p-adic admissible measures attached to Siegel modular forms of arbitrary genus

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Mesures p-adiques admissibles associées aux Formes modulaires de Siegel de genre arbitraire

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Introduction

Version française

Les objets centraux de cette thèse sont les formes modulaires de Siegel (et en particulier les séries d'Eisenstein) et les fonction L p-adiques. Les principaux objectifs sont de donner une construction conceptuelle des mesures p-adiques admissibles associées aux formes modulaires de Siegel. Ces mesures fournissent des fonctions L p-adiques à croissance logarithmique $o(\log^h)$ d'invariant principal h (voir le théorème 2). Le demi-plan supérieur de Siegel de genre n est l'ensemble de toutes les matrices complexes symétriques de partie imaginaire définie positive:

$$\mathbb{H}_n = \{z = {}^t z = x + iy | x, y \in M_n(\mathbb{R}), y > 0\}$$

Le groupe symplectique $\mathrm{Sp}_n(\mathbb{Z})$ agit sur l'espace $\mathbb{H}_n$ via:

$$\gamma(z) = (az + b)(cz + d)^{-1}$$

où $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z})$ et $z \in \mathbb{H}_n$. Une fonction holomorphe $f : \mathbb{H}_n \longrightarrow \mathbb{C}^r$ est dite forme modulaire de Siegel de genre n et de poids l sur $\mathrm{Sp}_n(\mathbb{Z})$ lorsqu'elle satisfait

$$\det(cz + d)^{-l} f(\gamma(z)) = f(z) \quad \forall \gamma \in \mathrm{Sp}_n(\mathbb{Z}).$$

Dans le cas $n = 1$, on demande aussi à f d'être holomorphe à l'infini. Les formes modulaires de Siegel de genre 1 sont les formes modulaires classiques sur le demi-plan supérieur pour le groupe $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_1(\mathbb{Z})$ et ses sous-groupes de congruence.
L'étude des fonctions zêta standard amène à considérer les paramètres de Satake, un invariant lié aux formes modulaires de Siegel. Les p-paramètres de Satake associés à une forme propre $f \in \mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$ sont les composants du $(n+1)$-uplet $(\alpha_0, \alpha_1, \cdots, \alpha_n) \in [(A^\times)^{n+1}]^{W_n}$ qui est l'image de l'application $f \longmapsto \lambda_f(X)$ à travers l'isomorphisme $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}_n, \mathbb{C}) \cong [(A^\times)^{n+1}]$, qui est défini au quotient près par l'action de W_n, où W_n est le groupe de Weyl et $\mathcal{L}_n$ une algèbre de Hecke locale. Les p-paramètres de Satake satisfont la relation suivante:

$$\alpha_0^2 \alpha_1 \cdots \alpha_n = \psi(p)^n p^{ln - \frac{n(n+1)}{2}}.$$

Soit p un nombre premier et $f \in S_n^l(\Gamma_0(N), \psi)$ une forme propre de Siegel de genre n et poids l, de p-paramètres de Satake $\alpha_0(p), \alpha_1(p), \cdots, \alpha_n(p)$. On considère la fonction zêta standard $D^{(Np)}(f, s, \chi)$, qui prend des valeurs algébriques aux points critiques (après normalisation). Cette fonction zêta standard de f est définie à l'aide des p-paramètres de Satake et du produit d'Euler suivant:

$$D^{(M)}(f, s, \chi) := \prod_{q \nmid M} \left(\frac{1}{1 - \chi\psi(q) q^{-s}} \prod_{i=1}^{n} \frac{1}{(1 - \chi\psi(q)\alpha_i(q) q^{-s})(1 - \chi\psi(q)\alpha_i^{-1}(q) q^{-s})} \right)$$

où χ est un caractère de Dirichlet arbitraire. On définit les fonctions normalisées suivantes:

$$\mathcal{D}^{(M\star)}(s,f,\chi) = (2\pi)^{-n(s+l-(n+1)/2}\Gamma((s+\delta)/2)\prod_{j=1}^{n}\left(\Gamma(s+l-j)\right)\mathcal{D}^{(M)}(s,f,\chi)$$

$$\mathcal{D}^{(M+)}(s,f,\chi) = \Gamma((s+\delta)/2)\mathcal{D}^{(M\star)}(s,f,\chi)$$
$$\mathcal{D}^{(M-)}(s,f,\chi) = \frac{i^{\delta}\pi^{1/2-s}}{\Gamma((1-s+\delta)/2)}\mathcal{D}^{(M\star)}(s,f,\chi)$$

où l'on écrit $\delta = 0$ ou $\delta = 1$ suivant si $\chi(-1) = 1$ ou $\chi(-1) = -1$. Rappelons d'abord le résultat suivant sur les propriétés d'algébricité des valeurs spéciales des fonctions zêta standard :

Théorème 1 (Algébricité des valeurs spéciales des fonctions zêta standard).

(a) Pour tous les entiers s tels que $1 \le s \le l-\delta-n$ et $s \equiv \delta \mod 2$ et tout caractère de Dirichlet χ tel que χ^2 n'est pas trivial si $s = 1$, on sait:

$$\langle f,f\rangle^{-1}D^{(M+)}(f,s,\chi) \in K = \mathbb{Q}(f,\Lambda_f,\chi),$$

où $K = \mathbb{Q}(f,\Lambda_f,\chi)$ désigne le corps engendré par les coefficients de Fourier de f, par les valeurs propres $\Lambda_f(X)$ des opérateurs de Hecke X agissant sur f et par les valeurs de χ.

(b) Pour tous les entiers s tels que $1-l+\delta+n \le s \le 0$ et $s \not\equiv \delta \mod 2$, on sait:

$$\langle f,f\rangle^{-1}D^{(M-)}(f,s,\chi) \in K.$$

Les propriétés analytiques de ces fonctions L complexes ont été étudiées par plusieurs auteurs (voir [2], [4] et [27]) et sont plus ou moins bien connues. Elles ont pour conséquence que les valeurs critiques $D(s,f,\chi)$ de la fonction zêta standard normalisée peuvent être explicitement réécrites en termes d'intégrales à valeurs dans $\mathbb{C}_p$ le long de mesures admissibles (sur le groupe $\mathbb{Z}_p^\times$ des unités p-adiques), au moins dans le cas où χ est non trivial.

Le domaine de définition des fonctions zêta p-adiques est le groupe de Lie p-adique

$$X_p = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$$

de tous les caractères p-adiques continus du groupe profini $\mathbb{Z}_p^\times$, où $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ note le corps de Tate (la complétion d'une clôture algébrique du corps p-adique $\mathbb{Q}_p$), que l'on munit de l'unique norme $|.|_p$ telle que $|p|_p = 1/p$, de telle sorte que chaque entier k peut être identifié au caractère $x_p^k; y \mapsto y^k$. On travaillera tout au long de la thèse avec un plongement

$$i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$$

et on identifiera $\mathbb{Q}$ avec un sous-corps de $\mathbb{C}$ et de $\mathbb{C}_p$. Un caractère de Dirichlet $\chi : (\mathbb{Z}/p^N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ est alors un élément du sous-groupe de torsion

$$X_p^{\mathrm{tors}} \subset X_p = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times).$$

Pour énoncer le résultat principal, on utilise aussi certains facteurs élémentaires tels que $E_p(s,\psi)$, $\Lambda_\infty(s)$ et $A(\chi)$, mais aussi des sommes de Gauss, les p-paramètres de Satake et le conducteur c_χ de χ. Précisément, on note:

$$E_p(s,\psi) := \prod_{j=1}^{n} \frac{(1-\psi(p)\alpha_j^{-1}p^{s-1})}{(1-\bar{\psi}(p)\alpha_j^{-1}p^{-s})}.$$

Définissons encore:

$$\Lambda_\infty^{+}(s) := \frac{(2i)^s \cdot \Gamma(s)}{(2\pi i)^s} \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s+l-j)$$
$$\Lambda_\infty^{-}(s) := (2i)^s \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(1-s+l-j)$$

où $\Gamma_{\mathbb{C}}(s) := 2\cdot(2\pi)^{-s}\Gamma(s)$. De plus pour tout caractère χ de conducteur une puissance de p c_χ on note:

$$A^{-}(\chi) := c_\chi^{nl-\frac{n(n+1)}{2}} \alpha(c_\chi)^{-2} \cdot (\chi^0([p,c_\chi]) \cdot \chi(-1)G(\chi))$$
$$A^{+}(\chi) := (\overline{\chi^0\varphi})_o(c_\chi) \cdot \frac{A^{-}(\chi)}{\chi(-1)G(\chi)},$$

où $[a,b]$ est le plus petit multiple commun de a et b. Enfin, on écrit:

$$E_p^{+} := (1-(\overline{\varphi\chi\chi^0})_0(p)p^{t-1}) \cdot E_p(s,\chi\chi^0)$$
$$E_p^{-} := E_p(p,\chi\chi^0).$$

On construit deux mesures admissibles μ^+ et μ^- avec les propriétés suivantes:

Théorème 2 (Théorème principal sur les mesures admissibles, théorème 4.4.1).

(i) Pour tous les couples (s,χ) tels que $\chi \in X_p^{\mathrm{tors}}$ est un caractère de Dirichlet non trivial, $s \in \mathbb{Z}$ avec $1 \le s \le l-\delta-n$ et $s \equiv \delta \mod 2$; de plus dans le cas $s=1$ on demande que χ^2 ne soit pas trivial. Dans ces conditions, on a:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^{+} = i_p\left(c_\chi^{s(n+1)} A^{+}(\chi) \cdot E_p^{+}(s,\chi\chi^0) \frac{\Lambda_\infty^{+}(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f,s,\overline{\chi\chi^0})\right).$$

(ii) Pour tous les couples (s,χ) tels que $\chi \in X_p^{\mathrm{tors}}$ est un caractère de Dirichlet non trivial, $s \in \mathbb{Z}$ avec $l-\delta+n \le s \le 0$ et $s \not\equiv \delta \mod 2$, on a:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^{-} = i_p\left(c_\chi^{n(1-s)} A^{+}(\chi) \cdot E_p^{-}(1-s,\chi\chi^0) \frac{\Lambda_\infty^{-}(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f,1-s,\overline{\chi\chi^0})\right).$$

(iii) Si $\mathrm{ord}_p(\alpha_0(p)) = 0$ (i.e. f est p-ordinaire), alors les mesures dans (i) et (ii) sont bornées.

(iv) Dans le cas général (mais en supposant que $\alpha_0(p) \neq 0$) avec $x \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ les fonctions holomorphes

$$\mathcal{D}^+(x) = \int x d\mu^+$$
$$\mathcal{D}^-(x) = \int x d\mu^-$$

ont une croissance $o(\log(x_p)^h)$ où $h = [4\mathrm{ord}_p(\alpha_0(p))] + 1$ et peuvent être interprétées comme des transformées de Mellin de mesures h-admissibles.

(v) Si $h \leq k - m - 1$, alors les fonctions $\mathcal{D}^\pm$ sont déterminées de façon unique par les conditions (i) et (ii) ci-dessus.

Des cas particuliers ont été traités par différents auteurs:

1. Le cas du genre $n = 1$ a été étudié par B. Gorsse (2006) dans sa thèse de doctorat pour les carrés symétriques et aussi par Dabrowski et Delbourgo dans [11] ; ils ont prouvé qu'il existe une mesure h-admissible à valeurs dans $\mathbb{C}_p$ qui interpole les valeurs spéciales du carré symétrique.

2. Le cas où f n'est pas ordinaire de genre pair n a été étudié par A. Panchishkin et M. Courtieu LNM 1471(2002) par la méthode de Rankin-Selberg telle que définie par Andrianov en utilisant l'action d'opérateurs différentiels de Shimura sur les formes modulaires de Siegel.

3. Le cas où f est ordinaire de genre arbitraire n a été étudié par S. Böcherer et C.-G. Schmidt (Annales de l'Institut Fourier 2000) par la méthode de doublement.

L'assertion (iii) (i.e le cas ordinaire) qui a été prouvée par Panchishkin (voir [24]) dans le cas de genre n pair et Böcherer, Schmidt dans [7] pour un genre quelconque est aussi une conséquence facile des congruences principales de (i) et (ii).

La preuve de (iv) est similaire à la preuve dans [14], [3] et [30].

Enfin, si $h \leq l - n - 1$ alors les conditions dans (i) et (ii) déterminent de façon unique des fonctions analytiques $\mathcal{D}^\pm$ de croissance $o(\log(x_p)^h)$ par leurs valeurs spéciales – c'est une propriété générale des mesures admissibles (voir [3] et [30]). Dans le cas $h > l-n-1$, il existe beaucoup de fonctions analytiques $\mathcal{D}^\pm$ qui satisfont les conditions de (i) et (ii) suivant un choix de prolongement analytique (interpolation) pour les valeurs $\mathcal{D}^\pm(\chi x_p^s)$ si $s > l-n-1-\nu$ mais on montre dans le théorème qu'il en existe au moins une (par exemple celle construite dans la preuve de (iv)).

Pour construire les mesures admissibles μ^+ and μ^- qui vérifient (i) et (ii) on procède en quatre étapes:

1. Construction de certaines suites de distributions modulaires à valeurs dans les formes modulaires de Siegel ($\mathcal{H}^{(j)}_{L,\chi}$).

2. Application d'une forme linéaire algébrique convenable (représentée par un double produit scalaire de Petersson).

3. Vérification de l'admissibilité (les mesures sont h-admissibles).

4. Preuve que certaines intégrales coincident avec les valeurs spéciales de la fonction zêta standard.

Soit $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ le corps de Tate. Pour $h \in \mathbb{N}^*$, on note $\mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p)$ l'espace des fonctions définies sur $\mathbb{Z}_p^\times$ qui sont localement polynomiales en x_p de degré strictement plus petit que h. En particulier, $\mathcal{C}^1(\mathbb{Z}_p^\times, \mathbb{C}_p)$ est l'espace des fonctions localement constantes. Rappelons la définition des mesures admissibles à valeurs scalaires et vectorielles; voir [3], [30], [23]. Une mesure h-admissible sur $\mathbb{Z}_p^\times$ est une application $\mathbb{C}_p$-linéaire:

$$\phi : \mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p) \to V$$

telle que pour tout $t = 0, 1, \ldots, h-1$:

$$\left| \int_{a+(p^\nu)} (x_p - a_p)^t d\phi \right|_p = o(p^{\nu(h-t)}) \quad \text{lorsque } \nu \to \infty,$$

où $a_p = x_p(a)$.

Pour prouver que certaines intégrales coincident avec les valeurs spéciales de la fonction zêta standard, on utilise la méthode de doublement de Böcherer dans [5]. En général, cette méthode est appelée la méthode du pullback, et peut être décrite par la formule suivante:

$$\Lambda(k,\chi) D(k-r,f,\chi) E_k^{n,r}(z,f) = \left\langle f(w), E_k^{n+r}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \right\rangle$$

où $f \in S_k^r(\mathrm{Sp}_{2r}(\mathbb{Z}))$, $\Lambda(k,\chi)$ est un produit de fonctions L de Dirichlet, $E_k^{n,r}(z,f)$ est une série de Klingen-Eisenstein et $D(k-r,f,\chi)$ est la fonction zêta standard en le point critique $k-r$. Dans notre cas, on considère seulement $r = n$ pour lequel on a les formules plus simples suivantes. Soit φ un caractère de Dirichlet $\mod M > 1$, χ est un caractère de Dirichlet modulo N, $N^2|M$, $l = k+\nu$, $\nu \in \mathbb{N}$ et $f \in S_n^l(\Gamma_0(M)^n, \bar{\varphi})$. On a:

$$\Lambda(k+2s,\chi) D(k+2s-n,f,\bar{\chi}) f =$$

$$\langle f, \det(v)^s \det(y)^s \mathring{\mathfrak{D}}^\nu_{n,k+s} \left(\sum_{X \in \mathbb{Z}^{(n,n)}, X \mod N} \chi(\det X) \hat{\mathbb{F}}^k_{2n}(-,M,\psi,s) \mid_k \begin{pmatrix} 1_{2n} & S(\frac{X}{N}) \\ 0_{2n} & 1_{2n} \end{pmatrix} \right) \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} \rangle$$

où $S(X)$ note la matrice symétrique de dimension $2n$ $\begin{pmatrix} 0_n & X \\ {}^tX & 0_n \end{pmatrix}$, $z = x+iy$, $w = u+iv$, $\Lambda(k+2s,\chi)$ est le produit de fonctions L de Dirichlet et $D(k+2s-n,f,\bar{\chi})$ est la fonction zêta standard attachée à f. Ici on utilise un opérateur différentiel $\mathring{\mathfrak{D}}^\nu_{n,k+s}$ qui agit sur les séries de Siegel-Eisenstein de degré $2n$ sous des hypothèses que nous présenterons plus précisément plus tard. On obtient alors la valeur de la fonction L en un point critique. On passe ensuite aux autres points critiques en appliquant l'opérateur différentiel. On obtient ainsi les valeurs de la fonction L en tous les points critiques.

Dans une autre situation, Panchishkin et Courtieu utilisent la méthode de Rankin-Selberg pour trouver une représentation intégrale de la fonction zêta standard. L'idée principale, basée sur un résultat d'Andrianov, s'exprime comme une égalité entre la fonction zêta standard $D(s, f, \chi)$ et une fonction zêta de Rankin, c'est-à-dire la convolution entre la forme f donnée et une fonction thêta de caractère de Dirichlet χ modulo M. Plus précisément, la fonction zêta de Rankin $R(s, f, \chi)$ peut s'écrire comme la convolution de Rankin de f et une fonction thêta. La convolution de Rankin $L(s, f, g)$ peut s'exprimer comme le produit scalaire de Petersson entre f et le produit de g et une série d'Eisenstein idoine. Sinon, on a aussi l'identité suivante entre la fonction zêta de Rankin et la fonction zêta standard:

$$D(s, f, \chi) = L(s + (n/2), \chi\psi) \left(\prod_{i=0}^{n/2-1} L(2s + 2i, \chi^2\psi^2) \right) R(s, f, \chi).$$

Cette formule a l'inconvénient de faire intervenir des séries d'Eisenstein de poids entiers et demi-entiers selon la parité de n ; c'est la raison pour laquelle Panchishkin et Courtieu n'ont considéré que le cas où le genre n est pair. Dans le cas que nous considérons, f est non triviale et de genre n arbitraire, donc on a besoin d'une méthode différente ; on utilise la méthode de doublement qui produit une bonne représentation intégrale pour les torsions de la fonction zêta standard. Pour prouver les congruences principales (i) et (ii), nous devons établir que les mesures μ^+ et μ^- vérifient les conditions de croissance caractéristiques des mesures h-admissibles. Tout d'abord on écrit les intégrales comme sommes:

$$\int_{a+(L)} (x_p - a_p)^r d\mu^+ := \gamma(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^+(L, j+1, \chi) \qquad (1)$$

$$\int_{a+(L)} (x_p - a_p)^r d\mu^- := \gamma'(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^-(L, -j, \chi).$$

où $v^+(L, j+1, \chi)$ et $v^-(L, -j, \chi)$ sont les coefficients de Fourier de distributions à valeurs dans les formes modulaires de Siegel. On voit que ces sommes portent à la fois sur j et le caractère χ donc il est difficile de prouver les congruences par la méthode habituelle, mais grâce à la méthode de V. Q. My ces sommes peuvent être transformées en intégrales et exprimées comme les dérivées d'un produit:

$$\begin{aligned} \int_{a+(L)} (x_p - a_p)^r d\mu^+ &= \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{(j+i+1)!}{(j+1)!} x^{j+1} d\mu^+(T_2, \omega) \\ &= \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i x^{-1} \cdot \frac{\partial^i}{\partial x^i} \left(x^{i+1} (x-a)^r \right) d\mu^+(T_2, \omega). \end{aligned}$$

Citons le lemme de V. Q. My qui est la clef de sa méthode:

Lemme 1 (Lemme 5.2 dans [22], page 158). *Soient h et q des entiers naturels avec $h > q$, et $d \equiv -Cd'a \mod m$. Alors le nombre*

$$B_q = \sum_{j=0}^{h} \binom{h}{j} (-a)^{h-j} (-C)^{h-j} d'^{h-j} d^{j-i} j^q \frac{\Gamma(j+1)}{\Gamma(j+1+i)} \tag{2}$$

est divisible par m^{h-i-q}.

En utilisant les relations d'orthogonalité des caractères χ et la congruence $x \equiv a \mod L$ on obtient la congruence $\frac{\partial^i}{\partial x^i}(x^{i+1}(x-a)^r) \equiv 0 \mod L^{r-i}$ ce qui prouve les congruences principales.

On voit que les sommations (1) dépendent de deux autres facteurs. Le premier facteur sont $\gamma(L)$ et $\gamma'(L)$ qui sont liés à une non nuls p-paramètres de Satake associés à une forme propre f et le deuxième sont les coefficients de Fourier de distributions à valeurs dans les formes modulaires de Siegel qui concerne un polynôme différentiel. Pour les congruences, on donne un théorème qui exprime ce polynôme sous la forme suivante:

Théorème 3 (Sur un polynôme différentiel, théorème 4.9.2). *En utilisant les notations ci-dessus et aussi les relations élémentaires $l = k + \nu, k = n + j, j \geq 0$ où l est le poids de la forme modulaire de Siegel f et $T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix}$ avec $T \in \Lambda_{2n}^+, T_1, T_4 \in \Lambda_n^+$ et L un nombre strictement positif donné, on a les formules suivantes:*

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(L^4T_1T_4)^{\frac{\nu}{2}} \sum_{|M| \leq \frac{\nu}{2}} C_M(k) Q_M(L^{-2}D)$$

si ν est pair ; et si ν est impair:

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(T_2)\det(L^4T_1T_4)^{\frac{\nu-1}{2}} \sum_{|M| \leq \frac{\nu-1}{2}} C_M(k) Q_M(L^{-2}D),$$

où M balaie l'ensemble des $(e_0, \cdots, e_{n-1}) \neq 0$ tels que $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$ et $C_M(k)$ est un polynôme en la variable k, avec $k = n + j$, de degré $|M|$ et $Q_M(L^{-2}D)$ est un polynôme homogène en les variables $L^{-2}d_i^2, i = 1, \ldots, n$ de degré $|M|$.

Le polynôme $\mathfrak{P}_{n,k}^{\nu}(T)$ a été introduit par Böcherer, et provient de la composition et restriction de certains opérateurs différentiels sur les séries d'Eisenstein. Dans le cas ordinaire, S. Böcherer-C.-G. Schmidt utilisaient uniquement le terme principal $c_{n,\alpha}^{\nu}\det(T_2)^{\nu}$ de ce polynôme (où $c_{n,\alpha}^{\nu}$ est une certaine constante) ; dans ce travail, nous calculons tous les termes de ce polynôme. Pour simplifier, on écrira $P(T_1, T_4, T_2)$ plutôt que $\mathfrak{P}_{n,\alpha}^{\nu}(T)$. On voit que pour chaque $(T_1, T_4, T_2) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$ et tout $A, B \in GL(n, \mathbb{R})$, on a la propriété suivante:

$$P(AT_1{}^tA, BT_4{}^tB, AT_2{}^tB) = \det(AB)^{\nu} P(T_1, T_4, T_2). \tag{3}$$

En utilisant la théorie des invariants classique, on prouve que ce polynôme est déterminé par ses valeurs lorsque T_1 et T_4 sont des matrices identité de taille n et T_2 une matrice diagonale D de coefficients $d_1, \ldots, d_n$. On exprime alors $P(1_n, 1_n, D)$ comme un polynôme homogène en les coefficients de D, ce qui donne une expression de $\mathfrak{P}^{\nu}_{n,k}(T)$ qui suffit à prouver les congruences principales. C'est un point clef de ce travail et un résultat nouveau. Dans d'autres situations, des polynômes similaires ont été étudiés par M. Courtieu.

Pour illustrer ce théorème, on calcule explicitement le polynôme $\mathfrak{P}^{\nu}_{n,k}$ dans les cas $n = 1$ et $n = 2$.

Organisation de la thèse

Cette thèse comporte quatre chapitres. Dans le premier, on rappelle des généralités sur les formes modulaires de Siegel et les algèbres de Hecke. On définit l'algèbre de Hecke pour le groupe symplectique Sp_n et les formes modulaires de Siegel pour le groupe de niveau 1 $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. On définit alors les paramètres de Satake, qui fournissent une correspondance entre les algèbres de Hecke locales et certaines algèbres de polynômes. Le produit scalaire de Petersson est aussi mentionné à la fin de ce chapitre.

Dans le second chapitre on étudie les opérateurs différentiels et les polynômes ; l'origine de cette théorie est le travail [4] de Böcherer.

Définition 1. *On définit pour tout $\alpha \in \mathbb{C}$*

$$\mathfrak{D}_{n,\alpha} = \sum_{r+q=n} (-1)^r \binom{n}{r} C_r(\alpha - n + \frac{1}{2}) \Delta(r, q),$$

où le polynôme $\Delta(r, q)$, pour $p + q = n$ est donné par:

$$\Delta(p, q) = \sum_{a+b=q} (-1)^b \binom{n}{b} z_2^{[a]} \partial_4^{[a]} \sqcap \left(\left(1_n^{[r]} \sqcap z_2^{[b]} \partial_3^{[b]}\right) \left(Ad^{[r+b]} \partial_1\right) \partial_2^{[r+b]} \right),$$

leurs coefficients sont des polynômes en les coefficients de z_2, et la notation $\sqcap$ est celle de Böcherer dans [4] et E. Freitag dans [13]. Rappelons la définition de cette multiplication: à chaque couple d'applications $A : \bigwedge^p V \to \bigwedge^p V$ et $B : \bigwedge^q V \to \bigwedge^q V$ on associe l'application $A \sqcap B : \bigwedge^{p+q} V \to \bigwedge^{p+q} V$ dont les coefficients sont donnés par:

$$(A \sqcap B)^a_b = \frac{1}{\binom{p+q}{p}} \sum_{\substack{a=a' \cup a'' \\ b=b' \cup b''}} \epsilon(a', a'') \epsilon(b', b'') A^{a'}_{b'} B^{a''}_{b''}.$$

Pour $a' = \{a'_1, \ldots, a'_p\}$ et $a" = \{a_1", \ldots, a_q"\}$ tels que $a'_1 < \ldots < a'_p$ et $a_1" < \ldots < a_q"$ on note $\epsilon(a', a'')$ le signe de la permutation qui trie le $(p + q)$-uplet $(a'_1, \ldots, a'_p, a_1", \ldots, a_q")$. L'opérateur $\sqcap$ est bilinéaire, associatif et commutatif.
Pour $\nu \in \mathbb{N}$ on pose:

$$\mathfrak{D}^{\nu}_{n,\alpha} = \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,\alpha}$$
$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha} = (\mathfrak{D}^{\nu}_{n,\alpha}) \,|_{z_2=0} \,.$$

Définition 2. *Pour $T \in \mathbb{C}^{2n,2n}_{\mathrm{sym}}$ le polynôme $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ en les coefficients $(t_{ij})_{1\le i\le j\le 2n}$ de T est donné par:*

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(e^{\mathrm{tr}(TZ)}) = \mathfrak{P}^{\nu}_{n,\alpha}(T)e^{\mathrm{tr}(T_1z_1+T_4z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}.$$

On voit que les polynômes $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ sont homogènes de degré $n\nu$. Dans ce chapitre, on étudie leurs propriétés et on en donne des formules explicites dans certains cas ($n = 1$ et $n = 2$). Ces formules sont nécessaires pour prouver le théorème principal.

Le troisième chapitre présente les séries d'Eisenstein et le procédé du twist supérieure. Au début de ce chapitre on discute le procédé du twist supérieure. Il y a essentiellement deux représentations intégrales différentes des fonctions zêta standard pour les formes automorphes sur le groupe symplectique $\mathrm{Sp}_n(\mathbb{Z})$: la méthode d'Adrianov et Kalinin dans [2] (et sa version en théorie des représentations par Piateski-Shapiro et Rallis), qui se généralise facilement aux tordues par les caractères de Dirichlet, mais qui a le désavantage de faire intervenir des séries d'Eisenstein de poids entiers et demi-entiers suivant la parité de n, ce qui nécessite de traiter les deux sous-cas. C'est la principale raison qui explique que Panchishkin et Courtieur dans [8] ne traitent que le cas n pair. La méthode de doublement, elle, peut être modifiée pour fournir une représentation intégrale convenable pour les tordues des fonctions L standard. À la fin de ce chapitre on utilise la définition des fonctions $\mathcal{H}$ et leurs développement de Fourier pour définir les distributions modulaires en vue de notre théorème principal dans le chapitre 4.

Le chapitre final de ce rapport de thèse est consacré à l'application des résultats des chapitres 2 et 3 à la construction de mesure h-admissibles. En utilisant la méthode de doublement et l'expression du polynôme $P^{\nu}_{n,k}(T)$, nous prouvons le théorème principal suivant la méthode de V.Q. My dans [22]. À la fin de ce chapitre, on compare la fonction L p-adique construite avec notre méthode avec celle obtenue via une autre méthode. On considère la fonction L standard $L(s, F_{12}, st, \chi)$ où χ est un caractère de Dirichlet et F_{12} est la forme modulaire de Siegel parabolique de degré 3 et poids 12 construite par Miyawaki dans [21]. Miyawaki et Ikea ont prouvé:

$$L(s, F_{12}, \chi, St) = L_{2,\Delta}(s + 11, \chi)L(s + 10, g_{20}, \chi)L(s + 9, g_{20}, \chi),$$

où $L_{2,\Delta}(s + 11, \chi)$ est le carré symétrique de la forme parabolique classique Δ et $L(s, g_{20}, \chi)$ est la fonction L de la forme parabolique g_{20}.
On dispose alors de la fonction L p-adique associée:

$$\mathfrak{L}(\chi x_p^s, F_{12}) = \mathfrak{L}_G(\chi x_p^{s_1}, \Delta)\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})\mathfrak{L}_V(\chi x_p^{s_2+1}, g_{20})$$

où $\mathfrak{L}_G(\chi x_p^{s_1}, \Delta)$ est le carré symétrique h-admissible que Gorsse a étudié en détails dans sa thèse et $\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})$ est la fonction L p-adique construite par Višik dans [30]. On voit que si les p-paramètres de Satake $\alpha_0(c_\chi^{-2})$ et la somme de Gauss des deux méthodes sont égaux alors les deux fonctions L p-adiques sont égales. Pour illustrer la preuve générale, les cas plus faciles $n = 1$ et $n = 2$ sont présentés.

Les prérequis nécessaires pour ce travail sont exposés dans les articles et livres suivants:

- Andrianov "Quadratic forms and Hecke algebra", [1]
- Courtieu and Panchishkin "Non-Archimedean L-functions and Arithmetical Siegel modular forms", [8]
- Lang "Introduction to Modular forms", [18]
- Miyake "Modular forms", [20]
- Maass "Siegel's modular forms and Dirichlet series" [19]
- Freitag "Siegelsche Modulfunktionen", [13]

on peut aussi citer plusieurs articles de S. Böcherer et C.-G. Schmidt, A. Panchishkin, P. Feit, T. Ibukiyama et D. Zagier.

Les résultats de ce travail ont aussi fait l'objet de présentations lors de divers séminaires et conférences:

- Seminar of the institute Fourier Grenoble in 2011, 2012
- Congress for Vietnam mathematic University Paris 13, 2011
- Journées arithmétiques Grenoble 2013.

Introduction

English version

The central objects of this thesis are Siegel modular forms, in particular Eisenstein series and p-adic L-functions. Our main objective is to give a conceptual construction of p-adic admissible measures attached to Siegel modular forms. These measures produce p-adic L-functions of logarithmic growth $o(\log^h)$ with a certain invariant h (see Theorem 2). The Siegel upper half plane in genus n is the set of all $n \times n$ complex symmetric matrices with positive definite imaginary part

$$\mathbb{H}_n = \{z = {}^t z = x + iy | x, y \in M_n(\mathbb{R}), y > 0\}$$

The symplectic group $\mathrm{Sp}_n(\mathbb{Z})$ acts on the space $\mathbb{H}_n$ by

$$\gamma(z) = (az + b)(cz + d)^{-1},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z})$ and $z \in \mathbb{H}_n$. A holomorphic function $f : \mathbb{H}_n \longrightarrow \mathbb{C}^r$ is called a genus n Siegel modular form of weight l on $\mathrm{Sp}_n(\mathbb{Z})$ if f satisfies

$$\det(cz + d)^{-l} f(\gamma(z)) = f(z) \quad \forall \gamma \in \mathrm{Sp}_n(\mathbb{Z}).$$

In the case of $n = 1$ we also require that f be holomorphic at ∞. Siegel modular form of genus 1 are classical modular forms on the upper half plane for the group $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_1(\mathbb{Z})$ and its congruence subgroups.
A study of standard zeta functions leads to the study of Satake parameters, invariants related to Siegel modular forms. The Satake p-parameters associated to the eigenform $f \in \mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$ are the elements of the $(n+1)$-tuple $(\alpha_0, \alpha_1, \cdots, \alpha_n) \in [(A^\times)^{n+1}]^{W_n}$, which is the image of the map $f \longmapsto \lambda_f(X)$ under the isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}_n, \mathbb{C}) \cong [(A^\times)^{n+1}]$. This is defined up to the action of W_n where W_n is the Weyl group and $\mathcal{L}_n$ is a local p algebra. The Satake p-parameters of f satisfy the following relation

$$\alpha_0^2 \alpha_1 \cdots \alpha_n = \psi(p)^n p^{ln - \frac{n(n+1)}{2}}.$$

Let p be a prime number and $f \in S_n^l(\Gamma_0(N), \psi)$ be a Siegel cusp eigenform of genus n and weight l with Satake p-parameters $\alpha_0(p), \alpha_1(p), \cdots, \alpha_n(p)$. We consider the standard zeta function $D^{(Np)}(f, s, \chi)$ which takes algebraic values at critical points after normalization. The standard zeta function of f is defined by means of the Satake p-parameters as the following Euler product

$$D^{(M)}(f, s, \chi) := \prod_{q \nmid M} \left(\frac{1}{1 - \chi\psi(q) q^{-s}} \prod_{i=1}^{n} \frac{1}{(1 - \chi\psi(q)\alpha_i(q) q^{-s})(1 - \chi\psi(q)\alpha_i^{-1}(q) q^{-s})} \right)$$

where χ is an arbitrary Dirichlet character. We introduce the following normalized functions:

$$\mathcal{D}^{(M\star)}(s, f, \chi) = (2\pi)^{-n(s+l-(n+1)/2}\Gamma((s+\delta)/2) \prod_{j=1}^{n} (\Gamma(s + l - j)) \mathcal{D}^{(M)}(s, f, \chi)$$

$$\mathcal{D}^{(M+)}(s,f,\chi) = \Gamma((s+\delta)/2)\mathcal{D}^{(M\star)}(s,f,\chi)$$
$$\mathcal{D}^{(M-)}(s,f,\chi) = \frac{i^{\delta}\pi^{1/2-s}}{\Gamma((1-s+\delta)/2)}\mathcal{D}^{(M\star)}(s,f,\chi).$$

Here, $\delta = 0$ or 1 according to whether $\chi(-1) = 1$ or $\chi(-1) = -1$. We recall first the following result about the algebraic properties of the special values of standard zeta functions.

Theorem 1 (**On algebraic properties of special values of standard zeta functions**).

(a) *For all integers s with $1 \le s \le l - \delta - n, s \equiv \delta \mod 2$ and Dirichlet character χ such that χ^2 is non-trivial for $s = 1$, we have that:*

$$\langle f, f\rangle^{-1} D^{(M+)}(f,s,\chi) \in K = \mathbb{Q}(f, \Lambda_f, \chi),$$

where $K = \mathbb{Q}(f, \Lambda_f, \chi)$ denotes the field generated by Fourier coefficients of f, by the eigenvalues $\Lambda_f(X)$ of the Hecke operator X on f, and by the values of the character χ.

(b) *For all integers s with $1 - l + \delta + n \le s \le 0, s \not\equiv \delta \mod 2$, we have that:*

$$\langle f, f\rangle^{-1} D^{(M-)}(f,s,\chi) \in K.$$

The analytic properties of these complex L-functions have been investigated by several authors (see [2],[4], [27]) and are more or less well-known. It follows that the normalized critical values $D(s,f,\chi)$ of standard zeta functions can be explicitly rewritten in terms of certain $\mathbb{C}_p$-valued integrals of admissible measure (over a profinite group $\mathbb{Z}_p^\times$ of p-adic units), provided the character χ is non-trivial.
The domain of definition of a p-adic zeta function is the p-adic analytic Lie group

$$X_p = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$$

of all continuous p-adic characters of the profinite group $\mathbb{Z}_p^\times$, where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$), which is endowed with a unique norm $|.|_p$ such that $|p|_p = p^{-1}$. So that all integers k can be viewed as the characters $x_p^k : y \mapsto y^k$. Throughout the thesis we fix an embedding

$$i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p,$$

and we identify $\mathbb{Q}$ with a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$. Then a Dirichlet character $\chi : (\mathbb{Z}/p^N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ becomes an element of the torsion subgroup

$$X_p^{\mathrm{tors}} \subset X_p = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times).$$

In order to state the main result we also use certain elementary factors such as $E_p(s,\psi), \Lambda_\infty(s), A(\chi)$ including Gauss sum, Satake p-parameters, and the conductor c_χ of χ. Precisely,

$$E_p(s,\psi) := \prod_{j=1}^{n} \frac{(1-\psi(p)\alpha_j^{-1}p^{s-1})}{(1-\bar{\psi}(p)\alpha_j^{-1}p^{-s})}.$$

To formulate our result, let

$$\Lambda_\infty^+(s) := \frac{(2i)^s \cdot \Gamma(s)}{(2\pi i)^s} \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s+l-j)$$
$$\Lambda_\infty^-(s) := (2i)^s \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(1-s+l-j),$$

where $\Gamma_{\mathbb{C}}(s) := 2 \cdot (2\pi)^{-s}\Gamma(s)$. Further for any character χ of p-power conductor c_χ we let

$$A^-(\chi) := c_\chi^{nl-\frac{n(n+1)}{2}} \alpha(c_\chi)^{-2} \cdot (\chi^0([p, c_\chi]) \cdot \chi(-1)G(\chi))$$
$$A^+(\chi) := (\overline{\chi^0\varphi})_o(c_\chi) \cdot \frac{A^-(\chi)}{\chi(-1)G(\chi)},$$

where $[a, b]$ denotes the least common multiple of the integers a, b. Finally, define

$$E_p^+ := (1 - (\overline{\varphi\chi\chi^0})_0(p)p^{t-1}) \cdot E_p(s, \chi\chi^0)$$
$$E_p^- := E_p(p, \chi\chi^0).$$

We construct two admissible measures μ^+ and μ^- with the following properties:

Theorem 2 (**Main theorem on admissible measures, theorem 4.4.1).**

(i) For all pairs (s, χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $1 \le s \le l - \delta - n, s \equiv \delta \mod 2$ and for $s = 1$ the character χ^2 is non-trivial, the following equality holds

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p\Bigg(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s, \chi\chi^0)\frac{\Lambda_\infty^+(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, s, \overline{\chi\chi^0})\Bigg).$$

(ii) For all pairs (s, χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $l - \delta + n \le s \le 0, s \not\equiv \delta \mod 2$ the following equality holds

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p\Bigg(c_\chi^{n(1-s)} A^+(\chi) \cdot E_p^-(1-s, \chi\chi^0)\frac{\Lambda_\infty^-(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f, 1-s, \overline{\chi\chi^0})\Bigg).$$

(iii) If $\mathrm{ord}_p(\alpha_0(p)) = 0$ (i.e. f is p-ordinary), then the measures in (i) and (ii) are bounded.

(iv) In the general case (but assuming that $\alpha_0(p) \neq 0$) with $x \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ the holomorphic functions

$$\mathcal{D}^+(x) = \int x d\mu^+$$
$$\mathcal{D}^-(x) = \int x d\mu^-$$

belong to type $o(\log(x_p)^h)$ where $h = [4\mathrm{ord}_p(\alpha_0(p))] + 1$. Furthermore, they can be represented as the Mellin transforms of certain h-admissible measures.

(v) *If $h \le k-m-1$, then the functions $\mathcal{D}^{\pm}$ are uniquely determined by the above conditions (i) and (ii).*

Special cases were treated by several authors:

1. The case of genus $n = 1$ was treated by B. Gorsse (2006) in his Ph. D. thesis on symmetric squares and also by Dabrowski and Delbourgo in [11], who proved that there exist h-admissible measures with values in $\mathbb{C}_p$, interpolating the special values of symmetric squares.

2. The case of f is non-ordinary, and the genus n is even was treated by A. Panchishkin and M. Courtieu LNM 1471(2002) by the Rankin-Selberg method in the form of Andrianov, using the Shimura differential operator action on Siegel modular forms.

3. The case of f is ordinary, arbitrary genus n was treated by S. Böcherer and C-G. Schmidt Annales de Institut Fourier 2000, by the doubling method.

The assertion (iii) (i.e the ordinary case) which was proved by Panchishkin (see [24]) for even genus n and by Böcherer, Schmidt in [7] for arbitrary genus. It also follows easily from the main congruence in (i) and (ii).
The proof of (iv) is similar to proofs in [14], [3], and [30].
Finally, if $h \le l - n - 1$ then the conditions in (i) and (ii) uniquely determine the analytic functions $\mathcal{D}^{\pm}$ of type $o(\log(x_p)^h)$ by their values following a general property of admissible measures (See [3], [30]). In the case $h > l - n - 1$, there exist many analytic functions $\mathcal{D}^{\pm}$ satisfying the conditions in (i) and (ii), which depend on a choice of analytic continuation (interpolation) for the values $\mathcal{D}^{\pm}(\chi x_p^s)$ if $s > l - n - 1 - \nu$. But one shows in the Theorem 2 that there exists at least one such continuation (for example the one which was described in the proof of (iv)).
To construct the admissible measures μ^+ and μ^- satisfying (i) and (ii) we follow four steps:

1. Construct certain sequence of modular distributions with values in the Siegel modular forms $(\mathcal{H}_{L,\chi}^{(j)})$.

2. Apply a suitable algebraic linear form (represented by a double Petersson scalar product).

3. Check the admissibility properties (h-admissible measures).

4. Prove certain integrals coincide with the special values of the standard zeta function.

Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ be the Tate field. For $h \in \mathbb{N}^*$ we denote $\mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p)$ the space of $\mathbb{C}_p$-valued functions which can be locally represented by polynomials of degree less than a natural number h of variable in x_p. In particular, $\mathcal{C}^1(\mathbb{Z}_p^\times, \mathbb{C}_p)$ is the space of locally constant functions. Let us recall the definition of admissible measures with scalar and vector values; see [3], [30],

[23]:
An h-admissible measure on $\mathbb{Z}_p^\times$ is a $\mathbb{C}_p$-linear map:

$$\phi : \mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p) \to V$$

for all $t = 0, 1, \ldots, h-1$

$$\mid \int_{a+(p^\nu)} (x_p - a_p)^t d\phi \mid_p = o(p^{\nu(h-t)}) \quad \text{for } \nu \to \infty,$$

where $a_p = x_p(a)$.
In order to prove that certain integrals coincide with the special values of the standard zeta function we use the doubling method which given by Böcherer in [5]. In general, this method is called the pull-back method and can be written as the following formula

$$\Lambda(k,\chi)D(k-r,f,\chi)E_k^{n,r}(z,f) = \langle f(w), E_k^{n+r}\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}\rangle$$

where $f \in S_k^r(\mathrm{Sp}_{2r}(\mathbb{Z})), \Lambda(k,\chi)$ is a product of Dirichlet L-functions, $E_k^{n,r}(z,f)$ is Klingen-Eisenstein series and $D(k-r,f,\chi)$ is the standard zeta function at critical points $k-r$. In our case we consider only $r = n$. Then we have the following formula:
Let φ be a Dirichlet character $\mod M > 1, \chi$ a Dirichlet character $\mod N, N^2|M, l = k+\nu, \nu \in \mathbb{N}$ and $f \in S_n^l(\Gamma_0(M)^n, \bar{\varphi})$ we have

$$\Lambda(k+2s,\chi)D(k+2s-n,f,\bar{\chi})f =$$

$$\langle f, \det(v)^s \det(y)^s \overset{\circ}{\mathfrak{D}}{}^\nu_{n,k+s}\left(\sum_{X \in \mathbb{Z}^{(n,n)}, X \mod N} \chi(\det X)\hat{\mathbb{F}}^k_{2n}(-,M,\psi,s) \mid_k \begin{pmatrix} 1_{2n} & S(\frac{X}{N}) \\ 0_{2n} & 1_{2n} \end{pmatrix}\right)\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}\rangle,$$

where $S(X)$ denotes the $2n$-rowed symmetric matrix $\begin{pmatrix} 0_n & X \\ {}^tX & 0_n \end{pmatrix}$, and where $z = x+iy, w = u+iv, \Lambda(k+2s,\chi)$ is the product of Dirichlet L-function, $D(k+2s-n,f,\bar{\chi})$ is the standard zeta function attached to f. Here we use certain differential $\overset{\circ}{\mathfrak{D}}{}^\nu_{n,k+s}$ acts on Siegel Eisenstein series of degree $2n$ with the restriction which we shall describe more detail later. Therefore, we obtain the values of the L-function at a critical point. We move to other critical points by application of the differential operator, to obtain values of the L-function at all of its critical points.
In the other situation, Panchishkin and Courtieu use the Rankin-Selberg method to find the integral representation for the standard zeta function. The main idea, based on a result of Andrianov, can be stated as a certain identity expressing the standard zeta function $D(s,f,\chi)$ as a Rankin zeta function, which is the convolution for the given form f and a theta function with the Dirichlet character $\chi \mod M$. More precisely, the Rankin zeta function $R(s,f,\chi)$ can be written as the Rankin convolution of f and a theta function. The Rankin convolution $L(s,f,g)$ can be expressed as the Petersson scalar product of f and the multiplication of g and a suitable Eisenstein series. Otherwise, we have the identity between the Rankin zeta function and the standard zeta function

$$D(s,f,\chi) = L(s+(n/2),\chi\psi)\left(\prod_{i=0}^{n/2-1} L(2s+2i,\chi^2\psi^2)\right)R(s,f,\chi).$$

This method has the disadvantage that it involves Eisenstein series of integral and half-integral weight depending on the parity of n, therefore Panchishkin and Courtieu treated only the case of even genus n. For our case, f is non-trivial and n is arbitrary , so we need a different method. We use the doubling variables, giving a modification which produces a good integral representation for twists of the standard zeta function. To prove the main congruences (i) and (ii), we have to prove that the measures μ^+ and μ^- satisfy the growth condition of an h-admissible measure. First, we write the integrals as sums

$$\int_{a+(L)} (x_p - a_p)^r d\mu^+ := \gamma(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^+(L, j+1, \chi)$$

$$\int_{a+(L)} (x_p - a_p)^r d\mu^- := \gamma'(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^-(L, -j, \chi).$$

where $v^+(L, j+1, \chi)$ and $v^-(L, -j, \chi)$ are Fourier coefficients of distribution in Siegel modular forms. We see that these are summations over j and characters χ so it is difficult to prove the congruence directly. But by the method of V. Q. My these summations can be put into integrals and composed as the derivative of a product:

$$\int_{a+(L)} (x_p - a_p)^r d\mu^+ = \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{(j+i+1)!}{(j+1)!} x^{j+1} d\mu^+(T_2, \omega)$$

$$= \int_{x \equiv a \mod L} \sum_{i=0}^{|M|} \mu_i x^{-1} \cdot \frac{\partial^i}{\partial x^i} \left(x^{i+1} (x-a)^r \right) d\mu^+(T_2, \omega).$$

We quote a lemma of V. Q. My, which is the key point of his method.

Lemma 1 (Lemma 5.2 in [22], page 158)**.** *Suppose that h and q are natural numbers, $h > q$, and $d \equiv -Cd'a \mod m$. Then the number*

$$B_q = \sum_{j=0}^{h} \binom{h}{j} (-a)^{h-j} (-C)^{h-j} d'^{h-j} d^{j-i} j^q \frac{\Gamma(j+1)}{\Gamma(j+1+i)} \tag{4}$$

is divisible by m^{h-i-q}.

Using the orthogonal relations of the character χ and the congruence $x \equiv a \mod L$, which gives the congruence $\frac{\partial^i}{\partial x^i} (x^{i+1}(x-a)^r) \equiv 0 \mod L^{r-i}$, we can prove the main congruences.

We see that the summations (1) depend on two other factors. The first factors are $\gamma(L)$ and $\gamma'(L)$, which are related to non-zero Satake p-parameters of the eigenform f. The second factors are the Fourier coefficients of certain distributions in the Siegel modular forms which concern the differential polynomial. For the congruence, we give a theorem about the expression for this polynomial in the following form

Theorem 3 (About a differential polynomial, Theorem 4.9.2). *Using the notations defined as above and also some basic relations $l = k+\nu, k = n+j, j \geq 0$ with l the weight of a*

Siegel modular form f and $T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix} \in \Lambda_{2n}^+, T_1, T_4 \in \Lambda_n^+, L$ fixed positive number, we have that the following expressions hold:

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(L^4T_1T_4)^{\frac{\nu}{2}} \sum_{|M|\leq \frac{\nu}{2}} C_M(k)Q_M(L^{-2}D) \text{ if } \nu \text{ is even,}$$

$$\mathfrak{P}_{n,k}^{\nu}(T) = \det(T_2)\det(L^4T_1T_4)^{\frac{\nu-1}{2}} \sum_{|M|\leq \frac{\nu-1}{2}} C_M(k)Q_M(L^{-2}D) \text{ if } \nu \text{ is odd.}$$

In the sums, M runs over the set of $(e_0, \cdots, e_{n-1}) \neq 0$ such that $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$, $C_M(k)$ is a polynomial of variable k, $k = n+j$ degree $|M|$, and $Q_M(L^{-2}D)$ is a homogeneous polynomial of degree $|M|$ in variables $L^{-2}d_i^2, i = \overline{1,n}$.

The polynomial $\mathfrak{P}_{n,k}^{\nu}(T)$ introduced by Böcherer, comes from composition and restriction of certain differential operators on Eisenstein series. In the ordinary case S. Böcherer-C.G. Schmidt only needs the main term $c_{n,\alpha}^{\nu}\det(T_2)^{\nu}$ with a certain constant $c_{n,\alpha}^{\nu}$ of this polynomial. In present work we find all terms of this polynomial. For simplicity, we write $P(T_1, T_4, T_2)$ instead of $\mathfrak{P}_{n,\alpha}^{\nu}(T)$. We see that for each $(T_1, T_4, T_2) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$ the following property is satisfied for any $A, B \in GL(n, \mathbb{R})$.

$$P(AT_1{}^tA, BT_4{}^tB, AT_2{}^tB) = \det(AB)^{\nu}P(T_1, T_4, T_2). \tag{5}$$

By classical invariant theory we prove that this polynomial is determined by its values at T_1 and T_4 are identity matrix of size n and T_2 is a diagonal matrix with diagonal elements $d_1, d_2, \cdots, d_n$. Then we express $P(1_n, 1_n, D)$ as the homogenous polynomial of d_i^2 with $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$. We obtain the expression for the polynomial $\mathfrak{P}_{n,k}^{\nu}(T)$ which is sufficient for the proof of the main congruence. This is a key point of present work which was not known. In the other situation, such polynomials were studied by M. Courtieu.
We also compute explicitly the polynomial $\mathfrak{P}_{n,k}^{\nu}$ in certain cases for $n = 1$ and $n = 2$ to give an illustration for this theorem.

Organization of the thesis

The thesis contains four chapters. In the first chapter we give general information on Siegel modular forms and Hecke algebras. We define the Hecke algebra for the symplectic group Sp_n and Siegel modular forms for the whole group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. Then we describe Satake parameters, which provide a correspondence between the local Hecke algebra and a certain polynomial algebra. Petersson scalar product is also mentioned at the end of this chapter.
We study the differential operator in the second chapter. Originally, the differential operator and the polynomial comes from the work of Böcherer in [4].

Definition 1. *We define for any* $\alpha \in \mathbb{C}$

$$\mathfrak{D}_{n,\alpha} = \sum_{r+q=n} (-1)^r \binom{n}{r} C_r(\alpha - n + \frac{1}{2})\Delta(r,q).$$

The polynomial $\Delta(r,q), (p+q=n)$ is defined by the following formula:

$$\Delta(p,q) = \sum_{a+b=q} (-1)^b \binom{n}{b} z_2^{[a]} \partial_4^{[a]} \sqcap \left(\left(1_n^{[r]} \sqcap z_2^{[b]} \partial_3^{[b]}\right) \left(Ad^{[r+b]} \partial_1\right) \partial_2^{[r+b]} \right).$$

Their coefficients are polynomials in the entries of z_2. The notation $\sqcap$ used by Böcherer in [4], and by E. Freitag in [13]. We recall the definition of this multiplication:
To each pair of mappings $A : \bigwedge^p V \to \bigwedge^p V$ and $B : \bigwedge^q V \to \bigwedge^q V$, we attach the mapping $A \sqcap B : \bigwedge^{p+q} V \to \bigwedge^{p+q} V$, defined in coordinates by the relation

$$(A \sqcap B)^a_b = \frac{1}{\binom{p+q}{p}} \sum_{\substack{a=a' \cup a'' \\ b=b' \cup b''}} \epsilon(a', a'')\epsilon(b', b'') A^{a'}_{b'} B^{a''}_{b''}.$$

For $a' = \{a'_1, \ldots, a'_p\}, a" = \{a_1", \ldots, a_q"\}$ with $a'_1 < \ldots < a'_p$ and $a_1" < \ldots < a_q"$ we use the notation $\epsilon(a', a'')$ for the sign of the permutation which takes the $(p+q)$-tuple into its natural order $(a'_1, \ldots, a'_p, a_1", \ldots, a_q")$. The operator $\sqcap$ is bilinear, associative and commutative.
For $\nu \in \mathbb{N}$, we put

$$\mathfrak{D}^\nu_{n,\alpha} = \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,\alpha}$$
$$\mathring{\mathfrak{D}}^\nu_{n,\alpha} = (\mathfrak{D}^\nu_{n,\alpha}) \mid_{z_2=0} .$$

Definition 2. *For* $T \in \mathbb{C}^{2n,2n}_{\mathrm{sym}}$ *we define a polynomial* $\mathfrak{P}^\nu_{n,\alpha}(T)$ *in the entries* $t_{ij} (1 \le i \le j \le 2n)$ *of* T *by*

$$\mathring{\mathfrak{D}}^\nu_{n,\alpha}(e^{\mathrm{tr}(TZ)}) = \mathfrak{P}^\nu_{n,\alpha}(T) e^{\mathrm{tr}(T_1 z_1 + T_4 z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}.$$

We see that $\mathfrak{P}^\nu_{n,\alpha}(T)$ are homogenous polynomials of degree $n\nu$. In this chapter we study the properties of this polynomial, and give the explicit formula for certain cases when $n = 1$ and $n = 2$. To prove the main theorem we need the expression for this polynomial.

The third chapter presents the Eisenstein series and twisting process. At the beginning of this chapter, we discuss the twisting process. There are basically two different integral representations of standard zeta functions for automorphic forms on the symplectic group $\mathrm{Sp}_n(\mathbb{Z})$: the method of Andrianov/ Kalinin in [2] (and its representation theoretic version by Piateski-Shapiro/Rallis in [27]) and the method immediately generalizes to twists by Dirichlet characters, but it has the disadvantage that it involves the Eisenstein series of integral and half-integral weight depending on the parity of n. Therefore, the case n even or odd must be treated separately. This is the main reason why Panchishkin and Courtieu in [8] only treats the case of even n. The doubling variables admit a modification which produces a good integral representation for twists of the standard L-functions. At the end of this chapter we use the definition of $\mathcal{H}$-functions and also their Fourier expansions as the modular distribution for our main theorem in chapter 4.

The final part of the dissertation is devoted to applying chapter 2 and chapter 3 to construct h-admissible measures . Using the doubling method and the expression for the polynomial $P^{\nu}_{n,k}(T)$, we prove the main theorem in the way of V. Q. My in [22]. At the end of this chapter, we compare the p-adic L-function constructed by our method with that constructed by a different method. We consider the standard L-function $L(s, F_{12}, st, \chi)$ for all Dirichlet character χ, where F_{12} is the Siegel cusp form of degree 3 and weight 12. This was constructed by Miyawaki in [21]. Due to Miyawaki and Ikeda,

$$L(s, F_{12}, \chi, St) = L_{2,\Delta}(s+11, \chi)L(s+10, g_{20}, \chi)L(s+9, g_{20}, \chi),$$

where $L_{2,\Delta}(s+11, \chi)$ is the symmetric square of cusp form Δ and $L(s, g_{20}, \chi)$ is the Dirichlet L-function of a cusp form g_{20}.
Then we have its associated p-adic L-function

$$\mathfrak{L}(\chi x_p^s, F_{12}) = \mathfrak{L}_G(\chi x_p^{s_1}, \Delta)\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})\mathfrak{L}_V(\chi x_p^{s_2+1}, g_{20}),$$

where $\mathfrak{L}_G(\chi x_p^{s_1}, \Delta)$ is the h-admissible case for symmetric squares which was investigated carefully by Gorsse in his thesis, and $\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})$ is the p-adic L-function constructed by Višik in [30]. We see that the Satake p-parameter $\alpha_0(c_\chi^{-2})$ and the Gauss sum in two methods are equal, so the p-adic L-functions constructed by these two methods coincide. Some easier cases $n = 1, n = 2$ are also given as an illustration of the general proof.

The essential theoretical background for the presented work is given by some papers and some books as follows:

- Andrianov "Quadratic forms and Hecke algebra", [1]
- Courtieu and Panchishkin "Non-Archimedean L-functions and Arithmetical Siegel modular forms", [8]
- Lang "Introduction to Modular forms", [18]
- Miyake "Modular forms", [20]
- Maass "Siegel's modular forms and Dirichlet series", [19]

- Freitag "Siegelsche Modulfunktionnen", [13],

as well as the several articles by S. Böcherer and G. Schmidt, A. Panchishkin, P. Feit, T. Ibukiyama, D. Zagier. The main results were presented at the following seminars and conferences:

- Seminar of the Institute Fourier Grenoble in 2011, 2012
- Congress for Vietnam mathematic University Paris 13, 2011
- Journées arithmétiques Grenoble 2013.

Notations

$\mathbb{N} = \{1, 2, \cdots\}$ is the set of natural numbers
$\mathbb{Z}$ is the ring of rational integers
$\mathbb{Q}$ is the field of rational numbers
$\mathbb{R}$ is the field of real numbers
$\mathbb{C}$ is the field of complex numbers
$\mathbb{Z}_p$ is the ring of p-adic integers
$\mathbb{Q}_p$ is the field of p-adic numbers
$\mathbb{C}_p$ is the Tate field
1_n is the identity matrix of order n
0_n is the null matrix of order n
J_n is the antisymmetric matrix of order $2n$:

$$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$$

${}^tM$ is the transpose of the matrix M
$\mathrm{tr}(M)$ is the trace of the matrix M
$M > 0$ means that M is a positive definite matrix
$M_n(K)$ is the set of $n \times n$-matrices with entries in K
The symplectic group

$$\mathrm{Sp}_n(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}) : MJ^tM = J\}$$

GL_n is the group of $n \times n$ invertible matrices with entries in K
$(M) = \Gamma M \Gamma$ is the double coset modulo the group Γ
For $r, n \in \mathbb{N}$ two positive integers, the symbol

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

denotes the classical binomial coefficient (it is a polynomial in n of degree r).
We denote by $\Gamma(s)$ the gamma function defined by the integral formula

$$\Gamma(s) = \int_0^\infty y^{s-1} e^{-y} dy \quad (s \in \mathbb{C} | R(s) > 0).$$

We also have the general gamma function $\Gamma_n(s)$, defined by

$$\Gamma_n(s) = \int_Y \det(y)^{s-\frac{n(n+1)}{2}} e^{-\mathrm{tr}(y)} dy = \pi^{n(n-1)/4} \prod_{j=0}^{n-1} \Gamma(s - \frac{j}{2}).$$

The notation $e_n(z)$ means that $e_n(z) = \exp(2\pi i \mathrm{tr}(z))$ and in the classical case, $e(z) = e_1(z)$.
The Siegel upper half plane is defined to be

$$\mathbb{H}_n = \{Z = {}^tZ = X + iY : X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any $f \in \mathbb{H}_n$, $M \in G^+\mathrm{Sp}(n,\mathbb{R})$ we write

$$f|_{\alpha,\beta}M(Z) = \det(M)^{\frac{\alpha+\beta}{2}} f(M\langle Z\rangle)\det(CZ+D)^{-\alpha}\det(C\bar{Z}+D)^{-\beta}, \tag{6}$$

where $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.
We denote by Λ_n the set of all half-integral symmetric matrices of size n and by Λ_n^*, Λ_n^+ the subsets of matrices of maximal rank and of positive definite matrices, respectively.
For two natural numbers L and R, we define an operator $U_L(R)$ acting on L-periodic function f with Fourier expansion $f(Z) = \sum_{T\in\Lambda_n} a(T,Y)e^{2\pi i\frac{1}{L}\mathrm{tr}(TX)}$ by

$$(f|U_L(R))(Z) = \sum_{T\in\Lambda_n} a(RT, \frac{1}{R}Y)e^{2\pi i\frac{1}{L}\mathrm{tr}(TX)}.$$

We write $U(R)$ for $U_1(R)$.

Abstract

Let p be a prime number, and let $f \in S_n^l(\Gamma_0(N), \psi)$ be a Siegel cusp eigenform of genus n and weight l with Satake p-parameters $\alpha_0(p), \alpha_1(p), \cdots, \alpha_n(p)$. We consider the standard zeta function $D^{(Np)}(f, s, \chi)$, which takes algebraic values at critical points after normalization. We construct two admissible measures μ^+ and μ^-, with the following properties:

(i) For all pairs (s, χ) such that $\chi \in X_p^{\text{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $1 \le s \le l - \delta - n, s \equiv \delta \mod 2$, and for $s = 1$ the character χ^2 is non-trivial, the following equality holds:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p\left(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s, \chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle f_0, f_0 \rangle} \cdot D^{(Np)}(f, s, \overline{\chi\chi^0}) \right).$$

(ii) For all pairs (s, χ) such that $\chi \in X_p^{\text{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $l - \delta + n \le s \le 0, s \not\equiv \delta \mod 2$, the following equality holds:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p\left(c_\chi^{n(1-s)} A^+(\chi) \cdot E_p^-(1-s, \chi\chi^0) \frac{\Lambda_\infty^-(s)}{\langle f_0, f_0 \rangle} \cdot D^{(Np)}(f, 1-s, \overline{\chi\chi^0}) \right).$$

Here $\Lambda_\infty(s), A(\chi)$ and $E_p(s, \psi)$ are certain elementary factors including Gauss sum, Satake p-parameters, the conductor c_χ of the Dirichlet character χ, etc. Special cases were treated by Böcherer, Schmidt for arbitrary genus in the ordinary case (Annales Inst.Fourier, 2000, by doubling method), Courtieu, Panchishkin (LNM 1471, 2004, 1990) for even genus in the general h-admissible cases, by Ranking-Selberg method in the form of Andrianov.

Résumé

Soit p un nombre premier et $f \in \mathcal{S}_n^l(\Gamma_0(N), \psi)$ une forme parabolique de Siegel de genre n et de poids l avec p-paramètres de Satake $\alpha_0(p), \cdots, \alpha_n(p)$. Nous considèrons la fonction zêta standard $D^{(Np)}(f, s, \chi)$ qui prend des valeurs algébriques aux points critiques après normalisation. On construit deux mesures p-adiques admissibles μ^+ et μ^- ayant les propriétés suivantes:

(i) Pour tout couple (s, χ) tel que $\chi \in X_p^{\text{tors}}$ est un caractère de Dirichlet non-trivial, $s \in \mathbb{Z}$ avec$1 \leq s \leq k - \delta - n, s \equiv \delta(\mod 2)$ et pour $s = 1$ le caractère χ^2 soit non-trivial, l'égalité suivante est vérifiée

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p\left(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s, \chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle f_0, f_0 \rangle} \cdot D^{(Np)}(f, s, \overline{\chi\chi^0})\right).$$

(ii) Pour tout couple (s, χ) tel que $\chi \in X_p^{\text{tors}}$ est un caractère de Dirichlet non-trivial, $s \in \mathbb{Z}$ avec $1 \leq k - \delta + n \leq s \leq 0, s \not\equiv \delta(\mod 2)$ l'égalité suivante est vérifiée

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p\left(c_\chi^{n(s-1)} A^+(\chi) \cdot E_p^-(1-s, \chi\chi^0) \frac{\Lambda_\infty^-(s)}{\langle f_0, f_0 \rangle} \cdot D^{(Np)}(f, 1-s, \overline{\chi\chi^0})\right).$$

Où $\Lambda_\infty(s), A(\chi), E_p(s, \psi)$ sont certains facteurs élémentaires, incluant la somme de Gauss, les p-paramètres de Satake, le conducteur c_χ du caractère de Dirichlet etc. Les cas particuliers ont été traités par Böcherer, Schmidt pour un genre arbitraires dans le cas ordinaire (Annales Inst.Fourier, 2000, par la méthode doublement), Courtieu, Panchishkin (LNM 1471, 2004, 1990) dans les cas h-admissibles générales, par la méthode de Ranking-Selberg dans la forme de Andrianov avec un genre pair.

Contents

Chapter 1

Siegel modular forms and the Hecke algebra

This chapter contains some preparatory facts which we shall use in the construction of non-Archimedean standard zeta functions in the final chapter. We recall main properties of Siegel modular forms, Satake p-parameters, the Hecke algebra, and the action of the Hecke algebra on Siegel modular forms, as well as the definition of standard zeta functions.

1.1 The symplectic group and the Siegel upper half plane

Let $G = \mathrm{GSp}_n$ be the algebraic subgroup of GL_{2n} defined by

$$G_A = \{\gamma \in GL_{2n}(A) \mid {}^t\gamma J_n \gamma = \nu(\gamma) J_n, \nu(\gamma) \in A^\times\},$$

for any commutative ring A, where

$$J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.$$

The elements of G_A are characterized by the conditions

$$b^t a - a^t b = d^t c - c^t d = 0_n, \quad d^t a - c^t b = 1_n,$$

and if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_A, \text{ then } \gamma^{-1} = \nu(\gamma)^{-1} \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}.$$

The multiplier ν defines a homomorphism $\nu : G_A \longrightarrow A^\times$ so that $\nu(\gamma)^{2n} = \det(\gamma)^2$. Its kernel $\ker(\nu)$ is denoted by $\mathrm{Sp}(A)$. We also put

$$G_\infty = G_\mathbb{R}, \quad G_\infty^+ = \{\gamma \in G_\infty \mid \nu(\gamma) > 0\}, G_\mathbb{Q}^+ = G_\infty^+ \cap G_\mathbb{Q}.$$

The group G_∞^+ acts transitively on the upper half plan $\mathbb{H}_n$ by the rule

$$z \longmapsto \gamma(z) = (az+b)(cz+d)^{-1} \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+, z \in \mathbb{H}_n\right)$$

so that scalar matrices act trivially, so $\mathbb{H}_n$ can be identified with a homogeneous space of the group $\mathrm{Sp}_n(\mathbb{R})$. Let K_n denote the stabilizer of the point $i1_n \in \mathbb{H}_n$ in the group $\mathrm{Sp}_n(\mathbb{R})$,

$$K_n = \{\gamma \in \mathrm{Sp}_n(\mathbb{R}) \mid \gamma(i1_n) = i1_n\}.$$

There is a bijection $\mathrm{Sp}_n(\mathbb{R})/K_n \simeq \mathbb{H}_n$ and $K_n = \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{SO}_{2n}$. The group K_n is a maximal compact subgroup of the Lie group $\mathrm{Sp}_n(\mathbb{R})$, and it can be identified with the group $\mathrm{U}(n)$ of all unitary $n \times n$-matrices via the map

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a + ib.$$

We adopt also the notations

$$dx = \prod_{i \le j} dx_{ij}, dy = \prod_{i \le j} dy_{ij}, dz = dxdy,$$
$$d^\times y = \det(y)^k dy, d^\times z = \det(y)^{-k} dz,$$

where $z = x + iy, x = (x_{ij}) = {}^t x, y = (y_{ij}) = {}^t y > 0$. Then $d^\times z$ is a differential on $\mathbb{H}_n$ invariant under the action of the group G_∞^+, and the measure $d^\times y$ is invariant under the action of elements $a \in GL_n(\mathbb{R})$ on

$$Y = \{y \in M_n(\mathbb{R}) \mid {}^t y = y > 0\}$$

defined by the rule $y \longmapsto {}^t a y a$.

1.2 Siegel modular forms

We denote by $\mathbb{H}_n = \{z = {}^t z = x + iy : x, y \in M_n(\mathbb{R}), y > 0\}$ the set of $n \times n$ complex symmetric matrices with positive definite imaginary part. The symplectic group

$$\mathrm{Sp}_n(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}) : M J_n {}^t M = J_n\}$$

for $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$ acts on the space $\mathbb{H}_n$ by

$$\gamma(z) = (az+b)(cz+d)^{-1}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}), z \in \mathbb{H}_n$$

and

$$(f|_k\gamma)(z) = \det(cz+d)^{-k} f(\gamma(z)).$$

Definition 1.2.1. *A holomorphic function $f : \mathbb{H}_n \longrightarrow \mathbb{C}^r$ is called a genus n Siegel modular form of weight l on $\mathrm{Sp_n}(\mathbb{Z})$ if f satisfies*

$$(f|_l\gamma)(z) = f(z) \quad \forall \gamma \in \mathrm{Sp}_n(\mathbb{Z}).$$

When $n = 1$ we also require that f be holomorphic at ∞.

We denote the vector space of Siegel modular forms of weight k with genus n on $\mathrm{Sp}_n(\mathbb{Z})$ by $\mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$. When $n = 1$, the Siegel modular forms are classical modular forms. For each $f \in \mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$, there is the Fourier expansion

$$f(z) = \sum_{\xi} a(\xi) e_n(\xi z),$$

where ξ run over all $\xi = {}^t\xi, \xi \geq 0$. We put

$$\begin{aligned} A &= \{\xi = (\xi_{ij}) \in M_n(\mathbb{R}) \mid \xi = {}^t\xi, \xi_{ij}, 2\xi_{ii} \in \mathbb{Z}\}, \\ B &= \{\xi \in A \mid \xi \geq 0\}, \\ C &= \{\xi \in A \mid \xi > 0\}. \end{aligned}$$

For each $\gamma \in G_{\mathbb{Q}}^+$ we have the Fourier expansion

$$(f|_l\gamma)(z) = \sum_{\xi \in M_\gamma^{-1} B} a_\gamma(\xi) e_n(\xi z)$$

with $a_\gamma(\xi) \in \mathbb{C}^r, M_\gamma \in \mathbb{N}$. A form f is called a cusp form if for all ξ with $\det(\xi) = 0$ one has $a_\gamma(\xi) = 0$ for all $\gamma \in G_{\mathbb{Q}}^+$. This means that we have

$$(f|_l\gamma)(z) = \sum_{\xi \in M_\gamma^{-1} C} a_\gamma(\xi) e_n(\xi z).$$

We denote by $\mathcal{S}_n^l(\mathrm{Sp}_n(\mathbb{Z})) \subset \mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$ the subspace of cusp forms.
Definition of the vector spaces $\mathcal{M}_n^l(N, \psi)$. Let us consider congruence subgroup $\Gamma_1^n(N) \subset \Gamma_0^n(N) \subset \Gamma^n(N) = \mathrm{Sp}_n(\mathbb{Z})$, defined by

$$\begin{aligned} \Gamma_0^n(N) &= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}) \mid c \equiv 0_n \mod N \right\} \\ \Gamma_1^n(N) &= \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z}) \mid c \equiv 0_n \mod N, \det(a) \equiv 1 \mod N \right\}. \end{aligned}$$

Then we set

$$\mathcal{M}_n^l(N, \psi) = \left\{ f \in \mathcal{M}_n^l(\Gamma_1^n(N)) \mid f|_l\gamma = \psi(\det(a)) f \right\}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^n(N)$.
Put

$$\mathcal{S}_n^l(N, \psi) = \mathcal{M}_n^l(N, \psi) \cap \mathcal{S}_l(\Gamma_1^l(N)).$$

The Petersson scalar product. For $f \in \mathcal{S}_n^k(N,\psi)$ and $h \in \mathcal{M}_n^k(N,\psi)$ the Petersson scalar product is defined by

$$\langle f, h\rangle_N = \int_{\Phi_0(N)} \overline{f(z)} h(z) \det(y)^k d^{\times} z,$$

where $\Phi_0(N) = \Gamma_0^n(N)/\mathbb{H}_n$ is a fundamental domain for the group $\Gamma_0^n(N)$ and $d^{\times} z = \det(y)^{-\frac{n(n+1)}{2}} dz$.

Estimates for Fourier coefficients. If $f \in \mathcal{S}_n^k(N,\psi)$, then there is the following upper estimate

$$|f(z)| = O\left(\det(y)^{-k/2}\right) \quad (z = x + iy \in \mathbb{H}_n).$$

This provides us also with the estimate

$$|c(\xi)| = O\left(\det(\xi)^{k/2}\right).$$

For modular (not necessary cusp) forms

$$f(z) = \sum_{\xi \in B} c(\xi) e_n(\xi z) \in \mathcal{M}_n^k(N,\psi),$$

there is the upper estimate of their growth

$$|c(\xi)| \le c_1 \prod_{j=1}^{n} (1 + \lambda_j^k),$$

with $\lambda_1, \cdots, \lambda_n$ being eigenvalues of the matrix $y, z = x + iy$. In this situation one has also the estimate

$$|c(\xi)| \le c_2 \det(\xi')^k,$$

in which c_2 is a positive constant depending only on f, and $\xi = {}^t u \begin{pmatrix} \xi' & 0 \\ 0 & 0 \end{pmatrix} u$, where

$$u \in \mathrm{SL}_n(\mathbb{Z}), \xi \in B_r, \det(\xi') > 0, r < n.$$

1.3 Hecke Algebras

(See [8], [7]). Let q be a prime number, $q \nmid N$. We denote by

$$\Delta = \Delta_q^n(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}}^{+} \cap GL_{2n}(\mathbb{Z}[q^{-1}]) \mid \nu(\gamma)^{\pm} \in \mathbb{Z}[q^{-1}], c \equiv 0_n \mod N \right\}$$

be a subgroup in $G_{\mathbb{Q}}^{+}$ containing $\Gamma = \Gamma_0^n(N)$. The Hecke algebra over $\mathbb{Q}$ denoted by $\mathcal{L} = \mathcal{L}_q^n(N) = \mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta)$ is then defined as a $\mathbb{Q}$-linear space generated by the double cosets $(g) = (\Gamma g \Gamma), g \in \Delta$ of the group Δ, with respect to the subgroup Γ for which multiplication

is defined by the standard rule.
A double coset can be represented as a disjoint union of left cosets.

$$(\Gamma g\Gamma) = \cup_{j=1}^{t(g)} \Gamma g_j$$

where $t(g) = \sharp \{\Gamma/(\Gamma g\Gamma)\}$. We also can write

$$\mathcal{L} = \Big\langle \sum_{j=1}^{t(g)} (\Gamma g_j) \Big\rangle_{\mathbb{Q}}, g \in \Delta.$$

Therefore, any element $X \in \mathcal{L}$ takes the form of a finite linear combination

$$X = \sum_{i=1}^{t(X)} \mu_i (\Gamma g_i),$$

with $\mu_i \in \mathbb{Q}, g_i \in \Delta$.
Now, we recall the multiplication in the Hecke algebra denoted by $\mathcal{L} = \mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta)$ is defined by use of the larger vector space $\mathcal{V} = \mathcal{V}_{\mathbb{Q}}(\Gamma, \Delta)$ over $\mathbb{Q}$ consisting of all $\mathbb{Q}$-linear combinations of left cosets of the form (Γg) where $g \in \Delta = \Delta_q^n(N)$. Let us consider two elements

$$X = \sum_{i=1}^{t(X)} a_i (\Gamma g_i), Y = \sum_{j=1}^{t(Y)} b_j (\Gamma g_j)$$

in $\mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta) \in \mathcal{V}$ the element

$$X \cdot Y = \sum_{i=1}^{t(X)} \sum_{j=1}^{t(Y)} a_i b_j (\Gamma g_i g_j) \in \mathcal{V}_{\mathbb{Q}}(\Gamma, \Delta)$$

is well defined and also belongs to $\mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta) \subset \mathcal{V}$.
For each $j, 1 \leq j \leq n$ let us denote by W_j an automorphism of the algebra $\mathbb{Q}[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ defined by the rule:

$$X_0 \mapsto X_0 X_j, X_j \mapsto X_j^{-1}, X_i \mapsto X_i, 1 \leq i \leq n, i \neq j.$$

Then the automorphisms W_j and the permutation group S_n of the variables $X_i (1 \leq i \leq n)$ generate together the Weyl group $W = W_n$ and there is the Satake isomorphism:

$$\text{Sat} : \mathcal{L} \to \mathbb{Q}[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{W_n}.$$

For any commutative $\mathbb{Q}$-algebra A the group W_n acts on the set $(A^\times)^{n+1}$, therefore any homomorphism of $\mathbb{Q}$-algebras $\lambda : \mathcal{L} \to A$ can be identified with some element

$$(\alpha_0, \alpha_1, \cdots, \alpha_n) \in [(A^\times)^{n+1}],$$

which is defined up to the action of W_n.

Definition 1.3.1. *The Satake p-parameters associated to the eigenform $f \in \mathcal{M}_n^l(\mathrm{Sp}_n(\mathbb{Z}))$ are the elements of the $(n+1)$-tuple $(\alpha_0, \alpha_1, \cdots, \alpha_n) \in [(A^\times)^{n+1}]^{W_n}$ which is the image of the map $f \longmapsto \lambda_f(X)$ under the isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathbb{C}) \cong [(A^\times)^{n+1}]$, defined up to the action of W_n.*

Example 1.3.2. *If f is the Siegel-Eisenstein series of weight l genus n, the Satake p-parameters are*

$$\alpha_0 = 1, \alpha_i = p^{k-i}, i = \overline{1,n}.$$

Remark 1.3.3. *If $f \in \mathcal{M}_n^l(N,\psi)$, then the Satake p-parameters of f satisfy the relation*

$$\alpha_0^2 \alpha_1 \cdots \alpha_n = \psi(q)^n q^{ln - \frac{n(n+1)}{2}}.$$

Next, we define the Hecke operator. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$, we used the notation

$$(f \mid_{k,\psi} g)(z) = \det(g)^{k-\kappa} \psi(\det(a)) \det(cz+d)^{-k} f(g(z)).$$

(This convenient notation was suggested by Petersson and Andrianov; compare with the notation (6)). For any Hecke algebra X we define the action of X as follows:

$$f \mid X = \sum_{i=1}^{t(X)} \mu_i f \mid_{k,\psi} g_i.$$

Now we want to study the abstract $\mathbb{C}$-Hecke algebra associated to the Hecke pair $(\mathrm{Sp}(n,\mathbb{Q}), \Gamma_0^n(M))$. As a vector space, this is the set of finite formal linear combinations of all double cosets $\Gamma_0^n(M) g \Gamma_0^n(M), g \in \mathrm{Sp}(n,\mathbb{Q})$. We only consider special double cosets of the type

$$\Gamma_0^n(M) \begin{pmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{pmatrix} \Gamma_0^n(M), W \in M_n(\mathbb{Z})^*, \tag{1.1}$$

where $M_n(\mathbb{Z})^*$ denotes the non-singular integral matrices of size n, and where W is chosen as an elementary divisor matrix. We denote by $\mathcal{L}^\circ$ the $\mathbb{C}$-linear span of all these double cosets. Actually, we shall soon see that $\mathcal{L}^0$ is a subalgebra of $\mathcal{L}$.
We easily see that the above double cosets have "upper triangular" representatives

$$\Gamma_0^n(M) \begin{pmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{pmatrix} \Gamma_0^n(M) = \bigcup \Gamma_0^n(M) g_i$$

where $g_i = \begin{pmatrix} * & * \\ 0_n & * \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{Q})$. From this, we obtain for all $V, W \in M_n(\mathbb{Z})^*$ with coprime determinants

$$\Gamma_0^n(M) \begin{pmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{pmatrix} \Gamma_0^n(M) \cdot \Gamma_0^n(M) \begin{pmatrix} {}^tV^{-1} & 0 \\ 0 & V \end{pmatrix} \Gamma_0^n(M)$$
$$= \Gamma_0^n(M) \begin{pmatrix} {}^t(WV)^{-1} & 0 \\ 0 & WV \end{pmatrix} \Gamma_0^n(M). \tag{1.2}$$

Lemma 1.3.4 (Corollary 3.1. in [7])**.** *The set of upper triangular matrices in $\mathrm{Sp}(n,\mathbb{Q})$ which occur in the above doubles cosets is equal to*

$$\left\{ \begin{pmatrix} {}^t\omega_3^{-1} {}^t\omega_1 & {}^t\omega_3^{-1} B \omega_1^{-1} \\ 0 & \omega_3 \omega_1^{-1} \end{pmatrix} \mid \omega_1, \omega_3 \in M_n(\mathbb{Z})^*, \omega_1 \text{ coprime to } \omega_3 \right\} \tag{1.3}$$

with $B \in \mathbb{Q}_{sym}^{(n,n)}, \det(\omega_1)$ and $\nu(B)$ both coprime to M.

We only have the following relation between the upper triangular matrix and the elementary divisor matrix W :

$$\det(W) = \pm\det(\omega_1)\det(\omega_3)\nu(B).$$

The set (1.3) is a semigroup, therefore $\mathcal{L}^\circ$ is a Hecke algebra. Otherwise, as a consequence of (1.1) we get a decomposition of our Hecke algebra into p-components:

$$\mathcal{L}^\circ \simeq \bigotimes_p \mathcal{L}^\circ_{M,p}$$

the p-component being defined by double cosets with $\det(W) =$ power of the prime p.
From the Satake isomorphism we know that for p coprime to M,

$$\mathcal{L}^\circ_{M,p} \simeq \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, \cdots, X_n^{\pm 1}]^{W_n}$$

with W_n denoting the Weyl group generated by the permutation of the X_i and the rule:

$$X_j \mapsto X_j^{-1}, X_i \mapsto X_i, 1 \leq i \leq n, i \neq j.$$

For $w \in \mathbb{Z}$, define

$$(w)_M := \prod_{p|M} p^{\nu_p(w)}.$$

Note that

$$\Gamma_0^n(M)\begin{pmatrix} {}^tW^{-1} & 0 \\ 0 & W \end{pmatrix}\Gamma_0^n(M) \mapsto \det(W)$$

does not define an algebra homomorphism from $\mathcal{L}^\circ$ to $\mathbb{C}$, but that $(\det W)_M$ does.
For a double coset (1.1), we define a Hecke operator $T_M(W)$ acting on $C^\infty M_n^l(\Gamma_0^n(M),\psi)$ by

$$f \mid T_M(W) = \sum_i \psi(\det(W)_M)\cdot\det(\alpha_i) f \mid_l \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix},$$

with

$$\Gamma_0^n(M)\begin{pmatrix} {}^tW'^{-1} & 0 \\ 0 & W \end{pmatrix}\Gamma_0^n(M) = \bigcup_i \Gamma_0^n(M)\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

For $r \mid M$ and a cusp form $f \in S_n^l(\Gamma_0^n(M),\psi)$ consider the summation

$$\sum_D f \mid T_M(D)\det(D)^{-s},$$

where $D = \operatorname{diag}(d_1,\cdots,d_n), d_i > 0$ runs over all elementary divisor matrices of size n, and $d_i \mid d_{i+1}$ with $(\det D, r) = 1$. Assume now that f is an eigenform of all Hecke operators $T_M(D)$, and that $(\det D, r) = 1$.Then

$$f \mid T_M(D) = \lambda_D(f) f.$$

The mapping

$$\Gamma_0^n(M)\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}\Gamma_0^n(M) \mapsto \lambda_D(f)$$

induces (for all p coprime to r) homomorphisms $\chi_p : \mathcal{L}^{\circ}_{M,p} \to \mathbb{C}$, which are parameterized by

$$\begin{aligned} &\alpha^{\pm 1}_{1,p}, \cdots, \alpha^{\pm 1}_{n,p} \text{ for } p \nmid M \\ &\beta_{1,p}, \cdots, \beta_{n,p} \text{ for } p \mid M, p \nmid r. \end{aligned}$$

To compute the summation defined above we use the following theorem:

Theorem 1.3.5 ([6], page 37)**.** *Let p be a prime number. Then the following holds in the ring of formal power series $\mathcal{L}^{\circ}_{M,p}[[T]]$ over $\mathcal{L}^{\circ}_{M,p}$ via the isomorphism Q*

$$\sum_D Q\left(\Gamma^n \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \Gamma^n\right) T^{\sum_{i=1}^n \alpha_i} = \frac{1-T}{1-p^nT} \prod_{i=1}^n \frac{(1-p^{2i}T^2)}{(1-X_ip^nT)(1-X_i^{-1}p^nT)}.$$

The summation runs over all elementary matrices

$$D = \operatorname{diag}(p^{\alpha_1}, \cdots, p^{\alpha_n}) \textit{ with } 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$$

and Q is the isomorphism $\mathcal{L}^{\circ}_{M,p} \to \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$.

Proof. First, we write the sum in the following form:

$$\sum_{v,\omega,B} Q\left(\Gamma^n \begin{pmatrix} \omega'^{-1}v' & \omega'^{-1}Bv^{-1} \\ 0 & \omega v^{-1} \end{pmatrix}\right) T^{\nu_p(\det(v)\det(\omega)\mu(B))}.$$

The summation runs over all

$$\begin{aligned} &\begin{pmatrix} v \\ \omega \end{pmatrix} \in \begin{pmatrix} 1_n & 0 \\ 0 & GL(n,\mathbb{Z}) \end{pmatrix} \backslash \mathcal{L}^n_p / GL(n,\mathbb{Z}) \\ &B \in \mathbb{Z}[p^{-1}]^{(n,n)}_{\text{sym}} \mod \omega' \mathbb{Z}^{(n,n)}_{\text{sym}} \omega. \end{aligned}$$

Next we study the sum over B. For a fixed ω we decompose

$$\begin{aligned} &B = B_0 + B_1 \\ &B_0 \in \mathbb{Z}[p^{-1}]^{(n,n)}_{\text{sym}} \mod 1 \\ &B_1 \in \mathbb{Z}^{(n,n)}_{\text{sym}} \mod \omega' \mathbb{Z}^{(n,n)}_{\text{sym}} \omega. \end{aligned}$$

Then we have $\mu(B) = \mu(B_0)$. It Follows from the definition of the mapping Q that we can decompose it as follows:

$$\sum_{B_0} T^{\nu_p(\mu(B_0))} = \frac{1-T}{1-p^nT} \prod_{i=1}^n \frac{1-p^{2i}T^2}{1-p^{n+i}T^2}.$$

The summation over B_1 gives the following additional factor

$$\mid \det(\omega) \mid^{n+1} = \sharp \mathbb{Z}^{(n,n)}_{\text{sym}} / \omega' \mathbb{Z}^{(n,n)}_{\text{sym}} \omega.$$

The sum over v and ω can be described from the summation over $v \in M^n_p/GL(n,\mathbb{Z}), \omega \in GL(n,\mathbb{Z})/M^n_p$ under the condition that $\begin{pmatrix} v \\ \omega \end{pmatrix}$ is primitive. Here we denote $M^n_p = GL(n,\mathbb{Z}[p^{-1}]) \cap$

$\mathbb{Z}^{(n,n)}$.

Notice now each pair $\begin{pmatrix} v \\ \omega \end{pmatrix}$ can be written in the form

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} v_0 \\ \omega_0 \end{pmatrix} R,$$

where R is defined up to the unique unimodular left side factor.

When we multiply the summation with

$$\sum_{R \in GL(n,\mathbb{Z}) \backslash M_p^n} |\det(R)|^{n+1} T^{2\nu_p(\det(R))} = \prod_{i=1}^{n} \frac{1}{(1-p^{n+1}T^2)},$$

then we reduce it to the following series:

$$\sum_{v \in M_p^n / GL(n,\mathbb{Z})} \sum_{\omega \in GL(n,\mathbb{Z})/M_p^n} |\det(\omega)|^{n+1} Q\left(\Gamma^n \begin{pmatrix} \omega'^{-1} v' & \omega'^{-1} B v^{-1} \\ 0 & \omega v^{-1} \end{pmatrix}\right) T^{\nu_p(\det(v)\det(\omega))}.$$

To study (under the condition $\begin{pmatrix} v \\ \omega \end{pmatrix}$ primitive), one can choose without loss the generality v and ω in triangular image. So one can follow the definition of isomorphism Q immediately and the contribution of v and of ω separately:

$$\begin{aligned} &\sum_{v \in M_p^n / GL(n,\mathbb{Z})} Q\left(\Gamma^n \begin{pmatrix} v' & 0 \\ 0 & v^{-1} \end{pmatrix}\right) T^{\nu_p(\det(v))} \\ &= \sum_{t_1=0}^{\infty} \cdots \sum_{t_n=0}^{\infty} p^{t_2} \cdots p^{(n-1)t_n} Q\left(\Gamma^n \begin{pmatrix} v' & 0 \\ 0 & v^{-1} \end{pmatrix}\right) T^{t_1+\cdots+t_n}. \end{aligned}$$

For v a diagonal matrix with entries $p^{t_1}, \cdots, p^{t_n}$, we have

$$\begin{aligned} &\sum_{t_1=0}^{\infty} \cdots \sum_{t_n=0}^{\infty} p^{t_2} \cdots p^{(n-1)t_n} \left(\frac{X_1}{p}\right)^{t_1} \cdots \left(\frac{X_n}{p^n}\right)^{t_n} p^{(n+1)(t_1+\cdots+t_n)} T^{t_1+\cdots+t_n} \\ &= \prod_{i=1}^{n} \frac{1}{1 - X_i p^n T}. \end{aligned}$$

Similarly for the summation over ω, we have

$$\sum_{\omega \in GL(n,\mathbb{Z})/M_p^n} |\det(\omega)|^{n+1} Q\left(\Gamma^n \begin{pmatrix} \omega'^{-1} & 0 \\ 0 & \omega \end{pmatrix}\right) T^{\nu_p(\det(\omega))} = \prod_{i=1}^{n} \frac{1}{1 - X_i^{-1} p^n T}.$$

Multiplying of the summations over B, v, and ω, we have the proof of the theorem. □

1.4 Hecke polynomials

Let us consider polynomials $\tilde{Q}(z) \in \mathbb{Q}[x_0, \cdots, x_n][z]$ and $\tilde{R}(z) \in \mathbb{Q}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}][z]$:

$$\begin{aligned}\tilde{Q}(z) &= \tilde{Q}(z) \in \mathbb{Q}(x_0, \cdots, x_n; z) \\ &= (1 - x_0 z) \prod_{r=1}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - x_0 x_{i_1} \cdots x_{i_n} z),\end{aligned}$$

$$\tilde{R}(z) = \prod_{i=1}^{n} (1 - x_i^{-1} z)(1 - x_i z).$$

From the definition it follows that the coefficients of the powers of the variable z all belong to the subring $\mathbb{Q}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}]^{W_n}$. Therefore, by the Satake isomorphism there exist polynomials

$$Q(z) = \sum_{i=0}^{2^n} (-1)^i T_i z^i,$$

$$R(z) = \sum_{i=0}^{2^n} (-1)^i R_i z^i \in \mathcal{L}[z],$$

over the associative commutative ring $\mathcal{L} = \mathcal{L}_q^n(N)$, such that

$$\tilde{Q}(z) = \sum_{i=0}^{2^n} (-1)^i \tilde{T}_i z^i,$$

$$\tilde{R}(z) = \sum_{i=0}^{2^n} (-1)^i \tilde{R}_i z^i,$$

with $\tilde{X} = \mathrm{Sat} X, X \in \mathcal{L}$. The polynomials $\tilde{\Delta}_M'^{\pm 1}, R_i (1 \le i \le n-1)$ and T_1 with

$$\begin{aligned}\tilde{\Delta}_M'^{\pm 1} &= x_0^2 x_1^2 \cdots x_n^2, \\ R_i &= S_i(x_1, \cdots, x_n; x_1^{-1}, \cdots, x_n^{-1}), \\ \tilde{T}_1 &= x_0 \sum_{i=1}^{n} S_i(x_1, \cdots, x_n) = x_0 \prod_{i=1}^{n} (1 + x_i),\end{aligned}$$

can be taken as generators of the Hecke algebra, where S_i denotes the elementary symmetric polynomial of degree i in the corresponding set of variables.

1.5 The standard zeta function

Let $f \in \mathcal{M}_n^k(N, \psi)$ be an eigenfunction of all Hecke operators $f \longmapsto f|X, X \in \mathcal{L}_q^n(N)$, with q a prime number, $q \nmid N$, so that $f|X = \lambda_f(X) f$. Then the number $\lambda_f(X) \in \mathbb{C}$ defines a homomorphism $\lambda_f : \mathcal{L} \longrightarrow \mathbb{C}$ which is uniquely determined by the $(n+1)$-tuple of numbers

$$(\alpha_0, \alpha_1, \cdots, \alpha_n) = (\alpha_{0,f}(q), \alpha_{1,f}(q), \cdots, \alpha_{n,f}(q)) \in [(\mathbb{C})^{n+1}]^{W_n}.$$

These are called the Satake q-parameters of the modular form f.
Now let the variables $x_0, x_1, \cdots, x_n$ be equal to the corresponding Satake q-parameters $\alpha_{0,f}(q), \alpha_{1,f}(q), \cdots, \alpha_{n,f}(q)$. Then

$$\tilde{R}_{f,q}(z) = \prod_{i=1}^{n}(1-\alpha_i^{-1}z)(1-\alpha_i z) \in \mathbb{Q}[\alpha_0^{\pm 1}, \cdots, \alpha_n^{\pm 1}].$$

The standard zeta function of f is defined by means of the Satake p-parameters as the following Euler product:

$$\mathcal{D}(s,f,\chi) = \prod_{q \nmid N} \mathcal{D}^{(q)}(s,f,\chi),$$

with

$$\mathcal{D}^{(q)}(s,f,\chi) = (1-\chi(q)\psi(q)q^{-s})^{-1} R_{f,q}(\chi(q)\psi(q)q^{-s})^{-1}$$

$$\mathcal{D}(s,f,\chi) = \prod_{p} \left\{ (1 - \frac{\chi(p)}{p^s}) \prod_{i=1}^{m} (1 - \frac{\chi(p)\alpha_i(p)}{p^s})(1 - \frac{\chi(p)\alpha_i(p)^{-1}}{p^s}) \right\}^{-1}.$$

For $n=1$ and a normalized cusp eigenform $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$, we have that

$$\mathcal{D}(s,f,\chi) = L_{2,f}(s+k-1,\chi),$$

where

$$L_{2,f}(s+k-1,\chi) = L_{NM}(2s-2k+2,\chi^2\psi^2) \sum_{n=1}^{\infty} \chi(n)a(n^2)n^{-s}$$

is the symmetric square of the modular form f (see [2], [8]).

Chapter 2

Differential operators

In this chapter we study a class of differential operators introduced by Böcherer in [4]. These operators preserve automorphy for the groups $\mathrm{Sp}(n,\mathbb{R})^{\uparrow}, \mathrm{Sp}(n,\mathbb{R})^{\downarrow}$. They map automorphic forms of type (α, β) on $\mathbb{H}_{2n}$ to functions on $\mathbb{H}_n \times \mathbb{H}_n$, which are automorphic forms of type $(\alpha+\nu, \beta)$. This class of differential operators and their connection with pluriharmonic polynomials was carefully investigated by Ibukiyama in [16] and [15]. The operators of Böcherer have stronger properties than those of Ibukiyama. In particular, they can be iterated, but we must allow these operators to have non constant coefficients. By representation theory, we can see that two kinds of operators differ only up to a constant. In the first section, we recall the definition of differential operators and also its construction by Böcherer. The second section is devoted to an exposition of the invariant properties of differential operators, in comparison with the works of Ibukiyama. In the third section, we give an explicit description of differential operators which we need to prove the main congruences in Chapter 4. The last section presents some examples for the polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ in several cases.

2.1 Differential operators

2.1.1 Exterior algebra

The exterior algebra is the algebra of the wedge product. The exterior algebra of a vector space can be discribed as a quotient vector space

$$\bigwedge^{p} V = \bigotimes^{p} V / W_p$$

where W_p is the subspace of p-tensors generated by transpositions $W_2 =< x \otimes y + y \otimes x >$, and $\otimes$ denotes the vector space tensor product. The equivalence class $[x_1 \otimes \cdots \otimes x_p]$ is denoted by $x_1 \wedge \cdots \wedge x_p$.

Example 2.1.1. *Let V be a vector space with the basis $\{e_1, e_2, e_3, e_4\}$. We have*

$$\bigwedge^0 V =< 1 >$$
$$\bigwedge^1 V =< e_1, e_2, e_3, e_4 >$$
$$\bigwedge^2 V =< e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >$$
$$\bigwedge^3 V =< e_1 \wedge e_2 \wedge e_3 \wedge e_4 >$$

$\bigwedge^k V = \{0\}$ when $k > \dim V$.

For a general vector space V of dimension n, the space $\bigwedge^p V$ has dimension $\binom{n}{p}$. If $T : V \to W$ is a linear transformation, there is a map

$$T_{*,p} : \bigwedge^p V \to \bigwedge^p W$$
$$v_1 \wedge \cdots \wedge v_p \mapsto T(v_1) \wedge \cdots \wedge T(v_p).$$

If $n = \dim V$, and $T(v) = Av$ with A a square matrix then

$$T_{*,p}(e_1 \wedge \cdots \wedge e_p) = \det(A) e_1 \wedge \cdots \wedge e_n.$$

Wedge product

The wedge product is the product in an exterior algebra. If α, β are differential K-forms of degree p and q then

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

It has the following properties:

. Associative: $(\alpha \wedge \beta) \wedge u = \alpha \wedge (\beta \wedge u)$

. Non commutative: $\alpha \wedge \alpha = 0$ with $\alpha \in V$

. Bilinear

$$(c_1\alpha_1 + c_2\alpha_2) \wedge \beta = c_1(\alpha_1 \wedge \beta) + c_2(\alpha_2 \wedge \beta)$$
$$\alpha \wedge (c_1\beta_1 + c_2\beta_2) = c_1(\alpha \wedge \beta_1) + c_2(\alpha \wedge \beta_2).$$

The wedge product can be defined using a basis e_i for V.

$$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_q}) = e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_1} \wedge \cdots \wedge e_{j_q}.$$

The wedge product can be used to calculate determinants. Write $\det A = \det(c_1, \cdots, c_n)$ where c_i are the columns of A. Then

$$c_1 \wedge \cdots \wedge c_n = \det(c_1, \cdots, c_n) e_1 \wedge \cdots \wedge e_n.$$

2.1.2 $\sqcap$ multiplication

In this section we follow [5], p.84 and [13], chapter 3, paragraph 6. We consider a commutative ring k and a free module $V = k^n$ with basis $e_1, e_2, \ldots, e_n$. $e_a = e_{a_1} \wedge \ldots \wedge e_{a_p}$ are basis of $\bigwedge^p V$ $(0 \leq p \leq n)$ where $a = \{a_1, \ldots, a_p\}$ with $a_1 < \ldots < a_p, N = \{1, \ldots, n\}$.
Every linear mapping $A : \bigwedge^p V \to \bigwedge^p V$ corresponds to a $\binom{n}{p}$-matrix $(A^a_b)_{a,b \in \binom{N}{p}}$ using the relation

$$Ae_a = \sum_x A^a_x e_x.$$

We make no distinction between a mapping A and the corresponding matrix. To each pair of mappings $A : \bigwedge^p V \to \bigwedge^p V$ and $B : \bigwedge^q V \to \bigwedge^q V$, we attach the mapping $A \sqcap B : \bigwedge^{p+q} V \to \bigwedge^{p+q} V$ defined in coordinates by the relation

$$(A \sqcap B)^a_b = \frac{1}{\binom{p+q}{p}} \sum_{\substack{a=a' \cup a'' \\ b=b' \cup b''}} \epsilon(a', a'')\epsilon(b', b'') A^{a'}_{b'} B^{a''}_{b''}.$$

For $a' = \{a'_1, \ldots, a'_p\}, a" = \{a_1", \ldots, a_q"\}$ with $a'_1 < \ldots < a'_p$ and $a_1" < \ldots < a_q"$ we use the notation $\epsilon(a', a'')$ for the sign of permutation which takes the $(p+q)$-tuple into its natural order $(a'_1, \ldots, a'_p, a_1", \ldots, a_q")$. The operator

$$\sqcap : \mathrm{End}_k(\bigwedge^p V) \times \mathrm{End}_k(\bigwedge^q V) \to \mathrm{End}_k(\bigwedge^{p+q} V)$$

is bilinear, associative and commutative.
To a linear mapping $A : V \to V$, one can attach $A^{[p]}$ and $\mathrm{Ad}^{[p]} A$ on the space $\bigwedge^p V$ using the following relations

$$\begin{aligned}(A^{[p]})^a_b &= |A|^a_b \\ (\mathrm{Ad}^{[p]} A)^a_b &= \epsilon(a, N \backslash a)\epsilon(b, N \backslash b)|A|^{N \backslash b}_{N \backslash a}.\end{aligned}$$

By the notation $|A|^a_b$, we mean the following p-row subdeterminant:

$$|A|^a_b = \det((A_{ij})_{i \in a, j \in b}).$$

For $A : V \to V, B : V \to V, C : \bigwedge^p V \to \bigwedge^p V$ and $D : \bigwedge^q V \to \bigwedge^q V$, the following rules hold

$$A^{[p]} = A \sqcap \ldots \sqcap A \tag{2.1}$$

$$(A + B)^{[p]} = \sum_{\alpha+\beta=p} \binom{p}{\alpha} A^{[\alpha]} \sqcap B^{[\beta]} \tag{2.2}$$

$$A^{[p+q]}(C \sqcap D) = (A^{[p]}C) \sqcap (A^{[q]}D) \tag{2.3}$$

$$(C \sqcap D)A^{[p+q]} = (CA^{[p]}) \sqcap (DA^{[q]}) \tag{2.4}$$

$$(\mathrm{Ad}^{[p]} A)A^{[p]} = \det(A)1_{\binom{n}{p}}. \tag{2.5}$$

2.1.3 Differential operators on $\mathbb{H}_{2n}$

We consider a holomorphic function $f : \mathbb{H}_{2n} \to \mathbb{C}$ and $g : \mathbb{H}_{2n} \to \mathrm{End}_{\mathbb{C}}(\bigwedge^p \mathbb{C}^n)$. We denote:

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}, Z \in \mathbb{H}_{2n}, z_1, z_4 \in \mathbb{H}_n, z_2 = {}^t z_3 \in \mathbb{C}^{(n,n)}.$$

We use the operator

$$\partial_{ij} = \frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial z_{ij}}, \partial_3 = {}^t\partial_2.$$

We will put together in the symmetric $2n \times 2n$ matrix

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}, \partial_3 = {}^t\partial_2.$$

We recall multiplication $\sqcap$:

$$(\partial_i^{[p]} f)_b^a = |\partial_i|_b^a f$$
$$(\partial_i^{[p]} \sqcap g)_b^a = \frac{1}{\binom{p+q}{p}} \sum_{\substack{a=a'\cup a'' \\ b=b'\cup b''}} \epsilon(a',a'')\epsilon(b',b'')|\partial_i|_{b'}^{a'} g_{b''}^{a''}.$$

In particular, if $p+q=n$ then

$$\partial_i^{[p]} \sqcap \partial_j^{[q]} = \mathrm{tr}(\partial_{\mathrm{i}}^{[\mathrm{p}]}\mathrm{Ad}^{[\mathrm{q}]}\partial_{\mathrm{j}}).$$

Example 2.1.2. *We consider the test function*

$$f_{\mathfrak{T}}(Z) = e^{\mathrm{tr}(\mathfrak{T})Z} = e^{\mathrm{tr}(t_1 z_1 + 2t_2' z_2 + t_4 z_4)},$$

where

$$\mathfrak{T} = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}, t_3 = {}^t t_2,$$

$$\partial_i^{[p]} f_{\mathfrak{T}} = t_i^{[p]} f_{\mathfrak{T}} \quad (1 \le i \le 4),$$

and

$$\partial_2^{[p]} e^{\mathrm{tr}(tz_2)} = 2^{-p}(t')^{[p]} e^{\mathrm{tr}(tz_2)}, t \in \mathbb{C}^{(n,n)}.$$

Moreover, for $f, g \in \mathrm{Hol}(\mathbb{H}_{2n})$ we have the product

$$\partial_i^{[p]}(fg) = \sum_{\alpha+\beta=p} \binom{p}{\alpha} (\partial_i^{[\alpha]} f) \sqcap (\partial_i^{[\beta]} g),$$

as well as following formula:

$$\partial_4^{[p]} \det(z_4)^s = C_p(s) \det(z_4)^s (z_4)^{-[p]},$$

with

$$C_p(s) = s\big(s+\frac{1}{2}\big)\dots\big(s+\frac{p-1}{2}\big).$$

2.1.4 Construction of the operator $\mathcal{D}_k$

In this section, we will construct the operator $\mathcal{D}_k$ mentioned in the introduction in three steps.

a. Let α, β, p be nonnegative integers such that $\alpha + \beta + p = n$, and let $\delta(p, \alpha, \beta)$ be the operator given by

$$\delta(p, \alpha, \beta) = z_2^{[\alpha]}(\mathrm{Ad}^{[p+\beta]}\partial_1)\partial_2^{[p+\beta]}).$$

For $f \in \mathrm{Hol}(\mathbb{H}_{2n})$, denote a function of the following special form

$$f(Z) = g(z_4, z_2)e^{\mathrm{tr}(tz_1)}, t = {}^t t, \det(t) \neq 0.$$

$\delta(p, \alpha, \beta)$ is defined as

$$\delta(p, \alpha\beta) = t^{[\alpha+\beta]} z_2^{[\alpha+\beta]}(\partial_4^{[\alpha]} \sqcap \partial_3^{[\beta]} t^{-[\beta]} \partial_2^{[\beta]}) \sqcap \partial_2^{[p]} f. \tag{2.6}$$

For a justification one has to check the following identity

$$z_2^{[\alpha]}\partial_4^{[\alpha]} \sqcap (1_n^{[p]} \sqcap z_2^{[\beta]}\partial_3^{[\beta]})(Ad^{[p+\beta]}t)\partial_2^{[p+\beta]} = t^{[\alpha+\beta]} z_2^{[\alpha+\beta]}(\partial_4^{[\alpha]} \sqcap \partial_3^{[\beta]} t^{-[\beta]} \partial_2^{[\beta]}) \sqcap \partial_2^{[p]}; \tag{2.7}$$

This is done by repeated use of the rules (2.3), (2.4), as well as (2.5).

b. For $p + q = n$ $(p, q \geq 0)$ one puts

$$\Delta(p, q) = \sum_{\alpha+\beta=q} (-1)^\beta \delta(p, \alpha, \beta).$$

For

$$f(Z) = g(z_4, z_2)e^{\mathrm{tr}(tz_1)} \text{ with } t = {}^t t, \det(t) \neq 0,$$

it follows from (2.2) and (2.6) that

$$\Delta(p, q)f = t^{[q]} z_2^{[q]}(\partial_4 - \partial_3 t^{-1}\partial_2)^{[q]} \sqcap \partial_2^{[p]} f. \tag{2.8}$$

Using the commutativity rule of the $\sqcap$-multiplication, one rewrites this as

$$\Delta(p, q)f = \partial_2^{[p]} \sqcap t^{[q]} \underline{z}_2^{[q]}(\partial_4 - \partial_3 t^{-1}\partial_2)^{[q]} f$$

where $\underline{z}_2^{[q]}$ means that z_2 is considered as a constant in the derivation $\partial_2^{[p]}$. The operator $\Delta(p, q)$ has an additional symmetric property:

$$Z \mapsto V < Z >= Z\left[\begin{pmatrix} 0 & 1 \\ 1 & t_0 \end{pmatrix}\right] = \begin{pmatrix} z_4 & z_3 \\ z_2 & z_1 \end{pmatrix}.$$

For all $f \in \mathrm{Hol}(\mathbb{H}_{2n})$,

$$\Delta(p, q)(f \mid V) = (\Delta(p, q)f) \mid V. \tag{2.9}$$

To prove this formula, it is sufficient to check it for the test functions (as in [13], chapter 3, paragraph 6). One may admit also that t_1, t_2, t_4 are of maximal rank. In particular, one may use $\Delta(p,q)$ in the form (2.8). Then (2.9) follows from the identity

$$t_4^{[q]} z_2^{[q]} (t_1 - t_2 t_4^{-1} t_3)^{[q]} \sqcap t_3^{[p]} = t_1^{[q]} z_3^{[q]} (t_4 - t_3 t_1^{-1} t_2)^{[q]} t_2^{[p]}. \tag{2.10}$$

For proof one uses additional (2.10) to the rule (2.3), (2.4) the equation

$$A^{[p]} \sqcap B^{[q]} = A'^{[p]} \sqcap B'^{[q]}, A, B \in \mathbb{C}^{(n,n)}, p + q = n.$$

We need information on the operators $\Delta(p,q)$ with respect to the mapping behavior of

$$Z \mapsto I^{\downarrow} < Z >= \begin{pmatrix} z_1 - z_2 z_4^{-1} z_3 & z_2 z_4^{-1} \\ z_4^{-1} z_3 & -z_4^{-1} \end{pmatrix},$$

where

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}).$$

Theorem 2.1.3 (Proposition 1. in [5]). *Let $f = f_{\mathfrak{T}}$ be the test function with* $\det(t_1) \neq 0$*. We have*

$$\Delta(p,q)(f \mid_k I^{\downarrow}) = (-1)^{\alpha} \sum_{p+q=n} \sum_{\alpha+\beta=q} (-1)^{\alpha} \binom{q}{\alpha} C_{\alpha} \left(-k + \frac{n+\beta}{2}\right)$$
$$.(t_1 z_2 - t_2)^{[p]} \sqcap t_1^{[q]} z_2^{[q]} .(1_n^{[\alpha]} \sqcap z_4^{-[\beta]} (t_1^{-1}[t_2] - t_4)^{[\beta]}) f \mid_{k+1} I^{\downarrow}. \tag{2.11}$$

c. We denote, with not yet determined coefficients $\lambda_k(q)$, the operator

$$\mathcal{D}_k = \sum_{p+q=n} \lambda_k(q) \Delta(p,q).$$

One obtains from (2.11) the following decomposition:

$$(t_1 z_2 - t_2)^{[p]} = \sum_{\gamma+\delta=p} (-1)^{\delta} \binom{p}{\gamma} (t_1 z_2)^{[\gamma]} \sqcap t_2^{[\delta]}.$$

We also have the following simplification for the test function $f_{\mathfrak{T}}, \det(t_1) \neq 0$:

$$\mathcal{D}_k(f_{\mathfrak{T}} \mid_k I^{\downarrow}) = \sum_{p+q=n} \sum_{\gamma+\delta=p} \sum_{\alpha+\beta=q} (-1)^{\alpha+\beta+\delta+n} \lambda_k(q) \binom{p}{\gamma} \binom{q}{\alpha} C_{\alpha} \left(-k + \frac{n+\beta}{2}\right)$$
$$.(t_1 z_2)^{[\gamma+q]} (1_n^{[\alpha+\gamma]} \sqcap z_4^{-[\beta]} (t_4 - t_1^{-1}[t_2])^{\beta}) \sqcap t_2^{[\delta]} f_{\mathfrak{T}} \mid_k I^{\downarrow}. \tag{2.12}$$

Set $t = \alpha + \gamma$, and

$$a_k(t, \beta, \delta) = \sum_{\alpha+\gamma=t} (-1)^{\alpha+\beta+\delta+n} \lambda_k(\alpha+\beta) \binom{\alpha+\beta}{\alpha} \binom{\gamma+\delta}{\gamma} C_{\alpha} \left(-k + \frac{n+\beta}{2}\right).$$

Then (2.12) becomes the following

$$\mathcal{D}_k(f_{\mathfrak{T}} \mid_k I^{\downarrow}) = \sum_{t+\beta+\delta=n} a_k(t,\beta,\delta)(t_1 z_2)^{[t+\beta]}$$
$$.(1_n^{[t]} \sqcap z_4^{-[\beta]}(t_4 - t_1^{-1}[t_2])^{\beta}) \sqcap t_2^{[\delta]} f_{\mathfrak{T}} \mid_{k+1} I^{\downarrow}.$$

On the other hand

$$(\mathcal{D}_k f_{\mathfrak{T}}) \mid_{k+1} I^{\downarrow} = \sum_{p+q=n} \lambda_k(q)(t_1 z_2 z_4^{-1})^{[q]}(t_4 - t_1^{-1}[t_2])^{[q]} \sqcap t_2^{[p]} f_{\mathfrak{T}} \mid_{k+1} I^{\downarrow}.$$

Thus we have the following:

$$\mathcal{D}_k(f_{\mathfrak{T}} \mid_k I^{\downarrow}) = (\mathcal{D}_k f_{\mathfrak{T}}) \mid_{k+1} I^{\downarrow}.$$

When the coefficients $\lambda_k(q)$ for $t+\beta+\delta=n$ satisfy the following

$$a_k(t,\beta,\delta) = 0 \quad (1 \le t \le n) \tag{2.13}$$
$$a_k(0,\beta,\delta) = \lambda_k(\beta). \tag{2.14}$$

In the special case $\beta = 0$, the equation (2.13) says that

$$\sum_{\alpha+\delta=t} (-1)^{\alpha} \lambda_k(\alpha) \binom{\gamma+n-t}{\gamma} C_{\alpha}(-k+\frac{n}{2}) = 0 \quad (1 \le t \le n).$$

Assuming k is different from $\frac{n}{2}, \frac{n+1}{2}, \ldots, \frac{2n-1}{2}$, and $C_{\alpha}\left(-k+\frac{n}{2}\right) \neq 0$ for $0 \le \alpha \le n$ the following holds:

$$\lambda_k(\alpha) = \binom{n}{\alpha} \frac{\lambda_k(0)}{C_{\alpha}\left(-k+\frac{n}{2}\right)} \quad (0 \le \alpha \le n).$$

Equation (2.13) is then satisfied; in fact,

$$C_{\alpha+\beta}\left(-k+\frac{n}{2}\right) = C_{\beta}\left(-k+\frac{n}{2}\right) C_{\alpha}\left(-k+\frac{n+\beta}{2}\right),$$

and

$$\binom{n}{\alpha+\beta}\binom{\alpha+\beta}{\alpha}\binom{\gamma+\delta}{\gamma} = \frac{n!}{t!\beta!\delta!}\binom{t}{\alpha} \text{ with } t = \alpha+\gamma.$$

For $t \ge 1$:

$$a_k(t,\beta,\delta) = (-1)^{\beta+\delta+n} \frac{n!}{t!\beta!\delta!} \frac{1}{C_{\beta}\left(-k+\frac{n}{2}\right)} \sum_{\alpha+\gamma=t} (-1)\alpha \binom{t}{\alpha} = 0.$$

Setting

$$\tilde{C}_q(x) = \prod_{0 \le p \le n, p \neq q} C_p(x),$$

we obtain from the normalization

$$\lambda_k(0) = \tilde{C}_0\left(-k+\frac{n}{2}\right).$$

Theorem 2.1.4 (Theorem 2. in [5])**.**

$$\mathcal{D}_k = \sum_{p+q=n} \binom{n}{q} \tilde{C}_q\left(-k+\frac{n}{2}\right)\Delta(p,q)$$
$$= \sum_{\alpha+\beta+p=n} (-1)^\beta \binom{n}{\alpha+\beta} \tilde{C}_{\alpha+\beta}\left(-k+\frac{n}{2}\right) z_2^{[\alpha]}\partial_4^{[\alpha]} \sqcap ((1_n^{[p]} \sqcap z_2^{[\beta]}\partial_3^{[\beta]})(\mathrm{Ad}^{[p+\beta]}\partial_1)\partial_2^{[p+\beta]})$$

satisfies the following important relations:

$$\mathfrak{D}_k(f|_k M^\uparrow) = (\mathfrak{D}_k f)|_{k+1} M^\uparrow$$
$$\mathfrak{D}_k(f|_k M^{\downarrow)} = (\mathfrak{D}_k F)|_{k+1} M^\downarrow$$
$$\mathfrak{D}_k(f \mid V) = (\mathfrak{D}_k f) \mid V$$

for all $f \in Hol(\mathbb{H}_{2n}), M \in \mathrm{Sp}(n,\mathbb{R})$ *and* V *denotes the operator*

$$(f \mid V)(Z) = f\left(\begin{pmatrix} z_4 & z_3 \\ z_2 & z_1 \end{pmatrix}\right).$$

2.1.5 Definition of differential operators

We review the basic properties of certain holomorphic differential operators. These operators act on functions defined on $\mathbb{H}_{2n}$ and have some automorphy properties for the two copies of $\mathrm{Sp}(n,\mathbb{R})$ embedded in $\mathrm{Sp}(2n,\mathbb{R})$ in the usual way:

$$\mathrm{Sp}(n,\mathbb{R})^\uparrow = \left\{ \begin{pmatrix} a & 0_n & b & 0_n \\ 0_n & 1_n & 0_n & 0_n \\ c & 0_n & d & 0_n \\ 0_n & 0_n & 0_n & 1_n \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \right\},$$

$$\mathrm{Sp}(n,\mathbb{R})^\downarrow = \left\{ \begin{pmatrix} 1_n & 0_n & 0 & 0_n \\ 0_n & a & 0_n & b \\ 0_n & 0_n & 1_n & 0_n \\ 0_n & c & 0_n & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \right\}.$$

The differential operators are built up from the operators (with $1 \le i,j \le 2n$)

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial z_{ij}} & i=j \\ \frac{1}{2}\frac{\partial}{\partial z_{ij}} & i \neq j, \end{cases}$$

which we put together in the symmetric $2n \times 2n$ matrix

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix},$$

where ∂_i are block matrices of size n which correspond to the decomposition

$$\mathfrak{Z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

of $\mathbb{H}_{2n}$ into block matrices (with $z_3 = {}^t z_2$). We consider the polynomial $\Delta(r,q), r+q=n$, their coefficients being polynomials in the entries of z_2

$$\Delta(r,q) = \sum_{a+b=q} (-1)^b \binom{n}{b} z_2^{[a]} \partial_4^{[a]} \sqcap \Big(\big(1_n^{[r]} \sqcap z_2^{[b]} \partial_3^{[b]}\big) \big(\mathrm{Ad}^{[r+b]} \partial_1\big) \partial_2^{[r+b]} \Big).$$

In particular,

$$\Delta(n,0) = \det(\partial_2)$$
$$\Delta(0,n) = \det(z_2)\det(\partial).$$

Using the notation

$$C_q(s) = s\big(s+\frac{1}{2}\big)\dots\big(s+\frac{q-1}{2}\big) = \frac{\Gamma_q(s+\frac{q+1}{2})}{\Gamma_q(s+\frac{q-1}{2})},$$

we have the following definition.

Definition 2.1.5. *Define for any* $\alpha \in \mathbb{C}$

$$\mathfrak{D}_{n,\alpha} = \sum_{r+q=n} (-1)^r \binom{n}{r} C_r(\alpha - n + \frac{1}{2}) \Delta(r,q).$$

Example 2.1.6 ($n=1$)**.**

$$\begin{aligned}
\mathfrak{D}_{1,\alpha} &= \sum_{r+q=1} (-1)^r C_r(\alpha - 1 + \frac{1}{2}) \Delta(r,q) \\
&= -C_1(\alpha - \frac{1}{2}) \cdot \det(\partial_2) + C_0(\alpha - \frac{1}{2}) \det(z_2) \cdot \det(\partial) \\
&= -\frac{\partial_1(\alpha+\frac{1}{2})}{\partial_1(\alpha-\frac{1}{2})} \det(\partial_2) + \frac{\Gamma_0(\alpha)}{\Gamma_0(\alpha-1)} \cdot \det(z_2) \cdot \det(\partial) \\
&= -\frac{\partial_1(\alpha+\frac{1}{2})}{\partial_1(\alpha-\frac{1}{2})} \frac{\partial}{\partial z_2} + \frac{\Gamma_0(\alpha)}{\Gamma_0(\alpha-1)} \cdot z_2 \big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_4} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_3}\big) \\
&= (-\alpha + \frac{1}{2})\partial_2 + z_2(\partial_1\partial_4 - \partial_2\partial_2).
\end{aligned}$$

Example 2.1.7 ($n=2$)**.**

$$\begin{aligned}
\mathfrak{D}_{1,\alpha} &= \sum_{r+q=2} (-1)^r C_r(\alpha - \frac{3}{2}) \Delta(r,q) \\
&= C_0(\alpha - \frac{3}{2}) \Delta(0,2) - 2C_1(\alpha - \frac{3}{2}) \Delta(1,1) + C_2(\alpha - \frac{3}{2}) \Delta(2,0) \\
&= \det(z_2)\det(\partial) - 2(\alpha - \frac{3}{2}) \Delta(1,1) + (\alpha - \frac{3}{2})(\alpha - 1)\det(\partial_2),
\end{aligned}$$

where

$$\begin{aligned}
\Delta(1,1) &= \sum_{a+b=1} (-1)^b \binom{2}{b} z_2^{[a]} \partial_4^{[a]} \sqcap \Big(\big(1_2^{[1]} \sqcap z_2^{[b]} \partial_3^{[b]}\big) \big(\mathrm{Ad}^{[1+b]} \partial_1\big) \partial_2^{[1+b]} \Big) \\
&= z_2^{[1]} \partial_4^{[1]} \sqcap \Big(\big(1_2^{[1]} \sqcap z_2^{[0]} \partial_3^{[0]}\big) \big(\mathrm{Ad}^{[1]} \partial_1\big) \partial_2^{[1]} \Big) - 2 z_2^{[0]} \partial_4^{[0]} \sqcap \Big(\big(1_2^{[1]} \sqcap z_2^{[1]} \partial_3^{[1]}\big) \big(\mathrm{Ad}^{[2]} \partial_1\big) \partial_2^{[2]} \Big) \\
&= \mathrm{tr}(\mathrm{z}_2 \partial_4 (\mathrm{Ad}\partial_1)\partial_2) - 2\mathrm{tr}(\mathrm{z}_2\partial_3)\det(\mathrm{Ad}\partial_1)\det(\partial_2).
\end{aligned}$$

This operator satisfies the important relations:

$$\mathfrak{D}_{n,\alpha}(F|_{\alpha,\beta}M^{\uparrow}) = (\mathfrak{D}_{n,\alpha}F)|_{\alpha+1,\beta}M^{\uparrow}$$
$$\mathfrak{D}_{n,\alpha}(F|_{\alpha,\beta}M^{\downarrow)} = (\mathfrak{D}_{n,\alpha}F)|_{\alpha+1,\beta}M^{\downarrow}$$
$$\mathfrak{D}_{n,\alpha}(F \mid V) = (\mathfrak{D}_{n,\alpha}F) \mid V$$

for all $\beta \in \mathbb{C}$, all $F \in C_\infty(\mathbb{H}_{2n})$, and all $M \in \mathrm{Sp}(n,\mathbb{R})$.
For $v \in \mathbb{N}$, we put

$$\mathfrak{D}^{\nu}_{n,\alpha} := \mathfrak{D}_{n,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{n,k}$$
$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha} := \mathfrak{D}^{\nu}_{n,\alpha} \mid_{z_2=0} .$$

In particular, $\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}$ maps ($C^\infty-$) automorphic forms of type (α,β) on $\mathbb{H}_{2n}$ to functions on $\mathbb{H}_n \times \mathbb{H}_n$ which are automorphic of type $(\alpha+v,\beta)$ with respect to z_1 and z_4. If F is a holomorphic modular form on $\mathbb{H}_{2n}$, then $\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}F$ becomes a cusp form with respect to z_1 and z_2 (if $\nu > 0$).

2.2 The polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ and its properties

Definition 2.2.1. *For $T \in \mathbb{C}^{2n,2n}_{\mathrm{sym}}$, we define a polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ in the entries $t_{ij}(1 \leq i \leq j \leq 2n)$ of T by*

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(e^{\mathrm{tr}(TZ)}) = \mathfrak{P}^{\nu}_{n,\alpha}(T)e^{\mathrm{tr}(T_1z_1+T_4z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}.$$

Remark 2.2.2. *The $\mathfrak{P}^{\nu}_{n,\alpha}$ are homogenous polynomials of degree $n\nu$.*

Proof. We consider the action of $\mathrm{diag}(\lambda,\cdots,\lambda,\lambda^{-1},\cdots,\lambda^{-1}) \in \mathrm{Sp}(n,\mathbb{R})^{\uparrow} \times \mathrm{Sp}(n,\mathbb{R})^{\downarrow}$.
We have

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}\left(e^{\mathrm{tr}(TZ)})|_{\alpha}\mathrm{diag}(\lambda,\cdots,\lambda,\lambda^{-1},\cdots,\lambda^{-1})\right) = \mathring{\mathfrak{D}}^{\nu}_{n,\alpha}\left(e^{\mathrm{tr}(TZ)})|_{\alpha+\nu}\mathrm{diag}(\lambda,\cdots,\lambda,\lambda^{-1},\cdots,\lambda^{-1})\right).$$

Then

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(\lambda^{-2n\alpha}e^{\mathrm{tr}(\lambda^2TZ)}) = \lambda^{-2n(\alpha+\nu)}\mathfrak{P}^{\nu}_{n,\alpha}(T)e^{\mathrm{tr}(\lambda^2TZ)})$$
$$\lambda^{-2n\alpha}\mathfrak{P}^{\nu}_{n,\alpha}(\lambda^2T) = \lambda^{-2n(\alpha+\nu)}\mathfrak{P}^{\nu}_{n,\alpha}(T).$$

Hence

$$\mathfrak{P}^{\nu}_{n,\alpha}(\lambda^2T) = \lambda^{2n\nu}\mathfrak{P}^{\nu}_{n,\alpha}(T).$$

Therefore, $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ are homogeneous polynomials of degree $n\nu$. □

Remark 2.2.3. *For $X, Y \in \mathbb{C}^{m,n}$, m even, the polynomial*

$$Q(X,Y) = \mathfrak{P}^{\nu}_{n,\frac{m}{2}}\begin{pmatrix} X^tX & X^tY \\ Y^tX & Y^tY \end{pmatrix}$$

is a harmonic form of degree ν in both matrix variables X and Y, and it is symmetric in X and Y.

$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}$ is a polynomial in the ∂_{ij}, homogenous of degree $n\nu$ with at most one term free of the entries of ∂_1 and ∂_4, namely the term $C^{\nu}_{n,\alpha}\det(\partial_2)^{\nu}$ with a certain constant $C^{\nu}_{n,\alpha}$.
To determine the constant $C^{\nu}_{n,\alpha}$ explicitly, we first observe that (for arbitrary $\alpha, s \in \mathbb{C}$)

$$\mathfrak{D}^{\nu}_{n,\alpha}(\det(z_2)^s) = (-1)^n C_n(\frac{s}{2})C_n(\alpha - n + \frac{s}{2})\det(z_2)^{s-1},$$

which implies

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(\det(z_2)^{\nu}) = \left(\prod_{\mu=1}^{\nu} C_n(\frac{\mu}{2})\right) c^{\nu}_{n,\alpha} = (-1)^{n\nu}\prod_{\mu=1}\left(C_n(\frac{\mu}{2})C_n(\alpha - n + \nu - \frac{\mu}{2})\right).$$

In particular, we shall apply these differential operators to functions of type

$$f_s(\mathfrak{Z}) = \det(z_1 + z_2 + z_2' + z_4)^{-s}, s \in \mathbb{C}.$$

The following formulas will be used

$$\begin{aligned} \Delta(r,q)f_s &= 0 \text{ for } q > 0 \\ \Delta(n,0)f_s &= C_n(-s)f_{s+1} \\ \mathfrak{D}^{\nu}_{n,\alpha}f_s &= \frac{\Gamma_n(s+\nu)}{\Gamma_n(s)}.\frac{\Gamma_n(s+\nu-\frac{n}{2})}{\Gamma_n(s-\frac{n}{2})}.f_{s+\nu}. \end{aligned}$$

We would like to study further the properties of the polynomial $Q(X,Y)$, and also the polynomial $\mathfrak{P}^{\nu}_{n,\frac{m}{2}}$. First, we see that the polynomial $Q(X,Y)$ satisfies the following properties:

(i) $Q(AX, BY) = \det(AB)^{\nu}Q(X,Y)$ for any $A, B \in GL(n,\mathbb{C})$.

(ii) $Q(Xh, Yh) = Q(X,Y)$ for any $h \in O(d)$.

(iii) $Q(X,Y)$ are plurihamormonic for each X and Y:

$$\Delta_{ij}(X)Q = \Delta_{ij}(Y)Q = 0, (i,j = 1,\cdots,n),$$

where $\Delta_{ij}(X) = \sum_{\mu=1}^{d}(\partial^2/\partial x_{i\mu}\partial x_{j\mu})$ and $\Delta_{ij}(Y) = \sum_{\mu=1}^{d}(\partial^2/\partial y_{i\mu}\partial y_{j\mu})$ for $X = (x_{ij}), Y = (y_{ij})$. Under the condition (i) and (iii), it is equivalent to say that $Q(X,Y)$ is harmonic for each X and Y. We assume that $d \geq n$. If we write Q as $Q = Q(T)$, where T is a $2n \times 2n$ symmetric matrix, then by (i) we have

$$Q\left(\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} T \begin{pmatrix} {}^tA & 0 \\ 0 & {}^tB\end{pmatrix}\right) = \det(AB)^{\nu}Q(T) \text{ for any } A, B \in GL(n,\mathbb{C}).$$

We denote by $\mathbb{P}_{n,\nu}$ the set of all such polynomials Q, and we call ν an index of the polynomial $Q \in \mathbb{P}_{n,\nu}$. The total degree of Q as a polynomial is $n\nu$. Here note that the space $\mathbb{P}_{n,\nu}$ does not depend on d but the harmonicity condition does.
Now we study the generators of $\mathbb{P}_{n,\nu}$. We denote by $\mathrm{Sym}_n(\mathbb{R})$ the set of $n \times n$ symmetric matrices with coefficients in $\mathbb{R}$. We can regard $\mathbb{P}_{n,\nu}$ as the set of polynomials $P(R,S,W)$ in

the components of $(R,S,W) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$ such that the following relation is satisfied for any $A, B \in GL(n,\mathbb{R})$.

$$P(AR^tA, BS^tB, AW^tB) = \det(AB)^\nu P(R,S,W).$$

Here in the (X,Y) coordinates of the last section, we have $R = X^tX, S = Y^tY, W = X^tY$. The direct sum $\mathbb{P}_n = \oplus_{\nu=0}^\infty \mathbb{P}_{n,\nu}$ becomes a graded ring by natural multiplication. We also define the graded subring of even indices by $\mathbb{P}_{n,even} = \oplus_{\nu=0}^\infty \mathbb{P}_{n,2\nu}$. In order to give generators of these graded rings, we introduce the following notation. For each $0 \leq \alpha \leq n$, we define polynomial $P_\alpha(R,S,W) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$ by

$$\det \begin{pmatrix} xR & W \\ {}^tW & S \end{pmatrix} = \sum_{\alpha=0}^{n} P_\alpha(R,S,W)x^\alpha,$$

where x is an indeterminate. For example, $P_0(R,S,W) = (-1)^n \det(W)^2$ and $P_n(R,S,W) = \det(RW)$.

Theorem 2.2.4 (Proposition 3.1. in [16])**.** *The graded ring $\mathbb{P}_{n,even}$ is generated by the polynomials $P_\alpha (0 \leq \alpha \leq n)$ and $\mathbb{P}_n = \mathbb{P}_{n,even} \oplus \det(W)\mathbb{P}_{n,even}$. The $n+1$ polynomials $\det(W), P_1, \cdots, P_n$ are algebraically independent.*

Proof. We take $P(R,S,W) \in \mathbb{P}_{n,\nu}$. The polynomial P is determined by its values at $R = S = 1_n$ and $W =$ diagonal matrices. Indeed, this polynomial is determined by its values on any non-empty open subset, e.g. the open set consisting of (R,S,W) such that $R > 0, S > 0$ (positive definite symmetric matrices) and $W \in GL(n,\mathbb{R})$. For these matrices R, S, W we can take $A, B \in GL(n,\mathbb{R})$, so that $AR^tA = BS^tB = 1_n$. We put $W_0 = AW^tB$. Since we assumed that $\det(W) \neq 0$, there exist orthogonal matrices h_1, h_2 such that $h_1W_0h_2 = D$, where D is the diagonal matrix with diagonal elements $d_i (1 \leq i \leq n)$ with $d_i \neq 0$. So by (2), we have

$$P(1_n, 1_n, D) = \det(h_1h_2)^\nu P(1_n, 1_n, W_0) = \det(h_1h_2AB)^\nu P(R,S,W).$$

This shows that P is determined by $P(1_n, 1_n, D)$. Now, since $P(1_n, 1_n, V^{-1}DV) = P(1_n, 1_n, D)$ for any permutation matrix V, the polynomial $P(1_n, 1_n, D)$ is a polynomial in elementary symmetric polynomials of $d_1, \cdots, d_n$. For each i with $1 \leq i \leq n$, take a diagonal matrix ϵ_i such that (i,i)-component is -1 and that other diagonal components are 1. Then we see $P(1_n, 1_n, \epsilon_i D) = (-1)^\nu P(1_n, 1_n, D)$. So if ν is even, then $P(1_n, 1_n, D)$ is a polynomial in the elementary symmetric polynomials of $d_1^2, \cdots, d_n^2$. If ν is odd, then P changes sign if we change d_i into $-d_i$ for i. This means that $P(1_n, 1_n, D)$ is divisible by $d_1 \cdots d_n$ and $P(1_n, 1_n, D)/(d_1 \cdots d_n)$ is a symmetric polynomial of $d_1^2, \cdots, d_n^2$. Put $\det(x1_n - W_0{}^tW_0) = \sum_{\alpha=0}^n P'_\alpha(W_0)$. By the relation

$$\det(x1_n - W_0{}^tW_0) = \det(x1_n - D^2),$$

we see that $P(1_n, 1_n, W_0)$ is a polynomial in $P'_\alpha(W_0)$ when ν is even. When ν is odd, we have

$$P(1_n, 1_n, D)/\det(D) = P(1_n, 1_n, W_0)\det(h_1h_2)^\nu/\det(D) = P(1_n, 1_n, W_0)\det(h_1h_2)^\nu/\det(W_0).$$

We see also that $P(1_n, 1_n, W_0)$ is $\det(W_0)$ times a polynomial of P'_α. Since we have

$$\begin{aligned}\det(x1_n - W_0{}^tW_0) &= \det(x1_n - B^tWR^{-1}W^tB) = \det(x1_n - S^{-1t}WR^{-1}W) \\ &= \det(x1_n - R^{-1}WS^{-1t}W),\end{aligned}$$

and

$$\begin{vmatrix} R^{-1} & 0 \\ 0 & S^{-1} \end{vmatrix} \begin{vmatrix} xR & W \\ {}^tW & S \end{vmatrix} \begin{vmatrix} x1_n - R^{-1}WS^{-1t}W & R^{-1}W \\ 0 & 1_n \end{vmatrix},$$

we get

$$\det(RS)\det(x1_n - W_0{}^tW_0) = \begin{vmatrix} xR & W \\ {}^tW & S \end{vmatrix}.$$

Hence, $P_\alpha(R, S, W) = P'_\alpha(W_0)\det(RS)$. First, assume that ν is even. Since

$$\det(RS)^{\nu/2}P(1_n, 1_n, W_0) = \det(AB)^{-\nu}P(1_n, 1_n, W_0) = P(R, S, W),$$

$P(R, S, W)$ is a linear combination of the following functions:

$$\det(RS)^{\nu/2}\prod_{\alpha=0}^{n-1} P'_\alpha(AW^tB)^{e_\alpha} = \prod_{\alpha=0}^{n-1} P_\alpha(R, S, W)^{e_\alpha}\det(RS)^{\nu/2-\sum_{\alpha=0}^{n-1} e_\alpha}.$$

We will show that $\nu/2 - \sum_{\alpha=0}^{n-1} e_\alpha$ is non-negative. Consider the degree of this polynomial P. We write $R = (r_{ij}), S = (s_{ij}), W = (w_{ij})$ and put

$$P(R, S, W) = \sum_{\substack{1\le i_1\le i_2\le n \\ 1\le i_3\le i_4\le n \\ 1\le i_5\le i_6\le n}} c_{i_1i_2i_3i_4i_5i_6} r_{i_1i_2}^{l_{i_1i_2}} s_{i_3i_4}^{m_{i_3i_4}} w_{i_5i_6}^{n_{i_5i_6}}.$$

For simplicity, we put $l_{ij} = l_{ji}$ and $m_{ij} = m_{ji}$. Taking diagonal matrices $A = \mathrm{diag}(a_1, \cdots, a_n), B = \mathrm{diag}(b_1, \cdots, b_n)$, we get

$$P(AR^tA, BS^tB, AW^tB) = (\prod_{i=1}^n a_ib_i)^\nu P(R, S, W).$$

This means that for a fixed i or j, we have

$$2l_{ii} + \sum_{i_2\neq i} l_{i,i_2} + \sum_{i_6=1}^n n_{i,i_6} = \nu,$$

or

$$2m_{jj} + \sum_{i_1\neq j} m_{i_1,j} + \sum_{i_5=1}^n n_{i_5,j} = \nu.$$

Hence if we denote by N_{11} the degree of $P(R, S, W)$ with respect to w_{11}, then $N_{11} \le \nu$. If we assume that ν is even then we may write

$$P(1_n, 1_n, D) = P(1_n, 1_n, W_0) = \sum c(e_0, \cdots, e_n)\prod_{\alpha=0}^{n-1} P'_\alpha(W_0)^{e_\alpha}.$$

Here, $P'_\alpha(W_0)$ is an elementary symmetric polynomial in d_i^2. By Lemma 2.2.5, which follows below, we see that the degree of $P(1_n, 1_n, W_0)$ with respect to d_1 is the maximum of $2\sum_{\alpha=0}^{n-1} e_\alpha$ for $c(e_0, \cdots, e_{n-1}) \neq 0$. On the other hand, the degree of $P(1_n, 1_n, D) = P(1_n, 1_n, D)$ with respect to d_1 is at most $N_{11} \leq \nu$. So

$$2\sum_{\alpha=0}^{n-1} e_\alpha \leq \nu.$$

Next, we assume that ν is odd. Then

$$P(1_n, 1_n, W_0) = \det(W_0)p(P'_0(W_0), \cdots, P'_{n-1}(W_0)),$$

where p is a polynomial in n variables. Since $\det(W_0) = \det(AB)\det(W)$,

$$\begin{aligned} P(R, S, W) &= \det(W)\det(AB)^{-\nu+1}p(P'_0(W_0), \cdots, P'_{n-1}(W_0)) \\ &= \det(W)\det(RS)^{(\nu-1)/2}p(P'_0(W_0), \cdots, P'_{n-1}(W_0)). \end{aligned}$$

The last polynomial is a linear combination of monomials

$$\det(W)\det(RS)^{(\nu-1)/2-\sum_{\alpha=0}^{n-1} e_\alpha}\prod_{\alpha=0}^{n-1} P_\alpha(R, S, W)^{e_\alpha}.$$

Hence by the same argument as in the case of even ν, we have

$$(\nu - 1)/2 \geq \sum_{\alpha=0}^{n-1} e_\alpha.$$

Finally, by the restriction of $P_0, \cdots, P_{n-1}$ to $P(R, S, W) = (1_n, 1_n, D)$ is algebraically independent, and since $P_0, \cdots, P_n$ are homogeneous polynomials of the same degree, this also implies that $P_0, \cdots, P_n$ are algebraically independent. □

Now we show the lemma we used above. Let $F(z_1, \cdots, z_n)$ be a polynomial. We write $F(z_1, \cdots, z_n) = \sum_\beta c_\beta z^\beta$ where β runs over $\beta = (\beta_1, \cdots, \beta_n) \in (\mathbb{Z})^n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. We put $|\beta| = \beta_1 + \cdots + \beta_n$. For i with $1 \leq i \leq n$, we denote by s_i the elementary symmetric polynomial in independent variables $d_1, \cdots, d_n$ of degree i.

Lemma 2.2.5 (Lemma 3.2. in [16]). *Notation being as above, assume that $F(s_1, \cdots, s_n)$ is of degree a with respect to d_1. Then total degree of $F(z_1, \cdots, z_n)$ is a.*

Proof. See lemma 3.2. in [16] □

For later use we state here some arithmetic properties of $\mathfrak{P}^\nu_{n,\alpha}$. They are immediate consequences of the results above when combined with the simple observation that the operator

$$2^{n\nu}.\mathring{\mathfrak{D}}^\nu_{n,\alpha}$$

is a polynomial with integer coefficients in α and the $\partial_{i,j}(1 \leq i \leq j \leq 2n)$ evaluated at $z_2 = 0$.

Remark 2.2.6. *For all $\alpha \in \mathbb{Z}$ the $4^{n\nu}\mathfrak{P}^{\nu}_{n,\alpha}(T)$ are polynomials in the entries $t_{ij}(1 \leq i \leq j \leq 2n)$ of T with coefficients in $\mathbb{Z}$. They satisfy the congruence*

$$4^{n\nu}\mathfrak{P}^{\nu}_{n,\alpha}(T) \equiv (2^{n\nu}C^{\nu}_{n,\alpha}) \det(2T_2)^{\nu} \mod L$$

for any integer L and any half-integral

$$T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}$$

with $\frac{1}{L}T_1, \frac{1}{L}T_4$ both half-integral. We also mention that the integer $(2^{n\nu}C^{\nu}_{n,\alpha})$ is certainly nonzero for $\alpha > n$.

2.3 An explicit description of differential operators

For $(T_1, T_4, T_2) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathrm{Sym}_n(\mathbb{R}) \times M_n(\mathbb{R})$, we regard the polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ with $T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix} \in \Lambda^+_{2n}$ as a polynomial in variables (T_1, T_4, T_2). We denote this polynomial by $P(T_1, T_4, T_2)$.
For each (T_1, T_4, T_2) we can take matrices $A, B \in GL(n, \mathbb{R})$ such that

$$AT_1{}^tA = 1_n$$
$$BT_4{}^tB = 1_n.$$

We put

$$W_0 = AT_2{}^tB.$$

Since we assumed that $\det(W_0) \neq 0$, there exist two orthogonal matrices h_1, h_2 such that

$$h_1 W_0 h_2 = D,$$

where D is the diagonal matrix with diagonal elements d_i $(1 \leq i \leq n), d_i \neq 0$.

Theorem 2.3.1. *For the matrix $T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix} \in \Lambda^+_{2n}$ with $T_1, T_2, T_4 \in \Lambda^+_n$ and the matrices D defined as above, we have:*
If ν is even, then

$$\mathfrak{P}^{\nu}_{n,\alpha}(T) = \sum_{(e_0,\cdots,e_{n-1})\neq 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} \sum_{1\leq j_1<\cdots<j_k\leq n} \left(d^2_{j_1} \cdots d^2_{j_k}\right)^{e_\alpha}.$$

If ν is odd, then

$$\mathfrak{P}^{\nu}_{n,\alpha}(T) = (d_1 \cdots d_n) \sum_{(e_0,\cdots,e_{n-1})\neq 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} \sum_{1\leq j_1<\cdots<j_k\leq n} \left(d^2_{j_1} \cdots d^2_{j_k}\right)^{e_\alpha}.$$

Here, $P_j(D)$ is the elementary symmetric polynomial in d_i^2.

Proof. By the property of the polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$,

$$P(AT_1{}^tA, BT_4{}^tB, AT_2{}^tB) = \det(AB)^{\nu} P(T_1, T_4, T_2).$$

Then

$$P(T_1, T_4, T_2) = \det(AB)^{-\nu} P(AT_1{}^tA, BT_4{}^tB, AW{}^tB) = \det(AB)^{-\nu} P(1_n, 1_n, W_0).$$

We also have $T_2 \in \Lambda_n^+$, so there exist two orthogonal matrices h_1, h_2 such that

$$h_1 W_0 h_2 = D,$$

where D is the diagonal matrix with diagonal elements d_i $(1 \le i \le n), d_i \neq 0$. Therefore,

$$P(T_1, T_4, T_2) = \det(AB)^{-\nu} P(1_n, 1_n, W_0) = \det(ABh_1h_2)^{-\nu} P(1_n, 1_n, D).$$

This shows that the polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ is determined by its values at $T_1 = T_4 = 1_n$ and T_2 diagonal matrices.

Now, since $P(1_n, 1_n, V^{-1}DV) = P(1_n, 1_n, D)$ for any permutation matrix V, the polynomial $P(1_n, 1_n, D)$ is a polynomial in elementary symmetric polynomials of $d_1, \cdots, d_n$. For each i with $1 \le i \le n$, take a diagonal matrix ϵ_i such that (i,i)-components is -1 and that other diagonal components are 1. Then we see $P(1_n, 1_n, \epsilon_i D) = (-1)^{\nu} P(1_n, 1_n, D)$. So if ν is even, then $P(1_n, 1_n, D)$ is a polynomial in elementary symmetric polynomials of $d_1^2, \cdots, d_n^2$. If ν is odd, then P changes sign if we change d_i into $-d_i$ for i. This means that $P(1_n, 1_n, D)$ is divisible by $d_1 \cdots d_n$ and $P(1_n, 1_n, D)/(d_1 \cdots d_n)$ is a symmetric polynomial of $d_1^2, \cdots, d_n^2$.
Hence we can write the following:
If ν is even, then

$$\mathfrak{P}^{\nu}_{n,\alpha}(T) = \sum_{(e_0, \cdots, e_{n-1}) \neq 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} \sum_{1 \le j_1 < \cdots < j_k \le n} \left(d_{j_1}^2 \cdots d_{j_k}^2\right)^{e_\alpha}.$$

If ν is odd, then

$$\mathfrak{P}^{\nu}_{n,\alpha}(T) = (d_1 \cdots d_n) \sum_{(e_0, \cdots, e_{n-1}) \neq 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} \sum_{1 \le j_1 < \cdots < j_k \le n} \left(d_{j_1}^2 \cdots d_{j_k}^2\right)^{e_\alpha}.$$

□

2.4 An explicit formula for the polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ when $n = 1$ and $n = 2$

2.4.1 The polynomial $\mathfrak{P}^{\nu}_{1,\alpha}(T)$

For $n = 1$, denote

$$T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix}, Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_4 \end{pmatrix}, \partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2 & \partial_4 \end{pmatrix}$$

where

$$\partial_1 = \frac{\partial}{\partial z_1}, \partial_2 = \frac{1}{2}\frac{\partial}{\partial z_2}, \partial_4 = \frac{\partial}{\partial z_4}$$

and the test function

$$f = e^{t_1 z_1 + 2t_2 z_2 + t_4 z_4} = e^{\mathrm{tr}(TZ)}.$$

By the formula of the differential operator for $n = 1$ we have

$$\mathfrak{D}_{1,\alpha} = (-\alpha + \frac{1}{2})\partial_2 + z_2(\partial_1\partial_4 - \partial_2\partial_2).$$

Then composing ν times and specializing to $z_2 = 0$,

$$\mathfrak{D}^{\nu}_{1,\alpha} := \mathfrak{D}_{1,\alpha+\nu-1} \circ \ldots \circ \mathfrak{D}_{1,\alpha}$$
$$\mathring{\mathfrak{D}}^{\nu}_{1,\alpha} := \mathfrak{D}^{\nu}_{1,\alpha} \mid_{z_2=0},$$

we obtain an explicit formula for $\mathfrak{P}^{\nu}_{1,\alpha}(T)$ in certain cases.
With $\nu = 1$, we have:

$$\mathfrak{P}_{1,\alpha}(T) = (-\alpha + \frac{1}{2})2t_2.$$

With $\nu = 2$, we have:

$$\mathfrak{P}^2_{1,\alpha}(T) = (\alpha^2 - \frac{1}{4})4t_2^2 + (-\alpha - \frac{1}{2})(t_1 t_4 - t_2^2).$$

With $\nu = 3$, we have:

$$\begin{aligned}\mathfrak{P}^3_{1,k}(T) = &(-\alpha - \frac{3}{2})(\alpha^2 - \frac{1}{4})8t_2^3 + (-\alpha - \frac{3}{2})(-\alpha - \frac{1}{2})(t_1t_4 - t_2^2)2t_2 + (-\alpha - \frac{3}{2})(-\alpha + \frac{1}{2})t_1t_4t_2 \\ &- (-\alpha - \frac{3}{2})(-\alpha + \frac{1}{2})8t_2^3 + (-\alpha - \frac{3}{2})4t_2(t_1t_4 - t_2^2).\end{aligned}$$

By induction, we obtain the general formula

$$\mathfrak{P}^{\nu}_{1,k}(T) = \sum_{i+2i'=\nu} C_{\nu}(k) t_2^i (t_1t_4)^{i'},$$

where $C_\nu(k)$ is a polynomial in variable k ($k = j + 1$) of degree ν.

2.4.2 The polynomial $\mathfrak{P}^{\nu}_{2,\alpha}(T)$

For $T \in \mathbb{C}^{(4,4)}_{\mathrm{sym}}$, we quote from [7] the definition of the polynomial $\mathfrak{P}^{\nu}_{2,k}(T)$ in the entries $t_{i,j}(1 \le i \le j \le 4)$ of T by

$$\mathring{\mathfrak{D}}^{\nu}_{2,k}\left(e^{\mathrm{tr}(TZ)}\right) = \mathfrak{P}^{\nu}_{2,k}(T) e^{\mathrm{tr}(T_1 z_1 + T_4 z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}.$$

The $\mathfrak{P}^{\nu}_{2,k}(T)$ are homogenous polynomials of degree 2ν.
We give an explicit formula for the polynomials $\mathfrak{P}^{\nu}_{2,k}(T)$.
We denote

$$T = \begin{pmatrix} t_1 & t_2 \\ {}^t t_2 & t_4 \end{pmatrix} \in \mathbb{C}^{(4,4)}_{\text{sym}}, Z = \begin{pmatrix} z_1 & z_2 \\ {}^t z_2 & z_4 \end{pmatrix} \in \mathbb{H}_4, \partial = \begin{pmatrix} \partial_1 & \partial_2 \\ {}^t \partial_2 & \partial_4 \end{pmatrix}$$

where

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial z_{ii}} & i = j \\ \frac{1}{2} \frac{\partial}{\partial z_{ij}} & i \neq j, \end{cases}$$

which we put together in the symmetric 4×4 matrix ∂, and the test function

$$f = e^{t_1 z_1 + 2t_2 z_2 + t_4 z_4} = e^{\text{tr}(TZ)}.$$

Compare with the polynomial defined by Ibukiyama in [15] and [16], we obtain an explicit formula for the polynomials $\mathfrak{P}^{\nu}_{2,k}(T)$

$$\mathfrak{P}^{\nu}_{2,k}(T) = \sum_{i+2j+2k=2\nu} C_{\nu}(k)(\det t_2)^i (\det t_1 \det t_4)^j (\det T)^k,$$

where $C_{\nu}(k)$ is a polynomial of variable α of degree 2ν.

Chapter 3

Eisenstein series and the twisting process

This chapter is organized as five sections. In the first section, we recall the definitions of Siegel Eisenstein series and its Fourier expansion. To present Fourier coefficients of Siegel Eisenstein series we have to give an exposition of some results of Shimura and Feit on real analytic Siegel Eisenstein series and their analytic continuation in terms of confluent hypergeometric functions (see [25], [26], [12], [19]). These results extend previous results of Kalinin and Langlands. It is remarkable that we only need information about the Fourier coefficients of maximal rank. This allows us to stay essentially selfcontained and to avoid the use of more sophisticated results of Shimura in [25], [26] and [12]. In the second section, we introduce the twisted Eisenstein series and doubling method. We emphasize that the method of doubling the variables admits a modification which produces a good integral representation for twists of the standard L-function. The third section presents the trace and the shift operators. The fourth and the fifth section are devoted to introducing the notation of $\mathcal{H}$-functions as well as their Fourier expansions which we shall use to produce the modular distributions in chapter 4.

3.1 Eisenstein series

3.1.1 Definition

We call two matrices $C, D \in M(\mathbb{Z})$ coprime if

$$\{G \in M(\mathbb{Q}) \mid GC, GD \in M(\mathbb{Z})\} = M(\mathbb{Z}).$$

A couple (C, D) is called a symmetric couple if $C^t D = D^t C$. Two couples $(C_1, D_1), (C_2, D_2)$ of coprime matrices are called equivalent if and only if for some matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ we have

$$(C_1, D_1) = (UC_2, UD_2).$$

We denote by $\Delta = \Delta_n$ the set of equivalence classes of symmetric couples of coprime matrices. Let k, N be integers, s a complex number and χ a Dirichlet character $\mod N$ such that $\chi(-1) = (-1)^k$. For $z \in \mathbb{H}_n$ (the Siegel upper half plane degree n), we define the Siegel-Eisenstein series as follows:

$$E(Z, s, k, \chi, N) = E(z, s) = \det(Y)^s \sum_{(C,D)} \chi(\det(D)) \cdot \det(Cz + D)^{-k} |\det(CZ + D)|^{-2s}. \tag{3.1}$$

The summation is taken over all $(C, D) \in \Delta$ with the condition $C \equiv 0 \mod N$. This series is absolutely convergent for $k + 2Re(s) > n + 1$, and it admits a meromorphic analytic continuation over the whole complex s-plane.
A more conceptual definition is as follows:

$$E(Z, s, k, \chi, N) = E(z, s) = \det(Y)^s \sum_{T^n(N)_\infty \backslash T^n(M)} \chi(\det(D)) j(R, Z)^{-k} \det(\mathrm{Im}(R < Z >))^s, \tag{3.2}$$

with

$$\begin{aligned} T^n(M) &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid A \equiv 0 \mod M \right\} \\ T^n(M)_\infty &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid C = 0, B \equiv 0 \mod M \right\} \\ R < Z > &= (AZ + B)(CZ + D)^{-1}, R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ j(R, Z) &= \det(CZ + D). \end{aligned}$$

The aim of this text is to explain the Fourier expansion of the Siegel Eisenstein series. First, we consider some easier cases.
For the full symplectic modular group $\Gamma = \mathrm{Sp}(\mathbb{Z})$, Siegel defined the series

$$E(Z) = E_k^{(m)}(Z) = \sum_{T^n(N)_\infty \backslash T^n(M)} \chi(\det(D)) j(R, Z)^{-k}, \tag{3.3}$$

where $Z \in \mathbb{H}_n, k$ is even , $k > n+1, j(R, Z) = \det(CZ+D)$, and $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We consider the Fourier expansion of the series $E(Z)$ in the case $n = 1$:

$$\begin{aligned} E_k^{(1)}(z) &= 1 - \frac{2k}{B_k} \sum_{m=1}^{\infty} \sigma_{k-1}(m) e(mz) \\ &= 1 + \sum_{m=1}^{\infty} \left(\frac{2\sigma_{k-1}(m)}{\zeta(1-k)} \right) e(mz), \end{aligned} \tag{3.4}$$

where $\sigma_{k-1}(m) = \sum_{d|m} d^{k-1}$, B_k are Bernoulli numbers, and $\zeta(s)$ is the Riemann zeta function.

3.1.2 Formula for Fourier coefficients of Siegel Eisenstein series in the general case

For a detailed description of the Fourier expansion of Siegel Eisenstein series in the general case, we need the notation of the confluent hypergeometric function. First, we define the function $\zeta(z, \alpha, \beta)$ on

$$\mathbb{H}'_n = \{z \in M_n(\mathbb{C}) \mid iz \in \mathbb{H}_n\},$$

by the integral

$$\zeta(z, \alpha, \beta) = \int_Y e^{-\mathrm{tr}(zx)} \det(x + 1_n)^{\alpha-\kappa} \det x^{\beta-\kappa} dx. \tag{3.5}$$

Here, $\alpha - \kappa \in \mathbb{Z}_{\geq 0}, \beta \in \mathbb{Z}_{\leq 0}$, and $\kappa = \frac{n+1}{2}$. Let

$$w(z, \alpha, \beta) = \Gamma_n(\beta)^{-1} \det(z)^\beta \zeta(z, \alpha, \beta). \tag{3.6}$$

This function was used by Shimura ([26], theorem 3.1) for computing the Fourier expansion of the series

$$S(z, L, \alpha, \beta) = \sum_{a \in L} \det(z + a)^{-\alpha} \det(\bar{z} + a)^{-\beta}, \tag{3.7}$$

where

$$L \subset V, V = \{h \in M_n(\mathbb{R}) \mid {}^t h = h\}, Y = \{h \in V \mid h > 0\}.$$

For each matrix $T \in M_n(\mathbb{R})$, let $\delta_+(T)$ denote the product of all positive eigenvalues of T, and let $\delta_-(T) = \delta_+(-T)$.
We have

$$\mu(V/L) S(z, L, \alpha, \beta) = \sum_{h \in L'} \xi(y, h, \alpha, \beta) e_n(hx), \tag{3.8}$$

where

$$L' = \{h \in V \mid \mathrm{tr}(hl) \in \mathbb{Z}\}, \mu(V/L) = \int_{V/L} dy.$$

The Fourier coefficients of this series have the form:

$$\begin{aligned} \xi(y, h, \alpha, \beta) = {} & i^{n\beta - n\alpha} 2^r \pi^\sigma \Gamma_{n-r}(\alpha + \beta - \kappa) \Gamma_{n-q}(\alpha)^{-1} \Gamma_{n-p}(\beta)^{-1} \\ & \cdot \det(y)^{\kappa-\alpha-\beta} \delta_+(hy)^{\alpha-\kappa+q/4} \delta_-(hy)^{\beta-\kappa+q/4} w(2\pi y, h, \alpha, \beta). \end{aligned} \tag{3.9}$$

The confluent hypergeometric function has the following properties:

$$\begin{aligned} w(z, \kappa - \beta, \kappa - \alpha) &= w(z, \alpha, \beta) \\ w(z, \alpha, \beta) &\leq A(1 + \mu(y)^{-B}) \\ w(y, h, \alpha, \beta) &= 2^{\frac{m\kappa}{2}} e^{-\mathrm{tr}(y)} w(2ahya^{-1}, \alpha, \beta), \end{aligned} \tag{3.10}$$

for constants $A, B > 0$ and $h \in \mathbb{H}'_n$. This function also relates to the Maass differential operator by the polynomial defined by:

$$R_n(z, m, \beta) = (-1)^{mn} e^{\mathrm{tr}(z)} \det(z)^{m+\beta} \Delta_n^m [e^{-\mathrm{tr}(z)} \det(z)]. \tag{3.11}$$

Here, the differential operator Δ_n defined as follows:

$$\begin{aligned} \Delta_n &= \det(\partial_{ij}) \\ \partial_{ij} &= 2^{-1}(1 + \delta_{ij})\partial/\partial_{ij}. \end{aligned}$$

The polynomial $R_n(z, m, \beta)$ has rational coefficients, and is of degree mn. The term of the highest degree coincides with $\det(z)^m$.

Example 3.1.1. *For $n = 1$, we can easily compute the polynomial $R_1(z, m, \beta)$*

$$R_1(z, m, \beta) = \sum_{k=0}^{m} \binom{m}{k} \beta(\beta + 1) \cdots (\beta + k - 1) z^{m-k}.$$

Now we can see the relation between the polynomial $R_n(z, m, \beta)$ and the confluent hypergeometric function via the following proposition of Shimura:

Proposition 3.1.2 (Proposition 3.2. in [26])**.** *For any non negative integer m the function $\det(z)^m w(z, m+\kappa, \beta)$ and $\det(z)^m w(z, \alpha, -m)$ are polynomial function of z. More precisely,*

$$\begin{aligned} w(z, m + \kappa, \beta) &= \det(z)^{-m} R_n(z, m, \beta) \\ w(z, \alpha, -m) = w(z, m + \kappa, \kappa - \alpha) &= \det(z)^{-m} R_n(z, m, \kappa - \alpha). \end{aligned} \tag{3.12}$$

In conclusion from this relation, we have that

$$\begin{aligned} w(2\pi y, h, m + \kappa, \beta) &= w(2\pi y, h, \kappa - \beta, -m) \\ &= 2^{-n\kappa} e_n(ihy) \det(4\pi hy)^{-m} R_n(4\pi y, m, \beta). \end{aligned} \tag{3.13}$$

Now, we study the behavior of general Siegel Eisenstein series $E_m(Z, s, k, \chi, N)$. We consider here $m = 2n, k = n + t, t \geq 1$, χ a Dirichlet character mod $N, N > 1$ with $\chi(-1) = (-1)^k$, $Z \in \mathbb{H}_m, Z = X + iY$. Λ_m is the set of all half integral symmetric matrices of size m.
We define the normalized Eisenstein series

$$E_m^*(Z, s, k, \chi, N) := \mathfrak{L}(k + 2s, \chi) \times E_m(Z, s, k, \chi, N) \tag{3.14}$$

where $\mathfrak{L}(s, \chi) = L(s, \chi) \prod_{i=1}^n L(2s - 2i, \chi^2)$, and $L(s, \chi)$ is the Dirichlet L-function with character χ.
With the above conditions, the Siegel Eisenstein series has the following Fourier expansion:

$$E_m^*(z, s, k, \chi, N) = \sum_{T \in \Lambda_m} a_m^k(Y, s, \chi, T, N) e^{2\pi i \mathrm{tr}(TX)}, \tag{3.15}$$

with

$$a_m^k(Y, s, \chi, T, N) = \frac{(-1)^{kn} 2^m \pi^{m(k+2s)}}{\Gamma_m(k + s)\Gamma_m(s)} \det(Y)^s h_{k+s,s}^{(m)}(Y, T) \mathrm{Sing}_m(T, k + 2s, \chi). \tag{3.16}$$

Here, $h_{k+s,s}^{(m)}(Y,T)$ is the following function defined by Maass:

$$h_{\alpha,\beta}(Y,T) = \int_{H\pm T>0}\cdots\int e^{-2\pi \mathrm{tr}(YH)} \mid H+T \mid^{\alpha-1/2(n+1)} \mid H-T \mid^{\beta-1/2(n+1)} [dH]. \tag{3.17}$$

This function gives the Fourier coefficients of the following series:

$$\psi = \psi(Z,\bar{Z}) = \sum_{S\in\Lambda_m} \mid Z+S \mid^{-\alpha} \mid \bar{Z}+S \mid^{-\beta} . \tag{3.18}$$

Actually, this function is related to the confluent hypergeometric function $w(z,\alpha,\beta)$ via the following formula:

$$h_{\alpha,\beta}(Y,T) = e^{\frac{1}{2}\pi irn} w(2Y,T,\alpha,\beta) \frac{i^{m\beta-m\alpha}}{\delta(Y)^{\alpha+\beta-\kappa}\mu(V/L)}. \tag{3.19}$$

To describe the series $\mathrm{Sing}_m(T,k+2s,\chi)$, we need some notation. For $T\in\Lambda_m^*$ (the set of $T\in\Lambda_m$ of maximal rank), we denote by ϵ_T the quadratic character

$$\epsilon_T(*) = \left(\frac{(-1)^n \det(2T)}{*}\right), \tag{3.20}$$

and by $\mathbb{D}(T)$ the "set of divisor of T":

$$\mathbb{D}(T) = \{G \in M_m(\mathbb{Z}^*) \mid T[G^{-1}] \in \Lambda_m\}.$$

Proposition 3.1.3 (Proposition 5.1. in [7]). *For all $T\in\Lambda_m^*$ and for all $s\in\mathbb{C}$, with* $\mathrm{Re}(s)>0$, *we have*

$$\begin{aligned}\mathrm{Sing}_m(T,s,\chi) = \sum_{G\in \mathrm{GL}(m,\mathbb{Z})/\mathbb{D}(T)} \sum_{b|\det(2T[G^{-1}]),b>0} \chi^2(\det G)|\det G|^{m+1-2s} \\ \times L(s-n,\epsilon_{T[G^{-1}]}\chi) b^{-s} d(b,T).\end{aligned} \tag{3.21}$$

Here, $b^{-s}d(b,T)$ is an integer such that

$$\prod_q B_q^m(q^{-s},T) = \sum_{b|\det(2T[G^{-1}]),b>0} \chi(b) b^{-s} d(b,T)$$

and $B_q^m(x,T)$ is a polynomial in $\mathbb{Z}[X]$ of degree $\leq m-1$ with the following properties:

(i) $B_q^m(x,T)$ depends only on $T \mod q$.

(ii) $B_q^m(x,T)=1$ if $q\nmid \det(2T)$.

(iii) The degree of $B_q^m(x,T)\leq g$, where $m-g=$ rank of T over $\mathbb{F}_q$.

(iv) It satisfies the important relation

$$\begin{aligned}\mathrm{Sing}_m(T,s,\chi) = \sum_{G\in \mathrm{GL}(m,\mathbb{Z})\mathbb{D}(T)} \chi^2(\det G) \mid \det G \mid^{m+1-2s} \\ \times L(s-n,\epsilon_{T[G^{-1}]}\chi) \prod_q B_q^m(\chi(q)q^{-s}, T[G^{-1}]).\end{aligned} \tag{3.22}$$

3.2 Twisted Eisenstein series

We recall the definition of Siegel Eisenstein series: For a Dirichlet character $\psi \mod M, M > 1$, a weight $k \in \mathbb{N}$ with $\psi(-1) = (-1)^k$, and a complex parameter s with $Re(s) > 0$, we define an Eisenstein series

$$\hat{\mathbb{F}}_n^k(Z, M, \psi, s) \text{ and } \mathbb{F}_n^k(Z, M, \psi, s) = \det(Y)^s \hat{\mathbb{F}}_n^k(Z, M, \psi, s) \tag{3.23}$$

of degree n (with $Z = X + iY \in \mathbb{H}_n$) by

$$\hat{\mathbb{F}}_n^k(Z, M, \psi, s) = \sum_{(C,D)} \psi(\det(C)) \det(CZ + D)^{-k} |\det(CZ + D)|^{-2s}. \tag{3.24}$$

Here, (C, D) run over all "non-associated coprime symmetric pairs" with $\det C$ coprime to M.
A more conceptual definition is as follows:

$$\mathbb{F}_n^k(Z, M, \psi, s) = \sum_{R \in \Gamma^n(M)_\infty \backslash \Gamma^n(M)} \tilde{\psi}(R) j(R, Z)^{-k} \det(\mathrm{Im}(R < Z >)^s), \tag{3.25}$$

with

$$\begin{aligned} T^n(M) &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid A \equiv 0 \mod M \right\} \\ T^n(M)_\infty &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{Z}) \mid C = 0, B \equiv 0 \mod M \right\} \\ R < Z > &= (AZ + B)(CZ + D)^{-1}, R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ j(R, Z) &= \det(CZ + D). \end{aligned}$$

A key ingredient in our subsequent calculations is the following proposition:

Proposition 3.2.1 (Proposition 2.1. in [7]). *A complete set of representatives for $T^{2n}(M)_\infty \backslash T^{2n}(M)$ is given by*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\uparrow \begin{pmatrix} {}^tW & 0_{2n} \\ 0_{2n} & W^{-1} \end{pmatrix}^\uparrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\downarrow \mid (i), (ii), (iii) \right\} \tag{3.26}$$

with

(i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T^n(M)_\infty \backslash T^n(M)$,

(ii) $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in T^n(M)_\infty \backslash T^n(M)$,

(iii) $W \in \left\{ \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \in GL(2n, \mathbb{Z}) | w_2 \equiv 0 \mod M, (\det w_1, M) = 1 \right\}$
$/ \left\{ \begin{pmatrix} GL(n, \mathbb{Z}) & M \cdot \mathbb{Z}^{(n,n)} \\ 0_n & GL(n, \mathbb{Z}) \end{pmatrix} \right\}$.

From Proposition 3.2.1, we obtain an expression for the Eisenstein series of degree $2n$ (essentially the Fourier Jacobi-expansion for the decomposition of

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{H}_{2n}$$

into n-rowed block matrices):

$$\hat{\mathbb{F}}_{2n}^k(Z, M, \psi, s) = \sum_{R \in T^n(M)_\infty \backslash T^n(M)} \sum_{\begin{pmatrix} w_1 \\ w_3 \end{pmatrix} \in \mathbb{Z}^{(2n,n)}/\mathrm{GL}(n,\mathbb{Z})} \psi^2(\det w_1)\tilde{\psi}(R)$$
$$j(R, z_4)^{-k}|j(R, z_4)|^{-2s} \times \hat{\mathbb{F}}_n^k\left(R^{\downarrow} < Z > \begin{bmatrix} w_1 \\ w_3 \end{bmatrix}, M, \psi, s\right). \tag{3.27}$$

As mentioned above, $\begin{pmatrix} w_1 \\ w_3 \end{pmatrix}$ must satisfy the additional conditions that $\begin{pmatrix} w_1 \\ w_3 \end{pmatrix}$ is primitive and that $\det w_1$ is coprime to M. We want to twist these Eisenstein series of degree $2n$ in a certain way by a Dirichlet character. To do this, we first observe that for

$$R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}), X \in \mathbb{R}^{(n,n)} \tag{3.28}$$

$$W = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{Z}), \tag{3.29}$$

the relation

$$\begin{pmatrix} {}^tW & 0_{2n} \\ 0_{2n} & W^{-1} \end{pmatrix} R^{\downarrow} \begin{pmatrix} 1_{2n} & \begin{pmatrix} 0_n & X \\ {}^tX & 0_n \end{pmatrix} \\ 0_{2n} & 1_{2n} \end{pmatrix} = \begin{pmatrix} 1_{2n} & S \\ 0_{2n} & 1_{2n} \end{pmatrix} \begin{pmatrix} {}^t\tilde{W} & 0_{2n} \\ 0_{2n} & \tilde{W}^{-1} \end{pmatrix} R^{\downarrow}$$

holds with

$$\tilde{W} = \begin{pmatrix} w_1 & w_2 \\ -\gamma^t X w_1 + w_3 & -\gamma^t X w_2 + w_4 \end{pmatrix} \tag{3.30}$$

$$S = {}^tW \begin{pmatrix} -X^t\alpha\gamma^t X & X^t\alpha \\ \alpha^t X & 0_n \end{pmatrix} W. \tag{3.31}$$

In particular, the symmetric matrix S has integral entries if $W \in \mathrm{GL}(2n, \mathbb{Z}), X = \frac{\tilde{X}}{N}$ with $\tilde{X} \in \mathbb{Z}^{(n,n)}, N \in \mathbb{N}$ and $R \in \mathrm{Sp}(n, \mathbb{Z})$ with $\alpha \equiv 0 \mod N^2$. This implies that for any $\tilde{X} \in \mathbb{Z}^{(n,n)}$ and any $N \in \mathbb{N}$ with $N^2 \mid M$, we have

$$\hat{\mathbb{F}}_{2n}^k(Z, M, \psi, s)\Big|_k \begin{pmatrix} 1_{2n} & \begin{pmatrix} 0_n & X \\ {}^tX & 0_n \end{pmatrix} \\ 0_{2n} & 1_{2n} \end{pmatrix}$$
$$= \sum_{R=\begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in T^n(M)_\infty / T^n(M)} \sum_{w_1 \in \mathbb{Z}^{(2n,n)}/\mathrm{GL}(n,\mathbb{Z})} \sum_{w_3 \in \mathbb{Z}^{(n,n)}} \psi^2(\det w_1)\tilde{\psi}(R)$$
$$j(R, z_4)^{-k}|j(R, z_4)|^{-2s} \times \hat{\mathbb{F}}_n^k\left(R^{\downarrow} < Z > \begin{bmatrix} w_1 \\ w_3 - \gamma\frac{{}^t\tilde{X}}{N}w_1 \end{bmatrix}, M, \psi, s\right). \tag{3.32}$$

We use the fact that the matrix S does not contribute anything because $\hat{\mathbb{F}}_n^k(Z, M, \psi, s)$ is a periodic function of $z_1 \in \mathbb{H}_n$.
Now, let χ be a Dirichlet character $\mod N, N \mid M$, and consider

$$\sum_{\tilde{X} \in \mathbb{Z}^{(n,n)} \mod N} \chi(\det \tilde{X}) \hat{\mathbb{F}}_{2n}^k(-, M, \psi, s)\Big|_k \begin{pmatrix} 1_{2n} & S\left(\frac{\tilde{X}}{N}\right) \\ 0_{2n} & 1_{2n} \end{pmatrix}, \tag{3.33}$$

where $S(X)$ denotes the $2n$-rowed symmetric matrix

$$\begin{pmatrix} 0_n & X \\ {}^tX & 0_n \end{pmatrix}. \tag{3.34}$$

We put $\tilde{w_3} := Nw_3 - \gamma^t \tilde{X} w_1$. If w_1, γ are fixed and $w_3, \tilde{X}$ are varying, then $\tilde{w_3}$ runs through all elements of $\mathbb{Z}^{(n,n)}$ with the properties

$$\begin{pmatrix} w_1 \\ \tilde{w_3} \end{pmatrix} \text{ primitive }, \det \tilde{w_3} \text{ coprime to } N\ , \tag{3.35}$$

and we have

$$\chi(\det \tilde{X}) = \chi(\det \tilde{w_3}) \bar{\chi}(\det \gamma) \bar{\chi}(\det(-w_1)). \tag{3.36}$$

Hence we obtain

Proposition 3.2.2 (Proposition 2.2. in [7]). *For a Dirichlet character $\psi \mod M, M > 1$, a Dirichlet character $\chi \mod N, N^2 \mid M$ and $k \in \mathbb{N}$ with $\psi(-1) = (-1)^k$, the twisted Eisenstein series can be written as:*

$$\sum_{R \in \Gamma^n(M)_\infty \backslash \Gamma^n(M)} \sum_{\omega_1 \in \mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})} \sum_{\omega_3 \in \mathbb{Z}^{(n,n)}} \psi^2(\det \omega_1) \bar{\chi}(\det(-\omega_1)) \chi(\det \omega_3) \bar{X}(R) \tilde{\psi}(R) j(R, z_4)^{-k} \mid j(R, z_4) \mid^{-2s}$$
$$\times \hat{\mathbb{F}}_n^k \left(R^{\downarrow} < \mathfrak{Z} > \left[\begin{pmatrix} \omega_1 \\ \frac{w_3}{N} \end{pmatrix} \right], M, \psi, s \right).$$

Here, ω_1, ω_3 satisfy the additional conditions $\begin{pmatrix} \omega_1 \\ \omega_3 \end{pmatrix}$ primitive, $(\det(\omega_1), M) = 1$, $(\det(\omega_3), N) = 1$.

For future purposes, we now change notation.
We put

$$\varphi := \psi \bar{\chi}$$

and we work with φ and χ (instead of ψ and χ). The expression

$$\psi^2(\det w_1) \bar{\chi}(\det(-w_1)) \chi(\det w_3) \bar{\chi}^{\sim}(R) \tilde{\psi}(R)$$

becomes

$$\bar{\chi}(-1)^n (\chi \varphi^2)(\det w_1) \chi(\det w_3) \tilde{\varphi}(R)$$

and $\psi(-1)=(-1)^k$ becomes $\chi(-1)=(-1)^k\varphi(-1)$. We put $\varphi=\psi\bar\chi$ and $l=k+\nu,\nu\geq 0$. Then we define a function on $\mathbb{H}_n\times\mathbb{H}_n$ (with $z=x+iy, w=u+iv$) by

$$\begin{aligned}
&\mathfrak{C}_{2n}^{k,\nu}(w,z,M,N,\varphi,\chi,s)\\
&=\det(v)^s\det(y)^s\mathring{\mathfrak{D}}_{n,k+s}^{\nu}\left(\sum_{X\in\mathbb{Z}^{(n,n)},X \mod N}\chi(\det X)\hat{\mathbb{F}}_{2n}^{k}(-,M,\psi,s)\mid_k\begin{pmatrix}1_{2n} & S(\frac{X}{N})\\ 0_{2n} & 1_{2n}\end{pmatrix}\right)\\
&\times\begin{pmatrix}z & 0\\ 0 & w\end{pmatrix}.
\end{aligned}\tag{3.37}$$

An elementary calculation using the properties of the differential operator $\mathring{\mathfrak{D}}_{n,k+s}^{\nu}$ shows

$$\begin{aligned}
\mathfrak{C}_{2n}^{k,\nu}(w,z,M,N,\varphi,\chi,s)&=\frac{\Gamma_n(k+\nu+s)}{\Gamma_n(k+s)}\cdot\frac{\Gamma_n(k+\nu+s-\frac{n}{2})}{\Gamma_n(k+s-\frac{n}{2})}\\
&\times\det(v)^s\det(y)^s\bar\chi(-1)^n\sum_{R\in\Gamma^n(M)_\infty\backslash\Gamma^n(M)}\ \sum_{\omega_1\in\mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})}\ \sum_{\omega_3\in\mathbb{Z}^{(n,n)}}\\
&(\chi\varphi^2)(\det\omega_1)\chi(\det\omega_3)\tilde\varphi(R)\det(w_1)^\nu\det(w_3)^\nu N^{-n\nu}\\
&\times\hat{\mathbb{F}}_n^{k,\nu}\left(z[w_1]+R < w > \left[\frac{w_3}{N}\right],M,\psi,s\right),
\end{aligned}\tag{3.38}$$

where $\hat{\mathbb{F}}_n^{k,\nu}$ is defined on $\mathbb{H}_n$ by

$$\begin{aligned}
\hat{\mathbb{F}}_n^{k,\nu}(z,M,\varphi\chi,s)&=\sum_{\begin{pmatrix}* & *\\ \gamma & \delta\end{pmatrix}\in T^n(M)_\infty/T^n(M)}(\varphi\chi)(\det c)^\nu\det(cz+d)^{-k-\nu}|\det(cz+d)|^{-2s}\\
&=\sum_{\mathfrak{R}}(\varphi\chi)(\nu(\mathfrak{R})^{-k-2s}\det(z+\mathfrak{R})^{-k-\nu}|\det(z+\mathfrak{R})|^{-2s}.
\end{aligned}\tag{3.39}$$

Here, $\mathfrak{R}=\mathfrak{R}^t=c^{-1}d$ runs through all rational symmetric matrices, and $\nu(\mathfrak{R})=|\det c|$ is the absolute value of the product of the denominators of the elementary divisor of $\mathfrak{R}$. For a cusp form $g\in S_n^l(\Gamma_0(M),\varphi)$, we want to compute the scalar product

$$\langle g,\mathfrak{C}_{2n}^{k,\nu}(\star,-\bar z,M,N,\varphi,\chi,\bar s)\rangle_{\Gamma_0(M)}.\tag{3.40}$$

Summarizing all these computations, we obtain

$$\begin{aligned}
&\left\langle g,\mathfrak{C}_{2n}^{k,\nu}(\star,-\bar z,M,N,\varphi,\chi,\bar s)\right\rangle_{\Gamma_0(M)}\\
&=(-1)^{\frac{nl}{2}}2^{1+\frac{n(n+1)}{2}-2ns}\pi^{\frac{n(n+1)}{2}}\frac{\Gamma_n(l+s-\frac{n}{2})\Gamma_n(l+s-\frac{n+1}{2})}{\Gamma_n(k+s)\Gamma_n(k+s-\frac{n}{2})}\\
&\times\chi(-1)^n(N^n)^{2k+\nu+2s-n-1}M^{\frac{n(n+1)}{2}-\frac{nl}{2}}\\
&\times\sum_{\omega_1\in\mathbb{Z}^{(n,n)}/GL(n,\mathbb{Z})}\ \sum_{\omega_3\in\mathbb{Z}^{(n,n)}}\ \sum_{\mathfrak{R}_0\in\mathbb{Q}_{sym}^{(n,n)} \mod \mathbb{Z}_{sym}^{(n,n)}[\omega_3]}(\bar\chi\bar\varphi^2)(\det\omega_1)\\
&\bar\chi(\det\omega_3)(\bar\varphi\bar\chi)(v(\mathfrak{R}_0))v(\mathfrak{R}_0)^{-k-2s}\det(\omega_3)^{-k-2s}\\
&\times g\mid_l\begin{pmatrix}0_n & -1_n\\ M & 0_n\end{pmatrix}\mid U(\frac{M}{N^2})\mid_l\begin{pmatrix}{}^t\omega_3^{-1}{}^t\omega_1 & {}^t\omega_3^{-1}\mathfrak{R}_0\omega_1^{-1}\\ 0_n & \omega_3\omega_1^{-1}\end{pmatrix}.
\end{aligned}$$

We mention again that ω_1, ω_3 satisfy the additional condition $\begin{pmatrix}\omega_1\\ \omega_3\end{pmatrix}$ primitive, $(\det(\omega_1), M) = 1$, $(\det(\omega_3), N) = 1$.

3.3 Trace and shift operators

Let S be a square free number, $p \mid S$ and f_0 an eigenform for the Hecke algebra

$$\otimes_{q\nmid NS}\mathcal{L}^{\circ}_{NS,q} \text{ and } \otimes_{q\mid S}\mathcal{L}^{\circ}_{NS,q}, \tag{3.41}$$

and also an eigenform of $U(L)$ for all $L \mid S^{\infty}$:

$$f_0 \mid U(L) = \alpha(L) f_0. \tag{3.42}$$

Let χ be a Dirichlet character mod RN with $\varphi(-1) = (-1)^k\chi(-1)$ and $R_0 \mid S$, where $R_0 := \prod_{q\mid R} q$.
We put $M = R^2N^2\frac{S}{R_0}$ and

$$g := f_0 \mid_l \begin{pmatrix}0 & -1\\ M & 0\end{pmatrix} \in \mathcal{S}^l_n(\Gamma_0(M), \varphi). \tag{3.43}$$

We move the whole situation from $\Gamma_0(M)$ to $\Gamma^0(M)$ by applying $\begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}$. We have the relation

$$\begin{aligned}
g \mid_l \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix} &= f_0 \mid \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, g \mid_l \begin{pmatrix}0 & -1\\ M & 0\end{pmatrix} = (-1)^{nl} f_0\\
g \mid_l \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} &= (-1)^{nl} f_0 \mid_l \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}.
\end{aligned}$$

For $r|M$ and a cusp form $f \in S^l_n(\Gamma_0(M), \psi)$, let us denote $r' := \prod_{\substack{p|M\\ p\nmid r}} p$, and

$$D^{(M,r')}(f,s,\chi) = \prod_{p|r'}\left(\prod_{i=1}^{n}\frac{1}{(1-\beta_{i,p}\chi(p)p^{-s})}\right) D^{(M)}(f,s,\chi).$$

Then we have

$$\begin{aligned}
&\left\langle f_0 \mid_l \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \mathfrak{E}^{k,\nu}_{2n}(\star, -\bar{z}, M, RN, \varphi, \chi, \bar{s}) \mid^w_l \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix} \mid^z_l \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}\right\rangle_{\Gamma^0(M)}\\
&\quad= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\chi}\bar{\varphi})}(RN)^{n(2k+\nu+2s-n-1)}(R^2N^2\frac{S}{R_0})^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl}\\
&\qquad\qquad \times D^{(M,\frac{S}{R_0})}(f, k+2s-n, \bar{\chi}) f_0 \mid_l \begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix},
\end{aligned}$$

with

$$\Omega_{l,\nu}(s) = (-1)^{\frac{nl}{2}} 2^{1+\frac{n(n+1)}{2}-2ns} \pi^{\frac{n(n+1)}{2}} \frac{\Gamma_n(l+s-\frac{n}{2})\Gamma_n(l+s-\frac{n+1}{2})}{\Gamma_n(k+s)\Gamma_n(k+s-\frac{n}{2})},$$

and $\mathfrak{L}(s,\psi) = L(s,\psi)\prod_{i=1}^{n} L(2s-2i,\psi^2)$.

Lemma 3.3.1 (Lemma 4.1. in [7])**.** *For f_0, M as above, and for any $h \in \mathcal{S}_n^l(\Gamma^0(N^2S),\bar{\varphi})$, we have*

$$\left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}, h \right\rangle_{\Gamma^0(M)} = \left(\frac{R^2}{R_0}\right)^{\frac{-nl+n(n+1)}{2}} \alpha\left(\frac{R^2}{R_0}\right) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)} \tag{3.44}$$

Proof. We have

$$\left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}, h \right\rangle_{\Gamma^0(M)} = \left\langle \sum_{\gamma} f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \mid_l \gamma, h \right\rangle_{\Gamma^0(N^2S)}, \tag{3.45}$$

where γ runs over $\Gamma^0(M)/\Gamma^0(N^2S)$. The set of γ can be represented as

$$\left\{ \begin{pmatrix} 1_n & N^2ST \\ 0 & 1_n \end{pmatrix} \mid T = {}^tT \in \mathbb{Z}^{(n,n)} \mod \frac{R^2}{R_0} \right\}. \tag{3.46}$$

The proof of this lemma follows from the relation

$$\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & N^2ST \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & T \\ 0 & \frac{R^2}{R_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}. \tag{3.47}$$

□

Lemma 3.3.2 (Lemma 4.2. in [7])**.** *For all $h \in \mathcal{S}_n^l(\Gamma^0(N^2S),\bar{\varphi})$ we have*

$$\left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{C}(*,*) \mid^z \mathfrak{K} \mid_l^z \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \mid_l^w \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \right\rangle^w_{\Gamma^0(M)}, h \right\rangle^z_{\Gamma^0(M)}$$
$$= \left(\frac{R^2}{R_0}\right)^{n(n+1)-nl} \left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)}, \tag{3.48}$$

with

$$\mathfrak{F}(z,w) = \mathfrak{C}(z,w) \mid^z U\left(\frac{R^2}{R_0}\right) \mid^w U\left(\frac{R^2}{R_0}\right) \mid_l^z \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \mid_l^w \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix},$$

and the operator $(f \mid \mathfrak{K})(z) = f(-\bar{z})$.

For any $h \in \mathcal{S}_n^l(\Gamma^0(N^2S),\bar{\varphi})$ we obtain a level N^2S identity

$$\left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)}$$
$$= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s,\bar{\chi}\bar{\varphi})} (RN)^{n(2k+\nu+2s-n-1)} (N^2S)^{\frac{n(n+1)-nl}{2}} \chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SR^2}{R_0^2}\right) D^{(M,\frac{S}{R_0})}(f_0, k+2s-n, \bar{\chi}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)}.$$

Using the properties of the Petersson scalar product and the fact that f_0 is an eigenfunction of $U(R_1)$, we obtain

$$\begin{aligned}
&\alpha(R_1)\left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \right\rangle_{\Gamma^0(N^2S)} \\
&= \left\langle f_0 \mid U(R_1) \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \right\rangle_{\Gamma^0(N^2S)} \\
&= \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*, -\bar{z}) \mid^w U_{N^2S}(R_1) \right\rangle_{\Gamma^0(N^2S)} .
\end{aligned} \tag{3.49}$$

On the other hand, writing just c we get

$$\begin{aligned}
&\alpha(R_2)c\left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)} \\
&= c\left\langle f_0 \mid U(R_2) \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)} \\
&= c\left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix} \mid U_{N^2S}(R_2) h \right\rangle_{\Gamma^0(N^2S)} \\
&= c\left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \mid U_{N^2S}(R_2)^* \right\rangle_{\Gamma^0(N^2S)} \\
&= \left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \mid U_{N^2S}(R_2)^* \right\rangle^z_{\Gamma^0(N^2S)} \\
&= \left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)} \mid^z U_{N^2S}(R_2), h \right\rangle^z_{\Gamma^0(N^2S)} \\
&= \left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{F}(*,*) U_{N^2S}(R_2) \mid^z \mathfrak{K} \right\rangle_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)} .
\end{aligned} \tag{3.50}$$

Here, $U_{N^2S}(R_2)^*$ denotes the adjoint operator of $U_{N^2S}(R_2)$. Summarizing these results, we get for $R_1 \mid S^\infty, R_2 \mid S^\infty$ and any $h \in \mathcal{S}_n^l(\Gamma^0(N^2S), \bar{\varphi})$ the identity

$$\begin{aligned}
&\left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{g}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)} \\
&= \frac{\Omega_{l,\nu}(s)}{\mathfrak{L}(k+2s, \bar{\chi}\bar{\varphi})} (RN)^{n(2k+\nu+2s-n-1)} (N^2S)^{\frac{n(n+1)-nl}{2}} \\
&\times \chi(-1)^n (-1)^{nl} \alpha\left(\frac{SR^2}{R_0^2}\right) \alpha(R_1)\alpha(R_2) \\
&D^{(M,\frac{S}{R_0})}(f_0, k+2s-n, \bar{\chi}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, h \right\rangle_{\Gamma^0(N^2S)} ,
\end{aligned} \tag{3.51}$$

with

$$\begin{aligned}\mathfrak{g}(w,z) &= \mathfrak{g}^{k,\nu}\left(w,z,R^2N^2\frac{S}{R_0},RN,\varphi,\chi,s,R_1,R_2\right)\\ &= \mathfrak{C}^{k,\nu}_{2n}(w,z,R^2N^2\frac{S}{R_0},RN,\varphi,\chi,s) \mid^z U\left(\frac{R^2}{R_0}R_1\right) \mid^w\\ & U\left(\frac{R^2}{R_0}R_2\right) \mid^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} \mid^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}. \end{aligned} \tag{3.52}$$

3.4 Definition of $\mathcal{H}$-functions

Let S be a square free number, $p \mid S$, L a natural number, χ a Dirichlet character mod RN, φ a Dirichlet character mod M with $\varphi(-1) = (-1)^k\chi(-1)$ and $R_0 \mid S$ where $R_0 = \prod_{q|R} q$. We put $M = R^2N^2\frac{S}{R_0}$. Now we define the function $\mathcal{H}^{(t)}_{L,\chi}$ as follows:
If $\chi \neq 1$ then

$$\mathcal{H}^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi)\mathfrak{C}^{k,\nu}_{2n}(w,z,R^2N^2\frac{S}{R_0},RN,\varphi,\chi,s) \\ \mid^z U(L^2) \mid^w U(L^2) \mid^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} \mid^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

If $\chi = 1$, we assume that N is coprime to S, $S = R'\cdot p$, R' coprime to p, $R \mid S^\infty$. Let χ' be a Dirichlet character mod $R'N$. With some natural number such that $R \mid L$, we define:

$$\mathcal{H}^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi')\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}p^{-in}\mathfrak{C}^{k,\nu}(w,z,(R'p)^2N^2\frac{S}{R'_0},R'N,\varphi,\chi',s,i) \\ \mid^z U(L^2) \mid^w U(L^2) \mid^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} \mid^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

Then we define the function $\mathcal{H'}^{(t)}_{L,\chi}$ as follows:
If $\chi \neq 1$, then

$$\mathcal{H'}^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi)\mathring{\mathfrak{D}}^v_{n,k}\left(\mathbb{F}^k_{2n}\left(--,R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi)}\right) \\ \mid^z U(L^2) \mid^w U(L^2) \mid^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} \mid^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

Similarly, if $\chi = 1$, we define

$$\mathcal{H'}^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi')\mathring{\mathfrak{D}}^v_{n,k}\left(\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}p^{-in}\sum_j \right. \\ \left. \mathbb{F}^k_{2n}\left(--,R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi')} \mid \begin{pmatrix}1_{2n} & S(g^*_{ij})\\ 0_{2n} & 1_{2n}\end{pmatrix}\right)_{z_2=0} \\ \mid^z U(L^2) \mid^w U(L^2) \mid^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} \mid^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix},$$

where

$$\mathfrak{L}(s, \varphi\chi') = L(s, \varphi\chi') \prod_{i=1}^{n} L(2s - 2i, (\varphi\chi')^2). \tag{3.53}$$

We set

$$\mathcal{H}_{(a,L)}(z, w) := (A_{2n}^k)^{-1} \cdot (2\pi i)^{-n\nu - t} \cdot p^{ln} \cdot (t-1)! \cdot \frac{p}{L} \sum_{\chi} \chi(a) c_{\chi}^{t-1-\frac{n(n-1)}{2}} \cdot (\overline{\varphi\chi^0})\chi^0(c_\chi)^{-n}$$
$$\frac{G(\bar{\chi})}{G(\chi)^n}(1 - (\overline{\varphi\chi^0\chi})_0(p)p^{t-1})\mathcal{H}_{L,\chi}^{(t)},$$

and

$$\mathcal{H}'_{(a,L)}(z, w) \ := \ (B_{2n}^k)^{-1} \cdot \frac{p^{ln}}{(2\pi i)^{n\nu}} \cdot p^{ln} \cdot \frac{p}{L} \sum_{\chi} \chi(a) c_{\chi}^{-\frac{n(n-1)}{2}} \cdot G(\chi)^{-n} \cdot \chi^0(c_\chi)^{-n} \cdot \mathcal{H}'^{(t)}_{L,\chi}.$$

Here,

$$A_{2n}^k = (-1)^{nk} \frac{2^{2n}}{\Gamma_m(k)} \pi^{2nk},$$
$$B_{2n}^k = (-1)^{nk} \frac{2^{n+2nt}}{\Gamma_m(n + \frac{1}{2})} \pi^{n+2n^2},$$

and

$$G_m(\chi) = \sum_{h \in M_m(\mathbb{Z}) \mod C_\chi} \chi(\det(h)) e_m(h/C_\chi)$$

denotes the Gauss sum of degree m of the primitive Dirichlet character $\chi \mod C_\chi, G(\chi) = G_1(\chi)$.

3.5 Fourier expansion of $\mathcal{H}$-functions

We would like to study the Fourier expansion of $\mathcal{H}$-functions at two values of s, namely

$$s_0 := 0 \text{ and } s_1 := \frac{m+1}{2} - k = \frac{1}{2} - t, k = n + t, t \geq 1.$$

For our purpose, we need two additional assumptions; the first one

N is coprime to S.

This implies that the Dirichlet character $\chi \mod RN$ may be written as a product

$$\chi = \chi^0 \cdot \chi_1,$$

where

$$\chi^0 \text{ is a Dirichlet character} \quad \bmod N \text{ and}$$
$$\chi_1 \text{ is a Dirichlet character} \quad \bmod R.$$

Therefore,

$$G_n(T, RN, \chi) = \chi^0(R)^n \chi_1(N)^n \cdot G_n(T, N, \chi^0) \cdot G_n(T, R, \chi_1).$$

The second assumption is:

$$\chi_1 \text{ is primitive} \quad \bmod R.$$

As a new ingredient we introduce a natural number L with $L \mid S^\infty$. Starting now from a primitive Dirichlet character χ_1 with conductor $c(\chi_1) = R \mid L$, we want to compute the Fourier expansion of $\mathcal{H}$-functions as explicitly as possible with

$$R_1 = R_2 = \left(\frac{L}{R}\right)^2 \cdot R_0.$$

We are mainly interested in $s = 0$ and $s = s_1$.
We first observe that for an arbitrary function on $\mathbb{H}_{2n}$ of the form

$$\mathfrak{F}(Z) = \sum_{T \in \Lambda_{2n}} a(T, N) e^{2\pi i \operatorname{tr}(TX)}$$

with

$$\mathfrak{F}^{(\chi)} = \text{ twist of } \mathfrak{F} \text{ in the above sense,}$$

the Fourier expansion of

$$\mathfrak{F}^{(\chi)} \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} |^z\, U(L^2) \,|^w\, U(L^2) \,|_l^z \begin{pmatrix} 1 & 0 \\ 0 & N^2 S \end{pmatrix} |_l^w \begin{pmatrix} 1 & 0 \\ 0 & N^2 S \end{pmatrix}$$

equals

$$R^{\frac{n(n+1)}{2}} G(\chi_1)^n (N^2 S)^{-ln} \chi^0(R)^n \chi_1(N)^n$$
$$\times \sum_{T_1 \in \Lambda_n^+} \sum_{T_4 \in \Lambda_n^+} \sum_{2T_2 \in \mathbb{Z}^{(n,n)}} G_n(2T_2, N, \chi^0) \bar{\chi}_1(\det(2T_2)) a\left(\begin{pmatrix} L^2 T_1 & T_2 \\ {}^t T_2 & L^2 T_4 \end{pmatrix}\right),$$
$$\begin{pmatrix} (L^2 N^2 S)^{-1} y_1 & 0 \\ 0 & (L^2 N^2 S)^{-1} y_4 \end{pmatrix} \exp\left(\frac{2\pi i}{N^2 p} \operatorname{tr}(T_1 z + T_4 w)\right).$$

The Fourier expansion of $\mathcal{H}_{L,\chi}^{(t)}$ in the case $\chi \neq 1$ at $s = 0$ is

$$A_{2n}^k (2\pi i)^{n\nu} R^{\frac{n(n-1)}{2}} (N^2 S)^{-ln} G(\chi_1)^n \chi^0(R)^n \chi_1(N)^n \sum_{T_1 \in \Lambda_n^+} \sum_{T_4 \in \Lambda_n^+}$$
$$\sum_{2T_2 \in \mathbb{Z}^{(n,n)},\, T = \begin{pmatrix} L^2 T_1 & T_2 \\ {}^t T_2 & L^2 T_4 \end{pmatrix} \in \Lambda_{2n}^+} \mathfrak{P}_{n,k}^{\nu}(T) G_n(2T_2, N, \chi^0) \bar{\chi}_1(\det(2T_2))$$
$$\sum_{G \in GL(2n,\mathbb{Z}) \backslash \mathbb{D}(T)} (\varphi\chi)^2 (\det G) \det(2T[G^{-1}])^{k - \frac{2n+1}{2}} L(k-n, \epsilon_{T[G^{-1}]} \varphi\chi)$$
$$\sum_{\substack{b \mid \det(2T[G^{-1}]) \\ b > 0}} \times (\varphi\chi^0)(b) \cdot b^{-k} \cdot d(b, T[G^{-1}]).$$

At $s = s_1$, we look at the Fourier expansion of $\mathcal{H'}^{(t)}_{L,\chi}(z,w)$ with $\chi \neq 1$

$$\mathcal{H'}^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi)\mathring{\mathfrak{D}}^{v}_{n,k}\left(\mathbb{F}^{k}_{2n}\left(--, R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi)}\right)$$
$$|^{z}\, U(L^2)\, |^{w}\, U(L^2)\, |^{z}_{l} \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} |^{w}_{l} \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

The Fourier expansion of $\mathcal{H'}^{(t)}_{L,\chi}(z,w)$ at $s = s_1$ is

$$B^{k}_{2n}(2\pi i)^{n\nu}R^{\frac{n(n-1)}{2}}(N^2S)^{-ln}G(\chi_1)^n\chi^0(R)^n\chi_1(N)^n \sum_{T_1\in\Lambda^+_n}\sum_{T_4\in\Lambda^+_n}$$
$$\sum_{2T_2\in\mathbb{Z}^{(n,n)},T=\begin{pmatrix}L^2T_1 & T_2\\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda^+_{2n}} \mathfrak{P}^{\nu}_{n,k}(T)G_n(2T_2,N,\chi^0)\bar{\chi}_1(\det(2T_2))$$
$$\sum_{G\in GL(2n,\mathbb{Z})\backslash\mathbb{D}(T)} (\varphi\chi)^2(\det G)\det(2T[G^{-1}])^{2t-1}L(k-n,\epsilon_{T[G^{-1}]}\varphi\chi)$$
$$\sum_{\substack{b|\det(2T[G^{-1}])\\ b>0}} \times(\varphi\chi^0)(b)\cdot b^{-k}\cdot d(b,T[G^{-1}]).$$

The Fourier expansion of $\mathcal{H}^{(t)}_{L,\chi}$ at $s = 0$ is given by

$$A^{k}_{2n}(2\pi i)^{n\nu}R^{\frac{n(n-1)}{2}}(N^2S)^{-ln}G(\chi'_1)^n\chi^0(R')^n\chi'_1(N)^n \sum_{T_1\in\Lambda^+_n}\sum_{T_4\in\Lambda^+_n}$$
$$\sum_{2T_2\in\mathbb{Z}^{(n,n)},T=\begin{pmatrix}L^2T_1 & T_2\\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda^+_{2n}} \mathfrak{P}^{\nu}_{n,k}(T)G_n(2T_2,N,\chi^0)\bar{\chi}'_1(\det(2T_2))$$
$$\sum_{G\in GL(2n,\mathbb{Z})\backslash\mathbb{D}(T)} (\varphi\chi')^2(\det G)\det(2T[G^{-1}])^{k-\frac{2n+1}{2}}L(k-n,\epsilon_{T[G^{-1}]}\varphi\chi')$$
$$\sum_{\substack{b|\det(2T[G^{-1}])\\ b>0}} \times(\varphi\chi')(b)\cdot b^{-k}\cdot d(b,T[G^{-1}]),$$

where

χ' is a Dirichlet character $\mod R'N$ and
$\chi' = \chi^0\cdot\chi'_1$ where χ^0 is a Dirichlet character $\mod N$.
χ'_1 is a primitive Dirichlet character $\mod R'$.

The Fourier expansion of ${\mathcal{H}'}^{(t)}_{L,\chi}(z,w)$ at $s=0$ is

$$B_{2n}^{k}(2\pi i)^{n\nu}{R'}^{\frac{n(n-1)}{2}}(N^2S)^{-ln}G(\chi_1')^n\chi^0(R')^n\chi_1'(N)^n\sum_{T_1\in\Lambda_n^+}\sum_{T_4\in\Lambda_n^+}$$

$$\sum_{2T_2\in\mathbb{Z}^{(n,n)},T=\begin{pmatrix}L^2T_1 & T_2\\ {}^tT_2 & L^2T_4\end{pmatrix}\in\Lambda_{2n}^+}\mathfrak{P}_{n,k}^{\nu}(T)G_n(2T_2,N,\chi^0)\bar{\chi}_1'(\det(2T_2))$$

$$\sum_{G\in GL(2n,\mathbb{Z})\backslash\mathbb{D}(T)}(\varphi\chi')^2(\det G)\det|G|^{2t-1}L(1-t,\epsilon_{T[G^{-1}]}\varphi\chi)$$

$$\sum_{\substack{b|\det(2T[G^{-1}])\\ b>0}}\times(\varphi\chi')(b)\cdot b^{-k}\cdot d(b,T[G^{-1}]).$$

For future purposes, we also need the Fourier expansion of functions $\mathcal{H}_{(a,L)}(z,w)$ and ${\mathcal{H}'}_{(a,L)}(z,w)$. The function $\mathcal{H}_{(a,L)}(z,w)$ has a Fourier expansion of the form

$$\mathcal{H}_{(a,L)}(z,w)=\sum_{T_1,T_4\in\Lambda_n^+}\alpha_{a,L}(T_1,T_4)\cdot\exp\left(\frac{2\pi i}{N^2p}\mathrm{tr}(T_1z+T_4w)\right),$$

where

$$\begin{aligned}\alpha_{a,L}(T_1,T_4)&=(2\pi i)^{-t}\cdot N^{-2ln}\cdot(t-1)!\\ &\sum_{T(T_2),G,b}\mathfrak{P}_{n,k}^{\nu}(T)\cdot G_n(2T_2,N,\chi^0)\cdot(\varphi\chi^0)^2(\det G)\cdot\det(2T[G^{-1}])^{k-\frac{2n+1}{2}}\\ &\cdot(\varphi\chi^0)(b)\cdot b^{-k}\cdot d(b,T[G^{-1}])\cdot\frac{p}{L}\sum_{\chi}\chi(aN^n)c_{\chi}^{t-1}(\overline{\varphi\chi^0})_0(c_\chi)\cdot G(\bar{\chi})\cdot\bar{\chi}(\det(2T_2))\\ &\cdot\chi(\det(G^2))\cdot b)\cdot(1-(\overline{\varphi\chi^0\chi})_0(p)p^{t-1})\cdot L(t,\epsilon_{T[G^{-1}]}\varphi\chi^0\chi)\end{aligned}$$

and the Fourier expansion of ${\mathcal{H}'}_{(a,L)}(z,w)$ is as follows:

$${\mathcal{H}'}_{(a,L)}(z,w)=\sum_{T_1,T_4\in\Lambda_n^+}\alpha'_{a,L}(T_1,T_4)\cdot\exp\left(\frac{2\pi i}{N^2p}\mathrm{tr}(T_1z+T_4w)\right),$$

where

$$\begin{aligned}\alpha'_{a,L}(T_1,T_4)&=N^{2ln}\sum_{T,G,b}\mathfrak{P}_{n,k}^{\nu}(T)\cdot G_n(2T_2,N,\chi^0)\\ &\times(\varphi\chi^0)(\det(G)^2b)\cdot\mid\det G\mid^{2t-1}\cdot b^{t-(n+1)}\cdot d(b,T[G^{-1}])\\ &\times\frac{p}{L}\sum_{\chi}\chi(abN^n\det G^2\cdot\det(2T_2)^{-1})\cdot L(1-t,\epsilon_{T[G^{-1}]}\varphi\chi^0\chi).\end{aligned}$$

Chapter 4

p-adic admissible measures attached to Siegel modular forms of arbitrary genus

The purpose of this chapter is to give a new conceptual construction of admissible measures (in the sense of Amice-Vélu) attached to a standard L-function of a Siegel cusp eigenform. For this purpose, we use the theory of p-adic integration in spaces of holomorphic Siegel modular forms (in the sense of Shimura) over an $\mathcal{O}$-algebra A, where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. Often, we simply assume that $A = \mathbb{C}_p$. We study the action of certain differential operators on Siegel Eisenstein distributions with values in spaces of modular forms. In order to obtain from them numerically valued distributions interpolating critical values attached to standard L-functions of Siegel modular forms, one applies a suitable linear form coming from the Petersson scalar product.

In previous work some special cases were treated by Böcherer, Schmidt for arbitrary genus in the ordinary case (Annales Inst.Fourier, 2000, by doubling method), Courtieu, Panchishkin (LNM 1471, 2004, 1990) for even genus in the general h-admissible cases, by Ranking-Selberg method in the form of Andrianov.

In the present chapter, we give a conceptual explanation of these p-adic properties satisfied the special values of the standard L-function $D^{(Np)}(f, s, \chi)$, where f is a Siegel cusp form of weight l and of arbitrary genus.

4.1 Non-Archimedean integration and admissible measures

4.1.1 Measures associated with Dirichlet characters

Let $\omega \mod M$ be a fixed primitive Dirichlet character such that $(M, M_0) = 1$ with $M_0 = \prod_{q\in S} q$. This section gives a construction of the direct image of the Mazur measure under the natural map $\mathbb{Z}_{\bar{S}}^{\times} \longrightarrow \mathbb{Z}_{\bar{S}}^{\times}$, where $\bar{S} = S \cup S(M)$, $\bar{M}_0 = \prod_{q\in\bar{S}} q$. We show that for any positive integer c with $(c, \bar{M}_0) = 1, c > 1$, there exist $\mathbb{C}_p$-measures $\mu^+(c,\omega)$, $\mu^-(c,\omega)$ on $\mathbb{Z}_{\bar{S}}^{\times}$ which are determined by the following conditions, for $s \in \mathbb{Z}$, $s > 0$:

$$\begin{aligned} i_p\left(\int_{\mathbb{Z}_{\bar{S}}^{\times}} \chi x_p^s d\mu^+(c,\omega)\right) &= (1-\bar{\chi}\omega(c)c^{-s})\frac{C_{\omega\bar{\chi}}}{G(\omega\bar{\chi})} \\ &\times \prod_{q\in S\setminus S(\chi)} \left(\frac{1-\chi\bar{\omega}(q)q^{s-1}}{1-\chi\bar{\omega}(q)q^{-s}}\right) L^+_{M_0}(s,\bar{\chi}\omega), \end{aligned} \tag{4.1}$$

and for $s \in \mathbb{Z}, s \le 0$,

$$i_p\left(\int_{\mathbb{Z}_{\bar{S}}^{\times}} \chi x_p^s d\mu^-(c,\omega)\right) = (1-\bar{\chi}\omega(c)c^{s-1})L^+_{M_0}(s,\bar{\chi}\omega), \tag{4.2}$$

where

$$\begin{aligned} L^+_{M_0}(s,\bar{\chi}\omega) &= L_{\bar{M}}(s,\bar{\chi}\omega)2i^{\delta}\frac{\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s} \\ L^-_{M_0}(s,\bar{\chi}\omega) &= L_{\bar{M}}(s,\bar{\chi}\omega) \end{aligned} \tag{4.3}$$

are normalized Dirichlet L-functions with $\delta \in \{0,1\}$ and $\bar{\chi}\omega(-1) = (-1)^{\delta}$. The function $G(\omega\bar{\chi})$ denotes the Gauss sum of the Dirichlet character $\omega\bar{\chi}$. The functions satisfy the functional equation

$$L^-_{M_0}(1-s,\bar{\chi}\omega) = \prod_{q\in S\setminus S(\chi)} \left(\frac{1-\chi\bar{\omega}(q)q^{s-1}}{1-\chi\bar{\omega}(q)q^{-s}}\right) L^+_{M_0}(s,\bar{\chi}\omega). \tag{4.4}$$

Indeed, by the definition of the $\bar{S}$-adic Mazur measure μ^c on $\mathbb{Z}_S^{\times}$, the distributions (4.1) and (4.2) are given by

$$\begin{aligned} \int_{\mathbb{Z}_{\bar{S}}^{\times}} d\mu^+(c,\omega) &= \int_{\mathbb{Z}_{\bar{S}}^{\times}} x x_p^{-1}\omega^{-1} d\mu^c, \\ \int_{\mathbb{Z}_{\bar{S}}^{\times}} d\mu^-(c,\omega) &= \int_{\mathbb{Z}_{\bar{S}}^{\times}} x^{-1}\omega d\mu^c, \end{aligned}$$

where $x \in X_S$ and $X_{\bar{S}}$ is viewed as a subgroup of X_S.

4.1.2 Non-Archimedean integration

The set on which our non-Archimedean zeta functions are defined is the $\mathbb{C}_p$-adic analytic Lie group

$$X_S = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_S, \mathbb{C}_p^\times),$$

where Gal_S is the Galois group of the maximal abelian extension of $\mathbb{Q}$-unramified outside S and infinity. We recall the notation of h-admissible measures on Gal_S, and properties of their Mellin transforms. These Mellin transforms are certain p-adic analytic functions on the $\mathbb{C}_p$-analytic group X_S.

$$\mathrm{Gal}_S = \varprojlim_M (\mathbb{Z}/M\mathbb{Z})^\times = \mathbb{Z}_S^\times, \tag{4.5}$$

where M runs over integers with support in the set of primes S. The canonical $\mathbb{C}_p$-analytic structure on X_S is obtained by shift from the obvious $\mathbb{C}_p$-analytic structure on the subgroup $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. We regard the elements of finite order $\chi \in X_S^{\mathrm{tors}}$ as Dirichlet character whose conductor c_χ may contain only primes in S, by means of the decomposition

$$\chi : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{Z}_S^\times \longrightarrow \overline{\mathbb{Q}}^\times \longrightarrow \mathbb{C}^\times. \tag{4.6}$$

The character $\chi \in X_S^{\mathrm{tors}}$ forms a discrete subgroup X_S^{tors}. We shall need also the natural homomorphism

$$x_p : \mathbb{Z}_S^\times \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{C}_p^\times, x_p \in X_S, \tag{4.7}$$

so that all integers $k \in \mathbb{Z}$ can be regarded as characters of the type $x_p^k : y \mapsto y^k$. Recall that a p-adic measure on $\mathbb{Z}_S^\times$ may be regarded as a bounded $\mathbb{C}_p^\times$-linear form μ on the space $\mathcal{C}(\mathbb{Z}_S^\times, \mathbb{C}_p)$ of all continuous $\mathbb{C}_p$-valued functions

$$\mathcal{C}(\mathbb{Z}_S^\times, \mathbb{C}_p) \longrightarrow \mathbb{C}_p$$
$$\varphi \longmapsto \mu(\varphi) = \int_{\mathbb{Z}_S^\times} \varphi d\mu,$$

which are uniquely determined by its restriction to the subspace $\mathcal{C}^1(\mathbb{Z}_S^\times, \mathbb{C}_p)$ of locally constant functions.

The Mellin transform L_μ of μ is defined by

$$L_\mu : X_S \longrightarrow \mathbb{C}_p$$
$$\chi \longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi d\mu.$$

It is a bounded analytic function on X_S, uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in X_S^{\mathrm{tors}}$.

4.1.3 h-admissible measure

A more delicate notation of an h-admissible measure was introduced by Y. Amice, J. Vélu and M.M. Višik (see [3], [30]). Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ be the Tate field. For $h \in \mathbb{N}^*$, we denote $\mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p)$

the space of $\mathbb{C}_p$-valued functions which can be locally represented by polynomials of degree less than a natural number h of variable in x_p. In particular, $\mathcal{C}^1(\mathbb{Z}_p^\times, \mathbb{C}_p)$ is the space of locally constant functions. Let us recall the definition of admissible measures with scalar and vector values; see [3], [30], [23].

Definition 4.1.1. *An h-admissible measure on $\mathbb{Z}_p^\times$ is a $\mathbb{C}_p$-linear map:*

$$\phi : C^h(\mathbb{Z}_p^\times, \mathbb{C}_p) \longrightarrow V$$

with the following growth condition: for all $t = 0, 1, \ldots, h-1$,

$$| \int_{a+(p^m)} (x_p - a_p)^t d\phi \,|_p = o(p^{m(h-t)}) \quad \text{for } m \longrightarrow \infty, \tag{4.8}$$

where $a_p = x_p(a)$.

We know that each h-admissible measure can be uniquely extended to a linear form on the $\mathbb{C}_p$-space of all locally analytic functions, so that one can associate to its Mellin transform

$$L_\mu : X_S \longrightarrow \mathbb{C}_p$$
$$\chi \longmapsto L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi d\mu,$$

which is a $\mathbb{C}_p$-analytic function X_S of the type $o(\log(x_p^h))$. Moreover, the measure μ is uniquely determined by the special values of the type $L_\mu(\chi x_p^r)$ with $\chi \in X_S^{\text{tors}}$ and $r = 0, 1, \cdots, h-1$.

Example 4.1.2. *For any $s \in \mathbb{N}$, we define a distribution μ_s by*

$$\mu_s(\chi) = \mu(\chi x_p^s) = s \log(1 + p^v), \forall \chi \in X_S^{\text{tors}}, X_S = \text{Hom}_{\text{cont}}(\text{Gal}_S, \mathbb{C}_p^\times).$$

This sequence of distributions turns out to be a 2-admissible measure. Indeed, for $r = 0, 1$, we have

$$\int_{a+(M)} (x_p - a_p)^r d\mu = \sum_{\chi \mod M} \chi^{-1}(a) \int_{\mathbb{Z}_S^\times} \chi(x)(x_p - a_p)^r d\mu$$
$$a_p = x_p(a)$$
$$a + (M) = \{x \in \mathbb{Z}_S^\times | x \equiv a(\ mod\ M)\} \subset \mathbb{Z}_S^\times$$
$$\mathbb{Z}_S^\times = \varprojlim_M (\mathbb{Z}/M\mathbb{Z})^\times \text{ a projective limit .}$$

- *If $r = 0$,*

$$\int_{a+(M)} (x_p - a_p)^r d\mu = 0. \tag{4.9}$$

- *If $r = 1$,*

$$\int_{a+(M)} (x_p - a_p)^r d\mu = \sum_{\chi \mod M} \chi^{-1}(a) \log(1 + p^v).$$

1. $a \neq 1 \to \chi^{-1}(a) = 0 \to \int_{a+(M)} (x_p - a_p)^r d\mu = 0$

2. $a = 1$

$$\to \sum_{\chi \mod M} \chi^{-1}(a) = \varphi(M) \to \int_{a+(M)} (x_p - a_p)^r d\mu = \varphi(M) \tag{4.10}$$

from (4.9) and (4.10) we have the condition of 2-admissible are satisfied. Its Mellin transform is

$$L_\mu(x) = \int_{\mathbb{Z}_S^\times} x d\mu.$$

We have $L_\mu(\chi x_p) = \log(x_p)$ *which is a* $\mathbb{C}_p$*-analytic function on* $X_S^\times$ *of type* $o(\log(x_p)^2)$*.*

Example 4.1.3. *Case* $h = 1$*. Every bounded measure defines an 1-admissible measure because every measure satisfies the growth condition with* $j = 0, t = 0$ *and* $h = 1$*.*

In order to construct an h-admissible measure, we follow two steps:

(1) Construct certain modular distributions with values in the Siegel modular forms.

(2) Apply a suitable algebraic linear form to the submodule of Siegel modular forms of finite dimension.

We will explain how to construct an h-admissible measure $\phi : C^h(\mathbb{Z}_p^\times, \mathbb{C}_p) \longrightarrow V$ out of a sequence of distributions

$$\phi_j : \mathcal{C}^1(\mathbb{Z}_p^\times, \mathbb{C}_p) \longrightarrow \mathcal{M}$$

with values in an A-module $\mathcal{M} = \mathcal{M}(A)$ of holomorphic modular forms over A (for all $j \leq h - 1$). Then we define a $\mathbb{C}_p$-linear map

$$\phi : \mathcal{C}^h(\mathbb{Z}_p^\times, \mathbb{C}_p) \longrightarrow \mathcal{M}$$

on local monomials x_p^j by

$$\int_{a+(p^m)} x_p^j d\phi = \phi_j(a + (p^m)).$$

4.1.4 Mellin transforms of h-admissible measures

In this section we study some properties of Mellin transform of h-admissible measures. We recall a theorem of Višik in [30] about Mellin transforms of certain h-admissible measures, and a theorem of Ha in [14] which proves that any analytic function of class $o(\log^h)$ is the Mellin-Mazur transform of an h-admissible measure.

Theorem of Višik

Let $X(.) = \mathrm{Hom}_{\mathrm{const}}(\cdot, \mathbb{C}_p^\times)$ be the continuous characters of a topological group. We set $q = \begin{cases} p & \text{if } p > 2 \\ 4 & \text{if } p = 2 \end{cases}$, $U = \{x \in \mathbb{Z}_p^\times | x \equiv 1 \bmod q\}$ and $T = \{u \in \mathbb{C}_p^\times, |U - 1| < 1\}$.

An analytic function $F : T \to V_p$ is, by definition, the sum of power series $\sum_{j=0}^{\infty} a_j (u-1)^j$, which converges in $T (a_j \in V_p)$ where V_p is a finite dimensional linear space over $\mathbb{C}_p$.

We have $\log_p \sup_{|u-1|<r} |F(u)|$ is a linear function of the variable $t = \log_p r$. We denote

$$M_F(t) = \log_p \sup_{|u-1|<r} |F(u)|. \tag{4.11}$$

Take a system of representatives of $U \mathrm{mod}\ qp^m$ of a special type $\Lambda_m = \{g^j\}, 0 \leq j \leq p^m - 1$. Let $\chi_u \in X(U)$ be such that $\chi_U(g) = u$. Then for $x \in U$,

$$\chi_U(x) = u^{\log x / \log g}. \tag{4.12}$$

We write the Riemann sum for $\chi_U(x)$:

$$S_{\Lambda_m}(\chi_u) = \sum_{j=0}^{p^m-1} \sum_{i=0}^{h-1} \frac{\chi_u^{(i)}(g^j)}{i!} \int_{g^j(1+qp^m)} (x_p - g_p^j)^i d\mu. \tag{4.13}$$

Theorem 4.1.4 (see [30])**.** *Suppose that $h > 0$, and that μ is an h-admissible measure on $\mathbb{Z}_S^\times$. Then the function $\chi \mapsto \int_{\mathbb{Z}_S^\times} \chi d\mu$ ($\chi \in X(\mathbb{Z}_S^\times)$) is analytic and equals $o(\log^h(.))$.*

Proof. We set $S_m(u) = S_{\Lambda_m}(\chi_u)$ the Riemann sum of $\chi_U(x)$. First, we have to proof $S_m(u)$ converges uniformly on any disc $|u - 1| \leq r < 1$. Let $t_N = \dfrac{-1}{\varphi(p^N)}$ with N sufficiently large. Let $M_m(t)$ be the Newton polygon of difference $S_{m-1}(u) - S_m(u)$. We have

$$\frac{dM_m(t)}{dt} \geq \frac{dM_F(t)}{dt},$$

where $F(u) = \log^h u$. Hence

$$M_m(t_m) - M_m(t_N) \geq h(m - N), \text{ for } m > N.$$

But we have $M_m(t_m) = hm - d(m)$ where $d(m) \to \infty$ as $m \to \infty$. Consequently,

$$M_m(t_N) \to -\infty \text{ as } m \to \infty.$$

Secondly, we have to prove the limit of $S_m(u)$ is $o(\log^h u)$. Let $F(u) = \lim_{m\to\infty} S_m(u)$. We have

$$M_F(t_N) \leq \max_{0 \leq m < \infty} M_m(t_N) = \sup\Big(\max_{m \leq N} M_m(t_N), \max_{m > N} M_m(t_N)\Big).$$

Case $m \leq N$:

$$M_m(t_N) \leq \max_{0 \leq l \leq h-1} \Big(Nl - \frac{lp}{p-1} + mh - ml - d(m) \Big),$$

where $d(m) \to \infty$ as $m \to \infty$. We rewrite this last expression in the form

$$\max_{0 \leq l \leq h-1} \Big(Nh + (m-N)(h-l) - \frac{lp}{p-1} - d(m) \Big) \leq Nh - (N-m) - d(m).$$

Case $m > N$: We have $M_m(t) - M_m(t_N) \geq h(m-N)$, $M_m(t_m) = mh - c(m)$ where $c(m) \to 0$ as $N \to \infty$. Hence

$$M_m(t_N) \leq Nh - c(m).$$

Then from two cases:

$$M_F(t_N) = Nh - e(N)$$

where $e(m) \to 0$ as $N \to \infty$. Therefore, the theorem is proved. □

p-adic Mellin-Mazur transform

Theorem 4.1.5 (Theorem 1.2 in [14]). *For any function $f(z) \in \mathfrak{H}$ with $f(z) \in o(\log^h)$ there exists an h-admissible measure on U such that*

$$f(\chi) = \int_U \chi d\mu. \tag{4.14}$$

Proof. First, define the measure μ as follows:

$$\mu(z^k \psi_a^{(m)}) = \frac{1}{\varphi(p^m)} \sum_{\chi} \chi^{-1}(a) f(z^k x), \tag{4.15}$$

where $0 \leq k \leq h-1$ and χ runs over the Dirichlet characters modulo p^m. We easily see that μ is additive linear. Second, we must prove that μ satisfies the growth condition

$$\left| \sup_{a \in U} \int_U (z-a)^j \psi_a^{(m)}(z) d\mu \right| = o\Big(p^{m(h-j)}\Big), j = 0, 1, \cdots, h-1. \tag{4.16}$$

For every $g(z) \in \mathfrak{H}$ and every $t_0 > 0$ we set

$$\| g \|_{t_0} = \sup_{v(z)=t_0} |g(z)|. \tag{4.17}$$

We know that

$$\| \log^h(1+z) \|_{t_m} = p^{mh},$$

where $t_m = \frac{1}{p}(p^m)$. This implies

$$\| f \|_{t_m} = o(p^{mh}), m \to \infty.$$

We let $S_m(z)$ be the sequence of interpolating polynomial for $f(z)$ between the points

$$\{g^{i\gamma} - 1\}, i = 0, \cdots, h-1; \gamma \in M_{p^m},$$

where M_{p^m} is the set of p^m-th roots of unity. We have

$$\deg(S_m(z)) \leq hp^m - 1.$$

By Lazard's Lemma we have

$$S_m(z) = \varphi(z) \prod_{i=0,\cdots,h-1} \Big(1 - \frac{z}{g^{i\gamma-1}}\Big) + Q_m(z) \tag{4.18}$$

where $\deg Q_m(z) \leq hp^m - 1$. Then

$$v(v_m, t_m) \geq v(f, t_m) \Rightarrow S_m(z) \equiv Q_m(z) \text{ and } \| S_m \|_{t_m} = o(p^{mh}).$$

We write $S_m(z)$ in the form

$$S_m(z) = \sum_{l=0}^{hp^m-1} b_l^{(m)} z^l, \tag{4.19}$$

with $|b_l^{(m)}| = o(p^{mh})$, for every l. If we write

$$S_m(z-1) = \sum_{l=0}^{hp^m-1} a_l^{(m)} z^l, \tag{4.20}$$

then we obtain $a_l^{(m)} = o(p^{mh})$, for every l. By the definition of μ, we have

$$\begin{aligned}\int_U (z-a)^j \psi_a^{(m)}(z) d\mu &= \sum_{k=0}^{j} (-a)^{j-k} \binom{j}{k} \frac{1}{\varphi(p^m)} \sum_{\chi} \chi^{-1}(a) f(z^k \chi) \\ &= \sum_{k=0}^{j} (-a)^{j-k} \binom{j}{k} \frac{1}{\varphi(p^m)} \sum_{\chi} f(g^k \chi(g) - 1) \\ &= \sum_{k=0}^{j} (-a)^{j-k} \binom{j}{k} \frac{1}{\varphi(p^m)} \sum_{\chi} S_m(g^k \chi(g) - 1) \end{aligned} \tag{4.21}$$

$$\Rightarrow \int_U (z-a)^j \psi_a^{(m)} d\mu = \sum a_l^{(m)} (g^l - a)^j$$

$$\Rightarrow \sup_{a \in U} \left| \int_U (z-a)^j \psi_a^{(m)}(z) d\mu \right| = \sup_{a \in U} \left| a_l^{(m)} (g^l - a)^j \right| = o(p^{m(h-j)}), j = 0, \cdots, h-1,$$

because $\left(a_l^{(m)}\right) = o(p^{hm})$ and $|g^l - a| \leq p^{-m}$.

Finally, we prove that

$$f(z^k\chi) = \int_U z^k \chi d\mu, \tag{4.22}$$

where χ is a Dirichlet character and $0 \leq k \leq h-1$.

We have

$$\begin{aligned}\int_U z^k \chi d\mu &= \int_U \sum_{a \mod p^m} \chi(a) z^k \psi_a^{(m)}(z) d\mu \\ &= \sum_{a \mod p^m} \chi(a) \sum_{\bar{\chi}} \frac{1}{\varphi(p^m)} \overline{\chi^{-1}}(a) f(z^k \bar{\chi}) = f(z^k\chi)\end{aligned}$$

From three stages we have the proof of theorem. □

4.2 The standard L-function of a Siegel cusp eigenform and its critical values

(See [8]). For a Siegel modular form $f(z)$ of genus n and weight l, which is an eigenfunction of the Hecke algebra, and for each prime number p, one can define the Satake p-parameters of f, denoted by $\alpha_i(p)$ $(i = 0, 1, \ldots, n)$. In this introduction, we assume for simplicity that f is a modular form with respect to the full Siegel modular group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$. The standard zeta function of f is defined by means of the Satake p-parameters as the following Euler product:

$$\mathcal{D}(s, f, \chi) = \prod_p \left\{ \left(1 - \frac{\chi(p)}{p^s}\right) \prod_{i=1}^n \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right)\left(1 - \frac{\chi(p)\alpha_i(p)^{-1}}{p^s}\right) \right\}^{-1}, \tag{4.23}$$

where χ is an arbitrary Dirichlet character. We introduce the following normalized functions:

$$\mathcal{D}^\star(s, f, \chi) = (2\pi)^{-n(s+l-(n+1)/2}\Gamma((s + \delta)/2) \prod_{j=1}^n \left(\Gamma(s+k-j)\right) \mathcal{D}(s, f, \chi)$$

$$\begin{aligned}\mathcal{D}^+(s, f, \chi) &= \Gamma((s+\delta)/2)\mathcal{D}^\star(s, f, \chi) \\ \mathcal{D}^-(s, f, \chi) &= \frac{i^\delta \pi^{1/2-s}}{\Gamma((1-s+\delta)/2)} \mathcal{D}^\star(s, f, \chi),\end{aligned}$$

where $\delta = 0$ or 1 according to whether $\chi(-1) = 1$ or $\chi(-1) = -1$. Let

$$f(z) = \sum_{\xi > 0} a(\xi) e_n(\xi z) \in \mathcal{S}_n^k$$

be the Fourier expansion of the Siegel cusp form $f(z)$ of weight l. The sum is extended over all positive definite half integral $n \times n$ matrices, and

$$z \in \mathbb{H}_n = \{z \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t z = z, \mathrm{Im}(z) > 0\},$$

the Siegel upper half plane of degree n, and $e_n(z) = e^{2\pi i \mathrm{tr}(z)}$.

Theorem 4.2.1. *(a) For all integers s with $1 \leq s \leq l - \delta - n, s \equiv \delta \mod 2$ and Dirichlet character χ such that χ^2 is non-trivial for $s = 1$, we have that:*

$$\langle f, f\rangle^{-1}\mathcal{D}^+(s, f, \chi) \in K = \mathbb{Q}(f, \Lambda_f, \chi),$$

where $K = \mathbb{Q}(f, \Lambda_f, \chi)$ denotes the field generated by Fourier coefficients of f, by the eigenvalues $\Lambda_f(X)$ of the Hecke operator X on f, and by the values of the character χ.

(b) For all integer s with $1 - l + \delta + n \leq s \leq 0, s \not\equiv \delta \mod 2$, we have that:

$$\langle f, f\rangle^{-1}\mathcal{D}^-(s, f, \chi) \in K.$$

This theorem was proved by M.Harris in 1981 for n even and by S.Bocherer-C.G.Schmidt for arbitrary n.

4.3 Integral representations for the standard zeta function

(See [7]). Now we recall the definition of the $\mathcal{H}$-functions in some cases as follows:
If $\chi \neq 1$, then we set

$$\mathcal{H}^{(t)}_{L,\chi}(z, w) = \mathfrak{L}(k + 2s, \varphi\chi)\mathfrak{E}^{k,\nu}_{2n}(w, z, R^2N^2\frac{S}{R_0}, RN, \varphi, \chi, s)$$
$$|^z\, U(L^2)\, |^w\, U(L^2)\, |^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} |^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

If $\chi = 1$, then we set

$$\mathcal{H}^{(t)}_{L,\chi}(z, w) = \mathfrak{L}(k + 2s, \varphi\chi')\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}} p^{-in}\mathfrak{E}^{k,\nu}(w, z, (R'p)^2N^2\frac{S}{R'_0}, R'N, \varphi, \chi', s, i)$$
$$|^z\, U(L^2)\, |^w\, U(L^2)\, |^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} |^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

Then we define the functions $\mathcal{H}'^{(t)}_{L,\chi}$ as follows:
If $\chi \neq 1$, then

$$\mathcal{H}'^{(t)}_{L,\chi}(z, w) = \mathfrak{L}(k + 2s, \varphi\chi)\mathring{\mathfrak{D}}^v_{n,k}\left(\mathbb{F}^k_{2n}\left(--, R^2N^2\frac{S}{R_0}, \varphi, s\right)^{(\chi)}\right)$$
$$|^z\, U(L^2)\, |^w\, U(L^2)\, |^z_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix} |^w_l \begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}.$$

If $\chi = 1$, then

$$\mathcal{H}'^{(t)}_{L,\chi}(z,w) = \mathcal{L}(k+2s,\varphi\chi')\mathring{\mathfrak{D}}^{v}_{n,k}\left(\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}p^{-in}\sum_j \right.$$
$$\mathbb{F}^{k}_{2n}\left(--,R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi')} \mid \begin{pmatrix} 1_{2n} & S(g^*_{ij}) \\ 0_{2n} & 1_{2n}\end{pmatrix}_{z_2=0}$$
$$|^z\ U(L^2)\ |^w\ U(L^2)\ |^z_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix} |^w_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}.$$

We want to compute the following double product:

$$\left\langle \left\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \mathfrak{g}(*,*) \mid^z \mathfrak{K} \right\rangle^w_{\Gamma^0(N^2S)}, h \right\rangle^z_{\Gamma^0(N^2S)}, \tag{4.24}$$

with $g(*,*)$ the function $\mathcal{H}^{(t)}_{L,\chi}(z,w), \mathcal{H}'^{(t)}_{L,\chi}(z,w)$ defined as above.

<u>Case 1</u> $g = \mathcal{H}^{(t)}_{L,\chi}(z,w), \chi \neq 1$:

We get

$$\Omega_{l,\nu}(0)(RN)^{n(2k+\nu-n-1)}(N^2S)^{\frac{n(n+1)-nl}{2}} \times \chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SL^4}{R^2}\right) D^{(NS,\frac{S}{R_0})}(f_0,k-n,\bar{\chi}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}, h\right\rangle_{\Gamma^0(N^2S)}. \tag{4.25}$$

<u>Case 2</u> $g = \mathcal{H}^{(t)}_{L,\chi}(z,w), \chi = 1$:

We get

$$\Omega_{l,\nu}(0)(R'pN)^{n(2k+\nu-n-1)}(N^2S)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SL^2}{(R'p)^2}\right) D^{(NS,\frac{S}{R'_0})}(f_0,k-n,\bar{\chi})(-1)^n p^{-\frac{n^2+n}{2}}$$
$$\prod_{i=1}^{n}(1-\beta_i\bar{\chi}(p)p^{-n+k}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}, h\right\rangle_{\Gamma^0(N^2S)}. \tag{4.26}$$

<u>Case 3</u> $g = \mathcal{H}'^{(t)}_{L,\chi}(z,w), \chi \neq 1$:

We get

$$\left(\Omega_{l,\nu}(s)\cdot\frac{p_s(k)}{d_s(k)}\right)_{s=s_1}(R'pN)^{n(\nu+n)}(N^2S)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SL^4}{R^2}\right) D^{(NS,\frac{S}{R_0})}(f_0,n+1-k,\bar{\chi}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}, h\right\rangle_{\Gamma^0(N^2S)}. \tag{4.27}$$

<u>Case 4</u> $g = \mathcal{H}'^{(t)}_{L,\chi}(z,w), \chi = 1$:

We get

$$\left(\Omega_{l,\nu}(s)\cdot\frac{p_s(k)}{d_s(k)}\right)_{s=s_1}(R'pN)^{n(\nu+n)}(N^2S)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl}$$
$$\alpha\left(\frac{SL^4}{(R'p)^2}\right) D^{(NS,\frac{S}{R'_0})}(f_0,n+1-k,\bar{\chi})(-1)^n p^{-\frac{n^2+n}{2}}$$
$$\prod_{i=1}^{n}(1-\beta_i\bar{\chi}(p)p^{-n+k}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S\end{pmatrix}, h\right\rangle_{\Gamma^0(N^2S)}. \tag{4.28}$$

4.4 Main theorem

Let $f \in S_n^l(\Gamma_0(N), \bar{\varphi})$ be a Siegel modular form of genus n and weight l with Satake p-parameters $\beta_0, \beta_1, \cdots, \beta_n$. We recall here some notations that we shall use in the statement of our main theorem. For an arbitrary Dirichlet character ψ we introduce the modified p-Euler factor

$$E_p(s,\psi) := \prod_{j=1}^{n} \frac{(1-\psi(p)\beta_j^{-1}p^{s-1})}{(1-\bar{\psi}(p)\beta_j^{-1}p^{-s})}$$

$$\chi = \chi^0 \cdot \chi_1,\ \chi \text{ is a Dirichlet character modulo } RN\ .$$

To formulate our result, let

$$\Lambda_\infty^+(s) := \frac{(2i)^s \cdot \Gamma(s)}{(2\pi i)^s} \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(s+l-j)$$

$$\Lambda_\infty^-(s) := (2i)^s \cdot \prod_{j=1}^{n} \Gamma_{\mathbb{C}}(1-s+l-j),$$

where $\Gamma_{\mathbb{C}}(s) := 2 \cdot (2\pi)^{-s}\Gamma(s)$. Further, for any character χ of p-power conductor c_χ we let

$$A^-(\chi) := c_\chi^{nl-\frac{n(n+1)}{2}} \alpha(c_\chi)^{-2} \cdot (\chi^0([p,c_\chi]) \cdot \chi(-1)G(\chi))$$

$$A^+(\chi) := (\overline{\chi^0\varphi})_o(c_\chi) \cdot \frac{A^-(\chi)}{\chi(-1)G(\chi)},$$

where $[a,b]$ denotes the least common multiple of the integers a, b. Finally, we let

$$E_p^+ := (1-(\overline{\varphi\chi\chi^0})_0(p)p^{t-1}) \cdot E_p(s,\chi\chi^0)$$

$$E_p^- := E_p(p,\chi\chi^0).$$

Theorem 4.4.1 (Main theorem). *For each prime number p there exist two p-adic admissible measures μ^+, μ^- on $\mathbb{Z}_p^\times$ with values in $\mathbb{C}_p$ verifying the following properties:*

(i) For all pairs (s,χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet characters, $s \in \mathbb{Z}$ with $1 \leq s \leq l-\delta-n, s \equiv \delta \mod 2$ and for $s=1$ the character χ^2 is non-trivial, the following equality holds:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p\left(c_\chi^{s(n+1)} A^+(\chi) \cdot E_p^+(s,\chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f,s,\overline{\chi\chi^0})\right).$$

(ii) For all pairs (s,χ) such that $\chi \in X_p^{\mathrm{tors}}$ is a non-trivial Dirichlet character, $s \in \mathbb{Z}$ with $l-\delta+n \leq s \leq 0, s \not\equiv \delta \mod 2$ the following equality holds:

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{s-1} d\mu^- = i_p\left(c_\chi^{n(1-s)} A^+(\chi) \cdot E_p^-(1-s,\chi\chi^0) \frac{\Lambda_\infty^-(s)}{\langle f_0, f_0\rangle} \cdot D^{(Np)}(f,1-s,\overline{\chi\chi^0})\right).$$

(iii) If $\mathrm{ord}_p(\alpha_0(p)) = 0$ *(i.e. f is p-ordinary), then the measures in (i) and (ii) are bounded.*

(iv) In the general case (but assuming that $\alpha_0(p) \neq 0$) with $x \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ the holomorphic functions

$$\mathcal{D}^+(x) = \int x d\mu^+$$
$$\mathcal{D}^-(x) = \int x d\mu^-$$

belong to type $o(\log(x_p)^h)$ where $h = [4\mathrm{ord}_p(\alpha_0(p))] + 1$, and they can be represented as the Mellin transforms of certain h-admissible measures.

(v) If $h \le k - m - 1$, then the functions $\mathcal{D}^\pm$ are uniquely determined by the above conditions (i) and (ii).

4.5 Further properties of $\mathcal{H}$-functions

(See [7]). Let S be a square free number, $p \mid S$, L a natural number, χ a Dirichlet character mod RN, φ a Dirichlet character mod M with $\varphi(-1) = (-1)^k\chi(-1)$, and $R_0 \mid S$ where $R_0 = \prod_{q|R} q$. We put $M = R^2N^2\frac{S}{R_0}$. Now we can define the function $\mathcal{H}^{(t)}_{L,\chi}$ as follows:
If $\chi \neq 1$ then

$$\mathcal{H}^{(t)}_{L,\chi}(z,w) = \mathcal{L}(k+2s,\varphi\chi)\mathfrak{E}^{k,\nu}_{2n}(w,z,R^2N^2\frac{S}{R_0},RN,\varphi,\chi,s) \Big|^{z} U(L^2)\Big|^{w} U(L^2)\Big|^{z}_{l}\begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}\Big|^{w}_{l}\begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix},$$

where we recall the definition of Siegel Eisenstein series: For a Dirichlet character ψ mod M, $M > 1$, a weight $k \in \mathbb{N}$ with $\psi(-1) = (-1)^k$ and a complex parameter s with $Re(s) > 0$, we define an Eisenstein series

$$\hat{\mathbb{F}}^k_n(Z,M,\psi,s) \text{ and } \mathbb{F}^k_n(Z,M,\psi,s) = \det(Y)^s\hat{\mathbb{F}}^k_n(Z,M,\psi,s)$$

of degree n (with $Z = X + iY \in \mathbb{H}_n$) by

$$\hat{\mathbb{F}}^k_n(Z,M,\psi,s) = \sum_{(C,D)} \psi(\det(C))\det(CZ+D)^{-k}|\det(CZ+D)|^{-2s}.$$

Here, (C,D) run over all "non-associated coprime symmetric pairs" with $\det C$ coprime to M. It is well-known that this series converges for $k+2Re(s) > n+1$, and has a meromorphic continuation to the whole plane. We put $\varphi = \psi\bar{\chi}$ and $l = k+\nu, \nu \ge 0$. Then we define a function on $\mathbb{H}_n \times \mathbb{H}_n$ (with $z = x+iy, w = u+iv$) by

$$\begin{aligned}&\mathfrak{E}^{k,\nu}_{2n}(w,z,M,N,\varphi,\chi,s)\\ &= \det(v)^s\det(y)^s\mathring{\mathfrak{D}}^{\nu}_{n,k+s}\left(\sum_{X\in\mathbb{Z}^{(n,n)},X \bmod N}\chi(\det X)\hat{\mathbb{F}}^k_{2n}(-,M,\psi,s)\Big|_k\begin{pmatrix}1_{2n} & S(\frac{X}{N})\\ 0_{2n} & 1_{2n}\end{pmatrix}\right)\\ &\times\begin{pmatrix}z & 0\\ 0 & w\end{pmatrix}.\end{aligned}$$

This function defines an element of $C^\infty M_n^l(\Gamma_0(M),\varphi)$.

If $\chi=1$, we assume that N coprime to S, $S=R'\cdot p, R'$ coprime to $p, R\mid S^\infty$. Let χ' be a Dirichlet character $\mod R'N$. With some natural number such that $R\mid L$, we define:

$$\mathcal{H}^{(t)}_{L,\chi}(z,w)=\mathfrak{L}(k+2s,\varphi\chi')\sum_{i=0}^{n}(-1)^i p^{\frac{i(i-1)}{2}}p^{-in}\mathfrak{E}^{k,\nu}(w,z,(R'p)^2N^2\frac{S}{R_0'},R'N,\varphi,\chi',s,i)$$
$$\Big|^{z}U(L^2)\Big|^{w}U(L^2)\Big|^{z}_{l}\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\Big|^{w}_{l}\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}.$$

Then we define the functions ${\mathcal{H}'}^{(t)}_{L,\chi}$ as follows:
If $\chi\neq 1$, then

$$ {\mathcal{H}'}^{(t)}_{L,\chi}(z,w)=\mathfrak{L}(k+2s,\varphi\chi)\mathring{\mathfrak{D}}^{v}_{n,k}\left(\mathbb{F}^k_{2n}\left(--,R^2N^2\frac{S}{R_0},\varphi,s\right)^{(\chi)}\right)$$
$$\Big|^{z}U(L^2)\Big|^{w}U(L^2)\Big|^{z}_{l}\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\Big|^{w}_{l}\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}.$$

We recall the definition of the differential operator:
We define for any $\alpha\in\mathbb{C}$

$$\mathfrak{D}_{n,\alpha}=\sum_{r+q=n}(-1)^r\binom{n}{r}C_r(\alpha-n+\frac{1}{2})\Delta(r,q),$$

where

$$C_q(s)=s\big(s+\frac{1}{2}\big)\ldots\big(s+\frac{q-1}{2}\big)=\frac{\Gamma_q(s+\frac{q+1}{2})}{\Gamma_q(s+\frac{q-1}{2})},$$

and we use the notation $\Delta(r,q), r+q=n$ are the polynomials in ∂_{ij}, their coefficients being polynomials in the entries of z_2.
For $\nu\in\mathbb{N}$ we put

$$\mathfrak{D}^{\nu}_{n,\alpha}=\mathfrak{D}_{n,\alpha+\nu-1}\circ\ldots\circ\mathfrak{D}_{n,\alpha}$$
$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}=(\mathfrak{D}^{\nu}_{n,\alpha})\mid_{z_2=0}.$$

We also recall the twist of Eisenstein series as follows:

$$\mathbb{F}^k_{2n}(-,M,\psi,s)^{(\chi)}=\sum_{X\in\mathbb{Z}^{(n,n)},X\mod N}\chi(\det X)\mathbb{F}^k_{2n}(-,M,\psi,s)\Big|_k\begin{pmatrix}1_{2n}&S(\frac{X}{N})\\0_{2n}&1_{2n}\end{pmatrix},$$

where $S(X)$ denotes the $2n$-rowed symmetric matrix

$$\begin{pmatrix}0_n&X\\{}^tX&0_n\end{pmatrix}.$$

Similarly, if $\chi = 1$, we define

$$\mathcal{H}'^{(t)}_{L,\chi}(z,w) = \mathfrak{L}(k+2s,\varphi\chi')\mathring{\mathfrak{D}}^{v}_{n,k}\Bigg(\sum_{i=0}^{n}(-1)^{i}p^{\frac{i(i-1)}{2}}p^{-in}\sum_{j}$$
$$\mathbb{F}^{k}_{2n}\bigg(--, R^2N^2\frac{S}{R_0},\varphi,s\bigg)^{(\chi')}\bigg|\begin{pmatrix}1_{2n} & S(g^*_{ij})\\ 0_{2n} & 1_{2n}\end{pmatrix}\Bigg)_{z_2=0}$$
$$\Big|^{z}U(L^2)\Big|^{w}U(L^2)\Big|^{z}_{l}\begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix}\Big|^{w}_{l}\begin{pmatrix}1 & 0\\ 0 & N^2S\end{pmatrix},$$

where

$$\mathfrak{L}(s,\varphi\chi') = L(s,\varphi\chi')\prod_{i=1}^{n}L(2s-2i,(\varphi\chi')^2).$$

We set

$$\mathcal{H}_{(a,L)}(z,w) := (A^k_{2n})^{-1}\cdot(2\pi i)^{-n\nu-t}\cdot p^{ln}\cdot(t-1)!\cdot\frac{p}{L}\sum_{\chi}\chi(a)c_{\chi}^{t-1-\frac{n(n-1)}{2}}\cdot(\overline{\varphi\chi^0})\chi^0(c_\chi)^{-n}$$
$$\frac{G(\bar\chi)}{G(\chi)^n}(1-(\overline{\varphi\chi^0\chi})_0(p)p^{t-1})\mathcal{H}^{(t)}_{L,\chi},$$

and

$$\mathcal{H}'_{(a,L)}(z,w) := (B^k_{2n})^{-1}\cdot\frac{p^{ln}}{(2\pi i)^{n\nu}}\cdot p^{ln}\cdot\frac{p}{L}\sum_{\chi}\chi(a)c_{\chi}^{-\frac{n(n-1)}{2}}\cdot G(\chi)^{-n}\cdot\chi^0(c_\chi)^{-n}\cdot\mathcal{H}'^{(t)}_{L,\chi}.$$

Here,

$$A^k_{2n} = (-1)^{nk}\frac{2^{2n}}{\Gamma_m(k)}\pi^{2nk},$$
$$B^k_{2n} = (-1)^{nk}\frac{2^{n+2nt}}{\Gamma_m(n+\frac{1}{2})}\pi^{n+2n^2},$$

and

$$G_m(\chi) = \sum_{h\in M_m(\mathbb{Z}) \mod C_\chi}\chi(\det(h))e_m(h/C_\chi)$$

denotes the Gauss sum of degree m of the primitive Dirichlet character $\chi \mod C_\chi, G(\chi) = G_1(\chi)$.

4.6 Fourier coefficients of $\mathcal{H}$-functions

(See [7]). For simplicity, we replace the index χ by χ_1 as in the Section 3.5. The function $\mathcal{H}_{(a,L)}(z,w)$ has a Fourier expansion of the form:

$$\mathcal{H}_{(a,L)}(z,w) = \sum_{T_1,T_4\in\Lambda_n^+}\alpha_{a,L}(T_1,T_4)\cdot\exp\left(\frac{2\pi i}{N^2p}\mathrm{tr}(T_1z+T_4w)\right),$$

where

$$\alpha_{a,L}(T_1,T_4) = (2\pi i)^{-t}\cdot N^{-2ln}\cdot (t-1)!$$
$$\sum_{T(T_2),G,b} \mathfrak{P}^{\nu}_{n,k}(T)\cdot G_n(2T_2,N,\chi^0)\cdot(\varphi\chi^0)^2(\det G)\cdot \det(2T[G^{-1}])^{k-\frac{2n+1}{2}}$$
$$\cdot(\varphi\chi^0)(b)\cdot b^{-k}\cdot d(b,T[G^{-1}])\cdot \frac{p}{L}\sum_{\chi}\chi(aN^n)c_{\chi}^{t-1}(\overline{\varphi\chi^0})_0(c_\chi)\cdot G(\bar{\chi})\cdot\bar{\chi}(\det(2T_2))$$
$$\cdot\chi(\det(G^2))\cdot b)\cdot(1-\overline{\varphi\chi^0\chi})_0(p)p^{t-1})\cdot L(t,\epsilon_{T[G^{-1}]}\varphi\chi^0\chi)$$

and the Fourier expansion of $\mathcal{H}'_{(a,L)}(z,w)$ is as follows:

$$\mathcal{H}'_{(a,L)}(z,w) = \sum_{T_1,T_4\in\Lambda_n^+} \alpha'_{a,L}(T_1,T_4)\cdot\exp\left(\frac{2\pi i}{N^2p}\mathrm{tr}(T_1 z + T_4 w)\right),$$

where

$$\alpha'_{a,L}(T_1,T_4) = N^{2ln}\sum_{T,G,b}\mathfrak{P}^{\nu}_{n,k}(T)\cdot G_n(2T_2,N,\chi^0)$$
$$\times(\varphi\chi^0)(\det(G)^2 b)\cdot\mid\det G\mid^{2t-1}\cdot b^{t-(n+1)}\cdot d(b,T[G^{-1}])$$
$$\times\frac{p}{L}\sum_{\chi}\chi(abN^n\det G^2\cdot\det(2T_2)^{-1})\cdot L(1-t,\epsilon_{T[G^{-1}]}\varphi\chi^0\chi).$$

4.7 Algebraic linear forms

(See [7]). For a modular form $g(z,w)$, which as a function of z (or w) belongs to $M_n^l(\Gamma^0(N^2p),\varphi)$, we consider the following $\mathbb{C}$-valued function:

$$\mathcal{F}(g) = \frac{\langle\langle f_0|_l\begin{pmatrix}0&-1\\1&0\end{pmatrix}, g|^z\mathfrak{K}\rangle^w_{\Gamma_0(N^2p)}, f_0|_l\begin{pmatrix}1&0\\0&N^2S\end{pmatrix}\rangle^z_{\Gamma_0(N^2p)}}{\langle f_0,f_0\rangle^2_{\Gamma_0(N^2p)}}, \tag{4.29}$$

where $(f\mid\mathfrak{K})(z) = f(-\bar{z})$.
We want to know the action of the linear form $\mathcal{F}$ on $\mathcal{H}^{(t)}_{L,\chi}(z,w)$ and $\mathcal{H}'^{(t)}_{L,\chi}(z,w)$.
First we define the p-Euler factor

$$E_p(s,\psi) = \prod_{j=1}^{n}\frac{(1-\psi(p)\beta_j^{-1}p^{s-1})}{(1-\bar{\psi}(p)\beta_j p^{-s})},$$

where ψ is an arbitrary Dirichlet character and $\beta_0,\beta_1,\cdots,\beta_n$ are Satake p-parameters of eigenform f_0.

Then computing $\mathcal{F}(\mathcal{H}^{(t)}_{L,\chi}(z,w))$, we have

$$\begin{aligned}
\mathcal{F}(\mathcal{H}^{(t)}_{L,\chi}(z,w)) &= \frac{\langle\langle f_0 \mid_l \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathcal{H}^{(t)}_{L,\chi}(z,w) \mid^z \mathfrak{K}\rangle^w_{\Gamma_0(N^2p)}, f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\rangle^z_{\Gamma_0(N^2p)}}{\langle f_0, f_0\rangle^2_{\Gamma_0(N^2p)}} \\
&= \langle f_0, f_0\rangle^{-2}_{\Gamma_0(N^2p)} \cdot \Omega_{l,\nu}(0)(RN)^{n(2k+\nu-n-1)}(N^2S)^{\frac{n(n+1)-nl}{2}}\chi(-1)^n(-1)^{nl} \\
&\alpha\left(\frac{SL^4}{R^2}\right) D^{(NS,\frac{S}{R_0})}(f_0, k-n, \bar{\chi}) \left\langle f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}, f_0 \mid_l \begin{pmatrix} 1 & 0 \\ 0 & N^2S \end{pmatrix}\right\rangle^z_{\Gamma^0(N^2S)} \\
&=< f_0, f_0 >^{-1} \cdot\Omega_{l,\nu}(0) \cdot (N^2p)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\chi\chi^0(-1)^n(-1)^{ln} \\
&\cdot (Nc_\chi)^{n(l+t-1)}\alpha(pL^4c_\chi^{-2}) \cdot E_p(t,\chi^0\chi) \cdot \overline{\chi^0}(\frac{p}{(p,c_\chi)})^n \cdot D^{(Np)}(f,t,\overline{\chi^0\chi}), \quad (4.30)
\end{aligned}$$

for any character χ whose conductor c_χ is a power of p. Similarly, we have:

$$\begin{aligned}
\mathcal{F}(\mathcal{H'}^{(t)}_{L,\chi}) &=< f_0, f_0 >^{-1} \cdot\Omega_{l,\nu}(s_1)\frac{p_{s_1}(k)}{d_{s_1}(k)} \cdot (N^2p)^{\frac{n(n+1)}{2}-\frac{nl}{2}}\chi\chi^0(-1)^n(-1)^{ln} \\
&\cdot (Nc_\chi)^{n(l-t)}\alpha(pL^4c_\chi^{-2}) \cdot E_p(1-t,\chi^0\chi) \cdot \overline{\chi^0}(\frac{p}{(p,c_\chi)})^n \cdot D^{(Np)}(f,1-t,\overline{\chi^0\chi}). \quad (4.31)
\end{aligned}$$

$\mathcal{F}(\mathcal{H}_{L,\chi}), \mathcal{F}(\mathcal{H}'_{L,\chi})$ depend only on L by the factor $\alpha(L^4)$.

4.8 Distributions in Siegel modular forms

Consider the $\mathbb{C}_p$-linear forms $\mathcal{D}^\pm : C^{l-n}(\mathbb{Z}_S^\times) \to \mathbb{C}_p$ defined on the local monomials x_p^j for $j = 0, 1, \cdots, l-n-1$ by:

$$\begin{aligned}
\int_{a+(L)} x_p^j d\mathcal{D}^+ &:= \int_{a+(L)} d\mathcal{D}^+_{j+1,L} \\
\int_{a+(L)} x_p^j d\mathcal{D}^- &:= \int_{a+(L)} d\mathcal{D}^-_{-j,L}, \quad (4.32)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}^+_{j+1,L} &= \mathcal{F}\left(\mathcal{H}^{(j+1)}_{L,\chi}\right) \\
\mathcal{D}^-_{-j,L} &= \mathcal{F}\left(\mathcal{H'}^{(-j)}_{L,\chi}\right). \quad (4.33)
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_{a+(L)} (x_p - a_p)^r d\mathcal{D}^+ &= \sum_{j=0}^r \binom{t}{j} (-a)^{r-j} \int_{a+(L)} d\mathcal{D}^+_{j+1,L} \\
&= \sum_{j=0}^r \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a)\mathcal{D}^+_{j+1,L}(\chi) \\
&= \gamma(L) \sum_{j=0}^r \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a)\mathcal{F}(\mathcal{H}^{(j+1)}_{L,\chi}). \quad (4.34)
\end{aligned}$$

Similarly we have

$$
\begin{aligned}
\int_{a+(L)} (x_p - a_p)^r d\mathcal{D}^- &= \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \int_{a+(L)} d\mathcal{D}^{c-}_{-j,L} \\
&= \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \mathcal{D}^{-}_{-j,L}(\chi) \\
&= \gamma'(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \mathcal{F}(\mathcal{H'}^{(-j)}_{L,\chi}). \qquad (4.35)
\end{aligned}
$$

We denote

$$
\begin{aligned}
A^+ &:= \gamma(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^+(L, j+1, \chi) \\
A^- &:= \gamma'(L) \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^-(L, -j, \chi). \qquad (4.36)
\end{aligned}
$$

Then we need to prove that A^+, A^- also satisfy the growth conditions

$$
\begin{aligned}
A^+ &= o(|L|_p^{r-h}) \\
A^- &= o(|L|_p^{r-h}). \qquad (4.37)
\end{aligned}
$$

We state here the main congruences

$$
\left| \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^+(L, j+1, \chi) \right|_p \leq C \cdot p^{-\nu r} \qquad (4.38)
$$

and

$$
\left| \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^-(L, -j, \chi) \right|_p \leq C \cdot p^{-\nu r} \qquad (4.39)
$$

with $r = 0, 1, \cdots, l-n-1, L = p^\nu$.

$$
A^+ = \gamma(L) \sum_{j=0}^{t} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) v^+(L, j+1, \chi), \qquad (4.40)
$$

where

$$
\begin{aligned}
v^+(L, j+1, \chi) = {} & A^k_{2n} (2\pi i)^{n\nu} \cdot R^{\frac{n(n-1)}{2}} (N^2 S)^{-ln} G(\chi_1)^n \chi^0(R)^n \chi_1(N^n) \\
& \sum_{T_1 \in \Lambda_n^+} \sum_{T_4 \in \Lambda_n^+} \sum_{2T_2 \in \mathbb{Z}^{(n,n)}} \mathfrak{P}^{\nu}_{n,k}(T) G_n(2T_2, N, \chi^0) \bar{\chi}_1(\det(2T_2)) \\
& \sum_{G \in GL(2n,\mathbb{Z}) \backslash D(T)} (\varphi\chi)^2 (\det G) \det(2T[G^{-1}])^{k - \frac{2n+1}{2}} \cdot L(k-n, \epsilon_{\det(T[G^{-1}])} \varphi\chi) \\
& \cdot \sum_{b | \det T[G^{-1}]} (\varphi\chi)(b) b^{-k} d(b, T[G^{-1}]). \qquad (4.41)
\end{aligned}
$$

4.9 Criterion for admissibility of the Fourier coefficients of $\mathcal{H}$-functions

First, we recall the definition of the polynomial defined by Böcherer in [7]

Definition 4.9.1. *For $T \in \mathbb{C}^{2n,2n}_{\text{sym}}$ we define a polynomial $\mathfrak{P}^{\nu}_{n,\alpha}(T)$ in the entries $t_{ij}(1 \leq i \leq j \leq 2n)$ of T by*

$$\mathring{\mathfrak{D}}^{\nu}_{n,\alpha}(e^{\text{tr}(TZ)}) = \mathfrak{P}^{\nu}_{n,\alpha}(T)e^{\text{tr}(T_1 z_1 + T_4 z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_4 \end{pmatrix}. \tag{4.42}$$

The $\mathfrak{P}^{\nu}_{n,k}(T)$ are homogenous polynomials of degree $n\nu$.
$\mathfrak{D}^{\nu}_{n,k}$ is a polynomial in the ∂_{ij}, homogenous of degree $n\nu$ with at most one term free of the entries of ∂_1 and ∂_4, namely the term $C^{\nu}_{n,k}det(\partial_2)^{\nu}$ with a certain constant $C^{\nu}_{n,k}$.
To determine the constant $C^{\nu}_{n,k}$ explicitly we first observe that (for arbitrary $k, s \in \mathbb{C}$)

$$\mathfrak{D}_{n,k}(\det(z_2)^s) = (-1)^n C_n(\frac{s}{2})C_n(\alpha - n + \frac{s}{2}) \det(z_2)^{s-1}. \tag{4.43}$$

which implies that

$$\mathring{\mathfrak{D}}^{\nu}_{n,k}(\det(z_2)^{\nu}) = \left(\prod_{\mu=1} C_n(\frac{\mu}{2})\right) C^{\nu}_{n,k} = (-1)^{n\nu} \prod_{\mu=1} \left(C_n(\frac{\mu}{2})C_n(k - n + \nu - \frac{\mu}{2})\right). \tag{4.44}$$

We would like to give an explicit formula for the polynomials $\mathfrak{P}^{\nu}_{n,k}(\mathfrak{T})$, but for the purpose of proving the main congruences, we need only the following expression for this polynomial.

Theorem 4.9.2. *Using the notations defined as above and also some basic relations $l = k + \nu, k = n + j, j \geq 0$ with l the weight of Siegel modular form f and $T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix} \in \Lambda^+_{2n}, T_1, T_4 \in \Lambda^+_n, L$ fixed positive number, we have that the following expression holds:*

$$\mathfrak{P}^{\nu}_{n,k}(T) = \det(L^4T_1T_4)^{\frac{\nu}{2}} \sum_{|M| \leq \frac{\nu}{2}} C_M(k)Q_M(L^{-2}D) \text{ if } \nu \text{ is even} \tag{4.45}$$

and if ν is odd,

$$\mathfrak{P}^{\nu}_{n,k}(T) = \det(T_2)\det(L^4T_1T_4)^{\frac{\nu-1}{2}} \sum_{|M| \leq \frac{\nu-1}{2}} C_M(k)Q_M(L^{-2}D), \tag{4.46}$$

where M runs over the set of $(e_0, \cdots, e_{n-1}) \neq 0$ such that $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$ and $C_M(k)$ is a polynomial in variable k, $k = n + j$ of degree $|M|$, $Q_M(L^{-2}D)$ is a homogeneous polynomial in variables $L^{-2}d_i^2, i = \overline{1, n}$ of degree $|M|$.

Proof. The summation in (4.45) and (4.46) runs over over the set of $(e_0, \cdots, e_{n-1}) \neq 0$ such that $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$ because of the condition which is given in the proof of theorem

2.2.4. We apply Theorem 2.3.1 for the matrix $T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix} \in \Lambda_{2n}^+, T_1, T_4 \in \Lambda_n^+, L$ a fixed positive number. We can take $\hat{A} = L^{-1}A$ and $\hat{B} = L^{-1}B \in GL(n, \mathbb{R})$ such that

$$\hat{A}T_1{}^t\hat{A} = 1_n$$
$$\hat{B}T_4{}^t\hat{B} = 1_n.$$

We put

$$\hat{W}_0 = \hat{A}T_2{}^t\hat{B}.$$

Since we assumed that $\det(T_2) \neq 0$, there exist two orthogonal matrices h_1, h_2 such that

$$h_1\hat{W}_0h_2 = \hat{D}.$$

Comparing with the previous statement, we have

$$\hat{W}_0 = L^{-2}W_0$$
$$\hat{D} = L^{-2}D.$$

Then $P(T_1, T_4, T_2)$ is determined by $P(1_n, 1_n, \hat{D})$, where $\hat{D}$ is a diagonal matrix with diagonal elements $L^{-2}d_i^2$, $i = \overline{1, n}$. We put

$$\det(x1_n - \hat{D}^2) = \sum_{j=0}^{n} P_j(\hat{D})x^j.$$

Then similarly to Theorem 2.3.1 we can write the polynomial in this case in the following form:
If ν is even, then

$$\mathfrak{P}_{n,\alpha}^{\nu}(T) = \det(L^4T_1T_4)^{\nu/2} \sum_{(e_0,\cdots,e_{n-1})\neq 0} c(e_0, \cdots, e_{n-1}) \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j}. \tag{4.47}$$

If ν is odd, then

$$\mathfrak{P}_{n,\alpha}^{\nu}(T) = \det(\hat{D}) \det(L^4T_1T_4)^{\nu/2} \sum_{(e_0,\cdots,e_{n-1})\neq 0} c(e_0, \cdots, e_{n-1}) \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j}, \tag{4.48}$$

where $P_j(\hat{D})$ is a polynomial in elementary symmetric polynomials in $L^{-2}d_i^2$, $i = 1, \cdots n$. We define for each set of multi-indices $M = (e_0, \cdots, e_{n-1}) \neq 0$ the polynomial

$$Q_M(L^{-2}D) = \prod_{j=0}^{n-1} P_j(\hat{D})^{e_j}. \tag{4.49}$$

We easily see that $Q_M(L^{-2}D)$ is a homogeneous polynomial of variables $L^{-2}d_i^2, i = \overline{1, n}$ of degree $|M|$ with $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$. Otherwise, we know that the coefficient $C_M(k)$ is a polynomial in variable k, $k = n + j$ of degree $|M|$ and with coefficients in the ring $\mathbb{Z}(1/2)$. Therefore, we have the expression for this polynomial. Concluding, after all the above statements we can write the polynomial in following form:

$$\mathfrak{P}_{n,k}^{\nu}(T) = \sum_M C_M(k) \cdot Q_M(L^{-2}D), \tag{4.50}$$

where M runs over the set of $(e_0, \cdots, e_{n-1}) \neq 0$ such that $|M| = \sum_{\alpha=0}^{n-1} e_\alpha \leq \left[\frac{\nu}{2}\right]$ and $C_M(k)$ is a polynomial of variable k, $k = n+j$ degree $|M|$, $Q_M(L^{-2}D)$ is a homogeneous polynomial of variables $L^{-2}d_i^2, i = \overline{1, n}, D = D(T_1, T_4, T_2)$. $\square$

4.10 Proof of the main theorem

(i) We denote $\omega = \bar{\epsilon}_{T[G^{-1}]}$. For $s \in \mathbb{Z}, s > 0$, we use the Mazur measure and the functional equation of L-functions associated to Dirichlet characters

$$\begin{aligned} i_p\left(\int_{\mathbb{Z}_p^\times} \chi x_p^s d\mu^+(\omega)\right) &= \frac{C_{\omega\bar{\chi}}}{G(\omega\bar{\chi})} \\ &\times \prod_q \left(\frac{1-\chi\bar{\omega}(q)q^{s-1}}{1-\chi\bar{\omega}(q)q^{-s}}\right) L^+_{M_0}(s, \bar{\chi}\omega), \end{aligned}$$

and for $s \in \mathbb{Z}, s \leq 0$,

$$i_p\left(\int_{\mathbb{Z}_p^\times} \chi x_p^s d\mu^-(\omega)\right) = L^+_{M_0}(s, \bar{\chi}\omega),$$

where

$$\begin{aligned} L^+_{M_0}(s, \bar{\chi}\omega) &= L_{\bar{M}}(s, \bar{\chi}\omega) 2 i^\delta \frac{\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s} \\ L^-_{M_0}(s, \bar{\chi}\omega) &= L_{\bar{M}}(s, \bar{\chi}\omega) \end{aligned}$$

are normalized Dirichlet L-functions with $\delta \in \{0, 1\}$ and $\bar{\chi}\omega(-1) = (-1)^\delta$. The function $G(\omega\bar{\chi})$ denotes the Gauss sum of the Dirichlet character $\omega\bar{\chi}$. The functions satisfy the functional equation

$$L^-_{M_0}(1-s, \bar{\chi}\omega) = \prod_{q \in S \setminus S(\chi)} \left(\frac{1-\chi\bar{\omega}(q)q^{s-1}}{1-\chi\bar{\omega}(q)q^{-s}}\right) L^+_{M_0}(s, \bar{\chi}\omega).$$

Otherwise, we can write the factor $\sum_{b|\det T[G^{-1}]}(\varphi\chi)(b)b^{-k}d(b, T[G^{-1}])$ as a finite linear combination with integer coefficients $b_i \in \mathbb{Z}$:

$$\begin{aligned} \sum_{b|\det T[G^{-1}]} (\varphi\chi)(b)b^{-k}d(b, T[G^{-1}]) &= \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \chi(\alpha_i)\alpha_i^k \delta_{\alpha_i} \\ &= \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \int_{\mathbb{Z}_p^\times} \chi(x) x^k \delta_{\alpha_i}, \end{aligned} \quad (4.51)$$

where δ_{α_i} is the Dirac measure at the point $\alpha_i \in \mathbb{Z}_p^\times$. We define the measure μ_{T_2} by

$$\int_{\mathbb{Z}_p^\times} \chi(x)x^s d\mu_{T_2} = \int_{\mathbb{Z}_p^\times} \chi(x)x^k \sum_{\alpha_i \in \mathbb{Z}_p^\times} b_i \delta_{\alpha_i}. \quad (4.52)$$

We also have the followings measures obtained by convolution:

$$\begin{aligned} \mu^+(T_2, \omega) &= \mu^+(\omega) * \mu_{T_2} \\ \mu^-(T_2, \omega) &= \mu^-(\omega) * \mu_{T_2}. \end{aligned} \quad (4.53)$$

As in Section 4.8, we know that to prove μ^+ is an h-admissible measures, we have to prove $A^\pm = o(|L|_p^{r-h})$.
Actually, we have the factor

$$\gamma(L) =< f_0, f_0 > \cdot\Omega_{l,\nu}^{-1}(0) \cdot (N^2p)^{\frac{nl}{2}-\frac{n(n+1)}{2}}(-1)^{-n}(-1)^{-ln}N^{n(1-j-l)}\alpha_0(pL^4)^{-1}. \quad (4.54)$$

We easily see that

$$\gamma(L) \equiv 0 \mod |L|_p^{[-4\mathrm{ord}_p(\alpha_0(p))]-1}. \quad (4.55)$$

On the other hand, we use interpolation for the polynomial

$$C_M(j) = \sum_{i=0}^{|M|} \mu_i \cdot \frac{(j+i+1)!}{(j+1)!}. \quad (4.56)$$

Therefore,

$$\begin{aligned} B^+ = &\sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \sum_{i=0}^{|M|} \mu_i \cdot \frac{(j+i+1)!}{(j+1)!} \\ &G_n(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2)) \int_{\mathbb{Z}_p^\times} \chi x^{j+1} d\mu^+(T_2, \omega). \end{aligned} \quad (4.57)$$

Here, we fix T_1 and T_4 and study only the dependence on T_2. Then

$$\begin{aligned} B^+ &= \int_{x\equiv a \mod L} \sum_{i=0}^{|M|} \mu_i \sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{(j+i+1)!}{(j+1)!} x^{j+1} d\mu^+(T_2, \omega) \\ &= \int_{x\equiv a \mod L} \sum_{i=0}^{|M|} \mu_i x^{-1} \cdot \frac{\partial^i}{\partial x^i}\left(x^{i+1}(x-a)^r\right) d\mu^+(T_2, \omega). \end{aligned} \quad (4.58)$$

Then $(x-a)^{|M|} \equiv 0 \mod L^{|M|}$, giving the congruence

$$B^+ \equiv 0 \mod L^{r-i} \equiv 0 \mod L^{r-|M|}. \quad (4.59)$$

On the other hand, $Q_M(L^{-2}D)$ is a homogeneous polynomial in L^{-2} of degree $|M|$. So $Q_M(L^{-2}) \equiv 0 \mod L^{-2|M|}$. Thus

$$\begin{aligned} A^+ &\equiv 0 \mod L^{-h} \cdot L^{2n\nu} L^{r-|M|} \cdot L^{-2|M|} \\ &\equiv 0 \mod L^{r-h-3|M|+2n\nu}. \end{aligned} \quad (4.60)$$

Therefore,

$$A^+ \equiv 0 \mod L^{r-h}.$$

This congruence says that with h as above, μ^+ is an h-admissible measure.

(ii) The proof for the negative case is actually similar to the proof for the positive case in (i).

(iii) The assertion (iii) (i.e the ordinary case), which was proved by A.A.Panchishkin (see [24]) with even genus and Böcherer-Schmidt with arbitrary genus (see [7]), also follows easily from the main congruence.

(iv) In the general case $h > k - n - 1$, the integer s runs over $\{0, 1, \cdots, h-1\}$ and one can extend the values of our functions $\mathcal{D}^{\pm}(\chi x_p^s)$ by the equality $\mathcal{D}^{\pm}(\chi x_p^s) = 0$ (for all $\chi \in X_S^{\text{tors}}$ of a conductor divisible by all the prime divisors of Np) for $s > l - n - 1 - \nu$ (but keeping the same values for $0 \le s \le l - n - 1 - \nu$) the verification of the h-admissibility goes without change in this situation. Also, one obtains again the h-admissible measures $\mu^{\pm}$ with $h = [4\mathrm{ord_p}(\alpha_0(\mathrm{p}))] + 1$. The functions $\mathcal{D}^{\pm}$ therefore coincide with the Mellin transforms of these h-admissible measures. We also can find the proof of (iv) in [14], [3], and [30].

(v) Finally, if $h \le l - n - 1$, then the condition in (i) and (ii) uniquely determines the analytic functions $\mathcal{D}^{\pm}$ of type $o(\log(x_p^h))$ by their values following a general property of admissible measures (see [3], [30]). In the case $h > l - n - 1$, there exist many analytic functions $\mathcal{D}^{\pm}$ verifying the condition in (i) and (ii) which depend on a choice of analytic continuation (interpolation) for the values $\mathcal{D}^{\pm}(\chi x_p^s)$ if $s > l - n - 1 - \nu$. But one shows in the Theorem 4.4.1 that there exists at least one such continuation (for example the one which was described in the proof of (iv)).

4.11 Congruences for $n = 1$

For $T \in \mathbb{C}_{sym}^{2n,2n}$ we quote from [7] the definition of the polynomial $\mathfrak{P}_{n,k}^{\nu}(T)$ in the entries $t_{i,j} (1 \le i \le j \le 2n)$ of T by

$$\overset{\circ}{\mathfrak{D}}{}_{n,k}^{v}\left(e^{\mathrm{tr(TZ)}}\right) = \mathfrak{P}_{n,k}^{\nu}(T) e^{\mathrm{tr}(T_1 z_1 + T_4 z_4)}, T = \begin{pmatrix} T_1 & T_2 \\ {}^t T_2 & T_4 \end{pmatrix}.$$

The $\mathfrak{P}_{n,k}^{\nu}(T)$ are homogenous polynomials of degree $n\nu$.
Now we try to give an explicit formula for the polynomials $\mathfrak{P}_{n,k}^{\nu}(T)$.
Considering first the case $n = 1$, we denote

$$T = \begin{pmatrix} t_1 & t_2 \\ t_2 & t_4 \end{pmatrix}, Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_4 \end{pmatrix}, \partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2 & \partial_4 \end{pmatrix},$$

where

$$\partial_1 = \frac{\partial}{\partial z_1}, \partial_2 = \frac{1}{2}\frac{\partial}{\partial z_2}, \partial_4 = \frac{\partial}{\partial z_4}$$

and the test function

$$f = e^{t_1 z_1 + 2 t_2 z_2 + t_4 z_4} = e^{\mathrm{tr}(TZ)}.$$

After computation, we obtain

$$\mathfrak{P}_{1,k}^{\nu}(T) = \sum_{i + 2i' = \nu} C_{\nu}(k) t_2^i (t_1 t_4)^{i'}$$

where $C_\nu(k)$ is a polynomial of variable k ($k = j+1$) of degree ν. Then

$$\sum_{2t_2\in\mathbb{Z},T=\begin{pmatrix} L^2t_1 & t_2 \\ t_2 & L^2t_4\end{pmatrix}\in\Lambda_2^+} \mathfrak{P}_{1,k}^{\nu}(T) = \sum_{2t_2\in\mathbb{Z},t_2^2\leq L^4t_1t_4}\sum_{t_1,t_4}\sum_{i+2i'=\nu} C_\nu(k)t_2^i(L^4t_1t_4)^{i'}. \tag{4.61}$$

We consider the summation

$$\begin{aligned}
A^+ = &\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \mod L}\chi^{-1}(a)\sum_{2t_2\in\mathbb{Z},T=\begin{pmatrix} L^2t_1 & t_2 \\ t_2 & L^2t_4\end{pmatrix}\in\Lambda_2^+}\mathfrak{P}_{1,k}^{\nu}(T)\\
&G_1(2t_2,N,\chi^0)\bar{\chi}_1(2t_2)\int_{\mathbb{Z}_p^\times}\chi x^{j+1}d\mu_{T_2}\\
=&\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \mod L}\chi^{-1}(a)\sum_{2t_2\in\mathbb{Z},t_2^2\leq L^4t_1t_4}\sum_{t_1,t_4}\sum_{i+2i'=\nu}C_\nu(k)t_2^i(L^4t_1t_4)^{i'}\\
&G_1(2t_2,N,\chi^0)\bar{\chi}_1(2t_2)\int_{\mathbb{Z}_p^\times}\chi x^{j+1}d\mu_{T_2}\\
=&\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \mod L}\chi^{-1}(a)\sum_{i+2i'=\nu}C_\nu(k)\sum_{2t_2\in\mathbb{Z},t_2^2\leq L^4t_1t_4}t_2^i\sum_{t_1,t_4}(L^4t_1t_4)^{i'}\\
&G_1(2t_2,N,\chi^0)\bar{\chi}_1(2t_2)\int_{\mathbb{Z}_p^\times}\chi x^{j+1}d\mu_{T_2}.
\end{aligned}\tag{4.62}$$

To satisfy the admissibility we have to prove that $A^+ \equiv 0 \mod L^r$.
We have the interpolation

$$C_\nu(k) = \sum_{t=0}^{\nu}\mu_t\cdot\frac{(j+t+1)!}{(j+1)!}. \tag{4.63}$$

Then

$$\begin{aligned}
B^+ &= \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(L)}\sum_{\chi \mod L}\chi^{-1}(a)\sum_{t=0}^{\nu}\mu_t\cdot\frac{(j+t+1)!}{(j+1)!}G_1(2t_2,N,\chi^0)\bar{\chi}_1(2t_2)\int_{\mathbb{Z}_p^\times}\chi x^{j+1}d\mu_{T_2}\\
&= \int_{x\equiv a \mod L}\sum_{t=0}^{\nu}\mu_t x^{-1}\cdot\frac{\partial^t}{\partial x^t}\left(x^{t+1}(x-a)^r\right)d\mu_{T_2}.
\end{aligned}\tag{4.64}$$

Therefore $(x-a)^r\equiv 0 \mod L^r$ giving the congruence

$$B^+\equiv 0 \mod L^{r-t}\equiv 0 \mod L^{r-\nu}. \tag{4.65}$$

On the other hand, we have

$$\sum_{2t_2\in\mathbb{Z},t_2^2\leq L^4t_1t_4} t_2^i = 2\sum_{t_2=0}^{n_TL^2}t_2^i, \tag{4.66}$$

with $n_T = [\sqrt{t_1 t_4}]$. Using the Bernoulli formula for this summation we get

$$\sum_{2t_2 \in \mathbb{Z}, t_2^2 \le L^4 t_1 t_4} t_2^i = 2 \sum_{k=0}^{i} \binom{i+1}{k} B_k (n_T L^2)^{i+1-k}, \tag{4.67}$$

where B_k are the Bernoulli numbers. Then

$$\begin{aligned} A^+ &\equiv 0 \mod L^{r-\nu} \cdot L^{4i'} \cdot L^{i+1-k} \\ &\equiv 0 \mod L^{r-\nu+2(2i'+i)+2-2k} \\ &\equiv 0 \mod L^{r-\nu+2\nu+2-2k} \\ &\equiv 0 \mod L^{r+\nu+2-2k}. \end{aligned} \tag{4.68}$$

Therefore,

$$A^+ \equiv 0 \mod L^r \quad \text{with } (\nu + 2 - 2k \ge 0).$$

Thus the main congruence holds for $n = 1$.

4.12 Congruences for $n = 2$

From Theorem 4.9.2 we know that if ν is even, then

$$\begin{aligned} P(T_1, T_4, T_2) &= \det(T_1 T_4)^{\nu/2} P(1_n, 1_n, W_0) \\ &= \det(T_1 T_4)^{\nu/2} \sum_{(e_0, \cdots, e_{n-1}) \ne 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} P'_\alpha(W_0)^{e_\alpha}. \end{aligned}$$

If ν is odd, then

$$\begin{aligned} P(T_1, T_4, T_2) &= \det(T_2) \det(T_1 T_4)^{\frac{\nu-1}{2}} P(1_n, 1_n, W_0) \\ &= \det(T_2) \det(T_1 T_4)^{\nu/2} \sum_{(e_0, \cdots, e_{n-1}) \ne 0} c(e_0, \cdots, e_{n-1}) \prod_{\alpha=0}^{n-1} P'_\alpha(W_0)^{e_\alpha} \end{aligned}$$

where $P'_\alpha(W_0)^{e_\alpha}$ is a polynomial in elementary symmetric polynomials. We now consider the case $n = 2$ with $T = \begin{pmatrix} L^2 T_1 & T_2 \\ {}^t T_2 & L^2 T_4 \end{pmatrix} \in \Lambda_4^+$ and ν even. We have

$$\mathfrak{P}_{2,k}^\nu(T) = \det(L^4 T_1 T_4) \sum_{|M| \le \frac{\nu}{2}} c_M(k) Q_M(L^{-2} D).$$

Here $Q_M(L^{-2}D)$ are the homogeneous polynomials of $L^{-2}d_i^2, i=1,2$ degree $|M|$ and $c_M(k)$ is the polynomial of $k=2+j$ of degree $|M|$. We consider the summation

$$\begin{aligned} A^+ = &\sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \sum_{2T_2 \in \mathbb{Z}^{(2,2)}, T = \begin{pmatrix} L^2T_1 & T_2 \\ {}^tT_2 & L^2T_4 \end{pmatrix} \in \Lambda_4^+} \mathfrak{P}_{2,k}^{\nu}(T) \\ &G_2(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2)) \int_{\mathbb{Z}_p^\times} \chi x^{j+1} d\mu_{T_2} \\ = &\sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \det(L^4T_1T_4) \sum_{|M| \le \frac{\nu}{2}} c_M(k) Q_M(T, L^{-2}D) \\ &G_2(2T_2, N, \chi^0)\bar{\chi}_1(\det(2T_2)) \int_{\mathbb{Z}_p^\times} \chi x^{j+1} d\mu_{T_2}. \end{aligned} \tag{4.69}$$

To satisfy the admissibility we have to prove that $A^+ \equiv 0 \mod L^r$.
We have the interpolation

$$c_M(k) = \sum_{t=0}^{|M|} \mu_t \cdot \frac{(j+t+1)!}{(j+1)!}. \tag{4.70}$$

Then

$$\begin{aligned} B^+ = &\sum_{j=0}^{r} \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(L)} \sum_{\chi \mod L} \chi^{-1}(a) \sum_{t=0}^{|M|} \mu_t \cdot \frac{(j+t+1)!}{(j+1)!} G_2(2T_2, N, \chi^0) \\ &\bar{\chi}_1(\det(2T_2)) \int_{\mathbb{Z}_p^\times} \chi x^{j+1} d\mu_{T_2} \\ = &\int_{x \equiv a \mod L} \sum_{t=0}^{|M|} \mu_t x^{-1} \cdot \frac{\partial^t}{\partial x^t} \left(x^{t+1}(x-a)^r\right) d\mu_{T_2}. \end{aligned} \tag{4.71}$$

Therefore, $(x-a)^r \equiv 0 \mod L^r$, giving the congruence

$$B^+ \equiv 0 \mod L^{r-t} \equiv 0 \mod L^{r-|M|}. \tag{4.72}$$

Then

$$\begin{aligned} A^+ &\equiv 0 \mod L^{r-|M|} \cdot L^{4\nu} \cdot L^{-2|M|} \\ &\equiv 0 \mod L^{r-3|M|+4\nu}. \end{aligned} \tag{4.73}$$

Therefore,

$$A^+ \equiv 0 \mod L^r,$$

because $|M| \le \frac{\nu}{2}$. Then the main congruence holds for the case ν is even.
Similarly, the congruence holds when ν is odd.

4.13 Example for $n = 3$

We consider the standard L-function $L(s, F_{12}, st, \chi)$ for all Dirichlet characters χ where F_{12} is the Siegel cusp form of degree 3 and weight 12, constructed by Miyawaki in [21]. Due to Miyawaki and Ikeda,

$$L(s, F_{12}, \chi, St) = L_{2,\Delta}(s + 11, \chi)L(s + 10, g_{20}, \chi)L(s + 9, g_{20}, \chi), \tag{4.74}$$
$$\text{the critical strip } s \in \{-8, -6, -4, -2, 0, 1, 3, 5, 7, 9\},$$

where Δ is Ramanujan's discriminant cusp form and g_{20} is the cusp form of weight 20 of level 1. The first term $L_{2,\Delta}(s + 11, \chi)$ is the symmetric square of cusp form Δ which is defined as follows:

$$L_{2,f}(s, \chi) = L(2s - 2k + 2, (\psi\chi)^2) \sum_{n=1}^{\infty} \chi(n)a(n^2)n^{-s}, \tag{4.75}$$

where $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in \mathcal{M}_1^k(N, \psi)$ and χ is a Dirichlet character. The functional equation for this standard L-function in the general case was proved by Böcherer in [4] but only for trivial character. For F_{12}, it is as follows:

$$\psi(s, F_{12}, st) = \psi(1 - s, F_{12}, st), \tag{4.76}$$

where

$$\psi(s, F_{12}, st) = \gamma(s)L(s, F_{12}, st) \tag{4.77}$$
$$\gamma(s) = c \cdot 2^{-3s}\pi^{-7s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma(s + 9)\Gamma(s + 10)\Gamma(s + 11). \tag{4.78}$$

For each h-admissible measure μ associated to a cusp eigenform f, we denote

$$\mathfrak{L}(\chi x_p^s, f) = \int_{\mathbb{Z}_p^\times} \chi(x)x_p^s d\mu. \tag{4.79}$$

Then from (4.74) we can write

$$\mathfrak{L}(\chi x_p^s, F_{12}) = \mathfrak{L}_G(\chi x_p^{s_1}, \Delta)\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})\mathfrak{L}_V(\chi x_p^{s_2+1}, g_{20}) \tag{4.80}$$

where $\mathfrak{L}_G(\chi x_p^{s_1}, \Delta)$ is the h-admissible case for symmetric squares which was investigated carefully by Gorsse in his thesis, and $\mathfrak{L}_V(\chi x_p^{s_2}, g_{20})$ is the p-adic L-function constructed by Višik in [30]. We see that this expression is true for all plus (+) measures and also minus (-) measures. For simplicity, we consider only the plus measures.
First, we recall the measure which was constructed by Gorsse:

$$\mathfrak{L}_G(\chi x_p^s, \Delta) = J^+(\Delta, \alpha, s + d, \chi, c)\frac{D^+(2k - 2 - (s + d), \Delta^0, \chi')}{\langle \Delta, \Delta \rangle_C}, \tag{4.81}$$

where

$$\begin{aligned} J^+(\Delta, \alpha, s, \chi, c) = {} & \frac{i^{s+1-\nu}G(\bar{\chi})c_\chi^{2k-3-s}\lambda^{-\nu}2^{2k-2s-3}}{\gamma'\alpha(c_\chi^2)} \\ & \times \tilde{\tau}(k - s - 1)(1 - (\psi\chi)^2(2)2^{-2(k-s-1)})^\eta(1 - (\psi\chi)^2(c)c^{-2(k-s-1)})) \\ & \times \frac{1 - (\bar{\psi}\overline{\chi'})(p)p^{k-s-2}}{1 - (\psi\chi')(p)p^{s-k+1}}\frac{(c_{\chi\psi\chi_{-1}^{s-\nu}})^{k-s-1}}{G(\chi\psi\chi_{-1}^{s-\nu})}, \end{aligned}$$

$$\tilde{\tau}(s) = p^{1-k/2} \begin{cases} \alpha_p^2 & \text{if } \chi \text{ is non trivial} \\ (1-p^{-2s})(\alpha_p^2 - p^{s+k-2}) & if \chi \text{ is trivial} \end{cases}$$

$$\eta = \begin{cases} 1 & \text{if } C \text{ is odd} \\ 0 & \text{if } C \text{ is even} \end{cases}$$

$$\gamma' = \frac{\langle \Delta^0, \Delta_0 \rangle_{4Cp}}{\langle \Delta, \Delta \rangle_C} \in \overline{\mathbb{Q}}^\times.$$

We consider the L-function which was studied by Višik:

$$\mathfrak{L}_V(\chi x_p^s, g_{20}) = \frac{c_\chi^s}{G(\chi)\alpha(c_\chi)} i^s \frac{\Gamma(s)}{(2\pi)^s} L(g_{20}, s, \chi) \tag{4.82}$$
$$\times \prod_{q \in S \setminus \mathrm{div}(c_\chi)} (1 - \chi(q)\alpha^{-1}(q)q^{-s})(1 - \chi^{-1}(q)\alpha^{-1}(q)q^{s-1}),$$

where div(w)={primes l: $l|w$} for $w \in \mathbb{Z}$ and $L(f, s, \chi) = \sum_{n=1}^\infty \chi(n) a_n n^{-s}$.
Then we multiply two L-functions (4.81) and (4.82) we obtain the L-function for F_{12}.
In the last step, we compare the L-function $\mathfrak{L}(\chi x_p^s, F_{12})$ with the p-adic L-function constructed by our method for F_{12} in the case $n = 3$. We know that with the positive measure, for $s = 0, \cdots, 8$ we have

$$\int_{\mathbb{Z}_p^\times} \chi x_p^{-s} d\mu^+ = i_p \left(c_\chi^{4s} A^+(\chi) \cdot E_p^+(s, \chi\chi^0) \frac{\Lambda_\infty^+(s)}{\langle (F_{12})_0, (F_{12})_0 \rangle} \cdot D^{(Np)}(F_{12}, s, \overline{\chi\chi^0}) \right), \tag{4.83}$$

where

$$\Lambda_\infty^+(s) := \frac{(2i)^s \cdot \Gamma(s)}{(2\pi i)^s} \cdot \prod_{j=1}^3 \Gamma_\mathbb{C}(s + 12 - j)$$

with $\Gamma_\mathbb{C}(s) := 2 \cdot (2\pi)^{-s}\Gamma(s)$. For any character χ of p-power conductor c_χ, we let

$$A^+(\chi) := (\overline{\chi^0\varphi})_o(c_\chi)^{3l-6} \cdot \frac{\alpha_0(c_\chi)^{-2} \cdot (\chi^0([p, c_\chi]) \cdot \chi(-1)G(\chi))^{-3}}{\chi(-1)G(\chi)},$$

where $[a, b]$ denotes the least common multiple of the integers a, b. We define the p-Euler factor

$$E_p(s, \psi) = \prod_{j=1}^3 \frac{(1 - \psi(p)\beta_j^{-1}p^{s-1})}{(1 - \bar{\psi}(p)\beta_j p^{-s})}.$$

Here, ψ is an arbitrary Dirichlet character and $\beta_1, \beta_2, \beta_3$ are Satake p-parameters of eigenform $(F_{12})_0$ denotes by the function associated to F_{12}. Finally, we let

$$E_p^+ := (1 - (\overline{\varphi\chi\chi^0})_0(p)p^{s-1}) \cdot E_p(s, \chi\chi^0).$$

Computing the Satake p-parameter $\alpha_0(p)$ in the L-functions (4.80), we have

$$\alpha_{0,\mathfrak{L}(\chi x_p^s, F_{12})}(p) = \alpha_{0,\mathfrak{L}_G(\chi x_p^{s_1}, F_{12})}(p) \cdot \alpha^2_{0,\mathfrak{L}_V(\chi x_p^s, F_{12})}(p).$$

We see that the Satake p-parameter $\alpha_0(c_\chi^{-2})$ in (4.80) coincides with the $\alpha_0(c_\chi^{-2})$ in (4.83). The Gauss sum in (4.80) and in (4.83) are equal to $G(\chi)^{-4}$. Therefore, the two p-adic L-functions constructed by these two methods coincide.

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