$ EFT has the following field content, with all fields depending on the $5+27$ coordinates (x^μ, Y^M),

$$g_{\mu\nu}\,, \quad \mathcal{M}_{MN}\,, \quad \mathcal{A}_\mu{}^M\,, \quad \mathcal{B}_{\mu\nu M}\,. \tag{1.5.18}$$

Here $g_{\mu\nu}$ is the external, five-dimensional metric, $\mathcal{M}_{MN}$ is the *generalised* internal metric, while the tensor fields $\mathcal{A}_\mu{}^M$ and $\mathcal{B}_{\mu\nu M}$ describe off-diagonal field components that encode, in particular, the interconnection between the external and internal generalised geometries. The dynamics of the bosonic fields (1.5.18) is governed by the following action

$$\begin{aligned} S \;=\; & \int d^5x\, d^{27}Y\, e\, \Big(\widehat{R} + \frac{1}{24}\, e\, g^{\mu\nu}\, D_\mu \mathcal{M}_{MN}\, D_\nu \mathcal{M}^{MN} \\ & -\frac{1}{4}\, e\, \mathcal{F}_{\mu\nu}{}^M \mathcal{F}^{\mu\nu\, N}\, \mathcal{M}_{MN} + e^{-1}\mathcal{L}_{\rm top} - V(\mathcal{M}, g)\Big)\,. \end{aligned} \tag{1.5.19}$$

The structure of the action is simple and can be divided into five different contribution. The first contribution resembles an Einstein-Hilbert term, the second and third terms are the gauge invariant kinetic terms for the scalars and the gauge connection, respectively. The fourth term is a Chern-Simons topological term. Finally, the last term is the scalar potential. We will come back to the $\mathrm{E}_{6(6)}$ action in a more detailed manner in Chapter 3.

1.6 Consistent truncation in EFT

It is a notoriously difficult problem to establish the consistency of Kaluza-Klein truncations. Consistency requires that any solution of the lower-dimensional theory can be lifted to a solution of the original higher-dimensional theory [51]. While this condition is trivially satisfied for torus compactifications, the compactification on curved manifolds is generically inconsistent except for very specific geometries and matter content of the theories. Even in the case of maximally symmetric spherical geometries, consistency only holds for a few very special cases [52] and even then the proof is often surprisingly laborious. An example for a Kaluza-Klein truncation for which a complete proof of consistency was out of reach until recently is that of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which was believed to have a consistent truncation to the maximal SO(6) gauged supergravity in five dimensions constructed in [53, 54, 7].

The manifestly covariant formulation of EFT described in the previous sections has proven a rather powerful tool in order to describe consistent truncations by means of a generalisation of the Scherk-Schwarz ansatz [55] to the exceptional space-time [56]. This relates to gauged supergravity theories in lower dimensions (in this case to $D = 5$ supergravities), formulated in the embedding tensor formalism. Via the explicit dictionary of EFT to $D = 11$ and type IIB supergravity, this ansatz then provides the full Kaluza-Klein embedding of various consistent truncations.

The generalised Scherk-Schwarz ansatz in EFT is governed by a group-valued twist matrix $U \in \mathrm{E}_{6(6)}$, depending on the internal coordinates, which rotates each fundamental group index. For instance, for the generalised metric the ansatz reads

$$\mathcal{M}_{MN}(x, Y) \;\; = \;\; U_M{}^{\underline{K}}(Y)\, U_N{}^{\underline{L}}(Y)\, M_{\underline{KL}}(x) \;, \tag{1.6.1}$$

where $M_{\underline{MN}}$ becomes the $\mathrm{E}_{6(6)}$-valued scalar matrix of five-dimensional gauged supergravity. This ansatz is invariant under a global $\mathrm{E}_{6(6)}$ symmetry acting on the underlined indices. Indeed, gauged supergravity in the embedding tensor formalism is covariant w.r.t. a global duality group ($\mathrm{E}_{6(6)}$ in the present case), although this is not a physical symmetry but rather relates different gauged supergravities to each other. In addition to the group valued twist matrix, consistency requires that we also introduce a scale factor ρ, depending only on the internal coordinates, for fields carrying a non-zero density weight λ, for which the ansatz contains $\rho^{-3\lambda}$. We thus write the general reduction ansatz for all bosonic fields of the $\mathrm{E}_{6(6)}$ EFT (1.5.18) as [56]

$$\begin{aligned}
\mathcal{M}_{MN}(x, Y) &= U_M{}^{\underline{K}}(Y)\, U_N{}^{\underline{L}}(Y)\, M_{\underline{KL}}(x) \;, \\
g_{\mu\nu}(x, Y) &= \rho^{-2}(Y)\, \mathbf{g}_{\mu\nu}(x) \;, \\
\mathcal{A}_\mu{}^M(x, Y) &= \rho^{-1}(Y) A_\mu{}^{\underline{N}}(x) (U^{-1})_{\underline{N}}{}^M(Y) \;, \\
\mathcal{B}_{\mu\nu\, M}(x, Y) &= \rho^{-2}(Y) U_M{}^{\underline{N}}(Y)\, B_{\mu\nu\, \underline{N}}(x) \;.
\end{aligned} \tag{1.6.2}$$

We will call the above ansatz consistent if the twist matrix U and the function ρ factor out of all covariant expressions in the action, the gauge transformations or the equations of motion. If this is established, it follows that the reduction is consistent in the strong Kaluza-Klein sense that any solution of the lower-dimensional theory can be uplifted to a solution of the full theory, with the uplift formulas being (1.6.2). Let us explain the required consistency conditions for the gauge transformations under internal generalised diffeomorphisms, for which the gauge parameter is subject to the same ansatz as the one-form gauge field,

$$\Lambda^M(x,Y) \;=\; \rho^{-1}(Y)(U^{-1})_{\underline{N}}{}^M(Y)\,\boldsymbol{\Lambda}^{\underline{N}}(x)\;. \tag{1.6.3}$$

We start with the field $g_{\mu\nu}$ that transforms as a scalar density of weight $\lambda = \frac{2}{3}$. Consistency of the ansatz (1.6.2) requires that under gauge transformations we have

$$\delta_\Lambda g_{\mu\nu}(x,Y) \;=\; \rho^{-2}(Y)\delta_\Lambda \mathbf{g}_{\mu\nu}(x)\;, \tag{1.6.4}$$

where the expression for $\delta_\Lambda \mathbf{g}_{\mu\nu}$ is Y-independent and can hence consistently be interpreted as the gauge transformation for the lower-dimensional metric. The variation on the left-hand side yields, upon insertion of (1.6.3),

$$\begin{aligned}
\delta_\Lambda g_{\mu\nu} &= \Lambda^N\partial_N g_{\mu\nu} + \tfrac{2}{3}\,\partial_N\Lambda^N g_{\mu\nu} \\
&= \rho^{-1}(U^{-1})_{\underline{K}}{}^N\boldsymbol{\Lambda}^{\underline{K}}\,\partial_N(\rho^{-2}\mathbf{g}_{\mu\nu}) + \tfrac{2}{3}\,\partial_N(\rho^{-1}(U^{-1})_{\underline{K}}{}^N)\boldsymbol{\Lambda}^{\underline{K}}\,\rho^{-2}\mathbf{g}_{\mu\nu} \\
&= \tfrac{2}{3}\,\rho^{-3}\Big[\partial_N(U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\,\rho^{-1}\partial_N\rho\Big]\boldsymbol{\Lambda}^{\underline{K}}\,\mathbf{g}_{\mu\nu}\;.
\end{aligned} \tag{1.6.5}$$

If we now demand that

$$\partial_N(U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\,\rho^{-1}\partial_N\rho \;=\; 3\,\rho\,\vartheta_{\underline{K}}\;, \tag{1.6.6}$$

where $\vartheta_{\underline{K}}$ is *constant*, then the ansatz (1.6.4) is established with

$$\delta_\Lambda\mathbf{g}_{\mu\nu} \;=\; 2\,\boldsymbol{\Lambda}^{\underline{M}}\,\vartheta_{\underline{M}}\,\mathbf{g}_{\mu\nu}\;. \tag{1.6.7}$$

This corresponds to a gauging of the so-called trombone symmetry that rescales the metric and the other tensor fields of the theory with specific weights. Here, $\vartheta_{\underline{K}}$ is the embedding tensor component for the trombone gauging, as introduced in [57]. An important consistency condition is that (1.6.6) is a covariant equation under internal generalised diffeomorphisms. Treating the (inverse) twist matrix as a vector of weight zero, its divergence $\partial_N(U^{-1})_{\underline{M}}{}^N$ (recalling that the underlined index is inert) is not a scalar. Indeed, a quick computation with (1.5.2) using the section constraint shows that it transforms as a scalar density of weight $\lambda = -\frac{1}{3}$, except for the following anomalous term in the transformation

$$\Delta^{\rm nc}_\Lambda(\partial_N(U^{-1})_{\underline{M}}{}^N) \;=\; -\tfrac{4}{3}\,\partial_N(\partial\cdot\Lambda)(U^{-1})_{\underline{M}}{}^N\;. \tag{1.6.8}$$

This contribution is precisely cancelled by the anomalous variation of the second term in (1.6.6), provided ρ is a scalar density of weight $\lambda(\rho) = -\frac{1}{3}$. Then both sides of (1.6.6) are scalar densities of weight $\lambda = -\frac{1}{3}$ and the equation is gauge covariant.

Let us now turn to the consistency conditions required for fields with a non-trivial tensor structure under internal generalised diffeomorphisms, as the generalised metric.

In parallel to the above discussion we require that the twist matrices consistently factor out, i.e.

$$\delta_\Lambda \mathcal{M}_{MN}(x,Y) \;=\; U_M{}^{\underline{K}}(Y) U_N{}^{\underline{L}}(Y) \delta_\Lambda M_{\underline{KL}}(x)\;. \tag{1.6.9}$$

Using the explicit form of the gauge transformations given by generalised Lie derivatives (1.5.2) one may verify by direct computation that this leads to consistent gauge transformations

$$\delta_\Lambda M_{\underline{MN}}(x) \;=\; 2\,\mathbf{\Lambda}^{\underline{L}}(x)\left(\Theta_{\underline{L}}{}^{\boldsymbol{\alpha}} + \tfrac{9}{2}\,\vartheta_{\underline{R}}\,(t^{\boldsymbol{\alpha}})_{\underline{L}}{}^{\underline{R}}\right)(t_{\boldsymbol{\alpha}})_{(\underline{M}}{}^{\underline{P}}\, M_{\underline{N})\underline{P}}(x)\;, \tag{1.6.10}$$

provided we assume the consistency conditions

$$\left[(U^{-1})_{\underline{M}}{}^{K}(U^{-1})_{\underline{N}}{}^{L}\partial_K U_L{}^{P}\right]_{\mathbf{351}} \;=\; \tfrac{1}{5}\,\rho\,\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_{\underline{N}}{}^{\underline{P}}\;, \tag{1.6.11}$$

where the *constant* $\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}}$ is the embedding tensor encoding conventional (i.e. non-trombone) gaugings, and the left-hand side is projected onto the **351** sub-representation. Specifically, writing the derivatives of U in terms of

$$\mathcal{X}_{\underline{MN}}{}^{\underline{K}} \;\equiv\; (U^{-1})_{\underline{M}}{}^{K}(U^{-1})_{\underline{N}}{}^{L}\partial_K U_L{}^{\underline{K}} \;\equiv\; \mathcal{X}_{\underline{M}}{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_{\underline{N}}{}^{\underline{K}}\;, \tag{1.6.12}$$

where we used that since U is group valued, $U^{-1}\partial U$ is Lie algebra valued (in the indices $\underline{N}$, $\underline{K}$), so that we can expand it in terms of generators as done in the second equality, the projector acts as,

$$\begin{aligned}\left[\,\mathcal{X}_{\underline{M}}{}^{\boldsymbol{\alpha}}\,\right]_{\mathbf{351}} \;&\equiv\; (\mathbb{P}_{\mathbf{351}})_{\underline{M}}{}^{\boldsymbol{\alpha}}{}^{\underline{N}}{}_{\boldsymbol{\beta}}\,\mathcal{X}_{\underline{N}}{}^{\boldsymbol{\beta}} \\ &=\; \tfrac{1}{5}\Big(\,\mathcal{X}_{\underline{M}}{}^{\boldsymbol{\alpha}} - 6\,(t^{\boldsymbol{\alpha}})_{\underline{P}}{}^{\underline{N}}\,(t_{\boldsymbol{\beta}})_{\underline{M}}{}^{\underline{P}}\,\mathcal{X}_{\underline{N}}{}^{\boldsymbol{\beta}} + \tfrac{3}{2}\,(t^{\boldsymbol{\alpha}})_{\underline{M}}{}^{\underline{P}}\,(t_{\boldsymbol{\beta}})_{\underline{P}}{}^{\underline{N}}\,\mathcal{X}_{\underline{N}}{}^{\boldsymbol{\beta}}\Big)\;.\end{aligned} \tag{1.6.13}$$

Let us emphasize that solving the consistency equations (1.6.6) and (1.6.11) for U and ρ in general is a rather non-trivial problem. It would be important to develop a general theory for doing this, which plausibly may require a better understanding of large generalised diffeomorphisms, as in [58, 59, 60, 61].

The consistency conditions (1.6.6) and (1.6.11) can equivalently be encoded in the structure of a 'generalised parallelization', see [25]. To this end, the twist matrix U and the scale factor ρ are combined into a vector of weight $\frac{1}{3}$,

$$(\widehat{U}^{-1})_{\underline{M}}{}^{N} \;\equiv\; \rho^{-1}\,(U^{-1})_{\underline{M}}{}^{N}\;. \tag{1.6.14}$$

Both consistency conditions (1.6.6) and (1.6.11) can then be encoded in the single manifestly covariant equation

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\widehat{U}^{-1}_{\underline{N}} \;\equiv\; -X_{\underline{MN}}{}^{\underline{K}}\,\widehat{U}^{-1}_{\underline{K}}\;, \tag{1.6.15}$$

with $X_{\underline{MN}}{}^{\underline{K}}$ constant and related to the $D=5$ embedding tensor as

$$X_{\underline{MN}}{}^{\underline{K}} \;=\; \left(\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}} + \tfrac{9}{2}\,\vartheta_{\underline{L}}(t^{\boldsymbol{\alpha}})_{\underline{M}}{}^{\underline{L}}\right)(t_{\boldsymbol{\alpha}})_{\underline{N}}{}^{\underline{K}} - \delta_{\underline{N}}{}^{\underline{K}}\,\vartheta_{\underline{M}}\;, \tag{1.6.16}$$

as we briefly verify in the following. In particular, equation (1.6.15) implies that

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\rho \;=\; -\vartheta_{\underline{M}}\,\rho\,. \tag{1.6.17}$$

It is straightforward to verify that subject to (1.6.15), the gauge transformations of all bosonic fields in (1.6.2) reduce to the correct gauge transformations in gauged supergravity. Let us illustrate this for a vector of generic weight λ, for which the Scherk-Schwarz ansatz reads

$$V^M(x,Y) \;=\; \rho^{-3\lambda}(U^{-1})_{\underline{N}}{}^M(Y)\,V^{\underline{N}}(x) \;=\; \rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M(Y)\,V^{\underline{N}}(x)\,. \tag{1.6.18}$$

Using (1.6.15) and (1.6.17), its gauge transformation then takes the form

$$\begin{aligned}
\delta_\Lambda V^M \;&=\; \mathbb{L}_{\mathbf{\Lambda}^{\underline{K}}\widehat{U}^{-1}_{\underline{K}}}\big(\rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M\big)\,V^{\underline{N}} \\
&=\; \mathbf{\Lambda}^{\underline{K}}\Big((-3\lambda+1)\big(\mathbb{L}_{\widehat{U}^{-1}_{\underline{K}}}\,\rho\big)\rho^{-3\lambda}\,(\widehat{U}^{-1})_{\underline{N}}{}^M+\rho^{-3\lambda+1}\mathbb{L}_{\widehat{U}^{-1}_{\underline{K}}}(\widehat{U}^{-1})_{\underline{N}}{}^M\Big)\,V^{\underline{N}} \\
&=\; \rho^{-3\lambda+1}(\widehat{U}^{-1})_{\underline{N}}{}^M\,\Big((3\lambda-1)\mathbf{\Lambda}^{\underline{K}}\,\vartheta_{\underline{K}}\,V^{\underline{N}}-\mathbf{\Lambda}^{\underline{K}}\,X_{\underline{KL}}{}^{\underline{N}}\,V^{\underline{L}}\Big)\,,
\end{aligned} \tag{1.6.19}$$

from which we read off, inserting (1.6.16),

$$\delta_{\mathbf{\Lambda}}V^{\underline{N}} \;=\; -\mathbf{\Lambda}^{\underline{K}}\,\big(\Theta_{\underline{K}}{}^{\boldsymbol{\alpha}}+\tfrac{9}{2}\,\vartheta_{\underline{P}}(t^{\boldsymbol{\alpha}})_{\underline{K}}{}^{\underline{P}}\big)(t_{\boldsymbol{\alpha}})_{\underline{L}}{}^{\underline{N}}\,V^{\underline{L}}+3\,\lambda\,\mathbf{\Lambda}^{\underline{K}}\,\vartheta_{\underline{K}}\,V^{\underline{N}}\,. \tag{1.6.20}$$

This is the expected transformation in gauged supergravity with general trombone gauging and in particular is compatible with (1.6.10) and (1.6.7) for $\lambda=0$ and $\lambda=\frac{2}{3}$, respectively. As the covariant derivatives and field strengths are defined in terms of generalised Lie derivatives (or its antisymmetrisation, the E-bracket), it follows immediately that also these objects reduce 'covariantly' under Scherk-Schwarz, e.g.,

$$\begin{aligned}
\mathcal{D}_\mu g_{\nu\rho}(x,Y) \;&=\; \rho^{-2}\,\big(\partial_\mu-A_\mu{}^N\vartheta_N\big)\,\mathbf{g}_{\nu\rho}\,, \qquad\qquad (1.6.21)\\
\mathcal{D}_\mu\mathcal{M}_{MN}(x,Y) \;&=\; U_M{}^{\underline{P}}U_N{}^{\underline{Q}}\Big(\partial_\mu M_{\underline{PQ}}-2A_\mu{}^{\underline{L}}\,\big(\Theta_{\underline{L}}{}^{\boldsymbol{\alpha}}\;+\tfrac{9}{2}\,\vartheta_{\underline{R}}\,(t^{\boldsymbol{\alpha}})_{\underline{L}}{}^{\underline{R}}\big)\,(t_{\boldsymbol{\alpha}})_{(\underline{M}}{}^{\underline{P}}M_{\underline{N})\underline{P}}\Big)\,.
\end{aligned}$$

In addition, the covariant two-form field strength reduces consistently,

$$\mathcal{F}_{\mu\nu}{}^M(x,Y) \;=\; \rho^{-1}\,(U^{-1})_{\underline{N}}{}^M\,F_{\mu\nu}{}^{\underline{N}}(x)\,, \tag{1.6.22}$$

with the $D=5$ covariant field strength $F_{\mu\nu}{}^N$ given by

$$F_{\mu\nu}{}^{\underline{M}} \;\equiv\; 2\partial_{[\mu}A_{\nu]}{}^{\underline{M}}+X_{\underline{KL}}{}^{\underline{M}}\,A_{[\mu}{}^{\underline{K}}A_{\nu]}{}^{\underline{L}}-2d^{\underline{MKL}}X_{\underline{KL}}{}^{\underline{N}}B_{\mu\nu\,\underline{N}}\,, \tag{1.6.23}$$

and similarly for the three-form curvature. To summarize, the reduction ansatz (1.6.2) describes a consistent truncation of $E_{6(6)}$ EFT to a $D=5$ maximal gauged supergravity, provided the twist matrices satisfy the consistency conditions (1.6.6) and (1.6.11). It is intriguing, that the match with lower-dimensional gauged supergravity, does in fact not explicitly use the section constraint (provided the initial scalar potential is written in an appropriate form) [62, 27, 56]. Formally this allows to reproduce all $D=5$

maximal gauged supergravities, and it is intriguing to speculate about their possible higher-dimensional embedding upon a possible relaxation of the section constraints that would define a genuine extension of the original supergravity theories. For the moment it is probably fair to say that our understanding of a consistent extension of the framework is still limited. If on the other hand the twist matrices U do obey the section constraint (1.5.1), the reduction ansatz (1.6.2) translates into a consistent truncation of the original $D = 11$ or type IIB supergravity, respectively, depending on to which solution of the section constraint the twist matrices U belong. With the explicit dictionary between EFT and the original supergravities, given in this thesis for type IIB and in [33] for $D = 11$ supergravity, the simple factorization ansatz (1.6.2) then translates into a highly non-linear ansatz for the consistent embedding of the lower-dimensional theory. This requires the precise interplay between various identities whose validity appears somewhat miraculous from the point of view of conventional geometry but which find a natural interpretation within the extended geometry of exceptional field theory.

Chapter 2

Consistent Pauli Reduction of the bosonic string

In the last section of the previous chapter, we have introduced the generalised Scherk-Schwarz ansatz within EFT. This is not a proprietary feature of EFT, and the same type of ansatz holds in Double Field Theory. There has already been several interesting results of generalised Scherk-Schwarz reduction in DFT, mainly regarding non-geometric flux compactifications and gauged supergravity [63, 26]. In this chapter, we will show another example of the usefulness of the generalised Scherk-Schwarz ansatz.

It was observed in [6] that in a reduction of the $(n+d)$-dimensional bosonic string on a group manifold G of dimension d, the potentially dangerous trilinear coupling of a massive spin-2 mode to bilinears built from the Yang-Mills gauge bosons of $G_L \times G_R$ was in fact absent. On that basis, it was conjectured in [6] that there exists a consistent Pauli reduction of the $(n+d)$-dimensional bosonic string on a group manifold G of dimension d, yielding a theory in n dimensions containing the metric, the Yang-Mills gauge bosons of $G_L \times G_R$, and d^2+1 scalar fields which parameterise $\mathbb{R} \times SO(d,d)/(SO(d) \times SO(d))$. Further support for the conjectured consistency was provided in [64], where it was observed that the $K = SO(d) \times SO(d)$ maximal compact subgroup of the enhanced $O(d,d)$ global symmetry of the T^d reduction of the bosonic string is large enough to contain the $G_L \times G_R$ gauge group as a subgroup.

We shall present a complete and constructive proof of the consistency of the Pauli reduction of the bosonic string on the group manifold G. Our construction makes use of the recent developments realising non-toroidal compactifications of supergravity via generalised Scherk-Schwarz-type reductions [55] on an extended spacetime within duality covariant reformulations of the higher-dimensional supergravity theories [65, 63, 62, 27, 66, 67, 56, 68]. In this language, consistency of a truncation ansatz translates into a set of differential equations to be satisfied by the group-valued Scherk-Schwarz twist matrix U encoding all dependence on the internal coordinates. We then explicitly construct the $SO(d,d)$ valued twist matrix describing the Pauli reduction of the bosonic string on a group manifold G in terms of the Killing vectors of the group manifold. We show that it satisfies the relevant consistency equations thereby establishing consistency of the truncation. From the Scherk-Schwarz reduction formulas we then read off the explicit truncation ansätze for all fields of the bosonic string. We find agreement with the linearised ansatz proposed in [6] and we confirm the non-linear reduction ansatz conjectured in [64] for the metric.

Our solution for the twist matrix straightforwardly generalises to the case when G is a non-compact group. In this case, the construction describes the consistent reduction of the bosonic string on an the internal manifold M_d whose isometry group is given by the maximally compact subgroup $K_L \times K_R \subset G_L \times G_R$. The truncation retains not only the gauge bosons of the isometry group, but the gauge group of the lower-dimensional theory enhances to the full non-compact $G_L \times G_R$. At the scalar origin, the gauge group is broken down to its compact part. This is a standard scenario in supergravity with non-compact gauge groups: for the known sphere reductions the analogous generalisations describe the compactification on hyperboloids $H^{p,q}$ and lower-dimensional theories with $SO(p,q)$ gauge groups [69, 70, 56, 71]. We will come back to this in the next chapter.

2.1 $O(d,d)$ covariant formulation of the $(n+d)$-dimensional bosonic string

Our starting point is the $(n+d)$-dimensional bosonic string (or NS-NS sector of the superstring)

$$S \;=\; \int dX^{n+d}\,\sqrt{|\hat{G}|}\,e^{-2\phi}\Big(R+4\,\hat{G}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi-\frac{1}{12}\,H^{\hat{\mu}\hat{\nu}\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}}\Big)\,, \tag{2.1.1}$$

with dilaton ϕ and three-form field strength $H_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv 3\,\partial_{[\hat{\mu}}C_{\hat{\nu}\hat{\rho}]}$. As described in the introduction of this chapter, the conjecture of [6] states this theory admits a consistent Pauli reduction to n dimensions on a d-dimensional group manifold G retaining the full set of $G_L \times G_R$ non-abelian gauge fields, according to the isometry group of the bi-invariant metric on G. In the following, for the explicit reduction formulas we will use the metric in the Einstein frame

$$G_{\hat{\mu}\hat{\nu}} \;\equiv\; e^{-4\beta\phi}\,\hat{G}_{\hat{\mu}\hat{\nu}}\,, \tag{2.1.2}$$

with $\beta = 1/(n+d-2)$, and split coordinates according to

$$\{X^{\hat{\mu}}\} \to \{x^{\mu}, y^{m}\}\,, \qquad \mu = 0,\ldots,n-1\,, \quad m = 1,\ldots,d\,. \tag{2.1.3}$$

The key tool in the following construction is double field theory (DFT) [21, 24, 72, 45], introduced in the previous chapter. Most suited for our purpose, is the reformulation of (2.1.1) in which an $O(d,d)$ subgroup of the full duality group is made manifest [73]. This is obtained by Kaluza-Klein decomposing all fields according to n external and d internal dimensions (keeping the dependence on all $(n+d)$ coordinates) and rearranging the various components into $O(d,d)$ objects. Formally, this theory lives on an extended space of dimension $(n+2d)$ with coordinates $\{x^\mu, Y^M\}$, with all fields subject to the section constraint $\partial^M \otimes \partial_M \equiv 0$ which effectively removes the d non-physical coordinates. In this sense, this version of DFT, invariant under both external and generalised diffeomorphisms, is close in spirit to EFT. Fundamental $SO(d,d)$ indices M, N are raised and lowered with the $SO(d,d)$ invariant metric η_{MN}. Regarding the field content, in addition to $\mathcal{H}_{MN}$ and the dilaton Φ already found in the standard form of DFT, here a symmetric $SO(d,d)$ group matrix and a scalar of weight $\frac{1}{2}$ under generalised diffeomorphisms, one has the external space-time metric $\mathrm{g}_{\mu\nu}$, a Kaluza-Klein vector $\mathcal{A}_\mu{}^M$ and a two-form potential $\mathcal{B}_{\mu\nu}$. As in EFT, the vector field acts as a gauge field for the generalised diffeomorphism

$$\delta_\Lambda \mathcal{A}_\mu{}^M \;=\; \left(\partial_\mu - \mathbb{L}_{A_\mu}\right)\Lambda \equiv D_\mu \Lambda \;, \tag{2.1.4}$$

where we have introduced the covariant derivative D_μ w.r.t. to generalised diffeomorphisms on the extended space. The naive field strength of the vector field

$$F_{\mu\nu}{}^M = 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^M - [\mathcal{A}_\mu, \mathcal{A}_\nu]_C^M \;, \tag{2.1.5}$$

here given in terms of the C-bracket (1.4.11), is not covariant. Again, this is very similar to the EFT case and one need to introduce a two-form $\mathcal{B}_{\mu\nu}$ with the appropriate gauge transformation such that the field strength

$$\mathcal{F}_{\mu\nu}{}^M \;=\; 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^M - [\mathcal{A}_\mu, \mathcal{A}_\nu]_C^M - \partial^M \mathcal{B}_{\mu\nu} \;, \tag{2.1.6}$$

is now covariant under

$$\begin{aligned} \delta\mathcal{A}_\mu{}^M &= D_\mu \Lambda^M + \partial^M \Lambda_\mu \;, \\ \Delta\mathcal{B}_{\mu\nu} &\equiv \delta\mathcal{B}_{\mu\nu} - \mathcal{A}_{[\mu}{}^N \delta\mathcal{A}_{\nu]N} = 2D_{[\mu}\Lambda_{\nu]} - \Lambda^N \mathcal{F}_{\mu\nu\,N} \;. \end{aligned} \tag{2.1.7}$$

The two form potential $\mathcal{B}_{\mu\nu}$ also comes with its covariant field strength

$$\mathcal{H}_{\mu\nu\rho} \;=\; 3\,D_{[\mu}\mathcal{B}_{\nu\rho]} + 3\,\mathcal{A}_{[\mu}{}^N \partial_\nu \mathcal{A}_{\rho]N} - \mathcal{A}_{[\mu N}\left[\mathcal{A}_\nu, \mathcal{A}_{\rho]}\right]_C^N \;, \tag{2.1.8}$$

which can be read from the modified Bianchi identity

$$3D_{[\mu}\mathcal{F}_{\nu\rho]}{}^M + \partial^M \mathcal{H}_{\mu\nu\rho} \;=\; 0 \;. \tag{2.1.9}$$

With this formalism, one can show that the action (2.1.1) can be rewritten in the form

$$\begin{aligned} S \;=\; \int dx^n dY^{2d}\,\sqrt{|\mathrm{g}|}\,e^{-2\Phi}\Big(&\widehat{\mathcal{R}} + 4\,\mathrm{g}^{\mu\nu} D_\mu\Phi D_\nu\Phi - \frac{1}{12}\mathcal{H}^{\mu\nu\rho}\mathcal{H}_{\mu\nu\rho} + \frac{1}{8}\,\mathrm{g}^{\mu\nu} D_\mu \mathcal{H}^{MN} D_\nu \mathcal{H}_{MN} \\ &- \frac{1}{4}\,\mathcal{H}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N + \frac{1}{4}\,\mathcal{H}^{MN}\partial_M \mathrm{g}^{\mu\nu}\,\partial_N \mathrm{g}_{\mu\nu} + \mathcal{R}(\Phi,\mathcal{H})\Big) \;. \end{aligned} \tag{2.1.10}$$

with the covariant derivatives explicitly given by

$$
\begin{aligned}
D_\mu \Phi &= \partial_\mu \Phi - \mathcal{A}_\mu{}^M \partial_M \Phi + \frac{1}{2}\,\partial_M \mathcal{A}_\mu{}^M \,, \\
D_\mu \mathcal{H}_{MN} &= \partial_\mu \mathcal{H}_{MN} - \mathcal{A}_\mu{}^K \partial_K \mathcal{H}_{MN} - 2\,\partial_{(M} \mathcal{A}_\mu{}^K \mathcal{H}_{N)K} + 2\,\partial^K \mathcal{A}_{\mu\,(M} \mathcal{H}_{N)K} \,,
\end{aligned}
\tag{2.1.11}
$$

the improved Ricci scalar

$$
\widehat{\mathcal{R}} = \widehat{\mathcal{R}} + e^{a\mu} e^{b\nu} \mathcal{F}_{\mu\nu}{}^M e_a{}^\rho \partial_M e_{\rho b} \,, \tag{2.1.12}
$$

and $\mathcal{R}(\Phi, \mathcal{H})$ the scalar DFT curvature (1.4.15).

The section constraint $\partial^M \otimes \partial_M \equiv 0$ is solved by splitting the internal coordinates according to

$$
\{Y^M\} \to \{y^m, y_m\} \,, \tag{2.1.13}
$$

in a light-cone basis where

$$
\eta_{MN} \;\equiv\; \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix} \,, \tag{2.1.14}
$$

and restricting the dependence of all fields to the physical coordinates y^m by imposing $\partial^m \equiv 0$, thereby reducing the extended space-time in (2.1.10) back to $(n+d)$ dimensions. Upon breaking the DFT field content accordingly, and rearranging of fields, the $O(d,d)$ covariant form (2.1.10) then reproduces the bosonic string (2.1.1). The precise dictionary can be straightforwardly worked out by matching the gauge and diffeomorphism transformations of the various fields on the bosonic string side (after the split (2.1.3)) to the generalised gauge transformations (2.1.7) on the DFT side (with $\partial^m = 0$). In the following, we will only give the the result of the dictionary. In the next chapter, we will review in detail how to build the dictionary in the richer $\mathrm{E}_{6(6)}$ case. For the DFT p-forms and metric this yields

$$
\begin{aligned}
&\mathcal{A}_\mu{}^m = A_\mu{}^m \equiv G^{mn} G_{\mu n} \,, \quad \mathcal{A}_{\mu\, m} = -\left(C_{\mu\, m} - A_\mu{}^n C_{nm}\right) \,, \\
&\mathcal{B}_{\mu\nu} = C_{\mu\nu} + 2 A_{[\mu}{}^m C_{\nu]\, m} + A_{[\mu}{}^m A_{\nu]}{}^n C_{mn} + A_{[\mu}{}^m A_{\nu]m} \,, \\
&\mathrm{g}_{\mu\nu} = e^{4\beta\phi} \left(G_{\mu\nu} - A_\mu{}^m A_\nu{}^n G_{mn}\right) .
\end{aligned}
\tag{2.1.15}
$$

The dictionary for the DFT scalar fields is most conveniently obtained by comparing the transformation of the DFT vector fields under generalised external diffeomorphisms

$$
\delta_\xi \mathcal{A}_\mu{}^M \;=\; \xi^\nu \mathcal{F}_{\mu\nu}{}^M + e^{4\gamma\Phi} \mathcal{H}^{MN} g_{\mu\nu} \partial_N \xi^\nu \,, \tag{2.1.16}
$$

to the transformations in the original theory (2.1.1) and yields

$$
\begin{aligned}
\mathcal{H}^{mn} &= e^{-4\beta\phi}\, G^{mn} \,, \qquad \mathcal{H}_m{}^n = e^{-4\beta\phi}\, G^{nk} C_{km} \,, \\
\mathcal{H}_{mn} &= e^{-4\beta\phi}\, G^{kl} C_{km} C_{ln} + e^{4\beta\phi}\, G_{mn} \,, \\
e^\Phi &= e^{\frac{\beta}{\gamma}\phi} \left(\det\, G_{mn}\right)^{-1/4} \,,
\end{aligned}
\tag{2.1.17}
$$

with $\gamma = \frac{1}{n-2}$. With the dictionary (2.1.15), (2.1.17), and imposing $\partial^m \equiv 0$, the $O(d,d)$ covariant action (2.1.10) reduces to the original action (2.1.1) of the bosonic string. The reduction ansatz on the other hand will be most compactly formulated in terms of the $O(d,d)$ objects.

2.2 Generalised Scherk-Schwarz ansatz and consistency equations

An important property of the $O(d,d)$ covariant form of the action (2.1.10) is the fact that particular solutions and truncations of the theory take a much simpler form in terms of the $O(d,d)$ objects $\mathcal{A}_\mu{}^M$, $\mathcal{H}_{MN}$, etc., as opposed to the original fields of the bosonic string (2.1.1). In particular, consistent truncations to n dimensions can be described by a generalised Scherk-Schwarz ansatz in which the dependence on the compactified coordinates Y^M is carried by an $SO(d,d)$ twist matrix $U_M{}^A$ and scalar functions u and ρ (which respectively take care of the weight of the fields under the $\mathbb{R}^+$ scaling and generalised diffeomorphisms), according to [65, 63][1]

$$
\begin{aligned}
\mathcal{H}_{MN} &= U_M{}^A(y) M_{AB}(x) U_N{}^B(y)\,, \qquad e^{\Phi} \;=\; u^{(n-2)/2}(y)\, e^{\varphi(x)}\,,\\
\mathcal{A}_\mu{}^M &= u(y)\rho^{-1}(y)(U^{-1})_A{}^M(y)\, A_\mu{}^A(x)\,, \qquad \mathcal{B}_{\mu\nu} \;=\; u^2(y)\rho^{-2}(y) B_{\mu\nu}(x)\,,\\
\mathrm{g}^{SF}_{\mu\nu} &\equiv e^{4\gamma\Phi}\mathrm{g}^{EF}_{\mu\nu} = u^2(y)\rho^{-2}(y) e^{4\gamma\varphi(x)}\, g_{\mu\nu}(x)\,.
\end{aligned}
\tag{2.2.1}
$$

Here, $A_\mu{}^M$ $B_{\mu\nu}$ and $g_{\mu\nu}$ are the gauge vectors, two-form and space-time metric of the reduced theory. The symmetric $SO(d,d)$ group valued matrix $M_{AB}(x)$ can be thought of as parametrizing the coset space $SO(d,d)/(SO(d)\times SO(d))$, and together with $e^{\varphi}(x)$ carries the d^2+1 scalar fields of the reduced theory. In the following, we will choose u=ρ such that no shift symmetries on the vector field remain after applying the generalised Scherk-Schwarz ansatz $\delta A_\mu{}^M = \partial^M \Lambda_\mu(x,y) = \partial^M(u^2(y)\rho^{-2}(y))\Lambda_\mu(x)$. In this case, the ansatz (2.2.1) describes a consistent truncation of (2.1.10), provided $U_M{}^A$ and ρ satisfy the consistency equations

$$
\eta_{D[A}\,(U^{-1})_B{}^M (U^{-1})_{C]}{}^N \partial_M U_N{}^D \;=\; f_{ABC} \;=\; \text{const.}\,, \tag{2.2.2}
$$

$$
\rho^{-1}\,\partial_M\rho \;=\; -\gamma\,(U^{-1})_A{}^N \partial_N U_M{}^A\,, \tag{2.2.3}
$$

with the $SO(d,d)$ invariant constant matrix η_{AB} and $\gamma = \frac{1}{n-2}$. If $U_M{}^A$ and ρ in addition depend only on the physical coordinates on the extended space (2.1.13)

$$
\partial^m U_M{}^A \;=\; 0 \;=\; \partial^m\rho\,, \tag{2.2.4}
$$

[1] Since with (2.1.10) we use DFT in its split form with internal and external coordinates, the reduction ansatz (2.2.1) resembles the corresponding ansatz in exceptional field theory [56] for the p-forms and metric.

the ansatz (2.2.1) likewise describes a consistent truncation of the original theory (2.1.1). As a consequence of this section condition, the Jacobi identity is automatically satisfied for f_{ABC} upon using its explicit expression (2.2.2)

$$[X_A, X_B] = -X_{AB}{}^C X_C \tag{2.2.5}$$

where we have introduced the generalised structure constant $X_{AB}{}^C = f_{[ABD]}\eta^{DC}$. Then, for a given solution of (2.2.2), (2.2.3), the explicit reduction formulas for the original fields are obtained by combining (2.2.1) with the dictionary (2.1.15), (2.1.17), as we will work out shortly.

In order to explicitly solve the generalised Scherk-Schwarz consistency conditions (2.2.2)–(2.2.4), let us first note that with the index split (2.1.13), and the parametrization

$$U_M{}^A \;=\; \eta^{AB}\,\{\mathcal{Z}_{B\,m}, \mathcal{K}_B{}^m\}\;, \qquad (U^{-1})_A{}^M \;=\; \{\mathcal{K}_A{}^m, \mathcal{Z}_{A\,m}\}\;, \tag{2.2.6}$$

of the $SO(d,d)$ matrix, equation (2.2.2) turns into

$$\begin{aligned} \mathcal{L}_{\mathcal{K}_A}\mathcal{K}_B{}^m \;&=\; -X_{AB}{}^C\,\mathcal{K}_C{}^m\;, \\ \mathcal{L}_{\mathcal{K}_A}\mathcal{Z}_{B\,m} + \mathcal{K}_B{}^n\,(\partial_m\mathcal{Z}_{A\,n} - \partial_n\mathcal{Z}_{A\,m}) \;&=\; -X_{AB}{}^C\,\mathcal{Z}_{C\,m}\;. \end{aligned} \tag{2.2.7}$$

The $SO(d,d)$ property of $U_M{}^A$ translates into

$$2\,\mathcal{K}_{(A}{}^m\mathcal{Z}_{B)\,m} \;=\; \eta_{AB} \;\equiv\; \begin{pmatrix} 0 & \delta_a{}^b \\ \delta^a{}_b & 0 \end{pmatrix}\,. \tag{2.2.8}$$

In the following, we will construct an explicit solution of (2.2.7), (2.2.8) in terms of the Killing vectors of the bi-invariant metric on a $d-$dimensional group manifold G. For compact G, the resulting reduction describes the Pauli reduction of the bosonic string on G, for non-compact G, this describes a consistent truncation on an internal space M_d with isometry group given by two copies of the maximally compact subgroup $K \subset G$. Specifically, we choose the $\mathcal{K}_A$ as linear combinations of the $G_L \times G_R$ Killing vectors $\{L_a^m, R_a^m\}$, in the following way

$$\mathcal{K}_A{}^m \;\equiv\; \{L_a{}^m + R_a{}^m, L^{a\,m} - R^{a\,m}\}\;, \tag{2.2.9}$$

with their algebra of Lie derivatives given by

$$\mathcal{L}_{L_a}L_b = -f_{ab}{}^c\,L_c\;, \qquad \mathcal{L}_{L_a}R_b = 0\;, \qquad \mathcal{L}_{R_a}R_b = f_{ab}{}^c\,R_c\;, \tag{2.2.10}$$

in terms of the structure constants $f_{ab}{}^c$ of $\mathfrak{g} \equiv \mathrm{Lie}\,G$, and with indices $a, b, \ldots$, raised and lowered by the associated Cartan-Killing form $\kappa_{ab} \equiv f_{ac}{}^d f_{bd}{}^c$. Moreover, the bi-invariant metric on the group manifold can be expressed by

$$\tilde{G}^{mn} \;\equiv\; -4\,L_a{}^m L^{a\,n} \;=\; -\,4\,R_a{}^m R^{a\,n}\;. \tag{2.2.11}$$

With (2.2.10), the ansatz (2.2.9) solves the first equation of (2.2.7), with structure constants $X_{AB}{}^{C}$ given by

$$X_{abc} \;=\; f_{abc}\;, \quad X_{a}{}^{bc} \;=\; f_{a}{}^{bc}\;, \quad X^{a}{}_{b}{}^{c} \;=\; f^{a}{}_{b}{}^{c}\;, \quad X^{ab}{}_{c} \;=\; f^{ab}{}_{c}\;, \tag{2.2.12}$$

and all other entries vanishing. Indeed, these structure constants are of the required form $X_{AB}{}^{C} = f_{[ABD]}\eta^{DC}$, c.f. (2.2.5). We may define the $G_L \times G_R$ invariant Cartan-Killing form of the algebra (2.2.5)

$$\kappa_{AB} \;\equiv\; \frac{1}{2} X_{AC}{}^{D} X_{BD}{}^{C} \;=\; \begin{pmatrix} \kappa_{ab} & 0 \\ 0 & \kappa^{ab} \end{pmatrix}\;, \tag{2.2.13}$$

such that the Killing vectors (2.2.9) satisfy

$$\kappa^{AB}\,\mathcal{K}_{A}{}^{m}\mathcal{K}_{B}{}^{n} \;=\; -\,\tilde{G}^{mn}\;, \qquad \eta^{AB}\,\mathcal{K}_{A}{}^{m}\mathcal{K}_{B}{}^{n} \;=\; 0\;, \tag{2.2.14}$$

and moreover $\kappa^{AB}\eta_{AB} = 0$.

In order to solve the second equation of (2.2.7), with the same structure constants (2.2.12), we start from the ansatz[2]

$$\mathcal{Z}_{A\,m} \;=\; -\,\kappa_{A}{}^{B}\,\mathcal{K}_{B\,m} + \mathcal{K}_{A}{}^{n}\,\tilde{C}_{nm}\;. \tag{2.2.15}$$

Here, the space-time index in the first term has been lowered with the group metric $\tilde{G}_{mn}$ from (4.2.12), and $\tilde{C}_{mn} = \tilde{C}_{[mn]}$ represents an antisymmetric 2-form, such that the $SO(d,d)$ property (2.2.8) is identically satisfied. With this ansatz for $\mathcal{Z}_{A\,m}$, the second equation of (2.2.7) turns into

$$\kappa_{A}{}^{C}\mathcal{K}_{B}{}^{n}\left(\partial_n\mathcal{K}_{C\,m} - \partial_m\mathcal{K}_{C\,n}\right) - 3\,\mathcal{K}_{A}{}^{k}\,\mathcal{K}_{B}{}^{n}\,\partial_{[k}\tilde{C}_{mn]} \;=\; 2\,\eta^{DE}\,X_{A(E}{}^{C}\,\kappa_{B)C}\,\mathcal{K}_{D\,m}\,. \tag{2.2.16}$$

The right-hand side of (2.2.16) vanishes by invariance of the Cartan-Killing form κ_{AB}. From (2.2.14), one derives the following identity

$$\partial_{[m}\mathcal{K}_{A\,n]} = X_{AC}{}^{B}\kappa^{CD}\mathcal{K}_{B\,m}\mathcal{K}_{D\,n}\,, \tag{2.2.17}$$

for the derivative of the Killing vectors. Inserting this relation in (2.2.16) gives

$$3\,\mathcal{K}_{A}{}^{k}\partial_{[k}\tilde{C}_{mn]} = 2\,X_{A}{}^{BC}\mathcal{K}_{B\,m}\mathcal{K}_{C\,n}\,, \tag{2.2.18}$$

where we have used $\kappa_{A}{}^{E}X_{ED}{}^{C}\kappa^{DB} = X_{A}{}^{BC}$. We note that both sides of this equation vanish under projection with $\eta^{DA}\mathcal{K}_{A\,p}$ as a consequence of (2.2.14). Projecting instead with $\kappa^{DA}\mathcal{K}_{A\,p}$, equation (2.2.16) reduces to an equation for $\tilde{C}_{mn}$

$$3\,\partial_{[k}\tilde{C}_{mn]} \;=\; \tilde{H}_{kmn} \;\equiv\; -\,2X^{ABD}\kappa_{D}{}^{C}\mathcal{K}_{A\,k}\mathcal{K}_{B\,m}\mathcal{K}_{C\,n}\;. \tag{2.2.19}$$

[2] Let us stress that our notation is such that adjoint G indices $a, b, \ldots$ are raised and lowered with the Cartan-Killing form κ_{ab}, whereas fundamental $SO(d,d)$ indices $A, B, \ldots$ are raised and lowered with the $SO(d,d)$ invariant metric η_{AB} from (2.2.8) and *not* with the G-dependent Cartan-Killing form κ_{AB} from (2.2.13).

Explicitly, the flux $\tilde{H}_{kmn}$ takes the form

$$\tilde{H}_{kmn} \;=\; -16\, f^{abc}\, L_{a\,k} L_{b\,m} L_{c\,n} \;=\; -\,16\, f^{abc}\, R_{a\,k} R_{b\,m} R_{c\,n}, \qquad (2.2.20)$$

and can be integrated since $\partial_{[k}\tilde{H}_{lmn]} = 0$, due to the Jacobi identity on f_{abc}. We have thus solved the second equation of (2.2.7).

With (2.2.9), (2.2.15), the remaining consistency equation (2.2.3) reduces to

$$\begin{aligned}(n-2)\,\mathcal{K}_A{}^m \partial_m \log\rho \;&=\; \partial_m \mathcal{K}_A{}^m \;=\; -\,\tilde{\Gamma}_{mn}{}^m \mathcal{K}_A{}^n\,, \\ \Longrightarrow \qquad \rho \;&=\; (\det \tilde{G}_{mn})^{-\gamma/2}\,. \end{aligned} \qquad (2.2.21)$$

We have thus determined the $SO(d,d)$ matrix $U_M{}^A$ and the scalar function ρ solving the system (2.2.2), (2.2.3) in terms of the Killing vectors on a group manifold G, and a two-form determined by (2.2.19). The resulting structure constants are given by (2.2.12) such that the gauge group of the reduced theory is given by $G_L \times G_R$.

2.3 Reduction ansatz and reduced theory

We now have all the ingredients to read off the full non-linear reduction ansatz of the bosonic string (2.1.1). Combining the DFT reduction formulas (2.2.1) with the dictionary (2.1.15), (2.1.17), and the explicit expressions (2.2.9), (2.2.15) for the Scherk-Schwarz twist matrix, we obtain

$$\begin{aligned} ds^2 \;=\;& \Delta^{-2\gamma}(x,y)\, g_{\mu\nu}(x)\, dx^\mu dx^\nu \\ &+ G_{mn}(x,y)\left(dy^m + \mathcal{K}_A{}^m(y) A_\mu^A(x) dx^\mu\right)\left(dy^n + \mathcal{K}_B{}^n(y) A_\nu^B(x) dx^\nu\right)\,, \end{aligned} \qquad (2.3.1)$$

for the metric in the Einstein frame, with $G_{mn}(x,y)$ given by the inverse of

$$G^{mn}(x,y) \;=\; \Delta^{2\gamma}(x,y)\, \mathcal{K}_A{}^m(y) \mathcal{K}_B{}^n(y) e^{4\gamma\varphi(x)} M^{AB}(x)\,. \qquad (2.3.2)$$

The dilaton and the original two-forms are given by

$$\begin{aligned} e^{4\beta\phi} &= \Delta^{2\gamma}(x,y)\, e^{4\gamma\varphi(x)} \\ C_{mn} &= \widetilde{C}_{mn}(y) + \Delta^{2\gamma}(x,y)\, \kappa_A{}^D \mathcal{K}_{D\,m} \mathcal{K}_B{}^p G_{pn}(x,y)\, e^{4\gamma\varphi(x)} M^{AB}(x)\,, \\ C_{\mu\,m} &= \left(\kappa_A{}^D \mathcal{K}_{D\,m} + \Delta^{2\gamma}(x,y)\, \kappa_C{}^E \mathcal{K}_A{}^n \mathcal{K}_{E\,n} \mathcal{K}_D{}^p G_{pm}(x,y)\, e^{4\gamma\varphi(x)} M^{CD}(x)\right) A_\mu{}^A(x)\,, \\ C_{\mu\nu} &= B_{\mu\nu}(x) - \kappa_B{}^C \mathcal{K}_A{}^m\, \mathcal{K}_{C\,m}\, A_{[\mu}{}^A(x) A_{\nu]}{}^B(x) \\ &\quad - \Delta^{2\gamma}(x,y)\, \kappa_C{}^E\, \mathcal{K}_B{}^n \mathcal{K}_{E\,n}\, \mathcal{K}_D{}^p\, \mathcal{K}_A{}^m\, G_{pm}(x,y)\, e^{4\gamma\varphi(x)} M^{CD}(x)\, A_{[\mu}{}^A(x) A_{\nu]}{}^B(x)\,. \end{aligned} \qquad (2.3.3)$$

where we have introduced $\Delta^2 = \det(\tilde{G}_{mn}(y))^{-1} \det(G_{mn}(x,y))$. In these expressions, all space-time indices on the Killing vectors $\mathcal{K}_A{}^m$ are raised and lowered with the metric

$\tilde{G}_{mn}(y)$ from (4.2.12), rather than with the full metric $G_{mn}(x,y)$. For the group manifold $G = SU(2)$, the construction describes the S^3 reduction of the bosonic string, for which the full reduction ansatz has been found in [52]. For general compact groups, the reduction ansatz for the internal metric (2.3.2) was correctly conjectured in [64].[3]

In order to compare our formulas to the linearised result given in [6], we first note that for compact G, we may normalise the Cartan-Killing form as $\kappa_{AB} = -\delta_{AB}$, such that the background (at $M_{AB}(x) = \delta_{AB}$) is given by

$$\mathring{G}_{mn} = \tilde{G}_{mn}\,, \qquad \mathring{C}_{mn} = \tilde{C}_{mn}\,, \qquad \mathring{\phi} = 0\,. \tag{2.3.4}$$

We then linearise the reduction formulas (2.3.1)–(2.3.3) around the scalar origin

$$M_{AB}(x) = \delta_{AB} + m_{AB}(x) + \dots\,, \tag{2.3.5}$$

and (back in the string frame) obtain

$$\hat{G}_{mn}(x,y) = \tilde{G}_{mn}(y) + \hat{h}_{mn}(x,y) + \dots\,, \quad C_{mn}(x,y) = \tilde{C}_{mn}(y) + \hat{k}_{mn}(x,y) + \dots\,, \tag{2.3.6}$$

with

$$\begin{aligned} \hat{h}_{mn}(x,y) &= -m_{AB}(x)\,\mathcal{K}^A{}_m(y)\,\mathcal{K}^B{}_n(y)\,, \\ \hat{k}_{mn}(x,y) &= m_{AB}(x)\,\kappa^{AD}\mathcal{K}_{D\,m}(y)\,\mathcal{K}^B{}_n(y)\,, \end{aligned} \tag{2.3.7}$$

as well as

$$\phi = \varphi(x) + \tfrac{1}{4}\tilde{G}^{mn}\hat{h}_{mn} + \dots\,, \tag{2.3.8}$$

for the dilaton, where we have used the linearisation $\Delta(x,y) = 1 + \frac{1}{2}\tilde{G}^{mn}\hat{h}_{mn} - 2d\beta\phi + \dots$. Parametrizing the scalar fluctuations (2.3.5) as

$$m_{AB} \equiv \begin{pmatrix} a & -b \\ b & -a \end{pmatrix}_{AB}\,, \tag{2.3.9}$$

with symmetric a and antisymmetric b, in accordance with the $SO(d,d)$ property of M_{AB}, we finally obtain the fluctuations

$$\begin{aligned} \hat{h}_{mn} + \hat{k}_{mn} &= S^{ab}(x)L_{a\,n}(y)R_{b\,m}(y)\,, \\ \phi &= \varphi(x) + \frac{1}{4}S^{ab}(x)L_a{}^m(y)R_{b\,m}(y)\,, \end{aligned} \tag{2.3.10}$$

[3] The translation uses an explicit parametrization of the $SO(d,d)$ matrix M_{AB} in a basis where η_{AB} is diagonal, as

$$\tilde{M}_{AB} = \begin{pmatrix} (1+PP^t)^{1/2} & P \\ P^t & (1+P^tP)^{1/2} \end{pmatrix}\,,$$

in terms of an unconstrained $d \times d$ matrix $P_a{}^b$.

with $S^{ab} \equiv 4\left(a^{ab}+b^{ab}\right)$. These precisely reproduces the linearised result given in [6].

After the full non-linear reduction (2.3.1)–(2.3.3), the reduced theory is an n-dimensional gravity coupled to a 2-form and $2d$ gauge vectors with gauge group $G_L \times G_R$. The (d^2+1) scalar fields couple as an $\mathbb{R} \times SO(d,d)/(SO(d) \times SO(d))$ coset space sigma model, and come with a scalar potential [74, 75]

$$V(x) \;=\; \frac{1}{12}\, e^{4\gamma\varphi(x)} X_{AB}{}^{C} X_{DE}{}^{F} M^{AD}(x) \left(M^{BE}(x) M_{CF}(x) + 3\,\delta_C^E \delta_F^B\right) \;, \quad (2.3.11)$$

with the structure constants $X_{AB}{}^{C}$ from (2.2.12). Due to the dilaton prefactor, this potential cannot support (A)dS geometries, but only Minkowski or domain wall solutions.

Let us finally comment on adding a cosmological term $e^{4\beta\phi}\Lambda$ in the higher-dimensional theory (2.1.1). E.g. for the bosonic string such a term would arise as conformal anomaly in dimension $n+d \neq 26$. In the Einstein frame, the modified action takes the form

$$S \;=\; \int dX^{n+d}\, \sqrt{|G|}\Big(R + 4\, G^{\hat\mu\hat\nu}\partial_{\hat\mu}\phi\partial_{\hat\nu}\phi - \frac{1}{12}\, e^{-8\beta\phi}\, H^{\hat\mu\hat\nu\hat\rho} H_{\hat\mu\hat\nu\hat\rho} + e^{4\beta\phi}\Lambda\Big) \;, \quad (2.3.12)$$

with constant Λ. With the $O(d,d)$ dictionary (2.1.17), it follows that the effect of this term in the $O(d,d)$ covariant action (2.1.10) is a similar term

$$\mathcal{L}_c \;=\; \sqrt{|\mathrm{g}|}\, e^{-2\Phi}\Lambda \;, \quad (2.3.13)$$

manifestly respecting $O(d,d)$ covariance. The presence of this term thus does not interfere with the consistency of the truncation ansatz and simply results in a term

$$\mathcal{L}_c \;=\; \sqrt{|g|}\, e^{4\gamma\varphi}\Lambda \;, \quad (2.3.14)$$

in the reduced theory, as already argued in [6, 52].

2.4 Summary

We have in this chapter given a complete and constructive proof of the consistency of the Pauli reduction of the low-energy effective action of the bosonic string on the group manifold G, proving the conjecture of [6]. The construction is based on the $O(d,d)$ covariant reformulation of the original theory in which the consistent truncations of the latter are rephrased as generalised Scherk-Schwarz reductions on an extended spacetime. We have explicitly constructed the relevant $SO(d,d)$ valued twist matrix, carrying the dependence on the internal variables, in terms of the Killing vectors of the group manifold G. From the twist matrix, we have further read off the full non-linear reduction ansätze for all fields of the bosonic string. The construction is the first example of the power of the generalised Scherk-Schwarz reductions on extended spacetime we will see in this thesis and hints towards a more systematic understanding of the conditions under which consistent Pauli reductions are possible.

Chapter 3

Type IIB supergravity within the $\mathrm{E}_{6(6)}$ EFT

In this chapter, we establish the precise embedding of type IIB supergravity into the $\mathrm{E}_{6(6)}$ EFT. We start by a review of the $\mathrm{E}_{6(6)}$ EFT action and derive the field equations of the 2-form which will translate to the self-duality relations for the field strength of the 4-form on the type IIB side. We then decompose the EFT field content under the appropriate solution of the section constraint. This constitutes the preliminary work needed to prove the consistency of the Kaluza-Klein reduction of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$, one of the application we will see in the next chapter.

3.1 Review and type IIB decomposition of the $\mathrm{E}_{6(6)}$ EFT

3.1.1 Covariant $\mathrm{E}_{6(6)}$ dynamics

In the introduction, we have seen that the theory is invariant under generalised diffeomorphisms, generated by a parameter $\Lambda^M = \Lambda^M(x, Y)$. We also gave the action, which is manifestly invariant under generalised diffeomorphisms. Let us now define the dynamics of the $\mathrm{E}_{6(6)}$ EFT by giving the unique action principle on the extended space, which decomposed into the five terms

$$S_{\mathrm{EFT}} \;=\; S_{\mathrm{EH}} + \mathcal{L}_{\mathrm{sc}} + \mathcal{L}_{\mathrm{VT}} + S_{\mathrm{top}} - V\ . \tag{3.1.1}$$

The first term formally takes the same form as the standard Einstein-Hilbert term,

$$S_{\mathrm{EH}} \;=\; \int d^5x\, d^{27}Y\, e\, \widehat{R} \;=\; \int d^5x\, d^{27}Y\, e\, e_a{}^{\mu} e_b{}^{\nu}\, \widehat{R}_{\mu\nu}{}^{ab}\ , \tag{3.1.2}$$

except that in the definition of the Riemann tensor all partial derivatives are replaced by $\mathcal{A}_\mu$ covariant derivatives and one adds an additional term to make it properly local Lorentz invariant, $\widehat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^M e^{\rho[a}\partial_M e_\rho{}^{b]}$. The second term is the 'scalar-kinetic' term defined by

$$\mathcal{L}_{\rm sc} \;=\; \frac{1}{24}\, e\, g^{\mu\nu}\, \mathcal{D}_\mu \mathcal{M}_{MN}\, \mathcal{D}_\nu \mathcal{M}^{MN}\;. \tag{3.1.3}$$

The third term is the kinetic term for the gauge-vectors, written in terms of the gauge covariant curvature (1.5.11),

$$\mathcal{L}_{\rm VT} \;\equiv\; -\frac{1}{4}\, e\, \mathcal{F}_{\mu\nu}{}^M \mathcal{F}^{\mu\nu\, N}\, \mathcal{M}_{MN}\;. \tag{3.1.4}$$

The fourth term is a Chern-Simons-type topological term, which is only gauge invariant up to boundary terns. It is most conveniently defined by writing it as a manifestly gauge invariant action in one higher dimension, where it reduces to a total derivative, reducing it to the boundary integral in one dimension lower. Using form notation it reads

$$\begin{aligned} S_{\rm top} \;&=\; \int d^5x\, d^{27}Y\, \mathcal{L}_{\rm top} \\ &=\; \frac{\sqrt{10}}{6}\int d^{27}Y \int_{\mathcal{M}_6} \left(d_{MNK}\, \mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40\, d^{MNK} \mathcal{H}_M \wedge \partial_N \mathcal{H}_K\right), \end{aligned} \tag{3.1.5}$$

Under a general variation of $\mathcal{A}$ and $\mathcal{B}$ the topological Lagrangian varies as

$$\delta\mathcal{L}_{\rm top} \;=\; \frac{1}{8}\sqrt{10}\,\varepsilon^{\mu\nu\rho\sigma\tau}\Big(d_{MNK}\, \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma}{}^N \delta A_\tau{}^K + \frac{20}{3}\, d^{MNK}\, \partial_N \mathcal{H}_{\mu\nu\rho\, M}\, \Delta B_{\sigma\tau\, K}\Big)\;. \tag{3.1.6}$$

The final term in the action is the 'scalar potential' that involves only internal derivatives ∂_M and reads

$$\begin{aligned} V \;=\; &-\frac{1}{24}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\, \partial_N \mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\partial_L \mathcal{M}_{NK} \\ &-\frac{1}{2} g^{-1}\partial_M g\, \partial_N \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN} g^{-1}\partial_M g\, g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}\;. \end{aligned} \tag{3.1.7}$$

Its form is uniquely determined by the internal generalised diffeomorphism invariance (up to the relative coefficient between the last two terms in the second line that is universal for all EFTs).

The field equations of the $\mathrm{E}_{6(6)}$ EFT follow by varying (3.1.1) naively w.r.t. all fields. For now we focus on the field equations for two-form only, because they will be significant below. The 2-form $\mathcal{B}_{\mu\nu M}$ does not enter with a kinetic term, but appears inside the Yang-Mills-type kinetic term, c.f. the definition (1.5.11), and the topological term (3.1.5). Therefore, its field equations are first order and read

$$d^{MNK}\partial_K\left(e\mathcal{M}_{NL}\mathcal{F}^{\mu\nu L} + \frac{1}{6}\sqrt{10}\,\varepsilon^{\mu\nu\rho\sigma\tau}\, \mathcal{H}_{\rho\sigma\tau N}\right) \;=\; 0\,. \tag{3.1.8}$$

These equations take the same form as the standard duality relations in five dimensions between vectors and two-forms. However, here they appear only under a differential operator, which thus leads to different sets of duality relations for different solutions of the section constraint.

In the action above (3.1.1), all terms are independently gauge invariant. Therefore, they could appear in the action with arbitrary relative coefficients. It was shown in [33] that all the coefficient are fixed by demanding invariance of the action under external diffeomorphisms, generated by a parameter $\xi^\mu = \xi^\mu(x, Y)$

$$\begin{aligned}
\delta_\xi e_\mu{}^a &= \xi^\nu D_\nu e_\mu{}^a + D_\mu \xi^\nu e_\nu{}^a \ , \\
\delta_\xi \mathcal{M}_{MN} &= \xi^\mu D_\mu \mathcal{M}_{MN} \ , \\
\delta_\xi A_\mu{}^M &= \xi^\nu \mathcal{F}_{\mu\nu}{}^M + \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu \ , \\
\Delta_\xi \mathcal{B}_{\mu\nu\mathcal{M}} &= \frac{1}{2\sqrt{10}} \xi^\rho \, e\varepsilon_{\mu\nu\rho\sigma\tau} \, \mathcal{F}^{\sigma\tau\, N} \mathcal{M}_{MN} \ .
\end{aligned} \tag{3.1.9}$$

They take the same form as standard diffeomorphisms generated by conventional Lie derivatives, except that all partial derivatives are replaced by gauge covariant derivatives. Moreover, in $\delta\mathcal{A}_\mu$ there is an additional $\mathcal{M}$-dependent term and in $\Delta\mathcal{B}_{\mu\nu}$ the naively covariant form $\xi^\rho \mathcal{H}_{\mu\nu\rho}$ has been replaced according to the duality relation (3.1.8).

3.1.2 IIB solution of the section constraint

In [33], it was shown with extensive details that upon breaking $\mathrm{E}_{6(6)}$ to GL(6), according to,

$$\mathrm{GL}(6) \ = \ \mathrm{SL}(6) \times \mathrm{GL}(1) \in \mathrm{SL}(6) \times \mathrm{SL}(2) \in \mathrm{E}_{6(6)} \tag{3.1.10}$$

one recovers 11-dimensional supergravity in a 5+6 split formulation. In this chapter, we will focus the $\mathrm{GL}(5) \times \mathrm{SL}(2)$ invariant solution of the section condition and we will show that it is on-shell equivalent to type IIB supergravity after a 5+5 split. We consider the following embedding into $\mathrm{E}_{6(6)}$.

$$\mathrm{GL}(5) \times \mathrm{SL}(2) \ \subset \ \mathrm{SL}(6) \times \mathrm{SL}(2) \ \subset \ \mathrm{E}_{6(6)} \ . \tag{3.1.11}$$

In this case, the fundamental and the adjoint representation of $\mathrm{E}_{6(6)}$ break as

$$\bar{\mathbf{27}} \ \rightarrow \ (5,1)_{+4} + (5',2)_{+1} + (10,1)_{-2} + (1,2)_{-5} \ , \tag{3.1.12}$$

$$\mathbf{78} \ \rightarrow \ (5,1)_{-6} + (10',2)_{-3} + (1+15+20)_0 + (10,2)_{+3} + (5',1)_{+6} \ , \tag{3.1.13}$$

with the subscripts referring to the charges under $\mathrm{GL}(1) \subset \mathrm{GL}(5)$. An explicit solution to the section condition (1.5.1) is given by restricting the Y^M dependence of all fields to the five coordinates in the $(5,1)_{+4}$. Explicitly, splitting the coordinates Y^M and the fundamental indices according to (3.1.12) into

$$\{Y^M\} \;\;\rightarrow\;\; \{\, y^m \,,\; y_{m\alpha} \,,\; y^{mn} \,,\; y_\alpha \,\} \;, \tag{3.1.14}$$

with internal indices $m, n = 1, \dots, 5$ and SL(2) indices $\alpha = 1, 2$, the non-vanishing components of the d-symbol are given by

$$\begin{aligned}
d^{MNK} \;&:\; d^{m}{}_{n\alpha,\beta} = \frac{1}{\sqrt{10}}\,\delta^m_n\varepsilon_{\alpha\beta}\;, \quad d^{mn}{}_{k\alpha,l\beta} = \frac{1}{\sqrt{5}}\,\delta^{mn}_{kl}\,\varepsilon_{\alpha\beta}\;, \quad d^{mn,kl,p} = \frac{1}{\sqrt{40}}\,\varepsilon^{mnklp}\;, \\
d_{MNK} \;&:\; d_m{}^{n\alpha,\beta} = \frac{1}{\sqrt{10}}\,\delta^n_m\varepsilon^{\alpha\beta}\;, \quad d_{mn}{}^{k\alpha,l\beta} = \frac{1}{\sqrt{5}}\,\delta^{kl}_{mn}\,\varepsilon^{\alpha\beta}\;, \quad d_{mn,kl,p} = \frac{1}{\sqrt{40}}\,\varepsilon_{mnklp}\;,
\end{aligned} \tag{3.1.15}$$

and all those related by symmetry, $d^{MNK} = d^{(MNK)}$. In particular, the GL(1) grading guarantees that all components $d^{m\,n\,k}$ vanish, such that the section condition (1.5.1) indeed is solved by restricting the coordinate dependence of all fields according to

$$\{\partial^{m\,\alpha}A = 0\;,\; \partial_{mn}A = 0\;,\; \partial^\alpha A = 0\} \qquad \Longleftrightarrow \qquad A(x^\mu, Y^M) \;\longrightarrow\; A(x^\mu, y^m)\;. \tag{3.1.16}$$

3.1.3 Decomposition of EFT fields

In this subsection we analyse various objects of EFT, e.g., the generalised metric and the gauge covariant curvatures, in terms of the component fields originating under the above decomposition of $\mathrm{E}_{6(6)}$, together with their gauge symmetries. This sets the stage for our analysis in sec. 4, where we start from type IIB supergravity and perform the complete Kaluza-Klein decomposition in order to match it to the fields and symmetries discussed here. Thus, here we split tensor fields and indices according to (3.1.12)–(3.1.15), assuming the explicit solution (3.1.16) of the section condition.

To begin, let us consider the p-form field content of the $\mathrm{E}_{6(6)}$ EFT under the split (3.1.12). This yields

$$\mathcal{A}_\mu{}^M \;:\; \{\mathcal{A}_\mu{}^m, \mathcal{A}_{\mu\,m\,\alpha}, \mathcal{A}_{\mu\,kmn}, \mathcal{A}_{\mu\,\alpha}\}\;, \qquad \mathcal{B}_{\mu\nu\,M} \;:\; \{\mathcal{B}_{\mu\nu}{}^\alpha, \mathcal{B}_{\mu\nu\,mn}, \mathcal{B}_{\mu\nu}{}^{m\,\alpha}, \mathcal{B}_{\mu\nu\,m}\}\;, \tag{3.1.17}$$

where we have defined $\mathcal{A}_{\mu\,kmn} \equiv \frac{1}{2}\varepsilon_{kmnpq}\mathcal{A}_\mu{}^{pq}$. However, the EFT Lagrangian actually depends on the two-forms only under certain derivatives,

$$\left\{\, \partial_m\mathcal{B}_{\mu\nu}{}^\alpha\;,\; \partial_{[k}\mathcal{B}_{|\mu\nu|\,mn]}\;,\; \partial_m\mathcal{B}_{\mu\nu}{}^{m\,\alpha} \,\right\}\;, \tag{3.1.18}$$

introducing an additional redundancy in the two-form field content, which will be important for the match with type IIB.

Let us now work out the general formulas of the $\mathrm{E}_{6(6)}$-covariant formulation with (3.1.15) and imposing the explicit solution of the section condition (3.1.16) on all fields.

We then obtain, by inserting (3.1.15) into (1.5.11), the following covariant field strengths of the different vector fields in (3.1.17),

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{m} &= 2\partial_{[\mu}\mathcal{A}_{\nu]}{}^{m} - \mathcal{A}_{\mu}{}^{n}\partial_{n}\mathcal{A}_{\nu}{}^{m} + \mathcal{A}_{\nu}{}^{n}\partial_{n}\mathcal{A}_{\mu}{}^{m} \;, \\
\mathcal{F}_{\mu\nu\, m\alpha} &= 2D^{\rm KK}_{[\mu}\,\mathcal{A}_{\nu]\, m\alpha} + \varepsilon_{\alpha\beta}\,\partial_{m}\tilde{\mathcal{B}}_{\mu\nu}{}^{\beta} \;, \\
\mathcal{F}_{\mu\nu\, kmn} &= 2D^{\rm KK}_{[\mu}\,\mathcal{A}_{\nu]\, kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\mathcal{A}_{[\mu\,[k|\alpha|}\partial_{m}\mathcal{A}_{\nu]n]\beta} + 3\,\partial_{[k}\tilde{\mathcal{B}}_{|\mu\nu|\, mn]} \;, \\
\mathcal{F}_{\mu\nu\,\alpha} &= 2D^{\rm KK}_{[\mu}\,\mathcal{A}_{\nu]\,\alpha} - 2(\partial_{k}\mathcal{A}_{[\mu}{}^{k})\,\mathcal{A}_{\nu]\alpha} - \sqrt{2}\,\mathcal{A}_{[\mu}{}^{mn}\partial_{n}\mathcal{A}_{\nu]\, m\alpha} \\
&\quad - \sqrt{2}\,\mathcal{A}_{[\mu|m\alpha|}\partial_{n}\mathcal{A}_{\nu]}{}^{mn} - \varepsilon_{\alpha\beta}\,\partial_{k}\tilde{\mathcal{B}}_{\mu\nu}{}^{k\beta} \;, \qquad (3.1.19)
\end{aligned}
$$

with the redefined two-forms

$$
\begin{aligned}
\tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha} &\equiv \sqrt{10}\,\mathcal{B}_{\mu\nu}{}^{\alpha} - \varepsilon^{\alpha\beta}\,\mathcal{A}_{[\mu}{}^{n}\mathcal{A}_{\nu]\, n\beta} \;, \\
\tilde{\mathcal{B}}_{\mu\nu\, mn} &\equiv \sqrt{10}\,\mathcal{B}_{\mu\nu\, mn} + \mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu]\, kmn} \;, \\
\tilde{\mathcal{B}}_{\mu\nu}{}^{k\alpha} &\equiv \sqrt{10}\,\mathcal{B}_{\mu\nu}{}^{k\alpha} + \varepsilon^{\alpha\beta}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu]\beta} \;. \qquad (3.1.20)
\end{aligned}
$$

Here all covariant derivatives are $D^{\rm KK}_{\mu} \equiv \partial_{\mu} - \mathcal{L}_{\mathcal{A}_{\mu}}$, covariantized w.r.t. to the action of the five-dimensional internal diffeomorphisms reviewed above. The corresponding vector gauge transformations, obtained from (1.5.15), are given by

$$
\begin{aligned}
\delta\mathcal{A}_{\mu}{}^{m} &= D^{\rm KK}_{\mu}\,\Lambda^{m} \;, \\
\delta\mathcal{A}_{\mu\, m\alpha} &= D^{\rm KK}_{\mu}\,\Lambda_{m\alpha} + \mathcal{L}_{\Lambda}\mathcal{A}_{\mu\, m\alpha} - \varepsilon_{\alpha\beta}\,\partial_{m}\tilde{\Xi}_{\mu}{}^{\beta} \;, \\
\delta\mathcal{A}_{\mu\, kmn} &= D^{\rm KK}_{\mu}\,\Lambda_{kmn} + \mathcal{L}_{\Lambda}\mathcal{A}_{\mu\, kmn} - 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\partial_{[k}\mathcal{A}_{|\mu|m|\alpha|}\Lambda_{n]\beta} - 3\,\partial_{[k}\tilde{\Xi}_{|\mu|\, mn]} \;, \qquad (3.1.21)
\end{aligned}
$$

with

$$
\tilde{\Xi}_{\mu}{}^{\alpha} \equiv \sqrt{10}\,\Xi_{\mu}{}^{\alpha} - \varepsilon^{\alpha\beta}\,\Lambda^{n}\mathcal{A}_{\mu\, n\beta} \;, \qquad \tilde{\Xi}_{\mu\, mn} \equiv \sqrt{10}\,\Xi_{\mu\, mn} + \Lambda^{k}\mathcal{A}_{\mu\, kmn} \;. \qquad (3.1.22)
$$

For the vector fields $\mathcal{A}_{\mu\,\alpha}$ we observe that its gauge variation contains the contribution

$$
\delta\mathcal{A}_{\mu\,\alpha} = \cdots + \varepsilon_{\alpha\beta}\,\partial_{k}\tilde{\Xi}_{\mu}{}^{k\beta} \;. \qquad (3.1.23)
$$

This implies that it can entirely be gauged away by the tensor gauge symmetry associated with the two-forms $\mathcal{B}_{\mu\nu}{}^{k\beta}$. Consequently, it will automatically disappear from the Lagrangian upon integrating out $\partial_{k}\mathcal{B}_{\mu\nu}{}^{k\beta}$. The remaining two-form field strengths in turn come with gauge transformations

$$
\begin{aligned}
\delta\tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha} &= 2D^{\rm KK}_{[\mu}\,\tilde{\Xi}_{\nu]}{}^{\alpha} + \mathcal{L}_{\Lambda}\tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha} - \varepsilon^{\alpha\beta}\,\Lambda_{n\beta}\mathcal{F}_{\mu\nu}{}^{n} + \tilde{\mathcal{O}}_{\mu\nu}{}^{\alpha} \;, \\
\delta\tilde{\mathcal{B}}_{\mu\nu\, mn} &= 2D^{\rm KK}_{\mu}\left(\tilde{\Xi}_{\nu\, mn} + \frac{1}{\sqrt{2}}\,\varepsilon^{\alpha\beta}\,\mathcal{A}_{\nu\, m\alpha}\,\Lambda_{n\beta}\right) + \sqrt{2}\,\partial_{m}\mathcal{A}_{\mu\, n\alpha}\,\tilde{\Xi}_{\nu}{}^{\alpha} \\
&\quad + \mathcal{L}_{\Lambda}\tilde{\mathcal{B}}_{\mu\nu\, mn} - \frac{1}{\sqrt{2}}\,\Lambda_{[m|\alpha|}\,\partial_{n]}\tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha} + \Lambda_{mnk}\,\mathcal{F}_{\mu\nu}{}^{k} \\
&\quad + \frac{1}{\sqrt{2}}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{\mu\nu\, m\alpha}\,\Lambda_{n\beta} + \tilde{\mathcal{O}}_{\mu\nu mn} \;, \qquad (3.1.24)
\end{aligned}
$$

where

$$\tilde{\mathcal{O}}_{\mu\nu}{}^{\alpha} \;\equiv\; \sqrt{10}\,\mathcal{O}_{\mu\nu}{}^{\alpha}\;, \tag{3.1.25}$$

$$\tilde{\mathcal{O}}_{\mu\nu\,mn} \;\equiv\; \sqrt{10}\,\mathcal{O}_{\mu\nu\,mn}+\partial_m\left(2\Lambda^k\,\tilde{\mathcal{B}}_{\mu\nu\,nk}+\sqrt{2}\mathcal{A}_{\mu\,n\alpha}\,\Xi_\nu{}^\alpha+\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathcal{A}_{\mu\,n\alpha}\mathcal{A}_{\nu\,k\beta}\right)\;.$$

Finally, the associated three-form field strengths are obtained from (1.5.17) and read

$$\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\alpha} \;\equiv\; \sqrt{10}\,\mathcal{H}_{\mu\nu\rho}{}^{\alpha} \;=\; 3\,D^{\rm KK}_{[\mu}\tilde{\mathcal{B}}_{\nu\rho]}{}^{\alpha}+3\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{[\mu\nu}{}^{n}\mathcal{A}_{\rho]\,n\beta}\;, \tag{3.1.26}$$

$$\begin{aligned}\tilde{\mathcal{H}}_{\mu\nu\rho\,mn} \;&\equiv\; \sqrt{10}\,\mathcal{H}_{\mu\nu\rho\,mn}\\ &=\; 3\,D^{\rm KK}_{\mu}\tilde{\mathcal{B}}_{\nu\rho\,mn}-3\,\mathcal{F}_{\mu\nu}{}^{k}\mathcal{A}_{\rho\,kmn}-3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathcal{A}_{\mu m\alpha}D_\nu\mathcal{A}_{\rho n\beta}+3\sqrt{2}\,\mathcal{A}_{\mu m\alpha}\partial_n\tilde{\mathcal{B}}_{\nu\rho}{}^{\alpha}\;.\end{aligned}$$

More precisely, this holds up to terms that are projected out from the Lagrangian under y-derivatives. The expressions on the r.h.s. in (3.1.24)–(3.1.26) are understood to be projected onto the corresponding antisymmetrizations in their parameters, i.e. $[mn]$, $[\mu\nu]$, $[\mu\nu\rho]$, etc.

It is also instructive to give the component form of the Bianchi identities originating from (1.5.17). We obtain the components

$$4\,D^{\rm KK}_{[\mu}\tilde{\mathcal{H}}_{\nu\rho\sigma]}{}^{\alpha} \;=\; 6\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{[\mu\nu}{}^{n}\mathcal{F}_{\rho\sigma]\,n\beta}\;. \tag{3.1.27}$$

After a straightforward but somewhat tedious computation one finds

$$\begin{aligned}4\,D^{\rm KK}_{[\mu}\tilde{\mathcal{H}}_{\nu\rho\sigma]\,mn}+4\sqrt{2}\,\mathcal{A}_{\mu m\alpha}\partial_n\tilde{\mathcal{H}}_{\nu\rho\sigma}{}^{\alpha} \;&=\; -6\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{F}_{\rho\sigma]\,kmn}-3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{[\mu\nu\,|m\alpha|}\mathcal{F}_{\rho\sigma]\,n\beta}\\ &\quad+3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha}\partial_n\tilde{\mathcal{B}}_{\rho\sigma}{}^{\beta}\right)-12\,\partial_m\left(\mathcal{F}_{\mu\nu}{}^{k}\tilde{\mathcal{B}}_{\rho\sigma\,kn}\right)\\ &\quad-6\sqrt{2}\,\partial_m\left(\mathcal{A}_{\mu n\alpha}\varepsilon^{\alpha\beta}\mathcal{F}_{\nu\rho}{}^{k}\mathcal{A}_{\sigma\,k\beta}\right)\;.\end{aligned} \tag{3.1.28}$$

Again, the indices m,n and μ,ν,ρ,σ in here are totally antisymmetrized, which we did not indicate explicitly in order not to clutter the notation.

Let us now move to the scalar field content of the theory. In the EFT formulation, they parametrize the symmetric matrix $\mathcal{M}_{MN}$. We now need to choose a parametrization of this matrix in accordance with the decomposition (3.1.13). In standard fashion [76], we build the matrix as $\mathcal{M}_{MN}=(\mathcal{V}\mathcal{V}^T)_{MN}$ from a 'vielbein' $\mathcal{V}\in{\rm E}_{6(6)}$ in triangular gauge

$$\mathcal{V} \;\equiv\; \exp\left[\varepsilon^{klmnp}\,c_{klmn}\,t_{(+6)\,p}\right]\,\exp\left[b_{mn}{}^{\alpha}\,t_{(+3)}{}_{\alpha}^{mn}\right]\,\mathcal{V}_5\,\mathcal{V}_2\exp\left[\Phi\,t_{(0)}\right]\;. \tag{3.1.29}$$

Here, $t_{(0)}$ is the ${\rm E}_{6(6)}$ generator associated to the GL(1) grading of (3.1.13), $\mathcal{V}_2$, $\mathcal{V}_5$ denote matrices in the SL(2) and SL(5) subgroup, respectively, parametrized by vielbeins ν_2, ν_5. The $t_{(+n)}$ refer to the ${\rm E}_{6(6)}$ generators of positive grading in (3.1.13), with non-trivial commutator

$$\left[t_{(+3)}{}_{\alpha}^{kl},t_{(+3)}{}_{\beta}^{mn}\right] \;=\; \varepsilon_{\alpha\beta}\,\varepsilon^{klmnp}\,t_{(+6)\,p}\;. \tag{3.1.30}$$

All generators are evaluated in the fundamental **27** representation (3.1.12), such that the symmetric matrix $\mathcal{M}_{MN}$ takes the block form

$$\mathcal{M}_{KM} \;=\; \begin{pmatrix} \mathcal{M}_{km} & \mathcal{M}_{k}{}^{m\beta} & \mathcal{M}_{k,mn} & \mathcal{M}_{k}{}^{\beta} \\ \mathcal{M}^{k\alpha}{}_{m} & \mathcal{M}^{k\alpha,m\beta} & \mathcal{M}^{k\alpha}{}_{mn} & \mathcal{M}^{k\alpha,\beta} \\ \mathcal{M}_{kl,m} & \mathcal{M}_{kl}{}^{m\beta} & \mathcal{M}_{kl,mn} & \mathcal{M}_{kl}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_{m} & \mathcal{M}^{\alpha,m\beta} & \mathcal{M}^{\alpha}{}_{mn} & \mathcal{M}^{\alpha\beta} \end{pmatrix} . \tag{3.1.31}$$

Explicit evaluation of (3.1.29) determines the various blocks in (4.5.3). For instance,

$$\mathcal{M}_{mn,kl} \;=\; e^{2\Phi/3}\, m_{m[k}m_{l]n} + 2e^{5\Phi/3}\, b_{mn}{}^{\alpha} b_{kl}{}^{\beta}\, m_{\alpha\beta}\;, \tag{3.1.32}$$

while the components in the last line are given by[1]

$$\begin{aligned} \mathcal{M}^{\alpha\beta} &= e^{5\Phi/3}\, m^{\alpha\beta}\;, \qquad \mathcal{M}^{\alpha}{}_{mn} = \sqrt{2}\, e^{5\Phi/3}\, m^{\alpha\beta}\varepsilon_{\beta\gamma}\, b_{mn}{}^{\gamma}\;, \\ \mathcal{M}^{\alpha,m\beta} &= \frac{1}{2}\, e^{5\Phi/3}\, m^{\alpha\gamma}\varepsilon_{\gamma\delta}\, \varepsilon^{mklpq}\, b_{kl}{}^{\beta} b_{pq}{}^{\delta} - \frac{1}{24}\, e^{5\Phi/3}\, m^{\alpha\beta}\, \varepsilon^{mklpq}\, c_{klpq}\;, \\ \mathcal{M}^{\alpha}{}_{m} &= \frac{2}{3}\, e^{5\Phi/3}\, m_{\beta\gamma}\, \varepsilon^{kpqrs} \left(b_{mk}{}^{[\alpha} b_{pq}{}^{\beta]} b_{rs}{}^{\gamma} + \frac{1}{8}\, \varepsilon^{\alpha\beta}\, b_{mk}{}^{\gamma}\, c_{pqrs} \right)\;, \end{aligned} \tag{3.1.33}$$

with the symmetric matrix $m^{\alpha\beta} = (\nu_2)^{\alpha}{}_{u}(\nu_2)^{\beta\,u}$ built from the SL(2) vielbein from (3.1.29). We will also need the following combinations of the matrix entries of $\mathcal{M}_{MN}$ (that emerge after integrating out some of the fields),

$$\tilde{\mathcal{M}}_{MN} \;\equiv\; \mathcal{M}_{MN} - \mathcal{M}_{M}{}^{\alpha}(\mathcal{M}^{\alpha\beta})^{-1}\mathcal{M}_{N}{}^{\beta}\;, \tag{3.1.34}$$

for which we find

$$\begin{aligned} \tilde{\mathcal{M}}_{mn,kl} &= e^{2\Phi/3}\, m_{m[k}m_{l]n}\;, \\ \tilde{\mathcal{M}}_{mn}{}^{k\alpha} &= \frac{1}{\sqrt{2}}\, e^{2\Phi/3}\, \varepsilon_{mnpqr} m^{kp} m^{qu} m^{rv} b_{uv}{}^{\alpha}\;, \\ \tilde{\mathcal{M}}_{mn,k} &= -\frac{1}{6\sqrt{2}}\, e^{2\Phi/3}\, \varepsilon^{uvpqr}\, m_{mu}m_{nv} \left(c_{kpqr} - 6\varepsilon_{\alpha\beta}\, b_{kp}{}^{\alpha} b_{qr}{}^{\beta} \right)\;, \\ \tilde{\mathcal{M}}^{m\alpha,n\beta} &= e^{-\Phi/3}\, m^{mn}m^{\alpha\beta} + 2\, e^{2\Phi/3}\, m^{kp} \left(m^{mn}m^{lq} - 2\, m^{ml}m^{nq} \right) b_{kl}{}^{\alpha} b_{pq}{}^{\beta}\;, \end{aligned} \tag{3.1.35}$$

etc., with $m_{mn} = (\nu_5)_{m}{}^{a}(\nu_5)_{n}{}^{a}$.

Next, we can work out the covariant derivatives of the various 'scalar components' of the generalised metric. Using (3.1.15) we find for the covariant derivatives of the matrix parameters in (4.5.3)

$$\begin{aligned} \mathcal{D}_{\mu}\Phi &= D^{\rm KK}_{\mu}\,\Phi + \tfrac{4}{5}\,\partial_k\mathcal{A}_{\mu}{}^{k}\;, \\ \mathcal{D}_{\mu}m_{mn} &= D^{\rm KK}_{\mu}\,m_{mn} + \tfrac{2}{5}\,\partial_k\mathcal{A}_{\mu}{}^{k}\,m_{mn}\;, \\ \mathcal{D}_{\mu}b_{mn}{}^{\alpha} &= D^{\rm KK}_{\mu}\,b_{mn}{}^{\alpha} - \varepsilon^{\alpha\beta}\partial_{[m}\mathcal{A}_{n]\beta\,\mu}\;, \\ \mathcal{D}_{\mu}c_{klmn} &= D^{\rm KK}_{\mu}\,c_{klmn} + 4\sqrt{2}\,\partial_{[k}\mathcal{A}_{lmn]\mu} + 12\, b_{[kl}{}^{\alpha}\,\partial_m\mathcal{A}_{n]\,\alpha\,\mu}\;, \end{aligned} \tag{3.1.36}$$

[1] The explicit expressions (3.1.33) and (3.1.35) for the matrix components of $\mathcal{M}_{MN}$ and $\tilde{\mathcal{M}}_{MN}$ correct some typos in equations (5.22) and (5.24), respectively, in the published version of [33].

where we recall that $D_\mu^{\rm KK}$ denotes the covariant derivatives w.r.t. ${\cal A}_\mu{}^m$ (that below will be identified with the Kaluza-Klein vector $A_\mu{}^m$) without the density terms, which here have been indicated explicitly, thereby defining the weight of all fields. The form of these covariant derivatives implies in particular that we have the following gauge symmetries on these fields,

$$
\begin{aligned}
\delta\Phi &= {\cal L}_\Lambda\Phi-\tfrac{4}{5}\,\partial_k\Lambda^k\;,\\
\delta m_{mn} &= {\cal L}_\Lambda m_{mn}-\tfrac{2}{5}\,\partial_k\Lambda^k\,m_{mn}\;,\\
\delta b_{mn}{}^\alpha &= {\cal L}_\Lambda b_{mn}{}^\alpha+\varepsilon^{\alpha\beta}\partial_{[m}\Lambda_{n]\beta}\;,\\
\delta c_{klmn} &= {\cal L}_\Lambda c_{klmn}-4\sqrt{2}\,\partial_{[k}\Lambda_{lmn]}-12\,b_{[kl}{}^\alpha\,\partial_m\Lambda_{n]\,\alpha}\;.
\end{aligned}
\tag{3.1.37}
$$

We close this section by giving some relevant formulas for the decompositions of various terms in the action upon putting the solution of the section constraint. The scalar kinetic term (3.1.3) yields

$$
\begin{aligned}
\frac{1}{24}\,D_\mu{\cal M}_{MN}D^\mu{\cal M}^{MN} &= -\frac{5}{6}\,{\cal D}_\mu\Phi{\cal D}^\mu\Phi+\frac{1}{4}\,{\cal D}_\mu m_{\alpha\beta}{\cal D}^\mu m^{\alpha\beta}+\frac{1}{4}\,{\cal D}_\mu m_{mn}{\cal D}^\mu m^{mn}\\
&\quad -e^{\Phi}\,{\cal D}_\mu b_{mn}{}^\alpha{\cal D}^\mu b_{kl}{}^\beta\,m^{mk}m^{nl}m_{\alpha\beta}\\
&\quad -\frac{1}{48}\,e^{2\Phi}\,\widehat{\cal D}_\mu c_{klmn}\widehat{\cal D}^\mu c_{pqrs}m^{kp}m^{lq}m^{mr}m^{ns}\;,
\end{aligned}
\tag{3.1.38}
$$

where we defined

$$
\widehat{\cal D}_\mu c_{klmn} \;\equiv\; {\cal D}_\mu c_{klmn}+12\varepsilon_{\alpha\beta}\,b_{kl}{}^\alpha{\cal D}_\mu b_{mn}{}^\beta\;.
\tag{3.1.39}
$$

The 'scalar potential' (3.1.7) takes the form

$$
\begin{aligned}
V &= 3\,e^{7\Phi/3}\,\partial_{[k}b_{mn]}{}^\alpha\partial_l b_{pq}{}^\beta\,m^{kl}m^{mp}m^{nq}m_{\alpha\beta}\\
&\quad +\frac{5}{48}\,e^{10\Phi/3}\,X_{klmnp}X_{qrstu}m^{kq}m^{lr}m^{ms}m^{nt}m^{pu}+V_\Phi(\partial_k\Phi,\partial_k m_{mn})\;,
\end{aligned}
\tag{3.1.40}
$$

where the last term combines all contributions with the internal derivative acting on Φ and m_{mn}, and

$$
X_{klmnp} \;\equiv\; \partial_{[k}c_{lmnp]}+12\,\varepsilon_{\alpha\beta}\,b_{[kl}{}^\alpha\partial_m b_{np]}{}^\beta\;.
\tag{3.1.41}
$$

Finally, we give the topological term (3.1.5) in this parametrization,

$$
\begin{aligned}
\mathcal{L}_{\rm top} \;=\; & \frac{1}{8}\,\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\Big(\frac{\sqrt{2}}{6}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{\mu\nu\,m\alpha}\mathcal{F}_{\rho\sigma\,n\beta}\,A_{\tau\,pkl}+\frac{1}{6}\,\mathcal{F}_{\mu\nu mnq}F_{\rho\sigma}{}^{q}\,A_{\tau klp} \\
& -\frac{\sqrt{2}}{2}\,\varepsilon^{\alpha\beta}\,A_{\mu\,m\alpha}\partial_n A_{\nu\,p\beta}F_{\rho\sigma}{}^{q}\,A_{\tau klq}+\frac{1}{2}\,\partial_p\tilde{B}_{\mu\nu\,mn}F_{\rho\sigma}{}^{q}\,A_{\tau klq} \\
& +\sqrt{2}\,\varepsilon^{\alpha\beta}\,A_{\mu m\alpha}D_\nu A_{\rho n\beta}\,\partial_p\tilde{B}_{\sigma\tau\,kl}-\sqrt{2}\,A_{\mu m\alpha}\partial_n\tilde{B}_{\nu\rho}{}^{\alpha}\,\partial_p\tilde{B}_{\sigma\tau\,kl} \\
& +\frac{2}{3}\,\varepsilon^{\alpha\beta}\,A_{\mu\,m\alpha}\partial_n A_{\nu\,k\beta}A_{\rho\,l\gamma}\partial_p\tilde{B}_{\sigma\tau}{}^{\gamma}-\varepsilon^{\alpha\beta}\,\varepsilon^{\gamma\delta}A_{\mu\,m\alpha}\partial_n A_{\nu\,k\beta}A_{\rho\,l\gamma}D_\sigma A_{\tau\,p\delta} \\
& +\frac{\sqrt{2}}{9}\,\partial_m\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\alpha}\,A_{\sigma\,n\alpha}A_{\tau klp}-D_\mu\tilde{B}_{\nu\rho\,mn}\partial_p\tilde{B}_{\sigma\tau\,kl}-\frac{2}{3}\,\varepsilon_{\alpha\beta}\,\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\beta}\partial_k\tilde{B}_{\sigma\tau}{}^{k\alpha} \\
& +\mathcal{O}(A_{\mu\,\alpha})\Big)\;. \qquad (3.1.42)
\end{aligned}
$$

3.1.4 External diffeomorphisms

Let us finally turn to the action of the external diffeomorphisms (3.1.9) under the type IIB decomposition. On scalar-densities such as $e_\mu{}^a$ and ξ^μ the gauge-covariant derivative of EFT simply reduces to the Kaluza-Klein covariant derivative w.r.t. $\mathcal{A}_\mu{}^m$. Therefore, the external diffeomorphisms acts on the vielbein $e_\mu{}^a$ as in (3.1.9), with the EFT covariant derivatives replaced by Kaluza-Klein covariant derivatives. For the internal generalised metric $\mathcal{M}_{MN}$ the external diffeomorphism transformations on the various components in (4.5.3) are read off from (3.1.9), with the EFT covariant derivatives written out in (3.1.36).

Next, we consider the external diffeomorphism transformations of the vector fields, which are more subtle due to the presence of the term involving the inverse of the generalised metric $\mathcal{M}$. From (3.1.33) we determine the relevant components of the matrix $\mathcal{M}^{MN}$,

$$
\begin{aligned}
\mathcal{M}^{m,n} &= e^{4\Phi/3}\,m^{mn}\;, \\
\mathcal{M}_{m\alpha,}{}^{n} &= 2\,e^{4\Phi/3}\,\varepsilon_{\alpha\beta}\,m^{nk}b_{km}{}^{\beta}\;, \\
\mathcal{M}^{mn,k} &= -\frac{\sqrt{2}}{12}\,e^{4\Phi/3}\,\varepsilon^{mnpqr}\,m^{ks}\left(c_{pqrs}-6\,\varepsilon_{\alpha\beta}\,b_{pq}{}^{\alpha}b_{rs}{}^{\beta}\right)\;. \qquad (3.1.43)
\end{aligned}
$$

This in turn determines the following gauge variations of the vector field components in (3.1.17),

$$
\begin{aligned}
\delta_\xi\mathcal{A}_\mu{}^m &= \xi^\nu\mathcal{F}_{\nu\mu}{}^m+\mathcal{M}^{m,n}g_{\mu\nu}\partial_n\xi^\nu\;, \\
\delta_\xi\mathcal{A}_{\mu\,m\alpha} &= \xi^\nu\mathcal{F}_{\nu\mu m\alpha}+\mathcal{M}_{m\alpha,}{}^{n}g_{\mu\nu}\partial_n\xi^\nu\;, \qquad (3.1.44) \\
\delta_\xi\mathcal{A}_{\mu\,mnk} &= \tfrac{1}{2}\epsilon_{mnkpq}\xi^\nu\mathcal{F}_{\nu\mu}{}^{pq}+\tfrac{1}{2}\epsilon_{mnkpq}\mathcal{M}^{pq,n}\partial_n\xi^\nu\;,
\end{aligned}
$$

with the field strengths given in (3.1.19). This closes the type IIB decomposition of the EFT field content. We now move to type IIB supergravity side, where we will perform the appropriate field redefinitions to be able to establish the dictionary between the two theories.

3.2 Type IIB supergravity and its Kaluza-Klein decomposition

In this section, we review ten-dimensional IIB supergravity and bring it into a convenient form that allows for the translation of its field content into the various components of the EFT fields identified above.

3.2.1 Type IIB supergravity

Denoting ten-dimensional curved indices by $\hat{\mu}, \hat{\nu}, \ldots$, the type IIB field content is given by

$$E_{\hat{\mu}}{}^{\hat{a}}\ , \quad m_{\alpha\beta}\ , \quad \hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha}\ , \quad \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\ , \quad \alpha, \beta = 1, 2\ , \tag{3.2.1}$$

i.e., the zehnbein, the two SL(2)/SO(2) coset scalars parametrizing the symmetric SL(2) matrix $m_{\alpha\beta}$, a doublet of 2-forms and a 4-form. The 2-forms combine RR 2-form and the NS B-field, with the abelian field strengths given by

$$\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\alpha} \ = \ 3\, \partial_{[\hat{\mu}} \hat{C}_{\hat{\nu}\hat{\rho}]}{}^{\alpha}\ . \tag{3.2.2}$$

The Chern-Simons (CS)-modified curvature of the 4-form is given in components by

$$\hat{F}_{\hat{\mu}_1 \ldots \hat{\mu}_5} \ \equiv \ 5\, \partial_{[\hat{\mu}_1} \hat{C}_{\hat{\mu}_2 \ldots \hat{\mu}_5]} - \frac{5}{4}\, \varepsilon_{\alpha\beta}\, \hat{C}_{[\hat{\mu}_1\hat{\mu}_2}{}^{\alpha} \hat{F}_{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5]}{}^{\beta}\ , \tag{3.2.3}$$

such that they satisfy the Bianchi identities

$$6\, \partial_{[\hat{\mu}_1} \hat{F}_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5\hat{\mu}_6]} = -\frac{5}{2}\, \varepsilon_{\alpha\beta} \hat{F}_{[\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}^{\alpha} \hat{F}_{\hat{\mu}_4\hat{\mu}_5\hat{\mu}_6]}{}^{\beta}\ , \tag{3.2.4}$$

and transform as

$$\begin{aligned} \delta \hat{C}_{\hat{\mu}\hat{\nu}}{}^{\alpha} \ &= \ 2\, \partial_{[\hat{\mu}} \hat{\lambda}_{\hat{\nu}]}{}^{\alpha}\ , \\ \delta \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \ &= \ 4\, \partial_{[\hat{\mu}} \hat{\lambda}_{\hat{\nu}\hat{\rho}\hat{\sigma}]} + \frac{1}{2}\, \varepsilon_{\alpha\beta} \hat{\lambda}_{[\hat{\mu}}{}^{\alpha} \hat{F}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^{\beta}\ , \end{aligned} \tag{3.2.5}$$

under tensor gauge transformations. The IIB field equations have been constructed in [77, 78, 79]. They can be described by a pseudo-action which in our conventions is given by

$$
\begin{aligned}
S \;=\; & \int d^{10}x\sqrt{|G|}\,\Big(\hat{R}+\frac{1}{4}\partial_{\hat\mu}m_{\alpha\beta}\partial^{\hat\mu}m^{\alpha\beta}-\frac{1}{12}\hat{F}_{\hat\mu_1\hat\mu_2\hat\mu_3}{}^{\alpha}\hat{F}^{\hat\mu_1\hat\mu_2\hat\mu_3\,\beta}m_{\alpha\beta}\\
& \qquad -\frac{1}{30}\hat{F}_{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4\hat\mu_5}\hat{F}^{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4\hat\mu_5}\Big) \\
& \qquad -\frac{1}{864}\int d^{10}\hat{x}\,\varepsilon_{\alpha\beta}\,\varepsilon^{\hat\mu_1\dots\hat\mu_{10}}C_{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4}\hat{F}_{\hat\mu_6\hat\mu_7\hat\mu_8}{}^{\alpha}\hat{F}_{\hat\mu_8\hat\mu_9\hat\mu_{10}}{}^{\beta}\;,
\end{aligned}
\qquad (3.2.6)
$$

and which after variation of the fields has to be supplemented with the standard self-duality equations for the 5-form field strength

$$
\hat{F}_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau} \;=\; \frac{1}{5!}\sqrt{|G|}\,\varepsilon_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4\hat\mu_5}\,\hat{F}^{\hat\mu_1\hat\mu_2\hat\mu_3\hat\mu_4\hat\mu_5}\;, \qquad (3.2.7)
$$

with $|G| \equiv |\det G_{\hat\mu\hat\nu}| = |\det E_{\hat\mu}{}^{\hat a}|^2$. It is straightforward to verify that the integrability conditions of the self-duality equations together with the Bianchi identities (3.2.4) coincide with the second-order field equations obtained by variation of (3.2.6). Our SL(2) conventions can be translated into the SU(1,1)/U(1) conventions of [78], by combining the real components of the doublet $\hat{F}_{\hat\mu\hat\nu\hat\rho}{}^{\alpha}$ into a complex F

$$
F_{\hat\mu\hat\nu\hat\rho} \;\equiv\; \hat{F}_{\hat\mu\hat\nu\hat\rho}{}^{1}+i\,\hat{F}_{\hat\mu\hat\nu\hat\rho}{}^{2}\;, \qquad (3.2.8)
$$

and parametrizing the symmetric SL(2) matrix $m_{\alpha\beta}$ in terms of a complex scalar B as

$$
m_{\alpha\beta} \;\equiv\; (1-BB^*)^{-1}\begin{pmatrix} (1-B)(1-B^*) & i(B-B^*) \\ i(B-B^*) & (1+B)(1+B^*) \end{pmatrix}\;. \qquad (3.2.9)
$$

In terms of the complex combinations

$$
G_{\hat\mu\hat\nu\hat\rho} \;\equiv\; f(F_{\hat\mu\hat\nu\hat\rho}-BF^*_{\hat\mu\hat\nu\hat\rho})\;,\quad P_{\hat\mu}\;\equiv\; f^2\partial_{\hat\mu}B\;,\qquad \text{with}\;\; f=(1-BB^*)^{-1/2}\;, \qquad (3.2.10)
$$

charged under the $U(1)\subset SU(1,1)$, the kinetic terms of (3.2.6) translate into those of [78] with

$$
\begin{aligned}
m_{\alpha\beta}\hat{F}_{\hat\mu\hat\nu\hat\rho}{}^{\alpha}\hat{F}^{\hat\mu\hat\nu\hat\rho\,\beta} \;&=\; G^*_{\hat\mu\hat\nu\hat\rho}G^{\hat\mu\hat\nu\hat\rho}\;,\\
\frac{1}{4}\,\partial_{\hat\mu}m_{\alpha\beta}\partial^{\hat\mu}m^{\alpha\beta} \;&=\; -2\,P^*_{\hat\mu}P^{\hat\mu}\;.
\end{aligned}
\qquad (3.2.11)
$$

In the following, we will perform the standard $5+5$ Kaluza-Klein redefinitions of the IIB fields but keeping the dependence on all ten coordinates.

3.2.2 Kaluza-Klein decomposition and field redefinitions

We now split the the coordinates according to a $5+5$ Kaluza-Klein decomposition into

$$
x^{\hat\mu} = (x^\mu, y^m)\;, \qquad (3.2.12)
$$

and similarly for the flat indices $\hat{a} = (\underline{a}, \underline{\alpha})$. The μ and $\underline{a}$ indices range from $0, \ldots, 4$ and respectively represent the curved and flat indices of what we will refer to as the external space. Similarly, the indices m and $\underline{\alpha}$ range from $1, \ldots, 5$ and are associated with the internal space. After partial fixation of the Lorentz gauge symmetry, the vielbein may be brought into triangular form

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} \phi^{-1/3}\, e_\mu{}^{\underline{a}} & A_\mu{}^m \phi_m{}^{\underline{\alpha}} \\ 0 & \phi_m{}^{\underline{\alpha}} \end{pmatrix} , \tag{3.2.13}$$

parametrized in terms of two 5 by 5 matrices $e_\mu{}^{\underline{a}}$ and $\phi_m{}^{\underline{\alpha}}$ with $\phi \equiv \det(\phi_m{}^{\underline{\alpha}})$, and the Kaluza-Klein vectors $A_\mu{}^m$. We stress again that all fields depend on all ten coordinates, such that we are still describing the full IIB theory. We next perform an analogous decomposition of the remaining gauge symmetries, i.e., of the ten-dimensional diffeomorphisms $x^{\hat{\mu}} \to x^{\hat{\mu}} - \xi^{\hat{\mu}}$ and local Lorentz transformations parametrized by $\lambda^{\hat{a}}{}_{\hat{b}}$, acting on the vielbein as

$$\delta E_{\hat{\mu}}{}^{\hat{a}} \;=\; \xi^{\hat{\nu}} \partial_{\hat{\nu}} E_{\hat{\mu}}{}^{\hat{a}} + \partial_{\hat{\mu}} \xi^{\hat{\nu}} E_{\hat{\nu}}{}^{\hat{a}} + \lambda^{\hat{a}}{}_{\hat{b}} E_{\hat{\mu}}{}^{\hat{b}} \; . \tag{3.2.14}$$

Specifically, we decompose the diffeomorphism parameter as

$$\xi^{\hat{\mu}} \;=\; (\xi^\mu \, , \; \Lambda^m) \; , \tag{3.2.15}$$

and refer to the diffeomorphisms generated by ξ^μ as 'external' and those generated by Λ^m as 'internal'. Inserting (3.2.13) into (3.2.14) we read off the following action of the internal diffeomorphisms,

$$\begin{aligned} \delta_\Lambda e_\mu{}^a &= \Lambda^m \partial_m e_\mu{}^a + \frac{1}{3}\, \partial_m \Lambda^m \, e_\mu{}^a \; , \\ \delta_\Lambda \phi_m{}^\alpha &= \Lambda^n \partial_n \phi_m{}^\alpha + \partial_m \Lambda^n \, \phi_n{}^\alpha \; , \\ \delta_\Lambda A_\mu{}^m &= \partial_\mu \Lambda^m - A_\mu{}^n \partial_n \Lambda^m + \Lambda^n \partial_n A_\mu{}^m \; . \end{aligned} \tag{3.2.16}$$

We will also use the notation $\mathcal{L}_\Lambda$ for the conventional Lie derivative of the purely internal space, acting in the standard fashion on tensors (of weight zero). Thus, the above transformations read

$$\begin{aligned} \delta_\Lambda e_\mu{}^a &= \mathcal{L}_\Lambda e_\mu{}^a + \frac{1}{3}\, \partial_m \Lambda^m \, e_\mu{}^a \; , \qquad \delta_\Lambda \phi_m{}^\alpha \;=\; \mathcal{L}_\Lambda \phi_m{}^\alpha \; , \\ \delta_\Lambda A_\mu{}^m &= \partial_\mu \Lambda^m - \mathcal{L}_{A_\mu} \Lambda^m \;\equiv\; \partial_\mu \Lambda^m + \mathcal{L}_\Lambda A_\mu{}^m \; . \end{aligned} \tag{3.2.17}$$

Note that here we employ the convention in which the density term is not part of the Lie derivative. Analogously to the discussion in EFT, we can define derivatives and non-abelian field strengths that are covariant under these transformations,

$$\mathcal{D}_\mu^{\rm KK} \;\equiv\; \partial_\mu - \mathcal{L}_{A_\mu} - \lambda \, \partial_m A_\mu{}^m \; , \qquad F_{\mu\nu} \;\equiv\; 2\, \partial_{[\mu} A_{\nu]} - [A_\mu, A_\nu] \; , \tag{3.2.18}$$

where λ is the density weight, e.g., $\lambda = \frac{1}{3}$ for the external vielbein, and $[\;,\;]$ the conventional Lie bracket. Sometimes we will use the notation $D_\mu^{\rm KK} = \partial_\mu - \mathcal{L}_{A_\mu}$ for the part of the covariant derivative without the density term.[2] Specifically, for (3.2.16) we have

$$\begin{aligned} \mathcal{D}_\mu^{\rm KK} e_\nu{}^a &= \partial_\mu e_\nu{}^a - A_\mu{}^m \partial_m e_\nu{}^a - \frac{1}{3}\,\partial_n A_\mu{}^n\, e_\nu{}^a \;, \\ \mathcal{D}_\mu^{\rm KK} \phi_m{}^\alpha &= \partial_\mu \phi_m{}^\alpha - A_\mu{}^n \partial_n \phi_m{}^\alpha - \partial_m A_\mu{}^n \phi_n{}^\alpha \;, \\ F_{\mu\nu}{}^m &= \partial_\mu A_\nu{}^m - \partial_\nu A_\mu{}^m - A_\mu{}^n \partial_n A_\nu{}^m + A_\nu{}^n \partial_n A_\mu{}^m \;. \end{aligned} \tag{3.2.19}$$

Let us now turn to the external diffeomorphisms. These are obtained from (3.2.14) by inserting (3.2.13), switching on only the ξ^μ component, and adding the compensating Lorentz transformation with parameter $\lambda^a{}_\beta = -\phi^\gamma \phi_\beta{}^m \partial_m \xi^\nu\, e_\nu{}^a$, which is necessary in order to preserve the gauge choice in (3.2.13). For instance, on the Kaluza-Klein vectors this yields

$$\delta_\xi A_\mu{}^m \;=\; \xi^\nu \partial_\nu A_\mu{}^m + \partial_\mu \xi^\nu A_\nu{}^m - A_\mu{}^n \partial_n \xi^\nu A_\nu{}^m + \phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu} \partial_n \xi^\nu \;, \tag{3.2.20}$$

where $G^{mn} \equiv \phi_\alpha{}^m \phi^{\alpha n}$. This gauge transformation can more conveniently be written in the form of 'improved' or 'covariant' diffeomorphisms by adding an *internal* diffeomorphism (3.2.16) with field-dependent parameter $\Lambda^m = -\xi^\nu A_\nu{}^m$. The gauge-field-dependent terms then organize into the covariant field strength in (3.2.19),

$$\delta_\xi A_\mu{}^m \;=\; \xi^\nu F_{\nu\mu}{}^m + \phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu} \partial_n \xi^\nu \;. \tag{3.2.21}$$

We infer that this is of the same structural form as the external diffeomorphism transformation of the EFT gauge vector in (3.1.9), and we may already verify that they can be matched precisely upon picking the type IIB solution of the section constraint. Indeed, the external diffeomorphism variation of the EFT vector field is

$$\delta_\xi \mathcal{A}_\mu{}^m \;=\; \xi^\nu \mathcal{F}_{\nu\mu}{}^m + e^{4\Phi/3}\, m^{mn} g_{\mu\nu} \partial_n \xi^\nu \;, \tag{3.2.22}$$

where we have used (3.1.43) in (3.1.9) . We see the field strength components $\mathcal{F}_{\mu\nu}{}^m$ reduce to the Kaluza-Klein components $F_{\mu\nu}{}^m$, see (3.1.19) and (3.2.19), and the metric-dependent terms coincide upon identifying

$$e^{4\Phi/3}\, m^{mn} \;=\; \phi^{-2/3} G^{mn} \;, \tag{3.2.23}$$

which relates the matrix $m^{mn} \in \mathrm{SL}(5)$ and the scale factor Φ to the metric G^{mn} with dynamical determinant ϕ^2. (This relation can be fixed, for instance, by noting with (3.1.37) that both sides transform in the same way under internal diffeomorphisms.) The precise match for the remaining vector field components will be the subject of the following section.

[2] We emphasize that this is introduced for purely notational convenience. In general, acting with $D_\mu^{\rm KK}$ is not a covariant operation.

Similarly, these improved external diffeomorphisms act on the internal and external vielbein as

$$\begin{aligned} \delta_\xi e_\mu{}^a &= \xi^\nu \mathcal{D}^{\rm KK}_\nu e_\mu{}^a + \mathcal{D}^{\rm KK}_\mu \xi^\nu\, e_\nu{}^a \,, \\ \delta_\xi \phi_m{}^\alpha &= \xi^\nu \mathcal{D}^{\rm KK}_\nu \phi_m{}^\alpha \,, \end{aligned} \tag{3.2.24}$$

again in structural agreement with the corresponding transformations (3.1.9) in EFT.

We now move on to the Kaluza-Klein decomposition of the p-forms. We introduce in standard Kaluza-Klein manner the projector $P_\mu{}^{\hat\nu} = E_\mu{}^{\underline{a}} E_{\underline{a}}{}^{\hat\nu}$. It converts 10-dimensional curved indices into 5-dimensional ones such that the resulting fields transform covariantly (i.e. according to the structure of their internal indices) under internal diffeomorphisms. We denote its action by a bar on the corresponding p-form components,

$$\overline{C}_\mu \;\equiv\; P_\mu{}^{\hat\nu}\, \hat C_{\hat\nu} \,, \qquad \text{etc.} \,, \tag{3.2.25}$$

such that the IIB two- and four-form give rise to the components

$$\begin{aligned} \overline{C}_{mn}{}^\alpha &= \hat C_{mn}{}^\alpha \,, \\ \overline{C}_{\mu m}{}^\alpha &= \hat C_{\mu m}{}^\alpha - A_\mu{}^p \hat C_{pm}{}^\alpha \,, \\ \overline{C}_{\mu\nu}{}^\alpha &= \hat C_{\mu\nu}{}^\alpha - 2A_{[\mu}{}^p \hat C_{|p|\nu]}{}^\alpha + A_\mu{}^p A_\nu{}^q \hat C_{pq}{}^\alpha \,, \\ \overline{C}_{mnkl} &= \hat C_{mnkl} \,, \\ \overline{C}_{\mu nkl} &= \hat C_{\mu nkl} - A_\mu{}^p \hat C_{pnkl} \,, \\ \overline{C}_{\mu\nu kl} &= \hat C_{\mu\nu kl} - 2A_{[\mu}{}^p \hat C_{|p|\nu]kl} + A_\mu{}^p A_\nu{}^q \hat C_{pqkl} \,, \\ \overline{C}_{\mu\nu\rho l} &= \hat C_{\mu\nu\rho l} - 3A_{[\mu}{}^p \hat C_{|p|\nu\rho]\,l} + 3A_{[\mu}{}^p A_\nu{}^q \hat C_{|pq|\rho]\,l} - A_\mu{}^p A_\nu{}^q A_\rho{}^r \hat C_{pqrl} \,, \\ \overline{C}_{\mu\nu\rho\sigma} &= \hat C_{\mu\nu\rho\sigma} - 4A_{[\mu}{}^p \hat C_{|p|\nu\rho\sigma]} + 6A_{[\mu}{}^p A_\nu{}^q \hat C_{|pq|\rho\sigma]} - 4A_{[\mu}{}^p A_\nu{}^q A_\rho{}^r \hat C_{|pqr|\sigma]} \\ &\quad + A_\mu{}^p A_\nu{}^q A_\rho{}^r A_\sigma{}^s \hat C_{pqrs} \,. \end{aligned} \tag{3.2.26}$$

The same redefinition applies to field strengths and gauge parameters. The redefined fields now transform covariantly under internal diffeomorphisms. Indeed, separating ten-dimensional diffeomorphisms into $\xi^{\hat\mu} = (\xi^\mu, \Lambda^m)$, we find together with (3.2.5)

$$\begin{aligned} \delta\overline{C}_{mn}{}^\alpha &= 2\partial_{[m}\overline{\lambda}_{n]}{}^\alpha + \mathcal{L}_\Lambda \overline{C}_{mn}{}^\alpha \,, \\ \delta\overline{C}_{\mu m}{}^\alpha &= D^{\rm KK}_\mu \overline{\lambda}_m{}^\alpha - \partial_m \overline{\lambda}_\mu{}^\alpha + \mathcal{L}_\Lambda \overline{C}_{\mu m}{}^\alpha \,, \\ \delta\overline{C}_{\mu\nu}{}^\alpha &= 2\, D^{\rm KK}_{[\mu}\, \overline{\lambda}_{\nu]}{}^\alpha + F_{\mu\nu}{}^k \overline{\lambda}_k{}^\alpha + \mathcal{L}_\Lambda \overline{C}_{\mu\nu}{}^\alpha \,, \end{aligned} \tag{3.2.27}$$

for the transformation behaviour of the redefined 2-forms under gauge transformations and internal diffeomorphisms. As in the previous section, derivatives $D^{\rm KK}_\mu$ are covariantized w.r.t. the action of internal diffeomorphisms, i.e.

$$D^{\rm KK}_\mu \overline{\lambda}_m{}^\alpha \;\equiv\; \partial_\mu \overline{\lambda}_m{}^\alpha - A_\mu{}^n \partial_n \overline{\lambda}_m{}^\alpha - \partial_m A_\mu{}^n \overline{\lambda}_n{}^\alpha \,, \qquad \text{etc.}\,. \tag{3.2.28}$$

In contrast to $D=11$ supergravity for which these redefinitions and covariant gauge transformations have been explicitly worked out in [33], the presence of Chern-Simons terms in the IIB field strengths (3.2.3) requires a further redefinition for the components of the 4-form in order to establish the dictionary to the fields of EFT. This is related to the fact that tensor gauge transformations for the EFT p-forms that we have discussed in the previous section do not mix these forms with the scalar fields of the theory. This motivates the following and final field redefinition[3]

$$
\begin{aligned}
C_{klmn} &\equiv \overline{C}_{klmn} \,, \\
C_{\mu\,kmn} &\equiv \overline{C}_{\mu\,kmn} - \frac{3}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{\mu\,[k}{}^{\alpha}\overline{C}_{mn]}{}^{\beta} \,, \\
C_{\mu\nu\,mn} &\equiv \overline{C}_{\mu\nu\,mn} - \frac{1}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{\mu\nu}{}^{\alpha}\overline{C}_{mn}{}^{\beta} \,, \\
C_{\mu\nu\rho\,m} &\equiv \overline{C}_{\mu\nu\rho\,m} - \frac{3}{8}\,\varepsilon_{\alpha\beta}\overline{C}_{[\mu\nu}{}^{\alpha}\overline{C}_{\rho]\,m}{}^{\beta} \,, \\
C_{\mu\nu\rho\sigma} &\equiv \overline{C}_{\mu\nu\rho\sigma} \,.
\end{aligned}
\tag{3.2.29}
$$

For the components of the two-form $\overline{C}_{\mu\nu}{}^{\alpha}$, etc., there is no further redefinition, so for simplicity of the notation, we will simply drop their bars in the following

$$
C_{mn}{}^{\alpha} \equiv \overline{C}_{mn}{}^{\alpha} \,, \qquad C_{\mu m}{}^{\alpha} \equiv \overline{C}_{\mu m}{}^{\alpha} \,, \qquad C_{\mu\nu}{}^{\alpha} \equiv \overline{C}_{\mu\nu}{}^{\alpha} \,. \tag{3.2.30}
$$

Although we have not seen the 3-form and the 4-form in the tensor hierarchy of the $\mathrm{E}_{6(6)}$ EFT, we will show later that it is possible to test their expressions by comparing the reduced $D=10$ self duality equations (3.2.7) to the first order duality equations (3.1.8) from EFT. The redefined 4-forms (3.2.29) continue to transform covariantly under internal diffeomorphisms with their total gauge transformations given by

$$
\begin{aligned}
\delta C_{mnkl} &= 4\partial_{[m}\overline{\lambda}_{nkl]} + \frac{3}{2}\,\epsilon_{\alpha\beta}\partial_{[m}\overline{\lambda}_{n}C_{kl]}{}^{\beta} + \mathcal{L}_{\Lambda}C_{mnkl} \,, \\
\delta C_{\mu\,kmn} &= D_{\mu}^{\mathrm{KK}}\overline{\lambda}_{kmn} - 3\partial_{[k}\overline{\lambda}_{|\mu|mn]} + \mathcal{L}_{\Lambda}C_{\mu\,kmn} \\
&\quad + \frac{3}{4}\,\varepsilon_{\alpha\beta}\left(\overline{\lambda}_{[k}{}^{\alpha}\partial_{m}C_{|\mu|n]}{}^{\beta} - \partial_{[m}\overline{\lambda}_{k}{}^{\alpha}C_{|\mu|n]}{}^{\beta}\right) \,, \\
\delta C_{\mu\nu\,mn} &= 2\,D_{[\mu}^{\mathrm{KK}}\overline{\lambda}_{\nu]\,mn} + 2\partial_{[m}\overline{\lambda}_{n]\mu\nu} + F_{\mu\nu}{}^{k}\overline{\lambda}_{kmn} + \mathcal{L}_{\Lambda}C_{\mu\nu mn} \\
&\quad + \frac{1}{4}\,\varepsilon_{\alpha\beta}\left(-2\partial_{[m}C_{|\mu|n]}{}^{\alpha}\overline{\lambda}_{\nu}{}^{\beta} + F_{\mu\nu\,[m}{}^{\alpha}\overline{\lambda}_{n]}{}^{\beta} - \overline{\lambda}_{[m}{}^{\alpha}\partial_{n]}C_{\mu\nu}{}^{\beta}\right) \,.
\end{aligned}
\tag{3.2.31}
$$

We see that after the redefinitions (3.2.29), the variation of $\delta C_{\mu\,kmn}$ and $\delta C_{\mu\nu\,mn}$ no longer carry any scalar fields $\overline{C}_{mn}{}^{\alpha}$ and are thus of the form to be matched with the fields and

[3] Similar redefinitions have been discussed in [80] in order to recover part of the $\mathrm{E}_{6(6)}$ tensor hierarchy structure from the IIB supersymmetry variations.

transformations of EFT. The field strengths appearing on the r.h.s. of (3.2.31) are the Kaluza-Klein field strength (3.2.18) and the modified three-form field strength

$$\begin{aligned} F_{\mu\nu\, n}{}^{\alpha} &\equiv \overline{F}_{\mu\nu\, n}{}^{\alpha} - F_{\mu\nu}{}^{k} C_{kn}{}^{\alpha} \,, \\ &= 2D_{[\mu} C_{\nu]m}{}^{\alpha} + \partial_m C_{\mu\nu}{}^{\alpha} \,, \end{aligned} \tag{3.2.32}$$

again redefined such that the scalar contribution is split off. For completeness we also give the remaining components of the three-form field strength

$$\begin{aligned} F_{kmn}{}^{\alpha} &\equiv \overline{F}_{kmn}{}^{\alpha} = 3\partial_{[k} C_{mn]}{}^{\alpha} \,, \\ F_{\mu\, mn}{}^{\alpha} &\equiv \overline{F}_{\mu\, mn}{}^{\alpha} = D^{\rm KK}_{\mu} C_{mn}{}^{\alpha} - 2\partial_{[m} C_{|\mu|\, n]}{}^{\alpha} \,, \\ F_{\mu\nu\rho}{}^{\alpha} &\equiv \overline{F}_{\mu\nu\rho}{}^{\alpha} = 3\, D^{\rm KK}_{[\mu} C_{\nu\rho]}{}^{\alpha} - 3\, F_{[\mu\nu}{}^{k} C_{\rho]k}{}^{\alpha} \,, \end{aligned} \tag{3.2.33}$$

as well as the properly redefined components of the five-form field strength, expressed in terms of the components (3.2.29) according to

$$\begin{aligned} F_{mpqrs} &\equiv \overline{F}_{mpqrs} = 5\,\partial_{[m} C_{pqrs]} - \frac{5}{4}\,\varepsilon_{\alpha\beta}\, C_{[mp}{}^{\alpha} \overline{F}_{qrs]}{}^{\beta} \,, \\ F_{\mu pqrs} &\equiv \overline{F}_{\mu pqrs} \\ &= D^{\rm KK}_{\mu} C_{pqrs} - 4\partial_{[p} C_{|\mu|qrs]} - \frac{3}{4}\,\varepsilon_{\alpha\beta} C_{[pq}{}^{\alpha} F_{|\mu|rs]}{}^{\beta} + \frac{3}{2}\,\varepsilon_{\alpha\beta} C_{[pq}{}^{\alpha} \partial_r C_{|\mu|s]}{}^{\beta} \,, \\ F_{\mu\nu kmn} &\equiv \overline{F}_{\mu\nu kmn} - \frac{3}{4}\,\varepsilon_{\alpha\beta} F_{\mu\nu[k}{}^{\alpha} C_{mn]}{}^{\beta} - F_{\mu\nu}{}^{p} (\overline{C}_{pkmn} - \frac{3}{8}\,\varepsilon_{\alpha\beta} C_{[km}{}^{\alpha} C_{|p|n]}{}^{\beta}) \\ &= 2\, D^{\rm KK}_{[\mu} C_{\nu]kmn} + 3\partial_{[k} C_{|\mu\nu|mn]} - \frac{3}{2}\,\varepsilon_{\alpha\beta} C_{\mu[k}{}^{\alpha} \partial_m C_{|\nu|n]}{}^{\beta} \,, \\ F_{\mu\nu\rho mn} &\equiv \overline{F}_{\mu\nu\rho mn} - \frac{1}{4} \varepsilon_{\alpha\beta} \overline{F}_{\mu\nu\rho}{}^{\alpha} C_{mn}{}^{\beta} \\ &= 3D^{\rm KK}_{[\mu} C_{\nu\rho]mn} - 2\partial_{[m} C_{|\mu\nu\rho|n]} - 3F_{[\mu\nu}{}^{k} C_{\rho]kmn} \\ &\quad - \frac{3}{2}\,\varepsilon_{\alpha\beta} (\partial_{[m} C_{[\mu\nu}{}^{\alpha} C_{\rho]n]}{}^{\beta} + C_{[\mu|m|}{}^{\alpha} D_{\nu} C_{\rho]n}{}^{\beta}) \,, \\ F_{\mu\nu\rho\sigma m} &\equiv \overline{F}_{\mu\nu\rho\sigma m} \\ &= 4D^{\rm KK}_{[\mu} C_{\nu\rho\sigma]m} + \partial_m C_{\mu\nu\rho\sigma} + 6F_{[\mu\nu}{}^{p} C_{\rho\sigma]pm} \\ &\quad + \frac{3}{2}\,\varepsilon_{\alpha\beta} F_{[\mu\nu}{}^{k} C_{\rho|m|}{}^{\alpha} C_{\sigma]k}{}^{\beta} - \frac{3}{4}\,\varepsilon_{\alpha\beta} C_{[\mu\nu}{}^{\alpha} \partial_{|m|} C_{\rho\sigma]}{}^{\beta} + \varepsilon_{\alpha\beta}\, C_{\mu\, m}{}^{\alpha} \mathcal{F}_{\nu\rho\sigma}{}^{\beta} \,, \\ F_{\mu\nu\rho\sigma\tau} &\equiv \overline{F}_{\mu\nu\rho\sigma\tau} = 5D^{\rm KK}_{[\mu} C_{\nu\rho\sigma\tau]} - 10F_{[\mu\nu}{}^{m} C_{\rho\sigma\tau]m} - \frac{15}{4}\,\varepsilon_{\alpha\beta} C_{[\mu\nu}{}^{\alpha} D^{\rm KK}_{\rho} C_{\sigma\tau]}{}^{\beta} \end{aligned} \tag{3.2.34}$$

3.2.3 External diffeomorphisms

In the previous subsection we have decomposed the IIB fields according to a 5+5 Kaluza-Klein split (without giving up the dependency on the 5 internal coordinates) and spelled out their transformations under internal diffeomorphisms and tensor gauge transformations after suitable redefinitions of the various components. Before fully establishing the dictionary of the fields in the EFT basis, we will now compute the behaviour of the redefined IIB fields under external diffeomorphisms ξ^μ, whose parameter may in general also depend on all 10 coordinates.

Above, we have already discussed the transformation of the KK vector fields under external diffeomorphisms

$$\delta_\xi^{\rm cov} A_\mu{}^m \;=\; \xi^\nu F_{\nu\mu}{}^m + \phi^{-\frac{2}{3}} G^{mn} g_{\mu\nu}\partial_n \xi^\nu \;, \tag{3.2.35}$$

c.f. (3.2.21), which is in agreement with the EFT gauge vector transformations reduced to this component. Let us now test the remaining vector components from the IIB p-forms. For $C_{\mu m}{}^\alpha$, as redefined in (3.2.26), a straightforward calculation gives

$$\begin{aligned}\delta_\xi C_{\mu m}{}^\alpha \;&=\; \mathcal{L}_\xi C_{\mu m}{}^\alpha - \phi^{-\frac{2}{3}} G^{nk} C_{nm}{}^\alpha g_{\mu\nu}\partial_k \xi^\nu \\ &\qquad + \partial_m \xi^\nu A_\nu{}^n C_{\mu n}{}^\alpha - A_\mu{}^n \partial_n \xi^\nu C_{\nu m}{}^\alpha + \partial_m \xi^\nu C_{\mu\nu}{}^\alpha \;,\end{aligned} \tag{3.2.36}$$

under external diffeomorphisms. The origin of the second term is the corresponding variation of the Kaluza-Klein vector (3.2.35) which enters the redefined fields in (3.2.26). As for the Kaluza-Klein vector field, it follows that the last three terms are eliminated by field dependent gauge transformations with parameters (parameter redefinition)

$$\Lambda^m \;=\; -\xi^\nu A_\nu{}^m \;, \quad \overline{\lambda}_m{}^\alpha \;=\; -\xi^\nu C_{\nu m}{}^\alpha \;, \quad \overline{\lambda}_\mu{}^\alpha \;=\; -\xi^\nu C_{\nu\mu}{}^\alpha \;, \tag{3.2.37}$$

which render the action of the diffeomorphism manifestly gauge covariant. Together, the variation takes the form

$$\delta_\xi^{\rm cov} C_{\mu m}{}^\alpha \;=\; \xi^\nu F_{\nu\mu m}{}^\alpha - \phi^{-\frac{2}{3}} G^{nk} C_{nm}{}^\alpha g_{\mu\nu}\partial_k \xi^\nu \;. \tag{3.2.38}$$

Note in particular that the field strength entering this formula is the one defined in (3.2.32) which does not carry any scalar contributions. This is the form of the variation that we will be able to match with the corresponding variation for the fields in the EFT basis.

Next let us consider the variation of the 4-form component $C_{\mu mnk}$. After standard Kaluza-Klein redefinition (3.2.26), some straightforward calculation yields

$$\begin{aligned}\delta_\xi \overline{C}_{\mu mnk} \;&=\; \xi^\nu \left(2 D^{\rm KK}_{[\nu} \overline{C}_{\mu] mnk} + 3\partial_{[m} \overline{C}_{|\nu\mu| nk]}\right) + \mathcal{L}_{\xi^\nu A_\nu} \overline{C}_{\mu mnk} \\ &\qquad + D^{\rm KK}_\mu (\xi^\nu \overline{C}_{\nu mnk}) - 3\,\partial_{[m}(\xi^\nu \overline{C}_{|\nu\mu| nk]}) + \phi^{-\frac{2}{3}} G^{lp}\, C_{mnkl}\, g_{\mu\nu}\partial_p \xi^\nu \;,\end{aligned} \tag{3.2.39}$$

for the variation under external diffeomorphisms in terms of the redefined fields. In the first term we recognize the covariant field strength $F_{\nu\mu mnk}$ from (3.2.34) up to its bilinear contributions. These will be completed once we consider the variation of the redefined four form

$$\delta_\xi C_{\mu mnk} \;=\; \delta_\xi \overline{C}_{\mu mnk} - \frac{3}{8}\,\varepsilon_{\alpha\beta}\delta_\xi C_{\mu[m}{}^\alpha C_{nk]}{}^\beta - \frac{3}{8}\,\varepsilon_{\alpha\beta} C_{\mu[m}{}^\alpha \delta_\xi C_{nk]}{}^\beta \;, \tag{3.2.40}$$

with the second term obtained via (3.2.38), and the third term carrying

$$\delta_\xi C_{mn}{}^{\alpha} \;=\; \xi^\nu F_{\nu mn}{}^{\alpha} + 2\partial_{[m}(\xi^\nu C_{|\nu|n]}{}^{\alpha}) + \mathcal{L}_{\xi^\nu A_\nu} C_{mn}{}^{\alpha} \,. \tag{3.2.41}$$

Combining all these contributions and supplementing the variation by the gauge transformations with parameters (3.2.37), we arrive at the final form

$$\delta_\xi^{\rm cov} C_{\mu mnk} \;=\; \xi^\nu\, F_{\nu\mu mnk} + \phi^{-\frac{2}{3}} G^{lp} \left(C_{mnkl} + \frac{3}{8}\,\varepsilon_{\alpha\beta} C_{l[m}{}^{\alpha} C_{nk]}{}^{\beta} \right) g_{\mu\nu}\partial_p \xi^\nu \,. \tag{3.2.42}$$

In the next section, we will provide the complete dictionary between the Kaluza-Klein redefined fields of type IIB supergravity and the fundamental fields in the $E_{6(6)}$ EFT. In particular, matching the EFT equations against the IIB self-duality equations (3.2.7), we will explicitly reconstruct the remaining 4-form components $C_{\mu\nu\rho m}$, $C_{\mu\nu\rho\sigma}$.

3.3 General embedding of type IIB into $\mathrm{E}_{6(6)}$ EFT and self-duality relations

In this section, we provide an explicit dictionary between the Kaluza-Klein redefined fields of type IIB supergravity and those of the $E_{6(6)}$ exceptional field theory after picking solution (3.1.16) of the section constraint. We first show that the fundamental EFT fields can be identified among the redefined IIB fields on a pure kinematical level by comparing the transformation behaviour under diffeomorphisms and gauge transformations. We then show that the equivalence also holds on the dynamical level by reproducing the IIB self-duality equations (3.2.7) from the EFT field equations. In particular, this will allow us to obtain explicit expressions for the remaining 4-form components $C_{\mu\nu\rho m}$, $C_{\mu\nu\rho\sigma}$ which do not show up among the fundamental EFT fields, but whose existence follows from the EFT dynamics.

3.3.1 Kinematics

Before identifying the details of the IIB embedding, let us first revisit the resulting field content of EFT after picking solution (3.1.16) of the section constraint. With the split (3.1.12), (3.1.13), the full p-form field content of the $\mathrm{E}_{6(6)}$ Lagrangian in this basis is given by (3.1.17)

$$\{\mathcal{A}_\mu{}^m, \mathcal{A}_{\mu\, m\,\alpha}, \mathcal{A}_{\mu\, kmn}, \mathcal{A}_{\mu\,\alpha}\} \;, \qquad \{\mathcal{B}_{\mu\nu}{}^{\alpha}, \mathcal{B}_{\mu\nu\, mn}, \mathcal{B}_{\mu\nu}{}^{m\,\alpha}\} \;, \tag{3.3.1}$$

where, more precisely, the Lagrangian depends on the 2-forms only under certain contractions with internal derivatives, c.f. (3.1.18). The EFT scalar sector is described by the fields parametrizing the $\mathrm{E}_{6(6)}$ generalised metric $\mathcal{M}_{MN}$ (4.5.3)

$$\{\Phi, m_{mn}, m_{\alpha\beta}, b_{mn}{}^{\alpha}, c_{klmn}\} \,. \tag{3.3.2}$$

Comparing the index structure of these fields to the field content of the Kaluza-Klein decomposition of IIB supergravity given in the previous section allows to give a first qualitative correspondence between the two formulations. With the discussion of the previous section in mind, it appears natural to relate the field $\mathcal{A}_\mu{}^m$ to the IIB Kaluza-Klein vector field $A_\mu{}^m$, and the scalars Φ, m_{mn}, to the remaining components of the internal IIB metric (3.2.13).

According to their index structure, the fields $\{b_{mn}{}^\alpha, \mathcal{A}_{\mu\, m\,\alpha}, \mathcal{B}_{\mu\nu}{}^\alpha\}$ from (3.3.1), (3.3.2) will relate to the different components of the SL(2) doublet of ten-dimensional two-forms. Similarly the fields $c_{klmn}, \mathcal{A}_{\mu\, kmn}, \mathcal{B}_{\mu\nu\, mn}$ will translate into the components of the (self-dual) IIB four-form. The remaining fields $\mathcal{A}_{\mu\,\alpha}, \mathcal{B}_{\mu\nu}{}^{m\,\alpha}$ descend from components of the doublet of dual six-forms. The two-form tensors $\mathcal{B}_{\mu\nu\, m}$ that complete the two-forms in (3.3.1) into the full **27** $\mathcal{B}_{\mu\nu\, M}$ of $\mathrm{E}_{6(6)}$ do not figure in the $\mathrm{E}_{6(6)}$ covariant Lagrangian. They represent the degrees of freedom on-shell dual to the Kaluza-Klein vector fields, i.e. descending from the ten-dimensional dual graviton.

Recall that in the EFT formulation, all vector fields in (3.3.1) appear with a Yang-Mills kinetic term whereas the two-forms couple via a topological term and are on-shell dual to the vector fields. In order to match the structure of IIB supergravity, we will thus have to trade the Yang-Mills vector fields $\mathcal{A}_{\mu\,\alpha}$ for a propagating two-form $\mathcal{B}_{\mu\nu}{}^\alpha$. Let us make this more explicit. The α-component of the EFT duality equations (3.1.8) yields

$$e\,\mathcal{M}^{\alpha\beta}\,\mathcal{F}^{\mu\nu}{}_\beta \;=\; -\frac{1}{6}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\tilde{\mathcal{H}}_{\rho\sigma\tau}{}^\alpha - e\,\mathcal{M}^\alpha{}_{\underline{M}}\,\mathcal{F}^{\mu\nu\,\underline{M}}\;, \qquad (3.3.3)$$

where we have introduced the index split

$$\{X^M\} \;\longrightarrow\; \{X^{\underline{M}}, X_\alpha\}\;. \qquad (3.3.4)$$

With the two-form fields $\tilde{\mathcal{B}}_{\mu\nu}{}^{k\beta}$ entering $\mathcal{F}_{\mu\nu\,\beta}$ on the l.h.s. of (3.3.3), this duality equation then allows to eliminate all $\tilde{\mathcal{B}}_{\mu\nu}{}^{k\beta}$ from the Lagrangian. The gauge symmetry (3.1.23) shows that in the process, the vector fields $\mathcal{A}_{\mu\,\alpha}$ also disappear from the Lagrangian.[4] We infer from (3.3.3) that the kinetic term for the remaining vector fields changes into the form

$$e^{-1}\,\mathcal{L}_{\rm kin,1} \;=\; -\frac{1}{4}\,\mathcal{F}_{\mu\nu}{}^M\mathcal{F}^{\mu\nu N}\tilde{\mathcal{M}}_{MN}\;, \qquad (3.3.6)$$

with $\tilde{\mathcal{M}}_{MN}$ from (3.1.34). At the same time, the two-forms $\tilde{B}_{\mu\nu}{}^\alpha$ are promoted into propagating fields with kinetic term

$$e^{-1}\,\mathcal{L}_{\rm kin,2} \;=\; -\frac{1}{12}\,e^{-5\Phi/3}\,m_{\alpha\beta}\,\tilde{\mathcal{H}}_{\mu\nu\rho}{}^\alpha\tilde{\mathcal{H}}^{\mu\nu\rho\,\beta}\;. \qquad (3.3.7)$$

[4] Strictly speaking, equation (3.3.3) only holds up to an x-dependent 'integration constant' $\mathcal{C}^{\mu\nu\,\alpha}(x)$, since it enters under y-derivative. To fix this freedom, we have to combine the equation with the vector field equations,

$$D_\nu\left(e\,\mathcal{M}^\alpha{}_M\,\mathcal{F}^{\nu\mu\,M}\right) \;=\; \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{\nu\rho}{}^k\mathcal{F}_{\sigma\tau\,k\beta}\;, \qquad (3.3.5)$$

and the Bianchi identity (3.1.27), leaving us with $D_\mu\mathcal{C}^{\mu\nu\,\alpha} = 0$. In the following we will directly set $\mathcal{C}^{\mu\nu\,\alpha} = 0$.

After this dualization, the remaining field content thus is given by

$$\{\Phi, m_{mn}, b_{mn}{}^{\alpha}, c_{klmn}, \mathcal{A}_{\mu}{}^{m}, \mathcal{A}_{\mu\, m\,\alpha}, \mathcal{A}_{\mu\, kmn}, \mathcal{B}_{\mu\nu}{}^{\alpha}, \mathcal{B}_{\mu\nu\, mn}\} \ , \tag{3.3.8}$$

with all except for the last field representing propagating degrees of freedom. In contrast, the two-form $\mathcal{B}_{\mu\nu\, mn}$ is related by a first order duality equation (3.1.8) to $\mathcal{A}_{\mu\, kmn}$, remnant of the IIB self-duality equations (3.2.7). In the following, we will make the dictionary fully explicit.

3.3.2 Dictionary and match of gauge symmetries

Having established the match of degrees of freedom between IIB supergravity and EFT upon choosing the IIB solution of the section condition, we can now make the map more precise by inspecting the gauge and diffeomorphism transformations on both sides. After Kaluza-Klein decomposition and redefinition of the IIB fields, as described in section 3.2.2, the resulting components turn out to be proportional to the EFT fields in their decomposition given in section 3.1.3 above. Specifically, comparing the variation of the EFT vector and two-form fields (3.1.21), (3.1.24), to the corresponding transformations in (3.2.27), (3.2.31), allows us to establish the dictionary

$$\begin{aligned} &A_{\mu}{}^{m} = \mathcal{A}_{\mu}{}^{m} \ , \qquad C_{\mu m}{}^{\alpha} = -\varepsilon^{\alpha\beta}\mathcal{A}_{\mu\, m\beta} \ , \qquad C_{\mu\nu}{}^{\alpha} = \tilde{\mathcal{B}}_{\mu\nu}{}^{\alpha} \ , \\ &C_{\mu\nu\, mn} = \frac{\sqrt{2}}{4}\,\tilde{\mathcal{B}}_{\mu\nu\, mn} \ , \qquad C_{\mu kmn} = \frac{\sqrt{2}}{4}\,\mathcal{A}_{\mu\, kmn} = \frac{\sqrt{2}}{8}\,\varepsilon_{mnkpq}\,\mathcal{A}_{\mu}{}^{pq} \ , \end{aligned} \tag{3.3.9}$$

respectively. The corresponding gauge parameters translate with the same proportionality factors, and also the redefined IIB field strengths (3.2.32), (3.2.34) precisely translate into the EFT analogues

$$F_{\mu\nu}{}^{m} = \mathcal{F}_{\mu\nu}{}^{m} \ , \qquad F_{\mu\nu\, m}{}^{\alpha} = -\varepsilon^{\alpha\beta}\mathcal{F}_{\mu\nu\, m\beta} \ , \qquad F_{\mu\nu\, kmn} = \frac{\sqrt{2}}{4}\,\mathcal{F}_{\mu\nu\, kmn} \ . \tag{3.3.10}$$

This dictionary may be further confirmed upon comparing the action of external diffeomorphisms on both sides. Indeed, the variations calculated in (3.2.35), (3.2.38), (3.2.42) above, precisely reproduce the EFT transformation law (3.1.9) for the vectors $\mathcal{A}_{\mu}{}^{M}$, provided we identify the components of the scalar matrix $\mathcal{M}^{MN}$ (3.1.43) with the IIB fields according to

$$\phi^{-\frac{2}{3}}G^{mn} \ = \ e^{4\Phi/3}m^{mn} \ , \qquad C_{mn}{}^{\alpha} \ = \ -2b_{mn}{}^{\alpha} \ , \qquad C_{mnkl} = -\frac{1}{4}\,c_{mnkl} \ . \tag{3.3.11}$$

This last identification is precisely compatible with the gauge transformation behaviour (3.1.37) as compared to the scalar components of (3.2.27), (3.2.31). Let us also note, that with this dictionary the EFT covariant derivatives (3.1.36) for the scalar fields precisely translate into the components of the IIB field strengths

$$\begin{aligned} \mathcal{D}_{\mu}b_{mn}{}^{\alpha} \ &= \ -\frac{1}{2}\,\overline{F}_{\mu mn}{}^{\alpha} \ , \\ \widehat{\mathcal{D}}_{\mu}c_{klmn} \ &= \ -4\,\overline{F}_{\mu klmn} \ , \end{aligned} \tag{3.3.12}$$

with $\widehat{\mathcal{D}}_\mu c_{klmn}$ from (3.1.39). Similarly, we have the identification

$$\partial_{[k}c_{lmnp]} + 12\,\varepsilon_{\alpha\beta}\,b_{[kl}{}^{\alpha}\partial_m b_{np]}{}^{\beta} \;=\; X_{klmnp} \;=\; -\frac{4}{5}\,\overline{F}_{klmnp}\,, \qquad (3.3.13)$$

with X_{klmnp} from (3.1.41).

We have thus identified the elementary EFT fields among the Kaluza-Klein components of the IIB fields. So far, the identification has been solely based on the matching of gauge symmetries on both sides. We will in the following show that the embedding of IIB into EFT also holds dynamically on the level of the equations of motion.

3.3.3 Dynamics and reconstruction of 3- and 4-forms

In this section, we will show how the full IIB self-duality equations (3.2.7) follow from the EFT dynamics. Along the way, we will establish explicit expressions for the remaining components of the ten-dimensional 4-form, thereby completing the explicit embedding of the IIB theory. To begin with, it is useful to first rewrite the various components of the IIB self-duality equations in terms of the Kaluza-Klein decomposed fields introduced in section 3.2.2 above. With the IIB metric (3.2.13) given in term of the EFT fields as

$$G_{\hat\mu\hat\nu} \;=\; \begin{pmatrix} e^{5\Phi/6}\,g_{\mu\nu} + \mathcal{A}_\mu{}^m\mathcal{A}_\nu{}^n\,\phi_{mn} & e^{-\Phi/2}\,m_{kn}\,\mathcal{A}_\mu{}^k \\ e^{-\Phi/2}\,m_{mk}\,\mathcal{A}_\nu{}^k & e^{-\Phi/2}\,m_{mn} \end{pmatrix}\,, \qquad (3.3.14)$$

the IIB self-duality equations (3.2.7) split into the following three components

$$\overline{F}_{\mu\nu\rho\,mn} \;=\; \frac{1}{12}\,e^{2\Phi/3}\,\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}\,\overline{F}^{\sigma\tau}{}_{qrs}\,m^{kq}m^{lr}m^{pq}\,, \qquad (3.3.15)$$

$$\overline{F}_{\mu\nu\rho\sigma\,m} \;=\; -\frac{1}{24}\,e^{2\Phi}\,\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau}m_{mn}\varepsilon^{nklpq}\,\overline{F}^{\tau}{}_{klpq}\,, \qquad (3.3.16)$$

$$\overline{F}_{\mu\nu\rho\sigma\tau} \;=\; \frac{1}{120}\,e^{10\Phi/3}\,\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{mnklp}\,\overline{F}_{mnklp}\,. \qquad (3.3.17)$$

On the r.h.s. all external indices are raised and lowered with the metric $g_{\mu\nu}$, and both ε-symbols denote the numerical tensor densities. All explicit appearance of Kaluza-Klein vectors $\mathcal{A}_\mu{}^m$ from (3.3.14) is absorbed in the redefined $\overline{F}$'s. We will now reproduce these equations one by one from the EFT dynamics.

Let us start from the $[mn]$ component of the EFT duality equations (3.1.8) which can be integrated to

$$\tilde{\mathcal{H}}_{\mu\nu\rho\,mn} + \mathcal{O}_{mn\,\mu\nu\rho} \;=\; \frac{1}{2}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\,\mathcal{M}_{mn,M}\,\mathcal{F}^{\sigma\tau\,M}\,, \qquad (3.3.18)$$

where the $\mathcal{O}_{mn\,\mu\nu\rho}$ keeps track of the integration ambiguity and satisfies

$$\partial_{[k}\mathcal{O}_{mn]\,\mu\nu\rho} = 0 \qquad \Longrightarrow \qquad \mathcal{O}_{mn\,\mu\nu\rho} \;\equiv\; \partial_{[m}\xi_{n]\,\mu\nu\rho} \quad \text{(locally)}\,. \qquad (3.3.19)$$

Eliminating $\mathcal{F}_{\mu\nu\,\alpha}$ on the r.h.s. of (3.3.18) by means of (3.3.3) turns $\mathcal{M}_{MN}$ into $\tilde{\mathcal{M}}_{MN}$, such that upon using the explicit expressions (3.1.35), we obtain

$$\begin{aligned}
\partial_{[m}\xi_{n]\,\mu\nu\rho} &= \frac{1}{12}\,e^{2\Phi/3}\,e\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}m^{kq}m^{lr}m^{ps}\,\widehat{\mathcal{F}}^{\sigma\tau}{}_{qrs} \\
&\quad -\tilde{\mathcal{H}}_{\mu\nu\rho\,mn}-\sqrt{2}\,\varepsilon_{\alpha\beta}\,b_{mn}{}^{\alpha}\tilde{\mathcal{H}}_{\mu\nu\rho}{}^{\beta}\;, \qquad (3.3.20)
\end{aligned}$$

with

$$\begin{aligned}
\widehat{\mathcal{F}}_{\mu\nu\,klm} &\equiv \mathcal{F}_{\mu\nu\,klm}+3\sqrt{2}\,b_{[kl}{}^{\alpha}\mathcal{F}_{|\mu\nu|m]\alpha}+3\sqrt{2}\,\varepsilon_{\alpha\beta}\,b_{n[k}{}^{\alpha}b_{lm]}{}^{\beta}\mathcal{F}_{\mu\nu}{}^{n}+\frac{1}{2}\sqrt{2}\,c_{klmn}\,\mathcal{F}_{\mu\nu}{}^{n}\;. \\
&= 2\sqrt{2}\,\overline{F}_{\mu\nu\,klm}\;, \qquad (3.3.21)
\end{aligned}$$

where the last identity is easily confirmed upon using the dictionary of field strengths (3.2.34), (3.3.10) and scalars (3.3.11). Together, the relation (3.3.20) then gives rise to

$$F_{\mu\nu\rho\,mn}-\frac{1}{4}\,\varepsilon_{\alpha\beta}\,C_{mn}{}^{\alpha}\,\overline{F}_{\mu\nu\rho}{}^{\beta} = \frac{1}{12}\,e^{2\Phi/3}\,e\,\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon_{mnklp}m^{kq}m^{lr}m^{ps}\,\overline{F}^{\sigma\tau}{}_{qrs}\;, \qquad (3.3.22)$$

and thus precisely reproduces (3.3.15) if we identify the 3-form component $C_{\mu\nu\rho\,m}$ from (3.2.29) as

$$C_{\mu\nu\rho\,m} = -\frac{1}{8}\sqrt{2}\,\xi_{m\,\mu\nu\rho}\;. \qquad (3.3.23)$$

We have thus reproduced the first of the components of the IIB self-duality equations and along the way identified one of the missing components (3.3.23) of the IIB four-form, that is not among the fundamental EFT fields. It is defined by the first order differential equations (3.3.22) in terms of the EFT fields up to a gradient

$$C_{\mu\nu\rho\,m} \longrightarrow C_{\mu\nu\rho\,m}+\partial_m\lambda_{\mu\nu\rho}\;, \qquad (3.3.24)$$

corresponding to a gauge transformation in the IIB theory.

Let us continue towards the other components (3.3.16), (3.3.17), of the self-duality relations. Consider the external curl of (3.3.18), which reads

$$4D_{[\mu}\tilde{\mathcal{H}}_{\nu\rho\sigma]\,mn}+4D^{\rm KK}_{[\mu}\,\mathcal{O}_{\nu\rho\sigma]\,mn} = 2\,e\varepsilon_{\tau\lambda[\nu\rho\sigma}\,D^{\rm KK}_{\mu]}\,\left(\mathcal{M}_{mn,N}\,\mathcal{F}^{\tau\lambda\,N}\right)\;, \qquad (3.3.25)$$

and use the Bianchi identity (3.1.28) to find

$$\begin{aligned}
4\,\partial_m\left(D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma]\,n}\right) &= 6\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{F}_{\rho\sigma]\,kmn}+3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{[\mu\nu\,|m\alpha|}\mathcal{F}_{\rho\sigma]\,n\beta} \\
&\quad +4\sqrt{2}\,\partial_m\tilde{\mathcal{H}}_{[\mu\nu\rho}{}^{\alpha}\,\mathcal{A}_{\sigma]n\alpha}-e\varepsilon_{\mu\nu\rho\sigma\lambda}\,D^{\rm KK}_{\tau}\,\left(\mathcal{M}_{mn,N}\,\mathcal{F}^{\tau\lambda\,N}\right) \\
&\quad -3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{[\mu\nu}{}^{\alpha}\partial_{|n|}\tilde{\mathcal{B}}_{\rho\sigma]}{}^{\beta}\right)+12\,\partial_m\left(\mathcal{F}_{[\mu\nu}{}^{k}\tilde{\mathcal{B}}_{\rho\sigma]\,kn}\right) \\
&\quad +6\sqrt{2}\,\partial_m\left(\varepsilon^{\alpha\beta}\mathcal{A}_{[\mu\,|n\alpha|}\mathcal{F}_{\nu\rho}{}^{k}\mathcal{A}_{\sigma]\,k\beta}\right)\;, \qquad (3.3.26)
\end{aligned}$$

where both, left and right hand side are supposed to be explicitly projected onto their part antisymmetric in $[mn]$.

In order to simplify the second line, we make use of the equations of motion obtained by varying the Lagrangian (3.1.1) w.r.t. the vector fields $\mathcal{A}_\mu{}^{mn}$ and using the duality equation (3.3.3) in order to eliminate $\mathcal{F}_{\mu\nu\,\alpha}$,

$$\begin{aligned} 0 \;=\; & -\frac{1}{24}\sqrt{2}\,\partial_{[m}\left(e^{2\Phi}m_{n]k}\,\widehat{\mathcal{D}}^\mu c_{pqrs}\varepsilon^{kpqrs}\right) + D_\nu^{\rm KK}\left(\mathcal{M}_{mn,M}\mathcal{F}^{\nu\mu\,M}\right) \\ & +\frac{1}{6}\sqrt{2}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\partial_{[m}\mathcal{A}_{|\nu|n]\alpha}\tilde{\mathcal{H}}_{\rho\sigma\tau}{}^\alpha - \frac{1}{12}\sqrt{2}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\mathcal{A}_{\nu\,[m|\alpha|}\partial_{n]}\tilde{\mathcal{H}}_{\rho\sigma\tau}{}^\alpha \\ & +\frac{3}{4}\,\varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{\sqrt{2}}{6}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{\nu\rho\,m\alpha}\mathcal{F}_{\sigma\tau\,n\beta} + \frac{1}{3}\,\mathcal{F}_{\nu\rho\,mnp}\mathcal{F}_{\sigma\tau}{}^p + \frac{\sqrt{2}}{9}\,\mathcal{A}_{\nu[m|\alpha|}\,\partial_{n]}\tilde{\mathcal{H}}_{\rho\sigma\tau}{}^\alpha\right) \end{aligned} \qquad (3.3.27)$$

Together we find for (3.3.26)

$$\begin{aligned} 4\,\partial_m\left(D_\mu^{\rm KK}\xi_{\nu\rho\sigma\,n}\right) \;=\; & -\frac{1}{24}\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,\partial_m\left(e^{2\Phi}m_{nk}\,\widehat{\mathcal{D}}^\lambda c_{pqrs}\varepsilon^{kpqrs}\right) \\ & -3\sqrt{2}\,\partial_m\left(\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{\mu\nu}{}^\alpha\partial_n\tilde{\mathcal{B}}_{\rho\sigma}{}^\beta\right) + 12\,\partial_m\left(\mathcal{F}_{\mu\nu}{}^k\tilde{\mathcal{B}}_{\rho\sigma\,kn}\right) \\ & +6\sqrt{2}\,\partial_m\left(\mathcal{A}_{\mu n\alpha}\varepsilon^{\alpha\beta}\mathcal{F}_{\nu\rho}{}^k\mathcal{A}_{\sigma\,k\beta}\right) - 4\sqrt{2}\,\partial_m\left(\mathcal{A}_{\mu\,n\alpha}\tilde{\mathcal{H}}_{\nu\rho\sigma}{}^\alpha\right), \end{aligned} \qquad (3.3.28)$$

again, projected onto the antisymmetric part $[mn]$. The entire equation thus takes the form of an internal curl and can be integrated to

$$\begin{aligned} -\frac{1}{24}\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,e^{2\Phi}m_{nk}\,\widehat{\mathcal{D}}^\lambda c_{pqrs}\varepsilon^{kpqrs} \;=\; & 4\,D^{\rm KK}_{[\mu}\,\xi_{\nu\rho\sigma]\,n} + 3\sqrt{2}\,\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{[\mu\nu}{}^\alpha\partial_{|n|}\tilde{\mathcal{B}}_{\rho\sigma]}{}^\beta \\ & -12\,F_{[\mu\nu}{}^k\tilde{\mathcal{B}}_{\rho\sigma]\,kn} - 6\sqrt{2}\,\varepsilon^{\alpha\beta}\,\mathcal{F}_{[\mu\nu}{}^k\,\mathcal{A}_{\rho\,|n\alpha|}\mathcal{A}_{\sigma]\,k\beta} \\ & +4\sqrt{2}\,\mathcal{A}_{\mu\,n\alpha}\tilde{\mathcal{H}}_{\nu\rho\sigma}{}^\alpha + \partial_n\xi_{\mu\nu\rho\sigma}\,, \end{aligned} \qquad (3.3.29)$$

up to an internal gradient $\partial_n\xi_{\mu\nu\rho\sigma}$. Applying the dictionary (3.3.9), (3.3.10) to translate all fields into the IIB components, this equation becomes

$$-\frac{1}{24}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\varepsilon^{kpqrs}\,e^{2\Phi}m_{nk}\,\overline{F}^\lambda{}_{pqrs} \;=\; \overline{F}_{\mu\nu\rho\sigma\,n} - \partial_n\left(C_{\mu\nu\rho\sigma} + \frac{1}{8}\sqrt{2}\,\xi_{\mu\nu\rho\sigma}\right)\,, \qquad (3.3.30)$$

i.e. reproduces equation (3.3.16), provided we identify the last missing component of the 4-form as

$$C_{\mu\nu\rho\sigma} \;=\; -\frac{1}{8}\sqrt{2}\,\xi_{\mu\nu\rho\sigma}\,. \qquad (3.3.31)$$

We have thus also reproduced the second component of the IIB self-duality equations and along the way identified the last missing components (3.3.31) of the IIB four-form, that is not among the fundamental EFT fields. It is defined by the first order differential equations (3.3.29) in terms of the EFT fields up to an additive function

$$C_{\mu\nu\rho\sigma} \;\longrightarrow\; C_{\mu\nu\rho\sigma} + \Lambda_{\mu\nu\rho\sigma}(x)\,, \qquad (3.3.32)$$

which we will fix in the following. In order to find the last component (3.3.17) of the self-duality equations, we take the external curl of (3.3.29)

$$\begin{aligned}
-\partial_n D^{\text{KK}}_{[\mu}\,\xi_{\nu\rho\sigma\tau]} \;&=\; -\frac{1}{120}\,\sqrt{2}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\,D^{\text{KK}}_{\lambda}\left(e^{2\Phi}m_{nk}\,\widehat{\mathcal{D}}^{\lambda}c_{pqrs}\varepsilon^{kpqrs}\right)+2\sqrt{2}\,\mathcal{F}_{[\mu\nu\,|n\alpha|}\tilde{\mathcal{H}}_{\rho\sigma\tau]}{}^{\alpha}\\
&\quad+4\,\mathcal{F}_{[\mu\nu}{}^{k}\left(\tilde{\mathcal{H}}_{\rho\sigma\tau]\,kn}+\partial_{[k}\xi_{\rho\sigma\tau]\,n]}\right)+2\sqrt{2}\,\varepsilon_{\alpha\beta}\partial_n\tilde{\mathcal{B}}_{[\mu\nu}{}^{\beta}\tilde{\mathcal{H}}_{\rho\sigma\tau]}{}^{\alpha}\\
&\quad-2\sqrt{2}\,\varepsilon_{\alpha\beta}\,\tilde{\mathcal{H}}_{[\mu\nu\rho}{}^{\alpha}\partial_{|n|}\tilde{\mathcal{B}}_{\sigma\tau]}{}^{\beta}-6\,\sqrt{2}\varepsilon^{\alpha\beta}\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho\,|n\alpha|}\mathcal{F}_{\sigma\tau]\,k\beta}\\
&\quad+6\sqrt{2}\,\varepsilon^{\alpha\beta}\mathcal{A}_{[\mu\,|n\alpha|}\mathcal{F}_{\nu\rho}{}^{k}\mathcal{F}_{\sigma\tau]k\beta}-3\sqrt{2}\,\partial_n\left(\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{[\mu\nu}{}^{\alpha}D_{\rho}\tilde{\mathcal{B}}_{\sigma\tau]}{}^{\beta}\right)\\
&\quad+2\partial_n\left(\mathcal{F}_{[\mu\nu}{}^{k}\xi_{\rho\sigma\tau]\,k}\right)\;, \qquad (3.3.33)
\end{aligned}$$

which after using the equations of motion for c_{klmn} turns into a full internal gradient and can be integrated to the equation

$$D^{\text{KK}}_{[\mu}\,\xi_{\nu\rho\sigma\tau]}+3\sqrt{2}\,\varepsilon_{\alpha\beta}\,\tilde{\mathcal{B}}_{[\nu\rho}{}^{\alpha}D_{\mu}\tilde{\mathcal{B}}_{\sigma\tau]}{}^{\beta}-2\mathcal{F}_{[\mu\nu}{}^{k}\xi_{\rho\sigma\tau]\,k} \;=\; \frac{\sqrt{2}}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,e^{10\Phi/3}\,X_{klmnp}\;, \qquad (3.3.34)$$

with X from (3.1.41), up to some y-independent function. The latter can be set to zero by properly fixing the freedom (3.3.32). After translating (3.3.34) into the IIB fields, we thus find

$$5D^{\text{KK}}_{[\mu}C_{\nu\rho\sigma\tau]}-\frac{15}{4}\,\varepsilon_{\alpha\beta}\,\overline{C}_{[\nu\rho}{}^{\alpha}D^{\text{KK}}_{\mu}\overline{C}_{\sigma\tau]}{}^{\beta}-10\mathcal{F}_{[\mu\nu}{}^{k}C_{\rho\sigma\tau]\,k} \;=\; \frac{1}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,e^{10\Phi/3}\,\overline{F}_{klmnp}\;. \qquad (3.3.35)$$

Thereby we find the last missing component (3.3.17) of the IIB self-duality equation. We have thus shown that the full IIB self-duality equations (3.2.7) follow from the EFT dynamics, provided we identify by (3.3.23), (3.3.31) the remaining components of the IIB 4-form. Together with the dictionary established in section (3.3.2), this defines all the IIB fields in terms of the fundamental fields from EFT.

3.4 Summary

We have reviewed the $E_{6(6)}$ exceptional field theory and established the precise embedding of ten-dimensional type IIB supergravity upon picking the GL(5)×SL(2) solution of the section constraint. We have done so by first matching the gauge symmetries on both sides. On the type IIB supergravity side, this requires a number of field redefinitions, which are largely analogous to those needed in conventional Kaluza-Klein compactifications. On the exceptional field theory side, this requires a suitable parametrization of the $E_{6(6)}$ valued '27-bein'. We have then given the explicit dictionary from the various components of the IIB fields to the EFT fields after solving the section constraint. We also established the on-shell equivalence of both theories and in particular showed how the three- and four-forms of type IIB, originating from components of the self-dual four-form in ten dimensions, are reconstructed on-shell in exceptional field theory in which these fields are not present from the start.

Chapter 4

Two applications of the EFT/Type IIB dictionary: reduction and deformation

In this chapter, we present two applications of the dictionary we have established in the previous chapter. The first application is the proof of the Kaluza-Klein consistency of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in type IIB. We will start by briefly laying out the material we will need on the gauged supergravity side to make the link with the $\mathrm{E}_{6(6)}$ EFT after the type IIB reduction ansatz. After analysing the twist equations in a general setting, we present the explicit and complete reduction formulas for a class of truncations of type IIB supergravity to maximal five-dimensional gauged supergravity, by working out the details of the construction of [56]. This includes the famous reduction on $\mathrm{AdS}_5 \times \mathrm{S}^5$ to the maximal $D = 5$ SO(6) gauged supergravity of [7], but also reductions to non-compact gaugings, corresponding to truncations with non-compact (hyperboloidal) internal manifolds. Consistency of the latter has first been conjectured in [69] and more recently been discussed in [70, 71]. Within the framework of EFT, the complicated geometric IIB reductions can very conveniently be formulated as Scherk-Schwarz reductions on an exceptional space-time. It was shown in [56] how sphere compactifications of the original supergravities and their non-compact cousins can be realized in EFT through generalised Scherk-Schwarz compactifications, which are governed by $\mathrm{E}_{d(d)}$ valued 'twist' matrices. In terms of the duality covariant fields of EFT the reduction formulas take the form of a simple Scherk-Schwarz ansatz (1.6.2), proving the consistency of the corresponding Kaluza-Klein truncation. Although this settles the issue of consistency it may nevertheless be useful to have the explicit reduction formulas in terms of the *conventional* supergravity fields, thus requiring the dictionary for identifying the original supergravity fields in the EFT formulation.

The second application regards a recently found deformation of type IIB, known in the literature as 'generalised type IIB' [81, 82, 5]. In section 4.4, we recall the bosonic field equations generalised IIB supergravity equation together with the modified Bianchi identities. By using a simple Scherk-Schwarz ansatz together with a different solution of the section constraint, we show in section 4.5 that the deformation induced by the factorisation ansatz match the deformation of generalised IIB.

4.1 Gauged maximal supergravity in five dimensions

The $D = 5$ gauged theory with gauge group $\mathrm{SO}(p,q)$ was originally constructed in [53, 54, 7]. For our purpose, the most convenient description is its covariant form found in the context of general gaugings [83] to which we refer for details.[1] In the covariant formulation, the $D = 5$ gauged theory features 27 propagating vector fields $A_\mu{}^M$ and up to 27 topological tensor fields $B_{\mu\nu\,M}$. The choice of gauge group and the precise number of tensor fields involved is specified by the choice of an embedding tensor $Z^{MN} = Z^{[MN]}$ in the **351** representation of $\mathrm{E}_{6(6)}$. E.g. the full non-abelian vector field strengths are given by

$$F_{\mu\nu}{}^M \;=\; 2\,\partial_{[\mu}A_{\nu]}{}^M + \sqrt{2}\,X_{KL}{}^M A_{[\mu}{}^K A_{\nu]}{}^L - 2\,\sqrt{2}\,Z^{MN} B_{\mu\nu\,N}\;, \tag{4.1.2}$$

with the tensor $X_{KL}{}^M$ carrying the gauge group structure constants and defined in terms of the embedding tensor Z^{MN} as

$$X_{MN}{}^P \;=\; d_{MNQ}Z^{PQ} + 10\,d_{MQS}d_{NRT}d^{PQR}Z^{ST}\;. \tag{4.1.3}$$

The $\mathrm{SO}(p,q)$ gaugings preserve the global $\mathrm{SL}(2)$ subgroup of the symmetry group $\mathrm{E}_{6(6)}$ of the ungauged theory, more specifically the centralizer of its subgroup $\mathrm{SL}(6)$. Accordingly, the vector fields in the **27** of $\mathrm{E}_{6(6)}$ can be split as

$$A_\mu{}^M \;\longrightarrow\; \left\{A_\mu{}^{ab}, A_{\mu\,a\alpha}\right\}\;, \qquad a,b = 0,\ldots,5\;, \quad \alpha = 1,2\;, \tag{4.1.4}$$

into 15 $\mathrm{SL}(2)$ singlets and 6 $\mathrm{SL}(2)$ doublets. The 27 two-forms $B_{\mu\nu\,M}$ split accordingly, with only the 6 $\mathrm{SL}(2)$ doublets $B_{\mu\nu}{}^{a\alpha}$ entering the supergravity Lagrangian. In the basis (4.1.4), the only non-vanishing components of the embedding tensor Z^{MN} are

$$Z_{a\alpha,b\beta} \;\equiv\; -\frac{1}{2}\,\sqrt{5}\,\varepsilon_{\alpha\beta}\eta_{ab}\;, \tag{4.1.5}$$

[1] To be precise, and to facilitate the embedding of this theory into EFT, we choose the normalization of [33] for vector and tensor fields which differs from [83] as

$$\mathcal{A}_\mu{}^M{}_{[1312.0614]} = \frac{1}{\sqrt{2}}A_\mu{}^M{}_{[\mathrm{hep-th}/0412173]}\;,\quad \mathcal{B}_{\mu\nu\,M[1312.0614]} = -\frac{1}{4}B_{\mu\nu\,M[\mathrm{hep-th}/0412173]}\;, \tag{4.1.1}$$

together with a rescaling of the associated symmetry parameters. Moreover, we have set the coupling constant of [83] to $g = 1$.

where the normalization has been chosen such as to match the later expressions. With (4.1.3), and the expression of the d-symbol of $\mathrm{E}_{6(6)}$ in the $\mathrm{SL}(6) \times \mathrm{SL}(2)$ basis given by

$$d^{MNK} \quad : \quad d^{ab}{}_{c\alpha,d\beta} = \frac{1}{\sqrt{5}}\,\delta^{ab}_{cd}\,\varepsilon_{\alpha\beta}\;, \quad d^{ab,cd,ef} = \frac{1}{\sqrt{80}}\,\varepsilon^{abcdef}\;, \tag{4.1.6}$$

we thus obtain

$$X_{MN}{}^{K} \quad : \quad \left\{ \begin{array}{l} X_{ab,cd}{}^{ef} = f_{ab,cd}{}^{ef} \\ X_{ab}{}^{c\alpha}{}_{d\beta} = -\delta_{[a}{}^{c}\eta_{b]d}\,\delta^{\alpha}_{\beta} \end{array} \right. \;, \tag{4.1.7}$$

with the $\mathrm{SO}(p, 6-p)$ structure constants

$$f_{ab\,,cd}{}^{ef} \;\equiv\; 2\,\delta_{[a}{}^{[e}\eta_{b][c}\delta_{d]}{}^{f]}\;. \tag{4.1.8}$$

The form of the field strength (4.1.2) is the generic structure of a covariant field strength in gauged supergravity, with non-abelian Yang-Mills part and a Stückelberg type coupling to the two-forms. In the present case, we can make use of the tensor gauge symmetry which acts by shift $\delta A_{\mu\,a\alpha} = \Xi_{\mu\,a\alpha}$ on the vector fields, to eliminate all components $A_{\mu\,a\alpha}$ from the Lagrangian and field equations. This is the gauge we are going to impose in the following, which brings the theory in the form of [7].[2] As a result, the covariant object (4.1.2) splits into components carrying the $\mathrm{SO}(p,q)$ Yang-Mills field strength, and the two-forms $B_{\mu\nu}{}^{a\alpha}$, respectively,

$$F_{\mu\nu}{}^{M} \;=\; \left\{ \begin{array}{lcl} F_{\mu\nu}{}^{ab} & \equiv & 2\,\partial_{[\mu}A_{\nu]}{}^{ab} + \sqrt{2}\,f_{cd,ef}{}^{ab}\,A_{\mu}{}^{cd}A_{\nu}{}^{ef} \\ F_{\mu\nu\,a\alpha} & \equiv & \sqrt{10}\,\varepsilon_{\alpha\beta}\eta_{ab}\,B_{\mu\nu}{}^{b\beta} \end{array} \right. . \tag{4.1.10}$$

In particular, fixing of the tensor gauge symmetry implies that the two-forms $B_{\mu\nu}{}^{a\alpha}$ turn into topologically massive fields, preserving the correct counting of degrees of freedom, [84]. The Lagrangian and field equations are still conveniently expressed in terms of the combined object $F_{\mu\nu}{}^{M}$. E.g. the first order duality equation between vector and tensor fields is given by

$$3\,D_{[\mu}B_{\nu\rho]}{}^{a\alpha} \;=\; \frac{1}{2\sqrt{10}}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}M^{a\alpha}{}_{N}\,F^{\sigma\tau\,N}\;, \tag{4.1.11}$$

which upon expanding around the scalar origin and with (4.1.10) yields the first order topologically massive field equation for the two-form tensors. The full bosonic Lagrangian

[2] To be precise: this holds with a rescaling of p-forms according to

$$\mathcal{A}_{\mu}{}^{ab}{}_{[1312.0614]} = -\sqrt{2}\,A_{\mu}{}^{ab}{}_{\mathrm{GRW}}\;, \qquad \sqrt{5}\,\mathcal{B}_{\mu\nu}{}^{a\alpha}{}_{[1312.0614]} = B_{\mu\nu}{}^{a\alpha}{}_{\mathrm{GRW}}\;, \tag{4.1.9}$$

and with their coupling constant set to $g_{\mathrm{GRW}} = 2$.

reads

$$
\begin{aligned}
\mathcal{L} \;=\;& \sqrt{|\mathrm{g}|}\,R - \frac{1}{4}\sqrt{|\mathrm{g}|}\,M_{MN}\,F_{\mu\nu}{}^{M}F^{\mu\nu\,N} + \frac{1}{24}\sqrt{|\mathrm{g}|}\,D_{\mu}M_{MN}D^{\mu}M^{MN} \\
&+ \varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{5}{4}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,B_{\mu\nu}{}^{a\alpha}D_{\rho}B_{\sigma\tau}{}^{b\beta} + \frac{1}{24}\sqrt{2}\,\varepsilon_{abcdef}\,A_{\mu}{}^{ab}\,\partial_{\nu}A_{\rho}{}^{cd}\,\partial_{\sigma}A_{\tau}{}^{ef}\right) \\
&+ \frac{1}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon_{abcdef}\,f_{gh,ij}{}^{ab}\,A_{\mu}{}^{cd}A_{\nu}{}^{gh}A_{\rho}{}^{ij}\Big(\partial_{\sigma}A_{\tau}{}^{ef} + \tfrac{1}{5}\sqrt{2}\,f_{kl,mn}{}^{ef}A_{\sigma}{}^{kl}A_{\tau}{}^{mn}\Big) \\
&- \sqrt{|\mathrm{g}|}\,V(M_{MN})\,.
\end{aligned}
\tag{4.1.12}
$$

Here, the 42 scalar fields parameterize the coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ via the symmetric $\mathrm{E}_{6(6)}$ matrix M_{MN} which can be decomposed in the basis (4.1.4) as

$$
M_{MN} \;=\; \begin{pmatrix} M_{ab,cd} & M_{ab}{}^{c\gamma} \\ \mathcal{M}^{a\alpha}{}_{bc} & M^{a\alpha,c\gamma} \end{pmatrix}\,, \tag{4.1.13}
$$

with the $\mathrm{SO}(p,6-p)$ covariant derivatives defined according to

$$
D_{\mu}X^{a} \;\equiv\; \partial_{\mu}X^{a} + \sqrt{2}\,A_{\mu}{}^{ab}\,\eta_{bd}\,X^{d}\,, \tag{4.1.14}
$$

and similarly on the different blocks of (4.1.13). The scalar potential V in (4.1.12) is given by the following contraction of the generalised structure constants (4.1.7) with the scalar matrix (4.1.13)

$$
V(M_{MN}) \;=\; \frac{1}{30}\,M^{MN}X_{MP}{}^{Q}\left(5\,X_{NQ}{}^{P} + X_{NR}{}^{S}\,M^{PR}M_{QS}\right)\,. \tag{4.1.15}
$$

For later use, let us explicitly state the vector field equations obtained from (4.1.12) which take the form

$$
\begin{aligned}
0 \;=\;& \sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\Big(\eta_{c[a}\,D^{\tau}M_{b]d,N}M^{N,cd} + \sqrt{2}\,D_{\lambda}\left(F^{\tau\lambda\,N}M_{N,ab}\right)\Big) \\
&+ \frac{3}{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{cd}F_{\rho\sigma]}{}^{ef} + 60\,\varepsilon_{\alpha\beta}\,\eta_{ac}\eta_{bd}\,B_{[\mu\nu}{}^{c\alpha}B_{\rho\sigma]}{}^{d\beta}\,.
\end{aligned}
\tag{4.1.16}
$$

We will also need part of the scalar field equations that are obtained by varying in (4.1.12) the scalar matrix (4.1.13) with an $\mathrm{SL}(6)$ generator $X_{a}{}^{b}$

$$
\begin{aligned}
0 \;=\;& \frac{1}{4}\,D^{\mu}(M^{ad\,K}D_{\mu}M_{K\,bd}) - \frac{1}{2}\,M_{bc\,N}\,F_{\mu\nu}{}^{ac}F^{\mu\nu\,N} + \frac{1}{4}\sqrt{10}\,\eta_{bc}\,\varepsilon_{\alpha\beta}\,M^{a\alpha}{}_{N}\,B_{\mu\nu}{}^{c\beta}F^{\mu\nu\,N} \\
&+ \left(2\,M^{ae,fc} + \frac{4}{15}\,M^{de,h(a}M^{c)j,fg}M_{dg,hj} + \frac{1}{15}\,M^{de,h(a}M^{c)\beta,f\alpha}M_{d\alpha,h\beta}\right)\eta_{bc}\eta_{ef} \\
&- \frac{2}{15}\left(M^{de,k(a}M^{c)\alpha}{}_{dg}M_{k\alpha}{}^{fg} + M^{de,h(a}M^{c)g}{}_{d\alpha}M^{f\alpha}{}_{hg}\right)\eta_{ef}\eta_{bc} \;-\; [\mathrm{trace}]_{b}{}^{a}\,.
\end{aligned}
\tag{4.1.17}
$$

4.2 Analysis of the twist equations

Here, we start with a general analysis of the Scherk-Schwarz twist equations. We focus on the four blocks of the twist matrix, given by the decomposition of the fundamental representation of $\mathrm{E}_{6(6)}$ under the type IIB solution of the section constraint. By making an ansatz on the form of one of these blocks, we are able to solve the twist equations in terms of a set of Killing tensors for an internal metric and a four-form. Further expliciting the tensors to the $SO(6,6-p)$ case, we get the analytic expression of the Killing tensors, the metric and the four-form in terms of the allowed EFT coordinates. This gives the explicit reduction ansatz for the EFT fields.

4.2.1 General analysis

In the introduction of this thesis, we showed that the consistency conditions of the generalised Scherk-Schwarz ansatz were given by

$$\begin{aligned} \partial_N (U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\,\rho^{-1}\partial_N\rho &= 3\,\rho\,\vartheta_{\underline{K}}\,, \\ \left[(U^{-1})_{\underline{M}}{}^K (U^{-1})_{\underline{N}}{}^L \partial_K U_L{}^P\right]_{\mathbf{351}} &= \tfrac{1}{5}\,\rho\,\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_{\underline{N}}{}^{\underline{P}}\,, \end{aligned} \tag{4.2.1}$$

or in a manifestly covariant form

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\widehat{U}^{-1}_{\underline{N}} \;\equiv\; -X_{\underline{MN}}{}^{\underline{K}}\,\widehat{U}^{-1}_{\underline{K}}\,, \tag{4.2.2}$$

with $X_{\underline{MN}}{}^{\underline{K}}$ constant and related to the $D=5$ embedding tensor and

$$(\widehat{U}^{-1})_{\underline{M}}{}^N \;\equiv\; \rho^{-1}\,(U^{-1})_{\underline{M}}{}^N\,. \tag{4.2.3}$$

We now would like to analyse these 'twist equations' and decompose them w.r.t. the subgroup appropriate for the type IIB solution of the section constraint, i.e.

$$\begin{aligned} \mathrm{E}_{6(6)} &\longrightarrow \mathrm{GL}(5)\times\mathrm{SL}(2)\,, \\ \mathbf{27} &\longrightarrow (\mathbf{5},\mathbf{1})\oplus(\mathbf{5}',\mathbf{2})\oplus(\mathbf{10},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\,. \end{aligned} \tag{4.2.4}$$

Accordingly, the fundamental index on the generalised vector $\widehat{U}^{-1}$ decomposes as

$$(\widehat{U}^{-1})_{\underline{M}}{}^M \;=\; \{\,\mathcal{K}_{\underline{M}}{}^m\,,\;\mathcal{R}_{\underline{M}\,m\alpha}\,,\;\mathcal{Z}_{\underline{M}\,mnk}\,,\;\mathcal{S}_{\underline{M}\,n_1\ldots n_5\alpha}\,\}\,, \tag{4.2.5}$$

in terms of GL(5) indices $m,n=1,\ldots,5$ and SL(2) indices $\alpha,\beta=1,2$. In order to give the decomposition of the twist equations (1.6.15) in terms of these objects we use the definition (1.5.2) of the generalised Lie derivative and the decomposition of the d-symbol (3.1.15). A straightforward computation, largely analogous to those in, e.g., sec. 3.3 of

[1], then yields

$$
\begin{aligned}
-X_{\underline{MN}}{}^{\underline{K}}\,\mathcal{K}_{\underline{K}}{}^{m} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{K}_{\underline{N}}{}^{m}\;, &(4.2.6)\\
-X_{\underline{MN}}{}^{\underline{K}}\,\mathcal{R}_{\underline{K}\,m\alpha} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{R}_{\underline{N}\,m\alpha} - \mathcal{L}_{\mathcal{K}_{\underline{N}}}\mathcal{R}_{\underline{M}\,m\alpha} + \partial_m\left(\mathcal{K}_{\underline{N}}{}^{n}\mathcal{R}_{\underline{M}\,n\alpha}\right)\;, &(4.2.7)\\
-X_{\underline{MN}}{}^{\underline{K}}\,\mathcal{Z}_{\underline{K}\,kmn} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{Z}_{\underline{N}\,kmn} - \mathcal{L}_{\mathcal{K}_{\underline{N}}}\mathcal{Z}_{\underline{M}\,kmn} + 3\,\partial_{[k}\left(\mathcal{K}_{\underline{N}}{}^{l}\mathcal{Z}_{\underline{M}\,mn]l}\right)\\
&\quad + 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\partial_{[k}\mathcal{R}_{\underline{M}\,m|\alpha|}\mathcal{R}_{\underline{N}\,n]\beta}\;, &(4.2.8)\\
-X_{\underline{MN}}{}^{\underline{K}}\,\mathcal{S}_{\underline{K}\,n_1\ldots n_5\alpha} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{S}_{\underline{N}\,n_1\ldots n_5\alpha}\\
&\quad + 20\sqrt{2}\left(\mathcal{Z}_{\underline{N}\,[n_1n_2n_3}\partial_{n_4}\mathcal{R}_{\underline{M}\,n_5]\alpha} - \partial_{[n_1}\mathcal{Z}_{\underline{M}\,n_2n_3n_4}\mathcal{R}_{\underline{N}\,n_5]\alpha}\right) &(4.2.9)
\end{aligned}
$$

We will now successively analyze these equations. We split the index as $\underline{M} \to \{A, u\}$, where A, B denote the 'gauge group directions' and u, v the remaining ones, and assume that the only non-vanishing entries of $X_{\underline{MN}}{}^{\underline{K}}$ are

$$
X_{AB}{}^{C} \;=\; -f_{AB}{}^{C}\;, \qquad X_{Au}{}^{v} \;=\; (D_A)_u{}^{v}\;, \tag{4.2.10}
$$

given in terms of structure constants and representation matrices of the underlying Lie algebra of the gauge group, c.f. [83]. Let us emphasize that $X_{\underline{MN}}{}^{\underline{K}}$ is not assumed to be antisymmetric. In particular, for this ansatz we have, e.g., $X_{uA}{}^{v} = 0$. Let us also stress that this ansatz is not the most general, but it is sufficient for the purposes of this thesis.

The first equation (4.2.6), specialized to external indices (A, B), implies that the vector fields $\mathcal{K}_A$ satisfy the Lie bracket algebra

$$
\left[\mathcal{K}_A, \mathcal{K}_B\right]^m \;\equiv\; \mathcal{L}_{\mathcal{K}_A}\mathcal{K}_B{}^{m} \;=\; f_{AB}{}^{C}\,\mathcal{K}_C{}^{m}\;. \tag{4.2.11}
$$

In view of standard Kaluza-Klein compactifications it is natural to interpret these vector fields as the Killing vectors of some internal geometry. We now *define* a metric w.r.t. which the $\mathcal{K}_A$ are indeed Killing vectors by setting for the inverse metric

$$
\tilde{G}^{mn} \;\equiv\; \mathcal{K}_A{}^{m}\,\mathcal{K}_B{}^{n}\,\eta^{AB}\;, \tag{4.2.12}
$$

with the Cartan-Killing metric $\eta_{AB} \equiv f_{AC}{}^{D} f_{BD}{}^{C}$. The internal metric $\tilde{G}_{mn}$ exists provided the Cartan-Killing metric is invertible and that there are sufficiently many vector fields $\mathcal{K}_A{}^{m}$ to make $\tilde{G}^{mn}$ invertible. This assumption, which we will make throughout the following discussion, is satisfied in the examples below. Since by (4.2.11) the $\mathcal{K}_A$ transform under themselves according to the adjoint group action, under which the Cartan-Killing metric is invariant, it follows that the vectors are indeed Killing:

$$
\mathcal{L}_{\mathcal{K}_A}\tilde{G}_{mn} \;\equiv\; \nabla_m\mathcal{K}_{An} + \nabla_n\mathcal{K}_{Am} \;=\; 0\;, \tag{4.2.13}
$$

where here and in the following ∇_m denotes the covariant derivative w.r.t. the metric (4.2.12), which is used to raise and lower indices. The other non-trivial components of

(4.2.6), with external indices (A, u), (u, A) and (u, v), imply that the remaining vector fields $\mathcal{K}_u{}^m$ satisfy

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{K}_u{}^m \;=\; -(D_A)_u{}^v\,\mathcal{K}_v{}^m \;=\; 0\;, \qquad \mathcal{L}_{\mathcal{K}_u}\mathcal{K}_v{}^m \;\equiv\; \big[\mathcal{K}_u,\mathcal{K}_v\big]^m \;=\; 0\;. \tag{4.2.14}$$

For non-vanishing $\mathcal{K}_u$ the first equation can only be satisfied if the representation encoded by the $(D_A)_u{}^v$ includes the trivial (singlet) representation. In the following we will analyze the remaining equations under the assumption that the representation does not contain a trivial part, which then requires

$$\mathcal{K}_u{}^m \;=\; 0\;. \tag{4.2.15}$$

We next consider the second equation (4.2.7), specialized to external indices (A, u) and (u, A) to obtain

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{R}_{u\,m\alpha} \;=\; -(D_A)_u{}^v\,\mathcal{R}_{v\,m\alpha} \;=\; \partial_m\left(\mathcal{K}_A{}^n\mathcal{R}_{u\,n\alpha}\right)\;. \tag{4.2.16}$$

Writing out the Lie derivative on the left-hand side we obtain in particular

$$\mathcal{K}_A{}^n\left(\partial_m\mathcal{R}_{u\,n\alpha}-\partial_n\mathcal{R}_{u\,m\alpha}\right) \;=\; 0\;. \tag{4.2.17}$$

With the above assumption that the metric (4.2.12) is invertible it follows that the curl of $\mathcal{R}$ is zero. Hence we can write it in terms of a gradient,

$$\mathcal{R}_{u\,m\alpha} \;\equiv\; \partial_m\mathcal{Y}_{u\,\alpha}\;. \tag{4.2.18}$$

As we still have to solve the first equation of (4.2.16), we must demand that the function $\mathcal{Y}$ transforms under the Killing vectors in the representation D_A,

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{Y}_{u\,\alpha} \;=\; -(D_A)_u{}^v\,\mathcal{Y}_{v\,\alpha}\;, \tag{4.2.19}$$

for then (4.2.16) follows with the covariant relation (4.2.18). Finally, specializing (4.2.7) to external indices (A, B), we obtain

$$f_{AB}{}^C\mathcal{R}_{C\,m\alpha} \;=\; \mathcal{L}_{\mathcal{K}_A}\mathcal{R}_{B\,m\alpha}-\mathcal{L}_{\mathcal{K}_B}\mathcal{R}_{A\,m\alpha}+\partial_m\big(\mathcal{K}_B{}^n\mathcal{R}_{A\,n\alpha}\big)\;. \tag{4.2.20}$$

This equation is solved by $\mathcal{R}_{A\,m\alpha}=0$, and the latter indeed holds for the SL(6) valued twist matrix to be discussed below. In addition, we will find that for these twist matrices also the components $\mathcal{Z}_u$ and $\mathcal{S}_A$ are zero, and therefore in the following we analyze the equations for this special case,

$$\mathcal{R}_{A\,m\alpha} \;=\; \mathcal{Z}_{u\,mnk} \;=\; \mathcal{S}_{A\,n_1\ldots n_5\,\alpha} \;=\; 0\;. \tag{4.2.21}$$

Let us now turn to the third equation (4.2.8), which will constrain the $\mathcal{Z}$ tensor. Specializing to external indices (A, B), we obtain

$$f_{AB}{}^C\,\mathcal{Z}_{C\,kmn} \;=\; \mathcal{L}_{\mathcal{K}_A}\mathcal{Z}_{B\,kmn}-\mathcal{L}_{\mathcal{K}_B}\mathcal{Z}_{A\,kmn}+3\,\partial_{[k}\big(\mathcal{K}_B{}^l\mathcal{Z}_{A\,mn]l}\big)\;, \tag{4.2.22}$$

where we used (4.2.21). Writing out the second Lie derivative on the right-hand side, this can be reorganized as

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{Z}_{B\,kmn} - 4\,\mathcal{K}_B{}^p\,\partial_{[p}\mathcal{Z}_{A\,kmn]} \;=\; f_{AB}{}^C\,\mathcal{Z}_{C\,kmn}\;. \tag{4.2.23}$$

In order to solve this equation we make the following ansatz

$$\mathcal{Z}_{A\,klm} \;\equiv\; -\frac{1}{4}\,\sqrt{2}\,\mathcal{K}_{A\,klm} - 2\,\sqrt{2}\,\mathcal{K}_A{}^p\,\tilde{C}_{pklm}\;, \tag{4.2.24}$$

in terms of a four-form $\tilde{C}$, where we chose the normalization for later convenience, and we defined the Killing tensor

$$\mathcal{K}_{A\,klm} \;\equiv\; \frac{1}{2}\,\tilde{\omega}_{klmpq}\,\mathcal{K}_A{}^{pq}\;, \qquad \mathcal{K}_{A\,mn} \;\equiv\; 2\,\nabla_{[m}\mathcal{K}_{A\,n]}\;, \tag{4.2.25}$$

with the volume form $\tilde{\omega}_{klmpq} \equiv |\tilde{G}|^{1/2}\,\varepsilon_{klmpq}$. We recall that all internal indices are raised and lowered with $\tilde{G}_{mn}$ defined in (4.2.12).

It remains to determine $\tilde{C}_{pklm}$ from the above system of equations. In order to simplify the result of inserting (4.2.24) into (4.2.23) we can use that the Killing tensor term transforms 'covariantly' under the Lie derivative,

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{K}_{B\,mnk} \;=\; f_{AB}{}^C\,\mathcal{K}_{C\,mnk}\;, \tag{4.2.26}$$

which follows from the corresponding property (4.2.11) of the Killing vectors. For the second term on the left-hand side of (4.2.23), however, we have to compute,

$$\begin{aligned}\mathcal{K}_B{}^p\nabla_{[p}\mathcal{K}_{A\,kmn]} \;&=\; \mathcal{K}_B{}^p\nabla_{[p}\big(\tfrac{1}{2}\,\tilde{\omega}_{kmn]lq}\,\mathcal{K}_A{}^{lq}\big) \;=\; \mathcal{K}_B{}^p\,\tilde{\omega}_{lq[kmn}\nabla_{p]}\nabla^{[l}\mathcal{K}_A{}^{q]}\\ &=\; -\tfrac{1}{2}\,\mathcal{K}_B{}^p\,\tilde{\omega}_{kmnpl}\,\nabla_q\nabla^{[l}\mathcal{K}_A{}^{q]} \;=\; \tfrac{1}{2}\,\mathcal{K}_B{}^p\,\tilde{\omega}_{kmnpl}\,\nabla_q\nabla^q\mathcal{K}_A{}^l\;.\end{aligned} \tag{4.2.27}$$

Here we used the $D=5$ Schouten identity $\tilde{\omega}_{[lqkmn}\nabla_{p]} \equiv 0$ and that the Killing tensor written as $\mathcal{K}_{A\,mn} = 2\,\nabla_m\mathcal{K}_{A\,n}$ is automatically antisymmetric as a consequence of the Killing equations (4.2.13). Using the latter fact again, the last expression simplifies as follows

$$\nabla_q\nabla^q\mathcal{K}_A{}^l \;=\; -\nabla_q\nabla^l\mathcal{K}_A{}^q \;=\; -\big[\nabla_q,\nabla^l\big]\,\mathcal{K}_A{}^q \;=\; -\,\tilde{\mathcal{R}}^{lp}\,\mathcal{K}_{A\,p}\;. \tag{4.2.28}$$

We will see momentarily that (4.2.23) can be solved analytically by the above ansatz (4.2.24) if the metric $\tilde{G}$ is Einstein. We thus assume this to be the case, so that the Ricci tensor reads $\tilde{\mathcal{R}}_{mn} = \lambda\,\tilde{G}_{mn}$, for some constant λ. Using this in (4.2.28) and inserting back into (4.2.27) we obtain

$$\mathcal{K}_B{}^p\nabla_{[p}\mathcal{K}_{A\,kmn]} \;=\; \frac{\lambda}{2}\,\tilde{\omega}_{kmnpl}\,\mathcal{K}_A{}^p\mathcal{K}_B{}^l\;. \tag{4.2.29}$$

Next, insertion of the second term in (4.2.24) into (4.2.23) yields the contribution

$$\mathcal{L}_{\mathcal{K}_A}\big(\mathcal{K}_B{}^p\,\tilde{C}_{pkmn}\big) + 4\,\mathcal{K}_B{}^p\,\partial_{[p}\big(\mathcal{K}_A{}^q\,\tilde{C}_{kmn]q}\big) \;=\; f_{AB}{}^C\mathcal{K}_C{}^p\,\tilde{C}_{pkmn} + 5\,\mathcal{K}_A{}^p\mathcal{K}_B{}^q\,\partial_{[p}\tilde{C}_{qkmn]}\;. \tag{4.2.30}$$

Here we used (4.2.11) and combined the terms from $\mathcal{L}_{\mathcal{K}_A}\tilde{C}_{pkmn}$ with those from the second term on the left-hand side. Employing now (4.2.29) and (4.2.30) we find that insertion of (4.2.24) into (4.2.23) yields

$$0 \;=\; \mathcal{K}_A{}^p\mathcal{K}_B{}^q\big(5\,\partial_{[p}\tilde{C}_{qkmn]} - \frac{1}{4}\,\lambda\,\tilde{\omega}_{pqkmn}\big)\;. \tag{4.2.31}$$

Thus, we have determined $\tilde{C}$, up to closed terms, to be

$$5\,\partial_{[p}\tilde{C}_{qkmn]} \;=\; \frac{1}{4}\lambda\,\tilde{\omega}_{kmnpq}\;, \tag{4.2.32}$$

which can be integrated to solve for $\tilde{C}_{klmn}$, since in five coordinates the integrability condition is trivially satisfied. In total we have proved that the (A,B) component of the third equation (4.2.8) of the system is solved by (4.2.24). We also note that the remaining components of (4.2.8) are identically satisfied under the assumption (4.2.21). (For the (u,v) component this requires using that the exterior derivative of $\mathcal{R}_{u\,m\alpha}$ vanishes by (4.2.18).) For the subsequent analysis it will be important to determine how $\tilde{C}$ transforms under the Killing vectors. To this end we recall that in the definition (4.2.24) $\tilde{C}$ is the only 'non-covariant' contribution, which therefore accounts for the second term on the left-hand side of the defining equation (4.2.23). From this we read off

$$\mathcal{L}_{\mathcal{K}_A}\tilde{C}_{mnkl} \;=\; -\sqrt{2}\,\partial_{[m}\mathcal{Z}_{A\,nkl]}\;. \tag{4.2.33}$$

Finally, we turn to the last equation (4.2.9), which determines $\mathcal{S}_u$. Under the assumptions (4.2.15), (4.2.21), the (u,v) and (u,A) components trivialize, while the (A,u) component implies

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{S}_{u\,n_1\ldots n_5\alpha} \;=\; -(D_A)_u{}^v\mathcal{S}_{v\,n_1\ldots n_5\alpha} + 20\sqrt{2}\,\partial_{[n_1}\mathcal{Z}_{A\,n_2n_3n_4}\,\mathcal{R}_{|u|n_5]\alpha}\;. \tag{4.2.34}$$

We will now show that this equation is solved by

$$\mathcal{S}_{u\,n_1\ldots n_5\alpha} \;=\; a\,\tilde{\omega}_{n_1\ldots n_5}\,\mathcal{Y}_{u\alpha} - 20\,\tilde{C}_{[n_1\ldots n_4}\,\partial_{n_5]}\mathcal{Y}_{u\alpha}\;, \tag{4.2.35}$$

in terms of the volume form of $\tilde{G}_{mn}$, the function defined in (4.2.18) and the four-form defined via (4.2.32). Here, a is an arbitrary coefficient, while we set the second coefficient to the value that is implied by the following analysis. We first note that $\mathcal{L}_{\mathcal{K}_A}\tilde{\omega}_{n_1\ldots n_5}=0$, which follows from the invariance under the Killing vectors of the metric $\tilde{G}$ defining $\tilde{\omega}$. Second, we recall (4.2.19), which states that the function $\mathcal{Y}_u$ transforms 'covariantly' under $\mathcal{L}_{\mathcal{K}_A}$ (i.e., w.r.t. the representation matrices D_A). Thus, all terms in (4.2.35) transform covariantly, except for the four-form $\tilde{C}$, whose 'anomalous' transformation must therefore account for the second term in $\mathcal{L}_{\mathcal{K}_A}\mathcal{S}_u$ on the right-hand side of (4.2.34). Using the anomalous transformations of $\tilde{C}$ given in (4.2.33), it then follows that (4.2.35) solves (4.2.34) for arbitrary coefficient a. This concludes our general discussion of the system of equations (4.2.6)–(4.2.9).

4.2.2 Explicit tensors

In [56], the twist equations (4.2.1) were solved for the sphere and hyperboloid compactifications, with gauge groups $\mathrm{SO}(p,6-p)$ and $\mathrm{CSO}(p,q,6-p-q)$, explicitly in terms of SL(6) group-valued twist matrices. Specifically, with the fundamental representation of $\mathrm{E}_{6(6)}$ decomposing as

$$\{Y^M\} \quad\longrightarrow\quad \{Y^{ab}, Y_{a\alpha}\}\;, \tag{4.2.36}$$

into $(15,1)\oplus(6',2)$ under $\mathrm{SL}(6)\times\mathrm{SL}(2)$, we single out one of the fundamental SL(6) indices $a\to(0,i)$ to define the SL(6) matrix $U_a{}^b$ as

$$\begin{aligned}
U_0{}^0 &\equiv (1-v)^{-5/6}\,(1+uK(u,v))\;,\\
U_0{}^i &\equiv -\eta_{ij}y^j\,(1-v)^{-1/3}\,K(u,v)\;,\\
U_i{}^0 &\equiv -\eta_{ij}y^j\,(1-v)^{-1/3}\;,\\
U_i{}^j &\equiv (1-v)^{1/6}\,\delta^{ij}\;,
\end{aligned}\tag{4.2.37}$$

with the combinations

$$u \equiv y^i\delta_{ij}y^j\;,\qquad v\equiv y^i\eta_{ij}y^j\;. \tag{4.2.38}$$

Here η_{ij} is the metric

$$\eta_{ij} = \mathrm{diag}\,(\underbrace{1,\dots,1}_{p-1},\underbrace{-1,\dots,-1}_{6-p})\;, \tag{4.2.39}$$

and we define similarly the $\mathrm{SO}(p,6-p)$ invariant metric η_{ab} with signature $(p,6-p)$. Note that in (4.2.38) we use two different metrics, one Euclidean, the other pseudo-Euclidean. The function $K(u,v)$ is the solution of the differential equation

$$2(1-v)\,(u\,\partial_vK+v\,\partial_uK) \;=\; ((7-2p)(1-v)-u)\,K-1\;, \tag{4.2.40}$$

which can be solved analytically. For instance, for $p=6$, i.e., for gauge group SO(6) relevant for the S^5 compactification, the solution reads

$$p=6\quad:\qquad K(u) \;=\; \frac{1}{2}\,u^{-3}\,\Big(u(u-3)+\sqrt{u(1-u)}\,\big(3\arcsin\sqrt{u}+c_0\big)\Big)\;, \tag{4.2.41}$$

with constant c_0. We refer to [56] for other explicit forms. The inverse twist matrix is given by

$$\begin{aligned}
(U^{-1})_0{}^0 &= (1-v)^{5/6}\;,\\
(U^{-1})_0{}^i &= \eta_{ij}y^j\,(1-v)^{1/3}\,K(u,v)\;,\\
(U^{-1})_i{}^0 &= \eta_{ij}y^j\,(1-v)^{1/3}\;,\\
(U^{-1})_i{}^j &= (1-v)^{-1/6}\,\big(\delta^{ij}+\eta_{ik}\eta_{jl}\,y^ky^l\,K(u,v)\big)\;.
\end{aligned}\tag{4.2.42}$$

Finally, the density factor ρ is given by

$$\rho \;=\; (1-v)^{1/6}\,. \tag{4.2.43}$$

Upon embedding the SL(6) twist matrix (4.2.37) into $\mathrm{E}_{6(6)}$, one may verify that it satisfies the consistency equations (4.2.1) with an embedding tensor that describes the gauge group $\mathrm{SO}(p,q)$, where the physical coordinates are embedded into the EFT coordinates via (4.2.36) according to

$$y^i \;\;=\;\; Y^{[0i]}\,. \tag{4.2.44}$$

With the above form of the generalised Scherk-Schwarz ansatz and the explicit form of the twist matrix and the scale factor we can give an explicit form of the geometric objects introduced in the previous section. To this end we have to split the $\mathrm{E}_{6(6)}$ indices further in order to make contact with the twist matrices given in (4.2.37), (4.2.42). As it turns out, for these twist matrices the split of indices $V_{\underline{M}} \;\equiv\; (V_A, V_u)$ discussed before (4.2.10), coincides with the split $27 = 15 + 12$ of (4.2.36)

$$V_{\underline{M}} \;\equiv\; (V_A, V_u) \;\equiv\; (V_{[ab]}, V^{a\alpha})\,, \qquad a,b \;=\; 0,\ldots,5\,, \quad \alpha,\beta \;=\; 1,2\,. \tag{4.2.45}$$

In several explicit formulas we will have to split $[ab]$ even further,

$$[ab] \;\equiv\; ([0i],[ij])\,, \qquad i,j = 1\,\ldots,5\,. \tag{4.2.46}$$

Similarly, we perform the same index split for the fundamental index M under $\mathrm{E}_{6(6)} \to \mathrm{SL}(6)$ (and then further to $\mathrm{GL}(5)\times\mathrm{SL}(2)$ according to (4.2.4)), thus giving up in the following the distinction between bare and underlined indices. Let us note that we employ the convention

$$V^{0i} \;\equiv\; \tfrac{1}{\sqrt{2}} V^i\,, \tag{4.2.47}$$

in agreement with the summation conventions of ref. [33]. In order to read off the various tensors from the twist matrices let us first canonically embed the SL(6) matrix $U_a{}^b$ into $\mathrm{E}_{6(6)}$. Under the above index split we have

$$U_M{}^{\underline{N}} \;\;=\;\; \begin{pmatrix} U_{[ab]}{}^{[cd]} & U_{[ab]}{}^{c\alpha} \\ U^{a\alpha,[cd]} & U^{a\alpha,}{}_{b\beta} \end{pmatrix} \;=\; \begin{pmatrix} U_{[a}{}^c\,U_{b]}{}^d & 0 \\ 0 & \delta^\alpha{}_\beta\,(U^{-1})_b{}^a \end{pmatrix}\,. \tag{4.2.48}$$

With this embedding, and recalling the convention (4.2.47), we can identify the Killing vector fields with components of the twist matrices as follows,

$$\mathcal{K}_{[ab]}{}^m \;\equiv\; \sqrt{2}\,(\widehat{U}^{-1})_{ab}{}^{m0}\,, \tag{4.2.49}$$

which yields

$$\mathcal{K}_{[0i]}{}^m(y) \;\;=\;\; -\frac{1}{2}\sqrt{2}\,(1-v)^{1/2}\,\delta_i^m\,, \qquad \mathcal{K}_{[ij]}{}^m(y) \;\;=\;\; \sqrt{2}\,\delta^m_{[i}\,\eta_{j]k}y^k\,. \tag{4.2.50}$$

It is straightforward to verify that these vectors satisfy the Lie bracket algebra (4.2.11). Specifically,

$$\big[\mathcal{K}_{ab}, \mathcal{K}_{cd}\big]^m \;=\; -\sqrt{2} f_{ab\,,cd}{}^{ef} \mathcal{K}_{ef}{}^m \;, \qquad f_{ab\,,cd}{}^{ef} \;\equiv\; 2\, \delta_{[a}{}^{[e} \eta_{b][c} \delta_{d]}{}^{f]} \;, \tag{4.2.51}$$

with the SO$(p, 6-p)$ metric η_{ab}. The Killing tensors defined in (4.2.25) are then found to be

$$\begin{aligned}
\mathcal{K}_{[0i]mnk} &= -\sqrt{2}\, \varepsilon_{mnkij}\, y^j \;, \\
\mathcal{K}_{[ij]mnk} &= -\sqrt{2}(1-v)^{-\frac{1}{2}}\, \varepsilon_{mnkpq} \big(\, \delta_i{}^p\, \delta_j{}^q - 2\, \delta_{[i}{}^p \eta_{j]l}\, y^q y^l \,\big) \;.
\end{aligned} \tag{4.2.52}$$

We can now define the metric $\tilde{G}$ as in (4.2.12) w.r.t. which these vectors are Killing, using the Cartan-Killing form $\eta^{ab,cd} = \eta^{a[c}\eta^{d]b}$. This yields for the metric and its inverse

$$\begin{aligned}
\tilde{G}_{mn} &= \eta_{mn} + (1-v)^{-1} \eta_{mp} \eta_{nq} y^p y^q \;, \\
\tilde{G}^{mn} &= \eta^{mn} - y^m y^n \;.
\end{aligned} \tag{4.2.53}$$

One may verify that this metric describes the homogeneous space SO(p,q)/SO$(p-1,q)$ with

$$\tilde{\mathcal{R}}_{mn} \;=\; 4\, \tilde{G}_{mn} \;, \tag{4.2.54}$$

determining the constant above, $\lambda = 4$. The associated volume form is given by

$$\tilde{\omega}_{mnklp} \;=\; (1-v)^{-\frac{1}{2}}\, \varepsilon_{mnklp} \;. \tag{4.2.55}$$

Next we give the function defining $\mathcal{R}$ in (4.2.18) w.r.t. the above index split,

$$\mathcal{R}_{u\,m\alpha} \;=\; \mathcal{R}^{a\beta}{}_{m\alpha} \;=\; \partial_m \mathcal{Y}^{a\beta}{}_\alpha \;, \tag{4.2.56}$$

for which we read off from the twist matrix

$$\mathcal{Y}^{a\beta}{}_\alpha \;=\; \mathcal{Y}^a \delta^\beta_\alpha \qquad \text{with} \qquad \mathcal{Y}^a(y) \;\equiv\; \begin{cases} (1-v)^{1/2} & a = 0 \\ y^i & a = i \end{cases} \;. \tag{4.2.57}$$

In agreement with (4.2.19) this transforms in the fundamental representation of the algebra of Killing vector fields (4.2.50). Specifically,

$$\mathcal{L}_{\mathcal{K}_{[ab]}} \mathcal{Y}^c \;=\; \mathcal{K}_{[ab]}{}^m\, \partial_m \mathcal{Y}^c \;=\; \sqrt{2}\, \delta^c{}_{[a}\, \mathcal{Y}_{b]} \;, \tag{4.2.58}$$

where $\mathcal{Y}_a$ is obtained from $\mathcal{Y}^a$ by means of η_{ab}. Let us also emphasize that the $\mathcal{Y}_a$ can be viewed as 'fundamental harmonics', satisfying

$$\Box \mathcal{Y}^a \;=\; -5\, \mathcal{Y}^a \;, \tag{4.2.59}$$

in that all higher harmonics can then be constructed from them. For instance, the Killing vectors themselves can be written as

$$\mathcal{K}_{[ab]m} \;=\; \sqrt{2}\big(\partial_m\mathcal{Y}_{[a}\big)\,\mathcal{Y}_{b]}\;. \tag{4.2.60}$$

Next we compute the four-form $\tilde{C}_{mnkl}$ by integrating (4.2.32). An explicit solution can be written in terms of the function K from (4.2.40) as

$$\tilde{C}_{mnkl} \;=\; \frac{\lambda}{16}\,(1-v)^{-1/2}\,\varepsilon_{mnklq}\,\big(K\delta^{qr}\eta_{rs}+\delta^q_s\big)y^s\;, \tag{4.2.61}$$

whose exterior derivative is indeed proportional to the volume form (4.2.55) for the metric $\tilde{G}_{mn}$. Together with the Killing vectors and tensors defined above, the $\mathcal{Z}$ tensor is now uniquely determined according to (4.2.24). Moreover, it is related to the twist matrix according to

$$\mathcal{Z}_{[ab]mnk} \;=\; \frac{1}{2}\,\varepsilon_{mnkpq}\,\big(\widehat{U}^{-1}\big)_{[ab]}{}^{[pq]} \;=\; \frac{1}{2}\,\varepsilon_{mnkpq}\,\rho^{-1}\,(U^{-1})_{[a}{}^{p}\,(U^{-1})_{b]}{}^{q}\;, \tag{4.2.62}$$

which agrees with (4.2.24) for $\lambda=4$.

Finally, let us turn to the tensor $\mathcal{S}_u$ whose general form is given in (4.2.35). Under the above index split it is convenient to write this tensor as

$$\mathcal{S}_{u\,n_1\ldots n_5\,\beta} \;\equiv\; \mathcal{S}^{a\alpha}{}_{n_1\ldots n_5\,\beta} \;\equiv\; \mathcal{S}^a\,\varepsilon_{n_1\ldots n_5}\,\delta^\alpha{}_\beta\;, \tag{4.2.63}$$

which is read off from the twist matrix as

$$\mathcal{S}^{a\alpha}{}_{n_1\ldots n_5\,\beta} \;=\; \varepsilon_{n_1\ldots n_5}\big(\widehat{U}^{-1}\big)^{a\alpha}{}_{0\beta} \;=\; \varepsilon_{n_1\ldots n_5}\,\rho^{-1}\,\delta^\alpha{}_\beta\,U_0{}^a\;, \tag{4.2.64}$$

leading with (4.2.37) to

$$\mathcal{S}^a \;=\; \left\{\begin{array}{ll} (1-v)^{-1}\,(1+uK) & a=0 \\ -\eta_{ij}y^j\,(1-v)^{-1/2}\,K & a=i \end{array}\right.\;. \tag{4.2.65}$$

One may verify that this agrees with (4.2.35) for

$$a \;=\; 1\;,\qquad \lambda \;=\; 4\;. \tag{4.2.66}$$

4.2.3 Useful identities

In this final paragraph we collect various identities satisfied by the above Killing-type tensors. These will be useful in the following sections when explicitly verifying the consistency of the Kaluza-Klein truncations. We find

$$
\begin{aligned}
\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{n} &= -\sqrt{2}\,f_{cd,ef}{}^{ab}\mathcal{K}^{[ef]}{}_{m} + 2\,\partial_m\left(\delta_{[c}{}^{[a}\mathcal{Y}^{b]}\mathcal{Y}_{d]}\right) && (4.2.67)\\
\mathcal{K}^{[ab]}{}_{n}\mathcal{K}_{[cd]}{}^{n} &= 2\,\delta_{[c}{}^{[a}\,\mathcal{Y}^{b]}\mathcal{Y}_{d]}\ , && (4.2.68)\\
\mathcal{K}_{[ab]}{}^{k}\mathcal{Z}_{[cd]\,kmn} + \mathcal{K}_{[cd]}{}^{k}\mathcal{Z}_{[ab]\,kmn} &= -\frac{1}{8}\,\varepsilon_{abcdef}\,\mathcal{K}^{[ef]}{}_{mn}\ , && (4.2.69)\\
\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{m}\mathcal{K}_{[ef]}{}^{n} &= 4\sqrt{2}\,\delta_{[c}{}^{[a}\,\mathcal{Y}_{d]}\mathcal{Y}_{[e}\,\delta_{f]}{}^{b]}\ , && (4.2.70)\\
\mathcal{K}_{[cd]}{}^{m}\mathcal{K}_{[ab]}{}^{n}\mathcal{K}_{[ef]}{}^{l}\,\partial_l\mathcal{K}^{[ab]}{}_{mn} &= -8\,\eta_{e[c}\,\mathcal{Y}_{d]}\mathcal{Y}_f + 8\,\eta_{f[c}\,\mathcal{Y}_{d]}\mathcal{Y}_e\ , && (4.2.71)
\end{aligned}
$$

which can be verified using the explicit tensors determined above.

4.3 The explicit IIB reduction ansatz

In terms of the $E_{6(6)}$ EFT fields, the reduction ansatz is given by the simple factorization (1.6.2) with the twist matrix U given by (4.2.42). In order to translate this into the original IIB theory, we may first decompose the EFT fields under (4.2.4), according to the IIB solution of the section constraint, and collect the expressions for the various components. We do this separately for EFT vectors, two-forms, metric, and scalars, and subsequently derive the expressions for three- and four-forms from the IIB self-duality equations, as outlined in the general case in section 3 of this chapter.

4.3.1 Vector and two-form fields

Breaking the 27 EFT vector fields according to (4.2.4) into

$$
\{\mathcal{A}_\mu{}^m, \mathcal{A}_{\mu\,m\alpha}, \mathcal{A}_{\mu\,kmn}, \mathcal{A}_{\mu\,\alpha}\}\ , \qquad (4.3.1)
$$

we read off the reduction ansatz from (1.6.2), (4.2.5), which in particular gives rise to

$$
\begin{aligned}
\mathcal{A}_\mu{}^m(x,y) &= \mathcal{K}_{[ab]}{}^m(y)\,A_\mu^{ab}(x)\ ,\\
\mathcal{A}_{\mu\,kmn}(x,y) &= \mathcal{Z}_{[ab]\,kmn}(y)\,A_\mu^{ab}(x)\ . && (4.3.2)
\end{aligned}
$$

The Kaluza-Klein vector field $\mathcal{A}_\mu{}^m = A_\mu{}^m$ thus reduces in the standard way with the 15 Killing vectors $\mathcal{K}_{[ab]}{}^m(y)$ whose algebra defines the gauge group of the $D=5$ theory. Note, however, that these extend to Killing vectors of the internal space-time metric only in case of the compact gauge group SO(6). In the general case, as discussed above, the $\mathcal{K}_{[ab]}{}^m(y)$ are the Killing vector fields of an auxiliary homogeneous Lorentzian metric (4.2.12), compare also [69, 70, 71]. The vector field components $\mathcal{A}_{\mu\,kmn}$ are expressed in terms of the same 15 $D=5$ vector fields. Their internal coordinate dependence is not exclusively carried by Killing vectors and tensors, but exhibits via the tensor

$\mathcal{Z}_{[ab]\,kmn}(y)$ an inhomogeneous term carrying the four-form $\tilde{C}_{mnkl}$ according to (4.2.24).[3] This is similar to reduction formulas for the dual vector fields in the S^7 reduction of $D = 11$ supergravity [85], which, however, in the present case already show up among the fundamental vectors.

For the remaining vector field components, the ansatz (1.6.2), (4.2.5), at first yields the reduction formulas

$$\begin{aligned} \mathcal{A}_{\mu\, m\alpha}(x,y) &= \mathcal{R}^{a\beta}{}_{m\alpha}(y)\, A_{\mu a\beta}(x) \;=\; \partial_m \mathcal{Y}^a(y)\, A_{\mu\, a\alpha}(x) \;, \\ \mathcal{A}_{\mu\,\alpha}(x,y) &= \mathcal{S}^a(y)\, A_{\mu\, a\alpha}(x) \\ &= |\tilde{G}|^{1/2} \left(\mathcal{Y}^a(y) - \frac{1}{6}\, \tilde{\omega}^{klmnp} \tilde{C}_{klmn}\, \partial_p \mathcal{Y}^a(y) \right) A_{\mu\, a\alpha}(x) \;, \end{aligned} \tag{4.3.3}$$

in terms of the 12 vector fields $A_{\mu\, a\alpha}$ in $D = 5$ and the tensors defined in (4.2.35) and (4.2.56). However, as discussed in the previous section, for the SO(p, q) gauged theories, a natural gauge fixing of the two-form tensor gauge transformations allows to eliminate these vector fields in exchange for giving topological mass to the two-forms. As a result, the final reduction ansatz reduces to

$$\mathcal{A}_{\mu\, m\alpha} \;=\; 0 \;=\; \mathcal{A}_{\mu\,\alpha} \;. \tag{4.3.4}$$

For the two-forms, upon breaking them into GL(5) components

$$\{\mathcal{B}_{\mu\nu}{}^{\alpha}, \mathcal{B}_{\mu\nu\, mn}, \mathcal{B}_{\mu\nu}{}^{m\alpha}, \mathcal{B}_{\mu\nu\, m}\} \;, \tag{4.3.5}$$

similar reasoning via (1.6.2) and evaluation of the twist matrix $\rho^{-2}\, U_M{}^{\underline{N}}$ gives the following ansatz for the SL(2) doublets

$$\begin{aligned} \mathcal{B}_{\mu\nu}{}^{\alpha}(x,y) &= \mathcal{Y}_a(y)\, B_{\mu\nu}{}^{a\alpha}(x) \;, \\ \mathcal{B}_{\mu\nu}{}^{m\alpha}(x,y) &= \mathcal{Z}_a{}^m(y)\, B_{\mu\nu}{}^{a\alpha}(x) \;, \end{aligned} \tag{4.3.6}$$

in terms of the 12 topologically massive two-form fields of the $D = 5$ theory. Here, $\mathcal{Z}_a{}^m(y)$ is the vector density, given by[3]

$$\mathcal{Z}_a{}^m \;=\; |\tilde{G}|^{1/2} \left(\tilde{G}^{mn} \partial_n \mathcal{Y}_a + \frac{1}{6}\, \tilde{\omega}^{mklpq} \tilde{C}_{klpq} \mathcal{Y}_a \right) \;, \tag{4.3.7}$$

in terms of the Lorentzian metric $\tilde{G}_{mn}$, vector field $\mathcal{Y}_a$, and four-form $\tilde{C}_{klmn}$. As is obvious from their index structure, the fields $\mathcal{B}_{\mu\nu}{}^{m\alpha}$ contribute to the dual six-form doublet of the IIB theory, but not to the original IIB fields. Accordingly, for matching the EFT Lagrangian to the IIB dynamics, these fields are integrated out from the theory [33, 1].

[3] This seems to differ from the ansatz derived in [80]. The precise comparison should take into account that the A_μ, $B_{\mu\nu}$ are non-gauge-invariant vector potentials. In the present discussion, the inhomogeneous term in $\mathcal{Z}_{[ab]\,kmn}(y)$ played a crucial role in the verification of the proper algebraic relations.

For the IIB embedding of $D = 5$ supergravity, we will thus only need the first line of (4.3.6).

For the remaining two-form fields, the reduction ansatz (1.6.2) yields the explicit expressions

$$
\begin{aligned}
\mathcal{B}_{\mu\nu\,m}(x,y) &= \mathcal{Z}^{[ab]}{}_m(y)\,B_{\mu\nu\,ab}(x)\;,\\
\mathcal{B}_{\mu\nu\,mn}(x,y) &= -\frac{1}{4}\,\sqrt{2}\,\mathcal{K}^{[ab]}{}_{mn}(y)\,B_{\mu\nu\,ab}(x)\;,
\end{aligned}
\tag{4.3.8}
$$

with the Killing tensor $\mathcal{K}^{[ab]}{}_{mn} = 2\,\partial_{[m}\mathcal{K}^{[ab]}{}_{n]}$, and the tensor density $\mathcal{Z}^{[ab]}{}_m$ given by

$$
\mathcal{Z}^{[ab]}{}_m = |\tilde{G}|^{1/2}\left(\mathcal{K}^{[ab]}{}_m + \frac{1}{12}\,\tilde{\omega}^{klnpq}\mathcal{K}^{[ab]}{}_{mk}\,\tilde{C}_{lnpq}\right)\;. \tag{4.3.9}
$$

Here, the 15 $D = 5$ two-forms $B_{\mu\nu\,ab}$ are in fact absent in the SO(p, q) supergravities, described in the previous section. In principle, they may be introduced on-shell, employing the formulation of these theories given in [83, 50], however, subject to an additional (three-form tensor) gauge freedom, which subsequently allows one to set them to zero. Hence, in the following we adopt $B_{\mu\nu\,ab}(x) = 0$, such that (4.3.8) reduces to

$$
\mathcal{B}_{\mu\nu\,m} = 0 = \mathcal{B}_{\mu\nu\,mn}\;. \tag{4.3.10}
$$

Within EFT, consistency of this choice with the reduction ansatz (4.3.8) can be understood by the fact that the fields $\mathcal{B}_{\mu\nu\,m}$ (related to the IIB dual graviton) do not even enter the EFT Lagrangian, while the fields $\mathcal{B}_{\mu\nu\,mn}$ enter subject to gauge freedom

$$
\delta\mathcal{B}_{\mu\nu\,mn} = 2\,\partial_{[m}\Lambda_{n]\,\mu\nu}\;, \tag{4.3.11}
$$

(descending from tensor gauge transformations of the IIB four-form potential), which allows us to explicitly gauge the reduction ansatz (4.3.8) to zero.

Combining the reduction formulas for the EFT fields with the explicit dictionary (3.3.9), we can use the results of this section to give the explicit expressions for the different components (3.2.26) of the type IIB form fields. This gives the following reduction formulae

$$
C_{\mu\nu}{}^{\alpha}(x,y) = \sqrt{10}\,\mathcal{Y}_a(y)\,B_{\mu\nu}{}^{a\alpha}(x)\;, \tag{4.3.12}
$$

$$
\begin{aligned}
C_{\mu m}{}^{\alpha}(x,y) &= 0\;,\\
C_{\mu\nu\,mn}(x,y) &= \frac{\sqrt{2}}{4}\,\mathcal{K}_{[ab]}{}^k(y)\mathcal{Z}_{[cd]\,kmn}(y)\,A_{[\mu}{}^{ab}(x)A_{\nu]}{}^{cd}(x)\;,\\
C_{\mu kmn}(x,y) &= \frac{\sqrt{2}}{4}\,\mathcal{Z}_{[ab]\,kmn}(y)\,A_{\mu}{}^{ab}(x)\;,
\end{aligned}
\tag{4.3.13}
$$

for two- and four form gauge potential in the basis after standard Kaluza-Klein decomposition. In the next subsection, we collect the expressions for the scalar components

$C_{mn}{}^{\alpha}$ and C_{klmn}, and in subsection 4.3.4 we derive the reduction formulas for the last missing components $C_{\mu\nu\rho\, m}$, and $C_{\mu\nu\rho\sigma}$ of the four-form.

Let us finally note that with the reduction formulas given in this section, also the non-abelian EFT field strengths of the vector fields factorize canonically, as can be explicitly verified with the identities given in (4.2.11), (4.2.22). Explicitly, we find

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{m} &\equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^{m} - \mathcal{A}_{\mu}{}^{n}\partial_{n}\mathcal{A}_{\nu}{}^{m} + \mathcal{A}_{\nu}{}^{n}\partial_{n}\mathcal{A}_{\mu}{}^{m} \\
&= \mathcal{K}_{[ab]}{}^{m}(y)\left(2\,\partial_{[\mu}A_{\nu]}{}^{ab}(x) + \sqrt{2}\,f_{cd,ef}{}^{ab}A_{\mu}{}^{cd}A_{\nu}{}^{ef}(x)\right) \\
&= \mathcal{K}_{[ab]}{}^{m}(y)\,F_{\mu\nu}{}^{ab}(x)\;, \\
\mathcal{F}_{\mu\nu\,kmn} &\equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]\,kmn} - 2\,\mathcal{A}_{[\mu}{}^{l}\partial_{l}\mathcal{A}_{\nu]\,kmn} - 3\,\partial_{[k}\mathcal{A}_{[\mu}{}^{l}\mathcal{A}_{\nu]\,mn]l} + 3\,\mathcal{A}_{[\mu}{}^{l}\partial_{[k}\mathcal{A}_{\nu]\,mn]l} \\
&= \mathcal{Z}_{[ab]\,kmn}(y)\,F_{\mu\nu}{}^{ab}(x)\;,
\end{aligned}
\tag{4.3.14}
$$

in terms of the non-abelian SO(p,q) field strength $F_{\mu\nu}{}^{ab}(x)$ from (4.1.10).

4.3.2 EFT scalar fields and metric

Similar to the discussion of the form fields, the reduction of the EFT scalars can be read off from (1.6.2) upon proper parametrization of the matrix $\mathcal{M}_{MN}$. We recall that $\mathcal{M}_{MN}$ is a real symmetric $\mathrm{E}_{6(6)}$ matrix parametrized by the 42 scalar fields

$$
\{G_{mn}, C_{mn}{}^{\alpha}, C_{klmn}, m_{\alpha\beta}\}\;,
\tag{4.3.15}
$$

where $C_{mn}{}^{\alpha} = C_{[mn]}{}^{\alpha}$, and $C_{klmn} = C_{[klmn]}$ are fully antisymmetric in their internal indices, $G_{mn} = G_{(mn)}$ is the symmetric 5×5 matrix, representing the internal part of the IIB metric, and $m_{\alpha\beta} = m_{(\alpha\beta)}$ is the unimodular symmetric 2×2 matrix parametrizing the coset space SL(2)/SO(2) carrying the IIB dilaton and axion. Decomposing the matrix $\mathcal{M}_{MN}$ into blocks according to the basis (4.3.1)

$$
\mathcal{M}_{KM} \;=\; \begin{pmatrix} \mathcal{M}_{k,m} & \mathcal{M}_{k}{}^{m\beta} & \mathcal{M}_{k,mn} & \mathcal{M}_{k}{}^{\beta} \\ \mathcal{M}^{k\alpha}{}_{m} & \mathcal{M}^{k\alpha,m\beta} & \mathcal{M}^{k\alpha}{}_{mn} & \mathcal{M}^{k\alpha,\beta} \\ \mathcal{M}_{kl,m} & \mathcal{M}_{kl}{}^{m\beta} & \mathcal{M}_{kl,mn} & \mathcal{M}_{kl}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_{m} & \mathcal{M}^{\alpha,m\beta} & \mathcal{M}^{\alpha}{}_{mn} & \mathcal{M}^{\alpha,\beta} \end{pmatrix}\;,
\tag{4.3.16}
$$

the scalar fields (4.3.15) can be read off from the various components of $\mathcal{M}_{MN}$ and its inverse $\mathcal{M}^{MN}$. We collect the final result

$$
\begin{aligned}
G^{mn} &= (\det G)^{1/3}\,\mathcal{M}^{m,n}\;, \qquad\qquad (4.3.17) \\
m^{\alpha\beta} &= (\det G)^{2/3}\mathcal{M}^{\alpha,\beta}\;, \\
C_{mn}{}^{\alpha} &= \sqrt{2}\,\varepsilon^{\alpha\beta}(\det G)^{2/3}\,m_{\beta\gamma}\,\mathcal{M}^{\gamma}{}_{mn} \;=\; -\,\varepsilon^{\alpha\beta}(\det G)^{1/3}\,G_{nk}\,\mathcal{M}_{m\beta}{}^{k}\;, \\
C_{klmn} &= \frac{1}{8}\,(\det G)^{2/3}\,\varepsilon_{klmnp}\,m_{\alpha\beta}\,\mathcal{M}^{\alpha,p\beta} \;=\; -\,\frac{\sqrt{2}}{16}\,(\det G)^{1/3}\varepsilon_{klmnp}G_{qr}\,\mathcal{M}^{pq,r}\;,
\end{aligned}
$$

where G^{mn} and $m^{\alpha\beta}$ denote the inverse matrices of G_{mn} and $m_{\alpha\beta}$ from (4.3.15). The last two lines represent examples how the $C_{mn}{}^{\alpha}$ and C_{klmn} can be obtained in different but equivalent ways either from components of $\mathcal{M}_{MN}$ or $\mathcal{M}^{MN}$. This of course does not come as a surprise but is a simple consequence of the fact that the 27×27 matrix $\mathcal{M}_{MN}$ representing the 42-dimensional coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ is subject to a large number of non-linear identities.

With (4.3.17), the reduction formulas for the EFT scalars are immediately derived from (1.6.2). For the IIB metric and dilaton/axion, this gives rise to the expressions

$$
\begin{aligned}
G^{mn}(x,y) &= \Delta^{2/3}(x,y)\, \mathcal{K}_{[ab]}{}^{m}(y)\mathcal{K}_{[cd]}{}^{n}(y)\, M^{ab,cd}(x) \;, \\
m^{\alpha\beta}(x,y) &= \Delta^{4/3}(x,y)\, \mathcal{Y}_{a}(y)\mathcal{Y}_{b}(y)\, M^{a\alpha,b\beta}(x) \;,
\end{aligned}
\tag{4.3.18}
$$

with the function $\Delta(x,y)$ defined by

$$
\Delta(x,y) \;\equiv\; \rho^{3}(y)\,(\det G)^{1/2} \;=\; (1-v)^{1/2}\,(\det G)^{1/2} \;,
\tag{4.3.19}
$$

and the 42 five-dimensional scalar fields parameterizing the symmetric $\mathrm{E}_{6(6)}$ matrix M_{MN} decomposed into an $\mathrm{SL}(6)\times\mathrm{SL}(2)$ basis as (4.1.13).

Similarly, the reduction formula for the internal components of the two-form $C_{mn}{}^{\alpha}$ is read off as

$$
\begin{aligned}
C_{mn}{}^{\alpha}(x,y) &= -\varepsilon^{\alpha\beta}\Delta^{2/3}(x,y)\, G_{nk}(x,y)\, \partial_{m}\mathcal{Y}^{c}(y)\, \mathcal{K}_{[ab]}{}^{k}(y)\, M^{ab}{}_{c\beta}(x) \\
&= -\frac{1}{2}\,\varepsilon^{\alpha\beta}\Delta^{4/3}(x,y)\, m_{\beta\gamma}(x,y)\, \mathcal{Y}_{c}(y)\, \mathcal{K}^{[ab]}{}_{mn}(y)\, M_{ab}{}^{c\gamma}(x) \;,
\end{aligned}
\tag{4.3.20}
$$

featuring the inverse matrices of (4.3.18), with the two alternative expressions corresponding to using the different equivalent expressions in (4.3.17). To explicitly show the second equality in (4.3.20) requires rather non-trivial quadratic identities among the components (4.1.13) of an $\mathrm{E}_{6(6)}$ matrix, together with non-trivial identities among the Killing vectors and tensors. In contrast, this identity simply follows on general grounds from the equivalence of the two expressions in (4.3.17), i.e., it follows from the group property of $\mathcal{M}_{MN}$ and the twist matrix $U_{M}{}^{\underline{N}}$. Let us also stress, that throughout all indices on the Killing vectors $\mathcal{K}_{[ab]}{}^{m}$ and tensors are raised and lowered with the Lorentzian x-independent metric $\tilde{G}_{mn}(y)$ from (4.2.12), not with the space-time metric $G_{mn}(x,y)$.

Eventually, the same reasoning gives the reduction formula for C_{mnkl}

$$
C_{klmn}(x,y) \;=\; \frac{1}{8}\,\varepsilon_{klmnp}\,\Delta^{4/3}(x,y)\, m_{\alpha\beta}(x,y)\, \mathcal{Y}_{a}(y)\, \mathcal{Z}_{b}{}^{p}(y)\, M^{a\alpha,b\beta}(x) \;,
\tag{4.3.21}
$$

with $\mathcal{Z}_{b}{}^{p}(y)$ from (4.3.7). Explicitly, this takes the form

$$
\begin{aligned}
C_{klmn}(x,y) \;=\; & \frac{1}{16}\,\tilde{\omega}_{klmnp}\,\Delta^{4/3}(x,y)\, m_{\alpha\beta}(x,y)\, \tilde{G}^{pq}(y)\, \partial_{q}\left(\Delta^{-4/3}(x,y)\, m^{\alpha\beta}(x,y)\right) \\
& + \tilde{C}_{klmn}(y) \;.
\end{aligned}
\tag{4.3.22}
$$

On the other hand, using the last identity in (4.3.17) to express C_{klmn}, the reduction formula is read off as

$$
\begin{aligned}
C_{klmn}(x,y) &= \frac{\sqrt{2}}{4}\,\Delta^{2/3}(x,y)\,\mathcal{Z}_{[ab]\,[klm}(y)G_{n]r}(x,y)\,\mathcal{K}_{[cd]}{}^{r}(y)\,M^{ab,cd}(x) \\
&= \tilde{C}_{mnkl}(y) - \frac{1}{8}\,\Delta^{2/3}(x,y)\,\mathcal{K}_{[ab]}{}^{p}(y)\,\mathcal{K}_{[cd]\,[klm}(y)\,G_{n]p}(x,y)\,M^{ab,cd}(x)\;,
\end{aligned}
\tag{4.3.23}
$$

where we have used the explicit expression (4.2.24) for $\mathcal{Z}_{[ab]\,klm}$. Again, the equivalence between (4.3.22) and (4.3.23) is far from obvious, but a consequence of the group property of $\mathcal{M}_{MN}$ and the twist matrix $U_{M}{}^{\underline{N}}$. For the case of the sphere S^5, several of these reduction formulas have appeared in the literature [86, 87, 88, 66, 80]. Here we find that they naturally generalize to the case of hyperboloids, inducing the $D=5$ non-compact $\mathrm{SO}(p,q)$ gaugings.

Let us finally spell out the reduction ansatz for the five-dimensional metric which follows directly from (1.6.2) as

$$
g_{\mu\nu}(x,Y) = \rho^{-2}(y)\,\mathrm{g}_{\mu\nu}(x)\;. \tag{4.3.24}
$$

Putting this together with the parametrization of the IIB metric in terms of the EFT fields, and the reduction (4.3.2) of the Kaluza-Klein vector field, we arrive at the full expression for the IIB metric

$$
\begin{aligned}
ds^2 &= \Delta^{-2/3}(x,y)\,\mathrm{g}_{\mu\nu}(x)\,dx^\mu dx^\nu \\
&\quad + G_{mn}(x,y)\left(dy^m + \mathcal{K}_{[ab]}{}^{m}(y)A_\mu^{ab}(x)dx^\mu\right)\left(dy^n + \mathcal{K}_{[cd]}{}^{n}(y)A_\nu^{cd}(x)dx^\nu\right)\;,
\end{aligned}
\tag{4.3.25}
$$

in standard Kaluza-Klein form [89], with G_{mn} given by the inverse of (4.3.18).

4.3.3 Background geometry

It is instructive to evaluate the above formulas at the particular point where all $D=5$ fields vanish, i.e. in particular the scalar matrix M_{MN} reduces to the identity matrix

$$
M_{MN}(x) = \delta_{MN}\;. \tag{4.3.26}
$$

This determines the background geometry around which the generalised Scherk-Schwarz reduction ansatz captures the fluctuations. Depending on whether or not the scalar potential of $D=5$ gauged supergravity has a stationary point at the origin — which is the case for the SO(6) and SO(3,3) gaugings [7] — this background geometry will correspond to a solution of the IIB field equations.

With (4.3.26) and the vanishing of the Kaluza-Klein vector fields, the IIB metric (4.3.25) reduces to

$$
\begin{aligned}
ds^2 &= \overset{\circ}{G}_{\hat\mu\hat\nu}\, dX^{\hat\mu} dX^{\hat\nu} \qquad (4.3.27)\\
&\equiv (1+u-v)^{1/2}\,\overset{\circ}{g}_{\mu\nu}(x)\, dx^\mu dx^\nu + (1+u-v)^{-1/2}\left(\delta_{mn} + \frac{\eta_{mi}\eta_{nj}y^i y^j}{1-v}\right) dy^m dy^n\,,
\end{aligned}
$$

where we have used the relations

$$
\begin{aligned}
\delta^{ac}\delta^{bd}\,\mathcal{K}_{[ab]}{}^m(y)\mathcal{K}_{[cd]}{}^n(y) &= (1+u-v)\,\delta^{mn} - \eta_{mi}\eta_{nj}y^i y^j\,,\\
\overset{\circ}{\Delta} &= (1+u-v)^{-3/4}\,. \qquad (4.3.28)
\end{aligned}
$$

The internal metric of (4.3.27) is conformally equivalent to the hyperboloid $H^{p,6-p}$ defined by the embedding of the surface

$$
z_1^2 + \dots + z_p^2 - z_{p+1}^2 - \dots - z_6^2 \;\equiv\; 1\,, \qquad (4.3.29)
$$

in $\mathbb{R}^6$. This is a Euclidean five-dimensional space with isometry group $\mathrm{SO}(p)\times\mathrm{SO}(6-p)$, inhomogeneous for $p=2,3,4$. Except for $p=6$, this metric differs from the homogeneous Lorentzian metric defined in (4.2.12) with respect to which the Killing vectors and tensors parametrizing the reduction ansatz are defined.

Using that $\mathcal{Y}_a\mathcal{Y}_b\,\delta^{ab} = 1+u-v$, it follows from (4.3.18) that the IIB dilaton and axion are constant

$$
\overset{\circ}{m}{}^{\alpha\beta} \;=\; \delta^{\alpha\beta}\,, \qquad (4.3.30)
$$

while the internal two-form (4.3.20) vanishes due to the fact that (4.3.26) does not break the SL(2). Eventually, the four-form C_{klmn} is most conveniently evaluated from (4.3.22) as

$$
\begin{aligned}
\overset{\circ}{C}_{klmn} &= \tilde{C}_{klmn} - \frac{1}{6}\,\tilde{\omega}_{klmnp}\,\tilde{G}^{pq}\,\overset{\circ}{\Delta}{}^{-1}\partial_q\overset{\circ}{\Delta}\\
&= \frac{1}{4}\,\varepsilon_{klmnp}\,\eta^{pq}y^q\,(1-v)^{-1/2}\,\left(K(u,v) + (1+u-v)^{-1}\right)\,, \qquad (4.3.31)
\end{aligned}
$$

which can also be confirmed from (4.3.23). In particular, its field strength is given by

$$
5\,\partial_{[k}\overset{\circ}{C}_{lmnp]} \;=\; \frac{1}{2}\,\varepsilon_{klmnp}\,\frac{p-4+(p-3)(u-v)}{(1-v)^{1/2}\,(1+u-v)^2}\,, \qquad (4.3.32)
$$

where we have used the differential equation (4.2.40) for the function $K(u,v)$. Again, it is only for $p=6$, that the background four-form potential $\overset{\circ}{C}_{klmn}$ coincides with the four-form $\tilde{C}_{klmn}$ that parametrizes the twist matrix $U_M{}^N$.

With this ansatz, the type IIB field equations reduce to the Einstein equations, which in this normalization take the form

$$
\overset{\circ}{R}_{mn} \;=\; \overset{\circ}{T}_{mn} \;\equiv\; \frac{25}{6}\,\partial_{[m}\overset{\circ}{C}_{klpq]}\partial_{[n}\overset{\circ}{C}_{rstu]}\,\overset{\circ}{G}{}^{kr}\overset{\circ}{G}{}^{ls}\overset{\circ}{G}{}^{pt}\overset{\circ}{G}{}^{qu}\,, \qquad (4.3.33)
$$

and similar for $\overset{\circ}{R}_{\mu\nu}$. With (4.3.27) and (4.3.32), the energy-momentum tensor takes a particularly simple form for $p = 6$ and $p = 3$:

$$\overset{\circ}{T}_{mn} \;=\; \begin{cases} 4\,\overset{\circ}{G}_{mn} & p=6 \\ (1+u-v)^{-5/2}\,\overset{\circ}{G}_{mn} & p=3 \end{cases} . \tag{4.3.34}$$

For the x-dependent background metric $\overset{\circ}{g}_{\mu\nu}(x)$ the most symmetric ansatz assumes an Einstein space (dS, AdS, or Minkowski)

$$R[\overset{\circ}{g}]_{\mu\nu} \;=\; k\,\overset{\circ}{g}_{\mu\nu}\,, \tag{4.3.35}$$

upon which the IIB Ricci tensor associated with (4.3.27) turns out to be blockwise proportional to the IIB metric for the same two cases $p = 6$ and $p = 3$

$$\overset{\circ}{R}_{mn} \;=\; \begin{cases} 4\,\overset{\circ}{G}_{mn} & p=6 \\ (1+u-v)^{-5/2}\,\overset{\circ}{G}_{mn} & p=3 \end{cases} ,$$

$$\overset{\circ}{R}_{\mu\nu} \;=\; \begin{cases} k\,\overset{\circ}{G}_{\mu\nu} & p=6 \\ -(1+u-v)^{-5/2}\,(1+(2-k)\,(1+u-v)^2)\,\overset{\circ}{G}_{\mu\nu} & p=3 \end{cases} . \tag{4.3.36}$$

Together it follows that (4.3.27), (4.3.31), (4.3.35) solve the IIB field equations for $p = 3, k = 2$ and $p = 6, k = -4$, c.f. [69]. The resulting backgrounds are $\mathrm{AdS}_5 \times S^5$ and $\mathrm{dS}_5 \times H^{3,3}$ and the induced $D = 5$ theories correspond to the SO(6) and the SO(3, 3) gaugings of [7], respectively. For $3 \neq p \neq 6$, the background geometry is not a solution to the IIB field equations. Let us stress, however, that also in these cases the reduction ansatz presented in the previous sections describes a consistent truncation of the IIB theory to an effectively $D = 5$ supergravity theory, but this theory does not have a simple ground state with all fields vanishing.

4.3.4 Reconstructing 3-form and 4-form

We have in the previous sections derived the reduction formulas for all EFT scalars, vectors, and two-forms. Upon using the explicit dictionary into the IIB fields [33, 1], this allows to reconstruct the major part of the original IIB fields. More precisely, among the components of the fundamental IIB fields only $\hat{C}_{\mu\nu\rho m}$ and $\hat{C}_{\mu\nu\rho\sigma}$ with three and four external legs of the IIB four-form potential remain undetermined from the previous analysis. These in turn can be reconstructed from the IIB self-duality equations, which are induced by the EFT dynamics. We refer to [1] for the details of the general procedure, which we work out in the following with the generalised Scherk-Schwarz reduction ansatz.

The starting point is the duality equation between EFT vectors and two-forms that follows from the Lagrangian

$$\partial_{[k}\left(\tilde{\mathcal{H}}_{|\mu\nu\rho|\,mn]} - \frac{1}{2}\,e\,\mathcal{M}_{mn],N}\,\mathcal{F}^{\sigma\tau\,N}\,\varepsilon_{\mu\nu\rho\sigma\tau}\right) \;=\; 0\,, \tag{4.3.37}$$

where $\mathcal{F}_{\mu\nu}{}^N$ is the non-abelian field strength associated with the vector fields $\mathcal{A}_\mu{}^N$, and $\tilde{\mathcal{H}}_{|\mu\nu\rho|\,mn}$ carries the field strength of the two-forms $\mathcal{B}_{\mu\nu\,mn}$. Taking into account the reduction ansatz (4.3.4), (4.3.10), it takes the explicit form

$$\begin{aligned}
\tilde{\mathcal{H}}_{\mu\nu\rho\,mn} &= -\partial_{[\mu}\mathcal{A}_\nu{}^k\mathcal{A}_{\rho]\,kmn} - \mathcal{A}_{[\mu}{}^k\partial_\nu\mathcal{A}_{\rho]\,kmn} - \mathcal{F}_{[\mu\nu}{}^k\mathcal{A}_{\rho]\,kmn} - \mathcal{A}_{[\mu}{}^k\mathcal{F}_{\nu\rho]\,kmn} \\
&\quad + 2\,\partial_{[m}\left(\mathcal{A}_{[\mu}{}^k\mathcal{A}_\nu{}^l\mathcal{A}_{\rho]\,n]kl}\right) \,,
\end{aligned} \tag{4.3.38}$$

in terms of the remaining vector fields and field strengths from (4.3.14). Since (4.3.37) is of the form of a vanishing curl, the equation can be integrated in the internal coordinates up to a curl $\partial_{[m}C_{n]\,\mu\nu\rho}$ related to the corresponding component of the IIB four-form, explicitly

$$\partial_{[m}C_{n]\,\mu\nu\rho} = \frac{1}{16}\sqrt{2}\,e\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\mathcal{M}_{mn,N}\,\mathcal{F}^{\sigma\tau\,N} - \frac{1}{8}\sqrt{2}\,\tilde{\mathcal{H}}_{\mu\nu\rho\,mn}\,. \tag{4.3.39}$$

It is a useful consistency test of the present construction, that with the reduction ansatz described in the previous sections, the r.h.s. of this equation indeed takes the form of a curl in the internal variables. Let us verify this explicitly. Since the reduction ansatz is covariant, the first term reduces according to the form of its free indices $[mn]$, c.f. (4.3.8)

$$e\,\mathcal{M}_{mn,N}\,\mathcal{F}^{\sigma\tau\,N} = -\frac{1}{2}\sqrt{2}\,\partial_{[m}\mathcal{K}^{[ab]}{}_{n]}\left(\sqrt{|\mathrm{g}|}\,M_{ab,N}\,F^{\sigma\tau\,N}\right)\,, \tag{4.3.40}$$

which indeed takes the form of a curl. We recall that the $D=5$ field strength $F_{\mu\nu}{}^N$ combines the 15 non-abelian field strengths $F_{\mu\nu}{}^{ab}$ and the 12 two-forms $B_{\mu\nu\,a\alpha}$ according to (4.1.10). The reduction of the second term on the r.h.s. of (4.3.39) is less obvious, since $\tilde{\mathcal{H}}_{\mu\nu\rho\,mn}$ is not a manifestly covariant object, and we have computed it explicitly by combining its defining equation (4.3.38) with the reduction of the vector fields (4.3.2) and field strengths (4.3.14). With the identity (4.2.69) among the Killing vectors and tensors, the second term on the r.h.s. of (4.3.39) then reduces according to

$$\tilde{\mathcal{H}}_{\mu\nu\rho\,mn} = \frac{1}{8}\,\varepsilon_{abcdef}\,\mathcal{K}^{[ef]}{}_{mn}\,\Omega^{abcd}_{\mu\nu\rho} + 2\,\partial_{[m}\left(\mathcal{A}_{[\mu}{}^k\mathcal{A}_\nu{}^l\mathcal{A}_{\rho]n]kl}\right)\,. \tag{4.3.41}$$

with the non-abelian $\mathrm{SO}(p,q)$ Chern-Simons form defined as

$$\Omega^{abcd}_{\mu\nu\rho} = \partial_{[\mu}A_\nu{}^{ab}A_{\rho]}{}^{cd} + F_{[\mu\nu}{}^{ab}A_{\rho]}{}^{cd}\,, \tag{4.3.42}$$

in terms of the $\mathrm{SO}(p,6-p)$ Yang-Mills field strength $F_{\mu\nu}{}^{ab}$. Again, (4.3.41) takes the form of a curl in the internal variables, such that equation (4.3.39) can be explicitly integrated to

$$\begin{aligned}
C_{m\,\mu\nu\rho} &= -\frac{1}{32}\,\mathcal{K}^{[ab]}{}_m\left(2\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,M_{ab,N}F^{\sigma\tau\,N} + \sqrt{2}\,\varepsilon_{abcdef}\,\Omega^{cdef}_{\mu\nu\rho}\right) \\
&\quad - \frac{1}{4}\sqrt{2}\,\mathcal{K}_{[ab]}{}^k\mathcal{K}_{[cd]}{}^l\mathcal{Z}_{[ef]\,mkl}\left(A_{[\mu}{}^{ab}A_\nu{}^{cd}A_{\rho]}{}^{ef}\right)\,.
\end{aligned} \tag{4.3.43}$$

This yields the full reduction ansatz for the component $C_{m\,\mu\nu\rho}$. Obviously, $C_{m\,\mu\nu\rho}$ is determined by (4.3.39) only up to a gradient $\partial_m\Lambda_{\mu\nu\rho}$ in the internal variables, which corresponds to a gauge transformation of the IIB four-form. Choosing the reduction ansatz (4.3.43), we have thus made a particular choice for this gauge freedom.

In a similar way, the last missing component $C_{\mu\nu\rho\sigma}$ can be reconstructed by further manipulating the equations and comparing to the IIB self-duality equations [1]. Concretely, taking the external curl of (4.3.39) and using Bianchi identities and field equations on the r.h.s. yields a differential equation that can be integrated in the internal variable to

$$\begin{aligned} -\frac{1}{6}\, e\varepsilon_{\mu\nu\rho\sigma\lambda}\,\varepsilon^{kpqrs}\,(\det G)^{-1}\,G_{nk}\,\widehat{\mathcal{D}}^{\lambda}C_{pqrs} \;&=\; 16\,D^{\rm KK}_{[\mu}C_{\nu\rho\sigma]\,n} - 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}\partial_n\mathcal{B}_{\rho\sigma]}{}^{\beta} \\ &\quad + 6\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,lkn} + 4\,\partial_n C_{\mu\nu\rho\sigma} \end{aligned} \tag{4.3.44}$$

up to an external gradient $\partial_n C_{\mu\nu\rho\sigma}$ which carries the last missing component of the IIB four-form. Here, $D^{\rm KK}_{\mu}$ denotes the Kaluza-Klein covariant derivative

$$D^{\rm KK}_{\mu}C_n \;\equiv\; \partial_\mu C_n - \mathcal{A}_\mu{}^k\partial_k C_n - \partial_n\mathcal{A}_\mu{}^k C_k\,, \qquad \text{etc.}\,, \tag{4.3.45}$$

and $\widehat{\mathcal{D}}_\mu C_{pqrs}$ is a particular combination of scalar covariant derivatives [1], which is most compactly defined via particular components of the scalar currents as

$$\mathcal{D}_\mu\mathcal{M}_{mn,N}\mathcal{M}^{N\,n} \;=\; \frac{\sqrt{2}}{3}\,(\det G)^{-1}\,G_{mn}\,\varepsilon^{npqrs}\,\widehat{\mathcal{D}}_\mu C_{pqrs}\,, \tag{4.3.46}$$

where $\mathcal{D}_\mu$ refers to the full EFT derivative, covariant under generalised diffeomorphisms. Again, it is a useful consistency check of the construction that with the reduction ansatz developed so far, equation (4.3.44) indeed turns into a total gradient, from which we may read off the function $C_{\mu\nu\rho\sigma}$. For the l.h.s. this is most conveniently seen by virtue of (4.3.46) and the reduction ansatz (1.6.2) for $\mathcal{M}_{MN}$, giving rise to

$$\begin{aligned} -4\,e\,(\det G)^{-1}\,G_{mk}\,\varepsilon^{kpqrs}\,\widehat{\mathcal{D}}^{\mu}C_{pqrs} \;&=\; 3\,\sqrt{|{\rm g}|}\,\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{n}D^{\mu}M_{ab,N}M^{N\,cd} \\ &=\; 6\,\sqrt{|{\rm g}|}\,\Big(\sqrt{2}\,\mathcal{K}^{[cb]}{}_{m}\eta_{ac} - \partial_m(\mathcal{Y}^b\mathcal{Y}_a)\Big)\,D^{\mu}M_{bd,N}M^{N\,da} \end{aligned} \tag{4.3.47}$$

where we have used (4.2.67). The derivatives D_μ on the r.h.s. now refer to the SO$(p,6-p)$ covariant derivatives (4.1.14). For the terms on the r.h.s. of (4.3.44), we find with (4.3.2), (4.3.6), and (4.2.60)

$$\begin{aligned} -30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}\partial_n\mathcal{B}_{\rho\sigma]}{}^{\beta} \;&=\; 15\,\sqrt{2}\,\varepsilon_{\alpha\beta}\,B_{[\mu\nu}{}^{a\alpha}B_{\rho\sigma]}{}^{b\beta}\,\mathcal{K}_{[ab]\,n}\,, \\ 6\,\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,lkn} \;&=\; -6\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma]}{}^{ef}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{Z}_{[ef]\,nkl}\,, \end{aligned} \tag{4.3.48}$$

as well as

$$\begin{aligned} 16\,D^{\rm KK}_{[\mu}C_{\nu\rho\sigma]\,m} \;&=\; \frac{1}{2}\,\mathcal{K}^{[ab]}{}_{m}\,\Big(\sqrt{|{\rm g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,D_\lambda\left(M_{ab,N}F^{\tau\lambda\,N}\right) + \sqrt{2}\,\varepsilon_{abcdef}\,D_{[\mu}\Omega_{\nu\rho\sigma]}^{cdef}\Big) \\ &\quad + 4\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,mkl} + 2\sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{F}_{\rho\sigma]\,mkl} \\ &\quad + \sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\left(2\,\mathcal{A}_{\rho}{}^{n}\partial_{|n|}\mathcal{A}_{\sigma]\,klm} + 3\,\partial_{[m}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kl]n} - 3\,\mathcal{A}_{\rho}{}^{n}\partial_{[m}\mathcal{A}_{\sigma]\,kl]n}\right) \\ &\quad - 2\sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu\,|kmn|}\left(\mathcal{A}_{\rho}{}^{l}\partial_{|l|}\mathcal{A}_{\sigma]}{}^{n}\right) - \sqrt{2}\,\partial_m\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kln}\right) \end{aligned} \tag{4.3.49}$$

where we have explicitly evaluated the Kaluza-Klein covariant derivative D_μ on $C_{\mu\nu\rho m}$, the latter given by (4.3.43). Moreover, we have arranged the $\mathcal{A}^4$ terms such that they allow for a convenient evaluation of their reduction formulae. Namely, in the last two lines we have factored out the quadratic polynomials that correspond to the $\mathcal{A}^2$ terms in the non-abelian field strengths (4.3.14) and thus upon reduction factor in analogy to the field strengths, leaving us with the A^4 terms

$$\begin{aligned}
\mathcal{A}\mathcal{A}\mathcal{A}\mathcal{A} \;\longrightarrow\; & -2\, f_{ef,gh}{}^{ij}\, \mathcal{K}^k_{[ab]} \left(\mathcal{Z}_{[cd]\,mkl}\mathcal{K}^l_{[ij]} + \mathcal{K}^l_{[cd]}\mathcal{Z}_{[ij]\,mkl}\right) A_{[\mu}{}^{ab} A_\nu{}^{cd} A_\rho{}^{ef} A_{\sigma]}{}^{gh} \\
& - \sqrt{2}\,\partial_m \left(\mathcal{A}_{[\mu}{}^k \mathcal{A}_\nu{}^l \mathcal{A}_\rho{}^n \mathcal{A}_{\sigma]\,kln}\right) \\
= \;& -\frac{1}{4}\sqrt{2}\, f_{ab,uv}{}^{xy} f_{ef,gh}{}^{ij}\, \varepsilon_{cdijxy}\, \mathcal{K}^{[uv]}_m A_{[\mu}{}^{ab} A_\nu{}^{cd} A_\rho{}^{ef} A_{\sigma]}{}^{gh} \\
& + \frac{1}{2}\, f_{ef,gh}{}^{ij}\, \varepsilon_{cdijau}\, \partial_m\left(\mathcal{Y}^u\mathcal{Y}_b\right) A_{[\mu}{}^{ab} A_\nu{}^{cd} A_\rho{}^{ef} A_{\sigma]}{}^{gh} \\
& - \sqrt{2}\,\partial_m \left(\mathcal{A}_{[\mu}{}^k \mathcal{A}_\nu{}^l \mathcal{A}_\rho{}^n \mathcal{A}_{\sigma]\,kln}\right) \;, \qquad (4.3.50)
\end{aligned}$$

upon using the identities (4.2.69), (4.2.67). While the last two terms are total gradients, the first term cancels against the corresponding contribution from the derivative of the Chern-Simons form $\Omega^{abcd}_{\mu\nu\rho}$ in (4.3.49)

$$\begin{aligned}
D_{[\mu}\Omega^{cdef}_{\nu\rho\sigma]}\,\varepsilon_{abcdef} \;=\; & \frac{3}{4}\, F_{[\mu\nu}{}^{cd} F_{\rho\sigma]}{}^{ef}\, \varepsilon_{abcdef} - \frac{1}{2}\sqrt{2}\, A_{[\mu}{}^{cd} A_\nu{}^{ef} F_{\rho\sigma]}{}^{gh}\, f_{ab,ef}{}^{uv}\, \varepsilon_{cdghuv} \\
& - \frac{1}{2}\, A_{[\mu}{}^{cd} A_\nu{}^{ef} A_\rho{}^{gh} A_{\sigma]}{}^{ij}\, f_{cd,ef}{}^{rs} f_{gh,ij}{}^{uv} \varepsilon_{abrsuv}\;. \qquad (4.3.51)
\end{aligned}$$

Similarly, the $\mathcal{F}\mathcal{A}\mathcal{A}$ terms in (4.3.49) combine with those of (4.3.48) according to

$$\begin{aligned}
\mathcal{F}\mathcal{A}\mathcal{A} \;\longrightarrow\; & -2\sqrt{2}\, F_{[\mu\nu}{}^{ab} A_\rho{}^{cd} A_{\sigma]}{}^{ef}\, \mathcal{K}^l_{[cd]} \left(\mathcal{K}^k_{[ab]}\mathcal{Z}_{[ef]\,mkl} + \mathcal{K}^k_{[ef]}\mathcal{Z}_{[ab]\,mkl}\right) \qquad (4.3.52) \\
= \;& \frac{1}{2}\, f_{cd,ij}{}^{gh}\, \mathcal{K}^{[ij]}{}_m F_{[\mu\nu}{}^{ab} A_\rho{}^{cd} A_{\sigma]}{}^{ef}\, \varepsilon_{abefgh} - \frac{1}{2}\sqrt{2}\, F_{[\mu\nu}{}^{ab} A_\rho{}^{cd} A_{\sigma]}{}^{ef}\, \varepsilon_{abefch}\, \partial_m(\mathcal{Y}^h\mathcal{Y}_d)\;.
\end{aligned}$$

Again, the first term cancels against the corresponding contribution from the derivative of the Chern-Simons form $\Omega^{abcd}_{\mu\nu\rho}$, given in (4.3.51).

Collecting all the remaining terms, equation (4.3.44) takes the final form

$$\begin{aligned}
0 \;=\; & \frac{1}{2}\, \mathcal{K}^{[ab]}{}_m \sqrt{|\mathrm{g}|}\, \varepsilon_{\mu\nu\rho\sigma\tau} \left(\frac{1}{2}\sqrt{2}\,\eta_{da}\, D^\tau M_{cb,N} M^{N\,cd} + D_\lambda\left(M_{ab,N} F^{\tau\lambda\,N}\right)\right) \\
& + \frac{3}{8}\sqrt{2}\, \mathcal{K}^{[ab]}{}_m \left(\varepsilon_{abcdef}\, F_{[\mu\nu}{}^{cd} F_{\rho\sigma]}{}^{ef} + 40\,\varepsilon_{\alpha\beta}\,\eta_{ac}\eta_{bd}\, B_{[\mu\nu}{}^{c\alpha} B_{\rho\sigma]}{}^{d\beta}\right) \\
& + \frac{1}{2}\, f_{ef,gh}{}^{ij}\, \varepsilon_{cdijay}\, \partial_m\left(\mathcal{Y}^y\mathcal{Y}_b\right) A_{[\mu}{}^{ab} A_\nu{}^{cd} A_\rho{}^{ef} A_{\sigma]}{}^{gh} \\
& - \frac{1}{4}\sqrt{|\mathrm{g}|}\, \varepsilon_{\mu\nu\rho\sigma\tau}\, \partial_m(\mathcal{Y}^b\mathcal{Y}_d)\, D^\tau M_{ab,N} M^{N\,ad} - \frac{1}{2}\sqrt{2}\, F_{[\mu\nu}{}^{ab} A_\rho{}^{cd} A_{\sigma]}{}^{ef}\, \varepsilon_{abefch}\, \partial_m(\mathcal{Y}^h\mathcal{Y}_d) \\
& - \sqrt{2}\,\partial_m \left(\mathcal{A}_{[\mu}{}^k \mathcal{A}_\nu{}^l \mathcal{A}_\rho{}^n \mathcal{A}_{\sigma]\,kln}\right) + 4\,\partial_m C_{\mu\nu\rho\sigma}\;. \qquad (4.3.53)
\end{aligned}$$

Now the first two lines of the expression precisely correspond to the vector field equations (4.1.16) of the $D = 5$ theory, which confirms that on-shell this equation reduces to a total gradient in the internal variables. Although guaranteed by the consistency of the generalised Scherk-Schwarz ansatz and the general analysis of [1], it is gratifying that this structure is confirmed by explicit calculation based on the $D = 5$ field equations and the non-trivial identities among the Killing vectors. We are thus in position to read off from (4.3.53) the final expression for the 4-form as

$$\begin{aligned} C_{\mu\nu\rho\sigma} \;\; = \;\; & -\frac{1}{16}\,\mathcal{Y}_a\mathcal{Y}^b\left(\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D^\tau M_{bc,N}M^{N\,ca}\;+2\,\sqrt{2}\,\varepsilon_{cdefgb}\,F_{[\mu\nu}{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{ga}\right) \\ & +\frac{1}{4}\,\left(\sqrt{2}\,\mathcal{K}_{[ab]}{}^k\mathcal{K}_{[cd]}{}^l\mathcal{K}_{[ef]}{}^n\mathcal{Z}_{[gh]\,kln}-\mathcal{Y}_h\mathcal{Y}^j\,\varepsilon_{abcegj}\,\eta_{df}\right)A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{gh} \\ & +\Lambda_{\mu\nu\rho\sigma}(x)\;, \end{aligned} \tag{4.3.54}$$

in terms of the $D = 5$ fields, up to an y-independent term $\Lambda_{\mu\nu\rho\sigma}(x)$, left undetermined by equation (4.3.44) and fixed by the last component of the IIB self-duality equations (3.2.7). This equation translates into

$$\begin{aligned} 4\,D^{\mathrm{KK}}_{[\mu}\,C_{\nu\rho\sigma\tau]} \;\; = \;\; & 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^\alpha D^{\mathrm{KK}}_\rho\mathcal{B}_{\sigma\tau]}{}^\beta+8\,\mathcal{F}_{[\mu\nu}{}^kC_{\rho\sigma\tau]\,k} \\ & -\frac{1}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp}\;, \end{aligned} \tag{4.3.55}$$

where X_{klmnp} is a combination of internal derivatives of the scalar fields, c.f. [1], that is most compactly given by

$$\frac{1}{120}\,\varepsilon^{kpqrs}\,X_{kpqrs} \;\; = \;\; -\frac{1}{20}\,\sqrt{2}\,(\det G)\,G^{ml}\,\partial_l\mathcal{M}_{mn,N}\mathcal{M}^{N\,n}\;, \tag{4.3.56}$$

in analogy to (4.3.46). It can be shown that equation (4.3.55) can be derived from the external curl of equations (4.3.44) upon using the EFT field equations and Bianchi identities, up to a y-independent equation that defines the last missing function $\Lambda_{\mu\nu\rho\sigma}$. For the general case this has been worked out in [1]. Alternatively, it can be confirmed by explicit calculation with the Scherk-Schwarz reduction ansatz, that equation (4.3.55) with the components $C_{\mu\nu\rho\,m}$ and $C_{\mu\nu\rho\sigma}$ from (4.3.43) and (4.3.54), respectively, decomposes into a y-dependent part, which vanishes due to the $D = 5$ scalar equations of motion, and a y-independent part, that defines the function $\Lambda_{\mu\nu\rho\sigma}$. The calculation is similar (but more lengthy) than the previous steps, requires the same non-trivial identities among Killing vectors derived above, but also some non-trivial algebraic identities among the components of the scalar $\mathrm{E}_{6(6)}$ matrix M_{MN}. We relegate the rather lengthy details to appendix A and simply report the final result from equation (A.20)

$$\begin{aligned}
D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= -\frac{1}{480}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{ac,N}\right)\\
&\quad+\frac{1}{240}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{ab,N}F_{\kappa\lambda}{}^{ab}-\frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,M^{a\,\alpha}{}_N\,B_{\kappa\lambda}{}^{b\,\beta}\right)\\
&\quad+\frac{1}{600}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\left(10\,\delta^d_h\delta^a_e+2\,M^{fd,ga}M_{gh,fe}-M_{e\alpha}{}^{ga}M_{gh}{}^{d\alpha}\right)M^{bh,ec}\,\eta_{cd}\eta_{ab}\\
&\quad+\frac{1}{32}\,\sqrt{2}\;\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}+\frac{1}{16}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\
&\quad+\frac{1}{40}\,\sqrt{2}\;A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_\sigma{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\;. \qquad (4.3.57)
\end{aligned}$$

Since there is no non-trivial Bianchi identity for (4.3.57), this equation can be integrated and yields the last missing term in the four-form potential (4.3.54). This completes the reduction formulae for the full set of fundamental IIB fields.

4.4 Generalised IIB supergravity

In the previous sections, we gave the embedding of type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ into the $E_{6(6)}$ EFT using a generalised Scherk-Schwarz ansatz together with specific solution of the section constraint. Here, we will show how one can obtain a generalised set of equations of motion and Bianchi identities known in the literature as generalised type IIB from EFT. This constitutes the second application of the EFT/Type IIB dictionary. To achieve this result, we use a factorisation ansatz [90] together with a different solution of the section constraint and work out explicitly various components of the deformed field strengths. Using the dictionary, we compare the deformed field strengths to their generalised type IIB analogue and find agreement for all the components tested.

In the introduction, we showed that the equations of motions of standard, bosonic type IIB could be obtained from the corresponding non-linear σ-model. This has been generalised to the case of the superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$ with coset superspace $\frac{SU(2,2|4)}{SO(4,1)\times SO(5)}$ by Metsaev and Tseytlin [91]. In this case, the action can be written in terms of the Maurer-Cartan 1 forms as a sum of the kinetic and Weiss-Zumino term, in the same spirit than the standard bosonic WZW action on a group manifold. The interesting built-in property of the Metsaev-Tseytlin action is its invariance under a fermionic gauge symmetry, κ-symmetry, which halves the number of fermions and fixes the value of the parameter in front of the WZ term. Intriguingly, this corresponds to the same value required for the integrability of the theory. It has been shown in [92, 93, 94] that it is possible to deform the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring while maintaining both properties of integrability and invariance under a local fermionic symmetry. The deformations are known in the literature as η-deformation [92, 93] and λ-deformation [93]. In the case of the

η-deformation, it was shown recently that the conditions for κ-symmetry do not correspond to standard type IIB supergravity equations [81, 82]. The resulting equations requires the presence of a Killing vector field K. When K vanishes, one recovers the standard type IIB supergravity equations. These are therefore known as generalised type IIB equations.

4.4.1 generalised field equations and Bianchi identities

Let us recall the bosonic generalised IIB supergravity equations which have been derived in [81]. Their fermionic completion has been found in [82]. The equations are expressed in string frame. The equations for the metric $G_{\hat{\mu}\hat{\nu}}$ and the B-field $B_{\hat{\mu}\hat{\nu}+}$ are

$$R_{\hat{\mu}\hat{\nu}} - \frac{1}{4}H_{\hat{\mu}\hat{\rho}\hat{\sigma}}H_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} - \mathcal{T}_{\hat{\mu}\hat{\nu}} + \nabla_{\hat{\mu}}X_{\hat{\nu}} + \nabla_{\hat{\nu}}X_{\hat{\mu}} = 0, \tag{4.4.1a}$$

$$\frac{1}{2}\,\nabla^{\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{1}{2}\mathcal{F}^{\hat{\rho}}\mathcal{F}_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{1}{12}\,\mathcal{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\mathcal{F}^{\hat{\rho}\hat{\sigma}\hat{\tau}} - X^{\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}} - \partial_{\hat{\mu}}X_{\hat{\nu}} + \partial_{\hat{\nu}}X_{\hat{\mu}} = 0, \tag{4.4.1b}$$

$$R - \frac{1}{12}\,H_{\hat{\mu}\hat{\nu}\hat{\rho}}H^{\hat{\mu}\hat{\nu}\hat{\rho}} + 4\,\nabla_{\hat{\mu}}X^{\hat{\mu}} - 4X_{\hat{\mu}}X^{\hat{\mu}} = 0\ , \tag{4.4.1c}$$

where $\hat{\mu},\hat{\nu} = 0..9$, $\nabla_{\hat{\mu}}$ denotes the space-time covariant derivative, $R_{\hat{\mu}\hat{\nu}}$ the Ricci tensor, R the Ricci scalar and $H_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\,\partial_{[\hat{\mu}}B_{\hat{\nu}\hat{\rho}]+}$ the field strength of the NS-NS B-field. The R-R fields enter via the currents $\mathcal{F}_{\hat{\mu}_1\cdots\hat{\mu}_n}$ and contribute to the stress tensor in (4.4.1a) via

$$\mathcal{T}_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\,\mathcal{F}_{\hat{\mu}}\mathcal{F}_{\hat{\nu}} + \frac{1}{4}\,\mathcal{F}_{\hat{\mu}\hat{\rho}\hat{\sigma}}\mathcal{F}_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} + \frac{1}{4\times 4!}\,\mathcal{F}_{\hat{\mu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}}\mathcal{F}_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}} - \frac{1}{4}\,G_{\hat{\mu}\hat{\nu}}\Big(\mathcal{F}_{\hat{\rho}}\mathcal{F}^{\hat{\rho}} + \frac{1}{6}\,\mathcal{F}_{\hat{\rho}\hat{\sigma}\hat{\tau}}\mathcal{F}^{\hat{\rho}\hat{\sigma}\hat{\tau}}\Big)\ . \tag{4.4.1d}$$

The equations (4.4.1a)–(4.4.1c) are based on the existence of a Killing vector field K and an additional vector field Z with $K^{\hat{\mu}}Z_{\hat{\mu}} = 0$, which enter the field equations in the combination $X \equiv K + Z$. The vector field Z satisfies the Bianchi type equations

$$\partial_{\hat{\mu}}Z_{\hat{\nu}} - \partial_{\hat{\nu}}Z_{\hat{\mu}} + K^{\hat{\rho}}H_{\hat{\mu}\hat{\nu}\hat{\rho}} \;=\; 0\ . \tag{4.4.2}$$

The ordinary type IIB equations are recovered in the limit where $K = 0$ such that Z can be integrated to the dilaton field $Z_{\hat{\mu}} = \partial_{\hat{\mu}}\phi$.

In the R-R sector, the generalised dynamical equations for the field strengths $\mathcal{F}_{\hat{\mu}_1\cdots\hat{\mu}_n}$ are given by

$$\nabla^{\hat{\mu}}\mathcal{F}_{\hat{\mu}} - Z^{\hat{\mu}}\mathcal{F}_{\hat{\mu}} - \frac{1}{6}\,H^{\hat{\mu}\hat{\nu}\hat{\rho}}\mathcal{F}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 0, \qquad K^{\hat{\mu}}\mathcal{F}_{\hat{\mu}} = 0, \tag{4.4.3a}$$

$$\nabla^{\hat{\rho}}\mathcal{F}_{\hat{\rho}\hat{\mu}\hat{\nu}} - Z^{\hat{\rho}}\mathcal{F}_{p\hat{\mu}\hat{\nu}\hat{\rho}} - \frac{1}{6}\,H^{\hat{\rho}\hat{\sigma}\hat{\tau}}\mathcal{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} - (K\wedge\mathcal{F}_1)_{\hat{\mu}\hat{\nu}} = 0, \tag{4.4.3b}$$

$$\nabla^{\hat{\tau}}\mathcal{F}_{\hat{\tau}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} - Z^{\hat{\tau}}\mathcal{F}_{\hat{\tau}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \frac{1}{36}\,\varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\kappa}\hat{\eta}\hat{\lambda}\hat{\xi}\hat{\theta}}H^{\hat{\tau}\hat{\kappa}\hat{\eta}}\mathcal{F}^{\hat{\lambda}\hat{\xi}\hat{\theta}} - (K\wedge\mathcal{F}_3)_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = 0, \tag{4.4.3c}$$

while their modified Bianchi identities can be cast into the compact form

$$d\mathcal{F}_{2n+1} - Z \wedge \mathcal{F}_{2n+1} + H_3 \wedge \mathcal{F}_{2n-1} \;=\; \star\left(K \wedge \star\mathcal{F}_{2n+3}\right) \,. \tag{4.4.4}$$

The Bianchi identities extend to the dual field strengths $-\mathcal{F}_7 \equiv \star\mathcal{F}_3$ and $\mathcal{F}_9 \equiv \star\mathcal{F}_1$. Furthermore, the selfduality property $\mathcal{F}_5 = \star\mathcal{F}_5$ of the five form continues to hold in the modified theory, relating its Bianchi identity and field equation. In the following, for simplicity of the formulas, we will often choose coordinates such that the Killing vector field points in a given direction $K^{\hat\mu} = \delta^{\hat\mu}_*$.

4.4.2 Solution of the Bianchi identities

It has been noted in [81] that equation (4.4.2) for the new vector $Z_{\hat\mu}$ may be interpreted as a modified "dilaton Bianchi identity" and locally integrated into

$$Z_{\hat\mu} \;=\; \partial_{\hat\mu}\phi + K^{\hat\nu} B_{\hat\nu\hat\mu\,+} \;=\; \partial_{\hat\mu}\phi - B_{\hat\mu *\,+} \,. \tag{4.4.5}$$

We will in the following stay in this picture and understand the combination $\partial_{\hat\mu}\phi - B_{\hat\mu *\,+}$ as a derivative $D_{\hat\mu}\phi$ on the dilaton that is covariantized in a suitable sense. As a related observation, one may straightforwardly check that the modified Bianchi identities (4.4.4) satisfied by the R-R field strengths allow for an explicit integration into $\mathcal{F} = e^{\phi}\, F$ with

$$\begin{aligned}
F_{\hat\mu} \;&=\; \partial_{\hat\mu}\chi + B_{\hat\mu *\,+}\,\chi + B_{\hat\mu *\,-} \;\equiv\; D_{\hat\mu}\chi \,,\\
F_{\hat\rho\hat\mu\hat\nu} \;&=\; 3\,\partial_{[\hat\rho} B_{\hat\mu\hat\nu]\,-} + \frac{3}{2}\, B_{[\hat\rho|*+|} B_{\hat\mu\hat\nu]\,-} - \frac{3}{2}\, B_{[\hat\rho|*-|} B_{\hat\mu\hat\nu]\,+} + C_{\hat\rho\hat\mu\hat\nu *} + \chi\, H_{\hat\rho\hat\mu\hat\nu} \,,\\
F_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau} \;&=\; 5\,\partial_{[\hat\mu} C_{\hat\nu\hat\rho\hat\sigma\hat\tau]} + 5\, B_{[\hat\mu|*+|} C_{\hat\nu\hat\rho\hat\sigma\hat\tau]} - 15\, B_{[\hat\mu\hat\nu\,|+|} \partial_{\hat\rho} B_{\hat\sigma\hat\tau]\,-}\\
&\qquad - 15\, B_{[\hat\mu\hat\nu\,|+|} B_{\hat\rho|*\,+|} B_{\hat\sigma\hat\tau]\,-} + 15\, B_{[\hat\mu\hat\nu\,|-|} \partial_{\hat\rho} B_{\hat\sigma\hat\tau]\,+} + C_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau *\,+} \,,\\
F_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\kappa\hat\eta} \;&=\; 7\,\partial_{[\hat\mu} C_{\hat\nu\hat\rho\hat\sigma\hat\tau\hat\kappa\hat\eta]\,+} + 7\, B_{[\hat\mu *+} C_{\hat\nu\hat\rho\hat\sigma\hat\tau\hat\kappa\hat\eta]\,+} + 35\, C_{[\hat\mu\hat\nu\hat\rho\hat\sigma} H_{\hat\tau\hat\kappa\hat\eta]}\\
&\qquad - 105\, B_{[\hat\mu\hat\nu|\,+|} B_{\hat\rho\hat\sigma|\,-|} H_{\hat\tau\hat\kappa\hat\eta]} + C_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\kappa\hat\eta *} \,.
\end{aligned} \tag{4.4.6}$$

All the terms, carrying indices '$_*$' represent the deformations from the standard IIB expressions. Again, in the following we will assign them a natural interpretation as the connection terms of covariantized derivatives, non-abelian field strengths and the Stückelberg type couplings among p-forms. These additional couplings precisely match the structure of general nine-dimensional gauged supergravities[95, 96] (recall that due to the existence of a Killing vector field, we are effectively describing a nine-dimensional theory). More precisely, equations (4.4.6) can be viewed as resulting from a gauging of nine-dimensional maximal supergravity in which a linear combination of the Cartan subgroup of the $\mathrm{SL}(2)_{\mathrm{IIB}}$ and the trombone symmetry which scales every field according to its Weyl weight has been gauged. The component $B_{\hat\mu *\,+}$ of the ten-dimensional NS-NS two-form serves as a gauge field.[4] An important consequence is the following. According

[4] To be precise, also a nilpotent generator of $\mathrm{SL}(2)_{\mathrm{IIB}}$ is gauged with the component $B_{\hat\mu *\,-}$ serving as the associated gauge field.

to (4.4.5), the dilaton ϕ is charged under the new local gauge symmetry. Translation to the Einstein frame via

$$G^{\rm st}_{\hat\mu\hat\nu} \;=\; e^{\phi/2}\, G^{\rm E}_{\hat\mu\hat\nu}\;, \tag{4.4.7}$$

thus implies that the metric in the Einstein frame is also charged. Translation of the Einstein field equations (4.4.1a) into the Einstein frame thus induces field equations which feature a covariantized Ricci tensor in the sense that all derivatives in its definition are replaced by properly covariantized ones. In particular, the Riemann tensor is calculated as curvature of the connection

$$\hat\Gamma_{\hat\mu\hat\nu}{}^{\hat\rho} \;\equiv\; \frac12\, G^{\hat\rho\hat\kappa}\left(D_{\hat\mu}G_{\hat\nu\hat\kappa}+D_{\hat\nu}G_{\hat\mu\hat\kappa}-D_{\hat\kappa}G_{\hat\mu\hat\nu}\right)\;,\quad D_{\hat\kappa}G_{\hat\mu\hat\nu}\equiv\partial_{\hat\kappa}G_{\hat\mu\hat\nu}+\frac12\,B_{\hat\kappa *+}\,G_{\hat\mu\hat\kappa}. \tag{4.4.8}$$

This is the generic structure of supergravities in which the trombone symmetry is gauged [57]. Upon transition to the Einstein frame, we may also regroup the field equations for NS-NS and R-R two-form (4.4.1b) and (4.4.3b) into the manifestly SL(2) covariant form

$$D_{\hat\rho}\left(F^{\hat\rho\hat\mu\hat\nu\,\alpha}\,m_{\alpha\beta}\right)-\frac16\,F^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\,F_{\hat\rho\hat\sigma\hat\tau}{}^{\alpha}\,\varepsilon_{\alpha\beta} \;=\; J^{\hat\mu\hat\nu}{}_{\beta}\;, \tag{4.4.9}$$

with the SL(2) doublet $F_{\hat\mu\hat\nu\hat\rho\pm}=\{H_{\hat\mu\hat\nu\hat\rho}, F_{\hat\mu\hat\nu\hat\rho}-\chi\, H_{\hat\mu\hat\nu\hat\rho}\}$, and the dilaton/axion matrix $m_{\alpha\beta}$ parametrized as

$$m_{\alpha\beta} \;=\; \begin{pmatrix} e^{\phi} & -e^{\phi}\chi \\ -e^{\phi}\chi & e^{\phi}\chi^2+e^{-\phi}\end{pmatrix}\;. \tag{4.4.10}$$

The current on the r.h.s. of (4.4.9) is given by the SL(2) doublet

$$J^{\hat\mu\hat\nu}{}_{\pm} \;=\; \left\{2\,e^{2\phi}\,K^{[\hat\mu}F^{\hat\nu]}\,,\; -4\,e^{2\phi}\,\nabla^{[\hat\mu}K^{\hat\nu]}-2\,\chi\, e^{2\phi}\,K^{[\hat\mu}F^{\hat\nu]}\right\}\;, \tag{4.4.11}$$

in terms of the Killing vector field K^m and the current $F_{\hat\mu}=D_{\hat\mu}\chi$. In the following, we will recover the non-abelian field-strengths (4.4.6) from a particular Scherk-Schwarz ansatz in exceptional field theory.

4.5 Generalised IIB from EFT

4.5.1 Section constraints and IIA/IIB/generalised supergravity

The section condition

$$d^{KMN}\,\partial_M\partial_N A \;=\; 0\,,\qquad d^{KMN}\,\partial_M A\,\partial_N B \;=\; 0\,, \tag{4.5.1}$$

that applies on any fields or gauge parameters A, B is solved by restricting the internal coordinate dependence of all fields to properly chosen subsets of coordinates. Breaking $E_{6(6)}$ down to its subgroup $SL(5)\times SL(2)\times GL(1)_{\rm IIB}$ according to

$$\begin{aligned} \mathbf{27} &\;\longrightarrow\; (5,1)_{+4}+(5',2)_{+1}+(10,1)_{-2}+(1,2)_{-5}\;,\\ \{Y^M\} &\;\longrightarrow\; \{y^m\,,\;\tilde y_{m\alpha}\,,\;\tilde y^{mn}\,,\;\tilde y_{\alpha}\}\;,\qquad m=1,\dots,5\;,\quad \alpha=\pm\;, \end{aligned} \tag{4.5.2}$$

and restricting all fields to depend only on the 5 coordinates $\{y^m\}$ of highest grading under $\mathrm{GL}(1)_{\mathrm{IIB}}$ solves the conditions (4.5.1). The EFT field equations derived from (3.1.1) then reproduce the IIB theory after decomposing the EFT fields (1.5.18) according to (4.5.2) and properly translating the various blocks into the various components of the IIB fields [1]. In particular, the scalar matrix $\mathcal{M}_{MN}$ decomposes into

$$\mathcal{M}_{KM} \;=\; \begin{pmatrix} \mathcal{M}_{mk} & \mathcal{M}_m{}^{k\beta} & \mathcal{M}_{m,kl} & \mathcal{M}_m{}^{\beta} \\ \mathcal{M}^{m\alpha}{}_k & \mathcal{M}^{m\alpha,k\beta} & \mathcal{M}^{m\alpha}{}_{kl} & \mathcal{M}^{m\alpha,\beta} \\ \mathcal{M}_{mn,k} & \mathcal{M}_{mn}{}^{k\beta} & \mathcal{M}_{mn,kl} & \mathcal{M}_{mn}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_k & \mathcal{M}^{\alpha,k\beta} & \mathcal{M}^{\alpha}{}_{kl} & \mathcal{M}^{\alpha\beta} \end{pmatrix} , \tag{4.5.3}$$

where the explicit form of the blocks is obtained by evaluating the matrix $\mathcal{M}_{MN} = \mathcal{V}_M{}^A\mathcal{V}_N{}^A$ from a vielbein $\mathcal{V}_M{}^A$ given by the product of matrix exponentials

$$\mathcal{V}_{\mathrm{IIB}} \;\equiv\; \exp\left[\varepsilon^{mnklp}\, c_{mnkl}\, t_{(+4)\,p}\right]\, \exp\left[b_{mn}{}^{\alpha}\, t_{(+2)}{}_{\alpha}^{mn}\right]\, \mathcal{V}_5\, \mathcal{V}_2 \exp\left[\Phi\, t_{\mathrm{IIB}}\right] , \tag{4.5.4}$$

with the relevant $\mathfrak{e}_{6(6)}$ generators t_{IIB}, $t_{(+2)}{}_{\alpha}^{mn}$, $t_{(+4)\,m}$ and their coefficients originating from the IIB metric, two-form and four-form, respectively. The matrices $\mathcal{V}_5$, $\mathcal{V}_2$ represent the $\mathrm{SL}(5)\times\mathrm{SL}(2)$ factor of the vielbein, related to the internal metric and the IIB dilaton/axion matrix, respectively. Similarly, vectors and two-forms are decomposed as (4.5.2)

$$\begin{aligned} \{\mathcal{A}_\mu{}^M\} &\;\longrightarrow\; \{\mathcal{A}_\mu{}^m\,,\; \mathcal{A}_{\mu\, m\alpha}\,,\; \mathcal{A}_\mu{}^{mn}\,,\; \mathcal{A}_{\mu\,\alpha}\}\,, \\ \{\mathcal{B}_{\mu\nu\, M}\} &\;\longrightarrow\; \{\mathcal{B}_{\mu\nu\, m}\,,\; \mathcal{B}_{\mu\nu}{}^{m\alpha}\,,\; \mathcal{B}_{\mu\nu\, mn}\,,\; \mathcal{B}_{\mu\nu\,\alpha}\}\,. \end{aligned} \tag{4.5.5}$$

In contrast, the IIA theory is recovered, if the physical coordinates are identified with the $\{\tilde{y}_{m+}\}$ in the decomposition (4.5.2) (which explicitly breaks the SL(2) factor), the EFT fields are decomposed accordingly and translated into the IIA fields. E.g. in this case, the proper parametrization of the matrix $\mathcal{M}_{MN} = \mathcal{V}_M{}^A\mathcal{V}_N{}^A$ is obtained via a vielbein $\mathcal{V}_M{}^A$

$$\mathcal{V}_{\mathrm{IIA}} \;\equiv\; \exp\left[\varphi\, t_{(+5)}\right]\, \exp\left[\,c_{mnk}\, t^{mnk}_{(+3)}\right]\, \exp\left[b_{mn}\, t^{mn}_{(+2)}\right]\, \exp\left[c_m\, t^{m}_{(+1)}\right]\, \mathcal{V}_5 \exp\left[\phi\, t_0 + \Phi\, t_{\mathrm{IIA}}\right] , \tag{4.5.6}$$

with the coefficients originating from the IIA metric, dilaton, and p-forms, respectively.

Here, we choose yet a different solution of the section constraint. First, we impose the existence of a Killing vector field in the IIB theory and accordingly split the coordinates $\{y^m\} = \{y^i, y^*\}$, $(i = 1,\ldots,4)$, such that $\partial_*\Phi = 0$ for all fields of the theory. Next, we relax the IIB solution, by allowing fields to depend on the 5 coordinates

$$\{y^i, \tilde{y}_{*+}\}\,, \quad i = 1,\ldots,4\,, \tag{4.5.7}$$

such that the section condition (4.5.1) is still satisfied. In the following, we will evaluate EFT in the IIB parametrization (4.5.3) however imposing a particular additional

$\tilde{y}_{*+}$-dependence according to a simple Scherk-Schwarz ansatz which will trigger the generalised IIB equations. It is important to note that the choice of coordinates (4.5.7) is equivalent (after rotation of the 27 coordinates) to selecting the IIA coordinates $\tilde{y}_{m+}$ in (4.5.2). Applying the same rotation to the IIA parametrization of EFT fields such as (4.5.6) we would simply recover the IIA theory. This is a manifestation of the fact that the generalised IIB supergravity equations can be obtained via T-duality from a sector of IIA supergravity. Since the framework of exceptional field theory is manifestly duality covariant, we can simply absorb the effect of this duality into a rotation of the extended coordinates. We will thus evaluate exceptional field theory in its IIB parametrization (4.5.3) however in coordinates (4.5.7) and with a proper Scherk-Schwarz ansatz in $\tilde{y}_{*+}$ in order to obtain directly the generalised IIB equations.

4.5.2 Scherk-Schwarz ansatz

Following the previous discussion, and having chosen physical coordinates according to (4.5.7) we now impose on the EFT fields (1.5.18) a specific $\tilde{y}_{*+}$-dependence, such that in particular the total $\tilde{y}_{*+}$-dependence consistently factors out from all the equations of motion. This is achieved by a Scherk-Schwarz ansatz [56]

$$
\begin{aligned}
\mathcal{M}_{MN} &= U_M{}^K(\tilde{y})\, U_N{}^L(\tilde{y})\, M_{KL}(x^\mu, y^i) \,, \\
g_{\mu\nu} &= \rho^{-2}(\tilde{y})\, \mathrm{g}_{\mu\nu}(x^\mu, y^i) \,, \\
\mathcal{A}_\mu{}^M &= \rho^{-1}(\tilde{y})\, A_\mu{}^N(x^\mu, y^i)\, (U^{-1})_N{}^M(\tilde{y}) \,, \\
\mathcal{B}_{\mu\nu\, M} &= \rho^{-2}(\tilde{y})\, U_M{}^N(\tilde{y})\, B_{\mu\nu\, N}(x^\mu, y^i) \,,
\end{aligned} \tag{4.5.8}
$$

where the $\tilde{y}_{*+}$-dependence of all fields is carried by an $\mathrm{E}_{6(6)}$-valued twist matrix $U_N{}^L$ and a scalar factor ρ. For simplicity of the notation, here and in the following we also use the notation $\tilde{y} \equiv \tilde{y}_{*+}$.[5]

The relevant Scherk-Schwarz ansatz for generalised IIB supergravity is based on a twist matrix $U_M{}^N$ living in an

$$
\mathrm{GL}(1) \subset \mathrm{SL}(2)_{\mathrm{diag}} \subset \mathrm{SL}(2) \times \mathrm{SL}(2) \subset \mathrm{SL}(2) \times \mathrm{SL}(6) \subset \mathrm{E}_{6(6)} \,, \tag{4.5.9}
$$

subgroup of the full duality group $\mathrm{E}_{6(6)}$. More precisely, upon decomposing

$$
\begin{aligned}
\mathrm{E}_{6(6)} \quad &\to \quad \mathrm{SL}(2) \times \mathrm{SL}(6) \,, \\
\mathbf{27} \quad &\longrightarrow \quad (1, 15) + (2, 6') \,, \qquad \left\{Y^M\right\} \;\longrightarrow\; \left\{Y^{ab}, \tilde{Y}_{a\alpha}\right\} \,,
\end{aligned} \tag{4.5.10}
$$

[5] Note that the ansatz (4.5.8) is slightly more general than the ones studied in [56] in that the fields multiplying the twist matrices on the r.h.s. do not only depend on the external coordinates x^μ but also on part of the internal coordinates y^i. In this sense, the ansatz (4.5.8) resembles the embedding of deformations of EFT studied in [90] (and in [97] in the context of double field theory), although here all fields and twist matrices respect the section constraint, so we remain within the original framework.

an (SL(2) × SL(6))-valued matrix U takes the form

$$U_M{}^N \;=\; \begin{pmatrix} U_a{}^{[c}U_b{}^{d]} & 0 \\ 0 & (U^{-1})_a{}^c U_\alpha{}^\beta \end{pmatrix}\;, \tag{4.5.11}$$

and we choose the matrix factors as

$$\begin{aligned} U_\alpha{}^\beta \;&=\; \begin{pmatrix} U_+{}^+ & 0 \\ 0 & U_-{}^- \end{pmatrix} \;=\; \begin{pmatrix} \rho(\tilde{y}) & 0 \\ 0 & \rho^{-1}(\tilde{y}) \end{pmatrix}\;, \\ U_a{}^b \;&=\; \begin{pmatrix} U_i{}^j & 0 & 0 \\ 0 & U_*{}^* & 0 \\ 0 & 0 & U_0{}^0 \end{pmatrix} \;=\; \begin{pmatrix} \delta_i{}^j & 0 & 0 \\ 0 & \rho(\tilde{y}) & 0 \\ 0 & 0 & \rho^{-1}(\tilde{y}) \end{pmatrix}\;, \end{aligned} \tag{4.5.12}$$

with scale factor given by a linear function $\rho(\tilde{y}) = \tilde{y} + c$. In order to check the effect of the Scherk-Schwarz ansatz (4.5.8) with (4.5.12) on the field equations of exceptional field theory, we consider the current

$$(\mathcal{X}_M)_N{}^K \;\equiv\; \rho^{-1}\,(U^{-1})_M{}^P\,(U^{-1})_N{}^Q\,\partial_P U_Q{}^K\;, \tag{4.5.13}$$

which encodes the combinations of the twist matrix and its derivatives that explicitly enter the field equations. With the explicit form of (4.5.12), this current lives in the algebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(6)$ with its only non-vanishing components given by

$$(\mathcal{X}^{*+})_\alpha{}^\beta \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;, \qquad (\mathcal{X}^{*+})_a{}^b \;=\; \begin{pmatrix} 0_{4\times 4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\;, \tag{4.5.14}$$

all constant, ensuring that the $\tilde{y}$-dependence factors out from all equations of motion.[6] We have thus presented a consistent Scherk-Schwarz ansatz on the EFT fields which moreover satisfies the section condition. Upon explicitly evaluating the field equations, the non-trivial $\tilde{y}$ dependence of the twist matrix induces a deformation of the original IB equations of motion. We shall work this out in the next section.

4.5.3 Induced deformation

In this section we will illustrate with several examples how the Scherk-Schwarz ansatz (4.5.8) induces a deformation of the resulting field equations which precisely coincides with the deformation of the IIB field equations and Bianchi identities discussed in section 4.4 above. Covariant derivatives in EFT carry vector fields $\mathcal{A}_\mu{}^M$ and internal derivatives ∂_M. Under (SL(2) × SL(6)), the coordinates (4.5.7) are embedded in the Y^M as

[6] Strictly speaking, for consistency of the Scherk-Schwarz ansatz a weaker condition is sufficient: only the projection of (4.5.13) onto the $\mathbf{27} \oplus \mathbf{351}$ representation of $E_{6(6)}$ appears in the field equations and is required to be constant. With (4.5.14) this is automatically guaranteed.

$\{Y^{i0}, \tilde{Y}_{*+}\}$, c.f. (4.5.10). With the ansatz (4.5.8), the relevant couplings then are obtained from

$$\mathcal{A}_\mu{}^{i0}\partial_{i0} \;=\; \rho^{-1}\rho\, A_\mu{}^{i0}\partial_{i0}\,, \qquad \mathcal{A}_{\mu*+}\partial^{*+} \;=\; \rho^{-1}\rho^2\, A_{\mu*+}\partial^{*+}\,. \tag{4.5.15}$$

Both operators give rise to additional $\tilde{y}$-independent couplings. Let us e.g. consider the covariant derivative on the external metric (4.5.16)

$$D_\mu g_{\nu\rho} \;\equiv\; \partial_\mu g_{\nu\rho} - \mathcal{A}_\mu{}^M\partial_M g_{\nu\rho} - \frac{2}{3}\,\partial_M\mathcal{A}_\mu{}^M\, g_{\nu\rho}\,. \tag{4.5.16}$$

With the Scherk-Schwarz ansatz (4.5.8), we obtain via (4.5.15)

$$D_\mu g_{\nu\rho} \;=\; \rho^{-2}(\tilde{y})\left(\partial_\mu \mathrm{g}_{\nu\rho} - 2\,A_\mu{}^{i0}\partial_{i0}\mathrm{g}_{\nu\rho} - \frac{4}{3}\,\partial_{0i}A_\mu{}^{0i}\,\mathrm{g}_{\nu\rho} + \frac{4}{3}\,A_{\mu*+}\mathrm{g}_{\nu\rho}\right)\,. \tag{4.5.17}$$

The first three terms on the r.h.s. correspond to the standard EFT result and upon translation into the IIB fields contribute to the standard IIB field equations [1]. We will thus employ the notation

$$D_\mu g_{\nu\rho} \;=\; \rho^{-2}(\tilde{y})\left(\mathring{D}_\mu \mathrm{g}_{\nu\rho} + \frac{4}{3}\,A_{\mu*+}\mathrm{g}_{\nu\rho}\right)\,. \tag{4.5.18}$$

The last term captures the effect of the Scherk-Schwarz twist matrix and shows that the IIB space-time metric acquires non-trivial covariant derivatives which is precisely in accordance with our discussion above regarding the charged IIB metric (4.4.8) after transition to the Einstein frame.[7] The Riemann tensor whose contraction appears in the Einstein field equations will thus correspond to the curvature of the modified connection (4.4.8) as in the generalised IIB equations.

In a similar way, we can work out the EFT field strengths (3.2.18) under the Scherk-Schwarz ansatz (4.5.8). As a general feature of the Scherk-Schwarz ansatz with consistent twist matrices, the $\tilde{y}$-dependence of these field strengths consistently factors out according to

$$\mathcal{F}_{\mu\nu}{}^M(x,Y) \;=\; \rho^{-1}(\tilde{y})(U^{-1})_N{}^M(\tilde{y})\,\mathcal{F}_{\mu\nu}{}^M(x,y)\,, \tag{4.5.20}$$

where

$$\mathcal{F}_{\mu\nu}{}^M(x,y) \;\equiv\; \mathring{\mathcal{F}}_{\mu\nu}{}^{NM} + X_{KL}{}^M\left(A_{[\mu}{}^K A_{\nu]}{}^L - 2\,d^{KLN}\,B_{\mu\nu\,N}\right)\,, \tag{4.5.21}$$

describes a deformation of the standard EFT field strength $\mathring{\mathcal{F}}_{\mu\nu}{}^M$ by non-abelian terms carrying the generic structure of five-dimensional gauged supergravity [83] encoded in

[7] To be precise, after identification $A_{\mu*+} = B_{\mu*+}$, the factor 4/3 in (4.5.18) comes via the standard 5 + 5 Kaluza-Klein decomposition

$$G_{mn} \;=\; \begin{pmatrix} (\det g_{ab})^{-1/3}\,g_{\mu\nu} + \ldots & A_\mu{}^b g_{ab} \\ g_{ab}A_\mu{}^b & g_{ab} \end{pmatrix}\,, \tag{4.5.19}$$

of the IIB metric.

an embedding tensor $X_{MN}{}^{K}$ living in the $\mathbf{351}+\mathbf{27}$ representation of $\mathrm{E}_{6(6)}$. Within the Scherk-Schwarz ansatz, the embedding tensor is obtained from projecting (4.5.13) onto the relevant $\mathrm{E}_{6(6)}$ representations. Again, the form of (4.5.21) resembles the deformations of EFT studied in [90], although here it simply results from a Scherk-Schwarz ansatz within the original EFT. Structure-wise, the new couplings (4.5.21) resemble those introduced in (4.4.6) in order to account for the deformed Bianchi identities in generalised IIB supergravity. In the rest of this section, we will make the agreement precise using the explicit dictionary between EFT and IIB fields [1].

Working out (3.2.18), it follows that the twist matrix (4.5.11)–(4.5.12) induces an embedding tensor

$$X_{MN}{}^{K} \;=\; (\tilde{X}_M)_N{}^K + \frac{2}{3}\,\delta_M{}^{*+}\,\delta_N{}^K\;, \tag{4.5.22}$$

in (4.5.21). Upon contraction with a gauge parameter Λ^M it identifies the gauged generators within $\mathfrak{e}_{6(6)}\oplus\mathbb{R}_{\text{tromb}}$. The second term in (4.5.22) refers to the gauging of the trombone symmetry under which the EFT fields $\left\{g_{\mu\nu},\mathcal{M}_{MN},\mathcal{A}_\mu{}^M,\mathcal{B}_{\mu\nu\,M}\right\}$ scale with weight $\{2,0,1,2\}$, respectively, whose effect we have already observed in (4.5.18). The first term in (4.5.22) identifies the gauged generators within $\mathfrak{e}_{6(6)}$, combining the diagonal generators

$$\left(\Lambda^M\tilde{X}_M\right)_{\mathfrak{sl}(2)} = \begin{pmatrix} \frac{1}{2}\,\Lambda_{*+} & 0\\ -\Lambda_{*-} & -\frac{1}{2}\,\Lambda_{*+}\end{pmatrix}\;,\quad \left(\Lambda^M\tilde{X}_M\right)_{\mathfrak{sl}(6)} = \begin{pmatrix} \frac{1}{6}\,\Lambda_{*+}\,\mathbb{I}_4 & 0 & 0\\ 0 & \frac{1}{6}\,\Lambda_{*+} & 0\\ 0 & \Lambda_{0+} & -\frac{5}{6}\,\Lambda_{*+}\end{pmatrix} \tag{4.5.23}$$

within $\mathfrak{sl}(2)\oplus\mathfrak{sl}(6)$ with the off-diagonal generators

$$\begin{aligned}
(\Lambda^M\tilde{X}_M)^{*+,ij} &= (\Lambda^M\,\tilde{X}_M)^{i+,j*} \;=\; -\,\Lambda^{ij}\;,\\
(\Lambda^M\tilde{X}_M)_{ij,0-} &= (\Lambda^M\,\tilde{X}_M)_{0i,j-} \;=\; -\,\frac{1}{2}\,\varepsilon_{*ijkl}\,\Lambda^{kl}\;,
\end{aligned} \tag{4.5.24}$$

in $\mathfrak{e}_{6(6)}\backslash(\mathfrak{sl}(6)\oplus\mathfrak{sl}(2))$. The Stückelberg-type couplings in (4.5.21) to the two-forms $B_{\mu\nu\,M}$ are read off from (4.5.23), (4.5.24) together with the explicit form of d^{MNP} in the decomposition (4.5.2), see [1]. The explicit result for the various components of the field strengths (4.5.21) is the following

$$\begin{aligned}
\mathcal{F}_{\mu\nu\,m+} &= \mathring{\mathcal{F}}_{\mu\nu\,m+}\;,\\
\mathcal{F}_{\mu\nu\,m-} &= \mathring{\mathcal{F}}_{\mu\nu\,m-} + A_{[\mu\,*+}A_{\nu]\,m-} - A_{[\mu\,*-}A_{\nu]\,m+} + \sqrt{2}\,\tilde{B}_{\mu\nu\,m*}\;,\\
\mathcal{F}_{\mu\nu\,kmn} &= \mathring{\mathcal{F}}_{\mu\nu\,kmn} + 2\,A_{[\mu*+}A_{\nu]kmn} + \frac{3}{2\sqrt{2}}\,\varepsilon_{kmnl*}\,\tilde{B}_{\mu\nu}{}^{l-}\;,\\
\mathcal{F}_{\mu\nu\,+} &= \mathring{\mathcal{F}}_{\mu\nu\,+} + 2\,A_{[\mu\,*+}A_{\nu]\,+}\;,
\end{aligned} \tag{4.5.25}$$

with the redefined two-forms

$$\begin{aligned}
\tilde{B}_{\mu\nu\,mn} &\equiv \sqrt{10}\,B_{\mu\nu\,mn} + A_{[\mu}{}^{k}A_{\nu]\,kmn}\;,\\
\tilde{B}_{\mu\nu}{}^{m\alpha} &\equiv \sqrt{10}\,B_{\mu\nu}{}^{m\alpha} + \varepsilon^{\alpha\beta}\,A_{[\mu}{}^{m}A_{\nu]\beta} + \frac{\sqrt{2}}{6}\,\varepsilon^{\alpha\beta}\,\varepsilon^{mnklp}\,A_{[\mu|n\beta|}A_{\nu]klp}\,.
\end{aligned} \tag{4.5.26}$$

Comparing the deformed field strengths (4.5.25) to the field strengths (4.4.6) solving the Bianchi identities of generalised IIB supergravity, we find precise agreement upon identifying the EFT components with the IIB field strengths (the precise dictionary between fields has been given in [1] and in particular takes care of the $\sqrt{2}$ factors that arise in the EFT expressions (4.5.25)).

Of course, the field strengths (4.5.25) only represent part of the full IIB field strengths, in which two of the ten-dimensional indices are chosen to be external. The remaining IIB components will appear among other EFT fields. E.g. let us consider the three-form field strength $\mathcal{H}_{\mu\nu\rho\,M}$ defined by (1.5.17). Evaluating this definition with the above Scherk-Schwarz ansatz in particular yields the components

$$\begin{aligned}\mathcal{H}_{\mu\nu\rho\,-} &= \mathring{\mathcal{H}}_{\mu\nu\rho\,-} + \sqrt{2}\,\mathcal{O}_{\mu\nu\rho}\,,\\ \mathcal{H}_{\mu\nu\rho\,*i} &= \mathring{\tilde{\mathcal{H}}}_{\mu\nu\rho\,*i} + 3\,A_{[\mu|\,*+|}\tilde{B}_{\nu\rho]\,*i} + \frac{3\sqrt{2}}{2}\,A_{[\mu|\,*+|}A_{\nu|\,*-|}A_{\rho]\,i+} - \partial_i\,\mathcal{O}_{\mu\nu\rho}\,.\end{aligned} \tag{4.5.27}$$

The second and third term of $\mathcal{H}_{\mu\nu\rho\,*i}$ reproduce the corresponding deformation terms in (4.4.6). The term $\mathcal{O}_{\mu\nu\rho}$ in (4.5.27) denotes the undetermined contribution in the field strength which vanishes under the projection $d^{KMN}\partial_N$ in (1.5.17).In the undeformed IIB theory, this term is already present in $\mathcal{H}_{\mu\nu\rho\,*i}$. It arises as an integration constant in the EFT field equations and is identified with a component of the IIB four-form according to

$$\sqrt{2}\,\mathcal{O}_{\mu\nu\rho} \;=\; C_{\mu\nu\rho*} + \frac{3}{2}\,B_{[\mu|*+|}B_{\nu\rho]\,-} - \frac{3}{2}\,B_{[\mu|*-|}B_{\nu\rho]\,+}\,, \tag{4.5.28}$$

in order to reconstruct the selfdual IIB five-form field strength from EFT. In the deformed case we are considering here, the same $\mathcal{O}_{\mu\nu\rho}$ arises as part of $\mathcal{H}_{\mu\nu\rho\,-}$ in (4.5.27) where it precisely acounts for the deformation of the IIB three-form field strength $F_{\mu\nu\rho}$, see (4.4.6). Again we thus find complete agreement.

In a similar way, the deformed scalar currents $\mathcal{M}^{MK}D_\mu\mathcal{M}_{KN}$ with the block decomposition (4.5.3) and parametrization (4.5.4) can be matched to the corresponding components of (4.4.6) in which one of the ten-dimensional indices is chosen to be external. Thus all the building blocks of the EFT Lagrangian (3.1.1) exhibit precisely the deformations of their IIB counterparts (4.4.6). Since equations (4.4.6) were derived as solution of the deformed IIB Bianchi identities, it follows that after imposing the Scherk-Schwarz ansatz (4.5.8), the EFT fields satisfy the deformed IIB Bianchi identities. Moreover, most of the generalised IIB field equations are obtained by covariantization of the standard IIB equations, i.e. by replacing the IIB field strengths by their deformed expressions (4.4.6). This is true for the Einstein field equations (upon taking into account the charged metric in the Einstein frame, c.f. (4.4.8)) and the self-duality equation $\mathcal{F}_5 = \star\mathcal{F}_5$ for the five-form field strength. Upon using the explicit dictionary between EFT fields and IIB fields [1] these equations thus follow from the EFT dynamics after imposing the Scherk-Schwarz ansatz. The two-form field equations (4.4.9) in generalised IIB supergravity on the other hand are not only covariantized via (4.4.6) but also acquire a source term $J^{\hat{\mu}\hat{\nu}}{}_{\beta}$. In EFT, the analogous term descends from variation of the Lagrangian (3.1.1) w.r.t. the gauge fields which upon implementing the Scherk-Schwarz ansatz gives rise to additional source terms from the Einstein-Hilbert term and the scalar kinetic term.

4.6 Summary

After a general analysis of the twist equations in section 1, we derived in section 2 the explicit reduction formulae for the full set of IIB fields in the compactification on the sphere S^5 and the inhomogeneous hyperboloids $H^{p,6-p}$. They were derived via the EFT formulation of the IIB theory by evaluating the formulas of the generalised Scherk-Schwarz reduction ansatz for the twist matrices (4.2.37) obtained in [56]. The Scherk-Schwarz origin also proves consistency of the truncation in the sense that all solutions of the respective $D = 5$ maximal supergravities lift to solutions of the type IIB fields equations. Upon some further computational effort we have also derived the explicit expressions for all the components of the IIB four-form. Along the way, we explicitly verified the IIB self-duality equations. Although their consistency is guaranteed by the general construction, this requires the precise interplay between various identities whose validity appears somewhat miraculous from the point of view of conventional geometry but which find a natural interpretation within the extended geometry of exceptional field theory.

Finally, in the last two sections of this chapter, we have shown how the equations of generalised IIB supergravity found in [81] can naturally be obtained from exceptional field theory upon imposing a simple Scherk-Schwarz type ansatz on all the fields that captures their non-isometric behavior in the IIA theory. The Scherk-Schwarz ansatz satisfies the consistency equations [56] and moreover the section constraints (4.5.1) and induces a deformation of the standard IIB supergravity equations. We have verified explicitly for most of their components that the deformed EFT fields coincide with the deformed IIB field strengths (4.4.6) which have been determined by solving the deformed IIB Bianchi identities. We should stress that although exceptional field theory admits a Lagrangian formulation(3.1.1) this does not allow to conclude the existence of an action underlying the generalised IIB equations, since the Scherk-Schwarz ansatz (4.5.8) is imposed on the level of the field equations and not on the action. The appearance of a trombone gauging (4.5.22) in the EFT formulation is in fact a sign that the resulting field equations cannot be obtained from an action [57].

Chapter 5

Supersymmetry in EFT: the peculiar case of $\mathrm{E}_{8(8)}$

It is well known that ungauged maximal supergravity in n dimensions can be obtained by dimensional reduction of the eleven dimension supergravity on a $(11-n)$-torus, where n stands for the number of external dimensions. For the $n = 4$ case, Cremmer and Julia first found in [46] that the equation of motions of the theory were invariant under the global exceptional group $\mathrm{E}_{7(7)}$. In the lower dimensional case $n = 3$, the exceptional symmetry can only be made apparent after dualisation of the vector fields of the theory [98]. This includes the Kaluza-Klein vector fields, which have components of the higher dimensional graviton as their origin. Therefore, the scalar sector of the dimensionally reduced theory carries degrees of freedom descending from the higher dimensional 'dual graviton'. In the $\mathrm{E}_{8(8)}$ EFT, this is taken care of by the introduction of an additional constrained gauge connection which is invisible in the dimensionally reduced theory. While this constrained connection may seem strange at first, the appearance of constrained $(n-2)$ forms is common to every known $\mathrm{E}_{d(d)}$ EFT with $d = 11-n$. However, the gauge symmetry associated with this constrained gauge connection is a new feature of the $\mathrm{E}_{8(8)}$ EFT in the sense that for lower rank groups, it only kicks in at the higher rank p-forms. As we will see in the next section, the additional gauge symmetry takes the form of a constrained $\mathrm{E}_{8(8)}$ rotation in the generalised Lie derivative. Together with the section constraint, this ensures the closure of the full symmetry algebra and a well-defined, consistent exceptional field theory.

Up to now, we have been interested in describing the bosonic sectors of various supergravity with extended field theories. EFT can be extended to describe the full higher dimensional supergravities with fermions transforming under the maximal compact subgroup $\mathrm{K}(\mathrm{E}_{d(d)})$. For $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{6(6)}$ the supersymmetric completions have been worked out in [37, 38]. In this chapter, we will construct the supersymmetric completion of the $\mathrm{E}_{8(8)}$ exceptional field theory. After reviewing the bosonic $\mathrm{E}_{8(8)}$ exceptional field theory, we introduce the fermions as tensors under the generalised Lorentz group $SO(1,2)\times SO(16)$, where $SO(16)$ is the maximal compact subgroup of $\mathrm{E}_{8(8)}$. We determine a torsion-free condition which partly solve the corresponding spin connection to the extent they are required to formulate the field equations and the supersymmetry transformation laws. We then establish the supersymmetry transformations of the field content of the theory and show the closure of the supersymmetry algebra. Finally, we give the full EFT lagrangian, and prove its invariance under supersymmetry.

5.1 Review of the bosonic $\mathrm{E}_{8(8)}$ EFT

In this section, we review the bosonic structures of the $\mathrm{E}_{8(8)}$ exceptional field theory. The theory is defined on a (3+248)-dimensional generalised spacetime. In addition to the usual dependency in spacetime ('external') coordinates x^μ, $\mu=0,1,2$, all fields and gauge parameters formally depend also on extended coordinates $Y^{\mathcal{M}}$, $\mathcal{M}=1,\ldots,248$, transforming in the adjoint representation of $\mathrm{E}_{8(8)}$. As in DFT and the previous $E_{(6)6}$ case, not all of these internal coordinates are physical. This is taken care of by the $\mathrm{E}_{8(8)}$ covariant section constraints,

$$\eta^{\mathcal{M}\mathcal{N}}\,\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}}\equiv 0\;,\quad f^{\mathcal{M}\mathcal{N}\mathcal{K}}\,\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}}\equiv 0\;,\quad (\mathbb{P}_{3875})_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}\mathcal{L}}\partial_{\mathcal{K}}\otimes\partial_{\mathcal{L}}\;\;\equiv\;\;0\;, \tag{5.1.1}$$

where $\eta^{\mathcal{M}\mathcal{N}}$ and $f^{\mathcal{M}\mathcal{N}\mathcal{K}}$ are respectively the Cartan-Killing form and the structure constants of $\mathrm{E}_{8(8)}$ (see appendix B for more details on the conventions used throughout this thesis), and $\mathbb{P}_{\mathbf{3875}}$ is the projector onto the irreducible representation **3875** in the tensor product of two adjoint representation

$$\mathbf{248}\otimes\mathbf{248}\;\;=\;\;\mathbf{1}\oplus\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{27000}\oplus\mathbf{30380}\;, \tag{5.1.2}$$

explicitly given by

$$(\mathbb{P}_{\mathbf{3875}})_{\mathcal{N}\mathcal{L}}{}^{\mathcal{M}\mathcal{K}}=\frac{1}{7}\,\delta_{(\mathcal{N}}{}^{\mathcal{M}}\,\delta_{\mathcal{L})}{}^{\mathcal{K}}-\frac{1}{56}\,\eta^{\mathcal{M}\mathcal{K}}\,\eta_{\mathcal{N}\mathcal{L}}-\frac{1}{14}f^{\mathcal{P}}{}_{\mathcal{N}}{}^{(\mathcal{M}}f_{\mathcal{P}\mathcal{L}}{}^{\mathcal{K})}\;. \tag{5.1.3}$$

The bosonic sector of the theory combines an external three-dimensional metric $g_{\mu\nu}$ (or dreibein $e_\mu{}^a$), an internal frame field $\mathcal{V}_{\mathcal{M}}{}^{\underline{\mathcal{K}}}$, parametrizing the coset space $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$, and the usual gauge connection $A_\mu{}^{\mathcal{M}}$ associated with generalised internal diffeomorphisms. In addition to these standard fields, common to every EFT, we will see latter that one need to introduce a constrained gauge connection $B_{\mu\,\mathcal{M}}$ associated with the additional constrained $\mathrm{E}_{8(8)}$ rotations that appear in the generalised internal diffeomorphisms.

The local symmetries of this exceptional field theory are generalised internal diffeomorphisms, constrained $\mathrm{E}_{8(8)}$ rotations, and external diffeomorphisms with respective parameters $\Lambda^{\mathcal{M}}$, $\Sigma_{\mathcal{M}}$, and ξ^μ. Let us first review the generalised internal diffeomorphisms. The generalised Lie derivative acting on a vector W^M of weight λ_W is defined by

$$\mathbb{L}_{(\Lambda,\Sigma)}W^{\mathcal{M}} = \Lambda^{\mathcal{K}}\partial_{\mathcal{K}}W^{\mathcal{M}} - 60\mathbb{P}^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}}\partial_{\mathcal{K}}\Lambda^{\mathcal{L}}W^{\mathcal{N}} + \lambda_W\partial_{\mathcal{N}}\Lambda^{\mathcal{N}}W^{\mathcal{M}} - \Sigma_{\mathcal{L}}f^{\mathcal{LM}}{}_{\mathcal{N}}W^{\mathcal{N}} \quad (5.1.4)$$

Here $\mathbb{P}^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}}$ projects onto the adjoint representation **248** and guarantees compatibility with the $E_{8(8)}$ structure, c.f. the explicit expression (B.3). The weight λ_W of the various fields in the theory coincides with the three-dimensional Weyl weight of the fields, i.e. weight 2 and 0 for the external and internal metrics $g_{\mu\nu}$ and $\mathcal{M}_{\mathcal{MN}}$, respectively, and weights 1 and 0 for the gauge connections $A_\mu{}^{\mathcal{M}}$ and $B_{\mu\,\mathcal{M}}$, respectively. Fermions (to be introduced in the next section) come with half-integer weight. This is summarized for all fields in Table 5.1.

Field	$e_\mu{}^a$	$\mathcal{V}_{\mathcal{M}}{}^{\underline{K}}$	$A_\mu{}^{\mathcal{M}}$, $\Lambda^{\mathcal{M}}$	$B_{\mu\,\mathcal{M}}$, $\Sigma_{\mathcal{M}}$	$\chi^{\dot{A}}$	$\psi_\mu{}^I$, ϵ^I
Weight (λ)	1	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$

Table 5.1: Weights of all fields and gauge parameters under gen. diffeomorphisms.

Unlike the lower-rank $E_{n(n)}$ cases with $n \leq 7$, the generalised Lie derivative (5.1.4) depends on two parameters, $\Lambda^{\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$, with the latter being subject to the section condition (5.1.1), i.e.

$$(\mathbb{P}_{3875})_{\mathcal{MN}}{}^{\mathcal{KL}}\,\Sigma_{\mathcal{K}}\otimes\Sigma_{\mathcal{L}} \;\equiv\; 0 \;\equiv\; (\mathbb{P}_{3875})_{\mathcal{MN}}{}^{\mathcal{KL}}\,\Sigma_{\mathcal{K}}\otimes\partial_{\mathcal{L}}\;, \qquad \text{etc.}\;. \qquad (5.1.5)$$

This is needed together with the section constraints (5.1.1) in order to ensure closure of the full symmetry algebra. Schematically, we have an algebra

$$[\delta_{(\Lambda_1,\Sigma_1)},\delta_{(\Lambda_2,\Sigma_2)}] \;=\; \delta_{(\Lambda_{12},\Sigma_{12})}\;, \qquad (5.1.6)$$

with notably the gauge parameter Σ_{12} given by

$$\Sigma_{12\,\mathcal{M}} \equiv -2\,\Sigma_{[2\,\mathcal{M}}\partial_{\mathcal{N}}\Lambda_{1]}^{\mathcal{N}} + 2\,\Lambda_{[2}^{\mathcal{N}}\partial_{\mathcal{N}}\Sigma_{1]\,\mathcal{M}} - 2\,\Sigma_{[2}^{\mathcal{N}}\partial_{\mathcal{M}}\Lambda_{1]\,\mathcal{N}} + f^{\mathcal{N}}{}_{\mathcal{KL}}\,\Lambda_{[2}^{\mathcal{K}}\,\partial_{\mathcal{M}}\partial_{\mathcal{N}}\Lambda_{1]}^{\mathcal{L}}\;, \quad (5.1.7)$$

confirming that the Λ transformations do not close among themselves.

Before we describe the associated gauge connections and curvatures, let us make a small digression to discuss connections and torsion compatible with the generalised diffeomorphisms (5.1.4). For an algebra-valued connection

$$\Gamma_{\mathcal{MN}}{}^{\mathcal{K}} \;=\; \Gamma_{\mathcal{M},\mathcal{L}}\,f^{\mathcal{LK}}{}_{\mathcal{N}}\;, \qquad (5.1.8)$$

the fact that pure Λ-transformations do not close into an algebra implies that the naive definition of torsion as

$$\bar{\mathcal{T}}(\Lambda,W)^{\mathcal{M}} \;=\; \bar{\mathcal{T}}_{\mathcal{NK}}{}^{\mathcal{M}}\,\Lambda^{\mathcal{N}}W^{\mathcal{K}} \;=\; \mathbb{L}^{\nabla}_{(\Lambda,\Sigma)}W^{\mathcal{M}} - \mathbb{L}_{(\Lambda,\Sigma)}W^{\mathcal{M}}\;, \qquad (5.1.9)$$

does no longer define a tensorial object. Here, $\mathbb{L}^\nabla$ refers to generalised Lie derivatives (5.1.4) with partial derivatives replaces by covariant ones $\nabla = \partial - \Gamma$. Following [99], this suggests to rather define torsion as the part of the Christoffel connection that transforms covariantly under the generalised diffeomorphisms. With the transformation of (5.1.8) under (5.1.4) given by

$$\delta_{(\Lambda,\Sigma)}\Gamma_{\mathcal{L},\mathcal{N}} \;=\; \delta^{\rm cov}_{(\Lambda,\Sigma)}\,\Gamma_{\mathcal{L},\mathcal{N}} + f_{\mathcal{Q}\mathcal{N}}{}^{\mathcal{P}}\,\partial_{\mathcal{L}}\partial_{\mathcal{P}}\Lambda^{\mathcal{Q}} + \partial_{\mathcal{L}}\Sigma_{\mathcal{N}}\;, \tag{5.1.10}$$

projection onto its irreducible $\mathrm{E}_{8(8)}$ representations according to (5.1.2) shows that only its components in the $\mathbf{1}\oplus\mathbf{3875}$ transform as tensors under (5.1.4). The proper definition of a torsionless connection thus corresponds to the condition

$$\big[\Gamma_{\mathcal{M},\mathcal{N}}\big]_{\mathbf{1}\oplus\mathbf{3875}} \;=\; 0\;, \tag{5.1.11}$$

which can be made explicit with the form of the projector (B.4). Let us note that such a torsionless connection gives rise to the identity

$$\mathbb{L}_{(\Lambda,\Sigma)}\,W^{\mathcal{M}} \;=\; \mathbb{L}^{\nabla}_{(\Lambda,\tilde\Sigma)}\,W^{\mathcal{M}}\;, \qquad \text{with}\qquad \tilde\Sigma_{\mathcal{M}} \equiv \Sigma_{\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\,\Lambda^{\mathcal{N}}\;. \tag{5.1.12}$$

With the r.h.s. of (5.1.12) manifestly covariant, this shows that the combination $\tilde\Sigma_{\mathcal{M}}$ behaves as a tensorial object under generalised diffeomorphisms. In this sense it may appear more natural to parametrise generalised diffeomorphisms in terms of the parameters $(\Lambda,\tilde\Sigma)$. The disadvantage of using $\tilde\Sigma$ w.r.t. the original formulation is the fact that the constraint (5.1.5) which $\Sigma_{\mathcal{M}}$ has to satisfy, takes a much less transparent form when expressed in terms of $\tilde\Sigma$ since the connection $\Gamma_{\mathcal{M},\mathcal{N}}$ in general will not be constrained in its first index and will not even be fully determined by covariant constraints. For the description of generalised diffeomorphisms we thus have the choice between a description with covariant parameters $(\Lambda,\tilde\Sigma)$ and a description in terms of parameters (Λ,Σ) in terms of which the constraints (5.1.5) are well defined and easily expressed. We will in general stick with the latter but observe that the existence of the covariant combination $\tilde\Sigma_{\mathcal{M}}$ gives rise to some compact reformulations of the resulting formulas.[1]

The various terms of the bosonic action are constructed as invariants under the generalised internal Lie derivatives (5.1.4). In the full theory, the gauge parameters $\Lambda^{\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$ depend not only on the internal $Y^{\mathcal{M}}$ but also on the external x^μ coordinates. From the three-dimensional perspective, these symmetries are implemented as (infinite-dimensional) gauge symmetries, such that external derivatives are covariantized with

[1] The existence of the covariant combination $\tilde\Sigma_{\mathcal{M}}$ may suggest to impose $\tilde\Sigma = 0$ in order to reduce the number of independent gauge parameters [99] while preserving closure of the algebra. In view of the constraints (5.1.5), this is only possible in case the connection $\Gamma_{\mathcal{M},\mathcal{N}}$ is identified with the Weitzenböck connection $\partial_{\mathcal{M}}\mathcal{V}_{\mathcal{L}}{}^{\underline{\mathcal{P}}}\,(\mathcal{V}^{-1})_{\underline{\mathcal{P}}}{}^{\mathcal{K}}\,f_{\mathcal{N}\mathcal{K}}{}^{\mathcal{L}}$ which itself is constrained in the first index. We will in the following keep both gauge parameters $\Lambda^{\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$ independent which seems important for the supersymmetric extension.

gauge connections $A_\mu{}^{\mathcal{M}}$, $B_{\mu\,\mathcal{M}}$

$$D_\mu = \partial_\mu - \mathbb{L}_{(A_\mu,B_\mu)}\;. \tag{5.1.13}$$

In accordance with (5.1.5), the connection $B_{\mu\,\mathcal{M}}$ is constrained to obey the same constraints as the gauge parameter $\Sigma_{\mathcal{M}}$. The commutator of the covariant derivatives (5.1.13) closes into the field strengths

$$[D_\mu, D_\nu] \;=\; -\mathbb{L}_{(\mathcal{F}_{\mu\nu},\mathcal{G}_{\mu\nu})}\;, \tag{5.1.14}$$

with

$$\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} \;&=\; 2\,\partial_{[\mu}A_{\nu]}{}^{\mathcal{M}} - 2\,A_{[\mu}{}^{\mathcal{N}}\partial_{\mathcal{N}}A_{\nu]}{}^{\mathcal{M}} + 14\,(\mathbb{P}_{3875})^{\mathcal{MN}}{}_{\mathcal{KL}}A_{[\mu}{}^{\mathcal{K}}\partial_{\mathcal{N}}A_{\nu]}{}^{\mathcal{L}} \\
&\quad + \frac{1}{4}\,A_{[\mu}{}^{\mathcal{N}}\partial^{\mathcal{M}}A_{\nu]\mathcal{N}} - \frac{1}{2}\,f^{\mathcal{MN}}{}_{\mathcal{P}}f^{\mathcal{P}}{}_{\mathcal{KL}}\,A_{[\mu}{}^{\mathcal{K}}\partial_{\mathcal{N}}A_{\nu]}{}^{\mathcal{L}} \;+\; \dots\;, \\
\mathcal{G}_{\mu\nu\mathcal{M}} \;&=\; 2\,D_{[\mu}B_{\nu]\mathcal{M}} - f^{\mathcal{N}}{}_{\mathcal{KL}}\,A_{[\mu}{}^{\mathcal{K}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}A_{\nu]}{}^{\mathcal{L}} \;+\; \dots\;.
\end{aligned} \tag{5.1.15}$$

The ellipsis denote additional two-form terms required for the proper transformation behavior of the field strengths, c.f. (5.1.18) below. As required for consistency, the section constraints (5.1.1) ensure that all these terms drop from the commutators of covariant derivatives where the field strengths are contracted with particular differential operators according to (5.1.14). Moreover, all the two-form terms drop out from the bosonic Lagrangian.

Under gauge transformations

$$\delta_{(\Lambda,\Sigma)}A_\mu{}^{\mathcal{M}} \;=\; D_\mu\Lambda^{\mathcal{M}}\;, \tag{5.1.16}$$

$$\delta_{(\Lambda,\Sigma)}B_{\mu\,\mathcal{M}} \;=\; D_\mu\Sigma_{\mathcal{M}} - \Lambda^{\mathcal{N}}\partial_{\mathcal{M}}B_{\mu\,\mathcal{N}} + f^{\mathcal{N}}{}_{\mathcal{KL}}\Lambda^{\mathcal{K}}\partial_{\mathcal{M}}\partial_{\mathcal{N}}A_\mu{}^{\mathcal{L}}\;, \tag{5.1.17}$$

(where just as the associated gauge connections, the parameters $\Lambda^{\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$ carry weight 1 and 0 under (5.1.4), respectively), the full field strengths (5.1.15) transform according to

$$\begin{aligned}
\delta_{(\Lambda,\Sigma)}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} \;&=\; \mathbb{L}_{(\Lambda,\Sigma)}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\;, \\
\delta_{(\Lambda,\Sigma)}\left(\mathcal{G}_{\mu\nu\,\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{N}}\right) \;&=\; \mathbb{L}_{(\Lambda,\Sigma)}\left(\mathcal{G}_{\mu\nu\,\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{N}}\right)\;,
\end{aligned} \tag{5.1.18}$$

i.e. not the $\mathcal{G}_{\mu\nu\,\mathcal{M}}$ but only the combination $\tilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}} \equiv \mathcal{G}_{\mu\nu\,\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{N}}$ behaves as a tensor under (5.1.4). This reflects the tensorial structure (5.1.12) of generalised diffeomorphisms. Pushing this structure further ahead, we are led to introduce the general 'covariant' variation of the connection $B_{\mu\,\mathcal{M}}$ as

$$\Delta B_{\mu\,\mathcal{M}} \;\equiv\; \delta B_{\mu\,\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\,\delta A_\mu{}^{\mathcal{N}}\;, \tag{5.1.19}$$

in order to cast the gauge transformations (5.1.16) into the more compact form

$$\begin{aligned}
\delta_{(\Lambda,\Sigma)}A_\mu{}^{\mathcal{M}} \;&=\; D_\mu\Lambda^{\mathcal{M}}\;, \\
\Delta_{(\Lambda,\Sigma)}B_{\mu\,\mathcal{M}} \;&=\; D_\mu\tilde{\Sigma}_{\mathcal{M}} + \Lambda^{\mathcal{N}}\,D_\mu\Gamma_{\mathcal{M},\mathcal{N}}\;,
\end{aligned} \tag{5.1.20}$$

with $\tilde{\Sigma}_{\mathcal{M}}$ from (5.1.12). This will turn out to be very useful in the following.

The action of bosonic $E_{8(8)}$ exceptional field theory is given by[2]

$$S_{\rm bos} \;\;=\;\; \int d^3x\, d^{248}Y \;\left(\mathcal{L}_{\rm EH} + \mathcal{L}_{\rm scalar} + \mathcal{L}_{\rm top} + \mathcal{L}_{\rm pot}\right)\;, \tag{5.1.21}$$

where each term is separately invariant under generalised internal diffeomorphisms (5.1.4). The Einstein-Hilbert Lagrangian is given by the Ricci scalar obtained from contraction of the improved Riemann tensor

$$\mathcal{L}_{\rm EH} \;\;=\;\; -e\, e_a{}^\mu e_b{}^\nu\, \widehat{\mathcal{R}}_{\mu\nu}{}^{ab}\;, \tag{5.1.22}$$

where e denotes the determinant of the dreibein $e_\mu{}^a$. The scalar kinetic term in (5.1.21) is given by

$$\mathcal{L}_{\rm scalar} \;\;=\;\; -\frac{1}{240}\, e\, D_\mu \mathcal{M}_{\mathcal{M}\mathcal{N}} D^\mu \mathcal{M}^{\mathcal{M}\mathcal{N}} \;\;=\;\; \mathcal{P}_\mu{}^A\, \mathcal{P}^{\mu\, A}\;. \tag{5.1.23}$$

where we have used the expression

$$\mathcal{M}^{\mathcal{K}\mathcal{P}} D_\mu \mathcal{M}_{\mathcal{P}\mathcal{L}} \;\;=\;\; 2\, f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{L}}\, \mathcal{V}_{\mathcal{M}}{}^A \mathcal{P}_\mu{}^A\;, \tag{5.1.24}$$

of the scalar currents (with covariant derivatives from (5.1.13)) in terms of the $E_{8(8)}$ structure constants $f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{L}}$ and the coset currents (5.2.15) to be introduced next section.

The topological term in (5.1.21) carries the non-abelian Chern-Simons couplings of the gauge connections according to

$$\begin{aligned}\mathcal{L}_{\rm CS} \;\;=\;\; -\frac{1}{2}\,\varepsilon^{\mu\nu\rho}\Big(&\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} B_{\rho\, \mathcal{M}} - f_{\mathcal{K}\mathcal{L}}{}^{\mathcal{N}} \partial_\mu A_\nu{}^{\mathcal{K}} \partial_{\mathcal{N}} A_\rho{}^{\mathcal{L}} - \frac{2}{3}\, f^{\mathcal{N}}{}_{\mathcal{K}\mathcal{L}} \partial_{\mathcal{M}} \partial_{\mathcal{N}} A_\mu{}^{\mathcal{K}} A_\nu{}^{\mathcal{M}} A_\rho{}^{\mathcal{L}} \\ &- \frac{1}{3}\, f_{\mathcal{M}\mathcal{K}\mathcal{L}} f^{\mathcal{K}\mathcal{P}}{}_{\mathcal{Q}} f^{\mathcal{L}\mathcal{R}}{}_{\mathcal{S}}\, A_\mu{}^{\mathcal{M}} \partial_{\mathcal{P}} A_\nu{}^{\mathcal{Q}} \partial_{\mathcal{R}} A_\rho{}^{\mathcal{S}}\Big)\;.\end{aligned} \tag{5.1.25}$$

Its covariance becomes manifest upon spelling out its variation as

$$\delta\mathcal{L}_{\rm CS} \;\;=\;\; -\frac{1}{2}\,\varepsilon^{\mu\nu\rho}\Big(\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\, \Delta B_{\rho\, \mathcal{M}} + \Big(\widetilde{\mathcal{G}}_{\mu\nu\, \mathcal{M}} + 2\, f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\, \nabla_{\mathcal{K}} \mathcal{F}_{\mu\nu}{}^{\mathcal{N}}\Big)\, \delta A_\rho{}^{\mathcal{M}}\Big)\;, \tag{5.1.26}$$

with the covariant field strengths from (5.1.15), (5.1.18) and the general covariant variation introduced in (5.1.19). As anticipated above, we note that the two-form contributions to the field strengths $\mathcal{F}$ and $\mathcal{G}$ (whose explicit form has been suppressed in (5.1.15)) drop out from this expression due to the section constraint. Moreover, the contributions to the Christoffel connection in $\nabla_{\mathcal{K}}$ that are left undetermined by the vanishing torsion condition cancel in this expression against the corresponding contributions in $\Delta B_{\rho\, \mathcal{M}}$.

[2] As usual, the integral over the 248 internal coordinates is to be taken in a formal sense since the section constraint (5.1.1) remains to be imposed by hand and eliminates the field dependence on most of these coordinates.

Finally, the last term in (5.1.21) carries only derivatives in the internal coordinates and is explicitly given by

$$\mathcal{L}_{\text{pot}} \;=\; -e\,V\,, \tag{5.1.27}$$

with the 'potential' V given in manifestly covariant form by

$$V \;=\; \mathcal{R} - \frac{1}{4}\mathcal{M}^{\mathcal{M}\mathcal{N}}\nabla_{\mathcal{M}}g^{\mu\nu}\nabla_{\mathcal{N}}g_{\mu\nu} + \nabla_{\mathcal{M}}I^{\mathcal{M}}\,, \tag{5.1.28}$$

with an internal curvature scalar $\mathcal{R}$ to be introduced in the next section and up to a boundary contribution $I^{\mathcal{M}}$ of weight $\lambda_I = -1$. This close the introduction on the bosonic structure of these EFT. In the next section, we will introduce the tools needed to incorporate the fermions in the EFT.

5.2 $\mathrm{E}_{8(8)} \times \mathrm{SO}(16)$ exceptional geometry

5.2.1 Generalised vielbein

Fermions enter the theory as spinors under the $\mathrm{SO}(1,2) \times \mathrm{SO}(16)$ generalised Lorentz group and transform as weighted scalars under generalised diffeomorphisms. Specifically, under SO(16), the gravitinos ${\psi_\mu}^I$ and fermions $\chi^{\dot{A}}$ transform in the fundamental vector **16** and spinor $\mathbf{128_c}$ representations, respectively. The field content of the full EFT is

$$\{{e_\mu}^a\,,\, {\mathcal{V}_{\mathcal{M}}}^{\underline{\mathcal{K}}}\,,\, {A_\mu}^{\mathcal{M}}\,,\, B_{\mu\,\mathcal{M}}\,, {\psi_\mu}^I\,, \chi^{\dot{A}}\}\,, \tag{5.2.1}$$

i.e. external and internal frame fields together with gauge connections ${A_\mu}^{\mathcal{M}}$, $B_{\mu\,\mathcal{M}}$. The 'dreibein' ${e_\mu}^a$ defines the external metric $g_{\mu\nu} = \eta_{ab}\,{e_\mu}^a{e_\nu}^b$. The '248-bein' ${\mathcal{V}_{\mathcal{M}}}^{\underline{\mathcal{K}}}$ is the internal analogue of the dreibein and parametrises the coset space $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$. Under SO(16), the collective index $\underline{\mathcal{K}}$ splits according to the decomposition of the algebra

$$\mathfrak{e}_{8(8)} \;\longrightarrow\; \mathfrak{so}(16) \oplus \mathbf{128}_s\,, \tag{5.2.2}$$

into the adjoint and the spinor of SO(16), i.e.

$${\mathcal{V}_{\mathcal{M}}}^{\underline{\mathcal{K}}} \;=\; \left\{{\mathcal{V}_{\mathcal{M}}}^{IJ}, {\mathcal{V}_{\mathcal{M}}}^{A}\right\}\,, \tag{5.2.3}$$

satisfying ${\mathcal{V}_{\mathcal{M}}}^{IJ} = {\mathcal{V}_{\mathcal{M}}}^{[IJ]}$ with SO(16) vector indices $I, J = 1, \ldots, 16$, and spinor indices $A, B = 1, \ldots, 128$.[3] In the same way the dreibein defines the external metric, the generalised vielbein defines the internal metric $\mathcal{M}_{\mathcal{M}\mathcal{N}}$

$$\mathcal{M}_{\mathcal{M}\mathcal{N}} \;=\; {\mathcal{V}_{\mathcal{M}}}^{\underline{\mathcal{K}}}{\mathcal{V}_{\mathcal{N}}}^{\underline{\mathcal{L}}}\,\delta_{\underline{\mathcal{K}\mathcal{L}}} \;\equiv\; {\mathcal{V}_{\mathcal{M}}}^{A}{\mathcal{V}_{\mathcal{N}}}^{A} + \frac{1}{2}{\mathcal{V}_{\mathcal{M}}}^{IJ}{\mathcal{V}_{\mathcal{N}}}^{IJ}\,, \tag{5.2.4}$$

[3] See appendix B for more details on the relevant group and algebra conventions.

in terms of which the bosonic theory can be formulated. The inverse 248-bein then is given by

$$(\mathcal{V}^{-1})_{\underline{\mathcal{K}}}{}^{\mathcal{M}} \;=\; \left\{\mathcal{V}^{\mathcal{M}}{}_{B}, -\mathcal{V}^{\mathcal{M}}{}_{IJ}\right\} \;\equiv\; \left\{\eta^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{N}}{}^{B}, -\eta^{\mathcal{M}\mathcal{N}}\mathcal{V}_{\mathcal{N}}{}^{IJ}\right\} \,, \tag{5.2.5}$$

where

$$\mathcal{V}_{\mathcal{M}}{}^{A}\mathcal{V}^{\mathcal{M}}{}_{B} = \delta^{A}{}_{B} \,, \quad \mathcal{V}_{\mathcal{M}}{}^{IJ}\mathcal{V}^{\mathcal{M}}{}_{KL} = -2\,\delta^{IJ}_{KL} \,. \tag{5.2.6}$$

Finally, the 248-bein is an $E_{8(8)}$ group-valued matrix, which results in the standard decomposition of the Cartan form

$$(\mathcal{V}^{-1})_{\underline{\mathcal{L}}}{}^{\mathcal{N}}\partial_{\mathcal{M}}\mathcal{V}_{\mathcal{N}}{}^{\underline{\mathcal{K}}} \;=\; \frac{1}{2}\,q_{\mathcal{M}}{}^{IJ}\,(X_{IJ})^{\underline{\mathcal{K}}}{}_{\underline{\mathcal{L}}} + p_{\mathcal{M}}{}^{A}\,(Y^{A})^{\underline{\mathcal{K}}}{}_{\underline{\mathcal{L}}} \,, \tag{5.2.7}$$

where X_{IJ} and Y^A denote the compact and non-compact generators of $E_{8(8)}$, respectively. With the explicit expressions for the structure constants in the SO(16) basis from (B.1), one finds the internal currents

$$q_{\mathcal{M}}{}^{IJ} \;=\; \frac{1}{64}\,\Gamma^{IJ}_{BA}\mathcal{V}^{\mathcal{N}}{}_{B}\partial_{\mathcal{M}}\mathcal{V}_{\mathcal{N}}{}^{A} \,, \qquad p_{\mathcal{M}}{}^{B} \;=\; -\frac{1}{120}\,\Gamma^{IJ}_{BA}\mathcal{V}^{\mathcal{N}}{}_{A}\partial_{\mathcal{M}}\mathcal{V}_{\mathcal{N}}{}^{IJ} \,, \tag{5.2.8}$$

which will be our building blocks for the internal spin connection and later the Ricci scalar. This sums up the basic properties of the generalised vielbein.

5.2.2 Spin connections

The coupling of fermions require four different blocks of the spin connection

$$\left\{ \begin{array}{cc} \omega_\mu & \omega_{\mathcal{M}} \\ \mathcal{Q}_\mu & \mathcal{Q}_{\mathcal{M}} \end{array} \right\} \tag{5.2.9}$$

that ensure covariance of both external and internal derivatives under SO(1,2) and SO(16), respectively. Via the generalised vielbein postulates

$$\begin{aligned} 0 \;&\equiv\; \nabla_\mu e_\nu{}^a \;\equiv\; D_\mu e_\nu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \Gamma_{\mu\nu}{}^{\rho}e_\rho{}^a \,, \\ 0 \;&\equiv\; \nabla_{\mathcal{M}}\mathcal{V}_{\mathcal{N}}{}^{\underline{K}} \;\equiv\; \partial_{\mathcal{M}}\mathcal{V}_{\mathcal{N}}{}^{\underline{K}} - \frac{1}{2}\mathcal{Q}_{\mathcal{M}}{}^{IJ}(X_{IJ})^{\underline{K}}{}_{\underline{L}}\mathcal{V}_{\mathcal{N}}{}^{\underline{L}} - \Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,\mathcal{V}_{\mathcal{P}}{}^{\underline{K}} \,, \end{aligned} \tag{5.2.10}$$

for the external and internal frame fields, the spin connections relate to the external and internal Christoffel connections

$$\left\{\,\Gamma_{\mu\nu}{}^{\rho},\, \Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\,\right\} \,. \tag{5.2.11}$$

Starting with the external sector, the SO(1,2) connection $\omega_\mu{}^{ab}$ is defined by the vanishing torsion condition of the external Christoffel connection

$$\Gamma_{[\mu\nu]}{}^{\rho} \;=\; 0 \,. \tag{5.2.12}$$

This leads to the standard expression for the spin connection in terms of the objects of anholonomity $\Omega_{abc} \equiv 2\, e_{[a}{}^{\mu} e_{b]}{}^{\nu}\, D_\mu e_{\nu c}$, where however derivatives are covariantized according to (5.1.13) with the dreibein transforming as a scalar of weight 1 under (5.1.4). The external SO(16) connection on the other hand is defined by imposing that the external current

$$(\mathcal{J}_\mu)^{\underline{\mathcal{K}}}{}_{\underline{\mathcal{L}}} \;\equiv\; (\mathcal{V}^{-1})_{\underline{\mathcal{L}}}{}^{\mathcal{N}} \mathcal{D}[A,\mathcal{Q}]_\mu \mathcal{V}_{\mathcal{N}}{}^{\underline{\mathcal{K}}}\,, \tag{5.2.13}$$

lives in the orthogonal complement of $\mathfrak{so}(16)$ within $\mathfrak{e}_{8(8)}$:

$$\mathcal{J}_\mu \;\equiv\; \mathcal{P}_\mu{}^A Y^A\,. \tag{5.2.14}$$

In analogy to (5.2.8) this yields the explicit expressions

$$\mathcal{Q}_\mu{}^{IJ} \;=\; \frac{1}{64}\,\Gamma^{IJ}_{BA}\mathcal{V}^{\mathcal{N}}{}_B\, D_\mu \mathcal{V}_{\mathcal{N}}{}^A\,, \qquad \mathcal{P}_\mu{}^B \;=\; -\,\frac{1}{120}\,\Gamma^{IJ}_{BA}\mathcal{V}^{\mathcal{N}}{}_A\, D_\mu \mathcal{V}_{\mathcal{N}}{}^{IJ}\,, \tag{5.2.15}$$

with covariant derivatives from (5.1.13). According to their definition, the currents $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ satisfy Maurer-Cartan integrability conditions

$$2\,\mathcal{D}_{[\mu}\mathcal{P}_{\nu]}{}^A \;=\; -\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} p_{\mathcal{M}A} + \mathcal{V}_{\mathcal{P}}{}^A f^{\mathcal{P}\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\mathcal{F}_{\mu\nu\,\mathcal{N}} + \mathcal{G}_{\mu\nu\,\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_A\,, \tag{5.2.16}$$

$$\begin{aligned}\mathcal{Q}_{\mu\nu}{}^{IJ} \;&\equiv\; 2\,\partial_{[\mu}\mathcal{Q}_{\nu]}{}^{IJ} + 2\,\mathcal{Q}_\mu{}^{K[I}\mathcal{Q}_\nu{}^{J]K} \\ &=\; -\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} q_{\mathcal{M}}{}^{IJ} + \mathcal{V}_{\mathcal{P}}{}^{IJ} f^{\mathcal{P}\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}}\mathcal{F}_{\mu\nu\,\mathcal{N}} + \mathcal{G}_{\mu\nu\,\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_{IJ} \\ &\quad -\frac{1}{2}\,\mathcal{P}_\mu{}^A\mathcal{P}_\nu{}^B\Gamma^{IJ}_{AB}\end{aligned} \tag{5.2.17}$$

W.r.t. the integrability relations of $D=3$ supergravity [100], these relations represent a deformation with additional terms in field strengths due to the introduction of the gauge fields $A_\mu{}^{\mathcal{M}}$ and $B_{\mu\,\mathcal{M}}$. We will see in the next section how these terms take a manifestly covariant form. In the fermionic sector, the full external covariant derivatives acting on the $\mathrm{SO}(1,2)\times\mathrm{SO}(16)$ spinors of the theory are given by

$$\begin{aligned}\mathcal{D}_\mu\psi^I \;&=\; D_\mu\psi^I + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\,\psi^I + \mathcal{Q}_\mu{}^{IJ}\,\psi^J\,, \\ \mathcal{D}_\mu\chi^{\dot A} \;&=\; D_\mu\chi^{\dot A} + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\,\chi^{\dot A} + \frac{1}{4}\,\mathcal{Q}_\mu{}^{IJ}\Gamma^{IJ}_{\dot A\dot B}\,\chi^{\dot B}\,,\end{aligned} \tag{5.2.18}$$

for spinors ψ^I and $\chi^{\dot A}$ transforming in the **16** and $\mathbf{128}_c$ of SO(16), respectively. Under generalised internal diffeomorphisms (5.1.4), the spinors ψ^I and $\chi^{\dot A}$ transform as scalars of weight 1/2 and $-1/2$, respectively, and the derivatives D_μ in (5.2.18) are covariantized accordingly.

Now, let us turn to the internal sector. Similar to (5.2.14) we derive the internal SO(1,2) spin connection by demanding that the internal current

$$(\mathcal{J}_{\mathcal{M}})^{ab} \;\equiv\; e^{b\mu}\,\mathcal{D}[\omega]_{\mathcal{M}} e_\mu{}^a\,, \tag{5.2.19}$$

lives in the orthogonal complement of $\mathfrak{so}(1,2)$ within $\mathfrak{gl}(3)$

$$(\mathcal{J}_{\mathcal{M}})^{ab} \equiv \pi_{\mathcal{M}}{}^{(ab)} \ . \tag{5.2.20}$$

Explicitly, this yields

$$\omega_{\mathcal{M}}{}^{ab} = e^{\mu[a}\partial_{\mathcal{M}} e_{\mu}{}^{b]} \ . \tag{5.2.21}$$

In order to define the internal SO(16) connection, we recall that the proper condition of vanishing torsion in the internal sector is given by setting to zero the tensorial part (5.1.11) of the Christoffel connection $\Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}$. Via (5.2.10) this condition determines a large part of the SO(16) connection. More precisely, the counting goes as follows [99]: decomposition of (5.1.11) into SO(16) irreducible representations

$$\mathbf{1} \oplus \mathbf{3875} \quad \longrightarrow \quad \mathbf{1} \oplus \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{1920}_c \ , \tag{5.2.22}$$

specifies the representation content of the vanishing torsion conditions. On the other hand, the various components of the SO(16) connection $(\mathcal{Q}_{\mathcal{M}})^{IJ}$ live in the SO(16) representations

$$\begin{aligned} \mathcal{Q}_{KL}{}^{IJ} \quad &: \quad \mathbf{120} \otimes \mathbf{120} = \mathbf{1} \oplus \mathbf{120} \oplus \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{5304} \oplus \mathbf{7020} \ , \\ \mathcal{Q}_{A}{}^{IJ} \quad &: \quad \mathbf{120} \otimes \mathbf{128}_s = \mathbf{128}_s \oplus \mathbf{1920}_c \oplus \mathbf{13312}_s \ . \end{aligned} \tag{5.2.23}$$

Comparison to (5.2.22) exhibits which SO(16) components of $(\mathcal{Q}_{\mathcal{M}})^{IJ}$ are not fixed by imposing vanishing torsion. For practical purposes, these undetermined parts $\mathbf{120} \oplus \mathbf{128}_s \oplus \mathbf{135} \oplus \mathbf{5304} \oplus \mathbf{7020} \oplus \mathbf{13312}_s$ do not pose a problem as they drop out of all physically relevant quantities such as the supersymmetry transformations, the Lagrangian etc., a property that all known supersymmetric exceptional field theories share.

Concretely, the four irreducible components (5.2.22) of the torsion-free condition (5.1.11) take the form

$$\begin{aligned} -\frac{1}{2}\Gamma_{IJ,IJ} + \Gamma_{A,A} &= 0 \ , \\ -\Gamma_{M(I,J)M} - \frac{1}{16}\delta_{IJ}\Gamma_{MN,MN} &= 0 \ , \\ \Gamma_{[IJ,KL]} + \frac{1}{24}\Gamma^{IJKL}_{AB}\,\Gamma_{A,B} &= 0 \ , \\ \Gamma^{J}_{A\dot{A}}\left(\Gamma_{IJ,A} + \Gamma_{A,IJ}\right) + \frac{1}{16}\left(\Gamma^{MN}\Gamma^{I}\right)_{A\dot{A}}\left(\Gamma_{MN,A} + \Gamma_{A,MN}\right) &= 0 \ . \end{aligned} \tag{5.2.24}$$

To explicitly solve these equations (5.2.24), we use (5.2.10), to express the internal Christoffel connection in terms of derivatives of the vielbein

$$\Gamma_{\mathcal{M},\mathcal{N}} = \frac{1}{60} f_{\mathcal{N}}{}^{\mathcal{K}\mathcal{P}}\left(\mathcal{V}_{\mathcal{P}}{}^{A}\mathcal{D}[\mathcal{Q}]_{\mathcal{M}}\mathcal{V}_{\mathcal{K}}{}^{A} - \frac{1}{2}\mathcal{V}_{\mathcal{P}}{}^{IJ}\mathcal{D}[\mathcal{Q}]_{\mathcal{M}}\mathcal{V}_{\mathcal{K}}{}^{IJ}\right) \ , \tag{5.2.25}$$

or, more explicitly

$$\Gamma_{\mathcal{M},\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{A} = -p_{\mathcal{M},A}\,, \qquad \Gamma_{\mathcal{M},\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{IJ} = \mathcal{Q}_{\mathcal{M}}{}^{IJ} - q_{\mathcal{M}}{}^{IJ}\,, \tag{5.2.26}$$

in terms of the Cartan form (5.2.8). Then, combining these equations with (5.2.24) translates conditions on the Christoffel connection into conditions on the spin connection. The solution for the SO(16) spin connection is then found to be

$$\mathcal{Q}_{\mathcal{M}}{}^{IJ} \;=\; \mathcal{V}_{\mathcal{M}}{}^{A}\,\mathcal{Q}_{A}{}^{IJ} - \frac{1}{2}\mathcal{V}_{\mathcal{M}}{}^{KL}\,\mathcal{Q}_{KL}{}^{IJ}\,, \tag{5.2.27}$$

with

$$\begin{aligned}
\mathcal{Q}_{IJ}{}^{KL} \;&=\; q_{IJ}{}^{KL} - \frac{1}{60}\,\delta_{IJ}^{KL}p_{A,A} + \frac{1}{14}\,\delta^{I[K}\,\Gamma_{AB}^{L]J}\,p_{A,B} \\
&\quad +\frac{1}{4!}\,\Gamma_{AB}^{IJKL}\,p_{A,B} + \frac{1}{7}\,\delta^{I[K}\,\mathcal{V}_{\mathcal{M}}{}^{L]J}\,\Gamma_{\mathcal{N}}{}^{\mathcal{M}\mathcal{N}} + U_{IJ,KL}\,, \\
\mathcal{Q}_{A}{}^{IJ} \;&=\; q_{A}{}^{IJ} + p_{IJ,A} - \frac{1}{56}\,\Gamma_{AB}^{IK}\,p_{KJ,B} + \frac{1}{56}\,\Gamma_{AB}^{JK}\,p_{KI,B} \\
&\quad +\frac{3}{364}\,\Gamma_{AB}^{IJKL}\,p_{KL,B} + \frac{1}{60}\,\mathcal{V}^{\mathcal{M}}{}_{B}\Gamma_{AB}^{IJ}\Gamma_{\mathcal{N}\mathcal{M}}{}^{\mathcal{N}} + (R_{13312})_{A}{}^{IJ}\,,
\end{aligned} \tag{5.2.28}$$

c.f. [99], in terms of the Cartan forms (5.2.8), whose first indices we have 'flattened' with the 248-bein $\mathcal{V}_{\mathcal{M}}{}^{\underline{\mathcal{K}}}$. The contributions $U_{IJ,KL}$, $(R_{13312})_{A}{}^{IJ}$ in (5.2.28) are constrained by

$$\begin{aligned}
U_{IJ,KL} \;&=\; U_{[IJ],[KL]}\,, \qquad U_{[IJ,KL]} \;=\; 0 \;=\; U_{IK,KJ}\,, \\
(R_{13312})_{A}{}^{IJ} \;&=\; (R_{13312})_{A}{}^{[IJ]}\,, \qquad \Gamma^{I}_{A\dot{A}}\,(R_{13312})_{A}{}^{IJ} \;=\; 0\,,
\end{aligned} \tag{5.2.29}$$

and not determined by the vanishing torsion condition, in accordance with (5.2.23). The undetermined parts in the $\mathbf{120} \oplus \mathbf{128}_s$ in (5.2.28) have been expressed via the trace $\Gamma_{\mathcal{N}\mathcal{M}}{}^{\mathcal{N}}$ of the Christoffel connection. The latter can be fixed by imposing as an additional condition that the determinant of the external vielbein $e \equiv \det e_\mu{}^a$ be covariantly constant

$$\nabla_{\mathcal{M}} e \;\equiv\; \partial_{\mathcal{M}} e - \frac{3}{2}\,\Gamma_{\mathcal{N}\mathcal{M}}{}^{\mathcal{N}} e \;\equiv\; 0\,, \qquad \Longrightarrow \qquad \Gamma_{\mathcal{N}\mathcal{M}}{}^{\mathcal{N}} \;=\; \frac{2}{3}\,e^{-1}\partial_{\mathcal{M}} e\,. \tag{5.2.30}$$

To summarize, the full internal covariant derivative act on an $\mathrm{E}_{8(8)} \times \mathrm{SO}(16)$ tensor $X_{\mathcal{M}}{}^{I}$ of weight λ_X as

$$\nabla_{\mathcal{M}} X_{\mathcal{N}}{}^{I} \;\equiv\; \partial_{\mathcal{M}} X_{\mathcal{N}}{}^{I} + \mathcal{Q}_{\mathcal{M}}{}^{IJ} X_{\mathcal{N}}{}^{J} - \Gamma_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}} X_{\mathcal{K}}{}^{I} - \frac{1}{2}\lambda_X \Gamma_{\mathcal{K}\mathcal{M}}{}^{\mathcal{K}} X_{\mathcal{N}}{}^{I}\,, \tag{5.2.31}$$

with the connections defined by (5.2.28) and (5.2.26), respectively. This covariant derivative transforms as a generalised tensor of weight $\lambda = \lambda_X - 1$ under generalised diffeomorphisms. In particular, for the spinor fields of the theory, the covariant internal derivatives take the form

$$\begin{aligned}
\nabla_{\mathcal{M}} \psi_\mu^I \;&\equiv\; \partial_{\mathcal{M}} \psi_\mu^I + \mathcal{Q}_{\mathcal{M}}{}^{IJ} \psi_\mu^I + \frac{1}{4}\,\omega_{\mathcal{M}}{}^{ab}\,\gamma_{ab}\,\psi_\mu^I - \frac{1}{4}\,\Gamma_{\mathcal{K}\mathcal{M}}{}^{\mathcal{K}}\,\psi_\mu^I\,, \\
\nabla_{\mathcal{M}} \chi^{\dot{A}} \;&\equiv\; \partial_{\mathcal{M}} \chi^{\dot{A}} + \frac{1}{4}\,\mathcal{Q}_{\mathcal{M}}{}^{IJ}\,\Gamma^{IJ}_{\dot{A}\dot{B}}\,\chi^{\dot{B}} + \frac{1}{4}\,\omega_{\mathcal{M}}{}^{ab}\,\gamma_{ab}\,\chi^{\dot{A}} + \frac{1}{4}\,\Gamma_{\mathcal{K}\mathcal{M}}{}^{\mathcal{K}}\,\chi^{\dot{A}}\,.
\end{aligned} \tag{5.2.32}$$

We conclude this section with a collection of the different covariant derivatives we have used and will use throughout this chapter:

$$
\begin{aligned}
&D_\mu = D[A]_\mu \ , \\
&\mathcal{D}_\mu = \mathcal{D}[A,\omega,\mathcal{Q}]_\mu \ , \qquad &&\mathcal{D}_\mathcal{M} = \mathcal{D}[\omega,\mathcal{Q}]_\mathcal{M} \ , \\
&\nabla_\mu = \nabla[A,\omega,\mathcal{Q},\Gamma]_\mu \ , \qquad &&\nabla_\mathcal{M} = \nabla[\omega,\mathcal{Q},\Gamma]_\mathcal{M} \ ,
\end{aligned} \tag{5.2.33}
$$

where $A_\mu{}^\mathcal{M}$ is the gauge field associated with generalised diffeomorphisms symmetry and the four blocks of the spin connection $\omega_\mu, \mathcal{Q}_\mu, \omega_\mathcal{M}, \mathcal{Q}_\mathcal{M}$ defined in (5.2.12), (5.2.15), (5.2.21), (5.2.28), respectively.

5.2.3 Curvatures

Having defined the various components of the spin connection (5.2.9), we can now discuss their curvatures which will be the building blocks for the bosonic Lagrangian and field equations. Moreover, we will require a number of identities for the commutators of covariant derivatives in order to prove the invariance of the full Lagrangian under supersymmetry.

Let us start with the commutator of two external covariant derivatives on an SO$(1,2)\times$ SO(16) spinor ϵ^I which is obtained straightforwardly from (5.2.18)

$$
[\mathcal{D}_\mu,\mathcal{D}_\nu]\,\epsilon^I \;=\; -\mathcal{F}_{\mu\nu}{}^\mathcal{M}\partial_\mathcal{M}\epsilon^I - \frac{1}{2}\partial_\mathcal{M}\mathcal{F}_{\mu\nu}{}^\mathcal{M}\epsilon^I + \mathcal{Q}_{\mu\nu}{}^{IJ}\epsilon^J + \frac{1}{4}\mathcal{R}_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon^I \ , \tag{5.2.34}
$$

with the field strength of the gauge field $A_\mu{}^\mathcal{M}$ introduced in (5.1.15), the usual external Riemann curvature defined by

$$
\mathcal{R}_{\mu\nu}{}^{ab} = 2D_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\,\omega_{\nu]\,c}{}^{b} \ , \tag{5.2.35}
$$

(with covariant derivatives (5.1.13)), and its analogue $\mathcal{Q}_{\mu\nu}{}^{IJ}$ from (5.2.17) for the SO(16) external spin connection. As the commutator of two external covariant derivatives, the left-hand side of (5.2.34) is covariant whereas this is not manifest from the r.h.s.. Embedding the internal derivatives on the r.h.s. into full covariant derivatives (5.2.32), the commutator can be rewritten as

$$
\begin{aligned}
[\mathcal{D}_\mu,\mathcal{D}_\nu]\,\epsilon^I \;=\;& -\mathcal{F}_{\mu\nu}{}^\mathcal{M}\nabla_\mathcal{M}\epsilon^I - \frac{1}{2}\nabla_\mathcal{M}\mathcal{F}_{\mu\nu}{}^\mathcal{M}\epsilon^I + \frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu}{}^{ab}\gamma_{ab}\epsilon^I \\
&+ \mathcal{Q}_{\mu\nu}{}^{IJ} + \mathcal{F}_{\mu\nu}{}^\mathcal{M}\mathcal{Q}_\mathcal{M}{}^{IJ}\epsilon^J
\end{aligned} \tag{5.2.36}
$$

with the improved Riemann tensor $\widehat{\mathcal{R}}_{\mu\nu}{}^{ab} \equiv \mathcal{R}_{\mu\nu}{}^{ab} + \omega_\mathcal{M}{}^{ab}\mathcal{F}_{\mu\nu}{}^\mathcal{M}$. The latter is covariant under local SO(1,2) Lorentz transformations, shows up in the gravitational field equations and whose contraction in particular gives rise to the improved Ricci scalar

$$
\widehat{R} \;=\; e_a{}^\mu e_b{}^\nu\,\widehat{\mathcal{R}}_{\mu\nu}{}^{ab} \ , \tag{5.2.37}
$$

that is part of the bosonic action. With the first line of (5.2.36) now manifestly covariant, the second line can be rewritten upon using the explicit expression (5.2.17) for $\mathcal{Q}_{\mu\nu}{}^{IJ}$ such that the commutator takes the manifestly covariant form

$$\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]\,\epsilon^I &= \frac{1}{2}\,\mathcal{P}_\mu{}^A\mathcal{P}_\nu{}^B\Gamma^{IJ}_{AB}\,\epsilon^J + \frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu}{}^{ab}\gamma_{ab}\,\epsilon^I + \mathcal{V}_{\mathcal{P}}{}^{IJ}f^{\mathcal{PMN}}\,\nabla_{\mathcal{M}}\mathcal{F}_{\mu\nu\,\mathcal{N}}\,\epsilon^J \\
&\quad + \widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_{IJ}\,\epsilon^J - \mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\nabla_{\mathcal{M}}\,\epsilon^I - \frac{1}{2}\nabla_{\mathcal{M}}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,\epsilon^I\;, \qquad (5.2.38)
\end{aligned}$$

with the tensorial combination of field strengths $\widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}$ from (5.1.18). Similarly, one may rewrite the second integrability relation (5.2.16) into the manifestly covariant form

$$2\,\mathcal{D}_{[\mu}\mathcal{P}_{\nu]}{}^A \;=\; \mathcal{V}_{\mathcal{P}}{}^A f^{\mathcal{PMN}}\nabla_{\mathcal{M}}\mathcal{F}_{\mu\nu\,\mathcal{N}} + \widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_A\;. \qquad (5.2.39)$$

We now turn to the mixed curvature, arising from the commutators of one external and one internal covariant derivatives. We will only be interested in those projections of this commutator, in which the undetermined part of the SO(16) connection drops out. Fortunately, they are the projections relevant to prove the invariance of the Lagrangian under supersymmetry. Evaluating different projections of such a commutator on an $\mathrm{SO}(1,2)\times\mathrm{SO}(16)$ spinor ϵ^I, we obtain the relations

$$\begin{aligned}
\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot{A}}\;[\nabla_{\mathcal{M}}, \mathcal{D}_\mu]\,\epsilon^I &= \frac{1}{4}\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot{A}}\,\mathcal{R}_{\mathcal{M}\mu}{}^{ab}\,\gamma_{ab}\epsilon^I \\
&\quad - \frac{3}{4}\,\Gamma^I_{A\dot{A}}\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^J + \frac{1}{8}\,\Gamma^{IJK}_{A\dot{A}}\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^K\;, \\
\mathcal{V}^{\mathcal{M}}{}_{IJ}\;[\nabla_{\mathcal{M}}, \mathcal{D}_\mu]\,\epsilon^J &= \frac{1}{4}\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\,\mathcal{R}_{\mathcal{M}\mu}{}^{ab}\,\gamma_{ab}\epsilon^J \\
&\quad - \frac{1}{8}\,\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^I - \frac{1}{4}\,\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^B\,\epsilon^J\;, \qquad (5.2.40)
\end{aligned}$$

where the mixed curvature tensor is defined by

$$\mathcal{R}_{\mathcal{M}\mu}{}^{\nu\rho} = e_a{}^\nu e_b{}^\rho\left(\partial_{\mathcal{M}}\omega_\mu{}^{ab} - \mathcal{D}_\mu\omega_{\mathcal{M}}{}^{ab}\right) = \left(\partial_{\mathcal{M}}\Gamma_{\mu\sigma}{}^{[\nu}\right)g^{\rho]\sigma}\;. \qquad (5.2.41)$$

One can show it constitutes a tensor under generalised diffeomorphisms (5.1.4), and satisfies a Bianchi identity

$$\mathcal{R}_{\mathcal{M}[\mu\,\nu\rho]} \equiv 0\;. \qquad (5.2.42)$$

Its contraction to a 'mixed Ricci tensor' yields the following current

$$\mathcal{R}_{\mathcal{M}\nu}{}^{\mu\nu} \;=\; -\frac{1}{2}\,\widehat{J}^\mu{}_{\mathcal{M}} \;\equiv\; e_a{}^\mu e_b{}^\nu\left(\partial_M\omega_\nu{}^{ab} - \mathcal{D}_\nu\left(e^{\rho[a}\partial_M e_\rho{}^{b]}\right)\right)\;, \qquad (5.2.43)$$

which is related to the improved Ricci scalar (5.2.37) by variation w.r.t. the vector fields

$$\delta_A\widehat{R} \;=\; \widehat{J}^\mu{}_{\mathcal{M}}\,\delta A_\mu{}^{\mathcal{M}} \;+\; \nabla_{\mathcal{M}}\mathcal{J}^{\mathcal{M}}_A + \mathcal{D}_\mu\mathcal{I}^\mu_A\;, \qquad (5.2.44)$$

up to a boundary currents $\mathcal{J}_A^{\mathcal{M}}$, $\mathcal{I}_A^{\mu}$ of respective weights $\lambda_{\mathcal{J}_A} = -1$, $\lambda_{\mathcal{I}_A} = -2$, that do not contribute under the integral.

Finally, for the internal curvature, we are again interested in specific projections of two internal covariant derivative in which the undetermined part of the connection drops out. The pertinent projection for the definition of an internal curvature scalar $\mathcal{R}$ in the $\mathrm{E}_{8(8)} \times \mathrm{SO}(16)$ exceptional geometry is given by

$$\begin{aligned}\left(16\,\mathcal{V}^{\mathcal{M}}{}_{KI}\mathcal{V}^{\mathcal{N}}{}_{JK} + 2\,\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{A}\,\delta_{IJ} + 2\,\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{B}\right)\nabla_{\mathcal{M}}\nabla_{\mathcal{N}}\,\epsilon^J \;=&\\ =\; -\frac{1}{8}\,\mathcal{R}\,\epsilon^I + \mathcal{V}^{\mathcal{M}}{}_{KI}\mathcal{V}^{\mathcal{N}}{}_{JK}\,\mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{ab}\gamma_{ab}\epsilon^J\;.&\qquad(5.2.45)\end{aligned}$$

On the l.h.s. the double derivative terms vanish by means of the section constraints (C.2), while a straightforward computation shows that also all linear derivative terms $\partial_{\mathcal{M}}\epsilon^I$ cancel. The curvature of the internal spin connection on the r.h.s. is defined in analogy to (5.2.35) and computed to be

$$\begin{aligned}\mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{ab} &= 2\,\partial_{[\mathcal{M}}\omega_{\mathcal{N}]}{}^{ab} + 2\omega_{[\mathcal{M}}{}^{ac}\,\omega_{\mathcal{N}]\,c}{}^{b}\\ &= -\frac{1}{2}\,e^{\mu[a}e^{b]\nu}g^{\sigma\tau}\nabla_{\mathcal{M}}g_{\mu\sigma}\nabla_{\mathcal{N}}g_{\nu\tau}\;.\qquad(5.2.46)\end{aligned}$$

Upon using the expressions for the SO(16) spin connection (5.2.28), the internal curvature scalar $\mathcal{R}$ in (5.2.45) can be calculated explicitly in terms of the Cartan forms (5.2.8) and the derivative of the external vielbein determinant e as

$$\begin{aligned}\mathcal{R} \;=\; & -\frac{2}{3}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}e^{-2}\partial_{\mathcal{M}}e\,\partial_{\mathcal{N}}e + \frac{4}{3}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}e^{-1}\partial_{\mathcal{M}}\partial_{\mathcal{N}}e + \frac{4}{3}\,\mathcal{V}^{(\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N})}{}_{IJ}\,\Gamma^{IJ}_{AB}\,p_{\mathcal{M}}{}^{B}\,e^{-1}\partial_{\mathcal{N}}e\\ & + \mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{IJ}\,\Gamma^{IJ}_{AB}\left(\partial_{(\mathcal{M}}p_{\mathcal{N})}{}^{B} + \frac{1}{4}\Gamma^{IJ}_{BC}\,q_{(\mathcal{M}}{}^{IJ}p_{\mathcal{N})}{}^{C}\right) + \mathcal{M}^{\mathcal{M}\mathcal{N}}\,p_{\mathcal{M}}{}^{A}\,p_{\mathcal{N}}{}^{A}\\ & + 2\,\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{B}\,p_{\mathcal{M}}{}^{B}p_{\mathcal{N}}{}^{A} - \frac{1}{8}\mathcal{V}^{\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}}{}_{KL}\,\left(\Gamma^{IJ}\Gamma^{KL}\right)_{AB}\,p_{\mathcal{M}}{}^{A}p_{\mathcal{N}}{}^{B}\\ & + \frac{1}{4}\,\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{B}\,\Gamma^{IJ}_{AC}\Gamma^{IJ}_{BD}\,p_{\mathcal{M}}{}^{C}p_{\mathcal{N}}{}^{D}\;.\qquad(5.2.47)\end{aligned}$$

By construction it transforms as a scalar (of weight $\lambda_{\mathcal{R}} = -2$) under generalised diffeomorphisms (5.1.4). Its dependence on the external metric is such that

$$\delta(e\mathcal{R}) \;=\; (\delta e)\,\mathcal{R}\; + \text{total derivatives}\;.\qquad(5.2.48)$$

The other relevant projection of two internal derivatives on a spinor is given by

$$\begin{aligned}\left(12\,\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{IJ}\,\Gamma^{I}_{A\dot{A}} + \left(\Gamma^{IJK}_{A\dot{A}} + 2\Gamma^{I}_{A\dot{A}}\delta^{JK}\right)\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{A}\right)\nabla_{\mathcal{M}}\nabla_{\mathcal{N}}\,\epsilon^J \;=&\\ =\; \frac{1}{8}\,\Gamma^{I}_{A\dot{A}}\mathcal{R}_A\,\epsilon^I + \frac{1}{16}\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}}{}_{A}\left(\Gamma^{IJK}_{A\dot{A}} - 14\,\delta^{JK}\Gamma^{I}_{A\dot{A}}\right)\mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{ab}\gamma_{ab}\,\epsilon^K\,.&\qquad(5.2.49)\end{aligned}$$

where again all double derivatives on the l.h.s. vanish due to the section constraints. The generalised curvature $\mathcal{R}_A$ on the r.h.s. plays the analogue of a Ricci tensor in this geometry and is most conveniently defined by variation of the curvature scalar $\mathcal{R}$ w.r.t. to a non-compact local $\mathfrak{e}_{8(8)}$ transformation of the internal frame field, i.e.

$$\delta_\Sigma \mathcal{R} \;\equiv\; \Sigma^A(Y)\,\mathcal{R}_A \;+\; \nabla_{\mathcal{M}} \mathcal{J}^{\mathcal{M}}_\Sigma\,, \qquad \text{under} \quad \delta_\Sigma \mathcal{V} \;=\; \mathcal{V}\, Y^A \Sigma^A(Y)\,. \tag{5.2.50}$$

up to a boundary current $\mathcal{J}^{\mathcal{M}}_\Sigma$ of weight $\lambda_{\mathcal{J}_\Sigma} = -1$. It can be explicitly given in terms of the Cartan forms (5.2.8) as

$$\begin{aligned}
\mathcal{R}_A \;=\; & -\frac{2}{3}\,\Gamma^{IM}_{AB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_B\partial_{\mathcal{M}}e\,\partial_{\mathcal{N}}e\,e^{-2} + \frac{1}{4}\,\Gamma^{IM}_{AB}\Gamma^{IMNP}_{CD}\mathcal{V}^{\mathcal{M}}{}_{NP}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^D \\
& -\Gamma^{IM}_{AB}\Gamma^{IN}_{CD}\mathcal{V}^{\mathcal{M}}{}_{MN}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^D - \frac{3}{2}\,\Gamma^{IM}_{AB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^C \\
& -2\,\Gamma^{IM}_{CB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_C p_{\mathcal{M}}{}^A p_{\mathcal{N}}{}^B + \frac{23}{16}\,\Gamma^{IM}_{AB}\Gamma^{IN}_{CD}\mathcal{V}^{\mathcal{M}}{}_{MN}\mathcal{V}^{\mathcal{N}}_C p_{\mathcal{M}}{}^D p_{\mathcal{N}}{}^B \\
& +\Gamma^{IM}_{CB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_A p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^B + 2\,\Gamma^{IM}_{AB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_C p_{\mathcal{M}}{}^B p_{\mathcal{N}}{}^C \\
& +\Big(-4\,\mathcal{V}^{\mathcal{N}}{}_A\mathcal{V}^{\mathcal{M}}{}_B - 3\,\delta_{AB}\,\mathcal{V}^{\mathcal{M}}{}_C\mathcal{V}^{\mathcal{N}}{}_C - \frac{1}{4}\,\Gamma^{IMNP}_{AB}\,\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_{NP} \\
& \qquad +\frac{1}{2}\,\Gamma^{IM}_{AC}\Gamma^{IM}_{BD}\,\mathcal{V}^{\mathcal{N}}{}_C\mathcal{V}^{\mathcal{M}}{}_D\Big)\left(\partial_{(\mathcal{M}}p_{\mathcal{N})}{}^B + \frac{1}{4}\Gamma^{IJ}_{BC}\,q_{(\mathcal{M}}{}^{IJ}p_{\mathcal{N})}{}^C\right).
\end{aligned} \tag{5.2.51}$$

This expression above is given in compact form, after simplification by various Fierz-like identities, some of which are collected in appendix D.

5.3 Supersymmetry algebra

In this section we establish the supersymmetry transformation of the various fields and verify that the supersymmetry algebra closes. Before discussing supersymmetry, we briefly review the bosonic symmetries of $E_{8(8)}$ exceptional field theory, since these are the transformations we are going to recover in the commutator of two supersymmetry transformations.

5.3.1 Bosonic symmetries of $E_{8(8)}$ exceptional field theory

In section 5.1 we have extensively discussed the structure of internal generalised Lie derivatives which depend on two parameters $\Lambda^{\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$ with associated gauge connections $\mathcal{A}_\mu{}^{\mathcal{M}}$ and $B_{\mu\,\mathcal{M}}$. A closer analysis [35] shows that these gauge connections come with additional shift symmetries which take the form

$$\begin{aligned}
\delta_\Xi A_\mu{}^{\mathcal{M}} &= \partial_{\mathcal{K}}\Xi_{\mu 3875}{}^{\mathcal{M}\mathcal{K}} + \eta^{\mathcal{M}\mathcal{N}}\Xi_{\mu\mathcal{N}} + f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\Xi_{\mu\mathcal{N}}{}^{\mathcal{K}}\,, \\
\delta_\Xi B_{\mu\,\mathcal{M}} &= \partial_{\mathcal{M}}\Xi_{\mu\mathcal{N}}{}^{\mathcal{N}} + \partial_{\mathcal{N}}\Xi_{\mu\,\mathcal{M}}{}^{\mathcal{N}}\,.
\end{aligned} \tag{5.3.1}$$

Here, the symmetry parameter $\Xi_{\mu 3875}{}^{\mathcal{M}\mathcal{N}}$ lives in the projection of the two adjoint indices $\mathcal{M}\mathcal{N}$ onto the **3875** representation, explicitly realized by (B.4). The parameter $\Xi_{\mu\mathcal{N}}$

is constrained in the same way as the fields $B_{\mu\,\mathcal{M}}$ and $\Sigma_{\mathcal{M}}$, c.f. (5.1.5). Similarly, the parameter $\Xi_{\mu\,\mathcal{N}}{}^{\mathcal{K}}$ is constrained as (5.1.5) in its first internal index $\mathcal{N}$. It is straightforward to check that the shift symmetries (5.3.1) leave the covariant derivatives (5.1.13) invariant. More precisely, they correspond to the tensor gauge transformations associated to the two-form gauge fields that complete the vector field strengths $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}$ and $\mathcal{G}_{\mu\nu\,\mathcal{M}}$ into fully covariant objects, but drop out from the Lagrangian of the theory.

Apart from the internal gauge symmetries, the full set of bosonic symmetries also includes a covariantized version of the (2+1)-external diffeomorphism with the parameter ξ^μ depending on both set of coordinates $\{x^\mu, Y^{\mathcal{M}}\}$. On the bosonic fields these act as[4]

$$\begin{aligned}
\delta_\xi e_\mu{}^a &= \xi^\nu D_\nu e_\mu{}^a + D_\mu \xi^\nu e_\nu{}^a\,, \\
\delta_\xi \mathcal{M}_{\mathcal{M}\mathcal{N}} &= \xi^\nu D_\nu \mathcal{M}_{\mathcal{M}\mathcal{N}}\,, \\
\delta_\xi A_\mu{}^{\mathcal{M}} &= -2\,\mathcal{V}^{\mathcal{M}A}\left(e\varepsilon_{\mu\nu\rho}\,\xi^\nu \mathcal{P}^{\rho\,A} + \mathcal{V}^{\mathcal{N}A} g_{\mu\nu} \nabla_{\mathcal{N}} \xi^\nu\right)\,, \\
\Delta_\xi B_{\mu\,\mathcal{M}} &= -e\varepsilon_{\mu\nu\rho}\Big(g^{\rho\lambda}\,\mathcal{D}^\nu\left(g_{\lambda\sigma}\nabla_{\mathcal{M}}\xi^\sigma\right) - \xi^\nu \widehat{J}^\rho{}_{\mathcal{M}}\Big)\,,
\end{aligned} \tag{5.3.2}$$

where the variation of $B_{\mu\,\mathcal{M}}$ is given in terms of the current $\widehat{J}^\rho{}_{\mathcal{M}}$ introduced in (5.2.43) and most compactly expressed via the general covariant variation $\Delta B_{\mu\,\mathcal{M}}$ introduced in (5.1.19). With (5.1.19), (5.2.26), and the explicit form of $\delta_\xi A_\mu{}^{\mathcal{M}}$ it is straightforward to verify that the variation $\delta_\xi B_{\mu\,\mathcal{M}}$ is uniquely determined and compatible with the constraints (5.1.5) this connection satifies. The external diffeomorphisms (5.3.2) take the expected form for the frame fields $e_\mu{}^a$, $\mathcal{M}_{\mathcal{M}\mathcal{N}}$. In contrast, for the gauge connections $A_\mu{}^{\mathcal{M}}$, $B_{\mu\,\mathcal{M}}$, they relate only on-shell to the standard diffeomorphism transformation of gauge fields.

5.3.2 Closure of the supersymmetry algebra

Let us now move on to the fermionic fields and the supersymmetry algebra. In addition to the bosonic fields introduced in section 5.2, the supersymmetric completion of the $\mathrm{E}_{8(8)}$ exceptional field theory contains the following spinor fields: sixteen gravitinos $\psi_\mu{}^I$ as well as 128 matter fermions $\chi^{\dot{A}}$, transforming in the vector and spinor representation of SO(16), respectively. With respect to generalised diffeomorphisms, they transform as scalar densities with half-integer weights given in Table 5.1. We are working in the Majorana representation and mostly minus signature, i.e. spinors are taken to be real and SO(1,2) gamma matrices γ_μ purely imaginary, c.f. [101] for our spinor conventions. In particular, we use $\gamma_{\mu\nu\rho} = -ie\varepsilon_{\mu\nu\rho}$.

[4] W.r.t. the form of these transformations given in [35], we have expressed the current bosonic current $j^{\rho\mathcal{M}}$ by the coset current $\mathcal{P}^{\rho\mathcal{M}}$, see (5.1.24) below, and furthermore changed the vector transformations by a shift transformation (5.3.1) with parameter $\Xi_{\mu\,\mathcal{M}} = -g_{\mu\nu}\partial_{\mathcal{M}}\xi^\nu$, in order to obtain a more compact presentation of the external diffeomorphisms. Also some signs differ from the formulas in [35] due to the fact that in this chapter we use mostly minus signature $(+--)$ for the external metric.

In this section, we present the supersymmetry transformation rules

$$\begin{aligned}
\delta_\epsilon e_\mu{}^a &= i\bar\epsilon^I\gamma^a\psi_\mu^I\,, \qquad \mathcal{V}^{-1}\delta_\epsilon\mathcal{V} \;=\; \Gamma^I_{A\dot A}\bar\chi^{\dot A}\epsilon^I Y^A\,,\\
\delta_\epsilon\psi_\mu^I &= \mathcal{D}_\mu\epsilon^I + 2\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}(i\gamma_\mu\epsilon^J) + 2\mathcal{V}^{\mathcal{M}}{}_{IJ}\,i\gamma_\mu\nabla_{\mathcal{M}}\epsilon^J\,,\\
\delta_\epsilon\chi^{\dot A} &= \frac{i}{2}\gamma^\mu\epsilon^I\Gamma^I_{A\dot A}\hat{\mathcal{P}}^A_\mu - 2\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\nabla_{\mathcal{M}}\epsilon^I\,,\\
\delta_\epsilon A_\mu{}^{\mathcal{M}} &= -4\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\,\bar\epsilon^I\psi_\mu{}^J + 2\Gamma^I_{A\dot A}\mathcal{V}^{\mathcal{M}}{}_A\,\bar\epsilon^I i\gamma_\mu\chi^{\dot A}\,,\\
\Delta_\epsilon B_{\mu\mathcal{M}} &= -2(\nabla_{\mathcal{M}}\bar\epsilon^I\psi_\mu^I - \bar\epsilon^I\nabla_{\mathcal{M}}\psi_\mu^I) + e\,\varepsilon_{\mu\nu\rho}g^{\rho\sigma}\nabla_{\mathcal{M}}(\bar\epsilon^I i\gamma^\nu\psi_\sigma^I)\,, \qquad (5.3.3)
\end{aligned}$$

and show its algebra closes into generalised diffeomorphisms and gauge transformations. The bosonic transformations (first and fourth line) precisely coincide with the supersymmetry transformations of standard D=3 supergravity [100, 101] with all fields now living on the exceptional space-time. The fermionic transformation rules on the other hand have been modified w.r.t. the three-dimensional theory with the addition of term containing internal covariant derivatives $\nabla_{\mathcal{M}}$ introduced in section 5.2.2. As in higher dimensions, the supersymmetry transformation rules only carry specific projections of these covariant derivatives, such that the undetermined part in the SO(16) connection $\mathcal{Q}_{\mathcal{M}}{}^{IJ}$ drops out. The supersymmetry variations of the gauge connection $B_{\mu\mathcal{M}}$ finally have no analogue in the three-dimensional theory and are entirely determined from closure of the supersymmetry algebra. Although its r.h.s. is such that not all undetermined parts of the SO(16) connection $\mathcal{Q}_{\mathcal{M}}{}^{IJ}$ drop out, these terms precisely cancel the corresponding contributions from the Christoffel connection in the covariant variation (5.1.19) on the l.h.s.. The resulting variation $\delta_\epsilon B_{\mu\mathcal{M}}$ is uniquely determined and compatible with the constraints (5.1.5) this field has to satisfy.

As a first test, we use this ansatz to calculate the commutator of two supersymmetry transformations on the dreibein $e_\mu{}^a$ to obtain

$$\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,e_\mu{}^a &= e\bar\epsilon_2^I\gamma^a\left(\mathcal{D}_\mu\epsilon^I + 2\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}(i\gamma_\mu)\epsilon^J + 4\mathcal{V}^{\mathcal{M}}{}_{IJ}i\gamma_\mu\nabla_{\mathcal{M}}\epsilon^J\right) - (1\leftrightarrow 2)\\
&= \mathcal{D}_\mu\left(\bar\epsilon_2^I\,i\gamma^a\,\epsilon_1^I\right) - 4\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\epsilon_2^I\,\epsilon_1^J\nabla_{\mathcal{M}}e_\mu{}^a + \nabla_{\mathcal{M}}\left(-4\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\epsilon_2^I\,\epsilon_1^J\right)e_\mu{}^a\\
&\quad - 4\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\left(\bar\epsilon_2^I\,\gamma^{ab}\,\nabla_{\mathcal{M}}\epsilon_1^J - \nabla_{\mathcal{M}}\bar\epsilon_2^I\,\gamma^{ab}\,\epsilon_1^J\right)e_{\mu\,b}\\
&\equiv \mathcal{D}_\mu(\xi^\nu e_\nu{}^a) + \Lambda^{\mathcal{M}}\partial_{\mathcal{M}}e_\mu{}^a + \partial_{\mathcal{M}}\Lambda^{\mathcal{M}}e_\mu{}^a + \tilde\Omega^{ab}e_{\mu\,b}\,. \qquad (5.3.4)
\end{aligned}$$

The first term reproduces the action of covariantized external diffeomorphisms, the second and third term describe the action of internal generalised diffeomorphisms on the dreibein, and the last term is an SO(1,2) Lorentz transformation, with the respective parameters given by

$$\begin{aligned}
\xi^\mu &= i\bar\epsilon_2^I\,\gamma^\mu\,\epsilon_1^I\,,\\
\Lambda^{\mathcal{M}} &= -4\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\epsilon_2^I\epsilon_1^J\,,\\
\tilde\Omega^{ab} &= -4\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\left(\bar\epsilon_2^I\,\gamma^{ab}\,\nabla_{\mathcal{M}}\epsilon_1^J - \nabla_{\mathcal{M}}\bar\epsilon_2^I\,\gamma^{ab}\,\epsilon_1^J\right) + \Lambda^{\mathcal{M}}\,\omega_{\mathcal{M}}{}^{ab}\,. \qquad (5.3.5)
\end{aligned}$$

Similarly, one can show closure of the supersymmetry algebra on the 248-bein. Using (5.3.3), we find the commutator

$$
\begin{aligned}
\mathcal{V}^{\mathcal{M}}{}_B\,[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,\mathcal{V}_{\mathcal{M}}{}^{KL} &= \left(-\frac{i}{2}\mathcal{P}_\mu{}^C\Gamma^J_{C\dot{A}}\bar{\epsilon}^J_1\gamma^\mu - 2\mathcal{V}^{\mathcal{N}}{}_C\Gamma^J_{C\dot{A}}\nabla_{\mathcal{N}}\bar{\epsilon}^J_1\right)\epsilon^I_2\Gamma^I_{A\dot{A}}(Y^A)^{KL}{}_B \\
&\quad -(1\leftrightarrow 2) \\
&= \xi^\mu\mathcal{P}_\mu{}^A\,(Y^A)^{KL}{}_B + 60\,\mathcal{V}^{\mathcal{M}}{}_B\mathbb{P}^{\mathcal{N}}{}_{\mathcal{M}}{}^{\mathcal{K}}{}_{\mathcal{L}}\mathcal{V}_{\mathcal{N}}{}^{KL}\nabla_{\mathcal{K}}\Lambda^{\mathcal{L}} \\
&\quad - 2\,\mathcal{V}^{\mathcal{M}}{}_B\left(\nabla_{\mathcal{N}}\bar{\epsilon}^I_2\epsilon^I_1 - \bar{\epsilon}^I_2\nabla_{\mathcal{N}}\epsilon^I_1\right)f^{\mathcal{N}\mathcal{P}}{}_{\mathcal{M}}\mathcal{V}_{\mathcal{P}}{}^{KL}\,,
\end{aligned}
\tag{5.3.6}
$$

with the adjoint projector from (B.3). We recognize the first term as the action of external diffeomorphisms on the 248-bein. The second term reproduces the action (5.1.4) of a generalised internal diffeomorphism with parameter $\Lambda^{\mathcal{L}}$ when parametrized covariantly as in (5.1.12) (note that the transport term $\Lambda^{\mathcal{N}}\nabla_{\mathcal{N}}\mathcal{V}_{\mathcal{M}}{}^{KL}$ vanishes due to the vielbein postulate (5.2.10)). The last term thus describes the covariantized $\mathrm{E}_{8(8)}$ rotation from which we read off the parameter $\tilde{\Sigma}_{\mathcal{N}}$

$$
\tilde{\Sigma}_{\mathcal{N}} \;=\; -2\left(\nabla_{\mathcal{N}}\bar{\epsilon}^I_2\epsilon^I_1 - \bar{\epsilon}^I_2\nabla_{\mathcal{N}}\epsilon^I_1\right)\,. \tag{5.3.7}
$$

As a consistency check, it is straightforward to verify that although the expression for the parameter (5.3.7) carries the full internal SO(16) spin connection $\mathcal{Q}_{\mathcal{N}}{}^{IJ}$ (including its undetermined parts), its form is such that the constrained parameter $\Sigma_{\mathcal{N}} = \tilde{\Sigma}_{\mathcal{N}} + \Gamma_{\mathcal{N},\mathcal{M}}\Lambda^{\mathcal{M}}$ which actually appears in the rotation term of (5.1.4) is uniquely determined (with the undetermined part from $\mathcal{Q}_{\mathcal{N}}{}^{IJ}$ cancelling the undetermined part from $\Gamma_{\mathcal{N},\mathcal{M}}$) and moreover satisfies the required constraints (5.1.5).

Also on the gauge field $A_\mu{}^{\mathcal{M}}$ we obtain closure of the supersymmetry algebra by a standard calculation which gives the explicit result

$$
\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,A_\mu{}^{\mathcal{M}} &= -4\,\mathcal{V}^{\mathcal{M}}{}_{IK}\,\bar{\epsilon}^I_2\left(\mathcal{D}_\mu\epsilon^K_1 + 2\,\mathcal{V}^{\mathcal{N}}{}_{KJ}\nabla_{\mathcal{N}}(i\gamma_\mu)\epsilon^J_1 + 4\mathcal{V}^{\mathcal{N}}{}_{KJ}\,i\gamma_\mu\nabla_{\mathcal{N}}\epsilon^J_1\right) \\
&\quad +2\,\Gamma^I_{A\dot{A}}\mathcal{V}^{\mathcal{M}}{}_A\bar{\epsilon}^I_2 i\gamma_\mu\left(\frac{i}{2}\,\gamma_\nu\epsilon^J_1\Gamma^J_{B\dot{A}}\mathcal{P}^{\nu\,B} - 2\,\mathcal{V}^{\mathcal{N}}{}_B\Gamma^J_{B\dot{A}}\nabla_{\mathcal{N}}\epsilon^J_1\right) - (1\leftrightarrow 2) \\
&= \mathcal{D}_\mu\Lambda^{\mathcal{M}} + \nabla_{\mathcal{N}}\left(-16i\,\mathcal{V}^{\mathcal{M}}{}_{K(I}\mathcal{V}^{\mathcal{M}}{}_{J)K}\bar{\epsilon}^I_2\gamma_\mu\epsilon^J_1\right) \\
&\quad + 8i\,f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\mathcal{V}^{\mathcal{K}}{}_{IJ}\left(\bar{\epsilon}^I_2\gamma_\mu\nabla_{\mathcal{N}}\epsilon^J_1 - \nabla_{\mathcal{N}}\bar{\epsilon}^I_2\gamma_\mu\epsilon^J_1\right) \\
&\quad -2i\,e\varepsilon_{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_A\mathcal{P}^{\rho\,A}\bar{\epsilon}^I_2 i\gamma^\nu\epsilon^I_1 + 4\mathcal{V}^{\mathcal{M}}{}_A\mathcal{V}^{\mathcal{N}}{}_A\xi^a\nabla_{\mathcal{N}}e_\mu{}^a - 4\mathcal{V}^{\mathcal{M}}{}_A\mathcal{V}^{\mathcal{N}}{}_A\nabla_{\mathcal{N}}\xi_\mu \\
&= \mathcal{D}_\mu\Lambda^{\mathcal{M}} - 2\mathcal{V}^{\mathcal{M}A}\left(e\varepsilon_{\mu\nu\rho}\,\xi^\nu\mathcal{P}^{\rho\,A} + \mathcal{V}^{\mathcal{N}A}g_{\mu\nu}\nabla_{\mathcal{N}}\xi^\nu\right) \\
&\quad +\partial_{\mathcal{N}}\Xi_{\mu\,3875}{}^{(\mathcal{M}\mathcal{N})} + f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\,\Xi_{\mu\mathcal{N}}{}^{\mathcal{K}} + \eta^{\mathcal{M}\mathcal{N}}\Xi_{\mu\mathcal{N}}
\end{aligned}
\tag{5.3.8}
$$

with the parameters $\Lambda^{\mathcal{M}}$ and ξ^μ from (5.3.5) and the shift parameters Ξ_μ of the last line defined as

$$
\begin{aligned}
\Xi_{\mu\mathcal{N}} &= -2\partial_{\mathcal{N}}\xi_\mu \;, \\
\Xi_{\mu\,3875}{}^{(\mathcal{MN})} &= -16\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{KJ}\bar{\epsilon}_2^{(I}i\gamma_\mu\epsilon_1^{J)} - \mathcal{V}^{\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}}{}_{IJ}\bar{\epsilon}_2^{K}i\gamma_\mu\epsilon_1^{K} \;, \\
\Xi_{\mu_{\mathcal{N}}}{}^{\mathcal{K}} &= -8\,\mathcal{V}^{\mathcal{K}}{}_{IJ}\left(\nabla_{\mathcal{N}}\bar{\epsilon}_2^{I}i\gamma_\mu\epsilon_1^{J} - \bar{\epsilon}_2^{I}i\gamma_\mu\nabla_{\mathcal{N}}\epsilon_1^{J}\right) \\
&\quad +\Gamma_{\mathcal{N},\mathcal{M}}\left(\Xi_{\mu\,3875}{}^{(\mathcal{MK})} - 2\eta^{\mathcal{MK}}\xi_\mu\right) \;, \qquad (5.3.9)
\end{aligned}
$$

corresponding to the shift symmetries (5.3.1) discussed above. The fact that $\Xi_{\mu\,3875}{}^{(\mathcal{MN})}$ lives in **3875** representations is an immediate consequence of its specific form

$$
\Xi_{\mu\,3875}{}^{(\mathcal{MN})} = -16\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{KJ}\,\xi_{\mu\,IJ} \;, \qquad \xi_{\mu\,IJ} \equiv i\bar{\epsilon}_2^{(I}\gamma_\mu\epsilon_1^{J)} - \frac{1}{16}\,\delta_{IJ}\,\xi_\mu \;, \qquad (5.3.10)
$$

with a parameter $\xi_{\mu\,IJ}$ in the **135** of SO(16), combined with the fact that the tensor product of two adjoint representations (5.1.2) contains only a single representation **135** of SO(16) which lives within the **3875** representation of $E_{8(8)}$. Moreover, the last term in (5.3.9) carrying the Christoffel connection ensures that the parameter $\Xi_{\mu_{\mathcal{N}}}{}^{\mathcal{K}}$ does not carry any of the undetermined parts of the SO(16) connection $\mathcal{Q}_{\mathcal{N}}{}^{IJ}$ and furthermore is constrained in its first index, as required by the shift symmetries (5.3.1).

We have at this point fully determined the supersymmetry algebra

$$
[\delta_{\epsilon_1},\delta_{\epsilon_2}] \;=\; \delta_\xi + \delta_{\tilde{\Omega}} + \delta_\Lambda + \delta_\Sigma + \delta_\Xi \;, \qquad (5.3.11)
$$

with parameters given in (5.3.5), (5.3.7), (5.3.9). As a consistency check of the construction it remains to verify that the algebra closes in the same form on the constrained connection $B_{\mu\,\mathcal{M}}$. This computation is greatly facilitated by the notation of the covariant variation (5.1.19) in terms of which its supersymmetry variation takes the covariant form (5.3.3). To lowest order in fermions, the supersymmetry algebra on $B_{\mu\,\mathcal{M}}$ is given by

$$
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\,B_{\mu\,\mathcal{M}} \;=\; 2\,\delta_{\epsilon_1}\Delta_{\epsilon_2}\,B_{\mu\,\mathcal{M}} + 2\,\Gamma_{\mathcal{M},\mathcal{N}}\,\delta_{\epsilon_1}\delta_{\epsilon_2}\,A_\mu{}^{\mathcal{N}} \;. \qquad (5.3.12)
$$

For the second term we may use the closure of the algebra on the vector fields $A_\mu{}^{\mathcal{M}}$ established above. The first term after some calculation yields

$$
\begin{aligned}
2\,\delta_{\epsilon_1}\Delta_{\epsilon_2}\,B_{\mu\,\mathcal{M}} &= \Delta_{\Lambda,\Sigma}B_{\mu\,\mathcal{M}} + \Delta_\xi B_{\mu\,\mathcal{M}} \\
&\quad +2\nabla_{(\mathcal{M}}\tilde{\Xi}_{\mu\mathcal{N})}{}^{\mathcal{N}} + \varepsilon_{\mu\nu\rho}\mathcal{R}_{\mathcal{MN}}{}^{\nu\rho}\Lambda^{\mathcal{N}} \\
&\quad +8\,\mathcal{V}^{\mathcal{N}}{}_{IJ}\left([\nabla_{\mathcal{M}},\nabla_{\mathcal{N}}]\bar{\epsilon}_2^{I}i\gamma_\mu\epsilon_1^{J} - \bar{\epsilon}_2^{I}i\gamma_\mu[\nabla_{\mathcal{M}},\nabla_{\mathcal{N}}]\epsilon_1^{J}\right) \;, \qquad (5.3.13)
\end{aligned}
$$

with the parameters given in (5.3.5), (5.3.7), (5.3.9) and the covariant combination

$$
\begin{aligned}
\tilde{\Xi}_{\mu\mathcal{N}}{}^{\mathcal{K}} &= -8\,\mathcal{V}^{\mathcal{K}}{}_{IJ}\left(\nabla_{\mathcal{N}}\bar{\epsilon}_2^{I}i\gamma_\mu\epsilon_1^{J} - \bar{\epsilon}_2^{I}i\gamma_\mu\nabla_{\mathcal{N}}\epsilon_1^{J}\right) \;, \\
&= \Xi_{\mu_{\mathcal{N}}}{}^{\mathcal{K}} - \Gamma_{\mathcal{N},\mathcal{M}}\left(\Xi_{\mu\,3875}{}^{(\mathcal{MK})} - 2\eta^{\mathcal{MK}}\xi_\mu\right) \;. \qquad (5.3.14)
\end{aligned}
$$

The first line of (5.3.13) reproduce the covariant variation of $B_{\mu\,\mathcal{M}}$ under generalised internal and external diffeomorphisms. For the supersymmetry algebra to close, the second and third line of (5.3.13) must reproduce the shift symmetries

$$\begin{aligned}
\Delta_{\Xi} B_{\mu\,\mathcal{M}} &= \delta_{\Xi} B_{\mu\,\mathcal{M}} - \Gamma_{\mathcal{M},\mathcal{N}}\, \delta_{\Xi} A_{\mu}{}^{\mathcal{M}} \,, \\
&= 2\,\nabla_{(\mathcal{M}} \tilde{\tilde{\Xi}}_{\mu\,\mathcal{N})}{}^{\mathcal{N}} + 2\Gamma_{[\mathcal{N}\mathcal{M}]}{}^{\mathcal{P}} \tilde{\tilde{\Xi}}_{\mu\,\mathcal{P}}{}^{\mathcal{N}} - \Gamma_{\mathcal{P}[\mathcal{N}}{}^{\mathcal{P}} \tilde{\tilde{\Xi}}_{\mu\,\mathcal{M}]}{}^{\mathcal{N}} \\
&\quad \left(\partial_{\mathcal{N}} \Gamma_{\mathcal{N},\mathcal{P}} - \Gamma_{\mathcal{N}\mathcal{P}}{}^{\mathcal{Q}} \Gamma_{\mathcal{M},\mathcal{Q}}\right) \left(\Xi_{\mu\,3875}{}^{\mathcal{N}\mathcal{P}} - 2\eta^{\mathcal{N}\mathcal{P}} \xi_{\mu}\right) \,, \\
&= 2\,\nabla_{(\mathcal{M}} \tilde{\tilde{\Xi}}_{\mu\,\mathcal{N})}{}^{\mathcal{N}} + \varepsilon_{\mu\nu\rho} \mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{\nu\rho} \Lambda^{\mathcal{N}} \\
&\quad +8\,\mathcal{V}^{\mathcal{N}}{}_{IJ} \left([\nabla_{\mathcal{M}}, \nabla_{\mathcal{N}}] \bar{\epsilon}_2^I i\gamma_{\mu} \epsilon_1^J - \bar{\epsilon}_2^I i\gamma_{\mu} [\nabla_{\mathcal{M}}, \nabla_{\mathcal{N}}] \epsilon_1^J\right) \,,
\end{aligned} \tag{5.3.15}$$

where we have obtained the last equality with the use of the following identity

$$\begin{aligned}
\left(2\partial_{[\mathcal{M}} \Gamma_{\mathcal{N}],\mathcal{P}} - \Gamma_{\mathcal{M},\mathcal{L}} \Gamma_{\mathcal{N},\mathcal{Q}} f^{\mathcal{L}\mathcal{Q}}{}_{\mathcal{P}}\right) \left(\mathcal{V}^{(\mathcal{N}}{}_{IK} \mathcal{V}^{\mathcal{P})}{}_{KJ} + \frac{1}{8} \mathcal{V}^{\mathcal{N}}{}_{A} \mathcal{V}^{\mathcal{P}}{}_{A}\, \delta_{IJ}\right) & \\
-\mathcal{V}^{\mathcal{N}}{}_{IK} \left(2\partial_{[\mathcal{M}} Q_{\mathcal{N}]}{}^{KJ} + 2Q_{[\mathcal{M}}{}^{KL} Q_{\mathcal{N}]}{}^{LJ}\right) &= 0 \,.
\end{aligned} \tag{5.3.16}$$

This is reminiscent of standard Riemannian geometry, where the curvature of the Christoffel symbols is the curvature of the spin connection

$$R_{\mu\nu}{}^{\rho\sigma}\,[\Gamma] = R_{\mu\nu}{}^{ab}\,[\omega]\, e_a{}^{\rho} e_b{}^{\sigma} \,, \tag{5.3.17}$$

albeit here, in a projected fashion.

This proves the closure of the supersymmetry algebra on $B_{\mu\,\mathcal{M}}$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, B_{\mu\,\mathcal{M}} = \delta_{(\Lambda,\Sigma)} B_{\mu\,\mathcal{M}} + \delta_{\xi} B_{\mu\,\mathcal{M}} + \delta_{\Xi} B_{\mu\,\mathcal{M}} \,, \tag{5.3.18}$$

and concludes the discussion on the consistency of the supersymmetry algebra (5.3.11).

5.4 Supersymmetric Lagrangian

We can now present the supersymmetric completion of the bosonic action (5.1.21). The fermionic field content comprises the gravitinos $\psi_{\mu}{}^{I}$ and spin 1/2 fermions $\chi^{\dot{A}}$ transforming in the fundamental vector **16** and spinor $\mathbf{128_c}$ representations of SO(16), respectively. The full $\mathrm{E}_{8(8)}$ Lagrangian is given by

$$\begin{aligned}
e^{-1}\mathcal{L} &= -\widehat{\mathcal{R}} + g^{\mu\nu} \mathcal{P}_{\mu}{}^{A} \mathcal{P}_{\nu}{}^{A} + e^{-1}\,\mathcal{L}_{\text{top}} - V \\
&\quad +2i\,\gamma^{\mu\nu\rho} \bar{\psi}_{\lambda}^{I} \mathcal{D}_{\mu} \psi_{\nu}^{I} - 2i\,\bar{\chi}^{\dot{A}} \gamma^{\mu} \mathcal{D}_{\mu} \chi^{\dot{A}} - 2\,\bar{\chi}^{\dot{A}} \gamma^{\mu} \gamma^{\nu} \psi_{\mu}^{I} \Gamma^{I}_{A\dot{A}} \mathcal{P}_{\nu}{}^{A} \\
&\quad +e^{-1}\,\mathcal{L}_{\text{quartic}} + 8\,\mathcal{V}^{\mathcal{M}}{}_{IJ} \bar{\psi}_{\mu}^{I} \gamma^{\mu\nu} \nabla_{\mathcal{M}} \psi_{\nu}^{J} - 8i\,\mathcal{V}^{\mathcal{M}}{}_{A}\, \Gamma^{I}_{A\dot{A}} \bar{\psi}_{\mu}^{I} \nabla_{\mathcal{M}} (\gamma^{\mu} \chi^{\dot{A}}) \\
&\quad -2\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\, \Gamma^{IJ}_{\dot{A}\dot{B}} \bar{\chi}^{\dot{A}} \nabla_{\mathcal{M}} \chi^{\dot{B}} \,.
\end{aligned} \tag{5.4.1}$$

The first line is the bosonic Lagrangian (5.1.21). The terms in the second line are obtained via a direct uplift (and proper covariantization) from $D = 3$ maximal supergravity [100, 101]: a Rarita-Schwinger term for the gravitinos ψ^I_μ, a kinetic term for the 128 matter fermions $\chi^{\dot{A}}$, and the Noether coupling between the coset current $\mathcal{P}_\mu{}^A$ and the fermions. The three last terms of (5.4.1) carrying internal covariant derivative $\nabla_\mathcal{M}$ have been added to ensure invariance of the Lagrangian under supersymmetry transformations. After proper Scherk-Schwarz reduction of the Lagrangian [56], these terms provide the Yukawa couplings of the gauged three-dimensional supergravity. Finally, $\mathcal{L}_{\text{quartic}}$ denotes the quartic fermion terms. We expect these to coincide with the corresponding terms of the three-dimensional theory [100, 101]

$$\begin{aligned} e^{-1}\,\mathcal{L}_{\text{quartic}} \;&=\; -\frac{1}{2}\Big(\overline{\chi}\gamma_\rho\Gamma^{IJ}\chi\,\big(\overline{\psi}{}^I_\mu\gamma^{\mu\nu\rho}\psi^J_\nu - \overline{\psi}{}^I_\mu\gamma^\rho\psi^{\mu J}\big) + \overline{\chi}\chi\,\overline{\psi}{}^I_\mu\gamma^\nu\gamma^\mu\psi^I_\nu\Big) \\ &\quad + \frac{1}{2}\Big((\overline{\chi}\chi)(\overline{\chi}\chi) - \frac{1}{12}\,\overline{\chi}\gamma^\mu\Gamma^{IJ}\chi\,\overline{\chi}\gamma_\mu\Gamma^{IJ}\chi\Big)\,, \end{aligned} \tag{5.4.2}$$

but as far as this thesis is concerned we will only deal with fermions at quadratic order.

For the proof of invariance of (5.4.1) under supersymmetry (5.3.3), we first note that all terms that do not carry internal derivatives cancel precisely as in the three-dimensional theory. Terms carrying internal derivatives arise in the bosonic sector from variation of the potential V and the topological term $\mathcal{L}_{\text{top}}$. In the fermionic sector, such terms arise from the corresponding terms in the supersymmetry transformations (5.3.3), from variation of the last three terms in (5.4.1), as well as from the modified integrability relations (5.2.38), (5.2.39).

We organise these terms according to their structure

$$\bar\psi\,\mathcal{D}_\mu\nabla_\mathcal{M}\epsilon\,,\quad \bar\chi\,\mathcal{D}_\mu\nabla_\mathcal{M}\epsilon\,,\quad \bar\psi\,\nabla_\mathcal{M}\nabla_\mathcal{N}\epsilon\,,\quad \bar\chi\,\nabla_\mathcal{M}\nabla_\mathcal{N}\epsilon \tag{5.4.3}$$

and show that they cancel against the contributions from the bosonic Lagrangian. In the rest of this section, we will only focus on the last two types of terms in (5.4.3), which carry two internal derivatives and thus exhibit an interesting geometric structure of the internal space. The cancellation of the remaining terms is described in detail in appendix E.

Let us start by collecting the terms in $\bar\psi\,\nabla_\mathcal{M}\nabla_\mathcal{N}\epsilon$ in the variation of the fermionic

Lagrangian

$$e^{-1}\delta\mathcal{L}_{\rm ferm}\Big|_{\bar\psi\nabla\nabla\epsilon} \;\longrightarrow\; 8i\Big(8\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}_{KJ}+\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{A}\delta_{IJ}\Big)\,\bar\psi^I_\mu\gamma^\mu\left\{\nabla_\mathcal{M},\nabla_\mathcal{N}\right\}\epsilon^J$$
$$+\,8i\Big(8\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{KJ}+\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_{A}\mathcal{V}^{\mathcal{N}}{}_{B}\Big)\,\bar\psi^I_\mu\gamma^\mu\left[\nabla_\mathcal{M},\nabla_\mathcal{N}\right]\epsilon^J$$
$$+\,32i\,\bar\psi^I_\mu\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{KJ}\Big(\gamma^{\mu\nu}\nabla_\mathcal{N}\gamma_\nu\nabla_\mathcal{M}\epsilon^J+2\gamma^{\mu\nu}\nabla_\mathcal{M}\gamma_\nu\nabla_\mathcal{N}\epsilon^J$$
$$+\,\nabla_\mathcal{M}(\gamma^{\mu\nu})\gamma_\nu\nabla_\mathcal{N}\epsilon^J\Big)$$
$$+\,16i\,\mathcal{V}^{\mathcal{M}}{}_{A}(\Gamma^I\Gamma^J)_{AB}\mathcal{V}^{\mathcal{N}}{}_{B}\,\bar\psi^I_\mu\,\nabla_\mathcal{M}\gamma^\mu\nabla_\mathcal{N}\epsilon^J$$
$$+\,32i\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{KJ}\bar\psi^I_\mu\Big(\gamma^{\mu\nu}\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma_\nu+\frac{1}{2}\nabla_\mathcal{M}\gamma^{\mu\nu}\nabla_\mathcal{N}\gamma_\nu\Big)\epsilon^J\;. \tag{5.4.4}$$

Upon use of the section constraints (C.2) and together with the identity (5.2.45), one can show that all the quadratic and linear terms in derivatives of ϵ vanish. Then, the remaining terms cancel the first two lines of the variation of the scalar potential (5.1.28) under a supersymmetry transformation (up to total derivatives)

$$\delta_\epsilon(eV) \;=\; \frac{1}{2}\,e\,\Big(g^{\mu\nu}\mathcal{R}-\frac{1}{4}g^{\mu\nu}\mathcal{M}^{\mathcal{M}\mathcal{N}}\nabla_\mathcal{M}g^{\rho\sigma}\nabla_\mathcal{N}g_{\rho\sigma}+\nabla_\mathcal{M}(\mathcal{M}^{\mathcal{M}\mathcal{N}}\nabla_\mathcal{N}g^{\mu\nu})$$
$$+g^{\mu\rho}\nabla_\mathcal{M}g_{\rho\sigma}\nabla_\mathcal{N}g^{\sigma\nu}\mathcal{M}^{\mathcal{M}\mathcal{N}}\Big)\,\delta_\epsilon g_{\mu\nu}$$
$$+\,e\,\Gamma^I{}_{A\dot A}\,\bar\chi^{\dot A}\epsilon^I\Big(\mathcal{R}_A+\frac{1}{4}\Gamma^{IJ}_{AB}\mathcal{V}^{(\mathcal{M}}{}_{B}\mathcal{V}^{\mathcal{N})}{}_{IJ}\nabla_\mathcal{M}g^{\mu\nu}\nabla_\mathcal{N}g_{\mu\nu}\Big)\;, \tag{5.4.5}$$

where for the cancellation we have used the following identity

$$\gamma^{\mu\nu}\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma_\nu+\frac{1}{2}\,\nabla_\mathcal{M}\gamma^{\mu\nu}\nabla_\mathcal{N}\gamma_\nu \;=\; \frac{1}{2}\,\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma^\mu-\frac{1}{2}\,g^{\mu\nu}\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma_\nu$$
$$-\frac{1}{4}\,\mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{ab}\gamma^\mu\gamma_{ab}-\frac{1}{8}\,\gamma^\mu\nabla_\mathcal{M}g^{\nu\rho}\nabla_\mathcal{N}g_{\nu\rho}\;. \tag{5.4.6}$$

The last line in (5.4.5) then cancels against the corresponding terms from the variation of the fermionic Lagrangian

$$e^{-1}\delta\mathcal{L}_{\rm ferm}\Big|_{\bar\chi\nabla\nabla\epsilon} \;\longrightarrow\; 4\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{A}\left(\Gamma^{IKJ}_{A\dot A}+12\,\Gamma^I_{A\dot A}\delta^{KJ}\right)\bar\chi^{\dot A}\left\{\nabla_\mathcal{M},\nabla_\mathcal{N}\right\}\epsilon^J$$
$$+\,4\,\mathcal{V}^{\mathcal{M}}{}_{IK}\mathcal{V}^{\mathcal{N}}{}_{A}(\Gamma^{IKJ}_{A\dot A}-10\Gamma^I_{A\dot A}\delta^{KJ})\bar\chi^{\dot A}\left[\nabla_\mathcal{M},\nabla_\mathcal{N}\right]\epsilon^J$$
$$+\,16\,\mathcal{V}^{\mathcal{N}}{}_{IJ}\mathcal{V}^{\mathcal{M}}{}_{A}\bar\chi^{\dot A}\Gamma^I_{A\dot A}\,\gamma^\mu\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma_\mu\,\epsilon^J\;. \tag{5.4.7}$$

Using the identity (5.2.49) and the section constraints (C.2) one finds that all quadratic and linear terms in ϵ vanish while the remaining terms precisely cancel the last line of (5.4.5). For this, the following relations are useful

$$\nabla_\mathcal{M}\gamma^{\mu\nu} \;=\; 2\,\gamma^{[\mu}\nabla_\mathcal{M}\gamma^{\nu]}\;,\qquad \gamma^\mu\nabla_\mathcal{M}\gamma_\mu \;=\; 0\;, \tag{5.4.8}$$

$$\gamma^\nu\nabla_\mathcal{M}\nabla_\mathcal{N}\gamma_\nu \;=\; -\frac{1}{2}\,\mathcal{R}_{\mathcal{M}\mathcal{N}}{}^{ab}\gamma_{ab}-\frac{1}{4}\,\gamma^\mu\nabla_\mathcal{M}g^{\nu\rho}\nabla_\mathcal{N}g_{\nu\rho}\;. \tag{5.4.9}$$

We have thus sketched the vanishing of all terms carrying two internal derivatives in the supersymmetry variation of (5.4.1). The cancellation of the remaining terms is described in detail in appendix E. To summarize the result, we have shown invariance of the action (5.4.1) up to quartic fermion terms.

5.5 A comment on the additional gauge connection

In contrast to the standard formulation of supergravities, in exceptional field theory the bosonic symmetries already uniquely determine the bosonic Lagrangian without any reference to fermions and supersymmetry. Nevertheless, it is important to establish that the resulting bosonic Lagrangian allows for a supersymmetric completion upon coupling of the proper fermionic field content as we have done in this chapter. In particular, in the context of generalised Scherk-Schwarz reductions [56] this construction provides the consistent reduction formulas for the embedding of the fermionic sector of lower-dimensional supergravities into higher dimensions.

A particular attribute of $E_{8(8)}$ exceptional field theory is the appearance of an additional constrained gauge connection $B_{\mu\,\mathcal{M}}$ related to an additional gauge symmetry which ensures closure of the algebra of generalised diffeomorphisms. Unlike all other fields of $E_{8(8)}$ exceptional field theory, this gauge connection is invisible in three-dimensional supergravity. More precisely, upon a consistent truncation of exceptional field theory down to three dimensions by means of a generalised Scherk-Schwarz reduction

$$\begin{aligned}
\mathcal{M}_{\mathcal{MN}}(x,Y) &= U_{\mathcal{M}}{}^{\mathcal{K}}(Y)\,U_{\mathcal{N}}{}^{\mathcal{L}}(Y)\,M_{\mathcal{KL}}(x)\;, \\
g_{\mu\nu}(x,Y) &= \rho^{-2}(Y)\,g_{\mu\nu}(x)\;, \\
A_\mu{}^{\mathcal{M}}(x,Y) &= \rho^{-1}(Y)A_\mu{}^{\mathcal{N}}(x)(U^{-1})_{\mathcal{N}}{}^{\mathcal{M}}(Y)\;,
\end{aligned} \tag{5.5.1}$$

with the Y-dependence carried by an $E_{8(8)}$ matrix U and a scaling factor ρ (satisfying their system of consistency equations), the constrained gauge connection $B_{\mu\,\mathcal{M}}$ reduces according to

$$B_{\mu\,\mathcal{M}}(x,Y) \;\propto\; \rho^{-1}(Y)\,(U^{-1})_{\mathcal{K}}{}^{\mathcal{P}}(Y)\,\partial_{\mathcal{M}}U_{\mathcal{P}}{}^{\mathcal{L}}(Y)\,f_{\mathcal{NL}}{}^{\mathcal{K}}\,A_\mu{}^{\mathcal{N}}(x)\;, \tag{5.5.2}$$

such that its fluctuations are expressed in terms of the same three-dimensional vector fields $A_\mu{}^{\mathcal{N}}(x)$ that parametrize the fluctuations of the $A_\mu{}^{\mathcal{M}}(x,Y)$. It is thus tempting to wonder if already in exceptional field theory, and before reduction, the constrained gauge connection can be considered as a function of the remaining fields such as [99]

$$B_{\mu\,\mathcal{M}} \;\overset{?}{=}\; \Gamma_{\mathcal{M},\mathcal{N}}\,A_\mu{}^{\mathcal{N}}\;, \tag{5.5.3}$$

c.f. (5.1.12). However, as seen above, coupling to fermions requires a connection $\Gamma_{\mathcal{M},\mathcal{N}}$ other than the Weitzenböck connection, such that (5.5.3) would obstruct compatibility with the constraints (5.1.5). Moreover, supersymmetry of the Lagrangian requires a non-trivial transformation law (5.3.3) for the constrained connection $B_{\mu\,\mathcal{M}}$. It is remarkable that as we have shown above this additional constrained connection consistently joins the remaining bosonic and fermionic fields into a single supermultiplet without the need of additional fermionic matter.

The fact that all transformation laws of $B_{\mu\,\mathcal{M}}$ are most compactly expressed in terms of the general covariant variation (5.1.19) is remnant of structures that appear in a general tensor hierarchy of non-abelian p-forms [50]. This may hint at a yet larger algebraic structure which in particular unifies the topological term and the generalised three-dimensional Einstein-Hilbert term of (5.1.21) into a single non-abelian Chern-Simons form on an enlarged algebra. If the present construction should allow for a generalization to the infinite-dimensional cases of E_9 [102, 103, 104], E_{10} [105, 106], (and maybe E_{11} [107, 108, 109]), this appearance of additional bosonic representations and their interplay with supersymmetry may play an essential role.

5.6 Summary

In this chapter we have constructed the supersymmetric completion of the bosonic $\mathrm{E}_{8(8)}$ exceptional field theory. The final result is given by the action (5.4.1) and the supersymmetry transformation laws (5.3.3). In particular, we have established the supersymmetry algebra which consistently closes into the generalised internal and external diffeomorphisms together with the tensor gauge transformations of the theory. The geometry of the internal space is constrained by the section condition (5.1.1) which admits (at least) two inequivalent solutions for which the action (5.4.1) reproduces the full $D = 11$ supergravity and full type IIB supergravity, respectively. The fermions of exceptional field theory can consistently accommodate the fermions of the type IIA and type IIB theory, since the $\mathrm{E}_{8(8)}$-covariant formulation (5.4.1) does not preserve the original $D = 10$ Lorentz invariance. The resulting $D = 10$ fermion chirality thus depends on the solution of the section constraint.

Chapter 6

Conclusion and outlook

To conclude, we have seen that extended field theories, besides rendering hidden symmetries of supergravity manifest, also provide a powerful tool: the generalised Scherk-Schwarz ansatz. In this setting, a reduction ansatz can be very conveniently spelled out in terms of generalised twist matrices. In addition, the often difficult question of consistency is solved provided the twist matrices satisfy a set of differential equations. Of course, finding such matrices satisfying these equations is still a very challenging problem. An important outlook would be to find a systematic method to solve the consistency equations under which consistent truncations are possible.

In most of this thesis, we focused on different applications of the generalised Scherk-Schwarz ansatz. In chapter 2, we proved an old conjecture [6] on the consistency of the Pauli reduction of the bosonic string in n+d dimensions on any d-dimensional group manifold. In contrast with a DeWitt reduction, where one keeps the fields that are singlet under G_L or G_R and therefore is automatically consistent, a Pauli reduction keeps the full isometry group $G_L \times G_R$ as gauge group. In this case, there is no group theoretical argument to tell which fields should be kept and which fields should be truncated. By using the formalism of DFT, a T-duality covariant rewriting of the NS-NS sector of supergravity, this question can be answered with the generalised Scherk-Schwarz ansatz.We constructed the $SO(d, d)$ twist matrices on which the GSS relies on, in terms of the Killing vectors of the bi-invariant metric on G and showed they satisfy the consistency equations. We then deduced the full non-linear reduction ansatz for all fields.

Chapter 3 constitutes the preliminary work needed such that one could extract the various type IIB reduction formulas shown in Chapter 4. The main topic of this chapter was the identification of the fundamental fields of EFT with those of type IIB in a dictionary. Having established the EFT/Type IIB dictionary, we presented two additional examples of the usefulness of the GSS in chapter 4. The first application of the GSS in the $\mathrm{E}_{6(6)}$ was the proof of the consistency Kaluza-Klein of type IIB on $\mathrm{AdS}_5\times\mathrm{S}^5$. The proof relies on the use of $\mathrm{SO}(p, q)$ twist matrices found in [56] satisfying the consistency equations. We evaluated the reduction formulas with these explicit twist matrices and translated the EFT fields into type IIB ones with the dictionary. We thus obtained the full type IIB reduction formulas. The last two sections of this chapter focused on the second application of the GSS: a deformation of type IIB within EFT. This deformation is known as 'generalised' type IIB supergravity. It is a set of equation resembling the standard type IIB field equations, but with a one-form, subject to a Bianchi-like identity, instead of the exterior derivative of the dilaton [82]. After reviewing the generalised field equation in section (4.4), we solved the deformed Bianchi identities, thus obtaining the explicit expression of the deformed field strengths in terms of the fundamental fields of type IIB. Finally, we showed in section (4.5) how the deformations of the field strengths can be obtained from a surprisingly simple Scherk-Schwarz ansatz upon picking a new solution of the section constraint. Since the gaugings generated by this ansatz contain the trombone generator, the resulting field equations cannot be obtained from an action.

In the first four chapters of this thesis, we have restricted the construction to the NS-NS and bosonic sectors of type IIB supergravity. EFT can be extended to describe the full higher dimensional supergravities with fermions transforming under the maximal compact subgroup $\mathrm{K}(\mathrm{E}_{d(d)})$. This has been done in [37, 38] for $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{6(6)}$ respectively. In the last chapter, we extended the supersymmetric completion of the bosonic EFT to the $\mathrm{E}_{8(8)}$ case. After a review of the bosonic EFT, we developed the tools needed to introduce the fermions in the theory, such as the generalised spin connection. In particular, a large part of the internal $SO(16)$ spin connection was determined from the torsion-free condition on the Christoffel connection, first found in [99]. As usual in EFT, the remaining undetermined parts in the spin connection always drop out of all physically relevant quantities. With the definition of proper internal and external covariant derivatives, we gave in section 3 the supersymmetry transformation rules and showed its algebra closes into generalised diffeomorphisms and gauge transformations. We then gave the supersymmetric lagrangian, whose full invariance under supersymmetry is proven in appendix D. Finally, a comment is made regarding the simplifications arising if one choose the connection to be of Weitzenböck type, a legitimate choice in the bosonic theory. However, coupling to fermions seems to require a different connection. Nevertheless, it is rather interesting that the transformations laws of the additional constrained connection are most conveniently expressed in terms of a general covariant variation where the algebra valued torsion-free connection appear. We speculate that the general covariant variation of the constrained $(n-2)$-forms can be written in a similar way in every known $\mathrm{E}_{d(d)}$ EFT $(n+d=11)$. For example, in the $\mathrm{E}_{7(7)}$ EFT, further simplifications should be achieved by rewriting the general covariant variation of the constrained 2-form as

$$\Delta\mathcal{B}_{\mu\nu\,M} = \delta\mathcal{B}_{\mu\nu M} - \Gamma_M{}^{\alpha}\mathcal{B}_{\mu\nu\,\alpha}\,. \tag{6.0.1}$$

Therefore, supersymmetrising the EFT action allowed us to unravel new bosonic structures and it could prove of vital importance for the generalisation of exceptional geometry to the infinite-dimensional algebras [102, 103, 104, 105, 106, 107, 108, 109].

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Appendices

A Finding $\Lambda_{\mu\nu\rho\sigma}$

In order to find the last missing contribution $\Lambda_{\mu\nu\rho\sigma}$ in the expression (4.3.54) for the four-form component $C_{\mu\nu\rho\sigma}$ let us study the reduction of the different terms of equation (4.3.55)

$$\begin{aligned}\frac{1}{120}\, e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp} \;&=\; 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}D^{\rm KK}_{\rho}\mathcal{B}_{\sigma\tau]}{}^{\beta}+8\,\mathcal{F}_{[\mu\nu}{}^{k}C_{\rho\sigma\tau]\,k}\\ &\quad -4\,D^{\rm KK}_{[\mu}C_{\nu\rho\sigma\tau]}\;. \end{aligned} \tag{A.1}$$

By construction, after imposing the generalised Scherk-Schwarz ansatz this equation should split into a y-dependent part proportional to the $D = 5$ scalar field equations (4.1.17), and a y-independent part which determines the function $\Lambda_{\mu\nu\rho\sigma}$.

The first term on the r.h.s. simply reduces according to the reduction ansatz (4.3.6)

$$30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}D^{\rm KK}_{\rho}\mathcal{B}_{\sigma\tau]}{}^{\beta} \;=\; 30\,\varepsilon_{\alpha\beta}\,\mathcal{Y}_a\mathcal{Y}_b\,B_{[\mu\nu}{}^{a\alpha}D_{\rho}B_{\sigma\tau]}{}^{b\beta}\;. \tag{A.2}$$

Note that the Kaluza-Klein covariant derivative turns into the $\mathrm{SO}(p, 6-p)$ covariant derivative by virtue of (4.2.58). With (4.3.43) and the identity (4.2.68), we find for the second term on the r.h.s. of (A.1)

$$
\begin{aligned}
8\,\mathcal{F}_{[\mu\nu}{}^{k} C_{\rho\sigma\tau]\,k} \;&=\; -\frac{1}{2}\,\mathcal{Y}_b\mathcal{Y}^a\, F_{[\mu\nu}{}^{cb}\left(2\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\rho\sigma\tau]\kappa\lambda}\, M_{ac,N}F^{\kappa\lambda\,N} + \sqrt{2}\,\Omega^{efgh}_{\rho\sigma\tau]}\,\varepsilon_{acefgh}\right) \\
&\quad + 2\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[ab]}{}^{m}\mathcal{K}_{[cd]}{}^{k}\mathcal{K}_{[ef]}{}^{l}\mathcal{Z}_{[gh]\,mkl}\;. \qquad \text{(A.3)}
\end{aligned}
$$

Next, we have to work out the covariant curl of $C_{\mu\nu\rho\sigma}$ with the explicit expression (4.3.54). To this end, we first note that for all terms with y-dependence proportional to $\mathcal{Y}^a\mathcal{Y}^b$, the Kaluza-Klein covariant derivative reduces to

$$
D_\mu^{\mathrm{KK}}\left(\mathcal{Y}^a\mathcal{Y}^b\,X_{ab}\right) \;=\; \mathcal{Y}^a\mathcal{Y}^b\,D_\mu X_{ab}\;, \qquad \text{(A.4)}
$$

in view of the property (4.2.58) of the harmonics $\mathcal{Y}^a$. We thus find

$$
\begin{aligned}
-4\,D^{\mathrm{KK}}_{[\mu}C_{\nu\rho\sigma\tau]} \;&=\; \frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ca}\,D^\lambda M_{bc,N}\right) - 4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} \\
&\quad + \frac{1}{2}\,\sqrt{2}\,\mathcal{Y}_b\mathcal{Y}^a\,\varepsilon_{acdefg}\,D_{[\mu}\left(F_{\nu\rho}{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{bg} + \sqrt{2}\,A_\nu{}^{cd}A_\rho{}^{eh}A_\sigma{}^{fj}A_{\tau]}{}^{bg}\,\eta_{hj}\right) \\
&\quad - \sqrt{2}\,D^{\mathrm{KK}}_{[\mu}\left(\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{K}_{[ef]}{}^{n}\mathcal{Z}_{[gh]\,kln}A_\nu{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\right)\;. \qquad \text{(A.5)}
\end{aligned}
$$

In order to evaluate the last term it is important to note that unlike in (A.4), the Kaluza-Klein covariant derivative here cannot just be pulled through the (non-covariant) y-dependent functions but has to be evaluated explicitly leading to

$$
\begin{aligned}
-\sqrt{2}\,D^{\mathrm{KK}}_{[\mu}\left(\mathcal{A}_\nu{}^{k}\mathcal{A}_\rho{}^{l}\mathcal{A}^n_\sigma\mathcal{A}_{\tau]\,kln}\right) \;&=\; -\frac{3}{2}\,\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{K}_{[ef]}{}^{n}\,\mathcal{Z}_{[gh]\,kln} \\
&\quad + \frac{1}{2}\,\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[cd]}{}^{k}\mathcal{K}_{[ef]}{}^{l}\mathcal{K}_{[gh]}{}^{n}\,\mathcal{Z}_{[ab]\,kln} \\
&\quad + \frac{3}{10}\,\sqrt{2}\,A_{[\mu}{}^{rs}A_\nu{}^{uv}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}f_{cd,rs}{}^{ab}\varepsilon_{abuvge}\mathcal{Y}_f\mathcal{Y}_h\;,
\end{aligned}
$$

after some manipulation of the functions $\mathcal{K}_{[ab]}$, $\mathcal{Z}_{[ab]}$. Putting everything together and again using once more the identity (4.2.69), the full r.h.s. of equation (A.1) is given by

$$
\begin{aligned}
(\mathrm{A.1})_{\mathrm{r.h.s}} \;&=\; \frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ca}\,D^\lambda M_{bc,N}\right) - 4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} \\
&\quad + \frac{1}{2}\,\sqrt{2}\,\mathcal{Y}_a\mathcal{Y}^b\,\varepsilon_{bcdefg}\,D_{[\mu}\left(F_{\nu\rho}{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{ag} + \sqrt{2}\,A_\nu{}^{cd}A_\rho{}^{eh}A_\sigma{}^{fj}A_{\tau]}{}^{ag}\,\eta_{hj}\right) \\
&\quad + \frac{1}{2}\,\varepsilon_{dfghce}\mathcal{Y}_a\mathcal{Y}_b\,F_{[\mu\nu}{}^{df}A_\rho{}^{ac}A_\sigma{}^{be}A_{\tau]}{}^{gh} + 30\,\varepsilon_{\alpha\beta}\,\mathcal{Y}_a\mathcal{Y}_b\,B_{[\mu\nu}{}^{a\,\alpha}D_\rho B_{\sigma\tau]}{}^{b\,\beta} \\
&\quad + \frac{3}{5}\,\sqrt{2}\,\varepsilon_{csuvge}\,\mathcal{Y}_a\mathcal{Y}_b\,\eta_{dr}\,A_{[\mu}{}^{rs}A_\nu{}^{uv}A_\rho{}^{cd}A_\sigma{}^{ae}A_{\tau]}{}^{bg} \\
&\quad - \frac{1}{2}\,\mathcal{Y}_b\mathcal{Y}^a\,F_{[\mu\nu}{}^{cb}\left(2\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\rho\sigma\tau]\kappa\lambda}\,M_{ac,N}F^{\kappa\lambda\,N} + \sqrt{2}\,\Omega^{efgh}_{\rho\sigma\tau]}\,\varepsilon_{acefgh}\right)\;. \qquad \text{(A.6)}
\end{aligned}
$$

Some calculation and use of the Schouten identity shows that all terms carrying explicit gauge fields add up precisely such that their y-dependence drops out due to $\mathcal{Y}_a\mathcal{Y}^a = 1$. Specifically, we find

$$
\begin{aligned}
(\mathrm{A.1})_{\mathrm{r.h.s}}\Big|_{FFA} &= \frac{1}{8}\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}\ , \\
(\mathrm{A.1})_{\mathrm{r.h.s}}\Big|_{FAAA} &= \frac{1}{4}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\ , \\
(\mathrm{A.1})_{\mathrm{r.h.s}}\Big|_{AAAAA} &= \frac{1}{10}\sqrt{2}\,A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma}{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\ .
\end{aligned}
\tag{A.7}
$$

In addition, we use the $D = 5$ duality equation (4.1.11) in order to rewrite the BDB term of (A.1) and arrive at

$$
\begin{aligned}
(\mathrm{A.1})_{\mathrm{r.h.s}} \;=\;& -\frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{bc,N}\right) - 4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} \\
&+\frac{1}{10}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{bc,N}F_{\kappa\lambda}{}^{ac} - \frac{1}{2}\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{db}\,M^{d\,\alpha}{}_N\,B_{\kappa\lambda}{}^{a\,\beta}\right) \\
&+\frac{1}{8}\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef} + \frac{1}{4}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh} \\
&+\frac{1}{10}\sqrt{2}\,A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma}{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\ .
\end{aligned}
\tag{A.8}
$$

Structurewise, the r.h.s. of equation (A.1) is thus of the form

$$
(\mathrm{A.1})_{\mathrm{r.h.s}} \;=\; \left(\mathcal{Y}_a(y)\mathcal{Y}_b(y) - \frac{1}{6}\,\eta_{ab}\right)\mathcal{E}_{1\,ab}(x) + \mathcal{E}_2(x)\ . \tag{A.9}
$$

Consistency of the reduction ansatz then implies that also the l.h.s. of (A.1) organizes into the same structure. The coefficients multiplying the y-dependent factor $\left(\mathcal{Y}_a(y)\mathcal{Y}_b(y) - \frac{1}{6}\,\eta_{ab}\right)$ must combine into a $D = 5$ field equation in order to reduce (A.1) to an y-independent equation which then provides the defining equation for $\Lambda_{\mu\nu\rho\sigma}$.

In order to see this explicitly, we recall, that the l.h.s. of (A.1) is defined by (4.3.56), which together with the reduction ansatz (1.6.2) for $\mathcal{M}_{MN}$ may be used to read off the form of this term after reduction. After some manipulation of the Killing vectors and tensors and use of the identities collected in section 4.2.3, we obtain

$$
\begin{aligned}
\frac{1}{120}\,e\,\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp} \;=\;& -\frac{1}{10}\sqrt{2}\,\sqrt{|\mathrm{g}|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q\mathcal{K}_{[cd]}{}^m\partial_m U_q{}^f \\
&-\frac{2}{5}\,\sqrt{|\mathrm{g}|}\,\mathcal{Y}_a\mathcal{Y}_b\,\eta_{cd}\,M^{ac,bd}\ .
\end{aligned}
\tag{A.10}
$$

in terms of the SL(6) twist matrix (4.2.37), and the combination

$$
\mathcal{X}^{(ab)cd,e}{}_f \;=\; \mathcal{X}^{(ab)[cd],e}{}_f \;\equiv\; 2\,M^{je,g(a}M^{b)h,cd}M_{gh,jf} - M_{f\alpha}{}^{g(a}M^{b)h,cd}M_{gh}{}^{e\alpha}\ , \tag{A.11}
$$

of matrix components of (4.1.13). At first view, the structure of this expression in no way ressembles the form of (A.9), with a far more complicated y-dependence in its first term.

This seemingly jeopardizes the consistency of the reduction of equation (A.1), which after all should be guaranteed by consistency of the ansatz. What comes to the rescue is some additional properties of the twist matrix together with some highly non-trivial non-linear identities among the components of an $\mathrm{E}_{6(6)}$ matrix. Namely the last factor in the first term of (A.10) drastically reduces upon certain index projections

$$\begin{aligned} (U^{-1})_a{}^q\mathcal{K}_{[bc]}{}^m\partial_m U_q{}^c + (U^{-1})_b{}^q\mathcal{K}_{[ac]}{}^m\partial_m U_q{}^c &= -\sqrt{2}\,\eta_{ab}\;, \\ (U^{-1})_a{}^q\mathcal{K}_{[bc]}{}^m\partial_m U_q{}^d + (U^{-1})_b{}^q\mathcal{K}_{[ca]}{}^m\partial_m U_q{}^d + (U^{-1})_c{}^q\mathcal{K}_{[ab]}{}^m\partial_m U_q{}^d &= 0\;, \end{aligned} \tag{A.12}$$

as may be verified by explicit computation. Moreover, the tensor $\mathcal{X}^{(ab)cd,e}{}_f$ defined in (A.11) is of quite restricted nature and satisfies

$$\mathcal{X}^{(ab)cd,e}{}_f = \mathcal{X}^{(ab)[cd,e]}{}_f - \frac{2}{5}\,\delta_f{}^{[c}\,\mathcal{X}^{(ab)d]g,e}{}_g - \frac{2}{45}\,\delta_f{}^{[c}\,\mathcal{X}^{(ab)d]e,g}{}_g + \frac{1}{9}\,\delta_f{}^e\,\mathcal{X}^{(ab)cd,g}{}_g\;, \tag{A.13}$$

implying in particular that

$$\mathcal{X}^{(ab)e[c,d]}{}_e = -\frac{1}{6}\,\mathcal{X}^{(ab)cd,e}{}_e\;. \tag{A.14}$$

The identity (A.13) is far from obvious and hinges on the group properties of the matrix (4.1.13). It can be verified by choosing an explicit parametrization of this matrix (e.g. as given in [1]), at least with the help of some computer algebra [110, 111, 112]. Combining this identity with the properties (A.12) of the twist matrix, we conclude that the first term on the r.h.s. of (A.10) simplifies according to

$$\begin{aligned} \mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q\mathcal{K}_{[cd]}{}^m\partial_m U_q{}^f &= \frac{2}{5}\,\mathcal{X}^{(ab)g(d,e)}{}_g\,(U^{-1})_e{}^q\mathcal{K}_{[fd]}{}^m\partial_m U_q{}^f \\ &= \frac{1}{5}\,\sqrt{2}\,\mathcal{X}^{(ab)gd,e}{}_g\,\eta_{de}\;, \end{aligned} \tag{A.15}$$

such that its y-dependence reduces to the harmonics $\mathcal{Y}_a\mathcal{Y}_b$.

As a consequence, together with (A.12), we conclude that the penultimate term in (A.10) reduces to

$$-\frac{1}{10}\,\sqrt{2}\,\sqrt{|g|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q\mathcal{K}_{[cd]}{}^l\partial_l U_q{}^f = -\frac{1}{25}\,\sqrt{|g|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)gc,d}{}_g\,\eta_{cd} \tag{A.16}$$

Together with (A.8), equation (A.1) then eventually reduces to

$$\begin{aligned} D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= -\frac{1}{80}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{bc,N}\right) \\ &\quad+\frac{1}{40}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{bc,N}F_{\kappa\lambda}{}^{ac} - \frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{db}\,M^{d\,\alpha}{}_N\,B_{\kappa\lambda}{}^{a\,\beta}\right) \\ &\quad+\frac{1}{100}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\mathcal{Y}_a\mathcal{Y}^b\left(10\,M^{ac,fd} + \mathcal{X}^{(af)ec,d}{}_e\right)\eta_{cd}\eta_{bf} \\ &\quad+\frac{1}{32}\,\sqrt{2}\;\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef} + \frac{1}{16}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh} \\ &\quad+\frac{1}{40}\,\sqrt{2}\;A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_\sigma{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\;, \end{aligned} \tag{A.17}$$

such that the y-dependence of the entire equation organizes into the form (A.9). Now the x-dependent coefficient of the traceless combination $\left(\mathcal{Y}_a\mathcal{Y}_b - \frac{1}{6}\,\eta_{ab}\right)$ precisely reproduces the $D = 5$ scalar equations of motion (4.1.17). In particular, the third line of (A.17) coincides with the SL(6) variation of the scalar potential (4.1.15). This match requires additional non-trivial relations among the components of an $\mathrm{E}_{6(6)}$ matrix (4.1.13)

$$
\begin{aligned}
\eta_{ef} M_{d\alpha}{}^{h(a} M^{b)c,de} M^{f\alpha}{}_{ch} &= \eta_{ef} M_{g\alpha}{}^{de} M^{fc,g(a} M^{b)\alpha}{}_{cd}\,, \qquad (A.18)\\
\eta_{ef} M^{de,c(a} M^{b)\gamma,f\alpha} M_{d\alpha,c\gamma} &= 2\,\eta_{ef}\, M^{de,c(a} M^{b)h,fg} M_{dg,ch} + \eta_{ef} M_{d\alpha}{}^{h(a} M^{b)c,de} M^{f\alpha}{}_{ch}\,,
\end{aligned}
$$

which can be proven similar to (A.13). From these it is straightforward to deduce that

$$
\begin{aligned}
\mathcal{X}^{(af)ec,d}{}_{e} &= -\frac{4}{3} M^{de,c(a} M^{b)h,fg} M_{dg,ch} \eta_{ef} - \frac{1}{3}\,\eta_{ef} M^{de,c(a} M^{b)\gamma,f\alpha} M_{d\alpha,c\gamma}\\
&\quad + \frac{2}{3}\,\eta_{de} M^{cd,g(a} M^{b\alpha}{}_{cf} M_{g\alpha}{}^{ef} + \frac{2}{3}\,\eta_{ef} M^{de,c(a} M^{b)h}{}_{d\alpha} M^{f\alpha}{}_{ch}\,, \qquad (A.19)
\end{aligned}
$$

thus matching the expression obtained from variation of the scalar potential in (4.1.17). As a consequence, the y-dependent part of equation (A.17) vanishes on-shell, such that the equation reduces to

$$
\begin{aligned}
D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= -\frac{1}{480}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau} D_\lambda \left(M^{N\,ac}\, D^\lambda M_{ac,N}\right)\\
&\quad + \frac{1}{240}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\, F^{\kappa\lambda\,N} \left(M_{ab,N} F_{\kappa\lambda}{}^{ab} - \frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\, M^{a\,\alpha}{}_{N}\, B_{\kappa\lambda}{}^{b\,\beta}\right)\\
&\quad + \frac{1}{600}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau} \left(10\, M^{ac,fd} + \mathcal{X}^{(af)ec,d}{}_{e}\right) \eta_{cd}\eta_{af}\\
&\quad + \frac{1}{32}\,\sqrt{2}\,\varepsilon_{abcdef}\, F_{[\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} A_{\tau]}{}^{ef} + \frac{1}{16}\, F_{[\mu\nu}{}^{ab} A_{\rho}{}^{cd} A_{\sigma}{}^{ef} A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\
&\quad + \frac{1}{40}\,\sqrt{2}\, A_{[\mu}{}^{ab} A_{\nu}{}^{cd} A_{\rho}{}^{ef} A_{\sigma}{}^{gh} A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\,. \qquad (A.20)
\end{aligned}
$$

This equation can be integrated to yield the function $\Lambda_{\mu\nu\rho\rho}$. This yields the last missing part in the reduction ansatz of the IIB four form (4.3.54) and establishes the full type IIB self-duality equation.

B $\mathrm{E}_{8(8)}$ conventions

The $\mathrm{E}_{8(8)}$ generators $t^{\mathcal{M}}$ split into 120 compact ones $X^{IJ} \equiv -X^{JI}$ and 128 non-compact ones Y^A, with $SO(16)$ vector indices $I, J, \dots \in \mathbf{16}$, spinor indices $A, \in \mathbf{128}$, and the collective label $\mathcal{M} = ([IJ], A)$. The conjugate $SO(16)$ spinors are labeled by dotted indices $\dot{A}, \dot{B}, \dots$. In this $SO(16)$ basis the totally antisymmetric $E_{8(8)}$ structure constants $f^{\mathcal{MNK}}$ possess the non-vanishing components:

$$
f^{IJ,\,KL,\,MN} = -8\,\delta^{\overline{I[K}}\,\delta^{\overline{L]J}}_{MN}\,, \qquad f^{IJ,\,A,\,B} = -\frac{1}{2}\Gamma^{IJ}_{AB}\,. \qquad (B.1)
$$

$E_{8(8)}$ indices are raised and lowered by means of the Cartan-Killing metric

$$\eta^{\mathcal{M}\mathcal{N}} = \frac{1}{60}\mathrm{Tr}\, t^{\mathcal{M}} t^{\mathcal{N}} = -\frac{1}{60} f^{\mathcal{M}}{}_{\mathcal{K}\mathcal{L}} f^{\mathcal{N}\mathcal{K}\mathcal{L}} \,, \tag{B.2}$$

with components $\eta^{AB} = \delta^{AB}$ and $\eta^{IJ\,KL} = -2\delta^{IJ}_{KL}$. When summing over antisymmetrized index pairs $[IJ]$, an extra factor of $\frac{1}{2}$ is always understood.

We will also need the projector onto the adjoint representation

$$\begin{aligned} \mathbb{P}^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}} &= \frac{1}{60}\, f^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}} f^{\mathcal{P}\mathcal{K}}{}_{\mathcal{L}} \\ &= \frac{1}{30}\, \delta^{\mathcal{M}}_{(\mathcal{N}}\delta^{\mathcal{K}}_{\mathcal{L})} + \frac{7}{30}\, (\mathbb{P}_{\mathbf{3875}})_{\mathcal{N}\mathcal{L}}{}^{\mathcal{M}\mathcal{K}} - \frac{1}{240}\, \eta^{\mathcal{M}\mathcal{K}}\eta_{\mathcal{N}\mathcal{L}} + \frac{1}{120}\, f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{P}} f^{\mathcal{P}}{}_{\mathcal{N}\mathcal{L}} \,, \end{aligned} \tag{B.3}$$

in terms of the Cartan-Killing form and structure constants of $\mathrm{E}_{8(8)}$ and the projector $(\mathbb{P}_{\mathbf{3875}})_{\mathcal{N}\mathcal{L}}{}^{\mathcal{M}\mathcal{K}}$ explicitly given by

$$(\mathbb{P}_{3875})^{\mathcal{M}\mathcal{K}}{}_{\mathcal{N}\mathcal{L}} = \frac{1}{7}\, \delta^{\mathcal{M}}_{(\mathcal{N}}\, \delta^{\mathcal{K}}_{\mathcal{L})} - \frac{1}{56}\, \eta^{\mathcal{M}\mathcal{K}}\, \eta_{\mathcal{N}\mathcal{L}} - \frac{1}{14}\, f^{\mathcal{P}}{}_{\mathcal{N}}{}^{(\mathcal{M}}\, f_{\mathcal{P}\mathcal{L}}{}^{\mathcal{K})} \,. \tag{B.4}$$

We refer to [113, 114] for other useful $\mathrm{E}_{8(8)}$ identities.

C $\mathrm{E}_{8(8)}$ section constraints under SO(16) decomposition

Since the section constraints (5.1.1) play a central role in the construction of the exceptional field theory, for the coupling of fermions it will be useful to spell out the decomposition of these constraints under the subgroup SO(16) according to (5.2.2). With the $\mathfrak{e}_{8(8)}$ representations of (5.1.1) decomposing as

$$\mathbf{1} \oplus \mathbf{248} \oplus \mathbf{3875} \quad \longrightarrow \quad \mathbf{1} \oplus \mathbf{120} \oplus \mathbf{128}_s \oplus \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{1920}_c \,, \tag{C.1}$$

the section constraints take the explicit form

$$\begin{aligned} \mathcal{M}^{\mathcal{M}\mathcal{N}}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= 2\, \mathcal{V}^{\mathcal{M}}{}_{A} \mathcal{V}^{\mathcal{N}}{}_{A}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} \,, \\ \mathcal{V}^{\mathcal{M}\, K[I} \mathcal{V}^{|\mathcal{N}|\, J]K}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= -\frac{1}{4}\Gamma^{IJ}_{CD} \mathcal{V}^{\mathcal{M}}{}_{C} \mathcal{V}^{\mathcal{N}}{}_{D}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} \,, \\ \mathcal{V}^{[\mathcal{M}}{}_{IJ} \mathcal{V}^{\mathcal{N}]}{}_{A} \Gamma^{IJ}_{AB}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= 0 \\ \mathcal{V}^{\mathcal{M}\, K(I} \mathcal{V}^{|\mathcal{N}|\, J)K}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= -\frac{1}{16}\delta^{IJ} \mathcal{V}^{\mathcal{M}}{}_{KL} \mathcal{V}^{\mathcal{N}}{}_{KL}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} \,, \\ \mathcal{V}^{\mathcal{M}\, [IJ} \mathcal{V}^{|\mathcal{N}|\, KL]}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= -\frac{1}{24}\Gamma^{IJKL}_{CD}\, \mathcal{V}^{\mathcal{M}}{}_{C} \mathcal{V}^{\mathcal{N}}{}_{D}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} \,, \\ \Gamma^{J}_{A\dot{A}} \mathcal{V}^{(\mathcal{M}}{}_{IJ} \mathcal{V}^{\mathcal{N})}{}_{A}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} &= -\frac{1}{16}\, (\Gamma^{MN}\Gamma^{I})_{A\dot{A}}\, \mathcal{V}^{(\mathcal{M}}{}_{MN} \mathcal{V}^{\mathcal{N})}{}_{A}\, \partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} \,, \end{aligned} \tag{C.2}$$

which we will use in the following. Following the above discussion, the same algebraic constraints hold for derivatives $\partial_{\mathcal{M}}$ replaced by the gauge connection $B_{\mu\,\mathcal{M}}$ or its gauge parameter $\Sigma_{\mathcal{M}}$.

Let us recall from [35] that these section constraints allow for (at least) two inequivalent solutions which break $E_{8(8)}$ to GL(8) or GL(7) $\times$ SL(2), and in which all fields depend on only eight or seven among the 248 internal coordinates $Y^{\mathcal{M}}$, respectively. The resulting theory then coincides with the bosonic sector of $D = 11$ and type IIB supergravity, respectively.

D SO(16) Gamma matrix identities

In this appendix, we give some of the SO(16) gamma matrices identities we have used to rewrite the curvature $\mathcal{R}_A$ in a more compact form. We started with 14 terms quadratic in the Cartan forms, where a simple counting gives only 12 independent terms. Then using an explicit representation of the $SO(16)$ gamma matrices together with the section constraints (C.2), we were able to write $\mathcal{R}_A$ with 7 independent terms quadratic in the Cartan forms.

The main identities behind this simplification are the following

$$\mathcal{V}^{[\mathcal{M}}{}_{IJ}\mathcal{V}^{\mathcal{N}]}{}_B(\Gamma^{IJ}\Gamma^{KL})_{BD}p_{\mathcal{M}}{}^A p_{\mathcal{N}}{}^C = 0\,, \tag{D.1}$$

$$\begin{aligned}\Gamma^{IM}_{B[A}\Gamma^{IMNP}_{D]C}\mathcal{V}^{\mathcal{M}}{}_{NP}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^D = {} & -4\Gamma^{IM}_{A[B}\Gamma^{IN}_{C]D}\mathcal{V}^{\mathcal{M}}{}_{MN}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^D \\ & +8\Gamma^{IM}_{AB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_C p_{\mathcal{M}}{}^{(B}p_{\mathcal{N}}{}^{C)} \\ & -\Gamma^{IM}_{BC}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^A p_{\mathcal{N}}{}^C \\ & -\Gamma^{IM}_{AB}\mathcal{V}^{\mathcal{M}}{}_{IM}\mathcal{V}^{\mathcal{N}}{}_B p_{\mathcal{M}}{}^C p_{\mathcal{N}}{}^C\,. \end{aligned} \tag{D.2}$$

E Supersymmetry of the full $E_{8(8)}$ Lagrangian

In this appendix, we give the remaining details for the invariance of the Lagrangian (5.4.1) under the supersymmetry transformations (5.3.3).

E.1 Cancellation of the terms carrying field strengths

We start with a simple check: all terms in $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}$ and $\mathcal{G}_{\mu\nu\,\mathcal{M}}$ from the supersymmetric variation of the fermionic terms in the Lagrangian should cancel against the corresponding contributions from variation of the kinetic and topological terms. The relevant contribution on the fermionic side are

$$
\begin{aligned}
\delta\left(-2\,e\bar{\chi}^{\dot{A}}\gamma^{\mu}\gamma^{\nu}\psi^{I}_{\mu}\,\Gamma^{I}_{A\dot{A}}\,\mathcal{P}^{A}_{\nu}\right) &\longrightarrow 2\,e\bar{\chi}^{\dot{A}}\gamma^{\mu\nu}\epsilon^{I}\,\Gamma^{I}_{A\dot{A}}\,\mathcal{D}_{\mu}\mathcal{P}^{A}_{\nu}\\
&= -i\varepsilon^{\mu\nu\rho}\bar{\chi}^{\dot{A}}\gamma_{\rho}\epsilon^{I}\,\Gamma^{I}_{A\dot{A}}\mathcal{V}^{\mathcal{M}A}\left(\widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}-f_{\mathcal{M}\mathcal{L}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}\mathcal{F}_{\mu\nu}{}^{\mathcal{L}}\right)\\
\delta\left(2\,\varepsilon^{\mu\nu\rho}\bar{\psi}^{I}_{\mu}\mathcal{D}_{\nu}\psi^{I}_{\rho}\right) &\longrightarrow 2\,\varepsilon^{\mu\nu\rho}\bar{\psi}^{I}_{\mu}\left[\mathcal{D}_{\nu}\,,\mathcal{D}_{\rho}\right]\epsilon^{I}\\
&\longrightarrow +2\,\varepsilon^{\mu\nu\rho}\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\left(\widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}-f_{\mathcal{M}\mathcal{L}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}\mathcal{F}_{\mu\nu}{}^{\mathcal{L}}\right)\bar{\psi}^{I}_{\rho}\epsilon^{J}\\
&\quad\; + \varepsilon^{\mu\nu\rho}\,\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\left(\nabla_{\mathcal{M}}\bar{\psi}^{I}_{\rho}\epsilon^{I}-\bar{\psi}^{I}_{\rho}\nabla_{\mathcal{M}}\epsilon^{I}\right)\;, \qquad \text{(E.1)}
\end{aligned}
$$

where we have used the commutator of two external covariant derivative (5.2.38). On the bosonic side, all terms with field strength come from the variation of kinetic and topological terms

$$
\begin{aligned}
\delta\mathcal{L} &\longrightarrow \varepsilon^{\mu\nu\rho}\tilde{\mathcal{E}}^{(B)}_{\mu\nu\,\mathcal{M}}\,\delta A_{\rho}{}^{\mathcal{M}}+\varepsilon^{\mu\nu\rho}\mathcal{E}^{(A)\,\mathcal{M}}_{\mu\nu}\,\Delta B_{\rho\mathcal{M}}\\
&\longrightarrow \varepsilon^{\mu\nu\rho}(-\frac{1}{2}\,\widetilde{\mathcal{G}}_{\mu\nu\,\mathcal{M}}+\frac{1}{2}\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}\mathcal{F}_{\mu\nu}{}^{\mathcal{N}})(-4\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar{\epsilon}^{I}\psi_{\rho}{}^{J}+2i\,\Gamma^{I}_{A\dot{A}}\mathcal{V}^{\mathcal{M}}{}_{A}\bar{\epsilon}^{I}\gamma_{\mu}\chi^{\dot{A}})\\
&\qquad \varepsilon^{\mu\nu\rho}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,\left(\nabla_{\mathcal{M}}\bar{\epsilon}^{I}\psi_{\rho}{}^{I}-\bar{\epsilon}^{I}\nabla_{\mathcal{M}}\psi_{\rho}{}^{I}\right)+\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,g^{\sigma\mu}\nabla_{\mathcal{M}}(\bar{\epsilon}^{I}i\gamma^{\nu}\psi_{\sigma}{}^{I})\;, \qquad \text{(E.2)}
\end{aligned}
$$

with the exception of an extra contribution from the improved Einstein-Hilbert term

$$
\begin{aligned}
\delta\left(-ee_{a}{}^{\mu}e_{b}{}^{\nu}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,\omega_{\mathcal{M}}{}^{ab}\right) &\longrightarrow -ee_{a}{}^{\mu}e_{b}{}^{\nu}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,\delta\omega_{\mathcal{M}}{}^{ab}\\
&= -ee_{a}{}^{\mu}e_{b}{}^{\nu}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\left(\delta e^{\rho[a}\nabla_{\mathcal{M}}e_{\rho}{}^{b]}+e^{\rho[a}\nabla_{\mathcal{M}}\delta e_{\rho}{}^{b]}\right)\\
&= -ee_{a}{}^{\mu}e_{b}{}^{\nu}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\left(e^{\sigma[a}e_{\tau}{}^{b]}\,\nabla_{\mathcal{M}}\left(e_{c}{}^{\tau}\delta e_{\sigma}{}^{c}\right)\right)\\
&= -ieg^{\mu\sigma}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\nabla_{\mathcal{M}}\left(\bar{\epsilon}^{I}\gamma^{\nu}\psi^{I}_{\sigma}\right)\;. \qquad \text{(E.3)}
\end{aligned}
$$

that cancels the last term of (E.2). Together, all terms with field strengths vanish.

E.2 Cancellation of the $\nabla_M\mathcal{D}_\mu\chi\epsilon$ terms

From the variation of the vector fields in the bosonic Lagrangian (we have now dropped all terms with field strengths), we have the following contribution

$$
\begin{aligned}
\delta\mathcal{L} &\longrightarrow +ej^{\mu\,\mathcal{M}}\,\Delta B_{\mu\,\mathcal{M}}-e\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}j^{\mu\mathcal{N}}\,\delta A_{\mu}{}^{\mathcal{M}}-e\,\widehat{J}^{\mu}{}_{\mathcal{M}}\,\delta A_{\mu}{}^{\mathcal{M}}\\
&\longrightarrow +4ie\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}(\mathcal{V}^{\mathcal{N}}{}_{B}\mathcal{P}^{\mu\,B})\,\mathcal{V}^{\mathcal{M}}{}_{A}\Gamma^{I}_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma_{\mu}\epsilon^{I}+2ie\,\widehat{J}^{\mu}{}_{\mathcal{M}}\,\mathcal{V}^{\mathcal{M}}{}_{A}\Gamma^{I}_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma_{\mu}\epsilon^{I}\\
&= -ie\,\mathcal{V}^{\mathcal{M}}{}_{KL}\nabla_{\mathcal{M}}\mathcal{P}^{\mu\,A}\,\Gamma^{IKL}_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma_{\mu}\epsilon^{I}-2ie\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}\mathcal{P}^{\mu\,A}\,\Gamma^{I}_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma_{\mu}\epsilon^{J}\\
&\quad\; +2ie\,\widehat{J}^{\mu}{}_{\mathcal{M}}\,\mathcal{V}^{\mathcal{M}}{}_{A}\Gamma^{I}_{A\dot{A}}\bar{\chi}^{\dot{A}}\gamma_{\mu}\epsilon^{I}\;. \qquad \text{(E.4)}
\end{aligned}
$$

On the fermionic side, the relevant contributions to this sector are

$$\begin{aligned}
\delta\left(-2\,e\bar\chi^{\dot A}\gamma^\mu\gamma^\nu\psi^I_\mu\,\Gamma^I_{A\dot A}\,\mathcal{P}^A_\nu\right) &\longrightarrow -4\,i\,e\bar\chi^{\dot A}\gamma^\mu\gamma^\nu\nabla_{\mathcal{M}}(\gamma_\mu\epsilon^J)\,\Gamma^I_{A\dot A}\,\mathcal{P}^A_\nu\mathcal{V}^{\mathcal{M}}{}_{IJ} \\
&\qquad + 4\,i\,e\bar\chi^{\dot A}\gamma^\mu\nabla_{\mathcal{M}}\epsilon^J\,\Gamma^I_{A\dot A}\,\mathcal{P}^A_\mu\mathcal{V}^{\mathcal{M}}{}_{IJ}\,, \qquad (E.5)\\
\delta\left(-2i\,e\,\bar\chi^{\dot A}\gamma^\mu\mathcal{D}_\mu\chi^{\dot A}\right) &\longrightarrow 8i\,e\,\bar\chi^{\dot A}\gamma^\mu\mathcal{D}_\mu\nabla_{\mathcal{M}}\epsilon^I\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A} \\
&\qquad - 2i\,e\,\bar\chi^{\dot A}\gamma^\mu\nabla_{\mathcal{M}}\epsilon^I\mathcal{P}^A_\mu\mathcal{V}^{\mathcal{M}}{}_{JK}\Gamma^{IJK}_{A\dot A} \\
&\qquad - 4i\,e\,\bar\chi^{\dot A}\gamma^\mu\nabla_{\mathcal{M}}\epsilon^J\mathcal{P}^A_\mu\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^I_{A\dot A}\,, \qquad (E.6)\\
\delta\left(-8\,e\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\bar\psi_\mu{}^I i\nabla_{\mathcal{M}}(\gamma^\mu\chi^{\dot A})\right) &\longrightarrow 8\,i\,e\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\nabla_{\mathcal{M}}\bar\chi^{\dot A}\gamma^\mu\mathcal{D}_\mu\epsilon^I \\
&= -8\,i\,e\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\bar\chi^{\dot A}\gamma^\mu\nabla_{\mathcal{M}}\mathcal{D}_\mu\epsilon^I\,, \qquad (E.7)\\
\delta\left(-2e\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^{IJ}_{\dot A\dot B}\bar\chi^{\dot A}\nabla_{\mathcal{M}}\chi^{\dot B}\right) &\longrightarrow -2i\,e\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^{IJ}_{\dot A\dot B}\bar\chi^{\dot A}\nabla_{\mathcal{M}}(\gamma^\mu\epsilon^K\Gamma^K_{A\dot B}\mathcal{P}^A_\mu) \\
&= 2i\,e\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^{IJK}_{A\dot A}\bar\chi^{\dot A}\nabla_{\mathcal{M}}(\gamma^\mu\epsilon^K\mathcal{P}^A_\mu) \\
&\qquad - 4i\,e\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^I_{A\dot A}\bar\chi^{\dot A}\nabla_{\mathcal{M}}(\gamma^\mu\epsilon^J\mathcal{P}^A_\mu)\,. \qquad (E.8)
\end{aligned}$$

Using the commutator

$$\begin{aligned}
\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\,[\nabla_{\mathcal{M}},\mathcal{D}_\mu]\,\epsilon^I &= \frac{1}{4}\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\,\mathcal{R}_{\mathcal{M}\mu}{}^{ab}\,\gamma_{ab}\epsilon^I \\
&\quad - \frac{3}{4}\,\Gamma^I_{A\dot A}\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^J + \frac{1}{8}\,\Gamma^{IJK}_{A\dot A}\mathcal{V}^{\mathcal{M}}{}_{IJ}\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^K\,, \qquad (E.9)
\end{aligned}$$

all of the above terms simply reduce to

$$\begin{aligned}
&\longrightarrow -4\,i\,e\bar\chi^{\dot A}\gamma^\mu\gamma^\nu\nabla_{\mathcal{M}}(\gamma_\mu)\epsilon^J\,\Gamma^I_{A\dot A}\,\mathcal{P}^A_\nu\mathcal{V}^{\mathcal{M}}{}_{IJ} - 4i\,e\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\Gamma^I_{A\dot A}\bar\chi^{\dot A}\nabla_{\mathcal{M}}(\gamma^\mu)\epsilon^J\mathcal{P}^A_\mu \\
&\qquad - 2ie\,\mathcal{V}^{\mathcal{M}}{}_{IJ}(\nabla_{\mathcal{M}}g^{\mu\nu})\mathcal{P}^A_\nu\,\Gamma^I_{A\dot A}\bar\chi^{\dot A}\gamma_\mu\epsilon^J \\
&\qquad - 2i\,e\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\,\mathcal{R}_{\mathcal{M}\mu}{}^{ab}\,\bar\chi^{\dot A}\gamma^\mu\gamma_{ab}\epsilon^I + 2ie\,\widehat{J}^\mu{}_{\mathcal{M}}\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\bar\chi^{\dot A}\gamma_\mu\epsilon^I \\
&= 4i\,e\,\mathcal{V}^{\mathcal{M}}{}_A\Gamma^I_{A\dot A}\left(\mathcal{R}_{\mathcal{M}\nu}{}^{\mu\nu} + \frac{1}{2}\,\widehat{J}^\mu{}_{\mathcal{M}}\right)\bar\chi^{\dot A}\gamma_\mu\epsilon^I \\
&= 0\,, \qquad (E.10)
\end{aligned}$$

where we have used (5.2.43) in the last equality.

E.3 Cancellation of the $\nabla_M \mathcal{D}_\mu \psi\epsilon$ terms

Similarly, we collect the vector field contributions in the bosonic Lagrangian

$$
\begin{aligned}
\delta\mathcal{L} \;\longrightarrow\;& +ej^{\mu\mathcal{M}}\,\Delta B_{\mu\mathcal{M}} - e\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\nabla_{\mathcal{K}} j^{\mu\mathcal{N}}\,\delta A_\mu{}^{\mathcal{M}} - e\,\widehat{J}^\mu{}_{\mathcal{M}}\,\delta A_\mu{}^{\mathcal{M}} \\
\longrightarrow\;& 2e\mathcal{P}^{\mu A}\mathcal{V}^{\mathcal{M}}{}_A\left(-4\bar\psi_\mu{}^I\nabla_{\mathcal{M}}\epsilon^I + 2\nabla_{\mathcal{M}}(\bar\psi_\mu{}^I\epsilon^I) + e\,\varepsilon_{\mu\nu\rho}g^{\rho\sigma}\nabla_{\mathcal{M}}(\bar\epsilon^I i\gamma^\nu\psi_\sigma{}^I)\right) \\
& -8e\,\mathcal{V}^{\mathcal{N}}{}_A\mathcal{V}^{\mathcal{M}}{}_{IJ}\,f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{K}}\nabla_{\mathcal{K}}\mathcal{P}^{\mu A}\bar\psi^I_\mu\epsilon^J - 4e\widehat{J}^\mu{}_{\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi^I_\mu\epsilon^J \\
=\;& -8e\mathcal{V}^{\mathcal{M}}{}_A\mathcal{P}^{\mu A}\bar\psi^I_\mu\nabla_{\mathcal{M}}\epsilon^I - 4e\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}(g^{\mu\nu}\mathcal{P}_\nu{}^A)\bar\psi^I_\mu\epsilon^I + 2\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}(\mathcal{P}_\mu{}^A)\bar\psi_\rho{}^I i\gamma^\nu\epsilon^I \\
& +2\varepsilon^{\mu\lambda\rho}\mathcal{V}^{\mathcal{M}}{}_A\mathcal{P}_\mu{}^A\nabla_{\mathcal{M}}(g_{\lambda\nu})\bar\psi_\rho{}^I i\gamma^\nu\epsilon^I \\
& -4e\,\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}\mathcal{P}^{\mu B}\bar\psi^I_\mu\epsilon^J - 4e\widehat{J}^\mu{}_{\mathcal{M}}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi^I_\mu\epsilon^J\;, \qquad \text{(E.11)}
\end{aligned}
$$

together with the relevant contributions from the fermionic Lagragian

$$
\begin{aligned}
\delta(2ie\gamma^{\mu\nu\rho}\bar\psi^I_\rho\mathcal{D}_\mu\psi^I_\nu) \;\longrightarrow\;& 8ie\gamma^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi^I_\rho\mathcal{D}_\mu(\nabla_{\mathcal{M}}(i\gamma_\nu\epsilon^J) + i\gamma_\nu\nabla_{\mathcal{M}}\epsilon^J) \\
& -4ie\gamma^{\mu\nu\rho}\bar\psi^I_\rho\mathcal{P}_\mu{}^A\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_B(\nabla_{\mathcal{M}}(i\gamma_\nu\epsilon^J) + i\gamma_\nu\nabla_{\mathcal{M}}\epsilon^J) \\
=\;& -16e\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi^I_\mu\gamma^{\mu\nu}\mathcal{D}_\nu\nabla_{\mathcal{M}}\epsilon^J + 8e\bar\psi^I_\mu\gamma^{\mu\nu}\nabla_{\mathcal{M}}\epsilon^J\mathcal{P}_\mu{}^A\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_B \\
& +8ie\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi^I_\mu\mathcal{D}_\nu(\nabla_{\mathcal{M}}(\gamma_\rho)\epsilon^J) \\
& -4ie\varepsilon^{\mu\nu\rho}\bar\psi^I_\mu\nabla_{\mathcal{M}}\gamma_\rho\epsilon^J\mathcal{P}_\mu{}^A\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_B\;, \qquad \text{(E.12)} \\
\delta(-2e\bar\chi^{\dot A}\gamma^\mu\gamma^\nu\psi^I_\mu\Gamma^I_{A\dot A}\mathcal{P}_\nu{}^A) \;\longrightarrow\;& 4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^J\Gamma^I)_{AB}\mathcal{P}_\nu{}^B\nabla_{\mathcal{M}}\bar\epsilon^J\gamma^\mu\gamma^\nu\psi^I_\mu \\
=\;& -4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^J\Gamma^I)_{AB}\mathcal{P}_\nu{}^B\bar\psi^I_\mu\gamma^{\mu\nu}\nabla_{\mathcal{M}}\epsilon^J \\
& +4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^J\Gamma^I)_{AB}\mathcal{P}^{\mu B}\bar\psi^I_\mu\nabla_{\mathcal{M}}\epsilon^J\;, \qquad \text{(E.13)} \\
\delta(8e\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\gamma^{\mu\nu}\nabla_{\mathcal{M}}\psi_\nu{}^J) \;\longrightarrow\;& -8ie\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\nabla_{\mathcal{M}}\gamma_\rho\mathcal{D}_\nu\epsilon^J \\
& +16e\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\gamma^{\mu\nu}\nabla_{\mathcal{M}}\mathcal{D}_\nu\epsilon^J \\
=\;& -8ie\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\mathcal{D}_\nu(\nabla_{\mathcal{M}}\gamma_\rho\epsilon^J) \\
& +8ie\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\mathcal{D}_\nu(\nabla_{\mathcal{M}}\gamma_\rho)\epsilon^J \\
& +16e\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar\psi_\mu{}^I\gamma^{\mu\nu}\nabla_{\mathcal{M}}\mathcal{D}_\nu\epsilon^J\;, \qquad \text{(E.14)} \\
\delta(-8ie\bar\psi_\mu{}^I\nabla_{\mathcal{M}}(\gamma^\mu\chi^{\dot A})\Gamma^I_{A\dot A}\mathcal{V}^{\mathcal{M}}{}_A) \;\longrightarrow\;& 4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^I\Gamma^J)_{AB}\bar\psi^I_\mu\nabla_{\mathcal{M}}(\gamma^\mu\gamma^\nu\epsilon^J\mathcal{P}_\nu{}^B) \\
=\;& 4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^I\Gamma^J)_{AB}\bar\psi^I_\mu\nabla_{\mathcal{M}}(\gamma^\mu\gamma^\nu)\epsilon^J\mathcal{P}_\nu{}^B \\
& +4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^I\Gamma^J)_{AB}\bar\psi^I_\mu\gamma^{\mu\nu}\nabla_{\mathcal{M}}\epsilon^J\mathcal{P}_\nu{}^B \\
& +4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^I\Gamma^J)_{AB}\bar\psi^I_\mu\nabla_{\mathcal{M}}\epsilon^J\mathcal{P}^{\mu B} \\
& +4e\mathcal{V}^{\mathcal{M}}{}_A(\Gamma^I\Gamma^J)_{AB}\bar\psi^I_\mu\gamma^\mu\gamma^\nu\epsilon^J\nabla_{\mathcal{M}}\mathcal{P}_\nu{}^B\;. \qquad \text{(E.15)}
\end{aligned}
$$

Upon using the commutator

$$\begin{aligned}\mathcal{V}^{\mathcal{M}}{}_{IJ}\,[\nabla_{\mathcal{M}},\mathcal{D}_\mu]\,\epsilon^J &= \frac{1}{4}\,\mathcal{V}^{\mathcal{M}}{}_{IJ}\,\mathcal{R}_{\mathcal{M}\mu}{}^{ab}\,\gamma_{ab}\epsilon^J \\ &\quad -\frac{1}{8}\,\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^I-\frac{1}{4}\,\Gamma^{IJ}_{AB}\mathcal{V}^{\mathcal{M}}{}_A\nabla_{\mathcal{M}}\mathcal{P}_\mu{}^A\,\epsilon^J\ ,\end{aligned}\tag{E.16}$$

this reduces to

$$\begin{aligned}\longrightarrow\ & -8e\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar{\psi}^I_\mu\epsilon^J(\mathcal{R}_{\mathcal{M}\nu}{}^{\mu\nu}+\frac{1}{2}\widehat{J}^\mu{}_{\mathcal{M}}) \\ & +8ie\varepsilon^{\nu\rho\sigma}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar{\psi}^I_\mu\gamma_\sigma\epsilon^J\mathcal{R}_{\mathcal{M}\nu\kappa\rho}g^{\kappa\mu} \\ & -8ie\varepsilon^{\nu\rho\sigma}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar{\psi}^I_\nu\gamma_\sigma\epsilon^J\mathcal{R}_{\mathcal{M}\mu\kappa\sigma}g^{\kappa\mu} \\ & -8ie\varepsilon^{\mu\nu\rho}\mathcal{V}^{\mathcal{M}}{}_{IJ}\bar{\psi}^I_\rho([\mathcal{D}_\mu,\nabla_{\mathcal{M}}]\gamma_\nu)\epsilon^J \\ =\ & 0\ ,\end{aligned}\tag{E.17}$$

where we have used the Schouten identity

$$\begin{aligned}\varepsilon^{\nu\rho\sigma}g^{\kappa\mu}(\bar{\psi}^I_\mu\gamma_\nu\epsilon^J\mathcal{R}_{\mathcal{M}\rho\kappa\sigma}-\bar{\psi}^I_\nu\gamma_\mu\epsilon^J\mathcal{R}_{\mathcal{M}\rho\kappa\sigma}+\bar{\psi}^I_\nu\gamma_\rho\epsilon^J\mathcal{R}_{\mathcal{M}\mu\kappa\sigma}) &= \varepsilon^{\nu\rho\sigma}\bar{\psi}^I_\nu\gamma_\rho\epsilon^J\mathcal{R}_{\mathcal{M}\sigma\kappa\mu}g^{\kappa\mu}\ , \\ &= 0\ .\end{aligned}\tag{E.18}$$

This completes the results obtained in section 4 and proves the invariance of the extended Lagrangian (5.4.1) under supersymmetry.