Resolutions and Castelnuovo-Mumford Regularity

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Résolutions et Régularité de Castelnuovo-Mumford

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Dedicated to

my wife

and to

my mother

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Abstract

In this thesis, we study square-free monomial ideals of the polynomial ring $S = K[x_1, \ldots, x_n]$ which have a linear resolution. By remarkable result of Bayer and Stilman [BS] and the technique of polarization, classification of homogeneous ideals with linear resolution is equivalent to classification of square-free monomial ideals with linear resolution. However, classification of square-free monomial ideals with linear resolution seems to be difficult because by Eagon-Reiner Theorem [ER], this is equivalent to classification of Cohen-Macaulay ideals.

It is worth to note that, square-free monomial ideals in S are in one-to-one correspondence to Stanley-Reisener ideals of simplicial complexes on one hand and the circuit ideal of clutters from another hand. This correspondence motivated mathematicians to use the combinatorial and geometrical properties of these objects in order to get the desired algebraic results.

Classification of square-free monomial ideals with 2-linear resolution, was successfully done by Fröberg [Fr] in 1990. Fröberg observed that the circuit ideal of a graph G has a 2-linear resolution if and only if G is chordal, that is, G does not have an induced cycle of length > 3. In [Em, ThVt, VtV, W] the authors have partially generalized the Fröberg's theorem for degree greater than 2. They have introduced several definitions of chordal clutters and proved that, their corresponding circuit ideals have linear resolutions.

Viewing cycles as geometric objects (triangulation of closed curves), in this thesis we try to generalize the concept of cycles to triangulation of pseudo-manifolds and get a partial generalization of Fröberg's theorem for higher dimensional hypergraphs.

All the results in Chapters 4 and 5 and some results in Chapter 3 are devoted to be original.

Keywords: Minimal Free Resolution, Castelnuovo-Mumford Regularity, Clutter, Betti Number, Pseudo-manifold, Triangulation, Simplicial Complex.

Mathematics Subject Classification[2010]: 13D14, 13D02, 13D45, 13F55, 16E05, 51H30.

Résumé

Le sujet de cette thèse, est l'étude d'idéaux monomiaux libres de carrés de l'anneau de polynômes $S = K[x_1, \dots, x_n]$, qui ont une résolution linéaire. D'après un résultat remarquable de Bayer et Stilman [BS] et en utilisant la polarisation, la classification des idéaux monomiaux ayant une résolution linéaire, est équivalente à la classification des idéaux monomiaux libres de carrés ayant une résolution linéaire. De plus le théorème de Eagon-Reiner, établit une dualité entre les idéaux monomiaux libres de carrés ayant une résolution linéaire et les idéaux monomiaux libres de carrés Cohen-Macaulay, ce qui montre que le problème de classification des idéaux monomiaux libres de carrés ayant une résolution linéaire est très difficile.

Nous rappelons que, les idéaux monomiaux libres de carrés sont en correspondance biunivoque avec les complexes simpliciaux d'une part, et d'autre part avec les clutters. Ces correspondances nous motivent pour utiliser les propriétés combinatoires des complexes simpliciaux et des clutters pour obtenir des résultats algébriques. La classification des idéaux monomiaux libres de carrés ayant une résolution linéaire engendrés en degré 2 a été faite par Fröberg [Fr] en 1990. Fröberg a observé que l'idéal des circuits d'un graphe G a une résolution 2-linéaire si et seulement si, G est un graphe de cordes, i.e. il n'a pas de cycles minimaux de longueur plus grande que 4. Dans [Em, ThVt, VtV, W] les auteurs ont partiellement généralisé les résultats de Fröberg à des idéaux engendrés en degré ≥ 3. Ils ont introduit plusieurs définitions de clutters de cordes et démontré que les idéaux de circuits correspondant ont une résolution linéaire.

Nous pouvons voir les cycles du point de vue topologique, comme la triangulation d'une courbe fermée, dans cette thèse nous utiliserons cette idée pour étudier des clutters associés à des triangulation de pseudo manifolds en vue d'obtenir une généralisation partielle des résultats de Fröberg à des idéaux engendrés en degré ≥ 3. Nous comparons notre travail à ceux de [Em, ThVt, VtV, W]. Nous présentons nos résultats dans le chapitres 4 et 5.

Mots clés: Résolution libre minimale, Régularité de Castelnuovo-Mumford, Idéaux monomiaux, Clutters, Nombres de Betti, Pseudo-manifold, Triangulation.

Classification AMS[2010]: 13D14, 13D02, 13D45, 13F55, 16E05, 51H30.

Contents

Introduction

Castelnuovo-Mumford regularity is one of the most fundamental invariants in Commutative Algebra and Algebraic Geometry. One of its first hidden appearances may be found in Castelnuovo's work on linear systems on smooth projective space curves [Cas, 1893]. Castelnuovo's result, gives a sharp upper bound on the largest degree r such that, the complete linear system of the r-fold plane sections on the given curve, is not cut out by surfaces of degree r. Although this result is of fairly geometric appearance, Castelnuovo's method of proof has a rather algebraic flavor.

Another early invisible occurrence of Castelnuovo-Mumford regularity was initiated in the work of Hermann [Her, 1926]. The results of Hermann show that, the minimal free resolution of an ideal generated by finitely many homogeneous polynomials, can be computed in a (finite) number of steps which depends only on the number of indeterminates of the ambient ring and the maximal degree of the given polynomials.

Hermann's work is not at all constructive, and so it does not give rise to an explicit algorithm. It was indeed only around 1980, when such algorithms became practicable, based on Gröbner base techniques, implemented in Computer Algebra Systems like Macaulay, CoCoA, Singular and powered by high performance computers. And indeed:

> Castelnuovo-Mumford regularity provides the ultimate bound of complexity for these algorithms.

D. Bayer and M. Stilman [BS2] showed that, an estimate of the regularity of an ideal, gives a bound on complexity of algorithms for computing syzygies.

In 1966, Mumford gave a first proper definition of Castelnuovo-Mumford regularity (see [M]), which he called Castelnuovo regularity. In fact, Mumford did define the notion of being m-regular in the sense of Castelnuovo for a coherent sheaf of ideals over a projective space and a given integer m. More precisely, a sheaf of ideals over a projective space is called m-regular, if for all positive values of i, the i-th Serre cohomology group of the $(m-i)$-fold twist of this sheaf vanishes. The minimal possible value of m is what today usually is called the Castelnuovo-Mumford regularity of the sheaf of ideals in question. Moreover, Mumford did prove a fundamental bounding result, namely:

> The Castelnuovo-Mumford regularity of a coherent sheaf of ideals over a projective space is bounded by the Hilbert polynomial of this ideal.

In fact Mumford's arguments allow to make this bound explicit. Although Castelnuovo-Mumford regularity was originally defined in terms of sheaf cohomology, it may be expressed in terms of degrees of syzygies and hence is of basic significance in classical Projective Algebraic Geometry.

Castelnuovo-Mumford regularity also found much interest in Commutative Algebra. In 1984, D. Eisenbud and S. Goto [EG] made explicit the link between this algebraic Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its minimal free resolution.

One of the aspects that makes the regularity very interesting, is that Castelnuovo-Mumford regularity can be computed in different ways. In pure algebraic setting, it is defined as follows:

Definition 0.0.1. Let K be a field and let S be a polynomial ring over K. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded S-module. In most interesting case, M is an ideal of S. For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t_i^S(M) = \max\{j\colon \quad \beta_{i,j}^K(M) \neq 0\}$$

where $\beta_{i,j}^K(M)$ is the i,j-th graded Betti number of M as an S-module, i.e.

$$\beta_{i,j}^K = \dim_K \operatorname{Tor}_i^S(K, M)_j$$

and $t_i^S(M) = -\infty$, if it happens that $\operatorname{Tor}_i^S(K, M) = 0$.

The *Castelnuovo-Mumford regularity* of M, $\operatorname{reg}(M)$, is given by:

$$\operatorname{reg}(M) = \sup\{t_i^S(M) - i\colon \quad i \in \mathbb{Z}\}.$$

We say that M has a d-linear resolution, if M is generated by homogeneous elements of degree d and $\operatorname{reg}(I) = d$. That is, the graded minimal free resolution of I is of the form:

$$0 \longrightarrow S^{\beta_s}(-d-s) \longrightarrow \cdots \longrightarrow S^{\beta_1}(-d-1) \longrightarrow S^{\beta_0}(-d) \longrightarrow I \longrightarrow 0.$$

Among all the interesting problems in Castelnuovo-Mumford regularity, classification of ideals with linear resolution is of great importance. Proving that a class of ideals has a d-linear resolution, is difficult in general. However, some classes of ideals with linear resolution may be found in [AHH, AHH2, ANH, CoH, Em, EO, EOS, Fr, HHZ, ThVt, Mo, MNYZ, VtV, W, Zh].

Classification of square-free monomial ideals with 2-linear resolution, was successfully done by Fröberg [Fr] in 1990. Fröberg observed that, the circuit ideal of a graph G has a 2-linear resolution if and only if G is chordal, that is, G does not have an induced cycle of length > 3.

Theorem 0.0.2 (Fröberg's Theorem [Fr]). *Let $G \neq \mathcal{C}_{n,2}$ be a graph on vertex set $[n]$ and $I = I(\bar{G})$ be the circuit ideal of G. The ideal I has a 2-linear resolution if and only if G is chordal.*

Fröberg's Theorem in particular implies that, having 2-linear resolution does not depend on the characteristic of the base field K. However, in general having linear resolution does depend on the characteristic of the base field (see for instance, Example 5.3.19).

Trying to generalize Fröberg's result for square-free ideals generated in degree greater than 2, some mathematicians have introduced various definitions of chordal hypergraphs and they proved that the corresponding circuit ideals have a linear resolution over any field K (see for example [Em, ThVt, VtV, W]).

In this thesis, we study square-free monomial ideal with linear resolution. Our method in this thesis, is to look at square-free monomial ideals in both aspects of combinatorics and geometrics.

In Chapter 1, we review basic notations and definitions concerning graded minimal free resolution, depth of module and local cohomology. The notions and remarks in this chapter are essential for the remaining of this thesis.

Chapter 2 of this thesis is devoted to introduce the main subject of this thesis. In this chapter, first we introduce the definition of Castelnuovo-Mumford regularity and then we study the basic properties of Castelnuovo-Mumford regularity. As it is mentioned before, the goal of this thesis is to study square-free monomial ideals with linear resolution, that is, the ideals with generators consist of square-free monomials of degree d and its Castelnuovo-Mumford regularity is again d. In Section 2.2, we will present a survey of known results on ideals with linear resolution.

Chapter 3 of this thesis is in fact the language of this thesis. First in Section 3.1, we outline that the classification of *homogeneous* ideal of $S =$

$K[x_1, \ldots, x_n]$ with linear resolution is equivalent to classification of *square-free* monomial ideal with linear resolution. That is why, in the remaining of this thesis, we consider only square-free monomial ideals of the polynomial ring $S = K[x_1, \ldots, x_n]$.

In Section 3.2, we deal with the notions of simplicial complexes. With a simplicial complex Δ, one can associate a square-free monomial ideals I_Δ whose generators correspond to the non-faces of Δ. This ideal is called Stanley-Reisner ideal of Δ. Note that, there exists a bijection between square-free monomial ideal $I \subset K[x_1, \ldots, x_n]$ and simplicial complexes Δ on vertex set $[n]$, given by $\Delta \leftrightarrow I_\Delta$. This correspondence, motivated us to investigate the interaction between the homological properties of these objects and algebraic properties of square-free monomial ideals.

Alexander duality plays an important role in study of minimal free resolution of Stanley-Reisner ideal. In particular, Eagon and Reiner used Alexander dual complexes and proved the following interesting theorem:

Theorem 0.0.3 (Eagon-Reiner)**.** *Let Δ be a simplicial complex on vertex set $[n]$. The ideal $I_\Delta \subset S = K[x_1, \ldots, x_n]$ has a q-linear resolution if and only if $\Delta^\vee$ is Cohen-Macaulay over K of dimension $n - q$.*

This theorem and Mayer-Vietoris long exact sequence on local cohomologies, will be frequently used in Chapter 4 and play a key role in many proofs.

We recall that, all the generators of an ideal with linear resolution have the same degree. So, it is worth to find a correspondence between square-free monomial ideals generated in same degree with some other combinatorial or geometrical objects rather than simplicial complexes. This leads us to investigate clutters and (pseudo-)manifolds rather than simplicial complexes.

With a d-uniform clutter $\mathcal{C}$ on vertex set $[n]$, we associate a square-free monomial ideal, $I(\bar{\mathcal{C}})$, whose generators are:

$$\left\{ \prod_{i \in F} x_i : \quad F \subset [n],\ |F| = d,\ F \notin \mathcal{C} \right\}.$$

This ideal is called the circuit ideal of $\mathcal{C}$. Note that, we have a bijection between square-free monomial ideal $I \subset K[x_1, \ldots, x_n]$ generated in the same degree d, with d-uniform clutters, given by $\mathcal{C} \leftrightarrow I(\bar{\mathcal{C}})$. Moreover, any triangulation of a manifold or a pseudo-manifold gives rise to a square-free monomial ideal in $K[x_1, \ldots, x_n]$.

The aim of this chapter is to study properties of simplicial complexes, clutters and triangulations.

Chapter 4 is the combinatorial core of this thesis. The goal of this chapter is to do some operations on a given graph (or clutter) to reduce it to a smaller graph (or clutter), such that the Castelnuovo-Mumford regularity of corresponding circuit ideals, does not change under these operations. As consequences of these operations:

- We will find some alternative proofs for Fröberg's Theorem.
- We will find an alternative proof for linearity of circuit ideals of generalized 3-uniform chordal clutter as defined by Emtander.
- We introduce a combinatorial criterion in order to check that if the circuit ideal of a given 3-uniform clutter has a linear resolution.
- We will find a large class of ideals with linear resolution.

Also, we compare several definitions of chordal clutters and some open problems for further studies are given in this chapter.

To attack to the problem of classification of ideas with d-linear resolution, in Chapter 5, we investigate clutters whose their circuit ideals do not have linear resolution, but any proper subclutter of them has a linear resolution. This chapter generalize many of the results in Chapter 4 for arbitrary d-uniform clutters. But the method in this chapter, is not algebraic combinatorics but is algebraic topology.

The circuit ideal of clutters which are minimal to linearity, is contained in the class of square-free monomial ideals I_Δ, with $\operatorname{indeg}(I_\Delta) = 1 + \dim \Delta$. So, first we deal with the class of square-free monomial ideals I_Δ with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$. The results in this section, enable us to find precisely, the minimal free resolution of ideals which are minimal to linearity. Some nice classes of clutters which are minimal to linearity are pseudo-manifolds, but unfortunately, pseudo-manifolds are strictly contained in this class. However, using the results in this chapter, we can compute the graded Betti numbers of the circuit ideals of an arbitrary pseudo-manifolds (like a triangulation of sphere, projective plane, Klein bottle, etc.).

Also, for two d-uniform clutters $\mathcal{C}_1$ and $\mathcal{C}_2$, we will prove that:

$$\operatorname{reg} I(\overline{\mathcal{C}_1 \cup \mathcal{C}_2}) = \max\{\operatorname{reg} I(\bar{\mathcal{C}}_1),\ \operatorname{reg} I(\bar{\mathcal{C}}_2)\}$$

whenever, $V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ is a clique or $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$. Again, this leads to an alternative proof for Fröberg Theorem as well as linearity of circuit ideal of generalized d-uniform chordal clutters as defined by Emtander.

Finally, in the last section, for a given square-free monomial I generated in degree d, we define a square-free monomial ideal $\hat{I}$, generated in degree $d+1$ which is very closed to I in regularity. In fact we have:

$$\operatorname{reg}(\hat{I}) = \begin{cases} \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) > d; \\ 1 + \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) = d. \end{cases}$$

This enables us to generate a square-free monomial ideal with $(d+1)$-linear resolution from a square-free ideal I with d-linear resolution.

The results in Chapter 5 have many hidden ideas for further studies in this area.

We acknowledge the support provided by the Computer Algebra Systems CoCoA and Singular [CCA, Si] for the extensive experiments which helped us to obtain some of the results in this thesis. Throughout this thesis, all known definitions and statements are quoted with a reference afterwards and all others with no references are supposed to be new. The results in Chapter 4 and Chapter 5 appear(ed) in [MNYZ, MYZ, MYZ2].

Chapter 1

Commutative Algebra

In this chapter, we recall some basic notions and results that will be used later. Throughout this thesis, all rings are considered to be commutative with the identity $1 \neq 0$.

1.1 Graded Modules, Hilbert Series

In commutative algebra, graded rings and graded modules are of great importance, especially local rings. That is, graded rings with just one graded maximal ideal. Graded local rings share many properties with polynomial rings. For example, consider the polynomial ring $S = K[x_1, \dots, x_n]$ in n variables over a field K; if $n > 0$, this has infinitely many maximal ideals but $(x_1, \dots, x_n) \subset S$ is the only graded maximal ideal of S.

Definition 1.1.1. A ring A is called *graded* (or more precisely, $\mathbb{Z}$-*graded*), if there exists a family of subgroups $\{A_n\}_{n\in\mathbb{Z}}$ of A such that,

(a) $A = \oplus_n A_n$ (as abelian groups), and

(b) $A_n \cdot A_m \subset A_{n+m}$, for all n, m.

Note that if $A = \oplus_n A_n$ is a graded ring, then A_0 is a subring of A, $1 \in A_0$ and A_n is an A_0-module for all n.

Let A be a ring and $x_1, \dots, x_n$ be indeterminates over A. For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$, let $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$. Then the polynomial ring $S = A[x_1, \dots, x_n]$ is a graded ring, where:

$$S_i = \{ \sum_{\mathbf{m}\in\mathbb{N}^n} r_{\mathbf{m}}\mathbf{x}^{\mathbf{m}} : \quad r_{\mathbf{m}} \in A \text{ and } m_1 + \cdots + m_n = i\}.$$

This is called the *standard grading* of the polynomial ring $A[x_1, \dots, x_n]$.

A product $x_1^{m_1} \cdots x_n^{m_n}$ with $m_i \in \mathbb{N}$ is called a *monomial.* The set of monomials of S, is denoted by $\mathrm{Mon}(S)$. A monomial $x_1^{m_1} \cdots x_n^{m_n}$ is called *square-free monomial*, if $m_i \leq 1$, for $i = 1, \dots n$.

An ideal $I \subset S$ is called (*square-free*) *monomial ideal*, if it is generated by (square-free) monomials.

Definition 1.1.2. Let A be a graded ring and M an A-module. We say that M is a *graded A-module* (or has an *A-grading*), if there exists a family of subgroups $\{M_n\}_{n\in\mathbb{Z}}$ of M such that,

(a) $M = \oplus_n M_n$ (as abelian groups), and

(b) $A_n \cdot M_m \subset M_{n+m}$, for all n, m.

If $u \in M \setminus \{0\}$ and $u = u_{i_1} + \cdots + u_{i_k}$ where $u_{i_j} \in M_{i_j} \setminus \{0\}$, then $u_{i_1}, \dots, u_{i_k}$ are called the *homogeneous components* of u.

For a non-zero element $u \in M_i$, the *degree* of u is denoted by $\deg(u)$ which we set to be i.

We let $\mathscr{M}(A)$ be the category of finitely generated graded A-modules. A *homogeneous homomorphism* $\varphi : M \longrightarrow N$ of graded A-modules of degree d is an A-module homomorphism such that $\varphi(M_i) \subset N_{i+d}$, for all i. For example, if $f \in A$ is homogeneous of degree d, then the multiplication map $A(-d) \longrightarrow A$, with $g \mapsto fg$ is a homogeneous homomorphism. Here, for a graded A-module W and an integer a, one denotes by $W(a)$ the graded A-module whose graded components are given by $W(a)_i = W_{a+i}$. One says that, $W(a)$ arises from W by applying the *shift* a. The morphisms being the homogeneous homomorphisms $M \longrightarrow N$ of degree 0, simply called *homogeneous homomorphisms.*

Definition 1.1.3. Let $M = \oplus M_n$ be a graded A-module and N a submodule of M. For each $n \in \mathbb{Z}$, let $N_n = N \cap M_n$. If the family of subgroups $\{N_n\}$ makes N into a graded A-module, we say that N is a *graded* (or *homogeneous*) *submodule* of M. The graded submodules of A is called *homogeneous* (or *graded*) ideal of A.

Note that for any submodule N of M, $A_n \cdot N_m \subset N_{n+m}$. Thus, N is graded if and only if $N = \oplus_n N_n$. In particular, every (square-free) monomial ideal of A is homogeneous ideal.

Proposition 1.1.4. *Let A be a graded ring, M a graded A-module and N a submodule of M. The following statements are equivalent:*

(i) *N is a graded A-module.*

(ii) $N = \sum\limits_n (N \cap M_n)$

(iii) *For every $u \in N$, all the homogeneous components of u are in N.*

(iv) *N has a homogeneous set of generators.*

Definition 1.1.5. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with the grading induced by $\deg(x_i) = d_i$, where d_i is a positive integer. If $M = \bigoplus\limits_{i=0}^{\infty} M_i$ is a finitely generated $\mathbb{N}$-graded module over S, its *Hilbert function* and *Hilbert series* are defined by:

$$H(M,i) = \dim_K(M_i) \qquad \text{and} \qquad F(M,t) = \sum_{i=0}^{\infty} H(M,i)t^i.$$

Theorem 1.1.6 (Hilbert-Serre)**.** *Let K be a field and $S = K[x_1, \dots, x_n]$ a polynomial ring graded by $\deg(x_i) = d_i \in \mathbb{N}^+$. If M is a finitely generated $\mathbb{N}$-graded S-module, then the Hilbert series of M is a rational function that can be written as:*

$$F(M,t) = \frac{h(t)}{\prod\limits_{i=0}^{n}(1 - t^{d_i})}, \qquad \textit{for some} \quad h(t) \in \mathbb{Z}[t].$$

In particular, if $d_i = 1$, for all i, then there is a unique polynomial $h(t) \in \mathbb{Z}[t]$ such that:

$$F(M,t) = \frac{h(t)}{(1-t)^d}, \qquad \textit{and} \quad h(1) \neq 0.$$

The number $e(M) = h(1)$ in the above theorem is called the *multiplicity* of the module M.

Definition 1.1.7. Let R be standard graded ring and M be a graded R-module such that,

$$h(t) = h_0 + h_1 t + \cdots + h_r t^r$$

is the (unique) polynomial with integer coefficients such that $h(1) \neq 0, h_r \neq 0$ and satisfying

$$F(M,t) = \frac{h(t)}{(1-t)^d},$$

where, $d = \dim(M)$. The **h**-*vector* of M is defined by $\mathbf{h}(M) = (h_0, \dots, h_r)$.

Example 1.1.8. If we consider the polynomial ring $S = K[x_1, \ldots, x_n]$ in n variables over the field K and $\deg(x_i) = 1$ (for $i = 1, \ldots, n$), then we have:

$$H(S, i) = \dim_K(S_i) = \binom{i+n-1}{n-1},$$

and for $n \geq 1$,

$$F(M, t) = \sum_{i=0}^{\infty} (\dim_K(S_i))\, t^i = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n}.$$

1.2 Graded Minimal Free Resolution

Throughout this thesis, we let K be a field, $(R, \mathfrak{m})$ a Noetherian graded local ring with residue field K or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. We write S for the polynomial ring $K[x_1, \ldots, x_n]$ with the standard grading.

We let M be a finitely generated R-module and will assume that M is graded, if R is graded.

Now, let M be a finitely generated graded R-module with homogeneous generators $m_1, \ldots, m_r$ and $\deg(m_i) = a_i$, for $i = 1, \ldots, r$. Then, there exists a surjective R-module homomorphism $F_0 = \bigoplus_{i=1}^{r} Re_i \to M$ with $e_i \mapsto m_i$. Assigning to e_i the degree a_i, for $i = 1, \ldots, r$ the map $F_0 \longrightarrow M$ becomes a morphism in $\mathscr{M}(R)$ and F_0 becomes isomorphic to $\oplus_{i=1}^{r} R(-a_i)$. Thus, we obtain the exact sequence:

$$0 \longrightarrow U \longrightarrow \bigoplus_{j} R(-j)^{\beta^R_{0,j}} \longrightarrow M \longrightarrow 0,$$

where $\beta^R_{0,j} = |\{i : a_i = j\}|$, and where $U = \operatorname{Ker}\left(\oplus_j R(-j)^{\beta^R_{0,j}} \to M\right)$.

The module U is a graded submodule of $F_0 = \oplus_j R(-j)^{\beta^R_{0,j}}$. By Hilbert's basis theorem for modules, we know that U is finitely generated and hence we find again an epimorphism $\oplus_j R(-j)^{\beta^R_{1,j}} \to U$. Composing this epimorphism with the inclusion map $U \to \oplus_j R(-j)^{\beta^R_{1,j}}$, we obtain the exact sequence:

$$\bigoplus_{j} R(-j)^{\beta^R_{1,j}} \longrightarrow \bigoplus_{j} R(-j)^{\beta^R_{0,j}} \longrightarrow M \longrightarrow 0$$

of graded R-modules. Proceeding in this way, we obtain a long exact sequence:

$$\mathscr{F} : \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of graded R-modules with $F_i = \oplus_j R(-j)^{\beta^R_{i,j}}$. Such an exact sequence is called a *graded free R-resolution of M.*

It is clear from our construction that, the resolution obtained is by no means unique. On the other hand, if we choose in each step of the resolution a minimal presentation, the resolution will be unique up to isomorphism, as we shall see later.

A set of homogeneous generators $m_1, \ldots, m_r$ of M is called *minimal*, if no proper subset of it generates M.

Lemma 1.2.1. *Let $m_1, \ldots, m_r$ be a homogeneous set of generators of the graded R-module M. Let $F_0 = \oplus_{i=1}^r Re_i$ and let $\varepsilon : F_0 \to M$ be the epimorphism with $e_i \mapsto m_i$, for $i = 1, \ldots, r$. Then, the following conditions are equivalent:*

(i) *$m_1, \ldots, m_r$ is a minimal system of generators of M.*

(ii) $\mathrm{Ker}\,(\varepsilon) \subset \mathfrak{m}F_0$, *where $\mathfrak{m}$ is the unique homogeneous maximal ideal of R.*

Let M be a finitely generated graded R-module. A graded free R-resolution $\mathscr{F}$ of M is called *minimal*, if for all i, the image of $F_{i+1} \to F_i$ is contained in $\mathfrak{m}F_i$. Lemma 1.2.1 implies at once that, each finitely generated graded R-module admits a minimal free resolution.

The next result shows that, the numerical data given by a graded minimal free R-resolution of M depends only on M and not on the particular chosen resolution.

Proposition 1.2.2. *Let M be a finitely generated graded R-module and*

$$\mathscr{F}: \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

a minimal graded free R-resolution of M with $F_i = \oplus_j R(-j)^{\beta^R_{i,j}}$, for all i. Then,

$$\beta^R_{i,j} = \dim_K \mathrm{Tor}^R_i\,(K, M)_j$$

for all i and j.

The numbers $\beta^R_{i,j}$ are called the *graded Betti numbers of M*, and $\beta^R_i = \sum_j \beta^R_{i,j} (= \mathrm{rank} F_i)$ is called the *i-th Betti number of M*. As long as we work with the polynomial ring $S = K[x_1, \ldots, x_n]$, we write $\beta^K_{i,j}$ instead of $\beta^S_{i,j}$. Also, the number

$$\mathrm{projdim}\, M = \sup\{i\colon \quad \mathrm{Tor}^R_i\,(K, M) \neq 0\}.$$

is called the *projective dimension of M*.

We close this section by stating that, not only are the graded Betti numbers determined by a minimal graded free resolution, but also, a minimal graded free resolution of M is unique up to isomorphisms.

Proposition 1.2.3. *Let M be a finitely generated graded R-module and let $\mathscr{F}$ and $\mathscr{G}$ be two minimal graded free R-resolutions of M. Then, the complexes $\mathscr{F}$ and $\mathscr{G}$ are isomorphic.*

Theorem 1.2.4 (Graded Hilbert syzygy theorem, [CLO, Theorem 3.8])**.** *Let $S = K[x_1, \dots, x_n]$. Then every finitely generated graded S-module has a finite graded resolution of length at most n.*

1.3 Tensor Algebra

Let A be a commutative ring and M an A-module. For every integer $n \geq 0$, the A-module *n-th tensor power of M* is denoted by $T^n(M)$ or $M^{\otimes n}$, where $T^0(M) = M^{\otimes 0} = A$. Sum of the tensors powers, forms a graded A-module:

$$\bigotimes M = \bigoplus_{j=0}^{\infty} M^{\otimes j}.$$

We shall define a graded A-algebra structure on $\bigotimes M$. By the assignment:

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) \longmapsto x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$$

we get an A-bilinear map $M^{\otimes m} \times M^{\otimes n} \to M^{\otimes(m+n)}$. Its additive extension to $\bigotimes M \times \bigotimes M$, gives $\bigotimes M$ the structure of a graded A-algebra with identity 1_A.

The A-algebra $\bigotimes M$ is called *tensor algebra of M*. Obviously, $\bigotimes M$ is not commutative in general. We identify M and $T^1(M)$. The injection $\varphi\colon M \to \bigotimes M$ is called the *canonical injection* of M into $\bigotimes M$. The tensor algebra is characterized by a universal property.

Proposition 1.3.1 (Universal property of tensor algebras, [Bou, Chapter III, §5, Proposition 1])**.** *Let E be an A-algebra and $f\colon M \to E$ an A-linear mapping. Then, there exists unique A-algebra homomorphism $g\colon \bigotimes M \to E$ such that, $f = g \circ \varphi$:*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "f"'] & \bigotimes M \arrow[d, dotted, "g"] \\ & E \end{tikzcd}$$

Proposition 1.3.2 (Functorial property of tensor algebras, [Bou, Chapter III, §5, Proposition 2])**.** *Let A be a commutative ring, M and N two A-modules and $u\colon M \to N$ an A-linear mapping. Then, there exists unique A-algebra homomorphism $u'\colon \bigotimes M \to \bigotimes N$ such that, the diagram*

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"] \\ \bigotimes M \arrow[r, dotted, "u'"'] & \bigotimes N \end{tikzcd}$$

is commutative.

The homomorphism u' in Proposition 1.3.2, will henceforth be denoted by $\bigotimes u$. If P is an A-module and $v\colon N \to P$ an A-linear mapping, then:

$$\bigotimes(v \circ u) = (\bigotimes v) \circ (\bigotimes u)$$

for $(\bigotimes v) \circ (\bigotimes u)$ is an A-algebra homomorphism rendering the commutative diagram:

$$\begin{tikzcd} M \arrow[r, "v \circ u"] \arrow[d, "\varphi_M"'] & P \arrow[d, "\varphi_P"] \\ \bigotimes M \arrow[r, "(\bigotimes v)\circ(\bigotimes u)"'] & \bigotimes P \end{tikzcd}$$

The homomorphism $\bigotimes u$ is sometimes called the *canonical extension* of u to $\bigotimes M$ (which contains $M = M^{\otimes 1}$). Note that the restriction $\bigotimes u|_{M^{\otimes n}}\colon M^{\otimes n} \to N^{\otimes n}$, is just the linear mapping $u^{\otimes n} = \underbrace{u \otimes u \otimes \cdots \otimes u}_{n \text{ times}}$, for

$$\bigotimes u|_{M^{\otimes n}}(x_1 \otimes \cdots \otimes x_n) = u(x_1) \otimes \cdots \otimes u(x_n).$$

Since $\bigotimes u$ is an A-algebra homomorphism and $u^{\otimes 1} = u$; the restriction $u^{\otimes 0}$ to A is the identity mapping. The homomorphism $u^{\otimes n}$ is called the *n-th tensor power of u.*

Proposition 1.3.3 ([Bou, Chapter III, §5, Proposition 3])**.** *If $u\colon M \to N$ is a surjective A-linear mapping, the homomorphism $\bigotimes u\colon \bigotimes M \to \bigotimes N$ is surjective and its kernel is the two-sided ideal of $\bigotimes M$, generated by the kernel $P \subset M \subset \bigotimes M$ of u.*

Proposition 1.3.4 ([Bou, Chapter III, §5, Proposition 5])**.** *Let A, B be two commutative rings, $\rho\colon A \to B$ a ring homomorphism and M an A-module. The canonical extension,*

$$\bigotimes_B (B \otimes_A M) \longrightarrow B \otimes_A \left(\bigotimes_A M\right)$$

of the B-linear mapping $1_B \otimes \varphi_M \colon B \otimes_A M \to B \otimes_A \left(\bigotimes_A M\right)$, is a graded B-algebra isomorphism.

1.4 Exterior Algebra

Definition 1.4.1. Let A be a commutative ring and M an A-module. The *exterior algebra* of M, denoted by $\bigwedge M$, is the A-algebra of the quotient of the tensor algebra $\bigotimes M$, over two-sided ideal $\mathfrak{J}$ generated by the elements $x \otimes x$, where x runs through M.

Since the ideal $\mathfrak{J}$ is generated by homogeneous elements of degree 2, it is a graded ideal. We write $\mathfrak{J}_n = \mathfrak{J} \bigcap T^n(M)$. The algebra $\bigwedge M$ is therefore graded by the graduation consisting of the $\bigwedge^n M = T^n(M)/\mathfrak{J}_n$. That is,

$$\bigwedge M = \bigoplus_{i \geq 0} \bigwedge^i M.$$

The A-module $\bigwedge^i M$ is called the *i-th exterior power of* M. Note that $\mathfrak{J}_0 = \mathfrak{J}_1 = 0$ and hence $\bigwedge^0 M$ and $\bigwedge^1 M$ are identified with A and $T^1(M) = M$, respectively. We shall always make these identification and the canonical injection $M \to \bigwedge M$, will be denoted by ψ or ψ_M.

The product in $\bigwedge M$ is denoted by $x \wedge y$, which comes from A-algebra structure on $\bigotimes M$, by passing to quotient. In general $\bigwedge M$ is not commutative. One has (c.f. [BH, Section 1.6]):

$$\begin{array}{ll} x \wedge y = (-1)^{(\deg x)(\deg y)} y \wedge x, & \text{for homogeneous } x, y \in \bigwedge M; \\ x \wedge x = 0, & \text{for homogeneous } x, \deg x \text{ is odd.} \end{array}$$

Let $x_1, \ldots, x_n$ be elements of M, and σ a permutation of $\{1, \ldots, n\}$. Then,

$$x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)} = |\sigma|\, x_1 \wedge \cdots \wedge x_n,$$

here $|\sigma|$ is the sign of σ. Furthermore $x_1 \wedge \cdots \wedge x_n = 0$, if $x_i = x_j$ for some indices $i \neq j$. For a subset I of $\{1, \ldots, n\}$, we set:

$$x_I = x_{i_1} \wedge \cdots \wedge x_{i_m}, \text{ when } I = \{i_1, \ldots, i_m\} \text{ with } i_1 < \cdots < i_m.$$

For subsets $J, T \subset \{1, \ldots, n\}$, let

$$\sigma(J, T) = \begin{cases} (-1)^i, & \text{if } J \cap T = \emptyset, \text{ where } i = |\{(j,t) \in J \times T \colon j > t\}| \\ 0, & \text{if } J \cap T \neq \emptyset. \end{cases}$$

Then,

$$x_J \wedge x_T = \sigma(J, T) x_{J \cup T}.$$

It is clear that, the notation x_I can be extended to the more general case in which $(x_g)_{g \in G}$ is a family of elements of M, indexed by a linearly ordered set G and I is a finite subset of G.

Let $(x_g)_{g \in G}$ be a system of generators of M. Then, $\bigwedge^j M$ is generated by the exterior products x_I, with $I \subset G$ and $|I| = j$. In particular, if M is generated by $x_1, \ldots, x_n$ then $\bigwedge^i M = 0$, for all $i > n$.

Proposition 1.4.2 (Universal property of exterior algebras, [Bou, Chapter III, §7, Proposition 1])**.** *Let E be an A-algebra and $f\colon M \to E$ an A-linear mapping such that $(f(x))^2 = 0$, for all $x \in M$. Then, there exists unique A-algebra homomorphism $g\colon \bigwedge M \to E$ such that, $f = g \circ \psi$:*

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, "f"'] & \bigwedge M \arrow[d, dotted, "g"] \\ & E \end{tikzcd}$$

Proposition 1.4.3 (Functorial property of exterior algebras, [Bou, Chapter III, §7, Proposition 2])**.** *Let A be a commutative ring, M and N two A-modules and $u\colon M \to N$ an A-linear mapping. Then, there exists unique A-algebra homomorphism $u'\colon \bigwedge M \to \bigwedge N$ such that, the diagram*

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "\psi_M"'] & N \arrow[d, "\psi_N"] \\ \bigwedge M \arrow[r, dotted, "u'"'] & \bigwedge N \end{tikzcd}$$

is commutative. Moreover, u' is a graded A-algebra homomorphism and one has:

$$u'(x_1 \wedge \cdots \wedge x_n) = u(x_1) \wedge \cdots \wedge u(x_n), \qquad \text{for all } x_1, \ldots, x_n \in M.$$

The homomorphism u' in Proposition 1.4.3, will henceforth be denoted by $\bigwedge u$. If P is an A-module and $v\colon N \to P$ an A-linear mapping, then:

$$\bigwedge(v \circ u) = (\bigwedge v) \circ (\bigwedge u),$$

for $(\bigwedge v) \circ (\bigwedge u)$ is an algebra homomorphism rendering the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{v \circ u} & P \\ {\scriptstyle \psi_M}\downarrow & & \downarrow{\scriptstyle \psi_P} \\ \bigwedge M & \xrightarrow[(\bigwedge v)\circ(\bigwedge u)]{} & \bigwedge P \end{array}$$

The homomorphism $\bigwedge u$ is sometimes called the *canonical extension* of u to $\bigwedge M$ (which contains $M = \bigwedge^1(M)$). The map $\bigwedge^i M \to \bigwedge^i N$ induced by $\bigwedge u$, is denoted by $\bigwedge^i u$ and is called the *n-th exterior power of u*. Note that, $\bigwedge^i u$ is obtained from $u^{\otimes i}\colon M^{\otimes i} \to N^{\otimes i}$ by passing to the quotients.

Proposition 1.4.4 ([Bou, Chapter III, §7, Proposition 3])**.** *If $u\colon M \to N$ is a surjective A-linear mapping, the homomorphism $\bigwedge u\colon \bigwedge M \to \bigwedge N$ is surjective and its kernel is the two-sided ideal of $\bigwedge M$ generated by the kernel $P \subset M \subset \bigwedge M$ of u.*

An important property of the exterior algebra is that it commutes with base extensions.

Proposition 1.4.5 ([Bou, Chapter III, §7, Proposition 5])**.** *Let A, B be two commutative rings, $\rho\colon A \to B$ a ring homomorphism and M an A-module. The canonical extension,*

$$\textstyle\bigwedge_B(B \otimes_A M) \longrightarrow B \otimes_A (\bigwedge_A M)$$

of the B-linear mapping $1_B \otimes \psi_M\colon B \otimes_A M \to B \otimes_A (\bigwedge_A M)$, is a graded B-algebra isomorphism.

Let M_1, M_2 be A-modules. On $(\bigwedge M_1) \otimes (\bigwedge M_2)$, one defines a multiplication by setting:

$$(x \otimes y)(x' \otimes y') = (-1)^{(\deg y)(\deg x')}(x \wedge x') \otimes (y \wedge y'),$$

for all homogeneous elements $x, x' \in M_1$ and $y, y' \in M_2$. It is straightforward to verify that, $(\bigwedge M_1) \otimes (\bigwedge M_2)$ is an A-algebra under this multiplication. Its degree one component, is $(M_1 \otimes A) \oplus (A \otimes M_2) \cong M_1 \oplus M_2$. By the universal property of the exterior algebra (Proposition 1.4.2), the natural map $M_1 \oplus M_2 \longrightarrow (\bigwedge M_1) \otimes (\bigwedge M_2)$, extends to an A-algebra homomorphism:

$$\Phi\colon \bigwedge(M_1 \oplus M_2) \longrightarrow (\bigwedge M_1) \otimes (\bigwedge M_2).$$

One gets an inverse $\Psi\colon (\bigwedge M_1) \otimes (\bigwedge M_2) \longrightarrow \bigwedge(M_1 \oplus M_2)$ to Φ, by setting:

$$\Psi(x \otimes y) = \Psi_1(x) \wedge \Psi_2(y),$$

where, $\Psi_i\colon \bigwedge M_i \longrightarrow \bigwedge(M_1 \oplus M_2)$ is the extension of the natural embedding $M_i \to M_1 \oplus M_2$. The compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identities on $(\bigwedge M_1) \otimes (\bigwedge M_2)$ and $\bigwedge(M_1 \oplus M_2)$. Therefore, we have an isomorphism:

$$(\bigwedge M_1) \otimes (\bigwedge M_2) \cong \bigwedge(M_1 \oplus M_2),$$

of graded A-algebras.

1.5 The Koszul Complex

Let A be any commutative ring (with unit) and $\mathbf{f} = f_1, \ldots, f_m$ a sequence of elements of A. The *Koszul complex* $K_{\bullet}(\mathbf{f}; A)$ attached to the sequence $\mathbf{f}$, is defined as follows:

Let F be a free A-module with basis $e_1, \ldots, e_m$. We let $K_j(\mathbf{f}; A)$ be the j-th exterior power of F, that is, $K_j(\mathbf{f}; A) = \bigwedge^j F$. A basis of the free A-module $K_j(\mathbf{f}; A)$ is given by the wedge products $e_F = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}$, where $F = \{i_1 < i_2 < \cdots < i_j\}$. In particular, it follows that:

$$\operatorname{rank} K_j(\mathbf{f}; A) = \binom{m}{j}.$$

We define the differential $\partial_j\colon K_j(\mathbf{f}; A) \to K_{j-1}(\mathbf{f}; A)$, by the formula:

$$\partial_j\left(e_{i_1} \wedge \cdots \wedge e_{i_j}\right) = \sum_{k=1}^{j} (-1)^{k+1} f_{i_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_j}.$$

One readily verifies that, $\partial_{j-1} \circ \partial_j = 0$. So that,

$$K_{\bullet}(\mathbf{f}; A) : \cdots \to K_j(\mathbf{f}; A) \xrightarrow{\partial_j} K_{j-1}(\mathbf{f}; A) \xrightarrow{\partial_{j-1}} \cdots \to K_1(\mathbf{f}; A) \xrightarrow{\partial_1} K_0(\mathbf{f}; A) \to 0$$

is indeed a complex.

Now, let M be an A-module. We define the complexes:

$$K_{\bullet}(\mathbf{f}; M) = K_{\bullet}(\mathbf{f}; A) \otimes_A M, \qquad K^{\bullet}(\mathbf{f}; M) = \operatorname{Hom}_A(K_{\bullet}(\mathbf{f}; A), M).$$

Also, we define:

$$\begin{aligned} H_i(\mathbf{f}; M) = H_i\left(K_{\bullet}(\mathbf{f}; M)\right) &= H_i\left(K_{\bullet}(\mathbf{f}; A) \otimes_A M\right) \\ &= \frac{\operatorname{Ker}\left(\partial_i \otimes \mathrm{id}_M\right)}{\operatorname{Im}\left(\partial_{i+1} \otimes \mathrm{id}_M\right)}, \end{aligned}$$

$$H^i(\mathbf{f}; M) = H^i\left(K^{\cdot}(\mathbf{f}; M)\right) = H^i\left(\mathrm{Hom}_A(K_{\cdot}(\mathbf{f}; A), M)\right)$$

$$= \frac{\mathrm{Ker}\ (\mathrm{Hom}(\partial_i, \mathrm{id}_M))}{\mathrm{Im}\ (\mathrm{Hom}(\partial_{i-1}, \mathrm{id}_M))}.$$

We call $H_i(\mathbf{f}; M)$ the *i-th Koszul homology* module of $\mathbf{f}$ with respect to M and $H^i(\mathbf{f}; M)$ the *i-th Koszul cohomology* module of $\mathbf{f}$ with respect to M.

Let $I \subset A$ be the ideal generated by $\mathbf{f} = f_1, \ldots, f_m$. Then,

$$H_0(\mathbf{f}; M) = M/IM \quad \text{and} \quad H_m(\mathbf{f}; M) \cong 0\colon {}_M I = \{x \in M\colon \quad Ix = 0\}.$$

The Koszul complex $K_{\cdot}(\mathbf{f}; A)$ is a graded A-algebra, namely the exterior algebra of F, with the multiplication of wedge product. The collection of the maps ∂_i defines a graded A-homomorphism,

$$\partial\colon \bigwedge F \to \bigwedge F.$$

We have the following rules (c.f. [BH, Section 1.6]):

(i) $a \wedge b = (-1)^{(\deg a)(\deg b)} b \wedge a$, for homogeneous elements $a, b \in \bigwedge F$.

(ii) $\partial(a \wedge b) = \partial(a) \wedge b + (-1)^{\deg a} a \wedge \partial(b)$, for $a, b \in \bigwedge F$ and a homogeneous.

We denote by $Z_{\cdot}(\mathbf{f}; A)$ the *cycles* of the Koszul complex and by $B_{\cdot}(\mathbf{f}; A)$ its *boundaries*. Rule (ii) has an interesting consequence.

Proposition 1.5.1 ([HH, Theorem A.3.1])**.** *The A-module $Z_{\cdot}(\mathbf{f}; A)$ is a graded subalgebra of $K_{\cdot}(\mathbf{f}; A)$ and $B_{\cdot}(\mathbf{f}; A) \subset Z_{\cdot}(\mathbf{f}; A)$ is a graded two-sided ideal in $Z_{\cdot}(\mathbf{f}; A)$. In particular, $H_{\cdot}(\mathbf{f}; A) = Z_{\cdot}(\mathbf{f}; A)/B_{\cdot}(\mathbf{f}; A)$ has a natural structure as graded $H_{\cdot}(\mathbf{f}; A)$-algebra. Moreover, if I is the ideal generated by the sequence $\mathbf{f}$, then*

$$IH_{\cdot}(\mathbf{f}; A) = 0.$$

For computing the Koszul homology there are two fundamental long exact sequences.

Theorem 1.5.2 ([HH, Theorem A.3.3])**.** *Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements in A, and denote by $\mathbf{g}$ the sequence $f_1, \ldots, f_{m-1}$. Let $0 \to U \to M \to N \to 0$ be a short exact sequence of A-modules. Then, we obtain the following long exact sequences:*

$$\begin{aligned} 0 \to H_m(\mathbf{f}; U) \to H_m(\mathbf{f}; M) \to H_m(\mathbf{f}; N) \to H_{m-1}(\mathbf{f}; U) \to \cdots \\ \cdots \to H_{i+1}(\mathbf{f}; N) \to H_i(\mathbf{f}; U) \to H_i(\mathbf{f}; M) \to H_i(\mathbf{f}; N) \to \cdots \\ \cdots \to H_1(\mathbf{f}; N) \to H_0(\mathbf{f}; U) \to H_0(\mathbf{f}; M) \to H_0(\mathbf{f}; N) \to 0. \end{aligned}$$

and

$$\begin{aligned} 0 \to H_m(\mathbf{f}; M) \to H_{m-1}(\mathbf{g}; M) \to H_{m-1}(\mathbf{g}; M) \to H_{m-1}(\mathbf{f}; M) \to \cdots \\ \cdots \to H_{i+1}(\mathbf{f}; M) \to H_i(\mathbf{g}; M) \to H_i(\mathbf{g}; M) \to H_i(\mathbf{f}; M) \to \cdots \\ \cdots \to H_1(\mathbf{f}; M) \to H_0(\mathbf{g}; M) \to H_0(\mathbf{g}; M) \to H_0(\mathbf{f}; M) \to 0. \end{aligned}$$

where for all i, the map $H_i(\mathbf{g}; M) \to H_i(\mathbf{g}; M)$ is multiplication by $\pm f_m$.

1.6 Depth

Let M be a module over a Noetherian local ring $(R, \mathfrak{m})$. We say that $x \in R$ is an *M-regular element*, if $xz = 0$ for $z \in M$, implies that $z = 0$. In other words, x is not a zero-divisor on M. Regular sequences are composed of successively regular elements:

Definition 1.6.1 (Regular sequences)**.** A sequence $\mathbf{x} = x_1, \ldots, x_m$ of elements of R is called an *M-regular sequence* or simply an *M-sequence*, if the following conditions are satisfied:

(a) x_i is an $\frac{M}{(x_1,\ldots,x_{i-1})M}$-regular element, for $i = 1, \ldots, m$ and

(b) $M \neq \mathbf{x}M$.

In this situation we shall sometimes say that M is an $\mathbf{x}$-*regular* module. A *regular sequence* is an R-sequence. A *weak M-sequence* is only required to satisfy condition (a).

The *depth* of M, denoted $\operatorname{depth} M$, is the common length of a maximal M-sequence contained in $\mathfrak{m}$ (consisting of homogeneous elements, if M is graded). In homological terms, the depth of M is given by:

$$\operatorname{depth} M = \min\{i \colon \quad \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

Definition 1.6.2 (Cohen-Macaulay module)**.** A finitely generated R-module $M \neq 0$ is called *Cohen-Macaulay module*, if $\operatorname{depth}(M) = \dim(M)$. If R itself is a Cohen-Macaulay module, then it is called *Cohen-Macaulay ring.*

Lemma 1.6.3 ([BH, Proposition 1.2.9])**.** *Let R be a Noetherian local ring, $I \subset R$ an ideal and $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ an exact sequence of finite R-modules. Then,*

(i) $\operatorname{depth} M \geq \min\{\operatorname{depth} U,\ \operatorname{depth} N\}$.

(ii) $\operatorname{depth} U \geq \min \{\operatorname{depth} M,\ \operatorname{depth} N + 1\}$.

(iii) $\operatorname{depth} N \geq \min \{\operatorname{depth} U - 1,\ \operatorname{depth} M\}$.

Proposition 1.6.4 ([HH, Proposition A.4.1]). *Let $\mathbf{f} = f_1, \ldots, f_m$ be an M-sequence contained in $\mathfrak{m}$ (consisting of homogeneous elements if M is graded). Then,*

$$\operatorname{depth} M/(\mathbf{f})M = \operatorname{depth} M - m.$$

Theorem 1.6.5 ([HH, Theorem A.3.4]). *Let $\mathbf{f} = f_1, \ldots, f_m$ be a sequence of elements in A and M an A-module.*

(i) *If $\mathbf{f}$ is an M-sequence, then $H_i(\mathbf{f}; M) = 0$, for all $i > 0$.*

(ii) *Suppose in addition that, M is a finitely generated A-module and that A is either* (i) *a Noetherian local ring with maximal ideal $\mathfrak{m}$, or* (ii) *a graded K-algebra with graded maximal ideal $\mathfrak{m}$, and that $(\mathbf{f}) \subset \mathfrak{m}$. In case* (ii) *we also assume that $\mathbf{f}$ is a sequence of homogeneous elements. Then, we have: if $H_1(\mathbf{f}; M) = 0$, then the sequence $\mathbf{f}$ is an M-sequence.*

Theorem 1.6.5 has the following important consequence.

Corollary 1.6.6. *Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables and M be a finitely generated graded S-module. Moreover, let $\mathbf{f} = f_1, \ldots, f_m$ be a homogeneous S-sequence. Then for each i, there exists an isomorphism of graded $S/(\mathbf{f})$-modules*

$$\operatorname{Tor}_i^S (S/(\mathbf{f}), M) \cong H_i(\mathbf{f}; M).$$

Proposition 1.6.7 ([HH, Proposition A.4.2]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. Let $\mathbf{x} = x_1, \ldots, x_n$ be a minimal system of generators of $\mathfrak{m}$. Then,*

$$\operatorname{depth} M = n - \max\{i : \quad H_i(\mathbf{x}; M) = 0\}.$$

Theorem 1.6.8 ([BH, Proposition 1.2.16]). *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings. Suppose M is a finite R-module, and N is a finite S-module which is flat over R. Then,*

$$\operatorname{depth}_S(M \otimes_R N) = \operatorname{depth}_R M + \operatorname{depth}_S N/\mathfrak{m}N.$$

The following theorem, the 'Auslander-Buchsbaum formula' is not only of theoretical importance, but also an effective instrument for the computation of the depth of a module.

Theorem 1.6.9 (Auslander-Buchsbaum [BH, Theorem 1.3.3])**.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring, and* $M \neq 0$ *a finite* R*-module. If* $\operatorname{projdim} M < \infty$*, then:*

$$\operatorname{projdim} M + \operatorname{depth} M = \operatorname{depth} R.$$

1.7 Local Cohomology

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. We set,

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M : \quad \mathfrak{m}^t x = 0, \text{ for some } t \in \mathbb{N}\}.$$

$\Gamma_{\mathfrak{m}}(M)$ is the largest submodule of M with support $\{\mathfrak{m}\}$. It is easy to check that, $\Gamma_{\mathfrak{m}}(-)$ is a left exact additive functor which is isomorphic to $\varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^k, -)$. The right derived functors $H^i_{\mathfrak{m}}(-)$ of $\Gamma_{\mathfrak{m}}(-)$ are called the local cohomology functors. Thus, if $\mathbb{I}$ is an injective resolution of M, it follows that:

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong H^i\left(\varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^k, \mathbb{I})\right) \cong \varinjlim H^i(\operatorname{Hom}_R(R/\mathfrak{m}^k, \mathbb{I})) \\ &\cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{m}^k, M). \end{aligned}$$

We quote the following fundamental vanishing theorem of Grothendieck.

Theorem 1.7.1 (Grothendieck, [BSh, Theorem 6.1.2])**.** *Let* $t = \operatorname{depth}(M)$ *and* $d = \dim(M)$*. Then,* $H^i_{\mathfrak{m}}(M) \neq 0$ *for* $i = t$ *and* $i = d$*, and* $H^i_{\mathfrak{m}}(M) = 0$ *for* $i < t$ *and* $i > d$*.*

By virtue of this theorem, we have similar definition for depth and dimension of an R-module M as following:

$$\begin{aligned} \dim M &= \max\{i : \quad H^i_{\mathfrak{m}}(M) \neq 0\}, \\ \operatorname{depth} M &= \min\{i : \quad H^i_{\mathfrak{m}}(M) \neq 0\}. \end{aligned} \tag{1.1}$$

Corollary 1.7.2. *M is Cohen-Macaulay if and only if* $H^i_{\mathfrak{m}}(M) = 0$*, for all* $i < \dim M$*.*

Chapter 2

Linear Resolution

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let M be a finitely generated graded S-module; in the most interesting case, M is an ideal of S. For a given natural number p, there is a great interest in the question:

> *Is it possible to generate M by (homogeneous) elements of degree $< p$?*

No simple answer, say in terms of the local cohomology of M, is known; but somewhat surprisingly the stronger question:

> *Can the j-th syzygy of M be generated by elements of degree $< p + j$, for all $j = 0, l, \ldots, n$?*

Castelnuovo-Mumford regularity was first defined by D. Mumford [M], who attributes the idea to G. Castelnuovo, for coherent sheaves on projective spaces. In a more algebraic setting, it was defined by D. Eisenbud and S. Goto [EG]. It comes out that, Castelnuovo-Mumford regularity gives an upper bound for the maximal degrees of the syzygies in a minimal free resolution. D. Bayer and M. Stilman [BS2], showed that an estimate of the regularity of an ideal, gives a bound of algorithms for computing syzygies.

In this chapter, we recall the definition of Castelnuovo-Mumford regularity and the notion of linear resolution. Then, basic properties of regularity and some known classes of ideals with linear resolution is given. The aim of this thesis, is to study ideals with linear resolution.

2.1 Castelnuovo-Mumford Regularity

Let K be a field and let S be a polynomial ring over K. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded S-module. For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t_i^S(M) = \max\{j : \quad \beta_{i,j}^K(M) \neq 0\}$$

where, $\beta_{i,j}^K(M)$ is the i, j-th graded Betti number of M as an S-module, i.e.

$$\beta_{i,j}^K = \dim_K \operatorname{Tor}_i^S (K, M)_j$$

and $t_i^S(M) = -\infty$, if it happens that $\operatorname{Tor}_i^S (K, M) = 0$.

The *Castelnuovo-Mumford regularity* of M, $\operatorname{reg}(M)$, is given by:

$$\operatorname{reg}(M) = \sup\{t_i^S(M) - i : \quad i \in \mathbb{Z}\}.$$

The *initial degree*, $\operatorname{indeg}(M)$, of a non-zero finitely generated graded S-module $M = \oplus_{i\in\mathbb{Z}} M_i$, is given by:

$$\operatorname{indeg}(M) = \inf\{i : \quad M_i \neq 0\}.$$

A finitely generated graded S-module M has a *d-linear resolution*, if its regularity is equal to $d = \operatorname{indeg}(M)$.

Lemma 2.1.1. *Let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a non-zero graded S-module and $m_1, \dots, m_r$ be a homogeneous minimal system of generator of M. Then,*

(i) $\operatorname{indeg}(M) = \min\{\deg(m_i) : \quad i = 1, \dots, r\}$.

(ii) $\operatorname{reg}(M) \geq \deg(m_i)$, *for all* $i = 1, \dots, r$.

(iii) $\operatorname{reg}(M) \geq \operatorname{indeg}(M)$.

Proof. (i) For $i = 1, \dots, r$ one has, $0 \neq m_i \in M_{\deg(m_i)}$. So that,

$$\operatorname{indeg}(M) = \inf\{i : \quad M_i \neq 0\} \leq \deg(m_i).$$

Hence, $\operatorname{indeg}(M) \leq \min\{\deg(m_i) : \quad i = 1, \dots, r\}$. Now, assume that $s = \operatorname{indeg}(M) < \min\{\deg(m_i) : \quad i = 1, \dots, r\}$. Then, there exists a non-zero homogeneous element $m \in M_s$. Since $m_1, \dots, m_r$ is a generator of M, one has:

$$m = \left(\sum_{i=0}^{n_1} f_{i,1}\right) m_1 + \cdots + \left(\sum_{i=0}^{n_r} f_{i,r}\right) m_r$$

where, $f_{i,j} \in S_i$. The degree of each element in the above sum is greater than s and this contradicts to the fact that $0 \neq m \in M_s$.

(ii) Let $\deg(m_j) = d_j$. Then, $\beta^K_{0,d_j}(M) \neq 0$. In particular,

$$\mathrm{reg}\,(M) = \max\{j - i\colon \quad \beta^K_{i,j}(M) \neq 0\} \geq d_j = \deg(m_j).$$

(iii) This is a consequence of (ii). □

Theorem 2.1.2. *Let $M \neq 0$ be a finitely generated graded S-module. The followings are equivalent:*

(i) $a = \mathrm{reg}\,(M)$.

(ii) $a = \max\{t\colon \quad \beta^K_{i,i+t}(M) \neq 0,$ *for some* $i \geq 0\}$.

(iii) $a = \max\{t\colon \quad \mathrm{Tor}^S_i(K, M)_{t+i} \neq 0,$ *for some* $i \geq 0\}$.

(iv) $a = \max\{t\colon \quad \mathrm{Ext}^i_S(K, M)_{-t-i} \neq 0,$ *for some* $i \geq 0\}$.

(v) $a = \max\{t\colon \quad H^i_{\mathfrak{m}}(M)_{t-i} \neq 0,$ *for some* $i \geq 0\}$.

(vi) $a = \max\{a_i(M) + i\colon \quad i \in \mathbb{N}\}$ *where* $a_i(M) = \max\{t\colon \quad H^i_{\mathfrak{m}}(M)_t \neq 0\}$.

Sketch of proof . The implications (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) are by definition. For the implication (iii) $\leftrightarrow$ (iv), see Proposition 20.16 in [E]. Also, (iv) $\leftrightarrow$ (v) $\leftrightarrow$ (vi) come from local duality theorem [BSh, Example 13.4.6]. □

Theorem 2.1.3. *Let $M \neq 0$ be a finitely generated graded S-module. The followings are equivalent:*

(i) *M has a d-linear resolution.*

(ii) $\beta^K_{i,i+j}(M) = 0$, *for all* $j \neq d$.

(iii) *The graded minimal free resolution of M is of the form,*

$$0 \to S^{\beta^K_\rho}(-d-\rho) \to \cdots \to S^{\beta^K_1}(-d-1) \to S^{\beta^K_0}(-d) \to M \to 0.$$

(iv) $\mathrm{reg}\,(M) = d$ *and M is minimally generated by homogeneous elements all of degree d.*

2.1.1 Basic Properties of Regularity

Below, we summarize some basic properties of Castelnuovo-Mumford regularity. In all of the following theorems, S is polynomial ring over a field K and all the modules are finitely generated graded S-modules. For a graded module $N = \oplus_{i\in\mathbb{N}} N_i$ of finite length, we set:

$$s(N) = \max\{s: \quad N_s \neq 0\}.$$

Lemma 2.1.4 ([E, Corollary 20.19]). *If $0 \to N \to M \to P \to 0$ is a short exact sequence, then:*

(i) $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(N), \ \operatorname{reg}(P)\}$.

(ii) $\operatorname{reg}(N) \leq \max\{\operatorname{reg}(M), \ \operatorname{reg}(P) + 1\}$.

(iii) $\operatorname{reg}(P) \leq \max\{\operatorname{reg}(M), \ \operatorname{reg}(N) - 1\}$.

(iv) *If N is of finite length, then $\operatorname{reg}(N) = s(N)$ and,*

$$\operatorname{reg}(M) = \max\{\operatorname{reg}(P), \ s(N)\}.$$

Lemma 2.1.5. *Let M be a graded S-module. Then,*

(i) $\operatorname{reg}(M(-d)) = \operatorname{reg}(M) + d$.

(ii) *If $y \in S_d$ is an M-regular element, then* $\operatorname{reg}\left(\frac{M}{yM}\right) = \operatorname{reg}(M) + d - 1$.

(iii) *If $\mathbf{y} = y_1, \ldots, y_r$ is an M-sequence with $y_i \in S_{d_i}$, then:*

$$\operatorname{reg}\left(\frac{M}{\mathbf{y}M}\right) = \operatorname{reg}(M) + (d_1 + \cdots + d_r) - r.$$

Proof. (i) One has, $\beta^K_{i,j}(M) = \beta^K_{i,j+d}(M(-d))$. So that,

$$\operatorname{reg}(M(-d)) = \operatorname{reg}(M) + d$$

by definition.

(ii) From short exact sequence, $0 \to M(-d) \xrightarrow{y} M \to M/yM \to 0$ and Lemma 2.1.4(iii), we have:

$$\begin{aligned} \operatorname{reg}\,(M/yM) &\leq \max\{\operatorname{reg}(M(-d)) - 1, \ \operatorname{reg}(M)\} \\ &= \max\{\operatorname{reg}(M) + d - 1, \ \operatorname{reg}(M)\} \\ &= \operatorname{reg}(M) + d - 1. \end{aligned} \tag{2.1}$$

Using (i) and Lemma 2.1.4(ii), we have:

$$\operatorname{reg}(M) + d - 1 = \operatorname{reg}(M(-d)) - 1 \leq \max\{\operatorname{reg}(M) - 1,\ \operatorname{reg}(M/yM)\}. \tag{2.2}$$

This implies that,

$$\max\{\operatorname{reg}(M) - 1,\ \operatorname{reg}(M/yM)\} = \operatorname{reg}(M/yM).$$

So that, $\operatorname{reg}(M) + d - 1 \leq \operatorname{reg}(M/yM)$, by (2.2). In addition to (2.1), we get the equality.

(iii) We use induction on r. For $r = 1$, we get the conclusion from (ii). Let $r > 1$ and the assertion is true for $r - 1$. For an S-sequence $y_1, \ldots, y_r$ of length r, let $M' = M/(y_1, \ldots, y_{r-1})M$. Then, $M/\mathbf{y}M \cong M'/y_rM'$ and by induction hypothesis,

$$\operatorname{reg}(M') = \operatorname{reg}(M) + (d_1 + \cdots + d_{r-1}) + (r - 1). \tag{2.3}$$

Using (ii) and (2.3), we have:

$$\begin{aligned} \operatorname{reg}\left(\frac{M}{\mathbf{y}M}\right) = \operatorname{reg}\left(\frac{M'}{y_rM'}\right) &= \operatorname{reg}(M') + d_r - 1 \\ &= \operatorname{reg}(M) + d_1 + \cdots + d_r - r. \end{aligned}$$

□

Theorem 2.1.6. *Let S be polynomial ring over a field K with standard grading and $\mathfrak{m}$ be irredundant maximal ideal of S. Then,*

(i) $\operatorname{reg}(S) = 0$.

(ii) *If $y \in S_d$, then* $\operatorname{reg}((y)) = d$.

(iii) *If I is a homogeneous ideal of S, then* $\operatorname{reg}(S/I) = \operatorname{reg}(I) - 1$.

(iv) *If $y_1, \ldots, y_r$ is an S-sequence with* $\deg(y_i) = d_i$ *and* $I = (y_1, \ldots, y_r)$, *then:*

$$\operatorname{reg}(I) = d_1 + \cdots + d_r - r + 1.$$

(v) $\operatorname{reg}(\mathfrak{m}) = 1$.

(vi) $\operatorname{reg}(K) = 0$.

(vii) *If $\prec$ is a monomial ordering and I is a homogeneous ideal of S, then:*

$$\mathrm{reg}\,(I) \leq \mathrm{reg}\,(\mathrm{in}_{\prec}(I))\,,$$

where, $\mathrm{in}_{\prec}(I)$ *is the initial degree of I with respect to monomial ordering* $\prec$*.*

(viii) *Let $\prec$ is a monomial ordering and I is a homogeneous ideal of S and $J = \mathrm{in}_{\prec}(I)$. If J has a linear resolution, then I has again a linear resolution.*

(ix) *If $M = N \oplus P$, is a finitely generated graded S-module, then:*

$$\mathrm{reg}\,(M) = \max\{\mathrm{reg}\,(N),\ \mathrm{reg}\,(P)\}.$$

(x) *If M is a finitely generated graded S-module, then:*

$$\mathrm{reg}\,(\mathfrak{m}M) \leq 1 + \mathrm{reg}\,(M).$$

(xi) *If M is a finitely generated graded S-module with d-linear resolution, then $\mathfrak{m}M$ has a $(d+1)$-linear resolution.*

Proof. (i) Clearly, $0 \to S \to S \to 0$ is a graded minimal free resolution of S. So that $\mathrm{reg}\,(S) = 0$, by definition.

(ii) For $y \in S_d$, the sequence

$$0 \to S(-d) \xrightarrow{y} (y) \to 0$$

is a graded minimal free resolution of (y). So that $\mathrm{reg}\,((y)) = d$, by definition.

(iii) If

$$0 \to F_\rho \to \cdots \to F_i = \oplus_j S^{\beta^K_{i,j}}(-j) \to \cdots \to F_0 \to I \to 0$$

is a graded minimal free resolution of I, then

$$0 \to F_\rho \to \cdots \to F_i = \oplus_j S^{\beta^K_{i,j}}(-j) \to \cdots \to F_0 \to S \to \frac{S}{I} \to 0$$

is a graded minimal free resolution of S/I. In particular, $\beta^K_{i,j}(I) = \beta^K_{i+1,j}(S/I)$. Hence,

$$\mathrm{reg}\,(I) = \max\{j - i : \quad \beta^K_{i,j}(I) \neq 0\} = \mathrm{reg}\,(S/I) + 1.$$

(iv) This a direct consequence of (iii) and Lemma 2.1.5(iii).

(v) The ideal $\mathfrak{m}$ is a graded ideal generated by regular sequence $x_1, \ldots, x_n$. Using (iv), we have $\operatorname{reg}(\mathfrak{m}) = 1$.

(vi) We have, $K \cong S/\mathfrak{m}$. So that, (iii) and (v) yield the conclusion.

(vii) By [HH, Corollary 3.3.3], we have $\beta_{i,j}^K(I) \leq \beta_{i,j}^K(\mathrm{in}_{\prec}(I))$. Hence,

$$\operatorname{reg}(I) = \max\{j - i : \quad \beta_{i,j}^K(I) \neq 0\} \leq \operatorname{reg}(\mathrm{in}_{\prec}(I)).$$

(viii) One has, $\operatorname{indeg}(I) = \operatorname{indeg}(\mathrm{in}_{\prec}(I))$.
Hence if $\operatorname{reg}(\mathrm{in}_{\prec}(I)) = \operatorname{indeg}(\mathrm{in}_{\prec}(I))$, then by (vii) we have, $\operatorname{reg}(I) \leq \operatorname{indeg}(I)$. On the other hand, $\operatorname{reg}(I) \geq \operatorname{indeg}(I)$, by Lemma 2.1.1(iii). That is, I has a linear resolution.

(ix) We have the graded isomorphism:

$$\operatorname{Tor}_i^S(K, M)_j \cong \operatorname{Tor}_i^S(K, N)_j \oplus \operatorname{Tor}_i^S(K, P)_j.$$

Hence, (ix) follows from the formula,

$$\operatorname{reg}(M) = \max\{t : \quad \operatorname{Tor}_i^S(K, M)_{t+i} \neq 0, \text{ for some } i \geq 0\}.$$

(x) From Lemma 2.1.5(iii), we get:

$$\operatorname{reg}(M/\mathfrak{m}M) = \operatorname{reg}(M).$$

So that, by Lemma 2.1.4(ii), we have:

$$\begin{aligned}\operatorname{reg}(\mathfrak{m}M) &\leq \max\{\operatorname{reg}(M),\ \operatorname{reg}(M/\mathfrak{m}M) + 1\} \\ &= \max\{\operatorname{reg}(M),\ \operatorname{reg}(M) + 1\} \\ &= \operatorname{reg}(M) + 1.\end{aligned}$$

(xi) First note that, $\operatorname{indeg}(\mathfrak{m}M) = 1 + \operatorname{indeg}(M)$. Now, assume that M has a d-linear resolution. Then, $d = \operatorname{indeg}(M) = \operatorname{reg}(M)$ and by Lemma 2.1.1(iii), $\operatorname{reg}(\mathfrak{m}M) \geq \operatorname{indeg}(\mathfrak{m}M) = 1 + d$. On the other hand, $\operatorname{reg}(\mathfrak{m}M) \leq 1 + d$, by (x). So that, $\operatorname{reg}(\mathfrak{m}M) = 1 + d = \operatorname{indeg}(\mathfrak{m}M)$. This means that, $\mathfrak{m}M$ has a $(d + 1)$-linear resolution. □

Below, we summarize some results on the regularity of Hom and product of modules which has been appeared in several papers. For more details on these results, one can refer to [Ca, CoH, H].

Theorem 2.1.7 ([H, Corollaries 3.2 and 3.3]). *Let I and J be monomial ideals. Then,*

(i) $\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$;

(ii) $\operatorname{reg}(I + J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) - 1$.

Theorem 2.1.8 ([Ca, Theorem 3.10]). *Let M be a finitely generated graded S-module with $\operatorname{indeg}(M) = d$ and let N be a finitely generated graded S-module such that $\operatorname{Ext}^i_S(M, N)$ is Cohen-Macaulay, for all $i > 0$. Then,*

$$\operatorname{reg}\ (\operatorname{Hom}_R(M, N)) \leq \operatorname{reg}(N) - d.$$

Theorem 2.1.9 ([Ca, Theorem 3.5]). *Let I be a homogeneous ideal and M a finitely generated graded S-module such that, $\dim \operatorname{Tor}^S_1(M, S/I) \leq 1$. Then,*

$$\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M).$$

Theorem 2.1.10 ([CoH, Theorem 2.5]). *Let I be a graded ideal of S with $\dim(S/I) \leq 1$. Then, for any finitely generated graded S-module M we have:*

$$\operatorname{reg}(IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I).$$

Definition 2.1.11 (Almost regular sequence). Let M be a finitely generated graded S-module. A homogeneous element $y \in S$ of degree d is called *almost regular* on M, if the multiplication map, $M_{i-d} \xrightarrow{y} M_i$ is injective for all large enough i.

One can easily check that, y is almost regular on M if and only if, y is a non-zerodivisor on $M/\Gamma_{\mathfrak{m}}(M)$. A sequence $y_1, \ldots, y_m$ of homogeneous elements of S is called an *almost regular M-sequence* if y_i is almost regular on $M/(y_1, \ldots, y_{i-1})M$, for $i = 1, \ldots, n$.

Theorem 2.1.12 ([CoH, Theorem 2.2]). *Let M be a finitely generated graded S-module and let I be an ideal of S generated by an almost regular M-sequence $y_1, \ldots, y_m$ with $\deg(y_i) = d_i$. Then,*

$$\operatorname{reg}(IM) \leq \operatorname{reg}(M) + d_1 + d_2 + \cdots + d_m - m + 1.$$

Corollary 2.1.13 ([CoH, Corollary 2.3]). *Let M be a finitely generated graded S-module and let I be a homogeneous ideal of S generated by a regular M-sequence. Then,*

$$\operatorname{reg}(IM) \leq \operatorname{reg}(I) + \operatorname{reg}(M).$$

Remark 2.1.14. Let M be a finitely generated graded S-module, $\mathbf{y} = y_1, \ldots, y_m$ a homogeneous M-sequence with $\deg(y_i) = d_i$ and I be the ideal generated by $y_1, \ldots, y_m$. Using Theorem 2.1.6(iv), we have:

$$\operatorname{reg}(I) = d_1 + \cdots + d_m - m + 1. \tag{2.4}$$

Since every M-sequence is obviously almost regular M-sequence, one may use Theorem 2.1.12, in order to obtain Corollary 2.1.13. However, we may directly prove Corollary 2.1.13, using our previous results. In fact, by Lemma 2.1.4(ii), 2.1.5(iii) and (2.4), we have:

$$\begin{aligned}\operatorname{reg}(IM) &\leq \max\{\operatorname{reg}(M),\ \operatorname{reg}(M/IM)\}\\ &= \max\{\operatorname{reg}(M),\ \operatorname{reg}(M) + (d_1 + \cdots + d_m) - m + 1\}\\ &= \operatorname{reg}(M) + (d_1 + \cdots + d_m) - m + 1\\ &= \operatorname{reg}(M) + \operatorname{reg}(I).\end{aligned}$$

2.2 Some Known Ideals with Linear Resolution

In this section, we will introduce some classes of ideals which have a linear resolution. These classes were found in several papers and one can find the details of the proofs in the the given references. These classes includes:

- Ideals with linear quotients (Polymatroidal ideals, Stable ideals),
- Product of ideals defined by Hankel matrices,
- Some special classes of lexsegment ideals,
- Square-free lexsegment ideals,
- Square-free stable ideals,
- Linearly joined ideals.

2.2.1 Ideals with Linear Quotients

Definition 2.2.1. A finitely generated graded R-module M has *linear quotients*, if M admits a minimal system of generators $m_1, \dots, m_k$ such that for every $t = 1, \dots, k$ one has that $\langle m_1, \dots, m_{t-1}\rangle :_R m_t$ is an ideal of R generated by linear forms.

Theorem 2.2.2 ([Mo] see also [ShV, Corollary 2.7])**.** *Let I be a homogeneous ideal with linear quotients with respect to $f_1, \dots, f_m$ where $\{f_1, \dots, f_m\}$ is a minimal system of homogeneous generators for I. For $1 \leq p \leq m$, let n_p be the minimal number of homogeneous generators of $\langle f_1, \dots, f_{p-1}\rangle : f_p$. Then,*

(i) $\operatorname{reg}(I) = \max\{\deg(f_p) : \quad 1 \leq p \leq m\}$.

(ii) $\operatorname{projdim}(I) = \max\{n_p : \quad 1 \leq p \leq m\}$.

(iii) $\beta_{i,i+j}(I) = \sum_{\substack{1 \leq p \leq m \\ \deg(f_p) = j}} \binom{n_p}{i}$.

(iv) $\beta_i(I) = \sum_{p=1}^{m} \binom{n_p}{i}$.

In particular, if all generators of I have the same degree, then I has a linear resolution.

In view of above theorem, it is natural to ask, "which classes of ideals have linear quotients?" Finding a class of ideals with linear quotients is not easy in general. However, two important classes with linear quotients are (weakly) polymatroidal ideals and stable ideals.

Polymatroidal ideals

For a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$, we denote by $G(I)$ the unique minimal set of monomial generators, and for a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$, we set $v_i(u) = a_i$, for $i = 1, \ldots, n$.

Definition 2.2.3. A monomial ideal $I \subseteq S$ is said to be *polymatroidal*, if all of its generators have the same degree and if it satisfies the following exchange property:

> For all $u, v \in G(I)$ and all i with $v_i(u) > v_i(v)$, there exists an integer j, with $v_j(v) > v_j(u)$ such that $x_j(u/x_i) \in G(I)$.

The name is explained by the fact that the elements of $G(I)$ correspond to the basis of a polymatroid, as defined in [We]. If I is a square-free ideal, then this set corresponds to the basis of a matroid. Hence, square-free polymatroidal ideals are also called *matroidal*.

Theorem 2.2.4 ([CoH, Theorem 5.2])**.** *A polymatroidal ideal I has linear quotients with respect to the reverse lexicographical order of the generators.*

Theorem 2.2.5 ([CoH, Theorem 5.3])**.** *Let I and J be polymatroidal monomial ideals. Then IJ is polymatroidal.*

Since ideals generated by subsets of the variables are obviously polymatroidal, Theorem 2.2.4 and 2.2.5 implies that,

Corollary 2.2.6. *Let $I_1, \dots, I_d$ be ideals generated by subsets of the variables. Then, $I = I_1 \cdots I_d$ has linear quotients.*

Definition 2.2.7. Let I and J be matroidal ideals. We let $I \star J$ be the ideal which is generated by all monomials uv with $u \in G(I)$ and $v \in G(J)$ such that, uv is square-free. We call $I \star J$, the *square-free product* of I and J.

Analogously to Theorem 2.2.5, we have:

Theorem 2.2.8 ([CoH, Theorem 5.5]). *Let I and J be matroidal monomial ideals. Then $I \star J$ is matroidal.*

Definition 2.2.9. A monomial ideal I is called *weakly polymatroidal*, if for every two monomials u and v in $G(I)$ such that $v_i(u) = v_i(v)(i = 1, \dots, t-1)$ and $v_t(u) > v_t(v)$ for some t, there exists $j > t$ such that $x_t(v/x_j) \in I$.

It is clear from the definition that a polymatroidal ideal is weakly polymatroidal. The converse is not true in general, as the following example shows.

Example 2.2.10. Let I be the polymatroidal ideal of $S = K[x_1, \dots, x_n]$ which is generated by all square-free monomials of degree 3. Let J denotes the monomial ideal of S generated by those monomials $u \in G(I)$ with $x_2x_4x_6 <_{\text{lex}} u$. Then, the monomial ideal J is weakly polymatroidal, but not polymatroidal.

Theorem 2.2.11 ([HH, Theorem 12.7.2]). *A weakly polymatroidal ideal I has linear quotients.*

Stable ideals

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring. We order the monomials lexicographically so that $x_1 > x_2 > \cdots > x_n$ and we denote by $\mathcal{M}_d$, the set of all monomials of S of degree d. For a monomial $u \in S$, we set:

$$m(u) = \max\{i : \quad x_i \text{ divides } u\}. \tag{2.5}$$

Definition 2.2.12. A subset $\mathcal{L} \subseteq \mathcal{M}_d$ is *strongly stable*, if one has:

$$x_i(u/x_j) \in \mathcal{L} \text{ for all } u \in \mathcal{L} \text{ and all } i < j \text{ such that } x_j \text{ divides } u.$$

A set $\mathcal{L} \subseteq \mathcal{M}_d$ is called *stable*, if $x_i(u/x_{m(u)}) \in \mathcal{L}$, for all $u \in \mathcal{L}$, and all $i < m(u)$. A monomial ideal I is called a *(strongly) stable* monomial ideal, if for each d, the monomials of degree d in I form a (strongly) stable set of monomials, respectively.

Lemma 2.2.13. *Any stable ideal I has linear quotients with respect to lexicographical ordering.*

Proof. Suppose I is a stable ideal with minimal generator $f_1, \ldots, f_m$. We order this generator such that:

(i) $\deg(f_1) \leq \cdots \leq \deg(f_m)$, and

(ii) If $\deg(f_i) = \deg(f_j)$, then $f_j <_{\text{lex}} f_i$.

Then, it is easy to see that $\langle f_1, \ldots, f_{i-1} \rangle : f_i = (x_1, \ldots, x_{m(f_i)-1})$, for all $i > 1$. $\square$

2.2.2 Products of Ideals Defined by Hankel Matrix

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring. Let X be a Hankel matrix with distinct entries $x_1, \ldots, x_n$. This means that, X is an $a \times b$ matrix (y_{ij}) with $y_{ij} = x_{i+j-1}$ and $a + b - 1 = n$. Let I_t be the ideal generated by the minors of size t of X. It is known that, it does not depend on the size of the matrix X (provided, of course, X contains t-minors); it depends only on t and n. For a given n, it follows that, t may vary from 1 to m, where $m = [(n+1)/2]$ is the integer part of $(n+1)/2$. Also, it is known that, the powers of I_2 have a linear resolution [Co]. Blum [Bl, Theorem 3.6] has recently shown that, if the Rees algebra of an ideal I, $\mathcal{R}(I) = \oplus_{n \geq 0} I^n t^n$, is Koszul, then all the powers of I have a linear resolution. As we know that $\mathcal{R}(I_t)$ is Koszul [Co], we have that I_t^k has a linear resolution for all t and k. There is also a stronger result:

Theorem 2.2.14 ([CoH, Theorem 6.2])**.** *Let X be a generic Hankel matrix. Let $t_1, \ldots, t_p$ be integers and I be the product of $I_{t_1} \cdots I_{t_p}$. Then, I has a linear resolution.*

For the proof of this theorem, we recall some definitions and results from [Co2] which have its own importance. Let τ be the lexicographic term order on the monomials of S and $>_1$ the partial order on $x_1, \ldots, x_n$ defined by $x_j >_1 x_i$ if and only if $j - i > 1$. A $>_1$*-chain of degree* k is a monomial $x_{i_1} \cdots x_{i_k}$ such that $x_{i_1} <_1 x_{i_2} <_1 \cdots <_1 x_{i_k}$. Denote by J the initial ideal of $I = I_{t_1} \cdots I_{t_p}$ and by J_k that of I_k with respect to τ. We know that:

$$J_k = (m\colon \quad m \text{ is a } >_1 -\text{chain of degree } k)$$

and that,

$$J = J_{t_1} \cdots J_{t_p}.$$

Since the regularity can only increase by passing to the initial ideal, to prove the theorem it suffices to know that:

Proposition 2.2.15 ([CoH, Theorem 6.2]). *The ideal J has linear quotients.*

2.2.3 Lexsegment Ideals

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring. We order the monomials lexicographically so that $x_1 > x_2 > \cdots > x_n$ and we denote by $\mathcal{M}_d$, the set of all monomials of S of degree d.

A *lexsegment* (of degree d), is a subset of $\mathcal{M}_d$ of the form:

$$L(u, v) = \{w \in \mathcal{M}_d\colon \quad u \geq_{\text{lex}} w \geq_{\text{lex}} v\}$$

for some $u, v \in \mathcal{M}_d$, with $u \geq_{\text{lex}} v$. An ideal generated by a lexsegment, is called a *lexsegment ideal.*

A lexsegment $\mathcal{L}$ is called *completely lexsegment*, if all the iterated shadows of $\mathcal{L}$ are again lexsegments. We recall that, the *shadow* of a set T of monomials is the set:

$$\text{Shad}(T) = \{vx_i\colon \quad v \in T,\ 1 \leq i \leq n\}$$

The i-th shadow is recursively defined as:

$$\text{Shad}^i(T) = \text{Shad}(\text{Shad}^{i-1}(T)).$$

Characterization of completely lexsegment ideals

Completely lexsegment ideals were characterized by De Negri and Herzog in [DH, 1998].

Theorem 2.2.16 ([DH, Theorem 2.3]). *Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d in S with $u \geq_{\text{lex}} v$ and $v \neq x_n^d$ and let I be the ideal generated by $L(u, v)$. Then, I is completely lexsegment if and only if $a_1 \neq 0$, and one of the following conditions holds:*

(i) *$u = x_1^a x_2^{d-a}$ and $v = x_1^a x_n^{d-a}$, for some a, $0 < a \leq d$;*

(ii) *$a_1 \neq b_1$ and for every $w <_{\text{lex}} v$, there exists an index $i > 1$ such that $x_i \mid w$ and $x_1 w / x_i \leq_{\text{lex}} u$.*

For $w, w' \in \mathcal{M}_d$, we set $w \succ w'$ if,

(a) $v_1(w) < v_1(w')$, or

(b) $v_1(w) = v_1(w')$ and $w >_{\text{lex}} w'$.

Then by [ANH, Theorem 1.3], [EOS, Theorem 1.2] and [EO, Theorem 2.10], we get the following equivalent statements,

Theorem 2.2.17. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d in S with $u \geq_{\text{lex}} v$ and let $I = (L(u,v))$ be a completely lexsegment ideal. Then, the following statements are equivalent:*

(i) *u and v satisfy one of the following conditions:*

1. *$u = x_1^a x_2^{d-a}$ and $v = x_1^a x_n^{d-a}$ for some a, $0 < a \leq d$;*
2. *$b_1 < a_1 - 1$;*
3. *$b_1 = a_1 - 1$ and for the largest monomial of degree d with $w <_{\text{lex}} v$, one has $x_1 w / x_{m(w)} \leq_{\text{lex}} u$.*

(ii) *I has a linear resolution.*

(iii) *I has linear quotients (with respect to $\succ$).*

(iv) *All the powers of I have linear quotients (with respect to $\succ$).*

(v) *All the powers of I have a linear resolution.*

Characterization of lexsegment ideals with linear resolution

In the following, we describe a procedure to determine whether or not a lexsegment ideal has a linear resolution. Let $I = (L(u,v))$ be a lexsegment ideal with $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$.

1. If $u = v$, then I has a linear resolution. In the next steps we may therefore assume that $u >_{\text{lex}} v$.

2. If I is completely lexsegment, see Theorem 2.2.17.

3. If I is not completely lexsegment, we let $m \geq 1$ be such that $a_i = b_i$ for $i = 1, \ldots, m-1$ and $a_m > b_m$. Set $f = x_1^{a_1} \cdots x_{m-1}^{a_{m-1}} x_m^{b_m}$ and let $\tilde{I}$ be the ideal in $K[x_m, \ldots, x_n]$ spanned by $L(u/f, v/f)$. It is clear that I has a linear resolution if and only $\tilde{I}$ has a linear resolution.

4. If $\tilde{I}$ is completely lexsegment, see Theorem 2.2.17, and if $\tilde{I}$ is not completely lexsegment, see Theorem 2.2.18 below.

Theorem 2.2.18 ([ANH, Theorem 2.4]). *Let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d in S with $a_1 \neq 0$ and let $I = (L(u, v))$ is not completely lexsegment. Then, I has a linear resolution if and only if u and v are of the form*

$$u = x_l x_{l+1}^{a_{l+1}} \cdots x_n^{a_n}, \quad v = x_l x_n^{d-1},$$

for some $l, 2 \leq l < n$.

Let $I = (L(u, v))$ be a lexsegment ideal with linear resolution which is not a completely lexsegment ideal. In [EO, Corollary 3.11], it was shown that the Rees ring of I, $\mathcal{R}(I)$, is Koszul. So by [Bl, Theorem 3.6], all the powers of I have linear resolution (see also [EOS, Theorem 1.2]). Moreover, in [EO, Corollary 3.9] it was shown that, all powers of I have linear quotients with respect to the increasing reverse lexicographic order.

To summarize our results about the resolution of lexsegment ideals, we have the following proposition.

Proposition 2.2.19. *Let $u = x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 > 0$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ be monomials of degree d in S with $u \geq_{\text{lex}} v$ and let $I = (L(u, v))$ be a lexsegment ideal. Then, the following statements are equivalent:*

(i) *I has a linear resolution.*

(ii) *I has linear quotients.*

(iii) *All the powers of I have linear quotients.*

(iv) *All the powers of I have a linear resolution.*

2.2.4 Square-free Lexsegment Ideals

We now introduce the concept of square-free lexsegment ideals. Let $\binom{V}{q}$ denote the set of all square-free monomials of degree $q \geq 1$ in the variables $V = \{x_1, x_2, \ldots, x_v\}$. A non-empty set $\mathcal{L} \subset \binom{V}{q}$ is called a *square-free lexsegment set of degree q*, if it satisfies in the following property:

$T \in \mathcal{L}, T' \in \binom{V}{q}$ and $T \leq_{\text{lex}} T'$, imply that $T' \in \mathcal{L}$.

An ideal I of $S = K[x_1, \ldots, x_v]$ is called a *square-free lexsegment ideal of degree* q, if I is generated by the square-free monomials belonging to a square-free lexsegment set of degree q.

For example, if $v = 5$ and $q = 3$, then

$$\mathcal{L} := \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4, x_1x_3x_5, x_1x_4x_5, x_2x_3x_4\}$$

is a square-free lexsegment set of degree 3.

More generally, we say that an ideal $I \subset S$ is *square-free lexsegment ideal*, if for every $1 \leq q \leq v$,

$$T \in I \cap \binom{V}{q}, T' \in \binom{V}{q} \text{ and } T <_{\text{lex}} T', \text{ imply that } T' \in I.$$

It follows immediately that, if $I \subset R$ is a square-free lexsegment ideal of degree q, then I is a square-free lexsegment ideal.

Theorem 2.2.20 ([AHH, Proposition 1.1, Theorem 1.4, Corollary 1.5])**.** *Suppose that $I \subset S = K[x_1, \ldots, x_v]$ is a square-free lexsegment ideal of degree q which is generated by square-free lexsegment set $\mathcal{L} \subset \binom{V}{q}$. Then,*

(i) *If $x_{\xi_1}x_{\xi_2}\cdots x_{\xi_q}$ $(1 \leq \xi_1 < \xi_2 < \cdots < \xi_q \leq v)$ is a unique minimal element (with respect to $<_{\text{lex}}$) of $\mathcal{L}$, then* $\dim(S/I) = v - \xi_1$.

(ii) *I has a q-linear resolution.*

(iii) $\beta_i^K(S/I) = \sum\limits_{T \in \mathcal{L}} \binom{m(T)-q}{i-1}$.

2.2.5 Square-free Stable Ideals

Let I be an ideal of $S = K[x_1, \ldots, x_v]$ which is generated by square-free monomials. the ideal I is called a *square-free stable ideal*, if for every square-free monomial $T \in I$, we have:

$(x_jT)/x_{m(T)} \in I$, for each $j < m(T)$ such that $x_j \nmid T$.

The ideal I is called a *square-free strongly stable ideal*, if for every square-free monomial $T \in I$, we have:

$(x_jT)/x_i \in I$, for all $i \in \operatorname{Supp}(T)$ and for all $j < i$ such that $x_j \nmid T$.

We remind that, for a monomial T, $\operatorname{Supp}(T) = \{i\colon \quad x_i \text{ divides } T\}$. By definition, we have the following implication:

square-free lexsegment $\Rightarrow$ *square-free strongly stable* $\Rightarrow$ *square-free stable.*

Hence all results of this section can be applied as well to square-free lexsegment ideals.

Theorem 2.2.21 ([AHH, Corollary 2.3, Corollary 2.4, Corollary 2.6]). *Let $I \subset S = K[x_1, \ldots, x_v]$ be a square-free stable ideal. Then,*

(i) $\operatorname{depth}(S/I) = v - \max\{m(T) - \deg(T): \quad T \in G(I)\} - 1.$

(ii) $\operatorname{reg}(I) = \max\{\deg(T): \quad T \in G(I)\}.$

(iii) $\beta_i^K(I) = \sum\limits_{T \in G(I)} \binom{m(T)-\deg(T)}{i}.$

Weakly stable ideals

For a monomial u, let $u' = u/x_{m(u)}$ where $m(u)$ is defined in (2.5).

Definition 2.2.22. A square-free monomial ideal I is called *weakly stable*, if for every square-free monomial $u \in I$, the following condition $(\star)$ is satisfied:

$(\star)$ For every integer $l \notin \operatorname{Supp}(u)$ such that $l < m(u')$, there exists an integer $i \in \operatorname{Supp}(u)$ with $i > l$ such that, $x_l(u/x_i) \in I$.

Note that if I is a square-free stable ideal, then I is weakly stable. But the converse is not true.

Example 2.2.23. Let I be the ideal of $S = K[x_1, x_2, x_3, x_4, x_5, x_6]$ which is generated by the square-free monomials

$$\langle x_1x_2x_3, x_1x_2x_5, x_1x_3x_4, x_1x_3x_6, x_1x_4x_5, x_1x_5x_6, x_2x_3x_4, x_2x_3x_6, x_2x_4x_5, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\rangle.$$

Then, I is a weakly stable ideal. We remark that there exists no permutation τ of $\{1, 2, 3, 4, 5, 6\}$, such that I^τ is a square-free stable ideal, where I^τ is the ideal generated by the square-free monomials $x_{\tau(a)}x_{\tau(b)}x_{\tau(c)}$ with $x_ax_bx_c \in I$.

Lemma 2.2.24 ([AHH2, Lemma 1.2]). *The ideal I is weakly stable, if for every $u \in G(I)$ condition $(\star)$ holds.*

Lemma 2.2.25 ([AHH2, Lemma 1.6]). *Let I be a weakly stable ideal with $G(I) = \{u_1, \ldots, u_m\}$ where, $u_1 \leq_{\text{deglex}} \cdots \leq_{\text{deglex}} u_m$. Then, $I_k = (u_1, \ldots, u_k)$ is weakly stable ideal for every k.*

Let I be a weakly stable ideal with $G(I) = \{u_1, \ldots, u_m\}$, where $u_1 \leq_{\text{deglex}} \cdots \leq_{\text{deglex}} u_m$. For $1 \leq k \leq m$, set

$$a_k = \deg u_k, \qquad \Lambda_1 = \varnothing$$

and for $2 \leq k \leq m$, define:

$$\Lambda_k = \{t \notin \operatorname{Supp}(u_k)\colon\ x_t u_k \in (u_1, \ldots, u_{k-1})\}, \quad R_k = (S/(x_t\colon t \in \Lambda_k))(-a_k).$$

Theorem 2.2.26 ([AHH2, Theorem 1.4])**.** *Let I be a weakly stable ideal with $G(I) = \{u_1, \ldots, u_m\}$ where $u_1 \leq_{\text{deglex}} \cdots \leq_{\text{deglex}} u_m$. Then, for all $i, j \geq 0$, one has:*

$$\beta_{i,j}^K(I) = \sum_{t=1}^{m} \beta_{i,j}^K(R_t),$$

where for $2 \leq t \leq m$,

$$\beta_{i,j}^K(R_t) = \begin{cases} 0, & j \neq a_t + i, \\ \binom{|\Lambda_t|}{i}, & j = a_t + i. \end{cases}$$

In particular, the Betti numbers of I are independent of the base field K.

Corollary 2.2.27. *If I be a weakly stable ideal of degree q, then I has a q-linear resolution.*

For a direct proof of this Corollary see also [AHH2, Theorem 2.1].

2.2.6 Linearly Joined Ideals

In this section, we study scrollers and linearly joined varieties. Scrollers were introduced in [BM] and linearly joined varieties are an extension of scrollers and were defined in [EGHP], where they proved that, scrollers are defined by homogeneous ideals having a 2-linear resolution.

Definition 2.2.28 (See also [EGHP, BM])**.** An ordered sequence $\mathcal{V}_1, \ldots, \mathcal{V}_l \subset \mathbb{P}^r$ of irreducible projective subvarieties is *linearly joined*, if for any $i = 1, \ldots, l-1$ we have:

$$\mathcal{V}_{i+1} \cap (\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_i) = \operatorname{span}(\mathcal{V}_{i+1}) \cap \operatorname{span}(\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_i)$$

where, $\operatorname{span}(\mathcal{V})$ is the smallest linear subspace of $\mathbb{P}^r$ containing $\mathcal{V}$.

We translate the above geometrical settings into algebraic language. Let V be a K-vector space of dimension $r+1, S = K[V]$, the polynomial ring corresponding to the projective space $\mathbb{P}^r$. For any set $\mathcal{Q} \subset V$, we will denote by $\langle \mathcal{Q} \rangle \subset V$ the K-vector space generated by $\mathcal{Q}$ and by $(\mathcal{Q}) \subset S$, the ideal generated by $\mathcal{Q}$. For all $m = 1, \dots, l$ let $\mathcal{J}_m \subset S$ be the defining ideal of $\mathcal{V}_m$ and $\mathcal{Q}_m \subset V$ be the linear space such that $(\mathcal{Q}_m)$ is the defining ideal of the linear variety $\mathcal{L}_m := \text{span}(\mathcal{V}_m)$. We can write $\mathcal{J}_m = (\mathcal{M}_m, (\mathcal{Q}_m))$, where $\mathcal{M}_m$ is an ideal containing no linear forms.

Definition 2.2.29. Let $\mathcal{J}_1, \dots, \mathcal{J}_l$ be a sequence of ideals such that for all $i, \mathcal{J}_i = (\mathcal{M}_i, (\mathcal{Q}_i))$ where $\mathcal{Q}_i \subset V$ is a vector space and $\mathcal{M}_m$ is an ideal containing no linear forms. Assume that the intersection $\mathcal{J}_1 \cap \cdots \cap \mathcal{J}_l$ is not redundant. The sequence of ideals $\mathcal{J}_1, \dots, \mathcal{J}_l$ is said to be *linearly joined*, if for all $k = 2, \dots, l$ we have:

$$\mathcal{J}_k + \bigcap_{i=1}^{k-1} \mathcal{J}_i = (\mathcal{Q}_i) + \left(\bigcap_{i=1}^{k-1} \mathcal{Q}_i \right).$$

Theorem 2.2.30 ([BM, Theorem 2.1])**.** *The following conditions are equivalent:*

(i) *The sequence of ideals $\mathcal{J}_1, \dots, \mathcal{J}_l \subset S := K[V]$ is linearly joined.*

(ii) *For all $i = 1, \dots, l$, there exist linear subspaces $\mathcal{D}_i, \mathcal{P}_i \subset V$, with $\mathcal{D}_l = 0, \mathcal{P}_1 = 0$, and ideals $\mathcal{M}_i \subset K[V]$, such that:*

(a) $\mathcal{J}_i = (\mathcal{M}_i, \mathcal{Q}_i)$, *for all $i = 1, \dots, l$;*

(b) $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i, \mathcal{P}_i \cap \mathcal{D}_{i-1} = 0$;

(c) $\mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots \supset \mathcal{D}_l$;

(d) $\mathcal{M}_i \subset (\mathcal{D}_{i-1})$, *for all $i = 2, \dots, l$;*

(e) $\mathcal{M}_i \subset \mathcal{P}_j$, *for all $i = 1, \dots, l-1$ and $j = i+1, \dots, l$;*

(f) $\bigcap_{j=1}^{k-1} (\mathcal{Q}_j) \subset (\mathcal{P}_k, \mathcal{D}_{k-1})$, *for all $k = 2, \dots, l$.*

Corollary 2.2.31 ([Mo, Corollary 1])**.** *For any sequence of ideals $\mathcal{J}_1, \dots, \mathcal{J}_l$ satisfying axioms* (a)-(e)*, we have:*

$$\bigcap_{j=1}^{k} \mathcal{J}_j = \left(\mathcal{M}_1, \dots, \mathcal{M}_k, \bigcap_{j=1}^{k} (\mathcal{Q}_j) \right),$$

for all $k = 1, \dots, l$.

Theorem 2.2.32 ([Mo, Theorem 2]). *Let $\mathcal{J}_1, \ldots, \mathcal{J}_l \subset S$ be a linearly joined sequence of ideals where $\mathcal{J}_i = (\mathcal{M}_i, \mathcal{Q}_i)$, $\mathcal{M}_i$ is not necessarily reduced and $\mathcal{J} = \bigcap_{j=1}^{l} \mathcal{J}_j$. Then,*

$$\operatorname{reg}(\mathcal{J}) = \max\{2, \operatorname{reg}(\mathcal{J}_1), \ldots, \operatorname{reg}(\mathcal{J}_l)\}.$$

In particular, if $\operatorname{reg}(\mathcal{J}_i) \leq 2$, *for all* $i = 1, \ldots, l$ *then,* $\operatorname{reg}(\mathcal{J}) = 2$.

In the reduced case, we can find the following equivalence in [EGHP] (the difficult part is the implication (i)⇒(ii)):

Theorem 2.2.33. *Suppose that we have a sequence $\mathcal{J}_1, \ldots, \mathcal{J}_l$ of prime (homogeneous) ideals. The following statements are equivalent:*

(i) *The ideal $\mathcal{J} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_l$ is 2-regular.*

(ii) $\operatorname{reg}(\mathcal{J}_j) \leq 2$, *for all $j = 1, \ldots, l$ and the sequence $\mathcal{J}_1, \ldots, \mathcal{J}_l$ is linearly joined.*

Linearly joined hyperplane

In this section, we consider a sequence of linear ideals $(\mathcal{Q}_1), \ldots, (\mathcal{Q}_l)$. We assume that the intersection $(\mathcal{Q}_1) \cap \cdots \cap (\mathcal{Q}_l)$ is not redundant. The following is immediate consequence of Theorem 2.2.30 and Theorem 2.2.33.

Corollary 2.2.34. *The following conditions are equivalent:*

(i) *The ideal $\mathcal{Q} := (\mathcal{Q}_1) \cap \cdots \cap (\mathcal{Q}_l)$ has a 2-linear resolution.*

(ii) *Modulo a permutation of the prime components of Q, the sequence of ideals $(\mathcal{Q}_1), \ldots, (\mathcal{Q}_l)$ is linearly joined.*

(iii) *For all $i = 1, \ldots, l$ there exist linear subspaces $\mathcal{D}_i, \mathcal{P}_i \subset V$, with $\mathcal{D}_l = 0, \mathcal{P}_1 = 0$, such that:*

  (a) $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$;

  (b) $\mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots \supset \mathcal{D}_l$;

  (c) $\bigcap_{j=1}^{k-1} (\mathcal{Q}_j) \subset (\mathcal{P}_k, \mathcal{D}_{k-1})$ *for all* $k = 2, \ldots, l$.

Proposition 2.2.35 ([Mo, Proposition 1]). *The following conditions are equivalent:*

(i) *The sequence of ideals* $(\mathcal{Q}_1), \ldots, (\mathcal{Q}_l)$ *is linearly joined,*

(ii) *For all* $i = 1, \ldots, l$ *there exist linear subspaces* $\langle\Delta_i\rangle, \mathcal{P}_i \subset V$, *with* $\langle\Delta_1\rangle = 0, \mathcal{P}_1 = 0$, *such that:*

(a) $\mathcal{D}_i = \bigoplus_{j=i+1}^{l} \langle\Delta_j\rangle$;

(b) $\mathcal{Q}_i = \mathcal{D}_i \oplus \mathcal{P}_i$;

(c) *for any* $k = 2, \ldots, l$ *and* $j < k$, *we have:*

$$\langle\Delta_j\rangle \times \mathcal{P}_j \subset (P_k);$$

where $\langle\Delta_j\rangle \times \mathcal{P}_j$ *is the ideal generated by all the products* fg, *with* $f \in \langle\Delta_j\rangle, g \in \mathcal{P}_j$.

For the Betti numbers and minimal free resolution of linearly joined hyperplane, see [Mo, Theorem 12].

Chapter 3

Algebraic Combinatorics

This chapter is devoted to be the language of this thesis. That is, the objects that we will consider for the purpose of this thesis, will be introduced in this chapter. These objects consist of simplicial complexes, clutters and manifolds. Each of these objects give rise to a square-free monomial ideal. First, in Section 3.1, we explain why we consider only square-free monomial ideals.

3.1 Passing to Square-free Ideals

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $<$ be a monomial order on S which satisfies $x_1 > x_2 > \cdots > x_n$. For a graded ideal $I \subset S$, there exists a non-empty open set U of linear automorphisms of S such that $\mathrm{in}_<(\alpha I)$ does not depend on $\alpha \in U$ [HH, Theorem 4.1.2]. The resulting initial ideal, which is a monomial ideal, is called the generic initial ideal of I with respect to $<$. Moreover, the remarkable result of Bayer and Stillman [BS], says that a graded ideal and its generic initial ideal have the same regularity. So that, in order to determine the regularity of arbitrary homogeneous ideals, it is enough to find a way for computing the regularity of monomial ideals.

On the other hand, a monomial ideal I and its polarization, share many homological and algebraic properties. By polarization (see Definition 3.1.1), many questions concerning monomial ideals can be reduced to square-free monomial ideals. Most important is that, the graded Betti numbers of I and its polarization are the same.

Definition 3.1.1. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K. Suppose $u = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in S. We define the *polarization* of M to be the square-free monomial,

$$\mathcal{P}(u) = x_{1,1}x_{1,2} \cdots x_{1,a_1}x_{2,1} \cdots x_{2,a_2} \cdots x_{n,1} \cdots x_{n,a_n}$$

in the polynomial ring $T = K[x_{i,j}\colon\ 1 \leq i \leq n,\ 1 \leq j \leq a_i]$. If I is an ideal of S generated by monomials $u_1, \dots, u_q$, then the *polarization* of I is defined as:

$$\mathcal{P}(I) = (\mathcal{P}(u_1), \dots, \mathcal{P}(u_q))$$

which is a square-free monomial ideal in a polynomial ring T.

Proposition 3.1.2 ([HH, Corollary 1.6.3])**.** *Let $I \subset S$ be a monomial ideal and $J \subset T$ its polarization. Then,*

(i) $\beta^K_{i,j}(I) = \beta^K_{i,j}(J)$.

(ii) $F(S/I, t) = (1-t)^\delta F(T/J, t)$, *where* $\delta = \dim(T) - \dim(S)$.

(iii) $\operatorname{projdim}(S/I) = \operatorname{projdim}(T/J)$.

(iv) $\operatorname{reg}(I) = \operatorname{reg}(J)$.

(v) *S/I is Cohen-Macaulay if and only if T/J is Cohen-Macaulay.*

To summarize, Proposition 3.1.2 together with the above argument implies that:

> Classifying homogeneous ideals in terms of their regularities, is equivalent to classifying square-free monomial ideals in terms of their regularities.

In this thesis, we deal with square-free monomial ideals with a d-linear resolution. By Theorem 2.1.3, these ideals are generated by square-free monomials in degree d with $\operatorname{reg}(I) = d$. So that, for the purpose of our work in this thesis, all the ideals are square-free monomial ideals generated in one degree, namely d.

Note that, square-free monomial ideals are in one-to-one correspondence with Stanley-Reisner ideals of simplicial complexes in one hand and the circuit ideals of clutters on the other hand. Many properties of these combinatorial objects have algebraic meaning and vice a versa.

For this reason, in the next sections, we will deal with the simplicial complexes and clutters and later in Chapter 4 and 5, we will use their combinatorial and topological properties in order to find the regularity of corresponding ideals.

3.2 Simplicial Complexes

Definition 3.2.1 (Simplicial complex)**.** A *simplicial complex* Δ over a set of vertices $V = \{v_1, \dots, v_n\}$, is a collection of subsets of V, with the property that:

(a) $\{v_i\} \in \Delta$, for all i;

(b) if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set).

An element of Δ is called a *face* of Δ and the *dimension* of a face F of Δ, $\dim F$, is $|F|-1$ where, $|F|$ is the number of elements of F and $\dim \varnothing = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim \Delta$, is the maximum of dimensions of its facets.

Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \dots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Γ is called a *subcomplex* of Δ, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

Definition 3.2.2. Let Δ be a simplicial complex over n vertices labelled $v_1, \dots, v_n$. For $F \subset \{v_1, \dots, v_n\}$, we set:

$$\mathbf{x}_F = \prod_{v_i \in F} x_i. \tag{3.1}$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F\colon \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F\colon \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ.

Theorem 3.2.3 ([St])**.** *Let Δ be a simplicial complex on vertex set $V = \{v_1, \dots, v_n\}$ and K a field. Then,*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}}$$

where $P_{\bar{F}}$ denotes the (prime) ideal generated by all $\{x_i\colon v_i \notin F\}$. In particular,

$$\dim K[\Delta] = 1 + \dim \Delta.$$

Definition 3.2.4 (**f**-vector)**.** Let Δ be a simplicial complex of dimension d on vertex set V and $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i. Thus, in particular $f_{-1} = 1, f_0 = n$. The sequence,

$$\mathbf{f}(\Delta) = (f_{-1}, f_0, \ldots, f_d)$$

is called the **f**-*vector* of Δ.

The **h**-*vector* of a simplicial complex Δ, is defined as follows:

$$\mathbf{h}(\Delta) := \mathbf{h}(K[\Delta]).$$

Theorem 3.2.5 ([St])**.** *Let Δ be a simplicial complex of dimension $d-1$ and $\mathbf{h} = (h_i)$ be the* **h**-*vector of Δ. Then $h_i = 0$, for $i > d$ and:*

$$h_k(\Delta) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}, \qquad \text{for } 0 \leq k \leq d.$$

Definition 3.2.6 (Alexander dual)**.** Given a simplicial complex Δ on V, we define $\Delta^\vee$, the *Alexander dual* of Δ, by

$$\Delta^\vee = \{V \setminus F : \quad F \notin \Delta\}.$$

Lemma 3.2.7 ([HH, Lemma 1.5.2])**.** *The collection of sets $\Delta^\vee$, is a simplicial complex and*

(i) $(\Delta^\vee)^\vee = \Delta$.

(ii) $\mathcal{F}(\Delta^\vee) = \{V \setminus F : F \in \mathcal{N}(\Delta)\}$.

Definition 3.2.8. Let Δ be a simplicial complex of dimension d on vertex set V. For each $0 \leq i \leq d$, the *i-skeleton* of Δ is the simplicial complex $\Delta^{(i)}$ on V, whose faces are those faces F of Δ, with $|F| \leq i+1$. That is,

$$\Delta^{(i)} = \{F \in \Delta : \quad \dim(F) \leq i\}.$$

Definition 3.2.9 (Cohen-Macaulay simplicial complex)**.** A simplicial complex Δ is said to be *Cohen-Macaulay* over K, if the Stanley-Reisner ring of Δ, $K[\Delta] = S/I_\Delta$, is Cohen-Macaulay ring.

Theorem 3.2.10 ([BH, Exercise 5.1.23])**.** *Let Δ be a simplicial complex of dimension d with r-skeleton $\Delta^{(r)}$. Then,*

(i) depth $K[\Delta] = \max\{r : \quad \Delta^{(r)}$ *is Cohen-Macaulay over* $K\} + 1$.

(ii) Δ *is Cohen-Macaulay if and only if $\Delta^{(r)}$ is Cohen-Macaulay for all r.*

3.2.1 Hochster Formula

Let Δ be a simplicial complex with vertex set V. An *orientation* on Δ, is a linear order on V. A simplicial complex together with an orientation is an *oriented simplicial complex*.

Suppose Δ is an oriented simplicial complex of dimension d, and $F \in \Delta$ an i-face. We write $F = [v_0, \ldots, v_i]$, if $F = \{v_0, \ldots, v_i\}$ and $v_0 < \cdots < v_i$ and by convention, $F = [\,]$, if $F = \varnothing$. Having introduced this notation, we define the *augmented oriented chain complex of* Δ,

$$\tilde{\mathscr{C}}(\Delta): \quad 0 \xrightarrow{\partial_{d+1}} \mathcal{C}_d \xrightarrow{\partial_d} \mathcal{C}_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \mathcal{C}_{-1} \longrightarrow 0$$

by setting:

$$\mathcal{C}_i = \bigoplus_{\substack{F\in\Delta \\ \dim F = i}} KF \qquad \text{and} \qquad \partial_i(F) = \sum_{j=0}^{i} (-1)^j F_j$$

for all $F \in \Delta$. Here $F_j = [v_0, \ldots, \hat{v}_j, \ldots, v_i]$, for $F = [v_0, \ldots, v_i]$. A straightforward computation shows that, $\partial_i \circ \partial_{i+1} = 0$. We set:

$$\tilde{H}_i(\Delta; K) = H_i(\tilde{\mathscr{C}}(\Delta)) = \frac{\operatorname{Ker} \partial_i}{\operatorname{Im} \partial_{i+1}}, \qquad i = -1, \ldots, d$$

and call $\tilde{H}_i(\Delta; K)$ the *i-th reduced simplicial homology of* Δ.

If Δ is a simplicial complex and Δ_1 and Δ_2 are subcomplexes of Δ, then there is an exact sequence:

$$\begin{aligned} \cdots \to \tilde{H}_j(\Delta_1 \cap \Delta_2; K) \to \tilde{H}_j(\Delta_1; K) \oplus \tilde{H}_j(\Delta_2; K) \to \\ \to \tilde{H}_j(\Delta_1 \cup \Delta_2; K) \to \tilde{H}_{j-1}(\Delta_1 \cap \Delta_2; K) \to \cdots \end{aligned} \tag{3.2}$$

with all coefficients in K, called the *reduced Mayer-Vietoris sequence* of Δ_1 and Δ_2 (see [HH, Proposition 5.1.8] for more details).

Definition 3.2.11. Let Δ be a simplicial complex of dimension d and $\mathbf{f}(\Delta) = (f_{-1}, f_0, \ldots, f_d)$ be the $\mathbf{f}$-vector of Δ. The number,

$$\chi(\Delta) = \sum_{i=0}^{d} (-1)^i f_i$$

is called the *Euler characteristic* of Δ.

In terms of simplicial homology, one has (see [Vil, Exercise 5.2.7]):

$$-1 + \chi(\Delta) = \sum_{i=0}^{d} (-1)^i \dim_K \tilde{H}_i(\Delta; K). \tag{3.3}$$

Hochster's formula describes the Betti number of a square-free monomial ideal I, in terms of the dimension of reduced homology of Δ, when $I = I_\Delta$.

Theorem 3.2.12 (Hochster formula, [HH, Theorem 8.1.1]). *Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$, K a field. Then,*

$$\beta_{i,j}^K(I_\Delta) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K)$$

where Δ_W is the simplicial complex with vertex set W and all faces of Δ with vertices in W.

For square-free monomial ideal I with $\operatorname{indeg}(I) \geq d$, we have the following theorem.

Proposition 3.2.13. *Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$ and d be an integer such that,* $\operatorname{indeg}(I_\Delta) \geq d$. *Then,*

(i) $\tilde{H}_i(\Delta_W; K) = 0$*, for all $i < d-2$ and $W \subset [n]$.*

(ii) *If $\beta_{i,j}^K(I_\Delta) \neq 0$, then $i + d \leq j \leq n$.*

Proof. (i) Let $\dim \Delta = r$ and

$$\begin{aligned} \tilde{\mathscr{C}}(\Delta): \quad 0 \longrightarrow \mathcal{C}_r \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_{d+1}} \mathcal{C}_d \xrightarrow{\partial_d} \mathcal{C}_{d-1} \\ \xrightarrow{\partial_{d-1}} \mathcal{C}_{d-2} \xrightarrow{\partial_{d-2}} \cdots \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \mathcal{C}_{-1} \longrightarrow 0 \end{aligned}$$

be the augmented chain complex of Δ and $\Delta^{(d-2)}$ be $(d-2)$-skeleton of Δ. Then, the augmented chain complex of $\Delta^{(d-2)}$ is:

$$\tilde{\mathscr{C}}(\Delta^{(d-2)}): \quad 0 \to \mathcal{C}_{d-2} \xrightarrow{\partial_{d-2}} \cdots \longrightarrow \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} \mathcal{C}_{-1} \longrightarrow 0.$$

So that,

$$\tilde{H}_i(\Delta; K) = \tilde{H}_i(\Delta^{(d-2)}; K), \quad \text{for all } i < d-2. \tag{3.4}$$

Since, $\operatorname{indeg}(I_\Delta) \geq d$, the facet set of the complex $\Delta^{(d-2)}$ contains all $(d-1)$-subsets of $[n]$. This implies that, $\tilde{H}_i(\Delta^{(d-2)}; K) = 0$, for $i < d-2$. Thus, by (3.4), we have:

$$\tilde{H}_i(\Delta; K) = \tilde{H}_i(\Delta^{(d-2)}; K) = 0, \quad \text{for all } i < d-2.$$

Now assume that, W is a subset of $[n]$. If $I_{\Delta_W} = 0$, then $\Delta_W = \langle W \rangle$ is a simplex and the assertion is clear in this case. So assume that, $I_{\Delta_W} \neq 0$.

Our assumption $\operatorname{indeg}(I_\Delta) \geq d$ implies that, $|W| \geq d$ and more generally, all $(d-1)$-subsets of W is again in Δ_W. This implies that, $\operatorname{indeg}(I_{\Delta_W}) \geq d$. Hence by what we have already proved, we conclude that $\tilde{H}_i(\Delta_W; K) = 0$, for all $i < d-2$.

(ii) If $\beta^K_{i,j}(I_\Delta) \neq 0$, then by Theorem 3.2.12, there exists $W \subset [n]$ with $|W| = j$ and $\tilde{H}_{j-i-2}(\Delta_W; K) \neq 0$. So that by part (i), $j - i - 2 \geq d - 2$. Moreover, by our choice of W, one has $j = |W| \leq n$. □

Proposition 3.2.14. *Let Δ be a simplicial complex on vertex set $[n] = \{1, \ldots, n\}$ and $0 \neq I = I_\Delta$ be its Stanley-Reisner ideal. Then, the followings are equivalent:*

(i) *I has a d-linear resolution in the polynomial ring $S = K[x_1, \ldots, x_n]$.*

(ii) *$\tilde{H}_i(\Delta_W; K) = 0$, for all $i \neq d-2$ and $W \subset [n]$.*

Proof. (i)→(ii) If $I = I_\Delta$ has a d-linear resolution, then $\beta^K_{i,j}(I_\Delta) = 0$, for all $j - i \neq d$. Now, let $i \neq d-2$ and W be any subset of $[n]$. Then, $(|W|) - (|W| - i - 2) = i + 2 \neq d$ and by Theorem 3.2.12, we have:

$$0 = \beta^K_{|W|-i-2,|W|} = \sum_{\substack{W' \subset [n] \\ |W'|=|W|}} \dim_K \tilde{H}_i(\Delta_{W'}; K) \geq \dim_K \tilde{H}_i(\Delta_W; K).$$

So that, $\tilde{H}_i(\Delta_W; K) = 0$.

(ii)→(i) If $j - i \neq d$, then our hypothesis implies that:

$$\beta^K_{i,j}(I_\Delta) = \sum_{\substack{W \subset [n] \\ |W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K) = 0. \tag{3.5}$$

In particular, $\beta^K_{0,j}(I_\Delta) = 0$, for $j \neq d$ while $\beta^K_{0,d}(I_\Delta) \neq 0$, because I is a non-zero ideal. This implies that, $\operatorname{indeg}(I_\Delta) = d$. Using (3.5), we have:

$$\operatorname{reg}(I) = \max\{j - i : \quad \beta^K_{i,j}(I) \neq 0\} = d = \operatorname{indeg}(I).$$

That is, I has a d-linear resolution. □

3.3 Eagon-Reiner Theorem

The Alexander dual of a square-free monomial ideals, plays an essential role in combinatorics and commutative algebra. One of the fundamental results is the Eagon-Reiner theorem, which says that, a square-free monomial ideal has a linear resolution if and only if its Alexander dual is Cohen-Macaulay.

Definition 3.3.1 (Alexander duality)**.** For a square-free monomial ideal $I = (M_1, \dots, M_q) \subset S = K[x_1, \dots, x_n]$, the *Alexander dual* of I, denoted by $I^\vee$, is defined to be:

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q}$$

where, P_{M_i} is prime ideal generated by $\{x_j\colon \quad x_j|M_i\}$.

If it happens that $I^\vee = I$, then the ideal I is called *self-dual* ideal.

It is clear that, for a square-free monomial ideal $I \subset S$, we have:

$$\dim S/I^\vee = n - \operatorname{indeg}(I). \tag{3.6}$$

Lemma 3.3.2 ([HH, Lemma 1.5.3])**.** *One has:*

$$(I_\Delta)^\vee = I_{\Delta^\vee} = I(\bar{\Delta})$$

where, $\bar{\Delta} = \{V \setminus F\colon F \in \mathcal{F}(\Delta)\}$.

Theorem 3.3.3 (Eagon-Reiner theorem [ER, Theorem 3])**.** *Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$. The ideal I has a q-linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.*

Proposition 3.3.4 ([HH, Proposition 8.1.10])**.** *Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$. Then,*

$$\operatorname{projdim}(S/I) = \operatorname{reg}(I^\vee).$$

Corollary 3.3.5. *Let I be a square-free monomial ideal in $S = K[x_1, \dots, x_n]$. Then,*

$$\dim \frac{S}{I^\vee} - \operatorname{depth} \frac{S}{I^\vee} = \operatorname{reg}(I) - \operatorname{indeg}(I).$$

Proof. By Proposition 3.3.4 and Theorem 1.6.9, we have:

$$n - \operatorname{depth} \frac{S}{I^\vee} = \operatorname{reg}(I).$$

Using (3.6), we get the conclusion. □

Remark 3.3.6. Let I and J be square-free monomial ideals in $S = K[x_1, \dots, x_n]$. By Proposition 3.3.4 and Theorem 1.6.9, we have:

$$\operatorname{reg}(I) = n - \operatorname{depth} \frac{S}{I^\vee}, \qquad \operatorname{reg}(J) = n - \operatorname{depth} \frac{S}{J^\vee}.$$

Therefore, $\operatorname{reg}(I) = \operatorname{reg}(J)$ if and only if $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$.

3.4 Clutters

Definition 3.4.1 (Clutter)**.** Let $[n] = 1, \ldots, n$. A *clutter* $\mathcal{C}$ on a vertex set $[n]$, is a collection of subsets of $[n]$ (called *circuits* of $\mathcal{C}$) such that, if e_1 and e_2 are distinct circuits of $\mathcal{C}$, then $e_1 \nsubseteq e_2$. A *d-circuit* is a circuit consisting of exactly d vertices and a clutter is *d-uniform*, if every circuit has exactly d vertices.

A clutter $\mathcal{C}$ is said to be *connected*, if for each two vertices u and v, there is a sequence of circuits $F_1, \ldots, F_r$ such that, $u \in F_1, v \in F_r$ and $F_i \cap F_{i+1} \neq \varnothing$.

For a non-empty clutter $\mathcal{C}$ on vertex set $[n]$, we define the ideal $I(\mathcal{C})$, as follows:

$$I(\mathcal{C}) = (\mathbf{x}_F : \quad F \in \mathcal{C})$$

and we define $I(\varnothing) = 0$.

Let n, d be positive integers with $d \leq n$. We define $\mathcal{C}_{n,d}$, the *maximal d-uniform clutter* on $[n]$, as follow:

$$\mathcal{C}_{n,d} = \{F \subset [n] : \quad |F| = d\}.$$

If $\mathcal{C}$ is a d-uniform clutter on $[n]$, we define $\bar{\mathcal{C}}$, the *complement* of $\mathcal{C}$, to be

$$\bar{\mathcal{C}} = \mathcal{C}_{n,d} \setminus \mathcal{C} = \{F \subset [n] : \quad |F| = d,\ F \notin \mathcal{C}\}.$$

Frequently in this thesis, we take a d-uniform clutter $\mathcal{C} \neq \mathcal{C}_{n,d}$ and we consider the square-free ideal $I = I(\bar{\mathcal{C}})$ in the polynomial ring $S = K[x_1, \ldots, x_n]$. We call $I = I(\bar{\mathcal{C}})$ the *circuit ideal* of $\mathcal{C}$.

Definition 3.4.2 (Clique)**.** Let $\mathcal{C}$ be a d-uniform clutter on $[n]$. A subset $G \subset [n]$ is called a *clique* in $\mathcal{C}$, if all d-subsets of G belongs to $\mathcal{C}$. The simplicial complex generated by cliques of $\mathcal{C}$, is called *clique complex* of $\mathcal{C}$ and is denoted by $\Delta(\mathcal{C})$.

Remark 3.4.3. Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $\Delta = \Delta(\mathcal{C})$ be its clique complex. Then by our definition, all the subsets of $[n]$ with less than d elements are also in $\Delta(\mathcal{C})$. In particular, this implies that indeg $(I_\Delta) \geq d$. So that by Proposition 3.2.13, we have:

$$\tilde{H}_i(\Delta_W; K) = 0, \qquad \text{for all } i < d-2 \text{ and } W \subset [n]. \tag{3.7}$$

Remark 3.4.4. Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $I = I(\bar{\mathcal{C}})$ be its circuit ideal. If G is a clique in $\mathcal{C}$ and $F \in \bar{\mathcal{C}}$, then $([n] \setminus G) \cap F \neq \varnothing$. This implies that, $\mathbf{x}_{[n]\setminus G} \in P_F$. Hence,

$$\mathbf{x}_{[n]\setminus G} \in \bigcap_{F\in\bar{\mathcal{C}}} P_F = I^{\vee}.$$

Example 3.4.5. In this example, we will show that $I(\mathcal{C}_{n,d})$ has a d-linear resolution.

Let Δ be a simplex $[n] = \{1, \ldots, n\}$. Then, clearly $I_\Delta = (0)$ and $K[\Delta] = K[x_1, \ldots, x_n]$ is Cohen-Macaulay. It follows from Theorem 3.2.10, that for any $r < n$, $\Delta^{(r)} = \langle F \subset [n] \colon \quad |F| \leq r+1\rangle$ is Cohen-Macaulay. Note that:

$$I^{\vee}_{\Delta^{(r)}} = I\left(\overline{\Delta^{(r)}}\right) = (\mathbf{x}_F \colon \quad |F| = n-(r+1))$$

which has linear resolution by Theorem 3.3.3.

Using this argument for $r = n-d-1$, we conclude that, $I^{\vee}_{\Delta^{(n-d-1)}} = I(\mathcal{C}_{n,d})$ has a d-linear resolution.

Proposition 3.4.6. *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $I = I(\bar{\mathcal{C}}) \subset K[x_1, \ldots, x_n]$ be the circuit ideal. Let $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Then,*

(i) $\mathcal{C} = \mathcal{F}\left(\Delta^{(d-1)}\right)$.

(ii) *For all $u \in G(I_\Delta)$, $\deg(u) = d$.*

(iii) $I_\Delta = I$.

Proof. In view of Theorem 3.2.3, we have:

$$I_\Delta = \bigcap_{F\in\mathcal{F}(\Delta)} P_{\bar{F}}.$$

So that,

$$\mathbf{x}_T \in I_\Delta \Longleftrightarrow T \cap ([n] \setminus F) \neq \emptyset, \text{ for all } F \in \mathcal{F}(\Delta). \tag{3.8}$$

(i) Clear.

(ii) Let $u = \mathbf{x}_T \in G(I_\Delta)$. By Remark 3.4.3, we know that $\deg(u) = |T| \geq d$. If $\deg(u) = |T| > d$, then for all d-subset T' of T, $\mathbf{x}_{T'} \notin I_\Delta$. This means that $T' \in \Delta$, for all d-subset T' of T (that is, T is a clique in $\mathcal{C}$). So that $T \in \Delta$, which is a contradiction to the fact that $u = \mathbf{x}_T \in G(I_\Delta)$.

(iii) Let $T \in \bar{\mathcal{C}}$ and $\mathbf{x}_T \notin I_\Delta$. Then, by (3.8), there exist $F \in \mathcal{F}(\Delta)$ such that $T \subset F$. Since T is a d-subset of F, so $T \in \mathcal{C}$, which is a contradiction. So that, $I(\bar{\mathcal{C}}) \subset I_\Delta$.

For the converse inclusion, let $\mathbf{x}_T \in G(I_\Delta)$. Then, $T \notin \Delta$. Using part (i), $T \notin \mathcal{C}$. Moreover, by (ii), we have $|T| = d$. Since $|T| = d$ and $T \notin \mathcal{C}$, we conclude that, $T \in \bar{\mathcal{C}}$. This means that, $I_\Delta \subset I(\bar{\mathcal{C}})$. $\square$

Definition 3.4.7 (Simplicial submaximal circuit)**.** Let $\mathcal{C}$ be a d-uniform clutter on $[n]$. A $(d-1)$-subset $e \subset [n]$ is called an *submaximal circuit* of $\mathcal{C}$, if there exists $F \in \mathcal{C}$, such that $e \subset F$. The set of all submaximal circuits of $\mathcal{C}$ is denoted by $\mathrm{SC}(\mathcal{C})$.

For $e \in \mathrm{SC}(\mathcal{C})$, let

$$\mathrm{N}[e] = e \cup \big\{c \in [n] : \quad e \cup \{c\} \in \mathcal{C}\big\} \subset [n].$$

We say that e is a *simplicial submaximal circuit*, if $\mathrm{N}[e]$ is a clique of $\mathcal{C}$. In the case of 3-uniform clutters, (simplicial) submaximal circuit is called (*simplicial*) *edge*.

Also, for $e \in \mathrm{SC}(\mathcal{C})$, we denote by $\deg_{\mathcal{C}}(e)$, the *degree* of a submaximal circuit e, to be:

$$\deg_{\mathcal{C}}(e) = |\{F \in \mathcal{C} : \quad e \subset F\}|.$$

3.5 Triangulation

An important class of d-uniform clutters that we will deal with the regularity of the corresponding circuit ideals, comes from triangulation of manifolds. This class is very important because, in d-uniform clutters ($d > 2$), triangulation of manifolds are examples of clutters which are minimal to d-linearity (c.f. Chapter 5). In this section, we briefly introduce some topological preliminaries in order to define the concept of triangulation. Some very good references for the readers on this subject are [Ma, Mau, Mu, Mu2, Sta].

Definition 3.5.1 (Manifolds)**.** An *n-dimensional manifold* (also called an *n-manifold*), is a Hausdorff space such that each point has an open neighborhood homeomorphic to the open n-dimensional disc,

$$U^n = \{(x_1, \ldots, x_n) : \quad \sum_{i=1}^{n} x_i^2 < 1\}.$$

That is, a space that locally behaves like $\mathbb{R}^n$.

The term *surface* will be used interchangeably with 2-manifold.

Example 3.5.2. (i) Euclidean n-space $\mathbb{R}^n$ is obviously an n-dimensional manifold. One can easily prove that, the unit n-dimensional sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \quad \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is an n-manifold.

(ii) If M is an m-dimensional manifold and N is an n-dimensional manifold, then the product space $M \times N$ is an $(m+n)$-dimensional manifold.

Definition 3.5.3. let $X \subset \mathbb{R}^s$. The *convex hull* of X, denoted by $\mathrm{conv}(X)$, is defined to be the set of all convex combinations of finite subsets of X, that is,

$$\mathrm{conv}(X) = \{\lambda_1 x_1 + \cdots + \lambda_s x_s : \quad x_i \in X,\ 0 \leq \lambda_i \in \mathbb{R},\ \sum_{i=1}^{s} \lambda_i = 1\}.$$

Definition 3.5.4. Let Δ be a finite simplicial complex on $[n]$ and $\mathbf{e}_i$ the i-th unit vector in $\mathbb{R}^n$. Given a face $F \in \Delta$, set:

$$\mathrm{conv}(F) = \mathrm{conv}(\{\mathbf{e}_j : \quad j \in F\}).$$

Define the *geometric realization* $|\Delta|$ of Δ, as:

$$|\Delta| = \bigcup_{F \in \Delta} \mathrm{conv}(F).$$

Thus, $|\Delta|$ is a topological space with the induced usual topology of $\mathbb{R}^n$.

Definition 3.5.5 (Triangulation)**.** Given a topological space X, a *triangulation* of X consists of a simplicial complex Δ and a homeomorphism $h : |\Delta| \to X$.

A space with a triangulation is called a *triangulated space*.

3.5.1 Classification of Triangulated 2-Manifolds

The classification theorem for compact surfaces is covered in most books on algebraic topology. This theorem, either appears at the beginning, in which case the presentation is usually rather informal because the machinery needed to give a formal proof has not been introduced yet (as in Massey [Ma]) or it is given as an application of the machinery as in [Mu2]. Munkres's proof appears in Chapter 12 and depends on material on the fundamental group from Chapters 9 and 11.

We outline that any compact, connected, triangulated 2-manifold is a sphere, the connected sum of tori, or the connected sum of projective planes. In order to state this theorem, first we need the definition of connected sum of two manifolds.

Let S_1 and S_2 be disjoint surfaces. Their *connected sum*, denoted by $S_1 \# S_2$, is formed by cutting a small circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes.

To be precise, we choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 and D_2 are closed discs (i.e., homeomorphic to $E^2 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$). Let S'_i denote the complement of the interior of D_i in S_i, for $i = 1, 2$. Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2. Then, $S_1 \# S_2$ is the quotient space of $S'_1 \cup S'_2$ obtained by identifying the points x and $h(x)$, for all points x in the boundary of D_1. It is clear that, $S_1 \# S_2$ is again a surface. Note that if S_2 is a 2-sphere, then $S_1 \# S_2$ is homeomorphic to S_1.

Theorem 3.5.6 ([Sta, Theorem 1.7])**.** *Any compact 2-manifold is homeomorphic to a compact, triangulated 2-manifold. In other words, all compact 2-manifolds are triangulable.*

It has been proved that 3-manifolds can be triangulated, as a result of E. Moise (1952) (see Massey [Ma]). Now, there is at least some hopes to have a classification theory of compact 3-manifolds (see section 6.5 of [GX]).

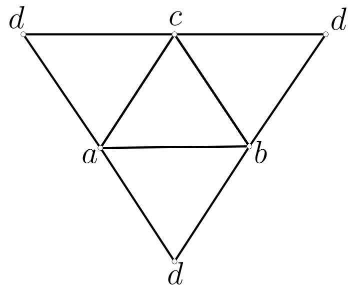
Theorem 3.5.7 ([Mau, Corollary 3.4.4])**.** *Let Δ be a triangulation of 2-manifold. Then,*

(i) $\dim(\Delta) = 2$.

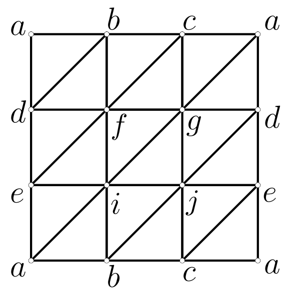
(ii) *For each edge e of Δ, there exists exactly two facets which contain e.*

Theorem 3.5.7 implies that, any triangulation of 2-manifold gives rise to a 3-uniform clutter. To be more precise, if Δ is a triangulation of 2-manifold, then $\mathcal{C} = \mathcal{F}(\Delta)$ is a 3-uniform clutter. In Chapter 4 and 5, we will find the regularity and resolution of the circuit ideals of such clutters.

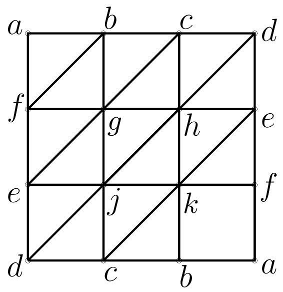
Example 3.5.8. The followings are triangulations of some 2-manifolds.



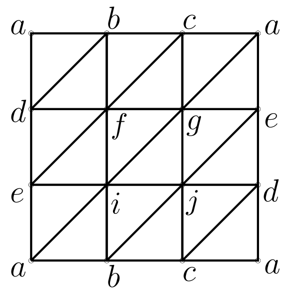
A Triangulation of Sphere



A Triangulation of Torus



A Triangulation of Projective Plane



A Triangulation of Klein Bottle

The Euler-Poincaré characteristic in one of the major ingredients in the classification of the compact surfaces.

Definition 3.5.9. For a triangulated surface X, if v is the number of vertices, e is the number of edges, and f is the number of triangles, then the *Euler-Poincaré characteristic* of X is defined by:

$$\chi(X) = v - e + f.$$

The Euler-Poincaré characteristic is an invariant of all the finite complexes corresponding to the same polytope X. It is known that, homeomorphic surfaces have the same Euler-Poincaré characteristic. Going back to the triangulations of the sphere, the torus, the projective plane, and the Klein bottle, we find that they have Euler-Poincaré characteristics 2 (sphere), 0 (torus), 1 (projective plane), and 0 (Klein bottle). It can also be shown that, the Euler-Poincaré characteristic of $S_1 \# S_2$ is given by the formula:

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Theorem 3.5.10 ([Ma, Theorem 5.1 (I)]). *Any connected compact* 2*-manifold is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes.*

3.5.2 Homology of 2-Manifolds

For any two Abelian groups A and B, we will use the notation $\mathrm{Tor}\,(A, B)$ to denote $\mathrm{Tor}_1^{\mathbb{Z}}\,(A, B)$. The definition and properties of this functor are given in most books on homological algebra (see for instance [Ro, Chapter 8]). Here is a list of some of the principal properties of this function:

(a) (**Symmetry**) $\mathrm{Tor}\,(A, B) \cong \mathrm{Tor}\,(B, A)$.

(b) If either A or B is torsion-free, then $\mathrm{Tor}\,(A, B) = 0$.

(c) Let $0 \to F_1 \to F_0 \to A \to 0$ be a short exact sequence with F_0 a free Abelian group; it follows that F_1 is also free. Then, there is an exact sequence, as follows:

$$0 \to \mathrm{Tor}\,(A, B) \to F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0.$$

Since any Abelian group A is the homomorphic image of some free Abelian group F_0, we can use this property to define $\mathrm{Tor}\,(A, B)$, or to determine it in specific cases.

(d) For any Abelian group G, $\mathrm{Tor}\,(\mathbb{Z}_n, G)$ is isomorphic to the subgroup of G consisting of all $x \in G$ such that $nx = 0$.
In particular, $\mathrm{Tor}\,(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{(m,n)}$, where (m, n) is the greatest common divisor of m and n.

(e) Tor is an additive functor in each variable, i.e., for direct sums

$$\mathrm{Tor}\left(\bigoplus_i A_i, B\right) \cong \bigoplus_i \mathrm{Tor}\,(A_i, B).$$

The following theorem enables us to compute the reduced homology of an oriented simplicial complex Δ in arbitrary field K, whenever we know the reduced homology of Δ with the coefficient in $\mathbb{Z}$.

Theorem 3.5.11 ([Ma, Theorem 6.2 (IX)]). *Let Δ be a simplicial complex and K be a field. Then,*

$$\tilde{H}_i(\Delta; K) \cong \left(\tilde{H}_i(\Delta; \mathbb{Z}) \otimes K\right) \oplus \mathrm{Tor}\,\left(\tilde{H}_{i-1}(\Delta; \mathbb{Z}), K\right).$$

It is a fundamental theorem in topology (see also [Mu, §34]) that the reduced singular homology $\tilde{H}_i(X; K)$ of a topological space X with triangulation Δ, can be computed by means of the reduced simplicial homology of Δ.

Theorem 3.5.12 ([BH, Theorem 5.3.2])**.** *Let X be a topological space with triangulation Δ. Then,*

$$\tilde{H}_i(X; K) \cong \tilde{H}_i(\Delta; K), \quad \textit{for all } i.$$

Example 3.5.13. In Example 3.5.8, we have seen a triangulation for some topological spaces like sphere, torus, projective plane and Klien bottle. It follows that:

$$\tilde{H}_i(\mathcal{S}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } i = 2, \\ 0, & \text{for } i \neq 2. \end{cases}$$

where $\mathcal{S}^2$ is the 2-dimensional sphere. For the complete list of the homology of surfaces, see Table 3.1.

A triangulable surface (2-manifold) is called *orientable*, if it has a triangulation such that, we can orient the boundary of all triangles in a directed 3-cycle in such a way that, if two triangles have an edge in common, they induce opposite directions on this edge.

Orientability is invariant under refinement, so orientability is also an invariant of the surface. The following table, summarizes the homology of 2-manifolds. For the details of this part, we refer the reader to Chapter I and VIII of [Ma].

<table>
<tr><td></td><td></td><td colspan="2">Oriented surfaces</td><td>Non-oriented surfaces</td></tr>
<tr><td></td><td>S</td><td>Sphere</td><td>Connected sum of r tori</td><td>Connected sum of r Projective Plane</td></tr>
<tr><td></td><td>$\chi(S)$</td><td>2</td><td>$2-2r$</td><td>$2-r$</td></tr>
<tr><td rowspan="3">$\tilde{H}_i(S, K)$</td><td>$i=0$</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$i=1$</td><td>0</td><td>K^{2r}</td><td>$(\mathbb{Z}_2 \otimes_{\mathbb{Z}} K) \oplus K^{r-1}$</td></tr>
<tr><td>$i=2$</td><td>K</td><td>K</td><td>$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K)$</td></tr>
</table>

Table 3.1: Homology of 2-manifolds

Note that, in Table 3.1, $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_2, K) \cong \{a \in K : 2a = 0\}$ and $\mathbb{Z}_2 \otimes_{\mathbb{Z}} K \cong \frac{K}{2K}$.

Chapter 4

Reduction Processes on Clutters

In this chapter, we introduce some operations (namely reductions) on the clutters and we prove that, the regularity of corresponding ideal is conserved under these operations. We apply these operations to reduce a d-uniform clutter to a smaller one, without changing the regularity. As a consequence, we can use this method to classify all 2-uniform clutters with 2-linear resolution (Fröberg's Theorem [Fr]). Also using these reductions method, we may prove that, the circuit ideal of any triangulation of the sphere $\mathcal{S}^2$ does not have linear resolution, while any proper subclutter of it has a linear resolution. Moreover, these reduction processes lead us to a large class of ideals with linear resolution.

It is worth to say that the reduction processes introduced in this chapter, have similarity to some operations in triangulations of surfaces as diagonal flips [N].

In order to obtain our desired results, we will use the remarkable Mayer-Vietoris long exact sequence on local cohomologies and its applications.

4.1 Mayer-Vietoris Sequence

Theorem 4.1.1 (Mayer-Vietoris sequence). *For any two ideals $I_1, I_2 \subset S = K[x_1, \ldots, x_n]$, the short exact sequence,*

$$0 \longrightarrow \frac{S}{I_1 \cap I_2} \longrightarrow \frac{S}{I_1} \oplus \frac{S}{I_2} \longrightarrow \frac{S}{I_1 + I_2} \longrightarrow 0$$

gives rise to long exact sequence:

$$\begin{aligned} \cdots \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{S}{I_1\cap I_2}\right) \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{S}{I_1}\right) \oplus H^{i-1}_{\mathfrak{m}}\left(\tfrac{S}{I_2}\right) \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{S}{I_1+I_2}\right) \to \\ \to H^{i}_{\mathfrak{m}}\left(\tfrac{S}{I_1\cap I_2}\right) \longrightarrow H^{i}_{\mathfrak{m}}\left(\tfrac{S}{I_1}\right) \oplus H^{i}_{\mathfrak{m}}\left(\tfrac{S}{I_2}\right) \longrightarrow H^{i}_{\mathfrak{m}}\left(\tfrac{S}{I_1+I_2}\right) \to \\ \to H^{i+1}_{\mathfrak{m}}\left(\tfrac{S}{I_1\cap I_2}\right) \to H^{i+1}_{\mathfrak{m}}\left(\tfrac{S}{I_1}\right) \oplus H^{i+1}_{\mathfrak{m}}\left(\tfrac{S}{I_2}\right) \to H^{i+1}_{\mathfrak{m}}\left(\tfrac{S}{I_1+I_2}\right) \to \cdots . \end{aligned} \tag{4.1}$$

Proposition 4.1.2. *Let I_1 and I_2 be ideals in a commutative Noetherian local ring $(R,\mathfrak{m})$ such that:*

$$\operatorname{depth}\frac{R}{I_1} \geq \operatorname{depth}\frac{R}{I_2} > \operatorname{depth}\frac{R}{I_1+I_2}.$$

Then,

$$\operatorname{depth}\frac{R}{I_1\cap I_2} = 1 + \operatorname{depth}\frac{R}{I_1+I_2}.$$

Proof. Let $r := 1 + \operatorname{depth} R/(I_1+I_2)$. Then, for all $i < r$,

$$H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1+I_2}\right) = H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) = H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) = 0. \tag{4.2}$$

From Theorem 4.1.1, we have the long exact sequence:

$$\begin{aligned} \cdots \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) \to H^{i-1}_{\mathfrak{m}}\left(\tfrac{R}{I_1+I_2}\right) \to H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap I_2}\right) \to \\ \to H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_2}\right) \to \cdots . \end{aligned}$$

So that, by (4.2), we have:

$$H^{i}_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap I_2}\right) = 0, \quad \text{for all } i < r$$

and $H^{r}_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap I_2}\right) \neq 0$. Therefore,

$$\operatorname{depth}\frac{R}{I_1\cap I_2} = r = 1 + \operatorname{depth}\frac{R}{I_1+I_2}.$$

□

Lemma 4.1.3. *Let I, I_1 and T be ideals in a commutative Noetherian local ring $(R,\mathfrak{m})$ such that, $I = I_1 + T$ and*

$$r := \operatorname{depth}\frac{R}{I_1\cap T} \leq \operatorname{depth}\frac{R}{T}.$$

Then, for all $i < r-1$ one has:

$$H^i_{\mathfrak{m}}\left(\frac{R}{I_1}\right) \cong H^i_{\mathfrak{m}}\left(\frac{R}{I}\right).$$

Proof. For $i < r-1$, our assumption implies that:

$$H^i_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap T}\right) = H^i_{\mathfrak{m}}\left(\tfrac{R}{T}\right) = H^{i+1}_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap T}\right) = 0. \tag{4.3}$$

From Theorem 4.1.1, we get the long exact sequence:

$$\cdots \longrightarrow H^i_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap T}\right) \longrightarrow H^i_{\mathfrak{m}}\left(\tfrac{R}{I_1}\right) \oplus H^i_{\mathfrak{m}}\left(\tfrac{R}{T}\right) \longrightarrow H^i_{\mathfrak{m}}\left(\tfrac{R}{I}\right) \longrightarrow$$
$$\longrightarrow H^{i+1}_{\mathfrak{m}}\left(\tfrac{R}{I_1\cap T}\right) \longrightarrow \cdots . \tag{4.4}$$

So that, (4.4) and (4.3) imply that:

$$H^i_{\mathfrak{m}}\left(\frac{R}{I_1}\right) \cong H^i_{\mathfrak{m}}\left(\frac{R}{I}\right), \qquad \text{for all } i < r-1$$

as desired. □

As a consequence of Proposition 4.1.2 and Lemma 4.1.3, we may get the following results.

Corollary 4.1.4. *Let $S = K[x_1,\ldots,x_n,y_1,\ldots,y_m]$ be the polynomial ring and I be an ideal in $K[y_1,\ldots,y_m]$. Then,*

$$\operatorname{depth}\frac{S}{(x_1\cdots x_n)\,IS} = \operatorname{depth}\frac{S}{IS}.$$

Proof. We use induction on n. For $n=1$, we have:

$$\frac{S}{(x_1)+I} \cong \frac{K[y_1,\ldots,y_m]}{I}.$$

Hence,

$$m = \operatorname{depth}\frac{S}{(x_1)} \geq \operatorname{depth}\frac{S}{I} = 1+\operatorname{depth}\frac{K[y_1,\ldots,y_m]}{I} > \operatorname{depth}\frac{S}{I+(x_1)}.$$

So that, Proposition 4.1.2 implies that:

$$\operatorname{depth}\frac{S}{(x_1)I} = \operatorname{depth}\frac{S}{(x_1)\cap I} = 1+\operatorname{depth}\frac{S}{(x_1)+I} = \operatorname{depth}\frac{S}{I}. \tag{4.5}$$

For $n>1$, (4.5) together with induction hypothesis yields the conclusion. □

Notation. For $n > 3$, let $T_{1,n}, T'_{1,n} \subset S = K[x_1, \ldots, x_n]$ denote the ideals:

$$T_{1,n} = \bigcap_{2 \leq i < j \leq n} (x_1, x_i, x_j),$$
$$T'_{1,n} = \bigcap_{2 \leq i < j \leq n} (x_i, x_j).$$

Proposition 4.1.5. *Let $n > 3$ and $S = K[x_1, \ldots, x_n]$ be the polynomial ring. Then,*

(i) $T'_{1,n} = \left(\prod_{\substack{2 \leq i \leq n \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leq i \leq n \\ i \neq n}} x_i \right).$

(ii) $T_{1,n} = \left(x_1, \prod_{\substack{2 \leq i \leq n \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leq i \leq n \\ i \neq n}} x_i \right).$

(iii) $\dfrac{S}{T'_{1,n}}$ *is Cohen-Macaulay of dimension $n-2$.*

(iv) $\dfrac{S}{T_{1,n}}$ *is Cohen-Macaulay of dimension $n-3$.*

Proof. All the statements are clear for $n = 3$. Let $n > 3$, we use induction on n, to prove both statements in (i) and (ii). Note that by our notation, it is clear that:

$$\begin{aligned} T'_{1,n} &= T'_{1,n-1} \bigcap \Big((x_2, x_n) \cap (x_3, x_n) \cap \cdots \cap (x_{n-1}, x_n) \Big) \\ &= T'_{1,n-1} \bigcap (x_2 \cdots x_{n-1}, x_n); \end{aligned}$$

and with the same argument, $T_{1,n} = T_{1,n-1} \bigcap (x_1, x_2 \cdots x_{n-1}, x_n)$.

(i) By induction, we have:

$$\begin{aligned} T'_{1,n} &= T'_{1,n-1} \bigcap (x_2 \cdots x_{n-1}, x_n) \\ &= \left(\prod_{\substack{2 \leq i \leq n-1 \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leq i \leq n-1 \\ i \neq n-1}} x_i \right) \bigcap (x_2 \cdots x_{n-1}, x_n) \\ &= \left(\prod_{\substack{2 \leq i \leq n \\ i \neq 2}} x_i, \ldots, \prod_{\substack{2 \leq i \leq n \\ i \neq n}} x_i \right). \end{aligned}$$

(ii) The proof is as the same as the proof of (i).
(iii) Our discussion in Example 3.4.5 for $r = 0$, gives the conclusion.
(iv) By part (ii), we have:

$$\frac{S}{T_{1,n}} \cong \frac{K[x_2, \dots, x_n]}{T'_{1,n}}.$$

Hence,

$$\operatorname{depth} \frac{S}{T_{1,n}} = \operatorname{depth} \frac{S}{T'_{1,n}} - 1 = n - 3.$$

This implies that, $S/T_{1,n}$ is Cohen-Macaulay of dimension $n-3$. □

Lemma 4.1.6. *Let $n > 3$, $S = K[x_1, \dots, x_n]$ be the polynomial ring and T_n be the ideal:*

$$T_n = (x_4 \cdots x_n, x_1x_2x_3\, \hat{x}_4 \cdots x_n, \dots, x_1x_2x_3\, x_4 \cdots \hat{x}_n).$$

Then, we have:

(i) $T_n = (T_{n-1} \cap (x_n)) + (x_1x_2x_3\, x_4 \cdots \hat{x}_n)$.

(ii) $\operatorname{depth} \frac{S}{T_n} = n - 2$.

Proof. (i) This is an easy computation.

(ii) The proof is on induction over n. For $n = 4$, every thing is clear. Let $n > 4$ and (ii) be true for $n-1$. By Corollary 4.1.4 and induction hypothesis, we have:

$$\operatorname{depth} \frac{S}{(x_n) \cap T_{n-1}} = n - 2. \tag{4.6}$$

One can easily check that:

$$\operatorname{depth} \frac{S}{(T_{n-1} \cap (x_n)) \cap (x_1x_2x_3\, x_4 \cdots \hat{x}_n)} = \operatorname{depth} \frac{S}{(x_1x_2x_3\, x_4 \cdots x_n)} = n - 1.$$

So by Lemma 4.1.3, we have:

$$H^i_{\mathfrak{m}}\left(\frac{S}{T_n}\right) = 0, \quad \text{for all } i < n - 2. \tag{4.7}$$

Moreover, from Theorem 4.1.1 and (i), we have the following exact sequence:

$$0 \to H^{n-2}_{\mathfrak{m}}\left(\frac{S}{(x_n) \cap T_{n-1}}\right) \oplus H^{n-2}_{\mathfrak{m}}\left(\frac{S}{(x_1x_2x_3\, x_4 \cdots \hat{x}_n)}\right) \to H^{n-2}_{\mathfrak{m}}\left(\frac{S}{T_n}\right).$$

Using (4.6) and (4.7), we conclude that:

$$\operatorname{depth} \frac{S}{T_n} = n - 2.$$

□

4.2 Reduction on 2-uniform Clutters

Let $G \neq \mathcal{C}_{n,2}$ be a graph (2-uniform clutter) on vertex set $[n]$ and $I = I(\bar{G})$ be the circuit ideal of G. In 1990, R. Fröberg [Fr] classified the ideals with 2-linear resolution.

Theorem 4.2.1 (Fröberg's Theorem [Fr])**.** *Let $G \neq \mathcal{C}_{n,2}$ be a graph on vertex set $[n]$ and $I = I(\bar{G})$ be the circuit ideal of G. The ideal I has a 2-linear resolution if and only if G is chordal.*

In this section, we introduce some reduction processes on graphs which preserve the regularity of related edge ideals. As a consequence, we will give two different proofs for Theorem 4.2.1, based on Dirac Theorem [Di].

4.2.1 Cycles

For the classification of square-free ideals with 2-linear resolution, cycles play a key role. First, we will show that the circuit ideal of cycles does not have linear resolution. This leads to the fact that, the circuit ideal of non-chordal graphs does not have linear resolution. Later, in Chapter 5, we will discuss about an analogue to a cycle for d-uniform clutters and also, we will find a minimal free resolution of circuit ideal of cycles.

Definition 4.2.2. A *cycle* of length q is a graph $C = (V, E)$ with $V = \{1, \ldots, q\}$ and

$$E = \big\{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{q-1}, i_q\}, \{i_q, i_1\}\big\},$$

where $i_j \neq i_k$, if $j \neq k$.

A graph G is called *chordal*, if it does not have an induced subgraph which is a cycle of length $q > 3$.

A connected graph with no cycle is called a *tree* and the disjoint union of some trees is called a *forest*. Consequently, forests are examples of chordal graphs.

Lemma 4.2.3. *Let G be a graph on vertex set $[n]$ such that, $\{1, 2\} \in E(G)$ and*

$$\big\{\{1, i\}, \{2, i\}\big\} \nsubseteq E(G), \qquad \text{for all } i > 2. \tag{4.8}$$

Let $I = I(\bar{G}) \subset S = K[x_1, \ldots, x_n]$ be the circuit ideal of G. Then,

(i) $\operatorname{depth} \frac{S}{I^\vee + (x_1, x_2)} \geq \operatorname{depth} \frac{S}{I^\vee} - 1$.

(ii) depth $\frac{S}{I^\vee \cap (x_1,x_2)} \geq$ depth $\frac{S}{I^\vee}$.

Proof. Let $t :=$ depth $S/I^\vee \leq \dim S/I^\vee = n-2$.
(i) One can easily check that, condition (4.8) is equivalent to say that:

$$\text{for all } r > 2, \text{ there exists } F \in E(\bar{G}) \text{ such that, } P_F \subset (x_1, x_2, x_r).$$

Therefore,

$$\begin{aligned} I^\vee = \bigcap_{F \in E(\bar{G})} P_F &= \left(\bigcap_{F \in E(\bar{G})} P_F \right) \cap ((x_1,x_2,x_3) \cap \cdots \cap (x_1,x_2,x_n)) \\ &= \left(\bigcap_{F \in E(\bar{G})} P_F \right) \cap (x_1, x_2, x_3 \cdots x_n) \\ &= I^\vee \cap (x_1, x_2, x_3 \cdots x_n). \end{aligned}$$

Clearly, $x_3 \cdots x_n \in I^\vee$. Thus, from Theorem 4.1.1, we get the long exact sequence:

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee}\right) \oplus H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{(x_1,x_2,x_3\cdots x_n)}\right) \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee + (x_1,x_2)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee}\right) \to \cdots.$$

So that,

$$H_{\mathfrak{m}}^{i-1}\left(\frac{S}{I^\vee + (x_1,x_2)}\right) = 0, \qquad \text{for all } i < t \leq n-2. \tag{4.9}$$

This proves (i).
(ii) From Theorem 4.1.1, we have the long exact sequence:

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee + (x_1,x_2)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee \cap (x_1,x_2)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee}\right) \oplus H_{\mathfrak{m}}^{i}\left(\tfrac{S}{(x_1,x_2)}\right) \to \cdots.$$

So that by (4.9), we have:

$$H_{\mathfrak{m}}^{i}\left(\frac{S}{I^\vee \cap (x_1,x_2)}\right) = 0, \qquad \text{for all } i < t \leq n-2,$$

which completes the proof of (ii). $\square$

Theorem 4.2.4. *Let G be a graph on vertex set $[n]$ such that, $\{1,2\} \in E(G)$ and $\big\{\{1,i\},\{2,i\}\big\} \nsubseteq E(G)$, for all $i > 2$. Let G' be a graph on $\{0\} \cup [n]$, with edges*

$$E(G') = (E(G) \setminus \{1,2\}) \cup \big\{\{0,1\},\{0,2\}\big\}.$$

If $I = I(\bar{G})$, $J = I(\bar{G}')$ be corresponding circuit ideals in $S = K[x_0, x_1, \ldots, x_n]$, then

$$\text{reg}\,(I) = \text{reg}\,(J).$$

Proof. By Remark 3.3.6, it is enough to show that, depth $S/I^\vee = $ depth $S/J^\vee$. Let G_1 be a graph on $[n]$ with $E(G_1) = E(G)\backslash\{1,2\}$ and $I_1 = I(\bar{G}_1)$. Clearly, $I_1^\vee = (x_1, x_2) \cap I^\vee$ and

$$\begin{aligned} J^\vee &= \left(\bigcap_{i=3}^{n}(x_0, x_i)\right) \cap I_1^\vee \\ &= (x_0, x_3 \cdots x_n) \cap I_1^\vee. \end{aligned}$$

Moreover, our assumption implies that for all $i > 2$, there exists $F \in E(\bar{G})$ such that, $P_F \subset (x_1, x_2, x_i)$. Therefore,

$$\begin{aligned} I_1^\vee + (x_0, x_3 \cdots x_n) &= (x_0, x_3 \cdots x_n, I_1^\vee) \\ &= (x_0) + \left(x_3 \cdots x_n, \left[(x_1, x_2) \cap \left(\bigcap_{F \in E(\bar{G})} P_F\right)\right]\right) \\ &= (x_0) + \left((x_1, x_2, x_3) \cap \cdots \cap (x_1, x_2, x_n) \cap \left(\bigcap_{F \in E(\bar{G})} P_F\right)\right) \\ &= (x_0) + \left(\bigcap_{F \in E(\bar{G})} P_F\right) = (x_0, I^\vee). \end{aligned} \tag{4.10}$$

Hence by Lemma 4.2.3(ii),

$$\operatorname{depth} \frac{S}{I_1^\vee + (x_0, x_3 \cdots x_n)} = \operatorname{depth} \frac{S}{I^\vee} - 1 \leq \operatorname{depth} \frac{S}{I_1^\vee} - 1.$$

Thus,

$$\operatorname{depth} \frac{S}{(x_0, x_3 \cdots x_n)} \geq \operatorname{depth} \frac{S}{I_1^\vee} > \operatorname{depth} \frac{S}{I_1^\vee + (x_0, x_3 \cdots x_n)}.$$

Using Proposition 4.1.2 and (4.10), we have:

$$\operatorname{depth} \frac{S}{J^\vee} = 1 + \operatorname{depth} \frac{S}{I_1^\vee + (x_0, x_3 \cdots x_n)} = \operatorname{depth} \frac{S}{I^\vee}.$$

□

Corollary 4.2.5. *Let C be a cycle of length $n > 3$ and $I = I(\bar{C}) \subset S = K[x_1, \ldots, x_n]$ be the circuit ideal of C. Then,* $\operatorname{reg}(I) = 3$. *In particular I does not have linear resolution.*

Proof. Let $E(C) = \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\big\}$. We use induction on n. For $n = 4$ an easy computation shows that, the minimal free resolution of $I(\bar{C})$ is:

$$0 \to S(-4) \to S^2(-2) \to I,$$

which is not linear. Assume that $n > 4$ and the theorem holds for cycles of length $n-1$. For a cycle C of length n, let C' be the graph

$$C' = \big(C \setminus \big\{\{1,2\},\{2,3\}\big\}\big) \cup \{1,3\}.$$

Then, C' is a cycle of length $n-1$ and by induction hypothesis, $\operatorname{reg} I(\bar{C'}) = 3$. Using Theorem 4.2.4, we have $\operatorname{reg} I(\bar{C}) = \operatorname{reg} I(\bar{C'}) = 3$. □

Proposition 4.2.6. *Let G be a graph on the vertex set $[n]$ and H be an induced subgraph of G. Then,*

$$\beta^K_{i,j}\left(I(\bar{H})\right) \leq \beta^K_{i,j}\left(I(\bar{G})\right).$$

Proof. Without loss of generality, we may assume that $V(H) = [m]$, with $m \leq n$. If $\Delta = \Delta(G)$ be the clique complex of G, then by Theorem 3.2.12, we have:

$$\begin{aligned}\beta^K_{i,j}\left(I(\bar{G})\right) = \sum_{\substack{W\subset[n]\\ |W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W;K) &\geq \sum_{\substack{W\subset[m]\\ |W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W;K)\\ &= \beta^K_{i,j}\left(I(\bar{H})\right).\end{aligned}$$

□

Corollary 4.2.7. *Let G be a graph such that the ideal $I(\bar{G})$ has a 2-linear resolution. Then G is chordal.*

Proof. If G is not chordal graph, then G has an induced subgraph C, which is a cycle of length > 3. Hence by Corollary 4.2.5 and Proposition 4.2.6, we have:

$$\operatorname{reg} I(\bar{G}) \geq \operatorname{reg} I(\bar{C}) = 3,$$

which contradicts to our assumption. □

Corollary 4.2.5 was independently proved in [OG, Proposition 3.1] and Proposition 4.2.6 was also proved in [J, Proposition 4.1.1] (see also [GHP, HHZ2]). However, the proofs in this section are based on our own results on reduction processes on graphs. In Sections 4.2.2 and 4.2.4 we will investigate the converse of Corollary 4.2.7 (see Corollary 4.2.12 and Corollary 4.2.19).

4.2.2 Simplicial Vertex

Definition 4.2.8. Let $G = (V, E)$ be a graph and $v \in V$. The *neighborhood* $N(v)$ of v, is defined to be:

$$N(v) = \{u \in V : \quad \{u, v\} \in E\}.$$

A vertex v is called *simplicial*, if $N(v)$ is complete subgraph of G.

Theorem 4.2.9. *Let G be a graph on $[n]$ and v be a simplicial vertex of G. Let $G_1 = G \setminus v$ and $I = I(\bar{G}), J = I(\bar{G}_1)$ be the corresponding non-zero circuit ideals in $S = K[x_1, \ldots, x_n]$. Then,*

$$\operatorname{reg}(I) = \operatorname{reg}(J).$$

Proof. By Remark 3.3.6, it is enough to show that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$. Without loss of generality, we may assume that, $N(v) = \{1, \ldots, v-1\}$ and $J \subset K[x_1, \ldots, \hat{x}_v, \ldots, x_n]$. Therefore, we have:

$$I = J + (x_v x_i : \quad v < i \leq n).$$

Moreover, since v is a simplicial vertex, we conclude that, $x_{v+1} \cdots x_n \in J^\vee$. Hence we have:

$$\begin{aligned} I^\vee &= J^\vee \cap \left(\bigcap_{i=v+1}^{n} (x_v, x_i) \right) \\ &= J^\vee \cap (x_v, x_{v+1} \cdots x_n) \\ &= ((x_v) \cap J^\vee) + (x_{v+1} \cdots x_n). \end{aligned}$$

Clearly, $((x_v) \cap J^\vee) \cap (x_{v+1} \cdots x_n) = (x_v \cdots x_n)$. Hence by Lemma 4.1.3,

$$H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{(x_v) \cap J^\vee}\right), \qquad \text{for all } i < n-2.$$

Since $\dim S/I^\vee = n - 2$, the above isomorphism and Corollary 4.1.4 imply that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$. ∎

Remark 4.2.10. Let $G \neq \mathcal{C}_{n,2}$ be a graph, v be a simplicial vertex of G and $G_1 = G \setminus v$. If G_1 is a complete graph, then the ideal $I = I(\bar{G}) = (x_v x_i : \quad \{v, i\} \in E(\bar{G}))$ is a non-zero ideal and

$$I^\vee = (x_v, \prod_{\{v,i\} \in E(\bar{G})} x_i).$$

In particular, $I^\vee$ is Cohen-Macaulay and the ideal I has a 2-linear resolution (Theorem 3.3.3).

If G_1 is not a complete graph, then Theorem 4.2.9 implies that $\operatorname{reg} I(\bar{G}) = \operatorname{reg} I(\bar{G}_1)$.

Theorem 4.2.11 ([LB], essentially [Di])**.** *A graph G is chordal if and only if every induced subgraph of G has a simplicial vertex.*

Corollary 4.2.12. *If $G \neq \mathcal{C}_{n,2}$ is a chordal graph, then the ideal $I = I(\bar{G})$ has a 2-linear resolution over any filed K.*

Proof. Let $G \neq \mathcal{C}_{n,2}$ be a chordal graph. By Theorem 4.2.11, G has simplicial vertex v. If $G_1 = G \setminus v$, then G_1 is again chordal graph. Now, the induction and Remark 4.2.10 yield the conclusion. ☐

4.2.3 Regularity of Union of Graphs

Let G_1 and G_2 be two graphs on the vertex sets V_1, V_2 and $G = G_1 \cup G_2$. In this section, we will find a formula for the regularity of $I(\bar{G})$ in terms of the regularity of $I(\bar{G}_1)$ and $I(\bar{G}_2)$, whenever $V_1 \cap V_2$ is a clique in G.

In the following, for convenience we use these notations:

$$\mathbf{x} = x_1, \ldots, x_n, \quad \mathbf{y} = y_1, \ldots, y_m, \quad \mathbf{z} = z_1, \ldots, z_r.$$

Recall that, for a subset $F \subset [n]$, we set:

$$\mathbf{x}_F = \prod_{i \in F} x_i, \qquad P_F = (x_j : \quad j \in F).$$

Lemma 4.2.13. *Let $I \neq 0$ be square-free monomial ideal generated in degree two in the polynomial ring $K[\mathbf{x}, \mathbf{z}]$ and J be the ideal*

$$J = I + (x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m) \subset S := K[\mathbf{x}, \mathbf{y}, \mathbf{z}].$$

Then, we have the followings:

(i) $J^\vee = I^\vee \cap \big(\mathbf{x}_{[n]}, \mathbf{y}_{[m]}\big)$.

(ii) *If $z_i z_j \notin I$ for all $1 \leq i < j \leq r$, then* $\operatorname{reg}(I) = \operatorname{reg}(J)$.

Proof. (i) This is an easy computation.

(ii) By Remark 3.3.6, it is enough to show that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$. We know that $I^\vee$ is intersection of prime ideals P_F, such that:

$$|F| = 2, \quad G(P_F) \subset \{\mathbf{x}, \mathbf{z}\} \quad \text{and} \quad \mathbf{x}_F \in I.$$

Since $z_iz_j \notin I$, for all $1 \le i < j \le r$, it follows that $P \not\subseteq \{\mathbf{z}\}$, for all $P \in \operatorname{Ass}(I)$. Hence $\mathbf{x}_{[n]} \in P$, for all $P \in \operatorname{Ass}(I)$. This means that, $\mathbf{x}_{[n]} \in I^\vee$. Now, by part (i) of this theorem, we have:

$$\begin{aligned} J^\vee &= I^\vee \cap \left(\mathbf{x}_{[n]}, \mathbf{y}_{[m]}\right) \\ &= \left(\mathbf{x}_{[n]}\right) + \left(\left(\mathbf{y}_{[m]}\right) I^\vee\right). \end{aligned} \tag{4.11}$$

Clearly, $\left(\mathbf{x}_{[n]}\right) \cap \left(\left(\mathbf{y}_{[m]}\right) I^\vee\right) = \left(\mathbf{x}_{[n]}\mathbf{y}_{[m]}\right)$. Hence by Lemma 4.1.3, we have:

$$H^i_{\mathfrak{m}}\left(\frac{S}{J^\vee}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{\left(\mathbf{y}_{[m]}\right) I^\vee}\right), \qquad \text{for all } i < (m+n+r)-2. \tag{4.12}$$

Since,

$$\dim \frac{S}{J^\vee} = (m+n+r) - 2 = \dim \frac{S}{I^\vee},$$

from (4.12) and Corollary 4.1.4 we conclude that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$.

□

Theorem 4.2.14. *Let G_1 and G_2 be graphs on two vertex sets V_1 and V_2 respectively, such that $V_1 \cap V_2 = \{\mathbf{z}\}$ and $\{z_i, z_j\} \in E(G_1) \cap E(G_2)$, for all $1 \le i < j \le r$. Let*

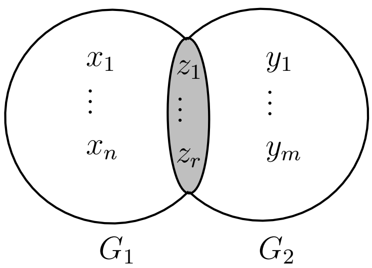
$$\begin{aligned} I_1 &= I(\bar{G}_1) \subset K[\mathbf{x}, \mathbf{z}], \\ I_2 &= I(\bar{G}_2) \subset K[\mathbf{y}, \mathbf{z}], \\ I &= I\left(\overline{G_1 \cup G_2}\right) \subset S = K[\mathbf{x}, \mathbf{y}, \mathbf{z}]. \end{aligned}$$

be corresponding non-zero circuit ideals. Then,

(i) $\operatorname{depth} \frac{S}{I^\vee} = \min\{\operatorname{depth} \frac{S}{I_1^\vee}, \operatorname{depth} \frac{S}{I_2^\vee}\}$.

(ii) $\operatorname{reg}(I) = \max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2)\}$.

(iii) *I has a 2-linear resolution if and only if both of I_1 and I_2 have a 2-linear resolution.*



Proof. (i) We know that:

$$I = I_1 + I_2 + (x_i y_j : \quad 1 \le i \le n, \quad 1 \le j \le m) \, .$$

Let,

$$\begin{aligned} J_1 &= I_1 + (x_i y_j : \quad 1 \le i \le n, \quad 1 \le j \le m) \, , \\ J_2 &= I_2 + (x_i y_j : \quad 1 \le i \le n, \quad 1 \le j \le m) \, . \end{aligned}$$

Then, $I^\vee = J_1^\vee \cap J_2^\vee$ and like the proof of Lemma 4.2.13(ii), we have:

$$J_1^\vee + J_2^\vee = \left(\mathbf{x}_{[n]}, \mathbf{y}_{[m]} \right) .$$

From Theorem 4.1.1, we have the long exact sequence:

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\frac{S}{\left(\mathbf{x}_{[n]}, \mathbf{y}_{[m]}\right)}\right) \to H_{\mathfrak{m}}^{i}\left(\frac{S}{I^\vee}\right) \to H_{\mathfrak{m}}^{i}\left(\frac{S}{J_1^\vee}\right) \oplus H_{\mathfrak{m}}^{i}\left(\frac{S}{J_2^\vee}\right) \to$$
$$\to H_{\mathfrak{m}}^{i}\left(\frac{S}{\left(\mathbf{x}_{[n]}, \mathbf{y}_{[m]}\right)}\right) \to \cdots .$$

Hence, for all $i < (m + n + r) - 2$, we have:

$$H_{\mathfrak{m}}^{i}\left(\frac{S}{J_1^\vee}\right) \oplus H_{\mathfrak{m}}^{i}\left(\frac{S}{J_2^\vee}\right) \cong H_{\mathfrak{m}}^{i}\left(\frac{S}{I^\vee}\right) .$$

This implies that,

$$\operatorname{depth} \frac{S}{I^\vee} = \min\{\operatorname{depth} \frac{S}{J_1^\vee}, \ \operatorname{depth} \frac{S}{J_2^\vee}\}. \tag{4.13}$$

By Lemma 4.2.13(ii) and Remark 3.3.6, we have:

$$\operatorname{depth} \frac{S}{I_i^\vee} = \operatorname{depth} \frac{S}{J_i^\vee}, \qquad \text{for } i = 1, 2.$$

Hence, (i) follows from (4.13) and the above equality.

(ii) This is an easy consequence of (i) and Remark 3.3.6.

(iii) This is a direct consequence of (ii). □

4.2.4 Decomposable Graphs

Definition 4.2.15. Let G be a graph on vertex set $[n]$. We say that G is *decomposable*, if there exists proper subsets P and Q of $[n]$ with $P \cup Q = [n]$ such that, $P \cap Q$ is a clique of G and that $\{i, j\} \notin E(G)$, for all $i \in P \setminus Q$ and $j \in Q \setminus P$.

Remark 4.2.16 (Regularity of Decomposable Graphs)**.** Let G be a decomposable graph and P, Q be proper subsets of $V(G) = [n]$ which satisfies in Definition 4.2.15.

- If both of G_P and G_Q are complete graphs, then:

$$I(\bar{G}) = (x_i y_j : \quad i \in P \setminus Q, \quad j \in Q \setminus P).$$

  Hence,

$$I(\bar{G})^{\vee} = \left(\prod_{i \in P \setminus Q} x_i, \prod_{i \in Q \setminus P} y_i \right)$$

  which is Cohen-Macaulay of dimension $n - 2$. Thus, $\operatorname{reg} I(\bar{G}) = 2$, by Theorem 3.3.3.

- If G_P is complete graph but G_Q is not complete graph, then all $v \in P \setminus Q$ are simplicial vertex. Hence by Theorem 4.2.9, $\operatorname{reg} I(\bar{G}) = \operatorname{reg} I \left(\overline{G \setminus v} \right)$.

  If $|P| = 1$, we conclude that $\operatorname{reg} I(\bar{G}) = \operatorname{reg} I(\bar{G}_Q)$. Otherwise, the graph $G' = G \setminus v$ is again decomposable with the components $P' = P \setminus v$ and Q. Note that, $G'_{P'}$ is again a complete graph. Going on this argument, we conclude that, $\operatorname{reg} I(\bar{G}) = \operatorname{reg} I(\bar{G}_Q)$.

- If non of G_P and G_Q are complete graphs, then Theorem 4.2.14 implies that, $\operatorname{reg} I(\bar{G}) = \max\{\operatorname{reg} I(\bar{G}_P), \ \operatorname{reg} I(\bar{G}_Q)\}$.

Definition 4.2.17. Let G be a graph on vertex set $[n]$. A subset $A \subset [n]$ is called a *separator* of G, if there exist subsets P and Q of $[n]$ with $P \cup Q = [n]$ and $P \cap Q = A$ and that $\{i, j\} \notin E(G)$, for all $i \in P \setminus Q$ and $j \in Q \setminus P$.

Clearly $[n] \setminus \{a, b\}$ is a separator of G, whenever $\{a, b\} \notin E(G)$.

The following theorem was proved in [HH, Lemma 9.2.1]. Because of importance of this theorem, we restate its proof here. Recall that, A *walk* of G of length q between $i, j \in V(G)$, is a sequence of edges of the form

$$\big\{\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{q-1}, i_q\}\big\}$$

where, $i_0, i_1, \ldots, i_q$ are vertices of G with $i_0 = i$ and $i_q = j$.

Theorem 4.2.18. *Every chordal graph which is not complete is decomposable.*

Proof. Let G be a chordal graph on $[n]$, which is not complete. Let a and b be vertices of G, with $\{a, b\} \notin E(G)$ and $B \subset [n] \setminus \{a, b\}$ be a minimal separator (with respect to inclusion) of G.

What we must prove is that, B is a clique of G. Let $|B| > 1$ and $x, y \in B$ with $x \neq y$. Let P and Q be subset of $[n]$ with $P \cup Q = [n]$ and $P \cap Q = B$ and that $\{i, j\} \notin E(G)$, for all $i \in P \setminus Q$ and $j \in Q \setminus P$. Let $C_1, \ldots, C_s$ be the connected components of the induced subgraph of G on $P \setminus B$ and $D_1, \ldots, D_t$ the connected components of the induced subgraph of G on $Q \setminus B$.

We claim that, there is a vertex x_0 of C_1 such that, $\{x_0, x\} \in E(G)$. To see why this is true, suppose that $\{z, x\} \notin E(G)$, for all vertices z of C_1. Let V denote the set of vertices of C_1 and $B' = B \setminus \{x\}$. Let $P_0 = V \cup B'$ and $Q_0 = [n] \setminus V$. Then $P_0 \cup Q_0 = [n]$ and $P_0 \cap Q_0 = B'$. Since $\{z, x\} \notin E(G)$, for all $z \in V$, it follows that $\{i, j\} \notin E(G)$, for all $i \in P_0 \setminus Q_0$ and $j \in Q_0 \setminus P_0$. Hence B' is a separator of G, which contradicts the minimality of B. Consequently, there is a vertex x_0 of C_1 such that, $\{x_0, x\} \in E(G)$. Similarly, there is a vertex y_0 of C_1 such that, $\{y_0, y\} \in E(G)$. In addition, there is a vertex x_1 of D_1 such that, $\{x_1, x\} \in E(G)$, and there is a vertex y_1 of D_1 such that, $\{y_1, y\} \in E(G)$.

Now, let W_1 be a walk of minimal length between x and y whose vertices belong to $V \cup \{x, y\}$ and W_2 a walk of minimal length between x and y whose vertices belong to $V' \cup \{x, y\}$, where V' is the set of vertices of D_1. Combining W_1 and W_2 yields a cycle C of G of length > 3. The minimality of W_1 and W_2 together with the fact that $\{i, j\} \notin E(G)$, for all $i \in V$ and $j \in V'$ guarantees that, except for $\{x, y\}$, the cycle C has no chord. Since G is chordal, the edge $\{x, y\}$ must belong to G.

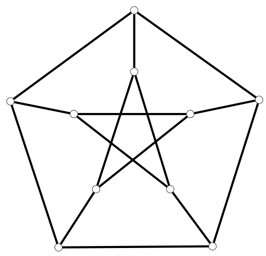
Hence $\{x, y\} \in E(G)$, for all x and y of B with $x \neq y$. Thus, B is a clique of G, as desired. □

Corollary 4.2.19. *If $G \neq \mathcal{C}_{n,2}$ is a chordal graph, then the ideal $I = I(\bar{G})$ has a 2-linear resolution over any filed K.*

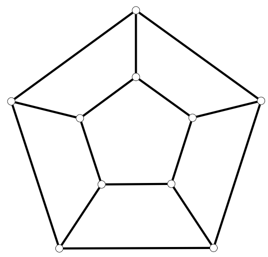
Proof. If $G \neq \mathcal{C}_{n,2}$ is a chordal graph, then Theorem 4.2.18 implies that, G is decomposable. So, we may use Remark 4.2.16 and induction to conclude that $\operatorname{reg}(I) = 2$, independent of the choice of base field. □

Remark 4.2.20. Let G be a (indecomposable) graph. After our reduction processes (Theorems 4.2.4 and 4.2.9), finally we get a graph G' with $\operatorname{reg} I(\bar{G}) = \operatorname{reg} I(\bar{G}')$ and G' has neither a simplicial vertex nor a subdivision. If at least one of the connected components of G' has cycle of length greater that 3, then $I(\bar{G})$ does not have a 2-linear resolution (Corollary 4.2.7).

But, sometimes we are not able to do more reduction on a graph. For example, if G is the Peterson graph or the following Hamiltonian graph, then we cannot apply our reduction process to further simplify G.



Peterson Graph



Hamiltonian Graph

4.3 Reduction on d-uniform Clutters

In this section, we try to generalize the analogous of reduction processes on 2-uniform clutters (graphs) in Section 4.2, for arbitrary d-uniform clutters. We extend Theorem 4.2.9 for d-uniform clutters and Theorem 4.2.4 will be extended for 3-uniform clutters. Also, in Chapter 5, we will find a generalization of Theorem 4.2.16, for the union of two d-uniform cutters as well.

Remark 4.3.1. Let $\mathcal{C}$ be a d-uniform clutter on the vertex set $[n]$. Surely, one can consider the clutter $\mathcal{C}$ as a d-uniform clutter on $[m]$, for any $m \geq n \geq d$. However, $\bar{\mathcal{C}}$ (and hence $I(\bar{\mathcal{C}})$) will be changed when we consider $\mathcal{C}$ either on $[n]$ or on $[m]$.

To be more precise, when we pass from $[n]$ to $[n+1]$, then the new generators,

$$\{x_{n+1}\mathbf{x}_F : \quad F \subset [n], \ |F| = d-1\}$$

will be added to $I(\bar{\mathcal{C}})$. Below, we will show that the regularity does not change when we pass from $[n]$ to $[m]$ or vice versa.

Lemma 4.3.2. *Let $I \subset S = K[x_1, \ldots, x_n, x_{n+1}]$ be a square-free monomial ideal generated in degree d, such that:*

$$\{x_{n+1}\mathbf{x}_F : \quad F \subset [n], \ |F| = d-1\} \subset I.$$

If $J = I \cap K[x_1, \ldots, x_n]$, then:

$$\operatorname{reg}(I) = \operatorname{reg}(J).$$

Proof. By our assumption, J is an ideal of $K[x_1, \ldots, x_n]$ and

$$I = J + (x_{n+1}\mathbf{x}_F : \quad F \subset [n],\ |F| = d-1).$$

So that,

$$\begin{aligned} I^\vee &= J^\vee \cap \left(\bigcap_{\substack{F \subset [n] \\ |F|=d-1}} (x_{n+1}, P_F) \right) \\ &= J^\vee \cap \left(x_{n+1}, \left(\bigcap_{\substack{F \subset [n] \\ |F|=d-1}} P_F \right) \right). \end{aligned} \tag{4.14}$$

By Remark 3.3.6, it is enough to show that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$. Let T be the ideal,

$$T = \bigcap_{\substack{F \subset [n] \\ |F|=d-1}} P_F.$$

Then,

(a) T is the Stanley-Reisner ideal of $\Delta = \langle [n]^{(n-d)} \rangle$.

(b) $\operatorname{depth} \frac{S}{T} = \dim \frac{S}{T} = n - d + 1$. (By (a), Theorems 3.2.10 and 3.2.3)

(c) if $\mathbf{x}_F \in T$, then $|F| > (n-d) + 1$. (By (a))

The ideal $J^\vee$ is intersection of some primes P_F, such that F is a d-subset of $[n]$, with $\mathbf{x}_F \in J$. So, (c) implies that, $T \subset J^\vee$. In particular,

$$\operatorname{depth} \frac{S}{J^\vee + (x_{n+1}, T)} = \operatorname{depth} \frac{S}{J^\vee} - 1. \tag{4.15}$$

Using (b) and (4.15), we have:

$$\operatorname{depth} \frac{S}{T} \geq \operatorname{depth} \frac{S}{J^\vee} > \operatorname{depth} \frac{S}{J^\vee + (x_{n+1}, T)}.$$

Hence (4.14), Proposition 4.1.2 and (4.15), implies that:

$$\operatorname{depth} \frac{S}{I^\vee} = 1 + \operatorname{depth} \frac{S}{J^\vee + (x_{n+1}, T)} = \operatorname{depth} \frac{S}{J^\vee}.$$

$\square$

Theorem 4.3.3 (Deletion)**.** *Let $\mathcal{C} \neq \mathcal{C}_{n,d}$ be a d-uniform clutter on $[n]$ and $e \in \mathrm{SC}(\mathcal{C})$ be a simplicial submaximal circuit. Let*

$$\mathcal{C}' = \mathcal{C} \setminus e = \{F \in \mathcal{C} : \quad e \nsubseteq F\},$$

and $I = I(\bar{\mathcal{C}}), J = I(\bar{\mathcal{C}}')$ be the corresponding non-zero circuit ideals. Then,

$$\operatorname{reg}(I) = \operatorname{reg}(J).$$

Proof. By Remark 3.3.6, it is enough to show that, $\operatorname{depth} S/I^\vee = \operatorname{depth} S/J^\vee$. Without loss of generality, we may assume that $e = \{1, \ldots, d-1\}$ and $N[e] = \{1, \ldots, r\}$.

Since $e = \{1, \ldots, d-1\}$ is a simplicial submaximal circuit, by Remark 3.4.4, we have:

$$\begin{aligned}
I^\vee &= (x_1, \ldots, x_{d-1}, x_{r+1} \cdots x_n) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \\
&= \left[(x_1, \ldots, x_{d-1}) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \right] + (x_{r+1} \cdots x_n), \\
J^\vee &= (x_1, \ldots, x_{d-1}, x_d \cdots x_n) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \\
&= \left[(x_1, \ldots, x_{d-1}) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \right] + (x_d \cdots x_n).
\end{aligned}$$

Note that, for the ideals:

$$\begin{aligned}
I_1 &= (x_1, \ldots, x_{d-1}) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \cap (x_{r+1} \cdots x_n) \\
&= (x_1 x_{r+1} \cdots x_n, \ldots, x_{d-1} x_{r+1} \cdots x_n), \\
J_1 &= (x_1, \ldots, x_{d-1}) \cap \left(\bigcap_{\substack{F \in \bar{\mathcal{C}} \\ \{1,\ldots,d-1\} \nsubseteq F}} P_F \right) \cap (x_d \cdots x_n) \\
&= (x_1 x_d \cdots x_n, \ldots, x_{d-1} x_d \cdots x_n)
\end{aligned}$$

we have:

$$\operatorname{depth}\frac{S}{I_1} = \operatorname{depth}\frac{S}{J_1} = n-(d-1).$$

Hence by Lemma 4.1.3, for all $i < n-d$, we have:

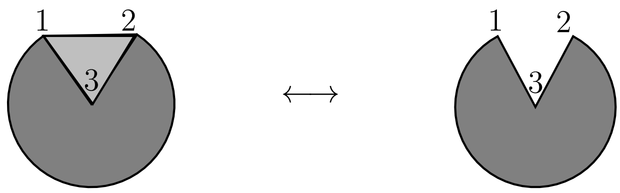
$$H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{(x_1,\dots,x_{d-1})\cap\left(\bigcap_{\substack{F\in\bar{\mathcal{C}}\\ \{1,\dots,d-1\}\nsubseteq F}} P_F\right)}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{J^\vee}\right).$$

The above equation implies that depth $S/I^\vee$ = depth $S/J^\vee$, for dim $S/I^\vee$ = dim $S/J^\vee = n-d$. ☐

For a clutter $\mathcal{C}$, if $e\in \mathrm{SC}(\mathcal{C})$ and $\deg_{\mathcal{C}}(e)=1$, then e is clearly a simplicial submaximal circuit. Hence, we have the following result:

Corollary 4.3.4. *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $I=I(\bar{\mathcal{C}})$ be the circuit ideal of $\mathcal{C}$. If F is the only circuit containing the submaximal circuit $e\in \mathrm{SC}(\mathcal{C})$, then:*

$$\operatorname{reg}(I) = \operatorname{reg}(I+(\mathbf{x}_F)).$$



$\mathcal{C}$ $\longleftrightarrow$ $\mathcal{C}'$

Let $\mathcal{C}$ be a 3-uniform clutter on $[n]$ such that:

$$\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\big\}\subset\mathcal{C}.$$

If there exist no other circuit which contains $e=\{1,2\}$, then e is a simplicial submaximal circuit. Hence by Theorem 4.3.3, we have the following result.

Theorem 4.3.5. *Let $\mathcal{C}$ be a 3-uniform clutter on $[n]$ and $I=I(\bar{\mathcal{C}})$ be the circuit ideal of $\bar{\mathcal{C}}$. Assume that,*

$$\big\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\big\}\subset\mathcal{C},$$

and there exist no other circuit which contains $\{1,2\}$*. If* $J = I + (x_1x_2x_3, x_1x_2x_4)$*, then:*

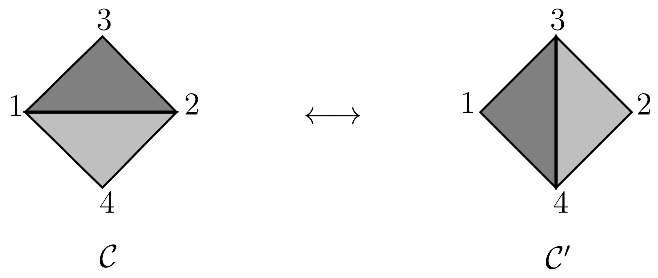
$$\operatorname{reg}(I) = \operatorname{reg}(J).$$

Definition 4.3.6. Let $\mathcal{C}$ be a 3-uniform clutter on vertex set $[n]$. Assume that $\{1,2,3\},\{1,2,4\} \in \mathcal{C}$ are the only circuits containing $\{1,2\}$ and there is no circuit in $\mathcal{C}$ containing $\{3,4\}$. Let

$$\mathcal{C}' = \mathcal{C} \cup \big\{\{1,3,4\},\{2,3,4\}\big\} \setminus \big\{\{1,2,3\},\{1,2,4\}\big\}.$$

Then, $\mathcal{C}'$ is called a *flip* of $\mathcal{C}$.

Clearly, if $\mathcal{C}'$ is a flip of $\mathcal{C}$, then $\mathcal{C}$ is a flip of $\mathcal{C}'$ too.



Corollary 4.3.7 (Flip)**.** *Let* $\mathcal{C}$ *be a* 3*-uniform clutter on* $[n]$ *and* $\mathcal{C}'$ *be a flip of* $\mathcal{C}$*. Then,*

$$\operatorname{reg} I(\bar{\mathcal{C}}) = \operatorname{reg} I(\bar{\mathcal{C}}').$$

Proof. With the same notation in the Definition 4.3.6, let

$$\mathcal{C}'' = \mathcal{C} \cup \big\{\{1,3,4\},\{2,3,4\}\big\}.$$

Theorem 4.3.5 applied to $\{3,4\}$, shows that $\operatorname{reg} I(\bar{\mathcal{C}}'') = \operatorname{reg} I(\bar{\mathcal{C}})$. Using Theorem 4.3.5 again applied to $\{1,2\}$, we conclude that $\operatorname{reg} I(\bar{\mathcal{C}}'') = \operatorname{reg} I(\bar{\mathcal{C}}')$. So that $\operatorname{reg} I(\bar{\mathcal{C}}) = \operatorname{reg} I(\bar{\mathcal{C}}')$ as desired. □

For our next theorem, we use the following important lemma.

Lemma 4.3.8. *Let* $\mathcal{C}$ *be a* 3*-uniform clutter on* $[n]$ *such that,* $F = \{1,2,3\} \in \mathcal{C}$ *and for all* $r > 3$*, we have:*

$$\big\{\{1,2,r\},\{1,3,r\},\{2,3,r\}\big\} \not\subseteq \mathcal{C}. \tag{4.16}$$

Let $\mathcal{C}_1 = \mathcal{C} \setminus F$ *and* $I = I(\bar{\mathcal{C}}), I_1 = I(\bar{\mathcal{C}}_1)$ *be the ideals in* $S = K[x_1, \ldots, x_n]$*. Then,*

(i) $\operatorname{depth} \dfrac{S}{I^\vee + (x_1, x_2, x_3)} \geq \operatorname{depth} \dfrac{S}{I^\vee} - 1.$

(ii) $\operatorname{depth} \dfrac{S}{I_1^\vee} \geq \operatorname{depth} \dfrac{S}{I^\vee}.$

Proof. Let $t := \operatorname{depth} S/I^\vee \leq \dim S/I^\vee = n-3$.
(i) One can easily check that, condition (4.16) is equivalent to say that:

$$\text{for all } r > 3, \text{ there exist } F \in \bar{\mathcal{C}} \text{ such that, } P_F \subset (x_1, x_2, x_3, x_r).$$

Therefore,

$$\begin{aligned} I^\vee = \bigcap_{F \in \bar{\mathcal{C}}} P_F &= \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right) \cap ((x_1, x_2, x_3, x_4) \cap \cdots \cap (x_1, x_2, x_3, x_n)) \\ &= \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right) \cap (x_1, x_2, x_3, x_4 \cdots x_n) \\ &= I^\vee \cap (x_1, x_2, x_3, x_4 \cdots x_n). \end{aligned}$$

Clearly, $x_4 \cdots x_n \in I^\vee$. Thus, from Theorem 4.1.1, we get the long exact sequence:

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee}\right) \oplus H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{(x_1,x_2,x_3,x_4\cdots x_n)}\right) \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee+(x_1,x_2,x_3)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee}\right) \to \cdots$$

So that,

$$H_{\mathfrak{m}}^{i-1}\left(\frac{S}{I^\vee + (x_1, x_2, x_3)}\right) = 0, \qquad \text{for all } i < t \leq n-3. \tag{4.17}$$

This proves inequality (i).
(ii) Clearly, $I_1^\vee = I^\vee \cap (x_1, x_2, x_3)$. from Theorem 4.1.1, we get the long exact sequence:

$$\cdots \to H_{\mathfrak{m}}^{i-1}\left(\tfrac{S}{I^\vee+(x_1,x_2,x_3)}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I_1^\vee}\right) \to H_{\mathfrak{m}}^{i}\left(\tfrac{S}{I^\vee}\right) \oplus H_{\mathfrak{m}}^{i}\left(\tfrac{S}{(x_1,x_2,x_3)}\right) \to \cdots.$$

So that by (4.17), we have:

$$H_{\mathfrak{m}}^{i}\left(\frac{S}{I_1^\vee}\right) = 0, \qquad \text{for all } i < t \leq n-3.$$

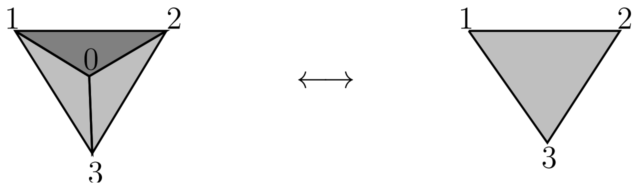
□

Theorem 4.3.9 (Subdivision)**.** *Let $\mathcal{C}$ be a 3-uniform clutter on $[n]$ such that $F = \{1,2,3\} \in \mathcal{C}$ and for all $r > 3$, $\big\{\{1,2,r\},\{1,3,r\},\{2,3,r\}\big\} \nsubseteq \mathcal{C}$. Let*

$$\mathcal{C}_1 = \mathcal{C} \setminus F, \quad \mathcal{C}' = \mathcal{C}_1 \cup \big\{\{0,1,2\},\{0,1,3\},\{0,2,3\}\big\},$$

and $I = I(\bar{\mathcal{C}}), J = I(\bar{\mathcal{C}}')$ be the circuit ideals in the polynomial ring $S = K[x_0, x_1, \dots, x_n]$. Then,

$$\operatorname{reg}(I) = \operatorname{reg}(J).$$



Proof. By Remark 3.3.6, it is enough to show that, depth $S/I^\vee =$ depth $S/J^\vee$.

Let $I_1 = I(\bar{\mathcal{C}}_1) \subset K[x_1, \dots, x_n]$. Clearly, $I_1^\vee = (x_1,x_2,x_3) \cap I^\vee$ and

$$\begin{aligned} J^\vee &= I_1^\vee \cap \left(\bigcap_{i=4}^{n} (x_0,x_1,x_i)\right) \cap \left(\bigcap_{i=4}^{n} (x_0,x_2,x_i)\right) \cap \left(\bigcap_{3 \le i < j \le n} (x_0,x_i,x_j)\right) \\ &= (x_0, x_4 \cdots x_n, x_1x_2x_3\ \hat{x}_4 \cdots x_n, \dots, x_1x_2x_3\ x_4 \cdots \hat{x}_n) \cap I_1^\vee. \end{aligned}$$

Let T be the ideal $T = (x_0, x_4 \cdots x_n, x_1x_2x_3\ \hat{x}_4 \cdots x_n, \dots, x_1x_2x_3\ x_4 \cdots \hat{x}_n)$. Then, $J^\vee = I_1^\vee \cap T$ and by Lemma 4.1.6, depth $\frac{S}{T} = n - 2$. Moreover, our assumption implies that for all $i > 3$, there exists $F \in \bar{\mathcal{C}}$ such that, $P_F \subset (x_1,x_2,x_3,x_r)$. So that:

$$\begin{aligned} I_1^\vee + T &= (x_0, x_4 \cdots x_n, I_1^\vee) \\ &= (x_0) + \left(x_4 \cdots x_n, \left[(x_1,x_2,x_3) \cap \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right)\right]\right) \\ &= (x_0) + \left((x_1,x_2,x_3,x_4) \cap \cdots \cap (x_1,x_2,x_3,x_n) \cap \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right)\right) \\ &= (x_0) + \left(\bigcap_{F \in \bar{\mathcal{C}}} P_F\right) = (x_0, I^\vee). \end{aligned} \tag{4.18}$$

Hence by Lemma 4.3.8(ii),

$$\operatorname{depth} \frac{S}{I_1^\vee + T} = \operatorname{depth} \frac{S}{I^\vee} - 1 \le \operatorname{depth} \frac{S}{I_1^\vee} - 1.$$

Thus,

$$\operatorname{depth}\frac{S}{T} \geq \operatorname{depth}\frac{S}{I_1^\vee} > \operatorname{depth}\frac{S}{I_1^\vee + T}.$$

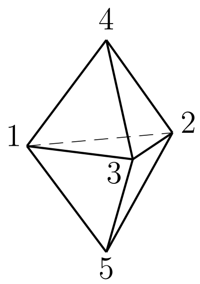
Using Proposition 4.1.2 and (4.18), we have:

$$\operatorname{depth}\frac{S}{J^\vee} = 1 + \operatorname{depth}\frac{S}{I_1^\vee + T} = \operatorname{depth}\frac{S}{I^\vee}.$$

□

Let $\mathcal{S}^2$ be a sphere in $\mathbb{R}^3$, Δ be a triangulation of $\mathcal{S}^2$ and $\mathcal{C} = \mathcal{F}(\Delta)$ be the corresponding 3-uniform clutter. We may use our reduction process on $\mathcal{C}$ to show that, the circuit ideal of $\mathcal{C}$ does not have 3-linear resolution, but the circuit ideal of any proper subclutter of $\mathcal{C}$ has a 3-linear resolution. In Chapter 5, we generalize this result to arbitrary triangulation of a Pseudo-manifold. First, we need the following lemma.

Lemma 4.3.10. *Let $\mathfrak{T}$ be a hexahedron. Then, the circuit ideal of $\mathfrak{T}$ does not have linear resolution. If $\mathfrak{T}'$ be a proper subclutter of $\mathfrak{T}$, then the circuit ideal of $\mathfrak{T}'$ has a 3-linear resolution.*



Proof. Let $I = I(\bar{\mathfrak{T}})$. We know that, $\bar{\mathfrak{T}} = \{145, 245, 345, 123\}$. So that,

$$I^\vee = (x_1x_2x_3, x_4, x_5) \cap (x_1, x_2, x_3) \subset S := K[x_1, \ldots, x_5].$$

It follows from Theorem 4.1.1 that, $H^1_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \neq 0$. Since $\dim S/I^\vee = 5-3 = 2$, we conclude that, $S/I^\vee$ is not Cohen-Macaulay. Hence, the ideal I does not have linear resolution (Theorem 3.3.3).

The second part of the theorem, is a direct conclusion of Corollary 4.3.4. □

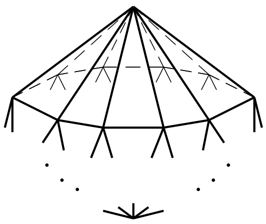
Theorem 4.3.11. *Let $S = K[x_1, \ldots, x_n]$. Let $\mathfrak{P}_n$ be the clutter defined by a triangulation of the sphere $\mathcal{S}^2$ with $n \geq 5$ vertices and let $I \subset S$ be the circuit ideal of $\overline{\mathfrak{P}}_n$. Then,*

(i) *For any proper subclutter $\mathcal{C}_1 \subset \mathfrak{P}_n$, the ideal $I(\bar{\mathcal{C}}_1)$ has a 3-linear resolution.*

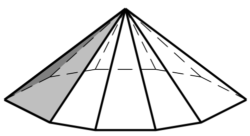
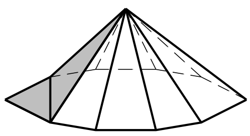
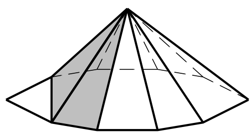
(ii) *S/I does not have linear resolution.*

Proof. (i) Let $r = |\mathcal{C}|$. Corollary 4.3.4 proves the claim for $r \leq 3$. We assume that $r > 3$ and the theorem is true for all pure subclutter of $\mathfrak{P}_n$ with less than r elements. If $\mathcal{C}_1$ is a proper subclutter of $\mathfrak{P}_n$ with r elements, then $\mathcal{C}_1$ has an edge e such that e is contained in only one circuit F of $\mathcal{C}_1$. Let $\mathcal{C}_2 = \mathcal{C}_1 \setminus F$. By induction hypothesis, $I(\bar{\mathcal{C}}_2)$ has a linear resolution. Using Corollary 4.3.4, the ideal $I(\bar{\mathcal{C}}_1)$ has a linear resolution too.

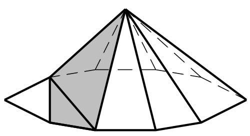
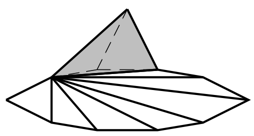
(ii) The proof is by induction on the number of vertices. First step of induction is Lemma 4.3.10. Let $n > 5$. If there is a vertex of degree 3, then by Theorem 4.3.9, we can remove this vertex and three circuits containing it and add a new circuit instead. Then we have a clutter with fewer vertices and by induction hypothesis and Theorem 4.3.9, S/I does not have linear resolution. Now, we take any vertex u of degree > 3 and the circuits containing this (See the following illustrations).
Using several flips (Corollary 4.3.7), we can reduce our triangulation to another one such that, one find only 3 circuits which contains the vertex u. Now, using Theorem 4.3.9 and Corollary 4.3.2, we get a triangulation of the sphere with $n-1$ vertices, which does not have linear resolution by induction hypothesis.



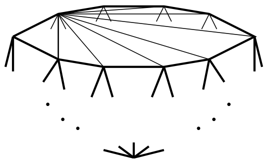
1. A triangulation of sphere with n vertices and m faces

2 3 4

 – – – 

5 6 7



8. A triangulation of sphere with $n-1$ vertices and $m-2$ faces

□

4.4 Chordal Clutters

Though that the problem of classification of monomial ideals with d-linear resolution is solved for $d = 2$ (Fröberg's Theorem [Fr]), it is still open for $d > 2$. In case $d = 2$, the linearity of resolution does not depend on the characteristics of the ground field. To generalize the Fröberg's result to higher dimensional clutters, we face that linearity of resolution of a circuit ideal of a d-uniform clutter for $d > 2$, depends on the characteristics of the ground field. For instance, the ideal corresponding to triangulation of the projective plane has a linear resolution in characteristics zero while it does not have linear resolution in characteristics 2 (c.f. Example 5.3.19).
In [Em, ThVt, VtV, W], the authors have partially generalized the Fröberg's Theorem. They have introduced several definitions of chordal clutters and proved that, the corresponding circuit ideals have a linear resolution. In this section, we compare these definition to each other and then we introduce a large class of ideals with d-linear resolution.

4.4.1 E-Chordal Clutters

E. Emtander [Em], has defined generalized chordal clutters as the following:

Definition 4.4.1. A *generalized chordal clutter* is a d-uniform clutter, obtained inductively as follows:

(a) $\mathcal{C}_{n,d}$ is a generalized chordal clutter.

(b) If $\mathcal{G}$ is generalized chordal clutter, then so is $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$, for all $0 \leq i < n$.

(c) If $\mathcal{G}$ is generalized chordal and $V \subset V(\mathcal{G})$ is a finite set with $|V| = d$ and at least one element of $\{F \subset V : |F| = d-1\}$ is not a subset of any element of $\mathcal{G}$, then $\mathcal{G} \cup V$ is generalized chordal.

Let us show the generalized chordal clutter, as defined above, by E-*chordal* clutter.

Emtander has proved that the circuit ideal of generalized chordal clutters have a d-linear resolution over any field K (c.f. [Em, Theorem 5.1]). For $d = 3$, we can recover this result as a consequence of our reduction process.

Theorem 4.4.2. *If $\mathcal{C} \neq \mathcal{C}_{n,3}$ is an* E-*chordal* 3-*uniform clutter, then its circuit ideal $I = I(\bar{\mathcal{C}})$ has a* 3-*linear resolution over any field K.*

Proof. If $\mathcal{C}$ has a circuit F, with the property (c) in Definition 4.4.1, then $\mathcal{C}$ has a simplicial edge. Hence, the induction and Theorem 4.3.3 yield the conclusion. So we may assume that, $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,3}} \mathcal{C}_{n,3}$. In this case, fix a vertex $a \in V(\mathcal{C}_{n,3}) \setminus V(\mathcal{C}_{i,3})$. If e_0 is an edge of $\mathcal{C}$ which contains a, then e_0 is a simplicial edge. Put $\mathcal{C}_1 = \mathcal{C} \setminus e_0 = \{F \in \mathcal{C} \colon e_0 \nsubseteq F\}$. If there exits a circuit in $\mathcal{C}_1$ which contains a, take an edge e_1 of $\mathcal{C}_1$ which contains a. It is clear that, e_1 is again a simplicial edge of $\mathcal{C}_1$ because, $N_{\mathcal{C}_1}[e_1] \subset [n] \setminus \{b\}$, where b is the unique element such that $e_0 = \{a, b\}$. Going on in this way, after finite step, we arrive to a clutter $\mathcal{C}_r$, which is obtained from $\mathcal{C}$ by successive removing of simplicial edges and a does not belong to any circuit of $\mathcal{C}_r$. Our inductive method, implies that $\mathcal{C}_r = \mathcal{C}_{n-1,3} \cup_{\mathcal{C}_{i,3}} \mathcal{G}$. Hence, Theorem 4.3.3 and induction hypothesis implies that, the ideal $\operatorname{reg} I(\bar{\mathcal{C}}) = 3$. That is, the ideal $I(\bar{\mathcal{C}})$ has a 3-linear resolution (over any field K). $\square$

Remark 4.4.3. The same proof of the Theorem 4.4.2, shows that any 3-uniform E-chordal clutter can be reduced to some isolated point after finite step of removing simplicial submaximal circuit.

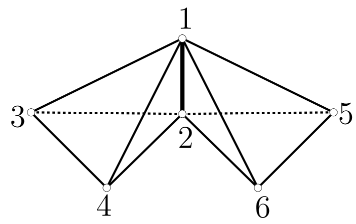
In [Em], the author asked the following question:

Question. Is there a clutter $\mathcal{C}$ such that the circuit ideal $I(\bar{\mathcal{C}})$ has linear resolution over any field K, but $\mathcal{C}$ is not chordal clutter?

To answer this question, consider the following example:

Example 4.4.4. Consider the following 3-uniform clutter,

$$\mathcal{C} = \{123, 124, 134, 234, 125, 126, 156, 256\}$$



Theorem 4.3.3 shows that, the ideal $I(\bar{\mathcal{C}})$ has a 3-linear resolution over any field K, but $\mathcal{C}$ is not generalized chordal clutter as in Definition 4.4.1.

Conjecture 4.4.5. *If $\mathcal{C}$ is a d-uniform E-chordal clutter, then the circuit ideal $I = I(\bar{\mathcal{C}})$ has linear quotients.*

4.4.2 W-Chordal Clutters

Another attempt for extending the definition of chordal graphs to chordal clutters, is the result of R. Woodroofe in [W]. To state his definition of chordal clutter, first we need the following prerequisites.

Given a clutter $\mathcal{C}$, there are two ways of removing a vertex that are of interest. Let $v \in V(\mathcal{C})$. The *deletion*, $\mathcal{C} \setminus v$, is the clutter on the vertex set $V(\mathcal{C}) \setminus \{v\}$, with circuits $\{F \in \mathcal{C} : \quad v \notin \mathcal{C}\}$. The *contraction*, $\mathcal{C}/v$, is the clutter on the vertex set $V(\mathcal{C}) \setminus \{v\}$, with circuits the minimal sets of $\{F \setminus \{v\} : \quad F \in \mathcal{C}\}$. Thus, $\mathcal{C} \setminus v$ deletes all circuits containing v, while $\mathcal{C}/v$ removes v from each circuit containing it and then removes any redundant circuits.

A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by repeated deletion and/or contraction is called a *minor* of $\mathcal{C}$. It is straightforward to prove that, if $v \neq w$ are vertices, then:

$$(\mathcal{C} \setminus v) \setminus w = (\mathcal{C} \setminus w) \setminus v, \quad (\mathcal{C}/v)/w = (\mathcal{C}/w)/v, \quad (\mathcal{C} \setminus v)/w = (\mathcal{C}/w) \setminus v.$$

Definition 4.4.6 (W-chordal)**.** Let $\mathcal{C}$ be a clutter. A vertex v of $\mathcal{C}$ is *simplicial*, if for every two circuits F_1 and F_2 of $\mathcal{C}$ that contain v, there is a third circuit F_3 such that, $F_3 \subset (F_1 \cup F_2) \setminus \{v\}$.

A clutter $\mathcal{C}$ is W-*chordal*, if every minor of $\mathcal{C}$ has a simplicial vertex.

In the case where G is a graph, Definition 4.4.6 obviously agrees with the previous definition of a simplicial vertex of G.

Example 4.4.7. The followings are examples of W-choral clutters:

1. **Chordal graphs**: If G is a graph, then G/v is (up to singleton circuits) the induced subgraph $G \setminus N[v]$. Hence, Theorem 4.2.11, implies that the definition of W-chordal clutter specializes in graphs to the usual definition of chordal graphs.

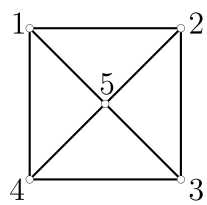
2. **Complete d-uniform clutter**: Since $\mathcal{C}_{n,d}/v \cong \mathcal{C}_{n-1,d-1}$ and $\mathcal{C}_{n,d} \setminus v \cong \mathcal{C}_{n-1,d}$ and every vertex is simplicial, the complete d-uniform clutter is W-chordal.

3. **Matroid circuits**: The simplicial vertex condition is exactly the weak circuit exchange property of matroids at a single vertex v (see [Du, Ox]). Thus every vertex of a matroid circuit clutter is simplicial. Since every deletion or contraction of a matroid gives another matroid, the circuit clutter of any matroid is W-chordal.
   For example, $\mathcal{C}_{n,d}$ is the circuit clutter of a uniform matroid.

If $\mathcal{C}$ is a clutter, then define $c_d(\mathcal{C})$ to be the clutter with the same vertex set V as $\mathcal{C}$ and circuit set $\{F \subset V : \quad |F| = d, F \notin \mathcal{C}\}$. In the special case that $\mathcal{C}$ is d-uniform, we have $c_d(\mathcal{C}) = \bar{\mathcal{C}}$.

Theorem 4.4.8 ([W, Corollary 6.10]). *If $\mathcal{C}$ is a W-chordal clutter with minimum circuit cardinality d, then the ideal $I = I(c_d(\mathcal{C}))$ has linear quotients, hence a linear resolution.*

Remark 4.4.9. Let $\mathcal{C} = \{125, 235, 345, 145\}$ be the a 3-uniform clutter and $I = I(\bar{\mathcal{C}})$ be the corresponding circuit ideal.



Then, we have the followings:

1. Theorem 4.3.3 shows that, the ideal I has a 3-linear resolution. Moreover, direct computation shows that I has linear quotients.

2. The clutter $\mathcal{C}$ is not W-chordal because, non of the vertices of $\mathcal{C}$ is simplicial. Hence, the converse of Theorem 4.4.8 is not true.

3. It is easy to check that, $\mathcal{C}$ is E-chordal. Hence, the class of d-uniform E-chordal clutters is not contained in the class of W-chordal clutters. Also, Example 4.4.4 is a 3-uniform clutter which is W-chordal but not E-chordal.

4.4.3 Open Problems

Let $\mathscr{C}_d$ be the class of d-uniform clutters which can be transformed to empty set after a sequence of deletions of simplicial submaximal circuits.

Using Theorem 4.3.3, we have:

Theorem 4.4.10. *If $\mathcal{C} \in \mathscr{C}_d$, then the ideal $I(\bar{\mathcal{C}})$ has a d-linear resolution over any field K.*

Also, Remark 4.4.3 shows that, the class of 3-uniform E-chordal clutter is contained in $\mathscr{C}_3$ and Example 4.4.4 shows that, this inclusion is strict.

All the examples and evidences shows that if $\mathcal{C}$ is a d-uniform W-chordal clutter, then $\mathcal{C} \in \mathscr{C}_d$. This leads us to the conjecture that:

Conjecture 4.4.11. *If $\mathcal{C}$ is a d-uniform W-chordal clutter, then $\mathcal{C} \in \mathscr{C}_d$.*

Note that Example 4.4.9, gives an example $\mathcal{C} \in \mathscr{C}_3$ which is not W-chordal.

More generally, our computations and intuition lead us to the following characterization of ideals with linear quotients.

Conjecture 4.4.12. *Let $2 \leq d \leq 3$ and $\mathcal{C}$ be a d-uniform clutter and $I = I(\bar{\mathcal{C}})$ be the circuit ideal of $\mathcal{C}$. The ideal I has linear quotients if and only if $\mathcal{C} \in \mathscr{C}_d$.*

Note that once we can prove the Conjecture 4.4.12, then we can conclude all Theorems and other conjectures in this section.

Some evidences. Conjecture 4.4.12 is true for $d = 2$. Below, we explain why this conjecture is true for $d = 2$. First we need the following easy lemma.

Lemma 4.4.13. *$\mathscr{C}_2$ is exactly the class of chordal graphs.*

Proof. Let $G \in \mathscr{C}_2$. If $G = \mathcal{C}_{n,2}$, then clearly G is chordal. So, assume that $G \neq \mathcal{C}_{n,2}$. Then by Theorem 4.3.3, the ideal $I(\bar{G})$ has a 2-linear resolution. So that, Corollary 4.2.7 implies that, G is chordal.

For the other direction, we use induction on the number of vertices of a graph G, $|V(G)|$ to prove that $G \in \mathscr{C}_2$. For a graph G with $|V(G)| = 1$ there exist nothing to prove. Assume that every chordal graph with less than n vertices is in $\mathscr{C}_2$ and G be a chordal graph with $n > 1$ vertices. By Theorem 4.2.11, the graph G has simplicial vertex v. we let $G_1 = G \setminus v$. Then G_1 is again a chordal graph with less than n vertices. Hence $G_1 \in \mathscr{C}_2$. Since v is a simplicial vertex of G, we conclude that $G \in \mathscr{C}_2$. □

We have the remarkable result of Herzog et al. in [HHZ] which states that:

Theorem 4.4.14 ([HHZ, Theorem 3.2])**.** *Let I be a monomial ideal generated in degree 2. The following conditions are equivalent:*

(i) *I has a linear resolution.*

(ii) *I has linear quotients.*

(iii) *Each power of I has a linear resolution.*

Using Lemma 4.4.13, Theorem 4.2.1, 4.4.14 implies that:

$$\begin{aligned} G \in \mathscr{C}_2 &\Leftrightarrow G \text{ is Chordal,} \\ &\Leftrightarrow I(\bar{G}) \text{ has a 2-linear resolution,} \\ &\Leftrightarrow I(\bar{G}) \text{ has linear quotient.} \end{aligned}$$

Chapter 5

Obstruction to d-linearity

Toward a partial classification of monomial ideals with d-linear resolution, in this chapter, some classes of d-uniform clutters which do not have linear resolution, but every proper subclutter of them has a d-linear resolution, are introduced and the regularity and Betti numbers of circuit ideals of such clutters are computed. Note that, in a new proof of Fröberg's Theorem in Chapter 4, the notion of cycle plays a key role in the proof. That means:

(1) Cycles are minimal to 2-linearity.

(2) The ideal $I(\bar{G})$ does not have 2-linear resolution if and only if G contains a cycle of length > 3, as induced subgraph.

The first problem to generalize Fröberg's Theorem, is to find a similar notion for cycles in arbitrary d-uniform clutters. Trying to find a similar notion for cycles, in this chapter, we introduce the notion of minimal to d-linearity in arbitrary d-uniform clutters. By Proposition 5.3.15, pseudo-manifolds have the property of minimal to d-linearity. However, Example 5.3.16, shows that the class of pseudo-manifolds is strictly contained in the class of minimal to linearity clutters.

We will show that, if $\mathcal{C}$ is a d-uniform clutter which has an induced subclutter isomorphic to a d-dimensional pseudo-manifold, then the ideal $I(\bar{\mathcal{C}})$ does not have linear resolution (see Remark 5.3.2).

Another difficulty for generalizing the Fröberg's Theorem, is the term 'induced' in (2). That is, there are clutters which do not have a linear resolution and do not have any **induced** subclutter which is minimal to d-linearity. For instance, see the Example 5.3.12

To attack to the problem of classification of ideas with d-linear resolution, in this chapter, we investigate clutters whose their circuit ideals do not have linear resolution, but any proper subclutter of them has a linear resolution. This chapter generalize many of the results in Section 4.2 for arbitrary d-uniform clutters. But the method here is not algebraic combinatorics but is algebraic topology.

5.1 Complexes Δ with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$

As we shall see later, the circuit ideals of clutters which are minimal to linearity are contained in the class of square-free monomial ideals I_Δ, with $\operatorname{indeg}(I_\Delta) = 1 + \dim \Delta$ (see Definition 5.3.1). So, in this section, we deal with the class of square-free monomial ideals I_Δ with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$.

Let Δ be a $(d-1)$-dimensional simplicial complex such that, $\operatorname{indeg}(I_\Delta) \geq d$. The main property of Δ is that, it contains all faces of dimension $d-2$. Hence, Δ contains all faces of dimension $-1, 0, \ldots, d-2$. So that,

$$f_i = \binom{n}{i+1}, \quad i = -1, \ldots, d-2. \tag{5.1}$$

As a consequence of Proposition 3.2.13, we have:

Corollary 5.1.1. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that,* $\operatorname{indeg}(I_\Delta) \geq d$. *Then,*

$$\dim_K \tilde{H}_{d-2}(\Delta; K) - \dim_K \tilde{H}_{d-1}(\Delta; K) = \sum_{i=0}^{d-1} (-1)^{d+i-1} \binom{n}{i} - e(S/I_\Delta). \tag{5.2}$$

Proof. Using (3.3), Proposition 3.2.13 and (5.1), we have:

$$(-1)^{d-2} \dim_K \tilde{H}_{d-2}(\Delta; K) + (-1)^{d-1} \dim_K \tilde{H}_{d-1}(\Delta; K) = -1 + (-1)^{d-1} f_{d-1} + \sum_{i=0}^{d-2} (-1)^i \binom{n}{i+1}.$$

Since $e(S/I_\Delta) = f_{d-1}$, we get the conclusion. □

The following theorems, extend some results of Terai and Yoshida (c.f. [TY]).

Cohen-Macaulayness and Regularity

Theorem 5.1.2. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that,* $\operatorname{indeg}(I_\Delta) \geq d$. *Then,*

(i) *If* $\beta^K_{i,j}(I_\Delta) \neq 0$, *then* $1 \leq j \leq n$ *and* $d \leq j - i \leq d+1$.

(ii) $d \leq \operatorname{reg}(I_\Delta) \leq d+1$.

(iii) $\operatorname{indeg}(I_\Delta) \leq d+1$ *and equality holds if and only if* I_Δ *has a* $(d+1)$-*linear resolution.*

(iv) $(n-d) - 1 \leq \operatorname{projdim}(I_\Delta) \leq n - d$.

Proof. (i) If $\beta^K_{i,j}(I_\Delta) \neq 0$, then by Theorem 3.2.12, there exists $W \subset [n]$, such that:

$$|W| = j \quad \text{and} \quad \tilde{H}_{j-i-2}(\Delta_W; K) \neq 0.$$

So that, $1 \leq j \leq n$ and by Proposition 3.2.13, $d-2 \leq j-i-2 \leq d-1$. That is, $d \leq j - i \leq d+1$.

(ii) By part (i), we have:

$$d \leq \operatorname{indeg}(I_\Delta) \leq \operatorname{reg}(I_\Delta) = \max\{j - i : \quad \beta^K_{i,j}(I_\Delta) \neq 0\} \leq d+1.$$

(iii) If $x_{i_1} \cdots x_{i_j} \in I_\Delta$, then $\beta^K_{0,j} \neq 0$. So that by (i), $j \leq d+1$. In particular, $\operatorname{indeg}(I_\Delta) \leq d+1$.

If $\operatorname{indeg}(I_\Delta) = d+1$, then $\operatorname{reg}(I_\Delta) \geq d+1$ and by (ii), I_Δ has a $(d+1)$-linear resolution. On the other hand, if I_Δ has $(d+1)$-linear resolution, then each generator of I_Δ has degree $d+1$. In particular, $\operatorname{indeg}(I_\Delta) = d+1$.

(iv) Let $\rho = \operatorname{projdim}(I_\Delta)$. By Theorem 1.6.9,

$$\rho + 1 = \operatorname{projdim} \frac{S}{I_\Delta} = n - \operatorname{depth} \frac{S}{I_\Delta} \geq n - \dim \frac{S}{I_\Delta} = n - d.$$

Hence, $\rho \geq (n-d) - 1$.

On the other hand, $\beta^K_\rho(I_\Delta) \neq 0$. Hence, there exists $1 \leq j \leq n$ such that, $\beta^K_{\rho,j} \neq 0$. So by (i), $j - \rho \geq d$. This implies that, $\rho \leq j - d \leq n - d$. ∎

Theorem 5.1.3. *Let* $S = K[x_1, \ldots, x_n]$ *be the polynomial ring over a field* K *and* Δ *be a* $(d-1)$-*dimensional simplicial complex such that,* $\operatorname{indeg}(I_\Delta) \geq d$. *Then,* S/I_Δ *is Cohen-Macaulay if and only if* $\tilde{H}_{d-2}(\Delta; K) = 0$.

Proof. We know that $\dim S/I_\Delta = d$. So that Theorem 1.6.9, implies that:

$$S/I_\Delta \text{is Cohen-Macaulay} \iff \operatorname{projdim} S/I_\Delta = (n-d).$$

In view of Theorem 5.1.2(iv), it is enough to prove that:

$$\operatorname{projdim} S/I_\Delta = (n-d) + 1 \Longleftrightarrow \tilde{H}_{d-2}(\Delta; K) \neq 0.$$

($\Leftarrow$) If $\tilde{H}_{d-2}(\Delta; K) \neq 0$, then by Theorem 3.2.12, $\beta^K_{(n-d)+1,n}(S/I_\Delta) \neq 0$. So that, $\operatorname{projdim} S/I_\Delta \geq (n-d)+1$. Hence by Theorem 5.1.2(iv), $\operatorname{projdim} S/I_\Delta = (n-d)+1$.

($\Rightarrow$) If $\operatorname{projdim} S/I_\Delta = (n-d)+1$, then $\beta^K_{(n-d)+1}(S/I_\Delta) \neq 0$. Hence there exists $1 \leq j \leq n$ such that, $\beta^K_{(n-d)+1,j}(S/I_\Delta) \neq 0$. So by Theorem 5.1.2(i), $j \geq n$. Hence $j = n$. Thus,

$$\begin{aligned}
0 \neq \beta^K_{(n-d)+1}\left(\frac{S}{I_\Delta}\right) &= \sum_{j=1}^{n} \beta^K_{(n-d)+1,j}\left(\frac{S}{I_\Delta}\right) \\
&= \beta^K_{(n-d)+1,n}\left(\frac{S}{I_\Delta}\right) \\
&= \dim \tilde{H}_{d-2}(\Delta; K). \qquad \text{(By Theorem 3.2.12)}
\end{aligned}$$

□

Now, let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that, $\operatorname{indeg}(I_\Delta) = d$. As a consequence of Theorem 5.1.2, we conclude that:

Corollary 5.1.4. *Let Δ be a a $d-1$-dimensional simplicial complex on $[n]$ such that, $\operatorname{indeg}(I_\Delta) = d$. Then, $I = I_\Delta$ has a d-linear resolution if and only if $\tilde{H}_{d-1}(\Delta; K) = 0$.*

Proof. If I has a d-linear resolution, then by Theorem 3.2.12, we have:

$$0 = \beta^K_{n-d-1,n}(I_\Delta) = \dim_K \tilde{H}_{d-1}(\Delta; K).$$

Assume that, I does not have d-linear resolution. Then by Theorem 5.1.2(ii), we have:

$$d + 1 = \operatorname{reg}(I) = \max\{j - i : \quad \beta^K_{i,j}(I_\Delta) \neq 0\}.$$

Let $d+1 = j_0 - i_0$ and $\beta^K_{i_0,j_0}(I_\Delta) \neq 0$. Then, by Theorem 3.2.12, there exists $W \subset [n]$, such that:

$$|W| = j_0 \quad \text{and} \quad \tilde{H}_{d-1}(\Delta_W; K) \neq 0.$$

This in particular implies that $\tilde{H}_{d-1}(\Delta; K) \neq 0$, for

$$\tilde{H}_{d-1}(\Delta_W; K) \subset \tilde{H}_{d-1}(\Delta; K).$$

□

h-Vector

Let Δ be a simplicial complex with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$. Using (5.1) and Theorem 3.2.5, we can find the Hilbert function of I_Δ, if we know the multiplicity $e(S/I_\Delta) = f_{d-1}(\Delta)$.

Corollary 5.1.5. *Let Δ be a a $(d-1)$-dimensional simplicial complex over $[n]$ such that,* $\operatorname{indeg}(I_\Delta) \geq d$. *Let $\mathbf{h}(\Delta) = (h_0, \dots, h_d)$ be the $\mathbf{h}$-vector of Δ. Then,*

$$h_k(\Delta) = \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} \binom{n}{i}, & 0 \leq k \leq d-1, \\ \\ f_{d-1} + \sum_{i=0}^{d-1} (-1)^{d-i} \binom{n}{i}, & k = d. \end{cases}$$

Minimal Free Resolution

Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ with $\operatorname{indeg}(I_\Delta) \geq d$. As it is proved in Theorem 5.1.2(i), the possible non-zero Betti numbers of I_Δ are $\beta^K_{i,i+d}(I_\Delta)$ and $\beta^K_{i,i+d+1}(I_\Delta)$, for $i = 0, \dots, \operatorname{projdim}(I_\Delta)$. This means that, in the graded minimal free resolution of I_Δ, in each position, there exist at most two degrees in which we may have non-zero Betti numbers. The following theorem, extends the well-known Herzog-Kühl equations [HK] in the case that $\beta_{i,d_{i+1}}(M) = 0$, for all $i \leq 0$. We will use this nice theorem to compute the Betti numbers of circuit ideals of clutters which are minimal to linearity (c.f. Theorem 5.3.4).

Theorem 5.1.6 ([DM]). *Let M be a $\mathbb{N}$-graded S-module, ρ its projective dimension and $\mathbf{d} = (d_0 < d_1 < \cdots < d_\rho < d_{\rho+1}) \in \mathbb{N}^{\rho+2}$, such that M has a free resolution with the following form:*

$$\begin{aligned} 0 \to S(-d_{\rho+1})^{\beta_{\rho,d_{\rho+1}}} \oplus S(-d_\rho)^{\beta_{\rho,d_\rho}} \to S(-d_\rho)^{\beta_{\rho-1,d_\rho}} \oplus S(-d_{\rho-1})^{\beta_{\rho-1,d_{\rho-1}}} \to \\ \to \cdots \to S(-d_2)^{\beta_{2,d_2}} \oplus S(-d_1)^{\beta_{1,d_1}} \to S(-d_1)^{\beta_{0,d_1}} \oplus S(-d_0)^{\beta_{0,d_0}} \to M \to 0. \end{aligned}$$

For $1 \leq i \leq \rho$, let $\beta'_i = \beta_{i,d_i} - \beta_{i-1,d_i}$. Then, we have:

(i) *If* $\operatorname{depth}(M) = \dim M$ and $\beta_{\rho,d_{\rho+1}} = 0$, *then for all $1 \leq i \leq \rho$,*

$$\beta'_i = \beta_0 (-1)^i \prod_{\substack{k=1 \\ k \neq i}}^{\rho} \left(\frac{d_k - d_0}{d_k - d_i} \right).$$

(ii) *If* depth(M) = dim M, $\beta_{\rho,d_{\rho+1}} \neq 0$ and $d_0 = 0$, *then for all* $1 \leq i \leq \rho+1$,

$$\beta_i' = (-1)^{i-1} \frac{\beta_0 \left(\prod_{\substack{k=1 \\ k\neq i}}^{\rho+1} d_k \right) - (\rho!)e(M)}{\prod_{\substack{k=1 \\ k\neq i}}^{\rho+1} (d_k - d_i)}.$$

(iii) *If* depth(M) = dim $M-1$, $\beta_{\rho,d_{\rho+1}} = 0$ and $d_0 = 0$, *then for all* $1 \leq i \leq \rho$,

$$\beta_i' = (-1)^{i-1} \frac{\beta_0 \left(\prod_{\substack{k=1 \\ k\neq i}}^{\rho} d_k \right) - (\rho-1)!e(M)}{\prod_{\substack{k=1 \\ k\neq i}}^{\rho} (d_k - d_i)}.$$

Example 5.1.7. Let G be a bipartite graph on vertex set $[n]$. That is, a graph whose vertices can be divided into two disjoint sets U and V such that, every edge connects a vertex in U to one in V. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

Let $I = I(\bar{G})$ be the circuit ideal of G. Since G does not have odd cycle, $\dim \Delta(G) = 1$. Note that, $I = I_{\Delta(G)}$ by Proposition 3.4.6(iii). Hence by Theorem 5.1.2(ii), $2 \leq \operatorname{reg}(I) \leq 3$. Moreover, by Corollaries 4.2.7 and 4.2.12 or 4.2.19, we have:

$$\operatorname{reg}(I) = \begin{cases} 2, & \text{if } G \text{ is a forest} \\ 3, & \text{O.W.} \end{cases}$$

5.2 Regularity of Union of Clutters

Definition 5.2.1. A d-uniform clutter $\mathcal{C}$ is called *decomposable*, if there exist proper subclutters $\mathcal{C}_1$ and $\mathcal{C}_2$ such that, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and either $V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ is a clique or $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$.

In this case, we write $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$. A d-uniform clutter is said to be *indecomposable*, if it is not decomposable. For $d = 2$, this definition coincides with the definition of decomposable graphs in [HH].

Below we will find the regularity of the circuit ideal of $\mathcal{C}$ in terms of circuit ideals of $\mathcal{C}_1$ and $\mathcal{C}_2$, whenever $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$. First we need the following lemma.

Lemma 5.2.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be d-uniform clutters on two vertex sets V_1 and V_2 and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let Δ (res. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$).*

(i) *If $F \in \Delta$, then $F \in \Delta_1 \iff F \subset V_1$.*

(ii) *If $G \subset V_1 \cup V_2$ and $G \cap (V_1 \setminus V_2) \neq \emptyset$, $G \cap (V_2 \setminus V_1) \neq \emptyset$, then $G \in \Delta \iff |G| \leq d-1$.*

(iii) *$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K)$, for all $i > d-2$.*

Proof. (i) This is clear.

(ii) Let G be a subset of $V_1 \cup V_2$, as in (ii). If $|G| \leq d-1$, then by definition, G is a clique in $\mathcal{C}$ and $G \in \Delta$.
Now, let $|G| \geq d$ and $x \in G \cap (V_1 \setminus V_2)$, $y \in G \cap (V_2 \setminus V_1)$. If F be a d-subset of G which contains x, y, then by part (i), $F \notin \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}$. Hence $G \notin \Delta$.

(iii) Let

$$\Delta_3 = \langle G \in \Delta : \quad G \cap (V_1 \setminus V_2) \neq \emptyset,\ G \cap (V_2 \setminus V_1) \neq \emptyset \rangle.$$

Then (i) and (ii) imply that:

$$\dim \Delta_3 = d-2, \qquad \Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3.$$

It is clear that, $\dim \Delta_1 \cap \Delta_3 = \dim \Delta_2 \cap \Delta_3 = d-3$. In particular,

$$\tilde{H}_i\left((\Delta_1 \cup \Delta_2) \cap \Delta_3; K\right) = 0, \qquad \text{for all } i > d-3.$$

Hence from (3.2), for all $i > d-2$, we have:

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K) \oplus \tilde{H}_i(\Delta_3; K) = \tilde{H}_i(\Delta_1 \cup \Delta_2; K).$$

□

Corollary 5.2.3. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a d-uniform clutter and Δ (res. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$). If $V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ is a clique in $\mathcal{C}$, then,*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K), \qquad \textit{for all } i > d-2.$$

Proof. By our assumption, $\Delta_1 \cap \Delta_2$ is a simplex. So that $\tilde{H}_i(\Delta_1 \cap \Delta_2; K) = 0$, for all i. Using (3.2), for all $i > 0$, we have:

$$\tilde{H}_i(\Delta_1 \cup \Delta_2; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K).$$

In addition to Lemma 5.2.2(iii), we get the conclusion. □

Corollary 5.2.4. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a d-uniform clutter and Δ (res. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$). If $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, then:*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K), \qquad \text{for all } i > d-2.$$

Proof. By our assumption, $\dim(\Delta_1 \cap \Delta_2) \leq d-2$. So that $\tilde{H}_i(\Delta_1 \cap \Delta_2; K) = 0$, for all $i > d-2$. Using (3.2), for all $i > d-1$, we have:

$$\tilde{H}_i(\Delta_1 \cup \Delta_2; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K)$$

and $\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \hookrightarrow \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$.

We claim that, $\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \cong \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$.

Proof of claim. Let $\mathscr{C}(\Delta, \partial)$ (res. $\mathscr{C}(\Delta_1, \partial^{(1)})$, $\mathscr{C}(\Delta_2, \partial^{(2)})$) be the chain complex of Δ (res. Δ_1, Δ_2). Since $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, we have:

$$\bigoplus_{\substack{F \in \Delta \\ \dim F = d-1}} KF = \left(\bigoplus_{\substack{F \in \Delta_1 \\ \dim F = d-1}} KF \right) \oplus \left(\bigoplus_{\substack{F \in \Delta_2 \\ \dim F = d-1}} KF \right). \tag{5.3}$$

Take $0 \neq F + \operatorname{Im} \partial_d \in \tilde{H}_{d-1}(\Delta; K)$. Then by (5.3), we can separate F as $F = (c_1 F_1 + \cdots + c_r F_r) + (c'_1 G_1 + \cdots + c'_s G_s)$, where $c_i, c'_i \in K$ and $F_i \in \mathcal{C}_1, G_i \in \mathcal{C}_2$. Let

$$\begin{aligned} \partial_{d-1}(c_1 F_1 + \cdots + c_r F_r) &= (d_1 e_1 + \cdots + d_{r'} e_{r'}) \\ \partial_{d-1}(c'_1 G_1 + \cdots + c'_s G_s) &= (d'_1 f_1 + \cdots + d'_{s'} f_{s'}) \end{aligned}$$

where, $d_i, d'_i \in K$ and $e_i \in \mathrm{SC}(\mathcal{C}_1)$, $f_i \in \mathrm{SC}(\mathcal{C}_2)$. Since

$$\begin{aligned} 0 = \partial_d(F) &= \partial_{d-1}(c_1 F_1 + \cdots + c_r F_r) + \partial_{d-1}(c'_1 G_1 + \cdots + c'_s G_s) \\ &= (d_1 e_1 + \cdots + d_{r'} e_{r'}) + (d'_1 f_1 + \cdots + d'_{s'} f_{s'}) \end{aligned}$$

and $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, we conclude that:

$$\partial_{d-1}(c_1 F_1 + \cdots + c_r F_r) = \partial_{d-1}(c'_1 G_1 + \cdots + c'_s G_s) = 0.$$

This means that the natural map

$$\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \hookrightarrow \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$$

is onto too.

By what we have already proved, we have:

$$\tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K), \quad \text{for all } i > d-2.$$

In addition to Lemma 5.2.2(iii), we get the conclusion. □

Remark 5.2.5. Let $\mathcal{C}_1, \mathcal{C}_2$ be d-uniform clutters on vertex set V_1, V_2 with $V_1 \cup V_2 = [n]$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. For all $W \subset [n]$, one can easily check that:

(i) $\mathcal{C}_W = (\mathcal{C}_1)_W \cup (\mathcal{C}_2)_W$.

(ii) $\Delta_W = \Delta(\mathcal{C}_W)$.

(iii) $\mathrm{SC}((\mathcal{C}_1)_W) \cap \mathrm{SC}((\mathcal{C}_2)_W) \subset \mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2)$.

Hence if $V_1 \cap V_2$ is a clique or $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, then (i)-(iii) together with Corollaries 5.2.3 and 5.2.4, imply that:

$$\tilde{H}_i(\Delta_W; K) \cong \tilde{H}_i((\Delta_1)_W; K) \oplus \tilde{H}_i((\Delta_2)_W; K), \quad \text{for all } i > d-2. \tag{5.4}$$

Now we present the main theorem of this section. This Theorem generalizes Theorem 4.2.14 in Section 4.2.

Theorem 5.2.6. *Let $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ be a d-uniform clutter and I (res. I_1, I_2) be the circuit ideals of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$). Then,*

(i) *$\beta^K_{i,j}(I) \geq \beta^K_{i,j}(I_1) + \beta^K_{i,j}(I_2)$, for $j - i > d$.*

(ii) *If I_1 and I_2 are non-zero ideals, then* $\mathrm{reg}\,(I) = \max\{\mathrm{reg}\,(I_1),\ \mathrm{reg}\,(I_2)\}$.

Proof. (i) Let Δ (res. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$). Then, by (5.4) and Theorem 3.2.12, for $j - i > d$, we have:

$$\begin{aligned}
\beta^K_{i,j}(I_\Delta) &= \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K) \\
&= \sum_{\substack{W \subset [n] \\ |W| = j}} \Big[\dim_K \tilde{H}_{j-i-2}((\Delta_1)_W; K) + \dim_K \tilde{H}_{j-i-2}((\Delta_2)_W; K)\Big] \\
&= \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}((\Delta_1)_W; K) + \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}((\Delta_2)_W; K) \\
&\geq \beta^K_{i,j}(I_{\Delta_1}) + \beta^K_{i,j}(I_{\Delta_2}).
\end{aligned} \tag{5.5}$$

Hence by Proposition 3.4.6(iii), $\beta^K_{i,j}(I) \geq \beta^K_{i,j}(I_1) + \beta^K_{i,j}(I_2)$, whenever $j - i > d$.

(ii) If I has a d-linear resolution, then $\beta_{i,j}^K(I) = 0$, for all $j - i > d$. So that (i) implies that,

$$\beta_{i,j}^K(I_1) = \beta_{i,j}^K(I_2) = 0, \quad \text{for all } j - i > d.$$

This means that, both of ideals I_1 and I_2 have a d-linear resolution and the equality $\mathrm{reg}\,(I) = \max\{\mathrm{reg}\,(I_1),\ \mathrm{reg}\,(I_2)\}$ holds.

Assume that I does not have d-linear resolution. Let

$$r = \mathrm{reg}\,(I) = \max\{j - i\colon \quad \beta_{i,j}^K(I) \neq 0\}$$

and j_0, i_0 be such that $r = j_0 - i_0$ with $\beta_{i_0,j_0}^K(I) \neq 0$. By Theorem 3.2.12, there exists a $W \subset [n]$, such that:

$$|W| = j_0 \quad \text{and} \quad \tilde{H}_{r-2}(\Delta_W; K) \neq 0.$$

Since $r - 2 > d - 2$, from (5.4), we conclude that, either

$$\tilde{H}_{r-2}((\Delta_1)_W; K) \neq 0 \quad \text{or} \quad \tilde{H}_{r-2}((\Delta_2)_W; K) \neq 0.$$

Without loss of generality, we may assume that $\tilde{H}_{r-2}((\Delta_1)_W; K) \neq 0$ and we put $W' = W \cap V(\Delta_1)$. Then, W' is a subset of the vertex set of Δ_1 with the property that, $\tilde{H}_{r-2}((\Delta_1)_{W'}; K) \neq 0$. Using Theorem 3.2.12 once again, we have:

$$\begin{aligned}
\beta_{|W'|-r,|W'|}^K(I_1) &= \sum_{\substack{T \subset V(\Delta_1) \\ |T| = |W'|}} \dim_K \tilde{H}_{r-2}\left((\Delta_1)_T; K\right) \\
&\geq \dim_K \tilde{H}_{r-2}\left((\Delta_1)_{W'}; K\right) > 0.
\end{aligned}$$

Hence, $\beta_{|W'|-r,|W'|}^K(I_1) \neq 0$ and

$$\begin{aligned}
\max\{\mathrm{reg}\,(I_1),\ \mathrm{reg}\,(I_2)\} \geq \mathrm{reg}\,(I_1) &= \max\{j - i\colon\ \beta_{i,j}^K(I_1) \neq 0\} \\
&\geq (|W'|) - (|W'| - r) = r.
\end{aligned}$$

The inequality, $\max\{\mathrm{reg}\,(I_1),\ \mathrm{reg}\,(I_2)\} \leq r$ comes from (i). Putting together these inequalities, we get the conclusion.

□

Remark 5.2.7. Let $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ be a d-uniform clutter on $[n]$ and I (res. I_1, I_2) be the circuit ideals of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$). Let Δ (res. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (res. $\mathcal{C}_1, \mathcal{C}_2$).

- If both of I_1 and I_2 are zero ideals, then Δ_1 and Δ_2 are simplices and they have zero reduced homologies in all degrees. So that

$$\tilde{H}_i(\Delta_W; K) = 0, \quad \text{for all } W \subset [n] \text{ and } i > d-2$$

by (5.4). Hence $\beta^K_{i,j}(I) = 0$, for all $j - i > d$. That is, the ideal I has a d-linear resolution.

- If only one of the ideals I_1 or I_2 is a zero ideal, say I_1, then Δ_1 is a simplex and all the reduced homologies of Δ_1 is zero. Using (5.4), we conclude that $\tilde{H}_i(\Delta_W; K) \cong \tilde{H}_i((\Delta_2)_W; K)$, for all $W \subset [n]$ and $i > d-2$. This implies that:

$$\mathrm{reg}\,(I) = \mathrm{reg}\,(I_2).$$

- If I_1 and I_2 are non-zero ideals, then Theorem 5.2.6(ii) implies that:

$$\mathrm{reg}\,(I) = \max\{\mathrm{reg}\,(I_1),\ \mathrm{reg}\,(I_2)\}.$$

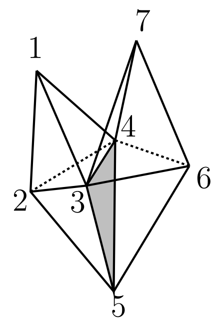
The following example shows that, the inequality

$$\beta_{i,j}(I) \geq \beta_{i,j}(I_1) + \beta_{i,j}(I_2), \quad \text{for } j - i > d$$

in Theorem 5.2.6, may be strict.

Example 5.2.8. Consider the following 3-uniform clutter:

$$\mathcal{C} = \{123, 124, 134, 235, 245, 345, 347, 367, 467, 356, 456\}.$$



$\mathcal{C}$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as follows:

$$\mathcal{C}_1 = \{123, 124, 134, 235, 245, 345\},$$
$$\mathcal{C}_2 = \{345, 347, 367, 467, 356, 456\}.$$

Then, $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ and a direct computation using CoCoA, shows that the minimal free resolution of the ideal $I(\bar{\mathcal{C}})$ is:

$$0 \to S^6(-7) \to S^{30}(-6) \oplus S^2(-7) \to S^{62}(-5) \oplus S^4(-6) \to$$
$$\to S^{61}(-4) \oplus S^2(-5) \to S^{24}(-3) \to I \to 0.$$

Note that $\beta^K_{2,6}(I(\bar{\mathcal{C}}_1)) = \beta^K_{2,6}(I(\bar{\mathcal{C}}_2)) = 0$, while $\beta^K_{2,6}(I(\bar{\mathcal{C}})) = 4$.

5.2.1 Some Applications

Fröberg's Theorem.

Recall that Fröberg's Theorem states that the ideal $I(\bar{G})$ has a 2-linear resolution if and only if G is chordal graph. The difficult part of this theorem is the "only if" part. We have already presented two proofs for this implication (See Corollary 4.2.12 and 4.2.19).

Using Theorem 5.2.6, we may find another proof for this implication. To be more precise, let $G \neq \mathcal{C}_{n,2}$ be a chordal graph. Then by Theorem 4.2.18, there exist proper induced subgraphs G_1 and G_2 such that $G = G_1 \uplus G_2$. Since G_1 and G_2 are induced subgraphs of a chordal graph G, we conclude that G_1 and G_2 are chordal. Hence induction and Remark 5.2.7 implies that, the ideal $I(\bar{G})$ has a 2-linear resolution.

E-chordal Clutter

Recall that E. Emtander [Em] has defined generalized chordal clutter (E-chordal) as the following.

Definition. A *generalized chordal clutter* is a d-uniform clutter, obtained inductively as follows:

(a) $\mathcal{C}_{n,d}$ is a generalized chordal clutter.

(b) If $\mathcal{G}$ is generalized chordal clutter, then so is $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$ for all $0 \leq i < n$.

(c) If $\mathcal{G}$ is generalized chordal and $V \subset V(\mathcal{G})$ is a finite set with $|V| = d$ and at least one element of $\{F \subset V : |F| = d-1\}$ is not a subset of any element of $\mathcal{G}$, then $\mathcal{G} \cup V$ is generalized chordal.

Emtander has proved that the circuit ideal of generalized chordal clutters has a d-linear resolution over any field K (c.f. [Em, Theorem 5.1]). We have already proved this result for $d = 3$ in Theorem 4.4.2. We can recover Emtander's result for arbitrary d, as an special case of Theorem 5.2.6.

Corollary 5.2.9. *If $\mathcal{C} \neq \mathcal{C}_{n,d}$ is a generalized chordal clutter, then the circuit ideal $I(\bar{\mathcal{C}}) \subset K[x_1, \ldots, x_n]$ has a d-linear resolution over K.*

Proof. Let $\mathcal{C}$ be a generalized chordal clutter. If $\mathcal{C}$ has a circuit F with property (c) in the above definition, then Theorem 4.3.3 together with induction, implies that $I(\bar{\mathcal{C}})$ has a d-linear resolution. So we may assume that $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$. Again, in this case, Remark 5.2.7 together with induction, implies that the ideal $I(\bar{\mathcal{C}})$ has a d-linear resolution over the field K. $\square$

5.3 Minimal to d-linearity

In this section, we define three classes of clutters which their circuit ideals do not have d-linear resolutions but the circuit ideal of any proper subclutter of them has a d-linear resolution.

Recall that a clutter $\mathcal{C}$ is said to be *connected*, if for each two vertices u and v, there is a sequence of circuits $F_1, \ldots, F_r$ such that, $u \in F_1, v \in F_r$ and $F_i \cap F_{i+1} \neq \varnothing$. A connected d-uniform clutter $\mathcal{C}$ is called a *tree*, if any subclutter of $\mathcal{C}$ has a submaximal circuit of degree one. A union of some trees is called a *forest*. Note that this definition of tree and forest, coincides with the usual definition of tree and forest in graph theory (See Definition 4.2.2).

By Theorem 4.4.10, the circuit ideal of any d-uniform forest has a d-linear resolution.

Definition 5.3.1. Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ and $\Delta = \Delta(\mathcal{C})$ be its clique complex. Suppose that $I = I(\bar{\mathcal{C}}) \subset K[x_1, \ldots, x_n]$, the circuit ideal of $\mathcal{C}$, does not have d-linear resolution.

(i) The clutter $\mathcal{C}$ is called *obstruction to d-linearity*, if for every proper subclutter $\mathcal{C}' \subsetneq \mathcal{C}$, the ideal $I(\bar{\mathcal{C}}')$ has a d-linear resolution.

(ii) The clutter $\mathcal{C}$ is called *minimal to d-linearity*, if it is obstruction to d-linearity and $\dim \Delta = d - 1$.

(iii) The clutter $\mathcal{C}$ is called *almost tree*, if every proper subclutter of $\mathcal{C}$ is a tree.

Let $\mathscr{C}_d^{\mathrm{obs}}$, $\mathscr{C}_d^{\min}$ and $\mathscr{C}_d^{\mathrm{a.tree}}$ denote the classes of clutters which are obstruction to d-linearity, minimal to d-linearity and almost tree, respectively.

Note that if $\mathcal{C} \in \mathscr{C}_d^{\min}$ and $\Delta = \Delta(\mathcal{C})$ is its clique complex, then we have:

$$\operatorname{indeg}(I_\Delta) = \operatorname{indeg} I(\bar{\mathcal{C}}) = d = 1 + \dim \Delta. \tag{5.6}$$

Remark 5.3.2. Let $\mathcal{C}$ be a d-uniform clutter on vertex set $[n]$ and $\mathcal{D}$ be an induced subclutter of $\mathcal{C}$. Without loss of generality, we may assume that, the vertex set of $\mathcal{D}$ is $[m]$ with $m \leq n$. Let $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Then $\Delta_{[m]}$ is a clique complex of $\mathcal{D}$ and by Theorem 3.2.12 and 3.4.6, we have:

$$\begin{aligned}\beta_{i,j}^K\left(I(\bar{\mathcal{C}})\right) = \sum_{\substack{W \subset [n] \\ |W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K) &\geq \sum_{\substack{W \subset [m] \\ |W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K) \\ &= \beta_{i,j}^K\left(I(\bar{\mathcal{D}})\right).\end{aligned}$$

This implies that, the ideal $I(\bar{\mathcal{C}})$ does not have linear resolution, if the ideal $I(\bar{\mathcal{D}})$ does not have linear resolution. Consequently, if a clutter $\mathcal{C}$, contains an induced subclutter $\mathcal{D}$ which is obstruction to d-linearity, then the circuit ideal of $\mathcal{C}$ does not have d-linear resolution.

Lemma 5.3.3. *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ which is minimal to d-linearity and $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Then,*

(i) $\dim_K \tilde{H}_{d-1}(\Delta; K) = 1$.

(ii) *If $W \subsetneq [n]$, then $\tilde{H}_{d-1}(\Delta_W; K) = 0$.*

Proof. (i) Let

$$0 \neq F = c_1F_1 + \cdots + c_rF_r \in \tilde{H}_{d-1}(\Delta; K), \quad \text{where } c_i \in K,\ F_i \in \mathcal{C}.$$

Then, $\mathrm{Supp}(F) := \{F_i\colon \quad c_i \neq 0\}$ is equal to $\mathcal{C}$, because every proper subclutter of $\mathcal{C}$ has a linear resolution.

If $\dim_K \tilde{H}_{d-1}(\Delta; K) > 1$ and

$$F = c_1F_1 + \cdots + c_rF_r, \quad G = d_1F_1 + \cdots + d_rF_r$$

be two basis element of $\tilde{H}_{d-1}(\Delta; K)$, then:

$$0 \neq c_1G - d_1F \in \tilde{H}_{d-1}(\Delta; K) \quad \text{and} \quad \mathrm{Supp}(c_1G - d_1F) \subsetneq \mathcal{C}$$

which is a contradiction.

(ii) One can easily check that $\Delta_W = \Delta(\mathcal{C}_W)$, for all $W \subset [n]$. By definition, for all $W \subsetneq [n]$, the induced clutter $\mathcal{C}_W$ has a linear resolution. So that by Theorem 3.2.12, $\tilde{H}_{d-1}(\Delta_W; K) = \tilde{H}_{d-1}(\Delta(\mathcal{C}_W); K) = 0$. $\square$

The following is the main theorem of this section which gives an explicit minimal free resolution for the circuit ideal of a clutter which is minimal to d-linearity.

Theorem 5.3.4. *Let $\mathcal{C}$ be a d-uniform clutter on $[n]$ which is minimal to d-linearity and $I = I(\bar{\mathcal{C}}) \subset K[x_1, \ldots, x_n]$ be the circuit ideal. Then, the minimal free resolution of I is:*

$$0 \to S^{\beta_{n-d,n}}(-n) \to S(-n) \oplus S^{\beta_{n-d-1,n-1}}(-(n-1)) \to S^{\beta_{n-d-2,n-2}}(-(n-2)) \\ \to \cdots \to S^{\beta_{1,d+1}}(-(d+1)) \to S^{\beta_{0,d}}(-d) \to I \to 0 \qquad (5.7)$$

where,

(i) $\beta_{n-d,n}(I) = 1 - e(S/I) + \sum_{i=0}^{d-1}(-1)^{d+i-1}\binom{n}{i}$.

(ii) $\beta_{i,i+d}(I) = \binom{n-d}{i}\left(\frac{d}{d+i}\binom{n}{d} - e(S/I)\right)$, *for* $0 \le i \le n-d-1$.

and $e(S/I) = \binom{n}{d} - \mu(I)$.

Proof. Let $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Since

$$\operatorname{indeg}(I_\Delta) = \operatorname{indeg} I(\bar{\mathcal{C}}) = d = 1 + \dim \Delta,$$

by Theorem 5.1.2(i) and Lemma 5.3.3(ii), $\beta_{i,j}(I) = 0$ either $j - i < d$ or $j - i > d + 1$ or $j - i = d + 1$ and $j < n$. Moreover, we have:

$$\beta_{n-(d+1),n} = \dim_K \tilde{H}_{d-1}(\Delta; K) = 1.$$

Hence, the minimal free resolution of I is in the form (5.7). The equation (ii) comes from Theorem 5.1.6. Using Theorem 3.2.12 once again, we have $\beta_{n-d,n}(I) = \dim_K \tilde{H}_{d-2}(\Delta; K)$. Hence (i) comes from Corollary 5.1.1. In order to find the multiplicity, note that:

$$e(S/I) = f_{d-1}(\Delta) = |\mathcal{C}| = \binom{n}{d} - \mu(I).$$

$\square$

Definition 5.3.5. Let $\mathcal{C}$ be a d-uniform clutter. The clutter $\mathcal{C}$ is called *strongly connected* (or *connected in codimension one*), if for any two circuits $F, G \in \mathcal{C}$, there exists a chain of circuits $F = F_0, \ldots, F_s = G$ in $\mathcal{C}$ such that $|F_i \cap F_{i+1}| = d - 1$, for $i = 0, \ldots, s - 1$.

Besides the algebraic properties of the clutters $\mathcal{C} \in \mathscr{C}_d^{\text{obs}}$, a combinatorial property of such clutters is that they are strongly connected.

Proposition 5.3.6. *If $\mathcal{C} \in \mathscr{C}_d^{\text{obs}}$ be a d-uniform clutter, then*

(i) *$\mathcal{C}$ is indecomposable.*

(ii) *$\mathcal{C}$ is strongly connected.*

Proof. (i) Let $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are proper subclutters of $\mathcal{C}$. By definition, the ideals $I_1 = I(\bar{\mathcal{C}}_1)$ and $I_2 = I(\bar{\mathcal{C}}_2)$ have d-linear resolutions. In view of Remark 5.2.7, the ideal $I(\bar{\mathcal{C}})$ has d-linear resolution which is a contradiction.

(ii) Let $\mathcal{C}_1 \subset \mathcal{C}$ be the maximal subclutter (w.r.t. inclusion) of $\mathcal{C}$ which is strongly connected. Clearly, $\mathcal{C}_1 \neq \varnothing$, because every clutter with one circuit is strongly connected.

Assume that $\mathcal{C}_1 \subsetneq \mathcal{C}$ and let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. By the maximality of $\mathcal{C}_1$, $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, that is $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ which contradicts to (i). So that, $\mathcal{C}_1 = \mathcal{C}$ is strongly connected. $\square$

Lemma 5.3.7. *Let $\mathcal{C}$ be a d-uniform clutter which is a tree or almost tree and $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Then, $\dim \Delta = d - 1$. In particular, $\mathscr{C}_d^{\text{a.tree}} \subset \mathscr{C}_d^{\text{min}}$.*

Proof. If $G \in \Delta$ and $|G| > d$ and V is the vertex set of G, then:

$$\mathcal{C}_V = \{F \in \mathcal{C} : \quad F \subset G\}.$$

Hence for all $e \in \mathrm{SC}(\mathcal{C}_V)$, $\deg_{\mathcal{C}_V}(e) \geq 2$. This contradicts to the fact that $\mathcal{C}_V$ has submaximal circuit of degree 1. So that, all faces of $\Delta(\mathcal{C})$ have at most d elements. Since $\mathcal{C} \subset \Delta$, we conclude that $\dim \Delta = d - 1$.

If $\mathcal{C} \in \mathscr{C}_d^{\text{a.tree}}$, then by what we have already proved, we know that $\dim \Delta(\mathcal{C}) = d - 1$. Also, Theorem 4.4.10 implies that, for every proper subclutter $\mathcal{C}' \subsetneq \mathcal{C}$, the ideal $I(\bar{\mathcal{C}}')$ has a linear resolution. Hence, $\mathcal{C} \in \mathscr{C}_d^{\text{min}}$. $\square$

We have shown that $\mathscr{C}_d^{\text{a.tree}} \subset \mathscr{C}_d^{\text{min}} \subset \mathscr{C}_d^{\text{obs}}$. All our evidences and computations lead us to make the following conjecture.

Conjecture 5.3.8. $\mathscr{C}_d^{\text{a.tree}} = \mathscr{C}_d^{\text{min}} = \mathscr{C}_d^{\text{obs}}$.

Example 5.3.9. In this example, we will find and explicit minimal graded free resolution for the circuit ideal of a cycle based on our own results in this section.

Let C_n be a cycle of length $n > 3$. Though that the Betti numbers of the circuit ideal of C_n is well-known (see e.g. [OG, Proposition 3.1]), we can recover them using results of this thesis.

Let $\Delta = \Delta(C_n)$ be the clique complex of C_n and $I = I(\bar{C}_n)$ be the circuit ideal. Then, $\operatorname{indeg}(I_\Delta) = 1+\dim(\Delta)$ and by Corollary 5.1.1, $\dim \tilde{H}_1(\Delta; K) = 1$. In particular, I does not have linear resolution (Corollary 5.1.4) and C_n is minimal to 2-linearity (Lemma 5.3.7). Moreover, By Theorem 5.3.4, the minimal free resolution of I is:

$$0 \to S(-n) \to S^{\beta_{n-4,n-2}}(-(n-2)) \to \cdots \to S^{\beta_{1,3}}(-3) \to S^{\beta_{0,2}}(-2) \to I \to 0$$

where,

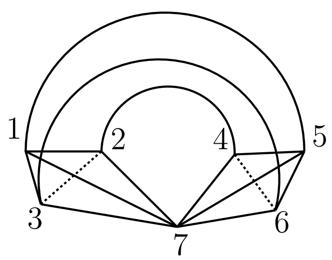
$$\beta_{i,i+2}(I) = n\binom{n-2}{i}\left(\frac{n-3-i}{2+i}\right), \quad \text{for } 0 \le i \le n-4.$$

5.3.1 Resolution of Pseudo-Manifolds

Definition 5.3.10. A d-uniform clutter $\mathcal{C}$ is called a *pseudo-manifold*, if $\mathcal{C}$ is strongly connected and each $e \in \mathrm{SC}(\mathcal{C})$ has degree 2.

For more details on pseudo-manifolds and the concept of orientability, refer to [Ma] Chapter IX.

Example 5.3.11. If Δ is a triangulation of a connected 2-manifold, then Theorem 3.5.7 implies that, the clutter $\mathcal{C} = \mathcal{F}(\Delta)$ is a 3-uniform pseudo-manifold. More generally, if Δ is a triangulation of a connected $(d-1)$-manifold (or homology manifold), then the clutter $\mathcal{C} = \mathcal{F}(\Delta)$ is a d-uniform pseudo-manifold (see [Mu, Excercise 2 in §43, §63]). It is not true that, every d-uniform pseudo-manifold is a triangulation of a $(d-1)$-manifold. For example, consider the 3-uniform clutter $\mathcal{C}$ as below:



The clutter $\mathcal{C}$ is organized as follows:

On the left side we take 3 triangles $127, 137, 237$ (but not 123).

On the right side we take 3 triangles $457, 467, 567$ (but not 456).

Now, connect the ends of these pyramids by a pipe and then triangulate this pipe as you want. For example triangulate by $124, 145,\ 234, 346,\ 135, 356$.

Then, $\mathcal{C}$ is a 3-uniform pseudo-manifold which is not a triangulation of a 2-manifold.

Example 5.3.12. In Remark 5.3.2, we have seen that:

> If a clutter $\mathcal{C}$ contains an induced subclutter $\mathcal{D}$ which is minimal to d-linearity, then the ideal $I(\bar{\mathcal{C}})$ does not have d-linear resolution.

In this example, we will show that the converse of this statement is not necessarily true. That is, we introduce a clutter $\mathcal{C}$ such that, the ideal $I(\bar{\mathcal{C}})$ does not have linear resolution but $\mathcal{C}$ does not contain any **induced** subclutter which is minimal to linearity.

Let $\mathcal{C}$ be a triangulation of sphere $\mathcal{S}^2$ (with large enough number of vertices), which is a pseudo-manifold. Let v_1, v_2, v_3 be vertices of $\mathcal{C}$ such that, v_1, v_2 belong to a circuit of $\mathcal{C}$ and neither v_1, v_3 nor v_2, v_3 belong to any other circuit. Then, add a new circuit $\{v_1, v_2, v_3\}$ to $\mathcal{C}$. The new clutter does not have any **induced** subclutter which is minimal to 3-linearity, however its circuit ideal does not have 3-linear resolution.

Lemma 5.3.13. *Let $\mathcal{C}$ be a d-uniform clutter such that $\deg_{\mathcal{C}}(e) = 2$, for all $e \in \mathrm{SC}(\mathcal{C})$. Then, every proper subclutter of $\mathcal{C}$ has a submaximal circuit of degree 1 if and only if $\mathcal{C}$ is strongly connected. In particular, every proper subclutter of a pseudo-manifold is a tree.*

Proof. ($\Rightarrow$) Let $F \in \mathcal{C}$ and $\mathcal{C}_1$ be a maximal subclutter of $\mathcal{C}$ which consists of all $G \in \mathcal{C}$ such that, there exists a chain $F = F_0, F_1, \ldots, F_r = G$ of circuits of $\mathcal{C}$ with $|F_i \cap F_{i+1}| = d - 1$, for $i = 0, \ldots, r-1$.

If $\mathcal{C}_1 \subsetneq \mathcal{C}$, then $\mathcal{C}_1$ has a submaximal circuit e of degree 1. By the maximality of $\mathcal{C}_1$, we have:

$$1 = \deg_{\mathcal{C}_1}(e) = \deg_{\mathcal{C}}(e).$$

This contradicts to our assumption on $\mathcal{C}$.

($\Leftarrow$) Let $\mathcal{C}' \subsetneq \mathcal{C}$ such that $\deg_{\mathcal{C}'}(e) = 2 = \deg_{\mathcal{C}}(e)$, for all $e \in \mathrm{SC}(\mathcal{C}')$. Take $F \in \mathcal{C}'$ and $G \in \mathcal{C} \setminus \mathcal{C}'$. By our assumption, there exists a chain

$F = F_0, F_1, \ldots, F_r = G$ of circuits of $\mathcal{C}$ such that $|F_i \cap F_{i+1}| = d-1$, for $i = 0, \ldots, r-1$.

Since $F_0 = F \in \mathcal{C}'$ and $|F_0 \cap F_1| = d-1$, we conclude that, $F_0 \cap F_1 \in \mathrm{SC}(\mathcal{C}')$. Hence, by our assumption, $\deg_{\mathcal{C}'}(F_0 \cap F_1) = 2$, which implies that, $F_1 \in \mathcal{C}'$. The same argument shows that, $F_0, F_1, \ldots, F_r$ are in $\mathcal{C}'$. This is a contradiction by our choice of $F_r = G$. □

Remark 5.3.14. Let $\mathcal{C}$ be a d-uniform pseudo-manifold and $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. In view of Lemma 5.3.13 and 5.3.7, we have:

(a) Every proper subclutter of $\mathcal{C}$ has a submaximal circuit of degree 1.

(b) $\operatorname{indeg}(I_\Delta) = 1 + \dim \Delta$.

Putting together these results, Corollary 5.1.4 implies that:

$$I(\bar{\mathcal{C}}) \text{ is minimal to } d\text{-linearity if and only if } \tilde{H}_{d-1}(\Delta; K) \neq 0.$$

The following proposition extends Theorem 4.3.11.

Proposition 5.3.15. *Let $\mathcal{C}$ be a d-uniform clutter.*

(i) *If $\mathcal{C}$ is oriented pseudo-manifold, then $\mathcal{C}$ is minimal to d-linearity.*

(ii) *If $\mathcal{C}$ is non-oriented pseudo-manifold, then $\mathcal{C}$ is minimal to d-linearity if and only if* $\mathrm{Char}(K) = 2$.

Proof. Let $\mathcal{C}$ be a d-uniform pseudo-manifold and $\Delta = \Delta(\mathcal{C})$ be its clique complex. In view of Lemma 5.3.7, we know that, $\dim \Delta = d-1$ and $\mathcal{C} = \mathcal{F}(\Delta)$. In particular,

$$\tilde{H}_{d-1}(\Delta; K) \cong \tilde{H}_{d-1}(\langle \mathcal{C} \rangle; K)$$

where, $\langle \mathcal{C} \rangle$ is the simplicial complex generated by $\mathcal{C}$. But we know that (see [Ma, Chapter X, Exercise 6.5] or [Mu, §43, Exercise 5]):

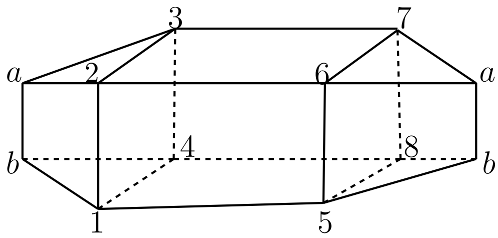
$$\tilde{H}_{d-1}(\langle \mathcal{C} \rangle; K) = \begin{cases} K, & \text{if } \mathcal{C} \text{ is oriented.} \\ \mathrm{Tor}(\mathbb{Z}_2, K), & \text{if } \mathcal{C} \text{ is non-oriented.} \end{cases}$$

where, $\mathrm{Tor}(\mathbb{Z}_2, K) = \{a \in K : \quad 2.a = 0\}$. Now, the conclusion follows from Remark 5.3.14. □

Using Lemma 5.3.13, and Proposition 5.3.15, one can conclude that, the class of d-uniform pseudo-manifolds is contained in $\mathscr{C}_d^{\text{a.tree}}$ in an appropriate choice of the ground field. It is worth to say that this inclusion is strict. (See the Example 5.3.16).

Example 5.3.16. Let Δ be a triangulation of the following shape and $\mathcal{C} = \mathcal{F}(\Delta)$. That is:

$$\Delta = \langle a23, b14, ab1, a12, ab4, a34, 236, 367, 125, 256, 145, 458, 348, 378, \\ a67, b58, ab5, a56, ab8, a78\rangle.$$



Then, $\mathcal{C}$ is not a pseudo-manifold, because $\deg_{\mathcal{C}}(ab) = 4$, but $\mathcal{C}$ is almost tree.

Remark 5.3.17. Let Δ be a triangulation of a 2-manifold X and $\mathcal{C} = \mathcal{F}(\Delta)$ be the corresponding 3-uniform clutter. Then $\mathcal{C}$ is a 3-uniform pseudo-manifold and by Theorem 5.3.15, it is minimal to linearity in a good choice of the characteristic of the base field. So that, one may use Theorem 5.3.4 to find the minimal free resolution of the circuit ideal, $I = I(\bar{\mathcal{C}})$, in terms of the multiplicity of I.

Note that the multiplicity of I can be computed by Table 3.1. To be more precise, if $\Delta' = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$, then:

$$e(I) = e(I_{\Delta'}) = f_2(\Delta') = f_2(\Delta) = |\mathcal{C}| = 2(n - \chi(X)).$$

The last equality comes from this fact that, each edge belongs to exactly 2 circuits.

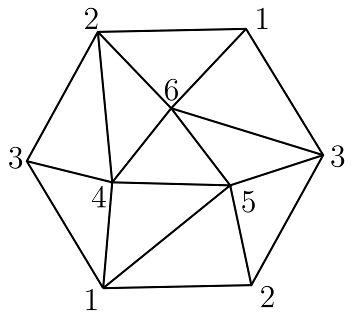
More generally, If $\mathcal{C}$ is a d-uniform pseudo-manifold, then in a good choice of the ground field, the clutter $\mathcal{C}$ is minimal to d-linearity. So that, one may use Theorem 5.3.4 to find the minimal free resolution of the circuit ideal, $I = I(\bar{\mathcal{C}})$, in terms of the the multiplicity of I. Again, in this case we have $e(I) = 2(n - \chi(\mathcal{C}))$.

Example 5.3.18. Let Δ be a triangulation of sphere $\mathcal{S}^2$ with $n > 4$ vertices and $\mathcal{C} = \mathcal{F}(\Delta)$ be the corresponding 3-uniform clutter. Then, $\mathcal{C}$ is a 3-uniform pseudo-manifold and the minimal free resolution of the ideal $I = I(\bar{\mathcal{C}}) \subset K[x_1, \ldots, x_n]$ in any characteristic of the base field is:

$$0 \to S^{\beta_{n-3,n}}(-n) \to S(-n) \oplus S^{\beta_{n-4,n-1}}(-(n-1)) \to S^{\beta_{n-5,n-2}}(-(n-2)) \\ \to \cdots \to S^{\beta_{0,3}}(-3) \to I \to 0.$$

where,
$$\beta^K_{i,i+3}(I) = \binom{n-3}{i}\left(\frac{3}{3+i}\binom{n}{3} - 2(n-2)\right).$$

Example 5.3.19. Let Δ be a triangulation of projective plane $\mathbb{P}^2$ with 6 vertices and $\mathcal{C} = \mathcal{F}(\Delta)$ be the corresponding 3-uniform clutter.



Using Theorem 5.3.4, Remark 5.3.17 and Table 3.1, the minimal free resolution of $I(\bar{\mathcal{C}}) \subset S = K[x_1, \ldots, x_6]$ in a field of characteristic two, is:
$$0 \to S(-6) \to S^6(-5) \oplus S(-6) \to S^{15}(-4) \to S^{10}(-3) \to I(\bar{\mathcal{C}}) \to 0$$
while with the same method as the proof of Theorem 5.3.4, one can verify that in any field K with $\mathrm{Char}(K) \neq 2$, the minimal free resolution is:
$$0 \to S^6(-5) \to S^{15}(-4) \to S^{10}(-3) \to I(\bar{\mathcal{C}}) \to 0.$$

5.4 Generating Ideals with Linear Resolution

Given a square-free monomial ideal I, in [TY] the authors introduced a new ideal constructed from I which is very close to I in Cohen-Macaulayness. We close this chapter by a dual to this concept. That is, we find a new ideal constructed from I which is very close to I in regularity. The method for the proof in this section is independent of the previous section of this chapter.

Notation. For a square-free monomial ideal
$$I = (\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}) \subset K[x_1, \ldots, x_n],$$
let $\bar{I}$ and $\hat{I}$ denote the ideals:
$$\bar{I} = \left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right),$$
$$\hat{I} = \left(x_j \mathbf{x}_{F_i} : \quad 1 \le i \le r,\ j \notin F_i\right).$$

The object of this section is to prove the following theorem. We will present two different proofs for this theorem.

Theorem 5.4.1. *Let* $I = (\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}) \subset K[x_1, \ldots, x_n]$ *be a square-free monomial ideal such that* $|F_i| = d$, *for* $i = 1, \ldots, r$. *Then,*

$$\operatorname{reg}(\hat{I}) = \begin{cases} \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) > d; \\ 1 + \operatorname{reg}(I), & \text{if } \operatorname{reg}(I) = d. \end{cases}$$

5.4.1 First Proof

The first proof of Theorem 5.4.1, uses the techniques in local cohomology. To present the first proof of this theorem, we need the following prerequisites.

Theorem 5.4.2. *Let* n, d *be non-negative integers with* $n > 0$ *and*

$$A_{n,d} = \{\mathbf{x}_{[n]\setminus A} : \quad A \subset [n], |A| \le d\} \subset S := K[x_1, \ldots, x_n].$$

Then,

(i) $A_{n,d} = x_n A_{n-1,d} \cup A_{n-1,d-1}$.

(ii) *If* $Q \subset A_{n,d}$, *then* depth $\frac{S}{(Q)} \ge n - (d+1)$.

Proof. (i) Clearly, $x_n A_{n-1,d} \cup A_{n-1,d-1} \subset A_{n,d}$.
For the converse, let $\mathbf{x}_{[n]\setminus A} \in A_{n,d}$. If $n \in A$, then:

$$\mathbf{x}_{[n]\setminus A} = \mathbf{x}_{[n-1]\setminus(A\setminus\{n\})} \in A_{n-1,d-1}$$

and if $n \notin A$, then:

$$\mathbf{x}_{[n]\setminus A} = x_n \mathbf{x}_{[n-1]\setminus A} \in x_n A_{n-1,d}.$$

So that, $A_{n,d} \subset x_n A_{n-1,d} \cup A_{n-1,d-1}$.

(ii) If $n = 1$ or $d = 0$, then $A_{n,d} = \{\mathbf{x}_{[n]}\}$ and there exist nothing to prove. So, we may assume that $n > 1$ and $d > 0$. We use induction on n, d to prove the theorem.

If $n = 2$, then for all $d > 0$, $A_{n,d} = \{x_1x_2, x_1, x_2\}$ and the assertion is trivial.
Assume that (ii) is true for $n-1, d-1$ and let $Q \subset A_{n,d}$, $J = (Q)$. By (i),

$$Q = x_n Q_1 \cup Q_2, \qquad \text{where} \quad Q_1 \subset A_{n-1,d}, \quad Q_2 \subset A_{n-1,d-1}.$$

Note that (Q_1) and (Q_2) are ideals in $K[x_1, \ldots, x_{n-1}]$ and by induction hypothesis, we have:

$$\operatorname{depth} \frac{S}{(Q_1)} = 1 + \operatorname{depth} \frac{K[x_1, \ldots, x_{n-1}]}{(Q_1)} \geq n - d - 1, \tag{5.8}$$

$$\operatorname{depth} \frac{S}{(Q_1)} = 1 + \operatorname{depth} \frac{K[x_1, \ldots, x_{n-1}]}{(Q_2)} \geq n - d. \tag{5.9}$$

Let $J_1 = (x_n Q_1) \cap (Q_2)$. One can easily check that, $G(J_1) \subset A_{n-1,d-1}$. So that by induction hypothesis,

$$\operatorname{depth} \frac{S}{J_1} = 1 + \operatorname{depth} \frac{K[x_1, \ldots, x_{n-1}]}{J_1} \geq n - d. \tag{5.10}$$

Since $\operatorname{depth} S/(x_n Q_1) = \operatorname{depth} S/(Q_1)$, from (5.8), (5.9) and (5.10), we conclude that:

$$H^i_{\mathfrak{m}}\left(\frac{S}{(x_n Q_1)}\right) = H^i_{\mathfrak{m}}\left(\frac{S}{(Q_2)}\right) = H^{i+1}_{\mathfrak{m}}\left(\frac{S}{J_1}\right) = 0, \qquad \text{for all } i < n - d - 1.$$

Hence, from Mayer-Vietoris long exact sequence,

$$\cdots \to H^i_{\mathfrak{m}}\left(\frac{S}{(x_n Q_1)}\right) \oplus H^i_{\mathfrak{m}}\left(\frac{S}{(Q_2)}\right) \to H^i_{\mathfrak{m}}\left(\frac{S}{J}\right) \to H^{i+1}_{\mathfrak{m}}\left(\frac{S}{J_1}\right) \to \cdots$$

we conclude that:

$$H^i_{\mathfrak{m}}\left(\frac{S}{J}\right) = 0, \qquad \text{for all } i < n - d - 1.$$

That is, $\operatorname{depth} S/J \geq n - (d+1)$. □

As a consequence of the Theorem 5.4.2, we have the following result:

Corollary 5.4.3. *Let $I = (\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}) \subset K[x_1, \ldots, x_n]$ be a square-free monomial ideal such that $|F_i| = d$, for $i = 1, \ldots, r$. Then,*

(i) $\operatorname{depth} \dfrac{S}{\bar{I}} \geq n - (d+1)$.

(ii) $\operatorname{depth} \dfrac{S}{I^{\vee} \cap \bar{I}} \geq n - d$.

Proof. (i) The ideal $\bar{I}$ is generated by a subset of $A_{n,d}$. Hence Theorem 5.4.2(ii) gives the conclusion.

(ii) Clearly, $I^\vee \cap \bar{I}$ is a square-free monomial ideal. By Theorem 5.4.2(ii), it is enough to show that the ideal $I^\vee \cap \bar{I}$ is generated by a subset of $A_{n,d-1}$.

Let $A \subset [n]$, $\mathbf{x}_{[n]\setminus A} \in I^\vee \cap \bar{I}$. Then there exists $1 \leq i \leq r$, such that $\mathbf{x}_{[n]\setminus F_i} | \mathbf{x}_{[n]\setminus A}$. In particular, $A \subset F_i$. If $A = F_i$, then:

$$\mathbf{x}_{[n]\setminus F_i} = \mathbf{x}_{[n]\setminus A} \in I^\vee \subset P_{F_i}$$

which is a contradiction. So that, $A \subsetneq F_i$ and $|A| < |F_i| = d$. This means that, $\mathbf{x}_{[n]\setminus A} \in A_{n,d-1}$. □

Lemma 5.4.4. *If $F_1, \ldots, F_r$ be subsets of $[n]$, then:*

$$\left(\bigcap_{i=1}^{r} P_{F_i}\right) + \left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right) = \bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right).$$

Proof. Clearly, for all j and i, $\bigcap_{i=1}^{r} P_{F_j} \subset \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right)$. So that,

$$\bigcap_{i=1}^{r} P_{F_i} \subset \bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right).$$

On the other hand, for all $j \neq i$, we have:

$$\mathbf{x}_{[n]\setminus F_j} \in P_{F_i} \subset \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right).$$

Hence, $\left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right) \subset \bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right)$. So that,

$$\left(\bigcap_{i=1}^{r} P_{F_i}\right) + \left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right) \subset \bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right).$$

For the other direction, let f be a square free monomials in $\bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right)$. If there exists an i such that $\mathbf{x}_{[n]\setminus F_i} \mid f$, then clearly:

$$f \in \left(\bigcap_{i=1}^{r} P_{F_i}\right) + \left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right).$$

So we may assume that, $\mathbf{x}_{[n]\setminus F_i} \nmid a$ for all i. In this case, for all $1 \le i \le r$, there exists $t_i \in P_{F_i}$ such that, $t_i \mid a$. Hence, the least common multiples of $t_1, \ldots, t_r$ divides a. This implies that:

$$a \in \bigcap_{i=1}^{r} P_{F_i} \subset \bigcap_{i=1}^{r} \left(P_{F_i}, \mathbf{x}_{[n]\setminus F_i}\right).$$

So, we get the other inclusion too. □

Theorem 5.4.5. *Let $I = (\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}) \subset K[x_1, \ldots, x_n]$ be a square-free monomial ideal such that $|F_i| = d$, for $i = 1, \ldots, r$. Then,*

$$H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{\hat{I}^\vee}\right), \quad \textit{for all } i < n-(d+1).$$

Proof. (i) By definition of ideal $\hat{I}$, we have:

$$\begin{aligned}
\hat{I}^\vee &= \bigcap_{i=1}^{r}\left(\bigcap_{j\notin F_i}(x_j, P_{F_i})\right)\\
&= \bigcap_{i=1}^{r}\left(\mathbf{x}_{[n]\setminus F_i}, P_{F_i}\right)\\
&= \left(\bigcap_{i=1}^{r} P_{F_i}\right) + \left(\mathbf{x}_{[n]\setminus F_1}, \ldots, \mathbf{x}_{[n]\setminus F_r}\right) \quad \text{(By Lemma 5.4.4)}\\
&= I^\vee + \bar{I}.
\end{aligned}$$

So that, we have the short exact sequence,

$$0 \longrightarrow \frac{S}{I^\vee \cap \bar{I}} \longrightarrow \frac{S}{I^\vee} \oplus \frac{S}{\bar{I}} \longrightarrow \frac{S}{\hat{I}^\vee} \longrightarrow 0$$

which gives rise to the long exact sequence,

$$\begin{aligned}
\cdots \longrightarrow H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee \cap \bar{I}}\right) \longrightarrow H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \oplus H^i_{\mathfrak{m}}\left(\frac{S}{\bar{I}}\right) \longrightarrow H^i_{\mathfrak{m}}\left(\frac{S}{\hat{I}^\vee}\right)\\
\longrightarrow H^{i+1}_{\mathfrak{m}}\left(\frac{S}{I^\vee \cap \bar{I}}\right) \longrightarrow \cdots.
\end{aligned}$$

Note that:

$$\operatorname{depth} \tfrac{S}{I^\vee \cap \bar{I}} \ge n-d, \qquad \text{(By Corollary 5.4.3(ii))}$$

$$\operatorname{depth} \tfrac{S}{\bar{I}} \ge n-(d+1), \quad \text{(By Corollary 5.4.3(i))}.$$

Hence, for $i < n-(d+1)$, we have:

$$H^i_{\mathfrak{m}}\left(\frac{S}{I^\vee}\right) \cong H^i_{\mathfrak{m}}\left(\frac{S}{\hat{I}^\vee}\right).$$

□

First proof of Theorem 5.4.1. Let $t = \text{reg}\,(I) \geq d$. Then, by Remark 3.3.6, $\text{depth}\,\frac{S}{I} = n - t$. Hence for all $i < n - t$, we have $H^i_{\mathfrak{m}}\,(S/I^{\vee}) = 0$ and $H^{n-t}_{\mathfrak{m}}\,(S/I^{\vee}) \neq 0$.

If $t > d + 1$, then Theorem 5.4.5 implies that:

$$H^{n-t}_{\mathfrak{m}}\left(\frac{S}{\hat{I}^{\vee}}\right) \neq 0, \qquad H^{i}_{\mathfrak{m}}\left(\frac{S}{\hat{I}^{\vee}}\right) = 0,\ \text{for all } i < n - t.$$

That is, $\text{depth}\, S/\hat{I}^{\vee} = n - t$. Using Remark 3.3.6 once again, we get:

$$\text{reg}\,(\hat{I}) = t = \text{reg}\,(I).$$

If $d \leq t \leq d + 1$, then Theorem 5.4.5 implies that:

$$H^{i}_{\mathfrak{m}}\left(\frac{S}{\hat{I}^{\vee}}\right) = 0, \quad \text{for all } i < n - (d + 1).$$

Since $\dim S/\hat{I}^{\vee} = n - (d + 1)$, we conclude that, $S/\hat{I}^{\vee}$ is Cohen-Macaulay of dimension $n - (d + 1)$. Hence, $\text{reg}\,(\hat{I}^{\vee}) = d + 1$ by Theorem 3.3.3. $\square$

5.4.2 Second Proof

The second proof of Theorem 5.4.1, uses the techniques of homologies of simplicial complexes. To state the second proof, all that we need is the following theorem.

Theorem 5.4.6. *Let $I = (\mathbf{x}_{F_1}, \ldots, \mathbf{x}_{F_r}) \subset K[x_1, \ldots, x_n]$ be a square-free monomial ideal such that $|F_i| = d$, for $i = 1, \ldots, r$ and Δ be Stanley-Reisner complex of I. Let*

$$\hat{\Delta} = \Delta \cup \langle \text{all } d\text{-subsets of } [n] \rangle = \Delta \cup \langle [n] \rangle^{(d-1)}.$$

Then,

(i) $I_{\hat{\Delta}} = \hat{I} = \hat{I}_{\Delta}$.

(ii) $\tilde{H}_i(\Delta_W; K) = \tilde{H}_i(\hat{\Delta}_W; K)$, *for all $W \subset [n]$ and $i > d - 1$.*

(iii) *If $j - i > d + 1$, then $\beta^K_{i,j}(I) = \beta^K_{i,j}(\hat{I})$.*

Proof. (i) Let $F \notin \Delta$, $j \notin F$, but $F \cup \{j\} \in \hat{\Delta}$. Then,

$$F \cup \{j\} \in \langle [n] \rangle^{(d-1)} \setminus \Delta.$$

This means that, $|F \cup \{j\}| \leq d$ and $F \cup \{j\} \notin \Delta$. Since $F \cup \{j\} \notin \Delta$ and $\operatorname{indeg}(I_\Delta) = d$, we conclude that, $|F \cup \{j\}| = d$. That is, $|F| = d-1$. Using the fact that $\operatorname{indeg}(I_\Delta) = d$ once again, we conclude that $F \in \Delta$, which is a contradiction. Thus,

$$\hat{I} = (x_k \mathbf{x}_F, \quad \mathbf{x}_F \in I,\ k \notin F) \subset I_{\hat{\Delta}}.$$

For the other direction, Let $\mathbf{x}_T \in I_{\hat{\Delta}}$. So that, $T \notin \hat{\Delta}$ which implies that $T \notin \Delta$, $|T| > d$. That is, $\mathbf{x}_T \in I_\Delta$ and $|T| > d$. Thus, there exists $F \subsetneq T$ such that, $\mathbf{x}_F \in G(I_\Delta)$ and $\mathbf{x}_F | \mathbf{x}_T$.

Take an element $t \in T \setminus F$. Then, $x_t \mathbf{x}_F | \mathbf{x}_T$, which implies that:

$$\mathbf{x}_T \in (x_k \mathbf{x}_F, \quad \mathbf{x}_F \in I,\ k \notin F).$$

So that,

$$I_{\hat{\Delta}} \subset (x_k \mathbf{x}_F, \quad \mathbf{x}_F \in I,\ k \notin F) = \hat{I}.$$

(ii) By the structure of $\hat{\Delta}$, we know that $\Delta_W^{(i)} = \hat{\Delta}_W^{(i)}$, for all $i > d-1$ and for all $W \subset [n]$. In particular, this implies that:

$$\tilde{H}_i(\Delta_W; K) = \tilde{H}_i(\hat{\Delta}_W; K), \quad \text{for all } W \subset [n] \text{ and for all } i > d-1.$$

(iii) Let $j - i > d+1$. By Theorem 3.2.12 and (ii), we conclude that:

$$\begin{aligned} \beta_{i,j}^K(I) &= \sum_{\substack{W \subset [n] \\ |W| = j}} \tilde{H}_{j-i-2}(\Delta_W; K) \\ &= \sum_{\substack{W \subset [n] \\ |W| = j}} \tilde{H}_{j-i-2}(\hat{\Delta}_W; K) = \beta_{i,j}^K(\hat{I}). \end{aligned}$$

□

Second proof of Theorem 5.4.1. First note that, the ideal $\hat{I}$ is generated by elements of degree $d+1$. Hence $\operatorname{reg}(\hat{I}) \geq d+1$, by Lemma 2.1.1.

If $d \leq \operatorname{reg}(I) \leq d+1$, then $\beta_{i,j}^K(I) = 0$, for all $j - i > d+1$. Hence, from Theorem 5.4.6(iii), $\beta_{i,j}^K(\hat{I}) = 0$, for all $j - i > d+1$. That is, $\operatorname{reg}(\hat{I}) = d+1$.

If $r = \operatorname{reg}(I) > d+1$ and i_0, j_0 be such that $r = j_0 - i_0$ with $\beta_{i_0,j_0}^K(I) \neq 0$ then Theorem 5.4.6(iii), implies that $\beta_{i_0,j_0}^K(\hat{I}) \neq 0$. So that,

$$\operatorname{reg}(\hat{I}) = \max\{j - i : \quad \beta_{i_0,j_0}^K(\hat{I}) \neq 0\} \geq r.$$

Using Theorem 5.4.6(iii) once again, we conclude that $\operatorname{reg}(\hat{I}) = r$. This completes the proof. □

Corollary 5.4.7. *Let $G \neq \mathcal{C}_{n,2}$ be a graph and $\mathcal{C}$ be the 3-uniform clutter*

$$\mathcal{C} = \{C\colon \quad C \text{ is a 3-cycle in } G\}.$$

Let $I = I(\bar{G}), J = I(\bar{\mathcal{C}})$. Then,

(i) $J = \hat{I}$.

(ii) *J has a 3-linear resolution if and only if* $\operatorname{reg}(I) \leq 3$.

(iii) *If G is a chordal graph, then the ideal J has a 3-linear resolution.*

Proof. (i) Let $A \in \mathcal{C}$. Then $B \in E(G)$, for all 2-subset $B \subset A$. Hence,

$$\mathbf{x}_A \notin (x_k \mathbf{x}_F\colon\ F \in E(\bar{G}),\ k \notin F).$$

This means that:

$$\hat{I} = \big(x_k \mathbf{x}_F\colon \quad F \in E(\bar{G}),\ k \notin F\big) \subset I(\bar{\mathcal{C}}) = J.$$

Now, let $x_{i_1} x_{j_1} x_{k_1} \in I(\bar{\mathcal{C}}) = J$. Then, $A = \{i_1, j_1, k_1\} \in \bar{\mathcal{C}}$ does not form a cycle in G. So we may assume that, $\{i_1, j_1\} \in E(\bar{G})$. Hence,

$$\mathbf{x}_A = x_{i_1} x_{j_1} x_{k_1} \in (x_k \mathbf{x}_F, \quad F \in E(\bar{G}),\ k \notin F).$$

So that,

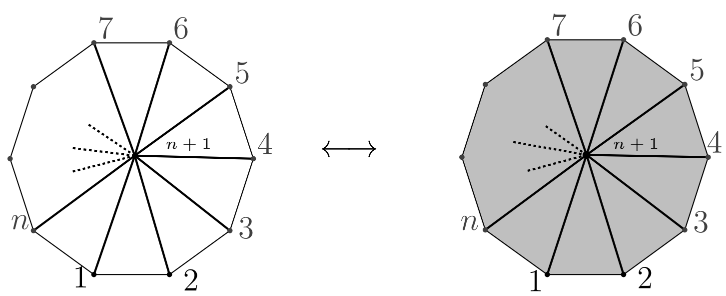
$$J \subset (x_k \mathbf{x}_F, \quad F \in E(\bar{G}),\ k \notin F) = \hat{I}.$$

(ii) This is a consequence of (i) and Theorem 5.4.1(ii).

(iii) If G is a chordal graph, then $\operatorname{reg}(I) = 2$. Hence (ii) implies that, J has a 3-linear resolution. □

Example 5.4.8. Let G be the the following wheel graph and $\mathcal{C}$ be the 3-uniform clutter consisting of all 3-cycles of G. That is:

$$G = C_n \cup \big\{\{i, n+1\}\colon \quad 1 \leq i \leq n\big\},$$
$$\mathcal{C} = \{C\colon \quad C \text{ is a 3-cycle in } G\}.$$



G $\qquad\qquad$ $\mathcal{C}$

Since G is a chordal graph, the ideal

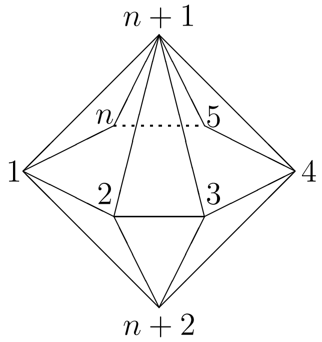
$$\begin{aligned} J &= I\left(\bar{\mathcal{C}}\right) = I\left(\mathcal{C}_{n+1,3} \setminus \mathcal{C}\right) \\ &= (x_i x_j x_k\colon \quad \{i,j\} \notin E(G),\ 1 \leq k \leq n+1,\ k \notin \{i,j\}) \end{aligned}$$

has a 3-linear resolution by Corollary 5.4.7.

Definition 5.4.9 (Bipyramid)**.** Let C_n be a cycle of length n and $\mathfrak{P}_n$ be 3-uniform clutter

$$\big\{\{x_{n+1},x_1,x_2\},\{x_{n+1},x_2,x_3\},\ldots,\{x_{n+1},x_n,x_1\}, \\ \{x_{n+2},x_1,x_2\},\{x_{n+2},x_2,x_3\},\ldots,\{x_{n+2},x_n,x_1\}\big\}$$

where, x_{n+1} and x_{n+2} are two new vertices. We call the clutter $\mathfrak{P}_n$ a *bipyramid on* C_n.



$\mathfrak{P}_n$

Since every bipyramid is a triangulation of a sphere $\mathcal{S}^2$, Theorem 4.3.11 or Proposition 5.3.15 implies that, the circuit ideal of any bipyramid does not have 3-linear resolution and $\operatorname{reg} I(\bar{\mathfrak{P}}_n) = 4$, over any field K. We can recover this result as a result of Corollary 5.4.7.

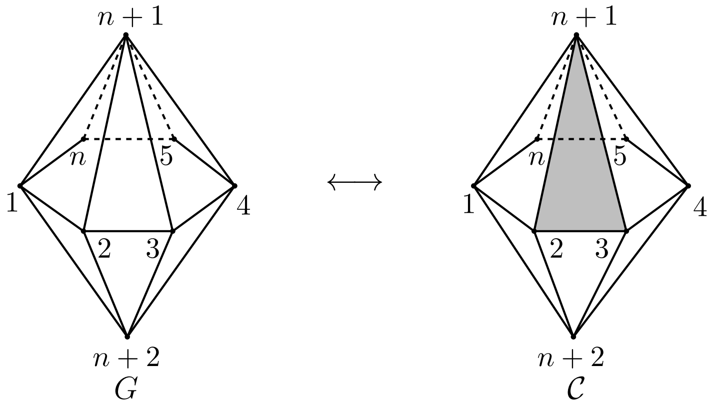
Corollary 5.4.10. *Let $\mathfrak{P}_n$ be a double pyramid on $C_n, n > 3$ and G be the following graph:*

$$G = C_n \cup \big\{\{i,n+1\}\colon i \in [n]\big\} \cup \big\{\{i,n+2\}\colon i \in [n]\big\}.$$

Let $I = I(\bar{G}), J = I(\bar{\mathfrak{P}}_n)$ be the corresponding circuit ideal in $S = K[x_1,\ldots,x_{n+2}]$. Then,

(i) depth $\frac{S}{I^\vee} = (n+2) - 4$*;*

(ii) *The ideal J does not have linear resolution and* $\operatorname{reg}(J) = 4$.



Proof. (i) Let $I_1 = I(\bar{C}_n) \subset K[x_1, \ldots x_n]$. By Example 5.3.9, $\operatorname{reg}(I_1) = 3$ and by Remark 3.3.6, we have:

$$\operatorname{depth} \frac{S}{I_1^{\vee}} = (n+2) - 3 = n - 1. \tag{5.11}$$

Clearly, $I = I_1 + (x_{n+1}x_{n+2})$ and $I^{\vee} = I_1^{\vee} \cap (x_{n+1}, x_{n+2})$. Since,

$$\frac{S}{I_1^{\vee} + (x_{n+1}x_{n+2})} \cong \frac{K[x_1, \ldots, x_n]}{I_1^{\vee}},$$

from (5.11), we get $\operatorname{depth} \frac{S}{I_1^{\vee}+(x_{n+1}x_{n+2})} = n - 3$. Hence, Proposition 4.1.2 yields the conclusion.

(ii) Note that Remark 3.3.6 and (i), imply that $\operatorname{reg}(I) = 4$. Since $\mathcal{C}$ is a 3-uniform clutter consisting of all 3-cycles of G, we get $\operatorname{reg}(J) = 4$, by Corollary 5.4.7. $\square$

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