Parallel Scheduling in the Cloud Systems : Approximate and Exact Methods

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Dedication

To my parents for their love, prays, and support. You put me through the best education possible. I appreciate your sacrifices and I wouldn't have been able to get to this stage without you.

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Table of contents

Résumé

Cette thèse porte sur la résolution exacte et heuristique de plusieurs problèmes ayant des applications dans le domaine de l'Informatique dématérialisé (cloud computing). L'Informatique dématérialisée est un domaine en plein extension qui consiste à mutualiser les machines/serveurs en définissant des machines virtuelles représentant des fractions des machines/serveurs. Il est nécessaire d'apporter des solutions algorithmiques performantes en termes de temps de calcul et de qualité des solutions. Dans cette thèse, nous nous sommes intéressés dans un premier temps au problème d'ordonnancement sur plusieurs machines (les machines virtuelles) avec contraintes de précédence, c-à-d., que certaines tâches ne peuvent s'exécuter que si d'autres sont déjà finies. Ces contraintes représentent une subdivision des tâches en sous tâchespouvant s'exécuter sur plusieurs machines virtuelles. Nous avons proposé plusieurs algorithmes génétiques permettant de trouver rapidement une bonne solution réalisable. Nous les avons comparés avec les meilleurs algorithmes génétiques de la littérature et avons défini les types d'instances où les solutions trouvées sont meilleures avec notre algorithme. Dans un deuxième temps, nous avons modélisé ce problème à l'aide de la programmation linéaire en nombres entiers permettant de résoudre à l'optimum les plus petites instances. Nous avons proposé de nouvelles inégalités valides permettant d'améliorer les performances de notre modèle. Nous avons aussi comparé cette modélisation avec plusieurs formulations trouvées dans la littérature. Dans un troisième temps, nous avons analysé de manière approfondie la sous-structure du sous-graphe d'intervalle ne possédant pas de clique de taille donnée. Nous avons étudié le polytope associé à cette sous-structure et nous avons montré que les facettes que nous avons trouvées sont valides pour le problème d'ordonnancement sur plusieurs machines avec contraintes de précédence mais elles le sont aussi pour tout problème d'ordonnancement sur plusieurs machines. Nous avons étendu la modélisation permettant de résoudre le précédent problème afin de résoudre le problème d'ordonnancement sur plusieurs machines avec des contraintes disjonctives entre les tâches, c-à-d., que certaines tâches ne peuvent s'exécuter en même temps que d'autres. Ces contraintes représentent le partage de ressources critiques ne pouvant pas être utilisées par plusieurs tâches. Nous avons proposé des algorithmes de séparation afin d'insérer de manière dynamique nos facettes dans la résolution du problème puis avons développé un algorithme de type Branch-and-Cut.

Nous avons analysé les résultats obtenus afin de déterminer les inégalités les plus intéressantes afin de résoudre ce problème.Enfin dans le dernier chapitre, nous nous sommes intéressés au problème d'ordonnancement d'atelier généralisé ainsi que la version plus classique d'ordonnancement d'atelier (open shop). En effet, le problème d'ordonnancement d'atelier généralisé est aussi un cas particulier du problème d'ordonnancement sur plusieurs machines avec des contraintes disjonctives entre les tâches. Nous avons proposé une formulation à l'aide de la programmation mathématique pour résoudre ces deux problèmes et nous avons proposé plusieurs familles d'inégalités valides permettant d'améliorer les performances de notre algorithme. Nous avons aussi pu utiliser les contraintes définies précédemment afin d'améliorer les performances pour le problème d'ordonnancement d'atelier généralisé. Nous avons fini par tester notre modèle amélioré sur les instances classiques de la littérature pour le problème d'ordonnancement d'atelier. Nous obtenons de bons résultats permettant d'être plus rapide sur certaines instances.

Résumé du chapitre 1 :

Dans ce chapitre, nous avons proposé un état de l'art portant dans un premier temps sur les problématiques de recherche opérationnelle que l'on peut trouver dans l'Informatique dématérialisée. Ensuite, nous avons rappelé quelques problématiques d'ordonnancement s'insérant dans le cadre de l'Informatique dématérialisée. Après cet état de l'art thématique, nous nous sommes intéressés aux méthodes permettant de résoudre ces problèmes. En introduction aux méthodologies, nous avons décrit ce qu'est un problème d'optimisation combinatoire, la modélisation par les graphes et expliqué la difficulté de résolution de certains problèmes en définissant la complexité. Ensuite, nous avons commencé par décrire les heuristiques et méta-heuristiques que sont les algorithmes gloutons, les méthodes de recherches locales et les algorithmes génétiques. Puis, nous avons rappelé les concepts de la programmation en nombres entiers. Ces concepts regroupent, la modélisation, l'approche polyédrale, les algorithmes de type Branch-and-Bound et ceux de type Branch-and-Cut.

Résumé du chapitre 2 :

Dans le chapitre 2 nous décrivons le problème d'ordonnancement sur plusieurs machines avec contraintes de précédence et nous donnons une formulation à l'aide de la programmation mathématique afin de comparer les heuristiques sur de petites instances. Nous discutons ensuite des algorithmes heuristiques et méta-heuristiques proposés dans la littérature et permettant de résoudre ce problème. Nous proposons un nouvel algorithme génétique basé sur l'affectation des jobs aux machines. Nous développons plusieurs va-

riantes basées sur cette idée, puis nous combinons plusieurs algorithmes génétiques différents afin d'améliorer la meilleure solution trouvée sur les instances de grandes tailles. Nous finissons par comparer les différents algorithmes sur un ensemble d'instances générées aléatoirement. Nous montrons que notre algorithme génétique obtient de bien meilleures performances sur de nombreuses instances.

Résumé du chapitre 3 :

Dans ce chapitre, nous décrivons plusieurs formulations mathématiques données dans la littérature. Nous proposons une nouvelle modélisation basée sur les graphes d'intervalles. Cette formulation possède un nombre exponentiel de contraintes. Nous proposons des séparations polynomiales pour ces inégalités nous permettant de résoudre efficacement les instances testées. Cette modélisation obtient de très bons résultats et outrepasse en termes de performance toutes les modélisations proposées dans la littérature à l'exception de la formulation basée sur les ordres linéaires. Nous avons proposé de nombreuses inégalités valides pour notre modèle basé sur les graphes d'intervalles nous permettant d'obtenir de meilleurs résultats et des résultats compétitifs sur de nombreuses instances en comparaison avec la formulation basée sur les ordres linéaires.

Résumé du chapitre 4 :

Dans le chapitre 4, nous avons analysé le problème du sous-graphe d'intervalle sans clique de taille supérieure à m. Ce sous-problème se retrouve dans de nombreux problèmes d'ordonnancement. Nous avons proposé des inégalités permettant de supprimer les sous-graphes interdits définis dans la littérature. Pour chacune de ces inégalités nous analysons leur aspect facial. Ces contraintes sont en nombres exponentiels et nous proposons plusieurs séparations, exactes et heuristiques pour chacune d'entre elle. Nous finissons par comparer les performances de chaque contrainte sur le problème d'ordonnancement sur plusieurs machines avec contraintes disjonctives. Cette comparaison nous permet de définir les contraintes les plus intéressantes et la force des séparations proposées.

Résumé du chapitre 5 :

Dans ce chapitre, nous étudions deux problématiques qui sont des cas particuliers du problème d'ordonnancement sur plusieurs machines avec contraintes disjonctives. Le premier problème consiste à définir le meilleur ordre des tâches sur plusieurs machines tout en respectant les contraintes disjonctives. Nous proposons une formulation par la programmation linéaire en nombres entiers pour résoudre ce problème ainsi que des contraintes spécifiques. Nous avons testé ce modèle en ajoutant les contraintes basées sur le sous-graphe d'intervalle sans clique de taille m et nous comparons les résultats sur des ins-

tances aléatoires. Le second problème est celui de l'ordonnancement d'atelier (open shop) largement étudié dans la littérature. Nous étendons le modèle précédent ainsi que les contraintes proposées précédemment. Ce modèle se base sur l'ordre linéaire des tâches sur une machine et appartenant à la même tâche. Nous testons la performance de notre modèle sur les instances utilisées dans la littérature.

Mots-clés: Ordonancement, programmation mathématique, heuristiques, Approche polyèdrale, Branch-and-cut.

Abstract

The Cloud Computing appears as a strong concept to share costs and resources related to the use of end-users. As a consequence, several related models exist and are widely used (IaaS, PaaS, SaaS...). In this context, our research focused on the design of new methodologies and algorithms to optimize performances using the scheduling and combinatorial theories. We were interested in the performance optimization of a Cloud Computing environment where the resources are heterogeneous (operators, machines, processors...) but limited. Several scheduling problems have been addressed in this thesis. Our objective was to build advanced algorithms by taking into account all these additional specificities of such an environment and by ensuring the performance of solutions. Generally, the scheduling function consists in organizing activities in a specific system imposing some rules to respect. The scheduling problems are essential in the management of projects, but also for a wide set of real systems (telecommunication, computer science, transportation, production...). More generally, solving a scheduling problem can be reduced to the organization and the synchronization of a set of activities (jobs or tasks) by exploiting the available capacities (resources). This execution has to respect different technical rules (constraints) and to provide the maximum of effectiveness (according to a set of criteria). Most of these problems belong to the NP-Hard problems class for which the majority of computer scientists do not expect the existence of a polynomial exact algorithm unless P=NP. Thus, the study of these problems is particularly interesting at the scientific level in addition to their high practical relevance. In particular, we aimed to build new efficient combinatorial methods for solving parallel-machine scheduling problems where resources have different speeds and tasks are linked by precedence constraints.

In our work we studied two methodological approaches to solve the problem under the consideration : exact and meta-heuristic methods.We studied three scheduling problems, where the problem of task scheduling in cloud environment can be generalized as unrelated parallel machines, and open shop scheduling problem with different constraints. For solving the problem of unrelated parallel machines with precedence constraints, we proposed a novel genetic-based task scheduling algorithms in order to minimize maximum completion time *(makespan)*. These algorithms combined the genetic algorithm approach with different techniques and batching rules such as list scheduling (LS) and earliest completion time (ECT). We reviewed, evaluated and compared the proposed algorithms against one of the well-known genetic algorithms available in the literature, which has been proposed for the task scheduling problem on heterogeneous computing systems. Moreover, this compari-

son has been extended to an existing greedy search method, and to an exact formulation based on basic integer linear programming. The proposed genetic algorithms show a good performance dominating the evaluated methods in terms of problems' sizes and time complexity for large benchmark sets of instances. We also extended three existing mathematical formulations to derive an exact solution for this problem. These mathematical formulations were validated and compared to each other by extensive computational experiments. Moreover, we proposed an integer linear programming formulations for solving unrelated parallel machine scheduling with precedence/disjunctive constraints, this model based on the intervaland $m-$clique free graphs with an exponential number of constraints. We developed a Branch-and-Cut algorithm, where the separation problems are based on graph algorithms. We also worked to hybridize the meta-heuristic with the mathematical program and improved our mathematical program by adding different classes and families of valid inequalities to strengthen the model. We also studied the polytope associated with our mathematical formulation. We discussed the separation algorithms associated with the valid inequalities and used them within branch-and-cut algorithm to solve the problem. Finally, we proposed a novel model for solving a generalized open shop task scheduling problem, and then, we adapted the model to solve the task scheduling problem in an open shop environment. We also identified some classes of valid inequalities to improve these models.

Keywords: scheduling, mathematical programming, heuristics, polyhedral study, Branch-and-cut

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General Introduction

Cloud computing is a natural development of the previous models of distributed and grid computing, beyond the technical innovations related to the idea of virtualization. Cloud providers provide the computing as a services, it is called infrastructure as a service (IaaS). Amazon Elastic Compute Cloud [13] is an example of IaaS. In cloud the providers deliver physical resources as virtual machines with different capacities to remote users as a service on pay-as-you-go basis. Remote users send their data (applications, programs, etc) to the cloud, the scheduler needs to place these data to its proper virtual machines. In scheduling theory, this problem belongs to the class of parallel machines and open shop. When the capacity of machines is different, then it becomes more precisely an unrelated parallel machines and a generalized case of open shop. In cloud computing most applications can be represented in a form of a directed cyclic graph (DAG). Therefore, precedence constraints and disjunctive constraints are found. Cloud management is responsible for resources allocation. When users send their applications (set of jobs with dependencies) to the cloud the scheduler aims to assign each dependent job to its virtual machines efficiently. The allocation of jobs to virtual machines is a complicated process in the cloud computing environment. Optimizing the maximum completion time normally affects the performance of the whole system. The main advantage of job scheduling algorithms in cloud environment is to achieve an excellent system throughput and high performance computing.

Scheduling is the allocation of resources over time to perform a collection of tasks, in which one or several objectives have to be optimized. From this general definition of the term, we could deduce that, scheduling is an important decision making function. We can say also, scheduling is a theory when it has a collection of principles, models and techniques. Scheduling plays a crucial role in manufacturing, as well as in services industries [113]. Effective scheduling becomes a necessity for survival in marketplace. For example, services companies have to schedule activities in such a way as to use the available resources in an efficient manner. Referring to Conway *et al* [4], scheduling is

classified according to four types of information : the operations to be processed, the number and types of machines, the constraints that restrict the assignment of jobs and the criteria by which the schedule can be evaluated. In the real life, there are tremendous number of scheduling applications in manufacturing [6], production systems [7] and in services industries [3], as well as in most information processing environment [1, 5]. The journey of scheduling theory starts by Henry Gantt and other pioneers. Different directions were pursued in academia and industry with an increasing amount of attention paid to scheduling problems. As a consequence, different approaches have been developed to solve the scheduling problems [8, 10, 11, 12]. These approaches, generally based on the optimization techniques including heuristics, meta-heuristics (approximate methods) and exact techniques, aim to design effective algorithms for attacking the considered scheduling problems.

Most of scheduling problems belong to the NP-Hard problems class for which the majority of computer scientists do not expect the existence of a polynomial exact algorithm. Thus, the study of these problems is particularly interesting at the scientific level in addition to their high industrial relevance.

Motivated by the optimization of the performances in cloud computing environment. The scheduling problem in cloud is generalized as an unrelated parallel machine and open shop scheduling problems according to the cloud environment. In this thesis, we proposed different optimization methods (approximate, and exact) to handle such scheduling problems. Our contribution is as follows :
- Several genetic algorithms have been proposed based on local search, list scheduling and some batching rules.
- Several mathematical formulations are developed to solve the parallel-machine and open shop scheduling problems.
- The proposed interval subgraph mathematical model have been investigated with the associated polytope and some facets are defined for this polytope.
- Several classes of valid inequalities have been derived.
- Several separation procedures are proposed to strengthen the model.

Many experimental computations have been applied for some generated benchmarks as well as for some known benchmarks.

Outlines of the thesis

This thesis consists of five chapters where each one could be a self-contained chapter based on the problem under the consideration and the combinatorial optimization methods used. The reader can access the chapter with the corresponding method of his interest directly. The manuscript is organized as follows :

Chapter 1 presents the preliminary and preparatory definitions and notations as a conceptual framework. This chapter includes also some state of the art on cloud computing, scheduling problem and some definitions about complexity theory, polyhedral and graph theory.

Chapter 2 focus on the approximate solutions of combinatorial problems. Greedy and genetic algorithms for solving the task scheduling problems in cloud computing are proposed in this chapter. Here, the problem is formulated as an unrelated parallel-machine with precedences and disjunctive constraints. Moreover, some related work on this area of research are presented and our results are compared with the existing works.

Chapter 3 presents the mathematical formulation of the studied problem. Our novel mathematical model, which is based on interval and m-clique free subgraphs for solving the unrelated parallel machines scheduling problem with precedence constraints is proposed. We also compared the proposed model against different other mathematical formulations found in the literature. At the end of this chapter computational experiments are presented and analysed.

Chapter 4 investigates our mathematical model and studies its associated polytope. We explore the subproblem of finding an interval graph and m-clique free subgraphs. Moreover, we present some facet definitions and we also describe the exact and heuristic separation algorithms to separate some forbidden subgraphs and we propose a branch-and-cut algorithm based on families of strong valid inequalities presented in this chapter.

Chapter 5 discusses two problems. The first one is the Generalized Open Shop problem and the second is the Open Shop scheduling problem. The structure of our model helps on solving such problems. By applying the idea of interval graph to propose other mathematical formulations for solving the considered problems, some classes of valid inequalities are presented. Some separation algorithms are proposed as well.

Most of the results of these chapters have been published in journals, and international conferences listed below :

Two articles in international journals :

– M-A. Hassan, I. Kacem, S. Martin, I.M. Osman, Genetic Algorithms for Job sche-

duling in cloud computing. Studies in Informatics and Control, 2015, Vol. 24, No. 4, December 2015. PP. 387-399.

- Mohammed-Albarra Hassan, ImedKacem, Sébastien Martin, Izzeldin M. Osman : m-clique free Interval sub-graph : Polyhedral analysis and Branch and Cut. (submitted to the Journal of Combinatoric Optinization (JOCO) in November 2016).

Four articles in international conferences :

- Mohammed-Albarra Hassan, ImedKacem, Sébastien Martin, Izzeldin M. Osman : Unrelated Parallel Machine Scheduling Problem with Precedence Constraints : Polyhedral Analysis and Branch-and-Cut. ISCO 2016 : Lecture Notes of Computer Science (SPRINGER) PP. 308-319 (DOI :10.1007/978-3-319-45587-727).
- M-A. Hassan, I. Kacem, S. Martin, I.M.Osman. Valid Inequalities for Unrelated Parallel Machines Scheduling with Precedence Constraints. Proceedings of IEEE CODIT'16, 6-8 April 2016, Saint Julian's – Malta : pp.677 - 682 (DOI : 10.1109/CoDIT.2016.7593644).
- M-A. Hassan, I. Kacem, S. Martin and I. M. Osman. Mathematical Formulations for the Unrelated Parallel Machines with Precedence Constraints. Proceedings of 45th International Conference on Computers & Industrial Engineering (CIE45), 28-30 October 2015, FRANCE.
- Mohammed-Albarra Hassan Abdel-Jabbar, ImedKacem, Sébastien Martin : Unrelated parallel machines with precedence constraints : application to cloud computing. IEEE CLOUDNET 2014 Luxembourg : pp.438-442 (DOI : 10.1109/CloudNet.2014.6969034).

One poster in poster session :

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1

State-of-the-Art

Contents

1.1 Cloud Computing

Cloud computing is a type of distributed and parallel system, which consists of physical and virtual resources. The physical resources in cloud shared virtually across a limited number of virtual machines to the end users according to their demand [18]. These virtual machines are dynamically presented as computing resources to the end user based on

what is called a service level agreements (SLA), which is a determined contract between end-user and service provider that defines the computing service expected from the service provider. When the computing resources are allocated to the users, they access the services such as applications and stored data from anywhere at any time. The request for virtualized resources is described through a list of parameters describing the processing, the memory and the disk needs. The hardware and software resources are allocated to the cloud applications on-demand basis. Execution of a task has a cost and this may vary depending on the resources allocated. Therefore, when the maximum completion time is minimized, that means the performance of the whole system will be improved. Cloud computing services are offered based on three-tier architecture. The challenge is that, for the cloud service providers it is difficult to allocate the virtual machines dynamically and efficiently [17, 18].

The cloud service providers receive simultaneously a lot of computing requests from different users with different requirements and preferences (see Figure 1.1). Some tasks need to use a lower cost and less computing resources, while some other tasks require more computing resources and take more bandwidth and CPU. In a cloud computing environment the tasks may be distributed across distinct computational resources nodes. In order to allocate the tasks to these nodes, the available computing resources are detected and analyzed. Hence, the quality of cloud computing service can be described by network bandwidth, task costs and the completion time. The importance of task scheduling in cloud environment arises from the previous description. Scheduling algorithms in cloud computing environment can be categorized into two main groups based on the cloud mode : batch mode scheduling algorithms and online mode scheduling algorithms. In batch scheduling algorithms, jobs are queued and collected into a buffer when they arrive in the system. Then, the scheduling algorithms will start after a fixed period of time. In the other mode, jobs are scheduled immediately when they are arrived to the system [17]. There are many different techniques used to allocate user requests to the cloud computing resources in order to optimize some objectives that affect the performance of the cloud services (See [15]).

1.2 Scheduling problems

Scheduling is a decision-making process, it is found in many real world applications. such as manufacturing and services industries. It is the process of allocating resources to

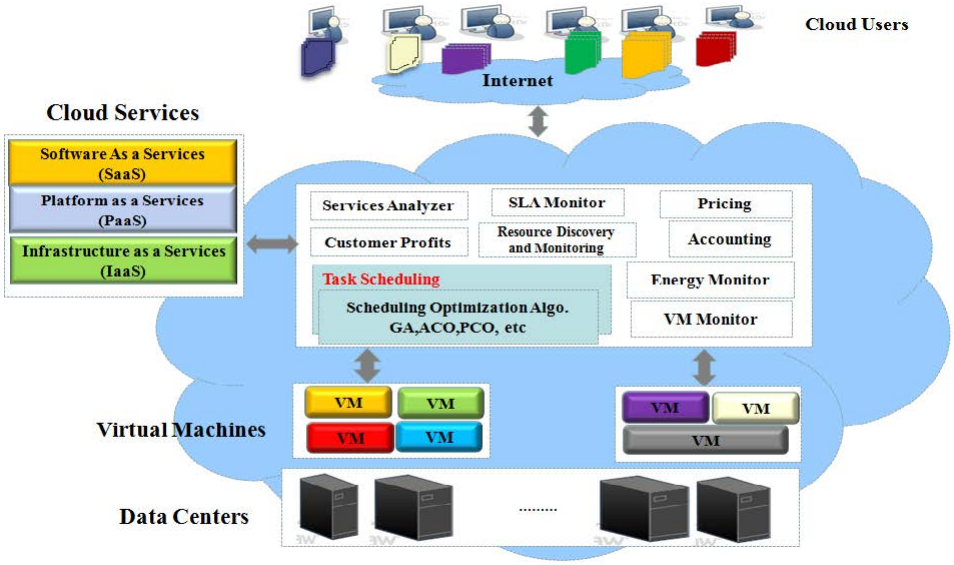


FIGURE 1.1 – Task scheduling in cloud computing model.

tasks over a given time period aiming to optimize one, two, or multiple objectives. The resources and tasks can take many different forms in services and in manufacturing. The resources may be machines in a factory, processing units in a computing environment, etc. The tasks may be operations in a production line, processing of a computer programs, etc. In machine environment each task may have some constraints such as a precedence constraints, a possible starting time and a due date. The objectives can also take many different forms. One objective may be a single objective such as the minimization of the completion time of the last task, and another may be bi-objective or multi-objectives. Scheduling, as a decision-making process, plays an important role in computing environments, especially in cloud computing. Suppose that m machines $M_i(i = 1, ..., m)$ have to process n jobs $J_j(j = 1, ..., n)$. A schedule is for each job an allocation of one or more time intervals to one or more machines.

The classes of scheduling problems are specified in terms of a three-field classification $\alpha|\beta|\gamma$ where α specifies the machine environment, β specifies the job characteristics and γ denotes the optimality criterion. Such a classification scheme was introduced by Graham *et al* [66].

1.3 Computational complexity

In this section, we present some definitions and principles about the computational complexity. Complexity theory provides a mathematical framework in which computational problems are studied so that they can be classified as "easy" or "hard". A more detailed presentation can be found in the book of Garey & Johnson [61]. The main issue of the theory of complexity is to determine the required resources needed (time, storage space) and to measure the performance of algorithms with respect to computational time. The notations P, NP and co-NP are collections of *decision problems* : problems that can be answered by 'yes' or 'no', like whether a given graph has a perfect matching or a Hamiltonian circuit. The *optimization problem* is not a decision problem, but often can be reduced to it in a certain sense [19]. An easy way to characterize the class NP is : NP is the collection of decision problems that can be reduced in polynomial time to the satisfiability problem. However, Cook in [16] defined NP as *the collection of all decision problems for which each input with positive answer, has a polynomial-time checkable of correctness of the answer (NP stands for nondeterministically polynomial-time).* The NP-complete problems are the problems that are the hardest in NP : every problem in NP can be reduced to them. Next description clarify. Problem $\prod \subset Sigma^*$ is said to be reducible to problem $\wedge \subset Sigma^*$ if there exists a polynomial-time algorithm that returns, for any input $w \in \Sigma^*$ an output $x \in \Sigma^*$ with the property : $w \in \prod \Leftrightarrow x \in \wedge$. This implies that if $\prod$ is reducible to $\wedge$ and $\wedge$ belongs to P, then also $\prod$ belongs to P. Similarly, if $\prod$ is reducible to $\wedge$ and $\wedge$ belongs to NP, then also $\prod$ belongs to NP. A problem $\prod$ is said to be NP-complete if each problem in NP is reducible to $\prod$. An optimization problem is NP-hard if the corresponding decision problem is NP-complete.

One of the most successful methods of attacking hard combinatorial optimization problems is the genetic algorithm, which will be discussed in this chapter. Genetic algorithm generally generates feasible solutions that are not guaranteed to be optimal. Any approach without formal guarantee of performance can be considered as a "heuristic". Such approaches are useful in practical situations if there is no better methods available. Important classes of problems which are polynomially solvable are linear programming problems [25] and integer linear programming problems with fixed number of variables [27].

1.4 Heuristics and meta-heuristics

In this section, we present some definitions for heuristics and metaheuristics widely used in combinatorial optimization.

1.4.1 Heuristics

Optimization techniques can be classified, in a heuristic, exact and approximation methods. The heuristic methods try to find optimal solutions or near-optimal solutions in a significantly reduced amount of time. The heuristic methods categorized into *constructive* methods and *local search* methods. Constructive algorithms obtain solutions from scratch by adding solution components to an initially empty list, until reaching the final solution. Local search algorithms start from an initial solution and iteratively replace the current solution by a better candidate from the neighbors of the current solution [22]. As defined in [18], a heuristic technique is a method, which tries to find good solutions at a reasonable computation cost without being able to guarantee optimality. Unfortunately, it may not even be possible to determine how close to the optimal solution a particular heuristic solution is.

1.4.2 Metaheuristics

The term meta-heuristic refers to a certain class of heuristic methods. As Fred Glover in [21], first used this term and defined it as follows : *A meta-heuristic refers to a master strategy that guides and modifies other heuristics to produce solutions beyond those that are normally generated in a quest for local optimality.* In another definition, "*meta-heuristics are solution methods that orchestrate an interaction between local improvement procedures and higher level strategies to create a process capable of escaping from local optima and performing a robust search of a solution space*" [23]. The heuristics guided by such a meta-strategy may be high level procedures or may include nothing more than a description of the strategies of moving from one solution to another with an associated evaluation rule (called fitness). To distinguish between heuristics and metaheuristic concepts, we can mention that heuristics are often problem dependent, heuristics normally defined for a given problem to find optimal or near to the optimal solutions for the problem under consideration, whereas the metaheuristics are problem independent techniques that can be applied for a wide range of problems. As an example, when we use simulated

annealing metaheuristic in scheduling, the decision of moving from current solution to another candidate one will be done by metaheuristic procedure whereas this method does not know nothing about scheduling. In the literature there is a vast amount of research that used a heuristic and metaheuristics to attack scheduling problems.

Genetic algorithm

Holland in 1975 developed the idea of applying the principles of natural evolution to optimization problems. This idea has been published in his book "Adaptation in natural and artificial systems". He built the first genetic algorithm. Holland's theory has been further developed. Now, genetic algorithms (GAs) considered as a powerful tool for solving optimization problems. Genetic algorithms are based on the principle of genetics and evolution. Todays, GAs are used to resolve complicated optimization problems, like timetabling, job shop scheduling, games playing and others [49].

Now, we give a brief introduction to simple genetic algorithms and associated terminology. GAs encode the decision variables of a search problem into finite length strings of alphabets or digits of a certain cardinality. The strings which are candidate solutions to the search problem are referred to as chromosomes. The chromosome represent a single solution, the alphabets or digits are referred to as genes and the values of genes are called alleles. For example, in scheduling problems, a chromosome represents a sequence and a gene may represent a job, and an allele is a value of processing time taken by a specific job.

In contrast to traditional optimization techniques, GAs work with coding of parameters, rather than the parameters themselves. The general procedures of the GA are as follows :

1. Initialization. The initial population of candidate solutions is usually generated randomly across the search space. It can be binary or non-binary chromosomes.

2. Evaluation. Once the population is initialized or an offspring population is created, the algorithm uses a fitness function to evaluate each chromosome in the population.

3. Selection. In the selection step, the algorithm works to prefer better solutions to worse ones. The algorithm selects a chromosome to mate the reproduction.

4. Recombination. Recombination combines parts of two or more parental solutions to create new ones. Here, the algorithm applies a genetic operator (crossover) on the selected chromosomes.

5. Mutation. Select one solution and apply a small random change to this solution.

6. Replacement. Replace the current population with the temporary population.
If stopping condition is met, then STOP with the best chromosome as the final solution for the problem. Otherwise, GOTO **2**.

The determination of a population size is a crucial element in the GAs. In most of GA applications, the *population size* can be considered as a constant. The *initialization of population* performed by using some suitable heuristics that are relevant to the considered problem or can be created randomly. Selecting a very small size of population increases the risk of not converging to a global optimal solution. Large size of population increases the chance to converge to obtain a good solution. The second operator of GAs is the *fitness function*, GA uses this function to evaluate the solution for each chromosome, then GA can determine if the *chromosome* can be kept or not. If the chromosome kept then it produces a new *offspring* or will be eliminated.The most important operator is the *selection of chromosomes,* which is ensure the *convergence* of the GA. When the genetic algorithm capable to select the best chromosome, then it will have a population of similar chromosomes, that led the GA to converge to a local optimum. Now, we give some selection methods : *roulette wheel* selection, *deterministic* selection, *ranking* selection, *tournament* selection and *etc.* In step four, the combination of two parents which combines the features of two fittest chromosomes and carries these features to the next generation by forming offsprings. Many well known crossover methods have been developed and applied. One of them is the two-position crossover method, which consists in selecting two crossover positions in two chromosomes and then making swapping segments between the chromosomes. Also, there is another crossover method, which is *multi-position* crossover method. This method changes the number of segments during the execution of GA. *Shuffle* crossover method first shuffles the crossover positions in the two selected chromosomes. Then, it exchanges the segments between the crossover positions and finally *un-shuffles* the chromosomes. The *uniform* crossover method is a mix between one position and multi-positions crossover methods. It produces two new children by exchanging genes in two chromosomes randomly. The fifth operator in GA steps is the mutation, which exchange one or more of the chromosome genes randomly to ensure search changement, which may lead to the global optimum.

Finally, the last GA step is the *stopping criterion.* There are many methods, which can be used for the stopping criteria. One of them is *the maximum number of generations.* The method based on the convergence is also used : the algorithm stops when the GA

converges after all chromosomes have reached a certain degree of homogeneity or, by another stopping criterion, after a chromosome with a certain level of fitness value is found.

1.5 Graph theory

In this section, we will introduce some basic definitions and notations of graph theory that will be used throughout the chapters of this dissertation. For more details, we refer the reader to [19].

A graph is denoted $G = (V, E)$ where V is the set of vertices or nodes and E is the set of edges. If $e \in E$ is an edge with initial node u and terminal node v, we may also use both notations uv or (u, v) to denote e.

The graphs considered here are directed, finite, loopless and may include multiple arcs.

A *directed graph or digraph* is denoted $G = (V, A)$ where V is the set of vertices or nodes and A is the set of arcs. If $a \in A$ is an arc with origin node u and destination node v, we may also use both notations uv or (u, v) to denote a. The graph G is said to be complete if there exists an arc between each pair of nodes (u, v).

A graph or *undirected graph* is a pair $G = (V, E)$, where V is a finite set and E is a family of unordered pairs from V. The elements of V are called the vertices, sometimes the nodes or the points. The elements of E are called the edges, sometimes the lines.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. If E' consists of all edges of G spanned by V', then G' is called an *induced subgraph*, or the subgraph induced by V'. In notation,
$G[V']$:= subgraph of G induced by V',
$E[V']$:= family of edges spanned by V'

The *complementary graph* or complement of a graph $G = (V, E)$ is the simple graph with vertex set V and edges all pairs of distinct vertices that are nonadjacent in G. In notation, $\bar{G}$:= the complementary graph of G.

1.6 Optimization problems

In mathematics, optimization is a branch of applied mathematics. It derives its importance from the wide variety of its applications and from the availability of efficient algorithms that have been used to solve such problems. Mathematically, it refers to the

minimization (or maximization) of a given objective function of several decision variables that satisfy functional constraints [113]. For example, let us consider the optimization model, which addresses the allocation of limited resources among possible alternative uses in order to maximize the total profit. *Objective function, decision variables, and constraints* are three essential elements of any optimization problem. If the decision variables in an optimization problem are restricted to integers, or to a discrete set of possibilities, there is an *integer* or *discrete optimization* problems. The problem is a *continuous optimization* problem, if there are no such restrictions on the variables. Some problems may have a mixture of discrete and continuous variables, that depends on the nature of the problem. We give the generic description of an optimization problem.
Given a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ and a set $S \subset \mathbb{R}^n$, the problem of finding an $x^* \in \mathbb{R}$ that solves

$$\begin{aligned} & min_x f(x) \\ s.t. \;\; & x \in S \end{aligned} \tag{1.1}$$

is called an optimization problem (OP). We denote by f the *objective function* and by S the *feasible region*. If S is empty, the problem is called *infeasible*. If it is possible to find a sequence $x^k \in S$ such that $f(x^k) \to -\infty$ as $k \to +\infty$, then the problem is unbounded. If the problem is neither infeasible nor unbounded, then it is often possible to find a solution $x^* \in S$.

1.6.1 Combinatorial optimization

Combinatorial Optimization is a subset of mathematical optimization that is related to operations research, algorithm theory, and computational complexity theory. Its purpose is to study the optimization problems where the set of feasible solutions can be represented as a discrete one.

The combinatorial optimization problems are the problems, which are formulated as follows. Let $E = \{e_1, ..., e_n\}$ be a finite set where each element e_i is associated with a weight $w(e_i)$. Let $\mathcal{F}$ be a family of subsets of E. If $F \in \mathcal{F}$, then $w(F) = \sum_{e_i \in F} w(e_i)$ denotes the weight of F. The problem consists in identifying an element F^* of $\mathcal{F}$ whose weight is minimum or maximum. The set $\mathcal{F}$ represents the set of feasible solutions of the problem. Such a problem is called combinatorial optimization problem.

The term *combinatorial* refers to the discrete structure of the representation of the feasible solution set $\mathcal{F}$. Generally, this structure is represented by a graph. The term

optimization tells that we are looking for the best element in the set of feasible solutions. This set may contain an exponential number of solutions. Thus, we cannot expect to solve a combinatorial optimization problem by checking or enumerating all its solutions one by one, which is not a reasonable option. Such a problem is then considered as a hard problem. Many effective techniques and approaches have been developed to attack combinatorial optimization problems. Some of these approaches use linear, integer programming, and polyhedral approach and others based on graph theory. combinatorial optimization is closely related to algorithm and computational complexity theory.

1.6.2 Linear programming

Linear programming deals with the OP with a linear function in the presence of linear inequalities. One of the most common optimization problems is linear optimization or linear programming (LP). It is the problem of optimizing a linear objective function subject to linear inequalities and equality constraints. Indeed, any combinatorial optimization problem can be reduced to solving a linear program. The standard form of the LP is given below :

$$\begin{aligned} min_x \;\; & C^T x \\ Ax &= b \\ x &\geq 0, \end{aligned} \tag{1.2}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is the variable vector to be determined. A wide variety of real life problems can be formulated as linear integer optimization problems. The combinatorial problems, such as the knapsack problem, resources allocation problem, TSP, network flow and graph problems, and many scheduling problems can also be solved as a linear integer optimization problems [24].

1.6.3 Integer programming

When the variables are integer, we call the formulation of the problem as integer programming. Integer programs are optimization problems that require some or all of the variables to take integer values. This restriction on the variables usually makes the

problems very hard to solve. A pure integer linear program is given by :

$$\begin{aligned} min_x \;\; & C^T x \\ Ax &\geq b \\ x &\geq 0 \;\; \text{and integral,} \end{aligned} \tag{1.3}$$

where $A \in \mathbb{R}^{\triangleright \times \kappa}$, $b \in \mathbb{R}^{\triangleright}$, $c \in \mathbb{R}^{\kappa}$ are given, and $x \in \mathbb{N}^{\kappa}$ is the variable vector to be determined.

A very common case occurs when the variables x_j represent binary decision variables, that is $x \in \{0,1\}^n$. The problem is then called a $0-1$ linear program (or discrete). When there are both integer constrained variables and continuous variables, the problem is called a *Mixed Integer Linear Program* (MILP) :

$$\begin{aligned} min_x \;\; & C^T x \\ Ax &\geq b \\ x &\geq 0 \\ x_j &\in \mathbb{N}, \;\; for \; j = 1, ..., p \end{aligned} \tag{1.4}$$

where A, b, c are given data and the integer p (with $1 \leq p < n$) is also part of the input.

1.7 Polyhedral approach

The development of polyhedral theory and the consequent introduction of strong valid inequalities led to a resurgence of cutting plane methods. The polyhedral method was initiated by Edmonds in 1965 for a matching problem. It consists in describing the convex hull of problem solutions by a system of linear inequalities. The problem reduces then to the resolution of a linear program. Normally, in most of the cases, it is not straightforward to obtain a complete characterization of the convex hull of the solutions for a combinatorial optimization problem. However, having a system of linear inequalities that partially describes the solutions polyhedron may often lead to solve the problem in polynomial time. This approach has been successfully applied to several combinatorial optimization problems. In this section, we present the basic notions of polyhedral theory. For detail, the reader is invited to consult [19, 20, 22].

First, we will recall some definitions, propositions, and properties related to polyhedral theory.

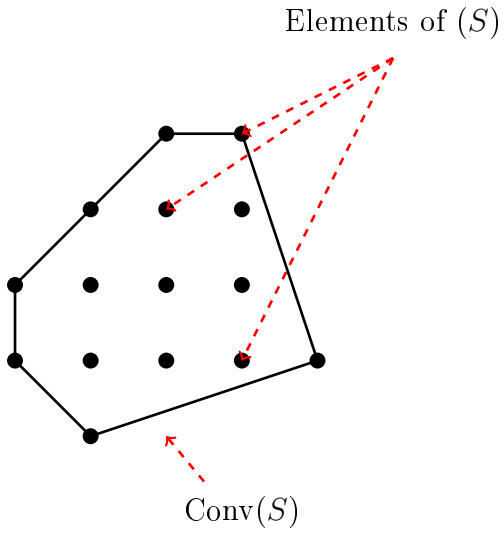


FIGURE 1.2 – A convex hull

1.7.1 Elements of polyhedral theory

Definition 1 *Given a set $S \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is a of points of A if there exists a finite set of points $\{x^i\}_{i=1}^t$ in S and a $\lambda \in R_+^t$ with $\sum_{i=1}^t \lambda_i = 1$ and $x = \sum_{i=1}^t \lambda_i x^i$.*

Figure 1.2 shows the convex hull of a set of integral points in $\mathbb{R}^2$. We see that conv(S) can be described by a finite set of a linear inequalities and that $max\{cx : x \in S\} = max\{cx : x \in conv(S)\}$. Moreover, the latter problem is a linear program.

Finding an inequality description of $conv(S)$ is not easy and questions such as the dimension of $conv(S)$ and so on, are very important. In this section, we give some results from linear algebra.

Definition 2 *A set of points $x_1, ..., x_k \in R^n$ is linearly independent if the unique solution of $\sum_{i=l}^k \lambda_i x^i = 0$ is $\lambda_i = 0, i = 1, ..., k$.*

Note that the maximum number of linearly independent points in R^n is n.

Definition 3 *$H \subset \mathbb{R}^n$ is a subspace if $x \in H$ implies $\lambda x \in H$ for all $\lambda \in \mathbb{R}^{\nVdash}$ and $x, y \in H$ implies $x + y \in H$.*

Definition 4 *A polyhedron $P \subset \mathbb{R}^n$ is the set of points that satisfy a finite number linear inequalities ; that is, $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where (A, b) is an $m \times (n + 1)$ matrix.*

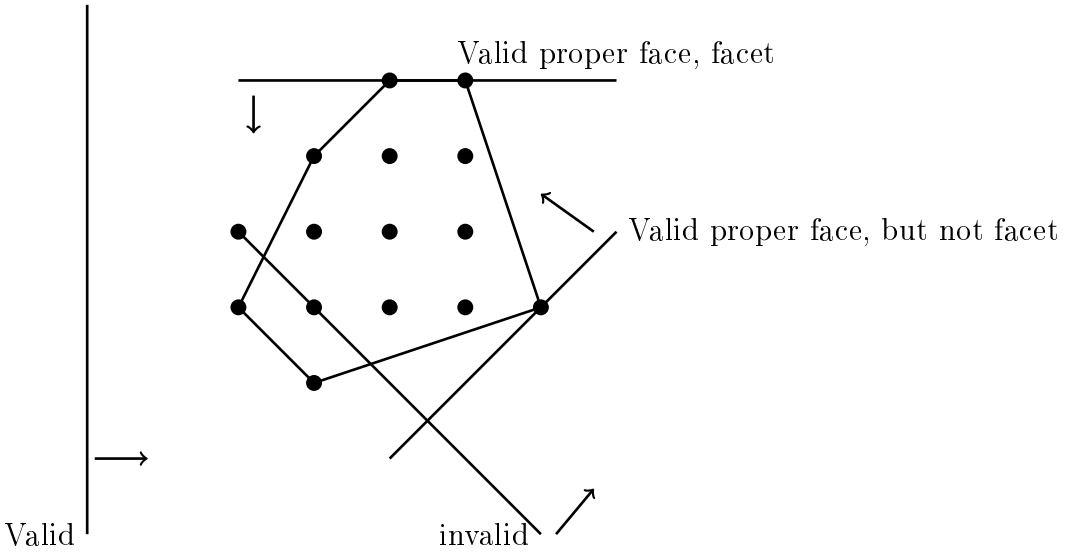


FIGURE 1.3 – Valid inequality, facet

Now, we describe the polyhedra by facets.

Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, the question is to find out which of the inequalities $a^i x \leq b_i$ are necessary in the description of P and which can be dropped. Indeed, we will show that those necessary to describe P are the same, whatever the initial inequality description of P.

Definition 5 *The inequality $\pi x \leq \pi_0 [or(\pi, \pi_0)]$ is called a valid inequality for P if it is satisfied by all points in P.*

Note that (π, π_0) is a valid inequality if and only if P lies in the half-space$\{x \in \mathbb{R}^n : \pi x \leq \pi_0\}$, or equivalently if and only if $\max\{\pi x : x \in P\} \leq \pi_0$.

Definition 6 *If (π, π_0) is a valid inequality for P and $F = \{x \in P : \pi x = \pi_0\}$, F is called a face of P, and we say that (π, π_0) represents F. A face F is said to be proper if $F \neq \emptyset$ and $F \neq P$.*

Definition 7 *A face F of P is a facet of P if dim(F)=dim(P)-1.*

1.7.2 Cutting plane methods

Many combinatorial optimization problems can be formulated as mixed integer linear programming problems. Then, they can be solved by branch-and-cut methods, which

are exact algorithms consisting of a combination of branch-and-bound algorithm with a cutting plane method. These methods work by solving a sequence of linear programming relaxations of the integer programming problem. *Cutting plane methods* improve the relaxation of the problem to more closely integer programming problem and branch-and-bound algorithms carry out by a sophisticated divide and conquer approach to solve problems. Cutting plane algorithms for general integer programming problems were first proposed by Gomory [27]. Thus, this method sometimes called "Gomory Cut", who proved that these algorithms terminate after a finite number of iterations with an optimum solution.

Now, let P be a combinatorial optimization problem, E its basic set, $w(.)$ the weight function, and $\mathcal{S}$ the set of feasible solutions. The problem P consists in finding an element of $\mathcal{S}$ whose weight is maximum/minimum. If $F \subseteq E$, then the $0-1$ vector $x^F \in \mathbb{R}^{\mathbb{E}}$ such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ otherwise, is called the incidence vector of F. The polyhedron $P(\mathcal{S}) = convx^S | S \in \mathcal{S}$ is the polyhedron of the solutions of P or polyhedron associated with P. P is thus equivalent to the linear program $\max\{cx | x \in P(\mathcal{S})\}$. Notice that the polyhedron $P(S)$ can be described by a set of a facet defining inequalities. And, when all the inequalities of this set are known, then solving P is equivalent to solve a linear program.

The objective of the polyhedral approach for combinatorial optimization problems is to reduce the resolution of P to that of a linear program. In order to reduce P we need a deep investigation of the polyhedron associated with P. It is generally not easy to characterize the polyhedron of a combinatorial optimization problem by a system of linear inequalities. In particular, when the problem is NP-hard it is difficult to find such a characterization. Moreover, the number of inequalities describing this polyhedron is exponential in most of time. Therefore, even if we know the complete description of that polyhedron, its resolution remains in practice a hard task because of the large number of inequalities.

Cutting plane method is based on the so-called *separation problem*. This consists, given a polyhedron P of $\mathbb{R}^n$ and a point $x^* \in \mathbb{R}^n$, in verifying whether if x^* belongs to P, and if this is not the case, to identify an inequality $a^T x \leq b$, valid for P and violated by x^*. In the later case, we say that the hyperplane $a^T x = b$ separates P and x^*. More precisely, the cutting plane method consists in solving successive linear programs, with possibly a large number of inequalities, by using the following steps. Let $LP = \max\{cx, Ax \leq b\}$ be a linear program and LP' a linear program obtained by considering a small number of

inequalities among $Ax \leq b$. Let x^* be the optimal solution of the latter system. We solve the separation problem associated with $Ax \leq b$ and x^*. This phase is called the separation phase. If every inequality of $Ax \leq b$ is satisfied by x^*, then x^* is also optimal for LP. If not, let $ax \leq \alpha$ be an inequality violated by x^*. Then, we add $ax \leq \alpha$ to LP' and repeat this process until an optimal solution is found.

1.8 Branch and cut algorithm

Branch and cut methods are often successful for finding an exact solution of hard optimization problems. For each instance, the method always maintains an upper bound (ub) and a lower bound (lb) for the optimum solution value. Iteratively, the value of the upper and lower bounds are improved, until they get the optimal solution, or a solution that is tight enough. It is known that, branch and cut is a special case of branch and bound method, where in branch and cut method the bounds are determined through LP and polyhedral theory. Informally, we summarize the basic concepts of branch and cut. We need a partial description Q of $Q_c(G)$ with the properties that the latter is contained in Q. We call such a polytope a relaxation polytope. The inequality system $A_p x \geq b_p$ describing Q is known and can be generated in polynomial time. We optimize over Q to solve the linear program, which is found in the form of the LP mentioned in this chapter. Fast algorithms for solving linear programs exist, for example the well known one is Simplex method. Within the branch and cut approach we start by some relaxation Q of $Q_c(G)$. Iteratively, we generate tighter description of the cut polytope. The upper bound (ub) on the optimum value of the maximum cut can be obtained by any heuristic. In the case where upper and lower bounds are the same we can stop and return an optimum solution, we can also stop where lower bound solution vector becomes integer. In branch and cut a sub problem is a *node* of the tree. The branch and cut algorithm is described as follows.

Algorithm 2: Branch and Cut Algorithm

1 Start with some $P \subseteq P_c(G)$.
2 Solve $(lb) = cx^* = max\{cx | x \in P\}$.
3 (ub) : value found heuristically.
4 **if** *ub=lb or x^* is cut* **then**
5 | Stop ;
6 **else**
7 | find a better description of $P_c(G)$;
8 | Goback to 2
9 **end**
10 **if** *no better description can be found* **then**
11 | *branch* : select a variable x_{ij} with $x^*_{ij} \notin \{1, 0\}$; Generate two sub problems in one of which x_{ij} is set to 0, and in the other is to 1.
12 **end**

Algorithm 1: Cutting Plane Algorithm

Input: A linear program LP and its system of inequalities $Ax \leq b$

Output: Optimal solution x^* of LP

1 Consider a linear program LP' with a small number of inequalities of LP ;
2 Solve LP' and let x^* be an optimal solution ;
3 Solve the separation problem associated with $Ax \leq b$ and x^* ;
4 **if** *an inequality* $ax \leq \alpha$ *of LP is violated by* x^* **then**
5 | Add $ax \leq \alpha$ to LP';
6 | Repeat step 2 ;
7 **else**
8 | x^* is optimal for LP;
9 | return x^*;
10 **end**

2

Heuristics and Meta-heuristics Solutions

Contents

Efficient job scheduling algorithms are addressed in this chapter to improve the resource utilization in cloud computing where the aim is to minimize the total completion time (Makespan). We present a genetic-based task scheduling algorithms in order to minimize Maximum Completion Time Makespan. These algorithms combines different techniques such as list scheduling and earliest completion time(ECT) with genetic algorithm. We reviewed, evaluated and compared the proposed algorithms against one of the well known GAs available in the literature, which has been proposed for optimizing the task scheduling on heterogeneous systems. After an exhaustive computational experiments the results identify that our proposed Genetic algorithms show a good performance dominating the evaluated algorithms in different problem sizes and complexity for a large benchmark set of instances. Moreover, greedy algorithm and ILP have been applied to attack the problem.

2.1 Introduction

This chapter discusses different solutions for the problem of unrelated parallel machine scheduling with precedence constraints based on heuristic and metaheuristic approaches. In Cloud environments a task scheduling is a process that maps and assign the inter-dependent tasks on the data centers (resources) [15]. It allocates the tasks its appropriate virtual resources which is virtual machines (VMs) inorder to satisfy objective functions imposed by end-users. The idea of virtualizing a computer system resources(processors, memory, input/output devices), aiming to improve the sharing of computer resources[29]. Generally,an efficient task scheduling algorithm will have an important impact to the performance of the system throughput [40, 36]. The scheduling problem in cloud computing can be generalized as an unrelated parallel machine with different speeds and precedence constraints. We consider VMs as an unrelated parallel machine because the cloud computing providers offer their services virtually by sharing their physical resources through a large number of virtual machines in parallel. These virtual machines, allocated with different CPU capacities, so it can be considered as an unrelated parallel machines. In cloud computing users may face hundreds of thousands of virtualized resources to utilize. It is hard to allocate userś tasks on the available resources. Due to the virtualization properties, cloud computing leaves task scheduling complexity to the virtual machine layer through resource virtualization.

Hence, to allocate the resources to each task efficiently, scheduling plays more important role in cloud computing [28]. It is difficult to obtain an optimal solution with traditional optimization methods. Mathematical optimization techniques can solve the problems optimally for a reasonable size of instances, however, in the case of a large scale problems, their application is limited [65]. Dispatching rules (EDD, SPT, LPT,...) are suitable only for small sized problems. It is also known that, no single dispatching rule guarantees optimal or near to the optimal solution in various problems [45]. Research efforts in scheduling are concentrated on heuristic approaches as well. Many heuristics and meta-heuristics have been proposed such as simulated annealing (SA), branch and bound (Branch and Bound), tabu search, and genetic algorithm (GA) [28]. Among these various approaches to different scheduling problems, there is an increasing interest in applying GAs, this interest comes from its characteristic, the ease of implementation, and the high adaptability. The important difference between GA and other heuristics is that GA generate a set of solutions (populations) rather than a unique solution, which can lead to a better diversity. In scheduling problems, C_{max} is equivalent to the completion time of

the last task leaving the system. The small *makespan* usually means a high utilization. Therefore, reducing the *makespan* should also lead to a higher throughput rate in the overall system [45]. Three genetic algorithms have been applied to solve this problem.

2.1.1 Literature Review

The problem of task scheduling in distributed systems is known as an NP-hard [46]. To allocate n jobs to m virtual machines (VMs), the number of allocation is $|n|^{|m|}$ and the number of states will be $n!$. One of the objectives of scheduling in cloud computing is to determine the assignment of jobs to VMs in order to optimize the completion time of the last task in the system. The job scheduling problem in distributed, and heterogeneous systems such as cloud computing [15], has been studied widely in the last few years. Generally, job scheduling problem can be found in two forms : dynamic and static. When all information needed for scheduling, such as data dependencies between jobs, and execution times of jobs are known by the scheduler, then the scheduling problem known as static scheduling problem. In static scheduling, jobs placed during the compile time. On the other hand, in the dynamic model, jobs are allocated to the processors upon their arrival to the scheduler, and scheduling decisions must be made immediately at run time [48, 49]. In this section we focus our attention on the available algorithms for static scheduling in cloud environment, as well the algorithms that haves been used to solve the unrelated parallel machine with precedence constraints scheduling problem. A survey on scheduling in cloud computing can be found in [15],[28],and [43]. Different methods for optimizing different objectives in cloud computing exist (See[32, 33, 34, 48, 35]). Some researchers proposed efficient meta-heuristics based on genetic algorithm : Zhou et al in [50] proposed a genetic algorithm based on earliest completion time (ECT) to minimize completion time (we represent this GA in the next sections. Arash and Yalda also developed hybrid genetic algorithm for work flow scheduling in cloud system (HSGA). It merges best-fit and Round Robin methods to obtain a good solution quickly by making an optimal initial population. It makes a job prioritization in complex graph. A particle swarm optimization (PSO) used in [51] for workflow scheduling in cloud environment, which considers not only execution cost but also the cost for transmitting dependent data. In [52] a PSO is also formulated as a model for the multi-objective task assignment to optimize the time and cost. To the best of our knowledge, none of the existing Genetic algorithms have considered the idea of scheduling jobs with a high number of successors in order to optimize the makespan. Shamsollah *et al* in [40] presented a novel approach for job scheduling in cloud computing

which was called priority based job scheduling algorithm. Their solution method based on multi-criteria decision-making model and multi-attribute decision-making model which was first developed by T.Saaty in 1980 and called the theory of Analytical Hierarchy Process. The proposed algorithm mainly focused on priority of job. The experimental results indicated that the algorithm has reasonable complexity, however the authors did not expect that this algorithm provides an optimal makespan. Xiaofeng Wang et al in [42] proposed a new max-min strategy specifically for GAs, which use a novel task criticalness. They modeled a workflow job as a Directed Acyclic Graph (DAG). The reliability driven scheduling of a workflow application applied to maximize the reliability and to minimize the makespan of the application. The literature on parallel machine scheduling is fairly large, we focus mainly on the non-preemptive unrelated parallel machine problem with precedence constraints to minimize makespan criterion. There are many applications for this scheduling problem specially in distributed computing systems [52],[50]. Several heuristics and meta-heuristics have been proposed to solve this problem for optimizing different objectives. In [38] Vallada and Ruiz proposed a genetic algorithm to minimize the makespan. Their GA includes a fast local search and a local search enhanced crossover operator. In [65] Balin proposed a new crossover operator for genetic algorithm to minimize makespan, his algorithm achieved a high computational speed for large-scale problems. In [53] Tavakkoli-Moghaddam *et al* proposed a genetic algorithm to solve bi-objective unrelated parallel machine scheduling problem. Jeffrey *et al*[44] considered the case where a task has at most one predecessor and at most one successor. They proposed a greedy search, which we use in this chapter for a comparison purpose, and they proposed also a heuristic based on a branch-and-bound procedure. They also applied a simulated annealing algorithm to solve the problem. Liu and Yang in [71] considered the case where the machines do not have specific speeds but every job has a processing time depending on the used machine, which is an extension of the constraint we used in our problem. They proposed an integer linear programming model and they provided a polynomial time algorithm that can schedule the prior job on the prior machine as early as possible for minimizing makespan. A hybrid genetic algorithm has been proposed in [71] for minimizing the total tardiness for the problem of unrelated parallel machines with precedence constraints. For other proposed approaches the reader could refer to see [37, 63, 30, 31, 39, 41].

In this chapter, we deal with the problem of job scheduling in cloud environment and we generalized this problem as an unrelated parallel machines scheduling with precedence

constraints for optimizing the makespan. We proposed three genetic algorithms, they show a very good performance for small and medium benchmarks.
In the next section we will present the problem formulation and a mathematical formulation associated to solve unrelated parallel machine with precedence constraints scheduling problem.

2.2 Problem formulation

The problem under consideration is to schedule n jobs on m machines which are arranged in parallel with the aim of minimizing the total completion time. Let J be the set of the jobs and M be the set of the parallel machines. A precedence constraint between two jobs j_1 and j_2 is denoted by $(j_1 \prec j_2)$ and it requires that job j_2 cannot start to be processed until job j_1 will finish its processing. The type of the precedence constraint is a graph type, which is denoted by $D = (V, A)$, where V is the set of vertices associated at each job ϑ and V denotes the set of edges associated with each precedence constraint. We called this graph the precedence graph. We take also the case where $\{v, \bar{w}, w\} \subseteq V$ such that v before $\bar{w}$ and $\bar{w}$ before w then v before w. We consider also the speeds for all machines denoted by s_i, where $i \in M$. Every job $j \in J$ has a processing time p_j and its effective processing time depends on the selected machine i, where $p_{ij} = \pi_j \times s_i$. Each machine $i \in M$ cannot process more than one job at a given time. Furthermore, machines have different speeds and preemption of jobs is not allowed. According to the well-known $\alpha|\beta|\gamma$ scheduling problem classification scheme proposed initially by Graham *et al* [66], scheduling problem classification scheme this problem can be denoted as $P|prec|C_{max}$ We denote by C_i the completion time of machine i, where $i \in M$, and denote by C_ϑ the completion time of job j, where $j \in J$, in the rest of this chapter. Thus, the problem can be reduced to the following mathematical formulation proposed in [71]

$$x_{jir} = \begin{cases} 1 & \text{if job } j \text{ is processed in position } r \text{ on machine } i \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in J, i \in M, r \in J$$

$C_j \in \mathbb{N}^+$ the completion time of job j.
$C_{max} \in \mathbb{N}^+$ is the total length of the schedule. That is, when all the jobs have finished processing.

The model can be stated as :

$$\min C_{max}$$

$$C_j \leq C_{max} \qquad \forall j \in J, \tag{2.1}$$

$$\sum_{i \in M} \sum_{r \in \{1,...,|J|\}} x_{jir} = 1 \qquad \forall j \in J, \tag{2.2}$$

$$\sum_{j \in J} x_{jir} \leq 1 \qquad \forall r \in \{1, ..., |J|\}, \forall i \in M, \tag{2.3}$$

$$\sum_{j_1 \in J} x_{j_1 ir} - \sum_{j_2 \in J} x_{j_2 ir-1} \leq 0 \qquad \forall r \in \{2, ..., |J|\}, \forall i \in M, \tag{2.4}$$

$$C_{j_2} - C_{j_1} + C(2 - x_{j_2 ir} - x_{j_1 ir-1}) \geq p_{j_2 i} \qquad \forall r \in \{2, ..., |J|\}, \forall i \in M, \forall j_1 \neq j_2 \in J, \tag{2.5}$$

$$C_j \geq \sum_{r \in \{1,...,|J|\}} p_{ji} x_{jir} \qquad \forall i \in M, \forall j \in J, \tag{2.6}$$

$$C_{j_2} - C_{j_1} \geq \sum_{r \in \{1,...,|J|\}} \sum_{i \in M} p_{j_2 i} x_{j_2 ir} \qquad \forall (j_1, j_2) \in A, \tag{2.7}$$

$$x_{jir} \in \{0, 1\}, \qquad \forall j \in J, \forall i \in M, \forall r \in \{1, ..., |J|\}, \tag{2.8}$$

$$C_j \geq 0, \qquad \forall j \in J, \tag{2.9}$$

$$C_{max} \geq 0, \tag{2.10}$$

The objective function is to minimize the makespan. Inequalities (2.1) ensure that the global makespan is greater than or equal to the completion time for all jobs. Inequalities (2.2) ensure that each job should be assigned to one position on one machine. Inequalities (2.3) guarantee that at most one job will be assigned to a position on all machines. Inequalities (2.4) guarantee that there must be no empty time slot between jobs in sequences. Inequalities (2.5) ensure that the job j_2 must start its processing time after the finishing of job j_1, if job j_1 is assigned on position $r-1$ and job j_2 on the position r on the same machine. Inequalities (2.6) bounding the completion time for all jobs, only if the job is in the first position, otherwise C_j is bounded with the inequalities (2.5). Inequalities (2.7) controls the precedence constraints. Inequalities (2.9) define the type of decision variables. Inequalities (2.10) bounds C_j. This mathematical model will be used later in computational experiments.

2.3 An existing algorithm

Greedy algorithms are simple and straightforward [42]. The described algorithm in this section is an adaptation from [44]. We report the main steps for the sake of comparison with our genetic algorithm. The algorithm first prepares the available tasks at each iteration, then uses the greedy technique to find the best local completion time hopefully to lead to the global minimal completion time. At each iteration the algorithm calculates the completion time for each available tasks on all machines. Then, it allocates the job to the machine on which we achieve the best minimum completion time. This algorithm is designed as follows :

Algorithm 3: Greedy algorithm description

Data: set of $M = \{1, ..., m\}$ machines, and set of $J = \{1, ..., n\}$ jobs

Result: $C_{\max}$

1 Let Av be the set of all available tasks that could be scheduled.
2 Starting : $C_i = 0$ for all $i \in M$.
3 **while** $|Av| > 0$ **do**
4 Let j' and i' be the job from the available tasks and the machine that can finish the earliest. Schedule j' on i' and update the availability $C_{i'}$ of machine i'. Update Av;
5 **end**

2.4 Genetic algorithm (GA)

The GA is a general search approach inspired by the process of the natural evolution. It has been widely exploited for solving combinatorial optimization problems [37]. It is introduced in the 1970s by Holland [38] The basic idea of our algorithm is to exploit the advantages of the both of the evolutionary and heuristic based algorithms. The solution of any problem using GAs will be represented as a chromosome containing a series of genes, its fitness value is related to its objective function and constraints for that solution.

The population P of generation g, denoted by (P_g), consists of a set of chromosomes. GA utilizes a population of solutions in its search in order to find a better solution. The efficiency of GA depends largely on the presentation of a chromosome which is composed of a series of genes. In this chapter we proposed two encoding methods random

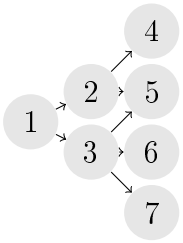


FIGURE 2.1 – Precedence Constraints

J	1	2	3	4	5	6	7
Processing Time	1	2	3	1	2	3	4

TABLE 2.1 – Processing Time

ordering method and list scheduling method to formulate the chromosome. During each iteration step (Generation), genetic operations, that is *crossover, mutation* and *selection* are processed to search potential better solutions. Crossover combines two chromosomes to generate the next generation(P_{g+1}). The Mutation operator reorganizes the structure of the chromosome by changing the positions of genes randomly so that a new combination of genes may appear in generation P_{g+1}. It manages the search by jumping form out of local optimal solutions. The reproduction process is to copy a chromosome to the next generation so that chromosomes from various generations could pull together in the evolution and the quality of the population may improved after each generation [65]. The general schema of GA may be illustrated as follows :

Algorithm 4: Genetic algorithm pseudo-code

```
1 Generate randomly an initial population of solutions.
2 Calculate the fitness of the initial population.
3 while Stopping Criteria Not Satisfied do
4 |   Select a pair of parents based on fitness.
5 |   Create two offspring using crossover.
6 |   Apply mutation to each child.
7 |   valuate the mutated offspring. All the offspring will be the new population.
8 end
```

The instances in Figure 2.1 and Table 2.1, will be considered for numerical example.

The rest of genetic algorithms tested and compared under the following proposed benchmark of instances. The processing times are uniformly distributed between 1 and 100 as it is common in the literature [78]. We keep the processing time for a specific size

of problem as in Table 2.2, and we changed the density of graph.

processing time		
	Average	Stdev
50	48.11	25.98
100	52.26	28.62
200	50.81	27.61
500	49.72	27.70

TABLE 2.2 – Average and Standard deviation for Instances Processing time

2.4.1 Modeling the problem using Genetic Algorithm

In this section, we present the modeling of our GAs for directed ascending graphs (DAGs) in cloud environment. These scheduling algorithms effectively addresses the issues of minimizing the makespan.

Task Scheduling Genetic Algorithm (GATS)

This GA has been proposed on heterogeneous computing systems by Zhou et al in [50]. They call it, *task scheduling based Genetic Algorithm (GATS)*. It has been modeled as follows : The linear order of all jobs forms the chromosome. Each chromosome represents a solution for the problem by scheduling the jobs in the order given by the permutation, the order of the jobs should be a valid topological order as the associated nodes in the DAG, where start nodes should be placed in the chromosome at the beginning position , while the last nodes should be placed at the end. The initial population is produced by making a random perturbation to the order of jobs in the first chromosome to produce a valid chromosome, until the desired size of the initial population reached. Indeed, a linear crossover from a single random position applied to the two selected parents. The mutation operation operated for all individuals of the new population considering the precedence constraints topologically. Then, the objective function is evaluated by using the Earliest Completion Time (ECT) technique, which schedules a candidate job onto machines (processors) on which the completion time of the job is the earliest. The robust characteristic in this GA is the generation of a valid chromosome in the initial population. At the next generation, we modify GATS in $GATS^{+}$ by just making a random mutation for two genes selected randomly and if the candidate chromosome is not valid, then we

throw it out by assigning a big value as C_{max} to this candidate. Since we have valid chromosomes in the initial population, the robust characteristics of the GATS can still be maintained and we will not spend a lot of computation time in the mutation operator. This small change increases the chance to find a best result, especially when the computation time is less than one minute, because GATS spends a lot of time in mutation procedure if the candidate is not valid. Table 2.3 shows the results obtained by GATS and $GATS^{+}$ in one second with Psize = 100, $P_c = 1.0$ and $P_m = 0.5$. The dashed results means that GATS does not find a solution during one second and also when we run the instances for 10 seconds GATS cannot find a solution with the problems of large number of instances in all of the three density sets.

Genetic Algorithm Based on Cut-point (GACP)

For this genetic algorithm (GACP), the chromosome coding composed of two rows : the first represents a valid order of jobs according to the precedence constraints, and the second row gives an information on job positioning according to the cut-point. We generate $m-1$ random cut-points $(cp) = \{cp_1, cp_2, ..., cp_{m-1},\}$ to assign jobs to its VMs. The solution provided as follows : The sequence of jobs from j_0 to j_{cp_1} will be assigned on VM1 and the sequence of jobs from $j_{cp_1} + 1$ to j_{cp_2} will be assigned on VM2 and the sequence of jobs from to on VM3 and so on. In other words, we assign a valid sub-sequence of a random length of jobs on a specific VM. In this genetic algorithm we carried out one point crossover between two parents and an exchange between two random points carried as mutation operator. However, this genetic algorithm gave bad results. The best result obtained by GACP is at least two times the C_{max} obtained by GATS.

Genetic Algorithm Based on The List of Available Jobs (GAAV)

In this section we will propose a simple idea to generate the population with lowest computational cost, where the chromosome coding depends on VMs and places the job in its associated VM, and the computational efforts will be taken in the evaluation function. In this section we will describe our second genetic algorithm, based on the list of available jobs (GAAV), which is depends mainly on the available-list scheduling heuristic.

Coding an initial population : The assignment of VMs to the list of jobs is a candidate solution to the problem. Therefore, the chromosome can be represented by a linear list of integers, each integer representing a VM, here mi considered as gene. The series of genes with the length of n are generated randomly by assigning each job of J to a random mi

Instances		L-Density		M-Density		H-Density	
m	n	GATS	GATS+	GATS	GATS+	GATS	GATS+
2	50	17274	17251	17251	17251	17274	17927
2	100	37667	37667	37713	37667	37667	38636
2	200	-	74590	-	74026	-	75702
2	500	-	181679	-	182170	-	183294
5	50	10349	10527	8163	8279	7813	7355
5	100	20791	19766	19240	18355	20049	19747
5	200	-	37731	-	36704	-	32981
5	500	-	88668	-	84741	-	82615
10	50	7820	7820	4850	4850	6460	6460
10	100	10860	10768	10858	11008	10970	10900
10	200	-	22236	-	20189	-	18182
10	500	-	48365	-	43323	-	40901
20	50	6238	6090	6430	6430	4540	4540
20	100	9775	9604	8693	8504	9628	9593
20	200	-	20892	-	15222	-	15189
20	500	-	39090	-	32557	-	30158

TABLE 2.3 – $GATS$ vs $GATS^{+}$, C_{max} Comparison

4	1	3	5	5	4	1	2	3	2

TABLE 2.4 – Chromosome representation for GAAV

from the set of M. Table 2.4 shows the chromosome representation for ten jobs on five VMs.

In GAAV there is no computational effort to produce the initial population (IP) because it is produced by making a random number of permutations to the integer-list to produce a chromosome until the size of IP ($IPsize$) reached. Hence, all chromosomes give a valid solution.

Fitness evaluation In GAAV, to evaluate the chromosome, first we search the virtual machine with minimum completion time C_i. For this machine we take its available jobs according to the precedence constraints, from these available jobs we schedule the job with the maximum number of successors will be placed to the selected virtual machine first. Then, we update Available list, and search again for machine with the minimum completion time and repeat this process until we finish the evaluation process. Simply, at each placement iteration we select the machine with minimum completion time and its available job with highest number of successors. Then, we assign the job which could lead to a late schedule of some jobs in the future to its VM, maybe this job will affect the C_{max} of the whole system. At the end of this process, a valid schedule will be obtained and the fitness function (C_{max}) also will be calculated. 5 illustrates the GAAV fitness function steps.

Algorithm 5: Fitness Evaluation Function for GAAV.

```
1 Let Available be the current set of jobs without predecessors
2 while |Available| > 0 do
3 |   Selectedmachine = the machine with minimum C_i.
4 |   Selectedjob= the job of greater number of successors in Selectedmachine.
5 |   Add Selectedjob to Selectedmachine Update C_i.
6 |   Update Available.
7 end
```

Crossover The process of replacing some of the genes of one parent by corresponding ones of the other parent is known as crossover. Here this operator is carried out based on a linear crossover from a single point. This operator is applied to the selected

parents ($parent1$, $parent2$), then a new two offsprings are obtained as $offspring1$, and $offspring2$, the crossover operated between two chromosomes one with higher fitness value and the other with the P_c ratio. Table 2.5 illustrates the crossover operator.

					↓					
Parent 1	1	4	2	1	3	3	5	2	1	3
					↓					
Parent 2	2	1	1	4	5	2	3	3	1	5
Child 1	1	4	2	1	3	2	3	3	1	5
Child 2	2	1	1	4	5	3	5	2	1	3

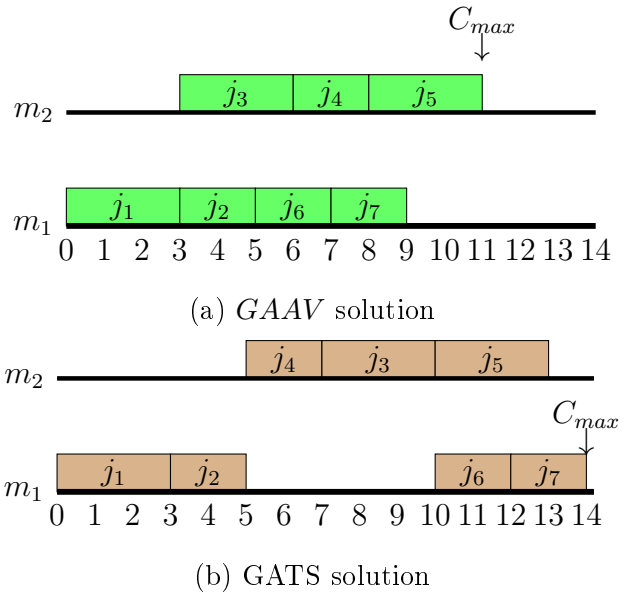
TABLE 2.5 – GAAV :Crossover Operator

Mutation Mutation can be thought as an effectively escape method for premature convergence by randomly change the value of an individual. For maintaining the feasibility of the new generated individual. During the mutation process, one gene selected randomly and we put it on a different random mi from the set of M to obtain a new offspring, Table 2.6 represents the mutation operator.

Offspring 1	1	4	2	1	3	**2**	3	3	1	5
Offspring 2	1	4	2	1	3	4	3	3	1	5

TABLE 2.6 – GAAV :Mutation Operator

Selection Finally, the best chromosome of the first population is stored as in a linear ranking.



(a) *GAAV* solution

(b) GATS solution

FIGURE 2.2 – Example of the GATS and GAAV encoding.

Stopping rule The GA is stopped when the execution time is greater than the maximum execution time allowed.

Figure 2.2, represents an example of chromosome encoding solution for GATS, which is (1-3-2-6-7-4-5), and a chromosome encoding solution for GAAV, which is (2-1-1-2-2-1-1), for the instances in Table 2.1 according to the precedence constraints in Figure2.1, run on two VMs of different speeds which are : $s_1 = 1$, $s_2 = 2$.

Genetic Algorithm ($GAAV^+$)

When GATS depends mainly on ECT technique and GAAV based on the local density of the DAG, according to the effectiveness of these two techniques in the optimization of scheduling unrelated parallel machine problem with precedence constraints, we combined these two techniques in $GAAV^+$.

In this genetic algorithm $GAAV^+$, the modification occurred in the Fitness Evaluation. Therefore, the chromosome representation as in Table 2.4, according to this coding we know the VM for each job, this fitness evaluation will select the job in machine that will give the minimum C_j from the available jobs j AV, where AV is the current set of jobs without predecessors, this is ECT technique. From the other hand, at the same time we considered the number of successors of this job, and this is the local density of the DAG technique. Thus, the evaluation can be taken by selecting job j from AV with the minimum

value produced by the following function : $\alpha(C_j) - (1-\alpha)|Succ_j|$ where $\alpha \in [1, 0]$ and $|Succ_j|$ is the number of successors of job j. We schedule the job with the minimum value of this function first to generate a valid schedule for calculating C_{max}.

Experimental Results

This section presents the experimental results of $GATS$, $GATS^+$, $GAAV$ and $GAAV^+$. A set of simulations have been performed on Dell Intel, core i5 running at 3.4 GHz, and 8 GB of RAM. The GAs have been coded in C++, compiled with g++ compiler, and tested under Ubuntu 14.02 64-OS. The entries in the Table 2.8 are :
m : number of machines,
n : number of jobs,
GATS : C_{max} value for GATS,
GATS+ : C_{max} value for GATS+,
GAAV : C_{max} value for GAAV,
GAAV+ : C_{max} value for GAAV+,
Genetic search is implemented through genetic operators. 2.8 show the results given by our proposed GAs ($GATS^+$, GAAV, and $GAAV^+$) compared to GATS. From this simulation study we fixed the parameters with the combination of (100, 1.0, 0.5, 600), Population size, Crossover ratio, mutation ratio and the computation time respectively. We have tested out different values of α in $GAAV^+$ to find the best value. Therefore, we took $\alpha = 0.5$. From the results we have noticed that when population size in GAAV is larger than 100, any increase of it has no significant influence on the performance of the genetic algorithm. In Table 2.8 we can see the genetic algorithm GAAV can improve 31% of the solutions obtained by GATS in low density problems, 43% in medium density problems and 43% in high density problems. One other interesting outcome is that GAAV can be considered as an efficient algorithm with the problems of small and medium number of VMs. $GAAV^+$ mostly outperforms GATS when the number of jobs 100 and 200 in high density. This may improve 50% of solutions obtained by GATS. It can also improve 50% of medium density problems and 31 of low density problems. If we focused our attention to genetic algorithm $GATS^+$, we can see that for low density showed a good performance and for medium and high density problems is really far from the best solutions, because when we thrown out the invalid candidates we lost some information about some generations. According to the GATS operators behaviors, it always needs more time than the specified stop criterion, with the problems of large numbers of machines and jobs.

Another interesting factor to study in the experiments is the count of best solutions obtained by GAs. In Figure 2.3 we can notice that $GATS^+$ can find maximum number of best solutions overall instances in one second and ten seconds, whereas GATS cannot find best solution within the specified computation time, we noticed that GATS cannot obtain solution when we run it for 1 and 10 seconds, it needs at least 77% seconds to obtain solution with few number of iterations. We also noticed that GATS need a lot of time to find the first population and for other genetic algorithm operators. We can also see the similarity of a behavior for GATS and $GATS^+$ when we run them for 10 and 60 seconds with the improvement of $GATS^+$. Therefore, we can say GATS+ outperforms GATS in terms of best solutions for sizes and densities. In Figure 2.3 it is clear that $GAAV^+$ has a positive relationship with the computation time, and has the ability to improve the counts of best solutions for different problems. The other positive thing is that, it can also obtain a solution within the specified computation time. The efficiency of GAAV appears when we run it for one second ; it can always obtain the best solution for the minimum and medium problems.

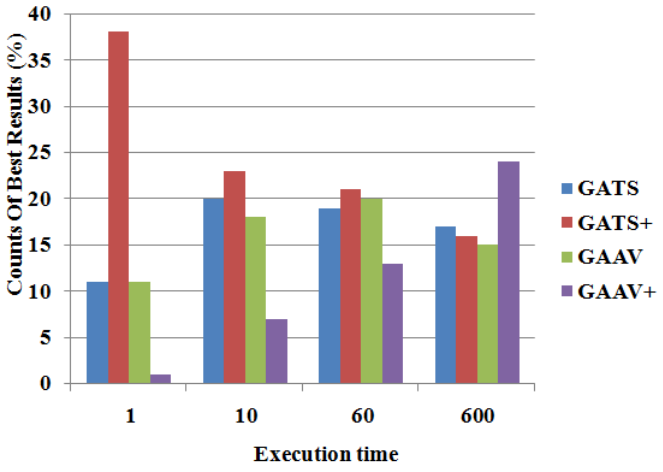


FIGURE 2.3 – Counts of best results .

Figure 2.5 indicates the Average Relative Percentage Gap (ARPG) between the first and the best solution for genetic algorithm. Indeed, ARPG is computed as follows $100 \times \frac{(C_{\max}^{\text{First}} - C_{\max}^{\text{Best}})}{C_{\max}^{\text{Best}}}$, where $C_{\max}^{\text{Best}}$ is the best known C_{max}, obtained by the given GA, and $C_{\max}^{\text{First}}$ is the first C_{max} obtained by this GA. We noticed that GATS with low, medium and high density DAG problems cannot improve its solutions. This means GATS starts with a good initial population and the computation time will not affect this solution positively.

This behavior inherited also by $GATS^+$, whereas GAAV and $GAAV^+$ start with random solutions, but they can obtain a better solutions when we increase the computation time. Figure 2.4 represents the convergence traces for processing the problem of high density of a randomly generated DAG with 5 VMs and 100 jobs. It can be observed from this figure $GAAV^+$ decreases quickly. GAAV also shows a quick decreasing and provides a best solution when it runs for 10 and 60 seconds. Whereas GATS remains in the same trend, this behavior followed with most of our problems. Hence, we can say that the techniques used in GAAV and $GAAV^+$ can improve the solution and we can find a better upper bound for this problem. The modification of GATS in $GATS^+$ also has a good outcome.

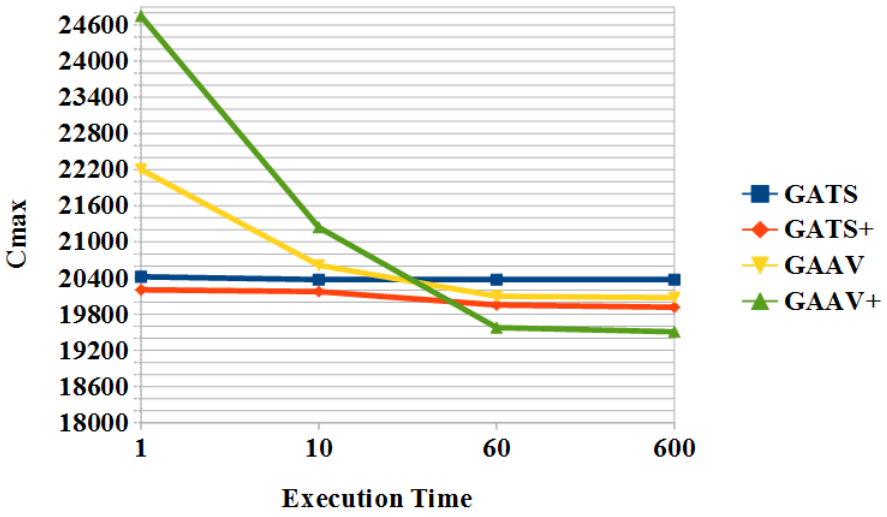


FIGURE 2.4 – Genetic algorithms convergence.

Integral Linear Programming Solution (ILP)

The mathematical model is applied for small instances. It is implemented with CPLEX 12.4, on an Intel, core i5 running at 3.4 GHz, and 8 GB of RAM under a computation time limit of one hour (if after one hour no optimal solution is obtained, the current integer solution is returned). In Table 2.7, columns LB and UB represent the lower bounds and the upper bounds respectively for some problems, for which CPLEX is not able to find the linear relaxation value. Indeed, we have limited the use of the RAM to 6 GB and for the most of instances this amount is not sufficient for the linear relaxation with all these constraints and variables in the model.

Instances		High Density		Medium Density		Low Density	
m	n	LB	UB	LB	UB	LB	UB
2	20	8892	38480	8008	38480	8190	38480
2	50	8887	40885	5482	40885	5575	40835
10	50	7263	45695	4746	45695	5888	45695

TABLE 2.7 – Results Obtained by the ILP

Transformations Between Genetic Algorithms

In spite of the variety between GATS and GAAV encodings, we tried to investigate the ability of each genetic algorithm to improve the solution obtained by the other genetic algorithm. We transformed the best population generated by the first genetic algorithm, to be the first population for the second genetic algorithm. This also provides interesting observations, about the differences between our proposed genetic algorithms and GATS, by doing all transformations from GATS to GAAV and $GAAV^+$, and from GAAV,$GAAV^+$ to GATS. We noticed that, because of the differences of the encoding and the genetic operators between GATS in comparing to GAAV and $GAAV^+$ the ARPG between the best solution obtained by the first GA before transformation and the best solution obtained by the second genetic algorithm after the transformation.

The ARPG is computed as follows : $100 \times \frac{(C_{\max}^{\text{FirstBest}} - C_{\max}^{\text{SecondBest}})}{C_{\max}^{\text{SecondBest}}}$, where $C_{max}^{FirstBest}$ is the best known C_{max}, obtained by the first GA, and $C_{max}^{SecondBest}$ is the best known C_{max}, obtained by the second GA. Table 2.9 shows the ARPG of the transformation processes : the negative values mean that the second best solution is worse than the first best. We observed that the behavior of the proposed GAs and GATS is not the same. From the transformations, GAAV and $GAAV^+$ cannot make an improvement to the best generation obtained by GATS. However, for the solutions obtained by the transformations from GAAV to GATS and from $GAAV^+$ to GATS sometimes these transformations can lead to solutions better than those obtained by GAAV, $GAAV^+$ and GATS. Table 2.10 represents a comparison between the best solutions obtained among all GAs in column labeled Best and the transformations solutions. We can see also, AV^+TS can improve the best solution obtained by the first genetic algorithm for instances of large number of jobs, but this improvement did not improve the best solution obtained among all genetic algorithms.

Instances		Low Density				Medium Density			
m	n	GATS	GATS+	GAAV	GAAV+	GATS	GATS+	GAAV	GAAV+
2	50	17927	17927	17927	17927	17251	17251	17251	17251
2	100	38577	38577	37790	37539	37661	37667	37505	37492
2	200	75595	75682	74659	74474	73847	73956	73210	74850
2	500	180908	180861	180849	195618	180375	180466	179120	190116
5	50	10349	9672	10073	9591	8318	8228	7997	8013
5	100	19921	19776	19921	19320	18467	18213	17793	17640
5	200	36521	36645	36036	36205	35621	35577	35575	34890
5	500	85015	85268	92482	114689	80365	81039	87801	113390
10	50	7820	7820	7820	7820	4850	4850	4887	4850
10	100	10494	10591	10378	10310	10580	10552	10118	9898
10	200	20656	21755	22980	21803	18858	19521	20010	18148
10	500	44935	46714	57585	73291	40309	41062	48073	70042
20	50	6156	6090	6156	6090	6430	6430	6430	6430
20	100	9310	9431	9801	9495	8206	8141	8186	8198
20	200	20112	19964	21151	20707	14477	14348	15395	14430
20	500	36364	36346	43463	54919	30117	31063	41631	51731

TABLE 2.8 – Makespan(C_{max})for the proposed algorithms and GATS, (Low& Medium Density)

Instances		High Density				Medium Density				Low Density			
m	n	$TSAV$	$TSAV^+$	$AVTS$	AV^+TS	$TSAV$	$TSAV^+$	$AVTS$	AV^+TS	$TSAV$	$TSAV^+$	$AVTS$	AV^+TS
2	50	1.14	-0.07	0.44	0.02	0.30	-3.18	3.32	0.08	2.03	0.02	0.30	0.30
2	100	2.12	2.54	0.90	0.26	0.86	-0.09	0.35	0.57	-1.97	0.48	1.47	0.00
2	200	1.63	1.23	0.90	0.66	0.74	-1.36	0.11	0.30	0.95	1.09	0.57	0.56
2	500	-0.28	-12.07	-0.51	12.72	0.06	-10.20	0.54	12.44	0.60	-17.32	0.28	17.46
5	50	10.86	5.88	1.72	8.85	16.06	6.60	-1.64	2.35	6.01	-0.26	6.96	4.61
5	100	4.1	3.03	6.65	5.14	11.16	-0.27	7.77	-3.34	14.82	2.70	3.49	3.57
5	200	-3.23	0.90	-0.74	0.04	1.08	-1.78	-3.83	1.25	-8.45	-0.93	-0.62	5.63
5	500	-9.79	-31.01	-0.14	31.59	-10.13	-37.78	-2.02	46.20	-9.76	-42.92	-6.85	52.27
10	50	1.36	-0.95	1.64	-3.47	-7.58	0.00	6.90	4.74	5.64	-1.97	3.90	1.19
10	100	-8.02	-1.11	2.79	2.78	3.87	4.00	3.22	3.39	1.19	6.03	6.98	4.61
10	200	-12.27	-5.81	0.19	1.30	-14.65	-2.90	-0.34	3.20	-15.55	-8.64	-1.30	-4.83
10	500	-27.8	-37.75	-14.43	52.79	-26.15	-42.14	-16.59	74.51	-26.10	-54.53	-15.66	93.17
20	50	4.29	1.33	-0.64	2.06	1.18	0.00	1.75	0.00	-6.02	0.00	1.46	3.66
20	100	0.38	1.60	10.09	0.84	3.34	-2.43	15.68	3.86	-2.95	-5.23	11.92	4.23
20	200	-1.46	-2.45	-2.65	5.09	-11.18	-4.11	-2.05	7.01	-7.41	-1.65	-2.50	10.00
20	500	-23.17	-39.09	-17.59	57.51	-28.99	-48.88	-12.75	83.44	-26.23	-52.70	-10.66	94.04

TABLE 2.9 – The Average Relative Percentage Gap(ARPG) for the transformation process between Genetic Algorithms

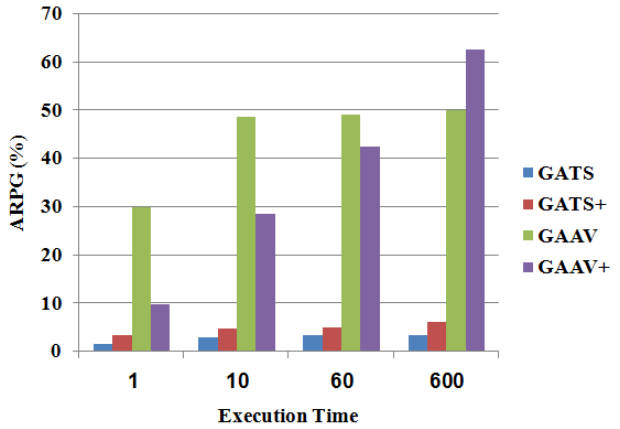
Instances		High Density			Medium Density			Low Density		
m	n	Best	$AVTS$	$AV^{+}TS$	Best	$AVTS$	$AV^{+}TS$	Best	$AVTS$	$AV^{+}TS$
2	50	17927	17927	17937	17251	17251	17251	17251	17251	17251
2	100	37539	**37488**	**37488**	37492	**37488**	**37488**	37667	**37488**	**37488**
2	200	74474	**74425**	**74438**	73210	**73112**	73358	73024	**72891**	**72891**
2	500	180849	**179768**	**180377**	179120	**178412**	**178919**	179612	**178334**	**179580**
5	50	9591	**9308**	**9217**	7997	**7365**	**7440**	7416	**7280**	7435
5	100	19320	**19318**	**18579**	17640	**16929**	18597	18020	**17973**	**17830**
5	200	36036	36011	36418	34890	**33995**	**34122**	31600	**30835**	**30796**
5	500	85015	86133	89414	80365	82007	84885	38147	81178	85589
10	50	7820	7820	8101	4850	4850	4850	6460	6460	6460
10	100	10310	10736	10306	9898	9952	9920	9544	9570	9704
10	200	20656	21045	21218	18148	18951	19301	17030	**16251**	19321
10	500	44935	45262	46536	40309	40841	41663	38147	38744	42580
20	50	6090	6156	6156	6430	6430	6430	4540	4540	4540
20	100	9310	9428	9262	8141	8249	8189	9320	**9054**	9372
20	200	19964	20208	20054	14348	**14318**	**13880**	13735	14235	**13054**
20	500	36346	38198	37252	30117	32289	30835	27765	28950	28707

TABLE 2.10 – Best Makespan/ C_{max} obtained among All GAs Comparing to AVtoTS and AV+toTS

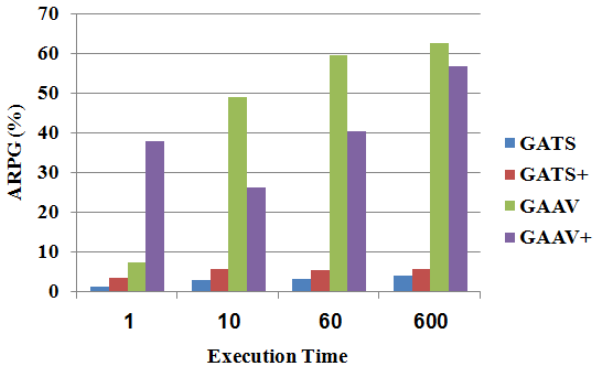
Conclusion

In this chapter we have proposed genetic algorithms for job scheduling problem in cloud computing with the objective of minimizing the makespan (C_{max}), which is considered as an unrelated parallel-machine scheduling problem under precedence constraints. GAAV includes a new local search procedure for local graph density to evaluate the chromosome. $GATS^{+}$ which is based on a permutation coding and ECT, and $GAAV^{+}$ which is combined the innovative characteristics of GAAV with the (ECT) technique. The performances of our proposed genetic algorithms have been compared against one of the best existing genetic algorithm for the same problem. After extensive comparisons, we can conclude that the proposed algorithms can improve the solutions obtained by GATS for small and medium problems. Moreover, they can get better results than GATS within the specific running time (stop criterion) for a high and medium DAG density problem. In the future work, we will enhance the mathematical model by adding new constraints for further improvement. Another interesting topic regarding scheduling problem in cloud computing is to consider the multiobjective optimization context.

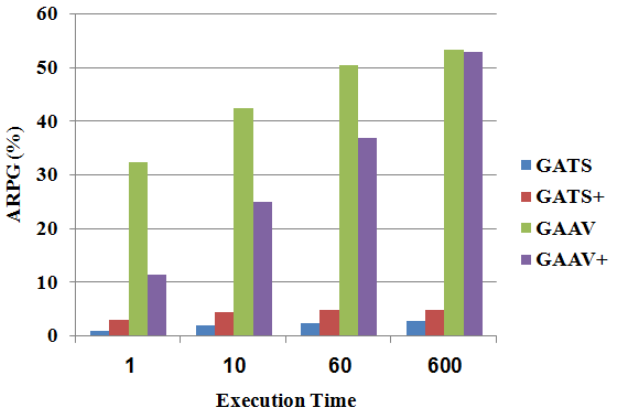
In the next chapter we will compare the available mathematical models with our proposed model, and will present some valid inequalities to our model.



(a) Low Density.



(b) Medium Density.



(c) High Density.

FIGURE 2.5 – Average Relatives Percentage Gap.

3

Mathematical Formulations

Contents

In this chapter we studied an unrelated parallel machine scheduling problem of minimizing makespan subject to precedence constraints $R_m|prec|C_{max}$. In this chapter we compared our proposed mathematical model with other mathematical formulations found in the literature. The main difference between these formulations is the way the makespan has been linearized. We generate sets of benchmark instances and compare the performance of the mathematical formulations with extensive computational testing. Moreover, three families of valid inequalities are proposed. The first two inequalities based on the idea of the precedence jobs and the third based on the shortest processing time(SPT). We studied the validity of the new inequalities and strength them by checking the linear combination. After an exhaustive computational and statistical analysis we can conclude that

the addition of these inequalities decreases the computational requirements to obtain the optimal solution in many cases.

3.1 Introduction

This chapter addresses the unrelated parallel machines scheduling problem with precedence constraints. However, parallel machine scheduling problems (PMSPs) have been a subject of continuing interest for researchers and practitioners since they were first introduced by McNaughton [56]. Many PMSPs for different manufacturing environments have been studied. Previous studies on parallel machine scheduling problems are generally classified into three categories McNaughton [56] : (1) identical parallelmachines (2) uniform parallelmachines and (3) unrelated parallel machines. Among these categories, machines that are non-identical to one another and cannot be fully correlated by simple rate adjustments are classified as unrelated parallel machines Pinedo [55]. This environment is common in different manufacturing and services domains, textile manufacturing, chemical, assembly lines, electronic manufacturing, the area of project management, service industries, and also in computing services. However, the case when machines are unrelated has been much less studied. Additionally, the consideration of precedence constraints between jobs has been considered in limited works (see e.g[74]). Several mathematical formulations are proposed to solve different types of unrelated parallel machine scheduling problems (UPMS)[74, 75].

The unrelated parallel machines is a generalization of the single machine from it is theoretical point of view, and a special case of the open shop. From a practical point of view, it is important because the occurrence of resources in parallel is common in the real world. Moreover, techniques for machines in parallel are often used in decomposition procedures for multi-stage systems[55]. Garey and Johnson in [61] showed that minimizing the makespan C_{max} considering two identical machines is an NP-hard problem. Indeed, The unrelated parallel machine scheduling problem (UPMSP) is more difficult than the identical case, Chiuh and Wei-Shung in [9]. The unrelated machines consist of multiple machines that have different speeds. This implies a different processing time for each job depending on the selected machine. It represents a lot of situations in several real world applications where each machine has a different capability (speed). For the solution of the unrelated parallel machine a variety of techniques have been developed and proposed in different cases [70, 69], especially in heuristics [62, 63, 64], meta-heuristics [38, 65, 59], and

exact solutions [60, 68]. In the last few decades there are numerous different models have been proposed to solve the problem of scheduling parallel machine considering different objective functions and different constraints [58, 7, 68, 67, 71, 72, 57]. A natural way to attack machine scheduling problems is to formulate them as mathematical programming models [73]. Therefore, in this chapter we proposed mathematical formulation for the problem of scheduling n jobs on m unrelated parallel machines with the objective of minimizing the makespan, considering the precedence constraints. We also adapted the formulations given by Chunfeng Liu and Shanlin Yang [71], and the formulations given in [67] by João Paulo *et al* in the first and the second models respectively. The objective of this chapter is to provide a mathematical formulation. Our mathematical model has been compared with the other mentioned models. From its promising results we proposed additional valid inequalities to strength the quality of the linear relaxation of the ILP. In addition the branch and cut approach and a separation algorithms have been applied to solve the ILP. All models are tested on large sets of instances. We enhanced the obtained results by applying branch and cut and separation algorithms to our ILP relaxation.

The chapter is organized as follows. Section 3.3 introduces the different mathematical formulations that are found in the literature and our proposed model with the separation algorithm for SPT inequality. 3.4 presents the classes of valid inequalities. Section 3.5 represent the experimental results.

3.2 Problem Description

Recall the description of the problem stated in the previous chapter : The problem under consideration is to schedule n jobs on m machines which are arranged in parallel with the aim of minimizing the total completion time. Let J be the set of the jobs and M be the set of the parallel machines. A precedence constraint between two jobs j_1 and j_2 is denoted by $(j1 \prec j2)$ and it requires that job j_2 cannot start to be processed until job j_1 finishes its processing. The type of the precedence constraint is a graph type, which is denoted by $D = (V, A)$, where V is the set of vertices associated at each job ϑ and V denotes the set of edges associated with each precedence constraint. We called this graph the precedence graph. We take also the case where $\{v, \bar{w}, w\} \subseteq V$ such that v before $\bar{w}$ and $\bar{w}$ before w then v before w. We consider also the speeds for all machines denoted by σ_i, where $i \in M$. Every job $\varphi \in \vartheta$ has a processing time π_φ and its effective processing time depends on the selected machine i, where $\pi_{i\vartheta} = \pi_\vartheta \times \sigma_i$. Each machine $i \in M$ cannot

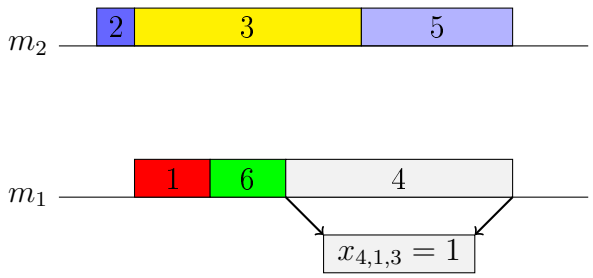


FIGURE 3.1 – Classical Formulation Illustration.

process more than one job at a given time. Furthermore, machines have different speeds and preemption of jobs is not allowed. According to the well-known $\alpha|\beta|\gamma$ scheduling problem classification scheme proposed initially by Graham *et al* [66], scheduling problem classification scheme this problem can be denoted as $P|prec|C_{max}$ We denote by C_i the completion time of machine i, where $i \in M$, and denote by C_ϑ the completion time of job ϑ, where $\vartheta \in \varphi$.

3.3 Mathematical Formulations

In this section we present different mathematical models for the UPMSP with precedence constraints, with the objective of minimizing the makespan C_{max}. The first model based on the position of job on the machine and the job completion time. The second model use idea of the flow formulation which is focus on the sequence of jobs on machines. The third model is based on the partition in linear orderings. Our proposed models are based on the interval and m-clique free graphs.
In the next sections we present these mathematical formulations.

3.3.1 Classical Formulation

The idea of this ILP given in [71] is based on the position of job on the machine and the completion time of each job. Figure 3.1, and Figure 3.2 illustrate the idea of this formulation.

In this Formulation we will denote by C a large positive number. For describing this model, let us introduce the following variables :

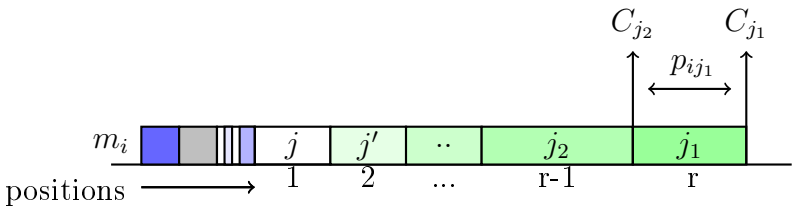


FIGURE 3.2 – Job in position r must start its processing after job in position $r-1$.

$$x_{jir} = \begin{cases} 1 & \text{if job } j \text{ is processed in the position } r \text{ on machine } i \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in J, i \in M, r \in J$$

$C_j \in \mathbb{N}^+$ is the completion time of job j.
$C_{max} \in \mathbb{N}^+$ is the total length of the schedule. That is, when all the jobs have finished processing.

The model can be stated as :

$$\min C_{max}$$

$$C_j \leq C_{max} \qquad \forall j \in J, \tag{3.1}$$

$$\sum_{i \in M} \sum_{r \in \{1,\dots,|J|\}} x_{jir} = 1 \qquad \forall j \in J, \tag{3.2}$$

$$\sum_{j \in J} x_{jir} \leq 1 \qquad \forall r \in \{1,\dots,|J|\}, \forall i \in M, \tag{3.3}$$

$$\sum_{j_1 \in J} x_{j_1 ir} - \sum_{j_2 \in J} x_{j_2 ir-1} \leq 0 \qquad \forall r \in \{2,\dots,|J|\}, \forall i \in M, \tag{3.4}$$

$$C_{j_2} - C_{j_1} + C(2 - x_{j_2 ir} - x_{j_1 ir-1}) \geq p_{j_2 i} \qquad \forall r \in \{2,\dots,|J|\}, \forall i \in M, \forall j_1 \neq j_2 \in J, \tag{3.5}$$

$$C_j \geq \sum_{r \in \{1,\dots,|J|\}} p_{ji} x_{jir} \qquad \forall i \in M, \forall j \in J, \tag{3.6}$$

$$C_{j_2} - C_{j_1} \geq \sum_{r \in \{1,\dots,|J|\}} \sum_{i \in M} p_{j_2 i} x_{j_2 ir} \qquad \forall (j_1, j_2) \in A, \tag{3.7}$$

$$x_{jir} \in \{0,1\}, \qquad \forall j \in J, \forall i \in M, \forall r \in \{1,\dots,|J|\}, \tag{3.8}$$

$$C_j \geq 0, \qquad \forall j \in J, \tag{3.9}$$

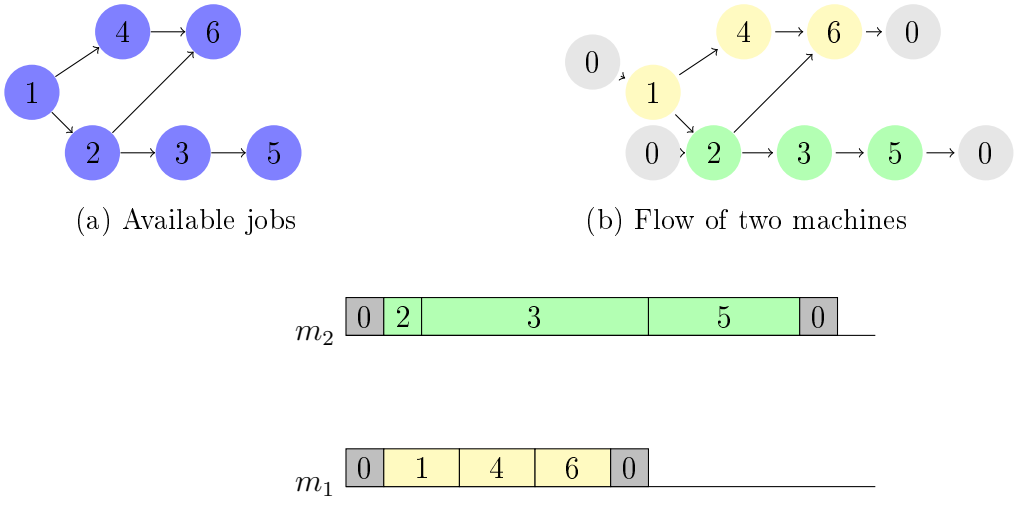
$$C_{max} \geq 0, \tag{3.10}$$

The objective function is to minimize the makespan. Inequalities (3.1) ensure that the

global makespan is greater than or equal to the completion time for all jobs. Inequalities (3.2) ensure that each job should be assigned to one position on one machine. Inequalities (3.3) guarantee that at most one job will be assigned to a position on all machines. Inequalities (3.4) guarantee that there must be no empty time slot between jobs in sequences. Inequalities (3.5) ensure that the job j_2 must start its processing time after the finishing of job j_1, if job j_1 is assigned on position $r-1$ and job j_2 on the position r on the same machine. Inequalities (3.6) bounding the completion time for all jobs, only if the job is in the first position, otherwise C_j is bounded with the inequalities (3.5). Inequalities (3.7) controls the precedence constraints.

3.3.2 Flow Formulation

In this ILP we consider a graph and we search m disjoint paths where each path represent the sequence of jobs on machines. Figure 3.3 illustrate the idea.



(a) Available jobs

(b) Flow of two machines

(c) Scheduling flows on two machines

FIGURE 3.3 – Flow Formulation Illustration.

This model is based on the Mixed Integer Programming model presented by João Paulo *et al* [67] for minimizing earliness and tardiness penalties. The model uses a dummy job j_0 to mark the beginning and the end of a sequence of jobs on each machine, or with another words, all paths begin and finish by a dummy job. The model involves the following variables :

$$x_{j_1,j_2,i} = \begin{cases} 1 & \text{if job } j_1 \text{ is precedes job } j_2 \text{ directly on machine } i \\ 0 & \text{otherwise} \end{cases} \qquad \forall j_1, j_2 \in J \cup \{j_0\}, \forall i \in$$

(a) An induced sub graph (b) Possible path

FIGURE 3.4 – Finding Path between sequences.

M

$C_{j,i} \in \mathbb{N}^+$ is the completion time of job j on machine i , $\forall j \in J \cup \{j_0\}$ and $\forall i \in M$.

$C_{max} \in \mathbb{N}^+$ is the total length of the schedule.

The mathematical model is :

$$\min C_{max}$$

$$C_{j,i} \leq C_{max} \qquad \forall j \in J, \forall i \in M \tag{3.11}$$

$$\sum_{i \in M} \sum_{j_1 \in (J \cup \{j_0\}) \setminus \{j_2\}} x_{j_1,j_2,i} = 1 \qquad \forall j_2 \in J, \tag{3.12}$$

$$\sum_{j \in J} x_{j_0,j,i} \leq 1 \qquad \forall i \in M, \tag{3.13}$$

$$\sum_{j_1 \in (J \cup \{j_0\}) \setminus \{j\}} x_{j_1,j,i} - \sum_{j_2 \in (J \cup \{j_0\}) \setminus \{j\}} x_{j,j_2,i} = 0 \quad \forall j \in J, \forall i \in M, \tag{3.14}$$

$$C_{j_2,i} \geq C_{j_1,i} - C + (p_{j_2,i} + C) x_{j_1,j_2,i} \qquad \forall i \in M, \forall j_1 \in J \cup j_0, \forall j_2 \in J, \tag{3.15}$$

$$C_{j_1,i_2} \leq C_{j_2,i_1} - \sum_{j \in (J \cup \{j_0\}) \setminus \{j_2\}} p_{j_2 i_1} x_{j_2,j,i_1} \qquad \forall (j_1, j_2) \in A, \forall i_1, i_2 \in M, \tag{3.16}$$

$$C_{j,i} \geq 0, \quad \forall (j_1, j_2) \in A, \tag{3.17}$$

$$x_{j_1,j_2,i} \in \{0, 1\} \quad \forall i \in M, \tag{3.18}$$

The objective function is to minimize the makespan. Inequalities (3.11) ensure that the global makespan is greater than or equal to the completion time for all jobs on all machines. Inequalities (3.12) ensure that for all jobs there exist a direct predecessor. Inequalities (3.13) limit the number of successors of the first dummy job j_0 for all machines. Furthermore, if $\sum_{j \in J} x_{j_0,j,i} = 0$ then we have no job on the machine i. Inequalities (3.14) ensure that we have several disjoint paths. For instance if $x_{j_1,j_2,i} = x_{j_2,j_4,i} = 1$ then we obtain the sequence $j_1 \rightarrow j_2 \rightarrow j_4$ on the machine i(see Figure 3.4). Inequalities (3.15) controls the completion times of the job at the machines, if a job j_2 assigned to machine i after j_1 (i.e. $x_{j_1,j_2,i} = 1$), it's completion time $C_{j_2,i}$ must be greater than the completion

time of j_1, $C_{j_1,i}$ plus the processing time of j_2. If ($x_{j_1,j_2,i}$=0), then the big constant C render the constrain redundant. Inequalities (3.16) ensure that in different machines the completion time of precedent job in an arc is less than or equal to the starting time of successor job in the same arc. Inequalities (3.17) and (3.18) bound the decision variables.

3.3.3 Order Formulation

This formulation proposed by Coll *et al* [76], this formulation reveals as a part of it a polytope of partition in linear orderings for solving the problem of multiprocessors scheduling with precedence constraints. The following variables used to describe the model :

$$x_{ji} = \begin{cases} 1 & \text{if job } j \text{ processed on machine } i \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in J, \forall i \in M$$

$$z_{j_1 j_2} = \begin{cases} 1 & \text{if job } j_1 \text{ scheduled before } j_2 \text{ on the same machine} \\ 0 & \text{otherwise} \end{cases} \qquad \forall j_1, j_2 \in J$$

- y_j the starting time of the execution of each job $j \in J$.
- $P_j = \{j_1 \in J$: there exists a path in G from j_1 to $j_2\}$, i.e., P_j is the set of predecessors of job j_1 ;
- $\Gamma_j = \{j_1 \in J : (j_1, j_2) \in A\}$, i.e.,$\Gamma_j$ is the set of immediate predecessors of job j_2 ;
- $Q_j = \{j_1 \in J$: there exists a path in G from j_2 to $j_1\}$, i.e.,
- Q_j is the set of successors of job j_2 ;
- $R_j = \{j_1 \in J$: there is no path in G from j_2 to j_1 or from j_1 to $j_2\}$. The model formulated

as follows :

$$\min C_{max}$$

$$\sum_{i \in M} x_j^i = 1 \qquad \forall j \in J, \tag{3.19}$$

$$y_{j_1} - y_{j_2} + \sum_{i \in M} d_{j_1 i}.x_{j_1 i} \leq \mu_{j_1 j_2}(1 - z_{j_1 j_2}) \qquad \forall j_2 \in J, \forall j_1 \in R_j, \tag{3.20}$$

$$y_{j_1} - y_{j_2} + \sum_{i \in M} d_{j_1 i}.x_{j_1 i} \leq 0, \qquad \forall j_2 \in J, \forall j_1 \in \Gamma_j, \tag{3.21}$$

$$y_j - C_{max} + \sum_{i \in M} d_{ji}.x_{ji} \leq 0, \qquad \forall j \in J, \tag{3.22}$$

$$z_{j_1 j_2} + z_{j_2 j_1} + x_{j_1 i} - x_{j_2 i} \leq 1 \qquad \forall j_1 \in J, \forall j_2 \in R_j, \forall i \in M, \tag{3.23}$$

$$z_{j_1 j_2} + x_{j_1 i} - x_{j_2 i} \leq 1 \qquad \forall j_1 \in P_j, \forall j_2 \in J, \forall i \in M, \tag{3.24}$$

$$x_{j_1 i} + x_{j_2 i} - z_{j_1 j_2} \leq 1 \qquad \forall j_1 \in R_j, \forall j_2 \in J, \forall i \in M, \tag{3.25}$$

$$y_j \leq \delta_j \qquad \forall j \in J, \tag{3.26}$$

$$y_{ji} \in \{0, 1\} \qquad \forall (j, i) \in J \times M, \tag{3.27}$$

$$z_{j_1 j_2} \in \{0, 1\} \qquad \forall j_2 \in J, \forall j_1 \in R_{j_2} \cup p_{j_2}, \tag{3.28}$$

Eq. (3.19) ensure that each job is processed and assigned to exactly one processor. Inequalities (3.21) express the precedence constraints. Inequalities (3.22) define the makespan. Inequalities (3.20) define the sequence of starting times of the jobs assigned to the same processor, ensuring that no overlap occurs. The constant $\mu_{j_1 j_2}$ is such that if job j_1 and j_2 are not executed in the same processor in that order, then inequality (3.20) is always satisfied. Inequality (3.26) δ_j is a lower bound to the earliest starting time of job j.

3.3.4 Interval Graph Formulation

The third Integer Linear Programming considers the beginning of the job and the relation between jobs if they run on the same machine, also verify if one job j_1 run before another job j_2 or they run the same time on different machines.
The graph induced by the relation where two jobs run in the same time must be an interval graph and check if all jobs can be schedule on this number of machines.

This model consists of the following decision variables :
$y_i \in \mathbb{N}^+$ the starting time of job j.

$x_j^i = \begin{cases} 1 & \text{if job } j \text{ on machine } i \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in J, \forall i \in M$

$z_{j_1,j_2} = \begin{cases} 1 & \text{if job } j_1 \text{ and job } j_2 \text{ run at the same time} \\ 0 & \text{otherwise} \end{cases} \quad \forall j_1, j_2 \in J$

$\bar{z}_{j_1,j_2} = \begin{cases} 1 & \text{if job } j_1 \text{ is before or runs at the same time with job } j_2 \\ 0 & \text{otherwise} \end{cases} \quad \forall j_1, j_2 \in J$

$C_{max} \in \mathbb{N}^+$ the total completion time

The model can be described by the following ILP (P')

$$\min C_{max}$$

$$y_j + \sum_{i \in M} p_{ij} x_j^i \leq C_{max}, \qquad \forall j \in J, \tag{3.29}$$

$$\sum_{i \in M} x_j^i = 1, \qquad \forall j \in J, \tag{3.30}$$

$$x_{j_1}^i + x_{j_2}^i \leq 2 - z_{j_1,j_2}, \qquad \forall j_1, j_2 \in J, \forall i \in M, \tag{3.31}$$

$$y_j + \sum_{i \in M} p_{ij} x_j^i \leq y_{j_1}, \qquad \forall (j, j_1) \in A, \tag{3.32}$$

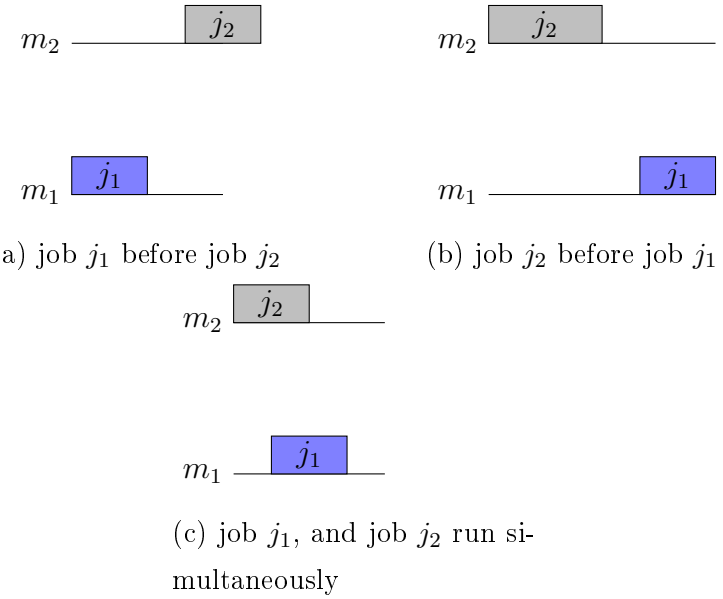
$$y_j + \sum_{i \in M} p_{ij} x_j^i \leq y_{j_1} + C\bar{z}_{j,j_1}, \qquad \forall j, j_1 \in J, \tag{3.33}$$

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_1} \leq 1 + z_{j_1,j_2}, \qquad \forall j_1, j_2 \in J, \tag{3.34}$$

$$\sum_{j_1,j_2 \in I} z_{j_1,j_2} \leq |E[I]| - 1, \qquad \forall I \subseteq \mathcal{I}, \tag{3.35}$$

$$\sum_{j_1,j_2 \in K} z_{j_1,j_2} \leq |E[K]| - 1, \qquad \forall K \subseteq \mathcal{K}, \tag{3.36}$$

The objective function is to minimize the makespan. Inequalities (3.29) ensure that the starting time for each job plus its processing time bound the makespan. Inequalities (3.30) controls each job to be processed on one machine. Inequalities (3.31) guarantee that there is no two jobs run on the same machine at the same time. Inequalities (3.32) controls the precedence constraints. Inequalities (3.33) ensure that the beginning of any job must began after the finishing of its predecessor. Inequalities (3.34) ensure that, if the job j_1 run before or at the same time with j_2, and j_2 run before or at the same time with job j_1 then job j_1 and j_2 run at the same time. Figure 3.5 illustrate the status of two jobs job j_1 and job j_2 on two machines, when $\bar{z}_{j_1,j_2}$=1, $\bar{z}_{j_2,j_1} = 0$, then $z_{j_1,j_2} = 0$, this is represented in (a), when $\bar{z}_{j_2,j_1}$=1, $\bar{z}_{j_1,j_2} = 0$, then $z_{j_2,j_1} = 0$, here in (b) job j_2 is before job j_1 but not at the same time, in (c) job j_1 and job j_2 run on the same time, here $\bar{z}_{j_1,j_2} = 1$, $\bar{z}_{j_2,j_1} = 1$, then $z_{j_1,j_2} = 1$, in this case the tow jobs run at the same time.



(a) job j_1 before job j_2

(b) job j_2 before job j_1

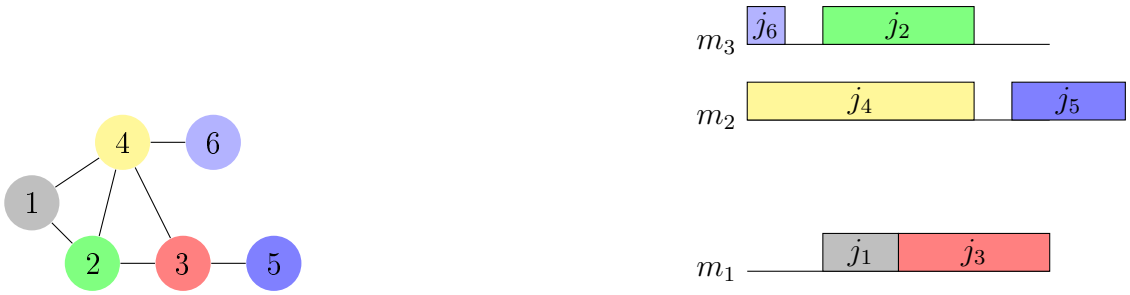
(c) job j_1, and job j_2 run simultaneously

FIGURE 3.5 – Illustration for the Status of Inequalities (3.34)

Remark that, If we consider a solution given by the vector $(\bar{z}, z, x)$ then the induced subgraph $G = (V, E)$ where for each job $j \in J$ we associate a vertex $v_j \in V$ and for all $z_{j_1,j_2} = 1$ we associate an edge $v_{j_1}v_{j_2} \in E$ must be an interval graph and the clique of maximum size must be less or equal to m. We denote by $\mathcal{I}$ the set of all no interval induced subgraph and by $\mathcal{K}$ the set of all cliques of size greater or equal to $m+1$. In Figure 3.6b the induced sub graph for this valid schedule is interval and 3-clique free where we have three machines. We notice that this sub graph contains a clique of size three, but if it has a clique of size four, then it can not be scheduled on three machines, i.e, if we have an edge between job 1 and job 3 in Figure 3.6b then we have a clique of size 4, which is not a valid schedule on three machines.

The inequalities (3.35) ensure that all induced subgraph are interval graphs. The inequalities (3.36) ensure that all induced subgraph has no clique of size greater or equal to $m + 1$. The number of inequalities (3.35) and (3.36) may be exponential. In order to solve this integer linear program using a branch-and-cut approach, we needs efficient algorithms for separating the inequalities (3.35) and (3.36). Remark that, to separate the inequalities (3.35) and (3.36) we need only the value of the vector z.

In the following we will deal with (UPMSPC) problem. Will propose different families of valid inequalities to the ILP as we shall see later the numerical results improved the performance of the ILP.



(a) An induced sub graph (b) A valid schedule

FIGURE 3.6 – An induced sub graph with it's valid schedule

For every solution generated by the ILP we associate the following graph, $G(V, E)$ where each job j associated with a vertex v and if two jobs run at the same time or share any time unit there exist an associated edge E between these two jobs. The graph induced by each solution to be valid must have two properties : Must be (i) interval graph, and (ii) m-clique free graph.

3.4 Valid Inequalities

In this section we will give new inequalities that are valid for P.

Proposition 1 *proposition Let j_1 and j_2 be two jobs in J then,*

$$y_{j_1} - y_{j_2} \leq p_{j_2}^{max} z_{j_1,j_2} + C_{3.37}(\bar{z}_{j_2,j_1} - z_{j_1,j_2}), \tag{3.37}$$

is valid inequality for P, where $C_{3.37} = C_{max}^{H} - p_{j_1}^{min}$.

proof 3.1 *proof Consider the different cases of job j_1 and job j_2 when there is one job is before other or they run at the same time :*
Case 1 : If job j_1 is before job j_2 without sharing any time unit, then ($z_{j_1,j_2} = \bar{z}_{j_2,j_1} = 0$). Thus, $y_{j_1} \leq y_{j_2}$, it is always true.
Case 2 : We will find $y_{j_1} \leq y_{j_2} + C_{3.37}$ when job j_2 is before job j_1 and they are not sharing any time unit, it is also a valid case when we have $C_{3.37}$ in the right side of the inequality.
Case 3 : When job j_1 and job j_2 run at the same time we deduce $y_{j_1} \leq y_{j_2} + p_{j_2}^{max}$. This case is valid because when the two jobs share any time unit that means the starting time of job j_1 is less than the starting time of job j_2 plus it is maximum processing time which is the worst case. Therefore (3.37) is valid for P in all cases.
$C_{3.37}$ appears when job j_2 run before job j_1.

Proposition 2 *Let j_1 and j_2 be two jobs in J then,*

$$y_{j_1} - y_{j_2} \leq C_{3.38}\bar{z}_{j_2,j_1} - p_{j_1}^{min}\bar{z}_{j_1,j_2}, \tag{3.38}$$

is valid inequality for P where $C_{3.38} = C_{max}^{H} - p_{j1}^{min}$.

proof 3.2 *proof Case 1 : Job j_1 is before job j_2 and they do not share any time unit. Then, $\bar{z}_{j_2,j_1} = 0$, and $\bar{z}_{j_1,j_2} = 1$. Thus, $y_{j_1} \leq y_{j_2}$ - $p_{j_1}^{min}$, this case is valid because the starting of job j_1 plus the smallest processing time is less than the starting of job j_2.*
Case 2 : Job j_2 is before job j_1 and they do not share any time unit. Then we deduce $y_{j_1} \leq y_{j_2} + C_{3.38}$, this case is also valid when we have $C_{3.38}$ in the right part of the inequality.
Case 3 : Job j_1 and job j_2 run at the same time, or share any time unit, then we deduce $y_{j_1} \leq y_{j_2} + C_{3.38} - p_{j_1}^{min}$, and it is true. Therefore (3.38) is valid for P.

We improved this inequality by adding $(p_{j_1}^{min} + p_{j_2}^{max} - C_{3.38})z_{j_1j_2}$ to the right side of the inequality is important to notice that, this part will not appear in case 1 and case 2 but will appear when job j_1 and job j_2 share any time unit. In case 3 when job j_1 and job j_2 run at the same time then we deduce that $y_{j_1} \leq y_{j_2} + p_{j_2}^{max}$ as we can see in Figure 3.38(3-a and 3-b) this inequality can be valid. Thus, we will get the following inequality :

Corollary 1 *corollary Let j_1 and j_2 be two jobs in J then,*

$$y_{j_1} - y_{j_2} \leq C_{3.38}\bar{z}_{j_2,j_1} - p_{j_1}^{min}\bar{z}_{j_1,j_2} + (p_{j_1}^{min} + p_{j_2}^{max} - C_{3.38})z_{j_1j_2}, \tag{3.39}$$

This inequality is valid for P in all cases.

Remark that if we add the following valid inequality $0 \leq p_{j_1}^{min}z_{j_1j_2} - p_{j_1}^{min}\bar{z}_{j_1j_2}$ to (3.39) then we obtain (3.37), we deduce that inequality (3.37) is dominated by (3.39). Then it is not necessary to add the inequalities (3.37) in the model.

Proposition 3 *Let j_1 and j_2 be two jobs in J then,*

$$y_{j_2} - y_{j_1} \leq C_{3.40}\bar{z}_{j_1,j_2} - \sum_{i \in M} p_{ij_2}x_{j_2}^{i} + p_{j_2}^{max}z_{j_1,j_2}, \tag{3.40}$$

is valid inequality for P where $C_{3.40} = C_{max}^{H}$.

proof 3.3 *proof Case 1 : If job j_1 is before job j_2 and they do not share any time unit, then $y_{j_2} \leq y_{j_1} - \sum_{i \in M} p_{ij_2} x^i_{j_2} + C_{3.40}$, which is valid.*
Case 2 : If job j_2 is before job j_1 and they do not share any time unit, then $y_{j_2} \leq y_{j_1} - \sum_{i \in M} p_{ij_2} x^i_{j_2}$, which is also valid when job j_1 starts after job j_2.
Case 3 : When job j_1 and job j_2 run at the same time we deduce $y_{j_2} \leq y_{j_1} - \sum_{i \in M} p_{ij_2} x^i_{j_2} + p^{max}_{j_1} + C_{3.40}$, which is valid when we have a positive $C_{3.40}$ in the right side of the equality.
Therefore (3.40) is valid for P in all cases.
Here, $C_{3.40}$ can be bounded by C^H_{max} where job j_1 runs before job j_2. In the worst case we have $y_{j_2} + p^{min}_{j_2} \leq y_{j_1} + C_{3.40}$.

Remark that in case 3 we can reduce the value of $C_{3.40}$ by this value $(p^{max}_{j_1} + p^{max}_{j_2} - C^H_{max})$ when job j_1 and job j_2 run at the same time, this value can be a coefficient for the variable $z_{j_1 j_2}$. Thus, we can improve (3.40) by inequality (3.41)

Corollary 2 *Let j_1 and j_2 be two jobs in J then,*

$$y_{j_2} - y_{j_1} \leq C_{3.40} \bar{z}_{j_1,j_2} - \sum_{i \in M} p_{ij_2} x^i_{j_2} + (p^{max}_{j_1} + p^{max}_{j_2} - C_{3.40}) z_{j_1,j_2}, \tag{3.41}$$

The coefficient of $z_{j_1 j_2}$ in (3.41) will not appear in case 1 and case 2 in (3.40). In case (3-a) when job j_1 and job j_2 share any time unit and job j_1 starts before job j_2 we obtain $y_{j_2} \leq y_{j_1} + p^{max}_{j_2}$, this inequality is valid where job j_2 starts before the completion of the processing time for job j_1. In case (3-b) it is clear $y_{j_2} \leq y_{j_1} + p^{max}_{j_2}$ is valid where job j_2 begin before job j_1. Thus, it will improve the inequality when we reduce the right side of the inequality.

There is no linear combination between inequalities (3.39) and (3.41), because when we subtract inequality (3.41) from inequality(3.39), we find inequality (3.42) which is invalid when job j_1 run before job j_2 and do not share any time unit

$$0 \leq C \bar{z}_{j_2,j_1} + \sum_{i \in M} p_i x^i_{j_2} + p^{min}_{j_1} z_{j_1,j_2} - p^{max}_{j_1} z_{j_1,j_2} - C_{3.41} \bar{z}_{j_1,j_2} - p^{min}_{j_1} z_{j_1,j_2}, \tag{3.42}$$

and when we subtract inequality (3.39) from inequality(3.41), we find inequality (3.43) which is invalid when job j_2 run before job j_1 and do not share any time unit

$$0 \leq (\bar{z}_{j_1,j_2} - \bar{z}_{j_2,j_1})C - \sum_{i \in M} p_i x^i_{j_2} + p^{max}_{j_1} z_{j_1,j_2} - p^{min}_{j_1} z_{j_1,j_2} + p^{min}_{j_1} \bar{z}_{j_1,j_2}, \tag{3.43}$$

Thus, its not possible to replace one inequality by another.

In the next section we will present another family of inequalities, these inequalities consider that, jobs are indexed in the non-decreasing order of their processing times. The principle of this inequality presented in [77]. This family of inequalities based on Shortest Processing Time (SPT).
We will define some notations to adapt the inequality to our problem :

- Let $J' \subseteq J$.
- J'_j denotes jobs in the SPT order of J' before j.
- $SPT(J')$ denotes the sum of the completion time when the jobs indexed in the non-decreasing order of their processing time in J'.
- pos_j denotes the position of job j in the SPT sequence of J'.
- $q_{ij} = (|J'| - pos_j + 1)p_{ij} + \sum_{j1 \in J'_j} p_{ij_1}$ which denotes the sum of the processing time for job j plus the processing time for the next jobs in the $SPT(J')$ for all $J' \subseteq J$,

Proposition 4

$$\sum_{j \in J'} y_j + \sum_{j \in J'} p_{ij} x^i_j + \sum_{j \in J'} \sum_{i1 \in M \setminus \{i\}} q_{ij} x^{i1}_j \geq SPT(J'), \tag{3.44}$$

is valid inequality for P,

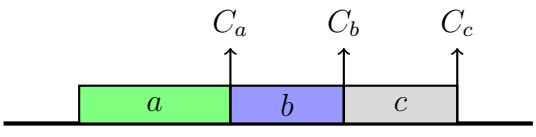


FIGURE 3.7 – Schedule of three jobs.

The idea of the proof is as the following. If we have the schedule which is illustrated in Figure 3.7 and this schedule on the same machine, having $p_a \leq p_b \leq p_c$. Then the $SPT \geq 3p_a + 2p_b + p_c$. Thus the $SPT(J')$ is bounding the schedule. Consider the case

where job b is runs on another machine, then the $SPT(J') \geq 2p_a + p_c$, and still bounding the schedule. Thus, in this case we must remove the value of q_{ij} from the $SPT(J')$. Thus, for all $j \in J'$, if we remove job j from this machine then $SPT(J')$ become : $SPT(J') - q_{ij}$. Remark that, if we remove more than one job(i.e.) two jobs (job j_1 and job j_2), then the value of SPT will be :
$SPT(J' \setminus \{j_1, j_2\}) \geq SPT(J') - qij_1 - qij_2$.

3.4.1 Separation Algorithm for SPT Inequality

In this section we give the idea of the heuristic used to separate inequality (3.44). Given a fractional solution $(\bar{x}, y^*, z^*, \bar{z}^*)$, the *separation problem* for inequalities (3.44) consists in determining whether $(\bar{x}, y^*, z^*, \bar{z}^*)$ satisfies inequalities (3.44). We run this algorithm for each machine. For all machine $i \in M$ we sort the jobs according to the SPT order for y^*, for each job j in the order if $\bar{x}^i_j \geq 0.99$ we take this job, and continue until the SPT inequality is violated.
The performance of the valid inequalities is evaluated in the experimental results section. In the following we will propose an updated formulation invented based on Interval graph formulation.

3.4.2 Reformulation of Interval Graph Formulation

In the following we will present another version of Interval Graph formulation, in this formulation we made some changes on the variable $\bar{z}_{j_1 j_2}$, by considering the case if one job processed before the other job by using the variable $\tilde{z}_{j_1 j_2}$ it is more nature for the precedence constraints, and this change enhanced the previous model by reducing the number of variables. We use the same variables in P with this change

$$\tilde{z}_{j_1,j_2} = \begin{cases} 1 \text{ if job } j_1 \text{processed before job } j_2 \\ 0 \text{ otherwise} \end{cases} \qquad \forall j_1, j_2 \in J.$$

The model can be described as follow :

$$\min C_{max}$$

$$y_j + \sum_i x_j^i p_{ij} \leq C_{max}, \qquad \forall j \in J, \tag{3.45}$$

$$\sum_{i \in M} x_j^i = 1, \qquad \forall j \in J \tag{3.46}$$

$$x_{j_1}^i + x_{j_2}^i \leq 2 - z_{j_1 j_2}, \qquad \forall j_1, j_2 \in J, i \in M, \tag{3.47}$$

$$y_{j_1} + \sum_{i \in M} p_{ij_1} x_{j_1}^i \leq y_{j_2}, \qquad \forall (j_1, j_2) \in A, \tag{3.48}$$

$$y_{j_1} - y_{j_2} + \sum_{i \in M} p_{ij_1} x_{j_1}^i \leq C(1 - \tilde{z}_{j_1 j_2}), \qquad \forall j_1, j_2 \in J \tag{3.49}$$

$$\tilde{z}_{j_1 j_2} + \tilde{z}_{j_2 j_1} + z_{j_1 j_2} = 1, \qquad \forall j_1, j_2 \in J, \tag{3.50}$$

$$y_{j_2} \leq y_{j_1} + \sum_{i \in M} p_{ij_1} x_{j_1}^i + (C - p_{j_1}^{min}) \tilde{z}_{j_1 j_2} \qquad \forall j_1, j_2 \in J, \tag{3.51}$$

$$\sum_{(j_1, j_2) \in E(\bar{I})} \tilde{z}_{j_1, j_2} - \sum_{(j_1, j_2) \in E(\bar{I})} \tilde{z}_{j_1, j_2} \leq |E(\bar{I})| - 1, \qquad \forall \bar{I} \subseteq \bar{\mathcal{I}}, \tag{3.52}$$

$$\sum_{(j_1, j_2) \in E(K)} \tilde{z}_{j_1, j_2} - \sum_{(j_1, j_2) \in E(K)} \tilde{z}_{j_1, j_2} \leq |E(K)| - 1, \qquad \forall K \subseteq \mathcal{K} \tag{3.53}$$

The objective function is to minimize the makespan. Inequalities (3.45) ensure that the beginning time for each job plus its processing time is less than or equal to the total completion time. Inequalities (3.46) controls each job to be processed on one machine. Inequalities (3.47) guarantee that there is no two jobs run on the same machine at the same time. Inequalities (3.48) controls the precedence constraints. Inequalities (3.49) ensure that the beginning of any job must began after the finishing of its predecessor. Inequalities (3.50) ensure that, if the job j_1 run before job j_2, and j_2 run before job j_1 then job j_1 and j_2 run at the same time, this inequality made the major modification. Inequalities (3.51) ensure that the beginning of any job must began after the finishing of its predecessor.

3.5 Experimental Results

The mathematical formulations tested and compared under the following proposed benchmark of instances.

The processing times are uniformly, distributed between 1 and 100 as it is common in the literature [78]. We generated five different sets of three subsets of DAG where the graph density is high, medium and low respectively, with the following combinations of number

of jobs $n = \{10, 12, 14, 16, 18, 20, 25\}$ and 2 machines . The speed of machines generated randomly between 10 and 20. In total $5 \times 7 \times 3$ instances are generated.
Regarding to the graph density(GD), is calculated as follow $GD = \frac{|E|}{|V|(|V|-1)}$ where E is the set of edges associated with precedence constraints between jobs, and V is the set of vertices associated with jobs, we generated the instances of three density sets (low, medium and high) density with the values (0.1, 0.15 and 0.25) respectively.
The Mathematical model implemented with CPLEX 12.4, on an Intel, core i5 running at 3.4 GHz, and 8 GB of RAM. The obtained results in this experiment are reported in table 3.1. Each line gives the average results obtained by solving 5 instances for each problem size. The entries in these tables are :

- n : the number of jobs,
- ILP : the Integer Linear Program used to solve :
  - 1=classical formulation,
  - 2=flow formulation,
  - 3=order formulation,
  - 4=interval graph formulation,
- CPU : the total CPU time in seconds,
- Gap : the Gap between the lower bounds and the upper bounds $(100 \times \frac{UB-LB}{LB})$,
- $GapH$: the Gap between the best solution given by the heuristic and the integer linear program $(100 \times \frac{Heuristique-UB}{UB})$,
- o/p : the number of problems solved to optimality over the number of instances tested.

Remark that, all instances are carried out for two machines and n jobs. We remark that the classical formulation give bad results and can solved only the smallest instances. We can observe that the interval graph formulation found 62%, 28% and 22% of optimal solutions in high density, medium and low density respectively. Whereas, the optimal solutions obtained by classical formulation is 17%, 11% and 11% in high density, medium and low density respectively, and flow formulation obtained 45%, 20%, and 20% in high density, medium and low density respectively, and order formulation obtained 68%, 57%, 45% in high density, medium, and low density. The dashed results means that, the model did not success to run the integer linear program due to the number of inequalities, we fixed the processing time to one hour.

Also we remark the Gap between the lower bounds and the upper bounds is sometimes better for the interval graph formulation. For the high density instances the Gap is always less than 40% and is acceptable for the instances with less than 20 jobs. Furthermore the interval graph formulation improve the value given by the heuristic on 11 instances and only 6 instances for the flow formulation where the classical formulation improve the value on 4 instances, and order improved 12 instances.

The ILP with the valid inequalities is tested under the following proposed benchmark of instances.
The processing times are uniformly, distributed between 1 and 100 as it is common in the literature [78]. We generated five different sets of instances where the graph density is equal to 0.15 which is calculated as follows $GD = \frac{|E|}{|V|(|V|-1)}$ where E is the set of edges associated with precedence constraints between jobs, and V is the set of vertices associated with jobs, with the following combinations of numbers of jobs $n \in \{10, 12, 14, 16, 18, 20\}$ and the number of machines $m \in 2, 3$. The speed of machines is generated randomly between 10 and 20. In total $5 \times 6 \times 2$ instances are generated. CPU time required is in seconds.
The ILP is implemented with CPLEX 12.4, on an Intel, core i5 running at 3.4 GHz, and 8 GB of RAM.
We first present an overview of the results of computational tests for the ILP when we add the inequalities. We limit the resolution time of each instance to 3600 seconds.

The number of obtained optimal solutions from the computational experiments are presented in Table 3.2 with the average of the CPU time required by the model when we add the inequality for the five sets of instances for each problem. We can observe that Cplex, within the established time, was not able to obtain the optimal solution for all instances. We can see that the solver founds the optimal solution for all five instances just for the problem of 2×10. We can notice that when we add the inequalities (3.39) the model found 53% optimal solutions, whereas, the optimal solutions obtained when we add the inequalities (3.41) is 43%, and when we add the two inequalities (3.39) and (3.41) the model found 53% optimal solutions with less CPU time. The inequalities (3.44) give 46% optimal solutions, however when we add all inequalities the model found 50% optimal solution. The results of Table 3.2 also indicate that when we add the two inequalities (3.39) and (3.41) the model needs less CPU time to find optimal solutions. We can notice also when we increase the number of machines the model is always able to find more optimal

Instances			Low Density				Medium Density				High Density			
	n	ILP	CPU	Gap	$GapH$	O/p	CPU	Gap	$GapH$	O/p	CPU	Gap	$GapH$	O/p
	10	1	228.8	0	0.32	**5/5**	2368.8	7.53	0	**4/5**	796.6	0	0.09	**5/5**
	10	2	3.4	0	0.32	**5/5**	182.4	0	0	**5/5**	25.6	0	0.09	**5/5**
	10	3	0	0	0	**5/5**	1	0	0	**5/5**	3	0	0	**5/5**
	10	4	0	0	0.32	**5/5**	23.8	0	0	**5/5**	0.8	0	0.09	**5/5**
	12	1	2923	31.83	0.12	**1/5**	3600	62.27	0	**0/5**	3600	81.22	0	0/5
	12	2	83.4	0	0.12	**5/5**	3600	29.7	0	**0/5**	2567.6	4.11	0	**2/5**
	12	3	0.6	0	0	**5/5**	6	0	0	**5/5**	60.6	0	0	**5/5**
	12	4	0.8	0	0.12	**5/5**	1418.6	0.18	0	**4/5**	748.8	0.06	0	**4/5**
	14	1	3600	40.15	0	**0/5**	3600	69.91	0	0/5	3600	80.52	0	0/5
	14	2	1400	7.05	0	**4/5**	2808.2	65.08	0	0/5	3506.2	64.93	0	0/5
	14	3	13.5	0	0	**5/5**	1030	0	0	**5/5**	788.2	0	0	**5/5**
	14	4	186	0	0	**5/5**	3600	9.3	0	0/5	2392.4	7.44	0	**2/5**
	16	1	3600	45.41	0	**0/5**	3600	49.27	0	0/5	3600	89.86	0	0/5
	16	2	736.2	38.18	0.06	**2/5**	2112.4	80.75	0	0/5	1016	71.77	0	0/5
	16	3	210	0.2	0.2	**5/5**	1605	11	0	**4/5**	3344.6	7.7	0.14	**1/5**
	16	4	1475.4	5.48	0.06	**3/5**	3600	27.09	0	0/5	3600	17.63	0	0/5
	18	1	-	-	-	**0/5**	-	-	-	0/5	-	-	-	0/5
	18	2	1016.4	50.75	1.2	**0/5**	1603	125.46	0	0/5	816.6	86.55	0	0/5
	18	3	1446.6	14.7	2.95	**3/5**	3039.6	10.4	0	**1/5**	3600	34.5	0.2	0/5
	18	4	1511.6	9.04	1.32	**3/5**	3600	56.49	0	0/5	3600	25.67	0	0/5
	20	1	-	-	-	**0/5**	3600	188.58	0	0/5	-	-	-	0/5
	20	2	868	75.48	0	**0/5**	569.4	166.67	0	0/5	914.25	93.94	0	0/5
	20	3	2882	10.05	0.9	**1/5**	3600	37.8	0	0/5	3600	53.9	0.24	0/5
	20	4	2888.4	19.37	0.15	**1/5**	3600	71.45	0	0/5	3600.2	49.16	0	0/5
	25	1	-	-	-	0/5	-	-	-	0/5	-	-	-	0/5
	25	2	947.3	101.75	0	0/5	691	227.09	0	0/5	865.3	117.82	0	0/5
	25	3	3600	37.53	0.3	0/5	3600	67.02	0	0/5	3600	97.7	0.45	0/5
	25	4	3600	29.78	0.71	0/5	3600	64.77	0	0/5	3600	70.69	0.07	0/5

TABLE 3.1 – Results

solutions within the time limit for all families of inequalities, with the best results obtained when we add the family of inequalities(3.39).

In Table 3.3 we can see the results for the average relative percentage gap between the lower bound and the upper bound when we add the inequalities. Indeed, Gap is computed as follows $100 \times \frac{\text{Upper bound}-\text{Lower bound}}{\text{Upper bound}}$. The average relative percentage gap is no more than 31% for all problems when we add the inequalities (3.39) and (3.41). However, the dashes in *Cplex* column indicates that the Cplex is not able to find a valid solution within the time limit and the limited use of the RAM. Furthermore, we remark that, when the number of tasks increases with 3 machines, the gap still reduces when we use the inequalities (3.39) and (3.41).

Other analysis can be displayed in Table 3.2. We can notice that when we add inequalities (3.39) and (3.41) together, the model gives the maximum counts of best obtained results for all problems, and its also interesting to notice the ability of inequalities (3.39) for obtaining the same number in comparing with other families of inequalities when they added individually.

Generally, we can notice the improvement of the model in terms of obtained results and CPU time occurred when we add the two families (3.39) and (3.41) together to the ILP. When we add the family of inequalities (3.39) alone to the ILP the improvement happen with the obtained optimal solutions but with more CPU time. We notice a limited improvement when we add inequalities (3.41) alone to the ILP . The family of inequalities (3.44) (the separation algorithm) may need more improvements to obtain better results. We can say there is no remarkable improvements when we add all families of inequalities to the model.

3.5.1 Conclusion

In this chapter we considered the problem of unrelated parallel machine scheduling, minimizing makespan subject to precedence constraints. The main difference between these formulations is the way the makespan has been linearized. To study and evaluate the effectiveness and efficiency of these models, 105 instances of numerical experiments are conducted. The obtained results show the effectiveness of the interval graph formulation in comparing with the available formulations except the order formulation, but the facial structure of the polytope generated by interval graph model can be investigated to define some facet defining inequalities, where there is an important piece of information from

Instances		*Cplex*		(3.39)		(3.41)		(3.39)&(3.41)		*SPT*		*ALL*	
m	n	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*	*Opt*	*CPU*
2	10	5/5	0.04	5/5	0.04	5/5	0.04	5/5	0.04	5/5	1.8	5/5	1.4
2	12	4/5	874.8	5/5	278.8	5/5	222	5/5	136.4	5/5	657.6	5/5	875.6
2	14	1/5	2924.8	2/5	2299.6	2/5	2248.6	2/5	2259.6	2/5	2993.6	2/5	3392.4
2	16	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600
2	18	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600
2	20	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600	0/5	3600
3	10	4/5	720.2	5/5	0.6	5/5	1.6	5/5	1	5/5	6.4	5/5	2.6
3	12	4/5	2430.4	5/5	45.8	5/5	55.2	5/5	50.4	5/5	412	5/5	310.8
3	14	0/5	3600	4/5	1340.6	2/5	1804	4/5	1355.2	3/5	1483.8	3/5	1461
3	16	0/5	3600	3/5	1471.4	1/5	2405	3/5	1520.8	3/5	1178.75	3/5	1760.4
3	18	0/5	3600	2/5	2291.5	1/5	2702.75	2/5	2269.25	1/5	2401.33	1/5	2700
3	20	0/5	3600	1/5	2400	0/5	3600	1/5	2700	0/5	3600	1/5	2400

TABLE 3.2 – Results

Instance	*Cplex*	(3.39)	(3.41)	(3.39)&(3.41)	*SPT*	*ALL*
2×10	0.0	0.00	0.00	0.00	0.00	0.00
2×12	4.52	0.00	0.00	0.00	3.29	0.64
2×14	15.67	3.41	3.62	2.22	47.42	8.91
2×16	46.52	7.97	7.72	7.73	47.65	14.42
2×18	47.60	12.38	11.80	14.39	50.04	19.23
2×20	85.91	29.54	29.42	30.60	-	33.81
3×10	20.00	0.00	0.00	0.00	0.00	0.00
3×12	20.00	0.00	0.00	0.00	20.02	0.00
3×14	82.04	0.34	20.96	0.26	-	3.48
3×16	-	2.27	41.23	2.66	80.06	4.93
3×18	-	23.54	23.68	22.34	-	24.98
3×20	-	40.80	45.70	25.31	-	46.01

TABLE 3.3 – Average Relative Percentage Gap between Optimal Solution and Obtained Solution.

our model where the model could know if the jobs run at the same time or not. There are some applications require such model, specially in some cloud computing security models.

We improved the interval graph formulation by adding valid inequalities based on the forbidden interval sub graph, and some Heuristics to solve the clique problem or find the smallest no interval subgraphs. Computational results show that, the addition of these inequalities decreases the computational requirements to obtain the optimal solution, in many cases. Furthermore, we proposed other inequalities based on SPT. Some heuristics used to separate the inequality based on the SPT. In the next chapter we will define the polytope associated with the interval model.

4

Polyedral study on interval graphs under m-clique free constraints

Contents

In this chapter we consider the problem of interval and m-clique free subgraphs and we show the relation between the graph problem and the unrelated parallel machines with disjunctives constraints (UPMSDC) problem. We study the facial structure of their polytope and we present some inequalities defining facets to bound the associated integer linear programming formulation. Moreover, we present exact and heuristic separation algorithms associated with a cutting-plane algorithm for solving the problem. Finally, we present some experimental results as well as their analysis.

4.1 Introduction

Interval, m-clique graphs have attracted the interest of researchers for many decades. The scope of current research in this area extends now to the mathematical and algorithmic properties of interval and m-clique graphs, their generalizations and the related graph parameters. One main reason for this increasing interest is that many real-world applications involve solving problems on graphs, which are either interval graphs themselves or are related to interval graphs or a clique in a natural way. Algorithmic aspects of interval graphs have been the subject of ongoing research for several decades, stimulated by their numerous applications ; see e.g. [97]. In some applications, interval representations with special properties are required.
Numerous applications of interval graphs have appeared in the literature including applications to genetic structure, sequential storage and scheduling (see [97, 58]). An application of the interval graphs arises on the context of scheduling jobs in cloud computing. In cloud computing, we not only have to determine how many, but also which jobs should be allocated to a virtual machine. In scheduling, for example, jobs can have certain durations that should be refected by the lengths of their intervals and two consecutive jobs can require a certain handover period that is determined by how much their intervals should intersect.

All graphs in this chapter are simple and have no self-loops. Let $G = (V, E)$ be a graph. An undirected graph G is called an interval graph if its vertices can be put into a one-to-one correspondence with a set of intervals I of a linearly ordered set (like the real line) such that two vertices are connected by an edge of G if their corresponding intervals have nonempty intersection.
An interval graph is the graph showing intersecting intervals on a line. Thus, we associate a set of intervals $I = \{I_1, ..., I_n\}$ on a line with the interval graph $G = (V, E)$, where

$V = \{1, ..., n\}$ and two vertices, x and y, are linked by an edge if and only if $I_x \cap I_y \neq \emptyset$. .

In parallel machines scheduling some jobs in different machines can share any time units, the jobs can be represented with multiple nodes and edges indicate there is a shared time units between jobs. Normally, the solution is mathematically formalized as a graph $G = (V, E)$ where V denotes the set of vertices (associated with jobs) and E denotes the set of relationships between vertices (intersections of jobs). However, when we assign jobs to parallel machines, the solution is valid for two types of graphs (i.e., interval graph and m-clique free, where m is the number of machines). This point will be discussed in Section 4.4.

Clique is a very common structure in many applications, which is composed of a subset of vertices as well as all the possible relationships among them. Therefore, clique detection is playing an important role in various applications, such as social recommendation [99] and network routing . The m-clique free graph is the graph that does not contain a clique of size greater than $m + 1$.

Definition 8 *(clique). Let $G = (V, E)$ be an undirected graph. A clique in G is a subset $S \subset V$ such that for any two vertices $v_i, v_j \in S$ there exists an edge $(v_i, v_j) \in E$.*

Definition 9 *(m-clique). Let $G = (V, E)$ be an undirected graph. An m-clique in G is a subset $S \subset V$ and $|S| = m$ such that for any two vertices $v_1, v_2 \in S$ there exists an edge $(v_1, v_2) \in E$.*

A graph is m-clique free if it does not contain a clique of size greater or equal than $m + 1$. The problem under consideration is to find an interval, and m-clique free graph.

FIGURE 4.1 – Cliques with 3, 4, 5 and 6 vertices.

Figure 4.1 shows the cliques of size 3, 4, 5 and 6. Motivated by practical applications, this chapter studies the interval and m-clique free sub-graphs as well as the associated polytope. In particular, the next section provides the polytopes of interval and m-clique graphs.

4.2 The polytopes of interval sub-graphs

In this section, we consider the problem of characterizing an interval graph m-clique free.
Let $\mathcal{I} := \{I \subseteq E \,|G[I]$ *induces an interval* m*-clique free subgraph*$\}$. The vector z^I is called the incidence vector associated with I. We define the Interval m-Clique Free Subgraph Problem (IMCFSP) polytope as follows :

$$\mathrm{P}_{\mathcal{I}}(G,m) := conv\{z^I \in \{0,1\}^{|E|} | I \in \mathcal{I}\} \ ,$$

Now, we analyze the dimension of this polytope.

Proposition 5 *Polytope $P_{\mathcal{I}}(G,m)$ has a full dimension.*

proof 4.1 *We need to show the existence of $|E|+1$ feasible solutions such that their incidence vectors are affinely independent.*
Let $I_0 = \emptyset$ be a valid solution, because it has no edge. We also define solutions $I_e = \{e\}$ for all $e \in E$. The incidence vectors of these solutions are clearly affinely independents. Thus, we have $|E|+1$ vectors of $P_{\mathcal{I}}(G,m)$ affinely independent and the proof is completed. ■

In the following proposition, we will prove that the trivial inequality is a facet.

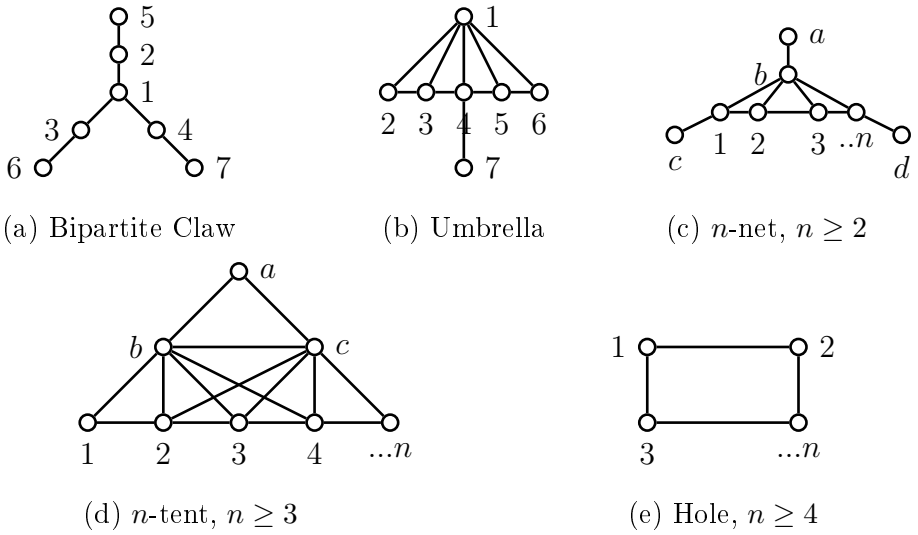
Proposition 6 *Let $e \in E$. The trivial inequality $z_e \geq 0$ defines a facet of $P_{\mathcal{I}}(G,m)$.*

proof 4.2 *Let us denote by $az \leq \alpha$ inequality $-z_e \leq 0$ associated with $e \in E$. Let $bz \leq \beta$ be a facet defining an equality of $\mathrm{P}_{\mathcal{I}}(G,m)$, such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*
Let $I_0 = \emptyset$ be a valid solution, and the associated incidence vector z^{I_0} verify $-z_e = 0$.
Let $e' \in E \setminus \{e\}$. The solution $I_{e'} = \{e'\}$ is valid and the incidence vector $z^{I_{e'}}$ associated with $I_{e'}$ verifies $-z_e = 0$.
Since, $az^{I_0} = az^{I_{e'}}$, then we deduce that $bz^{I_0} = bz^{I_{e'}}$ and this implies that $b(e') = 0$ for all $e' \in E$. Therefore, we set $b(e) = \rho$, and then $b = \rho a$. ■

4.2.1 Forbidden subgraphs inequalities

In this section we analyze the graph properties in order to propose valid inequalities. Properties on interval graphs have been studied in [80]. The authors give all forbidden

subgraphs. Indeed, if a graph does not contain at least one of the five subgraphs given in Figure 4.2, then it is an interval graph. The two first forbidden subgraphs are called *Bipartite Claw* and *Umbrella*. These subgraphs are defined on only seven nodes. The three last forbidden subgraphs are defined for different sizes. The n-net subgraphs are composed of $n+4$ nodes $\{1, ..., n, a, b, c, d\}$, where the edges are $\{a-b,\ 1-c,\ n-d\} \cup \{1-b,\ 2-b,\ 3-b, ...,\ n-b\} \cup \{1-2,\ 2-3, ...,\ (n-1)-n\}$. The n-tent subgraphs are defined as follows : the nodes a, b, c are connected by a triangle, the nodes $\{1, ..., n\}$ are connected in a line form, nodes 1 and b are connected, node n is linked to c and the nodes in $\{b, c, 2, ..., n-1\}$ define a clique. Finally, the fifth forbidden configuration is a hole of more than 3 nodes, which is a cycle without chord. These five forbidden subgraphs ensure that the graph is an interval one. In addition, to be m-clique free, all cliques of size greater or equal than $m+1$ must be forbidden.



(a) Bipartite Claw (b) Umbrella (c) n-net, $n \geq 2$

(d) n-tent, $n \geq 3$ (e) Hole, $n \geq 4$

FIGURE 4.2 – Forbidden Subgraphs Characterization

In the following subsection we propose some inequalities associated with all forbidden subgraphs and we prove that these inequalities define facets for $P_{\mathcal{I}}$.

Bipartite Claw

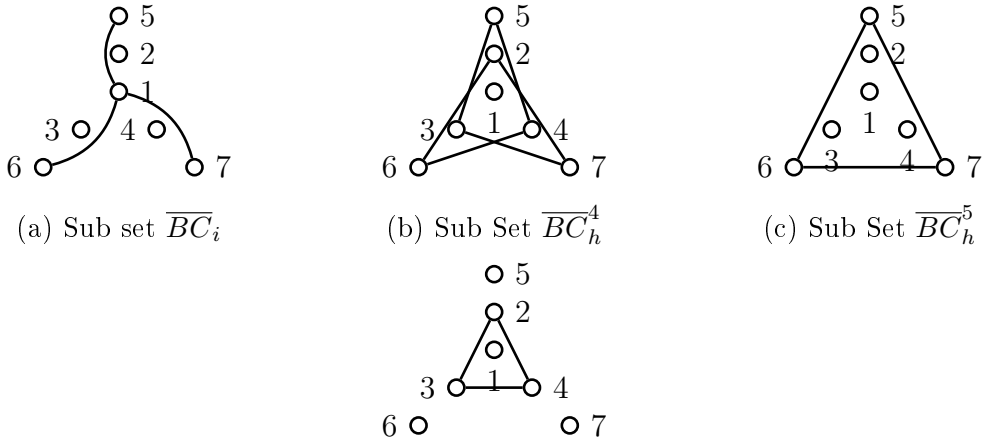
In this subsection, we give inequalities to avoid the *bipartite claw* forbidden subgraph. An example is given in Figure 4.3d.

We will give some notations to help in analyzing the *bipartite claw* forbidden subgraph. Let us consider the complete graph K_7 with seven nodes. We partition this graph to BC and $\overline{BC}$, where BC is the set of all edges that form the *bipartite claw* as in Figure 4.3d

and $\overline{BC}$ is the set of edges in the associated complementary graph of BC. Moreover, $\overline{BC}$ is partitioned as follows :

- Subset $\overline{BC}_h^4$ contains all the edges such that each of them enables to form a hole of size 4 in a bipartite claw.
- Subset $\overline{BC}_\triangle$ contains three edges such that when we add one of them to BC, then we obtain a central triangle.
- Subset $\overline{BC}_i$ contains the edges that are able to form a triangle with the inner vertex.
- Subset $\overline{BC}_h^5$ is composed of all edges such that each of them enables to form a hole of size 5 in the *bipartite claw*.

Figure 4.3 shows these subsets.



(a) Sub set $\overline{BC}_i$ (b) Sub Set $\overline{BC}_h^4$ (c) Sub Set $\overline{BC}_h^5$

(d) Sub Set $\overline{BC}_\triangle$

FIGURE 4.3 – Subsets of the complementary *Bipartite Claw*

As a consequence, the previous definitions lead explicitely to the following subsets :

- $BC = \{(1,2),(1,3),(1,4),(2,5),(4,7),(3,6)\}$.
- $\overline{BC} = \{ (7,1), (7,2), (7,3), (7,5), (7,6), (6,1), (6,2), (6,4), (6,5), (5,1), (5,3), (5,4), (4,2), (4,3), (3,2) \}$.
- $\overline{BC}_h^4 = \{(3,5),(2,6),(5,4),(2,7),(3,7),(4,6)\}$.
- $\overline{BC}_\triangle = \{(2,3),(2,4),(3,4)\}$,
- $\overline{BC}_i = \{(1,5),(1,6),(1,7)\}$.
- $\overline{BC}_h^5 = \{(5,6),(5,7),(6,7)\}$.

We consider two cases, when $m = 2$, and when $m \geq 3$.
If $m = 2$ then the following inequality is valid :

$$\sum_{e \in BC} z_e \leq 5. \tag{4.1}$$

Indeed, when we add an edge from $\overline{BC}_{\triangle}$ in Figure 4.3d, by definition, the resulting sub-graph will contain a clique of size 3, which is not m-clique free in this case, as well it is $2 - net$. Moreover, when we add an edge $e \in \overline{BC}_h$ then we obtain a *hole*. If we add another edge to break this hole then we obtain a clique of size 3.

Proposition 7 *The inequality (4.1) defines a facet, when $m = 2$.*

proof 4.3 *Let us denote by $az \leq \alpha$ the inequality (4.1). Let $bz \leq \beta$ be an inequality that defines a facet of $P_I(G, m)$, such that $\{z \in \mathrm{P}_{\mathcal{I}}(G, m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G, m) : bz = \beta\}$. Since $P_I(G, m)$ is of a full dimension, we need to prove that there exists ρ such that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $e_1, e_2 \in BC$ be two edges. The solutions $I_1 = BC \setminus \{e_1\}$, and $I_2 = BC \setminus \{e_2\}$ are valid. Their incidence vectors satisfy the inequality (4.1) to the equality. Since, $az^{I_1} = az^{I_2}$, therefore $bz^{I_1} = bz^{I_2}$ implying that $b(e_1) = b(e_2)$. We set $b(e_1) = \rho$. As e_1 and e_2 are arbitrary in BC, then $b(e) = b(e')$ $\forall e, e' \in BC$.

The solutions $I_3 = BC \setminus \{(1, 4)\}$, and $I_4 = I_3 \cup \{(2, 4)\}$, $I_5 = I_3 \cup \{(5, 4)\}$, and $I_6 = I_3 \cup \{(5, 7)\}$ are valid. Their incidence vectors satisfy the inequality (4.1) to the equality. Since, $az^{I_3} = az^{I_4} = az^{I_4} = az^{I_5} = az^{I_6}$, therefore $bz^{I_3} = bz^{I_4} = bz^{I_4} = bz^{I_5} = bz^{I_6}$ implying that $b((2, 4)) = b((5, 4)) = b((5, 7)) = 0$. By symmetry, $b(e) = 0$, $\forall e \in \overline{BC}^4_h \cup \overline{BC}^5_h \cup \overline{BC}_{\triangle}$.

The solutions $I_7 = BC \setminus \{(4, 7)\}$ and $I_8 = I_7 \cup \{(1, 7)\}$ are valid and verify the inequality (4.1) to the equality. Since, $az^{I_7} = az^{I_8}$, therefore $bz^{I_7} = bz^{I_8}$, implying that $b((1, 7)) = 0$. By symmetry, we have $b((1, 5)) = b((1, 6)) = 0$.

Let $e_3 \in E \setminus (BC \cup \overline{BC})$. The solutions I_3 and $I_9 = I_3 \cup \{e_3\}$ are valid and verify the inequality (4.1) to the equality. Since, $az^{I_3} = az^{I_9}$, therefore $bz^{I_3} = bz^{I_9}$, which implies that $b(e_3) = 0$. By symmetry, we have $b(e) = 0$ for all $e \in E \setminus (BC \cup \overline{BC})$. ■

Now, if $m \geq 3$ then the following inequality is valid.

$$\sum_{e \in BC} 2z_e - \sum_{e \in \overline{BC}_h^4 \cup \overline{BC}_\triangle} z_e - 2 \sum_{e \in \overline{BC}_i} z_e \leq 10 \tag{4.2}$$

This inequality is valid, if we add one edge of $\overline{BC}_\triangle$ to the bipartite claw, then the resulting subgraph contains $2-net$. If we add one edge of $\overline{BC}_h^5$ or $\overline{BC}_h^4$, then we obtain a hole of size 5 respectively 4. It is clear that when we add one, two or three edges of $\overline{BC}_i$, then the resulting graph becomes interval and m-clique free.

Proposition 8 *Inequality (4.2) defines a facet, when $m \geq 3$.*

proof 4.4 *Let us denote by $az \leq \alpha$ the inequality (4.2). Let $bz \leq \beta$ be an equality that defines a facet of $P_I(G,m)$, such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. Since $P_I(G,m)$ is of a full dimension, we need to prove that there exists ρ such that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $e_1, e_2 \in BC$ be two edges. The solution $I_1 = BC \setminus \{e_1\}$ and $I_2 = BC \setminus \{e_2\}$ are valid. Their incidence vectors satisfy the inequality (4.2) to the equality. Since $az^{I_1} = az^{I_2}$, therefore $bz^{I_1} = bz^{I_2}$ implying that $b(e_1) = b(e_2)$. We set $b(e_1) = 2\rho$. As e_1 and e_2 are arbitrary in BC, then $b(e) = b(e') = 2\rho$ $\forall e, e' \in BC$.

Let $e_3 \in \overline{BC}_i$ be one edge. The solutions $I_3 = BC \cup \{e_3\}$ and I_1 are valid. Their incidence vectors satisfy the inequality (4.2) to the equality. As $az^{I_3} = az^{I_1}$, then $bz^{I_3} = bz^{I_1}$ implying that $b(e_3) = -b(e_1)$. As e_3 is arbitrary in $\overline{BC}_i$ and $e_1 \in BC$, then $b(e) = -b(e'')$. We deduce that $b(e'') = -2\rho$, $e'' \in \overline{BC}_i$.

The solutions $I_4 = BC \cup \{(2,4),(3,4)\}$ and $I_5 = BC \cup \{(2,4),(2,3)\}$ are valid and verify the inequality (4.2) to equality. As $az^{I_4} = az^{I_5}$ then $bz^{I_4} = bz^{I_5}$, implying that $b((3,4)) = b((2,3))$. By symmetry, we have $b((3,4)) = b((2,4)) = b((2,3))$.
The two solutions $I_6 = BC \cup \{(2,4),(5,4)\}$ and $I_7 = BC \cup \{(2,4),(2,7)\}$ are valid and verify the inequality (4.2) to equality. Since $az^{I_6} = az^{I_7}$, then $bz^{I_6} = bz^{I_7}$, implying that $b((5,4)) = b((2,7))$. By symmetry, we have $b((2,6)) = b((3,5))$ and $b((4,6)) = b((3,7))$.

By considering the solutions I_4 and I_6 we can observe that $az^{I_4} = az^{I_6}$. For this reason $bz^{I_4} = bz^{I_6}$, which implies that $b((3,4)) = b((5,4))$. By symmetry we have $b(e) = b(e')$ for

all $e \in \overline{BC}_{\triangle}$ *and* $e' \in \overline{BC}_h^4$.

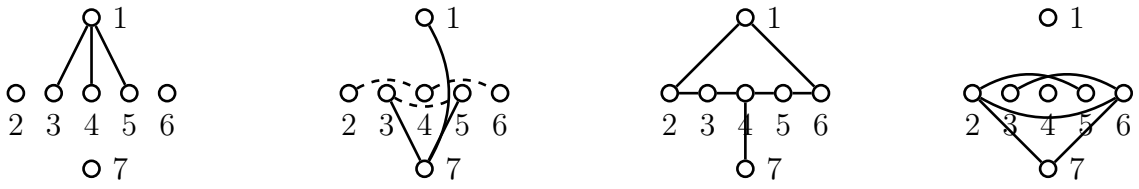
By considering the solutions I_1 *and* I_4 *are valid and verify the inequality* (4.2) *to equality. Hence,* $az^{I_1} = az_4^I$, *therefore* $bz^{I_1} = bz^{I_4}$, *this implying that* $b(e_1) = -b((2,4)) - b((5,4))$, *by the previous results and by the symmetry we deduce* $b(e) = -2b(e')$ *for all* $e \in BC$, *and for all* $e' \in \overline{BC}_{\triangle} \cup \overline{BC}_h^4$. *Therefore* $b(e') = -\rho$.

The solutions $I_{11} = BC \cup \{(5,7),(2,4),(5,4)\}$, *and the solution* I_6 *are valid and verify the inequality* (4.2) *to equality. As* $az^{I_{11}} = az^{I_9}$, *then* $bz^{I_{11}} = bz^{I_9}$ *implying that* $b(5,7) = 0$. *By symmetry, we have* $b(e) = 0$ *for all* $e \in \overline{BC}_h^5$.

Let $e_8 \in E \setminus (BC \cup \overline{BC})$. *The solutions* $I_{12} = (BC \setminus \{(1,2)\}) \cup \{e_8\}$ *and the solution* $I_{13} = BC \setminus \{(1,2)\}$ *are valid and verify the inequality* (4.2) *to equality. Since* $az^{I_{12}} = az^{I_{13}}$, *then* $bz^{I_{12}} = bz^{I_{13}}$ *implying that* $b(e_8) = 0$. *By symmetry, we have* $b(e) = 0$ *for all* $e \in E \setminus (BC \cup \overline{BC})$. ■

Umbrella Inequalities

For the umbrella subgraph as shown in Figure 4.4d, let $G_u = (U_u, E_u)$ be a graph that formulates the umbrella and let $\overline{E_u}$ be a set of the complementary edges for G_u. In the following, we will present a family of valid inequalities that delete the umbrella subgraphs. To analyze this forbidden subgraph we need the following notations :
Let $E_u^i \subset E_u$ be the set of the inner three edges in the umbrella subgraph. Let $E_u^t \subset \overline{E_u}$ be the set of the edges such that when we add one of these edges to the umbrella we create a new triangle. Subset $E_u^a \subset E_c$ is the dashed edges in Figure 4.6a. Finally, $E_u^a \subset E_u$ is the set of the around edges and $E_u^h \subset \overline{E_u}$ is the set of edges such that if they are connected, then they will form a hole of size 4 or of size 5.



(a) The Set E_u^i (b) The Set $E_u^t \cup E_u^c$ (c) The Set E_u^a (d) The Set E_u^h

FIGURE 4.4 – Subsets of umbrella and its complementary

- $E_u^i = \{ (1,3), (1,4), (1,5) \}$.
- $E_u^t = \{ (1,7), (3,7), (5,7) \}$.
- $E_u^c = \{(2,4), (3,5), (4,6)\}$.
- $E_u^a = \{(1,2), (2,3), (3,4), (4,5), (5,6), (1,6), (4,7) \}$.
- $E_u^h = \{ (2,7), (6,7), (2,5), (2,6), (3,6) \}$.

Remark that the graph induced by $H^u = \{E_u^i \cup E_u^t \cup E_u^c \cup E_u^a \cup E_u^h\}$ is a complete graph.

When $m = 2$, the triangle becomes a forbidden subgraph (and then it is not possible to find an umbrella). For this forbidden subgraph we focus on instance where $m \geq 3$.

When $m = 3$, in order to keep all edges of E_u it is necessary to add at least one edge of E_u^t. Moreover, when we add an edge from E_u^c in this case the subgraph contains a clique of size 4. If we add an edge from E_u^h, then the induced subgraph will contain a hole.
Thus, the valid inequalities when $m = 3$ will be :

$$\sum_{e \in E_u^a \setminus \{(4,7)\}} z_e + z_{(2,6)} + z_{(2,5)} + z_{(3,6)} \leq 5. \tag{4.3}$$

Proposition 9 *Inequality (4.3) defines a facet if $m = 3$.*

proof 4.5 *Let us denote by $az \leq \alpha$ the inequality (4.3) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.3) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $e_1, e_2 \in E_u^a \setminus \{(4,7)\}$. The solutions $I_1 = E_u \setminus \{e_1\}$ and $I_2 = E_u \setminus \{e_2\}$ are valid and the incidence vectors z^{I_1} and z^{I_2} verify the inequality (4.3) to equality. Moreover, we have $az^{I_1} = az^{I_2}$. Hence, $bz^{I_1} = bz^{I_2}$. This implies that $b(e_1) = b(e_2)$. We set $b(e_1) = \rho$. As e_1 and e_2 are arbitrary in $E_u^a \setminus \{(4,7)\}$, thus by symmetry, we have $b(e') = b(e) = \rho$ for all $e \in E_u^a \setminus \{(4,7)\}$.

The solution $I_3 = E_u^a \setminus \{(4,7),(1,6)\}$ and $I_4 = (I_3 \cup \{(2,5)\}) \setminus \{(4,5)\}$ are valid and the incidence vectors z^{I_3} and z^{I_4} verify the inequality (4.3) to equality. Moreover, we have $az^{I_3} = az^{I_4}$. Hence, $bz^{I_3} = bz^{I_4}$. This implies that $b((4,5)) = b((2,5))$. Thus, from the previous results we deduce that $b((2,5)) = b(e)$ for all $e \in E_u^a \setminus \{(4,7)\}$.

The solutions I_3 and $I_5 = (I_3 \cup \{(2,6)\}) \setminus \{(5,6)\}$ are valid, such that the incidence vectors z^{I_5} and z^{I_3} verify the inequality (4.3) to equality. Moreover, we have $az^{I_5} = az^{I_3}$.

Hence, $bz^{I_5}{=}bz^{I_6}$. *This implies that* $b((2,6)) = b((5,6))$. *Indeed, from the previous results we have* $b((2,6)) = b(e)$ *for all* $e \in E_u^a \setminus \{(4,7)\}$.

The solution I_3 *and* $I_6 = (I_3 \cup \{(3,6)\}) \setminus \{(5,6)\}$ *are valid and the incidence vectors* z^{I_3} *and* z^{I_6} *verify the inequality* (4.3) *to equality. Moreover, we have* $az^{I_3}{=}az^{I_6}$. *Hence,* $bz^{I_3}{=}bz^{I_6}$. *This implies that* $b((3,6)) = b((5,6))$. *Indeed, from the previous results we can deduce that* $b((3,6)) = b(e)$ *for all* $e \in E_u^a \setminus \{(4,7)\}$.

The solutions $I_7 = E_u^a \setminus \{(4,7) \cup (1,6)\}$ *and* $I_8 = (I_7 \cup \{(4,7)\} \cup E_u^i)$ *are valid and the incidence vectors* z^{I_7}, *and* z^{I_8} *verify the inequality* (4.3) *to equality. Moreover, we have* $az^{I_7}{=}az^{I_8}$. *Hence* $bz^{I_7}{=}bz^{I_8}$. *This implies that* $b(e) = 0\ \forall e \in E_u^i$.

Let $e \in \{(2,7),(3,7),(1,7),(5,7),(6,7)\}$. *The solutions* I_1 *and* $I_9 = I_1 \cup \{e\}$ *are valid and the incidence vectors* z^{I_1}, z^{I_9} *verify the inequality* (4.3) *to equality. Moreover, we have* $az^{I_1} = az^{I_9}$. *Hence,* $bz^{I_1} = bz^{I_9}$. *This implies that* $b((2,7)) = b((3,7)) = b((1,7)) = b((5,7)) = b((6,7)) = 0$. ■

When $m \geq 4$, to find a valid solution we can add also the edges from E_u^c. Then, the valid inequalities when $m \geq 4$ will be :

$$\sum_{e \in E_u^a} z_e - \sum_{e \in E_u^t \cup E_u^c} z_e \leq 6. \tag{4.4}$$

Proposition 10 *Inequality* (4.4) *defines a facet if* $m \geq 4$.

proof 4.6 *Let us denote by* $az \leq \alpha$ *inequality* (4.4) *associated with* e. *Let* $bz \leq \beta$ *be a facet defining an equality* (4.4) *such that* $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. *We show that* $b = \rho a$ *for some* $\rho \in \mathbb{R}$.

Let $e_1, e_2 \in E_u^t \cup E_u^c$ *be two edges, where* $e_1 \neq e_2$. *We consider the edge sets* $I_1 = E_u \cup \{e_1\}$ *and* $I_2 = E_u \cup \{e_2\}$ *where the incidence vectors* z^{I_1} *and* z^{I_2} *are solutions of* $\mathrm{P}_{\mathcal{I}}(G,m)$ *and satisfy the inequality* (4.4) *to equality. Moreover, we have* $az^{I_1}{=}az^{I_2}$. *Thus,* $bz^{I_1}{=}bz^{I_2}$. *This implies that* $b(e_1) = b(e_2)$. *As* e_1, e_2 *are arbitrary, then* $b(e) = b(e')$ *for all* $e, e' \in E_u^t \cup E_u^c$.

Let $e_3 \in E_u^a$. The solution $I_3 = E_u \setminus \{e_3\}$ is valid and the incidence vectors z^{I_3} verify the inequality (4.4) to equality. Moreover, we have $az^{I_1} = az^{I_3}$. Hence, $bz^{I_1} = bz^{I_3}$. This implies that $b(e_3) = -b(e_1)$. Thus, by symmetry we have $b(e') = -b(e)$ for all $e \in E_u^t \cup E_u^c, e' \in E_u^a$.

The solutions $I_4 = E_u \setminus \{(1,6)\}$, $I_5 = (E_u \setminus \{(1,6)\}) \cup E_u^i$, $I_6 = I_5 \setminus \{(1,5)\}$, $I_7 = I_6 \setminus \{(1,4)\}$ and $I_8 = I_7 \setminus \{(1,3)\}$ are valid. The incidence vectors z^{I_4}, z^{I_5}, z^{I_6}, z^{I_7} and z^{I_8} verify the inequality (4.3) to the equality. Since $az^{I_4} = az^{I_5} = az^{I_6} = az^{I_7} = az^{I_8}$, therefore $bz^{I_4} = bz^{I_5} = bz^{I_6} = bz^{I_7} = bz^{I_8}$, implying that $b(e) = 0 \ \forall e \in E_u^i$.

The solutions $I_9 = E_u \cup \{(2,7),(2,4)\}$, $I_{10} = E_u \cup \{(2,5),(2,4)\}$, $I_{11} = E_u \cup \{(2,4)\}$, $I_{12} = E_u \setminus \{(2,3)\}$, $I_{13} = I_{12} \cup \{(2,6)\}$, $I_{14} = I_{12} \cup \{(2,5)\}$, $I_{15} = E_u \setminus \{(5,6)\}$ and $I_{16} = I_{15} \cup \{(3,6)\}$ are valid and verify the inequality (4.4) to the equality. Since $az^{I_9} = az^{I_{10}} = az^{I_{11}} = az^{I_{12}} = az^{I_{13}} = az^{I_{14}} = az^{I_{15}} = az^{I_{16}}$, therefore $bz^{I_9} = bz^{I_{10}} = bz^{I_{11}} = bz^{I_{12}} = bz^{I_{13}} = bz^{I_{14}} = bz^{I_{15}} = bz^{I_{16}}$, implying that $b((2,6)) = b((2,5)) = b((2,7)) = b((6,7)) = b((3,6)) = 0$.

Let $e \in E \setminus (E_u \cup \bar{E_u})$. The solutions $I_{17} = E_u \setminus \{(4,7)\}$ and $I_{18} = I_{17} \cup \{e\}$ are valid and verify the inequality (4.4) to the equality. Since $az^{I_{17}} = az^{I_{18}}$, therefore $bz^{I_{17}} = bz^{I_{18}}$. By symmetry, we have $b(e) = 0$.

We set $b(e) = \rho$ for $e \in E_u$ and the proof is ended. ■

n-net Inequalities

The $n - net$ forbidden subgraph is shown in Figure 4.2c. We will give some notations to help in analyzing the $n - net$ forbidden subgraph.
Let $G_{net} = (U_{net}, E_{net})$ be the graph that forms a *net* of size n (i.e., $n - net$) and $\overline{E_{net}}$ be a set of complementary edges of G_{net}. To avoid to have a subgraph that represents an $n - net$, where $n \geq 2$ we need either to eliminate an edge from the $n - net$ without having a hole denoted by E_{net}^h, or to add an edge that does not construct a hole denoted by $E_{net}^{\bar{h}}$. To analyze this forbidden subgraph we will use the following notations. From Figure 4.2c let us consider :

- $E_{net}^{\bar{h}} = \{(a,c),(a,d)\} \cup \{(c,3),(c,4),...,(c,n)\} \cup \{(d,1),(d,2),...,(d,n-2)\} \cup \{(c,d)\}$.
- $E_{net}^h = \{(b,2),...,(b,n-1)\}$.

We propose valid inequalities that delete the $n-net$ forbidden subgraphs.

$$\sum_{e\in E_{net}\setminus E^h_{net}} z_e - \sum_{e\in \overline{E_{net}}\setminus E^{\bar{h}}_{net}} z_e \leq |E_{net}\setminus E^h_{net}| - 1. \tag{4.5}$$

Proposition 11 *Inequality (4.5) defines a facet.*

proof 4.7 *Let us denote by $az \leq \alpha$ the inequality (4.5) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.5) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $e_1, e_2 \in E_{net}\setminus E^h_{net}$ be two edges, where $e_1 \neq e_2$. We consider the edge sets $I_1 = E_{net}\setminus\{e_1\}$ and $I_2 = E^{net}\setminus\{e_2\}$. Their incidence vectors z^{I_1} and z^{I_2} are solutions of $\mathrm{P}_{\mathcal{I}}(G,m)$ and satisfy the inequality (4.5) at the equality. Moreover, we have $az^{I_1} = az^{I_2}$ and then $bz^{I_1} = bz^{I_2}$. This implies that $b(e_1) = b(e_2)$. As e_1, e_2 are arbitrary, then $b(e) = b(e')$ for all $e, e' \in E_{net}\setminus E^h_{net}$.

We consider the edge sets $I_3 = E_{net}\setminus\{(b,1)\}$ and $I_4 = E^{net}\setminus(\{(b,1)\}\cup E^h_{net})$.Their incidence vectors z^{I_3} and z^{I_4} are solutions of $\mathrm{P}_{\mathcal{I}}(G,m)$ and satisfy the inequality (4.5) at the equality. Moreover, we have $az^{I_3} = az^{I_4}$. Hence, we have $bz^{I_3} = bz^{I_4}$. This implies that $b(e) = 0$ for all $e \in E^h_{net}$.

Let $e_3 \in \overline{E_{net}}\setminus E^{\bar{h}}_{net}$. The solution $I_5 = E^{net}\cup\{e\}$ is valid and satisfies the inequality (4.5) to equality. Moreover, we have $az^{I_1} = az^{I_5}$ and then $bz^{I_1} = bz^{I_5}$. This implies that $b(e) = -b(e')$ for all $e \in E_{net}\setminus E^h_{net}$ and $e' \in \overline{E_{net}}\setminus E^{\bar{h}}_{net}$.

Considering the edge sets $I_6 = E^{net}\cup\{(c,2),(c,3),...,(c,n),(c,d)\}$, $I_7 = E^{net}\cup\{(c,2)\}$, $I_8 = E^{net}\cup\{(d,1),(d,2),...,(d,n-1)\}$, $I_9 = E^{net}\cup\{(d,n-1)\}$,$I_{10} = E^{net}\cup\{(a,1)\}$, $I_{11} = E^{net}\cup\{(a,1),(a,c)\}$,$I_{12} = E^{net}\cup\{(a,n)\}$, $I_{13} = E^{net}\cup\{(a,n),(a,d)\}$. These edge sets are solutions and satisfy the inequality (4.5) to the equality. Moreover, we have $az^{I_6} = az^{I_7} = az^{I_8} = az^{I_9} = az^{I_{10}} = az^{I_{11}} = az^{I_{12}} = az^{I_{13}}$. Hence, we have $bz^{I_6} = bz^{I_7} = bz^{I_8} = bz^{I_9} = bz^{I_{10}} = bz^{I_{11}} = bz^{I_{12}} = bz^{I_{13}}$. Thus, $b(e) = 0$ for all $e \in E^{\bar{h}}_{net}$.

Let $e \in E\setminus(E_{net}\cup\overline{E_{net}})$. By considering the valid solutions $I_{14} = E_{net}\setminus\{(a,b)\}$ and $I_{15} = E_{net}\setminus\{(a,b)\}\cup\{e\}$, we have $az^{I_{14}} = az^{I_{15}}$. Hence, $bz^{I_{14}} = bz^{I_{15}}$. This implies that $b(e) = 0$.

We set $b(e) = \rho$ for $e \in E_{net} \setminus E^h_{net}$ and the proof is completed. ■

n-tent Inequalities

Figure 4.2d shows the $n - tent$ forbidden subgraph, the graph is non-interval if it contains an $n - tent$ forbidden subgraph G_{tent}. The induced subgraph G_{tent} to be valid we need to add or remove one or more edges considering to not have a hole when we add or remove any edge.
Let the graph $G_{tent} = (U_{tent}, E_{tent})$ be a graph that formulate $n - tent$ for all $n \geq 3$, and $\overline{E_{tent}}$ be the set of complementary edges.
From Figure 4.2c

- $E^h_{tent} = \{(b, c), (c, 4), (b, 2)\}$,
- $E^{\bar{h}}_{tent} = \{(1, 4), (2, 5), ..., (n, n + 3)\}$.

Remark that, all n-tent where $n \geq 5$ contain a clique of size 5 then the clique inequality cut these n-tent sub graph if $m \leq 4$. It is the same idea for all n-tent where $n = 4$ (resp. $n = 3$) then the clique inequality cut these n-tent sub graph if $m = 3$(resp. $m = 2$).

In the following we will propose valid inequalities that delete the n-tent forbidden subgraphs.

$$\sum_{e \in E_{tent} \setminus E^h_{tent}} z_e - \sum_{e \in \overline{E_{tent}} \setminus E^{\bar{h}}_{tent}} z_e \leq |E_{tent} \setminus E^h_{tent}| - 1. \tag{4.6}$$

Proposition 12 *Inequality (4.6) defines a facet, when $m \geq 5$ or ($n = 4$ and $m = 3$) or ($n = 3$ and $m = 2$).*

proof 4.8 *Let us denote by $az \leq \alpha$ the inequality (4.6) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.6) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G, m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G, m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $e_1, e_2 \in E_{tent} \setminus E^h_{tent}$ be two edges, where $e_1 \neq e_2$. We consider the edge sets $I_1 = E_{tent} \setminus \{e_1\}$ and $I_2 = E^{tent} \setminus \{e_2\}$. Their incidence vectors z^{I_1} and z^{I_2} are solutions of $\mathrm{P}_{\mathcal{I}}(G, m)$ and satisfy the inequality (4.6) to equality. Moreover, we have $az^{I_1} = az^{I_2}$ and then $bz^{I_1} = bz^{I_2}$. This implies that $b(e_1) = b(e_2)$. As e_1, e_2 are arbitrary, then $b(e) = b(e')$ for all $e, e' \in E_{tent} \setminus E^h_{tent}$.

We consider the edge sets $I_3 = E_{tent} \setminus \{(a,c)\}$, *and* $I_4 = E^{tent} \setminus (\{(a,c)\} \cup \{(b,c)\})$ *there incidence vectors* z^{I_3}, *and* z^{I_4} *are solutions of* $\mathrm{P}_{\mathcal{I}}(G,m)$ *and satisfy the inequality* (4.6) *to equality. Moreover, we have* $az^{I_3} = az^{I_4}$ *and then* $bz^{I_3} = bz^{I_4}$. *This implies by symmetry that* $b((b,c)) = b((b,2)) = b((c,4)) = 0$.

Let $e_3 \in \overline{E_{tent}} \setminus E^{\bar{h}}_{tent}$. *The solution* $I_5 = E^{tent} \cup \{e\}$ *is valid and satisfies the inequality* (4.6) *to equality. Moreover, we have* $az^{I_1} = az^{I_5}$. *Hence,* $bz^{I_1} = bz^{I_5}$. *This implies that* $b(e) = -b(e')$ *for all* $e \in E_{tent} \setminus E^{h}_{tent}$ *and* $e' \in \overline{E_{tent}} \setminus E^{\bar{h}}_{tent}$.

Let $(i, i+3) \in E^{\bar{h}}_{tent}$. *The solutions* $I_6 = E^{tent} \cup \{(i,i+3),(i,i+2)\}$ *and* $I_7 = E^{tent} \cup \{(i,i+2)\}$ *are valid and satisfy the inequality* (4.6) *to equality. Moreover, we have* $az^{I_6} = az^{I_7}$ *and then* $bz^{I_6} = bz^{I_7}$. *Thus,* $b(e) = 0$ *for all* $e \in E^{\bar{h}}_{tent}$.

Let $e \in E \setminus (E_{tent} \cup \overline{E_{tent}})$. *By considering the valid solutions* $I_8 = E_{tent} \setminus \{(a,c)\}$ *and* $I_9 = E_{tent} \setminus \{(a,c)\} \cup \{e\}$, *we have* $az^{I_8} = az^{I_9}$ *and then* $bz^{I_8} = bz^{I_9}$. *This implies that* $b(e) = 0$.

We set $b(e) = \rho$ *for* $e \in E_{tent} \setminus E^{h}_{tent}$ *and the proof is completed.* ■

Hole inequalities

Here, it is convenient to define a hole as an induced subgraph of G isomorphic to C_k for some $k \geq 4$, [19]. The hole C is a forbidden subgraph as depicted in Figure 4.2e. Let C denote the set of edges that construct the hole, i.e., $C = \{ (u_1,u_2),(u_2,u_3),...,(u_{|C|-1},u_{|C|}),(u_{|C|},u_1) \}$. If $(i+k) > |C|$, then $u_{i+k} = u_{i'}$, $i' = (i+k) - |C|$. Let $\overline{C}$ denote the set of all chords of hole C.
Suppose we have a hole of size 4 this graph is non-interval graph. The induced subgraph by a hole is valid only if we add to it at least one chord.

Proposition 13 *For a hole* C, *the minimum number of necessary chords that should be added to the hole to be an interval graph is* $|C| - 3$, *when* $|C| \geq 4$.

proof 4.9 *We prove this proposition by induction. When* $|C| = 4$, *then we need one chord* $k_c = 1$, *and then the proposition is true.*
Let assume that the property is true until $|C| = l$ *and* $k_c = l - 3$. *We need to prove it for a cycle* C' *where* $|C'| = |C| + 1$ *and that* $k_{c'} = |C'| - 3$. *Let consider* $\tilde{C} = \{u_1, u_2, u_3, ..., u_{|C|}\}$ *be the hole graph given by* C *where we add* k_c *edges in order to make it hole free and*

(u_i, u_{i+1}) is an edge, where $i \in \{1, 2, ..., |C|-1\}$. Now, we will construct $\tilde{C}'$ from $\tilde{C}$ by adding the vertex $u_{|C|+1}$ in $\tilde{C}$, by replacing the edge (u_i, u_{i+1}) by $(u_i, u_{|C|+1})$ and $(u_{|C|+1}, u_{i+1})$. Then, we find a unique hole of a size 4 in the new cycle. Therefore, $|\tilde{C}'| = l + 1$. It is necessary to add one edge to break this hole. Thus, $k_{c'} = l' - 3$ is true. As C and k_c are arbitrary, then the property is true for every hole. ■

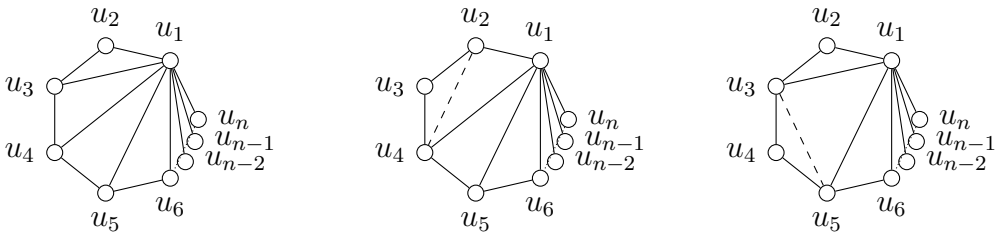


FIGURE 4.5 – Hole free subgraphs

In the following we will present valid inequalities for the hole forbidden subgraph. If $m = 2$, then the inequality (4.7) is valid.

$$\sum_{e \in C} z_e + \sum_{e \in \overline{C}} z_e \leq |C| - 1. \tag{4.7}$$

Indeed, if we add one chord to $\mathbf{C} \subset \{\mathbf{e}\}$, then we will obtain a triangle or another cycle and it is not valid for $m = 2$. Remark that this inequality for $m = 2$ is equivalent to the clique inequalities described in the next subsection.

If $m \geq 3$, then inequality (4.8) is valid.

$$\sum_{e \in C} (|C| - 3) z_e - \sum_{e \in \overline{C}} z_e \leq (|C| - 1)(|C| - 3). \tag{4.8}$$

Proposition 14 *Let C be a hole of size greater than 3, then inequality (4.8) associated with cycle C defines a facet if $m \geq 3$.*

proof 4.10 *Let us denote by $az \leq \alpha$ the inequality (4.8) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.8) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G, m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G, m) : bz = \beta\}$. We show that $b = \rho a$ for some $e \in \mathbb{R}$.*

Let e_1 and e_2 be two distinct edges of hole C. The solutions $I_1 = C \setminus \{e_1\}$ and $I_2 = C \setminus \{e_2\}$ are interval and m-clique graphs. We deduce that I_1, I_2 are solutions. Moreover, we have $az^{I_1} = az^{I_2}$. Hence, $bz^{I_1} = bz^{I_2}$. This implies that $b(e_1) = b(e_2)$. Set $b(e) = (|C| - 3)\rho$. As e_1, e_2 are arbitrary then $b(e) = b(e')$ $\forall e, e' \in C$. We deduce $b(e) = (|C| - 3)\rho$ for all $e \in C$.
Let $u_i \in V(C)$. The solution $I_3 = C \cup (\delta(u) \cap \overline{C})$, illustrated in Figure 4.5a, and the solution $I_4 = (I_3 \setminus \{(u_i, u_{i+2})\}) \cup (u_{i+1}, u_{i+3})$, illustrated in Figure 4.5b, are valid and verify the inequality (4.8) to equality. Moreover, we have $az^{I_3} = az^{I_4}$ and then $bz^{I_3} = bz^{I_4}$. This implies that $b(u_i, u_{i+2}) = b(u_{i+1}, u_{i+3})$. Thus, by symmetry $b((u_i, u_{i+2})) = b((u_{i+1}, u_{i+3}))$ are arbitrary for all $\forall i \in \{1, 2, ..., |C|\}$.

The solution $I_5 = (I_3 \setminus (u_i, u_{i+j})) \cup (u_{i+j-1}, u_{i+j+1}))$, illustrated in Figure 4.5c, is valid and verifies the inequality (4.8) to equality. Therefore, $az^{I_5} = az^{I_3}$. Hence $bz^{I_5} = bz^{I_3}$. This implies that $b(u_i, u_{i+j}) = b(u_{i+j-1}, u_{i+j+1})$. As these edges are arbitrary then $b(e) = b(e')$ for all $e' \in \overline{C}$.

Now, we will consider the two solutions I_1 and I_3. These solutions are valid and verify the inequalities (4.8) to equality. Hence, $az^{I_1} = az^{I_3}$ and therefore $bz^{I_1} = bz^{I_3}$. This implying that $b(e_1) = -\sum_{e' \in \delta(u_i) \setminus C} b(e')$. As e_1, and u_i are arbitrary for all $e \in C$ and $u_i \in V(C)$, then $b(e_1) = -(|C| - 3)b(e)$. We deduce that $b(e) = -\rho$ $\forall e \in \overline{C}$.
Let $e_3 \in E \setminus (C \cup \overline{C})$. The solutions $I_7 = I_3 \cup \{e_3\}$ and I_3 are valid and verify the inequalities (4.8) to equality. Hence, $az^{I_7} = az^{I_3}$ and then $bz^{I_7} = bz^{I_3}$. This implies that $b(e_3) = 0$. As $b(e_3)$ is arbitrary, then $b(e) = 0$ for all $e \in E \setminus (C \cup \overline{C})$. ■

Clique inequalities

In this section we will study the clique subgraph and we will propose valid inequalities and facets.

Proposition 15 *Let K be a clique and let $V(K)$ be its set of vertices. If $m = 2$ the inequality*

$$\sum_{e \in E(K)} z_e \leq |V(K)| - 1. \tag{4.9}$$

is valid and defines a facet of $\mathrm{P}_{\mathcal{I}}(G, m)$.

proof 4.11 *Remark that if $m = 2$, then a solution can be given only by a forest in the subgraph $G(K)$. We deduce that the maximum number of edges in this subgraph is equal to $|V(K)| - 1$.*

Let us denote by $az \leq \alpha$ the inequality (4.9) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.9) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.

Let $(u,v),(u,w) \in E(K)$, be two connected edges. Considering a line T_{uv} beginning by (u,v) and finishing by the vertex w and the line $T_{uw} = T_{uv} \setminus \{(u,v)\} \cup \{(u,w)\}$. These two solutions are valid and satisfy the inequality (4.9) to equality. Moreover, we have $az^{T_{uv}} = az^{T_{uw}}$. Hence, $bz^{T_{uv}} = bz^{T_{uw}}$. This implies that $b((u,v)) = b((u,w))$. By symmetry, we deduce then that $b(e) = b(e')$ for all $e, e' \in E(K)$.

Let $e \in E \setminus E(K)$. By considering the valid solution $T'_{uv} = T_{uv} \cup \{e\}$, we have $az^{T_{uv}} = az^{T'_{uv}}$. Hence, $bz^{T_{uv}} = bz^{T'_{uv}}$. This implies that $b(e) = 0$. ■

Proposition 16 *Let K be a clique of size $m+1$. The inequality*

$$\sum_{e \in E(K)} z_e \leq |E(K)| - 1 \tag{4.10}$$

defines a facet of $\mathrm{P}_{\mathcal{I}}(G,m)$.

proof 4.12 *Let us denote by $az \leq \alpha$ the inequality (4.10) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.10) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G,m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G,m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let e, e' be two edges in $E(K)$. The solution $I_1 = E(K) \setminus \{e\}$ and $I_2 = E(K) \setminus \{e'\}$ are valid and verify the inequality (4.10) to equality. Hence, $az^{I_1} = az^{I_2}$. Therefore, $bz^{I_1} = bz^{I_2}$. This implies that $b(e) = b(e')$. We set $b(e) = \rho$. As $b(e)$ and $b(e')$ are arbitrary, then $b(e) = b(e')$ for all $e, e' \in E(K)$.

Let e_1 be an edge in $E \setminus E(K)$. The solution $I_3 = I_1 \cup \{e_1\}$ and I_1 are valid and verify the inequality (4.10) to equality. Hence, $az^{I_1} = az^{I_3}$. Therefore $bz^{I_1} = bz^{I_3}$. This implies that $b(e_1) = 0$. As $b(e_1)$ is arbitrary we deduce that $b(e) = 0$, $\forall e \in E \setminus E(K)$. ■

Let $f(K,m)$ be a function giving the minimum number of edges necessary to be removed from $E(K)$ such that the resulting graph $G'(K)$ is m-clique free. Let $\alpha = \lceil \frac{|V(K)|}{m} \rceil$, $n_{\alpha-1} = m\alpha - |V(K)|$ and $n_\alpha = \frac{|V(K)| - (n_{\alpha-1})(\alpha-1)}{\alpha}$

Proposition 17 $f(K,m) = n_{\alpha-1}\frac{(\alpha-1)(\alpha-2)}{2} + n_{\alpha}\frac{(\alpha)(\alpha-1)}{2}$

proof 4.13 *Let E' be a set of $f(K,m)$ edges such that $E(K)\setminus E'$ is m-clique free. Remark that in the complementary graph $\overline{G}(K)$ we have a stable set of size $|K|$. If we add E', then there does not exist a stable of size $m+1$ (m-stable free). Let consider a minimum set of edges $E'' \in E(K)$ such that $|E''| < |E'|$ and $G(K)\setminus E''$ is m-clique free and thus $\overline{G}(K)\cup E''$ is m-stable free. Clearly E'' is a set of m disjoint cliques $\mathcal{K} = \{K_1, ..., K_m\}$, otherwise E'' is not minimal or contain a stable set of size $m+1$. To have the minimum set of edges, it is important to balance the size of cliques. Indeed, if we have two cliques K_i and K_j where $|K_i| \geq |K_j|+2$, then for $u \in K_i$ we obtain the following result $|E(K_i)|+|E(K_j)| > |E(K_i\setminus\{u\})|+|E(K_j\cup\{u\})|$. We deduce that the maximum difference between two cliques of $\mathcal{K}$ is less or equal to 1.* ■

Proposition 18 *Let K be a clique. The inequality*

$$\sum_{e\in E(K)} z_e \leq |E(K)| - f(K,m) \tag{4.11}$$

is valid.

proof 4.14 *By definition $f(K,m)$ is the minimum number of edges necessary to be removed from $E(K)$ such that the resulting graph $G'(K)$ is m-clique free. Thus, the inequality (4.11) is valid.* ■

In the next subsection, we will improve this family of inequalities.

Clique-Hole inequalities

Remark that if we remove m disjoint cliques in $G(K)$, then we obtain a complete bipartite subgraph between all pairs of two cliques. Let $H_{ij} = (K_i, K_j, E_{ij})$ be a complete bipartite graph. Remark that H contains a hole if $|K_i| \geq 2$ and $|K_j| \geq 2$. To remove every hole in H_{ij}, the minimum number of edges E' necessary to be removed to obtain a hole free graph is equal to $\max(|K_i|, |K_j|)-1$, otherwise we can always take 2 nodes in K_i or K_j such that these two nodes are not covered by E'. Note that E', with size $\max(|K_i|, |K_j|) - 1$, covers the maximum nodes of K_i and K_j. We can strength the inequality (4.11) by the following inequality. Let $\beta = \sum_{i\in m}(\max\{|K_i| - 1, 0\}) - \max_{i\in m}|K_i| - 1$.

Proposition 19 *Let K be a clique. The inequality*

$$\sum_{e\in E(K)} z_e \leq |E(K)| - (f(K,m) + \beta) \tag{4.12}$$

is valid and defines a facet.

proof 4.15 *Remark that if we unbalance two cliques K_i and K_j to reduce the value $\max(|K_i|, |K_j|) - 1$ of 1, then the number of edges necessary to be removed increases by $\max(|K_i|, |K_j|) - \min(|K_i|, |K_j|)$. Let us denote by $az \leq \alpha$ the inequality (4.12) associated with e. Let $bz \leq \beta$ be a facet defining an equality (4.12) such that $\{z \in \mathrm{P}_{\mathcal{I}}(G, m) : az = \alpha\} \subseteq \{z \in \mathrm{P}_{\mathcal{I}}(G, m) : bz = \beta\}$. We show that $b = \rho a$ for some $\rho \in \mathbb{R}$.*

Let $(u, v), (u, w) \in E(K)$ be two connected edges. Let consider a set of stable set where u is in the stable set i and v, w in the stable set j. We can easily find a solution where (u, v) is an edge and (u, w) is not an edge. These two solutions are valid and satisfy the inequality (4.12) to equality. This implies by symmetry that $b((u, v)) = b((u, w))$. We deduce then that $b(e) = b(e')$ for all $e, e' \in E(K)$.

Let $e \in E \setminus E(K)$. By considering a valid solution where e is not an edge and a second solution with the same edges plus e, we can deduce that $b(e) = 0$. ■

4.3 Cutting plane algorithms

Cutting plane method allows us to strengthen the linear relaxation by adding inequalities. Cutting plane algorithms mainly consist in generating constraints by means of a separation procedure (see, for example, [84], [81],[96], [19] and [67]). Let z^* be the incidence vector associated with the value of the variable z in the linear relaxation. The separation problem consists in finding if there exists a valid inequality $az \leq b_0$ that cuts off the solution z^*,i.e., $az^* > b_0$. The separation algorithm associated with a family of inequalities $a^{E'}x \leq b^{E'}$ for all $E' \in \mathcal{E}'$ consists in finding a set $E' \in \mathcal{E}'$ such that $a^{E'}z^* > b^{E'}$.

The results of the previous sections have allowed us to derive some exact separation algorithms and some heuristics separation algorithms. Furthermore, at the end of this section we propose "lazy separation procedure" to ensure that the integer solutions are interval and m-clique free subgraph. In the next paragraphs, we will describe these separation algorithms for all inequalities described in the previous section.

4.3.1 Bipartite claw separation

In this section we describe the BC separation algorithms. We propose three algorithms to separate the bipartite claw inequalities. One is an exact algorithm and two others are

heuristic procedures. We consider only $m \geq 3$ (note that it is easy to adapt the algorithm for $m = 2$.).

Let the vector $z^* \in \mathbb{R}^{|\mathbb{E}|}$ be a solution of a linear relaxation. We define a weight for each edge of the complete graph G as follows : $w(e) = z^*_e$ for all $e \in E$. The separation algorithm consists in finding one bipartite claw BC such that the associated inequality is violated by $z^*$and then we will add this inequality to the separation linear relaxation. This corresponds to find a bipartite claw $BC \subseteq E$ such that : $\sum_{e \in BC} 2z^*_e - \sum_{e \in \overline{BC}^4_h \cup \overline{BC}_\triangle} z^*_e - 2\sum_{e \in \overline{BC}_i} z^*_e > 10$.

From Figure 4.3d, the subsets are the following :
$BC = \{(1,2),(1,3),(1,4),(2,5),(4,7),(3,6)\}$.
$\overline{BC}^4_h = \{(3,5),(2,6),(5,4),(2,7),(3,7),(4,6)\}$.
$\overline{BC}_\triangle = \{(2,3),(2,4),(3,4)\}$.
$\overline{BC}_i = \{(1,5),(1,6),(1,7)\}$.

Proposition 20 *A partial bipartite claw $\tilde{BC}$ not violated, implies that each bipartite claw $\tilde{BC}$ including a partial bipartite claw is not violated.*

Remark that, from Figure 4.7a if we select the 3 first vertices 1, 2 and 3 such that $(2z^*_{1,2} + 2z^*_{1,3} - z^*_{2,3})$ is less than 2, then there does not exist a violated bipartite claw within these 3 vertices in this position. Indeed, if $z^*_e = 1, \forall e \in BC \setminus \{(1,2),(1,3)\}$ and $z'_e = 0, \forall e' \in \overline{BC}^4_h \cup \overline{BC}_\triangle$, then we obtain the best case, a left hand side value less than 10. Indeed, $\sum_{e \in BC \setminus \{(1,2),(1,3)\}} 2z^*_e = 8$. Thus, $8 + 0 + (2z^*_{1,2} + 2z^*_{1,3} - z^*_{2,3}) < 10$. In the same way, if we add the vertex 4 and the value of $(2z^*_{1,4} + 2z^*_{1,2} + 2z^*_{1,3} - z^*_{2,4} - z^*_{2,3} - z^*_{3,4})$ is less than 4, then there does not exist a violated bipartite claw within these 4 vertices in this position. In the best case we obtain a left side value that is $6 + 2z^*_{1,4} + 2z^*_{1,2} + 2z^*_{1,3} - z^*_{2,4} - z^*_{2,3} - z^*_{3,4}$. Thus, the left hand side is less than 10 and the value of $\sum_{e \in BC \setminus \{(1,2),(1,3),(1,4)\}} 2z^*_e = 6$. With the same argument we test the weight of all partial subgraphs to drop non interesting bipartite claws. This process is used for the exact and the first heuristic algorithms.

Exact Separation (ExBC-Sep)

Now, we will explain the exact separation algorithm. The exact algorithm consists in testing all the possible 7 vertices in this order (1,2,3,...,7). We select the nodes such that the values of the weighted edges in the incidence graph is maximized as follows : two times the weight of BC, minus the value of edges in $\overline{BC}^h_4 \cup \overline{BC}_\triangle$, minus two times the value of edges $\overline{BC}_i$. The running time of the exact algorithm in the worst case bounded by $O(n^7)$

even if in reality with the improvement presented before the running is better controlled.

Heuristic1 : Separation (H1BC-Sep)

In this heuristic we start by searching the vertices $1, 2, 3$, and 4 that maximize $2w(1,2)+2w(1,3)+2w(1,4)-w(2,4)-w(2,3)-w(3,4)$ from Figure 4.7a based on the results of **Bipartite Claw** in Section 4.2.1. If the value of $(2z^*_{1,2}+2z^*_{1,3}+2z^*_{1,4}-z^*_{2,4}-z^*_{2,3}-z^*_{3,4})$ is greater than 4, then it is possible to find a violated BC inequality with this set of vertices. After, using the greedy approach, we search to add the best weighted vertex 5 according to the incident weighted edge, if the partial BC is violated, then we will search for the best vertex 6 and with the same idea we will add vertex 7. If the BC induced by these vertices is violated by z^*, then we will add the inequality to the relaxation. Using this greedy approach the heuristic running time is $O(n^4)$.

Heuristic 2 : Separation (H2BC-Sep)

This heuristic follows a greedy approach to find a violated BC inequality. We search at each step the best next vertex to add in BC. The idea is to find the 'best weighted' edge in the graph. This edge is considered as $z_{1,2}$, if the weight of the edge $z^*_{1,2} > 0$. The heuristic tries to find with the greedy approach given in heuristic.1 the next best ordering of vertices 3, 4, ..., 7. This heuristic has $O(n^2)$ running time.

4.3.2 Umbrella separation

In this subsection we present the three algorithms for separating the umbrella forbidden subgraph. We propose an exact separation algorithm and two heuristics based on a greedy approach. Recall the inequality $\sum_{e\in E^a_u} z_e - \sum_{e\in E^t_u\cup E^c_u} z_e \leq 6$ where vector $z^* \in \mathbb{R}^{|E|}$. The separation algorithm consists in finding an umbrella, such that the associated inequality is violated by z^*. If the associated inequality violated by z^*, then we add this inequality to the linear relaxation. Thus, we search an umbrella, which is subset of E such that $\sum_{e\in E^a_u} z^*_e - \sum_{e\in E^t_u\cup E^c_u} z^*_e > 6$ where $m \geq 4$. Then, it is possible to adapt the algorithm for $m = 2$ and $m = 3$.

Exact separation algorithm

The exact separation algorithm starts by finding first edges $z_{1,2}$ and $z_{2,3}$. If $z^*_{1,2}+z^*_{2,3} < 1$, then the algorithm cannot find an umbrella within these two edges in this position. In the next step, the algorithm searches the best vertex 4, that satisfies $z^*_{1,2}+z^*_{2,3}+z_{3,4}-z_{2,4} \geq 2$, and then the algorithm adds vertex 5, then vertex 6 and finally it will add vertex 7. Figure 4.6 illustrates the umbrella and the complementary subgraps. The running time of the exact algorithm in the worst case is in $O(n^7)$.

(a) The coefficient of $E^t_u \cup E^c_u$

(b) The coefficient of E^a_u

FIGURE 4.6 – Coefficient of umbrella and its complementary edges

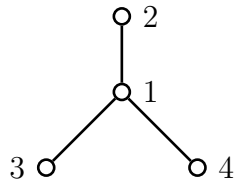
H1U-Sep separation

In this heuristic we start by searching the vertices 1, 2, 3, and 4 that maximize $(z_{1,2} + z_{2,3} + z_{3,4} - z_{2,4})$ in Figure4.6 the coefficient of each edge is illustrated, and Figure 4.7b shows these basic edges. If this value is greater than 4, then we search 6, and then 7. If the Umbrella induced by these vertices is violated by z^* we will add this inequality. Using the greedy approach the heuristic running time is in $O(n^4)$.

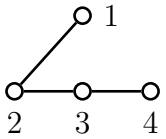
H2U-Sep Separation

This heuristic follow greedy approach to find a violated umbrella inequality we search at each step the best next edge to add in umbrella. The heuristic starts by the best weighted edge $z_{1,2}$, then with the greedy approach search the best next vertex 3 to add to the umbrella, by following this order the best weighted vertex 3, then the best weighted

vertex 4, after the best weighted 5 until the umbrella constructed. If it is violated by z^* we will add the inequality. This heuristic has $O(n^2)$ running time.

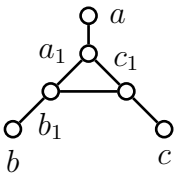


(a) Basic nodes for BC

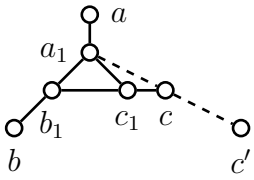


(b) Basic nodes for UH1

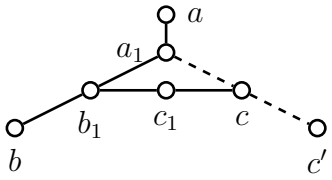
FIGURE 4.7 – Basic edges for Bipartite Claw and Umbrella



(a) 2-net



(b) From 2-net to 3-net



(c) 3-net

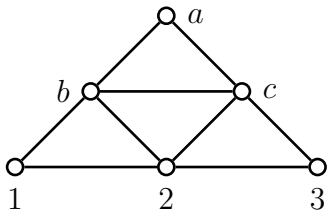
FIGURE 4.8 – Adding one vertex to 2-net to find 3-net

n-net separation

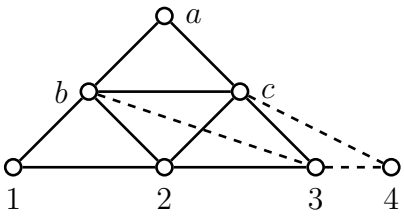
In this subsection we describe the n-net separation algorithms. Let the vector $z^* \in \mathbb{R}^{|\mathbb{E}|}$ be a solution of the linear relaxation. The weighted vector is defined in the previous sections. The separation algorithm consists in finding one n-net where $n \geq 2$, such that the associated inequality is violated by z^*. Then, we add this inequality to the separation linear program. This corresponds to find an n-net $\subseteq E$ such that : $\sum_{e \in E_{net} \setminus E^h_{net}} z^*_e - \sum_{e \in \overline{E_{net}} \setminus E^{\bar{h}}_{net}} z^*_e > |E_{net} \setminus E^h_{net}| - 1$. From Figure 4.2c, the sub sets are the following :
- $E_{net} = \{(a,b),(c,1),(d,n),(1,2)...,(n-1,n)\}$,
- $E^h_{net} = \{(b,2),...,(b,n-1)\}$.

Now, we give the n-net separation algorithm. Recall the inequalities (4.5). Let the vector $z^* \in \mathbb{R}^{|E|}$ be the solution of linear relaxation with the defined weighted graph G in the previous sections. The separation algorithm consists in finding one n-net such that the associated inequality is violated by z^* in order to add it to the linear relaxation. Therefore, we search an n-net such that : $\sum_{e \in E_{net} \setminus E^h_{net}} z^*_e - \sum_{e \in \overline{E_{net}} \setminus E^{\bar{h}}_{net}} z^*_e > |E_{net} \setminus E^h_{net}| - 1$.

Let us consider Figure 4.8a. We search the edge (a, a_1) with the maximum weight $w((a, a_1))$. Then, we search the best vertex b_1 such that $w((a_1, b_1))$ is maximum. By greedy search we continue for finding sequentially c_1 and c. By considering 2-net, if the value of $\sum_{e \in E_{net} \setminus E^h_{net}} z^*_e - \sum_{e \in \overline{E_{net}} \setminus E^{\bar{h}}_{net}} z^*_e$ is greater than 6, then we try to extend into 3-net by the following way : By adding a vertex (c') connected to the vertex c. Figure 4.8b explains the process, when the dashed edges are added.



(a) 3-tent



(b) 4-tent

FIGURE 4.9 – Adding one vertex to 3-tent to find 4-tent.

Remark that we remove the edge (a_1, c_1) (see. Figure 4.8c) if $n+1$ net is violated, then we add the n-net inequality (if $n \geq 2$) and we stop the algorithm.
The proposed heuristic running time is $O(n^2)$.

n-tent separation

For the n-tent separation algorithm we propose the following algorithm, which is based on a greedy approach.
The vector $z^* \in \mathbb{R}^{|E|}$. The n-tent separation algorithm consists in finding an n-tent, such that the associated inequality is violated by z^*. If the associated inequality is violated by z^*, then we add this inequality to the linear program. Thus, we search an n-tent where $n \geq 3$, n-tent$\subseteq E$ such that $\sum_{e \in E_{tent} \setminus E^h_{tent}} z_e - \sum_{e \in \overline{E_{tent}} \setminus E^{\bar{h}}_{tent}} z_e \leq |E_{tent} \setminus E^h_{tent}| - 1$.

We search for the best $z_{a,b}$, and $z_{a,c}$ in the first step (see 4.9a). In the next step, we search the best nodes $1, 2, 3$ one by one using a greedy approach to maximize the sum of n-tent edges, by considering the weight of the edges from n-tent and its complementary. If 2-tent is not found, that means 3-tent does not exist. If 3-tent is found, then we try to add the best node 4, which is connected with node c' and connecting c' to a_1 and remove (a_1, c) to have the structure of 4-tent. The heuristic continues for searching for $n+1$-net at each step. If the heuristic failed to find $n+1$-net, then the violated n-net inequality associated with the found n-net will be added. The proposed heuristic running time is

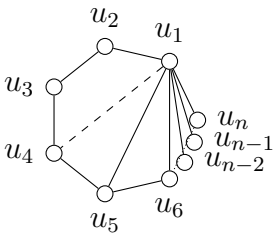


FIGURE 4.10 – Hole

$O(n^3)$.

4.3.3 Hole separation

Now, we explain the hole separation algorithm, which is based on a greedy approach. In the solution represented by vector $z^* \in \mathbb{R}^{|E|}$. The separation algorithm consists in finding a forbidden subgraph hole, such that the associated inequality is violated by z^*. If the associated inequality violated by z^* will be then added to the linear program. Thus, we search a hole of size C where $holeC \subseteq E$ such that $\sum_{e \in C}(|C| - 3)z_e - \sum_{e \in \overline{C}} z_e > (|C| - 1)(|C| - 3)$.

The algorithm starts by the edge (u_1,u_2) with $w((u_1, u_2))$ is maximum (see Figure 4.10). By a greedy search, we try to find the vertex u_3 such that $w((u_2, u_3)) - w((u_1, u_3))$ is maximum. With the same process, we search a sequence of vertices $u_4, ..., u_n$. In each step, we consider the cycle where we connect u_1 to u_n. We find a hole where the associated inequality is violated and then we add it.

If $\sum_{e \in C}(|C| - 3)z_e - \sum_{e \in \overline{C}} z_e \leq (|C| - 1)(|C| - 3)$ where C is the incident cycle, then we stop the algorithm since we cannot find a cycle where the associated inequality is violated.

4.3.4 Clique separation

In this subsection we explain the clique separation algorithm. The vector $z^* \in \mathbb{R}^{|E|}$ represents the solution. The clique separation algorithm consists in finding a clique, such that the associated inequality is violated by z^*. If the associated inequality is violated by z^*, then this inequality will be added to the linear program. Thus, we search a clique of size K, $E(K) \subseteq E$ such that $\sum_{e \in E(K)} z_e > |E(K)| - f(K, m)$.

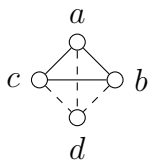


FIGURE 4.11 – Clique of size 4.

The heuristic starts by the edge (a,b) with $w((a,b))$ is maximum. By the greedy search we find the vertex c such that $w((a,c)) + w((a,b)) + w((c,b))$ is maximum. By the same process we search the vertex d such that $w((a,c)) + w((a,b)) + w((c,b)) + w((b,d)) + w((c,d)) + w((a,d))$ is maximum (see Figure 4.11). In each step we consider the clique where we connect a vertex with all other vertices in the clique. If we find a clique where the associated inequalities is violated, then the inequality will be added.

If $\sum_{e\in E(K)} z_e \leq |E(K)| - f(K,m)$, then we stop the algorithm, since we cannot find a clique where associated inequality is violated, and we add the inequality of clique of size $n-1$.

4.3.5 Lazy constraint approach

In this subsection, we propose some algorithms to ensure that a solution given by an integer value vector $\bar{z} \in \{0,1\}^{|V|\times|V|}$ induces an interval m-clique free subgraph. We consider the induced graph $\tilde{G} = (V, \tilde{E})$ where $\tilde{E}$ contains all edges such that $\bar{z}_e = 1$. For the interval graph detection, we use the algorithm given in [100] to check if $\tilde{G}$ is an interval graph. If $\tilde{G}$ is not an interval graph then we add an interdiction inequality

$$\sum_{e\in\tilde{E}} z_e - \sum_{e\in E\setminus\tilde{E}} z_e \leq |\tilde{E}| - 1,$$

This algorithm runs in $O(n + mlog(n))$.

For the clique inequalities, we search the clique of a maximum size. We use the integer linear program given in [101] to find the maximum clique in $\tilde{G}$. Finding the maximum clique in G is an NP-Hard problem. We use CPLEX to solve this problem. If $\tilde{G}$ contains a clique K of a size greater than m, then we add the following clique inequality associated with K, otherwise all inequalities (4.20) are valid.

$$\sum_{e\in E(K)} z_e \leq |E(K)| - 1,$$

Remark that, lazy constraints' call back is used only to check the validity of an integer solution.

4.4 Application to URPMDC problem.

In parallel machine scheduling, a graph can be used to represent a set of jobs competing for some resources such as a set of virtual machines in cloud computing. Here, the execution of a job when it is assigned to a virtual machine can be represented as an interval. Then, we may naturally assume that two jobs share any time unit if and only if there is an overlap between the corresponding intervals. Thus, the corresponding interval graph represents the jobs sharing execution intervals among these virtual machines. Figure 4.12 illustrates the use of interval graph in parallel machines scheduling. As one of the interval and m-clique free graphs' applications we will provide the following mathematical model for defining the problem of the unrelated parallel machines with incompatibility constraints. We will apply the results obtained from the previous sections to this problem. The unrelated parallel machine scheduling problem with disjunctive constraints is defined

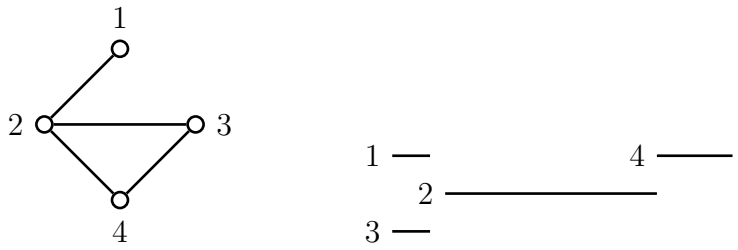


FIGURE 4.12 – Schedule of 4 jobs on 3 machines

as follows. We have n jobs and m machines, that are arranged in parallel with different speeds. The problem is to schedule these jobs on the machines, with the aim of minimizing the C_{max}. Let $G_I = (V_I, E_I)$ be the graph of incompatibility, where for each job $j \in J$, we associate a vertex $v_j \in V_I$ such that there exists an edge between v_{j_1} and v_{j_2} if j_1 and j_2 cannot be run at the same time. The case where we have to force certain jobs to not run at the same time with some others can be applied in some security issues in the cloud, and in constrained unrelated parallel machine.

4.4.1 Mathematical formulation

In this section we present an Integer Linear Programming Model (ILPM) to solve the unrelated parallel machine scheduling problem with disjunctive constraints. This ILP considers the starting of the job and the relation between jobs if they run on the same machine. Moreover, it verifies if one job runs before another job.

This model is based on interval graph and an m-clique free graph, because the graph induced by each solution must be interval graph, and m-clique free graph in order to be valid. A graph $G(V, E)$ is called interval graph if its vertices V can be represented by interval I_v of the real line such that two vertices are adjacent if and only if the corresponding intervals intersect [19]. Let K be a clique in G ($K \subseteq V$ such that every two distinct vertices of K are adjacent). This is equivalent to the condition that the subgraph of G induced by K is complete. Let $I \subseteq E$ be a subset of edges, the graph $G[I]$ is an m-clique free if and only if $G[I]$ does not contain a clique of size strictly greater than m.
Now, we introduce the ILPM :
For each job we consider a variable $y_j \in \mathbb{N}^+$ defining the starting of job j in J.
We consider binary variables for assigning the jobs on machines.
$x_j^i = \begin{cases} 1 \text{ if job } j \text{ is performed on machine } i, \\ 0 \text{ otherwise} \end{cases} \quad \forall i \in M, \forall j \in J.$
For every two jobs sharing a time unit the associated subgraph must be interval and m-clique free. For this reason we consider the following binary variables to know if two jobs share a time unit or if they are performed on disjonctive intervals :
$z_{j_1,j_2} = \begin{cases} 1 \text{ if job } j_1 \text{ and job } j_2 \text{ run at the same time}, \\ 0 \text{ otherwise} \end{cases} \quad \forall j_1, j_2 \in J.$
$\bar{z}_{j_1,j_2} = \begin{cases} 1 \text{ if job } j_1 \text{ runs before } j_2, \\ 0 \text{ otherwise} \end{cases} \quad \forall j_1, j_2 \in J.$
For all $j \in J$, we denote by $C_j \in \mathbb{N}^+$ the completion time of job j.
$C_{\max} \in \mathbb{N}^+$ is the maximum of C_j.

The URPMPC can be solved by the following ILPM denoted by (P) :

$$\min C_{max}$$

$$y_j + \sum_{i \in M} p_{ij} x_j^i \leq C_{max}, \qquad \forall j \in J, \tag{4.13}$$

$$\sum_{i \in M} x_j^i = 1, \qquad \forall j \in J, \tag{4.14}$$

$$x_{j_1}^i + x_{j_2}^i \leq 2 - z_{j_1,j_2} \qquad \forall j_1, j_2 \in J, \forall i \in M, \tag{4.15}$$

$$z_{j_1,j_2} = 0, \qquad \forall (j_1, j_2) \in E_I, \tag{4.16}$$

$$y_{j_1} + \sum_{i \in M} p_{ij_1} x_{j_1}^i \leq y_{j_2} + C - C\bar{z}_{j_1,j_2}, \qquad \forall j_1, j_2 \in J, \tag{4.17}$$

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_1} + z_{j_1,j_2} = 1, \qquad \forall j_1, j_2 \in J, \tag{4.18}$$

$$\sum_{(j_1,j_2) \in E(\bar{I})} z_{j_1,j_2} - \sum_{(j_1,j_2) \in E \setminus E(\bar{I})} z_{j_1,j_2} \leq |E(\bar{I})| - 1, \qquad \forall \bar{I} \subseteq \bar{\mathcal{I}}, \tag{4.19}$$

$$\sum_{(j_1,j_2) \in E(K)} z_{j_1,j_2} \leq |E(K)| - 1, \qquad \forall K \subseteq \mathcal{K}, \tag{4.20}$$

The objective function is to minimize the makespan. Inequalities (4.13) ensure that the starting time for each job plus its processing time is less than or equal to the total completion time. Inequalities (4.14) control each job to be processed on one machine. Inequalities (4.15) guarantee that there is no two jobs run on the same machine at the same time. Inequalities (4.16) control the disjunctive constraints. Inequalities (4.17) ensure that the starting of any job must be after the completion of its predecessor. Inequalities (4.18) ensure that, if the job j_1 runs before or at the same time with j_2, and j_2 runs before or at the same time with job j_1, then jobs j_1 and j_2 run at the same time. If we consider a solution given by the vector z, then we can define its induced subgraph $G = (V, E)$ where :

- for each job $j \in J$ we associate a vertex $v_j \in V$ and for all $z_{j_1 j_2} = 1$ we associate an edge $uv \in E$,
- G must be an interval graph and the clique of maximum size must be less or equal to m.

We denote by $\bar{\mathcal{I}}$ the set of all the induced non interval subgraphs and by $\mathcal{K}$ the set of all cliques of size greater or equal than $m + 1$. The inequalities (4.19) ensure that all induced subgraphs are interval graphs. The inequalities (4.20) ensure that all induced subgraphs have no clique of size greater or equal than $m + 1$.

4.4.2 Computational Results

To test the efficiency of the inequalities mentioned in Section 4.2.1 we developed the mentioned exact and heuristic separations. All computational results are obtained by using Cplex 12.6 and Java for implementing exact and heuristic algorithms. The ILPM with the valid inequalities is tested on the following proposed benchmark of instances. The processing times are uniformly distributed between 50 and 150 as it is common in the literature [78]. We generated five different sets of DAG where the graph density (GD) is equal to 0.5 and calculated as follows : $GD = \frac{|E|}{|V|(|V|-1)}$ where E is the set of edges associated with precedence constraints between jobs, and V is the set of vertices associated with jobs. The speeds of machines were generated randomly between 10 and 15.

The required CPU time is measured in seconds. We limit to 3600 seconds the algorithm running time for each instance, by using 4.0 GB of RAM.

The next tables provide the following information :

- $|J|$: number of jobs
- m : number of machines
- Method : 0 cplex only ; 1 bipartite claw inequalities with heuristic separation 1 ; 2 umbrella inequalities with heuristic separation 1 ; 3 hole inequalities ; 4 clique inequalities with heuristic separation ; 5 n-net inequalities ; 6 n-tent inequalities ; 7 all inequalities.
- CPU : cpu time (limited to 1 hour)
- Nb Nodes : number of nodes in the branching tree
- ct BC : number of bipartite claw inequalities added in the B&C
- ct UMB : number of umbrella inequalities added in the B&C
- ct H : number of hole inequalities added in the B&C
- ct Q : number of clique inequalities added in the B&C
- ct NN : number of n-net inequalities added in the B&C
- ct NT : number of n-tent inequalities added in the B&C
- o/p : number of instances solved (5 instances over 5 or 0 over 5)

In Table 4.1 we can see the that all the instances are solved in less than 20 seconds. These instances contain 10 jobs with 2 to 8 machines. Remark that the number of the generated inequalities is less than 200 for the bipartite claw inequalities and less than 70 for all the others. These values are relatively small. Moreover, we generate very small

number of clique inequalities, which is due to a small number of jobs. We did not generate n-net inequalities, which is due to the small average number of jobs per machine. Finally, we reduce in average the size of the search tree by adding these new valid inequalities.

Table 4.2 gives the results for 15 jobs with 4 to 8 machines. We easily solve the set of instances within less than 10 minutes. Adding the valid contraints reduces significatively the number of nodes in the tree search. The computation time is variable and adding the valid contraints does not reduce it systematically. This is due to the reduced size of the instances. Moreover, it can happen to generate a lot of contraints, as in the instances of 15 jobs on 6 machines, which explains the reduced performance. We observe that the method 3 allows us to reduce the computation times in average on the instances of 15 jobs and 6 machines. Finally, we remark that we generate few clique and n-net constraints for the same reasons previously mentioned.

Table 4.3 shows the results for 15 jobs and 2 machines. We did not solve the instances of this size, which demonstrates the hardness of the studied problem. Wa can notice that we improved the gap of the standard formulation by using the method 4 (exploiting the clique contraints).

For solving this problem the new valid inequalities dod not allow to significantly improve the computation time. Nevertheless, we will demonstrate in the next chapter the positive impact of these contraints. Indeed, the instances we try to solve are too small so that the valid inequalities will be able to improve the CPLEX routines. Moreover, our separation algorithms need certainly to be improved in order to quickly generate valid contraints.

4.5 Conclusion

In this chapter we presented a polyhedral study for the problem of interval under m-clique free subgraphs. A polyhedral investigation of the convex hull of these vectors yielded several results on facet defining inequalities for the defined polytope. We designed and implemented a branch-and-cut algorithms based on families of strong valid inequalities presented in this chapter. We separate some forbidden subgraphs, and we have also applied the obtained results to the problem of unrelated parallel machines with disjunctives/precedence constraints. The computational experiments on set of instances have shown that the algorithms are capable to solve all instances more less to the optimality within less CPU time when we add these separation algorithms. Further research in this

direction will be helpful to strengthen the integer programming formulations of a large variety of unrelated parallel machines problems as an application to the interval under m-clique free subgraphs. In the next chapter the properties of interval graph will be applied for solving generalized Open Shop.

$\|J\|$	m	Method	CPU	Nb Nodes	ct BC	ct UMB	ct H	ct Q	ct NN	ct NT	o/p
10	2	0	15,4	41866,8	0	0	0	0	0	0	5/5
10	2	1	15,2	35862,2	177,6	0	0	0	0	0	5/5
10	2	2	15,4	40972,6	0	35,6	0	0	0	0	5/5
10	2	3	13,6	35867,8	0	0	67,2	0	0	0	5/5
10	2	4	15,2	41925	0	0	0	10,8	0	0	5/5
10	2	5	15,4	41866,8	0	0	0	0	0	0	5/5
10	2	6	18	45625,2	0	0	0	0	0	42,4	5/5
10	2	7	13,8	30941,4	138,2	12,6	56,8	11	0	24,4	5/5
10	4	0	0,6	2189,6	0	0	0	0	0	0	5/5
10	4	1	0,8	2125,8	57,6	0	0	0	0	0	5/5
10	4	2	0,6	1864,2	0	40	0	0	0	0	5/5
10	4	3	0,8	1958	0	0	39,6	0	0	0	5/5
10	4	4	0,6	2189,6	0	0	0	0	0	0	5/5
10	4	5	0,6	2189,6	0	0	0	0	0	0	5/5
10	4	6	0,6	1801,8	0	0	0	0	0	38,4	5/5
10	4	7	0,8	1837,8	31,4	18,6	29,6	0	0	25,6	5/5
10	6	0	0,4	980	0	0	0	0	0	0	5/5
10	6	1	0,4	1046,8	38	0	0	0	0	0	5/5
10	6	2	0,4	1107,2	0	40,2	0	0	0	0	5/5
10	6	3	0,4	1245,4	0	0	31	0	0	0	5/5
10	6	4	0,4	980	0	0	0	0	0	0	5/5
10	6	5	0,4	980	0	0	0	0	0	0	5/5
10	6	6	0,2	1075,2	0	0	0	0	0	23,2	5/5
10	6	7	0,4	1294,2	23,2	18,8	28,4	0	0	23	5/5
10	8	0	0,6	1461,4	0	0	0	0	0	0	5/5
10	8	1	0,2	1060	28	0	0	0	0	0	5/5
10	8	2	0,2	1018	0	30,8	0	0	0	0	5/5
10	8	3	0,2	1104,4	0	0	33,4	0	0	0	5/5
10	8	4	0,6	1461,4	0	0	0	0	0	0	5/5
10	8	5	0,6	1461,4	0	0	0	0	0	0	5/5
10	8	6	0,4	1312,4	0	0	0	0	0	37,8	5/5
10	8	7	0,2	971,4	17,8	20,6	23,2	0	0	21,8	5/5

TABLE 4.1 – Results for 10 jobs with different methods

$\|J\|$	m	Method	CPU	Nb Nodes	ct BC	ct UMB	ct H	ct Q	ct NN	ct NT	o/p
15	4	0	116,2	91260,8	0	0	0	0	0	0	5/5
15	4	1	363,8	77416,4	3322,6	0	0	0	0	0	5/5
15	4	2	283,6	79595	0	2375,4	0	0	0	0	5/5
15	4	3	153,2	79613,4	0	0	794,6	0	0	0	5/5
15	4	4	105,6	81880,2	0	0	0	0,8	0	0	5/5
15	4	5	117,2	91260,8	0	0	0	0	0	0	5/5
15	4	6	142,2	82362,4	0	0	0	0	0	712	5/5
15	4	7	162,4	63070,2	1005	329,2	332,2	0,8	0	337,8	5/5
15	6	0	177,6	89917,4	0	0	0	0	0	0	5/5
15	6	1	306	48149	3606,2	0	0	0	0	0	5/5
15	6	2	283	36256,2	0	3391	0	0	0	0	5/5
15	6	3	133,4	39087,4	0	0	1067	0	0	0	5/5
15	6	4	168,8	89917,4	0	0	0	0	0	0	5/5
15	6	5	170,8	89917,4	0	0	0	0	0	0	5/5
15	6	6	148,8	39038	0	0	0	0	0	1326,4	5/5
15	6	7	507,6	42444,2	1538,4	1215,8	578,2	0	0	1197,8	5/5
15	8	0	56	29774,2	0	0	0	0	0	0	5/5
15	8	1	71,4	24297,2	1398,4	0	0	0	0	0	5/5
15	8	2	140,8	33379,6	0	2043,2	0	0	0	0	5/5
15	8	3	86,2	33893,8	0	0	638,6	0	0	0	5/5
15	8	4	63,6	29774,2	0	0	0	0	0	0	5/5
15	8	5	56,6	29774,2	0	0	0	0	0	0	5/5
15	8	6	77	29433,2	0	0	0	0	0	882,6	5/5
15	8	7	131,6	24945	1349	850	499,6	0	0	830	5/5

TABLE 4.2 – Results for 15 jobs with different methods

$\|J\|$	m	Method	CPU	Nb Nodes	ct BC	ct UMB	ct H	ct Q	ct NN	ct NT	o/p	gap
15	2	0	3600	2476280	0	0	0	0	0	0	0/5	0,2970
15	2	1	3600	535127,8	10931,8	0	0	0	0	0	0/5	0,3674
15	2	2	3600	1039497	0	5334,8	0	0	0	0	0/5	0,3248
15	2	3	3600	1492568,6	0	0	2921,6	0	0	0	0/5	0,3279
15	2	4	3600	2443961,6	0	0	0	43	0	0	0/5	0,2913
15	2	5	3600	2447396,2	0	0	0	0	0	0	0/5	0,2976
15	2	6	3600	1527756,2	0	0	0	0	0	1989	0/5	0,3177
15	2	7	3600	482432,2	9337	1070	1586	41,4	0	1173,4	0/5	0,3644

TABLE 4.3 – Results for hard instances with 15 jobs and different methods

5

Generalized Open Shop, and Open Shop Problems

Contents

In this chapter we continued the study of our mathematical formulation, which is based on the interval subgraph. The strong structure of its polytope encourage us to adapt this model to solve other scheduling problems. We addressed two open shop scheduling problems : the generalized version with disjunctive constraints and the standard one. We present two mathematical formulations for solving these problems. We derived different classes of valid inequalities to strength our models. We also add separation algorithms to the relaxed model of the generalized open shop problem. Exhaustive computational experiments on the well known sets of Taillard's benchmarks are presented. The derived valid inequalities show a good improvement to the computational time for the two models. Mo-

reover, the generalized open shop model shows the efficiency of adding the cutting plane inequalities.

5.1 Introduction

In this chapter, we consider a particular case related to the model of the previous chapter. This problem is called generalized open shop with disjunctive constraints. In the first part of this chapter, we propose a model for this problem and give some valid inequalities to improve the associated linear relaxation. Furthermore, we test experimentally the different inequalities. In the second part of this chapter we propose an integer linear programming model for the standard open shop problem. Indeed, the open shop is a restricted case of the generalized open shop with jobs disjunctive constraints. The Open-Shop problem is an important research branch of scheduling problems and it receives an important amount of attention because of their wide range of applications, such as modern transport and logistics, modern service industry, large-scale systematic maintenance, clothing industry, health care, and so on [102, 103, 104]. In cloud computing, there is a special case where the job can be divided into m sub tasks. Therefore, each sub task can be processed on a virtual machine. In general, this problem can be described by a set of n jobs to be processed by a set of m machines and there is no predetermined processing route for the jobs. Therefore in the open shop, there are two decisions to make : the determination of the processing route of each job as well as the job sequence at each stage [105]. The general $O||C_{max}$ problem is strongly NP-hard [66].

Among the many techniques proposed in the literature the problem has not been attacked a lot by the mathematical models. Masuda and Ishii [107] studied the open shop scheduling problem for two-machines and they proposed a bi-criteria linear program. Kis *et al* [108] described an integer program in two dimension.

We also propose new valid inequalities and we test our model on a well known Taillard's [109] instances. The experimental results show the efficiency of our model by solving all Taillard's instances of size 4×4, 5×5, 7×7 and 10×10. Furthermore, we solve all instances of size 15×15 and reduce all upper and lower bounds for all other instances.

5.2 Generalized open shop problem with jobs disjunctive constraints

The Generalized Open Shop with Disjunctive Constraints (GOSDC) can be formulated as follows. Let M be the set of machines. For all $i \in M$ we consider the set of jobs J_i must be running of the machine i and we denote by $\mathcal{J} = \{J_1, ..., J_m\}$ the set of all these sets and by $J = \bigcup_{i \in M} J_i$ the union of these sets. We denote by p_{ij} the processing time of job j on its machine i. We consider an incompatibility graph $G_I = (V_I, E_I)$, in this graph, for each job $j \in J$ we associate a vertex $v_j \in V_I$ and there exist an edge between v_{j_1} and v_{j_2} if j_1 and j_2 cannot run at the same time. Remark that, it is necessary to considering a linear ordering on each machine. To the best of our knowledge this special problem has not been studied before.

5.2.1 Integer linear programming formulation

In this section, we present the integer linear programming for solving the problem. We need a family of binary variables. In the following, we describe the variables used in the model :

$$\bar{z}_{j_1,j_2} = \begin{cases} 1 \text{ if the job } j_1 \text{ runs before the job } j_2 \\ 0 \text{ otherwise} \end{cases} \quad \forall j_1, j_2 \in J.$$

$$z_{j_1,j_2} = \begin{cases} 1 \text{ if } j_1 \text{ and } j_2 \text{ run at the same time} \\ 0 \text{ otherwise} \end{cases} \quad \forall j_1 \in J_i, j_2 \in J_{i'} | i \neq i' \in M.$$

For all $j \in J$, we consider the variable $y_j \in \mathbb{N}^+$ representing the starting time of job j.

$C_{\max} \in \mathbb{N}^+$ is the maximum completion time.

The GOSDC can be solved by the following ILP, denoted by (P_{GOS}) :

$$\min C_{max}$$

$$y_j + p_{ij} \leq C_{max}, \qquad \forall i \in M \qquad \forall j \in J_i, \tag{5.1}$$

$$y_{j_1} + p_{ij_1} \leq y_{j_2} + C\bar{z}_{j_2,j_1}, \qquad \forall i \in M \qquad \forall j_1 \in J_i \text{ and } j_2 \in J, \tag{5.2}$$

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_1} = 1, \qquad \forall i \in M \qquad \forall j_1, j_2 \in J_i, \tag{5.3}$$

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_1} = 1, \qquad \forall (v_{j_1}, v_{j_2}) \in E_I, \tag{5.4}$$

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_1} + z_{j_2,j_1} = 1, \quad \forall (v_{j_1}, v_{j_2}) \notin E_I, \tag{5.5}$$

$$\sum_{(j_1,j_2)\in E(\bar{I})} z_{j_1,j_2} - \sum_{(j_1,j_2)\in E\setminus E(\bar{I})} z_{j_1,j_2} \leq |E(\bar{I})| - 1, \quad \forall \bar{I} \subseteq \overline{\mathcal{I}}, \tag{5.6}$$

$$\sum_{(j_1,j_2)\in E(K)} z_{j_1,j_2} \leq |E(K)| - 1, \qquad \forall K \subseteq \mathcal{K}, \tag{5.7}$$

The objective function is to minimize the makespan. Inequalities (5.1) ensure that the beginning time for each job plus its processing time is less than or equal to the total completion time. Inequalities (5.2) and inequalities (5.3) guarantee that there is no two jobs run on the same machine at the same time and control the linear ordering. Inequalities (5.4) ensure that if two jobs are linked by an edge in the compatibility graph, then they do not run at the same time. Indeed, these two jobs j_1, j_2 either j_1 before j_2 or j_2 before j_1. Inequalities (5.5) ensure the three possibilities : j_1 before j_2 or j_2 before j_1 or they run at the same time. Inequalities (5.6) and (5.7) guarantee that the induced subgraphs are interval and m-clique free subgraphs. The number of inequalities may be exponential and thus we will use the separating algorithm presented in Chapter 4.

5.2.2 Valid inequalities

To strength the model in this section we propose some valid inequalities to (P_{GOS}).

Sequence inequalities

Considering every $i \in M$, we introduce the following valid inequalities :

$$\sum_{j\in J_i} (y_j + p_{ij}) \times p_{ij} \geq SC_i, \tag{5.8}$$

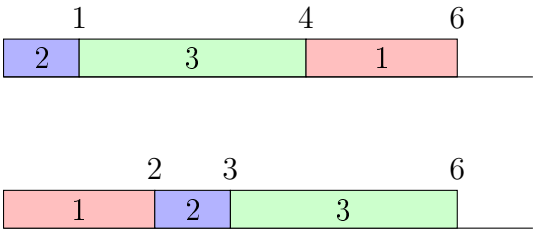


FIGURE 5.1 – Different sequences of jobs on a machine

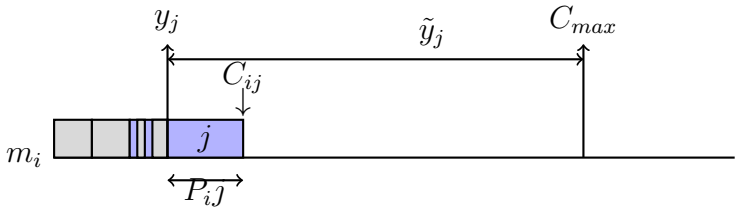


FIGURE 5.2 – Sequence inequalities.

with SC_i is equal to the sum of $C_j \times p_{ij}$ for all jobs in J_i given by a schedule of jobs in any sequence. We notice that this inequality remains valid for any subset. Figure 5.1 illustrates this relation. In this figure, the processing times of jobs are : $p_1 = 2$, $p_2 = 1$ and $p_3 = 3$, where the speed of machine equal to 1. In the first sequence, we have : $1 \times 1 + 3 \times 4 + 2 \times 6 = 25$. The second sequencing is $2 \times 2 + 1 \times 3 + 3 \times 6 = 25$. It is clear that the order will not affect the value $\sum_{j \in J_i}(y_j + p_{ij}) \times p_{ij}$.

Figure 5.2 illustrates the idea of this family of inequalities. Having $\tilde{y}_j = C_{max} - y_j$ and from the valid inequalities (5.8), we can derive that $\sum_{j \in J_i} p_{ij}(\tilde{y}_j) \geq SC_i$. Thus, we can establish that :

$$\sum_{j \in J_i}(C_{max} - y_j) \times p_{ij} \geq SC_i, \quad \forall j \in J, \tag{5.9}$$

Therefore, we deduce that :

$$\sum_{j \in J_i} C_{max} \times p_{ij} \geq SC_i + \sum_{j \in J_i} y_j \times p_{ij}, \quad \forall j \in J, \tag{5.10}$$

Previous job inequalities

Considering machine $i \in M$ and the job $j \in J_i$, we introduce the following valid inequality :

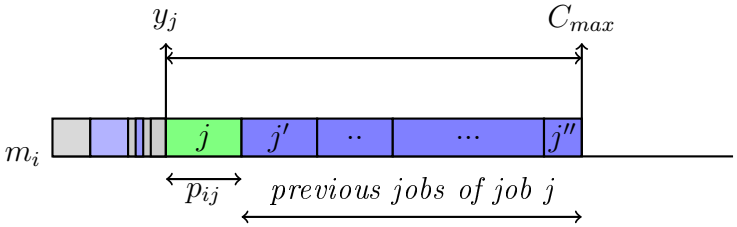


FIGURE 5.3 – Previous job inequalities.

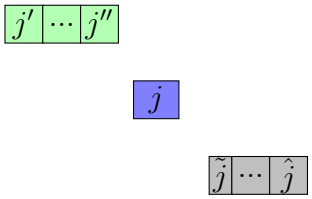


FIGURE 5.4 – Line job inequalities

$$\sum_{j' \in J_i} p_{ij'} \bar{z}_{j',j} \leq y_j, \tag{5.11}$$

Inequalities (5.13) ensure that the starting time of job j on machine i will be after the sum of the completion times of all previous jobs on the same machine. We derived other family of inequalities based on the idea of the previous jobs. Figure 5.3 helps to explain this family of inequalities. The C_{max} is bounded by the sum of the starting time of job j on machine i, plus the processing time of job j on machine i, plus the sum of the processing times of all next jobs on the same machine. Thus, job j processed previous these jobs on this machine.

$$C_{max} \geq p_{ij} + y_j + \sum_{j' \in J_i} p_{ij'} \bar{z}_{jj'}, \quad \forall i \in M, \forall j \in J_i, \tag{5.12}$$

Line job inequalities

This class of inequalities derived from the idea of the linear order of jobs on the same machine extended for 3 machines. Considering $i, i_1, i_2 \in M$ and a job $j \in J_i$, the following inequality holds.

$$\sum_{j' \in J_{i_1}} p_{i_1 j'} \bar{z}_{j',j} + p_{ij} + \sum_{j' \in J_{i_2}} p_{i_2 j'} \bar{z}_{j,j'} \leq C_{max}, , \tag{5.13}$$

C_{max} is bounded by the sum of the processing times of its previous jobs, plus its processing time, plus the sum of the processing times of all next jobs. This class of inequalities is very interesting when we have jobs with long processing times, because we can avoid some empty slots on other machine.

Logical implication inequalities

Considering these jobs $j_1, j_2, j_3 \in J$, we introduce the following valid inequalities :

$$\bar{z}_{j_1,j_2} + \bar{z}_{j_2,j_3} \leq 1 + \bar{z}_{j_1 j_3}, \tag{5.14}$$

In this valid inequalities, we apply the logical implication. If job j_1 is before job j_2 and job j_2 is before job j_3, then job j_1 is before job j_3.

5.2.3 Experimental results

The entries of the table are :

- $|J_i|$: Number of jobs by machine.
- m : Number of machines.
- opt : 0 : Basic model ; 1 : Sequence inequalities, previous job inequalities, line job inequalities ; 2 : Sequence inequalities, previous job inequalities, line job inequalities, logical implication
- IC : N basic model only ; Y using the inequalities explained in the previous chapter "Chapter.4" : bipartite claw inequalities with heuristic separation 1 ; umbrella inequalities with heuristic separation 1 ; hole inequalities ; clique inequalities with heuristic separation ; n-net inequalities ; n-tent inequalities.
- CPU : Computational time (limited to 1 hour).
- ct BC : The number of bipartite claw inequalities added in the B&C.
- ct UMB : The number of umbrella inequalities added in the B&C.
- ct H : The number of hole inequalities added in the B&C.
- ct Q : The number of clique inequalities added in the B&C.
- ct NN : The number of n-net inequalities added in the B&C.
- ct NT : The number of n-tent inequalities added in the B&C.
- Nodes : The number of nodes in the branching tree.
- gap : The gap between the lower bounds and the upper bounds ($100 \times \frac{UB-LB}{LB}$),
- o/p : the number of instances solved (5 instances over 5 or 0 over 5)

$\|J_i\|$	m	opt	IC	CPU	ct BC	ct UMB	ct H	ct Q	ct NN	ct NT	Nodes	gap	o/p
5	2	0	N	0,5	0	0	0	0	0	0	2550,75	0	5/5
5	2	0	Y	0,75	11,25	1,75	5,5	0	0	3	2773	0	5/5
5	2	1	N	0	0	0	0	0	0	0	106,75	0	5/5
5	2	1	Y	0	0,75	0,5	1	0	0	0,25	45	0	5/5
5	2	2	N	0	0	0	0	0	0	0	15	0	5/5
5	2	2	Y	0	0	0	0,5	0	0	0	17,5	0	5/5
5	4	0	N	3600	0	0	0	0	0	0	81065,75	0,6956	0/5
5	4	0	Y	3249,75	6489	2674,75	1785,25	0	599,5	1778,5	102119	0,2881	1/5
5	4	1	N	3600	0	0	0	0	0	0	224810	0,5756	0/5
5	4	1	Y	3600	7450,25	3656,75	1974,25	0	710,5	2187,25	90313,75	0,1169	0/5
5	4	2	N	3220	0	0	0	0	0	0	111017,25	0,0690	1/5
5	4	2	Y	3597,5	227,25	84,5	257,75	0	103,5	134,5	87498,5	0,0382	1/5
10	2	0	N	3600	0	0	0	0	0	0	986257,25	0,3880	0/5
10	2	0	Y	3600	3580	453,75	757,75	0	0	496,5	714914,5	0,4076	0/5
10	2	1	N	158,5	0	0	0	0	0	0	52605,75	0	5/5
10	2	1	Y	864,75	470	59,25	124,25	0	0	41,25	320811	0	5/5
10	2	2	N	4	0	0	0	0	0	0	339	0	5/5
10	2	2	Y	3	2	1	5,5	0	0	0	218	0	5/5

TABLE 5.1 – Results of the generalized open shop model

Table 5.1 presents the results of the generalized open shop problem. The entries of this table were shown before. From the obtained results, the model can solve many problems to the optimality within reasonable computation time (especially for 5 jobs per machine, for two machines, and for ten jobs per machine, for two machines). In overall, the model solved 61% of the problems to optimality within one hour. For the other problems the gap is reasonable within this time limit. We can also notice that from the structure of the polytope there is no clique inequality added. Interesting results could be extracted from this experiments. Indeed, when we add the sequence, previous, and line valid inequalities to our model, we obtain an optimal solution for the larger instances within good computational time. When we add sequence, previous, and logical valid inequalities, we obtain some good results. We can say that there is a noticeable improvement for the results when we add our derived valid inequalities to our model. The most interesting things come from the improvement of the gap when we add the inequalities given in chapter 4. These inequalities allow us to divide the gap for the harder instances by 2 to 5 and then show their efficiency. Remark that we add a lot of interval subgraph inequalities for the harder instances.

In the next section, we adapt this model to the standard open shop problem, where the job is divided into operations and the operations will be processed on a given set of parallel machines.

5.3 Open shop problem

In the second part of this chapter, we deal with the open shop problem.

The open shop scheduling problem can be described as follows. Having a set of parallel machines and a finite set of operations, these operations have to be processed on the given set of parallel machines. Preemption is not allowed. Each job has a processing time and each operation belongs to only one job. Operations are grouped in jobs. Moreover, the operations that belong to the same job and the operations that use the same machine cannot be processed at the same time. Furthermore, each operation is assigned to an only one machine. The objective of the open shop scheduling problem is to perform all operations, so as to minimize the maximum completion time (makespan).

More formally, we have a set $J = \{1, 2, .., n\}$ of n jobs to be performed on a set of machines M_i where $i = \{1, 2, ..., m\}$. Each job $j \in J$ consists of exactly m operations $O_{i,j} (i \in \{1, 2, ..., m\})$. For every job j and every index i, operation $O_{i,j}$ should be performed

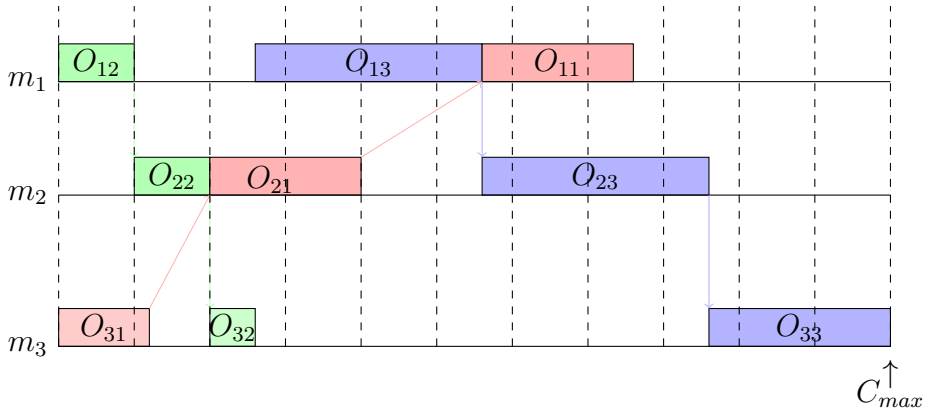


FIGURE 5.5 – Open shop for three jobs

on machine M_i. The processing time of each operation $O_{i,j}$ is denoted by $p_{i,j}$. At any time, a job can be processed by at most one machine. Moreover, any machine can process only one job at a time. Preemption of operations is not allowed. For every job j, its completion time C_j is defined as the completion time of its last operation. The objective is to find a feasible schedule that minimizes the maximum completion time C_{max}. Figure 5.5 illustrates the schedule of three jobs j_1, j_2 and j_3, where job j_2 does not share any time unit with job j_3 in the system, but job j_1 is processed at the same time with job j_2 and job j_3.

Open shop scheduling problem is NP-hard problem [111]. Using the standard notation $\alpha|\beta|\gamma$ of Graham *et al.* [66], we can denote the open shop problem as $O_m||C_{max}$. Numerous scheduling problems in real life applications can be modeled as an open shop : network flow has a lot of problems that can be immediately translated into an open shop scheduling problem [113]. A recent literature review of open shop is found in [112]

Let denote by $\mathcal{O}_i^J$ the set of operations of all jobs J assigned to machine $i \in M$ and by $\mathcal{O}_j^M$ the set of operations of job j assigned to all the machines. We denote by p_{ij} the processing time of the operation associated with $j \in J$ assigned to machine $i \in M$.

5.3.1 Integer linear programming formulation

$$\bar{z}_{O_1,O_2} = \begin{cases} 1 \text{ if } O_1 \text{ runs before } O_2 & \forall i \in M, \forall O_1, O_2 \in \mathcal{O}_i^J \vee \\ 0 \text{ otherwise} & \forall j \in J, \forall O_1, O_2 \in \mathcal{O}_j^M \end{cases}$$

For all $j \in J$ and $i \in M$:

$y_{i,j} \in \mathbb{N}^+$ is the starting time of operation $O_{i,j}$.

$C_{\max} \in \mathbb{N}^+$ is the maximum of $y_{i,j} + p_{ij}$.

The Open Shop can be solved by the following integer linear programming model, denoted

by (P_{OS}) :

$$\min C_{max}$$

$$y_{i,j} + p_{ij} \leq C_{max}, \qquad \forall i \in M \quad \forall j \in J, \tag{5.15}$$

$$y_{i,j_1} + p_{ij_1} \leq y_{i,j_2} + C\bar{z}_{O_{i,j_2},O_{i,j_1}}, \qquad \forall i \in M \quad \forall j_1, j_2 \in J, \tag{5.16}$$

$$y_{i_1,j} + p_{i_1 j} \leq y_{i_2,j} + C\bar{z}_{O_{i_2,j},O_{i_1,j}}, \qquad \forall j \in J \quad \forall i_1, i_2 \in M, \tag{5.17}$$

$$\bar{z}_{O_{i,j_1},O_{i,j_2}} + \bar{z}_{O_{i,j_2},O_{i,j_1}} = 1, \qquad \forall i \in M \quad \forall j_1, j_2 \in J, \tag{5.18}$$

$$\bar{z}_{O_{i_1,j},O_{i_2,j}} + \bar{z}_{O_{i_2,j},O_{i_1,j}} = 1, \qquad \forall j \in J \quad \forall i_1, i_2 \in M, \tag{5.19}$$

The objective function is to minimize the makespan.

Inequalities (5.15) ensure that the starting time for each operation plus its processing time is less than or equal to the total completion time. Inequalities (5.16) control that if operation O_{ij_1} is before operation O_{ij_2}, then operation O_{ij_2} must start after the completion of operation O_{ij_1} on the same machine. Inequalities (5.17) guarantee that if operation of job j runs on machine i_1 before its operation on machine i_2, then the starting time of $O_{i_2,j}$ must be after the completion of operation $O_{i_1,j}$. Inequalities (5.18) and (5.19) guarantee the linear ordering of the operations on the same machine and the linear ordering of the operations of the same job.

We also use the conditional constraints for inequalities (5.2) and (5.3). We can write these two constraints as follows :

$$if(\bar{z}_{O_{i,j_1},O_{i,j_2}} = 1), \text{ then } \; y_{i,j_1} + p_{ij_1} \leq y_{i,j_2}, \qquad \forall i \in M \quad \forall j_1, j_2 \in J, \tag{5.20}$$

(5.21)

$$if(\bar{z}_{O_{i_1,j},O_{i_2,j}} = 1), \text{ then } \; y_{i_1,j} + p_{i_1 j} \leq y_{i_2,j}, \qquad \forall j \in J \quad \forall i_1, i_2 \in M, \tag{5.22}$$

Inequalities (5.20) imply that if job j_1 is before job j_2, then it implies that, the starting time of job j_2 must be after the completion time of job j_1. The conditional inequalities 5.21 test if the operation of job j on machine i_1 is before the operation of the same job on machine i_2, then the starting time of the operation of job j on machine i_2 must be after the completion of the operation of job j on machine i_1. Adding the conditional inequalities will remove the big $M(C)$, but the solver must branch on these conditions. We will explain the results of using conditional inequalities in the section dedicated to the experimental results.

5.3.2 Valid inequalities

In this section, we derive various classes of valid inequalities for the integer linear programming model. In these derived inequalities, we consider the sequence of jobs on the machines, the relation between a job and its previous ones as in [110] and we consider a logical implication. We adapted the classes of valid inequalities introduced in Section 5.2.2.

Sequence inequalities

Considering job $j \in J$, we introduce the following valid inequalities :

$$\sum_{i \in M} (y_{i,j} + p_{ij}) \times p_{ij} \geq SC_j, \tag{5.23}$$

where SC_j is equal to the sum of $C_{i,j} \times p_{ij}$ where $C_{i,j}$ is the completion time of operation $O_{i,j}$ in any sequence. With the same idea, we can dervie similar valid inequalities for every subset of operations of job j.

By symmetry, let $i \in M$. The following valid inequalities hold :

$$\sum_{j \in J} (y_{i,j} + p_{ij}) \times p_{ij} \geq SC_i, \tag{5.24}$$

where SC_i is equal to the sum of $C_{i,j} \times p_{ij}$ by varying j, and $C_{i,j}$ is the completion time of $O_{i,j}$ in any sequence.

Having $\tilde{y}_{i,j} = C_{max} - y_{i,j}$, and form the valid inequalities (5.24), we can derive that $\sum_{j \in J} p_{ij}(\tilde{y}_{i,j}) \geq SC_i$. Thus, we deduce that :

$$\sum_{j \in J} p_{ij}(C_{max} - y_{i,j}) \geq SC_i, \quad \forall i \in M, \tag{5.25}$$

Therefore, the following relation holds :

$$\sum_{j \in J} p_{ij} C_{max} \geq \sum_{j \in J} p_{ij} y_{i,j} + SC_i, \quad \forall i \in M, \tag{5.26}$$

These valid inequalities can lead to similar inequalities (5.23) for all jobs $j \in J$:

$$\sum_{i \in M} p_{ij} C_{max} \geq \sum_{i \in M} p_{ij} y_{i,j} + SC_j, \tag{5.27}$$

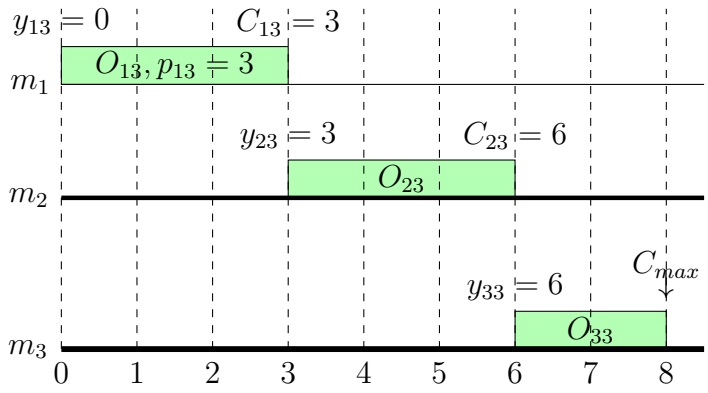


FIGURE 5.6 – Example : previous job inequalities between operations

Previous operations inequalities

Let consider the machine $i \in M$ and the job $j \in J$. The following inequalities are valid :

$$\sum_{j' \in J} p_{ij'} \bar{z}_{O_{i,j'}, O_{i,j}} \leq y_{i,j}, \tag{5.28}$$

By a similar reasoning, for every job $j \in J$ and every machine $i' \in M$, we introduce the following valid inequality :

$$\sum_{i \in M} p_{ij} \bar{z}_{O_{i,j}, O_{i',j}} \leq y_{i',j}, \tag{5.29}$$

By symmetry, we can derive the following two families of inequalities. Figure 5.6 presents a numerical example to illustrate the previous operations inequalities.

As we proposed in the first section, the following inequalities, based on precedence reasoning, hold :

$$C_{max} \geq p_{ij} + y_{ij} + \sum_{j' \in J} p_{ij'} \bar{z}_{o_{ij} o_{ij'}}, \quad \forall j \in J, \forall i \in M, \tag{5.30}$$

$$C_{max} \geq p_{ij} + y_{ij} + \sum_{i' \in M} p_{i'j} \bar{z}_{o_{ij} o_{i'j}}, \quad \forall j \in J, \forall i \in M, \tag{5.31}$$

FIGURE 5.7 – Logical implication on one machine.

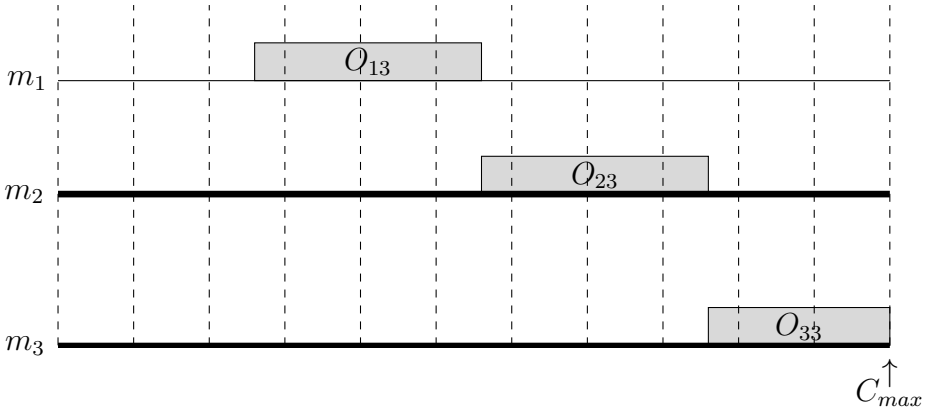


FIGURE 5.8 – Logical implication between operations.

Logical implication inequalities

Here, we adapt the logical implication for the operations.Le us consider machine $i \in M$ and jobs $j_1, j_2, j_3 \in J$. We introduce the following valid inequalities :

$$\bar{z}_{O_{i,j_1},O_{i,j_2}} + \bar{z}_{O_{i,j_2},O_{i,j_3}} \leq 1 + \bar{z}_{O_{i,j_1},O_{i,j_3}}, \tag{5.32}$$

This implication is related to the same machine for different operations. Figure 5.7 illustrates the idea behind : when operation O_{11} is before operation O_{12} and operation O_{12} is before operation O_{13}, then operation O_{11} is before operation O_{13}.

This implication can be also applied to the operations as in Figure 5.8.
Considering job $j \in J$ and the machine $i_1, i_2, i_3 \in M$, we introduce the following valid inequalities :

$$\bar{z}_{O_{i_1,j},O_{i_2,j}} + \bar{z}_{O_{i_2,j},O_{i_3,j}} \leq 1 + \bar{z}_{O_{i_1,j},O_{i_3,j}}, \tag{5.33}$$

In the next subsection, we present the computational experiments performed to test the effectiveness of the proposed inequalities.

5.3.3 Experimental results

The experimental results were performed for several sets of instances. First, we report the instances of Taillard. In these instances, every operation is assigned to a given machine without considering the order of the operations. Detailed description about this instances is available in [109]. The entries of the table are :

- name : name of the instance
- method : 0 basic model ;1 sequence inequalities, and logical implication ;2 conditional inequalities ; 3 previous inequalities ;4 logical implication ;5 all inequalities[1 to 4] ; 6 only conditional and previous job ; 7 : add(1,3,4) ; 8 : add(1,2,3,4) ; 9 :add(1,3) ; 10 : add(1,2,3).
- CPU : cpu time (limit 1 hour).
- LB ILP : lower bound found by our model.
- UB ILP : upper bound found by our model.
- LB : lower bound given in the file of the instance.
- UB : upper bound given in the file of the instance.
- Nb Nodes : number of nodes in the branching tree.
- Status : AbortTimeLim : No solution found within the 1 hour ; Optimal : found optimal solution ; MemLimInfeas : The limited memory not enough for more branching.

In order to show the efficiency of our inequalities and to see the difference between all inequalities we used CPLEX 12.4 solver with JAVA on DualCorei7, CPU 2.4 GHz. Tables 5.2, 5.3, 5.4 and 5.5 report the obtained results for the Open Shop model and its valid inequalities for different sized problems. In Table 5.2, it is clear that the basic model does not solve all problems within the limited resources (time and RAM). When we add the conditional and previous inequalities, the solutions for all instances are obtained within reasonable computation time. However, when we add the conditional inequalities alone, or the previous inequalities alone, they do not solve all the problems, but they just solve 50% of 77 problems. The logical implication inequalities do not show an important improvement to the basic model when these inequalities are added alone. We can also see that when we add all the inequalities, all problems are solved within the best CPU time. From these results, we can say that the addition of all the families of inequalities improved the computation time of the basic model. In Table 5.3, we tested another combination of inequalities to solve the instances (10 10). From the obtained results, we can also observe that all the problems have been solved except one problem (not solved within one hour)

when we add the combination of inequalities (sequence, previous and logical implication). It is also worthy to notice that the best CPU time obtained by the combination of the sequence and previous inequalities. Table 5.4 reports the results for instances (15 15). According to these results, the related problems are not easy to solve within the limited time and the memory space. Thus, few number of problems have been solved. The combination of the sequence and previous inequalities allows us to obtain some optimal solutions. It is also the same case, when we add to this combination the conditional inequalities (i.e., we obtained the same number of the optimal solutions). The gap between the obtained UB ILP and LB ILP does not exceed 5%.

Table 5.5 reports the results for instances of (20 20). Within the adjusted time to one hour, no problem has been solved for this size. But, we can see the small gaps between the UB and LB obtained by the ILP. All of them appear when we add the combination of sequence and previous inequalities.

5.4 Conclusion

In this chapter, we considered two parallel-machine scheduling problems. The first problem is the generalized open shop with disjunctive constraints. To the best of our knowledge there is no mathematical model for this problem. The second problem is the standard open shop problem. The aim is to minimize the maximum completion time. To deal with these problems, two mathematical models were proposed. We derived some classes of valid inequalities. We also added the interval subgraph separation algorithms for the generalized open shop problem. We adapted the derived valid inequalities to the open shop mathematical formulation. The results on some instances show that the basic models can solve the small size instances to optimality, and the generalized model shows the efficiency of adding the cutting plane inequalities proposed in Chapter 4, which can divide the gaps by 2 to 5. The derived valid inequalities show an important improvement of the computation time for the two models. We believe that the design of new classes of valid inequalities and the incorporation of further separation algorithms will improve the computation times of our models and it will lead to optimal solutions for some of the unsolved instances.

name	method	CPU	LB ILP	UB ILP	lb	ub	nodes	Status
taillard7 7 0	0	3600	410	435	435	438	2280194	AbortTimeLim
taillard7 7 0	1	12	435	435	435	438	20001	Optimal
taillard7 7 0	2	1288	435	435	435	438	1417451	OptimalTol
taillard7 7 0	3	136	435	435	435	438	126604	Optimal
taillard7 7 0	4	3600	427	435	435	438	3953672	AbortTimeLim
taillard7 7 0	5	2	435	435	435	438	1482	Optimal
taillard7 7 0	6	55	435	435	435	438	84226	Optimal
taillard7 7 1	0	3600	436	443	443	449	3877579	AbortTimeLim
taillard7 7 1	1	5	443	443	443	449	7161	Optimal
taillard7 7 1	2	3600	431	443	443	449	3141616	AbortTimeLim
taillard7 7 1	3	227	443	443	443	449	252160	Optimal
taillard7 7 1	4	3600	436	443	443	449	1755340	AbortTimeLim
taillard7 7 1	5	6	443	443	443	449	3029	Optimal
taillard7 7 1	6	305	443	443	443	449	276375	Optimal
taillard7 7 2	0	1394	468	468	468	479	1600501	OptimalTol
taillard7 7 2	1	470	468	468	468	479	1065452	Optimal
taillard7 7 2	2	3600	429	468	468	479	2763775	AbortTimeLim
taillard7 7 2	3	14	468	468	468	479	10898	Optimal
taillard7 7 2	4	109	468	468	468	479	91748	OptimalTol
taillard7 7 2	5	30	468	468	468	479	15622	Optimal
taillard7 7 2	6	26	468	468	468	479	14559	Optimal
taillard7 7 3	0	52	463	463	463	467	61095	OptimalTol
taillard7 7 3	1	25	463	463	463	467	55909	Optimal
taillard7 7 3	2	3600	457	463	463	467	3753894	AbortTimeLim
taillard7 7 3	3	3	463	463	463	467	3605	Optimal
taillard7 7 3	4	3600	439	463	463	467	1960948	AbortTimeLim
taillard7 7 3	5	5	463	463	463	467	2717	Optimal
taillard7 7 3	6	7	463	463	463	467	4017	Optimal
taillard7 7 4	0	96	416	416	416	419	113490	OptimalTol
taillard7 7 4	1	7	416	416	416	419	11797	Optimal
taillard7 7 4	2	111	416	416	416	419	108925	Optimal
taillard7 7 4	3	4	416	416	416	419	3190	Optimal
taillard7 7 4	4	1377	416	416	416	419	1467035	OptimalTol
taillard7 7 4	5	2	416	416	416	419	1659	Optimal
taillard7 7 4	6	5	416	416	416	419	4074	Optimal
taillard7 7 5	0	3600	439	451	451	460	2153465	AbortTimeLim
taillard7 7 5	1	893	451	451	451	460	1702917	Optimal
taillard7 7 5	2	104	451	451	451	460	157809	Optimal
taillard7 7 5	3	120	451	451	451	460	158977	Optimal
taillard7 7 5	4	3600	401	451	451	460	1839871	AbortTimeLim
taillard7 7 5	5	34	451	451	451	460	19285	Optimal
taillard7 7 5	6	33	451	451	451	460	20461	Optimal
taillard7 7 6	0	1143	422	422	422	435	962496	OptimalTol
taillard7 7 6	1	169	422	422	422	435	307675	Optimal
taillard7 7 6	2	169	422	422	422	435	152148	Optimal
taillard7 7 6	3	10	422	422	422	435	11140	Optimal
taillard7 7 6	4	3600	404	422	422	435	1694336	AbortTimeLim
taillard7 7 6	5	21	422	422	422	435	12358	Optimal
taillard7 7 6	6	37	422	422	422	435	28226	Optimal
taillard7 7 7	0	35	424	424	424	426	60107	OptimalTol
taillard7 7 7	1	0	424	424	424	426	0	Optimal
taillard7 7 7	2	3600	420	424	424	426	4260918	AbortTimeLim
taillard7 7 7	3	34	424	424	424	426	67727	Optimal
taillard7 7 7	4	26	424	424	424	426	28765	OptimalTol
taillard7 7 7	5	1	424	424	424	426	521	Optimal
taillard7 7 7	6	78	424	424	424	426	127959	Optimal
taillard7 7 8	0	254	458	458	458	460	198988	OptimalTol
taillard7 7 8	1	0	458	458	458	460	883	Optimal
taillard7 7 8	2	2401	458	458	458	460	2874228	Optimal
taillard7 7 8	3	1	458	458	458	460	1224	Optimal
taillard7 7 8	4	1422	458	458	458	460	931297	OptimalTol
taillard7 7 8	5	2	458	458	458	460	1433	Optimal
taillard7 7 8	6	2	458	458	458	460	1768	Optimal
taillard7 7 9	0	120	398	398	398	400	144714	OptimalTol
taillard7 7 9	1	0	398	398	398	400	731	Optimal
taillard7 7 9	2	3600	369	398	398	400	1984622	AbortTimeLim
taillard7 7 9	3	4	398	398	398	400	4252	Optimal
taillard7 7 9	4	15	398	398	398	400	15112	OptimalTol
taillard7 7 9	5	0	398	398	398	400	0	Optimal
taillard7 7 9	6	2	398	398	398	400	3047	Optimal

TABLE 5.2 – Number of optimal solutions obtained.

name	method	time	LB ILP	UB ILP	lb	ub	nodes	status
taillard10 10 0	7	370	637	637	637	652	60177	Optimal
taillard10 10 0	8	2767	637	637	637	652	509746	Optimal
taillard10 10 0	9	1320	637	637	637	652	716200	Optimal
taillard10 10 0	10	318	637	637	637	652	59542	Optimal
taillard10 10 1	7	3600	588	589	588	596	263243	AbortTimeLim
taillard10 10 1	8	138	588	588	588	596	22275	Optimal
taillard10 10 1	9	17	588	588	588	596	5115	Optimal
taillard10 10 1	10	27	588	588	588	596	7054	Optimal
taillard10 10 2	7	3502	598	598	598	617	207390	Optimal
taillard10 10 2	8	735	598	598	598	617	76285	Optimal
taillard10 10 2	9	128	598	598	598	617	68884	Optimal
taillard10 10 2	10	85	598	598	598	617	31821	Optimal
taillard10 10 3	7	36	577	577	577	581	4533	Optimal
taillard10 10 3	8	16	577	577	577	581	3099	Optimal
taillard10 10 3	9	18	577	577	577	581	8237	Optimal
taillard10 10 3	10	21	577	577	577	581	4493	Optimal
taillard10 10 4	7	136	640	640	640	657	21179	Optimal
taillard10 10 4	8	429	640	640	640	657	90436	Optimal
taillard10 10 4	9	27	640	640	640	657	10299	Optimal
taillard10 10 4	10	212	640	640	640	657	107261	Optimal
taillard10 10 5	7	45	538	538	538	545	4758	Optimal
taillard10 10 5	8	104	538	538	538	545	16536	Optimal
taillard10 10 5	9	15	538	538	538	545	3038	Optimal
taillard10 10 5	10	24	538	538	538	545	5524	Optimal
taillard10 10 6	7	29	616	616	616	623	2292	Optimal
taillard10 10 6	8	103	616	616	616	623	14769	Optimal
taillard10 10 6	9	269	616	616	616	623	146553	Optimal
taillard10 10 6	10	22	616	616	616	623	6938	Optimal
taillard10 10 7	7	80	595	595	595	606	11411	Optimal
taillard10 10 7	8	107	595	595	595	606	13793	Optimal
taillard10 10 7	9	17	595	595	595	606	4923	Optimal
taillard10 10 7	10	69	595	595	595	606	25884	Optimal
taillard10 10 8	7	59	595	595	595	606	9833	Optimal
taillard10 10 8	8	237	595	595	595	606	31298	Optimal
taillard10 10 8	9	57	595	595	595	606	27753	Optimal
taillard10 10 8	10	115	595	595	595	606	47695	Optimal
taillard10 10 9	7	48	596	596	596	604	6100	Optimal
taillard10 10 9	8	237	596	596	596	604	40515	Optimal
taillard10 10 9	9	11	596	596	596	604	4510	Optimal
taillard10 10 9	10	28	596	596	596	604	9797	Optimal

TABLE 5.3 – Number of optimal solutions obtained.

name	method	time	LB ILP	UB ILP	gap	lb	ub	nodes	status
taillard15 15 0	7	1233	937	956	1.99	937	956	16338	MemLimFeas
taillard15 15 0	8	3572	937	937	0.00	937	956	72686	Optimal
taillard15 15 0	9	3600	937	947	1.06	937	956	517809	AbortTimeLim
taillard15 15 0	10	2638	937	937	0.00	937	956	475219	Optimal
taillard15 15 1	7	3600	918	939	2.24	918	957	34301	AbortTimeLim
taillard15 15 1	8	3600	918	954	3.77	918	957	59266	AbortTimeLim
taillard15 15 1	9	1550	918	949	3.27	918	957	76726	MemLimFeas
taillard15 15 1	10	3600	918	933	1.61	918	957	610624	AbortTimeLim
taillard15 15 2	7	952	871	897	2.90	871	899	16983	MemLimFeas
taillard15 15 2	8	3600	871	896	2.79	871	899	54762	AbortTimeLim
taillard15 15 2	9	462	871	871	0.00	871	899	82995	Optimal
taillard15 15 2	10	3600	871	872	0.11	871	899	505148	AbortTimeLim
taillard15 15 3	7	1450	934	942	0.85	934	946	27800	MemLimFeas
taillard15 15 3	8	3600	934	937	0.32	934	946	52356	AbortTimeLim
taillard15 15 3	9	3600	934	940	0.64	934	946	364704	AbortTimeLim
taillard15 15 3	10	1518	934	934	0.00	934	946	255462	Optimal
taillard15 15 4	7	3600	946	990	4.44	946	992	19375	AbortTimeLim
taillard15 15 4	8	3600	946	990	4.44	946	992	54201	AbortTimeLim
taillard15 15 4	9	3600	946	949	0.32	946	992	222656	AbortTimeLim
taillard15 15 4	10	3600	946	969	2.37	946	992	577703	AbortTimeLim
taillard15 15 5	7	1336	933	956	2.41	933	959	20458	MemLimFeas
taillard15 15 5	8	3600	933	955	2.30	933	959	48663	AbortTimeLim
taillard15 15 5	9	299	933	933	0.00	933	959	54583	Optimal
taillard15 15 5	10	3525	933	933	0.00	933	959	557028	Optimal
taillard15 15 6	7	793	891	928	3.99	891	931	14393	MemLimFeas
taillard15 15 6	8	3600	891	925	3.68	891	931	92441	AbortTimeLim
taillard15 15 6	9	3600	891	893	0.22	891	931	948307	AbortTimeLim
taillard15 15 6	10	3600	891	897	0.67	891	931	993133	AbortTimeLim
taillard15 15 7	7	619	893	916	2.51	893	916	14173	MemLimFeas
taillard15 15 7	8	3600	893	900	0.78	893	916	94041	AbortTimeLim
taillard15 15 7	9	455	893	893	0.00	893	916	108151	Optimal
taillard15 15 7	10	1613	893	893	0.00	893	916	414890	Optimal
taillard15 15 8	7	3600	899	911	1.32	899	951	95415	AbortTimeLim
taillard15 15 8	8	3600	899	943	4.67	899	951	109120	AbortTimeLim
taillard15 15 8	9	3600	899	900	0.11	899	951	877150	AbortTimeLim
taillard15 15 8	10	3600	899	914	1.64	899	951	1046262	AbortTimeLim
taillard15 15 9	7	631		12226					MemLimInfeas
taillard15 15 9	8	3600	902	931	3.11	902	935	99270	AbortTimeLim
taillard15 15 9	9	431	902	902	0.00	902	935	98569	Optimal
taillard15 15 9	10	3600	902	912	1.10	902	935	1166219	AbortTimeLim

TABLE 5.4 – Number of optimal solutions obtained.

name	method	time	LB ILP	UB ILP	gap	lb	ub	nodes	status
taillard20 20 0	7	3600		1210		1155	1215		AbortTimeLim
taillard20 20 0	8	3091				1155	1215		MemLimInfeas
taillard20 20 0	9	440				1155	1215		MemLimInfeas
taillard20 20 0	10	3600	1155	1203	4.16	1155	1215	418184	AbortTimeLim
taillard20 20 1	7	894				1241	1332		MemLimInfeas
taillard20 20 1	8	3602				1241	1332		MemLimInfeas
taillard20 20 1	9	882	1241	1331	7.25	1241	1332	26232	MemLimFeas
taillard20 20 1	10	3600	1241	1331	7.25	1241	1332	95021	AbortTimeLim
taillard20 20 2	7	3600				1257	1294		AbortTimeLim
taillard20 20 2	8	731				1257	1294		MemLimInfeas
taillard20 20 2	9	270	1257	1292	2.78	1257	1294	17669	MemLimFeas
taillard20 20 2	10	3600				1257	1294		AbortTimeLim
taillard20 20 3	7	2412				1248	1310		MemLimInfeas
taillard20 20 3	8	929				1248	1310		MemLimInfeas
taillard20 20 3	9	689	1248	1309	4.89	1248	1310	20594	MemLimFeas
taillard20 20 3	10	3600	1248	1310	4.97	1248	1310	92297	AbortTimeLim
taillard20 20 4	7	3600				1256	1301		AbortTimeLim
taillard20 20 4	8	2929				1256	1301		MemLimInfeas
taillard20 20 4	9	3600	1256	1300	3.50	1256	1301	282077	AbortTimeLim
taillard20 20 4	10	3600				1256	1301		AbortTimeLim
taillard20 20 5	7	3600				1204	1252		AbortTimeLim
taillard20 20 5	8	2841				1204	1252		MemLimInfeas
taillard20 20 5	9	310	1204	1242	3.16	1204	1252	18846	MemLimFeas
taillard20 20 5	10	3262	1204	1250	3.82	1204	1252	415290	MemLimFeas
taillard20 20 6	7	3600				1294	1352		AbortTimeLim
taillard20 20 6	8	2778				1294	1352		MemLimInfeas
taillard20 20 6	9	400	1294	1352	4.48	1294	1352	18467	MemLimFeas
taillard20 20 6	10	3600	1294	1351	4.40	1294	1352	94174	AbortTimeLim
taillard20 20 7	7	3600				1169	1269		AbortTimeLim
taillard20 20 7	8	1295				1169	1269		MemLimInfeas
taillard20 20 7	9	3600				1169	1269		AbortTimeLim
taillard20 20 7	10	3600	1169	1258	7.61	1169	1269	107821	AbortTimeLim
taillard20 20 8	7	3600				1289	1322		AbortTimeLim
taillard20 20 8	8	2611				1289	1322		MemLimInfeas
taillard20 20 8	9	572	1289	1322	2.56	1289	1322	20666	MemLimFeas
taillard20 20 8	10	3600	1289	1322	2.56	1289	1322	102128	AbortTimeLim
taillard20 20 9	7	3600				1241	1284		AbortTimeLim
taillard20 20 9	8	806				1241	1284		MemLimInfeas
taillard20 20 9	9	413	1241	1280	3.14	1241	1284	18195	MemLimFeas
taillard20 20 9	10	3600	1241	1284	3.46	1241	1284	106041	AbortTimeLim

TABLE 5.5 – Number of optimal solutions obtained.

6

General Conclusion and Perspectives

Due to increasing demand, scheduling in cloud environment attracted much attention in recent years. Several scheduling problems have been recently addressed in cloud computing research field. Nonetheless, scheduling in cloud computing still lacks some research efforts, because of the everyday growing of this recent technology.

In this thesis, we consider the scheduling problem in cloud computing, which is formulated as an unrelated parallel-machine scheduling problem under precedence constraints (URPMPC) and as a generalized case of open shop problem under disjonctive constraints.

For solving these problems we applied different combinatorial optimization techniques. To attack URPMPC, we proposed several genetic algorithms for job scheduling problem in cloud computing with the objective of minimizing the makespan (C_{max}). We also worked on different exact approaches. We studied the mathematical formulations that are found in literature. Moreover, we proposed a novel mathematical formulations based on interval graph. The main difference between these formulations is the way the makespan has been linearized. The facial structure of the polytope generated by interval graph model is investigated to define some facets. There is an important piece of information contained in this model but it cannot know if the jobs run at the same time or not. There are some applications that require such an information, especially in some cloud computing security based models. We improved the interval graph formulation by adding valid inequalities based on the forbidden interval subgraph and some heuristics to solve the clique problem or find the smallest non interval subgraphs. Furthermore, we proposed other inequalities based on SPT (Shortest Processing Time). Some heuristics are used to separate the inequality based on the SPT. We also present a polyhedral study for the problem of interval and m-clique free graphs. A polyhedral investigation of the convex hull of

these vectors yielded several results on inequalities defining facets for this new polytope. We have also applied the obtained results to the problem of unrelated parallel-machine with disjunctive constraints. We designed and implemented a branch-and-cut algorithms based on families of strong valid inequalities presented in this chapter. We separate some forbidden subgraphs. The results in heuristics and metaheuristic show that the performances of our proposed genetic algorithms have been compared against one of the best existing genetic algorithm for the same problem. After extensive comparisons, we can conclude that the proposed algorithms can improve the solutions obtained. In the exact solutions, we studied and evaluated the effectiveness and efficiency of our model and the other models. Intensive numerical experiments are conducted. The obtained results show the effectiveness of the interval graph formulation compared to the available formulations except the order formulation. Computational results show that the addition of the valid inequalities decreases the computational requirements to obtain the optimal solution in many cases.

Finally, we considered the generalized open shop with disjunctive constraints and the open shop scheduling problem. To the best of our knowledge there is no mathematical model for this problem. To tackle these problems, two mathematical models are constructed. We derived some classes of valid inequalities. We also add the interval subgraph separation algorithms for the generalized open shop problem. We adapted the derived valid inequalities to the open shop mathematical formulation. The results on some instances show that the basic models can solve the small size instances to optimality. The generalized model shows the efficiency of adding the cutting plane inequalities and the reduction of the gap between upper and lower bounds. The derived valid inequalities show a good improvement of the computational time for the two models.

We can conclude that, in this thesis we developed different approaches to tackle the scheduling problem in cloud computing. Three novel genetic algorithms based and three novel mathematical models with interesting theoretical results in polyhedral analysis were proposed. The class of scheduling problems in cloud computing has different perspectives in terms of optimization criteria (minimization of total completion time and other objectives). Moreover, we need to develop other heuristics for the branch and cut, to derive new classes of valid inequalities and to incorporate further separation algorithms, in order to improve the computation time of our models. Furthermore, for the theoretical point of view, the identification of further relaxations to define more facets will be interesting in such problems. It will be also interesting to work on branch and price methods. Finally,

another interesting topic regarding scheduling problem in cloud computing is to consider the multiobjective optimization context.

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