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Motions of Julia sets and dynamical stability in several complex variables

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Abstract

In this thesis we study holomorphic dynamical systems depending on parameters. Our main goal is to contribute to the establishment of a theory of stability and bifurcation in several complex variables, generalizing the one for rational maps based on the seminal works of Mañé, Sad, Sullivan and Lyubich.

For a family of polynomial like maps, we prove the equivalence of several notions of stability, among the others an asymptotic version of the holomorphic motion of the repelling cycles and of a full-measure subset of the Julia set. This can be seen as a measurable several variables generalization of the celebrated λ-lemma and allows us to give a coherent definition of stability in this setting. Once holomorphic bifurcations are understood, we turn our attention to the Hausdorff continuity of Julia sets. We relate this property to the existence of Siegel discs in the Julia set, and give an example of such phenomenon. Finally, we approach the continuity from the point of view of parabolic implosion and we prove a two-dimensional Lavaurs Theorem, which allows us to study discontinuities for perturbations of maps tangent to the identity.

Résumé

Dans cette thèse, on s'intéresse aux systèmes dynamiques holomorphes dépendants de paramètres. Notre objectif est de contribuer à une théorie de la stabilité et des bifurcations en plusieurs variables complexes, généralisant celle des applications rationnelles fondées sur les travaux de Mañé, Sad, Sullivan et Lyubich.

Pour une famille d'applications d'allure polynomiale, on prouve l'équivalence de plusieurs notions de stabilité, entre autres une version asymptotique du mouvement holomorphe des cycles répulsifs et d'un sous-ensemble de l'ensemble de Julia de mesure pleine. Cela peut être considéré comme une généralisation mesurable à plusieurs variables du célèbre λ-lemme et nous permet de dégager un concept cohérent de stabilité dans ce cadre. Après avoir compris les bifurcations holomorphes, on s'intéresse à la continuité Hausdorff des ensembles de Julia. Nous relions cette propriété à l'existence de disques de Siegel dans l'ensemble de Julia, et donnons un exemple de ce phénomène. Finalement, on étudie la continuité du point de vue de l'implosion parabolique. Nous établissons un théorème de Lavaurs deux-dimensionel, ce qui nous permet d'étudier des phénomènes de discontinuité pour des perturbations d'applications tangentes à l'identité.

Sunto

In questa tesi, studiamo sistemi dinamici olomorfi dipendenti da un parametro, con l'obiettivo di contribuire ad una teoria di stabilità e biforcazione a più variabili complesse che generalizzi quella per frazioni razionali basata sui lavori di Mañé, Sad, Sullivan e Lyubich.

Per una famiglia di polynomial-like maps dimostriamo l'equivalenza di numerose nozioni di stabilità, tra le quali una versione asintotica del movimento olomorfo dei cicli repulsivi e di un sottoinsieme di misura piena dell'insieme di Julia. Questo può essere visto come una generalizzazione misurabile a più variabili del famoso λ-lemma e ci permette di dare una definizione coerente di stabilità in questo contesto. Dopo aver compreso le biforcazioni olomorfe, ci interessiamo alla continuità Hausdorff degli insiemi di Julia. Mettiamo in relazione questa proprietà con l'esistenza di dischi di Siegel nell'insieme di Julia, e diamo un esempio di questo fenomeno. Infine, studiamo la continuità dal punto di vista dell'implosione parabolica, e dimostriamo un teorema di Lavaurs a due variabili. Questo ci permette di studiare dei fenomeni di discontinuità per perturbazioni di mappe tangenti all'identità.

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You have your paintbrush, you have your colors.
Paint Paradise, then in you go...

NIKOS KAZANTZAKIS

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Introduction

This thesis is devoted to the study of holomorphic dynamical systems depending on parameters. Our main goal is to contribute to the establishment of a theory of stability and bifurcation in several complex variables, generalizing the one for rational maps based on the seminal works of Mañé, Sad, Sullivan and Lyubich.

The precise objects we will study are holomorphic families of polynomial-like maps. These maps naturally appear when dealing with semi-local problems within holomorphic dynamical systems and must be thought of as a generalization of the endomorphisms of $\mathbb{P}^k$ (the higher dimensional analogue of rational maps). Let us stress that they also provide a bridge towards the understanding of the (more complicated) dynamics of transcendental maps in several complex variables.

In the first part of this thesis we generalize to this setting results already due to Berteloot and Dupont for families of endomorphisms of $\mathbb{P}^k$. We prove that several reasonable notions of dynamical stability are actually equivalent, among the others the harmonicity of the Lyapounov function and the existence of *equilibrium webs*. This is a notion of dynamical stability for the equilibrium measures, which implies a weak version of holomorphic motion for the Julia sets. The main difficulty here is the lack of a Green function in this more general situation. While some proofs will be essentially adaptations of the ones on $\mathbb{P}^k$, others require to completely rethink the strategy of proof. We exploit here some tools introduced by Dinh-Sibony and Pham.

We then proceed to the main result of the thesis: we establish that the holomorphic motion of repelling points contained in the Julia set is enough to recover the holomorphic motion of a full-measure subset of the Julia sets. While this fact is ensured in dimension 1 by the celebrated λ-lemma, in several complex variables the failure of Hurwitz Theorem means that motions of different points may intersect. The goal of this part is thus to prove that, up to removing a negligible set of these motions, we can still recover a holomorphic motion of almost all the Julia set. This, together with the results of the previous part, allows us to give a coherent definition of stability and bifurcation for polynomial-like maps (and for endomorphisms of $\mathbb{P}^k$). The proof of this fact, and in general all our approach to stability and bifurcations, heavily relies on ergodic theory and on the use of plurisubharmonic functions, a point of view initiated in dimension 1 by Brolin (for single endomorphisms) and De Marco (for the study of bifurcations within families). We then show how to recover an *asymptotic* motion of the repelling cycles from the other equivalent notions of stability. Let us stress that the results in this part are new also in the setting of endomorphisms of $\mathbb{P}^k$, and constitute our main contribution to the subject.

Once holomorphic bifurcations are understood, we turn our attention to the Hausdorff continu-

ity of Julia sets. Following Berteloot and Dupont, we prove that higher dimensional Siegel discs – which are an obstruction to the existence of holomorphic motions, as in dimension 1 – may be seen as an obstruction to continuity, too. For some particular families we prove that, provided that these Siegel discs are outside the Julia set (as in dimension 1), the continuity of the Julia set would imply the holomorphic motion. Although the fact that Siegel discs are disjoint from the Julia set may seem a reasonable request, we prove that this is not the case in general. This shows that the question about the equivalence of continuity and holomorphic stability may be more difficult than expected.

Although the strategy based on Siegel discs seems to fail in the general situation, there is another way to attack this problem in dimension one: the theory of parabolic implosion. This rapidly became one of the cornerstones of the one-dimensional theory after the foundational works by Lavaurs and Douady. While this theory is today well developed and applied in dimension one, there is not at present an analogue in several complex variables. Here the theory begins with recent results in the semi-attracting setting, due to Bedford-Smillie-Ueda and Dujardin-Lyubich. The goal of the last part of this work is then to provide an analogous treatement in the completely parabolic setting, by a precise study of perturbations of endomorphisms of $\mathbb{C}^2$ tangent to the identity at the origin. As an application, we get estimates for the discontinuity of the large Julia set (i.e., the complement of the Fatou set) and (for regular polynomials) of the filled Julia set.

A brief survey of the context

The one-dimensional theory Given a dynamical system, it is reasonable to try to decompose it into two complementary subsets: a first one, where the dynamics is stable (i.e., the ω-limit of a point depends continuously on the point itself), and its complement, where the asymptotic dynamics is sensitive to the initial conditions. Without additional assumptions, it is difficult to say more about this decomposition. It is thus fair enough to say that *global holomorphic dynamics* as it is known today started when, in 1917, Fatou and Julia [Fat19, Fat20a, Fat20b, Jul68] had the intuition to try and apply Montel Theorem – stating that an equibounded family of holomorphic functions is equicontinuous – to the family of the iterates of a rational map. This gives a powerful criterion for stability, which leads to the following definition.

Definition 1. *Let f be a rational map. The* Fatou set *is the maximal open set where the family of iterates $\{f^n\}$ is normal. The* Julia *set J is the complement of the Fatou set.*

The Fatou and Julia sets are easily seen to be completely invariant. Moreover (see, e.g., [Zal98, BD00, Mil06]), an application of Montel Theorem and Zalcman renormalization Lemma [Zal75] gives the following equivalent characterizations of the Julia set.

Theorem 2 (Fatou, Julia). *Let f be a rational map. Then*

- *the repelling periodic points belong to the Julia set J, and J is the closure of these points; and*
- *with at most two exceptions, the preimages of any point on the Riemann sphere accumulate all the Julia set.*

Let us now consider a *holomorphic family* f_λ of rational maps, i.e., let us assume a holomorphic dependence of our system from a parameter λ in some complex manifold M. We want to study the following question:

$$\boxed{\text{how does the Julia set } J_\lambda \text{ vary with } \lambda?}$$

Since repelling points are dense in J, it is natural to approach this question by studying how the repelling points behave under perturbation. The central definition here is the following notion of *holomorphic motion*.

Definition 3. *Let E be a subset of the Riemann sphere, Ω be a complex manifold and $\lambda_0 \in \Omega$. A* holomorphic motion *of E over Ω centered at λ_0 is a map*

$$\begin{array}{rcl} h\colon \Omega \times E & \to & \mathbb{P}^1 \\ (\lambda, z) & \mapsto & h_\lambda(z) \end{array}$$

such that

- $h_{\lambda_0} = id|_E$*;*
- $E \ni z \mapsto h_\lambda(z)$ *is one-to-one for every* $\lambda \in \Omega$*;*
- $\Omega \ni \lambda \mapsto h_\lambda(z)$ *is holomorphic for every* $z \in E$*.*

A holomorphic motion is thus a *holomorphic family of injections* from E to $\mathbb{P}^1$, parametrized by the manifold Ω. In particular, it gives a lamination on the subset $\cup_\lambda (\{\lambda\} \times J_\lambda)$ of the product space $\Omega \times \mathbb{P}^1$. One important property of any holomorphic motion is the fact that it automatically extends to the closure of E. This is the content of the so-called λ-lemma, which is (once again) a consequence of Picard-Montel Theorem, combined with Hurwitz Theorem.

Lemma 4 (λ-lemma, Mañé-Sad-Sullivan [MSS83])**.** *A holomorphic motion h of E extends to a holomorphic motion $\overline{h}$ of $\overline{E}$ (and moreover $\overline{h}$ is continuous).*

The idea of this lemma is the following. Take any $z \in \overline{E}$. We want to define a *unique* holomorphic motion for this point, coherent with the ones of the points of E. To do this, we can approximate z with points $z_n \in E$, and consider the motions $h_{z_n}(\lambda)$ of these points. By means of Picard-Montel Theorem, the h_{z_n}'s form a normal family, and we can thus define a motion h_z of z as a limit of this sequence. To conclude, we just need to ensure that this limit is unique. But this, since we are in dimension 1, follows from an application of Hurwitz Theorem.

Let now $z_0, \dots, f_{\lambda_0}^{n-1}(z_0)$ be a repelling $n-$cycle for f_{λ_0}. By the implicit function theorem, there exists a holomorphic motion of this cycle on a neighbourhood Ω_{λ_0} of λ_0 that conjugates the dynamics ($f_\lambda \circ h_\lambda = h_\lambda \circ f_{\lambda_0}$). We say that the cycle *moves holomorphically* when this happens, or that there is a holomorphic motion of these points *as periodic repelling points*. The following theorem is then a consequence of the λ-lemma and the density of the repelling cycles in the Julia set.

Theorem 5 (Lyubich [Lyu83b], Mañé-Sad-Sullivan [MSS83])**.** *Let f_λ be a holomorphic family of rational maps. If the repelling cycles move holomorphically (as repelling points), then the Julia sets J_λ move holomorphically (and the motion conjugates the dynamics).*

This theorem allows one to give the following definition.

Definition 6. *The* stability locus *is the subset of the parameter space where the Julia set moves holomorphically. Its complement is the* bifurcation locus*.*

From the definition the stability locus is open, but it is not clear even that it should be non empty. It turns out that this set is actually dense in the parameter space. This is a consequence of the fact that the critical points are finite.

Dynamical stability as described above is strongly related to the critical dynamics. Indeed, it turns out that the behaviour of the critical orbits under perturbation completely determines the stability of a rational map. Let us assume for simplicity that the family has *marked critical points*. This means (since a rational map of degree d has $2d-2$ critical points, counting with multiplicity), having holomorphic functions $c_1, \dots, c_{2d-2}$ parametrizing the critical points. For every c_i, we can then consider the sequence of maps $\lambda \mapsto f_\lambda^n(c(\lambda))$.

Theorem 7 ([Lyu83b, MSS83])**.** *The bifurcation locus is the union of the non-normality loci of the critical orbits $f^n(c_i(\lambda))$.*

In order to continue the description of the picture in dimension one, we have to make a step back to 1965. In this year, Brolin proved the following quantitive version of the equidistribution of preimages for a polynomial map. This can be seen as the start of the use of potential-theoretic tools in holomorphic dynamics.

Theorem 8 (Brolin [Bro65])**.** *Let f be a polynomial map on $\mathbb{C}$ of degree $d \geq 2$. The equilibrium measure μ of the filled Julia set is ergodic, supported on the Julia set and*

$$\frac{1}{d^n} \sum_{b \in f^{-n}(a)} \delta_b \underset{n\to\infty}{\longrightarrow} \mu$$

for all $a \in \mathbb{C}$, with at most one exception.

This result was later generalized to rational maps by Lyubich, who also proved the analogous equidistribution result for the repelling cycles, as well as studied entropy properties of these systems.

Theorem 9 (Lyubich [Lyu83a])**.** *Let f be a rational map of degree $d \geq 2$. The equilibrium measure μ is the only measure of maximal entropy $\log d$. Moreover,*

$$\frac{1}{d^n} \sum_{p \in R_n} \delta_p \underset{n\to\infty}{\longrightarrow} \mu$$

where the sum is taken over n-periodic repelling points.

In order to explain the potential theoretic approach to bifurcations, we shall now focus on a polynomial family $(p_\lambda)_{\lambda \in M}$ of a given degree $d \geq 2$, i.e., a holomorphic function

$$\begin{array}{rccc} p\colon & M \times \mathbb{C} & \to & M \times \mathbb{C} \\ & (\lambda, z) & \mapsto & (\lambda, p_\lambda(z)) \end{array}$$

For simplicity, we assume that these polynomials are monic and that the critical points are marked as above. We need to introduce two objects. The first one is the *Green function of* p, defined as

$$G(\lambda, z) := \lim_{n\to\infty} d^{-n} \log^+ |p_\lambda^n(z)|\,,$$

where $\log^+(x) := \max(0, \log x)$. Since the convergence is locally uniform, the function G is plurisubharmonic (or psh, for brevity) on the product space $M \times \mathbb{C}$. Moreover, it is not difficult to check that $G(\lambda, z) = 0$ precisely on the *filled Julia sets*, where the sequence $p_\lambda^n(z)$ is bounded. The Julia set of p_λ is the boundary of the filled Julia set and its equilibrium measure μ_λ is given by $\mu_\lambda = dd_z^c G(\lambda, \cdot)$.

The second object we need is the *Lyapounov exponent*, defined as

$$L(\lambda) = \int \log |p'_\lambda(z)|\, \mu_\lambda(z).$$

It easily follows from Birkhoff Theorem that the Lyapounov exponent is the exponential rate of growth of $(p_\lambda^n)'(z)$, for μ_λ-almost every point z. A similar property holds also for the repelling cycles, although it is less obvious to establish. More precisely, we have

$$L(\lambda) = \lim_{n\to\infty} d^{-n} \sum_{z\in R_n} \frac{1}{n} \log \left|(p_\lambda^n)'(z)\right|$$

where the sum is taken over the repelling points of (exact) period n (see e.g., [Ber10]). Thus, the Lyapounov exponent is strictly related to the behaviour of the repelling cycles. It is then reasonable to try to detect their bifurcations by means of this function.

On the other hand, by its very definition, the Lyapounov exponent encodes a relation between the equilibrium measure (which is the laplacian of the Green function) and the critical orbit. This relation is made clear by the following formula, due to Przytycki [Prz85]:

$$L(\lambda) = \log d + \sum_{j=1}^{d-1} G(\lambda, c_j(\lambda)).$$

From this formula we readily deduce that L is psh, continuous and bounded from below by $\log d$. Applying dd_λ^c to it, we get the following fundamental relation.

$$dd_\lambda^c L(\lambda) = \sum_{j=1}^{d-1} dd_\lambda^c G(\lambda, c_j(\lambda))$$

deals with holomorphic motions of repelling cycles

detects the instability of the orbit of $c_j(\lambda)$

It follows immediately from this formula and Lyubich-Mañé-Sad-Sullivan Theorem that the support of $dd_\lambda^c L(\lambda)$ coincides with the bifurcation locus. Przytycki formula was later generalized to rational maps by De Marco [DeM01, DeM03], who also gave the following fundamental definition.

Definition 10. *The bifurcation current for a family or rational maps is*

$$T_{bif} = dd^c_\lambda L(\lambda).$$

Since L is psh, T_{bif} is a $(1,1)$-positive closed current on the parameter space M. For the quadratic family, it is a measure on the (one-dimensional) parameter space, which coincides with the equilibrium measure of the Mandelbrot set, whose boundary is the bifurcation locus. The bifurcation locus of a family of rational maps and the associated bifurcation current have been studied extensively. We refer to [Ber11] and [Duj11] for an account of techniques and results in this direction.

The Theorem that we seek to generalize to higher dimension is thus the following.

Theorem 11 (Lyubich, Mañé-Sad-Sullivan, De Marco)**.** *Let f be a holomorphic family of rational maps. Then the following are equivalent:*

1. *the repelling cycles move holomorphically;*
2. *the Julia sets move holomorphically;*
3. *the Lyapounov function is pluriharmonic.*

The higher dimensional setting Let us now move to several complex variables. The natural generalization of a rational map is an endomorphisms of $\mathbb{P}^k$. For such a map, we can still define (see [DS10]) a Green function G, but this time its laplacian is not a measure but a $(1,1)$-current. The object of interest here is the so-called Green current, given by $T := dd^c G + \omega_{FS}$, where ω_{FS} is the Fubini-Study form of $\mathbb{P}^k$. Since the Green function is locally continuous, it is meaningful to consider the measure given by

$$\mu := T^k.$$

This was first done by Fornaess and Sibony ([FS95, FS94]). Such a measure is ergodic and enjoys much of the properties of its one-dimensional counterpart. Indeed, by the work of Fornaess-Sibony, Briend-Duval [BD99, BD01] and Dinh-Sibony we know that μ is the only measure of maximal entropy and that both repelling cycles and preimages of generic points equidistribute this measure. We thus define the *Julia set* of an endomorphism of $\mathbb{P}^k$ to be the support of the measure μ. This is in general smaller than the complement of the Fatou set, which turns out to coincide with the support of the Green current. One difference with respect to the dimension 1 is worthy of particular attention: in general, we may have repelling points outside the Julia set. Examples of this phenomenon were given by Hubbard-Papadopol [HP94] and Fornaess-Sibony [FS01].

Let us now consider a family of endomorphisms of $\mathbb{P}^k$, parametrized by a complex manifold M. The first to adress the problem of studying bifurcations within such families have been Bassanelli and Berteloot [BB07], who generalized to this setting Przytycki-De Marco formula, thus relating the Lyapounov function and the critical dynamics. The study of the Lyapounov function had also been undertaken by Bedford and Smillie ([BS92, BS98]) for Hénon maps and by Bedford and Jonsson ([Jon99, BJ00]) for regular polynomial endomorphisms.

To introduce Bassanelli-Berteloot formula, let $G(\lambda, z)$ be the Green function for the family f. We can consider the k-power of its laplacian (added to the Fubini-Study form) on the product

space, given by

$$\mathcal{E}_{Green} = \left(dd^c_{\lambda,z} G + \omega_{FS}\right)^k .$$

This is a (k,k)-current on $M \times \mathbb{P}^k$, with the property that its *slice* at every λ (roughy speaking, the restriction to the vertical fiber) is the equilibrium measure of f_λ. Bassanelli-Berteloot formula is then the following (see [BB07]):

$$dd^c_\lambda L = (\pi_M)_* \left(\mathcal{E}_{Green} \wedge [C_f]\right).$$

Here $[C_f]$ denotes the integration current on the critical set of f (counting the topological multiplicity) and L is the sum of the Lyaponov exponents of f_λ. Heavily exploiting this formula, and the regularity properties of the Green function, Berteloot and Dupont proved the equivalence of various reasonable notions of stability, among the others the existence of a kind of holomorphic motion for the equilibrium measure, an *equilibrium web* (see Definition 18 below), implying the existence of a weak form of holomorphic motion for the Julia sets, where graphs corresponding to different points may a priori intersect. Here another crucial definition is the one of *Misiurewicz parameters*.

Definition 12. *A parameter $\lambda_0 \in M$ is called a* Misiurewicz parameter *if there exist a neighbourhood $N_{\lambda_0} \subset M$ of λ_0 and a holomorphic map $\sigma : N_{\lambda_0} \to \mathbb{P}^k$ such that:*

1. *for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ is a repelling periodic point;*
2. *$\sigma(\lambda_0) \in J_{\lambda_0}$;*
3. *there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0)) \in f^{n_0}(C)$;*
4. *$\sigma(N_{\lambda_0}) \not\subseteq f^{n_0}(C)$,*

where C is the critical set of f.

Their result can then be stated as follows.

Theorem 13 (Berteloot-Dupont [BD14b])**.** *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^k$. Then the following are equivalent:*

1. *the Lyapounov function is harmonic;*
2. *there are no Misiurewicz parameters;*
3. *there exists an equilibrium web.*

If $k = 2$, the above conditions are also equivalent to

4. *the repelling cycles in the Julia set move holomorphically.*

Let us mention here that, in the setting of dissipative Hénon maps, by completely differents methods a parallel study of stability has been undertaken by Berger, Dujardin and Lyubich (see [DL13, BD14a]).

Families of polynomial-like maps We now consider a holomorphic family of polynomial-like maps. These are proper holomorphic maps g from U to V, where $U \Subset V$ are open subsets of $\mathbb{C}^k$, with V convex (this assumption can be relaxed to V Stein). In this work we study bifurcations within families of such maps, being interested in the ones of *large topological degree*, i.e., the ones satisfying

$$d_{k-1}^* := \limsup_{n\to\infty} \sup_S \|(g^n)_* (S)\|_U^{1/n} < d_t,$$

where the sup is taken over all positive closed $(1,1)$-currents of mass less or equal than 1 on U. In particular, notice that the topological degree d_t dominates the volume growth of the hypersurfaces.

Polynomial-like maps are a natural generalization of endomorphisms of $\mathbb{P}^k$ (which in turn can be studied as polynomial-like maps by considering a lift to $\mathbb{C}^{k+1}$). By the work of Dinh and Sibony [DS03, DS10], we know that the ones of large topological degree enjoy much of the properties introduced in the previous sections, namely the existence of an equilibrium measure of maximal entropy describing the asymptotic distribution of repelling points and preimages of generic points. On the other hand, the main tools in the study of endomorphisms of $\mathbb{P}^k$ are not available here: the Green function and current. As a consequence, the generalization of Przytycki-De Marco formula in this context, heavily relying on the Green function, is not obvious at all. This has been done by Pham [Pha05], who proved the existence of a positive closed current $\mathcal{E}$ on the product space with the property that the slices are the equilibrium measures. This result in turn relies on work by Dinh and Sibony about the slicing of horizontal currents. Moreover, Pham proved that the intersection between such a current $\mathcal{E}$ and the integration current on the critical set C_f is well defined. This allowed him to get the fundamental formula

$$dd^c L = \pi_* (\mathcal{E} \wedge \mathsf{C}_f)$$

where the projection is as usual on the parameter space. This will be the starting point of our work.

Parabolic implosion In dimension one, a way to study bifurcation phenomena is given by the theory of parabolic implosion. Let us briefly recall the foundational results of this theory. We refer to [Dou94] for a more extended introduction to the subject, as well as to the original work by Lavaurs [Lav89]. Consider the polynomial map on $\mathbb{C}$ tangent to the identity given by $f(z) = z + z^2$. The origin is a *parabolic* fixed point for f. The dynamics is attracting near the negative real axis: there exists a parabolic basin $\mathcal{B}$ for 0, i.e., an open set of points converging to the origin after iteration. The origin is on the boundary of $\mathcal{B}$, and the convergence happens tangentially to the negative real axis. The iteration of f on $\mathcal{B}$ is semiconjugated to a translation by 1. More precisely, there exists an *incoming Fatou coordinate* $\varphi^\iota : \mathcal{B} \to \mathbb{C}$ such that, for every $z \in \mathcal{B}$, we have $\varphi^\iota \circ f(z) = f(z) + 1$.

The same happens for the inverse iteration near the positive real axis: we have a repelling basin $\mathcal{R}$ of points converging to 0 under some inverse iteration, and the convergence happens tangentially to the positive real axis. We can construct in this case an *outgoing Fatou parametrization*, i.e., a map $\psi^o : \mathbb{C} \to \mathcal{R}$ such that $f \circ \psi^o(z) = \psi^o(z+1)$. It is worth noticing here that the union of $\mathcal{B}$ and $\mathcal{R}$ gives a full pointed neighbourhood of the origin.

Notice that the incoming Fatou coordinate is a map from the dynamical plane to $\mathbb{C}$, while the

outgoing Fatou parametrization is a map from $\mathbb{C}$ to the dynamical plane. In particular, given any $\alpha \in \mathbb{C}$ and denoting by t_α the translation by α on $\mathbb{C}$, the composition $L_\alpha := \psi^o \circ t_\alpha \circ \varphi^\iota$ is well defined as a function from $\mathcal{B}$ to $\mathcal{R}$. Such a map is usually called a *Lavaurs map*, or a *transfer map*.

We consider now the perturbation $f_\varepsilon(z) = z + z^2 + \varepsilon^2$ of the system f, for ε real and positive. As $\varepsilon \neq 0$, the dynamics abruptly changes: the parabolic point splits in two (repelling) points $\pm i\varepsilon$, and the orbits of points in $\mathcal{B}$ can now pass through the "gate" between these two points. Using the Lavaurs map it is possible to give a very precise description of this phenomenon, by studying the dynamics of high iterates of the perturbed maps f_ε, as $\varepsilon \to 0$. The following definition plays a central role in this study.

Definition 14. *Given $\alpha \in \mathbb{C}$, an* α-sequence *is a sequence $(\varepsilon_\nu, n_\nu)_{\nu\in\mathbb{N}} \in (\mathbb{C} \times \mathbb{N})^{\mathbb{N}}$ such that $n_\nu \to \infty$ and $n_\nu - \frac{\pi}{\varepsilon_\nu} \to \alpha$ as $\nu \to \infty$.*

Notice in particular that the definition of α-sequence implies that ε_ν tends to the origin tangentially to the positive real axis. More precisely, there exists a constant c such that, for every ν sufficiently large, we have $|\operatorname{Im} \varepsilon_\nu| \leq c\,|\varepsilon_\nu|^2$. The following result gives the limit description of suitable high iterates of f_ε.

Theorem 15 (Lavaurs [Lav89])**.** *Let $f_\varepsilon(z) = z + z^2 + \varepsilon^2 + o(z^2, \varepsilon^2)$ and (ε_ν, n_ν) be an α-sequence. Then $f_{\varepsilon_\nu}^{n_\nu} \to L_\alpha$, locally uniformly on $\mathcal{B}$.*

One of the most direct consequences of this theorem is the fact that the set-valued functions $\varepsilon \mapsto J(f_\varepsilon)$ and $\varepsilon \mapsto K(f_\varepsilon)$ are discontinuous for the Hausdorff topology at $\varepsilon = 0$, where J and K denote the Julia set and the filled Julia set, as usual (recall – see e.g. [Dou94] – that $J(f_\varepsilon)$ is always lower semicontinuous, while $K(f_\varepsilon)$ is always upper semicontinuous). More precisely, define the *Lavaurs-Julia set* $J(f_0, L_\alpha)$ and the *filled Lavaurs-Julia set* $K(f_0, L_\alpha)$ by

$$J(f_0, L_\alpha) := \overline{\{\, z \in \mathbb{C} \colon \exists m \in \mathbb{N}, L_\alpha^m(z) \in J(f_0) \,\}}$$
$$K(f_0, L_\alpha) := \{\, z \in \mathbb{C} \colon \exists m \in \mathbb{N}, L_\alpha^m(z) \notin K(f_0) \,\}^c .$$

Notice that the Lavaurs-Julia set $J(f_0, L_\alpha)$ is in general larger than the Julia set of f_0. On the other hand, the set $K(f_0, L_\alpha)$ is in general smaller than $K(f_0)$. The following Theorem then gives an estimate of the discontinuity of the maps $\varepsilon \mapsto J(f_\varepsilon)$ and $\varepsilon \mapsto K(f_\varepsilon)$ at $\varepsilon = 0$.

Theorem 16 (Lavaurs [Lav89])**.** *Let $f_\varepsilon(z) = z + z^2 + \varepsilon^2 + o(z^2, \varepsilon^2)$ and (ε_ν, n_ν) be an α-sequence. Then*

$$\liminf J(f_{\varepsilon_\nu}) \supset J(f_0, L_\alpha) \text{ and } \limsup K(f_{\varepsilon_\nu}) \subset K(f_0, L_\alpha).$$

In particular, at $\varepsilon = 0$,

1. *the map $\varepsilon \to J(f_\varepsilon)$ is lower semicontinuous, but not continuous;*
2. *the map $\varepsilon \to K(f_\varepsilon)$ is upper semicontinuous, but not continuous.*

Recently, a similar study has been undertaken for semi-parabolic fixed points by Bedford-Smillie-Ueda [BSU12] (see also [DL13]). Their main result is as follows. Consider a family of polynomial diffeomorphisms of $\mathbb{C}^2$, holomorphic in ε^2, whose expression at the origin is given by

$$F_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^2 + \varepsilon^2 + \dots \\ b_\varepsilon y + \dots \end{pmatrix}. \tag{1}$$

Here $|b_\varepsilon| < 1$ and the omitted terms are of order at least 3 and 2 in the two lines, respectively. By results of Ueda [Ued86, Ued91], there exists an attracting (two-dimensional) basin $\mathcal{B}$ and a (one-dimensional) repelling leaf Σ for the origin. Moreover, there exist Fatou coordinates $\varphi^\iota : \mathcal{B} \to \mathbb{C}$ and $\varphi^o : \Sigma \to \mathbb{C}$ semiconjugating the system to a translation.

Theorem 17 (Bedford-Smillie-Ueda [BSU12]). *Let F_ε be as in* (1) *and let (ε_ν, n_ν) be an α-sequence. Then $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$, locally uniformly on $\mathcal{B}$, where $T_\alpha := (\varphi^o)^{-1} \circ t_\alpha \circ \varphi^\iota$.*

From this Theorem, they deduce the discontinuity at $\varepsilon = 0$ of various dynamically-defined sets, among others the (forward) Julia and filled Julia sets. However, notice that the limit map T_α in this context has a one-dimensional image (due to the exponential contraction of the system). Thus, since it is not open, they need new arguments to deduce the above discontinuities from Theorem 17.

Main results

In order to state our results, we have to give some definitions. First of all, the point of view here is the following: we consider the space of maps

$$\mathcal{J} := \{\gamma \in \mathcal{O}(M, \mathbb{C}^k) \colon \gamma(\lambda) \in J_\lambda \text{ for every } \lambda \in M\}$$

and want to study the action induced by the family f on this set. This is a compact metric space for the topology of local uniform convergence. It is the space of all candidates as holomorphic motions of points in the Julia set. The family f naturally induces a dynamical system on $\mathcal{J}$, by the natural action given by

$$(\mathcal{F} \cdot \gamma)(\lambda) := f_\lambda(\gamma(\lambda)).$$

In this work, we shall be mainly concerned with the study of the ergodic properties of the system $(\mathcal{J}, \mathcal{F})$. The following is then a key definition.

Definition 18. *An* equilibrium web *is a probability measure $\mathcal{M}$ on $\mathcal{J}$ such that:*

1. *$\mathcal{F}_*\mathcal{M} = \mathcal{M}$, and*
2. *$(p_\lambda)_* \mathcal{M} = \mu_\lambda$ for every $\lambda \in M$, where $p_\lambda : \gamma \mapsto \gamma(\lambda)$ is the evaluation map.*

The picture to have in mind is the following (see Figure 2.1): we have a set of graphs in the product space (to be thought of as the support of $\mathcal{M}$) and what the second condition says is that, if we slice this measure at any parameter λ, what we get is exactly the equilibrium measure of the map f_λ. Our goal can then be summarized as follows: given a holomorphic motion of the repelling points, we want to find an equilibrium web $\mathcal{M}$ and a subset $\mathcal{S} \subset \mathcal{J}$ such that

1. the graphs of any two elements in $\mathcal{S}$ do not intersect;
2. $\mathcal{M}(\mathcal{S}) = 1$.

This would imply the existence of a true *holomorphic motion* for a full-measure subset of the Julia set. In order to reach this goal, we have to make some preliminary steps. The first one is to generalize to this setting the work done by Berteloot and Dupont [BD14b] for families of endomorphisms of $\mathbb{P}^k$.

Theorem A. *Let f be a holomorphic family of polynomial-like maps of large topological degree. Then the following are equivalent:*

1. *the Lyapounov function is pluriharmonic;*
2. *there are no Misiurewicz parameters;*
3. *every parameter has a neighbourhood on which the family admits an equilibrium web $\mathcal{M} = \lim_n \mathcal{M}_n$, where the $\mathcal{M}_n$'s are measures on $\mathcal{J}$ such that the graph of any $\gamma \in \cup_n \operatorname{Supp} \mathcal{M}_n$ avoids the critical set of f.*

The proof of this Theorem is the content of the second chapter of this work. Although some parts of the proof are just adaptations of the arguments valid on $\mathbb{P}^k$, others require to completely rethink the strategy. In particular, the proof by Berteloot and Dupont that a Misiurewicz parameter is in the support of the current $dd^c L$ heavily relies on the existence of the Green current. We will thus give in Theorem 2.2.12 a completely different and more geometrical proof of this fact. An important step is given by the following theorem, which is of independent interest and can be seen as a generalization of Theorem 7.

Theorem B. *Let f be a holomorphic family of polynomial-like maps of large topological degree d_t. Then*

$$dd^c L \neq 0 \Leftrightarrow \limsup_{n\to\infty} \frac{1}{n} \log \|(f^n)_* \, C_f\| > \log d^*_{k-1}.$$

Once we have established the equivalence of the above notions of stability, we can proceed with the main result of this work, which is the core of the third chapter. Our higher-dimensional analogue of the holomorphic motion is the following (here Γ_γ denotes the graph in the product space of a map γ).

Definition 19. *An* equilibrium lamination *is a subset $\mathcal{L}$ of $\mathcal{J}$ such that*

1. *$\mathcal{F}(\mathcal{L}) = \mathcal{L}$,*
2. *$\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for every distinct $\gamma, \gamma' \in \mathcal{L}$,*
3. *$\mu_\lambda(\{\gamma(\lambda), \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M$,*
4. *Γ_γ does not meet the grand orbit of the critical set of f for every $\gamma \in \mathcal{L}$,*
5. *the map $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is d_t-to-1.*

Let us comment a bit this definition. The second and the third conditions ensure that we have a holomorphic motion of a full-measure subset of the Julia set. The other conditions say that $\mathcal{F}$ induces a covering, of degree d_t, on $\mathcal{L}$. In particular, we have a conjugacy of the dynamics on a full measure subset of the Julia sets. In Chapter 3 we prove that equilibrium laminations exist when the repelling points contained in the Julia set (called *J-repelling points*) move holomorphically. More precisely we have the following, which is our higher-dimensional analogue of Theorem 11. Let us stress that this result is new also for endomorphisms of $\mathbb{P}^k$.

Theorem C. *Let f be a holomorphic family of polynomial-like maps of large topological degree. Then the following are equivalent:*

1. *asymptotically all the J-repelling points move holomorphically;*
2. *there exists an equilibrium lamination;*
3. *the Lyapounov function is pluriharmonic.*

The first point deserves some explanation. It means that we have a subset of elements $\mathcal{P} = \cup_n \mathcal{P}_n$ of $\mathcal{J}$, of cardinality $\mathcal{P}_n \sim d_t^n$ (i.e., the same as the repelling points) such that on every compact subset of the parameter space the number of non-repelling elements of $\mathcal{P}_n$ is $o(d_t^n)$. It is a slightly weaker (and presumably equivalent) assumption than the motion of the J-repelling points. The theorem above can actually be improved to the motion of all J-repelling cycles if the family is in dimension 2 by a simple adaptation of arguments (sketched in Chapter 4) used by Berteloot and Dupont on $\mathbb{P}^2$ (see Theorem 13). While their idea is to study the relation between bifurcations and higher dimensional Siegel discs, our proof consists on a generalization to the metric space $\mathcal{J}$ of a strategy due to Briend and Duval to recover the existence of repelling points from the existence of a mixing measure with constant jacobian satisfying good asymptotic backward contraction properties.

Combined with the stronger result by Berteloot and Dupont, we get the following characterization of stability for endomorphisms of $\mathbb{P}^k$.

Theorem D. *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^2$. Then the following are equivalent:*

1. *the J-repelling cycles move holomorphically;*
2. *there exists an equilibrium lamination;*
3. *the Lyapounov function is pluriharmonic.*

In dimension one, all the equivalent notions of stability given so far are also equivalent to another one: the continuity of the Julia set for the Hausdoff topology. The simplest way to prove this is by means of Siegel discs: when a repelling cycle stops to move holomorphically its multiplier become of modulus one, creating (up to a small perturbation) a Siegel disc. Since such an object is contained in the Fatou set, the desired discontinuity follows. In Chapter 4 we investigate the relation between bifurcations and higher dimensional analogues of Siegel discs. In dimension two, Siegel points are defined as (periodic) points where the system is linearizable and conjugated to a rotation in one direction, while being repelling in the other one. A Siegel disc is then an invariant holomorphic disc through a Siegel point, where the dynamics is conjugated to a rotation. We prove that, at least in some particular families, the continuity of the Julia set would be equivalent to the holomorphic stability provided that Siegel discs are outside the Julia set. This motivates the question of whether a Siegel disc can be inside the Julia set. Naively speaking, the Julia set is the most repelling part of the system, while a Siegel disk has a direction of *Fatou type*: it is thus at least reasonable to expect that such an object should be disjoint from a Julia set. In Chapter 4 we prove that this is not the case in general.

Example E. *The polynomial endomorphism of $\mathbb{C}^2$ given by*

$$F\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z^3 \\ \varepsilon w^3 + \left(1 + \left(\frac{e^{i\theta}}{2} - 1\right)\frac{z-z_1}{z_0-z_1}\right)(w^2+2w) \end{pmatrix}, \tag{2}$$

where ε is a small enough parameter and θ is of Brjuno type, extends to $\mathbb{P}^2$ and has a Siegel disc at the point $(-1,0)$ contained in the Julia set.

This means that the general strategy to try to prove the equivalence between continuity and holomorphic motion by means of Siegel discs has at least to be reconsidered. On the other hand, there is another (much more involved) way to get the discontinuity in dimension one: the theory of parabolic implosion. This is the reason why we got interested in this latter subject.

As we saw before, while the one-dimensional theory is well-developed the study of parabolic implosion in higher dimension has just begun, with recent results only in the semi-attracting setting [BSU12, DL13]. The goal of the last chapter of this work is to provide a starting point in the completely parabolic setting, by a precise study of perturbations of germs of endomorphisms of $\mathbb{C}^2$ tangent to the identity at the origin. More specifically, we consider an endomorphism of $\mathbb{C}^2$ of the form

$$F_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^2(1 + (q+1)x + ry + O(x^2, xy, y^2)) \\ y(1 + \rho x + O(x^2, xy, y^2)) \end{pmatrix}, \tag{3}$$

where ρ is real and greater than 1 and $q, r \in \mathbb{C}$. For instance, F_0 may be the local expression of an endomorphism of $\mathbb{P}^2$ (e.g., if the two components of F_0 are polynomials of the same degree in (x,y) with 0 as the only common root of their higher-degree homogeneous parts). We shall primarily be interested in this situation.

The map F_0 has a fixed point tangent to the identity at the origin, and two invariant lines $\{\, x = 0 \,\}$ and $\{\, y = 0 \,\}$. By the work of Hakim [Hak97] (recalled in Section 5.1) we know that $[1:0]$ is a *non-degenerate characteristic direction*, and that there exists an open set $\mathcal{B}$ of initial conditions, with the origin on the boundary, such that every point in $\mathcal{B}$ is attracted to the origin tangentially to the direction $[1:0]$. Moreover there exists, on an open subset $\widetilde{C}_0$ of $\mathcal{B}$, a (one dimensional) Fatou coordinate $\widetilde{\varphi^\iota}$, with values in $\mathbb{C}$, such that $\widetilde{\varphi^\iota} \circ F_0(p) = \widetilde{\varphi^\iota}(p) + 1$.

A similar description holds for the inverse map. Indeed, after restricting ourselves to a neighbourhood U of the origin where F_0 is invertible, we can define the set $\mathcal{R}$ of points that are attracted to the origin tangentially to the direction $[1:0]$ by backward iteration. There is then a well defined map $\widetilde{\varphi^o} : -\widetilde{C}_0 \cap U \to \mathbb{C}$ such that $\widetilde{\varphi^o} \circ F_0(p) = \widetilde{\varphi^o}(p) + 1$ whenever the left hand side is defined. It is actually possible to construct two-dimensional Fatou coordinates (see [Hak97]), but we shall not need them in this work.

Consider now a perturbation F_ε of F_0 of the form

$$\begin{aligned} F_\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + (x^2+\varepsilon^2)\alpha_\varepsilon(x,y) \\ y(1 + \rho x + \beta_\varepsilon(x,y)) \end{pmatrix} \\ &= \begin{pmatrix} x + (x^2+\varepsilon^2)(1 + (q+1)x + ry + O(x^2, xy, y^2) + O(\varepsilon^2)) \\ y(1 + \rho x + O(x^2, xy, y^2) + O(\varepsilon^2)) \end{pmatrix}. \end{aligned} \tag{4}$$

Our goal is to study the dependence on ε of the large Julia set $J^1(F_\varepsilon)$ (i.e., the complement of the Fatou set which, for endomorphisms of $\mathbb{P}^2$, coincides with the support of the Green current and is in general larger than the Julia set) near the parameter $\varepsilon = 0$. Our main result is the following Theorem, which is a partial generalization of Theorem 15 to our setting. As in dimension 1, *α-sequences* (see Definition 14) play a crucial role. The set $\widetilde{C}_0$ introduced above will be precisely defined in Proposition 5.1.1, and the Fatou coordinates $\widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}$ in Lemma 5.1.2.

Theorem F. *Let F_ε be a holomorphic family of endomorphisms of $\mathbb{C}^2$ as in* (4)*. Let F_0 be invertible on a neighbourhood U of the origin and let $\widetilde{\varphi^\iota} : \widetilde{C}_0 \to \mathbb{C}$ and $\widetilde{\varphi^o} : -\widetilde{C}_0 \to \mathbb{C}$ be the (1-dimensional) Fatou coordinates for F_0. Let $\mathcal{B}$ be the attracting basin for the origin for the map F_0 with respect to the characteristic direction $[1:0]$ and $\mathcal{R}$ the repelling one. Let α be a complex number and (n_ν, ε_ν) be an α-sequence. Then every compact subset $\mathcal{C} \subset \mathcal{B} \cap \{ y = 0 \}$ has a neighbourhood $U_\mathcal{C}$ where, up to extracting a subsequence, we have*

$$F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$$

locally uniformly, where T_α is a well defined open holomorphic map from $U_\mathcal{C}$ to $\mathbb{C}^2$, with values in $\mathcal{R}$. Moreover,

$$\widetilde{\varphi^o} \circ T_\alpha(p) = \alpha + \widetilde{\varphi^\iota}(p) \tag{5}$$

whenever both sides are defined.

As a consequence, we shall deduce an estimate of the discontinuity of the large Julia set in this context (notice that the discontinuity itself follows from an application of Theorem 15 to the invariant line $\{ y = 0 \}$). We say that, given $\mathcal{U} \subset \widetilde{C}_0$, a map $T_\alpha : \mathcal{U} \to \mathbb{C}^2$ is a *Lavaurs map* if there exists an α-sequence (ε_ν, n_ν) such that $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$ on $\mathcal{U}$. We then have the following result (see Section 5.5.2 for the definition of the Lavaurs-Julia sets $J^1(F_0, T_\alpha)$ in this setting).

Theorem G. *Let F_ε be a holomorphic family of endomorphisms of $\mathbb{P}^2$ as in* (4) *and $T_\alpha : \mathcal{U} \to \mathbb{C}^2$ be a Lavaurs map such that $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$ on $\mathcal{U}$ for some α-sequence (ε_ν, n_ν). Then*

$$\liminf J^1(F_{\varepsilon_\nu}) \supset J^1(F_0, T_\alpha).$$

Finally, we consider a family of *regular polynomials*, i.e., polynomial endomorphisms of $\mathbb{C}^2$ admitting an extension to $\mathbb{P}^2(\mathbb{C})$. For these maps (which are in particular polynomial-like maps), it is meaningful to define the *filled Julia set* K as the set of points with bounded orbit. In analogy with the one-dimensional theory, we deduce from Theorem F an estimate for the discontinuity of the filled Julia set at $\varepsilon = 0$ (see Section 5.5.3 for the definition of the set $K(F_0, T_\alpha)$) and in particular deduce that $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$. Notice that, differently from the case of the large Julia set, this is not already a direct consequence of the 1-dimensional theory.

Theorem H. *Let F_ε be a holomorphic family of regular polynomial maps of $\mathbb{C}^2$ as in* (4) *and $T_\alpha : \mathcal{U} \to \mathbb{C}^2$ be a Lavaurs map such that $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$ on $\mathcal{U}$ for some α-sequence (ε_ν, n_ν). Then*

$$K(F_0, T_\alpha) \supset \limsup K(F_{\varepsilon_\nu}).$$

Moreover, $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.

Techniques and ideas of proof

We shall now give a brief description of the strategy of the proof of the main results of this work, focusing on the tecniques used to prove them.

Chapter 2: Misiurewicz parameters and bifurcations

Chapter 2 is devoted to the proof of Theorem A. A crucial role here is played by Misiurewicz parameters (see Definition 12): the core of Chapter 2 indeed consists in proving that Misiurewicz parameters are contained in the bifurcation locus, and are dense in it. We focus here on the first statement.

Theorem A'. *Let f be a holomorphic family of polynomial-like maps of large topological degree. Then Misiurewicz parameters are contained in the support of the bifurcation current $dd^c L$.*

Establishing this theorem is the main difference between the second chapter and the work of Berteloot and Dupont [BD14b] for endomorphisms of $\mathbb{P}^k$. Indeed, their approach to this statement relies on the existence of a potential – the Green function – for the Green current $\mathcal{E}_{Green}$. We thus need to adopt a different approach, completely rethinking the strategy of proof. The idea is the following: in dimension 1, a Misiurewicz parameter is responsible (because of the expansive behaviour of the system at the intersection between the repelling cycle and the postcritical set) of the non-normality of the critical orbit. Moreover, a Misiurewicz parameter is never isolated: it is quite straightforward to see that the existence of one Misiurewicz parameter implies the existence of many others nearby. This is related to a large growth of the mass of the postcritical set in this region of the parameter space. A crucial step in establishing the result above in thus proving Theorem B. The proof of this result, as most of the material in the first part of this work, relies on the theory of slicing of currents. We give in Appendix A.1 a brief account of this theory.

In view of Theorem B, the proof of Theorem A' thus consists in showing that, near a Misiurewicz parameter, the volume growth of the postcritical set is as close as we want to d_t^n. In order to do this, we construct (exploiting the mixing property of the equilibrium measure, and a procedure essentially due to Briend-Duval) a ball, contained in the dynamical space of the Misiurewicz parameter, with a lot (i.e., more than $\left(d_{k-1}^*\right)^n$) of preimages for f^n contained in it. Exploiting the contracting behaviour of these inverse branches, we can construct a cylinder T_0 in the product space with the same properties: more than c^n smaller tubes $T_{n,i}$, contained in it, sent biholomorphically to T_0 by f^n. But now we can arrange the local picture (see Figure 2.2) in such a way that the component of the postcritical hypersurface $f^{n_0}(C_f)$ intersecting the repelling cycle must cross all the $T_{n,i}$'s. By applying f^n, the intersections between $f^{n_0}(C_f)$ and the small tubes $T_{n,i}$ are all sent to analytic subsets of T_0, whose volume is thus uniformly bounded from below. We thus get that the volume of $f^{n_0+n}(C_f)$, is larger than $\left(d_{k-1}^*\right)^n$ (up to a constant). This gives the desired growth of the critical volume, and the assertion follows from Theorem B.

Let us spend two words on the density of the Misiurewicz parameters in the bifurcation locus. This is established by proving that, in absence of Misiurewicz parameters, we can construct a holomorphic graph in the product space avoiding the postcritical set. It is not difficult to prove, using the backward equidistribution property of preimages, that this implies the existence of an equilibrium web. The idea here is to construct a large hyperbolic set at a fixed parameter and consider a holomorphic motion of this set. Since there are no Misiurewicz parameters, all repelling cycles contained in this motion must avoid (or be contained in) the postcritical set. It follows that the same must be true for every holomorphic graph of the motion. It is thus enough to establish that the original hyperbolic set is not completely contained in the postcritical set. This can be

done by an entropy argument: indeed, an hyperbolic set of sufficiently large entropy cannot be contained in any hypersurface. This is easily seen for endomorphisms on $\mathbb{P}^k$, and the same holds for polynomial-like maps of large topological degree. The proof of this fact is essentially the same given by Dinh and Sibony to estimate the topological entropy of a polynomial-like map, combined with a relative version of the variational principle following from Brin-Katok theorem. Nevertheless, since we did not find the precise result we need in the literature, we included a complete proof of this statement in Appendix A.2, together with a brief summary of the notions of entropy dimensions that we need.

Chapter 3: holomorphic motions in higher dimension

The main result of Chapter 3 is the following higher-dimensional analogue of Theorem 5.

Theorem C'. *Let f be a holomorphic family of polynomial-like maps of large topological degree. If the repelling cycles in $\mathcal{J}$ move holomorphically then there exists an equilibrium lamination.*

We want to stress a difference with respect to the one-dimensional proof of this statement. As we saw, in dimension one this result is a consequence of Montel Theorem, combined with Hurwitz Theorem. The characterization of stability by means of the Lyapounov exponent can be seen there as a secondary (although very useful) way to look at bifurcations. Here the situation is different. We are not able to give a direct proof of the above result without using the results of the previous chapter, and in particular the characterization of stability by means of the Lyapounov exponent.

The first step in the proof consists indeed in proving that the holomorphic motion of the repelling cycles in $\mathcal{J}$ implies the existence of a particular equilibrium web $\mathcal{M}$: an acritical and ergodic one. Here acritical means that $\mathcal{M}$ gives no mass to the singular part of $\mathcal{J}$, defined as

$$\mathcal{J}_s := \left\{ \gamma \in \mathcal{J} \colon \Gamma_\gamma \cap \left(\bigcup_{m,n\geq 0} f^{-m}\left(f^n\left(C_f\right)\right)\right) \neq \emptyset \right\}.$$

To prove this property we need to use the results of the previous chapter, and in particular the condition on the harmonicity of L. Once we prove this, we can transform the system $(\mathcal{J} \setminus \mathcal{J}_s, \mathcal{F}, \mathcal{M})$ into an invertible one. This is done by means of the natural extension. This is a classical construction, and it allows us to somehow assume that the system was already invertible.

Our goal is then to prove that the set of elements of $\mathcal{J}$ whose graph intersects the graph of some other element has measure 0 for $\mathcal{M}$. The crucial step in the proof is to prove the following backward contraction property for graphs (see Figure 3.1).

Key proposition. *There exist a measurable function $\eta : \mathcal{X} = \mathcal{J} \setminus \mathcal{J}_s \to]0,1]$ and a positive constant A such that for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ the following hold:*

1. *f^{-n} is defined on a tubular neighbourhood $T(\gamma,\eta(\gamma))$ of Γ_γ; and*
2. *$f^{-n}\left(T(\gamma,\eta(\gamma))\right) \subset T(\mathcal{F}^{-n}\gamma, e^{-nA})$.*

Once we have established this property, the fact that $\mathcal{M}$ gives no mass to the set of intersecting graphs follows from an application of Poincaré recurrence Theorem. The existence of an equilibrium lamination then easily follows.

Let us thus describe the proof of the Key Proposition above. We exploit a method, developed by Briend-Duval [BD99] for endomorphisms of $\mathbb{P}^k$ (and generalized by Dinh-Sibony [DS03, DS10] to polynomial-like maps), which proves the analogous statement at a fixed parameter. Namely, given a polynomial-like map of large topological degree g almost every point x (with respect to the equilibrium measure μ) is contained in a ball $B(x,\eta(x))$ where a local inverse g^{-n} is defined for every n and satisfies the contraction property $g^{-n}(B(x,\eta(x))) \subset B(g^{-n}(x), e^{-n\chi_1})$. Here χ_1 denotes the smallest Lyapounov exponent of the system (J, g, μ), which is known ([DS03]) to be greater than 0. The main steps of the method are as follows:

- an asymptotic estimate of $\|dg^{-1}(\cdot)\|$ over the inverse orbit $\{g^{-j}(x)\}$ of a point x yields an estimate of the radius of a ball centered at x where g^{-n} is defined, for every n, and of the asymptotic rate of contraction of g^{-n} on this ball;
- the asymptotic estimate of $\|dg^{-1}\|$ is obtained from the fact that

$$\text{for } \mu-\text{a.e. } x\colon \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\left\|dg^{-1}(g^{-j}(x))\right\| = \int \log\left\|dg^{-1}(x)\right\|\mu(x) = -\chi_1 < 0,$$

where the first equality comes from the ergodicity of μ and Birkhoff Theorem.

In our setting, the same method reduces the problem to prove that

$$\text{for } \mathcal{M}-\text{a.e. } \gamma\in\mathcal{X}\colon \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\max_{\lambda}\left\|df_\lambda^{-1}(\mathcal{F}^{-j}\gamma(\lambda))\right\| < 0.$$

As $\mathcal{M}$ is ergodic, we still know that the limit equals $\int_{\mathcal{X}}\log\max_\lambda\left\|df_\lambda^{-1}(\gamma(\lambda))\right\|\mathcal{M}(\gamma)$. So, we only need to prove that this integral is negative. In order to get this, we show that

$$\lim_{n\to\infty}\frac{1}{n}\int_{\mathcal{X}}\log\max_{\lambda}\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|\mathcal{M}(\gamma) < 0 \tag{6}$$

and then get the desired estimate after replacing our system by a suitable high iterate f^N. Equation (6) can be thought of as an estimate of a Lyapounov exponent for the ergodic system $(\mathcal{J},\mathcal{F},\mathcal{M})$. Establishing this is the main technical part of the chapter, where we need to exploit both the ergodic properties of this system of graphs and properties of psh functions. Let $u_n(\gamma)$ be given by

$$u_n(\gamma) = \log\max_{\lambda}\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|.$$

First of all, we prove that $u_n \in L^1(\mathcal{M})$. This part is quite technical: we need to ensure that, roughly speaking, the elements of $\mathcal{J}$ approach the critical part $\mathcal{J}_s$ locally uniformly. Then, since the sequence u_n is easily seen to be subadditive (i.e., $u_{n+m}(\gamma) \le u_n(\mathcal{F}^m\gamma) + u_m(\gamma)$) we can apply the ergodic version of Kingman subadditive theorem and see that the limit above exists and is equal to $\lim_n u_n(\gamma)$, for almost every $\gamma\in\mathcal{J}$. We thus just need to prove that this limit is negative for a generic element of $\mathcal{J}$. By a Fubini-Tonelli argument (combined with Oseledets theorem) applied to the product space $M\times\mathcal{J}$ we can prove that, for almost every $\gamma\in\mathcal{J}$, the limit of $\log\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|$ is negative (uniformly in λ) for a full-measure subset of λ's in M. We thus prove

that the sequence is dominated (by arguments related to the ones giving the integrability of the u_n's), and the assertion can follow by an application of Lebesgue theorem.

Chapter 4: a Siegel disc in a Julia set

The main contribution in Chapter 4 is a simple example of an endomorphism of $\mathbb{P}^2$ with a Siegel disc contained in its Julia set. Consider the map $F(z,w) = (p(z), q_z(w))$ given in (2). It is straighforward to see that such an endomorphism extends to $\mathbb{P}^2$ (the only reason to have the εw^3 term is to ensure this extension). Moreover, the system is linearizable at $(-1,0)$, with a Siegel disc centered at the origin of the vertical fiber. It follows from a result of Jonsson [Jon99] that the Julia set of F is equal to

$$J(F) = \overline{\cup_{z\in J_p} \{ z \} \times J_z}.$$

Here J_z is the boundary (in the plane $\{z\} \times \mathbb{C}$) of the set $K_z := \{G_z = 0\}$, where the function G_z is defined by

$$G_z(w) := G(z,w) - \widetilde{G}(z),$$

G and $\widetilde{G}$ being the Green functions of F and z^3 respectively. It is immediate to see that, if $z \in K_p$, then

$$w \in K_z \Leftrightarrow \text{ the sequence } Q_z^n(w) := q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z(w) \text{ is bounded.}$$

By the form of F, for ε sufficiently small the Julia set of the map q_1 is a quasicircle passing through the origin and thus, by Jonsson formula, we have $\{1\} \times J_{q_1} \subset J(F)$. Since the preimages of 1 by z^3 are dense in the unit circle, we have preimages of this quasicircle accumulating the point $(-1,0)$. Moreover, since the system is linearizable at $(-1,0)$, these preimages are not contracted to a point, but rotate and accumulate an open neighbourhood of 0 in the vertical fiber of -1. Since the Julia set is closed, this implies the existence of a Siegel disc in the Julia set.

Chapter 5: a two-dimensional Lavaurs theorem

In Chapter 5 we study a phenomenon of parabolic implosion for endomorphisms of $\mathbb{C}^2$ tangent to the identity. Our main result here is Theorem F, from which we deduce the estimates on the discontinuity of the large Julia set and of the filled Julia set given in Theorems G and H. Our strategy is essentially an adaptation of the one used by Bedford-Smillie-Ueda [BSU12] in the semi-parabolic setting (i.e., for maps with a parabolic and an attracting eigenvalue at the fixed point). The main difference here is the need to carefully estimate the second coordinate of the points of the orbit, which in that case goes to zero exponentially with the number of iterations. Notice in particular an important difference between our result and Theorem 17. While, because of the contraction property just recalled, the image of the limit map in the semi-parabolic setting is one-dimensional (and in particular contained in the one-dimensional repelling petal at the origin), here our limit map is open. This allows us to recover the desired discontinuities more in the spirit of the one dimensional theory.

Let us depict the strategy of our proof. The first point is to get an horizontal convergence for the sequence of iterates $F_{\varepsilon_\nu}^{n_\nu}$ of perturbed maps (where (ε_ν, n_ν) is an α-sequence, as in Definition 14), i.e., a convergence for the first coordinate. We first construct an approximate Fatou coordinate $\widetilde{w}_\varepsilon$

for F_ε, satisfying

$$\widetilde{w}_\varepsilon(F_\varepsilon(x,y)) = \widetilde{w}_\varepsilon(x,y) + 1 + O(y, x^2, \varepsilon^2) = \widetilde{w}_\varepsilon(x,y) + 1 + A(x, y, \varepsilon).$$

Notice that the dependence of the error $A(x, y, \varepsilon)$ in y is linear. Then, we consider the ingoing and outgoing normalizations of $\widetilde{w}_\varepsilon$ given by

$$\widetilde{w}^\iota_\varepsilon := \widetilde{w}_\varepsilon + \frac{\pi}{2\varepsilon} \text{ and } \widetilde{w}^o_\varepsilon := \widetilde{w}_\varepsilon - \frac{\pi}{2\varepsilon}.$$

We focus now on the attracting basin, since the situation is symmetric on the other side. By means of the function $\widetilde{w}^\iota_\varepsilon$, we can define (denoting by (x_j, y_j) the orbit of $p = (x, y)$ under F_ε) the functions

$$\widetilde{\varphi^\iota}_{\varepsilon,m} = \widetilde{w}^\iota_\varepsilon(F^m_\varepsilon(p)) - m = \widetilde{w}^\iota_\varepsilon(p) - m + \sum_{j=1}^{m} A(x_j, y_j, \varepsilon).$$

The point here is to prove that, when $m_\varepsilon \sim \frac{\pi}{2\varepsilon}$, we have

$$\widetilde{\varphi^\iota}_{\varepsilon,m_\varepsilon} \to \widetilde{\varphi^\iota}.$$

This, together with the analogous convergence on the other side of the gate (i.e., in the repelling basin), allows us to recover the horizontal convergence to a Lavaurs map. By the construction of $\widetilde{w}_\varepsilon$, we can ensure that the difference between $\widetilde{\varphi^\iota}_{\varepsilon,m_\varepsilon}$ and $\widetilde{\varphi^\iota} + \sum_{j=0}^{m_\varepsilon} A(x_j, y_j, \varepsilon)$ goes to zero as $\varepsilon \to 0$. We thus need to prove that the series of the errors goes to 0, too, as $\varepsilon \to 0$. For the part of the error in x^2 and ε^2 we can essentially get the estimate as in [BSU12]. The main difference here is to get the convergence to zero for the series

$$\sum_{j=0}^{m_\varepsilon} |y_j| .$$

Notice that this would be true for $\varepsilon = 0$. Indeed, by results of Hakim we know that the second coordinate goes to zero under F_0 as $1/n^\rho$, where $\rho > 1$. The point here is thus to estimate the modulus of the points in the orbit, by means of the partial coordinates $\widetilde{w}_\varepsilon$ introduced above, to ensure that the same happens as $\varepsilon \to 0$ (notice that we are considering a sum of an increasing number of terms as ε goes to 0).

The second main point is to ensure the convergence in the second coordinate of the sequence of maps $F^{n_\nu}_{\varepsilon_\nu}$. Roughly speaking, in the first part of the orbit (before passing through the gate), the dynamics is contracting in the y direction, while, after the gate, the dynamics becomes expanding. Our strategy is thus the following:

1. we ensure that the number of points in the orbit in the expanding part is no more than the number of the points in the contracting one;

2. we then prove that each term in the expanding part is balanced by a suitable term in the contracting one.

The first point is easily adressed in the following way: we divide the orbit in three parts, where the central one roughly corresponds to the times where both coordinates are bounded by ε. We take

some extra care to ensure, by introducing a non-symmetry in this central region of the system, that we leave on the third part of the orbit less terms than in the first part.

The second point is the one really needing care. In order to compare the behaviour in the first and third parts of the orbit, it is convenient to introduce the family $H_\varepsilon(x,y) := (-1,0)\cdot F_\varepsilon^{-1}(-x,y)$ (where $\cdot$ denotes the componentwise multiplication). This just amounts to study both the first and the third part of the orbit in the same region of the space. We want to ensure that the orbit of a point under H_ε does not go to 0 too fast (as $\varepsilon \to 0$) with respect to the orbit of a (possibly different) point under F_ε. Denoting by (x_j^F, y_j^F) and (x_j^H, y_j^H) the two orbits, we thus want (by the expression of (4)) that

$$\prod_{j=1}^{J}(1+\rho x_j^F) \sim \prod_{j=1}^{J}(1+\rho x_j^H),$$

where J is the time the orbit spends in the first (and in the third) part. Since both x_j^F and x_j^H are approximately harmonic in the regions under consideration, and $\rho > 1$, it is enough to prove that

$$\left|x_j^F - x_j^H\right| \lesssim \frac{\log j}{j^2}.$$

This is done by estimating the distance of the two orbits in (a modification of) the chart $\widetilde{w}_\varepsilon$: this is bounded by the logarithm of the number of iterations j, and gets transformed to a quantity bounded by $\log j/j^2$ when passing to the dynamical space. This gives us the desired estimate in the vertical direction, and allows us to conclude.

1 Polynomial-like maps

In this preliminary chapter we introduce the main objects that we shall need in the sequel and fix the notations that we shall use in all this work. Unless otherwise stated, all the results presented here are due to Dinh and Sibony. In particular, we refer to the original paper [DS03] and to the survey [DS10] for the details. We just prove in Lemma 1.3.10 an approximation result that we will need in Section 2.2.3.

1.1. Definition and examples

The starting definition is the following.

Definition 1.1.1. *A* polynomial-like *map is a proper holomorphic map $g : U \to V$, where $U \Subset V$ are open subsets of $\mathbb{C}^k$ and V is convex.*

Notice in particular that a polynomial-like map is a (branched) holomorphic covering from U to V, of a certain degree d_t (the topological degree of g). We shall always assume that the topological degree satisfies $d_t \geq 2$. We shall denote by C_g the critical set of g, by Jac_g the determinant of the (complex) Jacobian matrix and by C_g the integration current $\mathsf{C}_g = dd^c \log |\mathrm{Jac}_g|$, supported on C_g (and taking into account the multiplicities).

Since polynomial-like maps share with polynomials the property of being a ramified covering, it is natural to ask whether there may always exist a conjugation (of a certain regularity) between a polynomial-like map and a polynomial. The following theorem by Douady and Hubbard [DH85] answers to this question in dimension 1, and allows one to reduce the dynamical study of polynomial-like maps in dimension 1 to that of polynomials.

Theorem 1.1.2 (Douady-Hubbard). *In dimension 1, every polynomial-like map $g : U \to V$ is conjugated to a polynomial by means of a Hölder homeomorphism, on a neighbourhood of its filled Julia set (see Definition 1.1.6).*

Notice that, in dimension $k > 1$, Theorem 1.1.2 does not hold (see [DS10, Example 2.25]). In view of Theorem 1.1.2, in the following we shall be only interested in the case $k > 1$.

We shall now give some examples of polynomial-like maps. Lemma 1.1.3 allows us to construct large families of examples by perturbation.

Lemma 1.1.3. *Let $g : U \to V \subset \mathbb{C}^k$ be a polynomial-like map and let V' open, convex and such that $U \Subset V' \Subset V$. Then, any holomorphic map $h : U \to \mathbb{C}^k$ sufficiently close to g (for the topology of uniform convergence on compact subsets) is a polynomial-like map from $h^{-1}(V')$ to V'.*

The following are basic examples to have in mind when working with polynomial-like maps.

Example 1.1.4. *Let $F : \mathbb{P}^k \to \mathbb{P}^k$ be any holomorphic endomorphism of $\mathbb{P}^k$ of a given degree d, i.e., let F be given, in homogeneous coordinates, by*

$$F([z_0 : \cdots : z_k]) = [F_0(z_0, \ldots, z_k) : \cdots : F_k(z_0, \ldots z_k)]$$

where the F_i's are homogeneous polynomials of degree d with the origin as only common zero. Consider the lift $\widehat{F} := (F_0, \ldots, F_k)$ of F to $\mathbb{C}^{k+1}$. Given any sufficiently large ball $B(0, R)$ (such that $B(0, R) \Subset \widehat{F}(B(0, R))$), the lift $\widehat{F}$ is a polynomial-like map from $B(0, R)$ to $\widehat{F}(B(0, R))$. Endomorphisms of $\mathbb{P}^k$ can then be seen as a particular case of polynomial-like maps.

Example 1.1.5. *Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial map such that $\|f(z)\| \geq 2\|z\|$ for $\|z\| \geq R$. Then, f is a polynomial-like map from $B(0, R)$ to $f(B(0, R))$. By Lemma 1.1.3, every sufficiently small perturbation h of f gives rise to new polynomial-like maps. In particular, the perturbation h can be transcendental. Thus, polynomial-like maps can be seen as an intermediate step between the polynomial behaviour and the (more complicated) transcendental one.*

In this work, we shall be mainly interested in the iteration of polynomial-like maps. In order to do this, it is natural to restrict our interest to the subset of U where all the iterates g^n are defined. This motivates the following definition.

Definition 1.1.6. *The* filled Julia set *K of a polynomial-like map is the subset of U given by*

$$K := \bigcap_{n \geq 0} g^{-n}(U).$$

Notice in particular that $g^{-1}(K) = K = g(K)$ and thus (K, g) is a well-defined dynamical system.

In the following section, we recall the main properties of a polynomial-like map that we shall need in the sequel. Then, in Section 1.3 we shall introduce the central objects of this work, the holomorphic families of polynomial-like maps.

1.2. Main properties

1.2.1. Dynamical degrees and entropy

When studying an endomorphism F of $\mathbb{P}^k$, the knowledge of just the (algebraic) degree d allows one to know the number of preimages of points (which is d^k), the degree of the images of

hypersurfaces (equal to d^{k-1} times the degree of the starting hypersurface), and more generally the degree of the induced map from an analytic set of dimension p to its image (which is d^p). All this follows from the cohomological properties of $\mathbb{P}^k$ – which is a compact Kähler manifold – by computing the degrees of the actions of F_* and F^* on the various cohomology groups, which are generated by the right power of the Fubini-Study form.

All of this does not hold (in general) for a polynomial-like map g: the knowledge of the topological degree is not enough to predict the volume growth of analytic subsets, and in general the action of g_* and g^* on forms (and currents) of all degrees. We are thus led to consider more general degrees than the topological one. In the following definitions, we denote by ω the standard Kähler form on $\mathbb{C}^k$. Moreover, recall that the mass of a positive (p,p)-current T on a Borel set X is given by $\|T\| = \int_X T \wedge \omega^{k-p}$.

Definition 1.2.1. *Given a polynomial-like map $g : U \to V$, the* dynamical degree of order p*, for $0 \leq p \leq k$, of g is given by*

$$d_p(g) := \limsup_{n\to\infty} \left\| (g^n)_* \left(\omega^{k-p}\right) \right\|_W^{1/n} = \limsup_{n\to\infty} \left\| (g^n)^* (\omega^p) \right\|_{g^{-n}(W)}^{1/n},$$

where $W \Subset V$ is a neighbourhood of K.

Definition 1.2.2. *Given a polynomial-like map $g : U \to V$, the* *-dynamical degree of order p*, for $0 \leq p \leq k$, of g is given by*

$$d_p^*(g) := \limsup_{n\to\infty} \sup_S \|(g^n)_* (S)\|_W^{1/n}$$

where $W \Subset V$ is a neighbourhood of K and the sup is taken over all positive closed $(k-p, k-p)$-currents of mass less or equal than 1 on a fixed neighbourhood $W' \Subset V$ of K.

It is quite straighforward to check that these definitions do not depend on the particular neighbourhoods W and W' chosen for the computations. Moreover, the following hold: $d_p \leq d_p^*$ for every p, $d_0^* = 1$ and $d_k = d_k^* = d_t$. We also have $d_p(g^m) = d_p^m$ and $d_p^*(g^m) = \left(d_p^*(g)\right)^m$.

The following refinement of Lemma 1.1.3 ensures that a relation $d_p^* < d_t$ is preserved by small perturbations. We shall give in Lemma 2.2.7 a proof of this fact.

Proposition 1.2.3. *Let $g\colon U \to V$ be a polynomial-like map such that $d_p^*(g) < d_t(g)$ for some p. Let V' be a convex open set such that $U \Subset V' \Subset V$. If $g' : U \to \mathbb{C}^k$ is a holomorphic map which is a sufficiently small perturbation of g and $U' := g'^{-1}(V')$, then $g' : U' \to V'$ satisfies $d_p^*(g') < d_t(g')$.*

The following theorem gives a bound on the topological entropy (see Section A.2) of a polynomial-like map and relates it with the dynamical degrees.

Theorem 1.2.4 (Dinh-Sibony [DS03],[DS10])**.** *Let $g : U \to V$ a polynomial-like map of topological degree $d_t \geq 2$. Then the topological entropy of g on K is less or equal than $\log d_t$. Moreover, all the dynamical degrees $d_p(g)$ are smaller or equal than $d_t(p)$.*

We do not detail here the proof of this theorem, since in Section 2.2.4 we will need a generalization of this result. Namely, we shall prove in Lemma A.2.6 that the topological entropy of (the intesection of K with) any analytic set of dimension p is smaller or equal than d_p^*.

By the Variational Principle (see Theorem A.2.1), in order to get the opposite inequality $h_t(g,K) \geq \log d_t$ it is enough to exhibit a measure whose *metric entropy* (see Section A.2) is equal to $\log d_t$. This is done in the next section. We notice however that the same conclusion may follow by an adaptation of the Misiurewicz-Przytycki estimate between the topological entropy and the degree of a map ([MP77]), or of the one by Yomdin with the volume growth of submanifolds ([Yom87]).

1.2.2. The equilibrium measure and the Julia set

Given a polynomial-like map g, it is possible to associate to it an ergodic measure μ_g, called the *equilibrium measure* of g, which is of constant Jacobian d_t and maximizes the entropy. We give here some ideas about its construction and introduce its main properties.

Theorem 1.2.5. *Let $g\colon U \to V$ be a polynomial-like map and ν be a probability measure supported on V which is defined by an L^1 form. Then $d_t^{-n}\,(g^n)^*\,\nu$ converge to a probability measure μ which does not depend on ν. Moreover, for any psh function φ on a neighbourhood of K the sequence $d_t^{-n}\,(g^n)_*\,\varphi$ converge to $\langle \mu,\varphi \rangle \in \{-\infty\} \cup \mathbb{R}$.*

For φ psh, the function $d_t^{-n}\,(g^n)_*\,\varphi$ is also psh. The convergence of $d_t^{-n}\,(g^n)_*\,\varphi$ in Theorem 1.2.5 is in L^p_{loc} for every $1 \leq p < \infty$ if $\langle \mu,\varphi \rangle$ is finite, locally uniform otherwise.

By definition, proving the first convergence in Theorem 1.2.5 amounts to show that, for every smooth function φ compactly supported on V, the sequence $d_t^{-n}\,(g^n)_*\,(\varphi)$ converge to a constant function c_φ. Since every smooth function with compact support is a difference of smooth psh functions, we can replace the test φ with a smooth psh function (but not compactly supported). The key point in the proof is then to notice that the desired convergence (for any psh function in a neighbourhood of K, even not continuous) is assured by the maximum principle (and Hartogs Lemma).

Lemma 1.2.6. *Let φ be a psh function on a neighbourhood of K. Then $d_t^{-n}\,(g^n)_*\,(\varphi)$ converge to a constant $c_\varphi \in \mathbb{R} \cup -\infty$. If $c_\varphi \in \mathbb{R}$, the convergence is in L^p_{loc} for every $1 \leq p < \infty$. If $c_\varphi = -\infty$, it is locally uniform. If φ is smooth, c_φ is finite.*

We refer to the original work by Dinh and Sibony for details. Here we shall content ourselves to give the main properties of the measure given by Theorem 1.2.5.

Definition 1.2.7. *The measure μ given by Theorem 1.2.5 is called the* equilibrium measure *of g. The support of μ is the* Julia set *of g, denoted with J_g.*

Theorem 1.2.8. *The equilibrium measure is ergodic and mixing.*

From the construction of μ it is immediate to see that this measure satisfies $g^*\mu = d_t\mu$. Moreover, by taking a ν supported outside K, we see that the Julia set is contained in the boundary of K. It is actually possible to see that the support of μ is contained in the Shilov boundary of K, which is in general smaller than the (topological) boundary, if the dimension is greater than 1. The assumption on ν to be defined by a L^1 form can be relaxed to just asking that ν does not charge pluripolar sets.

The following property of μ will be useful in the sequel (see Lemma 1.3.4). It states that μ maximizes (among all probability measures satisfying $g^*\nu = d_t\nu$) the momentum against psh functions.

Proposition 1.2.9. *Let $g : U \to V$ a polynomial-like map of topological degree $d_t \geq 2$. Let μ be the equilibrium measure and ν any probability measure satisfying $g^*\nu = d_t\nu$. Then for every psh function φ on a neighbourhood of K, we have $\langle \nu,\varphi \rangle \leq \langle \mu,\varphi \rangle$. Equality holds for pluriharmonic functions.*

The following Theorem ensures that μ does not charge the critical set of g. On the other hand, notice that μ may charge proper analytic subsets. This is a difference with respect to the case of endomorphisms of $\mathbb{P}^k$, where this possibility is excluded by Chern-Levine-Nirenberg inequality, because of the boundedness of the local potentials.

Theorem 1.2.10. *Let $f : U \to V$ be a polynomial-like map of degree d_t. Then $\langle \mu, \log|\mathrm{Jac}_g| \rangle \geq \frac{1}{2}\log d_t$.*

The factor $1/2$ comes from the fact that we are considering the complex jacobian matrix. Theorem 1.2.10 implies that the Jacobian of μ with respect to g (i.e., the Radon-Nikodym derivative of $g^*\mu$ with respect to μ) is well-defined. By Theorem 1.2.5, this derivative is constant, and equal to d_t. The following is then a consequence of Parry Theorem A.2.4, the Variational Principle A.2.1 and Theorem 1.2.4.

Corollary 1.2.11. *The metric entropy of μ is equal to $\log d_t$. So, $h_t(g, K) = \log d_t$ and μ is a measure of maximal entropy.*

1.2.3. Lyapounov exponents

In this Section we introduce the Lyapounov exponents of a polynomial-like map with respect to the equilibrium measure μ. These will play a crucial role in all this work. They can be thought of as the exponential rates of expansion (or contraction) of the map in the different directions at a generic point (with respect to the equilibrium measure) for the system. In order to introduce them, we need some preliminary definition.

Let X be a measure space, $T : X \to X$ a measurable map and ν a T-invariant probability measure. In the literature (see e.g. [Arn98]), the triple (X, T,ν) is often called *metric dynamical system*. In our applications, the system will usually be (K, g, μ). Notice in particular that we do not make (almost) any assumption on the regularity of X, in order to be able to apply the results to K. Let $A : X \to GL_k(\mathbb{C})$ be a a measurable map. The *multiplicative cocycle* generated by A is the sequence

$$A_n(x) := A(g^{n-1}(x)) \cdot \cdots \cdot A(x).$$

Notice in particular that $A(g^{m+n}(x)) = A_n(g^m(x))A_m(x)$. The following Theorem by Oseledets ([Ose68], see also [Arn98]) gives, under some reasonable hypotheses, an asymptotic description of the cocycle A_n.

Theorem 1.2.12 (Oseledets). *Let $T : X \to X$ be a measurable dynamical system and A_n a multiplicative cocycle. Let ν be an ergodic measure for T. Then, if*

$$\log^+ \|A(x)\| \in L^1(\nu)$$

there exists an integer m, real numbers $\chi_1 < \cdots < \chi_m$, a full measure subset Y of X and, for every $x \in Y$, a unique decomposition of $\mathbb{C}^k$ into a direct sum of linear subspaces $E_i(x)$ such that

1. *Y is forward invariant;*

2. *the decomposition is forward invariant, i.e., $A(x)\,(E_i(x)) \subseteq E_i(g(x))$;*

3. *for every $v \in E_i(x) \setminus \{0\}$,*
$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n(x)v\| = \chi_i.$$

Moreover, if we also have
$$\log^+ \left\|A^{-1}(x)\right\| \in L^1(\nu)$$
then

1'. *Y can be taken completely invariant;*

2'. *the decomposition is completely invariant, i.e., $A(x)\,(E_i(x)) = E_i(g(x))$.*

The second part of Theorem 1.2.12 is usually referred to as the *invertible Oseledets Theorem*.

Consider now a polynomial-like map $g : U \to V$. Let μ be the equilibrium measure of g, as in Definition 1.2.7. Notice that the function $\|dg_x\|$ is bounded from above on $\overline{U}$. This implies that the first part of Oseledec Theorem applies. Theorem 1.2.10 ensures that the same holds for the second part.

Corollary 1.2.13. *Let $g : U \to V$ be a polynomial-like map and let μ be the equilibrium measure of f. Then $\log^+ \|dg^{-1}\| \in L^1(\mu)$. In particular, the invertible form of Theorem 1.2.12 applies for $T = g$, $\nu = \mu$ and $A(x) = dg_x$.*

Proof. The first part of Oseledets Theorem 1.2.12 applies since the function $\|dg_x\|$ is bounded from above on $\overline{U}$. We have to prove that the fact that $\log|\mathrm{Jac}| \in L^1(\mu)$ implies that $\log^+ \|dg^{-1}\| \in L^1(\mu)$. This follows from the inequality, valid for any $A \in GL_k(\mathbb{C})$,
$$|\det A| \le \left\|A^{-1}\right\|^{-1} \cdot \|A\|^{k-1}. \tag{1.1}$$
This inequality follows from the fact that $|\det A|$ is the product of the k singular values of A, and $\left\|A^{-1}\right\|^{-1}$ and $\|A\|$ are, respectively, the smallest and the largest ones. Using (1.1) we deduce that, for every $x \in U$,
$$\log^+ \left\|dg_x^{-1}\right\| \le (k-1)\log^+ \|dg_x\| + |\log|\det dg_x||\,.$$
The assertion follows since the first term on the right side is bounded on $\overline{U}$ and the second belongs to $L^1(\mu)$ by assumption. □

Definition 1.2.14. *The numbers $\chi_i = \chi_i(g)$, counted with multiplicity, are the* Lyapounov exponents *of g with respect to μ. The* Lyapounov function *$L(g)$ is the sum*
$$L(g) = \sum \chi_i(g).$$

By Oseledets Theorem 1.2.12 and Birkhoff Theorem, it follows that $L(g) = \langle \mu, \log|\mathrm{Jac}| \rangle$. By Theorem 1.2.10, we thus have $L(g) \ge \frac{1}{2}\log d_t$ for every polynomial-like map g.

1.2.4. Maps of large topological degree

Let g be a polynomial-like map and μ its equilibrium measure. Even if the convergence of the sequence $d_t^{-n}\,(g^n)_*\,(\varphi)$ to the constant $\langle \mu,\varphi \rangle$ holds for any psh function on a neighbourhood of K, it is not true in general that that the integral $\langle \mu,\varphi \rangle$ is finite for any psh function. In this section we introduce a class of polynomial-like maps whose equilibrium measure precisely integrate psh function. Recall that the *-dynamical degrees were defined in Definition 1.2.2.

Definition 1.2.15. *A polynomial-like map is of* large topological degree *if* $d_{k-1}^* < d_t$.

The idea behind Definition 1.2.15 is the following. Consider any hypersurface in V. The associated integration current, suitably normalized, is a closed positive current of mass 1. Definition 1.2.15 in particular says that the volume growth of such an hypersurface is negligible with respect to the volume growth – with multiplicity – of the whole space (which is of order d_t^n). Notice that holomorphic endomorphisms of $\mathbb{P}^k$ (and thus their polynomial-like lifts as in Example 1.1.4) satify the above estimate. Morever, by Proposition 1.2.3, a small perturbation of a polynomial-like map of large topological degree still satisfy this property.

Polynomial-like maps of large topological degree share a lot of properties with endomorphisms of $\mathbb{P}^k$. We now give several equivalent definitions of this class of maps, and then state the main properties that we shall need in the sequel.

Theorem 1.2.16. *Let $g : U \to V$ be a polynomial-like map and μ be its equilibrium measure. The following are equivalent:*

1. *g has large topological degree;*
2. *psh functions are integrable with respect to μ;*
3. *μ can be extended to a linear continuous form of the cone of psh functions on V.*

In particular, the equilibrium measure of a polynomial-like map of large topological degree does not charge pluripolar sets. This is one of the most important properties of this class of maps.

Definition 1.2.17. *Let V be an open subset of $\mathbb{C}^k$. A measure ν, with compact support in V, is* moderate *if for any subset $\mathcal{P}$ of PSH(V) bounded in L^1_{loc} there exist two constants $\alpha > 0$ and $A > 0$ such that*

$$\left\langle \nu, e^{\alpha|\varphi|} \right\rangle < A$$

for every $\varphi \in \mathcal{P}$.

Moderate measures behave very much like the Lebesgue measure for what concern psh functions. Notice in particular that psh functions are in L^p_{loc} with respect to moderate measures, for every $1 \leq p < \infty$. The equilibrium measure of a polynomial-like map of large topological degree enjoys this property.

Theorem 1.2.18. *Let $g : U \to V$ be a polynomial-like map of large topological degree. Then the equilibrium measure μ of g is moderate.*

We end this section by recalling two fundamental equidistribution results about the equilibrium measure of a polynomial-like map of large topological degree that we shall repeatedly use in all this work. They can also be seen as alternative characterizations of the equilibrium measure. They are due to Briend-Duval and Dinh-Sibony[BD99, BD01, DS10] in the case of endomorphisms of $\mathbb{P}^k$, and to Dinh-Sibony [DS03, DS10] for polynomial-like maps of large topological degree.

Theorem 1.2.19. *Let $g\colon U \to V$ be a polynomial-like map of large topological degree $d_t \geq 2$. Let R_n denote the set of repelling n-periodic points in the Julia set J. Then*

$$\frac{1}{d_t^n} \sum_{a \in R_n} \delta_a \to \mu.$$

Notice that, differently from the case of endomorphisms of $\mathbb{P}^k$, here we cannot deduce the number of n-periodic points by Bezout Theorem. Nevertheless, it is possible to prove that they are d_t^n. In proving this we use the assumption that V is convex.

Theorem 1.2.20. *Let $g\colon U \to V$ be a polynomial-like map of large topological degree $d_t \geq 2$. There exists a proper analytic set $\mathcal{E}$ (possibly empty) contained in the postcritical set of g such that*

$$d_t^{-n} \left(g^n\right)^* \delta_a = \frac{1}{d_t^n} \sum_{g^n(b)=a} \delta_b \to \mu$$

if and only if a does not belong to the orbit of $\mathcal{E}$.

A remarkable consequence of the proof of Theorem 1.2.20 is the following estimate about the smallest Lyapounov exponent for polynomial-like maps of large topological degree. It ensures that the equilibrium measure of any such map is *hyperbolic*, i.e., all Lyapounov exponents are different from 0. More precisely, the Lyapounov exponents are all strictly positive. This property will play a crucial role in the proof of the main result of this work, namely the construction of an equilibrium lamination for the Julia sets starting from the motion of repelling points, given in Chapter 3.

Theorem 1.2.21. *Let $g\colon U \to V$ be a polynomial-like map of large topological degree d_t. Then all the Lyapounov exponents of g with respect to the equilibrium measure are at least equal to $\frac{1}{2}\log\left(\frac{d_t}{d_{k-1}}\right) > 0$.*

1.3. Holomorphic families

We now come to the main object of our study.

Definition 1.3.1. *Let M be a complex manifold and $\mathcal{U} \Subset \mathcal{V}$ be connected open subsets of $M \times \mathbb{C}^k$. Denote by π_M the standard projection $\pi_M : M \times \mathbb{C}^k \to M$. Suppose that for every $\lambda \in M$, the two sets $U_\lambda := \mathcal{U} \cap \pi^{-1}(\lambda)$ and $V_\lambda := \mathcal{V} \cap \pi^{-1}(\lambda)$ satisfy $\emptyset \neq U_\lambda \Subset V_\lambda \Subset \mathbb{C}^k$, that U_λ is connected and that V_λ is convex. Moreover, assume that U_λ and V_λ depend continuously on λ (in the sense of Hausdorff). A* holomorphic family of polynomial-like maps *is a proper holomorphic map $f : \mathcal{U} \to \mathcal{V}$ fibered over M, i.e., of the form*

$$\begin{aligned} f: \ \mathcal{U} &\to \mathcal{V} \\ (\lambda, z) &\mapsto (\lambda, f_\lambda(z)). \end{aligned}$$

Notice that, from the definition, f has a well defined topological degree, that we shall always denote with d_t and assume to be greater than 1. In particular, each $f_\lambda : U_\lambda \to V_\lambda$ may be viewed as a polynomial-like map, of degree d_t. We shall denote by μ_λ, J_λ and K_λ the equilibrium measure, the Julia set and the filled Julia set of f_λ, while C_f, Jac_f and C_f will be the critical set, the determinant of the (complex) jacobian matrix of f and the integration current $dd^c \log|\mathrm{Jac}_f|$. We may drop the subscript f if no confusion arises.

The main question that we shall adress in this work is the following: how does the Julia set J_λ vary with λ? It is immediate to see that the filled Julia set K_λ varies upper semicontinuously for the Hausdoff topology, and at the end of this section we shall see (Corollary 1.3.15) that the Julia set J_λ depends lower semicontinuously on the parameter for families of polynomial-like maps of large topological degree (see Definition 1.2.15).

In this section we present some general results about holomorphic families of polynomial-like maps, mainly following [DS10] and [Pha05]. We refer to the Appendix A.1 for the notions about horizontal currents and slicing that we shall use.

By Lemma 1.1.3, given a polynomial-like map $f_\lambda : U_\lambda \to V_\lambda$, if we replace V_λ by a slightly smaller convex open set $V'_\lambda \subset V_\lambda$ and U_λ by $U'_\lambda := f_\lambda^{-1}(V'_\lambda)$, the map $f_\lambda : U'_\lambda \to V'_\lambda$ is still polynomial-like. So, since the filled Julia set varies upper semicontinuously, when dealing with local problems we shall assume that V_λ does not depend on λ, i.e., that $\mathcal{V} = M \times V$, with V an open, convex and relatively compact subset of $\mathbb{C}^k$.

For endomorphisms of $\mathbb{P}^k$, the potential g of the Green current can be seen as a function in the variables $\lambda \in M$ and $z \in \mathbb{P}^k$. This allows one to construct a global object, a (k,k)-current on the product space, whose slices are exactly the equilibrium measures μ_λ. This global object is usually referred to as the *equilibrium current*. For polynomial-like maps, the lack of a potential for μ_λ does not permit to directly obtain an equilibrium current in the same way. Nevertheless, in [Pha05], by exploiting the theory of slicing of horizontal currents Pham constructs such an object in the setting of polynomial-like maps, i.e., a positive closed current on the space $\mathcal{V}$ whose slice at any λ is precisely the equilibrium measure μ_λ for f_λ. Lemma 1.3.4 below is the core of the proof of this result, which is given in Theorem 1.3.5. Here and in all this section we shall make a repeated use of the following result.

Theorem 1.3.2 (Dinh-Sibony, Pham). *Let M and V be relatively compact open subsets of $\mathbb{C}^m$ and $\mathbb{C}^k$, respectively. Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current and ψ a psh function on $M \times V$. Then*

1. *the slice $\langle \mathcal{R}, \pi, \lambda\rangle$ exists for every $\lambda \in M$, and its mass is independent from λ;*
2. *the function $g_{\psi,\mathcal{R}}(\lambda) := \langle \mathcal{R}, \pi, \lambda\rangle(\psi(\lambda,\cdot))$ is psh (or identically $-\infty$).*

If $\langle \mathcal{R}, \pi, \lambda_0\rangle(\psi(\lambda_0,\cdot)) > -\infty$ for some $\lambda_0 \in M$, then

3. *the product $\psi\mathcal{R}$ is well defined;*
4. *for every Ω continuous form of maximal degree compactly supported on M we have*

$$\int_M \langle \mathcal{R}, \pi, \lambda\rangle(\psi)\Omega(\lambda) = \langle \mathcal{R} \wedge \pi^*(\Omega), \psi\rangle. \tag{1.2}$$

In particular, the pushforward $\pi_*(\psi\mathcal{R})$ is well defined and coincides with the psh function $g_{\psi,\mathcal{R}}$.

Appendix A.1 is essentially dedicated to the proof of this statement, following the work of Dinh, Sibony and Pham. The four assertions are proved in Theorem A.1.11, Corollary A.1.12, Theorem A.1.14, and Proposition A.1.18, respectively. In the proof of Lemma 1.3.4 we shall also need the following fact (see [Hör07, Theorem 3.2.12]).

Lemma 1.3.3. *Let u_j be a sequence of psh functions in an open connected subset $M \subset \mathbb{C}^m$, converging (as distributions) to a distribution v. Then v is defined by a psh function u and $u_j \to u$ in $L^1_{loc}(M)$.*

Lemma 1.3.4. *Let $f : \mathcal{U} \to \mathcal{V} = M \times V$ be a holomorphic family of polynomial-like maps. Let $\mathcal{R}_n$ be a sequence of horizontal positive closed (k,k)-currents on $M \times V$ converging to a horizontal positive closed (k,k)-current R and such that for every $\lambda \in M$ we have*

$$\langle \mathcal{R}_n, \pi, \lambda \rangle \to \mu_\lambda, \tag{1.3}$$

where μ_λ is the equilibrium measure of f_λ. Then, $\langle \mathcal{R}, \pi, \lambda \rangle = \mu_\lambda$ for every $\lambda \in M$.

Notice that, since the $\mathcal{R}_n$'s and $\mathcal{R}$ are horizontal, positive and closed, the existence of their slices at any $\lambda \in M$ is ensured by Theorem A.1.11.

Proof. We have to show that, for every $\lambda \in M$ and for every smooth ψ, compactly supported in $M \times V$, we have

$$\langle \mathcal{R}, \pi, \lambda \rangle (\psi(\lambda, \cdot)) = \langle \mu_\lambda, (\psi(\lambda, \cdot)) \rangle .$$

Since any such ψ is a difference of two smooth psh functions, we can suppose that ψ is smooth and psh (but, obviously, not compactly supported). Define the following functions on M:

$$\begin{cases} u_n(\lambda) := \pi_* (\psi \mathcal{R}_n) = \langle \langle \mathcal{R}_n, \pi, \lambda \rangle , \psi(\lambda, \cdot) \rangle \\ u(\lambda) := \pi_* (\psi \mathcal{R}) = \langle \langle \mathcal{R}, \pi, \lambda \rangle , \psi(\lambda, \cdot) \rangle . \end{cases}$$

By Theorem 1.3.2, all these functions are well defined and psh. Moreover, since π_* is continuous and $\mathcal{R}_n \to \mathcal{R}$, we have that $u_n \to u$ as distributions, and thus in $L^1_{loc}(M)$ (by Lemma 1.3.3). By (1.3), we also have that the sequence u_n pointwise converge to the function

$$u'(\lambda) := \langle \mu_\lambda, \psi(\lambda, \cdot) \rangle .$$

Our goal is thus to prove that $u' = u$.

First, remark that this equality holds almost everywhere. Indeed, u' and u are the pointwise and the L^1_{loc} limit of the same sequence of psh functions, and this implies their coincidence on a set of full measure. Moreover, by Hartogs Lemma, the pointwise limit of a sequence of psh functions is (pointwise) smaller than the L^1_{loc} limit, and this implies that $u' \leq u$. We are thus left to prove that $u' \geq u$.

Fix $\lambda_0 \in M$. Since u is psh, we have $u(\lambda_0) = \limsup_{\lambda \to \lambda_0} u(\lambda)$, where the $\limsup$ can be taken over any full-measure subset $M' \subset M$. Thus, since $u = u'$ almost everywhere, there exists a sequence of points $\lambda_m \to \lambda_0$ in M such that $u(\lambda_m) = u'(\lambda_m)$ and $u(\lambda_m) \to u(\lambda_0)$. Since we are working locally and the filled Julia sets of the family vary upper-semicontinuously, we can suppose that the support of μ_{λ_0} and all the supports of μ_{λ_m} are contained in a common compact

set $L \Subset V$. So, up to a subsequence, the measures μ_{λ_m} converge to a measure ν_{λ_0}. Remark that, by construction, we have $u(\lambda_0) = \lim_m u'(\lambda_m) = \langle \nu_{\lambda_0}, \psi(\lambda_0, \cdot)\rangle$ and that, by continuity, ν_{λ_0} satisfies $f_{\lambda_0}^* \nu_0 = d_t \nu_0$.

The assertion follows since (by Proposition 1.2.9) the equilibrium measure μ_{λ_0} maximizes the integrals against psh functions (among the invariant measures for f_{λ_0}), and, by its very definition, $u'(\lambda_0) = \langle \mu_{\lambda_0}, \psi(\lambda_0, \cdot)\rangle$. □

Consider now a family of polynomial-like maps $f : \mathcal{U} \to \mathcal{V} = M \times V$ as in Lemma 1.3.4. Let θ be a smooth probability measure compactly supported in V and consider the (positive and closed) smooth (k,k)-currents on $M \times V$ defined by induction as

$$\begin{cases} S_0 = \pi_V^*(\theta) \\ S_n := \frac{1}{d_t} f^* S_{n-1} = \frac{1}{d_t^n} (f^n)^* S_0. \end{cases} \tag{1.4}$$

The currents S_n are in particular horizontal positive closed (k,k)-currents on $M \times V$, whose slice mass is equal to 1. Moreover, since by definition we have $\langle S_0, \pi, \lambda\rangle = \theta$ for every $\lambda \in M$, we have that $\langle S_n, \pi, \lambda\rangle = \frac{1}{d_t^n} (f_\lambda^n)^* \theta$. In particular, since every $f_\lambda : U_\lambda \to V$ is a polynomial-like map, for every $\lambda \in M$ we have $\langle S_n, \pi, \lambda\rangle \to \mu_\lambda$. The following Theorem is then a consequence of Lemma 1.3.4.

Theorem 1.3.5 ([Pha05], Proposition 2.1). *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. Then there exists a positive closed (k,k)-current $\mathcal{E}$ on $\mathcal{V}$, supported on $\cup_\lambda \{\lambda\} \times K_\lambda$, such that for every $\lambda \in M$ the slice $\langle \mathcal{E}, \pi, \lambda\rangle$ exists and is equal to μ_λ.*

Proof. We can solve the problem locally (on M) and then recover a global $\mathcal{E}$ by means of a partition of unity on M. We can thus assume from the beginning that the parameter space is a unit ball $B \Subset M$, and that $\mathcal{V} = B \times V$. In this way, we also give a sense to the slice of $\mathcal{E}$, otherwise defined only for currents on a product space (but see also Remark A.1.4). Let θ be any smooth positive measure on V and S_n be defined as in (1.4). Since the S_n's have bounded mass on $B \times V$ (recall that $B \Subset M$), we can extract a subsequence S_{n_i} converging to a horizontal positive closed (k,k)-current $\mathcal{E}$ on $B \times V$. We can also use the fact that the set of horizontal positive currents with bounded slice mass is compact, see [DS06]. The assertion now follows applying Lemma 1.3.4 to the sequence $S_{n_i} \to \mathcal{E}$. □

Remark 1.3.6. *Up to considering the Cesaro averages of the S_n's, we can construct a current $\mathcal{E}$ as above satisfying the extra property that $f^*\mathcal{E} = d_t \mathcal{E}$.*

Definition 1.3.7. *An* equilibrium current *for f is a positive closed current $\mathcal{E}$ on $\mathcal{V}$, supported on $\cup_\lambda \{\lambda\} \times K_\lambda$, such that $\langle \mathcal{E}, \pi, \lambda\rangle = \mu_\lambda$ for every $\lambda \in M$.*

The following result is now an immediate consequence of Theorem 1.3.2 (since, locally on M, an equilibrium current $\mathcal{E}$ is horizontal – by the upper semicontinuity of K_λ).

Corollary 1.3.8 ([Pha05], Proposition 2.1). *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial like maps. Let Jac denote the determinant of the Jacobian and C the critical set of f. Let $\mathcal{E}$ be an equilibrium current for the family. Then the product $\log|\mathrm{Jac}| \cdot \mathcal{E}$ is well defined. In particular, the intersection $\mathcal{E} \wedge C_f = dd^c(\log|\mathrm{Jac}| \cdot \mathcal{E})$ is well defined.*

Take now a $B \Subset M$ such that $B \times V' \subset \mathcal{V}$ and $\mathcal{E}$ is horizontal on $B \times V'$ and any psh function u on $B \times V'$ such that $\langle \mu_{\lambda_0}, u(\lambda_0, \cdot)\rangle > -\infty$ for some $\lambda_0 \in B$. By Theorem 1.3.2 we know that the distribution $\pi_*(u\mathcal{E})$ is represented by the (plurisubharmonic) function $\lambda \mapsto \langle \mu_\lambda, u(\lambda, \cdot)\rangle$. Notice that, while the product $u\mathcal{E}$ a priori depends on the particular equilibrium current $\mathcal{E}$, the pushforward is independent from the particular choice (by (1.2)).

This in particular applies with $u = \log|\mathrm{Jac}|$, since the hypothesis that $\langle \mu_\lambda, \log|\mathrm{Jac}(\lambda, \cdot)|\rangle > -\infty$ is satisfied at every λ by Theorem 1.2.10. By Oseledets theorem 1.2.12, the function $\lambda \mapsto \langle \mu_\lambda, \log|\mathrm{Jac}(\lambda, \cdot)|\rangle$ coincides with the Lyapounov function $L(\lambda)$, i.e., the sum of the Lyapounov exponents of f_λ with respect to μ_λ (see Definition 1.2.14). The following definition is then well posed.

Definition 1.3.9. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. The* bifurcation current *of f is the positive closed $(1,1)$-current on M given by*

$$T_{bif} := dd^c L(\lambda) = \pi_* (C_f \wedge \mathcal{E}), \tag{1.5}$$

where $\mathcal{E}$ is any equilibrium current for f.

The following result gives an approximation of the current $u\mathcal{E}$, for u psh, that we shall need in Section 2.2.3.

Lemma 1.3.10. *Let $f : \mathcal{U} \to \mathcal{V} = M \times V$ be a holomorphic family of polynomial-like maps. Let θ be a smooth positive measure compactly supported on V. Let S_n be as in* (1.4) *and $\mathcal{E}$ be any equilibrium current for f. Let u be a psh function on $M \times V$ and assume that there exists $\lambda_0 \in M$ such that $\langle \mu_{\lambda_0}, u(\lambda_0, \cdot)\rangle > -\infty$. Then, for every continuous form Ω of maximal degree and compactly supported on M, we have*

$$\langle uS_n, \pi^*(\Omega)\rangle \to \langle u\mathcal{E}, \pi^*(\Omega)\rangle, \tag{1.6}$$

where the right hand side is well defined by Theorem 1.3.2.

Notice that the assumption $\langle \mu_{\lambda_0}, u(\lambda_0, \cdot)\rangle > -\infty$ at some λ_0 is automatic if the family is of large topological degree, by Theorem 1.2.16. Moreover, notice that (1.6) holds without the need of taking the subsequence (and the right hand side is in particular independent from the subsequence used to compute $\mathcal{E}$). Finally, we do not need to restrict M to get the statement since Ω is compactly supported. This also follows from the compactness of horizontal positive closed currents with bounded slice mass, see [DS06].

Proof. We can suppose that Ω is a positive volume form, since we can decompose it in its positive and negative parts $\Omega = \Omega^+ - \Omega^-$ and prove the statement for Ω^+ and Ω^- separately. Moreover, by means of a partition of unity on M, we can also assume that $\mathcal{E}$ is horizontal. By Theorem 1.3.2, the product $u\mathcal{E}$ is well defined and the identity (1.2) holds with both $\mathcal{R} = \mathcal{E}$ or S_n and $\psi = u$. So, it suffices to prove that

$$\int_M \langle S_n, \pi, \lambda\rangle(u)\Omega(\lambda) \to \int_M \langle \mathcal{E}, \pi, \lambda\rangle(u)\Omega(\lambda). \tag{1.7}$$

The assertion then follows since the slices of $\mathcal{E}$, and thus also the right hand side, are independent from the particular equilibrium current chosen.

Set $\varphi_n(\lambda) := \langle S_n, \pi, \lambda\rangle\,(u(\lambda,\cdot))$ and $\varphi(\lambda) := \langle \mathcal{E}, \pi, \lambda\rangle\,(u(\lambda,\cdot)) = \langle \mu_\lambda, u(\lambda,\cdot)\rangle$. By Theorem 1.3.2, the φ_n's and φ are psh functions on M. Moreover, at λ fixed, we have (recalling the definition (1.4) of the S_n's and the fact that $d_t^{-n}\,(f_\lambda^n)_*\,u(\lambda,\cdot) \to \langle \mu_\lambda, u(\lambda,\cdot)\rangle$ since $u(\lambda,\cdot)$ is psh, see Theorem 1.2.5)

$$\begin{aligned}\varphi_n(\lambda) = \langle S_n, \pi, \lambda\rangle\, u(\lambda,\cdot) &= \left\langle \frac{1}{d_t^n}\,(f_\lambda^n)^*\,(\theta), u(\lambda,\cdot)\right\rangle = \left\langle \theta, \frac{1}{d_t^n}\,(f_\lambda^n)_*\,u(\lambda,\cdot)\right\rangle \\ &\to \langle \theta, \langle \mu_\lambda, u(\lambda,\cdot)\rangle\rangle = \langle \mu_\lambda, u(\lambda,\cdot)\rangle = \varphi(\lambda).\end{aligned}$$

Since u is upper semicontinuous (and thus locally bounded) all the φ_n's are bounded from above. This, together with the fact that they converge pointwise to φ, gives that the convergence happens in L^1_{loc}. So, we have

$$\int \varphi_n \Omega \to \int \varphi \Omega$$

which is precisely the assertion to prove. □

The following Corollary immediately follows from Lemma 1.3.10, using $u = \log|\mathrm{Jac}|$ and Theorem 1.2.10.

Corollary 1.3.11. *Let $f : \mathcal{U} \to \mathcal{V} = M \times V \subset \mathbb{C}^m \times \mathbb{C}^k$ be a holomorphic family of polynomial-like maps. Let $\mathcal{E}$ be an equilibrium current and S_n be a sequence of smooth forms as in* (1.4)*. Then for every smooth $(m-1, m-1)$-form Ω compactly supported on M we have*

$$\langle C_f \wedge S_n, \pi^*(\Omega)\rangle \to \langle C_f \wedge \mathcal{E}, \pi^*(\Omega)\rangle\,.$$

We end this chapter with two consequences of the following continuity result, valid for families of polynomial-like maps of large topological degree. Given an open subset of $\mathbb{C}^k$, the topology on the space of psh functions is the L^1_{loc} one (with respect to the Lebesgue measure). Recall that this topology coincides with the weak topology, as well as with the L^p_{loc} topology, for every $1 < p < \infty$.

Theorem 1.3.12 (Dinh-Sibony[DS10])**.** *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Let $\lambda_0 \in M$ and W be a neighbourhood of the filled Julia set K_{λ_0} of f_{λ_0}. Then, there exists a neighbourhood M_0 of λ_0 such that $\langle \mu_\lambda, \varphi\rangle$ depends continuously on (λ,φ) in $M_0 \times PSH(W)$.*

The first consequence is that the Lyapounov function $L(\lambda)$ is continuous in λ. It is actually possible to prove a stronger version of Theorem 1.3.12, implying the L is Hölder continuous.

Corollary 1.3.13. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Then, L is continuous on M.*

Finally, we deduce from Theorem 1.3.12 that the Julia set varies lower semicontinuously with the parameter. This follows from the following elementary Lemma. We say that a family of measures ν_λ is *continuous* if $\langle \nu_\lambda, \varphi\rangle$ is continuous for every smooth test form φ.

Lemma 1.3.14. *Let ν_λ, with $\lambda \in M$, be a continuous family of probability measures, compactly supported in a compact subset of $\mathbb{C}^k$. Then, the map*

$$\lambda \mapsto \mathrm{Supp}\,\nu_\lambda$$

is lower semicontinuous (with respect to the Hausdorff topology): given $\lambda_0 \in M$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{Supp} \nu_{\lambda_0}$ is contained in a ε-neighbourhood of $\operatorname{Supp} \nu_\lambda$ if $|\lambda - \lambda_0| < \delta$.

Proof. We denote by $(X)_\varepsilon$ the ε-neighbourhood of a compact set X. Let $p \notin (\operatorname{Supp} \nu_{\lambda_n})_\varepsilon$ for some $\lambda_n \to \lambda_0$. We prove that $p \notin \operatorname{Supp} \nu_{\lambda_0}$. By assumption, the ball $B(p,\varepsilon)$ does not intersect the support of any measure ν_{λ_n}, and this means that $\langle \nu_{\lambda_n} \rho \rangle = 0$ for every test function ρ supported in $B(p,\varepsilon)$. By the continuity of the measures, it follows that $\langle \nu_{\lambda_0}, \rho \rangle = 0$, too, and this gives the assertion. □

The lower semicontinuity of Julia sets then immediately follows from Theorem 1.3.12, since every smooth function with compact support is a difference of psh ones (not compactly supported).

Corollary 1.3.15. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Then the Julia set (i.e., the support of the equilibrium measure μ_λ) varies lower semicontinuously.*

2
First notions of stability

In this chapter we are going to prove the equivalence between several notions of stability for a holomorphic family of polynomial-like maps. This consists of a generalization to this more general setting of works by Berteloot and Dupont in the case of endomorphisms of $\mathbb{P}^k$. The main difference with respect to their work is Section 2.2.3, and in particular the proof of Theorem 2.2.12. While their approach exploits the existence of a Green function, we give here a different and more geometrical proof in our setting, motivated by the fact that a Green function does not exist in our situation.

2.1. Equilibrium webs

Recall that *holomorphic families* of polynomial-like maps were defined in 1.3.1. By the results of Chapter 1, we can associate to every polynomial-like map a well-defined *equilibrium measure*, whose support is, by definition, the Julia set. Our goal in this section is to introduce and study a notion of *holomorphic motion* for the equilibrium measures. We shall see in subsequent sections how this definition relates with other (more classical) definitions of stability (like the holomorphic motion of repelling cycles, or the absence of Misiurewicz parameters) for a family of polynomial-like maps.

In order to introduce this notion of stability for the equilibrium measure, we have to set some definitions. We shall be mainly concerned with the space of holomorphic maps

$$\mathcal{O}\left(M,\mathbb{C}^k,\mathcal{V}\right) := \{\,\gamma \in \mathcal{O}\left(M,\mathbb{C}^k\right) : \forall \lambda \in M, \gamma(\lambda) \in V_\lambda\,\}\,.$$

We endow this space with the topology of local uniform convergence. Since the family f is a map from $\mathcal{U}$ to $\mathcal{V}$, it is natural to consider the subset of $\mathcal{O}\left(M,\mathbb{C}^k,\mathcal{V}\right)$ given by

$$\mathcal{O}\left(M,\mathbb{C}^k,\mathcal{U}\right) := \{\,\gamma \in \mathcal{O}\left(M,\mathbb{C}^k\right) : \forall \lambda \in M, \gamma(\lambda) \in U_\lambda\,\}\,.$$

Notice that, by Montel Theorem, the subset $\mathcal{O}(M,\mathbb{C}^k,\mathcal{U})$ is relatively compact in $\mathcal{O}(M,\mathbb{C}^k,\mathcal{V})$.

The map f induces an action from this subset to all of $\mathcal{O}\left(M,\mathbb{C}^k,\mathcal{V}\right)$ by

$$\begin{array}{rcl} \mathcal{F}\colon \mathcal{O}\left(M,\mathbb{C}^k,\mathcal{U}\right) & \to & \mathcal{O}(M,\mathbb{C}^k,\mathcal{V}) \\ \gamma & \mapsto & \mathcal{F}\cdot\gamma, \end{array}$$

where $(\mathcal{F}\cdot\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$. The map $\mathcal{F}:\mathcal{O}(M,\mathbb{C}^k,\mathcal{U})\to\mathcal{O}(M,\mathbb{C}^k,\mathcal{V})$ gives a (partial) dynamical system, in the same way than every polynomial-like slice $f_\lambda : U_\lambda \to V_\lambda$. In order to recover an actual dynamical system, we directly restrict ouselves to the subset $\mathcal{J}$ of $\mathcal{O}(M,\mathbb{C}^k,\mathcal{U})$ given by

$$\mathcal{J} := \{\gamma\in\mathcal{O}(M,\mathbb{C}^k,\mathcal{U})\colon \gamma(\lambda)\in J_\lambda \text{ for every } \lambda\in M\}.$$

Notice that, according to Montel Theorem, $\mathcal{J}$ is a compact metrizable space for the topology of local uniform convergence. Moreover, since given any $\gamma\in\mathcal{J}$ the map $\mathcal{F}\cdot\gamma$ still belongs to $\mathcal{J}$, this set is $\mathcal{F}$-invariant and $\mathcal{F}$ induces a well-defined dynamical system on it.

We notice that nothing prevents the set $\mathcal{J}$ to be actually empty. The following lemma ensures that, for families of polynomial-like maps of large topological degree (see Definition 1.2.15), even if $\mathcal{J}$ may be empty over some parameter space M every point admits a neighbourhood where this set (for the induced family) is not empty. This is a quite direct consequence of the lower semicontinuity of the Julia sets (see Corollary 1.3.15).

Lemma 2.1.1. *Let $f:\mathcal{U}\to\mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t\geq 2$ and $\rho\in\mathcal{O}(M,\mathbb{C}^k,\mathcal{U})$ such that $\rho(\lambda)$ is n-periodic for every $\lambda\in M$. Then, the set*

$$J_\rho := \{\,\lambda\in M\colon \rho(\lambda) \textit{ is } n\textit{-periodic, repelling and belongs to } J_\lambda\,\}$$

is open.

On the other hand, it may happen that a repelling cycles leaves the Julia set (without stopping being repelling). An example of this phenomenon is given in Section 2.1.3.

Proof. Assume there exists $\lambda_0\in M$ such that $\rho(\lambda_0)$ is repelling and belongs to J_{λ_0}. Since being repelling is an open condition, we can suppose that $\rho(\lambda)$ is repelling for every λ. We are going to prove that, in a sufficiently small neighbourhood of λ_0, we have $\rho(\lambda)\in J_\lambda$, thus proving the assertion. Up to replacing f with f^n, we can suppose that $\rho(\lambda)$ is actually fixed and repelling for every λ and, up to locally change coordinates, we can also suppose that $\rho(\lambda)=0$ for every λ.

Since $\rho(\lambda)\equiv 0$, there exists a small polydisc $D=D_1\times D_2\subset M\times\mathbb{C}^k$ centered at $(\lambda_0,0)$ such that f is injective and uniformly expanding on D. Fix a $\varepsilon>0$ smaller than the radius of D_2. By the lower semicontinuity of J_λ (Corollary 1.3.15), there exists a positive δ such that, for every $\lambda\in B(\lambda_0,\delta)$, the Julia set J_{λ_0} is contained in the ε-neighbourhood $(J_\lambda)_\varepsilon$ of J_λ (and, without loss of generality, we can suppose that $B(\lambda_0,\delta)\subset D_1$). This implies that, for every $\lambda\in B(\lambda_0,\delta)$, there exists a point $p_\lambda\in B(0,\varepsilon)\cap J_\lambda$.

But now, by the expansivity of f on D, this implies that, for every λ, $0=\rho(\lambda)$ is accumulated by preimages of a point of J_λ, and thus by points of J_λ. Since J_λ is closed, this implies that $0=\rho(\lambda)$ belong to J_λ for every λ sufficiently close to λ. This proves the assertion. □

In the following sections we introduce and give the first properties of natural measures (strictly related to the equilibrium measure at every slice) that we can consider on $\mathcal{J}$.

2.1.1. Definition and basic properties

Let $\mathcal{M}$ be any probability measure compactly supported on $\mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$. Notice that we can naturally associate to it the (k,k)-current on $\mathcal{U}$ given by

$$W_{\mathcal{M}} := \int [\Gamma_\gamma] d\mathcal{M}(\gamma), \tag{2.1}$$

where $[\Gamma_\gamma]$ denotes the integration current on the graph $\Gamma_\gamma \subset \mathcal{U}$ of the map $\gamma \in \mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$. This is a well defined closed positive current on $\mathcal{U}$. We can also see $W_{\mathcal{M}}$ as a (positive closed) current on $\mathcal{V}$.

By definition of holomorphic family, every parameter $\lambda_0 \in M$ has a neighbourhood M_{λ_0} such that the $U_\lambda \Subset \widetilde{U} \Subset \widetilde{V} \Subset V_\lambda$. for every $\lambda \in M_{\lambda_0}$. This implies in particular that the current $W_{\mathcal{M}}$ is *horizontal* (see Definition A.1.9) on $M_{\lambda_0} \times \widetilde{V}$. Given $\lambda \in M$, we can thus consider the slice $\langle W_{\mathcal{M}}, \pi, \lambda \rangle$ (see Definition A.1.3), which can be seen as a measure on V_λ. We denote this measure by $\mathcal{M}_\lambda$. In this case, we have a very explicit description of $\mathcal{M}_\lambda$. Indeed, we have

$$\mathcal{M}_\lambda = \int \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma) = (p_\lambda)_* \mathcal{M},$$

where $p_\lambda : \mathcal{O}(M, \mathbb{C}^k, \mathcal{V}) \to V_\lambda$ is the evaluation map at λ given by $p_\lambda(\gamma) := \gamma(\lambda)$. Notice in particular that the operator p_λ is linear and continuous.

The map $\mathcal{F}\colon \mathcal{O}(M, \mathbb{C}^k, \mathcal{U}) \to \mathcal{O}(M, \mathbb{C}^k, \mathcal{V})$ is proper. This follows from Montel Theorem since, for any λ, p_λ is continuous and $f_\lambda : U_\lambda \to V_\lambda$ is proper. This means in particular that $\mathcal{F}$ induces a well defined notion of pushforward from the measures on $\mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$ to those on $\mathcal{O}(M, \mathbb{C}^k, \mathcal{V})$.

In the sequel, we shall be mainly interested on measures $\mathcal{M}$ supported on the compact metric space $\mathcal{J}$. In the following definition we introduce a central object in all our study.

Definition 2.1.2. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. An* equilibrium web *is a probability measure $\mathcal{M}$ on $\mathcal{J}$ such that:*

1. *$\mathcal{F}_* \mathcal{M} = \mathcal{M}$, and*
2. *$(p_\lambda)_* \mathcal{M} = \mu_\lambda$ for every $\lambda \in M$.*

We shall say that the equilibrium measures μ_λ move holomorphically *(over M) if f admits an equilibrium web.*

Notice in particular that, since $\mathcal{J}$ is compact, every equilibrium web is automatically compactly supported. Moreover given an equilibrium web $\mathcal{M}$, we can see the triple $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ as an invariant dynamical system.

The picture to have in mind in order to handle with this kind of objects is the following: we have a set of graphs in the product space (to be thought of as the support of $\mathcal{M}$) and what the second condition says is that what we see when we slice this picture at any parameter λ is precisely the equilibrium measure μ_λ of f_λ.

In the next lemma we give some elementary properties of the support of any equilibrium web.

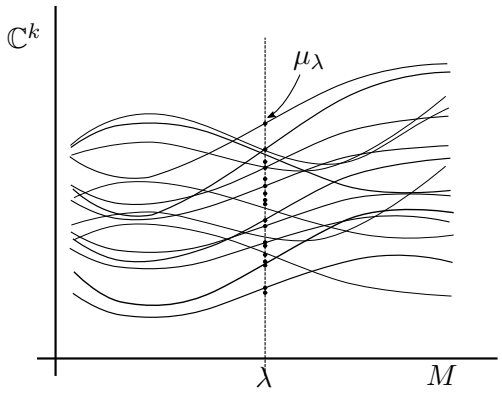


Figure 2.1.: an equilibrium web

Lemma 2.1.3. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of degree $d_t \geq 2$. Assume that f admits an equilibrium web $\mathcal{M}$. Then*

1. *for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ there exists an element $\gamma \in \operatorname{Supp} \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$, and*
2. *for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ such that z_0 is n-periodic and repelling for f_{λ_0} there exists a unique $\gamma \in \operatorname{Supp} \mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$ and $\gamma(\lambda)$ is n-periodic for f_λ for every $\lambda \in M$. Moreover, $\gamma(\lambda_0) \neq \gamma'(\lambda_0)$ for every $\gamma' \in \operatorname{Supp} \mathcal{M}$ different from γ.*

Proof. Since $(p_{\lambda_0})_* \mathcal{M} = \mu_{\lambda_0}$, there exist elements $\gamma_n \in \operatorname{Supp} \mathcal{M}$ such that $\gamma_n(\lambda_0) \to z_0$. Since $\mathcal{J}$ is compact (and thus the same is true for $\operatorname{Supp} \mathcal{M}$), any limit value γ of the sequence $(\gamma_n)_n$ belons to $\mathcal{J}$ and satisfies $\gamma(\lambda_0) = z_0$. This proves the first point.

For the second one we notice that, by the implicit function theorem, there exist a holomorphic function w, defined on some neighbourhood M' of λ_0 that gives a holomorphic motion of z_0 as a periodic point, i.e., such that $w(\lambda_0) = z_0$ and $f_\lambda^n(w(\lambda)) = w(\lambda)$ for every $\lambda \in M'$. By analytic continuation, it suffices to show that every $\gamma \in \operatorname{Supp} \mathcal{M}$ as in 1 coincides with w on M'. We can assume, without loss of generality, that $w(\lambda)$ is (uniformly) repelling on M'. This gives constants $A > 1$ and $r > 0$ such that

$$\|w(\lambda) - f_\lambda^n(z)\| = \|f_\lambda^n(w(\lambda)) - f_\lambda^n(z)\| \geq A \|w(\lambda) - z\|$$

for every $\lambda \in M'$ and z such that $\|w(\lambda) - z\| < r$. Up to shrinking M', we have that $\|w(\lambda) - \gamma(\lambda)\| < r$ on M' and, since $\operatorname{Supp} \mathcal{M}$ is compact, up to shrinking M' again we can also assume that the same is true for all the orbit of γ, i.e., that $\|w(\lambda) - f^{mn}\gamma(\lambda)\| < r$ on M' for every $m \geq 1$. This, combined with the expansion estimate above, gives the desired coincidence of w and γ. □

2.1.2. Construction of equilibrium webs

Since equilibrium webs will play a crucial role in our study, it will be very important to be able to build them. We present here two basic methods to do it. They are both applications of the following general result, which has to be thought of as a counterpart of the basic λ-lemma in our higher-dimensional setting.

Theorem 2.1.4. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of degree $d_t \geq 2$. Let μ_λ be the equilibrium measure of f_λ and assume that there exists a sequence of probability measures $\mathcal{M}_n$ on $\mathcal{O}\left(M, \mathbb{C}^k, \mathcal{U}\right)$ such that*

1. $\lim_n (p_\lambda)_* \mathcal{M}_n = \mu_\lambda$ *for every* $\lambda \in M$;
2. $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ *or* $\mathcal{F}_* \mathcal{M}_n = \mathcal{M}_n$ *for every* $n \geq n_0$.

Then the equilibrium measures μ_λ move holomorphically and any limit of $\left(\frac{1}{n}\sum_{l=1}^n \mathcal{M}_l\right)_n$ is an equilibrium web.

Proof. We set $\widetilde{\mathcal{M}}_n := \frac{1}{n}\sum_{l=1}^n \mathcal{M}_l$ and denote by $\widetilde{\mathcal{M}}$ any limit of the sequence $\left(\widetilde{\mathcal{M}}_n\right)$, say with $\widetilde{\mathcal{M}}_{n_j} \to \widetilde{\mathcal{M}}$. From 2 we get that $\widetilde{\mathcal{M}}$ is a $\mathcal{F}_*$-invariant probability measure on $\mathcal{O}(M, \mathbb{C}^k, \mathcal{V})$ (since all the $\widetilde{\mathcal{M}}_n$'s have locally bounded mass). Moreover, since $(p_\lambda)_*$ is continuous, condition 1 gives that the slice of $\widetilde{\mathcal{M}}$ at every λ is equal to μ_λ.

We are thus only left to showing that the support of $\widetilde{\mathcal{M}}$ is contained in $\mathcal{J}$. Let $\gamma_0 \in \mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$ such that $\gamma_0(\lambda_0) \notin J_{\lambda_0}$ for some $\lambda_0 \in M$. We are going to find an open neighbourhood $\mathcal{A}_0$ of γ_0 in $\mathcal{O}(M, \mathbb{C}^k, \mathcal{V})$ such that $\widetilde{\mathcal{M}}(\mathcal{A}_0) = 0$. To do so, consider a small neighbourhood A_0 of $\gamma(\lambda_0)$ such that $\mu_{\lambda_0}(A_0) = 0$ (it exists since $\gamma_0(\lambda_0) \notin J_{\lambda_0}$) and set $\mathcal{A}_0 := p_\lambda^{-1}(A_{\lambda_0})$, i.e., $\mathcal{A}_0 := \{\, \gamma \in \mathcal{O}(M, \mathbb{C}^k, \mathcal{V}) \colon \gamma(\lambda_0) \in A_0 \,\}$. Since $\mu_{\lambda_0} = (p_{\lambda_0})_* (\widetilde{\mathcal{M}})$, we get $\widetilde{\mathcal{M}}(\mathcal{A}_0) = \mu_{\lambda_0}(A_0) = 0$ and the assertion is proved. □

As in the case of families of endomorphisms of $\mathbb{P}^k$, Theorem 2.1.4 allows us to recover the existence of equilibrium webs from the existence of particular elements in $\mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$, at least for families of large topological degree (see Definition 1.2.15). The first way follows from Theorem 1.2.20.

Proposition 2.1.5. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Assume that the parameter space M is simply connected and that there exists $\gamma \in \mathcal{O}(M, \mathbb{C}^k, \mathcal{U})$ such that the graph Γ_γ does not intersect the post-critical set of f. Then, the equilibrium measures move holomorphically and any limit of $\left(\frac{1}{n}\sum_{l=1}^n \frac{1}{d_t^l}\sum_{\mathcal{F}^l\sigma=\gamma} \delta_\sigma\right)_n$ is an equilibrium web.*

Notice the assumption on the parameter space to be simply connected. This is needed to ensure the existence of the preimages.

Proof. Since the graph of γ does not intersect the postcritical set of f, for every n there exist d_t^n elements $\sigma_{n,j}$ such that $\mathcal{F}^n \sigma_{n,j} = \gamma$. By Theorem 1.2.20, this implies that the measures $\mathcal{M}_n := \frac{1}{d_t^n}\sum_{j=1}^{d_t^n} \delta_{\sigma_{n,j}} = \frac{1}{d_t^n}\sum_{\mathcal{F}^n\sigma=\gamma} \delta_\sigma$ satisfy the hypotheses of Theorem 2.1.4. The assertion follows. □

The second way exploits Theorem 1.2.19. In order to state this, we need to give a preliminary definition.

Definition 2.1.6. *For every $\lambda \in M$, a* repelling J-cycle is a repelling cycle which belongs to J_λ. *We say that the J-repelling cycles move holomorphically over M if, for every n, there exists a finite subset $\{\rho_{n,j}, 1 \le j \le N_n\}$ of $\mathcal{J}$ such that $\{\rho_{n,j}(\lambda), 1 \le j \le N_n\}$ is precisely the set of n-periodic repelling J-cycles of f_λ for every $\lambda \in M$.*

Proposition 2.1.7. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \ge 2$. Assume that the repelling J-cycles of f move holomorphically over the parameter space M and let $(\rho_{n,j})_{1\le j\le N_n}$ be the elements of $\mathcal{J}$ given by the motion of these n-periodic repelling cycles. Then, the equilibrium measures move holomorphically and any limit of $\left(\frac{1}{d_t^n}\sum_{j=1}^{N_n}\delta_{\rho_{n,j}}\right)_n$ is a equilibrium web.*

Proof. The result follows from Theorem 2.1.4 and from the equidistribution of repelling periodic points with respect to the equilibrium measure of a polynomial-like maps (see Theorem 1.2.19). Notice that, since for every n the measure $\frac{1}{d_t^n}\sum_{j=1}^{N_n}\delta_{\rho_{n,j}}$ is already $\mathcal{F}_*$-invariant, we do not need to take the Cesaro average in the proof of Theorem 2.1.4. □

2.1.3. A repelling point leaving the Julia set

In this section we give a simple example of a family of polynomial endomorphisms of $\mathbb{C}^2$ (which in particular are polynomial-like maps and extend to holomorphic endomorphisms of $\mathbb{P}^2$) with a repelling point leaving the Julia set. More precisely, we consider the family given by

$$F_\lambda(z,w) = (1-\lambda)\begin{pmatrix} z^2+\delta w \\ z-\varepsilon w^2\end{pmatrix} + \lambda\begin{pmatrix} w^2+2w \\ z^2+2z\end{pmatrix} \tag{2.2}$$

where δ is any fixed real number strictly greater than 1, and ε is any positive real number such that $\delta\varepsilon < \frac{1}{16}$. We claim that this family satisfies the following properties:

1. for (complex) λ in a neighbourhood of the real segment $[0,1]$, F_λ extends to an endomorphisms of $\mathbb{P}^2$;
2. $(0,0)$ is a repelling fixed point for each F_λ;
3. $(0,0) \notin J_0$;
4. $(0,0) \in J_1$.

This gives an example of a repelling cycles leaving the Julia set. Below we prove the four assertions.

1 - Extension to $\mathbb{P}^2$ We must verify that, denoting with $\widetilde{F}_\lambda^1$ and $\widetilde{F}_\lambda^2$ the homogeneous parts of maximal degree of the two components of F_λ, we have $\left(\widetilde{F}_\lambda^1\right)^{-1} \cap \left(\widetilde{F}_\lambda^2\right)^{-1} = \{(0,0)\}$ for every $\lambda \in [0,1]$. This amounts to prove that $(0,0)$ is the only solution of the system

$$\begin{cases}(1-\lambda)z^2 + \lambda w^2 = 0\\ -(1-\lambda)\varepsilon w^2 + \lambda z^2 = 0.\end{cases} \tag{2.3}$$

For $\lambda = 1$, the assertion is immediate. Otherwise, from the first equation we get $z^2 = -\frac{\lambda}{1-\lambda}w^2$ that, substituted in the second, gives $w^2\left(-\varepsilon(1-\lambda) - \frac{\lambda^2}{1-\lambda}\right) = 0$. So, we want to check that $-\varepsilon(1-\lambda) - \frac{\lambda^2}{1-\lambda} \neq 0$ under our hypotheses. But this is true, because

$$-(1-\lambda)\left(-\varepsilon(1-\lambda) - \frac{\lambda^2}{1-\lambda}\right) = \varepsilon(1-\lambda)^2 + \lambda^2 \tag{2.4}$$

is strictly positive for λ real, by the assumpion that ε is real and strictly positive.

So, the family F_λ extends to a family of endomorphisms of $\mathbb{P}^2$, for λ in a neighbourhood of the real line. Notice that here we used the fact that the term εw^2 in non zero, to extend F_0, as well as the choice $F_1(z,w) = (w^2, z^2)$ (the argument would not work with $F_1(z,w)$ equal to (z^2, w^2)).

2 - $(0,0)$ **is a repelling fixed point for each** F_λ The differential at $(0,0)$ of F_λ is

$$(DF_\lambda)_{(0,0)} = \begin{pmatrix} 0 & (1-\lambda)\delta + 2\lambda \\ (1-\lambda) + 2\lambda & 0 \end{pmatrix}. \tag{2.5}$$

So, being the trace equal to zero, we only need to verify that the modulus of the determinant is greater than 1 for $\lambda \in [0,1]$. But all the quantities in (2.5) are real, so also the determinant is real. It suffices then to show that the product of $(1+\lambda)$ and $(1-\lambda)\delta + 2\lambda$ is greater than one. But with $\lambda \in [0,1]$ the first is greater than or equal to 1, while the second is strictly greater than 1 (because of the assumption that $\delta > 1$), and the assertion follows.

3 - $(0,0) \notin J_0$ In this section it shall be useful to make it explicit the dependence of F_0 on ε. In particular, we shall denote by F_0^ε the map

$$F_0^\varepsilon \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z^2 + \delta w \\ z - \varepsilon w^2 \end{pmatrix}.$$

The argument presented here is essentially taken from [FS01, Section 4.1].

In [FM89], Friedland and Milnor introduced a decomposition of $\mathbb{C}^2$ defined as following: fix an $R > 0$ and define the three sets

$$\begin{aligned} V_R &= \{|z|, |w| \leq R\}, \\ V_R^+ &= \{|z| \geq R, |w| \leq |z|\}, \\ V_R^- &= \{|w| \geq R, |z| \leq |w|\}. \end{aligned} \tag{2.6}$$

They proved that, for a Hénon automorphism h of $\mathbb{C}^2$, there exists an R such that

$$h\left(V_R \cup V_R^+\right) \subset V_R \cup V_R^+. \tag{2.7}$$

In particular, this holds for the automorphism f^0 of $\mathbb{C}^2$ given by

$$f^0 \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z^2 + \delta w \\ z \end{pmatrix}.$$

Remark that $f^0 = F_0^0$. Property (2.7) holds also for F_0^ε, for sufficiently small ε, in the sense that there exist an $R > 0$ and an $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, we have

$$F_0^\varepsilon(V_R \cup V_R^+) \subset V_R \cup V_R^+. \tag{2.8}$$

More precisely, we have the following Proposition, essentially contained in [FS01]. Remark that this does not immediately follow from Friedland-Milnor result, since F_0^ε is not an automorphism of $\mathbb{C}^2$ for $\varepsilon \neq 0$.

Proposition 2.1.8 (cp. Lemma 4.1 [FS01]). *For every $R > 8\delta$ the property* (2.8) *holds for F_0^ε, with $0 < \varepsilon < \frac{1}{2R}$.*

Proof. We shall adopt the following notation:

$$\begin{cases} z_1 := z^2 + \delta w \\ w_1 := z - \varepsilon w^2. \end{cases}$$

We start proving the following: there exist a R and and ε_1 such that, for $0 < \varepsilon < \varepsilon_1$, in $V_{R/2}^+$ we have

$$\begin{aligned} &a) \quad |z_1| \geq \frac{|z|^2}{2}; \\ &b) \quad |w_1| \leq \frac{|z_1|^2}{2}. \end{aligned} \tag{2.9}$$

More precisely, this is true for

$$\begin{cases} \varepsilon_1 < \frac{1}{8} \\ R > 8\delta. \end{cases} \tag{2.10}$$

The existence of some such R sufficiently large and ε_1 sufficiently small follows from the definition of $V_{R/2}^+$. Let us verify that with the proposed values the requests are actually satisfied.

For a), we are asking that, on $V_{R/2}^+$, we have $|z^2 + \delta w| \geq \frac{|z|^2}{2}$. Since $|z^2 + \delta w| \geq |z|^2 - \delta|w|$, it suffices to find a R such that on $V_{R/2}^+$ we have $\frac{|z|^2}{2} > \delta|w|$. But on $V_{R/2}^+$ we have $|z| > |w|$, so it suffices to satisfy $\frac{|z|}{2} > \delta$, which gives $R > 4|\delta|$.

For the request b), we are asked that $\frac{|z^2+\delta w|}{2} \geq |z - \varepsilon w^2|$. Here it suffices to verify that $\frac{|z|^2-\delta|w|}{2} \geq |z| + \varepsilon|w|^2$. Recalling that $|z| \geq |w|$, we get $\frac{|z|^2-\delta|w|}{2} \geq \frac{|z|^2-\delta|z|}{2} = |z|\frac{|z|-\delta}{2}$ and $|z|(|z| + \varepsilon|z|) = |z|^2 + \varepsilon|z|^2 \geq |z|^2 + \varepsilon|w|^2$. So, it suffices to verify that $\frac{|z|-\delta}{2} \geq 1 + \varepsilon|z|$. This leads to

$$|z| > \frac{\delta + 2}{1 - 2\varepsilon}$$

and so to

$$\begin{cases} \varepsilon_1 < \frac{1}{2} \\ R > 2\frac{\delta+2}{1-2\varepsilon}. \end{cases}$$

Summing up, we have found

$$\begin{cases} \varepsilon_1 < \frac{1}{2} \\ R > \max\{4\delta, \frac{2(\delta+2)}{1-2\varepsilon}\}. \end{cases} \tag{2.11}$$

In particular, being $\delta > 1$, we see that the conditions (2.10) imply (2.11), and so we are done.

The next step will consist in proving that, after possibly increasing R and reducing ε_1, we have $F_0^\varepsilon(V_R^+) \subset V_R \cup V_R^+$ and $F_0^\varepsilon(V_R) \subset V_R \cup V_R^+$.

Let us start with the first. We shall see that this request does not imply to modify the constants. We shall prove that

$$\begin{cases} |z| \geq R \\ |z| \geq |w| \end{cases} \Rightarrow \begin{cases} |z_1| \geq R \\ |z_1| \geq |w_1|. \end{cases}$$

Remark that $V_R^+ \subset V_{R/2}^+$, and so we have the estimates in (2.9). To prove that $|z_1| \geq R$, we use the estimate $a)$, so that it suffices to verify that $\frac{|z|^2}{2} \geq R$. This is always true in V_R^+ if $R \geq 2$ (which is implied by the previous request that $R > 8\delta$), and so we are done.

To prove that $|z_1| \geq |w_1|$, we use the estimate $b)$, which immediately gives the assertion. Remark that in particular we have proved that $F_0^\varepsilon(V_R^+) \subset V_R^+$, i.e., V_R^+ is forward invariant (as in Friedland-Milnor Theorem).

So we are left to prove that, for $R > 8\delta$ and $\varepsilon < \frac{1}{2R}$, we have $F_0^\varepsilon(V_R) \subset V_R \cup V_R^+$. This means that we have to prove that, on V_R, if $|w_1| \geq R$ then $|z_1| \leq |w_1|$, i.e.,

$$\begin{cases} |z| \leq R \\ |w| \leq R \\ |w_1| \geq R \end{cases} \Rightarrow |z_1| \geq |w_1|.$$

We shall proceed in the following way:

1. if $|z| \leq \frac{R}{2}$ (and $|w| \leq R$), we show that $|w_1| \leq R$;
2. if $(z, w) \in V_{R/2}^+ \cap V_R$, we show that $|z_1| \geq |w_1|$;
3. on the missing "triangle" given by $\frac{R}{2} < |z| \leq |w| \leq R$, we show that $|z_1| \geq |w_1|$.

For the first zone, we have $|w_1| = |z - \varepsilon w^2| \leq |z| + \varepsilon|w|^2 \leq \frac{R}{2} + \varepsilon R^2$. We want to impose that the last term is smaller than R. So, we get $\varepsilon < \frac{1}{2R}$.

Remark 2.1.9. *The presence of the ε is the reason for which we cannot do a "uniform" estimate on V_R, but we have to treat the case with $|z|$ near R separatedly (in fact, using $|z| = R$, the estimate above becomes $R + \varepsilon R^2 < R$, which can never be satisfied, due to the positivity of ε).*

For the second zone, we can use the estimates in (2.9). In particular, using $b)$, we immediately get the assertion.

We are left with the "triangle"

$$\frac{R}{2} < |z| \leq |w| \leq R, \tag{2.12}$$

where we want to prove that $|z_1| \geq |w_1|$. Being $R > 8\delta$, we have $|z|^2 \geq \frac{R^2}{4} > 16\delta^2 > \delta(8\delta) \geq \delta|w|$, which gives $|z^2 - \delta w| \geq |z|^2 - \delta|w|$. It thus suffices to prove that $|z|^2 - \delta|w| \geq |z| + \varepsilon|w|^2$, which is equivalent to

$$|z|(|z| - 1) = |z|^2 - |z| \geq \delta|w| + \varepsilon|w|^2$$

By (2.12), it suffices to verify that $\frac{R}{2}\left(\frac{R}{2} - 1\right) \geq \delta R + \varepsilon R^2$. Recalling that we already have $R > 8\delta$, we finally come to

$$\varepsilon \leq \frac{1}{8} - \frac{1}{2R}.$$

But, again because of $R > 8\delta > 8$, we have that $\frac{1}{8} - \frac{1}{2R} \geq \frac{1}{2R}$, the previous estimate found for ε, and the Proposition is proved. □

Corollary 2.1.10. *The endomorphism F_0 of the family* (2.2) *has the following property: there exists a $R > 8\delta$, such that $F_0(V_R \cup V_R^+) \subset V_R \cup V_R^+$, where V_R and V_R^+ are as in* (2.6).

Since $F_0^{-1}(V_R^-) \subset V_R^-$, it follows from Theorem 1.2.5 that the Julia set of F_0 is contained in V_R^-. Thus, $(0,0)$ is outside the Julia set.

4 - $(0,0) \in J_1$ This immediately follows since $F_1(z,w) = (w^2 + 2w, z^2 + 2z)$ is conjugated to $H(z,w) = (w^2, z^2)$ by means of a translation and $(1,1)$ is in the Julia set of H.

2.2. Equivalent characterizations

2.2.1. Definitions and statement

In order to state the main Theorem of this chapter we have to give a preliminary definition.

Definition 2.2.1. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. A point $\lambda_0 \in M$ is called a* Misiurewicz parameter *if there exist a neighbourhood $N_{\lambda_0} \subset M$ of λ_0 and a holomorphic map $\sigma \in \mathcal{O}(N_{\lambda_0}, \mathbb{C}^k, \mathcal{V})$ such that:*

1. *for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ and is a repelling periodic point;*
2. *$\sigma(\lambda_0) \in J_{\lambda_0}$;*
3. *there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0)) \in f^{n_0}(C)$;*
4. *$\sigma(N_{\lambda_0}) \nsubseteq f^{n_0}(C)$.*

Notice that, by Lemma 2.1.1, condition 2 actually implies that, without loss of generality, we have $\sigma(\lambda) \in J_\lambda$ for every $\lambda \in N_{\lambda_0}$ (up to shriking N_{λ_0}).

Theorem 2.2.2. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Then the following are equivalent:*

I.1 for every $\lambda_0 \in M$ there exists a neighbourhood $M_0 \Subset M$ where the measures μ_λ move holomorphically and f admits a equilibrium web $\mathcal{M} = \lim_n \mathcal{M}_n$ such that the graph of any $\gamma \in \cup_n \operatorname{Supp} \mathcal{M}_n$ avoids the critical set of f;

I.2 the function L is pluriharmonic on M;

I.3 there are no Misiurewicz parameters in M;

I.4 for every $\lambda_0 \in M$ there exists a neighbourhood $M_0 \Subset M$ and a holomorphic map $\gamma \in \mathcal{O}\left(M_0, \mathbb{C}^k, \mathcal{U}\right)$ such that the graph of γ does not intersect the postcritical set of f.

Notice that Proposition 2.1.5 readily proves that I.4 imples I.1. Moreover, if all the J-repelling n-periodic points of sufficiently high period move holomorphically, Proposition 2.1.7 implies that I.1 holds.

In the following sections we shall prove the three implications I.1 $\Rightarrow$ I.2 $\Rightarrow$ I.3 $\Rightarrow$ I.4 of Theorem 2.2.2. We remark here that the assumption that the family is of large topological degree will be used to prove all the implications of Theorem 2.2.2.

2.2.2. Pluriharmonicity of L: I.1 $\Rightarrow$ I.2

The aim of this section is to prove the following theorem, which gives the implication I.1 $\Rightarrow$ I.2 of Theorem 2.2.2.

Theorem 2.2.3. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree such that the equilibrium measures μ_λ move holomorphically. Assume that an equilibrium web $\mathcal{M}$ for the family if given by $\mathcal{M} = \lim \mathcal{M}_n$, where $\Gamma_\gamma \cap C_f = \emptyset$ for every $\gamma \in \cup_n \operatorname{Supp} \mathcal{M}_n$. Then $dd^c_\lambda L = 0$ on M.*

The argument will closely follow the one for families of endomorphisms of $\mathbb{P}^k$ given in [BD14b] (see Proposition 3.5). The only small difference is how to get an estimate (Holder in ε) for the μ-measure of a ε-neighbourhood of an analytic set. In the case of endomorphisms of $\mathbb{P}^k$, this follows from the Hölder continuity of the potential of the Green current. Here, as observed by Dinh and Sibony ([DS10, p. 256]) we may exploit the fact that the equilibrium measure of a polynomial-like map of large topological degree is *moderate* (see 1.2.17 and Theorem 1.2.18). The desired estimate is given in the following lemma.

Lemma 2.2.4. *Let $f : U \to V$ be a polynomial-like map of large topological degree and let μ be the equilibrium measure of f. Let Z be a codimension 1 analytic subset of V and denote by Z_ε the ε-neighbourhood of Z. Then, there exist two positive constants A and α such that, for every sufficiently small ε, we have that $\mu(Z_\varepsilon) \leq A\varepsilon^\alpha$.*

Proof. Since V is convex, Z is given by $\{ h = 0 \}$, where h is an holomorphic function on V. If ε is small enough, we can suppose that $Z_\varepsilon \cap \operatorname{Supp} \mu \subset \{ |h| < C_1\varepsilon \}$ for some positive constant C_1. So, up to rescaling, we have to prove that there exist positive constants A and α such that $\mu(\{ |h| < \varepsilon \}) \leq A\varepsilon^\alpha$.

To do this, consider the psh function $\log|h|$. Since f is of large topological degree, $\log|h| \in L^1(\mu)$ (see Theorem 1.2.16). So, by Theorem 1.2.18 there exist positive A and α such that

$$\left\langle \mu, e^{-\alpha \log|h|} \right\rangle \leq A.$$

Since on $\{\,|h|<\varepsilon\,\}$ we have $\frac{1}{\varepsilon^\alpha}\le\frac{1}{|h|^\alpha}$, it follows that

$$\mu\left(\{\,|h|<\varepsilon\,\}\right)\frac{1}{\varepsilon^\alpha}=\left\langle\mu,\frac{1}{\varepsilon^\alpha}1_{\{\,|h|<\varepsilon\,\}}\right\rangle\le\left\langle\mu,\frac{1}{|h|^\alpha}1_{\{\,|h|<\varepsilon\,\}}\right\rangle\le\left\langle\mu,e^{-\alpha\log|h|}\right\rangle\le A,$$

which gives the assertion. □

Once we have established Lemma 2.2.4, the proof of Theorem 2.2.3 can follow the one for endomorphisms of $\mathbb{P}^k$. The following lemma gives the key estimate on $\mathcal{M}$ that we need.

Lemma 2.2.5. *Let $f:\mathcal{U}\to\mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Let Z be a codimension 1 analytic subset of $\mathcal{V}$ which does not contain any fiber $\mathcal{V}\cap\pi_M^{-1}(\{\lambda\})$. Assume that μ_λ move holomorphically. and that there exists an equilibrium web satisfying $\mathcal{M}=\lim\mathcal{M}_n$, where $\Gamma_\gamma\cap Z=\emptyset$ for $\gamma\in\cup_n\operatorname{Supp}\mathcal{M}_n$. Then, for every $\lambda_0\in M$ there exist a ball B_{λ_0} compactly contained in M and two constants $A(\lambda_0)>0$ and $a(\lambda_0)>0$ such that*

$$\mathcal{M}\{\,\gamma\in\mathcal{J}:\Gamma_{\gamma|B_{\lambda_0}}\cap Z_\varepsilon\neq\emptyset\,\}\le A\varepsilon^a.$$

for every sufficiently small $\varepsilon>0$, where Z_ε is the ε-neighbourhood of Z.

In the proof of the above lemma we need the following elementary result.

Lemma 2.2.6. *Let B be the unit ball in $\mathbb{C}^k$ and B' a relatively compact ball in B. There exists a positive α such that $\sup_{B'}|\varphi|\le|\varphi(t_0)|^\alpha$ for every $t_0\in B'$ and every holomorphic function φ from B to the punctured unit disc $\mathbb{D}^*$.*

Proof. We first notice that the set $\mathcal{D}:=\{\,h\in\mathcal{O}(B,\mathbb{D}):\exists t\in\overline{B}':h(t)=0\,\}$ is compact (for the topology of the local uniform convergence). Indeed, by Montel Theorem, every sequence in $\mathcal{D}$ must have a converging subsequence to a function $h_0:B\to\mathbb{C}$. If this map is open we are done (since $\overline{B}'$ is compact). Otherwise, notice that the only possibility is that $h_0\equiv0$, and the assertion follows anyway.

This implies that the set $\mathcal{H}:=\{\,h\in\mathcal{O}(B,\mathbb{H}):\exists t\in\overline{B}':h(t)=-1\,\}$ (where $\mathbb{H}$ is the left half plane), is compact, too. So, there exists a positive constant $\alpha\in(0,1]$ such that $\sup_{t\in\overline{B}'}\operatorname{Re}h(t)\le-\alpha$, for every $h\in\mathcal{H}$.

Take now any $\varphi:B\to\mathbb{D}^*$ as in the statement and let $t_0\in\overline{B}'$. Without loss of generality, we can assume that $\varphi(t_0)\in(0,1)$. Lifting φ to $\mathbb{H}$ by the exponential map, we get a function $h_0:B\to\mathbb{H}$ satisfying $h_0(t_0)=\log\varphi(t_0)\in(-\infty,0)$. So the normalization $h_1:=-h_0/h_0(t_0)$ belongs to $\mathcal{H}$. Since then $\operatorname{Re}h_1(t)\le-\alpha$ on $\overline{B}'$ we get, for any $t\in B'$, that $|\varphi(t)|=e^{\operatorname{Re}h_0(t)}\le e^{\alpha\log\varphi(t_0)}=|\varphi(t_0)|^\alpha$, and the assertion follows. □

Proof of Lemma 2.2.5. Since we are dealing with a local problem, we can assume that $V_\lambda\Subset\mathbb{C}^k$ is constant, i.e., that $\mathcal{V}=M\times V$. We consider two open balls $B_{\lambda_0}\Subset B\Subset M$, both centered at λ_0. We shall prove the statement on B_{λ_0}, possibly shrinking V and the balls B_{λ_0} and B. We can assume without loss of generality that $Z\cap(B\times V)$ is irreducible. Since V (and thus $B\times V$) is convex, we can find a holomorphic function h defining Z on $B\times V$, i.e., such that $Z=\{\,h=0\,\}$. Now, if ε is small enough, we can assume that there exist positive constants C_1,C_2,C_3,τ,τ_0 such that

1. $Z_\varepsilon \subset \{\, |h| < C_1\varepsilon \,\}$;
2. $\{\, |h| < \varepsilon \,\} \subset Z_{C_2\varepsilon^\tau}$;
3. $Z_\varepsilon \cap (\{\, \lambda_0 \,\} \times V_{\lambda_0}) \subset (Z \cap (\{\, \lambda_0 \,\} \times V_{\lambda_0}))_{C_3\varepsilon^{\tau_0}}$.

One usesŁ ojasiewicz inequality for the second and the third inclusions. Let now $\gamma \in \operatorname{Supp}\mathcal{M}$ be such that $\Gamma_{\gamma|B_{\lambda_0}} \cap Z_\varepsilon \neq \emptyset$ (i.e., an element in the set whose $\mathcal{M}$-mass has to be estimated). By the assumptions on the approximations $\mathcal{M}_n$ of $\mathcal{M}$ and Hurwitz Lemma, it follows that either $\Gamma_\gamma \subset Z$ or $\Gamma_\gamma \cap Z = \emptyset$. Even if we are in the second case, point 2 above and Lemma 2.2.6 (applied with $h \circ \gamma$, whose modulus is uniformly bounded in γ, since $\operatorname{Supp}\mathcal{M}$ is compact) imply that $\Gamma_{\gamma|B_{\lambda_0}} \subset Z_{C_4\varepsilon^{\tau\alpha}}$, for some positive C_4 (notice the restriction to the smaller ball B_{λ_0}). So, we get

$$\begin{aligned}
\mathcal{M}\{\, \gamma \in \mathcal{J} \colon \Gamma_{\gamma|B_{\lambda_0}} \cap Z_\varepsilon \neq \emptyset \,\} &\leq \mathcal{M}\{\, \gamma \in \mathcal{J} \colon \Gamma_{\gamma|B_{\lambda_0}} \subset Z_{C_4\varepsilon^{\tau\alpha}} \,\} \\
&\leq \mathcal{M}\{\, \gamma \in \mathcal{J} \colon\ (\lambda_0, \gamma(\lambda_0)) \in Z_{C_4\varepsilon^{\tau\alpha}} \,\} \\
&= \mu_{\lambda_0}\left(Z_{C_4\varepsilon^{\tau\alpha}} \cap (\{\, \lambda_0 \,\} \times V_{\lambda_0})\right) \\
&\leq \mu_{\lambda_0}\left((Z \cap (\{\, \lambda_0 \,\} \times V_{\lambda_0}))_{C_3(C_4\varepsilon^{\tau\alpha})^{\tau_0}}\right)
\end{aligned}$$

The assertion then follows from Lemma 2.2.4. □

We can now prove Theorem 2.2.3.

Proof of Theorem 2.2.3. Since the statement is local, we can prove it on a ball B compactly contained in an other ball B', still compactly contained in the parameter space M. Recalling Definition 1.3.9, we have that

$$dd^c_\lambda L = (\pi_B)_*\left(\mathcal{W}_\mathcal{M} \wedge dd^c_{\lambda,z} \log|\mathrm{Jac}|\right),$$

since

$$\mathcal{W}_\mathcal{M} = \int_\mathcal{J} [\Gamma_\gamma]\, d\mathcal{M}(\gamma).$$

is an equilibrium current for f (see Definition 1.3.7). We are going to show that $\mathcal{W}_\mathcal{M} \wedge dd^c_{\lambda,z} \log|\mathrm{Jac}| = 0$, i.e., since $\mathcal{W}_\mathcal{M}$ is dd^c-closed, that $\log|\mathrm{Jac}|\,\mathcal{W}_\mathcal{M}$ is dd^c-closed. Since we can assume that $\mathcal{W}_\mathcal{M}$ is a horizontal positive closed current (shrinking B and recalling that the filled Julia set varies upper semicontinuously), $\log|\mathrm{Jac}|\,\mathcal{W}_\mathcal{M}$ is the limit of the currents $v_n\mathcal{W}_\mathcal{M}$, where the v_n's are smooth psh functions decreasing to $\log|\mathrm{Jac}|$ (by the monotone convergence Theorem). We take as approximation of $\log|\mathrm{Jac}|$ the sequence $\log_\varepsilon|\mathrm{Jac}| = \chi_\varepsilon \circ \log|\mathrm{Jac}|$, where χ_ε is some sequence (decreasing as $\varepsilon \to 0$) of convex increasing functions such that $\chi_\varepsilon(x) = x$ for $x \geq \log\varepsilon$ and $\lim_{x\to-\infty}\chi_\varepsilon(x) = 2\log_\varepsilon$. Our goal is then to prove that

$$\lim_{\varepsilon\to 0} dd^c \log_\varepsilon|\mathrm{Jac}|\,\mathcal{W}_\mathcal{M} = 0.$$

Set $W_\varepsilon := \{\, |\mathrm{Jac}| < \varepsilon \,\}$ and $S_{\mathcal{M},\varepsilon} := \{\, \gamma \in \operatorname{Supp}\mathcal{M} \colon \Gamma_{\gamma|B'} \cap W_\varepsilon \neq \emptyset \,\}$. We decompose $\mathcal{W}_\mathcal{M}$ as $\mathcal{W}_{\mathcal{M},\varepsilon} + \mathcal{W}^*_{\mathcal{M},\varepsilon}$, where

$$\mathcal{W}_{\mathcal{M},\varepsilon} = \int_\mathcal{J} [\Gamma_\gamma]\, 1_{S_{\mathcal{M},\varepsilon}} d\mathcal{M}(\gamma)$$

and $\mathcal{W}^*_{\mathcal{M},\varepsilon} = \mathcal{W}_{\mathcal{M}} - \mathcal{W}_{\mathcal{M},\varepsilon}$. We thus have

$$\log_\varepsilon |\mathrm{Jac}|\, \mathcal{W}_{\mathcal{M}} = \log_\varepsilon |\mathrm{Jac}|\, \mathcal{W}_{\mathcal{M},\varepsilon} + \log_\varepsilon |\mathrm{Jac}|\, \mathcal{W}^*_{\mathcal{M},\varepsilon}.$$

The second term of the right hand side is dd^c closed, since $\log|\mathrm{Jac}|$ is pluriharmonic on the graphs not in $S_{\mathcal{M},\varepsilon}$. In order to conclude, it is then enough to show that

$$\lim_{\varepsilon\to 0} \log_\varepsilon |\mathrm{Jac}|\, \mathcal{W}_{\mathcal{M},\varepsilon} = 0.$$

But this follows from Lemma 2.2.5: since $S_{\mathcal{M},\varepsilon} \subset \{\, \gamma \in \mathcal{J} \colon \Gamma_{\gamma|B'} \cap (C_f)_{b\varepsilon^\beta} \neq \emptyset \,\}$ for some positive b and β (byŁ ojasiewicz inequality), we have

$$\|\log_\varepsilon |\mathrm{Jac}|\, \mathcal{W}_{\mathcal{M},\varepsilon}\| \lesssim |\log\varepsilon|\, \mathcal{M}\left(S_{\mathcal{M},\varepsilon}\right) \lesssim \varepsilon^a\, |\log\varepsilon|\,,$$

where $\|T\| = \left\|\sum_{I,J} T_{IJ} dz_I \wedge \overline{z}_J\right\| := \sum |T_{IJ}|$ denotes the sum of the masses of the distributional coefficients of the (order 0) current T. This completes the proof. □

2.2.3. Misiurewicz parameters belong to Supp dd^cL: I.2 ⇒ I.3

The goal of this section is to prove the implication I.2 ⇒ I.3 of Theorem 2.2.2, i.e., that Misiurewicz parameters (see Definition 2.2.1) belong to the support of the bifurcation current dd^cL. This will be done in Theorem 2.2.12 below. The idea is relate the mass of dd^cL on a given open set Λ of the parameter space with the growth of the mass of the currents $f^n_*[C]$ on the vertical set $\mathcal{V} \cap \pi_M^{-1}(\Lambda)$. This is done in Theorem 2.2.8. Then, we will show how the presence of a Misiurewicz parameter allows us to get the desired estimate for the growth of the critical volume, permitting to conclude.

We shall need the following lemma, whose proof is a simple adaptation of the one of [DS10, Proposition 2.7]. Recall that the mass of a positive (p,p)-current T on a relatively compact subset K of $\mathbb{C}^k$ is given by $\int_K T \wedge \omega^{k-p}$, where ω is the usual Kähler form. This is related with the sum of the masses of the distributional coefficients of T by $\|T\| \leq \sum_{I,J} |T_{IJ}| \leq 2^p \|T\|$.

Lemma 2.2.7. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. Let $\delta > d^*_p(f_{\lambda_0})$. There exists a constant C such that, for λ sufficiently close to λ_0, we have*

$$\|(f^n_\lambda)_*(S)\|_{U_\lambda} \leq C\delta^n.$$

for every $n \in \mathbb{N}$ and every closed positive $(k-p,k-p)$-current S of mass less or equal than to 1 on U_λ.

Proof. Since the problem is local, we can assume that the parameter space is the unit ball $B \subset \mathbb{C}^m$, that $\lambda_0 = 0$ and that $\mathcal{V} = M \times V$. Moreover, there exists some open and convex set $\widetilde{U}$ such that $U_\lambda \Subset \widetilde{U} \Subset V \Subset \mathbb{C}^k$ for every $\lambda \in B$. Let ω be the standard Kähler form on V.

Fix some δ_1 such that $d^*_p < \delta_1 < \delta$. By the definition of d^*_p, there exists some large N such that $\left\|\left(f_0^N\right)_*(S)\right\|_{\widetilde{U}} \leq \delta_1^N$ for every positive closed $(k-p,k-p)$-current S of mass less or equal than 1 on $\widetilde{U}$. Moreover, if λ is sufficiently close to 0 we have $f_\lambda^{-N}(U_\lambda) \Subset f_0^{-N}(\widetilde{U})$ and thus

$$\left\|\left(f_\lambda^N\right)^* \omega^p - \left(f_0^N\right)^* \omega^p\right\|_{L^\infty(f_\lambda^{-N}(U_\lambda))} \leq \varepsilon$$

for some given ε such that $2^{p+2}\varepsilon < \delta^N - \delta_1^N$. Since S is positive and of mass at most 1, writing $S = \sum_{I,J} S_{IJ} dz_I \wedge d\overline{z}_J$ we have that the coefficients S_{IJ} are complex measures such that $\sum_{I,J} |S_{IJ}| \leq 2^p$. Thus, given any smooth (p,p)-form α, by splitting S and α in their real and imaginary parts we get $\left|\int S \wedge \alpha\right| \leq 4 \|\alpha\|_{L^\infty} \sum_{I,J} |S_{IJ}| \leq 2^{p+2} \|\alpha\|_{L^\infty}$. This implies that

$$\begin{aligned}\left\|\left(f_\lambda^N\right)_*(S)\right\|_{U_\lambda} &= \int_{f_\lambda^{-N}(U_\lambda)} S \wedge \left(f_\lambda^N\right)^* \omega^p \\ &\leq \int_{f_0^{-N}(\widetilde{U})} S \wedge \left(f_0^N\right)^* \omega^p + \int_{f_\lambda^{-N}(U_\lambda)} S \wedge \left[\left(f_\lambda^N\right)^* \omega^p - \left(f_0^N\right)^* \omega^p\right] \\ &\leq \left\|\left(f_0^N\right)_*(S)\right\|_{\widetilde{U}} + 2^{p+2}\varepsilon \leq \delta_1^N + 2^{p+2}\varepsilon < \delta^N.\end{aligned}$$

By induction on m, this implies the statement for every n of the form $n = Nm$. Namely, for every λ sufficiently close to 0, we have

$$\left\|\left(f_\lambda^{Nm}\right)_*(S)\right\|_{U_\lambda} \leq \delta^{Nm}$$

for every positive closed $(k-p, k-p)$-current on U_λ of mass less of equal than 1. Notice in particular the uniformity of the neighbourhood of $\lambda = 0$ with respect to S.

Let us then prove the general case. Since every n can be written in the form $n = Nm + r$, with $0 \leq r < N$, we have

$$\left\|\left(f_\lambda^n\right)_*(S)\right\|_{U_\lambda} = \left\|\left(f_\lambda^{Nm}\right)_*\left(\left(f_\lambda^r\right)_*(S)\right)\right\|_{U_\lambda} \leq \delta^{Nm} \cdot \left\|\left(f_\lambda^r\right)_*(S)\right\|_{U_\lambda}$$

and it is thus enough to bound the terms $\left\|\left(f_\lambda^r\right)_*(S)\right\|_{U_\lambda}$, uniformly on S (of mass less or equal than 1), on some neighbourhood of $\lambda = 0$. Since r takes a finite number of values, we may assume that $r = 1$. We have

$$\left\|(f_\lambda)_*(S)\right\|_{U_\lambda} = \int_{U_\lambda} \left((f_\lambda)_* S\right) \wedge \omega^p = \int_{f_\lambda^{-1}(U_\lambda)} S \wedge (f_\lambda)^* \omega^p \leq \int_{f_0^{-1}(\widetilde{U})} S \wedge (f_\lambda)^* \omega^p$$

for λ sufficiently close to 0. Since all the S's are of mass bounded by 1, it is enough to bound the L^∞-norm of $(f_\lambda)^* \omega^p$, uniformly on λ on some neighbourhood of 0. This follows since the forms $(f_\lambda)^*(\omega^p)$ are continuous in λ, and the assertion is proved. □

The following Theorem gives the relation between the mass of $dd^c L$ and the growth of the mass of $(f^n)_* \mathsf{C}_f$. We recall that $\mathsf{C}_f = dd^c \log|\mathrm{Jac}_f|$ is the integration on the critical set of f, counting the multiplicity. We set $\mathcal{U}_\Lambda := \mathcal{U} \cap \left(\Lambda \times \mathbb{C}^k\right)$ and $\mathcal{V}_\Lambda := \mathcal{V} \cap \left(\Lambda \times \mathbb{C}^k\right)$.

Theorem 2.2.8. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. Set $d_{k-1}^* := \sup_{\lambda \in M} d_{k-1}^*(f_\lambda)$, and assume that d_{k-1}^* is finite. Let δ be greater than d_{k-1}^*. Then for any open subset $\Lambda \Subset M$ there exist positive constants c_1', c_1 and c_2 such that, for every $n \in \mathbb{N}$,*

$$c_1' \left\|dd^c L\right\|_\Lambda d_t^n \leq \left\|(f^n)_* \mathsf{C}_f\right\|_{\mathcal{U}_\Lambda} \leq c_1 \left\|dd^c L\right\|_\Lambda d_t^n + c_2 \delta^n.$$

In particular, if

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(f^n)_* \mathsf{C}_f\|_{\mathcal{U}_\Lambda} > \log d^*_{k-1},$$

then Λ intersects the bifurcation locus.

Notice that $(f^n)_* \mathsf{C}_f$ actually denotes the current on $\mathcal{U}_\Lambda$ which is the pushforward by the proper map $f^n : f^{-n}(\mathcal{U}_\Lambda) \to \mathcal{U}_\Lambda$ of the current C_f on $f^{-n}(\mathcal{U}_\Lambda)$.

Proof. The problem is local. We can thus assume that $\mathcal{V} = M \times V$, where V is convex. Moreover, there exists some open convex set $\widetilde{U}$ such that $U_\lambda \Subset \widetilde{U} \Subset V$ for every $\lambda \in M$.

Let us denote by ω_V and ω_M the standard Kähler forms on $\mathbb{C}^k$ and $\mathbb{C}^m$. By abuse of notation, we denote by $\omega_V + \omega_M$ the Kähler form $\pi_V^*\omega_V + \pi_M^*\omega_M$ on the product space $M \times V$. Since both ω_V^{k+1} and ω_M^{m+1} are zero, by the definition of mass we have

$$\begin{aligned}
\|(f^n)_* \mathsf{C}_f\|_{\mathcal{U}_\Lambda} &= \int_{\mathcal{U}_\Lambda} (f^n)_* \mathsf{C}_f \wedge (\omega_V + \omega_M)^{k+m-1} \\
&= \binom{k+m-1}{k} \int_{\mathcal{U}_\Lambda} (f^n)_* \mathsf{C}_f \wedge \omega_V^k \wedge \omega_M^{m-1} \\
&\quad + \binom{k+m-1}{k-1} \int_{\mathcal{U}_\Lambda} (f^n)_* \mathsf{C}_f \wedge \omega_V^{k-1} \wedge \omega_M^m.
\end{aligned}$$

We shall bound the two integrals by means of $\|dd^c L\|_\Lambda \, d_t^n$ and δ^n, respectively. Let us start with the first one. Let ρ be a positive smooth function, compactly supported on V and equal to a constant c_ρ on $\widetilde{U}$. Assume moreover that the integral of ρ is equal to 1. Notice in particular that ρ/c_ρ is equal to 1 on $\widetilde{U}$ and has total mass $1/c_\rho$. Then

$$\int_{\mathcal{U}_\Lambda} (f^n)_* \mathsf{C}_f \wedge \omega_V^k \wedge \omega_M^{m-1} \le \frac{1}{c_\rho} \int_{\mathcal{V}_\Lambda} (f^n)_* \mathsf{C}_f \wedge \left(\pi_V^*\rho \cdot \omega_V^k\right) \wedge \omega_M^{m-1}.$$

Consider the smooth (k,k)-forms $S_n := \frac{(f^n)^*}{d_t^n}\left(\pi_V^*\rho \cdot \omega_V^k\right)$. By Theorem 1.3.5, every subsequence of $(S_n)_n$ has a further subsequence S_{n_i} converging to an equilibrium current $\mathcal{E}_{(n_i)}$. On the other hand, the bifurcation current $dd^c L = \pi_*(\mathsf{C}_f \wedge \mathcal{E}_{(n_i)})$ (see Definition 1.3.9) is independent from the particular equilibrium current used to compute it. Since $f^*\omega_M = \omega_M$, we then have

$$\begin{aligned}
d_t^{-n_i} \int_{\mathcal{V}_\Lambda} (f^{n_i})_* \mathsf{C}_f \wedge \left(\pi_V^*\rho \cdot \omega_V^k\right) \wedge \omega_M^{m-1} &= \int_{f^{-n_i}(\mathcal{V}_\Lambda)} \mathsf{C}_f \wedge S_{n_i} \wedge \omega_M^{m-1} \\
&\to \int_{\mathcal{V}_\Lambda} \mathsf{C}_f \wedge \mathcal{E}_{(n_i)} \wedge \omega_M^{m-1} = \|dd^c L\|_\Lambda ,
\end{aligned}$$

where the convergence follows from Corollary 1.3.11 (by means of a partition of unity on Λ). Since the limit is independent from the subsequence, the convergence above happens without the need of extraction (see also Lemma 1.3.10). In particular we have

$$\int_{\mathcal{V}_\Lambda} (f^n)_* \mathsf{C}_f \wedge \left(\pi_V^*\rho \cdot \omega_V^k\right) \wedge \omega_M^{m-1} \le \widetilde{c}_1 \, \|dd^c L\|_\Lambda \, d_t^n$$

for some positive constant $\widetilde{c}_1$ and the desired bound from above follows. The bound from below is completely analogous, by means of a function ρ equal to 1 on a neighbourhood of $\cup_\lambda \{\lambda\} \times K_\lambda$.

Let us then estimate the second integral. We claim that

$$\int_{\Lambda\times\widetilde{U}} (f^n)_* \mathsf{C}_f \wedge \omega_V^{k-1} \wedge \omega_M^m = \int_\Lambda \|(f_\lambda^n)_* \mathsf{C}_{f_\lambda}\|_{\widetilde{U}}\, \omega_M^m. \tag{2.13}$$

where C_{f_λ} is the integration current (with multiplicity) on the critical set of f_λ. The assertion then follows since, by Lemma 2.2.7, the right hand side in (2.13) is bounded by $\widetilde{c}_2\delta^n$, for some positive $\widetilde{c}_2$.

Let us thus prove (2.13). By Lemmas A.1.7 and A.1.8, the slice $\langle (f^n)_* \mathsf{C}_f, \pi, \lambda\rangle$ of $(f^n)_* \mathsf{C}_f$ exists for almost every $\lambda \in \Lambda$ and is given by

$$\langle (f^n)_* \mathsf{C}_f, \pi, \lambda\rangle = (f_\lambda^n)_* \langle \mathsf{C}_f, \pi, \lambda\rangle = (f_\lambda^n)_* \mathsf{C}_{f_\lambda}.$$

Since ω_V^{k-1} is smooth, this implies that the slice of $(f^n)_* \mathsf{C}_f \wedge \omega_V^{k-1}$ exists for almost every λ and is equal to the measure $((f_\lambda^n)_* \mathsf{C}_{f_\lambda}) \wedge \omega_V^{k-1}$. The claim then follows from Theorem A.1.6 by integrating a partition of unity. □

Once we have proved the desired relation between the masses of $dd^c L$ and of the critical orbit, we need a way to bound from below a subsequence of $\left(\|(f^n)_* \mathsf{C}_f\|_{\mathcal{U}_\Lambda}\right)_n$ in presence of a Misiurewicz parameter. The main tool to achieve this goal is given by the next proposition.

Proposition 2.2.9. *Let $f : \mathcal{U} \to \mathcal{V} = \mathbb{D}\times V$ be a holomorphic family of polynomial-like maps of large topological degree d_t. Fix a ball $B \Subset V$ such that $B \cap J(f_0) \neq \emptyset$ and let δ be such that $0 < \delta < d_t$. There exists a ball $A_0 \subset B$, a $N > 0$ and a $\eta > 0$ such that f^N admits at least δ^N inverse branches defined on the cylinder $\mathbb{D}_\eta \times A_0$, with image contained in $\mathbb{D}_\eta \times A_0$.*

In the proof of the above proposition we shall first need to construct a ball $A \subset B$ with the required number of inverse branches for f_0. This is done by means of the following general lemma. Fix any polynomial-like map $g : U \to V$ of large topological degree. Given any $A \subset V$, $n \in \mathbb{N}$ and $\rho > 0$, denote by $C_n(A,\rho)$ the set

$$C_n(A,\rho) := \left\{ h \;\middle|\; \begin{array}{c} h \text{ is an inverse branch of } g^n \text{ defined on } \overline{A} \\ \text{and such that } h(\overline{A}) \subset A \text{ and } \operatorname{Lip} h_{|\overline{A}} \leq \rho \end{array} \right\}. \tag{2.14}$$

The following result, which is just a local version of [BD14b, Proposition 3.8], is essentially due to Briend-Duval (see [BD99]).

Lemma 2.2.10. *Let g be a polynomial-like map of large topological degree d_t. Let B be a ball intersecting J and ρ a positive number. There exists a ball A contained in B and a $\alpha > 0$ such that $\#C_n(A,\rho) \geq \alpha d_t^n$, for every n sufficently large.*

Proof. This is a standard argument. We follow the proof in [BD14b], just localizing the construction of the ball. The main tool is the following result, proved by Briend-Duval in the case of endomorphisms of $\mathbb{P}^k$, which extends to polynomial-like maps of large topological degree (see also [DS03, DS10]). We denote by $\widehat{X}$ the natural extension of $X := V \setminus GO(C_g)$, i.e., the set of

sequences $\widehat{x} = (x_n)_{n\in\mathbb{Z}} \subset V^{\mathbb{Z}}$ such that $x_n = g(x_{n-1})$ and x_n does not belong to the grand orbit of the critical set for every n. Let $g_{\widehat{x}}^{-n}$ be the inverse branch sending x_0 to x_{-n} (which is defined on some neighbourhood of x_0). Let p_0 be the projection $\widehat{x} \mapsto x_0$ and $\widehat{\mu}$ be the lift of the equilibrium measure μ. Notice that $\widehat{\mu}$ is mixing since μ is mixing. We denote by $\lambda_1 > 0$ the smallest Lyapounov exponent of g.

Theorem 2.2.11. *For every ε_0 sufficiently small there exists a subset $\widehat{Y}$ of $\widehat{X}$ and two measurable functions $\eta : \widehat{Y} \to [0,1]$ and $C : \widehat{Y} \to \mathbb{R}$ such that $\widehat{\mu}(Y) = 1$ and, for every point $\widehat{y} \in \widehat{Y}$ and $n \geq n_0$:*

- *$g_{\widehat{y}}^{-n}$ is defined on $B(y_0, \eta(\widehat{y}))$;*
- *$\operatorname{Lip} g_{\widehat{y}}^{-n} \leq C(\widehat{y}) e^{-n\lambda_1 + n\varepsilon_0/2}$ on $B(y_0, \eta(\widehat{y}))$.*

Let $\widehat{Y}, \varepsilon_0, \eta$ and C be given by Theorem 2.2.11. Fix a ball $B_0 \Subset B$ intersecting J, and a positive $\delta < \mu(B_0)/3$. There exists a subset $\widehat{Y}_\delta^1$ of $\widehat{Y}$ of measure $\geq 1-\delta$ such that, for every $\widehat{y} \in \widehat{Y}_\delta^1$, the function η is larger than some η_0. Moreover, there exists a subset $\widehat{Y}_\delta^2$ of $\widehat{Y}$, of measure $\geq 1-\delta$ such that, for every $\widehat{y} \in \widehat{Y}_\delta^2$, the function C is smaller than some C_0. This implies that $\operatorname{Lip} g_{\widehat{y}}^{-n} \leq e^{-n\lambda_1 + n\varepsilon_0}$ on $B(y_0, \eta(\widehat{y}))$, up to taking n_0 larger. Set $\widehat{Y}_\delta := \widehat{Y}_\delta^1 \cap \widehat{Y}_\delta^2$ and let $\widehat{B}_0 = p_0^{-1}(B_0)$. Since $\widehat{Y}_\delta$ has measure $\widehat{\mu}(\widehat{Y}_\delta) \geq 1 - 2\delta > 1 - \mu(B_0) = 1 - \widehat{\mu}(\widehat{B}_0)$, the intersection $\widehat{Y}_\delta \cap \widehat{B}_0$ has positive measure. Cover B_0 with (finite) balls $B_i \Subset B$ and of radii $\eta_i \leq \eta_0/4$. Take one such ball B_{i_0} such that $\widehat{B}_{i_0} \cap \widehat{Y}_\delta$ has positive measure, where $\widehat{B}_{i_0} = p_0^{-1}(B_{i_0})$. We claim that $B_{i_0}^\gamma$, the concentric ball of B_{i_0} with radius $\eta_{i_0} + \gamma \leq \eta_0/4 + \gamma < \eta_0/2$, can be taken as A. Let us see why. For every n, let C_n' be the collection of inverse branches $g_{\widehat{y}}^{-n}$, with $\widehat{y} \in \widehat{Y}_\delta \cap \widehat{B}_{i_0}^\gamma$ and such that $g_{\widehat{y}}^{-n}(B_{i_0}^\gamma) \cap B_{i_0} \neq \emptyset$. The estimate on the Lipschitz constant of the inverse branches corresponding to elements in $\widehat{Y}_\delta$ implies that $C_n' \subset C_n(B_{i_0}^\gamma, \rho)$, for every n sufficiently large. It is thus enough to prove that C_n' contains at least αd_t^n elements, with α independent from n. In order to do this, we exploit the mixing property of $\widehat{\mu}$. Since the lift of g to $\widehat{X}$ is the shift s by 1, we have

$$\widehat{\mu}\left(s^{-n}\left(\widehat{Y}_\delta \cap \widehat{B}_{i_0}^\gamma\right) \cap \widehat{B}_{i_0}\right) \to \widehat{\mu}\left(\widehat{Y}_\delta \cap \widehat{B}_{i_0}^\gamma\right) \widehat{\mu}(\widehat{B}_{i_0})$$

for $n \to \infty$. So, for n large enough, we have (recalling that $g^*\mu = d_t\mu$)

$$\begin{aligned} |C_n'|\, \mu(B_{i_0}^\gamma) d_t^{-n} &\geq \mu\left(\cup_{C_n'} g_{\widehat{y}}^{-n}(B_{i_0}^\gamma)\right) \\ &\geq \widehat{\mu}\left(s^{-n}\left(\widehat{Y}_\delta \cap \widehat{B}_{i_0}^\gamma\right) \cap \widehat{B}_{i_0}\right) \geq \widehat{\mu}\left(\widehat{Y}_\delta \cap \widehat{B}_{i_0}^\gamma\right) \widehat{\mu}(\widehat{B}_{i_0})/2 > 0 \end{aligned}$$

which gives the assertion. □

Proof of Proposition 2.2.9. Let $A \subset B$ be a ball given by an application of Lemma 2.2.10 to the map f_0, with $\rho = 1/4$. There thus exists a α such that, for every sufficiently large n, the set $C_n(A, 1/4)$ defined as in (2.14) has at least αd_t^n elements. Fix N sufficiently large such that $\delta^N < \alpha\, d_t^N$. Denote by h_i the elements of $C_N(A, 1/4)$ and by A_i the images $A_i := h_i(A) \subset A$. By definition of inverse branches, the A_i's are all disjoint and f_0^N induces a biholomorphism from every A_i to A.

Take as A_0 any open ball relatively compact in A and such that $\cup_i \overline{A}_i \Subset A_0$. Such an A_0 exists since $\cup_i \overline{A}_i \Subset A$. In particular, on A_0 the h_i's are well defined, with images (compactly) contained

in the A_i's. To conclude, it suffices to find a η such that these inverse branches for f_0^N extend to inverse branches for f^N on $\mathbb{D}_\eta \times A_0$, with images contained in $\mathbb{D}_\eta \times A_0$.

Define the sets A_i^ε by

$$A_i^\varepsilon := \{\, z \in A_i \colon d(z, A_i^c) > \varepsilon \,\}.$$

Since the A_i's are finite and $f_0^N(A_i) = A$, there exists a ε_0 such that, for every i, $A_0 \Subset f_0^N(A_i^{\varepsilon_0})$. This implies, since f is continuous and every f_λ is an open map, that $A_0 \Subset f_\lambda^N(A_i^{\varepsilon_0})$ for every λ sufficiently small. Indeed, notice that $f_\lambda^N(A_i^{\varepsilon_0})$ is open and meets A_0 (for small λ). We need then to show that $\partial f_\lambda^N(A_i^{\varepsilon_0}) \cap A_0 = \emptyset$. Since f_λ^N is open, we have $\partial f_\lambda^N(A_i^{\varepsilon_0}) \subset f_\lambda^N(\partial A_i^{\varepsilon_0})$, and the right term is close to $f_0^N(\partial A_i^{\varepsilon_0})$ for small λ, by continuity. But $f_0^N(\partial A_i^{\varepsilon_0})$ is equal to $\partial f_0^N(A_i^{\varepsilon_0})$ since f_0^N is a biholomorphism on A_i, and thus does not meet A_0.

We can thus consider, for $\mathbb{D}_\eta$ sufficiently small, the open sets $\widetilde{A}_i := \left(\left(f^N\right)_{|\mathbb{D}_\eta \times A_i^{\varepsilon_0}}\right)^{-1}(\mathbb{D}_\eta \times A_0) \Subset \mathbb{D}_\eta \times A_i^{\varepsilon_0}$. By the argument above, for every $\lambda \in \mathbb{D}_\eta$ the function f_λ^N is a proper holomorphic map from $\widetilde{A}_{\lambda,i} := \left(\left(f_\lambda^N\right)_{|A_i^{\varepsilon_0}}\right)^{-1}(A_0)$ to A_0. In order to conclude, we only need to check that, for λ in a neighbourhood of 0, the degree of $f_\lambda^N : \widetilde{A}_{\lambda,i} \to A_0$ is equal to 1. Since $\widetilde{A}_{\lambda,i} \Subset A_i^{\varepsilon_0}$, it is enough to find η such that the critical set of f^N does not intersect $\mathbb{D}_\eta \times A_i^{\varepsilon_0}$, for every i. The existence of such η follows from the Lipschitz estimate of the inverses h_i. Indeed, the fact that $\operatorname{Lip} h_i < 1/4$ on A implies that $\left\|\left(df_0^N\right)^{-1}\right\|^{-1} \geq 4$ on $\cup_i A_i$. It follows that $\left\|\left(df_\lambda^N\right)^{-1}\right\|^{-1} \geq 3$ on a neighbourhood of $\{\,0\,\} \times \cup_i A_i^{\varepsilon_0}$ of the form $\mathbb{D}_\eta \times \cup_i A_i^{\varepsilon_0}$. In particular, the critical set of f^N cannot intersect this neighbourhood, and the assertion follows. □

We can now prove the main result of this section.

Theorem 2.2.12. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Assume that λ_0 is a Misiurewicz parameter. Then, $\lambda_0 \in \operatorname{Supp} dd^c L$.*

Proof. We shall prove that the existence of a Misiurewicz parameter implies that the mass of $(f^n)_* \mathsf{C}_f$ is asymptotically larger than δ^n (up to considering a subsequence), for some $\delta > d_{k-1}^*$. The conclusion will then follow from Theorem 2.2.8.

Before starting proving the assertion, we make a few simplifications to the problem. Let $\sigma(\lambda)$ denote the repelling periodic point intersecting (but not being contained in) some component of $f^{n_0}(C)$ at $\lambda = 0$ and such that $\sigma(0) \in J_0$.

- We can suppose that $M = \mathbb{D} = \mathbb{D}_1$ and that $\lambda_0 = 0$. Doing this, we actually prove a stronger statement, i.e., that $dd^c L \neq 0$ on every complex disc passing through λ_0 such that $\sigma(\lambda)$ is not contained in $f_\lambda^{n_0}(C)$ for every λ is the selected disc. Moreover, we shall assume that $\mathcal{V} = \mathbb{D} \times V$.

- Without loss of generality, we can assume that $\sigma(\lambda)$ stays repelling for every $\lambda \in \mathbb{D}$. Up to considering an iterate of f, we can suppose that $\sigma(\lambda)$ is a repelling fixed point. Indeed, we can replace n_0 with $n_0 + r$, for some $0 \leq r < n(\sigma)$, where $n(\sigma)$ is the period of σ, to ensure that now the new n_0 is a multiple of $n(\sigma)$.

- Using a local biholomorphism (change of coordinates), we can suppose that $\sigma(\lambda)$ is a constant in V, and we can assume that this constant is 0.

- After possibly further rescaling, we can assume that $f^{n_0}(C)$ intersects $\{ z = 0 \}$ only at $\lambda = 0$.
- We denote by B a small ball in V centered at 0. By taking this ball sufficiently small (and up to rescaling the parameter space), we can assume that there exists some $b > 1$ such that, for every $\lambda \in \mathbb{D} = M$ and for every $z, z' \in B$,

$$\text{dist}\left(f_\lambda(z), f_\lambda(z')\right) \geq b \cdot \text{dist}(z, z'). \tag{2.15}$$

Fix a δ such that $d^*_{k-1} < \delta < d_t$. Proposition 2.2.9 gives the existence of a ball $A_0 \subset B$ and a η such that the cylinder $T_0 := \mathbb{D}_\eta \times A_0$ admits at least δ^N inverse branches h_i for f^N, with images contained in T_0. We notice that the images of T_0 under these inverse branches must be disjoint. Up to rescaling we can still assume that $\eta = 1$.

The cylinder T_0 is naturally foliated by the "horizontal" holomorphic graphs Γ_{ξ_z}'s, where $\xi_z(\lambda) \equiv z$, for $z \in A_0$. By construction, T_0 has at least δ^{Nn} inverse branches for f^{Nn}, with images contained in T_0. We denote these preimages by $T_{n,i}$, and we notice that every $T_{n,i}$ is biholomorphic to T_0, by the map f^{Nn}. In particular, f^{Nn} induces a foliation on every $T_{n,i}$, given by the preimages of the Γ_{ξ_z}'s by f^{Nn}.

The following lemma shows that there exists some n'_0 such that some component $\widetilde{C}$ of $f^{n_0+n'_0}(C)$ intersects the graph of every holomorphic map $\gamma : \mathbb{D} \to B$, and in particular every element of the induced foliation on $T_{n,i}$. This is a consequence of the expansivity of f on $\mathbb{D} \times B$ and the fact that $f^{n_0}(C) \cap \{ z = 0 \} = (0, 0)$.

Lemma 2.2.13. *Denote by $\mathcal{G}$ the set of holomorphic maps $\gamma : \mathbb{D} \to B$. There exists an n'_0 such that (at least) one irreducible component $\widetilde{C}$ of $f^{n_0+n'_0}(C)$ passing through $(0, 0)$ intersects the graph of every element of $\mathcal{G}$.*

Proof. Let $\widetilde{C}_0$ be an irreducible component of $f^{n_0}(C) \cap (\mathbb{D} \times B)$ passing through $(0, 0)$ and define by induction the set $\widetilde{C}_n$ as an irreducible component passing through $(0, 0)$ of $f(\widetilde{C}_{n-1}) \cap (\mathbb{D} \times B)$. We claim that there exists some n'_0 such that $\widetilde{C}_{n'_0} \subset \overline{\mathbb{D}}_{1/2} \times B$. Indeed, if this were not the case, for every n there would exist a continuous path $\delta_n \subset \widetilde{C}_n$ connecting $(0, 0)$ and some point in $\left(\mathbb{D} \setminus \overline{\mathbb{D}}_{1/2}\right) \times B$. By the definition of the $\widetilde{C}_n$'s, this would give the existence of a continuous path δ'_n contained in $\widetilde{C}_0$ and connecting $(0, 0)$ to some point in $\left(\mathbb{D} \setminus \overline{\mathbb{D}}_{1/2}\right) \times B$. Moreover, by the expansivity of f on $\mathbb{D} \times B$ and the fact that $f_\lambda(0) = 0$ for every λ, we would have $\delta'_n \subset \left(\mathbb{D} \times \frac{1}{b^n} B\right)$. Since $\widetilde{C}_0$ is closed in $\mathbb{D} \times B$, this would give the existence of points in $f^{n_0}(C) \cap \left(\left(\mathbb{D} \setminus \mathbb{D}_{1/2}\right) \times \{ 0 \}\right)$, which is excluded by the preliminary simplifications.

Let us then prove that the graph of every element in $\mathcal{G}$ must intersect $\widetilde{C}_{n'_0}$. Notice that $\mathcal{G}$ is path connected (actually star-shaped at the element $\xi_0 \equiv 0$). We shall prove that the subset $\mathcal{G}_0 \subset \mathcal{G}$ of elements whose graph intersects $\widetilde{C}_{n'_0}$ is open and closed in $\mathcal{G}$ for the topology of the local uniform convergence.

Take some sequence of elements $\gamma_n \in \mathcal{G}$ converging to some $\gamma_0 \in \mathcal{G}$. If all the Γ_{γ_n}'s intersect $\widetilde{C}_{n'_0}$ at parameters λ_n (necessarily contained in $\overline{\mathbb{D}}_{1/2}$), then also Γ_{γ_0} must intersect $\widetilde{C}_{n'_0}$ at any limit value $\lambda_0 \in \overline{\mathbb{D}}_{1/2} \subset \mathbb{D}$ for the sequence λ_n. This proves that $\mathcal{G}_0$ is closed in $\mathcal{G}$.

On the other hand, assume that Γ_{γ_0} intersects $\widetilde{C}_{n'_0}$ for some $\gamma_0 \in \mathcal{G}$. By Hurwitz Theorem, this implies that all Γ_γ's must intersect $\widetilde{C}_{n'_0}$, when γ is sufficently close to γ_0. This proves that $\mathcal{G}_0$ is

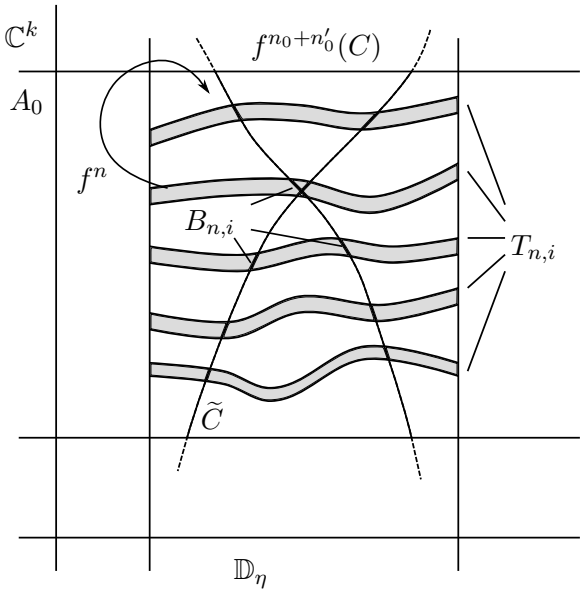


Figure 2.2.: estimating the postcritical mass

also open in $\mathcal{G}$, and proves the assertion. □

Let n_0' and $\widetilde{C}$ be given by Lemma 2.2.13. In particular, $\widetilde{C}$ intersects every element of the induced foliations on the $T_{n,i}$'s. Let $B_{n,i}$ denote the intersection $T_{n,i} \cap \widetilde{C}$ and set $D_{n,i} := f^{Nn}(B_{n,i}) \subset T_0$. The $D_{n,i}$'s are non-empty analytic subsets of T_0 (since $f^{Nn} : T_{n,i} \to T_0$ is a biholomorphism). Moreover, the graphs of the ξ_z's intersect every $D_{n,i}$, since their preimages in $T_{n,i}$ intersect every $B_{n,i}$. In particular, the projection of every $D_{n,i}$ on V is equal to A_0.

Let us finally estimate the mass of $\left(f^{n_0+n_0'+Nn}\right)_* [C]$ on $\mathcal{U}_\mathbb{D}$. First of all, notice that $\left(f^{n_0+n_0'}\right)_* \mathsf{C}_f \geq f_*^{n_0+n_0'}[C] \geq [f^{n_0+n_0'}(C)] \geq [\widetilde{C}]$ as positive currents on $\mathcal{U}_\mathbb{D}$. This implies that

$$\left\| f_*^{Nn+n_0+n_0'} \mathsf{C}_f \right\|_{\mathcal{U}_\mathbb{D}} \geq \left\| f_*^{Nn} \left[f^{n_0+n_0'}(C) \right] \right\|_{\mathcal{U}_\mathbb{D}} \geq \left\| f_*^{Nn} \left[\widetilde{C} \right] \right\|_{\mathcal{U}_\mathbb{D}} \geq \left\| f_*^{Nn} \left[\widetilde{C} \right] \right\|_{T_0}.$$

Now, since f^{Nn} gives a biholomorphism from every $T_{n,i}$ to T_0 and all the $T_{n,i}$'s are disjoint, we have

$$\left\| f_*^{Nn} \left[\widetilde{C} \right] \right\|_{T_0} \geq \left\| f_*^{Nn} \left(\sum_i [B_{n,i}] \right) \right\|_{T_0} = \sum_i \left\| f_*^{Nn} [B_{n,i}] \right\| = \sum_i \| [D_{n,i}] \|.$$

By Wirtinger formula, for every n and i the volume of $D_{n,i}$ is larger than the volume of its projection A_0 on V. Since by construction the last sum has at least δ^{Nn} terms, we have

$$\left\| f_*^{n_0+n_0'+Nn} \mathsf{C}_f \right\|_{\mathcal{U}_\mathbb{D}} \geq \text{volume}(A_0) \cdot \delta^{Nn} > \text{volume}(A_0) \cdot \left(d_{k-1}^*\right)^{Nn}$$

and the assertion follows from Theorem 2.2.8. □

2.2.4. Local existence of a good graph: I.3 ⇒ I.4

In this section we complete the proof of Theorem 2.2.2, by establishing the following result. This gives the last missing implication I.3 ⇒ I.4.

Theorem 2.2.14. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$, such that the parameter space M does not contain any Misiurewicz parameter. Then, for every $\lambda_0 \in M$, there exists a neighbourhood $M_0 \subset M$ and a holomorphic map $\gamma : M_0 \to \mathbb{C}^k$ such that the graph Γ_γ does not intersect the postcritical set of f.*

The proof that we give follows the main line of the one in the case of endomorphisms of $\mathbb{P}^k$ in [BD14b]. In the proof on $\mathbb{P}^k$, one needs to ensure that an hyperbolic set of sufficiently high entropy (see Section A.2) cannot be contained in the postcritical set and must, on the other hand, be contained in the Julia set. In our setting, the analogue of the first propery is given in Lemma A.2.9, while we will adress the second problem differently, by means of the following Lemma.

Lemma 2.2.15. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps with parameter space M. Let E_0 be an hyperbolic set for f_0 contained in J_0, such that repelling periodic points are dense in E_0 and $\left\|(df_\lambda)^{-1}\right\|^{-1} > K > 3$ on a neigbourhood of $(E_0)_\tau$ in the product space. Let h be a continuous holomorphic motion of E_{λ_0} as an hyperbolic set on some ball $B \subset M$, preserving the repelling cycles. Then $h_\lambda(E_0)$ is contained in J_λ, for λ sufficiently close to λ_0.*

Proof. We denote by γ_z the motion of a point $z \in E_0$ as part of the given holomorphic motion of the hyperbolic set.

First of all, notice that repelling points must be dense in E_λ for every λ, by the continuity of the motion and the fact that they are preserved by it. Moreover, by Lemma 2.1.1, every repelling cycle stays in J_λ for λ in a neighbourhood of 0. It is thus enough to ensure that this neighbourhood can be taken uniform for all the cycles, and the assertion then follows from the density of the repelling points in E_λ.

Since $\left\|df_\lambda^{-1}\right\|^{-1} > 3$ on a neighbourhood $(E_0)_\tau$ of E_0 in the product space, we can restrict ourselves to $\lambda \in B(0,\tau)$ and so assume that $\left\|df_\lambda^{-1}\right\|^{-1} > 3$ on a τ neighbourhood of every $z \in E_\lambda$, for every λ. Moreover, since the set of motions γ_z of points in E_0 is compact (by continuity), we can assume that $\gamma_z(\lambda) \in B(z, \tau/10)$ for every $z \in E_0$ and λ. Finally, by the lower semicontinuity of the Julia set, up to shrinking again the parameter space we can assume that $J_0 \subset (J_\lambda)_{\tau/10}$ for every λ. These two assumptions imply that, for every λ and every $z \in E_\lambda$, there exists at least a point of J_λ in the ball $B(z, \tau/2)$. Consider now any n-periodic repelling point p_0 in E_λ for f_λ, and let $\{ p_i \} = \{ f_\lambda^i(p_0) \}$ be its cycle (and thus with $p_0 = p_n$). Fix a point $z_0 \in J_\lambda \cap B(p_0, \tau)$. By hyperbolicity (and since without loss of generality we can assume that $\tau \leq (1 + \sup_{B_\tau} \|f_\lambda\|_{C^2})^{-1}$), every ball $B(p_i, \tau)$ has an inverse branch for f_λ defined on it, with image strictly contained in the ball $B(p_{i-1}, \tau)$ and strictly contracting. This implies that there exists an inverse branch g_0 for f_λ^n of $B(p_0, \tau)$, strictly contracting and with image strictly contained in $B(p_0, \tau)$ (and containing p_0). So, a sequence of inverse images of z_0 for f_λ must converge to p_0, and so $p_0 \in J_\lambda$. The assertion is proved. □

In the following theorem, we assume that the parameter space is the unit ball $B \subset \mathbb{C}^m$.

Theorem 2.2.16. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree d_t. Then there exists an integer N, a compact hyperbolic set $E_0 \subset J_0$ for f_0^N and a continuous holomorphic motion $h : B_r \times E_0 \to \mathbb{C}^k$ (defined on some small ball B_r of radius r and centered at 0) such that:*

1. *the repelling periodic points of f_0^N are dense in E_0 and E_0 is not contained in the postcritical set of f_0^N;*
2. *$h_\lambda(E_0) \in J_\lambda$ for every $\lambda \in B_r$;*
3. *if z is periodic repelling for f_0^N then $h_\lambda(z)$ is periodic and repelling for f_λ^N.*

The proof will follow the same lines as the one for endomorphisms of $\mathbb{P}^k$. The only differences will be the use of Lemma A.2.9 to get the first property and of Lemma 2.2.15 for the second one. In order to produce the hyperbolic set, we shall use Lemma 2.2.10.

Proof of Theorem 2.2.16. First of all, we need the hyperbolic set E_0. By Lemma 2.2.10, we can take a closed ball A, a constant $\rho > 0$ and a sufficiently large N such that the cardinality N' of $C_N(A,\rho)$ (see (2.14)) satisfies $N' \geq \left(d_{k-1}^*\right)^N$ (since by assumption $d_{k-1}^* < d_t$). We then consider the set E_0 given by the intersection $E_0 = \cap_{k\geq 0} E_k$, where E_k is given by

$$E_k := \{\, g_{i_1} \circ \cdots \circ g_{i_k}(A) \colon (i_1, \ldots, i_k) \in \{\, 1, \ldots, N' \,\}^k \,\}$$

where the g_i's are the elements of $C_N(A,\rho)$. The set E_0 is then hyperbolic, and contained in J_0 (since $A \cap J \neq \emptyset$, every point in E_0 is accumulated by points in the Julia set). Moreover, repelling cycles (for f_0^N) are dense in E_0.

Let $\Sigma : \{\, 1, \ldots, N' \,\}^{\mathbb{N}^*}$ and fix a point $z \in E_0$. Notice that the map $\omega : \Sigma \to E_0$ given by

$$\omega(i_1, i_2, \ldots) = \lim_{k\to\infty} g_{i_1} \circ \cdots \circ g_{i_k}(z)$$

satisfies the relation $f^N \circ \omega = \omega \circ s$, where s denotes the left shift

$$(i_1, \ldots, i_k, \ldots) \xrightarrow{s} (i_2, \ldots, i_{k+1}, \ldots).$$

We can thus pushforward with ω the uniform product measure on Σ. Since this is a s-invariant ergodic measure, its pushforward ν is an f^N-invariant ergodic measure on $E_0 \subset J_0$. By Lemma A.2.5, the metric entropy of ν satisfies $h_\nu \geq \log N' \geq \log \left(d_{k-1}^*\right)^N$, and this implies (by Lemma A.2.9) that ν gives no mass to analytic subsets. In particular, E_0 is not contained in the postcritical set of f_0.

We need to prove the points 2 and 3. It is a classical result (see, e.g., [Jon98]) that E_0 admits a continuous holomorphic motion that preserves the repelling cycles, and thus 3 follows. The second point then follows from Lemma 2.2.15 (and the density of the repelling cycles in E_0). □

Once we have established the existence of a hyperbolic set as in Theorem 2.2.16, we can prove Theorem 2.2.14, in the same way this is done on $\mathbb{P}^k$.

Proof of Theorem 2.2.14. This is a consequence of Theorem 2.2.16, exactly as in the case of endomorphisms of $\mathbb{P}^k$. Let $h : B_r \times E_0 \to \mathbb{C}^k$ and N be as provided there. First of all, without loss of generality we can assume that $N = 1$, since f and f^N have the same postcritical set. We then fix $z \in E_0$ and outside the postcritical set of f_0 (which exists by the first property of the motion h). Setting $\gamma(\lambda) := h_\lambda(z)$, we are going to prove that this graph does not intersect the postcritical set of f on all B_r, thus proving the assertion.

Assume that $\gamma(\lambda_0) \in f_{\lambda_0}^{n_0}(C_{f_{\lambda_0}})$, for some n_0 and $\lambda_0 \in B_r$ (necessarily different from 0). This implies the existence of Misiurewicz parameters in B_r. Indeed, by the first property of the motion, we have a sequence z_n of repelling periodic points contained in E_0 such that $z_n \to z$. Since $z \notin f_0^{n_0}(C_{f_0})$, we can assume that the same is true for all the z_n's. By the second and the third properties of the motion in Theorem 2.2.16, the maps $\lambda \mapsto h_\lambda(z_n)$ are holomorphic motions of repelling points, contained in the Julia set at every parameter. Since they converge (locally uniformy) to γ as $n \to \infty$, by Hurwitz theorem they must intersect $f^{n_0}(C_f)$ for $n \geq n_0$. Since at the parameter 0 they are disjoint from it, we get the existence of Misiurewicz parameters in B_r, and the theorem is proved. □

3
Holomorphic motions

In this chapter we prove the main result of this work: the holomorphic motion of the J-repelling cycles implies the holomorphic motion of almost every point of the Julia sets (with respect to the equilibrium measures). We also recover a partial converse of this implication: on every compact subset of the parameter space, asymptotically all the repelling cycles move holomrophically when the other equivalent conditions for the stability are satisfied.

3.1. Definitions and statements

In order to state our main theorem, we have to give some preliminary definitions. First of all, we define our higher-dimensional analogous of the holomorphic motion of the Julia set.

Definition 3.1.1. *An* equilibrium lamination *is a subset $\mathcal{L}$ of $\mathcal{J}$ such that*

1. $\mathcal{F}(\mathcal{L}) = \mathcal{L}$*,*
2. $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ *for every distinct* $\gamma, \gamma' \in \mathcal{L}$*,*
3. $\mu_\lambda(\{\gamma(\lambda), \gamma \in \mathcal{L}\}) = 1$ *for every* $\lambda \in M$*,*
4. Γ_γ *does not meet the grand orbit of the critical set of f for every* $\gamma \in \mathcal{L}$*,*
5. *the map* $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *is* d^k *to* 1.

The second and the third conditions ensure that we have a holomorphic motion of a full-measure subset of the Julia set. The other conditions say that $\mathcal{F}$ induces a covering, of degree d_t, on $\mathcal{L}$. In particular, conditions 4 and 5 allow us (by means of Proposition 2.1.5) to construct equilibrium webs by taking the preimages of any element in $\mathcal{L}$. So, even if equilibrium webs do not appear in the definition of an equilibrium lamination, the existence of this latter implies the existence of these measures.

To construct equilibrium laminations, it will be crucial to deal with equilibrium webs giving no mass to the subset $\mathcal{J}_s$ of $\mathcal{J}$ whose elements have a graph intersecting the grand orbit of the critical set of f (i.e., the elements that we want to avoid to recover the fourth condition).

Definition 3.1.2. *An equilibrium web is* acritical *if $\mathcal{M}(\mathcal{J}_s) = 0$, where*

$$\mathcal{J}_s := \{\, \gamma \in \mathcal{J} : \Gamma_\gamma \cap \left(\bigcup_{m,n\geq 0} f^{-m}\left(f^n\left(C_f\right)\right)\right) \neq \emptyset \,\}.$$

Notice that (on a simply connected parameter space) every element in $\mathcal{J} \setminus \mathcal{J}_s$ has d_t well-defined preimages. So, an acritical web $\mathcal{M}$ has the property that, in this setting, $\mathcal{M}$-almost every $\gamma \in \mathcal{J}$ has the maximal number of preimages.

Finally, we define a weaker (with respect to Definition 2.1.6) notion of holomorphic motion for the J-repelling points. We shall prove that this is enough to recover the existence of an equilibrium lamination, and on the other hand, this weak motion always exists in presence of an equilibrium web. We shall see in the next chapter how to recover the motion of all repelling cycles, by adding some assumption on the family.

Definition 3.1.3. *We say that* asymptotically all J-cycles move holomorphically *if there exists a subset $\mathcal{P} = \cup_n \mathcal{P}_n \subset \mathcal{J}$ such that*

1. *$\sharp\mathcal{P}_n = d^n + o(d^n)$;*
2. *every $\gamma \in \mathcal{P}_n$ is n-periodic; and*
3. *for every $M' \Subset M$, asymptotically every element of $\mathcal{P}$ is repelling, i.e.,*

$$\frac{\sharp\{ \text{ repelling cycles in } \mathcal{P}_n\}}{\sharp\mathcal{P}_n} \to 1.$$

The main result of this chapter is then the following.

Theorem 3.1.4. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Assume that the parameter space is simply connected. Then the following are equivalent:*

II.1 asymptotically all J-cycles move holomorphically;

II.2 there exists an acritical equilibrium web $\mathcal{M}$;

II.3 there exists an equilibrium lamination for f.

Moreover, if the previous conditions hold, the system admits a unique equilibrium web, which is ergodic and mixing.

The assumption of M being simply connected, as anticipated above, is to ensure the existence of the inverses of the graphs in $\mathcal{J} \setminus \mathcal{J}_s$.

After proving Theorem 3.1.4, we will relate these notions of stability with the ones given by Theorem 2.2.2. This is done in the following theorem.

Theorem 3.1.5. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Assume that the parameter space is simply connected. Then conditions I.1 – I.4 of Theorem 2.2.2 and II.1 – II.3 of Theorem 3.1.4 are all equivalent.*

This chapter is devoted to the proof of the two Theorems 3.1.4 and 3.1.5. In Section 3.2 we study ergodicity properties of webs that will be used in the sequel. In particular, we show that, any time that we can contruct an equilibrium web, it is not restrictive to assume that we have an ergodic one. The same applies in particular for acritical webs. In the subsequent sections, we prove the implications of Theorem 3.1.4. Finally, in Section 3.6, we show Theorem 3.1.5.

We explicitely notice that the proof of Theorem 3.1.4 heavily relies on results on the previous chapter. Indeed, as a general idea, the property of an equilibrium web of being acritical will follow from the absence of Misiurewicz parameters (see Definition 2.2.1) combined with the fact that the equilibrium measure of a polynomial-like map of large topological degree gives no mass to the postcritical set (see Theorem 1.2.16). More details on this will be given in Section 3.3.

3.2. Ergodic and acritical equilibrium webs

Assume that a holomorphic family of polynomial-like maps f admits an equilibrium web $\mathcal{M}$. The goal of this section is to prove that, under this assumption, there exist equilibrium webs that are ergodic.

We consider the space $\mathcal{P}_{inv}(\mathcal{J})$ of invariant probability measures on $\mathcal{J}$. It is clearly convex and, since we assume the existence of an equilibrium web, non empty. We consider on it the weak-* topology (with respect to the topology of the uniform convergence of continuous functions on $\mathcal{J}$). This topology is metrizable (since the space of continuous functions on $\mathcal{J}$ is separable), and this turns $\mathcal{P}_{inv}(\mathcal{J})$ into a metric space. Moreover, since $\mathcal{J}$ is compact, an application of Riesz representation theorem gives that also $\mathcal{P}_{inv}(\mathcal{J})$ is compact.

The first thing we do is to notice that the subspace $\mathcal{P}_{web}(\mathcal{J}) \subset \mathcal{P}_{inv}(\mathcal{J})$ of the equilibrium webs is closed in $\mathcal{P}_{inv}(\mathcal{J})$, and hence is a compact metric space, too. This follows from the continuity of $(p_\lambda)_*$, recalling that webs are characterized (among the invariant probability measures on $\mathcal{J}$) by the property that $(p_\lambda)_*(\mathcal{M}) = \mu_\lambda$ for every λ. Indeed, if $\mathcal{M}_n$ are webs converging to $\mathcal{M} \in \mathcal{P}_{inv}(\mathcal{J})$, then $(p_\lambda)_* \mathcal{M} = \lim_{n\to\infty} (p_\lambda)_* \mathcal{M}_n = \lim_{n\to\infty} \mu_\lambda = \mu_\lambda$, and the assertion is proved.

The following elementary lemma gives a semi-extremality property of the equilibrium webs.

Lemma 3.2.1. *Let $\mathcal{M} \in \mathcal{P}_{web}(\mathcal{J})$ be an equilibrium web such that $\mathcal{M} = \frac{\mathcal{N}_1}{2} + \frac{\mathcal{N}_2}{2}$, with $\mathcal{N}_1$ and $\mathcal{N}_2$ invariant probability measures. Then, also $\mathcal{N}_1$ and $\mathcal{N}_2$ are equilibrium webs.*

Proof. Since $(p_\lambda)_*$ is linear, for every λ we have $\mu_\lambda = (p_\lambda)_* \mathcal{M} = \frac{(p_\lambda)_* \mathcal{N}_1}{2} + \frac{(p_\lambda)_* \mathcal{N}_2}{2}$. Since $(p_\lambda) \circ \mathcal{F} = f_\lambda \circ p_\lambda$, we get the invariance of $(p_\lambda)_* \mathcal{N}_1$ and $(p_\lambda)_* \mathcal{N}_2$ by f_λ and so, by the ergodicity of μ_λ, we get $(p_\lambda)_* \mathcal{N}_i = \mu_\lambda$. Thus, the $\mathcal{N}_i$'s are webs. □

Remark that, since $\mathcal{P}_{web}(\mathcal{J})$ is compact and convex, we can find extremal elements (by Krein-Milman Theorem). From Lemma 3.2.1 we deduce that an extremal element in $\mathcal{P}_{web}(\mathcal{J})$ is also extremal in $\mathcal{P}_{inv}(\mathcal{J})$, i.e., it is an ergodic measure. In particular, we deduce the existence of ergodic webs.

Corollary 3.2.2. *An extremal element in $\mathcal{P}_{web}(\mathcal{J})$ is extremal also in $\mathcal{P}_{inv}(\mathcal{J})$. In particular, if there exists an equilibrium web there exists also an ergodic one.*

Proof. Let $\mathcal{M}$ be an extremal element in $\mathcal{P}_{web}(\mathcal{J})$ and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two invariant measures such that $\mathcal{M} = \frac{\mathcal{N}_1}{2} + \frac{\mathcal{N}_2}{2}$. First, from Lemma 3.2.1 it follows that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are webs. Then the assertion follows from the extremality of $\mathcal{M}$ in $\mathcal{P}_{web}(\mathcal{J})$. □

Equilibrium webs which are both acritical and ergodic will play a crucial role in the sequel. This motivates the following result.

Proposition 3.2.3. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps. Assume there exists an acritical equilibrium web $\mathcal{M}$. Then there exists an equilibrium web which is both acritical and ergodic.*

Proof. Since $\mathcal{P}_{web}(\mathcal{J})$ is a compact metric space, we can decompose $\mathcal{M}$ by means of Choquet Theorem, i.e, we can find a measure ν on the subset $\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))$ of extremal elements of $\mathcal{P}_{web}(\mathcal{J})$ (which coincides with the subset of ergodic equilibrium webs, by Corollary 3.2.2) such that

$$\mathcal{M} = \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \nu(\mathcal{E}).$$

This means that, for every element in the dual space of $\mathcal{P}_{web}(\mathcal{J})$ (and in particular for every continuous function φ on $\mathcal{J}$), we have

$$\langle \mathcal{M}, \varphi \rangle = \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \langle \mathcal{E}, \varphi \rangle \, \nu(\mathcal{E}).$$

Remark that $\mathcal{J}_s$ can be decomposed as a countable union $\cup_n \mathcal{J}_s^n$ of compact subsets of $\mathcal{J}$. In fact, for any $M' \Subset M$ and every component of the critical grand orbit, the set of graphs intersecting that given component over $\overline{M'}$ is compact.

We are going to prove that, for every $\mathcal{J}_s^n$, the set of ergodic equilibrium webs $\mathcal{E}$ such that $\mathcal{E}(\mathcal{J}_s^n) = 0$ has full ν-measure. Then, intersecting over n, it will follow that, for ν-almost every ergodic equilibrium webs $\mathcal{E}$, we have $\mathcal{E}(\mathcal{J}_s) = 0$. The assertion then follows.

So, let $\mathcal{J}_s^n$ be one such compact. Since $\mathcal{M}(\mathcal{J}_s^n) = 0$, it suffices to prove that

$$\mathcal{M}(\mathcal{J}_s^n) = \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \mathcal{E}(\mathcal{J}_s^n) \nu(\mathcal{E}). \tag{3.1}$$

Since $\mathcal{J}_s^n$ is a compact set in a metric space, we can find positive continuous functions ρ_j decreasing to $1_{\mathcal{J}_s^n}$. For every such ρ_j we have

$$\langle \mathcal{M}, \rho_j \rangle = \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \langle \mathcal{E}, \rho_j \rangle \, \nu(\mathcal{E})$$

and so, repeatedly applying the Lebesgue dominated convergence Theorem, we get

$$\begin{aligned}\mathcal{M}(\mathcal{J}_s^n) = \lim_j \langle \mathcal{M}, \rho_j \rangle &= \lim_j \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \langle \mathcal{E}, \rho_j \rangle \, \nu(\mathcal{E}) \\ &= \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \lim_j \langle \mathcal{E}, \rho_j \rangle \, \nu(\mathcal{E}) \\ &= \int_{\mathrm{Ex}(\mathcal{P}_{web}(\mathcal{J}))} \mathcal{E}(\mathcal{J}_s^n) \nu(\mathcal{E}),\end{aligned}$$

and (3.1) is proved. □

3.3. Building an acritical web: II.1 or II.3 ⇒ II.2

In this section we prove both the implications II.1 ⇒ II.2 and II.3 ⇒ II.2 in Theorem 3.1.4. The strategy is the same in both situations, and the idea is to prove that the equilibrium web constructed by using either Proposition 2.1.7 (for the first implication) or (a suitable modification of) Proposition 2.1.5 (for the second one) is actually acritical. In both situations, we shall use Theorem 2.2.2 to exclude the presence of intersections between graphs of elements of the equilibrium web and graphs of the ones in the singular set $\mathcal{J}_s$. The following lemma will be useful in the proof.

Lemma 3.3.1. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree. Let $\mathcal{M}$ be an equilibrium web. Then, for every $\lambda_0 \in M$,*

$$\mathcal{M}\left(\{\gamma \in \mathcal{J} : \ (\lambda_0, \gamma(\lambda_0)) \in \left(\cup_{k\geq 0} f^k(C_f)\right)\}\right) = 0.$$

Proof. The quantity to estimate is equal to $\mu_{\lambda_0}\left(\cup_{k\geq 0} f_{\lambda_0}^k(C_{f_{\lambda_0}})\right)$, by definition of equilibrium web. The assertion is then an immediate consequence of the fact that μ_{λ_0} does not charge pluripolar sets, since f_{λ_0} is of large topological degree (see Section 1.2.4). □

Proposition 3.3.2. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Assume that asymptotically all repelling J-cycles move holomorphically. Then f admits an acritical and ergodic equilibrium web.*

Proof. By assumption, for every $n \geq 1$ we have subsets $\mathcal{P}_n := \{\rho_{n,j} : \ 1 \leq j \leq N_n\}$ of $\mathcal{J}$ such that the $\rho_{n,j}(\lambda)$ are n-periodic points of f_λ for every $\lambda \in M$. Note that $\lim_n d_t^{-n} N_n = 1$. We define a sequence $(\mathcal{M}_n)_n$ of $\mathcal{F}$-invariant discrete probability measures on $\mathcal{J}$ by setting $\mathcal{M}_n := \frac{1}{N_n} \sum_{j=1}^{N_n} \delta_{\rho_{n,j}(\lambda)}$. The same proof of Proposition 2.1.7 gives that $(\mathcal{M}_n)_n$ converges to an equilibrium web $\mathcal{M}$ after taking a subsequence. Let us prove that $\mathcal{M}(\mathcal{J}_s) = 0$. By Theorem 2.2.3 we have $dd^c L = 0$ (since this is a local property, we can assume that the cycles are actually repelling on all the parameter space) and then Theorem 2.2.12 implies that M does not contain Misiurewicz parameters. We can now see that for every $k \in \mathbb{N}$ and every $\gamma \in \mathrm{Supp}\, \mathcal{M}$ one has:

$$\Gamma_\gamma \cap f^k(C_f) \neq \emptyset \Rightarrow \Gamma_\gamma \subset f^k(C_f).$$

Indeed, if this were not the case, by Hurwitz theorem, we could find some $\gamma' \in \cup_n \operatorname{Supp} \mathcal{M}_n$ such that $\Gamma_\gamma \cap f^k(C_f) \neq \emptyset$ and Γ_γ is not contained in $f^k(C_f)$. When $k = 0$ this is clearly impossible since $\gamma'(\lambda)$ is a repelling cycle of f_λ and when $k \geq 1$, this is impossible because M does not contain Misiurewicz parameter. So, fixing any $\lambda_0 \in M$, we get

$$\begin{aligned}\mathcal{M}\left(\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap \left(\cup_{k\geq 0} f^k(C_f)\right) \neq \emptyset\}\right) &= \mathcal{M}\left(\{\gamma \in \mathcal{J} : \Gamma_\gamma \subset \left(\cup_{k\geq 0} f^k(C_f)\right)\}\right) \\ &\leq \mathcal{M}\left(\{\gamma \in \mathcal{J} : (\lambda_0, \gamma(\lambda_0)) \in \left(\cup_{k\geq 0} f^k(C_f)\right)\}\right) = 0\end{aligned}$$

where the last equality follows from Lemma 3.3.1. The estimate $\mathcal{M}(\mathcal{J}_s) = 0$ follows from the $\mathcal{F}$-invariance of $\mathcal{M}$. Finally, Proposition 3.2.3 shows that there exists an ergodic equilibrium web $\mathcal{M}_0$ such that $\mathcal{M}_0(\mathcal{J}_s) = 0$. □

Proposition 3.3.3. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$, with parameter space M simply connected. Assume that there exists a holomorphic map $\gamma \in \mathcal{O}(M, \mathbb{C}^k, \mathcal{V})$ such that Γ_γ does not intersect the postcritical set of f. Then, f admits an acritical and ergodic equilibrium web.*

Proof. First of all, we apply Proposition 2.1.5 and thus find an equilibrium web $\mathcal{M}$ for f as a limit of the iterated preimages of Γ_γ. We have to check that this web is acritical. By invariance of $\mathcal{M}$, it is enough to prove that

$$\forall \widetilde{\gamma} \in \operatorname{Supp} \mathcal{M} \colon \Gamma_{\widetilde{\gamma}} \cap \cup_{k\geq 0} f^k(C_f) = \emptyset.$$

Assume on the contrary that we have $\Gamma_{\widetilde{\gamma}} \cap f^{n_0}(C_f) \neq \emptyset$, for some $\widetilde{\gamma} \in \operatorname{Supp} \mathcal{M}$ and $n \geq n_0$. By definition of $\mathcal{M}$, we have a sequence of maps γ_i such that $\mathcal{F}^{n_i}(\gamma_i) = \gamma$ and $\gamma_i \to \widetilde{\gamma}$. By Hurwitz theorem (since Γ_γ is not contained in $\cup_{k\geq 0} f^k(C_f)$) we thus have that, for sufficiently large i, $\Gamma_{\gamma_i} \cap f^{n_0}(C_f) \neq \emptyset$. But this would imply that $\Gamma_\gamma \cap f^{n_0+n_i}(C_f) \neq \emptyset$, which contradicts the assumption on γ. □

3.4. Building the equilibrium lamination: II.2 ⇒ II.3

Our goal here is to establish the implication II.2 ⇒ II.3 in Theorem 3.1.4, the uniqueness of the equilibrium web and its mixing property. We actually prove the following more precise result.

Theorem 3.4.1. *Let M be a simply connected complex manifold and $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. If f admits an acritical and ergodic equilibrium web then there exists an equilibrium lamination $\mathcal{L}$ for f. Moreover, f admits a unique equilibrium web $\mathcal{M}$ (which is ergodic and mixing) and $\mathcal{M}(\mathcal{L}_1 \Delta \mathcal{L}_2) = 0$ for any pair of equilibrium laminations $\mathcal{L}_1, \mathcal{L}_2$ of f.*

Given an acritical and ergodic equilibrium web $\mathcal{M}$ of f, our strategy will consist in first proving that the iterated inverse branches in $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ are exponentially contracting and then exploit this property to extract an equilibrium lamination from the support of $\mathcal{M}$. We notice that, by totally different methods, Berger and Dujardin ([BD14a]) have recently build measurable holomorphic motions in the context of polynomial automorphisms of $\mathbb{C}^2$.

3.4.1. On the rate of contraction of iterated inverse branches in $(\mathcal{J}, \mathcal{F}, \mathcal{M})$

We explain here how certain stochastic properties of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ allow to control the rate of contraction of the iterated inverse branches of $\mathcal{F}$ (see Proposition 3.4.2). We adapt to the context of $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ the tools which have been first introduced in [BD99] by Briend-Duval for the case of a single holomorphic endomorphism of $\mathbb{P}^k$, and generalized by Dinh-Sibony [DS03, DS10] in the setting of polynomial-like maps of large topological degree. Let us stress however that new arguments will be introduced in the next subsections.

Since all our statements here are local we may assume that the parameter space M is a simply connected open subset of $\mathbb{C}^m$ which we endow with the euclidean norm.

To study the inverse branches of the map $\mathcal{F}$, it is convenient to transform the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$ into an injective one. This is possible using a classical construction called the *natural extension* which we now describe (we refer to [CFS82] page 240 for more details).

Recall that $\mathcal{J}$ is a compact metric space and that $\mathcal{M}(\mathcal{J}_s) = 0$. Setting $\mathcal{X} := \mathcal{J} \setminus \mathcal{J}_s$, it is not difficult to check that the map $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is onto. We may therefore construct the natural extension $\left(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}}\right)$ of the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ in the following way. An element of $\widehat{\mathcal{X}}$ is a bi-infinite sequence $\widehat{\gamma} := (\cdots, \gamma_{-j}, \gamma_{-j+1}, \cdots, \gamma_{-1}, \gamma_0, \gamma_1, \cdots)$ of elements $\gamma_j \in \mathcal{X}$ such that $\mathcal{F}(\gamma_{-j}) = \gamma_{-j+1}$ and one defines the map $\widehat{\mathcal{F}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ by setting

$$\widehat{\mathcal{F}}(\widehat{\gamma}) := (\cdots \mathcal{F}(\gamma_{-j}), \mathcal{F}(\gamma_{-j+1}) \cdots).$$

The map $\widehat{\mathcal{F}}$ corresponds to the shift operator and is clearly bijective. There exists a unique measure $\widehat{\mathcal{M}}$ on $\widehat{\mathcal{X}}$ such that

$$(\pi_j)_\star \left(\widehat{\mathcal{M}}\right) = \mathcal{M}$$

for any projection $\pi_j : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}$ given by $\pi_j(\widehat{\gamma}) = \gamma_j$. The ergodicity of $\mathcal{M}$ implies the ergodicity of $\widehat{\mathcal{M}}$. We have thus obtained an invertible and ergodic dynamical system $\left(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}}\right)$.

For every $\gamma \in \mathcal{J}$ whose graph Γ_γ does not meet the critical set of f, we denote by f_γ the injective map which is induced by f on some neighbourhood of Γ_γ and by f_γ^{-1} the inverse branch of f_γ which is defined on some neighbourhood of $\Gamma_{\mathcal{F}(\gamma)}$. Thus, given $\widehat{\gamma} \in \widehat{\mathcal{X}}$ and $n \in \mathbb{N}$ we may define the iterated inverse branch $f_{\widehat{\gamma}}^{-n}$ of f along $\widehat{\gamma}$ and of depth n by

$$f_{\widehat{\gamma}}^{-n} := f_{\gamma_{-n}}^{-1} \circ \cdots \circ f_{\gamma_{-2}}^{-1} \circ f_{\gamma_{-1}}^{-1}.$$

Let us stress that $f_{\widehat{\gamma}}^{-n}$ is defined on a neighbourhood of Γ_{γ_0} with values in a neighbourhood of $\Gamma_{\gamma_{-n}}$. Moreover, since only a finite number of components of the grand critical orbit of f are involved for defining $f_{\widehat{\gamma}}^{-n}$, we may always shrink the parameter space M to some $\Omega \Subset M$ so that the domain of definition of $f_{\widehat{\gamma}}^{-n}$ for fixed n and $\widehat{\gamma}$ contains a tubular neighbourhood of $\Gamma_{\gamma_0} \cap \left(\Omega \times \mathbb{C}^k\right)$ of the form

$$T_\Omega(\gamma_0, \eta) := \{(\lambda, z) \in \Omega \times \mathbb{C}^k \,:\, d(z, \gamma_0(\lambda)) < \eta\}.$$

Our goal is to get a uniform η, independent from n, and to control the size of $f_{\widehat{\gamma}}^{-n}\left(T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))\right)$ for suitable $\widehat{\eta}_p(\widehat{\gamma}) > 0$ and $U_0 \subset M$. We will now explain how this boils down to estimating some

kind of Lyapounov exponent.

Let us denote by

$$F_{\gamma(\lambda)}(z) := f_\gamma(\lambda, z + \gamma(\lambda)) - f_\gamma(\lambda,\gamma(\lambda)). \tag{3.2}$$

This just amounts to a change of coordinates in order to have the origin sent to the origin for the restriction at every λ. We shall denote by $F^n_{\gamma(\lambda)}(z)$ the composition $F_{\mathcal{F}^{n-1}\gamma(\lambda)} \circ \cdots \circ F_{\gamma(\lambda)}(z)$.

As $F^n_{\gamma(\lambda)}$ is locally invertible at the origin when $\gamma \notin \mathcal{J}_s$, we may now define functions u_n on $\mathcal{X} \times \overline{W_0}$ by setting

$$u_n(\gamma,\lambda) := \log \|(DF^n_{\gamma(\lambda)}(0))^{-1}\|.$$

Let us stress that $(DF^n_{\gamma(\lambda)}(0))^{-1}$ depends holomorphically on $\lambda \in \overline{W_0}$.

From now on we consider three open balls $U_0 \Subset V_0 \Subset W_0$ centered at λ_0 in M. Let us introduce the function r_n on $\mathcal{X}$ and $\widehat{u}_n$ on $\widehat{\mathcal{X}}$ by setting

$$r_n(\gamma) := e^{-2\sup_{\lambda\in U_0} u_n(\gamma,\lambda)} \text{ and } \widehat{u}_n(\widehat{\gamma}) := \sup_{\lambda\in U_0} u_n(\gamma_0,\lambda) = -\frac{1}{2}\log r_n(\gamma_0). \tag{3.3}$$

We may now state the announced result.

Proposition 3.4.2. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps over M of large topological degree $d \geq 2$ which admits an acritical and ergodic equilibrium web $\mathcal{M}$. Assume that the functions $\widehat{u}_n$ are $\widehat{\mathcal{M}}$-integrable and that*

$$\lim_n \frac{1}{n}\int_{\widehat{\chi}} \widehat{u}_n \, d\widehat{\mathcal{M}} = L \text{ for some } L0. \tag{3.4}$$

Then there exist $p \geq 1$, a Borel subset $\widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$ such that $\widehat{\mathcal{M}}(\widehat{\mathcal{Y}}) = 1$, a measurable function $\widehat{\eta}_p : \widehat{\mathcal{Y}} \to]0,1]$ and a constant $A > 0$ which satisfy the following properties.

For every $\widehat{\gamma} \in \widehat{\mathcal{Y}}$ and every $n \in p\mathbb{N}^\star$ the iterated inverse branch $f^{-n}_{\widehat{\gamma}}$ is defined on the tubular neighbourhood $T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))$ of $\Gamma_{\gamma_0} \cap (U_0 \times \mathbb{C}^k)$ and

$$f^{-n}_{\widehat{\gamma}}\left(T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))\right) \subset T_{U_0}(\gamma_{-n}, e^{-nA}).$$

Moreover, the map $f^{-n}_{\widehat{\gamma}}$ is Lipschitz with Lip $f^{-n}_{\widehat{\gamma}} \leq \widehat{l}_p(\widehat{\gamma})e^{-nA}$ where $\widehat{l}_p(\widehat{\gamma}) \geq 1$.

The proof of Proposition 3.4.2 follows a strategy due to Briend-Duval [BD99]. We start recalling a technical lemma that we shall need later. A proof can be found in [Dup02, Lemma 1.1.30].

Lemma 3.4.3. *Let A be a metric space. Let φ, $(\psi_n)_{n\geq 0}$ measurable and strictly positive functions on A. Suppose that*

$$\forall x \in A, \lim_{n\to\infty} \frac{1}{n}\log \psi_n(x) = 0.$$

Then, for every $\varepsilon > 0$, there exist two measurable functions $\alpha,\beta : A \to \mathbb{R}^*_+$ such that $\alpha \leq \varphi \leq \beta$ and

$$\forall n \geq 0, \forall x \in A, \alpha(x)e^{-n\varepsilon} \leq \psi_n(x) \leq \beta(x)e^{n\varepsilon}.$$

The following is a classical quantitative version of the inverse mapping theorem (see [Dup02]).

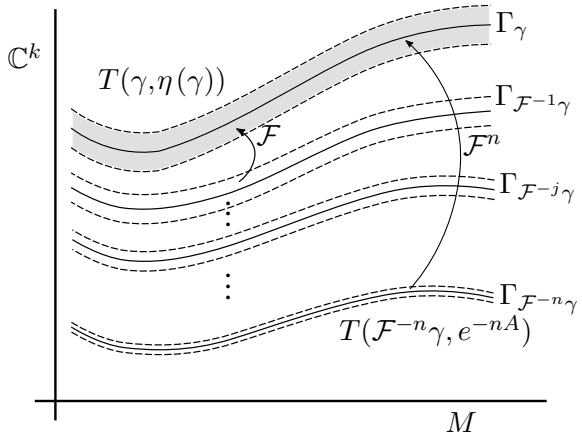


Figure 3.1.: the backward contraction

Lemma 3.4.4. *Let $g : U \to V$ a polynomial-like map. There exists $\varepsilon_0 > 0$ and, for every $\varepsilon \in]0, \varepsilon_0]$, a positive constant $c(g,\varepsilon)$ such that, for every $x \in J_g \setminus C_g$, the following are satisfied:*

1. *g is injective on $B\left(x, 2c(g,\varepsilon)\left\|Dg_x^{-1}\right\|^{-1}\right)$;*
2. *$B\left(g(x), c(g,\varepsilon)\left\|Dg_x^{-1}\right\|^{-2}\right) \subset g\left(B\left(x, 2c(g,\varepsilon)\left\|Dg_x^{-1}\right\|^{-1}\right)\right)$. So, on $B\left(g(x), c(g,\varepsilon)\left\|Dg_x^{-1}\right\|^{-2}\right)$ there is a well defined inverse branch for g, that we denote with g_x^{-1};*
3. *$\operatorname{Lip} g_x^{-1} \leq e^{\varepsilon/3}\left\|Dg_x^{-1}\right\|$ on $B\left(g(x), c(g,\varepsilon)\left\|Dg_x^{-1}\right\|^{-2}\right)$.*

Consider now a holomorphic family of polynomial like-maps. Remark that, in Lemma 3.4.4, if g depends continuously from some parameter, also the constant $c(g,\varepsilon)$ depends continuously from that parameter ($c(g,\varepsilon) = \frac{1-e^{\varepsilon/3}}{\|g\|_{\mathcal{C}^2}}$, see [BD99]). Let $p \geq 1$ and $r_p(\gamma)$ be defined as in (3.3). The next lemma shows that r_p measures the size of tubular neighbourhoods of Γ_γ on which f^p is invertible and contracting.

Lemma 3.4.5. *For every small $\varepsilon > 0$ there exists $C_p(\varepsilon) > 0$ such that for any $\gamma \in \mathcal{X}$ the map f^p admits an inverse branch $(f^p)_\gamma^{-1}$ on the tube $T_{U_0}(\mathcal{F}^p(\gamma), C_p(\varepsilon)r_p(\gamma))$ which maps $\Gamma_{\mathcal{F}^p(\gamma)} \cap (U_0 \times \mathbb{C}^k)$ to $\Gamma_\gamma \cap (U_0 \times \mathbb{C}^k)$ and satisfy $\operatorname{Lip}(f^p)_\gamma^{-1} \leq e^{\varepsilon/3} r_p(\gamma)^{-1/2}$.*

Proof. We use the quantitative version of the inverse mapping theorem given in Lemma 3.4.4. Let $M := \sup_{\lambda\in U_0, \gamma\in\mathcal{X}} \|F^p_{\gamma(\lambda)}\|_{\mathcal{C}^2,\overline{B(0,R_p)}}$ and let $\delta_p(\varepsilon) := R_p(1 - e^{-\varepsilon/3})/M$. Then for every $(\gamma,\lambda) \in \mathcal{X} \times U_0$:

- $(F^p_{\gamma(\lambda)})^{-1}$ is defined on $B_{\mathbb{C}^k}\left(0, C_p(\varepsilon)\|(DF^p_{\gamma(\lambda)}(0))^{-1}\|^{-2}\right)$,
- $\operatorname{Lip}(F^p_{\gamma(\lambda)})^{-1} \leq e^{\frac{\varepsilon}{3}}\|(DF^p_{\gamma(\lambda)}(0))^{-1}\|$

and the Lemma in proved. □

We can now prove Proposition 3.4.2.

Proof of Proposition 3.4.2. Recall that $\widehat{u}_p(\widehat{\gamma}) = -\frac{1}{2}\log r_p(\gamma_0)$. By assumption $\lim_n \frac{1}{n}\int_{\widehat{\mathcal{X}}} \widehat{u_n}\, d\widehat{\mathcal{M}} = L$ with $L \leq -\frac{\log d}{2}$. Let $p \geq 1$ such that $\frac{1}{p}\int_{\widehat{\mathcal{X}}} \widehat{u}_p\, d\widehat{\mathcal{M}} =: L' \leq L + \varepsilon$. By applying Birkhoff Ergodic Theorem there exists $\widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$ such that $\widehat{\mathcal{M}}(\widehat{\mathcal{Y}}) = 1$ and

$$\forall \widehat{\gamma} \in \widehat{\mathcal{Y}}\ ,\ \lim_n \frac{1}{n}\sum_{j=1}^{n} \widehat{u}_p\left(\widehat{\mathcal{F}}^{-j}(\widehat{\gamma})\right) = \int_{\widehat{\mathcal{X}}} \widehat{u}_p\, d\widehat{\mathcal{M}} = pL'. \tag{3.5}$$

Since $\widehat{u}_p(\widehat{\mathcal{F}}^{-n}(\widehat{\gamma})) = -\frac{1}{2}\log r_p(\gamma_{-n})$ we deduce from (3.5) that $\lim_n \frac{1}{n}\log r_p(\gamma_{-n}) = 0$. In particular there exists a measurable function $\widehat{r}_p : \widehat{\mathcal{Y}} \to]0,1]$ such that

$$C_p(\varepsilon) r_p(\gamma_{-n}) \geq \widehat{r}_p(\widehat{\gamma}) e^{-(n-1)\varepsilon/2}.$$

We also deduce from (3.5) that there exists $\widehat{l}_p : \widehat{\mathcal{Y}} \to [1, +\infty[$ such that

$$\prod_{j=1}^{n} \left(r_p(\gamma_{-j})\right)^{-1/2} \leq \widehat{l}_p(\widehat{\gamma}) e^{npL' + n\varepsilon/6}.$$

Now, setting $\widehat{\eta}_p := \widehat{r}_p/\widehat{l}_p$, one can verify by induction that:

- $(f^p)^{-n}_{\widehat{\gamma}}$ is defined on $T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))$,
- $\mathrm{Lip}(f^p)^{-n}_{\widehat{\gamma}} \leq \widehat{l}_p(\widehat{\gamma}) e^{n(pL' + \tau + \varepsilon/2)}$,
- $(f^p)^{-n}_{\widehat{\gamma}}\left[T_{U_0}(\gamma_0, \widehat{\eta}_p(\widehat{\gamma}))\right] \subset T_{U_0}(\gamma_{-n}, C_p(\varepsilon) r_p(\gamma_{-(n+1)}))$.

This completes the proof of Proposition 3.4.2. □

3.4.2. Estimating a Lyapounov exponent

In this subsection we prove that the assumption (3.4) of Proposition 3.4.2 is satisfied when f admits an acritical and ergodic equilibrium web $\mathcal{M}$. The statement is as follows.

Proposition 3.4.6. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. which admits an acritical and ergodic equilibrium web $\mathcal{M}$. Then the functions $\widehat{u}_n$ are $\widehat{\mathcal{M}}$-integrable, there exists a constant $L \leq -\frac{1}{2}\log\left(\frac{d_t}{d_{k-1}}\right) < 0$ such that*

$$\lim_n \frac{1}{n}\int_{\widehat{\mathcal{X}}} \widehat{u}_n\, d\widehat{\mathcal{M}} = L$$

and $\lim_n \frac{1}{n}\widehat{u}_n(\widehat{\gamma}) = L$ for $\widehat{\mathcal{M}}$-almost every $\widehat{\gamma} \in \mathcal{X}$.

Note that the constant L may be considered as a bound for a Lyapounov exponent of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$, and the estimate is precisely the (opposite of the) one in Theorem 1.2.21 for the smallest Lyapounov exponent for a polynomial-like map of large topological degre. The combination of Propositions 3.4.2 and 3.4.6 will allow us to prove Theorem 3.4.1 (see Section

3.4.3).

We keep here the assumptions and the notations introduced in the previous subsections. In the next lemma, we list some basic properties of the functions u_n and $\widehat{u}_n$.

Lemma 3.4.7. *Let W_0 be an open ball centered at λ_0 in M. Let $\chi_1(\lambda)$ be the smallest Lyapounov exponent of the system $(J_\lambda, f_\lambda, \mu_\lambda)$. The functions u_n and $\widehat{u}_n$ satisfy the following properties.*

1) $u_n(\gamma,\cdot)$ is continuous and psh *on W_0 for every $\gamma \in \mathcal{X}$.*

2) The sequence $(\widehat{u}_n)_n$ is subadditive on $\widehat{\mathcal{X}}$, i.e., $\widehat{u}_{m+n} \leq \widehat{u}_n + \widehat{u}_m \circ \widehat{\mathcal{F}}^n$.

3) For any fixed $\lambda \in W_0$, we have $\lim_n \frac{1}{n} u_n(\gamma,\lambda) = -\chi_1(\lambda)$ for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$.

4) For $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ we have $\lim_n \frac{1}{n} u_n(\gamma,\lambda) = -\chi_1(\lambda)$ for Lebesgue-almost every $\lambda \in W_0$.

Proof. 1) When $\gamma \in \mathcal{X}$ is fixed the function $u_n(\gamma,\cdot)$ is clearly continuous on W_0 and $u_n(\gamma,\lambda) = \sup_{\|e\|=1} \log \|(DF^n_{\gamma(\lambda)}(0))^{-1} \cdot e\|$. To see that $u_n(\gamma,\cdot)$ is *psh* it thus suffices to notice that $\lambda \mapsto \log \|(DF^n_{\gamma(\lambda)}(0))^{-1} \cdot e\|$ is *psh* for each unit vector $e \in \mathbb{C}^k$.

2) Let $\gamma \in \mathcal{X}$ and $m, n \geq 1$. It follows immediately from the definition of F_γ (3.2) that

$$\left(DF^{m+n}_{\gamma(\lambda)}(0)\right)^{-1} = \left(DF^n_{\gamma(\lambda)}(0)\right)^{-1} \circ \left(DF^m_{\mathcal{F}^n(\gamma)(\lambda)}(0)\right)^{-1} \quad \forall \lambda \in W_0. \tag{3.6}$$

Thus, if $\widehat{\gamma} \in \widehat{\mathcal{X}}$ we have $\widehat{u}_{m+n}(\widehat{\gamma}) \leq \log \sup_{\lambda\in U_0} (\|(DF^n_{\gamma_0(\lambda)}(0))^{-1}\| \, \|(DF^m_{\mathcal{F}^n(\gamma_0)(\lambda)}(0))^{-1}\|)$
$\leq \log \sup_{\lambda\in U_0} \|(DF^n_{\gamma_0(\lambda)}(0))^{-1}\| + \log \sup_{\lambda\in U_0} \|(DF^m_{\mathcal{F}^n(\gamma_0)(\lambda)}(0))^{-1}\| = \widehat{u}_n(\widehat{\gamma}) + \widehat{u}_m(\widehat{\mathcal{F}}^n(\widehat{\gamma}))$.

3) By Oseledets Theorem (see 1.2.12), the subset $J_{\lambda,1}$ of J_λ defined by

$$J_{\lambda,1} := \{x \in J_\lambda : \ \lim_n \tfrac{1}{n} \log \|(Df^n_\lambda)^{-1}_x\| = -\chi_1(\lambda,)\}$$

has full μ_λ measure. As $p_{\lambda\star}(\mathcal{M}) = \mu_\lambda$, this implies that $\gamma(\lambda) \in J_{\lambda,1}$ for $\mathcal{M}$-almost every γ in $\mathcal{X}$. The assertion follows.

4) Let us denote by $\mathscr{L}$ the Lebesgue measure on M. Let E be the subset of $\mathcal{X} \times W_0$ given by

$$E := \{(\gamma,\lambda) \in \mathcal{X} \times W_0 : \ \lim_n \frac{1}{n} u_n(\gamma,\lambda) = -\chi_1(\lambda)\}.$$

Notice that E is measurable, since 1_E is the indicatrix of the set

$$\left\{\liminf_{n\to\infty} \frac{u_n(\gamma,\lambda)}{n} = \limsup_{n\to\infty} \frac{u_n(\gamma,\lambda)}{n}\right\},$$

and the $u_n(\lambda,\gamma)$'s are continuous.

For every $\lambda \in W_0$ and every $\gamma \in \mathcal{X}$ we set

$$E^\lambda := \{\gamma \in \mathcal{X} : \ (\gamma,\lambda) \in E\} \quad \text{and} \quad E_\gamma := \{\lambda \in W_0 : \ (\gamma,\lambda) \in E\}.$$

We have to show that $\mathscr{L}(E_\gamma) = \mathscr{L}(W_0)$ for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$. This immediately follows from Tonelli's theorem:

$$\int_{\mathcal{X}} \mathscr{L}(E_\gamma)\, d\mathcal{M}(\gamma) = \int_{W_0} \mathcal{M}(E^\lambda)\, d\mathscr{L}(\lambda) = \mathscr{L}(W_0)$$

since, according to the above third assertion, $\mathcal{M}(E^\lambda) = 1$ for every $\lambda \in W_0$. □

Our strategy will be to transfer the estimates known for a fixed system $(J_{\lambda_0}, f_{\lambda_0}, \mu_{\lambda_0})$ to the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$. This will be possible because the graphs Γ_γ for $\gamma \in \mathcal{X}$ cannot approach the critical set C_f in a non uniform way, a phenomenon which simply relies on the compactness of the closure of $\mathcal{X}$ and the following basic property (see Lemma 2.2.6 for the proof).

Fact *There exist* $0 < \alpha \leq 1$ *such that* $\sup_{V_0} |\varphi| \leq |\varphi(\lambda)|^\alpha$ *for every* $\lambda \in V_0$ *and every holomorphic function* $\varphi : W_0 \to \mathbb{C}$ *such that* $0 < |\varphi| < 1$.

More specifically, the key uniformity property we need is given by the next lemma. In our proofs, we shall denote the smallest singular value of an invertible linear map L of $\mathbb{C}^k$ by $\delta(L)$. Let us recall that $\delta(L) = \|L^{-1}\|^{-1}$ and that $\delta(L)^k \leq |\mathrm{det}L| \leq \delta(L)\|L\|^{k-1}$.

Lemma 3.4.8. *Let* $U_0 \Subset V_0 \Subset W_0$ *be open balls centered at* λ_0 *in* M. *Then there exist* $\alpha > 0$ *and* $c > 0$ *such that* $\frac{1}{n}u_n(\gamma,\lambda) \leq \frac{k}{\alpha}\frac{1}{n}u_n(\gamma,\lambda_0') + \log c$ *for every* $n \geq 1$, *every* $\gamma \in \mathcal{X}$ *and every* $(\lambda_0', \lambda) \in V_0 \times V_0$.

Proof. By the compactness of $\overline{\mathcal{X}}$ and $\overline{V_0}$, we get $c_1 := \sup_{\gamma\in\mathcal{X},\lambda\in V_0} \|DF_{\gamma(\lambda)}(0)\|^{k-1} < +\infty$ and thus $|\mathrm{det}(DF^1_{\gamma(\lambda)}(0))| \leq c_1\delta(DF_{\gamma(\lambda)}(0))$ for every $\lambda \in \overline{V_0}$ and every $\gamma \in \mathcal{X}$.
Then, as $\mathrm{det}DF^n_{\gamma(\lambda)}(0) = \prod_{j=0}^{n-1} \mathrm{det}DF_{\mathcal{F}^j(\gamma)}(0)$ and $\prod_{j=0}^{n-1} \delta(DF_{\mathcal{F}^j(\gamma)}(0)) \leq \delta(DF^n_{\gamma(\lambda)}(0))$ we get

$$|\mathrm{det}DF^n_{\gamma(\lambda)}(0)| \leq c_1^n\delta(DF^n_{\gamma(\lambda)}(0)); \;\; \forall\gamma \in \mathcal{X},\; \forall\lambda \in \overline{V_0}. \tag{3.7}$$

Let us set $c_2 := \sup_{\lambda\in\overline{W_0},\gamma\in\overline{\mathcal{X}}} |\mathrm{det}DF_{\gamma(\lambda)}(0)|$. When $\gamma \in \mathcal{X}$, the holomorphic function $\varphi(\lambda) := \frac{1}{c_2^n}\mathrm{det}DF^n_{\gamma(\lambda)}(0)$ is non vanishing and its modulus is bounded by 1 on W_0. Applying Lemma 2.2.6 to φ, we get $0 < \alpha \leq 1$ (which only depends on V_0 and W_0) such that:

$$\sup_{\lambda'\in V_0} |\mathrm{det}DF^n_{\gamma(\lambda')}(0)| \leq c_2^{n(1-\alpha)}|\mathrm{det}DF^n_{\gamma(\lambda)}(0)|^\alpha; \;\; \forall n \geq 1,\; \forall\gamma \in \mathcal{X},\; \forall\lambda \in V_0. \tag{3.8}$$

Using successively (3.8) and (3.7) we get for any $\lambda,\lambda'_0 \in V_0$

$$\begin{aligned}\left[\delta(DF^n_{\gamma(\lambda_0')}(0))\right]^k \leq |\mathrm{det}DF^n_{\gamma(\lambda_0')}(0))| &\leq c_2^{n(1-\alpha)}|\mathrm{det}DF^n_{\gamma(\lambda)}(0)|^\alpha \\ &\leq c_2^{n(1-\alpha)}c_1^{n\alpha}\left[\delta(DF^n_{\gamma(\lambda)}(0))\right]^\alpha.\end{aligned}$$

Then, applying $\log$ and multiplying by $\frac{-1}{n}$ we get

$$k\frac{1}{n}u_n(\gamma,\lambda_0') \geq \alpha\frac{1}{n}u_n(\gamma,\lambda) - \alpha\left(\log c_1 + \frac{1-\alpha}{\alpha}\log c_2\right)$$

which is the desired estimate with $c := c_1 c_2^{(1-\alpha)/\alpha}$. □

The next Lemma gathers the properties of $(u_n)_n$ which will be crucial to end the proof.

Lemma 3.4.9. *Let U_0,V_0,W_0 be as in Lemma 3.4.8. Then the following properties occur.*

1) The sequence $(\frac{1}{n}u_n)_n$ is uniformly bounded from below on $\mathcal{X} \times V_0$.

2) The sequence $\left(\frac{1}{n}u_n(\gamma,\cdot)\right)_n$ is uniformly bounded on V_0 for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$.

3) The functions $\widehat{u}_n$ are $\widehat{\mathcal{M}}$-integrable.

Proof. 1) Using the properties of the smallest singular value we have

$$\begin{aligned}\frac{1}{n}u_n(\gamma,\lambda) &= -\frac{1}{n}\log\delta\left(DF^n_{\gamma(\lambda)}(0)\right) \geq -\frac{1}{nk}\log\left|\det\left(DF^n_{\gamma(\lambda)}(0)\right)\right| \\ &= -\frac{1}{k}\left(\frac{1}{n}\sum_{j=0}^{n-1}\log|\det DF_{(\mathcal{F}^j\gamma)(\lambda)}(0)|\right)\end{aligned}$$

and the assertion follows immediately from the definition and the continuity of $F_{\gamma(\lambda)}$.

2) We have just seen that $\frac{1}{n}u_n(\gamma,\cdot)$ is uniformly bounded from below on V_0. By the fourth assertion of Lemma 3.4.7, for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ there exists $\lambda_\gamma \in V_0$ such that $\lim_n \frac{1}{n}u_n(\gamma,\lambda_\gamma) = -\chi_1(\lambda_\gamma)$. On the other hand, by Lemma 3.4.8, we have $\frac{1}{n}u_n(\gamma,\lambda) \leq \frac{k}{\alpha}\frac{1}{n}u_n(\gamma,\lambda_\gamma) + \log c$ for every $n \in \mathbb{N}$ and every $\lambda \in V_0$ and thus $\frac{1}{n}u_n(\gamma,\cdot)$ is uniformly bounded from above on V_0.

3) By the above first assertion, we know that $\widehat{u}_n$ is bounded from below. It thus suffices to show that $\int \widehat{u}_n(\widehat{\gamma})\, d\widehat{\mathcal{M}}(\widehat{\gamma}) < +\infty$. By Lemma 3.4.8 we have

$$\begin{aligned}\int \widehat{u}_n(\widehat{\gamma})\, d\widehat{\mathcal{M}}(\widehat{\gamma}) &\leq n\log c + \frac{k}{\alpha}\int u_n(\pi_0(\widehat{\gamma}),\lambda_0)\, d\widehat{\mathcal{M}}(\widehat{\gamma}) \\ &= n\log c + \frac{k}{\alpha}\int u_n(\gamma,\lambda_0)\, d\mathcal{M}(\gamma) \\ &= n\log c + \frac{k}{\alpha}\int \log\|(DF^n_{\gamma(\lambda_0)}(0))^{-1}\|\, d\mathcal{M}(\gamma) \\ &= n\log c - \frac{k}{\alpha}\int \log\delta(DF^n_{\gamma(\lambda_0)}(0))\, d\mathcal{M}(\gamma).\end{aligned}$$

Using (3.7), we thus get

$$\begin{aligned}\int \widehat{u}_n(\widehat{\gamma})\, d\widehat{\mathcal{M}}(\widehat{\gamma}) &\leq -\frac{k}{\alpha}\int \log|\det(DF^n_{\gamma(\lambda_0)}(0))|\, d\mathcal{M}(\gamma) + \frac{kn}{\alpha}\log c_1 + n\log c \\ &= -\frac{k}{\alpha}\int \log|\det(Df^n_{\lambda_0})_{\gamma(\lambda_0)}|\ \ d\mathcal{M}(\gamma) + C_n \\ &= -\frac{k}{\alpha}\int \log|\det(Df^n_{\lambda_0})_x|\ (dp_{\lambda_0\star}\mathcal{M})(x) + C_n,\end{aligned}$$

and the conclusion follows because, since f_{λ_0} is of large topological degree, the psh function $\log|\det(Df^n_{\lambda_0})_x|$ is integrable with respect to $p_{\lambda_0\star}\mathcal{M} = \mu_{\lambda_0}$ (Theorem 1.2.16). □

We are now ready to establish Proposition 3.4.6. We shall make use of the following subadditive ergodic theorem due to Kingman (see [Arn98]).

Theorem 3.4.10 (Kingman subadditive Theorem). *Let A be a metric space, g an automorphism of A (i.e. an invertible and bimeasurable map) and ν an ergodic measure. Let $(\varphi_n)_{n\in\mathbb{N}}$ be measurable functions, $\varphi_n : A \to \mathbb{R}\cup\{-\infty\}$. Assume that $\max\{\varphi_1, 0\} \in L^1(\nu)$ and that the sequence $\{\varphi_n\}$ is* subadditive*, i.e., it satisfies*

$$\varphi_{m+n}(x) \leq \varphi_m(x) + \varphi_n \circ g^m(x)$$

for ν-almost every $x \in A$. Then, there exists a constant c such that

1. *for ν-almost every $x \in A$, $\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = c$;*
2. *$\lim_{n\to\infty} \frac{1}{n}\int_A \varphi_n d\nu = \inf_{n\in\mathbb{N}} \frac{1}{n}\int_A \varphi_n d\nu = c\nu(A)$.*

Proof of Proposition 3.4.6. We will apply Kingman subadditive ergodic theorem 3.4.10 to the sequence $(\widehat{u}_n)_n$. This is possible since the system $(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}})$ is ergodic, the sequence $(\widehat{u}_n)_n$ is subadditive (second assertion of Lemma 3.4.7) and $\widehat{u}_1 \in L^1(\widehat{\mathcal{M}})$ (last assertion of Lemma 3.4.9). According to this theorem, there exists $L \in \mathbb{R}$ such that $\lim_n \frac{1}{n}\widehat{u}_n(\widehat{\gamma}) = L$ for $\widehat{\mathcal{M}}$-almost every $\widehat{\gamma} \in \widehat{\mathcal{X}}$ and $\lim_n \frac{1}{n}\int_{\widehat{\mathcal{X}}} \widehat{u_n}\, d\widehat{\mathcal{M}} = L$. It remains to show that $L \leq -\frac{1}{2}\log\left(\frac{d_t}{d_{k-1}}\right)$.

Taking into account the fourth assertion of Lemma 3.4.7 and the second assertion of Lemma 3.4.9, we may thus pick $\widehat{\gamma} \in \widehat{\chi}$ such that:

i) $\lim_n \frac{1}{n}\widehat{u}_n(\widehat{\gamma}) = L$,

ii) $\frac{1}{n}u_n(\gamma_0, \cdot)$ is uniformly bounded on V_0,

iii) $\lim_n \frac{1}{n}u_n(\gamma_0, \lambda) = -\chi_1(\lambda)$ for Lebesgue-almost every $\lambda \in V_0$.

We are going to prove that $L \leq -\frac{1}{2}\log\left(\frac{d_t}{d_{k-1}}\right)$. Recalling that $\widehat{u}_n(\widehat{\gamma}) = \sup_{\lambda\in U_0} u_n(\gamma_0, \lambda)$, there exist $\lambda_n \in U_0$ such that $\lim_n \frac{u_n(\gamma_0,\lambda_n)}{n} = L$. Up to a subsequence, we may assume that $\lambda_n \to \lambda_0' \in \overline{U}_0$. Pick $r > 0$ such that $B(\lambda_n, r) \subset V_0$ for all $n \in \mathbb{N}$. By the subharmonicity of $u_n(\gamma_0, \cdot)$ on V_0 (first assertion of Lemma 3.4.7), we have, for every $n \in \mathbb{N}$,

$$\frac{u_n(\gamma_0, \lambda_n)}{n} \leq \frac{1}{|B(\lambda_n, r)|}\int_{B(\lambda_n,r)} \frac{u_n(\gamma_0, \lambda)}{n}.$$

Taking a limit in n, the left hand side converges to L, while the right hand side (by Lebesgue convergence Theorem, ii) and iii)), converges to $\frac{1}{|B(\lambda_0',r)|}\int_{B(\lambda_0',r)} -\chi_1(\lambda)$. The assertion follows since $\chi_1(\lambda) \geq \frac{1}{2}\log\left(\frac{d_t}{d_{k-1}}\right)$ for every λ (by Theorem 1.2.21). □

3.4.3. Proof of Theorem 3.4.1

Let $\mathcal{M}_0$ be an ergodic acritical equilibrium web for f. The proof is based on the following key property.

$$\mathcal{M}_0\left(\{\gamma\in\mathcal{J}\,:\;\exists k\in\mathbb{N},\exists\gamma'\in\mathcal{J}\text{ s.t. }\Gamma_{\mathcal{F}^k(\gamma)}\cap\Gamma_{\gamma'}\neq\emptyset\text{ and }\mathcal{F}^k(\gamma)\neq\gamma'\}\right)=0. \tag{3.9}$$

To prove (3.9), it is sufficient to show that for any fixed $k\in\mathbb{N}$ and any $\lambda_0\in M$ there exists a neighbourhood U_1 of λ_0 such that

$$\mathcal{M}_0\left(\{\gamma\in\mathcal{K}_0\,:\;\exists\gamma'\in\mathcal{K}_0'\text{ s.t. }\Gamma_{\mathcal{F}^k(\gamma)}\cap\Gamma_{\gamma'}\cap\left(U_1\times\mathbb{C}^k\right)\neq\emptyset\text{ and }\mathcal{F}^k(\gamma)\neq\gamma'\}\right)=0. \tag{3.10}$$

To this purpose, we shall work with the natural extension $\left(\widehat{\mathcal{X}},\widehat{\mathcal{F}},\widehat{\mathcal{M}_0}\right)$ of the system $(\mathcal{X},\mathcal{F},\mathcal{M}_0)$ and apply Proposition 3.4.2. We recall that, according to Proposition 3.4.6, all the assumptions of Proposition 3.4.2 are satisfied. Let U_0 be a neighbourhood of λ_0, we may assume that U_0 is simply connected and that $U_1\Subset U_0\Subset M$. Let p be the integer and $\widehat{\eta}_p:\widehat{\mathcal{Y}}\to]0,1]$ be the measurable function defined on the full $\widehat{\mathcal{M}_0}$-measure set $\widehat{\mathcal{Y}}$ given by Proposition 3.4.2. We recall that $\mathcal{X}=\mathcal{J}\setminus\mathcal{J}_s$ and by assumption $\mathcal{M}_0(\mathcal{J}_s)=0$.

For any $B\subset U_0$, we define the *ramification* function R_B by setting

$$R_B(\gamma):=\sup_{\gamma'\in\mathcal{J}:\Gamma_{\gamma'|B}\cap\Gamma_{\gamma|B}\neq\emptyset}\sup_B d\left(\gamma(\lambda),\gamma'(\lambda)\right),\quad\forall\gamma\in\mathcal{J}.$$

Let $\widehat{\mathcal{Y}}_\varepsilon:=\{\widehat{\gamma}\in\widehat{\mathcal{Y}}\,:\;R_{U_0}(\gamma_k)>\varepsilon\}$, it then suffices to prove that $\widehat{\mathcal{M}_0}\left(\widehat{\mathcal{Y}}_\varepsilon\right)=0$ for every $\varepsilon>0$ as it follows from the following observation:

$$\begin{aligned}&\mathcal{M}_0\left(\{\gamma\in\mathcal{J}\,:\;\exists\gamma'\in\mathcal{J}\text{ s.t. }\Gamma_{\mathcal{F}^k(\gamma)}\cap\Gamma_{\gamma'}\cap\left(U_0\times\mathbb{C}^k\right)\neq\emptyset\text{ and }\mathcal{F}^k(\gamma)\neq\gamma'\}\right)\\&=\mathcal{M}_0\left(\{\gamma\in\mathcal{J}\,:\;R_{U_0}(\mathcal{F}^k(\gamma))>0\}\right)=\mathcal{M}_0\left(\{\gamma\in\mathcal{X}\,:\;R_{U_0}(\mathcal{F}^k(\gamma))>0\}\right)\\&=\widehat{\mathcal{M}_0}\left(\{\widehat{\gamma}\in\widehat{\mathcal{Y}}\,:\;R_{U_0}(\gamma_k)>0\}\right)=\widehat{\mathcal{M}_0}\left(\cup_{s\in\mathbb{N}^*}\widehat{\mathcal{Y}}_{\frac{1}{s}}\right).\end{aligned}$$

Let us proceed by contradiction and assume that $\widehat{\mathcal{M}_0}\left(\widehat{\mathcal{Y}}_\varepsilon\right)>0$ for some $\varepsilon>0$. Then, after reducing $\varepsilon>0$, we may assume that $\widehat{\mathcal{M}_0}\left(\{\widehat{\gamma}\in\widehat{\mathcal{Y}}_\varepsilon\,:\;\widehat{\eta}_p(\widehat{\mathcal{F}}^k(\widehat{\gamma}))>\varepsilon\}\right)>0$. In the sequel we shall denote $\widehat{\gamma}_k:=\widehat{\mathcal{F}}^k(\widehat{\gamma})$. Owing to the equicontinuity of $\mathcal{X}$ we may cover U_1 with finitely many open sets $B_i\subset U_0$, say with $1\leq i\leq N$, such that

$$\forall(\gamma,\gamma')\in\mathcal{X}\times\mathcal{J},\;\forall\lambda_1\in B_i\;:\;\gamma(\lambda_1)=\gamma'(\lambda_1)\Rightarrow\sup_{\lambda\in B_i}d_{\mathbb{C}^k}\left(\gamma(\lambda),\gamma'(\lambda)\right)<\varepsilon\,. \tag{3.11}$$

As $R_{U_1}(\gamma)=0$ when $\max_{1\leq i\leq N}R_{B_i}(\gamma)=0$ (by analyticity we have $\gamma=\gamma'$ on U_1 if $\gamma=\gamma'$ on some B_i), there exist $1\leq j\leq N$ and $\alpha>0$ such that:

$$\widehat{\mathcal{M}_0}\left(\{\widehat{\gamma}\in\widehat{\mathcal{Y}}_\varepsilon\,:\;\widehat{\eta}_p(\widehat{\gamma}_k)>\varepsilon\text{ and }R_{B_j}(\gamma_k)>\alpha\}\right)>0.$$

Let us set $\widehat{\mathcal{Y}}_{\varepsilon,j,\alpha} := \{\widehat{\gamma} \in \widehat{\mathcal{Y}}_\varepsilon \,:\, \widehat{\eta}_p(\widehat{\gamma}_k) > \varepsilon \text{ and } R_{B_j}(\gamma_k) > \alpha\}$. Applying Poincaré recurrence theorem to $\widehat{\mathcal{F}}^{-p}$, we find $\widehat{\gamma} \in \widehat{\mathcal{Y}}_{\varepsilon,j,\alpha}$ and an increasing sequence of integers $(n_q)_q$ with $n_q \in p\mathbb{N}$ such that $\widehat{\mathcal{F}}^{-n_q}(\widehat{\gamma}) \in \widehat{\mathcal{Y}}_{\varepsilon,j,\alpha}$ for every $q \in \mathbb{N}$. In particular $\widehat{\gamma} \in \widehat{\mathcal{Y}}_{\varepsilon,j,\alpha}$ and $R_{B_j}(\gamma_{k-n_q}) > \alpha$ for every $q \in \mathbb{N}$. We will reach a contradiction by establishing that

$$\lim_{m\to+\infty} R_{B_j}(\gamma_{k-mp}) = 0, \;\; \forall \widehat{\gamma} \in \widehat{\mathcal{Y}}_{\varepsilon,j,\alpha}. \tag{3.12}$$

To this purpose we shall use Proposition 3.4.2 to show that $R_{B_j}(\gamma_{k-n}) \leq e^{-nA}$ when $n \in p\mathbb{N}$ and $\widehat{\gamma} \in \widehat{\mathcal{Y}}_{\varepsilon,j,\alpha}$. Let $\gamma' \in \mathcal{J}$ such that $\gamma'(\lambda_1) = \gamma_{k-n}(\lambda_1)$ for some $\lambda_1 \in B_j$. Then $(\mathcal{F}^n\gamma')(\lambda_1) = \gamma_k(\lambda_1)$ and thus, according to (3.11), $\sup_{\lambda\in B_j} d\left((\mathcal{F}^n\gamma')(\lambda), \gamma_k(\lambda)\right) < \varepsilon < \widehat{\eta}_p(\widehat{\gamma}_k)$. This means that

$$\Gamma_{\mathcal{F}^n\gamma'} \cap \left(B_j \times \mathbb{C}^k\right) \subset T_{B_j}\left(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k)\right). \tag{3.13}$$

Now, by Proposition 3.4.2, the inverse branch $f^{-n}_{\widehat{\gamma}_k}$ of f^n is defined on the tube $T_{U_0}\left(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k)\right)$ and maps it biholomorphically into $T_{U_0}\left(\gamma_{k-n}, e^{-nA}\right)$. As $B_j \subset U_0$, this yields:

$$f^{-n}_{\widehat{\gamma}_k}\left(T_{B_j}\left(\gamma_k, \widehat{\eta}_p(\widehat{\gamma}_k)\right)\right) \subset T_{B_j}\left(\gamma_{k-n}, e^{-nA}\right). \tag{3.14}$$

By construction, we have $f^{-n}_{\widehat{\gamma}_k}(\Gamma_{\gamma_k}) = \Gamma_{\gamma_{k-n}}$ and therefore $f^{-n}_{\widehat{\gamma}_k}\left((\mathcal{F}^n\gamma')(\lambda_1)\right) = f^{-n}_{\widehat{\gamma}_k}(\gamma_k(\lambda_1)) = \gamma_{k-n}(\lambda_1) = \gamma'(\lambda_1)$. This implies that $f^{-n}_{\widehat{\gamma}_k}\left(\Gamma_{\mathcal{F}^n\gamma'}\right) = \Gamma_{\gamma'}$ which in turns, by (3.13) and (3.14), implies that $\sup_{\lambda\in B_j} d(\gamma'(\lambda), \gamma_{k-n}(\lambda)) \leq e^{-nA}$. Then (3.12) follows and thus (3.9) and (3.10) are proved.

Let us now establish the existence of an equilibrium lamination $\mathcal{L}_0$. Consider the set

$$\mathcal{L}_0^+ := \{\gamma \in \mathcal{J} \setminus \mathcal{J}_s \,:\, \forall \gamma' \in \mathcal{J}, \forall k \in \mathbb{N}, \Gamma_{\mathcal{F}^k(\gamma)} \cap \Gamma_{\gamma'} \neq \emptyset \Rightarrow \mathcal{F}^k(\gamma) = \gamma'\}.$$

By (3.9), we have $\mathcal{M}_0\left(\mathcal{L}_0^+\right) = 1$ and, by construction, $\mathcal{L}_0^+$ satisfies the following properties:

1) $\mathcal{L}_0^+ \subset \mathcal{J} \setminus \mathcal{J}_s$,

2) $\mathcal{F}\left(\mathcal{L}_0^+\right) \subset \mathcal{L}_0^+$,

3) $\forall \gamma, \gamma' \in \mathcal{L}_0^+ \,:\, \Gamma_\gamma \cap \Gamma_{\gamma'} \neq \emptyset \Rightarrow \gamma = \gamma'$.

The set $\mathcal{L}_0 := \cup_{m\geq 0}\mathcal{F}^{-m}(\mathcal{L}_+)$ also satisfies the properties (1), (2) and (3). Moreover $\mathcal{F} : \mathcal{L}_0 \to \mathcal{L}_0$ is d^k-to-1.

Let us prove the uniqueness assertions. Let $\mathcal{M}_0'$ be an equilibrium web for f (or, more generally, a compactly supported probability measure on $\mathcal{J}$ such that $p_{\lambda_0\star}\mathcal{M}_0' = \mu_{\lambda_0}$ for some $\lambda_0 \in M$). Let $\mathcal{K}_0$ and $\mathcal{K}_0'$ be the supports of $\mathcal{M}_0$ and $\mathcal{M}_0'$. Let us fix $\lambda_0 \in M$ and recall that $\mu_{\lambda_0} = p_{\lambda_0\star}\mathcal{M}_0' = p_{\lambda_0\star}\mathcal{M}_0$. Then, for any Borel subset $\mathcal{A}$ of $\mathcal{J}$, we have $\mu_{\lambda_0}\left(p_{\lambda_0}(\mathcal{K}_0' \cap \mathcal{A})\right) = \mathcal{M}_0'\left(p_{\lambda_0}^{-1}\left(p_{\lambda_0}(\mathcal{K}_0' \cap \mathcal{A})\right)\right) \geq$

$\mathcal{M}_0'(\mathcal{K}_0' \cap \mathcal{A}) = \mathcal{M}_0'(\mathcal{A})$ and thus

$$\mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : \exists \gamma' \in \mathcal{K}_0' \cap \mathcal{A} \text{ s.t. } \gamma(\lambda_0) = \gamma'(\lambda_0)\}) = \mathcal{M}_0\left(p_{\lambda_0}^{-1}\left(p_{\lambda_0}(\mathcal{K}_0' \cap \mathcal{A})\right)\right) = \mu_{\lambda_0}\left(p_{\lambda_0}(\mathcal{K}_0' \cap \mathcal{A})\right) \geq \mathcal{M}_0'(\mathcal{A}).$$

But, according to (3.9) we have

$$\mathcal{M}_0(\{\gamma \in \mathcal{K}_0 : \exists \gamma' \in \mathcal{K}_0' \cap \mathcal{A} \text{ s.t. } \gamma(\lambda_0) = \gamma'(\lambda_0)\}) = \mathcal{M}_0(\mathcal{K}_0' \cap \mathcal{K}_0 \cap \mathcal{A}) = \mathcal{M}_0(\mathcal{K}_0' \cap \mathcal{A})$$

and therefore $\mathcal{M}_0(\mathcal{K}_0' \cap \mathcal{A}) \geq \mathcal{M}_0'(\mathcal{A})$. This implies that $\mathcal{M}_0(\mathcal{K}_0') = 1$ and that $\mathcal{M}_0 \geq \mathcal{M}_0'$. As both $\mathcal{M}_0$ and $\mathcal{M}_0'$ are probability measures, we have proved that $\mathcal{M}_0 = \mathcal{M}_0'$.

Let $\mathcal{L}'$ be an arbitrary equilibrium lamination for f. Let us pick $\lambda_0 \in M$ and set $\mathcal{L}'_{\lambda_0} := p_{\lambda_0}(\mathcal{L}')$. Using $\mathcal{M}_0(\mathcal{L}_0) = 1$, $\mu_{\lambda_0} = p_{\lambda_0 \star}\mathcal{M}_0$ and $\mu_{\lambda_0}\left(\mathcal{L}'_{\lambda_0}\right) = 1$ yields $\mathcal{M}_0\left(\{\gamma \in \mathcal{L}_0 : \gamma(\lambda_0) \in \mathcal{L}'_{\lambda_0}\}\right) = 1$. On the other hand, (3.9) implies that $\mathcal{M}_0\left(\{\gamma \in \mathcal{L}_0 : \gamma(\lambda_0) \in \mathcal{L}'_{\lambda_0}\}\right) = \mathcal{M}_0(\mathcal{L}_0 \cap \mathcal{L}')$. This shows that $\mathcal{M}_0(\mathcal{L}_0 \Delta \mathcal{L}') = 0$.

Finally, we establish the mixing property of the (unique) equilibrium web. Let $\mathcal{M}$ and $\mathcal{L}$ be the acritical web and an equilibrium lamination. Given $\mathcal{A}, \mathcal{B} \subset \mathcal{J}$, we want to prove that, as $n \to \infty$,

$$\mathcal{M}(\mathcal{F}^{-n}(\mathcal{A}) \cap \mathcal{B}) \to \mathcal{M}(\mathcal{A})\mathcal{M}(\mathcal{B}).$$

Without loss of generality, we can assume that both $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{L}$ (since $\mathcal{M}(\mathcal{L}) = 1$). So, given any $\lambda_0 \in M$, we have $\mathcal{M}(\mathcal{A}) = \mu_{\lambda_0}(p_{\lambda_0}(\mathcal{A}))$. Analogously, $\mathcal{M}(\mathcal{B}) = \mu_{\lambda_0}(p_{\lambda_0}(\mathcal{B}))$ and $\mathcal{M}(\mathcal{F}^{-n}(\mathcal{A}) \cap \mathcal{B}) = \mu_{\lambda_0}(p_{\lambda_0}(\mathcal{F}^{-n}(\mathcal{A}) \cap \mathcal{B})) = \mu_{\lambda_0}\left(f_{\lambda_0}^{-n} p_{\lambda_0}(\mathcal{A}) \cap p_{\lambda_0}\mathcal{B}\right)$. The assertion follows from the mixing property of μ_{λ_0}.

3.5. Motions of cycles (II.2 + II.3 ⇒ II.1)

In this section we prove how, starting with an acritical equilibrium web (whose existence is equivalent to that of an equilibrium lamination, by the previous sections) we can recover the weak notion of holomorphic motion for the repelling J-cycles given in Definition 3.1.3. We stress here that, in order to do this, we do not need to make any further assumption on the family we are considering. In Chapter 4, we shall see how we can recover an actual holomorphic motion for all J-repelling cycles in a family of *two-dimensional* polynomial-like maps.

We start noticing that just the existence of any equilibrium web implies the existence of a set $\mathcal{P} \subset \mathcal{J}$ satisfying all the properties required by Definition 3.1.3 but the last one. This is an immediate consequence of Lemma 2.1.3.

Lemma 3.5.1. *Let f be a holomorphic family of polynomial-like maps of large topological degree d_t. Assume that there exists an equilibrium web $\mathcal{M}$ for f. Then there exists a subset $\mathcal{P} = \cup_n \mathcal{P}_n \subset \mathcal{J}$ such that*

1. $\sharp \mathcal{P}_n = d_t^n + o(d_t^n)$;

2. *every element in $\mathcal{P}_n$ is n-periodic;*

3. *we have*
$$\sum_{\gamma\in\mathcal{P}_n}\delta_\gamma \to \mathcal{M}'.$$
where $\mathcal{M}'$ is a (possibly different) equilibrium web.

Notice that, if the equilibrium web $\mathcal{M}$ in the statement is acritical, by the uniqueness proved in Section 3.4.3 we have $\mathcal{M}=\mathcal{M}'$.

Proof. Let us fix λ_0 in the parameter space. Since f_{λ_0} has large topological degree, Theorem 1.2.19 gives $d_t^n+o(d_t^n)$ repelling periodic points for f_{λ_0} contained in the Julia set J_{λ_0}. By Lemma 2.1.3(2), for every such point p of period n there exists an element $\gamma_p\in\mathcal{J}$ such that $\gamma_p(\lambda_0)=p$ and $\mathcal{F}^n(\gamma_p)=\gamma_p$. This gives the first two assertions of the statement. The last one follows by Theorem 2.1.4, without the need of taking the Cesaro average (as in Proposition 2.1.7). We just need to ensure that, for every λ, we have $d_t^{-n}\sum_{\gamma_p\in\mathcal{P}_n}\delta_p(\lambda)\to\mu_\lambda$. This follows from Theorem 1.2.19, since (at every fixed λ), the number of n-periodic point is d_t^n, and thus only $o(d_t^n)$ of the $\gamma_p(\lambda)$'s can be non-repelling. □

In order to recover the asymptotic motion of the repelling cycles as in Definition 3.1.3, we thus just need to prove that, on any $M'\Subset M$, *asymptotically all* $\gamma_p\in\mathcal{P}_n$ given by Lemma 3.5.1 are repelling. This will be done by means of the following general lemma, which allows us to recover the existence of repelling points for a dynamical system from the information about backward contraction of balls along negative orbits. This can be seen as a generalization of a classical strategy [BD99] (see also [Ber10]).

We keep the notations introduced in Section 3.4.1 regarding the natural extension of a dynamical system and the inverse branches along negative orbits.

Lemma 3.5.2. *Let $\mathcal{K}$ be a compact metric space and $\mathcal{F}:\mathcal{K}\to\mathcal{K}$ be a continuous map. Assume that, for every n, the number of periodic repelling points of period dividing n is less than $d^n+o(d^n)$ for some integer $d\geq 2$. Let ν be a probability measure on $\mathcal{K}$ which is invariant and mixing for $\mathcal{F}$ and of constant Jacobian d (i.e., for every borelian set $A\subset\mathcal{K}$ on which $\mathcal{F}$ is injective we have $d\nu(A)=\nu(\mathcal{F}(A))$). Suppose that there exists an $\mathcal{F}$-invariant subset $\mathcal{L}\subset\mathcal{K}$ such that $\nu(\mathcal{L})=1$ and $\mathcal{F}:\mathcal{L}\to\mathcal{L}$ is a covering of degree d. Let $\left(\widehat{\mathcal{L}},\widehat{\mathcal{F}},\widehat{\nu}\right)$ be the natural extension of the induced system $(\mathcal{L},\mathcal{F},\nu)$ and suppose that for every $\widehat{x}\in\widehat{\mathcal{L}}$, the inverse branch $\mathcal{F}_{\widehat{x}}^{-n}$ is defined and Lipschitz on the open ball $B(x_0,\eta(\widehat{x}))$, with $\mathrm{Lip}\left(\mathcal{F}_{\widehat{x}}^{-n}\right)\leq l(\widehat{x})e^{-nL}$, for some positive measurable functions η and l and some positive constant L. Moreover, assume that*

$$\begin{gathered}\forall x_0\in\mathcal{L},\forall N\colon\ \textit{the preimages } \mathcal{F}_{\widehat{x}}^{-n}\left(\overline{B(x_0,1/N)}\right)\\ \textit{with } \pi_0(\widehat{x})=x_0,\eta(\widehat{x})>\tfrac{1}{N} \textit{ are disjoint for } n \textit{ large enough}.\end{gathered}\tag{3.15}$$

Then,
$$\sigma_n:=\frac{1}{d^n}\sum_{p\in R_n}\delta_p\to\nu$$
where R_n is the set of all repelling periodic points of period (dividing) n.

By *repelling periodic point* here we mean the following: a point x_0 such that, for some n, $\mathcal{F}^n(x_0) = x_0$ and there exists a local inverse branch $\mathcal{H}$ for $\mathcal{F}^n$ sending x_0 to x_0 and such that $\operatorname{Lip}\mathcal{H} < 1$.

Proof. We let $\widetilde{\sigma}$ be any limit value of the sequence σ_n. Remark that

$$\widetilde{\sigma}(\mathcal{K}) \leq \limsup_{n\to\infty} \sigma_n(\mathcal{K}) \leq \lim_{n\to\infty} \frac{d^n + o(d^n)}{d^n} = 1. \tag{3.16}$$

For every $N \in \mathbb{N}$, let $\widehat{\mathcal{L}}_N \subset \widehat{\mathcal{L}}$ be defined as

$$\widehat{\mathcal{L}}_N = \left\{ \widehat{x} \colon \eta(\widehat{x}) > \frac{1}{N} \text{ and } l(\widehat{x}) \leq N \right\}$$

and set $\widehat{\nu}_N := 1_{\widehat{\mathcal{L}}_N}\widehat{\nu}$ and $\nu_N = (\pi_0)_*\,\widehat{\nu}_N$. We also set $\mathcal{L}_N := \pi_0\left(\widehat{\mathcal{L}}_N\right)$. We are going to prove that

$$\widetilde{\sigma}(A) \geq \nu_N(A) \text{ for every borelian } A \quad \forall N \in \mathbb{N}. \tag{3.17}$$

As by hypothesis $\nu_N(A) \to \nu(A)$ as $N \to \infty$, the assertion will then follow from (3.16) and (3.17).

So we turn to prove (3.17). In order to do this, it suffices to prove the following:

$$\forall N \in \mathbb{N}, \forall \widehat{a} \in \widehat{\mathcal{L}}_N, \forall \text{ closed } C \subset B\left(a_0, \frac{1}{2N}\right) : \widetilde{\sigma}(C) \geq \nu_N(C). \tag{3.18}$$

Indeed, given any Borelian subset $A \subset \mathcal{K}$, since $\mathcal{K}$ is compact we can find a partition of $A \cap \mathcal{L}_N$ into finite borelian sets A_i, each of which contained in an open ball $B\left(a_0^i, \frac{1}{3N}\right)$, with $\widehat{a}^i \in \mathcal{L}_N$. The assertion thus follows from (3.18) since, for every A_i, the values $\widetilde{\sigma}(A_i)$ and $\nu_N(A_i)$ are the suprema of the respective measures on closed subsets of A_i (which by construction are contained in $B(a_0^i, \frac{1}{2N})$).

In the following we thus fix a closed subset $C \subset B(a_0, \frac{1}{2N})$. We shall denote by C_δ the closed δ-neighbourhood of C (in $\mathcal{K}$).

Take some δ such that $\delta < \frac{1}{2N}$ and notice that, since $\widehat{a} \in \widehat{\mathcal{L}}_N$, we have $C_\delta \subset B(a_0, \frac{1}{N}) \subset B(a_0, \eta(\widehat{a}))$. We can thus define the set:

$$\widehat{R}_n^\delta = \{\, \widehat{x} \in \widehat{C}_\delta \cap \widehat{\mathcal{L}}_N \colon x_0 = a_0 \text{ and } \mathcal{F}_{\widehat{x}}^{-n}(C_\delta) \cap C \neq \emptyset \,\}$$

and denote by S_n^δ the set of preimages of C_δ of the form $\mathcal{F}_{\widehat{x}}^{-n}(C_\delta)$, with $\widehat{x} \in \widehat{R}_n^\delta$. By the assumption (3.15), the elements of S_n^δ are mutually disjoint for $n \geq \widetilde{n}_0$ (and of course $\sharp S_n^\delta \leq d^n$). We claim that $\sharp S_n^\delta$ satisfies the following two estimates:

1. $\frac{1}{d^n}\sharp S_n^\delta \leq \sigma_n(C_\delta)$, for $n \geq n_0 \geq \widetilde{n}_0$, where n_0 depends only on C and δ;
2. $\frac{1}{d^n}\sharp S_n^\delta\, \nu(C_\delta) \geq \widehat{\nu}\left(\widehat{\mathcal{F}}^{-n}\left(\widehat{C}_\delta \cap \widehat{\mathcal{K}}_N\right) \cap \widehat{C}\right)$.

We first show how (3.18) follows from the estimates 1 and 2 and then prove the two inequalities. Combining the two we get

$$\widehat{\nu}\left(\widehat{\mathcal{F}}^{-n}\left(\widehat{C}_\delta \cap \widehat{\mathcal{K}}_N\right) \cap \widehat{C}\right) \leq \nu(C_\delta)\sigma_n(C_\delta)$$

and, since $\widehat{\nu}$ is mixing, letting $n \to \infty$ on a subsequence such that $\sigma_{n_i} \to \widetilde{\sigma}$ we find

$$\widehat{\nu}\left(\widehat{C}_\delta \cap \widehat{\mathcal{K}}_N\right) \widehat{\nu}\left(\widehat{C}\right) \leq \nu(C_\delta)\widetilde{\sigma}(C_\delta).$$

Since the left hand side is equal to $\nu_N\left(C_\delta\right)\nu\left(C\right)$ (and C is closed), (3.18) follows letting $\delta \to 0$.

We are thus left to proving the inequalities 1 and 2 above. We shall see that the first follows from the Lipschitz estimate on $\mathcal{F}_{\widehat{x}}^{-n}$, while the second is a consequence of the fact that ν is of constant Jacobian.

We start with 1. In order to prove this, we have to find a n_0 such that, for $n \geq n_0$, the neighbourhood C_δ contains al least $\sharp S_n^\delta$ repelling periodic points for $\mathcal{F}$. Take any $\widehat{x} \in \widehat{R}_n^\delta$. By definition, $\eta(\widehat{x}) \geq \frac{1}{N}$ and $l(\widehat{x}) \leq N$. This means that $\mathcal{F}_{\widehat{x}}^{-n}$ is well defined on $C_\delta \subset B(a_0, \frac{1}{N})$ and that Diam $\mathcal{F}_{\widehat{x}}^{-n}(C_\delta) \leq \frac{1}{N} \operatorname{Lip} \mathcal{F}_{\widehat{x}}^{-n} \leq \frac{1}{N} N e^{-nL} = e^{-nL}$. Since, by definition of $\widehat{R}_n^\delta$, we have that $\mathcal{F}_{\widehat{x}}^{-n}(C_\delta)$ intersects $C \subset C_\delta$, taking n_0 such that $3e^{-n_0 L} < \delta$ (which in particular also implies that $\operatorname{Lip} \mathcal{F}_{\widehat{x}}^{-n} < 1$ for $n \geq n_0$) it follows that $\mathcal{F}_{\widehat{x}}^{-n}(C_\delta) \subset C_\delta$ for every $\widehat{x} \in \widehat{R}_n^\delta$, with $n \geq n_0$. So, since C_δ is itself a compact metric space and $\mathcal{F}_{\widehat{x}}^{-n}$ is stricly contracting on it, we find a (unique) fixed point for it in C_δ. Since the elements of S_n^δ are disjoint, we have found at least $\sharp S_n^\delta$ periodic points (whose period divides n) for $\mathcal{F}$ in C_δ, which must be repelling by the Lipschitz estimate of the local inverse, and so 1 is proved.

For the second inequality, we have

$$\begin{aligned}
\widehat{\nu}\left(\widehat{\mathcal{F}}^{-n}\left(\widehat{C}_\delta \cap \widehat{\mathcal{K}}_N\right) \cap \widehat{C}\right) &= \nu\left(\pi\left(\widehat{\mathcal{F}}^{-n}\left(\widehat{C}_\delta \cap \widehat{\mathcal{K}}_N\right) \cap \widehat{C}\right)\right) \\
&\leq \nu\left(\bigcup_{\widehat{x} \in \widehat{R}_n^\delta} \left(\mathcal{F}_{\widehat{x}}^{-n}(C_\delta)\right) \cap C\right) \\
&\leq \nu\left(\bigcup_{\widehat{x} \in \widehat{R}_n^\delta} \left(\mathcal{F}_{\widehat{x}}^{-n}(C_\delta)\right)\right) \\
&= \sum_{\widehat{x} \in \widehat{R}_n^\delta} \nu\left(\mathcal{F}_{\widehat{x}}^{-n}(C_\delta)\right) \\
&= \sum_{\widehat{x} \in \widehat{R}_n^\delta} \frac{1}{d^n} \nu(C_\delta) = \frac{1}{d^n} \sharp S_n^\delta \nu(C_\delta)
\end{aligned}$$

where the last line follows from the fact that ν is of constant Jacobian. □

We can now show how conditions II.2 and II.3 (which we recall are equivalent by the results of Sections 3.3 and 3.4) imply condition II.1 in Theorem 3.1.4.

Theorem 3.5.3. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps, of degree $d_t \geq 2$. Assume that there exist an acritical equilibrium web $\mathcal{M}$ and an equilibrum lamination $\mathcal{L}$ for f. Then, there exists a subset $\mathcal{P} = \cup_n \mathcal{P}_n \subset \mathcal{J}$, such that*

1. *$\sharp \mathcal{P}_n = d_t^n + o(d_t^n)$;*
2. *every $\gamma \in \mathcal{P}_n$ is n-periodic; and*

3. $\forall M' \Subset M$*, asymptotically every element of* $\mathcal{P}$ *is repelling:* $\frac{\sharp\{\text{repelling cycles in } \mathcal{P}_n\}}{\sharp\mathcal{P}_n} \to 1$.

Moreover

$$\frac{1}{d_t^n}\sum_{\mathcal{P}_n}\delta_\gamma \to \mathcal{M}.$$

The need to restrict to compact subsets of M is essentially due to the fact that Propositions 3.4.2 and 3.4.6 ensure that the assumptions of Lemma 3.5.2 are satisfied on relatively compact subsets of M.

Proof. We consider the set $\mathcal{P} = \cup_n \mathcal{P}_n \subset \mathcal{J}$ given by Lemma 3.5.1. We just need to prove the third assertion. We thus fix $M' \Subset M$ and consider the compact metric space $\mathcal{O}(M', \overline{\mathcal{U}}, \mathbb{C}^k)$. By Propositions 3.4.2 and 3.4.6 and Theorem 3.4.1 all the assumptions of Lemma 3.5.2 are satisfied by the system $(\mathcal{O}(M', \overline{\mathcal{U}}, \mathbb{C}^k), \mathcal{F}, \mathcal{M})$, with $\mathcal{L}$ any equilibrium lamination for the system. The assumption (3.15) is verified since this is true at any fixed parameter. The statement follows from the following two assertions:

1. for every repelling periodic $\gamma \in R_n$ given by Lemma 3.5.2, the point $\gamma(\lambda)$ is repelling for every $\lambda \in M'$; and

2. asymptotically all elements of R_n coincide with elements of $\mathcal{P}_n$.

The first point is a consequence of the Lipschitz estimate of the local inverse of $\mathcal{F}^n$ at the points of R_n (since the Lipschitz constant of $\mathcal{F}^{-n}$ dominates the Lipschitz constant of f_λ^{-n}, for every λ), the second of the fact that both $\mathcal{P}_n$ and R_n have cardinality $d_t^n + o(d_t^n)$ and, at every λ, the number of n-periodic points is d_t^n. □

3.6. Equivalence with the previous notions of stability

In this section we show that the conditions stated in the Theorems 2.2.2 and 3.1.4 are all equivalent. It is immediate to see, by the definition of an equilibrium lamination, that condition II.3 (the existence of a lamination) implies condition I.4 (existence of a graph avoiding the postcritical set), since any element in the lamination satisfies the desired property. Viceversa, by Proposition 3.3.3, we see that condition I.4 directly implies a local version of Theorem 2.2.2. Using the uniqueness of the equilibrium lamination, we can nevertheless recover that the conditions in Theorem 2.2.2 imply the ones in Theorem 3.1.4 on all the parameter space. This is done in the following proposition.

Proposition 3.6.1. *Let* $f : \mathcal{U} \to \mathcal{V}$ *be a holomorphic family of polynomial-like maps of large topological degree* $d_t \geq 2$*. Assume that the parameter space* M *is simply connected and that every point* $\lambda_0 \in M$ *has a neighbourhood where the system admits an equilibrium lamination. Then* f *admits an equilibrium lamination on all the parameter space.*

In particular, if condition I.4 holds, the assumptions of Proposition 3.6.1 are satisfied (by Proposition 3.3.3 and Theorem 3.4.1) and thus condition II.3 holds, too. This completes the proof of Theorem 3.1.5.

Proof. Consider a countable cover $\{B_n\}$ by open balls of the parameter space M, with the property that on every B_n the system admits an equilibrium lamination $\mathcal{L}_n$. In particular, on every B_n the restricted system admits an acritical web. Consider two intersecting balls B_1 and B_2. By the uniqueness of the equilibrium web on the intersection (which is simply connected), both the corresponding webs induce the same one on $B_1 \cap B_2$. By analytic continuation, and up to removing a zero-measure (for the web on the intersection) subset of graphs from the laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ (and all their images and preimages, which are always of measure zero), we obtain a set of holomorphic graphs, defined on all of $B_1 \cup B_2$, that satisfy all the properties required in Definition 3.1.1, thus giving an equilibrium lamination there. The assertion follows repeating the argument, since the cover is countable and M is simply connected (and thus we do not have holonomy problems when glueing the laminations). □

4

Bifurcations, continuity, and Siegel disks

In this chapter we investigate the relation between *Siegel discs*, as defined below, and the existence of equilibrium webs for a holomorphic family of polynomial-like maps. This allows us to recover, in dimension 2, the equivalence between the notions of stability studied in the previous two chapters and the holomorphic motion of *all* repelling cycles in the Julia set. Since this is just an immediate adaptation of work done by Berteloot and Dupont for endomorphisms of $\mathbb{P}^2$, we will not give all the details of the proof. We then briefly discuss the relation between Siegel discs and the Hausdorff continuity of the Julia set. We end the chapter with an example of a Siegel disc contained in the Julia set, motivated by the results of the previous sections.

4.1. Definitions

Siegel points and discs in dimension two are defined as follows. We refer to [BD14b] for the definition in the general case.

Definition 4.1.1. *Let $g : U \to V \Subset \mathbb{C}^2$ be polynomial-like map. A point $z_0 \in U$ is a* Siegel fixed point *for g if*

1. *g is holomorphically linearizable at z_0;*
2. *the differential of g at z_0 has eigenvalues a and $e^{i\theta}$, where $|a| > 1$ and π and θ are linearly independent over $\mathbb{Q}$.*

The above definition means that there exists a local change of coordinate $\psi : \mathbb{D} \times \mathbb{D} \to U$ such that

1. $\psi((0,0)) = z_0$, and
2. $\psi^{-1} \circ g \circ \psi(x,y) = (ax, e^{i\theta}y)$.

Definition 4.1.2. *A* Siegel disk *at a Siegel fixed point z_0 is any set of the form $\psi(\{0\} \times \mathbb{D}_r)$, where ψ is the linearizing chart as above.*

Periodic Siegel points (and the associated Siegel disks) are defined analogously. Notice that we do not require any maximality property in the above definition of Siegel disk.

In the sequel we shall need the following Theorem by Brjuno [Brj71], giving sufficient conditions for the linearability of a holomorphic germ. We state it just in the two-dimensional setting that we need. Let G be (a germ of) holomorphic endomorphism of $\mathbb{C}^2$, fixing a point p. Let w_1 and w_2 denote the two eigenvalues of the differential of G at p. For every $n \in \mathbb{N}$, define the value ω_n to be

$$\omega_n := \min_{2 \leq m_1+m_2 \leq n, m_i \in \mathbb{N}, j \in \{1,2\}} |w_1^{m_1} w_2^{m_2} - w_j|$$

and the associated Brjuno sum

$$\mathcal{B} := \sum_{n=0}^{\infty} \frac{1}{2^n} \log \omega_{2^{n+1}}.$$

Brjuno Theorem then reads as follows.

Theorem 4.1.3 (Brjuno [Brj71]). *Let G be a (germ) of endomorphism of $\mathbb{C}^2$ and p a fixed point for G. Let w_1 and w_2 be the two eigenvalues of the differential of G at p. Then, if*

1. *DG_p is diagonalizable, and*
2. *the Brjuno sum at p satisfies $\mathcal{B} > -\infty$,*

the endomorphism G is locally linearizable at p.

We shall denote by $\mathcal{S}$ the set

$$\mathcal{S} := \{\theta \in \mathbb{R} \colon e^{i\theta} z + z^2 \text{ has a Siegel disk at } 0\}. \tag{4.1}$$

Recall that ([Yoc95]) if the polynomial $e^{i\theta} z + z^2$ is linearizable at 0, the same is true for every polynomial of the form $e^{i\theta} z + z^2 g(z)$, where g is any polynomial. Moreover, by Brjuno-Yoccoz Theorem $\theta \in \mathcal{S}$ if and only if

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \log \min_{2 \leq j \leq n} \left| e^{i\theta j} - e^{i\theta} \right| > -\infty. \tag{4.2}$$

The following immediate corollary then ensures that a fixed point at which the differential have eigenvalues a and $e^{i\theta}$, with $|a| > 1$ and $\theta \in \mathcal{S}$ is a Siegel point (and thus admits a Siegel disc through it).

Corollary 4.1.4. *Let G be an endomorphism of $\mathbb{C}^2$ and p a fixed point for G. Let w_1 and w_2 be the two eigenvalues of the differential of G at p. Assume that $|w_1| = 1$, that the argument of w_1 belongs to $\mathcal{S}$ and that $|w_2| > 1$. Then, the endomorphism G is locally linearizible at p.*

Proof. Since the two eigenvalues of DG_p have distinct modulus, this differential is diagonalizable. We thus only need to show that $\mathcal{B} > -\infty$.

Since the sequence $\{ w_1^n \}$ is dense in S^1, there exists some N such that $\left|w_1^N - w_1\right| < |w_2| - 1$. We claim that, for $n \geq N$, the term ω_n is of the form

$$\omega_n = \min_{2 \leq m \leq n} |w_1^m - w_1| \,,$$

i.e., the minimum is attained without using the second eigenvalue. The fact that $\mathcal{B} > -\infty$ then immediately follows from the assumption that the argument of w_1 belongs to $\mathcal{S}$ (by the characterization (4.2)).

Let us thus prove the claim. Let us start considering a term of the form $|w_1^{m_1} w_2^{m_2} - w_1|$ and prove that it is greater than or equal to $\min_{2 \leq m \leq m_1+m_2} |w_1^m - w_1|$. We can assume that $m_2 \geq 1$. We have

$$|w_1^{m_1} w_2^{m_2} - w_1| \geq |w_2^{m_2}| - 1 > |w_2| - 1 > \left|w_1^N - w_1\right| \geq \min_{2 \leq m \leq m_1+m_2} |w_1^m - w_1|$$

and in this situation the claim is proved.

It remains to consider a term of the form $|w_1^{m_1} w_2^{m_2} - w_2|$. Assume that $m_2 \geq 1$. Then, arguing as above (and recalling that $|w_2| > 1$), we see that

$$|w_1^{m_1} w_2^{m_2} - w_2| \geq \left|w_1^{m_1+1} w_2^{m_2-1} - w_1\right| \geq \min_{2 \leq m \leq m_1+m_2} |w_1^m - w_1|$$

and we are done. The only missing case is a term of the form $|w_1^{m_1} - w_2|$. But this term is clearly greater than or equal to $|w_2| - 1$ and the conclusion follows as in the other cases. □

Let us now consider a holomorphic family f of polynomial-like maps in dimension 2 and let z_0 be a Siegel fixed point for some f_{λ_0} in the family. By the implicit function theorem, it is possible to holomorphically follow the point z_0 with λ (at least on some small neighbourhood) as a fixed point $z(\lambda)$ for f_λ. It is then meaningful to distinguish between *persistent* Siegel points (i.e., Siegel points such that $z(\lambda)$ is still a Siegel point for λ close to λ_0), and, on the other hand, the ones which disappear after a small perturbation of the parameter. We shall be mostly interested in this second type.

Definition 4.1.5. *Let $f : \mathcal{U} \to \mathcal{V}$ be a holomorphic family of polynomial-like maps in dimension 2. A Siegel fixed point for the map f_{λ_0} is said* virtually repelling *if there exists an arbitrary small perturbation λ_1 of λ_0 such that the holomorphic motion $z(\lambda)$ of z_0 as a fixed point is repelling at λ_1.*

The above definition is clearly equivalent to ask that, for an arbitrary small perturbation λ_2 of λ_0, the fixed point $z(\lambda)$ is of saddle type.

In this chapter we prove that (virtually repelling) Siegel points are an obstruction to stability (Proposition 4.2.1). As a consequence, we shall deduce that, for a family of polynomial-like maps in dimension 2, the existence of an equilibrium web implies that the J-repelling points move holomorphically (Theorem 4.2.2). We then briefly discuss the relation between Siegel discs and (dis)continuity of the Julia set and give an example of a map admitting a Siegel disk contained in its Julia set.

4.2. Siegel disks and holomorphic motion of repelling points

Here we prove that Siegel discs are an obstruction to the existence of equilibrium webs. The following result is the core of the proof of the assertion.

Proposition 4.2.1. *Let $f : \mathcal{U} \to \mathcal{V} \subset M \times \mathbb{C}^2$ be a family of polynomial-like maps of large topological degree $d_t \geq 2$. Let z_0 be a virtually repelling Siegel point for f_{λ_0}. If f admits an equilibrium web, then every Siegel disk centered at z_0 is disjoint from the Julia set J_{λ_0}. In particular, if f admits an equilibrium web, then no f_λ can have a virtually repelling Siegel point in its Julia set.*

Proof. We can assume that the periodic Siegel point is actually fixed. We start proving that the punctured Siegel disk (i.e., the image by the linearizing chart of the pointed disk $\{0\} \times \mathbb{D}^*$) is outside the Julia set J_{λ_0}. Assume by contradiction that a point of the form $z_1 = \psi(0, w)$ belongs to J_{λ_0}. By Lemma 2.1.3, there exists an element $\gamma \in \operatorname{Supp} \mathcal{M}$ such that $\gamma(\lambda_0) = z_1$. Denote by σ the holomorphic motion of z_0, on some small neighbourhood of λ_0, as a fixed point of f_λ. Since z_0 is virtually repelling, there exist arbitrary small perturbations $\lambda_m \to \lambda_0$ such that $\sigma(\lambda_m)$ is repelling. In particular this implies that, for m sufficiently large, $f_{\lambda_m}^{-1}(\gamma(\lambda)) \to \sigma(\lambda_m)$ (where $f_{\lambda_m}^{-1}$ denotes the local inverse of f_{λ_m} sending $\sigma(\lambda_m)$ to itself). This gives a contradiction with the fact that, at λ_0, the backward local preimages of z_1 stay bounded away from z_0.

Once we have established that the punctured Siegel disk is outside the Julia set, we need to prove that also the Siegel point is outside J_{λ_0}. This is done in the following way. Notice that every point in the image of the linearizing chart (with the exception of the unstable manifold of the Siegel point z_0) accumulates, by the local inverse iteration, a subset of the punctured Siegel disk. This implies that all these point are outside the Julia set. To conclude, we need only to show that also the unstable manifold is outside J_{λ_0}. But this immediateley follows from the fact that the equilibrium measure μ_{λ_0} does not charge the analytic subsets since f_{λ_0} has large topological degree, see Section 1.2.4. □

Theorem 4.2.2. *Let $f : \mathcal{U} \to \mathcal{V} \subset M \times \mathbb{C}^2$ be a holomorphic family of polynomial-like maps of large topological degree $d_t \geq 2$. Assume that there exists an equilibrium web for f. Then the J-repelling cycles move holomorphically.*

Proof. The proof follows the one given in [BD14b, Proposition 6.3 and Lemma 6.4]. Consider a point $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ which is n-periodic and repelling for f_{λ_0}. We can clearly assume that $n = 1$. Since f admits an equilibrium web, Lemma 2.1.3 implies that there exists an element $\gamma \in \operatorname{Supp} \mathcal{M}$ such that $\gamma(\lambda)$ is n-periodic for f_λ, for every $\lambda \in M$ (and by definition contained in J_λ). Since the subset $M_r := \{ \lambda \in M : \gamma(\lambda)$ is repelling $\}$ is open in M, we only have to prove that it is also closed. Assume by contradiction that there exists a parameter λ_1 on the boundary of M_r (and thus not in M_r).

First of all notice that, although one of the two eigenvalues of the differential of f_λ stops to be greater than 1 in modulus there, up to slightly perturbing λ_1 we can assume that the other one remains bigger than 1. Indeed, assume that both eigenvalues at λ_1 are of modulus one. Since the locus where one of the two eigenvalues is of modulus one is the union of two analytic curves (one corresponding to each eigenvalue), we only have to ensure that these two curves do not coincide (and thus get the claim by slightly perturbing λ_1, as allowed). But if this were the case, we could otherwise perturb λ_1 to a new parameter λ_1' (outside M_r) which would then be attracting. But this contradicts the fact that $\gamma(\lambda_1') \in J_{\lambda_1'}$ (for a detailed proof see [BD14b, Lemma 6.4]).

Once we have established that $\gamma(\lambda_1)$ has one repelling eigenvalue and one neutral one $e^{i\theta}$, we can then perturb it to a new λ_2 to ensure that θ is in $\mathcal{S}$ as in (4.1) (since $\mathcal{S}$ is dense in $\mathbb{R}$). This gives a virtually repelling n-periodic Siegel point $\gamma(\lambda_2)$ at λ_2. Since by construction this point is contained in J_{λ_2}, we get a contradiction with Proposition 4.2.1 and the Theorem is proved.

We note that obtaining this Siegel point would be the main difficulty when working in higher dimension, since the validity of the Brjuno condition is in general far less easy to ensure. □

4.3. Continuity of Julia sets

In this section we briefly discuss the relation between holomorphic stability and continuity of the Julia set for the Hausdorff topology. First of all, we notice that the existence of an equilibrium web implies the continuity of the Julia set. Since the arguments are the same as on $\mathbb{P}^k$ (see [BD14b]), we just sketch the proof of this statement.

Proposition 4.3.1. *Let $f : \mathcal{U} \to \mathcal{V} \subset M \times \mathbb{C}^2$ be a holomorphic family of polynomial-like maps of large topological degree. Assume that f admits an equilibrium web. Then the Julia set depends continuously on the parameter λ.*

Proof. By Corollary 1.3.15, the Julia sets always depends lower semicontinuously on the parameter. We thus just need to prove that the existence of an equilibrium web $\mathcal{M}$ forces the dependence to be upper semicontinuous. We associate to $\mathcal{M}$ the equilibrium current $\mathcal{W}_{\mathcal{M}}$ as in (2.1). By [Dou94, Proposition 2.1], the intersection S_λ of the support of $\mathcal{W}_{\mathcal{M}}$ with the vertical fiber at λ depends upper semicontinuously on λ. Moreover, it is immediate to see that $J_\lambda \subset S_\lambda$ for every $\lambda \in M$. We thus just need to prove the reverse inclusion $S_\lambda \subset J_\lambda$. To do this we show that, if $z_0 \notin J_{\lambda_0}$, there exist $\varepsilon, r_0 > 0$ such that $\mathcal{M}\left(\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap (B(\lambda_0, \varepsilon) \times B(z_0, r_0)) \neq \emptyset\}\right) = 0$. This is a consequence of the compactedness of the support of $\mathcal{M}$. Indeed, take r_0 such that $B(z_0, 2r_0) \cap J_{\lambda_0} = \emptyset$. There exists some ε such that, for every $\gamma \in \operatorname{Supp} \mathcal{M}$, if $\Gamma_\gamma \cap (B(\lambda_0, \varepsilon) \times B(z_0, r_0)) \neq \emptyset$ then $\gamma(\lambda_0) \in B(z_0, 2r_0)$. Since this ball is disjoint from J_{λ_0}, the assertion follows. □

We now study the converse implication. The following statement says that the continuity forces Siegel discs centered at point of the Julia set to be contained in it.

Proposition 4.3.2. *Let $f : \mathcal{U} \to \mathcal{V} \subset M \times \mathbb{C}^2$ be a holomorphic family of polynomial-like maps of large topological degree. Let z_0 be a virtually repelling Siegel fixed point for f_{λ_0}. If the set valued function $\lambda \mapsto J_\lambda$ is continuous for the Hausdorff topology and $z_0 \in J_{\lambda_0}$, then any local Siegel disc centered at z_0 is contained in J_{λ_0}.*

Proof. We have to prove that every point of the form $\psi(0, w)$ is contained in the Julia set of f_{λ_0}, where ψ is the holomorphic change of coordinates given by the definition of Siegel disk. By hypothesis, we know that this is true for $w = 0$. Let us thus take $w \neq 0$ and assume by contradiction that $\psi(0, w) \notin J_{\lambda_0}$. We shall reach a contradiction by proving that this forces $\psi(0, 0) = z_0$ to be outside J_{λ_0}, too. We will use in a crucial way the assumption that f_{λ_0} is of large topological degree, and thus that equilibrium measure gives no mass to analytic subsets (Theorem 1.2.16).

Let so $\psi(0, w)$ be a point on the Siegel disc and not in J_{λ_0}. First of all, by invariance, this implies that a neighbourhood of $\psi(\{0\} \times |w| \cdot S^1)$ is outside J_{λ_0}. Moreover, by using the linearizing

coordinate and again by the total invariance of J_{λ_0}, we see that a subset of the form $B = \psi(\mathbb{D} \times \{ r_1 < |w| < r_2 \})$ is outside J_{λ_0}.

Since $\lambda \to J_\lambda$ is continuous, up to shrinking this neighbourhood, we can suppose that it remains outside the Julia set J_λ also for parameters λ sufficiently close to λ_0. But, since z_0 is virtually repelling, there exists such a small perturbations of λ_0 for which the holomorphic motion as a fixed point $z(\lambda)$ of z_0 becomes a saddle. For any such λ sufficiently small, the dynamics is now contracting in the direction of the (disappeared) Siegel disk. So, the f_λ-orbit of any point in a neighbourhood of $z(\lambda)$ which is not in the unstable manifold of this $z(\lambda)$ must pass through the set B (see [BD14b, Lemma 5.9 (2)] for a detailed proof of this fact). By invariance of J_λ, this implies that a set of the form $\psi(\mathbb{D} \times \mathbb{D}^*)$ (up to possibly reducing the linearization domain) is outside J_λ. Since μ_λ does not charge analytic sets, the same must be true for all the image of ψ, for a sequence of parameters λ converging to λ_0. Since $z_0 \in J_{\lambda_0}$, this contradicts the continuity of J_λ at λ_0. □

We can deduce from Proposition 4.3.2 that, *if Siegel discs are outside the Julia set*, the continuity of the Julia set implies all the equivalent characterizations of stability given in Theorems 2.2.2 and 3.1.4, at least for family of skew products extandable to $\mathbb{P}^2$. These are defined as follows.

Definition 4.3.3. *A polynomial skew-product is an endomorphism of $\mathbb{C}^2$ of the form $F(z,w) = (p(z), q(z,w))$, where p and q are polynomials of respectively one and two complex variables, that extends to a holomorphic map of $\mathbb{P}^2$.*

By [Jon99, Corollary 4.4], the Julia set of the skew product $F(z,w) = (p(z), q_z(w))$ is given by

$$J(F) = \overline{\cup_{z \in J_p} \{ z \} \times J_z}. \tag{4.3}$$

Here J_z is the boundary (in the plane $\{ z \} \times \mathbb{C}$) of the set $K_z := \{ G_z = 0 \}$, where the function G_z is defined by

$$G_z(w) := G(z,w) - G_p(z),$$

where the functions G and G_p are the Green functions of F and p respectively. It is immediate to see that, if $z \in K_p$, then

$$w \in K_z \Leftrightarrow \text{ the sequence } Q_z^n(w) := q_{p^{n-1}(z)} \circ \cdots \circ q_{p(z)} \circ q_z(w) \text{ is bounded.} \tag{4.4}$$

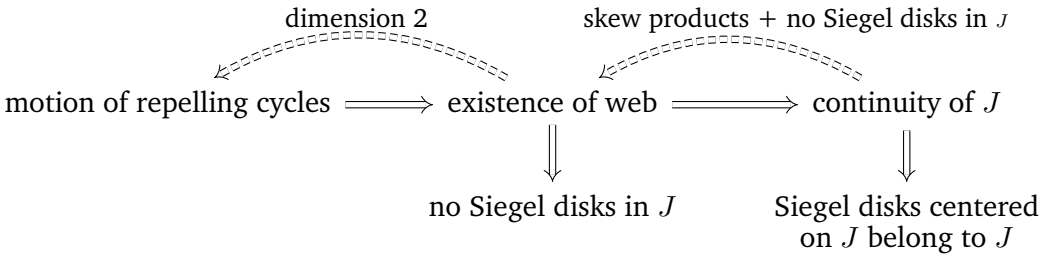
For a skew-product, all repelling points are contained in the Julia set. There is thus no difference in considering repelling or J-repelling points.

Corollary 4.3.4. *Let $f : \mathcal{U} \to \mathcal{V} \subset M \times \mathbb{C}^2$ be a holomorphic family of skew-products polynomials. Assume that any Siegel disc for any f_λ is outside J_λ. Then, the continuity of the Julia set is equivalent to the holomorphic motion of repelling points.*

Proof. Let $f_\lambda(z,w) = (p_\lambda(z), q_\lambda(z,w))$ be the family in the statement, and assume that some repelling point $(z_0(\lambda), w_0(\lambda))$ does not move holomorphically at λ_0. We are going to prove that J_λ is discontinuous at λ_0. We have two possibilities: at λ_0, one or both eigenvalues for the differential of f_λ at $(z_0(\lambda), w_0(\lambda))$ have modulus one. If we are in the first situation, up to a small perturbation, we can assume that $(z_0(\lambda_0), w_0(\lambda_0))$ is a Siegel point. Since by assumption every Siegel disk centered there is outside the Julia set J_{λ_0}, Proposition 4.3.2 gives the desired discontinuity of J_λ.

We thus only have to consider the case when both eigenvalues become of modulus 1. In this case, the polynomial p_{λ_0} must have (up to slightly perturbing again) a Siegel point at $z_0(\lambda_0)$ and thus a Siegel disc Δ there. By (4.3), this implies that all the column $\Delta \times \mathbb{C}$ is outside the Julia set. But by construction there exists λ arbitrarily close to λ_0 such that $(z_0(\lambda), w_0(\lambda))$ is repelling. Since for a skew product map all repelling points are contained in the Julia set, this gives the desired discontinuity of J_λ. □

The following diagram summerizes the results presented in this chapter so far.



4.4. A Siegel disk in a Julia set

It is clear that there are examples of endomorphisms of $\mathbb{P}^2$ with a Siegel disk disjoint from the Julia set. Consider by instance a product map on $\mathbb{C}^2$ of the form

$$F\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} p(z) \\ q(w) \end{pmatrix}$$

where p and q are polynomials on $\mathbb{C}$ of the same degree (so that F extends to an endomorphism of $\mathbb{P}^2$) and such that 0 is a fixed repelling point for p and a Siegel fixed point for q (i.e., q has a Siegel disk such that the image of the center of the disk by the linearizing coordinate is the origin). Since, for such a product map, the Julia set is the product of the Julia sets of the components, we see that the Siegel disk is outside the Julia set. The aim of this section is to prove that this is not true in general: there exist endomorphisms of $\mathbb{P}^2$ admitting a Siegel disk which is contained in the Julia set. Our statement is as follows.

Theorem 4.4.1. *Let $\theta \in \mathcal{S}$ and p and $\widetilde{q}$ be two polynomials such that $d = \deg p = \deg \widetilde{q} + 1$ and satisfying the following assumptions:*

1. *p has two fixed repelling points, that will be denoted by z_0 and z_1;*
2. *0 is a repelling fixed point for $\widetilde{q}$, with multiplier $\widetilde{q}'(0) =: a_0$;*
3. *$\widetilde{q}$ is hyperbolic, and its Julia set is path-connected.*

Then, for $\varepsilon > 0$ small enough, the endomorphism of $\mathbb{P}^2$ induced by the endomorphism F of $\mathbb{C}^2$ given by

$$F\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} p(z) \\ q_z(w) \end{pmatrix} = \begin{pmatrix} p(z) \\ \varepsilon w^d + \left(1 + \left(\frac{e^{i\theta}}{a_0} - 1\right)\frac{z - z_1}{z_0 - z_1}\right)\widetilde{q}(w) \end{pmatrix} \tag{4.5}$$

has the following properties:

1. *the point $(z_0, 0)$ is a Siegel fixed point for F, contained in the Julia set $J(F)$;*
2. *there exists a Siegel disk for $(z_0, 0)$ contained in the Julia set $J(F)$.*

Since the polynomials $p(z) = z^3$ and $\widetilde{q}(w) = (w+1)^2 - 1 = w^2 + 2w$ satisfy the assumpions of Theorem 4.4.1 (notice that the Julia set of $\widetilde{q}$ is the unit circle of center -1), the existence of Siegel disks contained in the Julia set follows.

Proof. First of all, we notice that the point $(z_0, 0)$ is a Siegel fixed point, by the assumption on z_0 and the form of F. This is an immediate consequence of Brjuno Theorem 4.1.3 and Corollary 4.1.4. Then, notice that any associated Siegel disk is contained in the vertical fiber $\{ z_0 \} \times \mathbb{C}$. We are going to prove that a neighbourhood of 0 in $\{ z_0 \} \times \mathbb{C}$ is contained in the Julia set F, thus proving the assertion. We divide the proof of the statement in the following three steps.

The inclusion $\{ z_1 \} \times J\left(\varepsilon w^d + \widetilde{q}(w)\right) \subset J(F)$ holds. Since by hypothesis z_1 is a fixed point for p contained in its Julia set J_p and by construction we have that $q_{z_1}(w) = \varepsilon w^d + \widetilde{q}(w)$, by the characterization (4.4) of K_z it follows that $K_{z_1} = K_{q_{z_1}} = K\left(\varepsilon w^d + \widetilde{q}(w)\right)$. The desired inclusion thus follows.

There exists $\widetilde{z}$ arbitrarily close to z_0 such that $J(F) \cap (\{ \widetilde{z} \} \times \mathbb{C})$ contains a continuous path connecting $(\widetilde{z}, 0)$ to $(\widetilde{z}, \widetilde{w})$, with $|\widetilde{w}| =: \widetilde{r} > 0$. Here we use a (weak) equidistribution property of the Julia set of a polynomial: the set of all the backward preimages of any point in the Julia set is dense in the Julia set.

Applying this to the point z_1, we can thus find preimages of this point by p arbitrarily close to z_0. So, since the line $\{ w = 0 \}$ is invariant by F, this implies the existence of some point of the form $(\widetilde{z}, 0)$ (that we can assume to be as close to z_0 as we like) such that $F^N((\widetilde{z}, 0)) = (z_1, 0)$ for some N. Since the Julia set of $\widetilde{q}$ is path-connected, this implies the existence, for every ε small enough, of a continuous (non trivial) path in $J(\varepsilon w^d + \widetilde{q}(w))$ containing 0. Indeed, for ε sufficiently small, the maps $g_\varepsilon(w) := \varepsilon w^d + \widetilde{q}(w)$ give a family of polynomial-like maps, of degree $d - 1 = \deg \widetilde{q}$, on a neighbourhood of the filled Julia set of $\widetilde{q} = g_0$. The claim follows considering a holomorphic motion (as part of the motion of the Julia set of g_0 as a hyperbolic set, which is contained in the Julia set of g_ε, see Lemma 2.2.15) of a non trivial path in $J(g_0) = J(\widetilde{q})$ containing 0.

There thus exists a continuous path joining 0 and an other point $0 \neq \widetilde{w} \in J_{\widetilde{z}}$. Indeed, the map $Q_{\widetilde{z}}^N$ is (locally near 0) topologically conjugated to some $z \mapsto z^n$, with $n \leq d^N$. The existence of the desired continuous path thus follows pulling back the path in the fiber at z_1 by this map.

$J(F)$ contains a Siegel disk centered at $(z_0, 0)$. Since, by the previous point, we can suppose that $\widetilde{z}$ is as close to z_0 as we like, we can assume that the point $(\widetilde{z}, 0)$ belongs to the image of the linearizing chart for the Siegel point $(z_0, 0)$. Moreover, the line $\{ y = 0 \}$ in the linearizing coordinate corresponds to the line $\{ w = 0 \}$ in the dynamical plane, since they are the unstable local manifolds associated to the fixed points. This means that the second coordinate of the representative of $(\widetilde{z}, 0)$ in the linearizing chart is 0.

By the local description of the dynamics near the Siegel point and the invariance of the Siegel disk by F we deduce that every sufficiently small Siegel disk at $(z_0, 0)$ is contained in the Julia set

of F, since it is accumulated by backward iterates of the continuous path from $(\widetilde{z}, 0)$ and $(\widetilde{z}, \widetilde{w})$ built in the previous point, and the Theorem is proved. □

Notice that we did not assume that the natural vertical foliation near the Siegel point coincides with the foliation induced by the linearizing chart. On the contrary, it was important to have the correspondence between the horizontal lines, to be able to accumulate a disk around 0 in the Siegel disk and not only an annulus.

An attracting 1-disk in the Julia set Once we have established the result above, it is easy to construct an example of an endomorphism of $\mathbb{P}^2$ having a fixed point of saddle type, where the system is linearizable, and whose stable manifold is contained in the Julia set.

Corollary 4.4.2. *There exists an endomorphisms of $\mathbb{P}^2$ admitting a saddle-type fixed point z_0 such that a neighbourhood of z_0 in its stable manifold (and thus all the stable manifold) is contained in the Julia set.*

Proof. Just assume that a_0 is real in (4.5) and replace $e^{i\theta}$ with $\rho e^{i\theta}$, with $\rho < 1$. In this way, the Siegel point now becomes a linearizable saddle point, and the same argument as before proves that a neighbourhood of this point in the stable manifold is accumulated by backward preimages of a continuous path in the Julia set. Notice that we still need to use an argument of Siegel type for the attracting eigenvalue, in order to accumulate a full neighbourhood of the fixed point. □

5

Parabolic implosion in dimension 2

This chapter is about the extension of parabolic implosion techniques from dimension 1 to 2.

Consider the family $f_\varepsilon(z) = z + z^2 + \varepsilon^2$ of polynomials in $\mathbb{C}$. For $\varepsilon = 0$ the point 0 is a parabolic fixed point. The fundamental result of Lavaurs is the following: if $\varepsilon_j \to 0$ and $n_j \to \infty$ are sequences such that $n_j - \frac{\pi}{\varepsilon_j} \to \alpha$, then $\lim_{j\to\infty} f_{\varepsilon_j}^{n_j} = L_\alpha$. The *Lavaurs map* L_α is defined on the parabolic basin of the origin and is given by $\psi \circ t_\alpha \circ \varphi$. Here φ and ψ are respectively the Fatou coordinate and parametrization and t_α is the translation by α on $\mathbb{C}$. This allows one to study the limit shape of Julia sets, as $\varepsilon \to 0$. In particular, it is possible to prove that the maps $\varepsilon \mapsto J(f_\varepsilon)$ are $\varepsilon \mapsto K(f_\varepsilon)$ are discontinuous at $\varepsilon = 0$.

Bedford-Smillie-Ueda recently generalized this results to systems, in two complex variables, admitting a fixed point having an eigenvalue equal to 1 and the second of norm strictly smaller. Their delicate arguments exploit the contracting direction in order to somehow reduce the problem to the one-dimensional situation.

In this chapter we give a partial generalization of the above Theorems in the case of maps tangent to the identity. This allows us to get an estimate of the discontinuity of the large Julia set (Theorem 5.5.5) as well as to prove the discontinuity of the filled Julia set for such perturbations of regular polynomials (Theorem 5.5.8 and Corollary 5.5.9).

Notation

The symbol $O(x)$ will stand for some element in the ideal generated by x. More generally, given any f, $O(f)$ will stand for some element in the ideal generated by f. Analogously, $O(f_1, \ldots, f_k)$ will stand for some element in the ideal generated by $f_1, \ldots, f_k$.

The notation $O_2(x, y)$ will be a shortcut for $O(x^2, xy, y^2)$. Given a point $p \in \mathbb{C}^2$, we shall denote its components as $x(p)$ and $y(p)$.

5.1. Preliminaries and Fatou coordinates

Following the work of Hakim [Hak97] (see also [AR14]), we start giving a description of the local dynamics near the origin for F_0 by recalling some classical notions in this setting. Let Φ be a germ of transformation tangent to the identity at the origin of $\mathbb{C}^2$. We can locally write it near the origin as

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + P(x,y) + \dots \\ y + Q(x,y) + \dots \end{pmatrix},$$

where P and Q are homogeneous polynomials of degree 2. In the following, we shall always assume that $P(x,y)$ is not identically zero. A *characteristic direction* is a direction $V = [x:y] \in \mathbb{P}^1(\mathbb{C})$ such that the complex line through the origin in the direction $[x:y]$ is invariant for (P,Q). The direction is *degenerate* if the restriction of (P,Q) is zero on it, *non degenerate* otherwise.

Consider now a non degenerate characteristic direction V and take coordinates such that $V = [1:u_0]$. Notice that the fact that $[1:u_0]$ is a characteristic direction is equivalent to u_0 being a zero of $r(u) := Q(1,u) - uP(1,u)$. The *director* of the characteristic direction $[1:u_0]$ is thus defined as

$$\frac{r'(u_0)}{P(1,u_0)}$$

(see [Aba15, Definition 2.4] for a more intrinsec – and equivalent – definition). Given a germ Φ and a non degenerate characteristic direction V for Φ we can assume, without loss of generality, that $V = [1:0]$ and that the coefficient of x^2 in $P(x,y)$ is 1 (notice that Hakim has the opposite normalization, i.e., with the term $-x^2$). The following result by Hakim ([Hak97, Proposition 2.6]) gives an explicit description of an invariant subdomain of $\mathcal{B}$. In all this work, we will restrict ourselves to points belonging to such an invariant domain.

Proposition 5.1.1 (Hakim). *Let Φ be a germ of transformation of $\mathbb{C}^2$ tangent to the identity (normalized as above), such that $V = [1:0]$ is a nondegenerate characteristic direction with director δ whose real part is greater than some $0 < \alpha \in \mathbb{R}$. Then, if γ, s and R are small enough positive constants, every point of the set*

$$\widetilde{C}_0(\gamma, R, s) := \{\, (x,y) \in \mathbb{C}^2 \colon\ |\operatorname{Im} x| \le -\gamma \operatorname{Re} x, |x| \le R, |y| \le s\,|x| \,\}$$

is attracted to the origin in the direction V and $x(\Phi^n(x,y)) \sim -\frac{1}{n}$. Moreover we have $|x_n| \le \frac{2}{n}$ and

$$|y(\Phi^n(x,y))|\,|x(\Phi^n(x,y))|^{-\alpha-1} \le |y|\,|x|^{-\alpha-1}\,. \tag{5.1}$$

Notice that, for a γ_1 slightly smaller than γ, we have $F_0(\widetilde{C}_0(\gamma,R,s)) \subseteq \widetilde{C}_0(\gamma,R,s)$.

Let us now consider F_0 as in (3), i.e., given by

$$F_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^2(1 + (q+1)x + ry + O(x^2, xy, y^2)) \\ y(1 + \rho x + O(x^2, xy, y^2)) \end{pmatrix}. \tag{5.2}$$

where ρ is real and larger than 1 and $q, r \in \mathbb{C}$. It is immediate to see that $[1:0]$ is a non-degenerate characteristic direction, with director equal to $\rho - 1$. This is the reason we made the assumption that $\rho > 1$. It will be even clearer later (Lemma 5.4.1) that this a crucial assumption.

An important feature of our setting is that the (local) inverse of a map tangent to the identity shares a lot of properties with the original map (this does not happen for instance in the semi-parabolic situation). In fact, it is immediate to see that the local inverse of an endomorphism tangent to the identity is still tangent to the identity, with the same characteristic directions and moreover the same Hakim directors. In our situation, $(0,0)$ is still a double fixed point for the local inverse G_0, which has the following form (see for example the explicit description of the coefficients of the inverse of an endomorphism tangent to the identity given in [AR13]),

$$G_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - x^2(1 + (q-1)x + ry + O_2(x,y)) \\ y(1 - \rho x + O_2(x,y)) \end{pmatrix}$$

and the stated properties are readily verified.

In the following, we will fix a neighbourhood U of the origin where F_0 is invertible, and consider an invariant domain $\widetilde{C}_0$ as in Proposition 5.1.1 for F_0 such that $-\widetilde{C}_0$ satisfies the same property for G_0 and both $\widetilde{C}_0$ and $-\widetilde{C}_0$ are contained in U.

We now briefly recall how to construct a (one dimensional) Fatou coordinate $\widetilde{\varphi^\iota}$ on $\widetilde{C}_0$ semi-conjugating F_0 to a translation by 1. We notice here that it is actually possible to construct a two-dimensional Fatou coordinate, on a subset of $\widetilde{C}_0$, with values in $\mathbb{C}^2$ and semiconjugating the system to the translation by $(1,0)$. Since we will not use it, we do not detail the construction here, but we refer the interested reader to [Hak97].

The first step of the construction of $\widetilde{\varphi^\iota}$ is to consider the map

$$\widetilde{w}_0^\iota(x,y) := -\frac{1}{x} - q\log(-x). \tag{5.3}$$

Notice that, in the chart $\widetilde{w}_0^\iota$, the map F_0 already looks like a translation by 1. Indeed, by (5.2), we have

$$\begin{aligned} w_0^\iota(F_0(x,y)) &= -\frac{1}{x(F_0(x,y))} - q\log(-x(F_0(x,y))) \\ &= -\frac{1}{x} - q\log(-x) + 1 + ry + O_2(x,y) \\ &= \widetilde{w}_0^\iota(x,y) + 1 + ry + O_2(x,y). \end{aligned} \tag{5.4}$$

In order to get an actual Fatou coordinate, we consider the functions

$$\widetilde{\varphi^\iota}_{0,n} := \widetilde{w}^\iota(F_0^n(x,y)) - n. \tag{5.5}$$

The following Lemma proves that the $\widetilde{\varphi^\iota}_{0,n}$'s converge to an actual Fatou coordinate $\widetilde{\varphi^\iota}$ as $n \to \infty$.

Lemma 5.1.2. *The functions $\widetilde{\varphi^\iota}_{0,n}$ converge, locally uniformly on $\widetilde{C}_0$, to an analytic function $\widetilde{\varphi^\iota} : \widetilde{C}_0 \to \mathbb{C}$ satisfying*

$$\widetilde{\varphi^\iota}(F_0(p)) = \widetilde{\varphi^\iota}(p) + 1.$$

Proof. Set $A_0(x,y) := \widetilde{w}_0^\iota(F_0(x,y)) - \widetilde{w}_0^\iota(x,y) - 1 = \widetilde{\varphi^\iota}_{0,1}(x,y) - \widetilde{\varphi^\iota}_{0,0}(x,y)$ and notice that $A_0(F_0^n(x,y)) = \widetilde{\varphi^\iota}_{0,n+1}(x,y) - \widetilde{\varphi^\iota}_{0,n}(x,y)$. In order to ensure the convergence of the $\widetilde{\varphi^\iota}_{0,n}$'s we can prove that the series of the $A_0(F_0^n(x,y))$'s converges normally on $\widetilde{C}_0$. It follows from (5.4)

that

$$A_0(F_0^n(x,y)) = ry(F_0^n(x,y)) + O_2(x(F_0^n(x,y)), y(F_0^n(x,y))).$$

By Proposition 5.1.1, we have $|x(F_0^n(x,y))| \leq 2/n$ and $|y(F_0^n(x,y))| \leq 1/n^{\alpha+1}$, for some $\alpha > 0$. This implies that the series $\sum_{n=0}^{\infty} |A_0(F^n(x,y))|$ converges normally to

$$\widetilde{\varphi^\iota}(x,y) := \widetilde{\varphi^\iota}_0(x,y) + \sum_{n=0}^{\infty} A_0(F_0^n(x,y)).$$

The functional relation is also easily verified, since $|A_0(F^n(x,y))| \to 0$. □

In the repelling basin the situation is completely analogous. Setting $\widetilde{w}_0^o := -\frac{1}{x} - q\log(x)$ on $-\widetilde{C}_0$ and $\widetilde{\varphi^o}_{0,n} := \widetilde{w}_0^o(F_0^{-n}(x,y)) + n$, we have $\widetilde{\varphi^o}_{0,n} \to \widetilde{\varphi^o}$ locally uniformly on $-\widetilde{C}_0$, where $\widetilde{\varphi^o} : \widetilde{C}_0 \to \mathbb{C}$ satisfies the functional relation $\widetilde{\varphi^o} \circ F_0(p) = \widetilde{\varphi^o}(p) + 1$.

We notice that the Fatou coordinates are not unique. For instance, we can add any constant to them and still have a coordinate satisfying the desired functional relation. In the following (and in Theorem F), we shall use as coordinate the one obtained in Lemma 5.1.2 above.

5.2. The perturbed Fatou coordinates

We consider now the perturbation

$$\begin{aligned} F_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + (x^2+\varepsilon^2)\alpha_\varepsilon(x,y) \\ y(1+\rho x + \beta_\varepsilon(x,y)) \end{pmatrix} \\ &= \begin{pmatrix} x + (x^2+\varepsilon^2)(1 + (q+1)x + ry + O_2(x,y) + O(\varepsilon^2)) \\ y(1+\rho x + O_2(x,y) + O(\varepsilon^2)) \end{pmatrix} \end{aligned} \tag{5.6}$$

of the system F_0 as in (5.2). The goal of this section is modify the Fatou coordinate $\widetilde{\varphi^\iota}$ built in Section 5.1 to an approximate coordinate for F_ε. More precisely, we are going to construct some coordinates $\widetilde{\varphi^\iota}_\varepsilon$ (with values in $\mathbb{C}$) that, on suitable subsets of $\widetilde{C}_0$:

1. almost conjugate F_ε to a translation by 1, in the sense that the error that we have in considering F_ε as a translation in this new chart will be bounded and explicitly estimated; and
2. tend to the one-dimensional Fatou coordinates $\widetilde{\varphi^\iota}$ for F_0 as $\varepsilon \to 0$.

We shall be only be concerned with ε small and satisfying

$$\begin{cases} \operatorname{Re}\varepsilon > 0 \\ |\operatorname{Im}\varepsilon| < c\,|\varepsilon^2|\,. \end{cases} \tag{5.7}$$

Notice that this means that ε is contained in the region, in a neighbouhood of the origin, of the points with positive real part and bounded by two circles of the same radius centered on the imaginary axis and tangent one to the other at the origin. Notice in particular that, by definition,

every sequence ε_ν associated to an α sequence (ε_ν, n_ν) satisfies the above property. We recall here the definition for convenience of the reader.

Definition 5.2.1. *Given $\alpha \in \mathbb{C}$, an α-*sequence *is a sequence $(\varepsilon_\nu, n_\nu)_{\nu\in\mathbb{N}} \in (\mathbb{C}\times\mathbb{N})^{\mathbb{N}}$ such that $n_\nu \to \infty$ and $n_\nu - \frac{\pi}{\varepsilon_\nu} \to \alpha$ as $\nu \to \infty$.*

First of all, we fix a small neighbourhood U of the origin, such that F_ε is invertible in U, for ε sufficiently small. In this section, we shall only be concerned with this local situation. Then, fix sufficiently small $\gamma < \gamma', R$ and s such that Proposition 5.1.1 holds on $\widetilde{C}_0(\gamma, R, s)$ and $\widetilde{C}_0(\gamma', R, s)$ for both F_0 and $H_0 := -F_0^{-1}$. By taking γ and γ' sufficiently close, we can assume that $F_0(\widetilde{C}_0(\gamma', R, s))$ and $H_0(\widetilde{C}_0(\gamma', R, s))$ are contained in $\widetilde{C}_0$. Denote by $\widetilde{C}_0, \widetilde{C}_0' \subset U$ (dropping for simplicity the dependence on the parameters) these sets and by C_0, C_0' their projections on the x-plane. We shall assume that $R\rho \ll 1$, and so that $\widetilde{C}_0 \subset \widetilde{C}_0' \Subset U$.

We consider the classical 1-variable change of coordinates on x (and depending on ε) given by

$$u_\varepsilon(x) = \frac{1}{\varepsilon}\arctan\left(\frac{x}{\varepsilon}\right) = \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon + x}\right). \tag{5.8}$$

The geometric idea behind this map is the following: for ε small as in (5.7), u_ε sends $i\varepsilon$ to the "infinity above" and $-i\varepsilon$ to the "infinity below". Circular arcs connecting these two points are sent to parallel (and almost vertical) lines. In particular, the image of the map u_ε is contained in the strip $\left\{-\frac{\pi}{2|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < \frac{\pi}{2|\varepsilon|}\right\}$ and the image of the disc of radius ε centered at the origin is the strip $\left\{-\frac{\pi}{4|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < \frac{\pi}{4|\varepsilon|}\right\}$. Notice the inverse of this function on $\left\{-\frac{\pi}{2|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < \frac{\pi}{2|\varepsilon|}\right\}$ is given by $w \mapsto \varepsilon\tan(\varepsilon w)$. We gather in the next Lemma the main properties of u_ε that we shall need in the sequel.

Lemma 5.2.2. *Let u_ε be given by* (5.8)*. Then the following hold.*

1. *For every compact subset $\mathcal{C} \subset C_0$ there exist two positive constants $M^-(\mathcal{C})$ and $M^+(\mathcal{C})$ such that, for every $x \in \mathcal{C}$, we have*

$$-\frac{\pi}{2\,|\varepsilon|} + M^- < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}u_\varepsilon(x)\right) < -\frac{\pi}{2\,|\varepsilon|} + M^+ \tag{5.9}$$

   for every ε sufficiently small.

2. *If $-\frac{\pi}{2|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}u_\varepsilon(x)\right) < -\frac{\pi}{4|\varepsilon|}$, then $|x| \le \frac{1}{\frac{\pi}{2|\varepsilon|} + \mathrm{Re}(\frac{\varepsilon}{|\varepsilon|}u_\varepsilon(x))}$.*

Proof. For the first assertion the main point is to notice that, by the compactness of $\mathcal{C}$, we have

$$u_\varepsilon(x) + \frac{\pi}{2\varepsilon} \to -\frac{1}{x}$$

uniformly on $\mathcal{C}$, as $\varepsilon \to 0$. From this we deduce the existence of constants M^-, M^+ such that (5.9) holds for every $x \in \mathcal{C}$.

For the second one, we exploit the inverse of u_ε on $\{-\frac{\pi}{2|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < \frac{\pi}{2|\varepsilon|}\}$, which is given by $w \mapsto \varepsilon\tan(\varepsilon w)$. We have

$$\frac{\pi}{4} < |\mathrm{Re}\, w| < \frac{\pi}{2} \Rightarrow |\tan w| \le \tan|\mathrm{Re}\, w| < \frac{1}{\frac{\pi}{2} - |\mathrm{Re}\, w|}$$

and the assertion follows putting $w = \varepsilon u_\varepsilon(x)$ □

We define now, by means of the functions u_ε, different regions in the dynamical plane. In order to do this, we have to define some constants (independent on ε) that we shall repeatedly use in the sequel.

First of all, fix some $1 < \rho' < \rho$. Then, fix some $1 < \rho'' < 5/4$ such that

$$\left| \frac{4\pi(\rho'' - 1)}{\tan\left(4\pi(\rho'' - 1)\right)} \right| > \frac{1}{\rho'}.$$

This is possible since $\rho' > 1$. In particular, ρ'' may be very close to 1. Finally, set

$$K := 2\pi(\rho'' - 1) \text{ and } \tau := \left| \tan\left(-\frac{\pi}{2} + \frac{K}{2}\right) \right|. \tag{5.10}$$

Without loss of generality, we can take ρ'' small enough to ensure that $K \leq \pi/4$. Moreover, we shall assume that γ' and s are small enough such that

$$\begin{cases} \rho' < \rho \frac{1-\gamma'}{\sqrt{1+\gamma'^2}}, \\ 4\tau s < 1. \end{cases} \tag{5.11}$$

Denote by D_ε the subset of $\mathbb{C}$ given by

$$x \in D_\varepsilon \Leftrightarrow -\frac{\pi}{2\,|\varepsilon|} + \frac{K}{|\varepsilon|} < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|} u_\varepsilon(x)\right) < \frac{\pi}{2\,|\varepsilon|} - \frac{K}{2\,|\varepsilon|}. \tag{5.12}$$

Notice the asymmetry in the definition of D_ε. This will be explained in Lemma 5.5.2.

Let us now move to $\mathbb{C}^2$. Let $\widetilde{D}_\varepsilon$ be the product $D_\varepsilon \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|} \subset \mathbb{C}^2$ (the constant $e^{4\pi\rho\tau}$ will be explained in Proposition 5.3.7). By definition, since $K \leq \pi/4$, we have

$$\mathbb{D}_{|\varepsilon|} \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|} \subset \widetilde{D}_\varepsilon \subset \mathbb{D}_{\tau|\varepsilon|} \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|}. \tag{5.13}$$

Notice in particular that the ratios τ and $2e^{4\pi\rho\tau}$ are independent of ε.

Set $C_\varepsilon := \frac{\varepsilon}{|\varepsilon|} C_0 \setminus D_\varepsilon$ and $\widetilde{C}_\varepsilon := \left(\frac{\varepsilon}{|\varepsilon|}, 1\right) \cdot \widetilde{C}_0 \setminus \widetilde{D}_\varepsilon$ the rotations of C_0 and $\widetilde{C}_0$ of $\frac{\varepsilon}{|\varepsilon|}$ around the y plane. Notice that $\widetilde{C}_\varepsilon \to \widetilde{C}_0$ and $\widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon \to \widetilde{C}_0$ as $\varepsilon \to 0$. Morevover, we have $\widetilde{C}_\varepsilon \subset \widetilde{C}_0'$ for ε sufficiently small (and satisfying (5.7)) The following Lemma will be very useful in the sequel.

Lemma 5.2.3. *For ε sufficiently small, we have $F_\varepsilon(\widetilde{C}_\varepsilon) \subset \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$.*

Proof. By the choice of $\widetilde{C}_0$ and $\widetilde{C}_0'$, we have $F_0(\widetilde{C}_0') \subset \widetilde{C}_0$. Moreover, $F_\varepsilon = F_0 + O(\varepsilon^2)$ and F_ε uniformly converges to F_0 on compact subsets of $\widetilde{C}_0'$. The assertion then follows from the the first inclusion in (5.13). □

The first step in the construction of the almost Fatou coordinates consists in considering the functions $\widetilde{u}_\varepsilon$ given by

$$\widetilde{u}_\varepsilon(x, y) := u_\varepsilon(x).$$

The following lemma gives the fundamental estimate on $\widetilde{u}_\varepsilon$: in this chart, the map $F_\varepsilon(x,y)$ approximately acts as a translation by 1 on the first coordinate. Here and in the following, it will be useful to consider the expression

$$\gamma_\varepsilon(x,y) := \frac{\alpha_\varepsilon(x,y)}{1+x\alpha_\varepsilon(x,y)}.$$

It is immediate to see that $\gamma_\varepsilon(x,y) = 1 + qx + ry + O_2(x,y) + O(\varepsilon^2)$.

Lemma 5.2.4. *Take $p=(x,y) \in \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$. Then*

$$\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) = 1 + qx + ry + O_2(x,y) + O(\varepsilon^2).$$

In particular, when γ, R, s and $\varepsilon(\gamma,R,s)$ are small enough, for $p=(x,y) \in \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$ we have

$$|\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) - 1| < \rho'' - 1 \text{ and } \left|\frac{\varepsilon}{|\varepsilon|}\left(\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p)\right) - 1\right| < \rho'' - 1.$$

Proof. Since $x(F_\varepsilon(x,y)) = x + (x^2+\varepsilon^2)\alpha_\varepsilon(x,y)$, it follows that

$$\frac{i\varepsilon - x(F_\varepsilon(x,y))}{i\varepsilon + x(F_\varepsilon(x,y))} = \frac{(i\varepsilon - x)\left(1 + (x+i\varepsilon)\alpha_\varepsilon(x,y)\right)}{(i\varepsilon + x)\left(1 + (x-i\varepsilon)\alpha_\varepsilon(x,y)\right)}$$

and so

$$\frac{i\varepsilon + x}{i\varepsilon - x}\frac{i\varepsilon - x(F_\varepsilon(x,y))}{i\varepsilon + x(F_\varepsilon(x,y))} = \frac{1 + i\varepsilon\gamma_\varepsilon(x,y)}{1 - i\varepsilon\gamma_\varepsilon(x,y)}.$$

The desired difference is then equal to

$$\begin{aligned}
\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) &= \frac{1}{2i\varepsilon}\log\frac{1+i\varepsilon\gamma_\varepsilon(x,y)}{1-i\varepsilon\gamma_\varepsilon(x,y)} \\
&= \frac{1}{i\varepsilon}\left[i\varepsilon\gamma_\varepsilon(x,y) + \frac{1}{3}\left(i\varepsilon\gamma_\varepsilon(x,y)\right)^3 + O(\varepsilon^4)\right] \\
&= \gamma_\varepsilon(x,y) + O(\varepsilon^2) \\
&= 1 + qx + ry + O_2(x,y) + O(\varepsilon^2)
\end{aligned}$$

and the assertion is proved. □

The next step is to slightly modify our coordinate $\widetilde{u}_\varepsilon$ to a coordinate $\widetilde{w}^\iota_\varepsilon$ satisfying the following two properties:

1. $\widetilde{w}^\iota_\varepsilon \to \widetilde{w}^\iota_0$ (with $\widetilde{w}^\iota_0$ as in (5.3)) as $\varepsilon \to 0$, and
2. $\widetilde{w}^\iota_\varepsilon(F^n_\varepsilon(p)) - n \to \widetilde{\varphi^\iota}$ when $\varepsilon \to 0$ and $n \to \infty$ satisfying some relation to be determined later.

We also look for functions $\widetilde{w}^o_\varepsilon$ satisfying analogous properties on $-\widetilde{C}_0$. Recall that the functions $\widetilde{w}^\iota_0(x,y)$ and $\widetilde{w}^o_0(x,y)$ almost semiconjugates the (first coordinate of the) system F_0 to a translation by 1 (by (5.4)).

We set

$$\widetilde{w}_\varepsilon(x,y) := \widetilde{u}_\varepsilon(x,y) - \frac{q}{2}\log(\varepsilon^2+x^2) = \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon + x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2).$$

and consider their incoming and outgoing normalizations $\widetilde{w}^\iota_\varepsilon$ and $\widetilde{w}^o_\varepsilon$ given by

$$\widetilde{w}^\iota_\varepsilon(x,y) := \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon + x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2) + \frac{\pi}{2\varepsilon},$$
$$\widetilde{w}^o_\varepsilon(x,y) := \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon + x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2) - \frac{\pi}{2\varepsilon}.$$

It is immediate to check that the first request is satisfied, i.e., that $\widetilde{w}^\iota_\varepsilon(x,y) \to \widetilde{w}^\iota_0$ on $\widetilde{C}_0$ (and $\widetilde{w}^o_\varepsilon(x,y) \to \widetilde{w}^o_0$ on $-\widetilde{C}_0$) as $\varepsilon \to 0$. In the next proposition we estimate the distance between the reading of F_ε in this new chart $\widetilde{w}_\varepsilon$ and the translation by 1. We want to prove, in particular, that now the error has no linear terms in the x variable. Indeed, notice that also for the system F_0 we had to remove this term (see Lemma 5.1.2) to ensure the convergence of the series of the $A_0(F_0^n(p))$'s, by the harmonic behaviour of $x(F_0^n(p))$. For convenience of notation, we denote this error by

$$A_\varepsilon(x,y) := \widetilde{w}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{w}(x,y) - 1.$$

We then have the following estimate.

Proposition 5.2.5. $A_\varepsilon(x,y) = ry + O_2(x,y) + O(\varepsilon^2)$.

Notice that, differently from [BSU12], here the error is still linear in y. The reason is that we do not add any correction term in y in the expression of $\widetilde{w}_\varepsilon$. On the other hand, by our assumptions we do not have any linear dipendence in ε.

Proof. The computation is analogous to the one in [BSU12]. By the definition of $\widetilde{w}_\varepsilon$ and the analogous property of $\widetilde{u}_\varepsilon$ (Lemma 5.2.4) we have

$$\begin{aligned}\widetilde{w}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{w_\varepsilon}(x,y) = {} & \widetilde{u}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{u_\varepsilon}(x,y) \\ & - \frac{q}{2}\log(\varepsilon^2 + x(F_\varepsilon(x,y))^2) + \frac{q}{2}\log(\varepsilon^2+x^2) \\ = {} & 1 + qx + ry + O_2(x,y) + O(\varepsilon^2) \\ & - \frac{q}{2}\log\frac{\varepsilon^2 + x(F_\varepsilon(x,y))^2}{\varepsilon^2+x^2}.\end{aligned}$$

It is thus sufficient to prove that

$$\frac{\varepsilon^2 + x(F_\varepsilon(x,y))^2}{\varepsilon^2+x^2} = 1 + 2x + O_2(x,y) + O(\varepsilon^2).$$

But

$$\begin{aligned}\varepsilon^2 + x(F_\varepsilon(x,y))^2 &= \varepsilon^2 + x^2 + (x^2+\varepsilon^2)^2\alpha^2_\varepsilon(x,y) + 2x(x^2+\varepsilon^2)\alpha_\varepsilon(x,y) \\ &= (x^2+\varepsilon^2)(1 + 2x\alpha_\varepsilon(x,y) + O(x^2,\varepsilon^2)) \\ &= (x^2+\varepsilon^2)(1 + 2x + O_2(x,y) + O(\varepsilon^2))\end{aligned}$$

and the assertion follows. □

Let us finally introduce the *incoming almost Fatou coordinate*, by means of the $\widetilde{w}^\iota_\varepsilon$, as it was done for the map F_0 in (5.5). Set

$$\widetilde{\varphi^\iota}_{\varepsilon,n}(p) := \widetilde{w}^\iota_\varepsilon(F^n_\varepsilon(p)) - n = \widetilde{w}^\iota_\varepsilon(p) + \sum_{j=0}^{n-1} A_\varepsilon(F^j_\varepsilon(p)). \tag{5.14}$$

We shall be particularly interested in the following relation between the parameter ε and the number of iterations.

Definition 5.2.6. *A sequence $(\varepsilon_\nu, m_\nu) \subset (\mathbb{C}\times\mathbb{N})^{\mathbb{N}}$ such that $\varepsilon_\nu \to 0$ will be said* of bounded type *if $\frac{\pi}{2\varepsilon_\nu} - m_\nu$ is bounded in ν.*

Notice that, given an α-sequence (ε_ν, n_ν), the sequence $(\varepsilon_\nu, n_\nu/2)$ is of bounded type.

The following result in particular proves that the coordinates $\widetilde{w}^\iota_\varepsilon$ satisfy the second request. This convergence will be crucial in order to prove Theorem F. Here $\widetilde{\varphi^\iota}$ denotes the Fatou coordinate on $\widetilde{C}_0$ given by Lemma 5.1.2.

Theorem 5.2.7. *Let $(\varepsilon_\nu, m_\nu)_{\nu\in\mathbb{N}}$ be a sequence of bounded type. Then*

$$\widetilde{\varphi^\iota}_{\varepsilon_\nu, m_\nu} \to \widetilde{\varphi^\iota}$$

locally uniformly on $\widetilde{C}_0$.

We can also define the outgoing almost Fatou coordinates on $-\widetilde{C}_0$ as

$$\widetilde{\varphi^o}_{\varepsilon,n}(p) := \widetilde{w}^o(F^{-n}_\varepsilon(p)) + n$$

(recall that by assumption $-\widetilde{C}_0$ is contained in a neighbourhood U of the origin where F_ε is invertible, for ε sufficiently small). The following convergence is then an immediate consequence of Theorem 5.2.7 applied to the inverse system.

Corollary 5.2.8. *Let (ε_ν, m_ν) be a sequence of bounded type. Then*

$$\widetilde{\varphi^o}_{\varepsilon_\nu, m_\nu} \to \widetilde{\varphi^o}$$

locally uniformly on $-\widetilde{C}_0$.

To prove Theorem 5.2.7, we need to estimate the series of the errors in (5.14). In particular, we need to bound the modulus of the two coordinates of the orbit $F^j_\varepsilon(p)$, for $p \in \widetilde{\mathbb{C}}_0$ and j up to (approximately) $\pi/2\,|\varepsilon|$. This is the content of the next section. The proof of Theorem 5.2.7 will be then given in Section 5.4.

In our study, we will need to carefully compare the behaviour of F_ε in $\widetilde{C}_0$ and the one of F^{-1}_ε on $-\widetilde{C}_0$. Notice that F^{-1}_ε is given by

$$F^{-1}_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - (x^2+\varepsilon^2)(1+(q-1)x + ry + +O(\varepsilon^2) + O_2(x,y)) \\ y(1-\rho x + O(\varepsilon^2) + O_2(x,y)) \end{pmatrix}.$$

In order to compare the behaviour of the orbits for F_ε^{-1} with the ones for F_ε, it will be useful to consider the change of coordinate $(x,y)\mapsto(-x,y)$ and thus study the maps

$$\begin{aligned} H_\varepsilon\begin{pmatrix} x\\ y\end{pmatrix} &= \begin{pmatrix} x+(x^2+\varepsilon^2)(1+(-q+1)x+ry++O(\varepsilon^2)+O_2(x,y))\\ y(1+\rho x+O(\varepsilon^2)+O_2(x,y))\end{pmatrix}\\ &= \begin{pmatrix} x+(x^2+\varepsilon^2)\alpha_\varepsilon^H(x,y)\\ y(1+\rho x+\beta_\varepsilon^H(x,y))\end{pmatrix}. \end{aligned} \tag{5.15}$$

In this way, we can study both F_ε and H_ε in the same region of space. Notice that the main difference between F_ε and H_ε is that the coefficient q has changed sign.

5.3. The estimates for the points in the orbit

In this section we are going to study the orbit of a point $p\in\widetilde{C}_0$ under the iteration of F_ε. In particular, since the main application we have in mind is the study of $F_{\varepsilon_\nu}^{n_\nu}$ when (ε_ν,n_ν) is an α-sequence, we shall be primarily interested in the study of orbit up to an order of $\pi/|\varepsilon|$ iterations.

Recall that the set $\widetilde{C}_0$ is given by Proposition 5.1.1 and in particular consists of points that converge to the origin under F_0 tangentially to the (negative) real axis of the complex direction $[1:0]$. We shall still assume (by taking $R\ll 1$ small enough) that $\widetilde{C}_0$ is contained in a small neighbourhood U of the origin where F_0 and F_ε are invertible, for ε sufficiently small.

By Lemma 5.2.2, for every compact $\mathcal{C}\subset\widetilde{C}_0$ there exist two constants $M^-(\mathcal{C})$ and $M^+(\mathcal{C})$ such that

$$-\frac{\pi}{2|\varepsilon|}+M^-(\mathcal{C})\le \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(p)\right)\le -\frac{\pi}{2|\varepsilon|}+M^+(\mathcal{C})\quad\forall p\in\mathcal{C},\forall\varepsilon\le\varepsilon_0. \tag{5.16}$$

Without loss of generality, we will assume that M^- and M^+ are integers and $\gg 1$ (since $R\ll 1$).

We shall divide the estimates of the coordinates of $F_\varepsilon^j(p)$ according to its position with respect to the set $\widetilde{D}_\varepsilon$, i.e., according to the position of $x(F_\varepsilon^j(p))$ with respect to D_ε as in (5.12). The following notation will be consistently used through all our study.

Definition 5.3.1. *Given $p\in\widetilde{C}_0$ and ε such that $p\in\widetilde{C}_\varepsilon$, we define the* entry time $n_p(\varepsilon)$ *and the* exit time $n'_p(\varepsilon)$ *by*

$$\begin{aligned} n_p(\varepsilon)&:=\min\{\,j\in\mathbb{N}\colon F_\varepsilon^j(p)\in\widetilde{D}_\varepsilon\,\}\\ n'_p(\varepsilon)&:=\min\{\,j\in\mathbb{N}\colon F_\varepsilon^j\notin\widetilde{C}_\varepsilon\cup\widetilde{D}_\varepsilon\,\}\end{aligned} \tag{5.17}$$

The next Proposition gives the bounds on $n_p(\varepsilon)$ that we shall need in the sequel.

Proposition 5.3.2. *Let $\mathcal{C}\subset\widetilde{C}_0$ be a compact subset and M^-,M^+ be as in* (5.16)*. Then, for every $p=(x,y)\in\mathcal{C}$ and ε sufficiently small,*

$$\frac{K}{\rho''|\varepsilon|}-\frac{M^+}{\rho''}\le n_p(\varepsilon)\le\frac{K}{(2-\rho'')|\varepsilon|}-\frac{M^-}{2-\rho''}.$$

In particular, $F_\varepsilon^j(p)\in\widetilde{C}_\varepsilon$ for $0\le j<\frac{K}{\rho''|\varepsilon|}-\frac{M^+}{\rho''}$.

Proof. Notice that, since $F_\varepsilon(\widetilde{C}_\varepsilon) \subset \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$ (by Lemma 5.2.3), we only have to study the first coordinate of the orbit. Since $\widetilde{C}_\varepsilon \to \widetilde{C}_0$, we have that $\mathcal{C} \subset \widetilde{C}_\varepsilon$ for ε sufficiently small. From Lemma 5.2.4 it follows that

$$2 - \rho'' < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon(p))\right) - \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(p)\right) < \rho''.$$

Thus, we deduce that

$$-\frac{\pi}{2\,|\varepsilon|} + M^- + (2-\rho'')j < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(q))\right) < -\frac{\pi}{2\,|\varepsilon|} + M^+ + \rho'' j \tag{5.18}$$

and the assertion follows from the definition of D_ε (see (5.12)). □

5.3.1. Up to $n_p(\varepsilon)$

Given p in some compact subset $\mathcal{C} \in \widetilde{C}_0$, here we study the modulus of the two coordinates of the points in the orbit for F_ε of p until they fall in $\widetilde{D}_\varepsilon$, i.e., for a number of iteration up to $n_p(\varepsilon)$. We start estimating the first coordinate. Here we shall make use of the definition of K (see (5.10)).

Lemma 5.3.3. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in* (5.16). *Then*

$$\left|x(F_\varepsilon^j(p))\right| \le \frac{2}{j + M^-}$$

for every $p \in \mathcal{C}$, for ε small enough and $j \le n_p(\varepsilon)$.

Proof. The statement follows from Lemma 5.2.2 (2) and the (first) inequality in (5.18). Indeed, we have (recall that $3/4 < 2 - \rho'' < 1$)

$$\begin{aligned}\left|x(F_\varepsilon^j(p))\right| &< \frac{1}{\frac{\pi}{2|\varepsilon|} + \operatorname{Re}(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p)))} \\ &\le \frac{1}{\frac{\pi}{2|\varepsilon|} - \frac{\pi}{2|\varepsilon|} + (2-\rho'')j + M^-} \le \frac{1}{2-\rho''}\frac{1}{j+M^-} \le \frac{2}{j+M^-}.\end{aligned}$$

and the inequality is proved. □

We now come to the second coordinate. Estimating this is the main difference between our setting and the semiparabolic one. Notice that, by (5.6), in order to bound the terms $|y(F_\varepsilon^j(p))|$, we will need to get an estimate from below of the first coordinate. This will be done by means of the following lemma.

Lemma 5.3.4. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in* (5.16). *Let $p, q \in \mathcal{C}$ and set $q_j := \varepsilon\,(\tan\varepsilon(\widetilde{u}_\varepsilon(q) + j))$ and $\widetilde{q}_j := \varepsilon\,(\tan\varepsilon(\widetilde{u}_\varepsilon(q) + |\varepsilon|\,j/\varepsilon))$. Then, for some positive constants C depending on $\mathcal{C}$ and C_ε depending on $\mathcal{C}$ and ε, and going to zero as $\operatorname{Re}\varepsilon \to 0$,*

$$\left|x(F_\varepsilon^j(p)) - q_j\right| < C\frac{1 + \log(M^- + j)}{(M^- + j)^2} \tag{5.19}$$

and

$$\left|x(F_\varepsilon^j(p)) - \widetilde{q}_j\right| < C\frac{1+\log(M^-+j)}{(M^-+j)^2} + C_\varepsilon\frac{1}{M^+ + j} \tag{5.20}$$

for every $0 \le j \le n_p(\varepsilon)$.

Notice in particular that the two estimates reduce to the same for ε real.

Proof. The idea is to first estimate the distance between the two sequences $\widetilde{u}_\varepsilon(F_\varepsilon^j(p))$ and $\widetilde{u}_\varepsilon(q)+j$ (and between $\widetilde{u}_\varepsilon(F_\varepsilon^j(p))$ and $\widetilde{u}_\varepsilon(q)+|\varepsilon|\,j/\varepsilon$) and then to see how this distance is transformed by the application of the inverse of u_ε. Notice that, since $j \le n_p(\varepsilon)$, by definition of $\widetilde{D}_\varepsilon$ (see (5.12)) we have $\mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p))\right) < -\frac{\pi}{4|\varepsilon|}$ for the points in the orbit under consideration (since $K \le \pi/4$).

We first prove that

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(q) - j\right| \le C_1\left(1+\log(M^-+j)\right). \tag{5.21}$$

Notice that this is an improvement with respect to the estimate obtained in Lemma 5.2.4, but that we shall need both that estimate and the bound from above obtained in Lemma 5.3.3 in order to get this one.

By the definition of M^-, we have that $|x(p)|$ and $|x(q)|$ are bounded above by $2/M^-$. Recalling that $|y| \le s\,|x|$ for every $(x,y) \in \widetilde{C}_\varepsilon$, Lemma 5.2.4 gives

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(p) - j\right| \le c_1\sum_{i<j}\left|x(F_\varepsilon^i(p))\right| + c_2\sum_{i<j}\left(\left|x(F_\varepsilon^i(p))\right|^2 + |\varepsilon|^2\right).$$

Since by Lemma 5.3.3 we have $\left|x(F_\varepsilon^j(p))\right| \le 2/(j+M^-)$ and the maximal number of iterations $n_p(\varepsilon)$ is bounded by a constant times $1/|\varepsilon|$, this gives

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(p) - j\right| \le C_2\left(1+\log(M^-+j)\right)$$

for some positive C_2, and the estimate (5.21) follows since the two sequences $\left(\widetilde{u}_\varepsilon(p)+j\right)_j$ and $\left(\widetilde{u}_\varepsilon(q)+j\right)_j$ obviously stay at constant distance.

We then consider the sequence $\widetilde{q}_j$. Using (5.21), it is immediate to see that

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(q) - |\varepsilon|\,j/\varepsilon\right| \le C_1\left(1+\log(M^-+j)\right) + |\arg(\varepsilon)|\,j, \tag{5.22}$$

since the distance between the two sequences $\widetilde{u}_\varepsilon(q)+j$ and $\widetilde{u}_\varepsilon(q)+|\varepsilon|\,j/\varepsilon$. is bounded by the last term.

We now need to estimate how the errors in (5.21) and (5.22) are transformed when passing to the dynamical space, and in particular recover the quadratic denominator in (5.19). By (5.22) we have

$$\begin{aligned}\mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p))\right) &\ge -\frac{\pi}{2\,|\varepsilon|} + M^- + j - C_1\left(1+\log(M^-+j)\right) - |\arg\varepsilon|\,j \\ &> -\frac{\pi}{2\,|\varepsilon|} + C_3(M^-+j)\end{aligned}$$

for ε sufficiently small (as in (5.7)), $j \leq n_p(\varepsilon)$ and some $C_3 > 0$. So, given $L > 0$, it is enough to bound from above the modulus of the derivative of the inverse of u_ε on the strip $\left\{-\frac{\pi}{2|\varepsilon|} + L \leq \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < -\frac{\pi}{4|\varepsilon|}\right\}$ by (a constant times) $1/|L|^2$. This can be done with a straightforward computation. Recall that $u_\varepsilon(z) = \frac{1}{\varepsilon}\arctan\left(\frac{z}{\varepsilon}\right)$, so that its inverse is given by $\varepsilon\tan(\varepsilon w)$. The derivative of this inverse at a point $-\pi/2\varepsilon + w$ is thus given by $\psi_\varepsilon(w) = \varepsilon^2\left(\cos(\varepsilon w)\right)^{-2}$. On the strip in consideration, ψ_ε takes its maximum at $w = -\frac{\pi}{2\varepsilon} + L$, where we have $\psi_\varepsilon(-\frac{\pi}{2\varepsilon} + L) = \varepsilon^2/\sin^2(\varepsilon L)$. The estimate then follows since $x \leq 2\sin(x)$ on $[0, \pi/4]$. □

Proposition 5.3.5. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset, M^-, M^+ be as in* (5.16) *and C, C_ε as in Lemma 5.3.4. Then*

$$\left(\frac{1}{\rho'} - C_\varepsilon\right)\frac{1}{M^+ + j} - C\frac{1 + \log(M^- + j)}{(M^- + j)^2} \leq \left|x(F_\varepsilon^j(p))\right| \leq \frac{2}{j + M^-}$$

for every $p \in \mathcal{C}$, for ε small enough and $j \leq n_p(\varepsilon)$.

Proof. The second inequality is the content of Lemma 5.3.3. Let us then prove the lower bound. By Lemma 5.3.4, it is enough to get the bound

$$\frac{1}{\rho'(M^+ + j)} \leq |\widetilde{q}_j|$$

where $\widetilde{q}_j := \varepsilon\tan\left(\varepsilon(\mathrm{Re}(\widetilde{u}_\varepsilon(p)) + |\varepsilon|\, j/\varepsilon)\right)$ as in Lemma 5.3.4. Notice that we arranged the points $\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(\widetilde{q}_j)$ to be on the real axis. Since we have $\mathrm{Re}\,\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(q_0) < -\frac{\pi}{2|\varepsilon|} + M^+$ (and thus $\mathrm{Re}\,\frac{\varepsilon}{|\varepsilon|}u_\varepsilon\left(\widetilde{q}_j\right) \leq -\frac{\pi}{2|\varepsilon|} + M^+ + j$), it follows that

$$|\widetilde{q}_j| \geq |\varepsilon|\left|\tan\left(\varepsilon\left(-\frac{\pi}{2\,|\varepsilon|} + (M^+ + j)\frac{|\varepsilon|}{\varepsilon}\right)\right)\right| = \frac{|\varepsilon|}{\tan\left(M^+\,|\varepsilon| + j\,|\varepsilon|\right)}.$$

We thus have to prove that, for ε sufficiently small and $j \leq n_p(\varepsilon)$,

$$\frac{|\varepsilon|\,M^+ + |\varepsilon|\,j}{\tan\left(M^+\,|\varepsilon| + j\,|\varepsilon|\right)} > \frac{1}{\rho'}.$$

The left hand side is decreasing in j, so we can evaluate it at $j = n_p(\varepsilon)$, which is less or equal than $\frac{K}{(2-\rho'')|\varepsilon|}$ by Proposition 5.3.2. We thus need to prove that, for ε sufficiently small,

$$\frac{|\varepsilon|\,M^+ + \frac{K}{2-\rho''}}{\left|\tan\left(M^+\,|\varepsilon| + \frac{K}{2-\rho''}\right)\right|} > \frac{1}{\rho'}.$$

This follows since $|\varepsilon|\,M^+ + \frac{K}{2-\rho''} < 2K$ for $|\varepsilon| \ll 1$ and, by assumption, K satisfies $\left|\frac{2K}{\tan(2K)}\right| > \frac{1}{\rho'}$. This concludes the proof. □

We can now give the estimate for the second coordinate.

Proposition 5.3.6. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^+ be as in* (5.16)*. There exists a positive constant c_1, depending on $\mathcal{C}$, such that for $p \in \mathcal{C}$ and $J \leq n_p(\varepsilon)$,*

$$\left|y(F_\varepsilon^J(p))\right| \leq c_1 \left|y(p)\right| \prod_{l=M^+}^{M^++J-1} \left(1 - \frac{\widetilde{\rho}}{l}\right)$$

for some $1 < \widetilde{\rho} < \frac{\rho}{\rho'}\frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$.

Notice that $1 < \frac{\rho}{\rho'}\frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$ by the assumption (5.11).

Proof. We shall make use of both estimates obtained in Proposition 5.3.5. Since the part of orbit which we are considering is in $\widetilde{C}_\varepsilon$ (at least) up to $J-1$, we have $\left|y(F_\varepsilon^j(p))\right| \leq s\left|x(F_\varepsilon^j(p))\right|$ and $\left|x(F_\varepsilon^j(p))\right| > |\varepsilon|$, for $j \leq J-1$. So, by the expression of $y(F_\varepsilon(p))$ in (5.6), we get

$$\begin{aligned}\left|y(F_\varepsilon^J(p))\right| &\leq |y(p)| \prod_{j=0}^{J-1} \left|1 + \rho x(F_\varepsilon^j(p)) + O(x^2(F_\varepsilon^j(p))\right| \\ &\leq |y(p)| \prod_{j=0}^{J-1} \left(\left|1 + \rho x(F_\varepsilon^j(p))\right| + \widetilde{c}_1 \left|x^2(F_\varepsilon^j(p))\right|\right)\end{aligned}$$

for some positive $\widetilde{c}_1$. For ε sufficiently small, we have $\widetilde{C}_\varepsilon \subset \widetilde{C}_0' = \widetilde{C}_0(\gamma', R, s)$ (see Proposition 5.1.1). This implies that $\left|\mathrm{Im}\left(x(F_\varepsilon^j(p))\right)\right| < \gamma' \left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right|$ for every $j < n_p(\varepsilon)$. Thus

$$\begin{aligned}\left|1 + \rho x(F_\varepsilon^j(p))\right| &\leq 1 - \rho\left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right| + \rho\left|\mathrm{Im}\left(x(F_\varepsilon^j(p))\right)\right| \\ &\leq 1 - \rho(1-\gamma')\left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right| \\ &\leq 1 - \rho\frac{1-\gamma'}{\sqrt{1+\gamma'^2}}\left|x(F_\varepsilon^j(p))\right|\end{aligned}$$

and thus, by the estimates on $x(F_\varepsilon^j(p))$ in Proposition 5.3.5 we deduce that (for ε sufficiently small)

$$\begin{aligned}\left|y(F_\varepsilon^J(p))\right| &\leq |y(p)| \prod_{j=0}^{J-1} \left(1 - \rho\frac{1-\gamma'}{\sqrt{1+\gamma'^2}}\left(\frac{1}{\rho'} - C_\varepsilon\right)\frac{1}{M^+ + j} + \widetilde{c}_1'\frac{1+\log(M^- + j)}{(M^- + j)^2}\right) \\ &\leq c_1 \left|y(p)\right| \prod_{j=0}^{J-1} \left(1 - \widetilde{\rho}\frac{1}{M^+ + j}\right)\end{aligned}$$

where $\widetilde{\rho}$ is some constant such that $1 < \widetilde{\rho} < \frac{\rho}{\rho'}\frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$, and the assertion follows. □

5.3.2. From $n_p(\varepsilon)$ to $n_p'(\varepsilon)$

Notice that $\widetilde{D}_\varepsilon$ needs not to be F_ε-invariant. In this section we estimate the second coordinate for points in an orbit entering $\widetilde{D}_\varepsilon$ (and in particular explain the constant $e^{4\pi\rho\tau}$ in the definition of $\widetilde{D}_\varepsilon$).

Our goal is prove a lower bound on $n'_p(\varepsilon)$ (and moreover to prove that the orbit cannot come back to $\widetilde{C}_\varepsilon$). This will in particular give an estimate for the coordinates of the point in the orbit for j up to the lower bound of $n'_p(\varepsilon)$ (since in $\widetilde{D}_\varepsilon$ both $|x|$ and $|y|$ are bounded by (a constant times) $|\varepsilon|$).

Proposition 5.3.7. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset. Then, for every $p \in \mathcal{C}$, and $n_p(\varepsilon) < j \leq n'_p(\varepsilon)$, we have*

$$\left|y(F_\varepsilon^j(p))\right| \leq e^{4\pi\rho\tau} \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \leq e^{4\pi\rho\tau} |\varepsilon|$$

Proof. Recall that $\tau = \tan\left(-\frac{\pi}{2} + \frac{K}{2}\right)$ and that by the assumption (5.11) we have $4s\tau < 1$. Since the part of orbit under consideration is contained in $\widetilde{D}_\varepsilon$ (and thus $|x(F_\varepsilon^j(p))| \leq \tau |\varepsilon|$, by (5.13)), we have

$$\begin{aligned}\left|y(F_\varepsilon^j(p))\right| &\leq \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \prod_{i=n_p(\varepsilon)}^{j-1} (1 + 2\rho\tau |\varepsilon|) \\ &\leq \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \prod_{i=n_p(\varepsilon)}^{\lfloor \frac{\pi - K/2}{(2-\rho'')|\varepsilon|} \rfloor} (1 + 2\rho\tau |\varepsilon|)\,.\end{aligned}$$

The product is bounded by $(1+2\rho\tau|\varepsilon|)^{2\pi/|\varepsilon|} \leq e^{4\pi\rho\tau}$ as $\varepsilon \to 0$. Moreover, we have $\left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \leq \left|y(F_\varepsilon^{n_p(\varepsilon)-1}(p))\right| \left|1 + \rho x(F_\varepsilon^{n_p(\varepsilon)-1})\right| \leq 4s\tau |\varepsilon| < |\varepsilon|$. This gives the assertion. □

We can now give the estimate on $n'_p(\varepsilon)$.

Proposition 5.3.8. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^-, M^+ be as in* (5.16)*. Then, for every $p \in \mathcal{C}$,*

$$\frac{\pi - K/2}{\rho'' |\varepsilon|} - \frac{M^+}{\rho''} \leq n'_p(\varepsilon) \leq \frac{\pi - K/2}{(2-\rho'')|\varepsilon|} - \frac{M^-}{2-\rho''}.$$

Moreover, we have $|y(F_\varepsilon^j(p))| \leq e^{4\pi\rho\tau} |\varepsilon|$ for $n_p(\varepsilon) \leq j < n'_p(\varepsilon)$ and

$$\mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|} \widetilde{u}_\varepsilon\left(F_\varepsilon^{n'_p(\varepsilon)}\right)\right) \geq \frac{\pi - K}{2|\varepsilon|}.$$

In particular, once entered in $\widetilde{D}_\varepsilon$, the orbit cannot come back to $\widetilde{C}_\varepsilon$.

Proof. By Proposition 5.3.7, the modulus of the second coordinate of the points of the orbit is bounded by $e^{4\rho\pi\tau} |\varepsilon|$ for $n_p(\varepsilon) < j \leq n'_p(\varepsilon)$. Since for $j \leq n_p(\varepsilon)$ it is bounded by $s\,|x(F_\varepsilon^j(p))|$, the assertion follows from Equation (5.18). □

5.3.3. After $n'_p(\varepsilon)$

In order to study the behaviour of F_ε after $\widetilde{D}_\varepsilon$, we shall make use of the family H_ε introduced in (5.15). The following proposition is an immediate consequence of the analogous results for F_ε (first assertion of Lemma 5.3.4). We denote by $n_p^H(\varepsilon)$ the entry time for H (see Definition 5.3.1).

Lemma 5.3.9. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in* (5.16)*. Let p, q be contained in some compact subset $\mathcal{C} \subset \widetilde{C}_0$. Then, for ε sufficiently small,*

$$\left|x(F_\varepsilon^j(p)) - x(H_\varepsilon^j(q))\right| < C\frac{1+\log(M^-+j)}{(M^-+j)^2}$$

for every $0 \leq j \leq \min(n_p(\varepsilon), n_q^H(\varepsilon))$, for some positive constant C.

We will get the estimates on the second coordinate in this part of the orbit directly in Section 5.5.1, when proving Theorem F, by applying Proposition 5.3.6 to both F_ε and H_ε.

5.4. A preliminary convergence: proof of Theorem 5.2.7

In this section we prove Theorem 5.2.7. Namely, given a sequence (ε_ν, m_ν) of bounded type (see Definition 5.2.6), we prove that $\widetilde{\varphi^\iota}_{\varepsilon_\nu,m_\nu} \to \widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}_{\varepsilon_\nu,m_\nu} \to \widetilde{\varphi^o}$, locally uniformly on $\widetilde{C}_0$ and $-\widetilde{C}_0$, where $\widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}$ are the Fatou coordinates for F_0 given by Lemma 5.1.2. Recall that by assumption these two sets are contained in a neighbourhood U of the origin where F_ε is invertible, for ε sufficiently small, ans thus in particular where $\widetilde{\varphi^o}$ is well defined. We shall need the following elementary Lemma.

Lemma 5.4.1. *Let $a \in \mathbb{R}$, be strictly greater than 1. Then, for every $j_0 \geq l_0 \geq 1$ such that $0 < 1 - \frac{a}{l} < 1$ for every $l \geq l_0$, the series*

$$\sum_{j=j_0}^{\infty} \prod_{l=l_0}^{j} (1 - \frac{a}{l})$$

converges.

Notice that the Lemma is false when $a = 1$, since the series reduces to an harmonic one. In our applications a will essentially be ρ, which we assume by hyphotesis to be greater than 1.

Proof. As in [Wei98, Lemma 4], let us set $P_j := \prod_{l=l_0}^{j}(1 - \frac{a}{l})$ and notice that the P_j's admit an explicit expression as

$$P_j = c\,\frac{\Gamma(j+1-a)}{\Gamma(j+1)}$$

for some constant $c = c(l_0)$, where Γ is the Euler Gamma function. Since $\Gamma(j+1-a) \sim \frac{1}{j^a} j!$ as $j \to \infty$, we deduce that $P_j \sim c\frac{1}{j^a}$, and so $\sum_j P_j$ converges. □

We can now prove Theorem 5.2.7. The proof follows the main ideas of the one of [BSU12, Theorem 2.6]. The major issue (and the main difference with respect to [BSU12]) will be to take into account the errors due the $O(y)$-terms in the estimates. This will be done by means of the following Lemma, which relies on Propositions 5.3.6 and 5.3.7.

Lemma 5.4.2. *Let $p \in \widetilde{C}_0$ and $n_p(\varepsilon)$ be as in* (5.17)*. Let $\overline{n}(\varepsilon)$ be such that $n_p(\varepsilon) \leq \overline{n}(\varepsilon) \leq \frac{3\pi}{5|\varepsilon|}$. Then the following hold:*

1. *the function* $\varepsilon \mapsto \sum_{j=1}^{\overline{n}(\varepsilon)} \left(\left| y\left(F_\varepsilon^j(p)\right)\right| + \left| y\left(F_0^j(p)\right)\right| \right)$ *is bounded, locally uniformly on* p*, for* ε *sufficiently small;*

2. $\lim_{\varepsilon\to 0} \sum_{j=n_p(\varepsilon)+1}^{\overline{n}(\varepsilon)} \left| y(F_\varepsilon^j(p))\right| = 0$*, locally uniformly on* p*.*

Notice that, by Proposition 5.3.8, $n_p'(\varepsilon) \geq \frac{\pi - K/2}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} \geq \frac{7\pi}{8}\frac{5}{4|\varepsilon|} - \frac{M^+}{\rho''} \geq \frac{3\pi}{5|\varepsilon|}$ for ε sufficiently small. So, in particular, the orbit up to time $\overline{n}(\varepsilon)$ is contained in $\widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$. On the other hand, we have $n_p(\varepsilon) + \frac{M^-}{2-\rho''} \leq \frac{K}{(2-\rho'')|\varepsilon|} \leq \frac{\pi/4}{(2-5/4)|\varepsilon|} - \frac{M^-}{2-\rho''} \leq \frac{\pi}{3|\varepsilon|}$. So, in particular, the assumption of Lemma 5.4.2 is satisfied when $(\varepsilon_\nu, \overline{n}(\varepsilon_\nu))$ is of bounded type.

Proof. We start with the first point. The convergence of the second part of the series is immediate from Proposition 5.1.1, by the harmonic behaviour of $x(F_0^j(p))$ and the estimate (5.1). Let us thus consider the first part. Here we split this series in a first part, with the indices up to $n_p(\varepsilon)$ and in the remaining part starting from $n_p(\varepsilon)+1$. The sum is thus given by

$$\sum_{j=1}^{n_p(\varepsilon)} \left| y\left(F_\varepsilon^j(p)\right)\right| + \sum_{j=n_p(\varepsilon)+1}^{\overline{n}(\varepsilon)} \left| y\left(F_\varepsilon^j(p)\right)\right|$$

and, by Propositions 5.3.6 and 5.3.7, this is bounded by (a constant times)

$$\sum_{j=1}^{n_p(\varepsilon)} \prod_{l=M^+}^{M^++j-1} \left(1 - \frac{\widetilde{\rho}}{l}\right) + \left(\prod_{j=M^+}^{n_p(\varepsilon)-1+M^+} \left(1 - \frac{\widetilde{\rho}}{j}\right)\right) \cdot \sum_{j=n_p(\varepsilon)}^{\overline{n}(\varepsilon)} e^{4\pi\rho\tau}$$

where M^+ is as in (5.16) and $\widetilde{\rho}$ is (as in Proposition 5.3.6) a constant greater than 1. By the lower estimates on $n_p(\varepsilon)$ in Proposition 5.3.2 and the asymptotic behaviour proved in Lemma 5.4.1, the last expression is bounded by

$$\sum_{j=1}^{\infty} \prod_{l=M^+}^{j-1+M^+} \left(1 - \frac{\widetilde{\rho}}{l}\right) + \frac{3\pi}{5\,|\varepsilon|} \cdot e^{4\pi\rho\tau} \cdot \left(\frac{1}{\frac{K}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} - 1 + M^+} \right)^{\widetilde{\rho}}.$$

The first term is bounded, again by Lemma 5.4.1, and the second one (which, up to a constant, is in particular a majorant for the sum in the second point in the statement) goes to zero as $\varepsilon \to 0$ (since $\widetilde{\rho} > 1$). This proves both statements. □

Proof of Theorem 5.2.7. First of all, recall that by Lemma 5.1.2 the sequence $\widetilde{\varphi^\iota}_{0,m_\nu} = \widetilde{w}_0^\iota + \sum_{j=0}^{m_\nu-1} A_0(F_0^j(p))$ converges to a (1-dimensional) Fatou coordinate $\widetilde{\varphi^\iota}$ (for this we just need that $m_\nu \to \infty$). It is then enough to show that the difference $\widetilde{\varphi^\iota}_{\varepsilon_\nu,m_\nu} - \widetilde{\varphi^\iota}_{0,m_\nu}$ goes to zero as $\nu \to \infty$. Here we shall make use of the hypothesis that the sequence (ε_ν, m_ν) is of bounded type. The difference is equal to

$$\widetilde{\varphi^\iota}_{\varepsilon_\nu,m_\nu}(p) - \widetilde{\varphi^\iota}_{0,m_\nu}(p) = \widetilde{w}_{\varepsilon_\nu}^\iota(p) - \widetilde{w}_0^\iota(p) + \sum_{j=0}^{m_\nu-1} \left(A_{\varepsilon_\nu}(F_{\varepsilon_\nu}^j(p)) - A_0(F_0^j(p))\right)$$

and we see that the first difference goes to zero as $\nu \to \infty$. We thus only have to estimate the second part, whose modulus is bounded by

$$\sum_I + \sum_{II} := \sum_{j=0}^{m_\nu - 1} \left| A_0(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_0(p)) \right| + \sum_{j=0}^{m_\nu - 1} \left| A_{\varepsilon_\nu}(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_{\varepsilon_\nu}(p)) \right|.$$

Let us consider the first sum. First of all, we prove that the majorant $\sum_{j=1}^{m_\nu - 1} \left(|A_0(F^j_{\varepsilon_\nu}(p))| + \left| A_0(F^j_0(p)) \right| \right)$ converges. This follows from the fact that $A_0(p) = O(x^2, y)$ by Proposition 5.2.5, the estimates on $\left| x(F^j_0(p)) \right|$ and $|x(F^j_{\varepsilon_\nu}(p))|$ in Propositions 5.1.1 and 5.3.5 and from Lemma 5.4.2 (1). Indeed, with M^+ as in (5.16), we have (for some positive constant K_0),

$$\begin{aligned}
\sum_I &\le \sum_{j=1}^{m_\nu - 1} \left(\left| A_0(F^j_{\varepsilon_\nu}(p)) \right| + \left| A_0(F^j_0(p)) \right| \right) \\
&\le K_0 \sum_{j=1}^{m_\nu - 1} \left(\left| x(F^j_{\varepsilon_\nu}(p)) \right|^2 + \left| x(F^j_0(p)) \right|^2 \right) + K_0 \sum_{j=1}^{m_\nu - 1} \left(\left| y(F^j_{\varepsilon_\nu}(p)) \right| + \left| y(F^j_0(p)) \right| \right) \\
&\le K_0 \sum_{j=1}^{m_\nu - 1} \left(\frac{8}{(j + M^+)^2} + |\varepsilon_\nu|^2 \right) + K_0 \sum_{j=1}^{m_\nu - 1} \left(\left| y(F^j_{\varepsilon_\nu}(p)) \right| + \left| y(F^j_0(p)) \right| \right) \le B
\end{aligned}$$

where in the last passage we used the assumption that the sequence (ε_ν, m_ν) is of bounded type to estimate the sum of the $|\varepsilon_\nu|^2$'s and in order to apply Lemma 5.4.2 (1) for the second sum.

We now prove that $\sum_I$ goes to zero, as $\nu \to \infty$. Given any small η, we look for a sufficiently large J such that the sum

$$\sum_{j=J}^{m_\nu - 1} \left| A_0(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_0(p)) \right|$$

is less than η for $|\varepsilon_\nu|$ smaller than some ε_0. The convergence to 0 of $\sum_I$ will then follow from the fact that $A_0(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_0(p)) \to 0$ as $\nu \to \infty$, for every fixed j. As above, this sum is bounded by

$$\sum_{j=J}^{m_\nu - 1} \left(\frac{8}{(j + M^+)^2} + |\varepsilon_\nu|^2 \right) + \sum_{j=J}^{m_\nu - 1} \left| y(F^j_{\varepsilon_\nu}(p)) \right| + \sum_{j=J}^{m_\nu - 1} \left| y(F^j_0(p)) \right|. \tag{5.23}$$

For J sufficiently large, the first sum is smaller than $\eta/3$ (uniformly in ε), since (ε_ν, m_ν) is of bounded type. The same is true for the third one, by the harmonic behavior of $x(F^j_0(p))$ and the estimate (5.1). We are thus left with the second sum of (5.23). We split it as in Lemma 5.4.2:

$$\sum_{j=J}^{m_\nu - 1} \left| y(F^j_{\varepsilon_\nu}(p)) \right| \le \sum_{j=J}^{n_p(\varepsilon_\nu)} \left| y(F^j_{\varepsilon_\nu}(p)) \right| + \sum_{j=n_p(\varepsilon_\nu)+1}^{m_\nu - 1} \left| y(F^j_{\varepsilon_\nu}(p)) \right|. \tag{5.24}$$

Lemma 5.4.2 (2) implies that the second sum of the right hand side goes to zero as $\varepsilon_\nu \to 0$. We

are thus left with the first sum in the right hand side of (5.24). We estimate it by applying twice Proposition 5.3.6 and Lemma 5.4.1:

$$\begin{aligned}\sum_{j=J}^{n_p(\varepsilon_\nu)} \left|y(F^j_{\varepsilon_\nu}(p))\right| &\le c_1 \sum_{j=J}^{n_p(\varepsilon_\nu)} \left|y(F^J_{\varepsilon_\nu}(p))\right| \prod_{l=J+M^+}^{j-1+M^+} \left(1-\frac{\widetilde{\rho}}{l}\right)\\ &\le c_1 \left|y(F^J_{\varepsilon_\nu}(p))\right| \sum_{j=J}^{\infty} \prod_{l=J+M^+}^{j-1+M^+} \left(1-\frac{\widetilde{\rho}}{l}\right)\\ &\le C_1 \left|y(F^J_{\varepsilon_\nu}(p))\right|\\ &\le C_2 \left|y(p)\right| \prod_{l=M^+}^{J-1+M^+} \left(1-\frac{\widetilde{\rho}}{l}\right).\end{aligned}$$

We can then take J large enough (and independent from ε) so that the last term is smaller than $\frac{\eta}{6}$. Notice in particular the independence of J from ε (for ε sufficiently small).

So, until now we have proved that $\sum_I$ goes to zero as $\nu \to \infty$. It is immediate to check that the same holds for $\sum_{II}$. Indeed,

$$\sum_{II} \le \sum_{j=0}^{m_\nu-1} \left|A_{\varepsilon_\nu}(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_{\varepsilon_\nu}(p))\right| \le \sum_{j=0}^{m_\nu-1} K_1 \left|\varepsilon_\nu\right|^2$$

for some positive constant K_1. The assertion then follows since (ε_ν, m_ν) is of bounded type. □

5.5. Proofs of the main results

5.5.1. The convergence to the Lavaurs map

In this section we prove Theorem F. We shall exploit the 1-dimensional Theorem 15, i.e., the convergence of the restriction of $F^{n_\nu}_{\varepsilon_\nu}$ on $C_0 = \widetilde{C}_0 \cap \{\, y = 0 \,\}$ to the 1-dimensional Lavaurs map L_α.

Lemma 5.5.1. *Let $p_0 \in \widetilde{C}_0 \cap \{\, y = 0 \,\}$ and (ε_ν, n_ν) an α-sequence. Assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0 \cap \{\, y = 0 \,\}$. Then for every δ there exists η such that (after possibly shrinking $\widetilde{C}_0$)*

$$\widetilde{\varphi^o}\left(-\widetilde{C}_0 \cap F^{n_\nu}_{\varepsilon_\nu}\left(\widetilde{C}_0 \cap \left(\widetilde{\varphi^\iota}\right)^{-1}\left(\mathbb{D}(\widetilde{\varphi^\iota}(p_0), \eta)\right)\right)\right) \subset \mathbb{D}(\widetilde{\varphi^o}(q_0), \delta)$$

for every ν sufficiently large.

The need of shrinking $\widetilde{C}_0$ is just due to the fact that Theorem 5.2.7 and Corollary 5.2.8 give the convergence on compact subsets of $\widetilde{C}_0$ (and $-\widetilde{C}_0$).

Proof. Let m^o_ν and m^ι_ν be sequences of bounded type such that $m^\iota_\nu + m^o_\nu = n_\nu$. By definition of

$\widetilde{\varphi^{\iota}}_{\varepsilon,n}$ and $\widetilde{\varphi^{o}}_{\varepsilon,n}$ we have

$$\begin{aligned}\widetilde{\varphi^{o}}_{\varepsilon_\nu,m^o_\nu} \circ F^{n_\nu}_{\varepsilon_\nu}(p) &= \widetilde{w}_{\varepsilon_\nu}\left(F^{-m^o}_{\varepsilon_\nu}(F^{n_\nu}_{\varepsilon_\nu}(p))\right) - \frac{\pi}{2\varepsilon_\nu} + m^o_\nu \\ &= \widetilde{w}_{\varepsilon_\nu}\left(F^{m^\iota_{\varepsilon_\nu}}_{\varepsilon_\nu}(p)\right) - \frac{\pi}{2\varepsilon_\nu} - m^\iota_\nu + n_\nu \\ &= \widetilde{\varphi^{\iota}}_{\varepsilon_\nu,m^o_\nu}(p) + n_\nu - \frac{\pi}{\varepsilon_\nu}\end{aligned} \tag{5.25}$$

whenever $F^{n_\nu}_{\varepsilon_\nu}(p) \in -\widetilde{C}_0$. The assertion follows from Theorem 5.2.7 and Corollary 5.2.8. □

Lemma 5.5.2. *Let $p_0 \in \widetilde{C}_0 \cap \{\, y = 0 \,\}$ and (ε_ν, n_ν) be a α-sequence. Assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0 \cap \{\, y = 0 \,\}$. Then, for every polydisc Δ_{q_0} centered at q_0 and contained in $-\widetilde{C}_0$ there exists a polydisc Δ_{p_0} centered at p_0 and contained in $\widetilde{C}_0$ such that $F^{n_\nu}_{\varepsilon_\nu}(\Delta_{p_0}) \subset \Delta_{q_0}$ for ν sufficiently large.*

Proof. Set $\Delta_{q_0} = \mathbb{D}^1_{q_0} \times \mathbb{D}^2_{q_0}$ and analogously $\Delta_{p_0} = \mathbb{D}^1_{p_0} \times \mathbb{D}^2_{p_0}$. By Lemma 5.5.1 it is enough to prove that, if Δ_{p_0} is sufficiently small, for every ν sufficiently large we have

$$\max_{\mathbb{D}^1_{p_0} \times \partial\mathbb{D}^2_{p_0}} \left| y(F^{m^\iota_\nu}_{\varepsilon_\nu}) \right| \le \frac{1}{2} \min_{\mathbb{D}^1_{q_0} \times \partial\mathbb{D}^2_{q_0}} \left| y(F^{-m^o_\nu}_{\varepsilon_\nu}) \right|.$$

We shall use the estimates collected in Section 5.3. First of all, notice that, by Proposition 5.3.7, it is enough to prove that

$$\max_{p \in \mathbb{D}^1_{p_0} \times \partial\mathbb{D}^2_{p_0}} \left| y(F^{n_p(\varepsilon_\nu)}_{\varepsilon_\nu}) \right| \le c \min_{q \in -\mathbb{D}^1_{q_0} \times \partial\mathbb{D}^2_{q_0}} \left| y(H^{n_\nu - n'_p(\varepsilon_\nu)}_{\varepsilon_\nu}) \right|$$

for some constant c, where H_ε is as in (5.15). Geometrically, we want to ensure that the vertical expansion in the third part of the orbit (i.e., after $n'_p(\varepsilon)$) is balanced by a suitable contraction during the first part (i.e., up to $n_p(\varepsilon)$).

This means proving that

$$\begin{aligned}&\left| \prod_{j=0}^{n_p(\varepsilon)} \left(1 + \rho x(F^j_{\varepsilon_\nu}(p)) + \beta_{\varepsilon_\nu}(x(F^j_{\varepsilon_\nu}(p)), y(F^j_{\varepsilon_\nu}(p)))\right) \right| \\ &\le c' \left| \prod_{j=0}^{n_\nu - n'_p(\varepsilon_\nu)} \left(1 + \rho x(H^j_{\varepsilon_\nu}(p)) + \beta^H_{\varepsilon_\nu}(x(H^j_{\varepsilon_\nu}(p)), y(F^j_{\varepsilon_\nu}(p)))\right) \right|\end{aligned} \tag{5.26}$$

for some positive c'. First of all, we claim that there exists a constant K_1 (independent from ν) such that $K_1 + n_p(\varepsilon_\nu) \ge n_\nu - n'_p(\varepsilon_\nu)$, i.e., the number of points in the orbit for F_ε before entering in $\widetilde{D}_{\varepsilon_\nu}$ (and thus in the contracting part) is at least the same (up to the constant) of the number of points in the expanding part. Indeed, recalling that the definition (5.10) of K, we have

$\frac{K}{\rho''|\varepsilon_\nu|} \geq \frac{\pi}{|\varepsilon_\nu|} - \frac{\pi - K/2}{\rho''|\varepsilon_\nu|}$. So, by Propositions 5.3.2 and 5.3.8 we have, with M^+ as in (5.16),

$$\begin{aligned} 1 + |\alpha| + \frac{M^+}{\rho''} + n_p(\varepsilon_\nu) &\geq 1 + |\alpha| + \frac{M^+}{\rho''} + \frac{K}{\rho''\,|\varepsilon_\nu|} - \frac{M^+}{\rho''} \\ &\geq n_\nu - \frac{\pi}{|\varepsilon_\nu|} + \frac{\pi}{|\varepsilon_\nu|} - \frac{\pi - K/2}{\rho''\,|\varepsilon_\nu|} \geq n_\nu - n'_p(\varepsilon_\nu) \end{aligned}$$

for ν sufficiently large, and the desired inequality is proved. The inequality (5.26) now follows from Lemma 5.3.9 (and Proposition 5.3.5), and the assertion follows. □

We can now prove Theorem F.

Proof of Theorem F. First of all, we can assume that p_0 belongs to $C_0 = \{\, y = 0 \,\} \cap \widetilde{C}_0$. Indeed, there exists some N_0 such that $F_0^{N_0}(p_0) \in \widetilde{C}_0$. So, we can prove the Theorem for the $(\alpha - N_0)$-sequence $(\varepsilon_\nu, n_\nu - N_0)$ and the base point $F_0^{N_0}(p_0)$ and the assertion then follows since $F_{\varepsilon_\nu}^{N_0} \to F_0^{N_0}$. For the same reason, we can assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0$.

By Lemma 5.5.2, there exists a polydisc Δ_{p_0} centered at p_0 such that the sequence $F_{\varepsilon_\nu}^{n_\nu}$ is bounded on Δ_{p_0}. In particular, up to a subsequence, this sequence converges to a limit map T_α, defined in Δ_{p_0} with values in $-\widetilde{C}_0$. Notice that the limit must be open, since the same arguments apply to the inverse system. The relation (5) then follows from (5.25) and the assertion follows. □

In the following, given a subset $\mathcal{U} \subset \widetilde{C}_0$, we denote by $\mathcal{T}_\alpha(\mathcal{U})$ the set

$$\mathcal{T}_\alpha(\mathcal{U}) := \{\, T : \mathcal{U} \to \mathbb{C}^2 \colon \exists (\varepsilon_\nu, n_\nu)\alpha - \text{sequence such that } F_{\varepsilon_\nu}^{n_\nu} \to T \text{ on } \mathcal{U} \,\}.$$

We denote by $\mathcal{T}_\alpha$ the union of all the $\mathcal{T}_\alpha(\mathcal{U})$'s, where $\mathcal{U} \subset \widetilde{C}_0$, and call the elements of $\mathcal{T}_\alpha$ *Lavaurs maps*. Theorem F can then be restated as follows: every compact subset $\mathcal{C}_0 \subset C_0$ has a neighbouhhood $\mathcal{U}_{\mathcal{C}_0} \subset \widetilde{C}_0$ such that every $\mathcal{T}_\alpha(\mathcal{U}_{\mathcal{C}_0})$ is not empty.

Remark 5.5.3. *Computer experiments suggest that given any α-sequence (ε_ν, n_ν) there is a neighbourhood of C_0 in $\widetilde{C}_0$ such that the sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a (unique) limit map T_α, without the need of extracting a subsequence.*

5.5.2. The discontinuity of the large Julia set

In this section, by means of the Lavaurs maps T_α, we first define a 2-dimensional analogue of the Julia-Lavaurs set $J^1(F_0, T_\alpha)$, and use this set to estimate the discontinuity of the Julia set at $\varepsilon = 0$.

Definition 5.5.4. *Let $\mathcal{U} \subset \widetilde{C}_0$ and $T_\alpha \in \mathcal{T}_\alpha(\mathcal{U})$. The Julia-Lavaurs set $J^1(F_0, \alpha)$ is the set*

$$J^1(F_0, T_\alpha) := \overline{\{\, z \in \mathbb{P}^2 | \exists m \in \mathbb{N} \colon T_\alpha^m(z) \in J^1(F_0) \,\}}.$$

The condition $T_\alpha^m(z) \in J^1(F_0)$ means that we require $T_\alpha^i(z)$ to be defined, for $i = 0, \dots m$. In particular, we have $z, \dots, T_\alpha^{m-1}(z) \in \mathcal{U}$.

From the definition it follows that $J^1(F_0) \subseteq J^1(F_0, T_\alpha)$, for every $T_\alpha \in \mathcal{T}_\alpha$. The following result gives the key estimate for the lower-semicontinuity of the large Julia sets at $\varepsilon = 0$. The proof is analogous to the 1-dimensional case, exploiting the fact that the maps T_α are open.

Theorem 5.5.5. *Let $T_\alpha \in \mathcal{T}_\alpha$ be defined on $\mathcal{U} \subset \widetilde{C}_0$ and (n_ν, ε_ν) be a α-sequence such that $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$ on $\mathcal{U}$. Then*

$$\liminf J^1(F_{\varepsilon_\nu}) \supseteq J^1(F_0, T_\alpha).$$

Proof. The key ingredients are the lower semicontinuity of $J^1(F_\varepsilon)$ and Theorem F. By definition, the set of all z's admitting an m such that $T_\alpha^m(z) \in J^1(F_0)$ is dense in $J^1(F_0, T_\alpha)$. Thus, given z_0 and m satisfying the previous condition, we only need to find a sequence of points $z_\nu \in J^1(F_{\varepsilon_\nu})$ such that $z_\nu \to z_0$, for some sequence $\varepsilon_\nu \to 0$.

Set $p_0 := T_\alpha^m(z_0)$. By the lower semicontinuity of $\varepsilon \mapsto J^1(F_\varepsilon)$ we can find a sequence of points $p_\nu \in J^1(F_{\varepsilon_\nu})$ such that $p_\nu \to p_0$. By Theorem F we have $F_{\varepsilon_\nu}^{mn_\nu} \to T_\alpha^m$ uniformly near z_0, and this (since T_α is open) gives a sequence z_ν converging to z_0 such that $F_{\varepsilon_\nu}^{mn_\nu}(z_\nu) = p_\nu \in J^1(F_{\varepsilon_\nu})$. This implies that $z_\nu \in J^1(F_{\varepsilon_\nu})$, and the assertion follows. □

Notice the function $\varepsilon \mapsto J^1(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$ since, by means of just the one-dimensional Lavaurs Theorem 15, we can create points in $\widetilde{C}_0 \cap \{ y = 0 \}$ (which is contained in the Fatou set) satisfying $L_\alpha(p) \in J^1(F_0)$. Indeed, the following property holds:

$$\forall p \in \widetilde{C}_0 \cap \{ y = 0 \} \text{ there exists } \alpha \text{ such that } p \in J^1((F_0)_{|y=0}, L_\alpha). \tag{5.27}$$

where L_α is the 1-dimensional Lavaurs map on the invariant line $\{ y = 0 \}$ associated to α. Indeed, since $\partial\mathcal{B} \subseteq J^1(F_0)$ and $\mathcal{B}$ intersects the repelling basin $\mathcal{R}$, we can find $q \in J^1(F_0) \cap \{ y = 0 \}$ in the image of the Fatou parametrization ψ^o for $(F_0)_{|\{ y=0 \}}$. The assertion follows considering α such that $L_\alpha(p) = q$.

In our context, given any $p \in C_0$ and $q \in -C_0$ as above, by means of Theorem F we can consider a neighbourhood of p where a sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a Lavaurs map T_α (necessarily coinciding with L_α on the line $\{ y = 0 \}$). Since T_α is open, we have that $T_\alpha^{-1}(J^1(F_0))$ is contained in the liminf of the Julia sets $J^1(f_{\varepsilon_\nu})$. This gives a two-dimensional estimate of the discontinuity.

5.5.3. The discontinuity of the filled Julia set

For regular polynomial endomorphism of $\mathbb{C}^2$ (which in particular are polynomial-like maps) it is meaningful to consider the filled Julia set, whose definition we recall here for convenience.

Definition 5.5.6. *Given a regular polynomial endomorphism F of $\mathbb{C}^2$, the* filled Julia set $K(F)$ *is the set of points whose orbit is bounded.*

Equivalently, given any sufficently large ball B_R, such that $B_R \Subset F(B_R)$, the filled Julia set is equal to

$$K(F) := \cap_{n \geq 0} F^{-n}(B_R).$$

In this section we shall prove that, if the family (5.6) is induced by regular polynomials, then the set-valued function $\varepsilon \to K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.

Recall that the function $\varepsilon \to K(F_\varepsilon)$ is always upper semicontinuous (see [Dou94]). Here the key definition will be the following analogous of the filled Lavaurs-Julia set in dimension 1 ([Lav89]).

Definition 5.5.7. *Given $\mathcal{U} \subset \widetilde{C}_0$ and $T_\alpha \in \mathcal{T}_\alpha(\mathcal{U})$, the filled Lavaurs-Julia set $K(F_0, T_\alpha)$ is the complement of the points p such that there exists $m \geq 0$ such that $T_\alpha^m(p)$ is defined and is not in $K(F_0)$.*

Notice in particular that $K(F_0, T_\alpha) \subseteq K(F_0)$ and coincides with $K(F_0)$ outside $\mathcal{U}$. Moreover, notice that $K(F_0, T_\alpha)$ is closed.

Theorem 5.5.8. *Let $T_\alpha \in \mathcal{T}_\alpha$ be defined on some $\mathcal{U} \subset \widetilde{C}_0$. and let (ε_ν, n_ν) be an α-sequence such that $F_{\varepsilon_\nu}^{n_\nu} \to T_\alpha$ on $\mathcal{U}$. Then*

$$K(F_0, T_\alpha) \supseteq \limsup K(F_{\varepsilon_\nu}).$$

Proof. Since the set-valued function $\varepsilon \mapsto K_\varepsilon$ is upper-semicontinuous, there exists a large ball B such that, for $\nu \geq \nu_0$, we have $\cup_\nu K(F_{\varepsilon_\nu}) \subset B$. Without loss of generality we can assume that $\nu_0 = 1$. Let us consider the space

$$P := \{\, \{\, 0 \,\} \cup \bigcup_\nu \{\varepsilon_\nu\} \,\} \times B$$

and its subset X given by

$$X := \{\, (0, z) \colon x \in K(F_0, T_\alpha) \,\} \cup \bigcup_\nu \{\, (\varepsilon_\nu, z) \colon z \in K(F_{\varepsilon_\nu}) \,\}.$$

By [Dou94, Proposition 2.1] and the fact that P is compact, it is enough to prove that X is closed in P. This follows from Theorem F. Indeed, let z be in the complement of $K(F_0, T_\alpha)$. Since this set is closed, a small ball B_z around z is outside $K(F_0, T_\alpha)$, too. By definition, this means that, for some m, we have $T^m(B_z) \subset K(F_0)^c$. Theorem F implies that, up to shrinking the ball B_z, we have $F_{\varepsilon_\nu}^{n_\nu}(B_z) \subset K(F_0)^c$ for ν sufficiently large. The upper semicontinuity of $\varepsilon \mapsto K(F_\varepsilon)$ then implies that $F_{\varepsilon_\nu}^{n_\nu}(B_z) \subset K(F_{\varepsilon_\nu})^c$, for ν large enough. So, $B_z \subset K(F_{\varepsilon_\nu})^c$ and this gives the assertion. □

Corollary 5.5.9. *Let F_ε be a holomorphic family of regular polynomials of $\mathbb{C}^2$ as in* (5.6)*. Then the set-valued function $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.*

Proof. The argument is the same used to prove the discontinuity of $J^1(F_\varepsilon)$ in Section 5.5.2. If the function $\varepsilon \to K(F_\varepsilon)$ were continuous, Theorem 5.5.8 and the fact that $K(F_0, T_\alpha) \subseteq K(F_0)$ for every α would imply that all the $K(F_0, T_\alpha)$'s were equal to $K(F_0)$. Since $\widetilde{C}_0 \subseteq K(F_0)$, it is enough to find $p \in \widetilde{C}_0$ and α such that $p \notin K(F_0, T_\alpha)$. To do this, it is enough to take any point q in $\{\, y = 0 \,\}$ not contained in $K(F_0)$ (recall that $K(F_0)$ is compact) and then consider a point $p \in \widetilde{C}_0 \cap \{\, y = 0 \,\}$ and α such that $L_\alpha(p) = q$. The existence of such points is a consequence of the property (5.27). Then, consider a neighbourhood $\mathcal{U}$ of p such that some sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a Lavaurs map T_α on $\mathcal{U}$. The assertion follows since T_α is open and coincides with L_α on the intersection with the invariant line $\{\, y = 0 \,\}$. □

A

Slicing and entropy

A.1. Horizontal currents and slicing

The aim of this appendix is introduce and study the operation of *slicing* on currents on a product space. This is the generalization to currents of the standard restriction of smooth forms to the fibers of a product space. We shall first give the main ideas in the general setting, and then focus on the situation we are more interested in. We refer to [Fed96] (see also [HS74]) for the details that we shall omit.

In the sequel, M and V will be two connected open relatively compact subsets of, respectively, $\mathbb{C}^m$ and $\mathbb{C}^k$. We shall always denote by π the standard projection $\pi : M \times V \to M$ and by $\mathscr{L}$ the Lebesgue measure on M. Since all the arguments needed can be reduced to local ones, we assume from the beginning that M is in fact the unit ball of $\mathbb{C}^m$.

Recall that a *locally flat current* is a current $\mathcal{R}$ that can locally be written as $\mathcal{R} = \mathcal{S} + d\mathcal{T}$, where the coefficients of $\mathcal{S}$ and $\mathcal{T}$ are L^1_{loc} functions. The following lemma (see e.g. [Fed96, 4.1.18] or [Siu74, p.120]) ensures that positive closed currents satisfy this assumption.

Lemma A.1.1. *Every current $\mathcal{R}$ such that both $\mathcal{R}$ and $d\mathcal{R}$ have measure coefficients is locally flat. In particular, every positive closed current is locally flat.*

Let ρ be a smooth positive function on $\mathbb{C}^m$, compactly supported in M and such that $\int_M \rho\, \mathscr{L} = 1$. Then, for every point $\lambda_0 \in M \subset \mathbb{C}^m$, the sequence of smooth functions $\rho_{\lambda_0,\varepsilon}(\lambda) = \frac{1}{\varepsilon^{2m}} \rho\left(\frac{\lambda - \lambda_0}{\varepsilon}\right)$ approximate δ_{λ_0} in the distributional sense, i.e., for every smooth function φ on M, we have

$$\int_M \rho_{\lambda_0,\varepsilon}(\lambda)\varphi(\lambda) d\mathscr{L}(\lambda) \underset{\varepsilon\to 0}{\longrightarrow} \int_M \delta_{\lambda_0}(\lambda)\varphi(\lambda) d\mathscr{L}(\lambda) = \varphi(\lambda_0). \tag{A.1}$$

The following Theorem by Lebesgue ensures that the same is true, up to a negligible set, for functions on M which are just L^1_{loc}.

Theorem A.1.2 (Lebesgue). *Let $\varphi \in L^1_{loc}(M)$. There exists a set $L(\varphi) \subset M$ of full measure such that, for every $\lambda_0 \in M$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2m} \int_{B(\lambda_0,\varepsilon)} |\varphi - \varphi(\lambda_0)| = 0$$

and thus

$$\lim_{\varepsilon \to 0} \int \rho_{\lambda_0,\varepsilon} \varphi \mathscr{L} = \varphi(\lambda_0)$$

for every $\lambda_0 \in L(\varphi)$ and every $\rho_{\lambda_0,\varepsilon}$ smooth approximation of δ_{λ_0}.

The *slice* of a (p,p)-current on the product $M \times V$ is then defined in the following way.

Definition A.1.3. *Let $\mathcal{R}$ be a (p,p)-current on $M \times V$, and let π denote the standard projection $M \times V \to M$. Let $\lambda_0 \in M$ and $\rho_{\lambda_0,\varepsilon}$ be a smooth approximation of δ_{λ_0}. The* slice of $\mathcal{R}$ at λ_0 with respect to π *(and $\rho_{\lambda_0,\varepsilon}$) is the limit (if it exists) for $\varepsilon \to 0$ of the currents $\mathcal{R} \wedge \pi^*(\rho_{\lambda_0,\varepsilon}\mathscr{L})$, where $\mathscr{L}$ denotes the standard Lebesgue measure on M. We shall denote the slice of $\mathcal{R}$ at λ_0 with $\langle \mathcal{R}, \pi, \lambda_0 \rangle$.*

Remark A.1.4. *We can also define the slice in the same way for π a submersion from a space $\widetilde{V}$ to M. The theory is the same. We shall restrict to the product situation for simplicity.*

By Definition A.1.3, the slice $\langle \mathcal{R}, \pi, \lambda_0 \rangle$ (if it exists) is a (p,p) current on $M \times V$, supported on $\pi^{-1}(\lambda_0)$. So, we can think of it as a (p,p) current on $\pi^{-1}(\lambda_0)$ (actually, we shall primarily think of slices in this way). If $\mathcal{R}$ is closed and positive, its slices are closed and positive, too.

Remark that, even when the slice exists, a priori it may depend on the function ρ chosen to approximate δ_{λ_0}. The following theorem ensures that, for almost every $\lambda_0 \in M$, the slice measure of a locally flat (p,p)-current exists and does not depend on the particular approximation of the δ chosen. The main ingredients are Theorem A.1.2 and the following basic Lemma (see e.g. [Dem, p. 17]).

Lemma A.1.5. *Let $\mathcal{R}$ be a locally flat current on $M \times V$ such that $\pi_{|\operatorname{Supp}\mathcal{R}}$ is proper. Then, the pushforward $\pi_*(\mathcal{R})$ is a locally flat current on M.*

Theorem A.1.6 (Theorem 4.3.2 in [Fed96]). *Let $\mathcal{R}$ be a positive closed (p,p)-current on a product space $M \times V$, with $p \leq k = \dim V$. Then the slice $\langle \mathcal{R}, \pi, \lambda \rangle$ exists for $\mathscr{L}$-almost every $\lambda \in M$ and*

$$\int_M \langle \mathcal{R}, \pi, \lambda \rangle \left(\psi(\lambda, \cdot)\right) \mathscr{L} = \langle \mathcal{R} \wedge \ \pi^* \mathscr{L}, \psi \rangle \tag{A.2}$$

for every smooth $(k-p, k-p)$-form ψ compactly supported on $M \times V$.

Sketch of proof. Fix some smooth $(k-p,k-p)$-form ψ_0 compactly supported on $M \times V$. We start proving the assertion using a fixed approximation ρ_ε of δ_0, giving approximations $\rho_{\lambda_0,\varepsilon}$ for every δ_{λ_0} by translation.

By Lemma A.1.5, the pushforward $\pi_*(\psi_0 \wedge \mathcal{R})$ is given by a L^1_{loc} function g_{ψ_0}. By Theorem A.1.2 we thus have

$$\lim_{\varepsilon \to 0} \langle \mathcal{R} \wedge \ \pi^* \left(\rho_{\lambda_0,\varepsilon} \mathscr{L}, \psi_0\right) \rangle = \lim_{\varepsilon \to 0} \int \rho_{\lambda_0,\varepsilon} g_{\psi_0} \mathscr{L} = g_{\psi_0}(\lambda_0)$$

for every $\lambda_0 \in L(g)$, i.e., for almost every λ_0. By a density argument (on the space of smooth $(k-p,k-p)$-forms ψ compactly supported on $M \times V$), we deduce that the limit above exists for

every ψ as in the statement, on a full measure subset M (independent from ψ). This allows us to define the slice of $\mathcal{R}$ with respect to the approximation ρ_ε. Denoting this slice by $\langle \mathcal{R}, \pi, \lambda \rangle_\rho$, we have

$$\int_M \langle \mathcal{R}, \pi, \lambda \rangle_\rho \, \psi \mathscr{L} = \int_M g_{\psi_0} = \langle \mathcal{R} \wedge \ \pi^* \Omega \mathscr{L}_\Lambda, \psi \rangle$$

and thus Equation A.2 is proved, with respect to this fixed approximation ρ.

The last step is to prove the independence from ρ. This follows from another density argument and the fact that slices $\langle \mathcal{R}, \pi, \lambda \rangle_\rho$ and $\langle \mathcal{R}, \pi, \lambda \rangle_{\rho'}$ obtained with different approximations must agree on a full measure subset of M. □

The following result (see e.g. [Siu74, p. 124]) shows that, for analytic subsets of $M \times V$, the operation of slice coincides with the restriction to the vertical fibers. This result is used in Section 2.2.3.

Lemma A.1.7. *Let X be an hypersurface on $M \times V$ such that $\pi_{|X}$ has rank $m = \dim M$. Then, for every $\lambda \in M$, the slice $\langle [X], \pi, \lambda \rangle$ exists and is equal to $[X \cap \pi^{-1}(\lambda)]$.*

The operation of slicing commutes with d,∂ and $\overline{\partial}$. The same holds with the pushforward by proper maps. We state this property only in the situation needed in Section 2.2.3. The Definition of a holomorphic family of polynomial-like map is given in 1.3.1.

Lemma A.1.8 (Lemma 1.19 in [HS74], Theorem 4.3.2(7) in [Fed96])**.** *Let $f : \mathcal{U} \to \mathcal{V} = M \times V$ be a family of polynomial-like maps. Let $\mathcal{R}$ be a positive closed (p,p)-current on $M \times V$, with $p \leq k = \dim V$. Then, for every λ such that the slice $\langle \mathcal{R}, \pi, \lambda \rangle$ exists, the slice of $f_* \mathcal{R}$ exists and*

$$\langle f_* \mathcal{R}, \pi, \lambda \rangle = (f_\lambda)_* \langle \mathcal{R}, \pi, \lambda \rangle .$$

Proof. Since $f : f^{-1}(M \times V) \to M \times V$ is proper, we have ([Dem, Theorem 2.14, p. 17])

$$f_* (\mathcal{R} \wedge f^* \alpha) = (f_* \mathcal{R}) \wedge \alpha$$

for every smooth form α compactly supported in $M \times V$. Given any smooth approximation ρ_ε of δ_λ (and since $\pi \circ f = \pi$), we thus have

$$(f_* \mathcal{R}) \wedge \pi^* (\rho_\varepsilon \cdot \mathscr{L}) = f_* (\mathcal{R} \wedge f^* \pi^* (\rho_\varepsilon \cdot \mathscr{L})) = f_* (\mathcal{R} \wedge \pi^* (\rho_\varepsilon \cdot \mathscr{L}))$$

and the assertion follows by taking the limit $\varepsilon \to 0$. □

In the remaining part of this section we focus of a particular kind of positive closed currents on the product space, defined as follows.

Definition A.1.9. *A* horizontal current *on the product space $M \times V$ is a current whose support is contained in $M \times K$, for some compact subset K of V.*

For horizontal positive closed currents, Theorem A.1.6 can be improved: the slice exists for every $\lambda \in M$, and moreover the mass is independent of the slice. This is the content of the next theorem, due to Dinh and Sibony. The proof relies on two main ingredients. The first is the fact that for psh functions Theorem A.1.2 holds for every $\lambda \in M$ (i.e., the Lebesgue set $L(\varphi)$ of a psh

function is the total space). This means that, for φ psh, property (A.1) holds for every $\lambda_0 \in M$. The second is the following representation property (see [Hör07, Theorem 3.2.11]).

Lemma A.1.10. *Let M be an open subset of $\mathbb{C}^m$. Let v be a distribution on M, such that $dd^c v \geq 0$. Then there exists a plurisubharmonic function u on M that represents v.*

Theorem A.1.11 (Theorem 2.1 in [DS06]). *Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current on $M \times V$ and let $\pi : M \times V \to M$ denote the standard projection. For every $\lambda \in M$ the slice $\langle \mathcal{R}, \pi, \lambda \rangle$ is well defined (and independent from the particular approximation of δ_λ used) and is a positive measure, whose mass is independent on λ. Moreover, for every continuous psh function ψ on $M \times V$, the function u_ψ on M given by*

$$\lambda \mapsto \langle \mathcal{R}, \pi, \lambda \rangle \left(\psi_{|\pi^{-1}(\lambda)}\right) \tag{A.3}$$

is psh and coincides with $\pi_* (\psi \mathcal{R})$ as a distribution on M.

Proof. Let ψ be a continuous psh function on $M \times V$ and consider the product $\mathcal{R}' = \psi \mathcal{R}$. Its support is contained in the support of $\mathcal{R}$, hence it is a horizontal current. It is then meaningful to consider its pushforward $\pi_* \mathcal{R}'$ by the projection π, which is a $(0,0)$-current (i.e., a distribution) on M. Since $\mathcal{R}$ is closed, we have

$$dd^c \left(\pi_* \mathcal{R}'\right) = \pi_* \left(dd^c \psi \mathcal{R}\right) = \pi_* \left(dd^c \psi \wedge \mathcal{R}\right).$$

By Lemma A.1.10, the distribution $\pi_* (\psi \mathcal{R})$ is then represented by a plurisubharmonic function g_ψ. It follows that, for any $\lambda_0 \in M$ and any approximation $\rho_{\lambda_0,\varepsilon}$ of δ_{λ_0}, (A.1) holds with $\varphi = g_\psi$.

Take now any smooth test function $\widetilde{\psi}$ on $M \times V$. It can be written as the difference of two continuous psh functions, $\widetilde{\psi} = \psi_1 - \psi_2$. This implies that for any $\lambda_0 \in M$, and any approximation $\rho_{\lambda_0,\varepsilon}$ of δ_{λ_0}, we have

$$\begin{aligned}
\left\langle \mathcal{R} \wedge \pi^*(\rho_{\lambda_0,\varepsilon} \mathscr{L}_M), \widetilde{\psi} \right\rangle &= \langle \mathcal{R} \wedge \ \pi^*(\rho_{\lambda_0,\varepsilon} \mathscr{L}_M), \psi_1 - \psi_2 \rangle \\
&= \langle \pi_* (\psi_1 \mathcal{R}), \rho_{\lambda_0,\varepsilon} \mathscr{L}_M \rangle - \langle \pi_* (\psi_2 \mathcal{R}), \rho_{\lambda_0,\varepsilon} \mathscr{L}_M \rangle \\
&= \int_M g_{\psi_1} \rho_{\lambda_0,\varepsilon} \mathscr{L}_M - \int_M g_{\psi_2} \rho_{\lambda_0,\varepsilon} \mathscr{L}_M \\
&\to g_{\psi_1}(\lambda_0) - g_{\psi_2}(\lambda_0) \text{ as } \varepsilon \to 0.
\end{aligned}$$

It is immediate to see that the last difference does not depend from the particular decomposition $\widetilde{\psi} = \psi_1 - \psi_2$ chosen. So, the limit in Definition A.1.3 exists and is equal to $g_{\psi_1} - g_{\psi_2}$ (and in particular it is independent on the particular ρ used). When the function ψ is already psh (but obviously not compactly supported), the function u_ψ of the statement coincides with the function g_ψ constructed above, and so the last assertion is proved.

We are thus only left to prove that the slice mass is independent on λ_0. Take a test function $\widetilde{\psi}$ which is equal to 1 in a neighbourhood $U_0 \times V_0$ of $\left(\pi^{-1}(\lambda_0)\right) \cap \mathrm{Supp}\, \mathcal{R}$, so that $g_{\widetilde{\psi}}(\lambda) = \pi_* \left(\widetilde{\psi} \mathcal{R}\right) (\lambda_0)$ is now the mass of the slice measure of $\mathcal{R}$, for λ in the neighbourhood U_0 of λ_0. Since $\widetilde{\psi} \mathcal{R} = \mathcal{R}$ on $\pi^{-1}(U_0)$, the current $\widetilde{\psi} \mathcal{R}$ is closed on $\pi^{-1}(U_0)$. So, $g_{\widetilde{\psi}} = \pi_* (\psi \mathcal{R})$ is closed on U_0, which means that it is (represented by a) locally constant function near λ_0. This proves the assertion. □

The second part of Theorem A.1.11 holds even if ψ is not continuous (but still psh), as noted in [Pha05].

Corollary A.1.12 ([Pha05], Proposition A.1). *Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current on $M \times V$ and ψ a psh function on a neighbourhood of the support of $\mathcal{R}$. Then, the function u_ψ on M defined as in* (A.3) *is psh, or identically equal to $-\infty$.*

Proof. Let ψ_n be a sequence of smooth psh function decreasing to ψ. For each ψ_n, the corresponding u_{ψ_n} given by Theorem A.1.11 is psh and, by their very definition, we have that the u_{ψ_n}'s decrease to u_ψ (since $\mathcal{R}$ and its slices are positive). So, u_ψ is a decreasing limit of smooth psh functions, and is thus psh (or identically equal to $-\infty$). □

From Theorem A.1.11 we can deduce the following very useful formula, which can also be seen as a characterization of the slice measures. It will play a central role in all our study.

Proposition A.1.13 ([DS06]). *Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current on $M \times V$, Ω a continuous form of maximal degree compactly supported on M and ψ a continuous function on $M \times V$. Then*

$$\int_M \langle \mathcal{R}, \pi, \lambda \rangle(\psi)\Omega(\lambda) = \langle \mathcal{R} \wedge \ \pi^*(\Omega), \psi \rangle. \tag{A.4}$$

Proof. First of all, the slices $\langle \mathcal{R}, \pi, \lambda \rangle$ of $\mathcal{R}$ exist at every $\lambda \in M$ by Theorem A.1.11. Since $\mathcal{R}$ is horizontal and Ω has compact support in M, we can assume that the support of ψ is compact. Moreover, we can suppose that ψ is smooth, and the general case will follow by approximation. Now, every smooth function with compact support is the difference of two smooth psh function, so we can suppose that ψ is smooth and psh (but not compactly supported). The result is then just a reformulation of the last statement of Theorem A.1.11. □

We will see later (A.1.18) that (A.4) still holds when ψ is a psh function, even if not continous. In order to prove this, we must first ensure that the product $\psi\mathcal{R}$ is still well-defined. This has been proved in [Pha05]. We give here a more detailed proof of this statement. Thanks to this result, it will then be possible to get the desired extension of (A.4).

Theorem A.1.14 ([Pha05]). *Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current and u be a psh function on $M \times V$. Assume that there exists a $\lambda_0 \in M$ such that $\langle \mathcal{R}, \pi, \lambda_0 \rangle(u) \neq -\infty$. Then the current $u\mathcal{R}$ is well defined on $M \times V$. In particular, the current $dd^c u \wedge \mathcal{R}$ is well defined, positive and closed.*

We will use here the mass on (order 0) currents given by $\|T\| = \sum |T_{IJ}|$, where T_{IJ} are the distributional coeffients of the current T. Proving that T is well defined means proving that every $|T_{IJ}|$ is (locally) finite. In our situation, after proving that the mass of $u\mathcal{R}$ is locally bounded, it will be possible to define $\langle u\mathcal{R}, \psi \rangle$ for a test (m,m)-form ψ (where m is the dimension of M), smooth and compactly supported in $M \times V$, by

$$\langle u\mathcal{R}, \psi \rangle := \langle \psi\mathcal{R}, u \rangle$$

since, by the local boundedness of the mass of $u\mathcal{R}$, the function u is integrable with respect to the measure $\psi\mathcal{R}$.

Proof. We can assume that M is an open subset of $\mathbb{C}^m$. We shall denote as usual with $\lambda = (\lambda_1, \ldots, \lambda_m)$ the variable(s) on M and with $z = (z_1, \ldots, z_k)$ the ones on V, and with ω_λ and ω_z the standard Kähler forms on $\mathbb{C}^m$ and $\mathbb{C}^k$, respectively.

Consider a compact subset of $M \times V$, of the form $M_0 \times V_0 \Subset \mathbb{C} \times \mathbb{C}^k$, such that $\lambda_0 \in M_0$ and $\mathcal{R}$ is still a horizontal (positive closed) current on $M_0 \times V_0$. We are going to prove that the product $u\mathcal{R}$ has finite mass on $M_0 \times V_0$. By the remarks above, this will prove that the product $u\mathcal{R}$ is well defined there. The idea is to exploit the formula (A.4) to prove that it is possible to define the duality coupling between $u\mathcal{R}$ and some vertical forms of the type $\widetilde{\pi}^*(\Omega)$, where the $\widetilde{\pi}$'s will be suitable projections of $M_0 \times V_0$ on M (and the Ω's some smooth form on M). We shall prove that we can do this for enough projections, in the sense that $(\omega_\lambda + \omega_z)^m$ can be written as a linear combination of forms of the type $\widetilde{\pi}^*\omega_\lambda$ (see Lemma A.1.17), for which we have the estimate on the mass.

The projections $M_0 \times V_0 \to M$ that we are going to consider are of the form $\pi_A(\lambda, z) = \lambda + Az$, with A a linear map from $\mathbb{C}^k$ to $\mathbb{C}$, i.e., a $1 \times k$ matrix with complex coefficients. We can think at A as a point in $\mathbb{C}^k$. Note that, for sufficiently small A's, we can still think at $\mathcal{R}$ as a horizontal current on the space $M \times V_0$ endowed with the new projection π_A. We are going to precise this in a moment. The important think is to remark that what we are going to do now is, instead of thinking of the various $(M, V_0, \mathcal{R}, \pi_A)$ as horizontal currents of different spaces, to collect them in a single object, with the parameter space given by the elements (A,λ). This is done in the following way. Consider the affine map $H : \mathbb{C}^k \times \mathbb{C} \times \mathbb{C}^k \to \mathbb{C} \times \mathbb{C}^k$ given by

$$H(A, \lambda, z) := (\lambda - Az, z).$$

The fact that $\mathcal{R}$ "remains horizontal for small A's" can now be made precise in the following way: there exists a $r \in \mathbb{R}$ such that, if $A \in B(0, r) \subset \mathbb{C}^k$, the current $\mathscr{R} := H^*(\mathcal{R})$ (which exists since H is a submersion) is a horizontal positive current on $\widetilde{M} \times V_0$, where we have denoted by $\widetilde{M}$ the product $B(0, r) \times M_0$. We shall denote by $\pi_{\widetilde{M}}$ the canonical projection of $\widetilde{M} \times V_0$ on $\widetilde{M}$.

Remark the following: H sends vertical copies of V to the fibers of π_A. This means that, given a point $(A,\lambda) \in \widetilde{M}$, the map H precisely sends the fiber $\pi^{-1}_{\widetilde{M}}(A,\lambda)$ in $\widetilde{M} \times V_0$ to the fiber $\pi_A^{-1}(\lambda)$ in $M \times V_0$. By the definition of $\mathscr{R}$, we have that the slice measure of $\mathscr{R}$ on $\pi^{-1}_{\widetilde{M}}(A,\lambda)$ is precisely induced by the slice measure of $\mathcal{R}$ on $\pi_A^{-1}(\lambda)$, that is

$$\langle \mathscr{R}, \pi_{\widetilde{M}}, (A,\lambda) \rangle = \left(H_{|\pi^{-1}_{\widetilde{M}}(A,\lambda)} \right)^* \langle \mathcal{R}, \pi_A, \lambda \rangle. \tag{A.5}$$

This can be seen as follows. Let $(\widetilde{A}, \widetilde{\lambda})$ be a fixed point in $\widetilde{M}$ and let $H_{\widetilde{A}} : \widetilde{M} \times V \to M \times V$ be given by

$$H_{\widetilde{A}}(A, \lambda, z) = (\lambda - \widetilde{A}z, z).$$

The idea behind the function $H_{\widetilde{A}}$ is the following: while H sends the vertical copy of V over (A,λ) to the fiber of λ for the projection π_A, the function $H_{\widetilde{A}}$ sends all these vertical copies always to the fibers of λ for the same projection $\pi_{\widetilde{A}}$.

Define the horizontal positive closed current $\mathscr{R}_{\widetilde{A}}$ on $\widetilde{M} \times V$ as $\mathscr{R}_{\widetilde{A}} := \left(H_{\widetilde{A}} \right)^* \mathcal{R}$. In particular, $\mathscr{R}_{\widetilde{A}}$ does not depend on A. Since the slices measures of both $\mathscr{R}$ and $\mathscr{R}_{\widetilde{A}}$ exist, (A.5) follows from the following two identities.

1. for every $(\widetilde{A},\widetilde{\lambda})\in\widetilde{M}$, we have $\left\langle \mathscr{R},\widetilde{\pi}_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\right\rangle = \left\langle \mathscr{R}_{\widetilde{A}},\widetilde{\pi}_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\right\rangle$;

2. $\langle \mathscr{R}_{\widetilde{A}},\pi_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\rangle = \left(\left(H_{\widetilde{A}}\right)_{|\pi^{-1}_{\widetilde{M}}(\widetilde{A},\widetilde{\lambda})}\right)^{*}\langle \mathcal{R},\pi_{\widetilde{A}},\widetilde{\lambda}\rangle$.

In the following two lemmas we prove the two identities above.

Lemma A.1.15. *With the notations as above, we have*

$$\langle \mathscr{R},\pi_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\rangle = \langle \mathscr{R}_{\widetilde{A}},\pi_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\rangle.$$

Proof. This follows from the fact that, for every smooth test form ψ on $\widetilde{M}\times V$, the value of the slice measure of a horizontal positive current depends only on the value of ψ on the fiber considered.

More precisely, here let ρ_ε be a smooth approximation of $\delta_{(\widetilde{A},\widetilde{\lambda})}$. We want to prove that

$$\lim_{\varepsilon\to 0}\left\langle \left(\mathscr{R}-\mathscr{R}_{\widetilde{A}}\right)\wedge\widetilde{\pi}^{*}(\rho_\varepsilon\mathscr{L}_{\widetilde{M}}),\psi\right\rangle = \lim_{\varepsilon\to 0}\left\langle \left(H^{*}\mathcal{R}-H^{*}_{\widetilde{A}}\mathcal{R}\right)\wedge\widetilde{\pi}^{*}(\rho_\varepsilon\mathscr{L}_{\widetilde{M}}),\psi\right\rangle = 0$$

for every test function ψ compactly supported in the product space $\widetilde{M}\times V$. So, it suffices to prove that

$$H_{*}\left(\psi\widetilde{\pi}^{*}(\rho_\varepsilon\mathscr{L}_{\widetilde{M}})\right)-\left(H_{\widetilde{A}}\right)_{*}\left(\psi\widetilde{\pi}^{*}(\rho_\varepsilon\mathscr{L}_{\widetilde{M}})\right)\to 0$$

as $\varepsilon\to 0$. But, by definition of H and $H_{\widetilde{A}}$, both this two terms tend to

$$\varphi\cdot[\pi^{-1}_{\widetilde{A}}(\widetilde{\lambda})]$$

where φ is the function on $[\pi^{-1}_{\widetilde{A}}(\widetilde{\lambda})]$ given by $\varphi(\lambda,z)=\psi(\widetilde{A},\widetilde{\lambda}+\widetilde{A},z)$. So the limit of the differences goes to zero and the assertion is proved. □

Lemma A.1.16. *With the notations as above, we have*

$$\langle \mathscr{R}_{\widetilde{A}},\pi_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\rangle = \left(\overline{H}_{|\pi^{-1}_{\widetilde{M}}(\overline{A},\overline{\lambda})}\right)^{*}\langle \mathcal{R},\pi_{\widetilde{A}},\widetilde{\lambda}\rangle.$$

Proof. Let φ be any test function compactly supported on $\widetilde{M}\times V$. We have to prove that

$$\langle \mathscr{R}_{\widetilde{A}},\pi_{\widetilde{M}},(\widetilde{A},\widetilde{\lambda})\rangle(\varphi(\widetilde{A},\widetilde{\lambda},\cdot)) = \langle \mathcal{R},\pi_{\widetilde{A}},\widetilde{\lambda}\rangle\,(H_{*}\varphi) = \langle \mathcal{R},\pi_{\widetilde{A}},\widetilde{\lambda}\rangle\left(\varphi(\widetilde{A},\widetilde{\lambda}+\widetilde{A}\cdot,\cdot)\right) \tag{A.6}$$

where the second equality comes form the definition of H (and H_{*}).

Consider a smooth approximation ρ_ε of $\delta_{\widetilde{A},\widetilde{\lambda}}$ of the form $\rho_\varepsilon(A,\lambda)=\rho^{A}_\varepsilon(A)\cdot\rho^{\lambda}_\varepsilon(\lambda)$. Then, the left hand side of (A.6) is equal to

$$\lim_{\varepsilon\to 0}\left\langle \mathscr{R}_{\widetilde{A}}\wedge\widetilde{\pi}^{*}\left(\rho^{\lambda}_\varepsilon\rho^{A}_\varepsilon\mathscr{L}_{M'}\right),\varphi\right\rangle.$$

Since this limit must only depend on the values of φ on $\widetilde{\pi}^{-1}(\widetilde{A},\widetilde{\lambda})$, we can assume that φ does not depend on A (at least on a neighbourhood of $\widetilde{\pi}^{-1}(\widetilde{A},\widetilde{\lambda})$ containing the support of $\widetilde{\pi}^{*}\left(\rho^{\lambda}_\varepsilon\rho^{A}_\varepsilon\mathscr{L}_{M'}\right)$,

for ε sufficiently small). So, since also $\mathscr{R}_{\widetilde{A}}$ and ρ^λ are independent from A, we can start integrating on the variable A, for each ε thus getting

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle \mathscr{R}_{\widetilde{A}}\wedge\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\rho_\varepsilon^A\mathscr{L}_{M'}\right),\varphi\right\rangle &= \lim_{\varepsilon\to 0}\left\langle \left(\mathscr{R}_{\widetilde{A}}\right)_{|\{\,A=\widetilde{A}\,\}}\wedge\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\mathscr{L}_M\right),\varphi_{|A=\widetilde{A}}\right\rangle\\ &= \lim_{\varepsilon\to 0}\left\langle \left(H_{\{\,A=\widetilde{A}\,\}}\right)^*\mathcal{R}\wedge\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\mathscr{L}_M\right),\varphi_{|A=\widetilde{A}}\right\rangle\\ &= \lim_{\varepsilon\to 0}\left\langle \mathcal{R},\left(H_{\{\,A=\widetilde{A}\,\}}\right)_*\left(\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\mathscr{L}_M\right)\varphi_{|A=\widetilde{A}}\right)\right\rangle.\end{aligned}$$

Since H is invertible on $\{A=\widetilde{A}\}$, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\langle \mathscr{R}_{\widetilde{A}}\wedge\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\rho_\varepsilon^A\mathscr{L}_{M'}\right),\varphi\right\rangle &= \lim_{\varepsilon\to 0}\left\langle \mathcal{R},\left(H_{\{\,A=\widetilde{A}\,\}}\right)_*\left(\widetilde{\pi}^*\left(\rho_\varepsilon^\lambda\mathscr{L}_M\right)\right)\cdot\left(H_{\{\,A=\widetilde{A}\,\}}\right)_*\varphi_{|A=\widetilde{A}}\right\rangle\\ &= \left\langle \mathcal{R},\pi_{\widetilde{A}}^*\left(\rho_\varepsilon^\lambda\mathscr{L}_M\right)\cdot\varphi\left(\widetilde{A},\lambda+\widetilde{A}\cdot,\cdot\right)\right\rangle\\ &= \left\langle \mathcal{R},\pi_{\widetilde{A}},\widetilde{\lambda}\right\rangle\left(\varphi\left(\widetilde{A},\lambda+\widetilde{A}\cdot,\cdot\right)\right).\end{aligned}$$

and the assertion follows. □

The next step consists in exploiting the fact that the integral of a psh function on the product space against the slice measures of a horizontal current is a a psh function on the parameter space (or is identically $-\infty$), see Theorem A.1.11 and Corollary A.1.12. In this case, we consider the psh function $\widetilde{u}$ on $\widetilde{M}\times V_0$ given by $\widetilde{u}:=H^*(u)$, i.e., $\widetilde{u}(A,\lambda,z):=u(\lambda-Az,z)$. Integrating it against the slice measures of $\mathscr{R}$ we get a function $v:\widetilde{M}\to\mathbb{R}\cup\{-\infty\}$ given by

$$v(A,\lambda):=\langle\mathscr{R},\pi_{\widetilde{M}},(A,\lambda)\rangle\widetilde{u}(A,\lambda,\cdot) \tag{A.7}$$

which, by what we have just recalled, is psh on $\widetilde{M}$ or identically $-\infty$. But this last possibility cannot happen, since (by hypothesis and because of (A.5)) we know that

$$\begin{aligned}v(0,\lambda_0) &= \int_{\pi_{\widetilde{M}}^{-1}(0,\lambda_0)}\langle\mathscr{R},\pi_{\widetilde{M}},(0,\lambda_0)\rangle\widetilde{u}(0,\lambda_0,\cdot)\\ &= \int_{\pi_M^{-1}(\lambda_0)}\langle\mathcal{R},\pi_M,(\lambda_0)\rangle u(\lambda_0,\cdot)\neq-\infty.\end{aligned}$$

So, we have found that $v(A,\lambda)$ is an actual psh function. This means that all the restrictions of v at fixed A (i.e., the functions $\lambda\mapsto v(A,\lambda)$) are psh or identically $-\infty$. We want to prove that, for A outside a pluripolar set of $B(0,r)$, we have that $\lambda\mapsto v(A,\lambda)$ is a psh function (not identically $-\infty$). To do this, we call P the set of the A's such that $\lambda\mapsto v(A,\lambda)$ is identically $-\infty$ and we prove that it is contained in a pluripolar subset of $B(0,r)$. We can characterize P in the following way: P is the intersection, over $\lambda\in M_0$, of the pluripolar set of the slices $A\to v(A,\lambda)$ with λ fixed. We get the assertion if we prove that for at least one value $\widetilde{\lambda}$ the restriction $A\mapsto v(A,\widetilde{\lambda})$ is not identically $-\infty$. But, by hypothesis, this is true with $\widetilde{\lambda}=\lambda_0$, since $v(0,\lambda)\neq-\infty$. So at least one of the terms of the intersection defining P is pluripolar, which means that P is contained in a pluripolar subset of $B(0,r)$, that we denote with $\mathcal{P}$. Notice in particular that $0\notin\mathcal{P}$.

We can now proceed with the conclusion of the proof. From (A.5) and the definition of $\widetilde{u}$ as

$H^*(u)$, and recalling the definition (A.7) of v, we get

$$\langle \mathcal{R}, \pi_A, \lambda \rangle(u) = v(A, \lambda) \tag{A.8}$$

for every $\lambda \in M_0$. So, it follows that, for $A \notin \mathcal{P}$, the function $\lambda \to \langle \mathcal{R}\pi_A, \lambda \rangle(u)$ is psh, not identically $-\infty$. This implies that this function $\lambda \to \langle \mathcal{R}\pi_A, \lambda \rangle(u)$ is integrable on M_0, which means that there exists a positive constant C_A such that

$$\int_{M_0} \langle \mathcal{R}, \pi_A, \lambda \rangle \left(u(\lambda, \cdot)\right) d\mathscr{L}_M > -C_A. \tag{A.9}$$

Suppose now to be able to apply formula (A.4) with $\psi = u$, i.e., with a psh function instead of a continuous one. Equation (A.9) would then precisely give a minoration on the modulus of the left hand side, thus giving an estimate of the form

$$\|u\mathcal{R} \wedge \pi_A^*(\omega_\lambda^m)\|_K < \infty$$

for every compact $K \subset M_0 \times V_0$ and every $A \in B(0, r) \notin \mathcal{P}$. Since, by Lemma A.1.17 below, it is possible to write $(\omega_\lambda + \omega_z)^m$ as a linear combination of forms of the type $\pi_A^*(\omega_\lambda^m)$, with $A \in B(0, r) \notin \mathcal{P}$, the assertion would follow. The problem is that we are not allowed not only to use, but even to write formula (A.4) directly with u, since the $u\mathcal{R}$ which appears in the right hand side is not even defined yet (defining this object is precisely what we are doing!). So, we are going to proceed in a slightly different way. Let u_n be smooth negative (recall that $u < 0$) psh functions decreasing to u. We are going to prove that for each of these u_n's we have an estimate, uniform in n, of the form

$$|\langle u_n\mathcal{R}, \psi \rangle| < C_\psi \tag{A.10}$$

for every smooth test form ψ. If we can prove this, then we can write

$$\langle u\mathcal{R}, \psi \rangle = \langle \mathcal{R}, u\,\psi \rangle = \langle \mathcal{R}, \lim_{n\to\infty} u_n\psi \rangle.$$

Now, by monotone convergence (decomposing ψ in its positive and negative part) we can switch the integral and the limit in the last term, thus obtaining

$$\langle u\mathcal{R}, \psi \rangle = \lim_{n\to\infty} \langle \mathcal{R}, u_n\psi \rangle.$$

Since now the terms in the limit in right hand side are uniformly bounded, we get that also $u\mathcal{R}$ has locally bounded mass, and so it is well defined.

We can assume that u is negative, and that the same holds for the u_n's. In order to prove (A.10), by Lemma A.1.17, it suffices to prove that, for every $A \notin P$, there exists a constant C'_A, uniform in n, such that

$$|\langle u_n\mathcal{R}, \pi_A^*(\omega_\lambda^m) \rangle| < C'_A$$

for every n. Using (A.4) we are left to prove that

$$\int_{M_0} \langle \mathcal{R}, \pi_A, \lambda \rangle \left(u_n(\lambda, \cdot)\right) d\mathscr{L}_M > -C'_A$$

and, since $u_n \geq u$, from (A.9) it follows that we can use $C'_A = C_A$, and we are done. □

Lemma A.1.17. *Let $x_1, \ldots, x_{k_x}, y_1, \ldots y_{k_y}$ be coordinates on $\mathbb{C}^{k_x+k_y}$. Let ω_x and ω_y be the standard Kähler forms on $\mathbb{C}^{k_x}$ and $\mathbb{C}^{k_y}$, respectively. For A a $k_y \times k_x$-matrix let $\pi_A : \mathbb{C}^{k_x+k_y}$ be given by $\pi_A(x,y) = y + Ax$. Then, for every $p \leq k_y$, $(\omega_x + \omega_y)^p$ is a sum of terms of type $\pi_A^*(\omega_y)$, whose number is polynomial in k_1. The A's involved can be taken (as elements of $\mathbb{C}^{k_x k_y}$) arbitrarily small and outside a given pluripolar set not containing lines through the origin.*

Proof. It is enough to prove that, for every $p \leq k_2$, any form of type

$$id x_{i_1} \wedge d\overline{x}_{i_1} \wedge \ldots \wedge id x_{i_p} \wedge d\overline{x}_{i_p} \tag{A.11}$$

is a sum of terms of type $\pi_A^*(\omega_y)$, whose number is polynomial in k_1, with the required conditions on A. The assertion follows since $(\omega_x + \omega_y)^p$ is a sum of terms

$$id w_1 \wedge d\overline{w}_1 \wedge \cdots \wedge id w_p \wedge d\overline{w}_p,$$

where each w_j is either a x_l or a y_l, whose number is polynomial in k_2.

We thus develop (A.11). Without loss of generality, we can assume that $i_j = j$. Notice that, for every complex numbers $r \neq 0, 1$ and $w \neq 0$, we have

$$\begin{aligned} id z_i \wedge d\overline{z}_i = \frac{r}{w\overline{w}} \Big[& id y_i \wedge d\overline{y_i} + \frac{1}{r-1} id(y_i + w z_i) \wedge d\overline{(y_i + w z_i)} \\ & - \frac{r}{r-1} id(y_i + \frac{w}{r} z_i) \wedge d\overline{(y_i + \frac{w}{r} z_i)} \Big] \end{aligned} \tag{A.12}$$

The assertion follows by taking the wedge product of the different terms (notice that we use a different y_i for every term $id x_i \wedge d\overline{x}_i$). Thus, we need the fact that $p \leq k_2$, since we need p different y_j's to build the combinations. □

We can now prove the desired generalization of Equation (A.4).

Proposition A.1.18. *Let $\mathcal{R}$ be a horizontal positive closed (k,k)-current on $M \times V$, Ω a continuous form of maximal degree compactly supported on M and u a plurisubharmonic function on a neighbourhood of the support of $\mathcal{R}$ such that there exists a λ_0 such that $\langle \mathcal{R}, \pi, \lambda_0 \rangle (u) > -\infty$. Then,* (A.4) *holds.*

Proof. Since $\mathcal{R} \wedge \pi^*\Omega$ has compact support, we can assume that u is negative. Moreover, writing Ω as a difference of two positive measures allows us to assume that Ω is a positive measure.

Take a sequence of negative smooth psh functions u_n decreasing to u. For each of them, (A.4) holds with $\psi = u_n$. Moreover, both terms in (A.4) are well defined with $\psi = u$, by Theorem A.1.14. We thus need to prove that

$$\int_M \langle \mathcal{R}, \pi, \lambda \rangle (u_n) \Omega(\lambda) \to \int_M \langle \mathcal{R}, \pi, \lambda \rangle (u) \Omega(\lambda) \tag{A.13}$$

and

$$\langle \mathcal{R} \wedge\ \pi^*(\Omega), u_n \rangle \to \langle \mathcal{R} \wedge \pi^*(\Omega), u \rangle$$

as $n \to \infty$. Both convergences follow from the monotone convergence theorem, since Ω and $\mathcal{R} \wedge \pi^*\Omega$ are positive measures on M and $M \times V$, respectively. □

A.2. Entropy dimensions

In this section we define the *topological entropy* and the *metric entropy* of a dynamical system, and discuss the relation between the two. We bound the topological entropy of a polynomial-like map on analytic subsets and deduce the fact, needed in Section 2.2.4, that measures of sufficiently large metric entropy cannot charge analytic subsets.

Given a metric space X (or actually even with less assumptions, but we shall restrict us to this setting for simplicity), there is a natural way to associate a *dimension* to a subset Z: we cover it by balls of a given diameter and, letting this diameter go to zero, we compute the exponential rate of growth of the number of balls we need in the cover. Letting $N_Z(\varepsilon)$ be the minimal cardinality of a cover with open sets of diameter less or equal than ε, we would like to define the dimension of Z as

$$\dim_Z = \lim_{\varepsilon \to 0} \varepsilon \log N_Z(\varepsilon)$$

Roughly speaking, this would mean that the required number of balls grows as $e^{\dim_Z / \varepsilon}$. More formally, a possible way to proceed, essentially due to Caratheodory (see [Pes08]), is the following.

Let (X, d) be a metric space. Consider a family $\mathcal{F}$ of open subsets and two real functions η and ψ satisfying the following properties:

1. $\eta(\emptyset) = \psi(\emptyset) = 0$;
2. $\eta(U) > 0$ and $\psi(U) > 0$ for every $U \neq \emptyset$;
3. $\forall \delta > 0 \exists \varepsilon$ such that $\eta(U) \leq \delta$ for every $U \in \mathcal{T}$ with $\psi(U) \leq \varepsilon$.
4. $\forall \varepsilon$ there exists a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\psi(U) \leq \varepsilon$ for all $U \in \mathcal{G}$.

Then, given any $Z \subset X$, define

$$M(Z, \alpha, \varepsilon) := \inf_{\mathcal{U}} \left\{ \sum \eta(U)^\alpha \right\}$$

where the infimum is taken over subfamilies of $\mathcal{F}$ covering Z such that $\psi(U) \leq \varepsilon$ for every $U \in \mathcal{U}$. Since $M_C(Z, \alpha, \varepsilon)$ is non-decreasing as $\varepsilon \to 0$, we can consider the limit

$$m(Z, \alpha) = \lim_{\varepsilon \to 0} M(Z, \alpha, \varepsilon).$$

This set-valued function satisfies some natural properties: $m(\emptyset, \alpha) = 0$ for $\alpha > 0$, it is increasing with respect to inclusion and is subadditive on countable unions. Actually, $m(\cdot, \alpha)$ is almost always 0 or infinity. Indeed, for every Z there exists a critical α_Z such that $m(Z, \alpha) = \infty$ for $\alpha < \alpha_Z$ and $m(Z, \alpha) = 0$ for $\alpha > \alpha_Z$. This critical α_Z is called the *Caratheodory dimension* $\dim_C(Z)$ *of* Z (with respect to η, ψ). It thus satisfies the following:

$$\dim_C(Z) = \inf \{\alpha \colon m(Z, \alpha) = 0\} = \sup \{\alpha \colon m(Z, \alpha) = \infty\}.$$

A first example of Caratheodory dimension is the Hausdoff dimension. This is constructed by taking all the open sets as $\mathcal{F}$ and $\eta(U) = \psi(U) = \operatorname{diam} U$ for every open set $U \in \mathcal{F}$. Given a dynamical system, we will now introduce its *topological entropy* as a Caratheodory dimension.

Let (X, d) be a compact metric space and $T : X \to X$ be a continuous map. Consider the *Bowen distance* on X, defined as follows:

$$d_n^B(x, y) := \max_{0 \le i \le n-1} d(T^i(x), T^i(y)).$$

Notice that the ball $B_n(x,\delta)$ or radius δ at a point x with respect to this distance consists of the points that we cannot distinguish from x, if observing the system for n iteration and with a resolution of δ. Consider the family $\mathcal{F}$ given by all the balls $B_n(x,\delta)$, with $n \in \mathbb{N}$ and $x \in X$ and the functions on $\mathcal{F}$ given by given by

$$\eta(B_n(x,\delta)) = e^{-n} \text{ and } \psi(B_n(x,\delta)) = \frac{1}{n}$$

(we assume for simplicity that $B_n(x,\delta) \neq B_m(x,\delta)$ for $n \neq m$). The δ-topological entropy of a subset Z is thus the Caratheodory dimension associated to $\mathcal{F}, \eta$ and ψ as above. The *topological entropy* is thus defined as

$$h_t(Z) := \limsup_{\delta \to 0} h_t(Z,\delta).$$

This definition of topological entropy coincides with the classical one given by means of (n,ε)-separated sets. Two points $x, y \in X$ are said (n,ε)-*separated* if $d_n^B(x, y) > \varepsilon$. Given any compact subset $Z \subset X$, we denote by $N(Z, n,\varepsilon)$ the maximal cardinality of a subset of Z of pairwise (n,ε)-separated points. We then have

$$h_t(Z) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log N(K, n,\varepsilon).$$

We now introduce the metric entropy. We first define it following the classical approach due to Kolmogorov-Sinai, by means of coverings, and then give an equivalent definition in the spirit of the Caratheodory dimension. From this second it will be even clearer the relation with the topological entropy introduced above.

Let ν be an invariant measure for the dynamical system X and $\mathscr{U} = \{ U_1, \dots, U_m \}$ be any measurable partition of X. The *entropy of the partition* $\mathscr{U}$ is defined as

$$H_{\mathscr{U}} := -\sum \nu(U_i) \log \nu(U_i).$$

Let now $T : X \to X$ be any measurable map. We can consider the partition $\mathscr{U}^{(n)}$ given by the sets $U_{i_1} \cap \dots T^{-1}(U_{i_2}) \cap \dots \cap T^{-(n-1)}(U_{i_{n-1}})$. Then the limit

$$h_\nu(T, \mathscr{U}) := \lim_{n \to \infty} \frac{1}{n} H\left(\mathscr{U}^{(n)}\right)$$

exists and the *metric entropy* of an invariant measure ν can be defined as

$$h_\nu(T) := \sup_{\mathscr{U}} h_\nu(T, \mathscr{U}).$$

An alternative definition of the metric entropy, by means of the Caratheodory dimensional approach, is the following. Given $\mathcal{F}, \eta, \psi$ as above defining a Caratheorory dimension, we let

$$\dim_C(\nu) = \inf \{\dim_C Z \colon \nu(Z) = 1\} = \lim_{\delta \to 0} \inf \{\dim Z \colon \nu(Z) \geq 1 - \delta\}.$$

When applied with $\mathcal{F}, \eta, \psi$ defining the topological entropy, the dimension of the measure ν is precisely the metric entropy $h_\nu(T)$. Thus, both the topological and the metric entropy are indeed *dimensions*, of a set and of a measure, respectively. Moreover, notice that the definition of metric entropy by the Caratheodory approach does not requires that ν is invariant (but we will only consider this situation in the sequel). The following theorem (due to Dinaburg [Din70], Goodman [Goo71] and Goodwyn [Goo69, Goo72]) relates the topological entropy of a compact metric space and the metric entropies of the invariant measures on it.

Theorem A.2.1 (Variational principle). *Let X be a compact metric space and $T : X \to X$ a continuous map. Then $\sup h_\nu(T) = h_t(T)$, where the sup is over all invariant probability measures ν.*

This results gives an effective way to bound the topological entropy from below. On the other hand, it motivates the question, given a dynamical system, of the existence of a measure of *maximal entropy*. For polynomial-like maps, such a measure is given in Section 1.2.2.

The relation between the two entropies is highlighted also by the following important theorem, due to Brin-Katok.

Theorem A.2.2 (Brin-Katok [BK83]). *Let X be a compact metric space and $T : X \to X$ be a continuous map. Let ν be an invariant measure of finite entropy. Then for ν-almost every $x \in X$, the two limits*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \nu\left(B_n(x, \varepsilon)\right) \text{ and } \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \nu\left(B_n(x, \varepsilon)\right)$$

exist and are equal. If ν is ergodic, both limits are equal to $h_\nu(T)$.

From this theorem, we get the following relative version of the variational principle.

Lemma A.2.3. *Let X be a compact metric space and $g : X \to X$ be a continuous map. Let ν be an invariant measure of finite entropy and Y a Borel subset such that $\nu(Y) > 0$. Then*

$$h_t(g, Y) \geq h_\nu(g).$$

Proof. First of all there exists a subset $Y' \subset Y$ of positive ν-measure such that, for every positive H, $x \in Y'$, $\varepsilon < \varepsilon_0(H)$ and $n \geq n_0(H)$ we have

$$h_\nu(g) - H \leq -\frac{1}{n} \log \nu\left(B_n^g(x, \varepsilon)\right),$$

which is equivalent to say that

$$\nu\left(B_n^g(x, \delta)\right) \leq e^{-n(h_\nu(g) - H)}.$$

This follows from two applications of Egorov Theorem to the family of functions $\frac{1}{n} \log \nu\left(B_n^g(x, \varepsilon)\right)$, since $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \inf_{m \geq n} \frac{1}{m} \log \nu\left(B_m^g(x, \varepsilon)\right) = h_\nu(g)$ for almost every $x \in X$, by Brin-Katok Theorem.

It follows that, by the definition of the topological entropy

$$\begin{aligned} h_t(g,Y) \geq \limsup_{n\to\infty} \frac{1}{n}\log N(Y',n,\varepsilon) &\geq \limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{\nu(Y')}{\sup_{x\in X'}\nu\left(B_n^g(x,\varepsilon)\right)}\right) \\ &\geq \limsup_{n\to\infty} -\frac{1}{n}\log\left(e^{n(h_\nu(g)-H)}\right) \\ &\geq h_\nu(g) - H. \end{aligned}$$

The assertion follows since H can be taken arbitrary small. □

The following two results are very useful in order to compute the metric entropy. The first one is minoration of the metric entropy of a measure in terms of the integral of its Jacobian. Recall that the *Jacobian* $J_T(\nu)$ of a measure ν with respect to the map T is the Radon-Nikodym derivative (when this exists) of $T^*\nu$ with respect to ν, i.e., $T^*\nu = J_T(\nu)\nu$.

Theorem A.2.4 (Parry [Par69]). *Let $T : X \to X$ be a measurable map and ν an invariant measure such that the Jacobian $J_T(\nu)$ of ν with respect to T exists. Then $h_\nu(T) \geq \log J_T(\nu)\nu$.*

This result is used in Section 1.2.1 to deduce that the entropy of the equilibrium measure is (at least) $\log d_t$. The following Lemma is used in Section 2.2.4.

Lemma A.2.5. *Let (X,T,ν) and (X',T',ν') be two invariant dynamical systems. Assume there exists a measurable invertible map $\pi : X \to X'$ such that $\pi \circ T = T' \circ \pi$ and $\pi_*(\nu) = \nu'$. Then $h_\nu(T) = h_{\nu'}(T')$.*

We now adress the problem of estimating the entropy on analytic subsets for a polynomial-like map. This is needed in Section 2.2.4 to ensure that an hyperbolic set with sufficiently large entropy cannot be contained in the postcritical set. This will follow from the relative variational principle stated above.

Lemma A.2.6. *Let $f : U \to V$ be a polynomial like map of topological degree d_t. Let K be the filled Julia set, X an analytic subset of V of dimension p, and δ_n be such that $\|f_*^n[X]\|_U \leq \delta_n$. Then*

$$h_t(f,\overline{U}\cap X) = h_t(f,K\cap X) \leq \limsup_{n\to\infty}\frac{1}{n}\log\delta_n \leq d_p^*.$$

We will closely follow the strategy used by Gromov [Gro03] to estimate the topological entropy of endomorphisms of $\mathbb{P}^k$, and adapted by Dinh and Sibony to the polynomial-like setting. We will need the following Lemma in order to get the crucial volume estimate.

Lemma A.2.7. *Let V be an open subset of $\mathbb{C}^k$, U a relatively compact subset of V and L be a compact subset of $\mathbb{C}$. Let X be an analytic subset of V of pure dimension p and Γ an analytic subset of pure dimension p of $\mathbb{C}^m \times V$, contained in $L^m \times V$. Let π denote the canonical projection $\pi : \mathbb{C}^m \times V \to V$ and assume that $\pi(\Gamma) = X$ and $\pi : \Gamma \to X$ is a ramified covering of degree d_Γ. Then there exist constants $c, s > 0$ (independent from X, Γ and m) such that*

$$\textit{volume}(\Gamma \cap (\mathbb{C}^m \times U)) \leq c \cdot \textit{volume}(X) \cdot m^s d_\Gamma.$$

Proof. The proof is essentially the same as that of [DS10, Lemma 2.5], which gives the case $X = V$. The idea is the following. Denote by x_i the coordinates on $\mathbb{C}^m$ and by y_i the ones on $\mathbb{C}^k$. Let ω_k denote the standard Kähler form on $\mathbb{C}^k$ and ω_m the one on $\mathbb{C}^m$. We have to bound the integral

$$\int_{\Gamma\cap\pi^{-1}(U)} (\omega_k + \omega_m)^p = \int_{\Gamma\cap\pi^{-1}(U)} \sum_{0\le j\le p} \binom{p}{j} \omega_k^j \wedge \omega_m^{p-j}.$$

Notice that, since $\pi : \Gamma \to X$ is a covering of degree d_Γ, we have $\int_{\Gamma\cap\pi^{-1}(U)} \pi^*\omega_k^p \le \text{volume}(X)\cdot d_\Gamma$. Moreover, the same is true replacing π with a sufficiently small perturbation π_A given by $\pi_A(x,y) = y + Ax$ (where A is a $k\times m$ matrix with sufficiently small coefficients). So, for A sufficiently small, we have $\int_{\Gamma\cap\pi_A^{-1}(U)} \pi_A^*\left(\omega_k^p\right) \le \text{volume}(X)\cdot d_\Gamma$. The idea is then to bound every term $\int_{\Gamma\cap\pi^{-1}(U)} \omega_k^j \wedge \omega_k^{p-j}$ by suitable combinations of integrals involving only $\pi_A^*(\omega_k^p)$. This in turns reduces to bound the form $\omega_k^j \wedge \omega_m^{p-j}$ with combinations (with $\le m^s$ terms) of wedge products of suitable forms of type $\pi_A^*(\omega_k)$ (which can be done by means of Lemma A.1.17). The details of the proof are as follows. We divide the proof in three steps.

Step 1: simplifications Since this is a local problem, we can assume that $V \subset \mathbb{C}^k$ is the unit ball, U is the concentric ball of radius $1/2$, and L is the unit disc. Denoting by π_A the projection $\pi_A(x,y) = y + Ax$, we will always assume that the entries of the matrix A are bounded by $1/8mk$ in modulus. Finally, set $\Gamma_A := \Gamma \cap \{\,(x,y)\colon\ \|\pi_A(x,y)\| < 3/4\,\}$ and $\Gamma_* := \Gamma \cap (L^m \times U) = \Gamma \cap \pi_0^{-1}(U)$. With these notations, notice that we want to prove that

$$\int_{\Gamma_*} (\omega_k + \omega_m)^p \le c\cdot \text{volume}(X)\cdot m^s d_\Gamma.$$

Step 2: the integral with π_A Here we get the desired estimate by replacing the integrated form by $\pi_A^*(\omega_k^p)$, i.e., we shall prove that

$$\int_{\Gamma_*} \pi_A^*(\omega_k^p) \le c'\cdot \text{volume}(X)\cdot d_\Gamma.$$

First of all, notice that $\Gamma_* \subset \Gamma_A$ (by the choice of A). Then, we have that for every a such that $\|a\| \le 3/4$ (still by the choice of A), $\pi_A^{-1}(a)\cap\Gamma$ contains exactly d_Γ points, counting the multiplicity. Indeed, for every $t \in [0,1]$ and every a and A as above the set $\pi_{tA}^{-1}(a)\cap\Gamma$ is contained in the compact set $\Gamma \cap \{\,(x,y)\colon\ \|\pi(x,y)\| \le 7/8\,\} \subset L^m \times \{\,\|y\| \le 7/8\,\}$, and thus $\#\left(\pi_{tA}^{-1}(a)\cap\Gamma\right)$ (which is equal to d_Γ for $t = 0$) is independent from t. This gives the desired inequality

$$\int_{\Gamma_*} \pi_A^*(\omega_k^p) \le \int_{\Gamma_A} \pi_A^*(\omega_k^p) = \left\langle (\pi_A)_*\left[\Gamma_A\right], \omega_k^p\right\rangle \le \text{volume}(X)\cdot d_\Gamma.$$

Step 3: the combination of the forms To conclude the proof, it suffices to bound $(\omega_k + \omega_m)^p$ by a linear combination of forms (whose number is polynomial in m) of type $\pi_A^*(\omega_k^p)$ with coefficients of order $\simeq m^{s_1}$ and then use the previous estimates. Recall that $\omega_k = dd^c\left\|y\right\|^2 =$

$dd^c\left(|y_1|^2+\cdots+|y_k|^2\right)$ and $\omega_m = dd^c\,\|x\|^2 = dd^c\left(|x_1|^2+\cdots+|x_m|^2\right)$ are the standard Kähler forms on $\mathbb{C}^k$ and $\mathbb{C}^m$, respectively. This follows from Lemma A.1.17, and the Lemma in proved. □

Proof of Lemma A.2.6. The inequality $\limsup_{n\to\infty}\frac{1}{n}\log\delta_n \le d_p^*$ follows by the definition of d_p^*. Moreover, we have $h_t(f,K_0)=0$ for every compact subset K_0 such that $K\cap K_0=\emptyset$. So, we just need to prove that $h_t(f,K\cap X)\le\limsup_{n\to\infty}\frac{1}{n}\log\delta_n$. Let Γ_n^X denote the subset of $V^n\subset(\mathbb{C}^k)^{n-1}\times V$ given by

$$\Gamma_n^X := \{\left(z,f(z),\ldots,f^{n-1}(z)\right) : z\in X\cap f^{-n}(V)\}.$$

Notice that Γ_n^X is an analytic subset of pure dimension p in $U^{n-1}\times V\subset V^n\subset(\mathbb{C}^k)^{n-1}\times V$. Let π be the canonical projection $(\mathbb{C}^k)^{n-1}\times V\to V$. Since $f:U\to V$ is a polynomial-like map, we have that $\pi:\Gamma_n^X\to f^n(X)\subset V$ is a union of ramified coverings of certain degrees δ_n^i. More precisely, for every component Y_i of $f^n(X)$ we have that $\pi:\Gamma_n^X\cap\pi^{-1}(Y_i)\to Y_i$ is a ramified covering of a certain degree δ_n^i. Setting

$$\text{lov}(f) := \limsup_{n\to\infty}\frac{1}{n}\log\text{volume}\left(\Gamma_n^X\cap\pi^{-1}(U)\right)$$

we shall prove the two inequalities

$$h_t(f,K\cap X)\le\text{lov}\,(f)\le\limsup_{n\to\infty}\frac{1}{n}\log\delta_n.$$

The second one follows from Lemma A.2.7 above. Indeed, we have (since $\pi:\Gamma_n^X\cap\pi^{-1}(Y_i)\to Y_i$ is a covering of degree δ_n^i and $\Gamma_n^X\subset U^{n-1}\times V$)

$$\text{volume}\left(\Gamma_n^X\cap\pi^{-1}(U)\right)\le c(kn)^s\sum_i\text{volume}(Y_i)\cdot\delta_n^i$$

and the inequality follows since the sum $\sum_i\text{volume}(Y_i)\cdot\delta_n^i$ is precisely equal to the mass of $f_*^n[X]$, which is less than δ_n by hypothesis.

Let us thus prove that $h_t(f,K\cap X)\le\text{lov}\,(f)$. This topological entropy is equal to

$$h_t(f,K\cap X) := \sup_{\varepsilon>0}\limsup_{n\to\infty}\frac{1}{n}\log N(K\cap X,n,\varepsilon)$$

where $N(K\cap X,n,\varepsilon)$ is the maximal cardinality of an (n,ε)-*separated* set $\mathcal{S}_{n,\varepsilon}$, i.e., a subset of $K\cap X$ such that

$$\sup_{0\le i\le n-1} d(f^i(x),f^i(y))>\varepsilon$$

for every $x\ne y\in\mathcal{S}_{n,\varepsilon}$. Notice that $N(K\cap X,n,\varepsilon)$ increases as $\varepsilon\to 0$. It is thus enough to prove that, for every $\varepsilon>0$ sufficiently small, we have

$$\frac{1}{n}\log N(K\cap X,n,\varepsilon)\le\frac{1}{n}\log\left(\text{volume }\Gamma_n^X\cap\pi^{-1}(U)\right)+O\left(\frac{1}{n}\right). \tag{A.14}$$

This is a standard argument, using the following classical estimate by Lelong.

Lemma A.2.8. *Let A be an analytic subset of dimension p in a ball B of radius r in $\mathbb{C}^N$. Assume that the center of the ball belongs to A. Then the $2p$-dimensional volume of A is at least equal to the volume of a ball of radius r in $\mathbb{C}^p$. In particular, we have*

$$\textit{volume}(A) \geq c_p r^{2p}$$

where $c_p > 0$ is a constant independent from N and r.

Fix any ε such that $d(K, U^c) > \varepsilon$ and let $\mathcal{S}_{n,\varepsilon}$ denote an (n,ε)-separeted subset of $X \cap K$. For $a \in \mathcal{S}_{n,\varepsilon}$, denote by $a^{(n)}$ the point $(a, f(a), \ldots, f^{n-1}(a)) \in \Gamma_n$. Let $B_{a,n}$ the Euclidean ball in V^n centered at $a^{(n)}$ and of radius $\varepsilon/2$. All the balls $B_{a,n}$ are pairwise disjoint (since $\mathcal{S}_{n,\varepsilon}$ is (n,ε)-separeted), and the volume of $(\Gamma_n \cap \pi^{-1}(U)) \cap (B_{a,n})$ is $\geq c_p \varepsilon^{2p}$ for some constant c_p (independend on n and ε, by Lemma A.2.8), since every $B(f^i(a), \varepsilon/2) \subset U$ and so $B_{a,n} \subset U^n$. This implies that the cardinality of $\mathcal{S}_{n,\varepsilon}$ is bounded from above by volume$(\Gamma_n^X \cap \pi^{-1}(U))\varepsilon^{-2p}/c_p$, and (A.14) follows. □

The following Lemma, needed in Section 2.2.4, is now an immediate consequence of Lemmas A.2.3 and A.2.6.

Lemma A.2.9. *Let g be a polynomial-like map of large topological degree. Let ν be an ergodic invariant probability measure for g whose metric entropy h_ν satisfies $h_\nu > \log d_p^*$. Then, ν gives no mass to analytic subsets of dimension $\leq p$.*

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Set thy heart upon thy work, but never on its reward.

Bhagavad Gītā

Abstract

In this thesis we study holomorphic dynamical systems depending on parameters. Our main goal is to contribute to the establishment of a theory of stability and bifurcation in several complex variables, generalizing the one for rational maps based on the seminal works of Mañé, Sad, Sullivan and Lyubich.

For a family of polynomial like maps, we prove the equivalence of several notions of stability, among the others an asymptotic version of the holomorphic motion of the repelling cycles and of a full-measure subset of the Julia set. This can be seen as a measurable several variables generalization of the celebrated λ-lemma and allows us to give a coherent definition of stability in this setting. Once holomorphic bifurcations are understood, we turn our attention to the Hausdorff continuity of Julia sets. We relate this property to the existence of Siegel discs in the Julia set, and give an example of such phenomenon. Finally, we approach the continuity from the point of view of parabolic implosion and we prove a two-dimensional Lavaurs Theorem, which allows us to study discontinuities for perturbations of maps tangent to the identity.

Résumé

Dans cette thèse, on s'intéresse aux systèmes dynamiques holomorphes dépendants de paramètres. Notre objectif est de contribuer à une théorie de la stabilité et des bifurcations en plusieurs variables complexes, généralisant celle des applications rationnelles fondées sur les travaux de Mañé, Sad, Sullivan et Lyubich.

Pour une famille d'applications d'allure polynomiale, on prouve l'équivalence de plusieurs notions de stabilité, entre autres une version asymptotique du mouvement holomorphe des cycles répulsifs et d'un sous-ensemble de l'ensemble de Julia de mesure pleine. Cela peut être considéré comme une généralisation mesurable à plusieurs variables du célèbre λ-lemme et nous permet de dégager un concept cohérent de stabilité dans ce cadre. Après avoir compris les bifurcations holomorphes, on s'intéresse à la continuité Hausdorff des ensembles de Julia. Nous relions cette propriété à l'existence de disques de Siegel dans l'ensemble de Julia, et donnons un exemple de ce phénomène. Finalement, on étudie la continuité du point de vue de l'implosion parabolique. Nous établissons un théorème de Lavaurs deux-dimensionel, ce qui nous permet d'étudier des phénomènes de discontinuité pour des perturbations d'applications tangentes à l'identité.

Sunto

In questa tesi, studiamo sistemi dinamici olomorfi dipendenti da un parametro, con l'obiettivo di contribuire ad una teoria di stabilità e biforcazione a più variabili complesse che generalizzi quella per frazioni razionali basata sui lavori di Mañé, Sad, Sullivan e Lyubich.

Per una famiglia di polynomial-like maps dimostriamo l'equivalenza di numerose nozioni di stabilità, tra le quali una versione asintotica del movimento olomorfo dei cicli repulsivi e di un sottoinsieme di misura piena dell'insieme di Julia. Questo può essere visto come una generalizzazione misurabile a più variabili del famoso λ-lemma e ci permette di dare una definizione coerente di stabilità in questo contesto. Dopo aver compreso le biforcazioni olomorfe, ci interessiamo alla continuità Hausdorff degli insiemi di Julia. Mettiamo in relazione questa proprietà con l'esistenza di dischi di Siegel nell'insieme di Julia, e diamo un esempio di questo fenomeno. Infine, studiamo la continuità dal punto di vista dell'implosione parabolica, e dimostriamo un teorema di Lavaurs a due variabili. Questo ci permette di studiare dei fenomeni di discontinuità per perturbazioni di mappe tangenti all'identità.