Inégalités de type Trudinger-Moser et applications

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Sujet

Inégalités de type Trudinger-Moser et applications

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Résumé

Cette thèse porte sur quelques inégalités de type Trudinger-Moser et leurs applications à l'étude des injections de Sobolev qu'elles induisent dans les espaces d'Orlicz et à l'analyse d'équations aux dérivées partielles non linéaires à croissance exponentielle.

Le travail qu'on présente ici se compose de trois parties. La première partie est consacrée à la description du défaut de compacité de l'injection de Sobolev 4D dans l'espace d'Orlicz dans le cadre radial.

L'objectif de la deuxième partie est double. D'abord, on caractérise le défaut de compacité de l'injection de Sobolev 2D dans les différentes classes d'espaces d'Orlicz. Ensuite, on étudie l'équation de Klein-Gordon semi-linéaire avec non linéarité exponentielle, où la norme d'Orlicz joue un rôle crucial. En particulier, on aborde les questions d'existence globale, de complétude asymptotique et d'étude qualitative.

Dans la troisième partie, on établit des inégalités optimales de type Adams, en étroite relation avec les inégalités de Hardy, puis on fournit une description du défaut de compacité des injections de Sobolev qu'elles induisent.

Mots clés : inégalités de Trudinger-Moser, injections de Sobolev, espaces d'Orlicz, défaut de compacité, équation de Klein-Gordon, inégalités de Hardy.

Abstract

This thesis focuses on some Trudinger-Moser type inequalities and their applications to the study of Sobolev embeddings they induce into the Orlicz spaces, and the investigation of nonlinear partial differential equations with exponential growth.

The work presented here includes three parts. The first part is devoted to the description of the lack of compactness of the 4D Sobolev embedding into the Orlicz space in the radial framework.

The aim of the second part is twofold. Firstly, we characterize the lack of compactness of the 2D Sobolev embedding into the different classes of Orlicz spaces. Secondly, we undertake the study of the nonlinear Klein-Gordon equation with exponential growth, where the Orlicz norm plays a crucial role. In particular, issues of global existence, scattering and qualitative study are investigated.

In the third part, we establish sharp Adams-type inequalities invoking Hardy inequalities, then we give a description of the lack of compactness of the Sobolev embeddings they induce.

Keywords : Trudinger-Moser inequalities, Sobolev embeddings, Orlicz spaces, lack of compactness, Klein-Gordon equation, Hardy inequalities.

TABLE DES MATIÈRES

CHAPITRE I

INTRODUCTION

Cette thèse s'inscrit dans le cadre des inégalités de type Trudinger-Moser avec leurs applications à l'étude des injections de Sobolev qu'elles induisent, ainsi qu'à l'analyse de quelques équations aux dérivées partielles non linéaires à croissance exponentielle.

Les inégalités de type Trudinger-Moser ont une longue histoire qui a commencé avec les travaux de S. I. Pohozaev ([84]), N. S. Trudinger ([101]) et V. I. Yudovich ([103]) : étant donné un domaine Ω de $\mathbb{R}^d$ avec $d \geq 2$, il est bien connu que si $p < d$, alors

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)\,, \quad \forall\, 1 \leq q \leq p^* := \frac{dp}{d-p}\,\cdot$$

Cette injection de Sobolev provient de l'inégalité suivante :

$$\sup_{u \in W_0^{1,p}(\Omega),\, \|\nabla u\|_{L^p} \leq 1} \int_\Omega |u(x)|^q\, dx < +\infty\,, \quad \forall\, 1 \leq q \leq p^*\,. \tag{I.1}$$

Notons que l'estimation (I.1) est optimale dans le sens où le supremum est infini pour $q > p^*$.

Si on s'intéresse maintenant au cas limite $p = d$, on sait que

$$W_0^{1,d}(\Omega) \hookrightarrow L^q(\Omega)\,, \quad \forall\, d \leq q < +\infty\,.$$

Cependant, $W_0^{1,d}(\Omega)$ ne s'injecte pas dans $L^\infty(\Omega)$.

En se basant sur le fait que, dans ce cas limite, toute croissance polynomiale est permise au sens de l'inégalité (I.1), S. I. Pohozaev, N. S. Trudinger et V. I. Yudovich ont cherché, dans leurs travaux pionners, à déterminer la fonction à croissance maximale $g:\ \mathbb{R} \to\ \mathbb{R}_+$ vérifiant

$$\sup_{u \in W_0^{1,d}(\Omega),\, \|\nabla u\|_{L^d} \leq 1} \int_\Omega g\big(u(x)\big)\, dx < +\infty\,,$$

et ils ont prouvé de manières indépendantes que la croissance maximale est de type exponentiel.

Ultérieurement, J. Moser ([78]) a amélioré ces résultats en établissant l'inégalité optimale suivante, connue sous le nom d'inégalité de Trudinger-Moser : étant donné un domaine borné Ω de $\mathbb{R}^d$, il existe une constante $C_d > 0$ telle que

$$\sup_{u \in W_0^{1,d}(\Omega),\, \|\nabla u\|_{L^d} \leq 1} \int_\Omega \mathrm{e}^{\alpha_d |u(x)|^{\frac{d}{d-1}}}\, dx \leq C_d |\Omega|\,, \tag{I.2}$$

où $\alpha_d := d\, \omega_{d-1}^{\frac{1}{d-1}}$, avec ω_{d-1} l'aire de la sphère unité de $\mathbb{R}^d$.

La preuve de cette inégalité s'appuie sur la symétrisation de Schwarz [1] qui préserve la norme de Lebesgue $\|u\|_{L^s}$ et minimise la norme de Dirichlet $\|\nabla u\|_{L^s}$ dans $W_0^{1,s}(\Omega)$, pour $s \geq 1$. Plus

1. Pour plus de détails sur la symétrisation de Schwarz, on peut consulter [63, 72, 98].

précisément, à toute fonction $u \in W_0^{1,s}(\Omega)$, on peut associer une fonction positive radiale et décroissante u^*, appelée réarrangement de u, dont les ensembles de sur-niveau vérifient

$$\Big|\{x \in \mathbb{R}^d;\ u^*(x) > t\}\Big| = \Big|\{x \in \mathbb{R}^d;\ |u(x)| > t\}\Big|, \quad \forall\, t \geq 0\,.$$

Comme le supremum dans l'inégalité de Trudinger-Moser (I.2) est infini pour un domaine Ω non borné, il a été naturel de penser à étendre cette inégalité à un domaine de mesure infinie. Parmi les résultats obtenus dans cette direction, rappelons celui de S. Adachi et K. Tanaka ([1]) dans le cas de la dimension 2 : pour tout $0 < \alpha < \alpha_2 = 4\pi$, il existe une constante $C_\alpha > 0$ telle que

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\alpha|u(x)|^2} - 1\Big)\, dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}\,, \quad \forall\, u \in H^1(\mathbb{R}^2) \text{ avec } \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1\,. \tag{I.3}$$

De plus, la constante 4π est optimale dans le sens où

$$\sup_{u \in H^1(\mathbb{R}^2),\, \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Big(\mathrm{e}^{4\pi|u(x)|^2} - 1\Big)\, dx = +\infty\,.$$

Pour rendre l'exposant $\alpha = 4\pi$ admissible, B. Ruf ([86]) [2] a remplacé la norme de Dirichlet par la norme de Sobolev classique

$$\|u\|^2_{H^1(\mathbb{R}^2)} = \|u\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u\|^2_{L^2(\mathbb{R}^2)}\,.$$

Plus précisément, il a montré que

$$\sup_{u \in H^1(\mathbb{R}^2),\, \|u\|_{H^1(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \Big(\mathrm{e}^{4\pi|u(x)|^2} - 1\Big)\, dx < +\infty\,.$$

Rappelons aussi que D. R. Adams ([2]) a obtenu une autre extension de (I.2) pour des dérivées d'ordre supérieur : étant donnés un domaine borné Ω de $\mathbb{R}^d$ et un entier $0 < m < d$, il existe une constante $C_{m,d} > 0$ telle que

$$\sup_{u \in W_0^{m,\frac{d}{m}}(\Omega),\, \|\nabla^m u\|_{L^{\frac{d}{m}}} \leq 1} \int_\Omega \mathrm{e}^{\beta_{m,d}|u(x)|^{\frac{d}{d-m}}}\, dx \leq C_{m,d}|\Omega|\,,$$

où [3]

$$\beta_{m,d} := \frac{d}{\omega_{d-1}} \begin{cases} \left[\dfrac{\pi^{\frac{d}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{d-m}{2})}\right]^{\frac{d}{d-m}} & \text{si m est pair,} \\[2ex] \left[\dfrac{\pi^{\frac{d}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{d-m+1}{2})}\right]^{\frac{d}{d-m}} & \text{si m est impair,} \end{cases}$$

2. On rappellera, dans l'Annexe A, la preuve de ce résultat.

3. La fonction Gamma, notée par Γ, est définie sur $]0, +\infty[$ par $\Gamma(t) = \displaystyle\int_0^\infty x^{t-1}\mathrm{e}^{-x}\, dx$.

et où

$$\nabla^m u = \begin{cases} \Delta^{\frac{m}{2}} u & \text{si } m \text{ est pair,} \\ \nabla \Delta^{\frac{m-1}{2}} u & \text{si } m \text{ est impair.} \end{cases}$$

En particulier, en prenant $m = N$ et $d = 2N$, il existe une constante $C_N > 0$ telle que

$$\sup_{u \in H_0^N(\Omega),\, \|\nabla^N u\|_{L^2} \leq 1} \int_{\mathbb{R}^{2N}} \mathrm{e}^{\beta_N |u(x)|^2}\, dx \leq C_N |\Omega|\,, \tag{I.4}$$

avec $\beta_N := \dfrac{2N(2\pi)^{2N}}{\omega_{2N-1}}$.

Comme dans le cas de la dérivée d'ordre 1, le supremum est infini lorsque le domaine Ω n'est pas borné. En remplaçant la norme $\|\nabla^N u\|_{L^2}$ dans (I.4) par la norme de Sobolev

$$\|u\|^2_{H^N(\mathbb{R}^{2N})} := \|u\|^2_{L^2(\mathbb{R}^{2N})} + \sum_{j=1}^{N} \|\nabla^j u\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

B. Ruf et F. Sani ([87]) ont montré, dans le cas où N est pair, l'inégalité optimale suivante : [4]

$$\sup_{u \in H^N(\mathbb{R}^{2N}),\, \|u\|_{H^N(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\beta_N |u(x)|^2} - 1\right) dx < +\infty\,. \tag{I.5}$$

Signalons que, dans le cadre des dérivées d'ordre supérieur, le problème ne peut être réduit au cas radial par la symétrisation de Schwarz. En effet, étant donnés un domaine Ω de $\mathbb{R}^{2N}$ avec $N \geq 2$ et une fonction u de $H^N(\Omega)$, le réarrangement u^* de u n'appartient pas nécessairement à $H^N(\Omega)$, et même si c'est le cas, aucune inégalité du type $\|\nabla^N u^*\|_{L^2} \leq \|\nabla^N u\|_{L^2}$ n'est connue. Pour surmonter cette difficulté, D. R. Adams a exprimé u comme un potentiel de Riesz en fonction de son gradient d'ordre N, puis il s'est restreint à un calcul unidimentionnel en utilisant un résultat de R. O'Neil ([79]) sur les réarrangements décroissants pour les produits de convolution. La même approche a été utilisée dans [87] pour établir l'inégalité (I.5).

Récemment, N. Masmoudi et F. Sani ([77]) ont montré une inégalité optimale analogue à (I.3) en dimension 4 : pour tout $0 < \beta < \beta_4 = 32\pi^2$, il existe une constante $C_\beta > 0$ telle que

$$\int_{\mathbb{R}^4} \left(\mathrm{e}^{\beta |u(x)|^2} - 1\right) dx \leq C_\beta \|u\|^2_{L^2(\mathbb{R}^4)}\,, \quad \forall\, u \in H^2(\mathbb{R}^4) \text{ avec } \|\Delta u\|_{L^2(\mathbb{R}^4)} \leq 1\,. \tag{I.6}$$

Terminons ce paragraphe par rappeler, qu'en plus des travaux mentionnés ci-dessus concernant les inégalités de type Trudinger-Moser, de nombreuses généralisations et extensions de ces inégalités ont été réalisées, telles que les inégalités de type Trudinger-Moser sur des variétés Riemanniennes compactes ([44, 47, 69, 70]) ou celles qui sont étroitement liées aux inégalités de Hardy ([5, 9, 67]). D'autres travaux ont été consacrés au même sujet. Entres autres, on peut citer [4, 36, 37, 40, 46, 59, 71, 73, 76, 83, 100, 102].

4. Dans [65], N. Lam et G. Lu ont généralisé ce résultat au cas d'une dimension quelconque.

Indiquons aussi que certaines inégalités de type Trudinger-Moser, telles que (I.3), (I.5) ou (I.6), jouent un rôle crucial dans l'étude des équations aux dérivées partielles à croissance exponentielle, que ce soit dans le cadre stationnaire ([3, 6, 7, 9, 11, 35, 43, 45, 66, 90, 91]) ou dans le cadre évolutif ([13, 23, 38, 41, 56, 57, 80, 81, 82, 88, 97]).

Il est à noter que l'inégalité de type Adams (I.5) implique l'injection de Sobolev suivante :

$$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})\,, \tag{I.7}$$

où $\mathcal{L}(\mathbb{R}^{2N})$, appelé l'espace d'Orlicz [5] associé à la fonction $\phi(s) := \mathrm{e}^{s^2} - 1$, est l'ensemble des fonctions mesurables $u : \mathbb{R}^{2N} \to \mathbb{C}$ pour lesquelles il existe un nombre réel $\lambda > 0$ tel que

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1\right) dx < +\infty\,.$$

Depuis les travaux de P.-L. Lions ([74, 75]), il est bien connu que l'injection de Sobolev

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2) \tag{I.8}$$

n'est pas compacte au moins pour deux raisons. En premier lieu, on a un défaut de compacité à l'infini comme le montre l'exemple suivant :

$$u_k(x) = \varphi(x + x_k)\,, \quad \text{où } \varphi \in \mathcal{D}(\mathbb{R}^2) \setminus \{0\} \text{ et } |x_k| \overset{k\to\infty}{\longrightarrow} \infty\,.$$

En second lieu, on a un défaut de compacité lié à un phénomène de concentration comme l'illustre l'exemple suivant, connu sous le nom d'exemple de Moser :

$$g_\alpha(x) := \begin{cases} 0 & \text{si } |x| \geq 1\,, \\ -\dfrac{\log|x|}{\sqrt{2\pi\alpha}} & \text{si } \mathrm{e}^{-\alpha} \leq |x| \leq 1\,, \\ \sqrt{\dfrac{\alpha}{2\pi}} & \text{si } |x| \leq \mathrm{e}^{-\alpha}\,. \end{cases} \tag{I.9}$$

En effet, cet exemple qui se formule comme suit :

$$g_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}}\,\mathbf{L}\Big(-\frac{\log|x|}{\alpha}\Big)\,, \tag{I.10}$$

avec $\mathbf{L}$ le profil de Moser défini par

$$\mathbf{L}(t) = \begin{cases} 1 & \text{si } t \geq 1 \\ t & \text{si } 0 \leq t \leq 1 \\ 0 & \text{si } t \leq 0\,, \end{cases} \tag{I.11}$$

5. Dans l'Annexe B, on donnera quelques propriétés élémentaires des espaces d'Orlicz.

satisfait

$$g_\alpha \rightharpoonup 0 \quad \text{dans } H^1(\mathbb{R}^2) \quad \text{et} \quad g_\alpha \to \frac{1}{\sqrt{4\pi}} \quad \text{dans } \mathcal{L}(\mathbb{R}^2)\,, \quad \text{lorsque } \alpha \to \infty\,.$$

Rappelons que, sous une hypothèse de compacité à l'infini, H. Bahouri, M. Majdoub et N. Masmoudi ([24]), ont montré dans le cas radial que les seuls éléments responsables du défaut de compacité de l'injection de Sobolev (I.8) sont du type (I.10). Plus précisément, ces éléments, qu'on appelera par la suite exemples généralisés de Moser, s'écrivent sous la forme

$$\sqrt{\frac{\alpha_n}{2\pi}}\,\psi\Big(\frac{-\log|x|}{\alpha_n}\Big)\,, \tag{I.12}$$

où $\underline{\alpha} = (\alpha_n)$, appelée échelle, est une suite de nombres réels positifs tendant vers l'infini et où ψ, appelé profil, appartient à l'ensemble

$$\mathcal{P}_2 = \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\ \Big\}.$$

Dans un premier article intitulé **Characterization of the lack of compactness of $H^2_{rad}(\mathbb{R}^4)$ into the Orlicz space**, en collaboration avec Ines Ben Ayed, publié dans *Communications in Contemporary Mathematics*, on a décrit le défaut de compacité de l'injection de Sobolev

$$H^2_{rad}(\mathbb{R}^4) \hookrightarrow \mathcal{L}(\mathbb{R}^4)\,. \tag{I.13}$$

Comme pour le cas de la dimension 2, cette injection de Sobolev présente un défaut de compacité à l'infini, ainsi qu'un défaut de compacité généré par des phénomènes de concentration. Dans [87], B. Ruf et F. Sani ont construit une suite de fonctions (f_α) vérifiant

$$f_\alpha \rightharpoonup 0 \quad \text{dans } H^2(\mathbb{R}^4) \quad \text{et} \quad f_\alpha \to \frac{1}{\sqrt{32\pi^2}} \quad \text{dans } \mathcal{L}(\mathbb{R}^4)\,, \quad \text{lorsque } \alpha \to \infty\,.$$

Cette suite de fonctions, qui illustre un phénomène de concentration, est définie comme suit :

$$f_\alpha(x) := \begin{cases} \sqrt{\dfrac{\alpha}{8\pi^2}} + \dfrac{1-|x|^2\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}} & \text{si } |x| \le \mathrm{e}^{-\alpha}\,, \\[2ex] \dfrac{-\log|x|}{\sqrt{8\pi^2\alpha}} & \text{si } \mathrm{e}^{-\alpha} \le |x| \le 1\,, \\[2ex] \eta_\alpha(x) & \text{si } |x| \ge 1\,, \end{cases}$$

où η_α est une fonction de $\mathcal{D}_{rad}(\mathbb{R}^4)$ dont toutes les dérivées jusqu'à l'ordre 2 sont contrôlées par $\dfrac{1}{\sqrt{\alpha}}$ et qui vérifie les conditions au bord suivantes :

$$\eta_{\alpha|\partial B_1} = 0\,, \quad \frac{\partial \eta_\alpha}{\partial \nu}_{|\partial B_1} = \frac{1}{\sqrt{8\pi^2\alpha}}\,,$$

avec B_1 la boule unité de $\mathbb{R}^4$. Il est facile de voir que la suite (f_α) peut s'écrire de la manière suivante :

$$f_\alpha(x) = h_\alpha(x) + R^1_\alpha(x)\,, \quad \|R^1_\alpha\|_{\mathcal{L}(\mathbb{R}^4)} \overset{\alpha\to\infty}{\longrightarrow} 0\,,$$

où $h_\alpha(x) := \sqrt{\frac{\alpha}{8\pi^2}}\,\mathbf{L}\Big(-\frac{\log|x|}{\alpha}\Big)$, avec $\mathbf{L}$ le profil de Moser défini par (I.11).

Cependant, contrairement au cas de la dimension 2, la suite (h_α) n'appartient pas à $H^2(\mathbb{R}^4)$. Pour surmonter cette difficulté, on a introduit une approximation de l'identité adaptée : étant donnée une fonction positive régulière ρ vérifiant

$$\operatorname{supp}\rho \subset [-1,1] \quad \text{et} \quad \int_{-1}^1 \rho(s)\,ds = 1\,,$$

on a décomposé la suite (h_α) de la manière suivante :

$$h_\alpha(x) = \tilde{h}_\alpha(x) + R^2_\alpha(x)\,, \quad \|R^2_\alpha\|_{\mathcal{L}(\mathbb{R}^4)} \overset{\alpha\to\infty}{\longrightarrow} 0\,,$$

où

$$\tilde{h}_\alpha(x) = \sqrt{\frac{\alpha}{8\pi^2}}\,(\mathbf{L}*\rho_\alpha)\Big(-\frac{\log|x|}{\alpha}\Big)\,, \tag{I.14}$$

avec $\rho_\alpha(s) := \alpha\,\rho(\alpha s)$.

Dans ce premier travail, on a montré que le défaut de compacité de l'injection de Sobolev (I.13) se décrit à l'aide d'éléments du type (I.14). Le résultat qu'on a obtenu se formule comme suit : étant donnée une suite $(u_n)_{n\in\mathbb{N}}$ bornée dans $H^2_{rad}(\mathbb{R}^4)$ vérifiant

$$u_n \rightharpoonup 0\,,$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(\mathbb{R}^4)} = A_0 > 0 \quad \text{et}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} |u_n(x)|^2\,dx = 0\,,$$

il existe alors une suite d'échelles $(\alpha_n^{(j)})_{j\geq1}$ deux à deux orthogonales[6] et une suite de profils $(\psi^{(j)})_{j\geq1}$ de l'ensemble

$$\mathcal{P}_4 = \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-4s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\Big\}$$

telles que, à extraction d'une sous-suite près, on a pour tout $\ell \geq 1$

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\Big(\psi^{(j)} * \rho_n^{(j)}\Big)\Big(\frac{-\log|x|}{\alpha_n^{(j)}}\Big) + \mathrm{r}_n^{(\ell)}(x)\,, \quad \limsup_{n\to\infty}\Big\|\mathrm{r}_\mathrm{n}^{(\ell)}\Big\|_{\mathcal{L}(\mathbb{R}^4)} \overset{\ell\to\infty}{\longrightarrow} 0\,, \tag{I.15}$$

avec $\rho_n^{(j)}(s) := \alpha_n^{(j)}\rho(\alpha_n^{(j)}s)$.

6. Deux échelles (α_n) et $(\tilde{\alpha}_n)$ sont dites orthogonales si $\Big|\log\Big(\frac{\tilde{\alpha}_n}{\alpha_n}\Big)\Big| \to \infty$.

La démarche qu'on a adoptée pour prouver ce résultat s'appuie sur une estimation L^∞ loin de l'origine. Cette estimation spécifique au cas radial nous a permis, en s'inspirant du travail [24], de mettre en place une stratégie pour extraire les éléments du type (I.14).

L'objectif de notre deuxième article intitulé **Description of the lack of compactness in Orlicz spaces and applications**, en collaboration avec Ines Ben Ayed, à paraître dans *Differential and Integral Equations*, est double. Dans un premier temps, on s'est intéressé à la classe d'injections de Sobolev

$$H^1(\mathbb{R}^2) \hookrightarrow L^{\phi_p}(\mathbb{R}^2)\,, \quad p \in \mathbb{N}^*\,, \tag{I.16}$$

où $L^{\phi_p}(\mathbb{R}^2)$, appelé l'espace d'Orlicz associé à la fonction $\phi_p(s) := \mathrm{e}^{s^2} - \sum_{k=0}^{p-1} \frac{s^{2k}}{k!}$, est l'ensemble des fonctions mesurables $u : \mathbb{R}^2 \to \mathbb{C}$ pour lesquelles il existe un nombre réel $\lambda > 0$ tel que

$$\int_{\mathbb{R}^2} \phi_p\Big(\frac{|u(x)|}{\lambda}\Big)\, dx < +\infty\,.$$

A cet effet, on a généralisé le travail de H. Bahouri, M. Majdoub et N. Masmoudi ([26]) traitant l'injection de Sobolev (I.16) dans le cas où $p = 1$. Plus précisément, on a fourni, pour tout entier $p \in \mathbb{N}^*$, une caractérisation du défaut de compacité de l'injection de Sobolev (I.16) en termes des exemples généralisés de Moser (I.12) concentrés autour de suites (x_n) de $\mathbb{R}^2$, appelées cœurs.

La preuve de ces décompositions en profils est complètement différente de celle du premier article qui est basée sur l'estimation L^∞ loin de l'origine. En effet, cette estimation radiale fait défaut dans le cas général comme le montre l'exemple de la suite :

$$u_\alpha(x) = \sum_{k=1}^{\infty} \frac{1}{k^2}\, g_\alpha(x - x_k)\,,$$

où (g_α) est l'exemple de Moser défini par (I.9) et (x_k) est une suite de $\mathbb{R}^2$ dont la norme tend vers l'infini. La méthode qu'on a développée ici pour extraire les cœurs de concentration repose sur des arguments de capacité comme dans [26] et utilise d'une manière cruciale la symétrisation de Schwarz.

Dans un second temps, on a abordé l'étude de l'équation de Klein-Gordon suivante :

$$\begin{cases} \Box u + u + u\,\phi_p(\sqrt{4\pi}u) = 0\,, \\ u_{|t=0} = u_0 \in H^1(\mathbb{R}^2)\,, \quad \partial_t u_{|t=0} = u_1 \in L^2(\mathbb{R}^2)\,, \end{cases} \tag{I.17}$$

où p est un entier supérieur ou égal à 1, où $u = u(t,x)$ est une fonction de $(t,x) \in \mathbb{R} \times \mathbb{R}^2$ à valeurs réelles et où $\Box = \partial_t^2 - \Delta$ est l'opérateur des ondes.

Rappelons que les solutions du problème de Cauchy (I.17) satisfont, formellement, la conservation de l'énergie :

$$\begin{aligned} E_p(u,t) &:= \|\partial_t u(t,.)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u(t,.)\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} F\Big(u(t,x)\Big)\,dx \\ &= E_p(u,0) := E_p^0\,, \end{aligned} \tag{I.18}$$

avec $F\Big(u(t,x)\Big) := \dfrac{1}{4\pi}\Big(\mathrm{e}^{4\pi u(t,x)^2} - 1 - \sum_{k=2}^{p} \dfrac{(4\pi)^k}{k!} u(t,x)^{2k}\Big)$.

Comme dans les travaux de S. Ibrahim, M. Majdoub et N. Masmoudi dans [56] et de S. Ibrahim, M. Majdoub, N. Masmoudi et K. Nakanishi dans [58], où les auteurs traitent le problème de Cauchy (I.17) dans le cas où $p = 1$, on a prouvé l'existence et l'unicité globale ainsi que la complétude asymptotique dans les cas sous-critique et critique. Ici, la notion de criticalité dépend de la taille de l'énergie E_p^0 par rapport à 1. Plus précisément, le problème (I.17) est dit sous-critique si $E_p^0 < 1$, critique si $E_p^0 = 1$ et sur-critique si $E_p^0 > 1$.

Par la suite, on s'est attaché à analyser les solutions du problème (I.17) dans les cas sous-critique et critique. L'approche qu'on a adoptée est celle introduite par P. Gérard dans [49] : elle consiste à comparer l'évolution des oscillations et des effets de concentration produits par les suites des solutions de l'équation non linéaire (I.17) et les suites des solutions de l'équation linéaire

$$\Box v + v = 0\,. \tag{I.19}$$

Comme pour le cas $p = 1$ étudié dans [24], on a montré que la nonlinéarité n'induit aucun effet sur le comportement des solutions dans le cas sous-critique, ainsi que dans le cas critique moyennant une hypothèse de la non concentration de la masse totale pour les solutions de (I.19).

Notre troisième article intitulé **Sharp Adams-type inequality invoking Hardy inequalities**, soumis pour publication (arXiv :1502.05154), est dédié en premier lieu à l'extension de l'inégalité (I.5) à l'espace fonctionnel

$$\mathcal{H}(\mathbb{R}^{2N}) := \left\{ u \in H^1(\mathbb{R}^{2N});\ \frac{\nabla u}{|\,.\,|^{N-1}} \in L^2(\mathbb{R}^{2N}) \right\}, \quad N \geq 2\,,$$

dans le cas radial. Cet espace qui contient strictement l'espace de Sobolev $H^N(\mathbb{R}^{2N})$, est en étroite relation avec les inégalités de Hardy (voir par exemple [16, 19, 52, 53]) :

$$\left\| \frac{u}{|\,.\,|^s} \right\|_{L^2(\mathbb{R}^d)} \leq C_{d,s} \|u\|_{\dot{H}^s(\mathbb{R}^d)}\,, \quad \forall\, s \in \Big[0, \frac{d}{2}\Big[\,.$$

Il est à noter que ces inégalités de Hardy sont fondamentales dans certains problèmes d'Analyse (on peut mentionner, par exemple, les méthodes d'explosion ou l'étude des opérateurs pseudo-différentiels à coefficients singuliers).

L'inégalité optimale de type Adams qu'on a obtenue, dans le cadre de l'espace $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ est donnée par

$$\sup_{u\in\mathcal{H}_{rad}(\mathbb{R}^{2N}),\, \|u\|_{\mathcal{H}(\mathbb{R}^{2N})}\leq 1} \int_{\mathbb{R}^{2N}} \Big(\mathrm{e}^{\gamma_N |u(x)|^2} - 1\Big)\, dx < +\infty\,, \tag{I.20}$$

où $\gamma_N := 2N\omega_{2N-1}$ et où $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})}$ désigne la norme suivante :

$$\|u\|^2_{\mathcal{H}(\mathbb{R}^{2N})} = \|u\|^2_{H^1(\mathbb{R}^{2N})} + \left\|\frac{\nabla u}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})}\,.$$

La stratégie qu'on a utilisée pour établir cette inégalité est inspirée de celle adoptée en dimension 2 par B. Ruf dans [86] : elle consiste à écrire l'intégrale dans (I.20) comme somme de deux intégrales I_1 et I_2, où

$$I_1 := \int_{B(r_0)} \Big(\mathrm{e}^{\gamma_N |u(x)|^2} - 1\Big)\, dx \quad \text{et} \quad I_2 := \int_{\mathbb{R}^{2N}\setminus B(r_0)} \Big(\mathrm{e}^{\gamma_N |u(x)|^2} - 1\Big)\, dx\,,$$

avec r_0 un certain nombre réel strictement positif, puis à montrer que ces deux intégrales peuvent être majorées par une constante qui dépend uniquement de r_0 et N. Le contrôle de I_2 s'appuie sur une estimation radiale loin de l'origine spécifique à la dimension $2N$, tandis que le contrôle de I_1 se réalise en effectuant un changement de variable qui permet par la suite d'appliquer l'inégalité de Trudinger-Moser (I.2) pour $d = 2$.

La preuve de l'optimalité a nécessité la construction d'une suite adéquate de fonctions. Cette suite, qui généralise au cas de la dimension $2N$ l'exemple de Moser (I.9), est donnée par

$$k_\alpha(x) = \sqrt{\frac{2N\alpha}{\gamma_N}}\, \mathbf{L}\Big(-\frac{\log|x|}{\alpha}\Big)\,, \tag{I.21}$$

avec $\mathbf{L}$ le profil de Moser défini par (I.11). Plus précisément, l'optimalité de l'inégalité (I.20) découle de la propriété de convergence suivante vérifiée par (k_α) :

$$\int_{\mathbb{R}^{2N}} \Big(\mathrm{e}^{\gamma \left|\frac{k_\alpha(x)}{\|k_\alpha\|_{\mathcal{H}(\mathbb{R}^{2N})}}\right|^2} - 1\Big)\, dx \overset{\alpha\to\infty}{\longrightarrow} \infty\,, \quad \forall\, \gamma > \gamma_N\,.$$

En second lieu, on a caractérisé, dans ce travail, le défaut de compacité de l'injection de Sobolev suivante impliquée par l'inégalité (I.20) :

$$\mathcal{H}_{rad}(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})\,. \tag{I.22}$$

Comme pour les injections de Sobolev étudiées dans les travaux précédents, on note que le défaut de compacité de l'injection de Sobolev (I.22) est dû à la fois à un défaut de compacité à l'infini et à un phénomène de concentration. Ce phénomène de concentration peut être illustré par la suite de fonctions (k_α) définie par (I.21). En effet, cette suite satisfait

$$k_\alpha \rightharpoonup 0 \quad \text{dans } \mathcal{H}(\mathbb{R}^{2N}) \quad \text{et} \quad k_\alpha \to \frac{1}{\sqrt{\gamma_N}} \quad \text{dans } \mathcal{L}(\mathbb{R}^{2N})\,, \quad \text{lorsque } \alpha\to\infty\,.$$

En s'inspirant de la méthode développée dans [24], on a montré que le défaut de compacité de l'injection de Sobolev (I.22) se décrit à l'aide d'éléments du type (I.21). Plus précisément, on a obtenu le résultat suivant : étant donnée une suite $(u_n)_{n\in\mathbb{N}}$ bornée dans $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ vérifiant

$$u_n \rightharpoonup 0\,,$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(\mathbb{R}^{2N})} = A_0 > 0 \quad \text{et}$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n(x)|^2\, dx = 0\,,$$

il existe alors une suite d'échelles $(\alpha_n^{(j)})_{j\geq 1}$ deux à deux orthogonales et une suite de profils $(\psi^{(j)})_{j\geq 1}$ de l'ensemble

$$\mathcal{P}_{2N} := \Big\{\, \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2Ns}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0 \Big\}$$

telles que, à extraction d'une sous-suite près, on a pour tout $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\, \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x)\,, \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{\ell\to\infty}{\longrightarrow} 0\,.$$

De plus, on a l'estimation de stabilité suivante :

$$\left\| \frac{\nabla u_n}{|\,.\,|^{N-1}} \right\|^2_{L^2(\mathbb{R}^{2N})} = \sum_{j=1}^{\ell} \left\| \psi^{(j)'} \right\|^2_{L^2(\mathbb{R})} + \left\| \frac{\nabla \mathrm{r}_n^{(\ell)}}{|\,.\,|^{N-1}} \right\|^2_{L^2(\mathbb{R}^{2N})} + \circ(1)\,, \quad n\to\infty\,.$$

Soulignons que récemment, H. Bahouri et G. Perelman ([27]) ont caractérisé le défaut de compacité de l'injection de Sobolev (I.7) dans le cadre général en utilisant une approche complètement différente basée sur l'analyse de Fourier.

Rappelons que les décompositions en profils ont pris leur origine dans le cadre elliptique dans les travaux de H. Brezis et J.-M. Coron ([32]) et M. Struwe ([94]). Depuis, ce problème n'a cessé de susciter l'attention des chercheurs, que ce soit dans la description des injections de Sobolev critiques invariantes par changement d'échelle ([20, 50, 60]) ou dans l'analyse d'équations aux dérivées partielles non linéaires ([17, 21, 22, 24, 48, 62, 68, 95, 99]).

Notons, cependant, que les éléments intervenant dans la caractérisation du défaut de compacité des injections de Sobolev dans les espaces d'Orlicz sont complètement différents de ceux apparaissant dans l'étude des injections de Sobolev critiques invariantes par changement d'échelle. En effet, les éléments responsables du défaut de compacité de ces dernières sont, selon la terminologie de P. Gérard dans [50], oscillants par rapport à des échelles (autrement dit, concentrés en fréquences), tandis que ceux qui décrivent le défaut de compacité des injections de Sobolev dans les espaces d'Orlicz sont étalés en fréquences (voir [26]).

Terminons cette introduction en rappelant que les différents travaux concernant l'étude du défaut de compacité d'injections de Sobolev ([12, 20, 24, 26, 27, 28, 50, 49, 60]) ont permis de fournir diverses informations sur les solutions d'équations aux dérivées partielles non linéaires. On peut mentionner, par exemple, les résultats de régularité concernant des systèmes de Navier-Stokes ([17, 21, 48]), l'étude qualitative d'équations d'évolution non linéaires ([22, 24, 62, 68, 99]) et l'estimation du temps de vie des solutions d'équations d'évolution semi-linéaires ([61]). Pour d'autres applications, on peut consulter [32, 74, 75, 94, 95] et les références associées.

CHAPITRE II

ON THE LACK OF COMPACTNESS OF A 4D SOBOLEV EMBEDDING

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II.1 Introduction

II.1.1 Development in critical Sobolev embedding

Due to the scaling invariance, the critical Sobolev embedding

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)\,, \tag{II.1}$$

when $0 \leq s < \frac{d}{2}$ and $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$, is not compact.
After the pioneering works of P. Lions [74] and [75], P. Gérard described in [49] the lack of compactness of (II.1) by means of profiles in the following terms : a sequence $(u_n)_n$ bounded in $\dot{H}^s(\mathbb{R}^d)$ can be decomposed, up to a subsequence extraction, on a finite sum of orthogonal profiles such that the remainder converges to zero in $L^p(\mathbb{R}^d)$ as the number of the sum and n tend to infinity. This question was later investigated by S. Jaffard in [60] in the more general case of $H^{s,q}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)\,,\, 0 < s < \frac{d}{q}$ and $\frac{1}{p} = \frac{1}{q} - \frac{s}{d}$ by the use of nonlinear wavelet and recently in [20] in an abstract frame $X \hookrightarrow Y$ including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder and BMO spaces. (One can consult [15] and the references therein for an introduction to these spaces). We also mention the work of Brezis-Coron [32] about H-systems. In addition, in [24], [25] and [26] H. Bahouri, M. Majdoub and N. Masmoudi characterized the lack of compactness of $H^1(\mathbb{R}^2)$ in the Orlicz space (see Definition II.1)

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2)\,,$$

in terms of orthogonal profiles generalizing the example by Moser :

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\;\psi\Big(\frac{-\log|x|}{\alpha_n}\Big)\,,$$

where $\underline{\alpha} := (\alpha_n)$, called the scale, is a sequence of positive real numbers going to infinity and ψ, called the profile, belongs to the set

$$\Big\{\;\psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\; \psi_{|]-\infty,0]} = 0\;\Big\}\,.$$

The study of the lack of compactness of critical Sobolev embedding was at the origin of several works concerning the understanding of features of solutions of nonlinear partial differential equations. Among others, one can mention [21], [22], [61], [62], [68] and [94].

II.1.2 Critical 4D Sobolev embedding

The Sobolev space $H^2(\mathbb{R}^4)$ is continuously embedded in all Lebesgue spaces $L^p(\mathbb{R}^4)$ for all $2 \leq p < \infty$. On the other hand, it is also known that $H^2(\mathbb{R}^4)$ embed in $BMO(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)\,$, where $BMO(\mathbb{R}^d)$ denotes the space of bounded mean oscillations which is the space of locally integrable functions f such that

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|}\int_B |f - f_B|\,dx < \infty \quad \text{with} \quad f_B = \frac{1}{|B|}\int_B f\,dx\,.$$

The above supremum being taken over the set of Euclidean balls B, $|\cdot|$ denoting the Lebesgue measure.

In this paper, our goal is to investigate the lack of compactness of the Sobolev space $H^2_{rad}(\mathbb{R}^4)$ in the Orlicz space $\mathcal{L}(\mathbb{R}^4)$ defined as follows :

Definition II.1 *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that*

$$\phi(0) = 0 = \lim_{s\to 0^+} \phi(s), \quad \lim_{s\to\infty} \phi(s) = \infty \,.$$

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right)\, dx < \infty \,.$$

We denote then

$$\|u\|_{L^\phi} = \inf\left\{\lambda > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right)\, dx \leq 1\right\}.$$

In what follows we shall fix $d = 4$, $\phi(s) = \mathrm{e}^{s^2} - 1$ and denote the Orlicz space L^ϕ by $\mathcal{L}$ endowed with the norm $\|\cdot\|_{\mathcal{L}}$ where the number 1 is replaced by the constant κ involved in (II.3). It is easy to see that $\mathcal{L} \hookrightarrow L^p$ for every $2 \leq p < \infty$.
The 4D Sobolev embedding in Orlicz space $\mathcal{L}$ states as follows :

$$\|u\|_{\mathcal{L}(\mathbb{R}^4)} \leq \frac{1}{\sqrt{32\pi^2}} \|u\|_{H^2(\mathbb{R}^4)} \,. \tag{II.2}$$

Inequality (II.2) derives immediately from the following proposition due to Ruf and Sani in [87] :

Proposition II.2 *There exists a finite constant $\kappa > 0$ such that*

$$\sup_{u\in H^2(\mathbb{R}^4), \|u\|_{H^2(\mathbb{R}^4)}\leq 1} \int_{\mathbb{R}^4} \left(\mathrm{e}^{32\pi^2|u(x)|^2} - 1\right)\, dx := \kappa < \infty \,. \tag{II.3}$$

Let us notice that if we only require that $\|\Delta u\|_{L^2(\mathbb{R}^4)} \leq 1$ then the following result established in [77] holds.

Proposition II.3 *Let $\beta \in [0, 32\pi^2[$, then there exists $C_\beta > 0$ such that*

$$\int_{\mathbb{R}^4} \left(\mathrm{e}^{\beta|u(x)|^2} - 1\right) dx \leq C_\beta \|u\|^2_{L^2(\mathbb{R}^4)} \quad \forall\, u \in H^2(\mathbb{R}^4) \textit{ with } \|\Delta u\|_{L^2} \leq 1 \,, \tag{II.4}$$

and this inequality is false for $\beta \geq 32\pi^2$.

Remarks II.4 *The well-known following properties can be found in [77] and [87].*

– *The inequality* (II.3) *is sharp.*

– *There exists a positive constant C such that for any domain $\Omega \subseteq \mathbb{R}^4$*

$$\sup_{u\in H^2(\Omega),\|(-\Delta+I)u\|_{L^2(\Omega)}\leq 1} \int_\Omega \left(\mathrm{e}^{32\pi^2|u(x)|^2} - 1\right) dx \leq C\,.$$

– *In dimension 2, the inequality* (II.4) *is replaced by the following Trudinger-Moser type inequality (see [1] and [86]) :*
Let $\alpha \in [0, 4\pi[$. A constant C_α exists such that

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - 1\right)\, dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}\,, \tag{II.5}$$

for any $u \in H^1(\mathbb{R}^2)$ with $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$. Moreover, if $\alpha \geq 4\pi$ then (II.5) *is false.*

II.1.3 Lack of compactness in 4D critical Sobolev embedding in Orlicz space

The embedding of $H^2(\mathbb{R}^4)$ into the Orlicz space is non compact. Firstly, we have a lack of compactness at infinity as shown by the following example :

$$u_k(x) = \varphi(x + x_k), \quad \varphi \in \mathcal{D}(\mathbb{R}^4) \setminus \{0\} \quad \text{and} \quad |x_k| \underset{k\to\infty}{\longrightarrow} \infty.$$

Secondly, we have a lack of compactness generated by a concentration phenomenon as illustrated by the following example (see [87] for instance) :

$$f_\alpha(x) = \begin{cases} \sqrt{\frac{\alpha}{8\pi^2}} + \frac{1-|x|^2\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}} & \text{if } |x| \leq \mathrm{e}^{-\alpha} \\ \frac{-\log|x|}{\sqrt{8\pi^2\alpha}} & \text{if } \mathrm{e}^{-\alpha} < |x| \leq 1 \\ \eta_\alpha(x) & \text{if } |x| > 1\,, \end{cases} \tag{II.6}$$

where $\eta_\alpha \in \mathcal{D}(\mathbb{R}^4)$ and satisfies the following boundary conditions :

$$\eta_{\alpha|\partial B_1} = 0,\ \frac{\partial\eta_\alpha}{\partial\nu}\Big|_{\partial B_1} = \frac{1}{\sqrt{8\pi^2\alpha}}\,,$$

with B_1 is the unit ball in $\mathbb{R}^4$. In addition, η_α, $\nabla\eta_\alpha$, $\Delta\eta_\alpha$ are all equal to $O\Big(\frac{1}{\sqrt{\alpha}}\Big)$ [1] as α tends to infinity.
By a simple calculation (see Appendix A), we obtain that

$$\|f_\alpha\|^2_{L^2} = O\Big(\frac{1}{\alpha}\Big),\ \|\nabla f_\alpha\|^2_{L^2} = O\Big(\frac{1}{\alpha}\Big) \text{ and } \|\Delta f_\alpha\|^2_{L^2} = 1 + O\Big(\frac{1}{\alpha}\Big) \quad \text{as } \alpha \to +\infty\,.$$

Also, we can see that $f_\alpha \underset{\alpha\to\infty}{\rightharpoonup} 0$ in $H^2(\mathbb{R}^4)$.
The lack of compactness in the Orlicz space $\mathcal{L}(\mathbb{R}^4)$ displayed by the sequence (f_α) when α goes to infinity can be stated qualitatively as follows :

1. The notation $g(\alpha) = O(h(\alpha))$ as $\alpha \to +\infty$, where g and h are two functions defined on some neighborhood of infinity, means the existence of positive numbers α_0 and C such that for any $\alpha > \alpha_0$ we have $|g(\alpha)| \leq C|h(\alpha)|$.

Proposition II.5 *The sequence* (f_α) *defined by* (II.6) *satisfies :*

$$\|f_\alpha\|_{\mathcal{L}} \to \frac{1}{\sqrt{32\pi^2}}\,,\ as\ \alpha \to +\infty\,.$$

Proof. Firstly, we shall prove that $\liminf\limits_{\alpha\to\infty} \|f_\alpha\|_{\mathcal{L}} \geq \frac{1}{\sqrt{32\pi^2}}\cdot$ For that purpose, let us consider $\lambda > 0$ such that

$$\int_{\mathbb{R}^4} \left(\mathrm{e}^{\frac{|f_\alpha(x)|^2}{\lambda^2}} - 1\right)\, dx \leq \kappa\,.$$

Then

$$\int_{|x|\leq \mathrm{e}^{-\alpha}} \left(\mathrm{e}^{\frac{|f_\alpha(x)|^2}{\lambda^2}} - 1\right) dx \leq \kappa\,.$$

But for $|x| \leq \mathrm{e}^{-\alpha}$, we have

$$f_\alpha(x) = \sqrt{\frac{\alpha}{8\pi^2}} + \frac{1-|x|^2\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}} \geq \sqrt{\frac{\alpha}{8\pi^2}}\cdot$$

So we deduce that

$$2\pi^2 \int_0^{\mathrm{e}^{-\alpha}} \left(\mathrm{e}^{\frac{\alpha}{8\pi^2\lambda^2}} - 1\right) r^3\, dr \leq \kappa\,.$$

Consequently,

$$2\pi^2 \left(\mathrm{e}^{\frac{\alpha}{8\pi^2\lambda^2}} - 1\right) \frac{\mathrm{e}^{-4\alpha}}{4} \leq \kappa\,,$$

which implies that

$$\lambda^2 \quad \geq \quad \frac{1}{32\pi^2 + \frac{8\pi^2}{\alpha}\log(\frac{2\kappa}{\pi^2} + \mathrm{e}^{-4\alpha})} \underset{\alpha\to\infty}{\longrightarrow} \frac{1}{32\pi^2}\cdot$$

This ensures that

$$\liminf_{\alpha\to\infty} \|f_\alpha\|_{\mathcal{L}} \geq \frac{1}{\sqrt{32\pi^2}}\cdot$$

To conclude, it suffices to show that $\limsup\limits_{\alpha\to\infty} \|f_\alpha\|_{\mathcal{L}} \leq \frac{1}{\sqrt{32\pi^2}}\cdot$ To go to this end, let us fix $\varepsilon > 0$ and use Inequality (II.4) with $\beta = 32\pi^2 - \varepsilon$. Thus, there exists $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^4} \left(\mathrm{e}^{(32\pi^2-\varepsilon)\frac{|f_\alpha(x)|^2}{\|\Delta f_\alpha\|^2_{L^2}}} - 1\right)\, dx \leq C_\varepsilon \frac{\|f_\alpha\|^2_{L^2}}{\|\Delta f_\alpha\|^2_{L^2}}\cdot$$

The fact that $\lim\limits_{\alpha\to\infty} \|f_\alpha\|_{L^2} = 0$ leads to

$$\limsup_{\alpha\to\infty} \|f_\alpha\|^2_{\mathcal{L}} \leq \frac{1}{32\pi^2-\varepsilon}\,,$$

which ends the proof of the result. ∎

The following result specifies the concentration effect revealed by the family (f_α) :

Proposition II.6 *With the above notation, we have*

$$|\Delta f_\alpha|^2 \to \delta(x=0) \quad \textit{and} \quad \mathrm{e}^{32\pi^2|f_\alpha|^2} - 1 \to \frac{\pi^2}{16}(\mathrm{e}^4+3)\delta(x=0)$$

in $\mathcal{D}'(\mathbb{R}^4)$ *as* $\alpha \to \infty$.

Proof. For any smooth compactly supported function φ, let us write

$$\int_{\mathbb{R}^4} |\Delta f_\alpha(x)|^2 \varphi(x)\, dx \;\;=\;\; I_\alpha + J_\alpha + K_\alpha\,,$$

with

$$\begin{aligned} I_\alpha &= \int_{|x|\leq \mathrm{e}^{-\alpha}} |\Delta f_\alpha(x)|^2 \varphi(x)\, dx\,, \\ J_\alpha &= \int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1} |\Delta f_\alpha(x)|^2 \varphi(x)\, dx \quad \text{and} \\ K_\alpha &= \int_{|x|\geq 1} |\Delta f_\alpha(x)|^2 \varphi(x)\, dx\,. \end{aligned}$$

Noticing that $\Delta f_\alpha(x) = \frac{-8\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}}$ if $|x| \leq \mathrm{e}^{-\alpha}$, we get

$$|I_\alpha| \leq \frac{\|\varphi\|_{L^\infty}}{\alpha} \underset{\alpha\to\infty}{\longrightarrow} 0\,.$$

On the other hand, as $\Delta f_\alpha = \frac{-2}{|x|^2\sqrt{8\pi^2\alpha}}$ if $\mathrm{e}^{-\alpha} \leq |x| \leq 1$, we get

$$\begin{aligned} J_\alpha &= \frac{1}{2\pi^2\alpha}\int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1} \frac{1}{|x|^4}\varphi(0)\, dx + \frac{1}{2\pi^2\alpha}\int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1} \frac{1}{|x|^4}\Big(\varphi(x)-\varphi(0)\Big)\, dx \\ &= \varphi(0) + \frac{1}{2\pi^2\alpha}\int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1} \frac{1}{|x|^4}\Big(\varphi(x)-\varphi(0)\Big)\, dx\,. \end{aligned}$$

Using the fact that $|\varphi(x)-\varphi(0)| \leq |x|\|\nabla\varphi\|_{L^\infty}$ we obtain that

$$|J_\alpha - \varphi(0)| \leq \frac{\|\nabla\varphi\|_{L^\infty}}{\alpha}(1-\mathrm{e}^{-\alpha}) \underset{\alpha\to\infty}{\longrightarrow} 0\,.$$

Finally, taking advantage of the existence of a positive constant C such that $\|\Delta\eta_\alpha\|_{L^\infty} \leq \frac{C}{\sqrt{\alpha}}$ and as φ is a smooth compactly supported function, we deduce that

$$|K_\alpha| \underset{\alpha\to\infty}{\longrightarrow} 0\,.$$

This ends the proof of the first assertion. For the second assertion, we write

$$\int_{\mathbb{R}^4} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2} - 1\Big)\varphi(x)\, dx \;\;=\;\; L_\alpha + M_\alpha + N_\alpha\,,$$

where

$$\begin{aligned} L_\alpha &= \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(x)\,dx\,,\\ M_\alpha &= \int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(x)\,dx \quad \text{and}\\ N_\alpha &= \int_{|x|\geq 1} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(x)\,dx\,. \end{aligned}$$

We have

$$L_\alpha = \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\Big(\varphi(x)-\varphi(0)\Big)\,dx + \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(0)\,dx\,.$$

Arguing as above, we infer that

$$\begin{aligned} &\left|L_\alpha - \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(0)\,dx\right|\\ &\qquad\qquad \leq 2\pi^2\|\nabla\varphi\|_{L^\infty}\left(\mathrm{e}^{32\pi^2\Big(\sqrt{\frac{\alpha}{8\pi^2}}+\frac{1}{\sqrt{32\pi^2\alpha}}\Big)^2}-1\right)\frac{\mathrm{e}^{-5\alpha}}{5}\,. \end{aligned}$$

As the right hand side of the last inequality goes to zero when α tends to infinity, we find that

$$\left|L_\alpha - \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(0)\,dx\right| \underset{\alpha\to\infty}{\longrightarrow} 0\,.$$

Besides,

$$\begin{aligned} \int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(0)\,dx &= 2\pi^2\mathrm{e}^{4(\alpha+1)}\mathrm{e}^{\frac{1}{\alpha}}\varphi(0)\int_0^{\mathrm{e}^{-\alpha}} \mathrm{e}^{\frac{\mathrm{e}^{4\alpha}}{\alpha}r^4-2\mathrm{e}^{2\alpha}(2+\frac{1}{\alpha})r^2}r^3\,dr\\ &- \frac{\pi^2}{2}\varphi(0)\mathrm{e}^{-4\alpha}\,. \end{aligned}$$

Now, performing the change of variable $s = r\mathrm{e}^\alpha$, we get

$$\int_{|x|\leq \mathrm{e}^{-\alpha}} \Big(\mathrm{e}^{32\pi^2|f_\alpha(x)|^2}-1\Big)\,\varphi(0)\,dx = 2\pi^2\mathrm{e}^{\frac{1}{\alpha}+4}\varphi(0)\int_0^1 s^3\mathrm{e}^{\frac{s^4}{\alpha}-2(2+\frac{1}{\alpha})s^2}\,ds - \frac{\pi^2}{2}\varphi(0)\mathrm{e}^{-4\alpha}\,,$$

which implies, in view of Lebesgue's theorem, that

$$\lim_{\alpha\to\infty} L_\alpha = 2\pi^2\mathrm{e}^4\varphi(0)\int_0^1 s^3\,\mathrm{e}^{-4s^2}\,ds = \frac{\pi^2}{16}(\mathrm{e}^4-5)\varphi(0)\,.$$

Also, writing

$$M_\alpha = \int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1}\Big(\varphi(x)-\varphi(0)\Big)\Big(\mathrm{e}^{\frac{4(\log|x|)^2}{\alpha}}-1\Big)\,dx + \int_{\mathrm{e}^{-\alpha}\leq|x|\leq 1}\varphi(0)\Big(\mathrm{e}^{\frac{4(\log|x|)^2}{\alpha}}-1\Big)\,dx\,,$$

we infer that M_α converges to $\frac{\pi^2}{2}\varphi(0)$ by using the following lemma the proof of which is similar to that of Lemma 1.9 in [24]. ■

Lemma II.7 *When α goes to infinity,*

$$\int_{e^{-\alpha}}^{1} r^4\, e^{\frac{4}{\alpha}\log^2 r}\, dr \longrightarrow \frac{1}{5} \quad \text{and} \quad \int_{e^{-\alpha}}^{1} r^3\, e^{\frac{4}{\alpha}\log^2 r}\, dr \longrightarrow \frac{1}{2}\cdot$$

Finally, in view of the existence of a positive constant C such that $\|\eta_\alpha\|_{L^\infty} \leq \frac{C}{\sqrt{\alpha}}$ and as φ is a smooth compactly supported function, we get

$$N_\alpha \underset{\alpha\to\infty}{\longrightarrow} 0\,,$$

which achieves the proof of the proposition.

II.1.4 Statement of the results

Before entering into the details, let us introduce some definitions as in [24] and [49].

Definition II.8 *We shall designate by a scale any sequence $\alpha := (\alpha_n)$ of positive real numbers going to infinity. Two scales α and β are said orthogonal if*

$$\left|\log\left(\frac{\beta_n}{\alpha_n}\right)\right| \to \infty\,.$$

The set of profiles is

$$\mathcal{P} := \Big\{\psi \in L^2(\mathbb{R}, e^{-4s}ds); \quad \psi' \in L^2(\mathbb{R}), \quad \psi_{|]-\infty,0]} = 0\Big\}.$$

Remark II.9 *The profiles belong to the Hölder space $C^{\frac{1}{2}}$. Indeed, for any profile ψ and real numbers s and t, we have by Cauchy-Schwarz inequality*

$$|\psi(s)-\psi(t)| = \left|\int_s^t \psi'(\tau)\, d\tau\right| \leq \|\psi'\|_{L^2(\mathbb{R})}|s-t|^{\frac{1}{2}}\,.$$

Our main goal is to establish that the characterization of the lack of compactness of critical Sobolev embedding

$$H^2_{rad}(\mathbb{R}^4) \hookrightarrow \mathcal{L}(\mathbb{R}^4)$$

can be reduced to the example (II.6). In fact, we can decompose the function f_α as follows :

$$f_\alpha(x) = \sqrt{\frac{\alpha}{8\pi^2}}L\Big(-\frac{\log|x|}{\alpha}\Big) + r_\alpha(x)\,,$$

where

$$L(t) = \begin{cases} 1 & \text{if} \quad t \geq 1 \\ t & \text{if} \quad 0 \leq t < 1 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

and

$$r_\alpha(x) = \begin{cases} \frac{1-|x|^2 e^{2\alpha}}{\sqrt{32\pi^2\alpha}} & \text{if} \quad |x| \leq e^{-\alpha} \\ 0 & \text{if} \quad e^{-\alpha} < |x| \leq 1 \\ \eta_\alpha(x) & \text{if} \quad |x| > 1\,. \end{cases}$$

The sequence α is a scale, the function L is a profile and the function r_α is called the remainder term.
We can easily see that $r_\alpha \underset{\alpha\to\infty}{\longrightarrow} 0$ in $\mathcal{L}$. Indeed, for all $\lambda > 0$, we have

$$\begin{aligned}\int_{|x|\leq e^{-\alpha}} \left(e^{\frac{|r_\alpha(x)|^2}{\lambda^2}} - 1\right) dx &\leq 2\pi^2 \int_0^{e^{-\alpha}} \left(e^{\frac{1+r^4 e^{4\alpha}}{16\pi^2\alpha\lambda^2}} - 1\right) r^3\, dr \\ &\leq \left[8\pi^4\lambda^2 e^{\frac{1}{16\pi^2\alpha\lambda^2}}\alpha e^{-4\alpha}\left(e^{\frac{1}{16\pi^2\alpha\lambda^2}} - 1\right) - \frac{\pi^2 e^{-4\alpha}}{2}\right] \underset{\alpha\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

Moreover, since η belongs to $\mathcal{D}(\mathbb{R}^4)$ and satisfies $\|\eta_\alpha\|_{L^\infty} \leq \frac{C}{\sqrt{\alpha}}$ for some $C > 0$, we get

$$\int_{|x|>1} \left(e^{\frac{|r_\alpha(x)|^2}{\lambda^2}} - 1\right) dx \underset{\alpha\to\infty}{\longrightarrow} 0\,.$$

Let us observe that $h_\alpha(x) := \sqrt{\frac{\alpha}{8\pi^2}} L\Big(-\frac{\log|x|}{\alpha}\Big)$ does not belong to $H^2(\mathbb{R}^4)$. To overcome this difficulty, we shall convolate the profile L with an approximation to the identity ρ_n where $\rho_n(s) = \alpha_n \rho(\alpha_n s)$ with ρ is a positive smooth compactly supported function satisfying

$$\operatorname{supp}\rho \subset [-1,1] \quad \text{and} \tag{II.7}$$

$$\int_{-1}^{1} \rho(s)\, ds = 1\,. \tag{II.8}$$

More precisely, we shall prove that the lack of compactness can be described in terms of an asymptotic decomposition as follows :

Theorem II.10 *Let $(u_n)_n$ be a bounded sequence in $H^2_{rad}(\mathbb{R}^4)$ such that*

$$u_n \underset{n\to\infty}{\rightharpoonup} 0\,, \tag{II.9}$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0\,, \quad \textit{and} \tag{II.10}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty} \int_{|x|>R} |u_n(x)|^2\, dx = 0\,. \tag{II.11}$$

Then, there exists a sequence $(\alpha^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\psi^{(j)})$ in $\mathcal{P}$ such that up to a subsequence extraction, we have for all $\ell \geq 1$

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}} \Big(\psi^{(j)} * \rho_n^{(j)}\Big)\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + r_n^{(\ell)}(x)\,, \tag{II.12}$$

where $\rho_n^{(j)}(s) = \alpha_n^{(j)} \rho(\alpha_n^{(j)} s)$ and $\limsup_{n\to\infty} \left\| r_n^{(\ell)} \right\|_{\mathcal{L}} \overset{\ell\to\infty}{\longrightarrow} 0$.

Remarks II.11

– *As in [49], the decomposition* (II.12) *is not unique.*

– *The assumption* (II.11) *means that there is no lack of compactness at infinity. We are particularly satisfied when the sequence* (u_n) *is supported in a fixed compact of* $\mathbb{R}^4$ *and also by the sequences*

$$g_n^{(j)}(x) := \sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\left(\psi^{(j)} * \rho_n^{(j)}\right)\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) \tag{II.13}$$

involved in the decomposition (II.12).

– *As it is mentioned above, the functions* $h_n^{(j)}(x) := \sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)$ *do not belong to* $H^2(\mathbb{R}^4)$. *However, we have*

$$\left\|g_n^{(j)} - h_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^4)} \underset{n\to\infty}{\longrightarrow} 0\,, \tag{II.14}$$

where the functions $g_n^{(j)}$ *are defined by* (II.13). *Indeed, by the change of variable* $s = -\frac{\log|x|}{\alpha_n^{(j)}}$ *and using the fact that, for any integer number* j, $\psi^{(j)} * \rho_n^{(j)}$ *is supported in* $[-\frac{1}{\alpha_n^{(j)}}, \infty[$ *and* $\psi^{(j)}$ *is supported in* $[0,\infty[$, *we infer that for all* $\lambda > 0$

$$\int_{\mathbb{R}^4}\left(\mathrm{e}^{\left|\frac{g_n^{(j)}(x)-h_n^{(j)}(x)}{\lambda}\right|^2} - 1\right)dx$$
$$= 2\pi^2\alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{\infty}\left(\mathrm{e}^{\frac{\alpha_n^{(j)}}{8\pi^2\lambda^2}\left|\left(\psi^{(j)}*\rho_n^{(j)}\right)(s)-\psi^{(j)}(s)\right|^2} - 1\right)\mathrm{e}^{-4\alpha_n^{(j)}s}\,ds\,.$$

Since

$$\left|\left(\psi^{(j)} * \rho_n^{(j)}\right)(s) - \psi^{(j)}(s)\right| \leq \int_{-1}^{1}\left|\psi^{(j)}\left(s - \frac{t}{\alpha_n^{(j)}}\right) - \psi^{(j)}(s)\right|\rho(t)\,dt\,,$$

we obtain, according to Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\left(\psi^{(j)} * \rho_n^{(j)}\right)(s) - \psi^{(j)}(s)\right|^2 &\lesssim \int_{-1}^{1}\left|\psi^{(j)}\left(s - \frac{t}{\alpha_n^{(j)}}\right) - \psi^{(j)}(s)\right|^2 dt\\ &\lesssim \alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{\frac{1}{\alpha_n^{(j)}}}\left|\psi^{(j)}(s-\tau) - \psi^{(j)}(s)\right|^2 d\tau\\ &\lesssim \alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{\frac{1}{\alpha_n^{(j)}}}\left(\int_{s-\tau}^{s}\left|\left(\psi^{(j)}\right)'(u)\right|du\right)^2 d\tau\,.\end{aligned}$$

Applying again Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\left(\psi^{(j)} * \rho_n^{(j)}\right)(s) - \psi^{(j)}(s)\right|^2 &\lesssim \alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{\frac{1}{\alpha_n^{(j)}}}\left(\int_{s-\tau}^{s}\left|\left(\psi^{(j)}\right)'(u)\right|^2 du\right)|\tau|\,d\tau\\ &\lesssim \frac{1}{\alpha_n^{(j)}}\sup_{|\tau|\leq\frac{1}{\alpha_n^{(j)}}}\int_{s-\tau}^{s}\left|\left(\psi^{(j)}\right)'(u)\right|^2 du\,.\end{aligned}$$

Then, there exists a positive constant C *such that*

$$\begin{aligned}
\int_{\mathbb{R}^4}\left(\mathrm{e}^{\left|\frac{g_n^{(j)}(x)-h_n^{(j)}(x)}{\lambda}\right|^2}-1\right)dx \\
&\lesssim \alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{\infty}\left(\mathrm{e}^{\frac{C}{\lambda^2}\sup\limits_{|\tau|\leq\frac{1}{\alpha_n^{(j)}}}\int_{s-\tau}^{s}|(\psi^{(j)})'(u)|^2\,du}-1\right)\mathrm{e}^{-4\alpha_n^{(j)}s}\,ds\\
&\lesssim I_n+J_n\,,
\end{aligned}$$

where

$$\begin{aligned}
I_n &= \alpha_n^{(j)}\int_{s_0}^{\infty}\left(\mathrm{e}^{\frac{C}{\lambda^2}\sup\limits_{s\in[s_0,\infty[,|\tau|\leq\frac{1}{\alpha_n^{(j)}}}\int_{s-\tau}^{s}|(\psi^{(j)})'(u)|^2\,du}-1\right)\mathrm{e}^{-4\alpha_n^{(j)}s}\,ds \quad \text{and}\\
J_n &= \alpha_n^{(j)}\int_{-\frac{1}{\alpha_n^{(j)}}}^{s_0}\left(\mathrm{e}^{\frac{C}{\lambda^2}\sup\limits_{s\in[-\frac{1}{\alpha_n^{(j)}},s_0],|\tau|\leq\frac{1}{\alpha_n^{(j)}}}\int_{s-\tau}^{s}|(\psi^{(j)})'(u)|^2\,du}-1\right)\mathrm{e}^{-4\alpha_n^{(j)}s}\,ds\,,
\end{aligned}$$

for some positive real s_0.
Noticing that

$$I_n\lesssim\left(\mathrm{e}^{\frac{C\left\|\left(\psi^{(j)}\right)'\right\|^2_{L^2(\mathbb{R})}}{\lambda^2}}-1\right)\frac{\mathrm{e}^{-4\alpha_n^{(j)}s_0}}{4}\,,$$

we infer that

$$\lim_{n\to\infty}I_n=0\,.$$

Moreover, the fact that

$$C_n:=C\sup_{s\in[-\frac{1}{\alpha_n^{(j)}},s_0],|\tau|\leq\frac{1}{\alpha_n^{(j)}}}\int_{s-\tau}^{s}\left|\left(\psi^{(j)}\right)'(u)\right|^2du\underset{n\to\infty}{\longrightarrow}0\,,$$

implies that

$$\lim_{n\to\infty}J_n=\lim_{n\to\infty}\left(\mathrm{e}^{\frac{C_n}{\lambda^2}}-1\right)\frac{\mathrm{e}^4-\mathrm{e}^{-4\alpha_n^{(j)}s_0}}{4}=0\,.$$

This leads to (II.14) *as desired.*

– *Similarly to the proof of Proposition 1.15 in [24], we get by using* (II.14)

$$\lim_{n\to\infty}\left\|g_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^4)}=\lim_{n\to\infty}\left\|h_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^4)}=\frac{1}{\sqrt{32\pi^2}}\max_{s>0}\frac{\left|\psi^{(j)}(s)\right|}{\sqrt{s}}\,.$$

e) *Setting* $\tilde{g}_n(x):=\sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\left(\psi^{(j)}*\tilde{\rho}_n^{(j)}\right)\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)$, *where* $\tilde{\rho}_n^{(j)}(s)=\alpha_n^{(j)}\tilde{\rho}\left(\alpha_n^{(j)}s\right)$ *with* $\tilde{\rho}$ *is a positive smooth compactly supported function satisfying* (II.7) *and* (II.8), *we notice that*

$$\left\|g_n^{(j)}-\tilde{g}_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^4)}\underset{n\to\infty}{\longrightarrow}0\,,\tag{II.15}$$

where the functions $g_n^{(j)}$ *are defined by* (II.13). *To prove* (II.15), *we apply the same lines of reasoning of the proof of* (II.14).

– *Compared with the decomposition in [49], it can be seen that there's no core in* (II.12). *This is justified by the radial setting.*

Theorem II.10 induces to

$$\|u_n\|_{\mathcal{L}} \to \sup_{j\geq 1}\left(\lim_{n\to\infty}\left\|g_n^{(j)}\right\|_{\mathcal{L}}\right).$$

This is due to the following proposition proved in [24].

Proposition II.12 *Let $(\alpha^{(j)})_{1\leq j\leq \ell}$ be a family of pairwise orthogonal scales and $(\psi^{(j)})_{1\leq j\leq \ell}$ be a family of profiles, and set*

$$g_n(x) \quad = \quad \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\left(\psi^{(j)} * \rho_n^{(j)}\right)\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) := \sum_{j=1}^{\ell} g_n^{(j)}(x)\,.$$

Then

$$\|g_n\|_{\mathcal{L}} \to \sup_{1\leq j\leq \ell}\left(\lim_{n\to\infty}\left\|g_n^{(j)}\right\|_{\mathcal{L}}\right).$$

II.1.5 Structure of the paper

The paper is organized as follows : Section 2 is devoted to the proof of Theorem II.10 by describing the algorithm construction of the decomposition of a bounded sequence (u_n) in $H^2_{rad}(\mathbb{R}^4)$, up a subsequence extraction, in terms of orthogonal profiles. In the last section, we deal with several complements for the sake of completeness.
We mention that C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. For simplicity, we shall also still denote by (u_n) any subsequence of (u_n).

II.2 Proof of the main theorem

II.2.1 Scheme of the proof

The first step of the proof is based on the extraction of the first scale and the first profile. As in [24], the heart of the matter is reduced to the proof of the following lemma :

Lemma II.13 *Let (u_n) be a sequence in $H^2_{rad}(\mathbb{R}^4)$ satisfying the assumptions of Theorem II.10. Then there exists a scale (α_n) and a profile ψ such that*

$$\|\psi'\|_{L^2(\mathbb{R})} \geq CA_0\,, \tag{II.16}$$

where C is a universal constant.

Then, the problem will be reduced to the study of the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence start which allows us to apply the lines of reasoning of the first step and extract a second scale and a second profile which verify the above key property (II.16). By contradiction arguments, we get the property of orthogonality between the two first scales. Finally, we prove that this process converges.

II.2.2 Preliminaries

To describe the lack of compactness of the Sobolev space $H^2_{rad}(\mathbb{R}^4)$ into the Orlicz space $\mathcal{L}(\mathbb{R}^4)$, we will make firstly the change of variable $s := -\log r$ with $r = |x|$ and associate to any radial function u on $\mathbb{R}^4$ a one space variable function v defined by $v(s) = u(\mathrm{e}^{-s})$. It follows that :

$$\|u\|^2_{L^2(\mathbb{R}^4)} = 2\pi^2 \int_{\mathbb{R}} \mathrm{e}^{-4s}|v(s)|^2\, ds\,, \tag{II.17}$$

$$\left\|\frac{\partial u}{\partial r}\right\|^2_{L^2(\mathbb{R}^4)} = 2\pi^2 \int_{\mathbb{R}} \mathrm{e}^{-2s}|v'(s)|^2\, ds\,, \tag{II.18}$$

$$\left\|\frac{1}{r}\partial_r u\right\|^2_{L^2(\mathbb{R}^4)} = 2\pi^2 \int_{\mathbb{R}} |v'(s)|^2\, ds \quad \textit{and} \tag{II.19}$$

$$\|\Delta u\|^2_{L^2(\mathbb{R}^4)} = 2\pi^2 \int_{\mathbb{R}} |-2v'(s) + v''(s)|^2\, ds\,. \tag{II.20}$$

The quantity (II.19) will play a fondamental role in our main result. Moreover, for a scale (α_n) and a profile ψ we define

$$g_n(x) := \sqrt{\frac{\alpha_n}{8\pi^2}}\,(\psi * \rho_n)\left(\frac{-\log|x|}{\alpha_n}\right)\,,$$

where $\rho_n(s) = \alpha_n\rho(\alpha_n s)$ with ρ is a positive smooth compactly supported function satisfying (II.7) and (II.8). Straightforward computations show that

$$\|g_n\|_{L^2(\mathbb{R}^4)} \lesssim \alpha_n\left(\int_0^\infty |\psi(s)|^2\mathrm{e}^{-4\alpha_n s}\, ds\right)^{\frac{1}{2}}\,, \tag{II.21}$$

$$\left\|\frac{\partial g_n}{\partial r}\right\|_{L^2(\mathbb{R}^4)} \lesssim \left(\int_{\mathbb{R}} |\psi'(s)|^2\mathrm{e}^{-2\alpha_n s}\, ds\right)^{\frac{1}{2}}\,, \tag{II.22}$$

$$\left\|\frac{1}{r}\partial_r g_n\right\|_{L^2(\mathbb{R}^4)} \lesssim \|\psi'\|_{L^2(\mathbb{R})} \quad \textit{and} \tag{II.23}$$

$$\|\Delta g_n\|_{L^2(\mathbb{R}^4)} \lesssim \|\psi'\|_{L^2(\mathbb{R})}\,. \tag{II.24}$$

Indeed, we have

$$\begin{aligned}\|g_n\|_{L^2(\mathbb{R}^4)} &= \frac{\alpha_n}{2}\left(\int_{\mathbb{R}} |(\psi * \rho_n)(s)|^2\mathrm{e}^{-4\alpha_n s}\, ds\right)^{\frac{1}{2}}\\ &= \left\|\tilde{\psi}_n * \tilde{\rho}_n\right\|_{L^2(\mathbb{R})}\,,\end{aligned}$$

where $\tilde{\psi}_n(\tau) = \frac{\alpha_n}{2}\psi(\tau)\mathrm{e}^{-2\alpha_n\tau}$ and $\tilde{\rho}_n(\tau) = \rho_n(\tau)\mathrm{e}^{-2\alpha_n\tau}$. According to Young's inequality, we get

$$\|g_n\|_{L^2(\mathbb{R}^4)} \leq \left\|\tilde{\psi}_n\right\|_{L^2(\mathbb{R})}\|\tilde{\rho}_n\|_{L^1(\mathbb{R})}\,.$$

Since $\left\|\tilde{\psi}_n\right\|_{L^2(\mathbb{R})} = \frac{\alpha_n}{2}\left(\int_0^\infty |\psi(\tau)|^2\mathrm{e}^{-4\alpha_n\tau}\, d\tau\right)^{\frac{1}{2}}$ and $\|\tilde{\rho}_n\|_{L^1(\mathbb{R})} = \int_{-1}^1 \rho(\tau)\mathrm{e}^{-2\tau}\, d\tau$, we obtain (II.21).

Similarly, writing

$$\begin{aligned}\left\|\frac{\partial g_n}{\partial r}\right\|_{L^2(\mathbb{R}^4)} &= \frac{1}{2}\left(\int_{\mathbb{R}} |(\psi' * \rho_n)(s)|^2 \mathrm{e}^{-2\alpha_n s}\, ds\right)^{\frac{1}{2}} \\ &= \left\|\tilde{\tilde{\psi}}_n * \tilde{\tilde{\rho}}_n\right\|_{L^2(\mathbb{R})},\end{aligned}$$

where $\tilde{\tilde{\psi}}_n(\tau) = \frac{1}{2}\psi'(\tau)\mathrm{e}^{-\alpha_n \tau}$ and $\tilde{\tilde{\rho}}_n(\tau) = \rho_n(\tau)\mathrm{e}^{-\alpha_n \tau}$ and using Young's inequality, we infer that

$$\begin{aligned}\left\|\frac{\partial g_n}{\partial r}\right\|_{L^2(\mathbb{R}^4)} &\leq \left\|\tilde{\tilde{\psi}}\right\|_{L^2(\mathbb{R})}\left\|\tilde{\tilde{\rho}}\right\|_{L^1(\mathbb{R})} \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}} |\psi'(\tau)|^2 \mathrm{e}^{-2\alpha_n \tau}\, d\tau\right)^{\frac{1}{2}} \int_{-1}^{1} \rho(\tau)\mathrm{e}^{-\tau}\, d\tau\,,\end{aligned}$$

which leads to (II.22).
Also, we have

$$\left\|\frac{1}{r}\partial_r g_n\right\|_{L^2(\mathbb{R}^4)} = \frac{1}{2}\|\psi' * \rho_n\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\|\psi'\|_{L^2(\mathbb{R})}\,.$$

Finally,

$$\begin{aligned}\|\Delta g_n\|_{L^2(\mathbb{R}^4)} &= \frac{1}{2}\left(\int_{\mathbb{R}} \left| -2(\psi' * \rho_n)(s) + \frac{1}{\alpha_n}(\psi' * \rho_n')(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\leq \|\psi' * \rho_n\|_{L^2(\mathbb{R})} + \frac{1}{2\alpha_n}\|\psi' * \rho_n'\|_{L^2(\mathbb{R})} \\ &\leq \|\psi'\|_{L^2(\mathbb{R})} + \frac{1}{2\alpha_n}\|\psi'\|_{L^2(\mathbb{R})}\|\rho_n'\|_{L^1(\mathbb{R})}\,.\end{aligned}$$

The fact that $\|\rho_n'\|_{L^1(\mathbb{R})} = \alpha_n \int_{-1}^{1} \rho'(\tau)\, d\tau$ ensures (II.24).

II.2.3 Extraction of the first scale and the first profile

Let us consider a bounded sequence (u_n) in $H^2_{rad}(\mathbb{R}^4)$ satisfying the assumptions (II.9), (II.10) and (II.11) and let us set

$$v_n(s) := u_n(e^{-s}).$$

We have the following lemma.

Lemma II.14 *Under the above assumptions, the sequence (u_n) converges strongly to 0 in $L^2(\mathbb{R}^4)$. Moreover, for any real number M, we have*

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(]-\infty, M[)} = 0\,. \tag{II.25}$$

Proof. For any $R > 0$, we have

$$\|u_n\|_{L^2(\mathbb{R}^4)} = \|u_n\|_{L^2(|x|<R)} + \|u_n\|_{L^2(|x|>R)}\,.$$

According to Rellich's theorem, the Sobolev space $H^2(|x| < R)$ is compactly embedded in $L^2(|x| < R)$. Thanks to (II.9), we get

$$\lim_{n\to\infty} \|u_n\|_{L^2(|x|<R)} = 0\,.$$

Now, taking advantage of the compactness at infinity of the sequence (u_n) given by (II.11), we deduce that

$$\lim_{n\to\infty} \|u_n\|_{L^2(\mathbb{R}^4)} = 0\,. \tag{II.26}$$

Besides, according to Proposition II.23, we infer that

$$|v_n(s)| \lesssim \mathrm{e}^{\frac{3}{2}s} \|u_n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^4)} \|\nabla u_n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^4)}\,. \tag{II.27}$$

For $s < M$, (II.25) derives immediately from (II.27) and the strong convergence of (u_n) to zero in $L^2(\mathbb{R}^4)$. ∎

Now, we shall determine the first scale and the first profile.

Proposition II.15 *For all* $0 < \delta < A_0$*, we have*

$$\sup_{s\geq 0}\left(\left|\frac{v_n(s)}{A_0-\delta}\right|^2 - 3s\right) \xrightarrow[n\to\infty]{} \infty\,.$$

Proof. To go to the proof of Proposition II.15, we shall proceed by contradiction by assuming that there exists a positive real δ such that, up to a subsequence extraction,

$$\sup_{s\geq 0, n\in\mathbb{N}}\left(\left|\frac{v_n(s)}{A_0-\delta}\right|^2 - 3s\right) \leq C\,, \tag{II.28}$$

where C is a positive constant. Thanks to (II.25) and (II.28), we get by virtue of Lebesgue's theorem

$$\lim_{n\to\infty}\int_{|x|<1}\left(\mathrm{e}^{\left|\frac{u_n(x)}{A_0-\delta}\right|^2} - 1\right) dx = \lim_{n\to\infty} 2\pi^2 \int_0^\infty \left(\mathrm{e}^{\left|\frac{v_n(s)}{A_0-\delta}\right|^2} - 1\right)\mathrm{e}^{-4s}\, ds = 0\,.$$

On the other hand, using Proposition II.23, the boundedness of (u_n) in $H^2(\mathbb{R}^4)$ ensures the existence of a positive constant C such that

$$|u_n(x)| \leq C, \quad \forall\, n \in \mathbb{N} \;\text{ and } |x| \geq 1\,.$$

By virtue of the fact that for any positive M there exists a finite constant C_M such that

$$\sup_{|t|\leq M}\left(\frac{\mathrm{e}^{t^2}-1}{t^2}\right) < C_M\,,$$

we obtain that

$$\int_{|x|\geq 1}\left(\mathrm{e}^{\left|\frac{u_n(x)}{A_0-\delta}\right|^2} - 1\right) dx \leq C\|u_n\|^2_{L^2(\mathbb{R}^4)}\,.$$

The strong convergence of (u_n) to 0 in $L^2(\mathbb{R}^4)$ leads to

$$\int_{\mathbb{R}^4} \left(\mathrm{e}^{\left|\frac{u_n(x)}{A_0-\delta}\right|^2} - 1 \right) dx \underset{n\to\infty}{\longrightarrow} 0\,.$$

Thus,

$$\lim_{n\to\infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \delta\,,$$

which is in contradiction with Hypothesis (II.10). ∎

Corollary II.16 *There exists a scale* $\left(\alpha_n^{(1)}\right)$ *such that*

$$4\left|\frac{v_n\left(\alpha_n^{(1)}\right)}{A_0}\right|^2 - 3\,\alpha_n^{(1)} \underset{n\to\infty}{\longrightarrow} \infty\,.$$

Proof. Let us set

$$W_n(s) := 4\left|\frac{v_n(s)}{A_0}\right|^2 - 3s \quad \textit{and} \quad a_n := \sup_{s\geq 0} W_n(s)\,.$$

Then, there exists a positive sequence $\left(\alpha_n^{(1)}\right)$ such that

$$W_n\left(\alpha_n^{(1)}\right) \geq a_n - \frac{1}{n}\,\cdot$$

According to Proposition II.15, a_n tends to infinity and then

$$W_n\left(\alpha_n^{(1)}\right) \underset{n\to\infty}{\longrightarrow} \infty\,.$$

It remains to show that $\alpha_n^{(1)} \underset{n\to\infty}{\longrightarrow} \infty$. If not, up to a subsequence extraction, the sequence $\left(\alpha_n^{(1)}\right)$ is bounded in $\mathbb{R}$ and so is $\left(W_n\left(\alpha_n^{(1)}\right)\right)$ thanks to (II.25). This yields a contradiction. ∎

Corollary II.17 *Under the above assumptions, we have for n big enough,*

$$\frac{\sqrt{3}}{2} A_0 \sqrt{\alpha_n^{(1)}} \leq \left|v_n\left(\alpha_n^{(1)}\right)\right| \leq C\sqrt{\alpha_n^{(1)}} + \circ(1)\,,$$

where $C = \dfrac{1}{\sqrt{8\pi^2}} \limsup_{n\to\infty} \|\Delta u_n\|_{L^2(\mathbb{R}^4)}\,.$

Proof. The left hand side inequality follows directly from Corollary II.16. On the other hand, for any $s \geq 0$ and according to Cauchy-Schwarz inequality, we obtain that

$$|v_n(s)| = \left|v_n(0) + \int_0^s v_n'(\tau)\, d\tau\right| \leq |v_n(0)| + \sqrt{s}\|v_n'\|_{L^2(\mathbb{R})}\,.$$

By virtue of (II.19) and Lemma II.22, we get

$$\|v_n'\|_{L^2(\mathbb{R})} = \Big(\int_0^\infty \Big|\frac{1}{r}u_n'(r)\Big|^2 r^3\, dr\Big)^{\frac12} \leq \frac{1}{\sqrt{8\pi^2}}\|\Delta u_n\|_{L^2(\mathbb{R}^4)}\,.$$

Using the boundedness of the sequence (Δu_n) in $L^2(\mathbb{R}^4)$ and the convergence of $\Big(v_n(0)\Big)$ to zero, we infer that

$$|v_n(s)| \leq o(1) + C\sqrt{s}\,,$$

where $C = \frac{1}{\sqrt{8\pi^2}}\limsup_{n\to\infty}\|\Delta u_n\|_{L^2(\mathbb{R}^4)}$, which ensures the right hand side inequality. ∎

Now we are able to extract the first profile. To do so, let us set

$$\psi_n(y) := \sqrt{\frac{8\pi^2}{\alpha_n^{(1)}}}\, v_n\Big(\alpha_n^{(1)}y\Big)\,.$$

The following lemma summarizes the principle properties of ψ_n.

Lemma II.18 *Under the same assumptions, we have*

$$\sqrt{6\pi^2}A_0 \leq |\psi_n(1)| \leq C + \circ(1)\,, \tag{II.29}$$

where $C = \limsup_{n\to\infty}\|\Delta u_n\|_{L^2(\mathbb{R}^4)}$. *Moreover, there exists a profile* $\psi^{(1)}$ *such that, up to a subsequence extraction,*

$$\psi_n' \underset{n\to\infty}{\rightharpoonup} (\psi^{(1)})' \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \Big\|(\psi^{(1)})'\Big\|_{L^2(\mathbb{R})} \geq \sqrt{6\pi^2}A_0\,.$$

Proof. According to Corollary II.17, we get (II.29). Besides, thanks to (II.19) and Lemma II.22 we obtain that

$$\|\psi_n'\|_{L^2(\mathbb{R})} = \sqrt{8\pi^2}\Big(\int_0^\infty \Big|\frac{1}{r}u_n'(r)\Big|^2 r^3\, dr\Big)^{\frac12} \leq \|\Delta u_n\|_{L^2(\mathbb{R}^4)}\,.$$

Then, (ψ_n') is bounded in $L^2(\mathbb{R})$. Consequently, up to a subsequence extraction, (ψ_n') converges weakly in $L^2(\mathbb{R})$ to some function $g \in L^2(\mathbb{R})$. Let us introduce the function

$$\psi^{(1)}(s) := \int_0^s g(\tau)\, d\tau\,.$$

It's obvious that, up a subsequence extraction, $\psi_n' \rightharpoonup (\psi^{(1)})'$ in $L^2(\mathbb{R})$. It remains to prove that $\psi^{(1)}$ is a profile.

Firstly, since

$$\Big|\psi^{(1)}(s)\Big| = \Big|\int_0^s g(\tau)\, d\tau\Big| \leq \sqrt{s}\|g\|_{L^2(\mathbb{R})}\,,$$

we get $\psi^{(1)} \in L^2(\mathbb{R}_+, \mathrm{e}^{-4s}ds)$.

Secondly, $\psi^{(1)}(s) = 0$ for all $s \leq 0$. Indeed, using the fact that

$$\|u_n\|^2_{L^2(\mathbb{R}^4)} = \frac{\Big(\alpha_n^{(1)}\Big)^2}{4}\int_{\mathbb{R}} |\psi_n(s)|^2 \mathrm{e}^{-4\alpha_n^{(1)}s}\, ds\,,$$

we obtain that

$$\int_{-\infty}^{0} |\psi_n(s)|^2 \, ds \leq \int_{-\infty}^{0} |\psi_n(s)|^2 \mathrm{e}^{-4\alpha_n^{(1)} s} \, ds \leq \frac{4}{\left(\alpha_n^{(1)}\right)^2} \|u_n\|^2_{L^2(\mathbb{R}^4)} \, .$$

By virtue of the boundedness of (u_n) in $L^2(\mathbb{R}^4)$, we deduce that ψ_n converges strongly to zero in $L^2(]-\infty, 0[)$. Consequently, for almost all $s \leq 0$, up to a subsequence extraction, $\Big(\psi_n(s)\Big)$ goes to zero. In other respects, as (ψ_n') converges weakly to g in $L^2(\mathbb{R})$ and ψ_n belongs to $H^1_{loc}(\mathbb{R})$, we infer that

$$\psi_n(s) - \psi_n(0) = \int_0^s \psi_n'(\tau) \, d\tau \underset{n\to\infty}{\longrightarrow} \int_0^s g(\tau) \, d\tau = \psi^{(1)}(s) \, .$$

This gives rise to the fact that

$$\psi_n(s) \underset{n\to\infty}{\longrightarrow} \psi^{(1)}(s) \, , \quad \forall \, s \in \mathbb{R} \, , \tag{II.30}$$

and ensures that $\psi^{(1)}{}_{|]-\infty,0]} = 0$.
Finally, knowing that $\left|\psi^{(1)}(1)\right| \geq \sqrt{6\pi^2} A_0$ and

$$\left\|(\psi^{(1)})'\right\|_{L^2(\mathbb{R})} \geq \int_0^1 \left|(\psi^{(1)})'(\tau)\right| \, d\tau = \left|\psi^{(1)}(1)\right| ,$$

we deduce that $\left\|(\psi^{(1)})'\right\|_{L^2(\mathbb{R})} \geq \sqrt{6\pi^2} A_0$. ■

Let us now consider the first remainder term :

$$r_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x) \, , \tag{II.31}$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{8\pi^2}} \Big(\psi^{(1)} * \rho_n^{(1)}\Big) \left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)$$

with $\rho_n^{(1)}(s) = \Big(\alpha_n^{(1)}\Big) \rho\Big(\alpha_n^{(1)} s\Big)$. Recalling that $u_n(x) = \sqrt{\frac{\alpha_n^{(1)}}{8\pi^2}} \psi_n \left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)$ and taking advantage of the fact that (ψ_n') converges weakly in $L^2(\mathbb{R})$ to $(\psi^{(1)})'$, we get the following result.

Proposition II.19 *Let $(u_n)_n$ be a sequence in $H^2_{rad}(\mathbb{R}^4)$ satisfying the assumptions of Theorem II.10. Then, there exist a scale $\Big(\alpha_n^{(1)}\Big)$ and a profile $\psi^{(1)}$ such that*

$$\left\|(\psi^{(1)})'\right\|_{L^2(\mathbb{R})} \geq \sqrt{6\pi^2} A_0 \, . \tag{II.32}$$

In addition, we have

$$\lim_{n\to\infty} \left\|\frac{1}{r}\partial_r r_n^{(1)}\right\|^2_{L^2(\mathbb{R}^4)} = \lim_{n\to\infty} \left\|\frac{1}{r}\partial_r u_n\right\|^2_{L^2(\mathbb{R}^4)} - \frac{1}{4}\left\|(\psi^{(1)})'\right\|^2_{L^2(\mathbb{R})} , \tag{II.33}$$

where $r_n^{(1)}$ is given by (II.31).

Proof. The inequality (II.32) is contained in Lemma II.18. Besides, noticing that

$$\Big\|\frac{1}{r}\partial_r r_n^{(1)}\Big\|_{L^2(\mathbb{R}^4)} = \frac{1}{2}\Big\|\psi_n' - \Big((\psi^{(1)})' * \rho_n^{(1)}\Big)\Big\|_{L^2(\mathbb{R})},$$

we get

$$\begin{aligned}
\lim_{n\to\infty}\Big\|\frac{1}{r}\partial_r r_n^{(1)}\Big\|^2_{L^2(\mathbb{R}^4)} &= \frac{1}{4}\lim_{n\to\infty}\|\psi_n'\|^2_{L^2(\mathbb{R})} + \frac{1}{4}\lim_{n\to\infty}\Big\|(\psi^{(1)})' * \rho_n^{(1)}\Big\|^2_{L^2(\mathbb{R})}\\
&- \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}}\psi_n'(s)\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s)\,ds\\
&= \lim_{n\to\infty}\Big\|\frac{1}{r}\partial_r u_n\Big\|^2_{L^2(\mathbb{R}^4)} + \frac{1}{4}\Big\|(\psi^{(1)})'\Big\|^2_{L^2(\mathbb{R})}\\
&- \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}}\psi_n'(s)\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s)\,ds\,.
\end{aligned}$$

We write

$$\begin{aligned}
\int_{\mathbb{R}}\psi_n'(s)\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s)\,ds &= \int_{\mathbb{R}}\psi_n'(s)\Big[\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s) - (\psi^{(1)})'(s)\Big]\,ds\\
&+ \int_{\mathbb{R}}\psi_n'(s)(\psi^{(1)})'(s)\,ds\,.
\end{aligned}$$

Since (ψ_n') converges weakly in $L^2(\mathbb{R})$ to $(\psi^{(1)})'$, we obtain that

$$\int_{\mathbb{R}}\psi_n'(s)(\psi^{(1)})'(s)\,ds \underset{n\to\infty}{\longrightarrow} \Big\|(\psi^{(1)})'\Big\|^2_{L^2(\mathbb{R})}. \tag{II.34}$$

Besides, according to Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\psi_n'(s)\Big[\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s) - (\psi^{(1)})'(s)\Big]\,ds\Big|\\
&\quad\leq \|\psi_n'\|_{L^2(\mathbb{R})}\Big\|\Big((\psi^{(1)})' * \rho_n^{(1)}\Big) - (\psi^{(1)})'\Big\|_{L^2(\mathbb{R})}\\
&\quad\leq 4\Big\|\frac{1}{r}\partial_r u_n\Big\|_{L^2(\mathbb{R}^4)}\Big\|\Big((\psi^{(1)})' * \rho_n^{(1)}\Big) - (\psi^{(1)})'\Big\|_{L^2(\mathbb{R})}.
\end{aligned}$$

The boundedness of $(\frac{1}{r}\partial_r u_n)$ in $L^2(\mathbb{R}^4)$ and the strong convergence of $\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)$ to $(\psi^{(1)})'$ in $L^2(\mathbb{R})$ imply that

$$\int_{\mathbb{R}}\psi_n'(s)\Big[\Big((\psi^{(1)})' * \rho_n^{(1)}\Big)(s) - (\psi^{(1)})'(s)\Big]\,ds \underset{n\to\infty}{\longrightarrow} 0\,. \tag{II.35}$$

Taking advantage of (II.34) and (II.35), we deduce (II.33). ∎

II.2.4 Conclusion

Our concern now is to iterate the previous process and to prove that the algorithmic construction converges. Thanks to the fact that $\left(\psi^{(1)} * \rho_n^{(1)}\right)$ is supported in $[-\frac{1}{\alpha_n}, \infty[$, we get for $R > \mathrm{e}$,

$$\begin{aligned}\left\|r_n^{(1)}\right\|^2_{L^2(|x|>R)} &= \frac{1}{4}\left(\alpha_n^{(1)}\right)^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n^{(1)}}} \left|\psi_n(t) - \left(\psi^{(1)} * \rho_n^{(1)}\right)(t)\right|^2 \mathrm{e}^{-4\alpha_n^{(1)} t}\, dt \\ &= \frac{1}{4}\left(\alpha_n^{(1)}\right)^2 \int_{-\infty}^{-\frac{\log R}{\alpha_n^{(1)}}} |\psi_n(t)|^2 \mathrm{e}^{-4\alpha_n^{(1)} t}\, dt \\ &= \|u_n\|^2_{L^2(|x|>R)}\,.\end{aligned}$$

This implies that $\left(r_n^{(1)}\right)$ satisfies the hypothesis of compactness (II.11). According to (II.33) and the inequalities (II.21), (II.22) and (II.23), we deduce that $\left(r_n^{(1)}\right)$ satisfies also (II.9).
Let us now define $A_1 = \limsup_{n\to\infty} \left\|r_n^{(1)}\right\|_{\mathcal{L}}$. If $A_1 = 0$, we stop the process. If not, since the sequence $\left(r_n^{(1)}\right)$ satisfies the assumptions of Theorem II.10, there exists a scale $\left(\alpha_n^{(2)}\right)$ satisfying the statement of Corollary II.16 with A_1 instead of A_0. In particular, there exists a constant C such that

$$\frac{\sqrt{3}}{2} A_1 \sqrt{\alpha_n^{(2)}} \leq \left|\tilde{r}_n^{(1)}\left(\alpha_n^{(2)}\right)\right| \leq C\sqrt{\alpha_n^{(2)}} + \circ(1)\,, \tag{II.36}$$

where $\tilde{r}_n^{(1)}(s) = r_n^{(1)}(\mathrm{e}^{-s})$. In addition, the scales $\left(\alpha_n^{(1)}\right)$ and $\left(\alpha_n^{(2)}\right)$ are orthogonal. Otherwise, there exists a constant C such that

$$\frac{1}{C} \leq \left|\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right| \leq C\,.$$

Using (II.31), we get

$$\tilde{r}_n^{(1)}\left(\alpha_n^{(2)}\right) = \sqrt{\frac{\alpha_n^{(1)}}{8\pi^2}}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \left(\psi^{(1)} * \rho_n^{(1)}\right)\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right)\,.$$

For any real number s, we have

$$\left|\psi_n(s) - \left(\psi^{(1)} * \rho_n^{(1)}\right)(s)\right| \leq \left|\psi_n(s) - \psi^{(1)}(s)\right| + \left|\left(\psi^{(1)} * \rho_n^{(1)}\right)(s) - \psi^{(1)}(s)\right|\,.$$

As $\psi^{(1)}$ belongs to the Hölder space $C^{\frac{1}{2}}$, we obtain that

$$\begin{aligned}\left|\left(\psi^{(1)} * \rho_n^{(1)}\right)(s) - \psi^{(1)}(s)\right| &= \left|\int_{-\frac{1}{\alpha_n}}^{\frac{1}{\alpha_n}} \rho_n^{(1)}(t)\left(\psi^{(1)}(s-t) - \psi^{(1)}(s)\right) dt\right| \\ &\lesssim \int_{-\frac{1}{\alpha_n}}^{\frac{1}{\alpha_n}} \rho_n^{(1)}(t)\sqrt{|t|}\, dt \\ &\lesssim \frac{1}{\sqrt{\alpha_n}} \underset{n\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

Thanks to (II.30), we infer that

$$\left|\psi_n(s) - \left(\psi^{(1)} * \rho_n^{(1)}\right)(s)\right| \underset{n\to\infty}{\longrightarrow} 0\,.$$

This gives rise to

$$\lim_{n\to\infty}\sqrt{\frac{8\pi^2}{\alpha_n^{(1)}}}\,\tilde{r}_n^{(1)}\Big(\alpha_n^{(2)}\Big) = \lim_{n\to\infty}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \left(\psi^{(1)} * \rho_n^{(1)}\right)\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right) = 0\,,$$

which is in contradiction with the left hand side inequality of (II.36).
Moreover, there exists a profile $\psi^{(2)}$ such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{8\pi^2}}\Big(\psi^{(2)} * \rho_n^{(2)}\Big)\left(\frac{-\log|x|}{\alpha_n^{(2)}}\right) + r_n^{(2)}(x)\,,$$

where $\rho_n^{(2)}(s) = \alpha_n^{(2)}\rho\Big(\alpha_n^{(2)}s\Big)$. Proceeding as the first step, we obtain that

$$\left\|(\psi^{(2)})'\right\|_{L^2(\mathbb{R})} \geq \sqrt{6\pi^2}A_1$$

and

$$\lim_{n\to\infty}\left\|\frac{1}{r}\partial_r r_n^{(2)}\right\|^2_{L^2(\mathbb{R}^4)} = \lim_{n\to\infty}\left\|\frac{1}{r}\partial_r r_n^{(1)}\right\|^2_{L^2(\mathbb{R}^4)} - \frac{1}{4}\left\|(\psi^{(2)})'\right\|^2_{L^2(\mathbb{R})}\,.$$

Consequently,

$$\lim_{n\to\infty}\left\|\frac{1}{r}\partial_r r_n^{(2)}\right\|^2_{L^2(\mathbb{R}^4)} \leq C - \frac{3\pi^2}{2}A_0^2 - \frac{3\pi^2}{2}A_1^2\,,$$

where $C = \limsup\limits_{n\to\infty}\left\|\frac{1}{r}\partial_r u_n\right\|^2_{L^2(\mathbb{R}^4)}$. At iteration ℓ, we get

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{8\pi^2}}\Big(\psi^{(j)} * \rho_n^{(j)}\Big)\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + r_n^{(\ell)}(x)\,,$$

with

$$\limsup_{\alpha\to\infty}\left\|\frac{1}{r}\partial_r r_n^{(\ell)}\right\|^2_{L^2} \lesssim 1 - A_0^2 - A_1^2 - \ldots - A_{\ell-1}^2\,.$$

Therefore $A_\ell \to 0$ as $\ell \to \infty$ and the proof of the main theorem is achieved.

II.3 Appendix

The first part of this appendix presents the proof of the following proposition concerning the convergence in $H^2(\mathbb{R}^4)$ of the sequence (f_α) defined by (II.6).

Proposition II.20 *We have*

$$\|f_\alpha\|^2_{L^2(\mathbb{R}^4)} = O\Big(\frac{1}{\alpha}\Big), \quad \|\nabla f_\alpha\|^2_{L^2(\mathbb{R}^4)} = O\Big(\frac{1}{\alpha}\Big) \quad \textit{and} \quad \|\Delta f_\alpha\|^2_{L^2(\mathbb{R}^4)} = 1 + O\Big(\frac{1}{\alpha}\Big)\cdot$$

Proof. Let us write

$$\|f_\alpha\|^2_{L^2(\mathbb{R}^4)} = I + II + III\,,$$

with

$$\begin{aligned} I &= \int_{|x|\le \mathrm{e}^{-\alpha}} |f_\alpha(x)|^2\,dx\,,\\ II &= \int_{\mathrm{e}^{-\alpha}<|x|\le 1} |f_\alpha(x)|^2\,dx \quad \text{and}\\ III &= \int_{|x|>1} |f_\alpha(x)|^2\,dx\,. \end{aligned}$$

It is easy to see that for α large enough

$$\begin{aligned} I &\le 2\pi^2 \int_0^{\mathrm{e}^{-\alpha}} r^3 \left(\sqrt{\frac{\alpha}{8\pi^2}} + \frac{1}{\sqrt{32\pi^2\alpha}}\right)^2 dr\\ &\le \left(\frac{\alpha}{8\pi^2} + \frac{1}{32\pi^2\alpha} + \frac{1}{8\pi^2}\right)\frac{\pi^2\mathrm{e}^{-4\alpha}}{2} = O\Big(\frac{1}{\alpha}\Big)\cdot \end{aligned}$$

Besides, by repeated integration by parts, we obtain that

$$\begin{aligned} II &= \frac{1}{4\alpha}\Big(-\frac{\alpha^2\mathrm{e}^{-4\alpha}}{4} - \int_{\mathrm{e}^{-\alpha}}^1 \frac{r^3}{2}\log r\,dr\Big)\\ &= \frac{1}{4\alpha}\Big(-\frac{\alpha^2\mathrm{e}^{-4\alpha}}{4} - \frac{\alpha\mathrm{e}^{-4\alpha}}{8} + \frac{1}{32}\big(1-\mathrm{e}^{-4\alpha}\big)\Big) = \mathrm{O}\Big(\frac{1}{\alpha}\Big)\cdot \end{aligned}$$

The fact that $\eta_\alpha \in \mathcal{D}(\mathbb{R}^4)$ and $\eta_\alpha = O\Big(\frac{1}{\sqrt{\alpha}}\Big)$ implies that $III = O\Big(\frac{1}{\alpha}\Big)\cdot$
Now, noticing that

$$\nabla f_\alpha(x) = \begin{cases} \frac{-2\,x\,\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}} & \text{si } |x| \le \mathrm{e}^{-\alpha}\,,\\[2mm] \frac{-x}{|x|^2\sqrt{8\pi^2\alpha}} & \text{si } \mathrm{e}^{-\alpha} < |x| \le 1\,,\\[2mm] \nabla\eta_\alpha(x) & \text{si } |x| > 1\,, \end{cases}$$

we easily get

$$\|\nabla f_\alpha\|^2_{L^2(\mathbb{R}^4)} = \frac{\mathrm{e}^{-2\alpha}}{24\alpha} + \frac{1-\mathrm{e}^{-2\alpha}}{8\alpha} + \int_{|x|>1} |\nabla\eta_\alpha(x)|^2\,dx\,.$$

This ensures the result knowing that $\eta_\alpha \in \mathcal{D}(\mathbb{R}^4)$ and $\|\nabla\eta_\alpha\|_{L^\infty} = O\Big(\frac{1}{\sqrt{\alpha}}\Big)\cdot$
Finally, since

$$\Delta f_\alpha(x) = \begin{cases} \frac{-8\mathrm{e}^{2\alpha}}{\sqrt{32\pi^2\alpha}} & \text{if } |x| \le \mathrm{e}^{-\alpha}\,,\\[2mm] \frac{-2}{|x|^2\sqrt{8\pi^2\alpha}} & \text{if } \mathrm{e}^{-\alpha} < |x| \le 1\,,\\[2mm] \Delta\eta_\alpha & \text{if } |x| > 1\,, \end{cases}$$

we get

$$\|\Delta f_\alpha\|^2_{L^2(\mathbb{R}^4)}dx = \frac{1}{\alpha} + 1 + \int_{|x|>1} |\Delta\eta_\alpha(x)|^2 \, dx \,,$$

which ends the proof of the last assertion in view of the fact that $\eta_\alpha \in \mathcal{D}(\mathbb{R}^4)$ and $|\Delta\eta_\alpha| = O\Big(\frac{1}{\sqrt{\alpha}}\Big) \cdot$ ■

In the following proposition, we recall the characterization of $H^2_{rad}(\mathbb{R}^4)$ which is useful in this article.

Proposition II.21 *We have*

$$H^2_{rad}(\mathbb{R}^4) = \Big\{u \in L^2(\mathbb{R}_+, r^3\,dr); \quad \partial_r u,\ \partial_r^2 u,\ \frac{1}{r}\partial_r u \in L^2(\mathbb{R}_+, r^3\,dr)\Big\}.$$

The proof of Proposition II.21 is based on the following lemma proved in [87] :

Lemma II.22 *For all $u \in H^2_{rad}(\mathbb{R}^4)$, we have*

$$\Big\|\frac{1}{r}\partial_r u\Big\|_{L^2(\mathbb{R}^4)} := \Big(2\pi^2 \int_0^\infty |u'(r)|^2 r\, dr\Big)^{\frac{1}{2}} \leq \frac{1}{2}\|\Delta u\|_{L^2(\mathbb{R}^4)} \,. \tag{II.37}$$

Proof. By density, it suffices to consider smooth compactly supported functions. Let us then consider $u \in \mathcal{D}_{rad}(\mathbb{R}^4)$. We have

$$\begin{aligned}
\|\Delta u\|^2_{L^2(\mathbb{R}^4)} &= 2\pi^2 \int_0^\infty |u''(r) + \frac{3}{r}u'(r)|^2 r^3\, dr \\
&= 2\pi^2\Big[\int_0^\infty \Big(u''(r) + \frac{1}{r}u'(r)\Big)^2 r^3\, dr + 8\int_0^\infty u'(r)^2 r\, dr \\
&+ 4\int_0^\infty u''(r)u'(r)r^2\, dr\Big] \\
&\geq 2\pi^2\Big(8\int_0^\infty u'(r)^2 r\, dr + 4\int_0^\infty u''(r)u'(r)r^2\, dr\Big).
\end{aligned}$$

By integration by parts, we deduce that

$$\|\Delta u\|^2_{L^2(\mathbb{R}^4)} \geq 8\pi^2 \int_0^\infty u'(r)^2 r\, dr \,,$$

which achieves the proof of (II.37). ■

It will be useful to notice, that in the radial case, we have the following estimate which implies the control of the L^∞-norm far away from the origin.

Proposition II.23 *Let $u \in H^1_{rad}(\mathbb{R}^4)$. For $r = |x| > 0$, we have*

$$|u(x)| \lesssim \frac{1}{r^{\frac{3}{2}}}\|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^4)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^4)} \,, \tag{II.38}$$

Proof. Let $u \in \mathcal{D}_{rad}(\mathbb{R}^4)$ and let us write for $r > 0$,

$$u(r)^2 = -2\int_r^\infty u(s)u'(s)ds = -2\int_r^\infty s^{\frac{3}{2}}u(s)s^{\frac{3}{2}}u'(s)\,\frac{ds}{s^3}\,\cdot$$

According to Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} u(r)^2 &\leq \left(\frac{2}{r^3}\int_r^\infty s^3|u(s)|^2\,ds\right)^{\frac{1}{2}}\left(\frac{2}{r^3}\int_r^\infty s^3|u'(s)|^2\,ds\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\pi^2 r^3}\|u\|_{L^2(\mathbb{R}^4)}\|\nabla u\|_{L^2(\mathbb{R}^4)}\,, \end{aligned}$$

which leads to (II.38) by density arguments. ∎

CHAPITRE III

ON THE LACK OF COMPACTNESS OF A 2D SOBOLEV EMBEDDING

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III.1 Introduction

III.1.1 Critical $2D$ Sobolev embedding

It is well known (see for instance [15]) that $H^1(\mathbb{R}^2)$ is continuously embedded in all Lebesgue spaces $L^q(\mathbb{R}^2)$ for $2 \leq q < \infty$, but not in $L^\infty(\mathbb{R}^2)$. It is also known that (for more details, we refer the reader to [85])

$$H^1(\mathbb{R}^2) \hookrightarrow L^{\phi_p}(\mathbb{R}^2)\,, \quad \forall p \in \mathbb{N}^*\,, \tag{III.1}$$

where $L^{\phi_p}(\mathbb{R}^2)$ denotes the Orlicz space associated to the function

$$\phi_p(s) = \mathrm{e}^{\mathrm{s}^2} - \sum_{\mathrm{k}=0}^{\mathrm{p}-1} \frac{\mathrm{s}^{2\mathrm{k}}}{\mathrm{k}!}\,\cdot \tag{III.2}$$

The embedding (III.1) is a direct consequence of the following sharp Trudinger-Moser type inequalities (see [1, 78, 86, 101]) :

Proposition III.1

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)}\leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1\right) dx := \kappa < \infty\,, \tag{III.3}$$

and states as follows :

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1(\mathbb{R}^2)}\,, \tag{III.4}$$

where the norm $\|.\|_{L^{\phi_p}}$ is given by :

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} = \inf \left\{ \lambda > 0, \int_{\mathbb{R}^2} \phi_p\left(\frac{|u(x)|}{\lambda}\right) dx \leq \kappa \right\}.$$

Note that (III.4) follows from (III.3) and the following obvious inequality

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} \leq \|u\|_{L^{\phi_1}(\mathbb{R}^2)}\,.$$

For our purpose, we shall resort to the following Trudinger-Moser inequality, the proof of which is postponed in the appendix.

Proposition III.2 *Let $\alpha \in [0, 4\pi[$ and p an integer larger than 1. There is a constant $c(\alpha, p)$ such that*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |u(x)|^{2k}}{k!}\right) dx \leq c(\alpha,p) \|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}\,, \tag{III.5}$$

for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$.

III.1.2 Development on the lack of compactness of Sobolev embedding in the Orlicz space in the case $p = 1$

In [24], [25] and [26], H. Bahouri, M. Majdoub and N. Masmoudi characterized the lack of compactness of $H^1(\mathbb{R}^2)$ into the Orlicz space $L^{\phi_1}(\mathbb{R}^2)$. To state their result in a clear way, let us recall some definitions.

Definition III.3 *We shall designate by a scale any sequence (α_n) of positive real numbers going to infinity, a core any sequence (x_n) of points in $\mathbb{R}^2$ and a profile any function ψ belonging to the set*

$$\mathcal{P} := \Big\{ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0 \Big\}.$$

Given two scales (α_n), $(\tilde{\alpha}_n)$, two cores (x_n), $(\tilde{x}_n)$ and tow profiles ψ, $\tilde{\psi}$, we say that the triplets $\big((\alpha_n),(x_n),\psi\big)$ and $\big((\tilde{\alpha}_n),(\tilde{x}_n),\tilde{\psi}\big)$ are orthogonal if

$$\textit{either} \qquad \Big|\log\left(\tilde{\alpha}_n/\alpha_n\right)\Big| \to \infty\,,$$

or $\tilde{\alpha}_n = \alpha_n$ and

$$-\frac{\log|x_n - \tilde{x}_n|}{\alpha_n} \longrightarrow a \geq 0 \ \textit{with}\ \psi\ \textit{or}\ \tilde{\psi}\ \textit{null for}\ s < a\,.$$

Remarks III.4

- *The profiles belong to the Hölder space $C^{\frac{1}{2}}$. Indeed, for any profile ψ and real numbers s and t, we have by Cauchy-Schwarz inequality*

$$|\psi(s) - \psi(t)| = \Big|\int_s^t \psi'(\tau)\, d\tau\Big| \leq \|\psi'\|_{L^2(\mathbb{R})}|s-t|^{\frac{1}{2}}\,.$$

- *Note also that (see [26])*

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad \textit{as} \quad s \to 0 \quad \textit{and} \quad \textit{as} \quad s \to \infty\,. \tag{III.6}$$

The asymptotically orthogonal decomposition derived in [26] is formulated in the following terms :

Theorem III.5 *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0\,, \tag{III.7}$$

$$\limsup_{n\to\infty} \|u_n\|_{L^{\phi_1}} = A_0 > 0 \qquad \textit{and} \tag{III.8}$$

$$\lim_{R\to\infty}\ \limsup_{n\to\infty}\ \|u_n\|_{L^{\phi_1}(|x|>R)} = 0\,. \tag{III.9}$$

Then, there exist a sequence of scales $(\alpha_n^{(j)})$*, a sequence of cores* $(x_n^{(j)})$ *and a sequence of profiles* $(\psi^{(j)})$ *such that the triplets* $(\alpha_n^{(j)}, x_n^{(j)}, \psi^{(j)})$ *are pairwise orthogonal and, up to a subsequence extraction, we have for all* $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x)\,, \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_1}} \stackrel{\ell\to\infty}{\longrightarrow} 0\,. \tag{III.10}$$

Moreover, we have the following stability estimates

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty\,. \tag{III.11}$$

Remarks III.6

– *It will be useful later on to point out that for any* $q \geq 2$*, we have*

$$\|g_n\|_{L^q} \stackrel{n\to\infty}{\longrightarrow} 0\,, \tag{III.12}$$

where g_n *is the elementary concentration defined by*

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\, \psi\left(\frac{-\log|x - x_n|}{\alpha_n}\right)\,. \tag{III.13}$$

Since the Lebesgue measure is invariant under translations, we have

$$\|g_n\|_{L^q}^q = (2\pi)^{-\frac{q}{2}}(\alpha_n)^{\frac{q}{2}} \int_{\mathbb{R}^2} \left|\psi\left(-\frac{\log|x|}{\alpha_n}\right)\right|^q dx\,.$$

Performing the change of variable $s = -\frac{\log|x|}{\alpha_n}$ *yields*

$$\|g_n\|_{L^q}^q = (2\pi)^{1-\frac{q}{2}}(\alpha_n)^{\frac{q}{2}+1} \int_0^\infty \left|\psi(s)\right|^q \mathrm{e}^{-2\alpha_n s}\, ds\,.$$

Fix $\varepsilon > 0$*. Then in view of* (III.6)*, there exist two real numbers* s_0 *and* S_0 *such that* $0 < s_0 < S_0$ *and*

$$|\psi(s)| \leq \varepsilon\sqrt{s}\,, \quad \forall\, s \in [0, s_0] \cup [S_0, \infty[\,.$$

This implies, by the change of variable $u = \alpha_n s$*, that*

$$\begin{aligned}(\alpha_n)^{\frac{q}{2}+1} \int_0^{s_0} |\psi(s)|^q\, \mathrm{e}^{-2\alpha_n s}\, ds \;&\leq\; \varepsilon^q \int_0^{\alpha_n s_0} u^{\frac{q}{2}} \mathrm{e}^{-2u}\, du \\ &\leq\; C_q\, \varepsilon^q\,.\end{aligned}$$

In the same way, we obtain

$$(\alpha_n)^{\frac{q}{2}+1} \int_{S_0}^\infty |\psi(s)|^q\, \mathrm{e}^{-2\alpha_n s}\, ds \;\leq\; C_q\, \varepsilon^q\,.$$

Finally, taking advantage of the continuity of ψ, we deduce that

$$(\alpha_n)^{\frac{q}{2}+1}\int_{s_0}^{S_0}|\psi(s)|^q\,\mathrm{e}^{-2\alpha_n s}\,ds \lesssim (\alpha_n)^{\frac{q}{2}+1}\int_{s_0}^{S_0}\mathrm{e}^{-2\alpha_n s}\,ds \lesssim (\alpha_n)^{\frac{q}{2}}\left(\mathrm{e}^{-2\alpha_n s_0}-\mathrm{e}^{-2\alpha_n S_0}\right)\stackrel{n\to\infty}{\longrightarrow}0\,,$$

which ends the proof of the assertion (III.12).

– *Setting*

$$g_n^{(j)}(x):=\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right) \tag{III.14}$$

the elementary concentration involved in Decomposition (III.10)*, we recall that it was proved in [24] that*

$$\|g_n^{(j)}\|_{L^{\phi_1}}\stackrel{n\to\infty}{\longrightarrow}\frac{1}{\sqrt{4\pi}}\max_{s>0}\frac{|\psi^{(j)}(s)|}{\sqrt{s}}$$

and

$$\Big\|\sum_{j=1}^{\ell}g_n^{(j)}\Big\|_{L^{\phi_1}}\stackrel{n\to\infty}{\longrightarrow}\sup_{1\le j\le\ell}\left(\lim_{n\to\infty}\|g_n^{(j)}\|_{L^{\phi_1}}\right)\,, \tag{III.15}$$

in the case when the scales $(\alpha_n^{(j)})_{1\le j\le\ell}$ are pairwise orthogonal. Note that Property (III.15) *does not necessarily remain true in the case when we have the same scales and the pairwise orthogonality of the couples $\left((x_n^{(j)}),\psi^{(j)}\right)$ (see Lemma* 3.6 *in [26]).*

III.1.3 Study of the lack of compactness of Sobolev embedding in the Orlicz space in the case $p>1$

Our first goal in this paper is to describe the lack of compactness of the Sobolev embedding (III.1) for $p>1$. Our result states as follows :

Theorem III.7 *Let $p>1$ be an integer and (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n\rightharpoonup 0\,, \tag{III.16}$$

$$\limsup_{n\to\infty}\|u_n\|_{L^{\phi_p}}=A_0>0\qquad\textit{and} \tag{III.17}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\|u_n\|_{L^{\phi_p}(|x|>R)}=0\,. \tag{III.18}$$

Then, there exist a sequence of scales $(\alpha_n^{(j)})$, a sequence of cores $(x_n^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that the triplets $(\alpha_n^{(j)},x_n^{(j)},\psi^{(j)})$ are pairwise orthogonal in the sense of Definition III.3 and, up to a subsequence extraction, we have for all $\ell\ge1$,

$$u_n(x)=\sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right)+\mathrm{r}_n^{(\ell)}(x)\,, \tag{III.19}$$

with $\limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \stackrel{\ell\to\infty}{\longrightarrow} 0$. *Moreover, we have the following stability estimates*

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|{\psi^{(j)}}'\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1)\,, \quad n\to\infty\,. \tag{III.20}$$

Remarks III.8

– *Arguing as in [24], we can easily prove that*

$$\|g_n\|_{L^{\phi_p}} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,, \tag{III.21}$$

where

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\left(\frac{-\log|x-x_n|}{\alpha_n}\right).$$

Indeed setting $L = \liminf_{n\to\infty} \|g_n\|_{L^{\phi_p}}$, *we have for fixed* $\varepsilon > 0$ *and* n *sufficiently large (up to subsequence extraction)*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(L+\varepsilon)^{2k}k!}\right) dx \le \kappa\,.$$

Therefore,

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - 1\right) dx \lesssim \kappa + \sum_{k=1}^{p-1} \|g_n\|_{L^{2k}}^{2k}\,. \tag{III.22}$$

Since

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - 1\right) dx = 2\pi \int_0^{+\infty} \alpha_n \mathrm{e}^{2\alpha_n s\left[\frac{1}{4\pi(L+\varepsilon)^2}\left(\frac{\psi(s)}{\sqrt{s}}\right)^2 - 1\right]} ds - \pi\,,$$

we obtain in view of (III.12) *and* (III.22) *that*

$$\int_0^{+\infty} \alpha_n \mathrm{e}^{2\alpha_n s\left[\frac{1}{4\pi(L+\varepsilon)^2}\left(\frac{\psi(s)}{\sqrt{s}}\right)^2 - 1\right]} ds \le C,$$

for some absolute constant C *and for* n *large enough. Using the fact that* ψ *is a continuous function, we deduce that*

$$L+\varepsilon \ge \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,,$$

which ensures that

$$L \ge \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,.$$

To end the proof of (III.21), *it suffices to establish that for any* $\delta > 0$

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(\lambda)^{2k}k!}\right) dx \stackrel{n\to\infty}{\longrightarrow} 0\,,$$

where $\lambda = \frac{1+\delta}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\cdot$ *Since*

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(\lambda)^{2k} k!} \Big)\, dx \leq \int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - 1 \Big)\, dx \,,$$

the result derives immediately from Proposition 1.15 *in [24], which achieves the proof of the result.*

- *Applying the same lines of reasoning as in the proof of Proposition 1.19 in [26], we obtain the following result :*
  Proposition III.9 *Let* $\Big((\alpha_n^{(j)}), (x_n^{(j)}), \psi^{(j)} \Big)_{1\leq j\leq \ell}$ *be a family of triplets of scales, cores and profiles such that the scales are pairwise orthogonal. Then for any integer p larger than* 1*, we have*

$$\Big\| \sum_{j=1}^{\ell} g_n^{(j)} \Big\|_{L^{\phi_p}} \stackrel{n\to\infty}{\longrightarrow} \sup_{1\leq j\leq \ell} \Big(\lim_{n\to\infty} \big\| g_n^{(j)} \big\|_{L^{\phi_p}} \Big) \,,$$

  where the functions $g_n^{(j)}$ *are defined by* (III.14)*.*

As we will see in Section 2, it turns out that the heart of the matter in the proof of Theorem III.7 is reduced to the following result concerning the radial case :

Theorem III.10 *Let p be an integer strictly larger than* 1 *and* (u_n) *be a bounded sequence in* $H^1_{rad}(\mathbb{R}^2)$ *such that*

$$u_n \rightharpoonup 0 \qquad \textit{and} \tag{III.23}$$

$$\limsup_{n\to\infty} \|u_n\|_{L^{\phi_p}} = A_0 > 0\,. \tag{III.24}$$

Then, there exist a sequence of pairwise orthogonal scales $(\alpha_n^{(j)})$ *and a sequence of profiles* $(\psi^{(j)})$ *such that up to a subsequence extraction, we have for all* $\ell \geq 1$*,*

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \stackrel{\ell\to\infty}{\longrightarrow} 0\,. \tag{III.25}$$

Moreover, we have the following stability estimates

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|{\psi^{(j)}}'\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n\to\infty\,.$$

Remarks III.11

- *Compared with the analogous result concerning the Sobolev embedding of* $H^1_{rad}(\mathbb{R}^2)$ *into* L^{ϕ_1} *established in [24], the hypothesis of compactness at infinity is not required. This is justified by the fact that* $H^1_{rad}(\mathbb{R}^2)$ *is compactly embedded in* $L^q(\mathbb{R}^2)$ *for any* $2 < q < \infty$ *which implies that*

$$\lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R}^2)} = 0\,, \quad \forall\, 2 < q < \infty\,. \tag{III.26}$$

– *In view of Proposition III.9, Theorem III.10 yields to*

$$\|u_n\|_{L^{\phi_p}} \to \sup_{j\geq 1} \left(\lim_{n\to\infty} \|g_n^{(j)}\|_{L^{\phi_p}} \right),$$

which implies that the first profile in Decomposition (III.25) *can be chosen such that up to extraction*

$$A_0 := \limsup_{n\to\infty} \|u_n\|_{L^{\phi_p}} = \lim_{n\to\infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \psi^{(1)} \left(-\frac{\log|x|}{\alpha_n^{(1)}} \right) \right\|_{L^{\phi_p}}. \qquad \text{(III.27)}$$

Note that the description of the lack of compactness in other critical Sobolev embeddings was achieved in [20, 29, 50] and has been at the origin of several prospectus. Among others, one can mention [21, 22, 23, 32, 61].

III.1.4 Layout of the paper

Our paper is organized as follows : in Section 2, we establish the algorithmic construction of the decomposition stated in Theorem III.7. Then, we study in Section 3 a nonlinear two-dimensional wave equation with the exponential nonlinearity $u\,\phi_p(\sqrt{4\pi}u)$. Firstly, we prove the global well-posedness and the scattering in the energy space both in the subcritical and critical cases, and secondly we compare the evolution of this equation with the evolution of the solutions of the free Klein-Gordon equation in the same space.
We mention that C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. For simplicity, we shall also still denote by (u_n) any subsequence of (u_n) and designate by $\circ(1)$ any sequence which tends to 0 as n goes to infinity.

III.2 Proof of Theorem III.7

III.2.1 Strategy of the proof

The proof of Theorem III.7 uses in a crucial way capacity arguments and is done in three steps : in the first step, we begin by the study of u_n^* the symmetric decreasing rearrangement of u_n. This led us to establish Theorem III.10. In the second step, by a technical process developed in [26], we reduce ourselves to one scale and extract the first core $(x_n^{(1)})$ and the first profile $\psi^{(1)}$ which enables us to extract the first element $\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\psi^{(1)}\left(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(j)}}\right)$. The third step is devoted to the study of the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence we start with which allows us to extract a second elementary concentration concentrated around a second core $(x_n^{(2)})$. Thereafter, we establish the property of orthogonality between the first two elementary concentrations and finally we prove that this process converges.

III.2.2 Proof of Theorem III.10

The main ingredient in the proof of Theorem III.10 consists to extract a scale and a profile ψ such that

$$\|\psi'\|_{L^2(\mathbb{R})} \geq C A_0 \,, \tag{III.28}$$

where C is a universal constant. To go to this end, let us for a bounded sequence (u_n) in $H^1_{rad}(\mathbb{R}^2)$ satisfying the assumptions (III.23) and (III.24), set $v_n(s) = u_n(\mathrm{e}^{-s})$. Combining (III.26) with the following well-known radial estimate :

$$|u(r)| \leq \frac{C}{r^{\frac{1}{p+1}}} \|u\|_{L^{2p}}^{\frac{p}{p+1}} \|\nabla u\|_{L^2}^{\frac{1}{p+1}} \,,$$

where $r = |x|$, we infer that

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(]-\infty,M])} = 0 \,, \quad \forall M \in \mathbb{R} \,. \tag{III.29}$$

This gives rise to the following result :

Proposition III.12 *For any $\delta > 0$, we have*

$$\sup_{s\geq 0} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \to \infty \,, \quad n \to \infty \,. \tag{III.30}$$

Proof. We proceed by contradiction. If not, there exists $\delta > 0$ such that, up to a subsequence extraction

$$\sup_{s\geq 0, n\in\mathbb{N}} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \leq C < \infty \,. \tag{III.31}$$

On the one hand, thanks to (III.29) and (III.31), we get by virtue of Lebesgue theorem

$$\begin{aligned} \int_{|x|<1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2} - \sum_{k=0}^{p-1} \frac{|u_n(x)|^{2k}}{(A_0-\delta)^{2k} k!} \right) dx &\leq \int_{|x|<1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2} - 1 \right) dx \\ &\leq 2\pi \int_0^\infty \left(\mathrm{e}^{|\frac{v_n(s)}{A_0-\delta}|^2} - 1 \right) \mathrm{e}^{-2s} \, ds \overset{n\to\infty}{\longrightarrow} 0 \,. \end{aligned}$$

On the other hand, using Property (III.29) and the simple fact that for any positive real number M, there exists a finite constant $C_{M,p}$ such that

$$\sup_{|t|\leq M} \left(\frac{\mathrm{e}^{t^2} - \sum_{k=0}^{p-1} \frac{t^{2k}}{k!}}{t^{2p}} \right) < C_{M,p} \,,$$

we deduce in view of (III.26) that

$$\int_{|x|\geq 1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2} - \sum_{k=0}^{p-1} \frac{|u_n(x)|^{2k}}{(A_0-\delta)^{2k} k!} \right) dx \lesssim \|u_n\|_{L^{2p}}^{2p} \to 0 \,.$$

Consequently,

$$\limsup_{n\to\infty} \|u_n\|_{L^{\phi_p}} \le A_0 - \delta\,,$$

which is in contradiction with Hypothesis (III.24). ∎

An immediate consequence of the previous proposition is the following corollary whose proof is identical to the proof of Corollaries 2.4 and 2.5 in [24].

Corollary III.13 *Under the above notations, there exists a sequence $(\alpha_n^{(1)})$ in $\mathbb{R}_+$ tending to infinity such that*

$$4\left|\frac{v_n(\alpha_n^{(1)})}{A_0}\right|^2 - \alpha_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty \tag{III.32}$$

and for n sufficiently large, there exists a positive constant C such that

$$\frac{A_0}{2}\sqrt{\alpha_n^{(1)}} \le |v_n(\alpha_n^{(1)})| \le C\sqrt{\alpha_n^{(1)}} + \circ(1)\,. \tag{III.33}$$

Now, setting

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}v_n(\alpha_n^{(1)}y)\,,$$

we obtain along the same lines as in Lemma 2.6 in [24] the following result :

Lemma III.14 *Under notations of Corollary III.13, there exists a profile $\psi^{(1)} \in \mathcal{P}$ such that, up to a subsequence extraction*

$$\psi_n' \rightharpoonup (\psi^{(1)})' \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \|(\psi^{(1)})'\|_{L^2} \ge \sqrt{\frac{\pi}{2}}A_0\,.$$

To achieve the proof of Theorem III.10, let us consider the remainder term

$$r_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x)\,, \tag{III.34}$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\psi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)\,.$$

By straightforward computations, we can easily prove that $(r_n^{(1)})$ is bounded in $H^1_{rad}(\mathbb{R}^2)$ and satisfies the hypothesis (III.23) together with the following property :

$$\lim_{n\to\infty} \|\nabla r_n^{(1)}\|^2_{L^2(\mathbb{R}^2)} = \lim_{n\to\infty}\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} - \left\|(\psi^{(1)})'\right\|^2_{L^2(\mathbb{R})}\,. \tag{III.35}$$

Let us now define $A_1 = \limsup_{n\to\infty}\|r_n^{(1)}\|_{L^{\phi_p}}$. If $A_1 = 0$, we stop the process. If not, arguing as above, we prove that there exist a scale $(\alpha_n^{(2)})$ satisfying the statement of Corollary III.13 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}}\psi^{(2)}\left(\frac{-\log|x|}{\alpha_n^{(2)}}\right) + r_n^{(2)}(x)\,,$$

with $\|(\psi^{(2)})'\|_{L^2} \geq \sqrt{\frac{\pi}{2}} A_1$ and

$$\lim_{n\to\infty} \|\nabla r_n^{(2)}\|^2_{L^2(\mathbb{R}^2)} = \lim_{n\to\infty} \|\nabla r_n^{(1)}\|^2_{L^2(\mathbb{R}^2)} - \left\|(\psi^{(2)})'\right\|^2_{L^2(\mathbb{R})} .$$

Moreover, as in [24] we can show that $(\alpha_n^{(1)})$ and $(\alpha_n^{(2)})$ are orthogonal. Finally, iterating the process, we get at step ℓ

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x) ,$$

with

$$\limsup_{n\to\infty} \|r_n^{(\ell)}\|^2_{H^1} \lesssim 1 - A_0^2 - A_1^2 - \cdots - A_{\ell-1}^2 ,$$

which implies that $A_\ell \to 0$ as $\ell \to \infty$ and ends the proof of the theorem.

III.2.3 Extraction of the cores and profiles

This step is performed as the proof of Theorem 1.16 in [26]. We sketch it here briefly for the convenience of the reader. Let u_n^* be the symmetric decreasing rearrangement of u_n. Since $u_n^* \in H^1_{rad}(\mathbb{R}^2)$ and satisfies the assumptions of Theorem III.10, we infer that there exist a sequence $(\alpha_n^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\varphi^{(j)})$ such that, up to subsequence extraction,

$$u_n^*(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \varphi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x) , \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \stackrel{\ell\to\infty}{\longrightarrow} 0 .$$

Besides, in view of (III.27), we can assume that

$$A_0 = \lim_{n\to\infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \varphi^{(1)} \left(-\frac{\log|x|}{\alpha_n^{(1)}}\right) \right\|_{L^{\Phi_p}} .$$

Now to extract the cores and profiles, we shall firstly reduce to the case of one scale according to Section 2.3 in [26], where a suitable truncation of u_n was introduced. Then assuming that

$$u_n^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\, \varphi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) ,$$

we apply the strategy developed in Section 2.4 in [26] to extract the cores and the profiles. This approach is based on capacity arguments : to carry out the extraction process of mass concentrations, we prove by contradiction that if the mass responsible for the lack of compactness of the Sobolev embedding in the Orlicz space is scattered, then the energy used would exceed that of the starting sequence. This main point can be formulated on the following terms :

Lemma III.15 (Lemma 2.5 in [26]) *There exist $\delta_0 > 0$ and $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$ there exists x_n such that*

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-\mathrm{b}\alpha_\mathrm{n}^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2 \,, \tag{III.36}$$

where $E_n := \{x \in \mathbb{R}^2; |u_n(x)| \geq \sqrt{2\alpha_n^{(1)}}(1 - \frac{\varepsilon_0}{10})A_0\}$ with $0 < \varepsilon_0 < \frac{1}{2}$, $B(x_n, \mathrm{e}^{-\mathrm{b}\,\alpha_\mathrm{n}^{(1)}})$ designates the ball of center x_n and radius $\mathrm{e}^{-\mathrm{b}\,\alpha_\mathrm{n}^{(1)}}$ with $b = 1 - 2\varepsilon_0$ and $|.|$ denotes the Lebesgue measure.

Once extracting the first core $(x_n^{(1)})$ making use of the previous lemma, we focus on the extraction of the first profile. For that purpose, we consider the sequence

$$\psi_n(y, \theta) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y, \theta) \,,$$

where $v_n(s, \theta) = (\tau_{x_n^{(1)}} u_n)(\mathrm{e}^{-\mathrm{s}} \cos\theta, \mathrm{e}^{-\mathrm{s}} \sin\theta)$ and $(x_n^{(1)})$ satisfies

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-(1-2\varepsilon_0)\alpha_\mathrm{n}^{(1)}}|}{|E_n|} \geq \delta_0 A_0^2 \,.$$

Taking advantage of the invariance of Lebesgue measure under translations, we deduce that

$$\begin{aligned} \|\nabla u_n\|_{L^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} |\partial_y \psi_n(y, \theta)|^2 \, dy \, d\theta \\ &+ \frac{\alpha_n^{(1)}}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} |\partial_\theta \psi_n(y, \theta)|^2 \, dy \, d\theta \,. \end{aligned}$$

Since the scale $\alpha_n^{(1)}$ tends to infinity and the sequence (u_n) is bounded in $H^1(\mathbb{R}^2)$, this implies that up to a subsequence extraction $\partial_\theta \psi_n \underset{n\to\infty}{\to} 0$ and $\partial_y \psi_n \underset{n\to\infty}{\to} g$ in $L^2(\mathbb{R} \times [0, 2\pi])$, where g only depends on the variable y. Thus introducing the function

$$\psi^{(1)}(y) = \int_0^y g(\tau) \, d\tau \,,$$

we obtain along the same lines as in Proposition 2.8 in [26] the following result :

Proposition III.16 *The function $\psi^{(1)}$ belongs to the set of profiles $\mathcal{P}$. Besides for any $y \in \mathbb{R}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_n(y, \theta) \, d\theta \to \psi^{(1)}(y) \,, \tag{III.37}$$

as n tends to infinity and there exists an absolute constant C such that

$$\|\psi^{(1)'}\|_{L^2} \geq C\, A_0 \,. \tag{III.38}$$

III.2.4 End of the proof

To achieve the proof of the theorem, we argue exactly as in Section 2.5 in [26] by iterating the process exposed in the previous section. For that purpose, we set

$$r_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x)\,,$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\psi^{(1)}\left(-\frac{\log|x-x_n^{(1)}|}{\alpha_n^{(1)}}\right).$$

One can easily check that the sequence $(r_n^{(1)})$ weakly converges to 0 in $H^1(\mathbb{R}^2)$. Moreover, since $\psi^{(1)}_{|]-\infty,0]} = 0$, we have for any $R \geq 1$

$$\|r_n^{(1)}\|_{L^{\phi_p}(|x-x_n^{(1)}|\geq R)} = \|u_n\|_{L^{\phi_p}(|x-x_n^{(1)}|\geq R)}\,. \tag{III.39}$$

But by assumption, the sequence (u_n) is compact at infinity in the Orlicz space L^{ϕ_p}. Thus the core $(x_n^{(1)})$ is bounded in $\mathbb{R}^2$, which ensures in view of (III.39) that $(r_n^{(1)})$ satisfies the hypothesis of compactness at infinity (III.18). Finally, taking advantage of the weak convergence of $(\partial_y\psi_n)$ to $\psi^{(1)'}$ in $L^2(y,\theta)$ as n goes to infinity, we get

$$\lim_{n\to\infty}\|\nabla r_n^{(1)}\|_{L^2}^2 = \lim_{n\to\infty}\|\nabla u_n^{(1)}\|_{L^2}^2 - \|\psi^{(1)'}\|_{L^2}^2\,.$$

Now, let us define $A_1 := \limsup_{n\to\infty}\|r_n^{(1)}\|_{L^{\phi_p}}$. If $A_1 = 0$, we stop the process. If not, knowing that $(r_n^{(1)})$ verifies the assumptions of Theorem III.7, we apply the above reasoning, which gives rise to the existence of a scale $(\alpha_n^{(2)})$, a core $(x_n^{(2)})$ satisfying the statement of Lemma III.15 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}}\,\psi^{(2)}\left(-\frac{\log|x-x_n^{(2)}|}{\alpha_n^{(2)}}\right) + r_n^{(2)}(x)\,,$$

with $\|\psi^{(2)'}\|_{L^2} \geq C\,A_1$ and

$$\lim_{n\to\infty}\|\nabla r_n^{(2)}\|_{L^2}^2 = \lim_{n\to\infty}\|\nabla r_n^{(1)}\|_{L^2}^2 - \|\psi^{(2)'}\|_{L^2}^2\,.$$

Arguing as in [26], we show that the triplets $\left(\alpha_n^{(1)}, x_n^{(1)}, \psi^{(1)}\right)$ and $\left(\alpha_n^{(2)}, x_n^{(2)}, \psi^{(2)}\right)$ are orthogonal in the sense of Definition III.3 and prove that the process of extraction of the elementary concentration converges. This ends the proof of Decomposition (III.10). The orthogonality equality (III.11) derives immediately from Proposition 2.10 in [26]. The proof of Theorem III.7 is then achieved.

III.3 Nonlinear wave equation

III.3.1 Statement of the results

In this section, we investigate the initial value problem for the following nonlinear wave equation :

$$\begin{cases} \Box u + u + u \left(\mathrm{e}^{4\pi u^2} - \displaystyle\sum_{k=0}^{p-1} \frac{(4\pi)^k u^{2k}}{k!} \right) = 0\,, \\ \\ u(0) = u_0 \in H^1(\mathbb{R}^2)\,, \quad \partial_t u(0) = u_1 \in L^2(\mathbb{R}^2)\,, \end{cases} \tag{III.40}$$

where $p \geq 1$ is an integer, $u = u(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$ and $\Box = \partial_t^2 - \Delta$ is the wave operator.

Let us recall that in [56, 58], the authors proved the global well-posedness for the Cauchy problem (III.40) when $p = 1$ and the scattering when $p = 2$ in the subcritical and critical cases (i.e when the energy is less or equal to some threshold). Note also that in [96, 97], M. Struwe constructed global smooth solutions to (III.40) with smooth data of arbitrary size in the case $p = 1$.

Formally, the solutions of the Cauchy problem (III.40) satisfy the following conservation law :

$$\begin{aligned} E_p(u,t) \;&:=\; \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \\ &+\; \frac{1}{4\pi} \left\| \mathrm{e}^{4\pi u(t)^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k}{k!} u(t)^{2k} \right\|_{L^1} = E_p(u,0) := E_p^0\,. \end{aligned} \tag{III.41}$$

This conducts us, as in [56], to define the notion of criticality in terms of the size of the initial energy E_p^0 with respect to 1.

Definition III.17 *The Cauchy problem* (III.40) *is said to be subcritical if*

$$E_p^0 < 1\,.$$

It is said to be critical if $E_p^0 = 1$ and supercritical if $E_p^0 > 1$.

We shall prove the following result :

Theorem III.18 *Assume that $E_p^0 \leq 1$. Then the Cauchy problem* (III.40) *has a unique global solution u in the space*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^2))\,.$$

Moreover, $u \in L^4(\mathbb{R}, \mathcal{C}^{1/4})$ and scatters.

III.3.2 Technical tools

The proof of Theorem III.18 is based on a priori estimates. This requires the control of the nonlinear term

$$F_p(u) := u\left(\mathrm{e}^{4\pi u^2} - \sum_{k=0}^{p-1} \frac{(4\pi)^k u^{2k}}{k!}\right) \tag{III.42}$$

in $L^1_t(L^2_x)$. To achieve our goal, we will resort to Strichartz estimates for the 2D Klein-Gordon equation. These estimates, proved in [51], state as follows :

Proposition III.19 *Let $T > 0$ and $(q,r) \in [4,\infty] \times [2,\infty]$ an admissible pair, i.e*

$$\frac{1}{q} + \frac{2}{r} = 1\,.$$

Then,

$$\|v\|_{L^q([0,T],\mathrm{B}^1_{r,2}(\mathbb{R}^2))} \lesssim \Big[\|v(0)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(0)\|_{L^2(\mathbb{R}^2)} + \|\Box v + v\|_{L^1([0,T],L^2(\mathbb{R}^2))}\Big]\,, \tag{III.43}$$

where $\mathrm{B}^1_{r,2}(\mathbb{R}^2)$ stands for the usual inhomogeneous Besov space (see for example [39] or [89] for a detailed exposition on Besov spaces).

Noticing that $(q,r) = (4, 8/3)$ is an admissible pair and recalling that

$$\mathrm{B}^1_{8/3,2}(\mathbb{R}^2) \hookrightarrow \mathcal{C}^{1/4}(\mathbb{R}^2)\,,$$

we deduce that

$$\|v\|_{L^4([0,T],\mathcal{C}^{1/4}(\mathbb{R}^2))} \lesssim \Big[\|v(0)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(0)\|_{L^2(\mathbb{R}^2)} + \|\Box v + v\|_{L^1([0,T],L^2(\mathbb{R}^2))}\Big]\,. \tag{III.44}$$

To control the nonlinear term $F_p(u)$ in $L^1_t(L^2_x)$, we will make use of the following logarithmic inequalities proved in [55, Theorem 1.3].

Proposition III.20 *For any $\lambda > \frac{2}{\pi}$ and any $0 < \mu \leq 1$, a constant $C_{\lambda,\mu} > 0$ exists such that for any function u in $H^1(\mathbb{R}^2) \cap \mathcal{C}^{1/4}(\mathbb{R}^2)$, we have*

$$\|u\|^2_{L^\infty} \leq \lambda \|u\|^2_\mu \log\left(C_{\lambda,\mu} + \frac{2\|u\|_{\mathcal{C}^{1/4}}}{\|u\|_\mu}\right)\,, \tag{III.45}$$

where $\|u\|^2_\mu := \|\nabla u\|^2_{L^2} + \mu^2 \|u\|^2_{L^2}$.

III.3.3 Proof of Theorem III.18

The proof of this result, divided into three steps, is inspired from the proofs of Theorems 1.8, 1.11, 1.12 in [56] and Theorem 1.3 in [58].

Local existence

Let us start by proving the local existence to the Cauchy problem (III.40). To do so, we use a standard fixed-point argument and introduce for any nonnegative time T the following space :

$$\mathcal{E}_T = \mathcal{C}([0,T], H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T], L^2(\mathbb{R}^2)) \cap L^4([0,T], \mathcal{C}^{1/4}(\mathbb{R}^2))$$

endowed with the norm

$$\|u\|_T = \sup_{0\leq t\leq T}\Big[\|u(t)\|_{H^1} + \|\partial_t u(t)\|_{L^2}\Big] + \|u\|_{L^4([0,T],\mathcal{C}^{1/4})}\,.$$

For a positive time T and a positive real number δ, we denote by $\mathcal{E}_T(\delta)$ the ball in the space $\mathcal{E}_T$ of radius δ and centered at the origin. On this ball, we define the map Φ by

$$v \longmapsto \Phi(v) = \widetilde{v}\,,$$

where

$$\Box\widetilde{v} + \widetilde{v} = -F_p(v+v_0)\,, \quad \widetilde{v}(0) = \partial_t\widetilde{v}(0) = 0$$

and v_0 is the solution of the free Klein-Gordon equation

$$\Box v_0 + v_0 = 0\,, \quad v_0(0) = u_0\,, \quad \textit{and} \quad \partial_t v_0(0) = u_1\,.$$

Now, the goal is to show that if δ and T are small enough, then the map Φ is well-defined from $\mathcal{E}_T(\delta)$ into itself and it is a contraction. To prove that Φ is well-defined, it suffices in view of the Strichartz estimates (III.43) to estimate $F_p(v+v_0)$ in the space $L^1([0,T], L^2(\mathbb{R}^2))$. Arguing as in [56] and using the Hölder inequality and the Sobolev embedding, we obtain for any $\epsilon > 0$

$$\begin{aligned}\int_{\mathbb{R}^2} |F_p(v+v_0)|^2\,dx &\leq \int_{\mathbb{R}^2} |F_1(v+v_0)|^2\,dx \\ &\lesssim \|v+v_0\|_{H^1}^2\, \mathrm{e}^{4\pi\|v+v_0\|_{L^\infty}^2}\, \Big\|\mathrm{e}^{4\pi(v+v_0)^2} - 1\Big\|_{L^{1+\epsilon}}\,.\end{aligned}$$

Note that the assumption $E_p^0 \leq 1$ implies that $\|\nabla u_0\|_{L^2} < 1$. Hence, we can choose $\mu > 0$ such that $\|u_0\|_\mu < 1$. Since v_0 is continuous in time, there exist a time T_0 and a constant $0 < c < 1$ such that for any t in $[0, T_0]$ we have

$$\|v_0(t)\|_\mu \leq c\,.$$

According to Proposition III.20, we infer that

$$\mathrm{e}^{4\pi\|v+v_0\|_{L^\infty}^2} \lesssim \left(1 + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\delta + c}\right)^{8\eta}\,,$$

for some $0 < \eta < 1$. Besides, applying the Trudinger-Moser inequality (III.5) for $p = 1$, the fact that

$$4\pi(1+\epsilon)(\delta+c)^2 \longrightarrow 4\pi c < 4\pi \quad \text{as } \epsilon,\, \delta \to 0 \quad \text{and} \quad \left\|\nabla\left(\frac{v+v_0}{\delta+c}\right)\right\|_{L^2} \leq 1$$

ensures that

$$\begin{aligned}\left\|e^{4\pi(v+v_0)^2}-1\right\|_{L^{1+\epsilon}}^{1+\epsilon} &\leq C_\epsilon \left\|e^{4\pi(1+\epsilon)(v+v_0)^2}-1\right\|_{L^1}\\ &\leq C_{\epsilon,\delta}\|v+v_0\|_{L^2}^2\\ &\leq C_{\epsilon,\delta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^2\,.\end{aligned}$$

Therefore, for any $0<T\leq T_0$, we obtain that

$$\|F_p(v+v_0)\|_{L^1([0,T],L^2(\mathbb{R}^2))}\lesssim T^{1-\eta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^{4\eta}\,.$$

Now, to prove that Φ is a contraction (at least for T small), let us consider two elements v_1 and v_2 in $\mathcal{E}_T(\delta)$. Notice that, for any $\epsilon>0$,

$$\begin{aligned}|F_p(v_1+v_0)-F_p(v_2+v_0)| &= |v_1-v_2|(1+8\pi\overline{v}^2)\left(e^{4\pi\overline{v}^2}-\sum_{k=0}^{p-2}\frac{(4\pi)^k\overline{v}^{2k}}{k!}\right)\\ &\leq C_\epsilon|v_1-v_2|\left(e^{4\pi(1+\epsilon)\overline{v}^2}-1\right)\,,\end{aligned}$$

where $\overline{v}=(1-\theta)(v_0+v_1)+\theta(v_0+v_2)$, for some $\theta=\theta(t,x)\in[0,1]$. Using a convexity argument, we get

$$\begin{aligned}|F_p(v_1+v_0)-F_p(v_2+v_0)| &\leq C_\epsilon\left|(v_1-v_2)\left(e^{4\pi(1+\epsilon)(v_1+v_0)^2}-1\right)\right|\\ &+ C_\epsilon\left|(v_1-v_2)\left(e^{4\pi(1+\epsilon)(v_2+v_0)^2}-1\right)\right|\,.\end{aligned}$$

This implies, in view of Strichartz estimates (III.44), that

$$\begin{aligned}\|\Phi(v_1)-\Phi(v_2)\|_T &\lesssim \|F_p(v_1+v_0)-F_p(v_2+v_0)\|_{L^1([0,T],L^2(\mathbb{R}^2))}\\ &\leq C_\epsilon\int_0^T\left\|(v_1-v_2)\left(e^{4\pi(1+\epsilon)(v_1+v_0)^2}-1\right)\right\|_{L^2}dt\\ &+ C_\epsilon\int_0^T\left\|(v_1-v_2)\left(e^{4\pi(1+\epsilon)(v_2+v_0)^2}-1\right)\right\|_{L^2}dt\,,\end{aligned}$$

which leads along the same lines as above to

$$\|\Phi(v_1)-\Phi(v_2)\|_T \lesssim T^{1-(1+\epsilon)\eta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^{4(1+\epsilon)\eta}\|v_1-v_2\|_T\,.$$

If the parameter ϵ is small enough, then $(1+\epsilon)\eta<1$ and therefore, for T small enough, Φ is a contraction map. This implies the uniqueness of the solution in $v_0+\mathcal{E}_T(\delta)$.
Now, we shall prove the uniqueness in the energy space. The idea here is to establish that, if $u=v_0+v$ is a solution of (III.40) in $\mathcal{C}([0,T],H^1(\mathbb{R}^2))\cap\mathcal{C}^1([0,T],L^2(\mathbb{R}^2))$, then necessarily $v\in\mathcal{E}_T(\delta)$ at least for T small. Starting from the fact that v satisfies

$$\Box v+v=-F_p(v+v_0)\,,\quad v(0)=\partial_t v(0)=0\,,$$

we are reduced, thanks to the Strichartz estimates (III.43), to control the term $F_p(v+v_0)$ in the space $L^1([0,T],L^2(\mathbb{R}^2))$. But $|F_p(v+v_0)|\leq|F_1(v+v_0)|$, which leads to the result arguing exactly as in [56].

Global existence

In this section, we shall establish that our solution is global in time both in subcritical and critical cases. Firstly, let us notice that the assumption $E_p^0 \leq 1$ implies that $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$, which ensures in view of Section 3.3.1 the existence of a unique maximal solution u defined on $[0, T^*)$ where $0 < T^* \leq \infty$ is the lifespan time of u. We shall proceed by contradiction assuming that $T^* < \infty$. In the subcritical case, the conservation law (III.41) implies that

$$\sup_{t \in (0,T^*)} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} < 1 \,.$$

Let then $0 < s < T^*$ and consider the following Cauchy problem :

$$\Box v + v + F_p(v) = 0 \,, \quad v(s) = u(s) \,, \quad \text{and} \quad \partial_t v(s) = \partial_t u(s) \,. \tag{III.46}$$

As in the first step of the proof, a fixed-point argument ensures the existence of $\tau > 0$ and a unique solution v to (III.46) on the interval $[s, s+\tau]$. Noticing that τ does not depend on s, we can choose s close to T^* such that $T^* - s < \tau$. So, we can prolong the solution u after the time T^*, which is a contradiction.

In the critical case, we cannot apply the previous argument because it is possible that the following concentration phenomenon holds :

$$\limsup_{t \to T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1 \,. \tag{III.47}$$

In fact, we shall show that (III.47) cannot hold in this case. To go to this end, we argue as in the proof of Theorem 1.12 in [56]. Firstly, since the first equation of the Cauchy problem (III.40) is invariant under time translation, we can assume that $T^* = 0$ and that the initial time is $t = -1$. Similarly to [56, Proposition 4.2, Corollary 4.4], it follows that the maximal solution u satisfies

$$\limsup_{t \to 0^-} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1 \,, \tag{III.48}$$

$$\lim_{t \to 0^-} \|u(t)\|_{L^2(\mathbb{R}^2)} = 0 \,, \tag{III.49}$$

$$\lim_{t \to 0^-} \int_{|x-x^*| \leq -t} |\nabla u(t,x)|^2 \, dx = 1 \,, \quad \text{and} \tag{III.50}$$

$$\forall t < 0, \quad \int_{|x-x^*| \leq -t} e_p(u)(t,x) \, dx = 1 \,, \tag{III.51}$$

for some $x^* \in \mathbb{R}^2$, where $e_p(u)$ denotes the energy density defined by

$$e_p(u)(t,x) := (\partial_t u)^2 + |\nabla u|^2 + \frac{1}{4\pi} \left(\mathrm{e}^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!} \right) .$$

Without loss of generality, we can assume that $x^* = 0$, then multiplying the equation of the problem (III.40) respectively by $\partial_t u$ and u, we obtain formally

$$\partial_t e_p(u) - div_x(2\partial_t u \nabla u) = 0 \,, \tag{III.52}$$

$$\partial_t(u\partial_t u) - div_x(u\nabla u) + |\nabla u|^2 - |\partial_t u|^2 + u^2 e^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!} = 0\,. \tag{III.53}$$

Integrating the conservation laws (III.52) and (III.53) over the backward truncated cone

$$K_S^T := \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^2 \ \text{ such that } S \leq t \leq T \text{ and } |x| \leq -t\Big\}$$

for $S < T < 0$, we get

$$\int_{B(-T)} e_p(u)(T,x)\, dx - \int_{B(-S)} e_p(u)(S,x)\, dx \tag{III.54}$$

$$= \frac{-1}{\sqrt{2}} \int_{M_S^T} \left[\left|\partial_t u \frac{x}{|x|} + \nabla u\right|^2 + \frac{1}{4\pi}\left(e^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!}\right)\, dx\, dt\right],$$

$$\int_{B(-T)} \partial_t u(T) u(T)\, dx - \int_{B(-S)} \partial_t u(S) u(S)\, dx + \frac{1}{\sqrt{2}} \int_{M_S^T} \left(\partial_t u + \nabla u . \frac{x}{|x|}\right) u\, dx\, dt \tag{III.55}$$

$$+ \int_{K_S^T} \left(|\nabla u|^2 - |\partial_t u|^2 + u^2 e^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!}\right)\, dx\, dt = 0\,,$$

where $B(r)$ is the ball centered at 0 and of radius r and

$$M_S^T := \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^2 \ \text{ such that } S \leq t \leq T \text{ and } |x| = -t\Big\}.$$

According to (III.51) and (III.54), we infer that

$$\int_{M_S^T} \left[\left|\partial_t u \frac{x}{|x|} + \nabla u\right|^2 + \frac{1}{4\pi}\left(e^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!}\right)\right]\, dx\, dt = 0\,.$$

This implies, using (III.55) and Cauchy-Schwarz inequality, that

$$\int_{B(-T)} \partial_t u(T) u(T)\, dx - \int_{B(-S)} \partial_t u(S) u(S)\, dx \tag{III.56}$$

$$+ \int_{K_S^T} \left(|\nabla u|^2 - |\partial_t u|^2 + u^2 e^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!}\right)\, dx\, dt = 0\,,$$

By virtue of Identities (III.48) and (III.49) and the conservation law (III.41), it can be seen that

$$\partial_t u(t) \underset{t\to 0}{\longrightarrow} 0 \quad \text{in } L^2(\mathbb{R}^2)\,, \tag{III.57}$$

which ensures by Cauchy-Schwarz inequality that

$$\int_{B(-T)} \partial_t u(T) u(T)\, dx \to 0\,. \tag{III.58}$$

Letting $T \to 0$ in (III.56), we deduce, from (III.58) and the fact that the term

$$u^2 \mathrm{e}^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!}$$

is positive,

$$-\int_{B(-S)} \partial_t u(S) u(S)\, dx \leq -\int_{K_S^0} |\nabla u|^2 \, dx\, dt + \int_{K_S^0} |\partial_t u|^2 \, dx\, dt \,. \tag{III.59}$$

Multiplying Inequality (III.59) by the positive number $-\frac{1}{S}$, we infer that

$$\int_{B(-S)} \partial_t u(S) \frac{u(S)}{S} \, dx \leq \frac{1}{S} \int_{K_S^0} |\nabla u|^2 \; dx \; dt - \frac{1}{S} \int_{K_S^0} |\partial_t u|^2 \; dx \; dt \,. \tag{III.60}$$

Now, Identity (III.57) leads to

$$\lim_{S \to 0^-} \frac{1}{S} \int_{K_S^0} |\partial_t u|^2 \; dx \; dt = 0 \,. \tag{III.61}$$

Moreover, using (III.50), it is clear that

$$\lim_{S \to 0^-} \frac{1}{S} \int_{K_S^0} |\nabla u|^2 \; dx \; dt = -1 \,. \tag{III.62}$$

Finally, since

$$\frac{u(S)}{S} = \frac{1}{S} \int_0^S \partial_t u(\tau) \; d\tau \,,$$

then $(\frac{u(S)}{S})$ is bounded in $L^2(\mathbb{R}^2)$ and hence

$$\lim_{S \to 0^-} \int_{B(-S)} \partial_t u(S) \frac{u(S)}{S} \; dx = 0 \,. \tag{III.63}$$

The identities (III.61), (III.62) and (III.63) yield a contradiction in view of (III.60). This achieves the proof of the global existence in the critical case.

Scattering

Our concern now is to prove that, in the subcritical and critical cases, the solution of the equation (III.40) approaches a solution of a free wave equation when the time goes to infinity. Using the fact that

$$|F_p(u)| \leq |F_2(u)| \,, \quad \forall p \geq 2 \,, \tag{III.64}$$

we can apply the arguments used in [58]. More precisely, in the subcritical case the key point consists to prove that there exists an increasing function $C : [0, 1[\longrightarrow [0, \infty[$ such that for any $0 \leq E < 1$, any global solution u of the Cauchy problem (III.40) with $E_p(u) \leq E$ satisfies

$$\|u\|_{X(\mathbb{R})} \leq C(E) \,, \tag{III.65}$$

where $X(\mathbb{R}) = L^8(\mathbb{R}, L^{16}(\mathbb{R}^2))$. Now, denoting by

$$E^* := \sup\left\{0 \le E < 1;\ \sup_{E_p(u)\le E} \|u\|_{X(\mathbb{R})} < \infty\right\},$$

and arguing as in [58, Lemma 4.1], we can show that Inequality (III.65) is satisfied if $E_p(u)$ is small, which implies that $E^* > 0$. Now our goal is to prove that $E^* = 1$. To do so, let us proceed by contradiction and assume that $E^* < 1$. Then, for any $E \in]E^*, 1[$ and any $n > 0$, there exists a global solution u to (III.40) such that $E_p(u) \le E$ and $\|u\|_{X(\mathbb{R})} > n$. By time translation, one can reduce to

$$\|u\|_{X(]0,\infty[)} > \frac{n}{2}\,. \tag{III.66}$$

Along the same lines as the proof of Proposition 5.1 in [58], we can show taking advantage of (III.64) that if E is close enough to E^*, then n cannot be arbitrarily large which yields a contradiction and ends the proof of the result in the subcritical case.
The proof of the scattering in the critical case is done as in Section 6 in [58] once we observed Inequality (III.64). It is based on the notion of concentration radius $r_\epsilon(t)$ introduced in [58].

Remark III.21 *Lower order nonlinear terms become more difficult when we look for global decay properties of the solutions. In [58], the authors avoid this problem by subtracting the cubic part from the nonlinearity $F_p(u)$ for $p = 1$, which is the lower critical power for the scattering problem in $\mathbb{R}^2$.*

III.3.4 Qualitative study

In this section we shall investigate the feature of solutions of the two-dimensional nonlinear Klein-Gordon equation (III.40) taking into account the different regimes. As in [24], the approach that we adopt here is the one introduced by P. Gérard in [49] which consists in comparing the evolution of oscillations and concentration effects displayed by sequences of solutions of the nonlinear Klein-Gordon equation (III.40) and solutions of the linear Klein-Gordon equation

$$\Box v + v = 0\,. \tag{III.67}$$

More precisely, let (φ_n, ψ_n) be a sequence of data in $H^1 \times L^2$ supported in some fixed ball and satisfying

$$\varphi_n \rightharpoonup 0 \quad \text{in } H^1\,, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2\,, \tag{III.68}$$

such that

$$E_p^n \le 1\,, \quad n \in \mathbb{N}\,, \tag{III.69}$$

where E_p^n stands for the energy of (φ_n, ψ_n) given by

$$E_p^n = \|\psi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 + \frac{1}{4\pi}\left\|\mathrm{e}^{4\pi\varphi_n^2} - 1 - \sum_{k=2}^{p}\frac{(4\pi)^k}{k!}\varphi_n^{2k}\right\|_{L^1},$$

and let us consider (u_n) and (v_n) the sequences of finite energy solutions of (III.40) and (III.67) such that

$$(u_n, \partial_t u_n)(0) = (v_n, \partial_t v_n)(0) = (\varphi_n, \psi_n)\,.$$

Arguing as in [49], the notion of linearizability is defined as follows :

Definition III.22 *Let T be a positive time. We shall say that the sequence (u_n) is linearizable on $[0,T]$, if*

$$\sup_{t\in[0,T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad as \quad n\to\infty\,,$$

where $E_c(w,t)$ denotes the kinetic energy defined by :

$$E_c(w,t) = \int_{\mathbb{R}^2} \Big[|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2\Big](t,x)\,dx\,.$$

For any time slab $I \subset \mathbb{R}$, we shall denote

$$\|v\|_{\mathrm{ST}(I)} := \sup_{(q,r)\ \text{admissible}} \|v\|_{L^q(I;\mathrm{B}^1_{r,2}(\mathbb{R}^2))}\,.$$

By interpolation argument, this Strichartz norm is equivalent to

$$\|v\|_{L^\infty(I;H^1(\mathbb{R}^2))} + \|v\|_{L^4(I;\mathrm{B}^1_{8/3,2}(\mathbb{R}^2))}\,.$$

As $\mathrm{B}^1_{r,2}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for all $r \le p < \infty$ (and $r\le p\le\infty$ if $r>2$), it follows that

$$\|v\|_{L^q(I;L^p)} \lesssim \|v\|_{\mathrm{ST}(I)}\,, \quad \frac{1}{q}+\frac{2}{p}\le 1\,. \tag{III.70}$$

As in [24], in the subcritical case (i.e $\limsup\limits_{n\to\infty}\ E_p^n < 1$), the nonlinearity does not induce any effect on the behavior of the solutions. But, in the critical case (i.e $\limsup\limits_{n\to\infty}\ E_p^n = 1$), it turns out that a nonlinear effect can be produced. More precisely, we have the following result :

Theorem III.23 *Let T be a strictly positive time. Then*

1. *If $\limsup\limits_{n\to\infty} E_p^n < 1$, the sequence (u_n) is linearizable on $[0,T]$.*
2. *If $\limsup\limits_{n\to\infty} E_p^n = 1$, the sequence (u_n) is linearizable on $[0,T]$ provided that the sequence (v_n) satisfies*

$$\limsup_{n\to\infty}\ \|v_n\|_{L^\infty([0,T];L^{\phi_p})} < \frac{1}{\sqrt{4\pi}}\,. \tag{III.71}$$

Proof. The proof of Theorem III.23 is similar to the one of Theorem 3.3 and 3.5 in [24]. Denoting by $w_n = u_n - v_n$, it is clear that w_n is the solution of the nonlinear wave equation

$$\Box w_n + w_n = -F_p(u_n)$$

with null Cauchy data.
Under energy estimate, we obtain

$$\|w_n\|_T \lesssim \|F_p(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))}\,,$$

where $\|w_n\|_T^2 \overset{\text{def}}{=} \sup_{t\in[0,T]} E_c(w_n,t)$. Therefore, it suffices to prove in the subcritical and critical cases that

$$\|F_p(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \longrightarrow 0 \quad \text{as} \quad n\to\infty\,. \tag{III.72}$$

Let us begin by the subcritical case. Our goal is to prove that the nonlinear term does not affect the behavior of the solutions. By hypothesis, there exists some nonnegative real ρ such that $\limsup_{n\to\infty} E_p^n = 1-\rho$. The main point for the proof is based on the following lemma, the proof of which is similar to the proof of Lemma 3.16 in [24] once we observed that

$$|F_p(u)| \leq |F_1(u)|, \quad \forall p \geq 1\,.$$

Lemma III.24 *For every $T>0$ and $E_p^0<1$, there exists a constant $C(T,E_p^0)$, such that every solution u of the nonlinear Klein-Gordon equation* (III.40) *of energy $E_p(u) \leq E_p^0$, satisfies*

$$\|u\|_{L^4([0,T];\mathcal{C}^{1/4})} \leq C(T,E_p^0)\,. \tag{III.73}$$

Now to establish the convergence property (III.72), it suffices to prove that the sequence $(F_p(u_n))$ is bounded in $L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))$ for some nonnegative ϵ and converges to 0 in measure in $[0,T]\times\mathbb{R}^2$. This can done exactly as in [24] using the fact that $|F_p(u_n)| \leq |F_1(u_n)|$.

Let us now prove (III.72) in the critical case. For that purpose, let $T>0$ and assume that

$$L := \limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];L^{\phi_p})} < \frac{1}{\sqrt{4\pi}}\,. \tag{III.74}$$

Applying Taylor's formula, we obtain

$$F_p(u_n) = F_p(v_n+w_n) = F_p(v_n) + F_p'(v_n)\,w_n + \frac{1}{2}\,F_p''(v_n+\theta_n\,w_n)\,w_n^2\,,$$

for some $0\leq\theta_n\leq 1$. Strichartz estimates (III.43) yield

$$\|w_n\|_{\mathrm{ST}([0,T])} \lesssim I_n + J_n + K_n\,,$$

where

$$\begin{aligned} I_n &= \|F_p(v_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))}\,,\\ J_n &= \|F_p'(v_n)\,w_n\|_{L^1([0,T];L^2(\mathbb{R}^2))}\,, \quad \text{and}\\ K_n &= \|F_p''(v_n+\theta_n\,w_n)\,w_n^2\|_{L^1([0,T];L^2(\mathbb{R}^2))}\,. \end{aligned}$$

As in [24], we have

$$\begin{aligned} I_n &\underset{n\to\infty}{\longrightarrow} 0 \quad \text{and}\\ J_n &\leq \varepsilon_n \|w_n\|_{ST([0,T])}\,, \end{aligned}$$

where $\varepsilon_n \to 0$. Besides, provided that

$$\limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1-L\sqrt{4\pi}}{2}\,, \tag{III.75}$$

we get

$$K_n \leq \varepsilon_n \|w_n\|^2_{ST([0,T])}\,, \quad \varepsilon_n \to 0\,.$$

Since $\|w_n\|_{ST([0,T])} \lesssim I_n + \varepsilon_n\|w_n\|^2_{ST([0,T])}$ wet obtain by bootstrap argument

$$\|w_n\|_{ST([0,T])} \lesssim \varepsilon_n\,,$$

which ends the proof of the result. ∎

III.4 Appendix : Proof of Proposition III.2

The proof uses in a crucial way the rearrangement of functions (for a complete presentation and more details, we refer the reader to [78]). By virtue of density arguments and the fact that for any function $f \in H^1(\mathbb{R}^2)$ and f^* the rearrangement of f, we have

$$\begin{aligned} \|\nabla f\|_{L^2} &\geq \|\nabla f^*\|_{L^2} , \\ \|f\|_{L^p} &= \|f^*\|_{L^p} , \\ \|f\|_{L^{\phi_p}} &= \|f^*\|_{L^{\phi_p}} , \end{aligned}$$

one can reduce to the case of a nonnegative radially symmetric and non-increasing function u belonging to $\mathcal{D}(\mathbb{R}^2)$. With this choice, let us introduce the function

$$w(t) = (4\pi)^{\frac{1}{2}} u(|x|) , \quad \text{where} \quad |x| = \mathrm{e}^{-\frac{t}{2}} .$$

It is then obvious that the functions $w(t)$ and $w'(t)$ are nonnegative and satisfy

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx &= \int_{-\infty}^{+\infty} |w'(t)|^2 \, dt , \\ \int_{\mathbb{R}^2} |u(x)|^{2p} \, dx &= \frac{1}{4^p \, \pi^{p-1}} \int_{-\infty}^{+\infty} |w(t)|^{2p} \, \mathrm{e}^{-t} \, dt \\ \int_{\mathbb{R}^2} \Big(\mathrm{e}^{\alpha |u(x)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |u(x)|^{2k}}{k!} \Big) dx &= \pi \int_{-\infty}^{+\infty} \Big(\mathrm{e}^{\frac{\alpha}{4\pi} |w(t)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |w(t)|^{2k}}{(4\pi)^k k!} \Big) \mathrm{e}^{-t} \, dt . \end{aligned}$$

So we are reduced to prove that for all $\beta \in [0,1[$, there exists $C_\beta \geq 0$ so that

$$\int_{-\infty}^{+\infty} \Big(\mathrm{e}^{\beta |w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!} \Big) \mathrm{e}^{-t} dt \leq C(\beta, p) \int_{-\infty}^{+\infty} |w(t)|^{2p} \mathrm{e}^{-t} \, dt , \quad \forall \, \beta \in [0,1[,$$

when $\int_{-\infty}^{+\infty} |w'(t)|^2 dt \leq 1$. For that purpose, let us set

$$T_0 = \sup \{ t \in \mathbb{R}, \ w(t) \leq 1 \} .$$

The existence of a real number t_0 such that $w(t_0) = 0$ ensures that $\{ t \in \mathbb{R}, \ w(t) \leq 1 \}$ is non empty. Then

$$T_0 \in]-\infty, +\infty] .$$

Knowing that w is nonnegative and increasing function, we deduce that

$$w :]-\infty, T_0] \longrightarrow [0,1] .$$

Therefore, observing that $\mathrm{e}^s - \sum_{k=0}^{p-1} \frac{s^k}{k!} \leq c_p \, s^p \, \mathrm{e}^s$ for any nonnegative real s, we obtain

$$\int_{-\infty}^{T_0} \Big(\mathrm{e}^{\beta |w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!} \Big) \mathrm{e}^{-t} dt \leq c_p \beta^p \, \mathrm{e}^{\beta} \int_{-\infty}^{T_0} |w(t)|^{2p} \mathrm{e}^{-t} dt .$$

To estimate the integral on $[T_0, +\infty[$, let us first notice that in view of the definition of T_0, we have for all $t \geq T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} w'(\tau) d\tau \\ &\leq w(T_0) + (t - T_0)^{\frac{1}{2}} \left(\int_{T_0}^{+\infty} w'(\tau)^2 d\tau \right)^{\frac{1}{2}} \\ &\leq 1 + (t - T_0)^{\frac{1}{2}} . \end{aligned}$$

Thus, using the fact that for any $\varepsilon > 0$ and any $s \geq 0$, we have

$$(1 + s^{\frac{1}{2}})^2 \leq (1 + \varepsilon)s + 1 + \frac{1}{\varepsilon} = (1 + \varepsilon)s + C_\varepsilon ,$$

we infer that for for any $\varepsilon > 0$ and all $t \geq T_0$

$$|w(t)|^2 \leq (1 + \varepsilon)(t - T_0) + C_\varepsilon . \tag{III.76}$$

Now β being fixed in $[0, 1[$, let us choose $\varepsilon > 0$ so that $\beta(1 + \varepsilon) < 1$. Then by virtue of (III.76)

$$\begin{aligned} \int_{T_0}^{+\infty} \Big(\mathrm{e}^{\beta |w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!} \Big) \mathrm{e}^{-t} \, dt &\leq \int_{T_0}^{+\infty} \mathrm{e}^{\beta |w(t)|^2} \mathrm{e}^{-t} \, dt \\ &\leq \frac{\mathrm{e}^{\beta C_\varepsilon - T_0}}{1 - \beta(1 + \varepsilon)} . \end{aligned}$$

But

$$\mathrm{e}^{-T_0} = \int_{T_0}^{+\infty} \mathrm{e}^{-t} \, dt \leq \int_{T_0}^{+\infty} |w(t)|^{2p} \, \mathrm{e}^{-t} \, dt ,$$

which gives rise to

$$\int_{T_0}^{+\infty} \Big(\mathrm{e}^{\beta |w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!} \Big) \mathrm{e}^{-t} dt \leq \frac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)} \int_{T_0}^{\infty} |w(t)|^{2p} \mathrm{e}^{-t} \, dt .$$

Choosing $C(\beta, p) = \max \left(c_p \mathrm{e}^{\beta} \beta^p, \dfrac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1 + \varepsilon)} \right)$ ends the proof of the proposition.

CHAPITRE IV

SHARP ADAMS-TYPE INEQUALITIES INVOKING HARDY INEQUALITIES

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Article soumis pour publication.

IV.1 Introduction

IV.1.1 Setting of the problem

The Trudinger-Moser type inequalities have a long history beginning with the works of Pohozaev [84] and Trudinger [101]. Letting $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n \geq 2$, the authors looked in these pioneering works for the maximal growth function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_{L^n} \leq 1} \int_\Omega g(u)\, dx < +\infty\,,$$

and they proved independently that the maximal growth is of exponential type. Thereafter, Moser improved these works by founding a sharp result known under the name Trudinger-Moser inequality (see [78]) and since that time, this subject has continued to interest researchers and Trudinger-Moser inequality has been extended in various directions (one can mention [1, 2, 77, 86, 87]) generating several applications. Among the results obtained concerning Trudinger-Moser type inequalities, we recall the so-called Adams' inequality in $\mathbb{R}^{2N}$.

Proposition IV.1 *[65, 87] There exists a finite constant $\kappa > 0$ such that*

$$\sup_{u \in H^N(\mathbb{R}^{2N}),\, \|u\|_{H^N(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\beta_N |u(x)|^2} - 1\right) dx := \kappa\,, \tag{IV.1}$$

where $\beta_N = N!\,\pi^N 2^{2N}$, and for any $\beta > \beta_N$

$$\sup_{u \in H^N(\mathbb{R}^{2N}),\, \|u\|_{H^N(\mathbb{R}^{2N})} \leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\beta |u(x)|^2} - 1\right) dx = +\infty\,. \tag{IV.2}$$

Remarks IV.2

– *In the above proposition, the norm $\|\,.\,\|_{H^N}$ designates the following Sobolev norm*

$$\|u\|^2_{H^N(\mathbb{R}^{2N})} := \|u\|^2_{L^2(\mathbb{R}^{2N})} + \sum_{j=1}^{N} \|\nabla^j u\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

where $\nabla^j u$ denotes the j-th order gradient of u, namely

$$\nabla^j u = \begin{cases} \Delta^{\frac{j}{2}} u & \text{if } j \text{ is even,} \\ \nabla \Delta^{\frac{j-1}{2}} u & \text{if } j \text{ is odd.} \end{cases}$$

– *The proof of Proposition IV.1, treated firstly in the radial case and generalized then by symmetrization arguments, is based on the following Trudinger-Moser inequality in a bounded domain.*

Proposition IV.3 ([2], Theorem 1) *Let Ω be a bounded domain in $\mathbb{R}^{2N}$. There exists a positive constant C_N such that*

$$\sup_{u \in H_0^N(\Omega),\, \|\nabla^N u\|_{L^2} \leq 1} \int_\Omega \mathrm{e}^{\beta_N |u(x)|^2}\, dx \leq C_N |\Omega|\,,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Furthermore, this inequality is sharp.

- *As emphasized above, Proposition IV.1 has been at the origin of numerous applications. Among others, one can mention the description of the lack of compactness of Sobolev embedding involving Orlicz spaces in [24, 26, 27, 29, 30], the analysis of some elliptic and biharmonic equations in [88, 90, 91] and the study of global wellposedness and the asymptotic completeness for evolution equations with exponential nonlinearity in dimension two in [13, 14, 23, 24, 41, 56, 58].*

Sobolev embedding inferred by Proposition IV.1 states as follows :

$$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N}) \,, \tag{IV.3}$$

where $\mathcal{L}$ is the so-called Orlicz space associated to the function $\phi(s) := \mathrm{e}^{s^2} - 1$ and defined as follows (for a complete presentation and more details, we refer the reader to [85] and the references therein) :

Definition IV.4 *We say that a measurable function* $u : \mathbb{R}^d \to \mathbb{C}$ *belongs to* $\mathcal{L}(\mathbb{R}^d)$ *if there exists* $\lambda > 0$ *such that*

$$\int_{\mathbb{R}^d} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) \, dx < \infty \,.$$

We denote then

$$\|u\|_{\mathcal{L}(\mathbb{R}^d)} = \inf \left\{ \lambda > 0, \int_{\mathbb{R}^d} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1 \right) \, dx \leq 1 \right\} . \tag{IV.4}$$

Remarks IV.5

- *It is easy to check that* $\| \, . \, \|_{\mathcal{L}}$ *is a norm on the* $\mathbb{C}$*-vector space* $\mathcal{L}$ *which is invariant under translations and oscillations.*
- *One can also verify that the number* 1 *in* (IV.1) *may be replaced by any positive constant. This changes the norm* $\| \, . \, \|_{\mathcal{L}}$ *to an equivalent one.*
- *In the sequel, we shall endow the space* $\mathcal{L}(\mathbb{R}^{2N})$ *with the norm* $\| \cdot \|_{\mathcal{L}(\mathbb{R}^{2N})}$ *where the number* 1 *is replaced by the constant* κ *involved in Identity* (IV.1)*. The Sobolev embedding* (IV.3) *states then as follows :*

$$\|u\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq \frac{1}{\sqrt{\beta_N}} \|u\|_{H^N(\mathbb{R}^{2N})} \,, \tag{IV.5}$$

where the Sobolev constant $\dfrac{1}{\sqrt{\beta_N}}$ *is sharp.*

- *Denoting by* L^{ϕ_p}*, the Orlicz space associated to the function*

$$\phi_p(s) := \mathrm{e}^{s^2} - \sum_{k=0}^{p-1} \frac{s^{2k}}{k!} \,,$$

with p *an integer larger than* 1*, we deduce from Proposition IV.1 the more general Sobolev embedding*

$$H^N(\mathbb{R}^{2N}) \hookrightarrow L^{\phi_p}(\mathbb{R}^{2N}) \,. \tag{IV.6}$$

– *Let us finally observe that* $\mathcal{L} \hookrightarrow L^p$ *for every* $2 \leq p < \infty$ *.*

In this article, our goal is twofold. Firstly obtain an analogue of Proposition IV.1 in the radial framework of a functional space $\mathcal{H}(\mathbb{R}^{2N})$ closely related to Hardy inequalities, which will easily lead to the following Sobolev embedding

$$\mathcal{H}_{rad}(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N}) . \tag{IV.7}$$

Secondly describe the lack of compactness of (IV.7), which could be at the origin of several applications as it has been the case by previous characterizations of defect of compactness of various Sobolev embeddings.

More precisely, for any integer $N \geq 2$, the space we will consider in this paper is defined as follows :

$$\mathcal{H}(\mathbb{R}^{2N}) := \left\{ u \in H^1(\mathbb{R}^{2N}); \frac{\nabla u}{|\,.\,|^{N-1}} \in L^2(\mathbb{R}^{2N}) \right\} .$$

In view of the well-known Hardy inequalities (see for instance [16, 19, 52, 53]) :

$$\left\| \frac{u}{|\,.\,|^s} \right\|_{L^2(\mathbb{R}^d)} \leq C_{d,s} \|u\|_{\dot{H}^s(\mathbb{R}^d)} , \quad \forall\, s \in \left[0, \frac{d}{2}\right[, \tag{IV.8}$$

the Sobolev space $H^N(\mathbb{R}^{2N})$ continuously embeds in the functional space $\mathcal{H}(\mathbb{R}^{2N})$ endowed with the norm

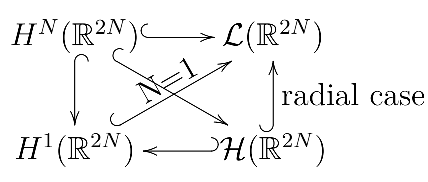
$$\|u\|^2_{\mathcal{H}(\mathbb{R}^{2N})} = \|u\|^2_{H^1(\mathbb{R}^{2N})} + \left\| \frac{\nabla u}{|\,.\,|^{N-1}} \right\|^2_{L^2(\mathbb{R}^{2N})} .$$

Actually, as shown by the example of function

$$x \longmapsto \log(1 - \log |x|)\, \mathbf{1}_{B_1(0)}(x) ,$$

with $B_1(0)$ the unit ball of $\mathbb{R}^{2N}$, the embedding of $H^N(\mathbb{R}^{2N})$ into $\mathcal{H}(\mathbb{R}^{2N})$ is strict for every $N \geq 2$.

For the convenience of the reader, the following diagram recapitulates the different embeddings including the spaces involved in this work.



The interest we take to the space $\mathcal{H}$ is motivated by the importance of Hardy inequalities in Analysis (among others, we can mention blow-up methods or the study of pseudodifferential operators with singular coefficients).

IV.1.2 Main results

The result we obtained concerning the sharp Adams-type inequality in the framework of the space $\mathcal{H}(\mathbb{R}^{2N})$ takes the following form :

Theorem IV.6 *For any integer N greater than 2, there exists a finite constant $\kappa' > 0$ such that*

$$\sup_{\|u\|_{\mathcal{H}_{rad}(\mathbb{R}^{2N})}\leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\gamma_N |u(x)|^2} - 1\right) dx := \kappa' , \tag{IV.9}$$

where $\gamma_N := \dfrac{4\pi^N N}{(N-1)!}$ *, and for any* $\gamma > \gamma_N$

$$\sup_{\|u\|_{\mathcal{H}_{rad}(\mathbb{R}^{2N})}\leq 1} \int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\gamma |u(x)|^2} - 1\right) dx = +\infty . \tag{IV.10}$$

Remarks IV.7

- *Note that the optimal constant involved in Identity* (IV.9) *is different from that appearing in Identity* (IV.1).
- *Usually, the proofs of Trudinger-Moser inequalities reduce to the radial framework under symmetrization arguments. In particular, in dimension two this question is achieved by means of Schwarz symmetrization (see [1]). The key point in that process is the preservation of Lebesgue norms and the minimization of energy.*

  Unfortunately, the quantities $\left\|\frac{\nabla u}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$ *cannot be minimized under Schwarz symmetrization as shown by the example* $u_k(x) := \varphi(|x|+k)$*, where* $\varphi \neq 0$ *is a smooth compactly supported function. The fact that* $u_k^* = \varphi$ *shows that the control of* $\left\|\frac{\nabla u_k^*}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$ *by* $\left\|\frac{\nabla u_k}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$ *fails.*
- *It is clear that, when the constant* 1 *in* (IV.4) *is replaced by* κ'*, Theorem IV.6 implies the following radial continuous embedding*

  $$\|u\|_{\mathcal{L}(\mathbb{R}^{2N})} \leq \frac{1}{\sqrt{\gamma_N}} \|u\|_{\mathcal{H}_{rad}(\mathbb{R}^{2N})} ,$$

  where the Sobolev constant $\dfrac{1}{\sqrt{\gamma_N}}$ *is optimal.*
- *Observe that due to the continuous embedding*

  $$H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{H}(\mathbb{R}^{2N}) ,$$

  Theorem IV.6 can be viewed as a generalization of Proposition IV.1 in the radial framework.

As mentioned above, our second aim in this paper is to describe the lack of compactness of the Sobolev embedding (IV.7). Actually, this embedding is non compact at least for two reasons. The first reason is a lack of compactness at infinity, as shown by the example $u_k(x) = \varphi(x + x_k)$ where $0 \neq \varphi \in \mathcal{D}$ and $|x_k| \to \infty$, which converges weakly to 0 in $\mathcal{H}(\mathbb{R}^{2N})$ and satisfies $\|u_k\|_{\mathcal{L}(\mathbb{R}^{2N})} = \|\varphi\|_{\mathcal{L}(\mathbb{R}^{2N})}$. The second reason is of concentration-type as illustrated by the following example derived by P.-L. Lions [74, 75] :

$$f_k(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1\,, \\ -\sqrt{\frac{2N}{k\,\gamma_N}}\log|x| & \text{if} \quad \mathrm{e}^{-k} \leq |x| < 1\,, \\ \sqrt{\frac{2Nk}{\gamma_N}} & \text{if} \quad |x| < \mathrm{e}^{-k}\,. \end{cases} \tag{IV.11}$$

Indeed, we have the following proposition the proof of which is postponed to Section IV.4 for the convenience of the reader.

Proposition IV.8 *The sequence $(f_k)_{k\geq 0}$ defined above converges weakly to 0 in $\mathcal{H}(\mathbb{R}^{2N})$ and satisfies*

$$\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{k\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}}\,.$$

It will be useful later on to emphasize that f_k can be recast under the following form :

$$f_k(x) = \sqrt{\frac{2Nk}{\gamma_N}}\,\mathbf{L}\Big(-\frac{\log|x|}{k}\Big)\,,$$

where

$$\mathbf{L}(t) = \begin{cases} 1 & \text{if} \quad t \geq 1\,, \\ t & \text{if} \quad 0 \leq t < 1\,, \\ 0 & \text{if} \quad t < 0\,, \end{cases}$$

and that

$$\|f_k\|_{H^1(\mathbb{R}^{2N})} \stackrel{k\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \Big\|\frac{\nabla f_k}{|\,.\,|^{N-1}}\Big\|_{L^2(\mathbb{R}^{2N})} = \big\|\mathbf{L}'\big\|_{L^2(\mathbb{R})} = 1\,. \tag{IV.12}$$

In order to state our second result in a clear way, let us introduce some objects as in [24].

Definition IV.9 *We shall designate by a scale any sequence $\underline{\alpha} := (\alpha_n)_{n\geq 0}$ of positive real numbers going to infinity and by a profile any function ψ belonging to the set*

$$\mathcal{P} := \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2Ns}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\ \Big\}.$$

Two scales $\underline{\alpha}$, $\underline{\beta}$ are said orthogonal if

$$\Big|\log\Big(\frac{\beta_n}{\alpha_n}\Big)\Big| \stackrel{n\to\infty}{\longrightarrow} \infty\,.$$

Remark IV.10 *Recall that each profile $\psi \in \mathcal{P}$ belongs to the Hölder space $C^{\frac{1}{2}}(\mathbb{R})$, and satisfies*

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad as \quad s \to 0\,. \tag{IV.13}$$

Indeed taking advantage of the fact that $\psi' \in L^2(\mathbb{R})$, we get for any $s_2 > s_1$

$$\Big|\psi(s_2) - \psi(s_1)\Big| = \Big|\int_{s_1}^{s_2} \psi'(\tau)\,d\tau\Big| \leq \sqrt{s_2 - s_1}\,\Big(\int_{s_1}^{s_2} \psi'^2(\tau)\,d\tau\Big)^{1/2}\,,$$

which ensures that $\psi \in C^{\frac{1}{2}}(\mathbb{R})$ and implies (IV.13) *by taking $s_1 = 0$.*

The result we establish in this paper highlights the fact that the lack of compactness of the Sobolev embedding (IV.7) can be described in terms of generalizations of the example by Moser (IV.11) as follows :

Theorem IV.11 *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ such that*

$$u_n \rightharpoonup 0\,, \tag{IV.14}$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(\mathbb{R}^{2N})} = A_0 > 0\,, \quad and \tag{IV.15}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty} \int_{|x|\geq R} |u_n(x)|^2\,dx = 0\,. \tag{IV.16}$$

Then, there exist a sequence of pairwise orthogonal scales $(\underline{\alpha}^{(j)})_{j\geq 1}$ and a sequence of profiles $(\psi^{(j)})_{j\geq 1}$ such that up to a subsequence extraction, we have for all $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\,\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x)\,, \tag{IV.17}$$

with $\limsup\limits_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{\ell\to\infty}{\longrightarrow} 0$. Moreover, we have the following stability estimate

$$\Big\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\Big\|^2_{L^2(\mathbb{R}^{2N})} = \sum_{j=1}^{\ell}\Big\|\psi^{(j)\prime}\Big\|^2_{L^2(\mathbb{R})} + \Big\|\frac{\nabla \mathrm{r}_n^{(\ell)}}{|\,.\,|^{N-1}}\Big\|^2_{L^2(\mathbb{R}^{2N})} + \circ(1)\,, \quad n\to\infty\,.$$

Remarks IV.12

- *The hypothesis of compactness at infinity* (IV.16) *is crucial : it allows to avoid the loss of Orlicz norm at infinity.*
- *Note that the elementary concentrations*

$$g_n^{(j)}(x) := \sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\,\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right)\,, \tag{IV.18}$$

  involved in Decomposition (IV.17) *are in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ whereas a priori, they do not belong to $H^N(\mathbb{R}^{2N})$.*

– *Actually, the lack of compactness of $H^N(\mathbb{R}^{2N}) \hookrightarrow \mathcal{L}(\mathbb{R}^{2N})$ was characterized in [27] by means of the following type of elementary concentrations :*

$$f_n(x) := \frac{C_N}{\sqrt{\alpha_n}} \int_{|\xi|\geq 1} \frac{\mathrm{e}^{i\,(x-x_n)\cdot\xi}}{|\xi|^{2N}}\, \varphi\Big(\frac{\log|\xi|}{\alpha_n}\Big)\, d\xi\,, \tag{IV.19}$$

with $(\alpha_n)_{n\geq 0}$ a scale in the sense of Definition IV.9, $(x_n)_{n\geq 0}$ a sequence of points in $\mathbb{R}^{2N}$ and φ a function in $L^2(\mathbb{R}_+)$. Note that (see Proposition 1.7 in [27])

$$f_n(x) = \widetilde{C}_N\, \sqrt{\alpha_n}\, \psi\Big(\frac{-\log|x|}{\alpha_n}\Big) + \mathrm{t}_n(x)\,,$$

with $\psi(y) = \displaystyle\int_0^y \varphi(t)\, dt$ and $\|\mathrm{t}_n\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{n\to\infty}{\longrightarrow} 0$.

– *Arguing as in [24], we have the following result :*
Proposition IV.13 *Let us consider*

$$g_n(x) := \sqrt{\frac{2N\alpha_n}{\gamma_N}}\; \psi\left(\frac{-\log|x|}{\alpha_n}\right)\,,$$

with ψ a profile and $(\alpha_n)_{n\geq 0}$ a scale. Then

$$\|g_n\|_{\mathcal{L}(\mathbb{R}^{2N})} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,. \tag{IV.20}$$

Proof. *Setting $L = \liminf\limits_{n\to\infty} \|g_n\|_{\mathcal{L}(\mathbb{R}^{2N})}$, we have for any fixed $\varepsilon > 0$ and any n sufficiently large (up to a subsequence extraction)*

$$\int_{\mathbb{R}^{2N}} \Big(\mathrm{e}^{\left|\frac{g_n(x)}{L+\varepsilon}\right|^2} - 1\Big)\, dx \leq \kappa'\,.$$

Therefore, there exists a positive constant C such that

$$\alpha_n \int_0^{+\infty} \mathrm{e}^{2N\alpha_n s\left[\frac{1}{\gamma_N (L+\varepsilon)^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2 - 1\right]}\, ds \leq C\,.$$

Using the fact that ψ is a continuous function, we deduce that

$$L + \varepsilon \geq \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,,$$

which ensures that

$$L \geq \frac{1}{\sqrt{\gamma_N}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\,.$$

To end the proof of (IV.20), it suffices to show that for any positive real number δ, the following estimate holds

$$\int_{\mathbb{R}^{2N}} \Big(\mathrm{e}^{\left|\frac{g_n(x)}{\lambda}\right|^2} - 1\Big)\, dx \stackrel{n\to\infty}{\longrightarrow} 0\,,$$

where $\lambda := \dfrac{1+\delta}{\sqrt{\gamma_N}} \max\limits_{s>0} \dfrac{|\psi(s)|}{\sqrt{s}}\,.$

Performing the change of variable $r = \mathrm{e}^{-\alpha_n s}$*, we easily get*

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\left|\frac{g_n(x)}{\lambda}\right|^2} - 1\right) dx = \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\infty \mathrm{e}^{-2N\alpha_n s\left(1 - \frac{1}{\gamma_N \lambda^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2\right)} ds - \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\infty \mathrm{e}^{-2N\alpha_n s} ds\,. \tag{IV.21}$$

Recalling that

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad as \quad s \to 0\,,$$

we infer that for any $\varepsilon > 0$*, there exists* $\eta > 0$ *such that*

$$\frac{1}{\gamma_N \lambda^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2 < \varepsilon \quad for\ any \quad 0 \leq s < \eta\,.$$

According to (IV.21)*, this gives rise to*

$$\begin{aligned}\frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\eta \mathrm{e}^{-2N\alpha_n s\left(1 - \frac{1}{\gamma_N \lambda^2}\left|\frac{\psi(s)}{\sqrt{s}}\right|^2\right)} ds \quad &- \quad \frac{2\pi^N \alpha_n}{(N-1)!} \int_0^\eta \mathrm{e}^{-2N\alpha_n s} ds \\ &\leq \quad \frac{\pi^N \varepsilon}{N!(1-\varepsilon)} + \circ(1)\,, \quad n \to \infty\,,\end{aligned}$$

which ensures the desired result. ∎

– *Arguing as in Proposition 1.18 in [24], we get*

$$\Big\|\sum_{j=1}^{\ell} g_n^{(j)}\Big\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{n\to\infty}{\longrightarrow} \sup_{1 \leq j \leq \ell} \left(\lim_{n\to\infty} \left\|g_n^{(j)}\right\|_{\mathcal{L}(\mathbb{R}^{2N})}\right)\,, \tag{IV.22}$$

where $g_n^{(j)}$ *is defined by* (IV.18).

IV.1.3 Layout

The paper is organized as follows : Section 2 is devoted to the proof of the sharp Adams-type inequality in the framework of the space $\mathcal{H}_{rad}(\mathbb{R}^{2N})$, namely Theorem IV.6. In Section 3, we establish Theorem IV.11 by describing the algorithm construction of the decomposition of a bounded sequence $(u_n)_{n\geq 0}$ in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$, up a subsequence extraction, in terms of asymptotically orthogonal profiles in the spirit of the example by Moser. The last section is devoted to the proof of Proposition IV.8.

Finally, we mention that, C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C. For simplicity, we shall also still denote by (u_n) any subsequence of (u_n).

IV.2 Proof of the Theorem IV.6

To establish Estimate (IV.9), we shall follow the 2D approach adopted in [86] by setting for a fixed $r_0 > 0$ (to be chosen later on)

$$I_1 := \int_{B(r_0)} \left(\mathrm{e}^{\gamma_N |u(x)|^2} - 1 \right) dx \quad \text{and} \quad I_2 := \int_{\mathbb{R}^{2N} \setminus B(r_0)} \left(\mathrm{e}^{\gamma_N |u(x)|^2} - 1 \right) dx \,,$$

where $B(r_0)$ denotes the ball centered at the origin and of radius r_0. The idea consists to show that it is possible to choose a suitable $r_0 > 0$ independently of u such that I_1 and I_2 are bounded by a constant only depending on r_0 and N.

Let us start by studying the part I_2. Using the power series expansion of the exponential, we can write

$$I_2 = \sum_{k=1}^{\infty} \frac{\gamma_N^k}{k!} I_{2,k} \,, \text{ where } \quad I_{2,k} := \int_{\mathbb{R}^{2N} \setminus B(r_0)} |u(x)|^{2k} \, dx \,.$$

In order to estimate $I_{2,k}$, we take advantage of the following radial estimate available for any function u in $H^1_{rad}(\mathbb{R}^{2N})$ (for further details, see [87]) :

$$|u(x)| \leq \sqrt{\frac{(N-1)!}{\pi^N}} \frac{\|u\|_{H^1(\mathbb{R}^{2N})}}{|x|^{N-\frac{1}{2}}} \quad \text{for a.e.} \quad x \in \mathbb{R}^{2N} \,, \tag{IV.23}$$

which for any integer $k \geq 2$, implies that

$$\begin{aligned} I_{2,k} &\leq \left(\frac{(N-1)!}{\pi^N} \right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{2\pi^N}{(N-1)!} \int_{r_0}^{\infty} \frac{dr}{r^{(k-1)(2N-1)}} \\ &\leq \frac{2\pi^N}{(N-1)!} \left(\frac{(N-1)!}{\pi^N} \right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{r_0^{k(1-2N)+2N}}{(2N-1)k - 2N} \\ &\leq \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \left(\frac{(N-1)!}{\pi^N} \right)^k \|u\|^{2k}_{H^1(\mathbb{R}^{2N})} \frac{1}{r_0^{(2N-1)k}} \,. \end{aligned}$$

This gives rise to

$$\begin{aligned} I_2 &\leq \gamma_N \|u\|^2_{L^2(\mathbb{R}^{2N})} + \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{\gamma_N (N-1)!}{\pi^N} \frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{(2N-1)}} \right)^k \\ &\leq \gamma_N + \frac{2\pi^N}{(N-1)!} \frac{r_0^{2N}}{2(N-1)} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\gamma_N \frac{(N-1)!}{\pi^N} \frac{1}{r_0^{(2N-1)}} \right)^k , \end{aligned}$$

under the fact that $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$, which ensures that I_2 is bounded by a constant only dependent of r_0 and N.

In order to estimate I_1, we shall make use of the following Adams-type inequality, the proof of which is postponed at the end of the section.

Proposition IV.14 *There exists a constant $C_N > 0$ such that for any positive real number R, we have*

$$\sup_{u\in\left(\mathcal{H}_{rad}\cap H_0^1\right)(B(R)),\,\left\|\frac{\nabla u}{|\,.\,|^{N-1}}\right\|_{L^2}\leq 1}\int_{B(R)} \mathrm{e}^{\gamma_N|u(x)|^2}\,dx \leq C_N R^{2N}\,,$$

and this inequality is sharp.

Let us admit this proposition for the time being, and continue the proof of the theorem. The key point consists to associate to a function u in $\mathcal{H}_{rad}(B(r_0))$ with $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$ an auxiliary function $w \in \left(\mathcal{H}_{rad} \cap H_0^1\right)(B(r_0))$ such that

$$\left\|\frac{\nabla w}{|\,.\,|^{N-1}}\right\|_{L^2(B(r_0))} \leq 1 \quad \text{and} \quad u^2 \leq w^2 + d(r_0)\,,$$

where the function $d(r_0) > 0$ depends only on r_0. To this end, let us first emphasize that if u belongs to $\mathcal{H}_{rad}(B(r_0))$ and satisfies $\|u\|_{\mathcal{H}(\mathbb{R}^{2N})} \leq 1$, then u is continuous far away from the origin. Indeed, for any real numbers $r_2 > r_1 > 0$, writing

$$u(r_2) - u(r_1) = \int_{r_1}^{r_2} u'(s)\,ds\,,$$

we get by Cauchy-Schwarz inequality

$$\begin{aligned} |u(r_2) - u(r_1)| &\leq \left(\int_{r_1}^{r_2} |u'(s)|^2 s^{2N-1}\,ds\right)^{\frac{1}{2}}\left(\int_{r_1}^{r_2} s^{-(2N-1)}\,ds\right)^{\frac{1}{2}} \\ &\leq C\|\nabla u\|_{L^2(\mathbb{R}^{2N})}\left(\int_{r_1}^{r_2} s^{-(2N-1)}\,ds\right)^{\frac{1}{2}}, \end{aligned}$$

which leads to the result. Thus, for any $0 < r < r_0$, we can define the function

$$v(r) := u(r) - u(r_0)\,,$$

which clearly belongs to $\left(\mathcal{H}_{rad}\cap H_0^1\right)(B(r_0))$. In light of the radial estimate (IV.23), this implies that

$$\begin{aligned} u^2(r) &\leq v^2(r) + v^2(r)u^2(r_0) + 1 + u^2(r_0) \\ &\leq v^2(r) + v^2(r)\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}} + 1 + \frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}} \\ &\leq v^2(r)\left(1 + \frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right) + d(r_0)\,, \end{aligned}$$

where $d(r_0) := 1 + \dfrac{(N-1)!}{\pi^N}\dfrac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\,\cdot$

Now by construction, the function

$$w(r) := v(r)\sqrt{1 + \frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}}\,,$$

belongs to $\Big(\mathcal{H}_{rad}\cap H^1_0\Big)(B(r_0))$, and easily satisfies

$$\begin{aligned}\int_{B(r_0)}\frac{|\nabla w(x)|^2}{|x|^{2(N-1)}}\,dx &= \left(1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right)\int_{B(r_0)}\frac{|\nabla u(x)|^2}{|x|^{2(N-1)}}\,dx\\ &\leq \left(1+\frac{(N-1)!}{\pi^N}\frac{\|u\|^2_{H^1(\mathbb{R}^{2N})}}{r_0^{2N-1}}\right)\Big(1-\|u\|^2_{H^1(\mathbb{R}^{2N})}\Big)\leq 1\,,\end{aligned}$$

provided that $\dfrac{\pi^N}{(N-1)!}\,r_0^{2N-1}\geq 1$.

Applying Proposition IV.14 with r_0 fixed so that $\dfrac{\pi^N}{(N-1)!}\,r_0^{2N-1}\geq 1$, we deduce that

$$I_1\leq \mathrm{e}^{\gamma_N d(r_0)}\int_{B(r_0)}\mathrm{e}^{\gamma_N|w(x)|^2}\,dx\leq C_N\,\mathrm{e}^{\gamma_N d(r_0)}\,r_0^{2N}\,,$$

which ensures the desired estimate, up to the proof of Proposition IV.14.

To achieve the proof of (IV.9), let us then establish Proposition IV.14. To this end, let us for a function u in $\Big(\mathcal{H}_{rad}\cap H^1_0\Big)(B(R))$ satisfying $\Big\|\frac{\nabla u}{|\cdot|^{N-1}}\Big\|_{L^2(\mathbb{R}^{2N})}\leq 1$, denote by

$$I(R):=\int_{B(R)}\mathrm{e}^{\gamma_N|u(x)|^2}\,dx\,.$$

Our aim is to show that

$$I(R)\leq C_N R^{2N}\quad\text{whenever}\quad \frac{2\pi^N}{(N-1)!}\int_0^R|v'(r)|^2 r\,dr\leq 1\,.$$

For that purpose, let us perform the change of variable $s=r^N$, and introduce the function $w(s)=\sqrt{\dfrac{N\pi^{N-1}}{(N-1)!}}\,v\Big(s^{\frac{1}{N}}\Big)$. Recalling that $\gamma_N=\dfrac{4\pi^N N}{(N-1)!}$, we infer that

$$\begin{aligned}I(R)=\frac{2\pi^N}{(N-1)!}\int_0^R\mathrm{e}^{\gamma_N|v(r)|^2}r^{2N-1}\,dr=\frac{2\pi^N}{N!}\int_0^{R^N}\mathrm{e}^{4\pi|w(s)|^2}s\,ds\quad\text{and}\\ \frac{2\pi^N}{(N-1)!}\int_0^R|v'(r)|^2 r\,dr=2\pi\int_0^{R^N}|w'(s)|^2 s\,ds\,.\end{aligned}$$

The conclusion stems then from the 2D radial framework of Proposition IV.3.

Now in order to prove the sharpness of the exponent γ_N, let us consider the sequence (f_k) defined by (IV.11). Since according to (IV.12), we have

$$\|f_k\|_{\mathcal{H}(\mathbb{R}^{2N})}=1+\circ(1)\,,\quad\text{as } k\to\infty\,,$$

we get for any $\gamma>\gamma_N$

$$\begin{aligned}\int_{\mathbb{R}^{2N}}\left(\mathrm{e}^{\gamma\left|\frac{f_k(x)}{\|f_k\|_{\mathcal{H}(\mathbb{R}^{2N})}}\right|^2}-1\right)dx &\geq \frac{2\pi^N}{(N-1)!}\int_0^{\mathrm{e}^{-k}}\Big(\mathrm{e}^{\frac{2Nk\gamma}{\gamma_N(1+\circ(1))}}-1\Big)r^{2N-1}\,dr\\ &\geq \frac{\pi^N}{N!}\Big(\mathrm{e}^{2Nk\frac{\gamma-\gamma_N(1+\circ(1))}{\gamma_N(1+\circ(1))}}-\mathrm{e}^{-2Nk}\Big)\overset{k\to\infty}{\longrightarrow}\infty\,,\end{aligned}$$

which ends the proof of the theorem.

IV.3 Proof of Theorem IV.11

IV.3.1 Scheme of the proof

The proof of Theorem IV.11 relies on a diagonal subsequence extraction and uses in a crucial way the radial setting and particularly the fact that we deal with bounded functions far away from the origin. The heart of the matter is reduced to the proof of the following lemma :

Lemma IV.15 *Let $(u_n)_{n\geq 0}$ be a bounded sequence in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ satisfying Assumptions* (IV.14)*,* (IV.15) *and* (IV.16)*. Then there exist a scale $(\alpha_n)_{n\geq 0}$ and a profile ψ in the sense of Definition IV.9, such that*

$$\|\psi'\|_{L^2(\mathbb{R})} \geq C_N A_0 \,, \tag{IV.24}$$

where C_N is a constant depending only on N.

Inspired by the strategy developed in [24], the proof is done in three steps. In the first step, according to Lemma IV.15, we extract the first scale and the first profile satisfying Inequality (IV.24). This reduces the problem to the study of the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence start which allows us to extract a second scale and a second profile which verifies the above key property (IV.24), by following the lines of reasoning of the first step. Thereafter, we establish the property of orthogonality between the two first scales. Finally, we prove that this process converges.

IV.3.2 Extraction of the first scale and the first profile

Let us consider a bounded sequence $(u_n)_{n\geq 0}$ in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ satisfying the assumptions of Theorem IV.11, and let us set $v_n(s) := u_n(\mathrm{e}^{-s})$. Then, we have the following lemma :

Lemma IV.16 *Under the above assumptions, the sequence $(u_n)_{n\geq 0}$ converges strongly to 0 in $L^2(\mathbb{R}^{2N})$, and we have for any real number M,*

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(]-\infty,M])} = 0 \,. \tag{IV.25}$$

Proof. Let us first observe that for any positive real number R, we have

$$\|u_n\|_{L^2(\mathbb{R}^{2N})} = \|u_n\|_{L^2(|x|\leq R)} + \|u_n\|_{L^2(|x|>R)} \,.$$

Now, invoking Rellich's theorem and the Sobolev embedding of $\mathcal{H}(\mathbb{R}^{2N})$ into $H^1(\mathbb{R}^{2N})$, we infer that the space $\mathcal{H}(|x| < R)$ is compactly embedded in $L^2(|x| < R)$. Therefore,

$$\limsup_{n\to\infty} \|u_n\|_{L^2(|x|<R)} \overset{n\to\infty}{\longrightarrow} 0 \,.$$

Taking advantage of the hypothesis of the compactness at infinity (IV.16), we deduce the strong convergence of the sequence $(u_n)_{n\geq 0}$ to 0 in $L^2(\mathbb{R}^{2N})$.

Finally, (IV.25) stems from the strong convergence to zero of $(u_n)_{n\geq 0}$ in $L^2(\mathbb{R}^{2N})$ and the following well-known radial estimate available for any function u in $H^1_{rad}(\mathbb{R}^{2N})$:

$$|u(x)| \leq \sqrt{\frac{(N-1)!}{\pi^N}}\, \frac{\|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})}\|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^{2N})}}{|x|^{N-\frac{1}{2}}}\,, \quad \text{for a.e.} \quad x \in \mathbb{R}^{2N}\,.$$

■

Now, arguing as in the proof of Proposition 2.3 in [24], we deduce the following result :

Proposition IV.17 *For any $\delta > 0$, we have*

$$\sup_{s\geq 0}\left(\left|\frac{v_n(s)}{A_0-\delta}\right|^2 - (2N-1)s\right) \to \infty\,, \quad n \to \infty\,. \tag{IV.26}$$

A byproduct of the previous proposition is the following corollary :

Corollary IV.18 *Under the above notations, there exists a sequence $(\alpha_n^{(1)})_{n\geq 0}$ in $\mathbb{R}_+$ tending to infinity such that*

$$4\left|\frac{v_n(\alpha_n^{(1)})}{A_0}\right|^2 - (2N-1)\,\alpha_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty\,, \tag{IV.27}$$

and for n sufficiently large, there exists a positive constant C such that

$$\frac{A_0}{2}\sqrt{(2N-1)\alpha_n^{(1)}} \leq |v_n(\alpha_n^{(1)})| \leq C\sqrt{\alpha_n^{(1)}} + \circ(1)\,, \tag{IV.28}$$

where $C = \sqrt{\frac{(N-1)!}{2\pi^N}} \limsup_{n\to\infty}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$.

Proof. In order to establish (IV.27), let us consider the sequences

$$W_n(s) := 4\left|\frac{v_n(s)}{A_0}\right|^2 - (2N-1)s \quad \text{and} \quad a_n := \sup_{s\geq 0} W_n(s)\,.$$

By definition, there exists a positive sequence $(\alpha_n^{(1)})_{n\geq 0}$ such that

$$W_n(\alpha_n^{(1)}) \geq a_n - \frac{1}{n}\cdot$$

Now, in view of (IV.26), $a_n \overset{n\to\infty}{\longrightarrow} \infty$ and then $W_n(\alpha_n^{(1)}) \overset{n\to\infty}{\longrightarrow} \infty$. It remains to prove that $\alpha_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty$. If not, up to a subsequence extraction, the sequence $(\alpha_n^{(1)})_{n\geq 0}$ is bounded and so is $(W_n(\alpha_n^{(1)}))_{n\geq 0}$ by (IV.25), which yields a contradiction.

Concerning Estimate (IV.28), the left hand side follows directly from (IV.27). Besides, for any positive real number s, we have

$$|v_n(s)| \leq \left|v_n(0) + \int_0^s v_n'(\tau)\,d\tau\right| \leq |v_n(0)| + s^{\frac{1}{2}}\|v_n'\|_{L^2(\mathbb{R})}\,,$$

which according to (IV.25) which implies that $v_n(0) \overset{n\to\infty}{\longrightarrow} 0$, and the following straightforward equality

$$\|v_n'\|_{L^2(\mathbb{R})} = \sqrt{\frac{(N-1)!}{2\pi^N}}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}\,,$$

gives the right hand side of Inequality (IV.28), and thus ends the proof of the result. ■

Corollary IV.18 allows to extract the first scale, it remains to extract the first profile. To do so, let us set

$$\psi_n(y) = \sqrt{\frac{\gamma_N}{2N\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y)\cdot$$

It will be useful later on to point out that, in view of Property (IV.25), $\psi_n(0) \overset{n\to\infty}{\longrightarrow} 0$.

The following result summarize the main properties of the sequence $(\psi_n)_{n\geq 0}$:

Lemma IV.19 *Under notations of Corollary IV.18, there exists a profile $\psi^{(1)} \in \mathcal{P}$ such that, up to a subsequence extraction*

$$\psi_n' \rightharpoonup \psi^{(1)\prime} \textit{ in } L^2(\mathbb{R}) \quad \textit{and} \quad \left\|\psi^{(1)\prime}\right\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,.$$

Proof. Noticing that $\|\psi_n'\|_{L^2(\mathbb{R})} = \left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})}$, we infer that the sequence $(\psi_n')_{n\geq 0}$ is bounded in $L^2(\mathbb{R})$. Thus, up to a subsequence extraction, $(\psi_n')_{n\geq 0}$ converges weakly in $L^2(\mathbb{R})$ to some function g. Let us now introduce the function

$$\psi^{(1)}(s) := \int_0^s g(\tau)\, d\tau\,.$$

Our aim is then to prove that $\psi^{(1)}$ is a profile and that $\left\|\psi^{(1)\prime}\right\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,.$

On the one hand, applying Cauchy-Schwarz inequality, we get

$$\left|\psi^{(1)}(s)\right| = \left|\int_0^s g(\tau)\ d\tau\right| \leq \sqrt{s}\|g\|_{L^2(\mathbb{R})}\,,$$

which ensures that $\psi^{(1)} \in L^2(\mathbb{R}_+, \mathrm{e}^{-2Ns} ds)\,.$

On the other hand, we have $\psi^{(1)}(s) = 0$ for all $s \leq 0$. Indeed, using the fact that

$$\|u_n\|^2_{L^2(\mathbb{R}^{2N})} = \left(\alpha_n^{(1)}\right)^2 \int_{\mathbb{R}} |\psi_n(s)|^2 \mathrm{e}^{-2N\alpha_n^{(1)}s}\ ds\,,$$

we obtain that

$$\int_{-\infty}^0 |\psi_n(s)|^2\ ds \leq \int_{-\infty}^0 |\psi_n(s)|^2 \mathrm{e}^{-2N\alpha_n^{(1)}s}\ ds \leq \frac{1}{\left(\alpha_n^{(1)}\right)^2}\|u_n\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

which implies that $(\psi_n)_{n\geq 0}$ converges strongly to zero in $L^2(]-\infty, 0[)$, and thus for almost all $s \leq 0$ (still up to the extraction of a subsequence).

But, we have

$$\psi_n(s) - \psi_n(0) = \int_0^s \psi_n'(\tau)\ d\tau \underset{n\to\infty}{\longrightarrow} \int_0^s g(\tau)\ d\tau = \psi^{(1)}(s)\,,$$

which, according to the fact that $\psi_n(0) \overset{n\to\infty}{\longrightarrow} 0$, implies that

$$\psi_n(s) \overset{n\to\infty}{\longrightarrow} \psi^{(1)}(s)\,, \quad \forall\, s \in \mathbb{R}\,.$$

We deduce that ${\psi^{(1)}}_{|]-\infty,0]} = 0$, which completes the proof of the fact that $\psi^{(1)} \in \mathcal{P}$.

Finally in light of (IV.28), we have

$$\left|\psi^{(1)}(1)\right| \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,.$$

Since

$$\left\|\psi^{(1)\prime}\right\|_{L^2(\mathbb{R})} \geq \int_0^1 \left|\psi^{(1)\prime}(\tau)\right| d\tau = \left|\psi^{(1)}(1)\right|,$$

this gives rise to

$$\left\|\psi^{(1)\prime}\right\|_{L^2} \geq \frac{A_0}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}\,,$$

which ends the proof of the key lemma IV.15. ■

IV.3.3 Study of the remainder term and iteration

Our concern is to iterate the previous process and to prove that the algorithmic construction converges. For that purpose, let us first consider the remainder term

$$\mathrm{r}_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x)\,, \tag{IV.29}$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{2N\alpha_n^{(1)}}{\gamma_N}}\,\psi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right).$$

It can be easily proved that $(\mathrm{r}_n^{(1)})_{n\geq 0}$ is a bounded sequence in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ satisfying (IV.14), (IV.16) and the following property :

$$\lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(1)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} = \lim_{n\to\infty}\left\|\frac{\nabla u_n}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} - \left\|\psi^{(1)\prime}\right\|^2_{L^2(\mathbb{R})}.$$

Let us now define $A_1 = \limsup_{n\to\infty}\|\mathrm{r}_n^{(1)}\|_{\mathcal{L}(\mathbb{R}^{2N})}$. If $A_1 = 0$, we stop the process. If not, arguing as above, we prove that there exists a constant C such that

$$\frac{A_1}{2}\sqrt{(2N-1)\,\alpha_n^{(2)}} \leq \left|\tilde{\mathrm{r}}_n^{(1)}\left(\alpha_n^{(2)}\right)\right| \leq C\sqrt{\alpha_n^{(2)}} + o(1)\,, \tag{IV.30}$$

where $\tilde{\mathrm{r}}_n^{(1)}(s) = \mathrm{r}_n^{(1)}(\mathrm{e}^{-s})$ and that there exist a scale $(\alpha_n^{(2)})$ satisfying the statement of Corollary IV.18 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$\mathrm{r}_n^{(1)}(x) = \sqrt{\frac{2N\alpha_n^{(2)}}{\gamma_N}}\,\psi^{(2)}\left(\frac{-\log|x|}{\alpha_n^{(2)}}\right) + \mathrm{r}_n^{(2)}(x)\,,$$

with $\left\|\psi^{(2)\prime}\right\|_{L^2} \geq \frac{A_1}{2}\sqrt{\frac{2N-1}{2N}\,\gamma_N}$ and

$$\lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(2)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} = \lim_{n\to\infty}\left\|\frac{\nabla \mathrm{r}_n^{(1)}}{|\,.\,|^{N-1}}\right\|^2_{L^2(\mathbb{R}^{2N})} - \left\|\psi^{(2)\prime}\right\|^2_{L^2(\mathbb{R})}.$$

Moreover, we claim that $(\alpha_n^{(1)})$ and $(\alpha_n^{(2)})$ are orthogonal in the sense of Definition IV.9. Otherwise, there exists a constant C such that

$$\frac{1}{C} \leq \left|\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right| \leq C\,.$$

Making use of Equality (IV.29), we get

$$\tilde{\mathrm{r}}_n^{(1)}\Big(\alpha_n^{(2)}\Big) = \sqrt{\frac{2N\alpha_n^{(1)}}{\gamma_N}}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \psi^{(1)}\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right)\cdot$$

This implies that, up to a subsequence extraction,

$$\lim_{n\to\infty}\sqrt{\frac{\gamma_N}{2N\alpha_n^{(1)}}}\;\tilde{\mathrm{r}}_n^{(1)}\Big(\alpha_n^{(2)}\Big) = \lim_{n\to\infty}\left(\psi_n\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right) - \psi^{(1)}\left(\frac{\alpha_n^{(2)}}{\alpha_n^{(1)}}\right)\right) = 0\,,$$

which is in contradiction with the left hand side of Inequality (IV.30).

Finally, iterating the process, we get at step ℓ

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{2N\alpha_n^{(j)}}{\gamma_N}}\;\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x)\,,$$

with

$$\limsup_{n\to\infty}\|\mathrm{r}_n^{(\ell)}\|^2_{\mathcal{H}(\mathbb{R}^{2N})} \lesssim 1 - A_0^2 - A_1^2 - \cdots - A_{\ell-1}^2\,.$$

This implies that $A_\ell \to 0$ as $\ell \to \infty$ and ends the proof of the theorem.

IV.4 Proof of Proposition IV.8

This section is devoted to the proof of Proposition IV.8. Actually, the fact that the sequence $(f_k)_{k\geq 0}$ converges weakly to 0 in $\mathcal{H}_{rad}(\mathbb{R}^{2N})$ stems from straightforward computations, and the heart of the matter consists to show that

$$\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \overset{k\to\infty}{\longrightarrow} \frac{1}{\sqrt{\gamma_N}}\cdot \tag{IV.31}$$

Firstly, let us prove that $\liminf_{k\to\infty}\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})} \geq \frac{1}{\sqrt{\gamma_N}}\cdot$ For that purpose, let us consider $\lambda > 0$ such that

$$\int_{\mathbb{R}^{2N}}\left(\mathrm{e}^{\left|\frac{f_k(x)}{\lambda}\right|^2} - 1\right)dx \leq \kappa'\,.$$

By definition, this gives rise to

$$\int_{|x|\leq \mathrm{e}^{-k}}\left(\mathrm{e}^{\left|\frac{f_k(x)}{\lambda}\right|^2} - 1\right)dx \leq \kappa'\,,$$

and thus consequently

$$\frac{\pi^N}{N!}\Big(\mathrm{e}^{\frac{2Nk}{\gamma_N\lambda^2}}-1\Big)\mathrm{e}^{-2Nk}\leq\kappa'\,.$$

We deduce that

$$\lambda^2\geq\frac{2Nk}{\gamma_N\log(1+\frac{N!}{\pi^N}\,\kappa'\,\mathrm{e}^{2Nk})}\underset{k\to\infty}{\longrightarrow}\frac{1}{\gamma_N}\,,$$

which ensures that

$$\liminf_{k\to\infty}\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})}\geq\frac{1}{\sqrt{\gamma_N}}\cdot$$

Now the fact that $\limsup\limits_{k\to\infty}\|f_k\|_{\mathcal{L}(\mathbb{R}^{2N})}\leq\dfrac{1}{\sqrt{\gamma_N}}$ derives from the following proposition the proof of which is postponed at the end of this section :

Proposition IV.20 *Let $\gamma\in]0,\gamma_N[$. A positive constant $C_{\gamma,N}$ exists such that*

$$\int_{\mathbb{R}^{2N}}\Big(\mathrm{e}^{\gamma|u(x)|^2}-1\Big)\,dx\leq C_{\gamma,N}\|u\|^2_{L^2(\mathbb{R}^{2N})}\,,\tag{IV.32}$$

for any non-negative function u belonging to $\mathcal{H}_{rad}(\mathbb{R}^{2N})$, compactly supported and satisfying $u(|x|):[0,\infty[\to\mathbb{R}$ is decreasing and $\Big\|\dfrac{\nabla u}{|\,.\,|^{N-1}}\Big\|_{L^2(\mathbb{R}^{2N})}\leq 1$. Besides, Inequality (IV.32) *is sharp.*

Assume indeed for the time being that the above proposition is true. Then, for any fixed $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\int_{\mathbb{R}^{2N}}\Big(\mathrm{e}^{(\gamma_N-\varepsilon)|f_k(x)|^2}-1\Big)\,dx\leq C_{\varepsilon,N}\|f_k\|^2_{L^2(\mathbb{R}^{2N})}\,,$$

which leads to the desired result, by virtue of the convergence of (f_k) to zero in $L^2(\mathbb{R}^{2N})$.

To end the proof of Proposition IV.8, it remains to establish Proposition IV.20 the proof of which is inspired from the one of Theorem 0.1 in [1].

Proof. Let u satisfying the assumptions of Proposition IV.20. Then there exists a function $v:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$u(x)=v(r)\,,\quad |x|=r\,,$$

$$v'(r)\leq 0\,,\quad\forall\, r\geq 0\,,\ \text{and}$$

$$\exists\, r_0>0 \text{ such that } v(r)=0\quad\forall\, r\geq r_0\,.$$

Setting $w(t)=\sqrt{\gamma_N}\,v\Big(\mathrm{e}^{-\frac{t}{2}}\Big)$, we can notice that w satisfies the following properties :

$$w(t)\geq 0\,,\quad\forall\, t\in\mathbb{R}\,,\tag{IV.33}$$

$$w'(t)\geq 0\,,\quad\forall\, t\in\mathbb{R}\,,\ \text{and}\tag{IV.34}$$

$$\exists\, t_0\in\mathbb{R}\text{ such that } w(t)=0\quad\forall\, t\leq t_0\,.\tag{IV.35}$$

Besides, we obtain by straightforward computations that

$$\|w'\|_{L^2(\mathbb{R})} = \sqrt{N}\left\|\frac{\nabla u}{|\,.\,|^{N-1}}\right\|_{L^2(\mathbb{R}^{2N})} \leq \sqrt{N}\,, \tag{IV.36}$$

$$\int_{\mathbb{R}} |w(t)|^2\, \mathrm{e}^{-Nt}\, dt = 4N\|u\|^2_{L^2(\mathbb{R}^{2N})}\,, \text{ and} \tag{IV.37}$$

$$\int_{\mathbb{R}} \left(\mathrm{e}^{\frac{\gamma}{\gamma_N}|w(t)|^2} - 1\right) \mathrm{e}^{-Nt}\, dt = \frac{(N-1)!}{\pi^N}\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\gamma|u(x)|^2} - 1\right) dx\,. \tag{IV.38}$$

Thus to prove (IV.32), it suffices to show that for any β belonging to $]0,1[$, there exists a positive constant C_β such that

$$\int_{\mathbb{R}} \left(\mathrm{e}^{\beta|w(t)|^2} - 1\right) \mathrm{e}^{-Nt}\, dt \leq C_\beta \int_{\mathbb{R}} |w(t)|^2\, \mathrm{e}^{-Nt}\, dt\,, \tag{IV.39}$$

where w satisfies (IV.33), (IV.34), (IV.35) and (IV.36). For that purpose, let us set

$$T_0 := \sup\{t \in \mathbb{R};\ w(t) \leq 1\} \in]-\infty, +\infty]$$

and write

$$\int_{\mathbb{R}} \left(\mathrm{e}^{\beta|w(t)|^2} - 1\right) \mathrm{e}^{-Nt}\, dt = I_1 + I_2\,,$$

where

$$I_1 := \int_{-\infty}^{T_0} \left(\mathrm{e}^{\beta|w(t)|^2} - 1\right) \mathrm{e}^{-Nt}\, dt \quad \text{ and } \quad I_2 := \int_{T_0}^{+\infty} \left(\mathrm{e}^{\beta|w(t)|^2} - 1\right) \mathrm{e}^{-Nt}\, dt\,.$$

In order to estimate I_1, let us notice that for any $t \leq T_0$, $w(t)$ belongs to $[0,1]$. Using the fact that there exists a positive constant M such that

$$\mathrm{e}^x - 1 \leq Mx\,, \quad \forall\, x \in [0,1]\,,$$

we deduce that

$$I_1 \leq M\,\beta \int_{-\infty}^{T_0} |w(t)|^2 \mathrm{e}^{-Nt}\, dt\,.$$

Let us now estimate I_2. By virtue of Cauchy-Schwarz inequality, we get for any $t \geq T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} w'(\tau)\, d\tau \\ &\leq 1 + \sqrt{t - T_0}\, \|w'\|_{L^2(\mathbb{R})}\,. \end{aligned}$$

This implies, in view of (IV.36), that

$$w(t) \leq 1 + \sqrt{(t - T_0)N}\,.$$

In addition, using the fact that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$1 + \sqrt{s} \leq \sqrt{(1+\varepsilon)s + C_\varepsilon}\,,$$

we deduce that for any $t \geq T_0$

$$w(t)^2 \leq (1+\varepsilon)(t-T_0)N + C_\varepsilon\,.$$

As $\beta \in]0,1[$, we can choose ε such that $\beta(1+\varepsilon)-1<0$. Hence,

$$\begin{aligned} I_2 &\leq \int_{T_0}^{+\infty} \mathrm{e}^{\beta(1+\varepsilon)(t-T_0)N+\beta C_\varepsilon - Nt}\,dt \\ &\leq \mathrm{e}^{\beta C_\varepsilon - NT_0} \int_{T_0}^{+\infty} \mathrm{e}^{(t-T_0)N[\beta(1+\varepsilon)-1]}\,dt \\ &\leq \frac{\mathrm{e}^{\beta C_\varepsilon - NT_0}}{N[1-\beta(1+\varepsilon)]}\,. \end{aligned}$$

Since $\displaystyle\int_{T_0}^{+\infty} |w(t)|^2\,\mathrm{e}^{-Nt}\,dt \geq \int_{T_0}^{+\infty} \mathrm{e}^{-Nt}\,dt = \frac{\mathrm{e}^{-NT_0}}{N}$, we infer that

$$I_2 \leq \frac{\mathrm{e}^{\beta C_\varepsilon}}{1-\beta(1+\varepsilon)} \int_{T_0}^{+\infty} |w(t)|^2\,\mathrm{e}^{-Nt}\,dt\,.$$

Now, setting $C_\beta = \max\left\{M\beta, \dfrac{\mathrm{e}^{\beta C_\varepsilon}}{1-\beta(1+\varepsilon)}\right\}$, we get (IV.39). This ends the proof of Inequality (IV.32).

Finally, note that the example by Moser f_k defined by (IV.11) illustrates the sharpness of Inequality (IV.32), since $\|f_k\|_{L^2(\mathbb{R}^{2N})} \overset{k\to\infty}{\longrightarrow} 0$ and

$$\int_{\mathbb{R}^{2N}} \left(\mathrm{e}^{\gamma_N |f_k(x)|^2} - 1\right) dx \geq \int_{|x|<\mathrm{e}^{-k}} \left(\mathrm{e}^{\gamma_N |f_k(x)|^2} - 1\right) dx = \frac{\pi^N}{N!}\left(1-\mathrm{e}^{-2Nk}\right) \underset{k\to\infty}{\longrightarrow} \frac{\pi^N}{N!}\,.$$

■

ANNEXE A

INÉGALITÉS DE TYPE TRUDINGER-MOSER

Dans cette section, on se propose de rappeler quelques inégalités de type Trudinger-Moser. Ces inégalités ont joué un rôle fondamental dans cette thèse, que ce soit dans l'étude du défaut de compacité des injections de Sobolev qu'elles induisent ou dans l'analyse d'une équation de Klein-Gordon semi-linéaire avec une non linéarité exponentielle.

Avant d'exposer ces inégalités, rappelons d'abord quelques injections de Sobolev.

Proposition A.1 *Soit Ω un domaine de $\mathbb{R}^d$, avec $d \geq 2$. On a les injections de Sobolev suivantes :*

$$\begin{aligned} W_0^{1,p}(\Omega) &\hookrightarrow L^{\frac{dp}{d-p}}(\Omega)\,, \quad \forall\, 1 \leq p < d\,, &\text{(A.1)}\\ W_0^{1,d}(\Omega) &\hookrightarrow L^{q}(\Omega)\,, \quad \forall\, d \leq q < +\infty\,, &\text{(A.2)}\\ W_0^{1,p}(\Omega) &\hookrightarrow L^{\infty}(\Omega)\,, \quad \forall\, p > d\,. &\text{(A.3)} \end{aligned}$$

Il est à noter que l'espace de Sobolev $W_0^{1,d}(\Omega)$ ne s'injecte pas dans $L^\infty(\Omega)$. Néanmoins, le résultat suivant, qui est connu sous le nom d'inégalité de Trudinger-Moser, montre que, dans le cas où Ω est borné, on a

$$W_0^{1,d}(\Omega) \hookrightarrow L^\phi(\Omega)\,,$$

où L^ϕ est l'espace d'Orlicz associé à la fonction $\phi(s) = \mathrm{e}^{s^{\frac{d}{d-1}}}$.

Théorème A.1 *[78, Theorem 1] Soit Ω un domaine borné de $\mathbb{R}^d$, avec $d \geq 2$. Il existe une constante $C_d > 0$ telle que*

$$\sup_{u \in W_0^{1,d}(\Omega),\, \|\nabla u\|_{L^d} \leq 1} \int_\Omega \mathrm{e}^{\alpha_d |u(x)|^{\frac{d}{d-1}}}\, dx \leq C_d |\Omega|\,, \tag{A.4}$$

où $\alpha_d := d\,\omega_{d-1}^{\frac{1}{d-1}}$, avec ω_{d-1} l'aire de la sphère unité de $\mathbb{R}^d$. De plus, l'exposant α_d est optimal.

Plus tard, S. Adachi et K. Tanaka ([1]) ont obtenu une extension de l'inégalité de Trudinger-Moser (A.4) pour un domaine quelconque de $\mathbb{R}^d$. Leur résultat concernant le cas de la dimension 2 se formule comme suit :

Théorème A.2 *[1, Theorem 0.1, Theorem 0.2] Pour tout $0 < \alpha < 4\pi$, il existe une constante $C_\alpha > 0$ telle que*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha |u(x)|^2} - 1 \right) dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}\,, \quad \forall\, u \in H^1(\mathbb{R}^2) \text{ avec } \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1\,. \tag{A.5}$$

De plus,

$$\sup_{u \in H^1(\mathbb{R}^2),\, \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi |u(x)|^2} - 1 \right) dx = +\infty\,.$$

Ultérieurement, B. Ruf ([86]) a montré que l'exposant 4π devient admissible lorsque la norme de Dirichlet $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ dans l'inégalité (A.5) est remplacée par la norme de Sobolev classique

$$\|u\|^2_{H^1(\mathbb{R}^2)} = \|u\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u\|^2_{L^2(\mathbb{R}^2)}\,.$$

Ce résultat s'énonce somme suit :

Théorème A.3 *[86, Theorem 1.1] On a*

$$\sup_{u\in H^1(\mathbb{R}^2),\, \|u\|_{H^1(\mathbb{R}^2)}\le 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2}-1\right) dx < +\infty\,. \tag{A.6}$$

De plus, cette inégalité est optimale.

Preuve. Commençons par établir l'inégalité (A.6). Sachant que si $u \in H^1(\mathbb{R}^2)$, alors son réarrangement symétrique décroissant u^* satisfait

$$\begin{aligned} \|u^*\|_{L^2(\mathbb{R}^2)} &= \|u\|_{L^2(\mathbb{R}^2)}\,, \\ \|\nabla u^*\|_{L^2(\mathbb{R}^2)} &\le \|\nabla u\|_{L^2(\mathbb{R}^2)} \quad \text{et} \\ \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u^*(x)|^2}-1\right) dx &= \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2}-1\right) dx\,, \end{aligned}$$

il suffit de se restreindre au cas des fonctions positives et décroissantes de $H^1_{rad}(\mathbb{R}^2)$.

Considérons maintenant, pour $r_0 > 0$ qui sera choisi ultérieurement, les intégrales suivantes :

$$I_1 := \int_{B(r_0)} \left(\mathrm{e}^{4\pi|u(x)|^2}-1\right) dx \quad \text{et} \quad I_2 := \int_{\mathbb{R}^2\setminus B(r_0)} \left(\mathrm{e}^{4\pi|u(x)|^2}-1\right) dx\,,$$

où $B(r_0)$ est la boule centrée à l'origine et de rayon r_0 et montrons qu'il est possible de trouver un $r_0 > 0$ indépendant de u tel que les intégrales I_1 et I_2 soient majorées par une constante qui dépend uniquement de r_0. Pour ce faire, on décompose d'abord I_2 comme suit :

$$I_2 = \sum_{k=1}^{\infty} \frac{(4\pi)^k}{k!} I_{2,k}\,, \text{ où } \quad I_{2,k} := \int_{\mathbb{R}^2\setminus B(r_0)} |u(x)|^{2k}\, dx\,.$$

Pour contrôler $I_{2,k}$, on utilise l'estimation suivante spécifique au cas radial décroissant (voir [31, Lemma A.IV]) :

$$|u(r)| \le \frac{1}{\sqrt{\pi}\, r} \|u\|_{L^2(\mathbb{R}^2)}\,, \quad \forall\, r > 0\,, \tag{A.7}$$

ce qui implique que, pour tout entier $k \ge 2$,

$$\begin{aligned} I_{2,k} &\le \frac{2\|u\|^{2k}_{L^2(\mathbb{R}^2)}}{\pi^{k-1}} \int_{r_0}^{\infty} \frac{dr}{r^{2k-1}} \\ &\le \frac{\|u\|^2_{L^2(\mathbb{R}^2)}}{k-1} \left(\frac{\|u\|^2_{L^2(\mathbb{R}^2)}}{\pi r_0^2}\right)^{k-1}. \end{aligned}$$

En utilisant le fait que $\|u\|_{L^2(\mathbb{R}^2)} \le 1$, on en déduit l'inégalité suivante :

$$\begin{aligned} I_2 &\le 4\pi\|u\|^2_{L^2(\mathbb{R}^2)} + 4\pi\|u\|^2_{L^2(\mathbb{R}^2)} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{4\|u\|^2_{L^2(\mathbb{R}^2)}}{r_0^2}\right)^{k-1} \\ &\le 4\pi + 4\pi \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{4}{r_0^2}\right)^{k-1} = C(r_0)\,. \end{aligned}$$

Pour estimer I_1, on considère la fonction

$$v(r) := u(r) - u(r_0)\,, \quad 0 < r < r_0\,.$$

Comme u appartient à $H^1_{rad}(\mathbb{R}^2)$ et $\|u\|_{H^1(\mathbb{R}^2)} \leq 1$, alors v est bien définie. En effet, pour tous nombres réels $r_2 > r_1 > 0$, en écrivant

$$u(r_2) - u(r_1) = \int_{r_1}^{r_2} u'(s)\,ds\,,$$

on obtient en utilisant l'inégalité de Cauchy-Schwarz

$$\begin{aligned} |u(r_2) - u(r_1)| &\leq \left(\int_{r_1}^{r_2} |u'(s)|^2 s\,ds\right)^{\frac{1}{2}} \left(\int_{r_1}^{r_2} \frac{1}{s}\,ds\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|_{L^2(\mathbb{R}^2)} \left(\int_{r_1}^{r_2} \frac{1}{s}\,ds\right)^{\frac{1}{2}}\,, \end{aligned}$$

ce qui implique que u est continue loin de l'origine.

En vertu de l'estimation radiale (A.7), on a

$$\begin{aligned} u^2(r) &\leq v^2(r) + 2v(r)u(r_0) + u^2(r_0) \\ &\leq v^2(r) + v^2(r)\frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)} + 1 + \frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)} \\ &\leq w(r)^2 + d(r_0)\,, \end{aligned}$$

avec $w(r) := v(r)\sqrt{1 + \frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)}}$ et $d(r_0) := 1 + \frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)}$. Il est clair que la fonction w appartient à $H^1_0(B(r_0))$. De plus, elle vérifie

$$\begin{aligned} \int_{B(r_0)} |\nabla w(x)|^2\,dx &= \left(1 + \frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)}\right) \int_{B(r_0)} |\nabla u(x)|^2\,dx \\ &\leq \left(1 + \frac{1}{\pi r_0^2}\,\|u\|^2_{L^2(\mathbb{R}^2)}\right)\left(1 - \|u\|^2_{L^2(\mathbb{R}^2)}\right) \leq 1\,, \end{aligned}$$

pour $r_0 \geq \frac{1}{\sqrt{\pi}}$. En appliquant l'inégalité de Trudinger-Moser (A.4) pour $d = 2$, on déduit l'existence d'une constante $c > 0$ telle que

$$I_1 \leq \mathrm{e}^{4\pi d(r_0)} \int_{B(r_0)} \mathrm{e}^{4\pi |w(x)|^2}\,dx \leq c\,\mathrm{e}^{4\pi d(r_0)}\,r_0^2\,, \quad \forall\, r_0 \geq \frac{1}{\sqrt{\pi}}\,.$$

Ceci termine la preuve de l'inégalité (A.6).

Montrons, maintenant, l'optimalité de l'exposant 4π dans l'inégalité (A.6). A cet effet, considérons la suite de fonctions de Moser $(u_k)_{k\in\mathbb{N}}$ donnée par

$$u_k(x) = \begin{cases} 0 & \text{si } |x| \geq 1\,, \\ -\dfrac{\log|x|}{\sqrt{2k\pi}} & \text{si } \mathrm{e}^{-k} \leq |x| \leq 1\,, \\ \sqrt{\dfrac{k}{2\pi}} & \text{si } |x| \leq \mathrm{e}^{-k}\,. \end{cases}$$

Par un simple calcul, on obtient que

$$\|u_k\|_{H^1(\mathbb{R}^2)} = 1 + \circ(1)\,, \quad \text{lorsque } k \to \infty\,,$$

ce qui implique que, pour tout $\alpha > 4\pi$,

$$\begin{aligned} \int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha \left| \frac{u_k(x)}{\|u_k\|_{H^1(\mathbb{R}^2)}} \right|^2} - 1 \right) dx &\geq 2\pi \int_0^{\mathrm{e}^{-k}} \left(\mathrm{e}^{\frac{k\alpha}{2\pi(1+\circ(1))}} - 1 \right) r\,dr \\ &\geq \pi \left(\mathrm{e}^{k\,\frac{\alpha - 4\pi(1+\circ(1))}{2\pi(1+\circ(1))}} - \mathrm{e}^{-2k} \right) \overset{k\to+\infty}{\longrightarrow} +\infty\,. \end{aligned}$$

Ceci achève la preuve du théorème.

■

ANNEXE B

ESPACES D'ORLICZ

Dans cette section, on va se limiter au rappel de quelques propriétés des espaces d'Orlicz. Pour plus de détails, on peut consulter [85].

B.1 Définition et propriétés élémentaires

Définition B.1 *Soit $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ une fonction convexe et croissante vérifiant*

$$\lim_{s\to 0^+} \phi(s) = \phi(0) = 0 \quad et \quad \lim_{s\to +\infty} \phi(s) = +\infty\,.$$

On dit qu'une fonction mesurable $u : \mathbb{R}^d \to \mathbb{C}$ appartient à l'espace d'Orlicz L^ϕ s'il existe $\lambda > 0$ tel que

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty\,. \tag{B.1}$$

On note alors

$$\|u\|_{L^\phi} = \inf\left\{\lambda > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}\cdot \tag{B.2}$$

Proposition B.2 *L'espace d'Orlicz L^ϕ est un $\mathbb{C}$-espace vectoriel et $\|\,.\,\|_{L^\phi}$ est une semi-norme.*

Preuve. Il est clair que la fonction identiquement nulle appartient à l'espace d'Orlicz L^ϕ, ce qui assure que cet espace est non vide.

Commençons par montrer que L^ϕ est un $\mathbb{C}$-espace vectoriel. Pour ce faire, étant donnés deux fonctions u et v de L^ϕ et un nombre complexe α, vérifions que $\alpha u + v$ appartient à L^ϕ. Par hypothèse, il existe deux nombres réels $\lambda_1 > 0$ et $\lambda_2 > 0$ tels que

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda_1}\right) dx < \infty \;\text{ et }\; \int_{\mathbb{R}^d} \phi\left(\frac{|v(x)|}{\lambda_2}\right) dx < \infty\,.$$

Par ailleurs, comme ϕ est croissante, on obtient pour tout $\lambda > 0$

$$\int_{\mathbb{R}^d} \phi\left(\frac{|\alpha u(x) + v(x)|}{\lambda}\right) dx \leq \int_{\mathbb{R}^d} \phi\left(\frac{|\alpha|\lambda_1}{\lambda}\frac{|u(x)|}{\lambda_1} + \frac{\lambda_2}{\lambda}\frac{|v(x)|}{\lambda_2}\right) dx\,.$$

En vertu de la convexité, on déduit que pour $\lambda = |\alpha|\lambda_1 + \lambda_2$

$$\int_{\mathbb{R}^d} \phi\left(\frac{|\alpha u(x) + v(x)|}{\lambda}\right) dx \leq \frac{|\alpha|\lambda_1}{\lambda}\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda_1}\right) dx + \frac{\lambda_2}{\lambda}\int_{\mathbb{R}^d} \phi\left(\frac{|v(x)|}{\lambda_2}\right) dx\,,$$

ce qui achève la preuve du résultat.

Démontrons à présent que $\|\,.\,\|_{L^\phi}$ est une semi-norme. Il est évident que si u est une fonction nulle presque partout, alors $\|u\|_{L^\phi} = 0$. Inversement, soit u une fonction de L^ϕ telle que $\|u\|_{L^\phi} = 0$. Il existe alors une suite décroissante de nombres réels strictement positifs $(\lambda_n)_{n\in\mathbb{N}}$ vérifiant à la fois

$$\lim_{n\to\infty} \lambda_n = 0 \quad \text{et} \tag{B.3}$$

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda_n}\right) dx \leq 1, \ \forall\, n \in \mathbb{N}\,. \tag{B.4}$$

Ceci implique facilement que la suite $v_n(x) := \phi\left(\frac{|u(x)|}{\lambda_n}\right)$ satisfait

$$\lim_{n\to\infty} v_n(x) = \begin{cases} +\infty & \text{si } u(x) \neq 0\,, \\ 0 & \text{si } u(x) = 0\,. \end{cases}$$

Supposons maintenant que la mesure de Lebesgue de l'ensemble $\Big\{x \in \mathbb{R}^d;\ |u(x)| > 0\Big\}$ est non nulle. En appliquant le théorème de convergence monotone, on trouve que

$$\lim_{n\to\infty} \int_{\{x\in\mathbb{R}^d;\ |u(x)|>0\}} v_n(x)\, dx = +\infty\,,$$

ce qui contredit (B.4), et donc entraîne que u est nulle presque pour tout $x \in \mathbb{R}^d$.

Comme pour tout $u \in L^\phi$ et pour tout $\alpha \in \mathbb{C}$ on a

$$\inf\left\{\lambda > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|\alpha u(x)|}{\lambda}\right) dx \leq 1\right\} = \inf\left\{\lambda' > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda'}\right) dx \leq 1\right\}\,,$$

on obtient

$$\|\alpha u\|_{L^\phi} = |\alpha|\|u\|_{L^\phi}\,.$$

Pour achever la preuve du lemme, il reste à établir l'inégalité triangulaire :

$$\|u+v\|_{L^\phi} \leq \|u\|_{L^\phi} + \|v\|_\phi\,, \quad \forall\, u\,, v \in L^\phi\,.$$

Par définition, il suffit de montrer que

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)+v(x)|}{\|u\|_{L^\phi} + \|v\|_{L^\phi}}\right) dx \leq 1\,. \tag{B.5}$$

En utilisant la convexité de la fonction ϕ, on obtient

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)+v(x)|}{\|u\|_{L^\phi} + \|v\|_{L^\phi}}\right) dx \\ &\qquad \leq \int_{\mathbb{R}^d} \phi\left(\frac{\|u\|_{L^\phi}}{\|u\|_{L^\phi} + \|v\|_{L^\phi}} \frac{|u(x)|}{\|u\|_{L^\phi}} + \frac{\|v\|_{L^\phi}}{\|u\|_{L^\phi} + \|v\|_{L^\phi}} \frac{|v(x)|}{\|v\|_{L^\phi}}\right) dx \\ &\qquad \leq \frac{\|u\|_{L^\phi}}{\|u\|_{L^\phi} + \|v\|_{L^\phi}} \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\|u\|_{L^\phi}}\right) dx + \frac{\|v\|_{L^\phi}}{\|u\|_{L^\phi} + \|v\|_{L^\phi}} \int_{\mathbb{R}^d} \phi\left(\frac{|v(x)|}{\|v\|_{L^\phi}}\right)\,, \end{aligned}$$

ce qui assure l'inégalité (B.5) sachant que les intégrales $\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\|u\|_{L^\phi}}\right) dx$ et $\int_{\mathbb{R}^d} \phi\left(\frac{|v(x)|}{\|v\|_{L^\phi}}\right) dx$ sont inférieures à 1. ∎

Dans la suite, l'espace quotient de L^ϕ par la relation d'équivalence d'égalité presque partout sera noté également par L^ϕ. Ainsi, l'espace $(L^\phi, \|\,.\,\|_{L^\phi})$ peut être considéré comme un espace normé.

Remarque B.3 *On peut remplacer la constante* 1 *dans la définition* (B.2) *par n'importe quelle constante positive. Ceci change la norme* $\|\,.\,\|_{L^\phi}$ *par une norme équivalente. En effet, soient* $0 < C < 1$ *un nombre réel et* $\|\,.\,\|_{\tilde{L^\phi}}$ *la norme définie par*

$$\|u\|_{\tilde{L^\phi}} = \inf\left\{\lambda > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \le C\right\}, \quad \forall\, u \in L^\phi .$$

D'une part, il est facile de voir que

$$\|u\|_{L^\phi} \le \|u\|_{\tilde{L^\phi}} .$$

D'autre part, la convexité de la fonction ϕ *entraîne que*

$$\int_{\mathbb{R}^d} \phi\left(\frac{C|u(x)|}{\|u\|_{L^\phi}}\right) dx \le C \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\|u\|_{L^\phi}}\right) dx \le C ,$$

ce qui implique que

$$\|u\|_{\tilde{L^\phi}} \le \frac{1}{C}\|u\|_{L^\phi}$$

et assure le résultat dans le cas où $C < 1$. *De la même manière, on obtient l'équivalence des normes* $\|\,.\,\|_{L^\phi}$ *et* $\|\,.\,\|_{\tilde{L^\phi}}$ *dans le cas où* $C \ge 1$.

Terminons ce paragraphe par les lemmes suivants qui seront utiles dans la section suivante.

Lemme B.4 *Soit* X *une partie Lebesgue-mesurable de* $\mathbb{R}^d$ *de mesure finie. Alors,*

$$L^\phi(X) \hookrightarrow L^1(X) .$$

Preuve. D'après les propriétés de la fonction ϕ, il existe deux nombres réels $a > 0$ et $b \ge 0$ tels que

$$as - b \le \phi(s) , \quad \forall\, s \ge 0 . \tag{B.6}$$

Par conséquent, si $u \in L^\phi(X)$ et $\lambda > 0$, alors

$$\frac{1}{\lambda}\int_X |u(x)|\, dx \le \frac{1}{a}\int_X \phi\left(\frac{|u(x)|}{\lambda}\right) dx + \frac{b}{a}|X| < +\infty ,$$

ce qui implique que $u \in L^1(X)$.

Pour achever la preuve du Lemme B.4, il suffit de montrer l'existence d'une constante $C > 0$ telle que

$$\int_X \phi\left(\frac{|u(x)|}{C\|u\|_{L^1(X)}}\right) dx \ge 1 .$$

Or, d'après l'inégalité (B.6), on a pour toute constante strictement positive C

$$\begin{aligned}\int_X \phi\left(\frac{|u(x)|}{C\|u\|_{L^1(X)}}\right) dx &\geq \int_X \frac{a|u(x)|}{C\|u\|_{L^1(X)}}\, dx - b|X| \\ &\geq \frac{a}{C} - b|X|\,,\end{aligned}$$

ce qui entraîne le résultat, par un choix convenable de C. ■

Lemme B.5 *Soit $(u_n)_n$ une suite de l'espace d'Orlicz L^ϕ. Alors, les assertions suivantes sont équivalentes :*

1. $\forall \lambda > 0$, $\displaystyle\limsup_{n\to\infty} \int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda}\right) dx \leq 1\,.$
2. $\displaystyle\lim_{n\to\infty} \|u_n\|_{L^\phi} = 0\,.$

Preuve. Supposons que, pour tout nombre réel $\lambda > 0$,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda}\right) dx \leq 1\,. \tag{B.7}$$

Puisque $\phi(0) = 0$ et ϕ est convexe, on a

$$\phi(s) \leq \varepsilon\, \phi\left(\frac{s}{\varepsilon}\right), \quad \forall\, s \geq 0\,, \quad \forall\, 0 < \varepsilon \leq 1\,.$$

Par conséquent,

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda}\right) dx \leq \varepsilon \int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda\,\varepsilon}\right) dx\,,$$

ce qui implique, grâce à l'hypothèse 1, que pour n assez grand

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda}\right) dx \leq \varepsilon\,.$$

Comme ε est arbitraire, il vient en vertu de la définition de la norme $\|\,.\,\|_{L^\phi}$ que

$$\lim_{n\to\infty} \|u_n\|_{L^\phi} = 0\,.$$

Inversement, supposons que $\lim_{n\to\infty} \|u_n\|_{L^\phi} = 0$. Donc pour tout nombre réel $\lambda > 0$, il existe un entier $N > 0$ tel que

$$\frac{1}{\lambda} < \frac{1}{\|u_n\|_{L^\phi}}\,, \quad \forall\, n \geq N\,.$$

Sachant que ϕ est une fonction croissante, on en déduit que pour tout $n \geq N$

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\lambda}\right) dx \leq \int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x)|}{\|u_n\|_{L^\phi}}\right) dx \leq 1\,,$$

ce qui conclut la preuve du résultat. ■

B.2 Complétude

Proposition B.6 *L'espace normé* $(L^\phi, \|\,.\,\|_{L^\phi})$ *est un espace de Banach.*

Preuve. Etant donnée une suite de Cauchy $(u_n)_{n\in\mathbb{N}}$ de l'espace $(L^\phi, \|\,.\,\|_{L^\phi})$, notre propos est de montrer qu'elle converge dans $(L^\phi, \|\,.\,\|_{L^\phi})$. Pour ce faire, écrivons d'abord l'ensemble $\mathbb{R}^d$ comme suit :

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{N}} X_k \,,$$

où $X_k := [-k, k]^d$. Comme $|X_k| < \infty$, le lemme B.4 assure que $L^\phi(X_k)$ s'injecte continûment dans $L^1(X_k)$. Par conséquent, la suite $(u_n)_{n\in\mathbb{N}}$ est une suite de Cauchy dans $(L^1(X_k), \|\,.\,\|_{L^1(X_k)})$ qui est un espace complet. On en déduit qu'elle converge dans $(L^1(X_k), \|\,.\,\|_{L^1(X_k)})$, et donc elle converge presque pour tout $x \in X_k$, à extraction d'une sous-suite près. En utilisant le procédé d'extraction diagonale, on peut extraire une sous-suite (u_{n_k}) de $(u_n)_{n\in\mathbb{N}}$ qui converge presque pour tout $x \in X_k$ vers u dans $\mathbb{R}^d$.

D'autre part, comme $(u_n)_{n\in\mathbb{N}}$ est une suite de Cauchy de $(L^\phi, \|\,.\,\|_{L^\phi})$, l'assertion 1 du Lemme B.5 entraîne que pour n et m assez grands, on a

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x) - u_m(x)|}{\lambda}\right) dx \leq 1\,, \quad \forall\, \lambda > 0\,.$$

En appliquant le lemme de Fatou, on obtient pour n assez grand

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x) - u(x)|}{\lambda}\right) dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^d} \phi\left(\frac{|u_n(x) - u_{n_k}(x)|}{\lambda}\right) dx \leq 1\,, \quad \forall\, \lambda > 0\,.$$

Ceci termine la preuve de la proposition, en vertu du Lemme B.5.

■

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Résumé

Cette thèse porte sur quelques inégalités de type Trudinger-Moser avec leurs applications à l'étude des injections de Sobolev qu'elles induisent dans les espaces d'Orlicz et l'analyse de quelques équations aux dérivées partielles non linéaires à croissance exponentielle. Le travail qu'on présente ici se compose de trois parties. La première partie est consacrée à la description du défaut de compacité de l'injection de Sobolev 4D dans l'espace d'Orlicz dans le cadre radial.

L'objectif de la deuxième partie est double. D'abord, on caractérise le défaut de compacité de l'injection de Sobolev 2D dans les différentes classes d'espaces d'Orlicz. Ensuite, on étudie l'équation de Klein-Gordon semi-linéaire avec non linéarité exponentielle, où la norme d'Orlicz joue un rôle crucial. En particulier, on aborde les questions d'existence globale, de complétude asymptotique et d'étude qualitative.

Dans la troisième partie, on établit des inégalités optimales de type Adams, en étroite relation avec les inégalités de Hardy, puis on fournit une description du défaut de compacité des injections de Sobolev qu'elles induisent.

Mots clés : inégalités de Trudinger-Moser, injections de Sobolev, espaces d'Orlicz, défaut de compacité, équation de Klein-Gordon, inégalités de Hardy.

Abstract

This thesis focuses on some Trudinger-Moser type inequalities with their applications to the study of Sobolev embeddings they induce into the Orlicz spaces, and the investigation of some nonlinear partial differential equations with exponential growth.

The work presented here includes three parts. The first part is devoted to the description of the lack of compactness of the 4D Sobolev embedding into the Orlicz space in the radial framework.

The aim of the second part is twofold. Firstly, we characterize the lack of compactness of the 2D Sobolev embedding into the different classes of Orlicz spaces. Secondly, we undertake the study of the nonlinear Klein-Gordon equation with exponential growth, where the Orlicz norm plays a crucial role. In particular, issues of global existence, scattering and qualitative study are investigated.

In the third part, we establish sharp Adams-type inequalities invoking Hardy inequalities, then we give a description of the lack of compactness of the Sobolev embeddings they induce.

Keywords : Trudinger-Moser inequalities, Sobolev embeddings, Orlicz spaces, lack of compactness, Klein-Gordon equation, Hardy inequalities.