A contribution to non-linear PDEs with applications to the level set method, non-Newtonian fluid flows and the Boltzmann equation

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Contribution à la théorie des EDP non linéaires avec applications à la méthode des surfaces de niveau, aux fluides non newtoniens et à l'équation de Boltzmann

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To my niece Sophia, to my grandmother Eleni

I remember, of all the fruits, when I was a kid, I loved the cherries. I used to put them in a bucket full of water, look at them close up and admire them -black or red, crunchy, as they would magnify when in the water. But as I would take them out, to my great dissappointment, I would see them get smaller, so I crammed them, huge as they looked, into my mouth.
This little thing reveals my perception of reality, even now in my old age. I make it brighter, better, more fit to my own purpose. My brain shouts, explains, proves, complains. But a voice inside me responds : "silence, brain, we want to hear the heart". But what heart, the escence of life, the madness, and the heart sings...
∼ Nikos Kazantzakis, Report to Greco.

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Résumé

Cette thèse comporte trois chapitres indépendants, consacrés à l'étude mathématique de trois problèmes physiques distincts, ayant pour modèles trois équations aux dérivées partielles différentes. Ces équations relèvent plus précisément de la méthode des surfaces de niveau, de la théorie de l'écoulement incompressible des matériaux non newtoniens et de la théorie cinétique des gaz raréfiés.

Le premier chapitre de la thèse porte sur la dynamique des frontières en mouvement et contient une justification mathématique de la procédure numérique dite de *ré-initialisation*, dont les applications sont nombreuses dans le contexte de la célèbre méthode des surfaces de niveau. Nous appliquons ces résultats pour une classe d'équations issues de la méthode des surfaces de niveau de premier ordre. Nous écrivons la procédure de *ré-initialisation* comme un algorithme de décomposition et nous étudions la convergence de l'algorithme en utilisant des techniques d'homogénéisation dans la variable temporelle. Grâce à cette analyse rigoureuse nous introduisons également une nouvelle méthode pour l'approximation de la fonction de distance dans le contexte de la méthode des surfaces de niveau. Dans le cas où l'on cherche seulement une fonction de l'ensemble de niveau avec un gradient minoré proche du niveau zéro, nous proposons une approximation plus simple. Dans le cas général, où le niveau zéro pourrait présenter des changements de topologie, nous introduisons une nouvelle notion de limites relâchées.

Dans le deuxième chapitre de la thèse, nous étudions un problème de frontière libre résultant de l'étude de l'écoulement incompressible d'un matériau non-newtonien, avec limite d'élasticité de type Drucker-Prager, sur un plan incliné et sous l'effet de la pesanteur. Nous obtenons une équation sous-différentielle, que nous formulons comme un problème variationnel avec un terme à croissance linéaire de type gradient, et nous étudions le problème dans un domaine non borné. Nous montrons que les équations sont bien posées et satisfont certaines propriétés de régularité. Nous sommes alors capables de relier les paramètres physiques avec le problème abstrait et de prouver des propriétés quantitatives de la solution. En particulier, nous montrons que la solution a un support compact, la limite de ce que nous appelons la frontière libre. Nous construisons également des solutions explicites d'une équation différentielle ordinaire qui peut estimer la frontière libre.

Enfin, le troisième et dernier chapitre de la thèse est dédié aux solutions de l'équation de Boltzmann homogène avec molécules maxwelliennes et énergie infinie. Nous obtenons de nouveaux résultats d'existence de solutions éternelles pour cette equation dans un espace de mesures de probabilité d'énergie infinie (*i.e.* de moment d'ordre deux infini). Elles permettent de décrire le comportement asymptotique en temps d'autres solutions d'énergie infinie, mais elles apparaissent aussi comme des états asymptotiques intermédiaires dans l'étude des solutions d'énergie finie, mais arbitrairement grande. Les méthodes issues de l'analyse harmonique sont utilisées pour étudier l'équation de Boltzmann, où la variable de vitesse est exprimée en Fourier. Enfin, un changement d'échelle logarithmique en la variable temporelle permet de déterminer le bon comportement asymptotique à l'infini des solutions.

Abstract

This thesis consists of three different and independent chapters, concerning the mathematical study of three distinctive physical problems, which are modelled by three non-linear partial differential equations. These equations concern the level set method, the theory of incompressible flow of non-Newtonian materials and the kinetic theory of rarefied gases.

The first chapter of the thesis concerns the dynamics of moving interfaces and contains a rigorous justification of a numerical procedure called *re-initialization*, for which there are several applications in the context of the level set method. We apply these results for first order level set equations. We write the *re-initialization* procedure as a splitting algorithm and study the convergence of the algorithm using homogenization techniques in the time variable. As a result of the rigorous analysis, we are also able to introduce a new method for the approximation of the distance function in the context of the level set method. In the case where one only looks for a level set function with gradient bounded from below near the zero level, we propose a simpler approximation. In the general case where the zero level might present changes of topology we introduce a new notion of relaxed limits.

In the second chapter of the thesis, we study a free boundary problem arising in the study of the flow of an incompressible non-Newtonian material with Drucker-Prager plasticity on an inclined plane. We derive a subdifferential equation, which we reformulate as a variational problem containing a term with linear growth in the gradient variable, and we study the problem in an unbounded domain. We show that the equations are well posed and satisfy some regularity properties. We are then able to connect the physical parameters with the abstract problem and prove some quantitative properties of the solution. In particular, we show that the solution has compact support and the support is the free boundary. We also construct explicit solutions of an ordinary differential equation, which we use to estimate the free boundary.

The last chapter of the thesis is dedicated to the study of infinite energy solutions of the homogeneous Boltzmann equation with Maxwellian molecules. We obtain new results concerning the existence of eternal solutions in the space of probability measure with infinite energy (i.e. the second order moment is infinite). These solutions describe the asymptotic behaviour of other infinite energy solutions but could also be useful in the study of intermediate asymptotic states of solutions with finite but arbitrarily large energy. We use harmonic analysis tools to study the equation, where the velocity variable is expressed in the Fourier space. Finally, a logarithmic scaling of the time variable allows to determine the correct asymptotic scaling of the solutions.

Sommaire

Introduction générale

Dans les trois sections qui suivent, nous allons donner une brève présentation des résultats obtenus dans les trois chapitres qui composent cette thèse et auxquels nous renvoyons le lecteur pour les démonstrations complètes et les références bibliographiques précises.

1 Ré-initialisation

Nous donnons ici un aperçu général de la théorie de la *ré-initialisation* des interfaces en mouvement avec une vitesse imposée, qui dépend des variables d'espace et de temps.

1.1 La méthode des surfaces de niveau

En sciences naturelles très souvent les chercheurs étudient des interfaces en mouvement. Celles-ci peuvent être, par exemple, des vagues dans l'océan, des flammes, des interfaces gazeuses, des bords de matériaux, etc. Suivre la trace de ces interfaces avec une méthode lagrangienne est difficile, parce que les interfaces peuvent développer des singularités, ou même des changements topologiques.

Ainsi, la notion de solutions à utiliser pour les équations aux dérivées partielles impliquées n'est pas très claire. Afin d'éviter ces problèmes, il a été introduite une *méthode des surfaces de niveau*, dont nous allons brièvement donner une présentation ci-dessous. Pour plus d'informations sur cette méthode voir par exemple [11], [15], [16].

On suppose que $\Gamma(t)$ est une hypersurface fermée de dimension $N-1$, qui se déplace avec une vitesse $V(x,t)$. L'objectif est de représenter l'ensemble de niveau zéro de la fonction $u_0(x) = u(x,0)$ par $\Gamma(0)$. Par exemple, on peut choisir $u_0 = d$, la distance signée à $\Gamma(0)$ définie comme $\mathrm{dist}(\cdot, \Gamma(0))$ pour les points à l'intérieur de $\Gamma(0)$ et $-\mathrm{dist}(\cdot, \Gamma(0))$ pour les points à l'extérieur. Le but est de construire une fonction $u(x,t)$ telle que $\{u(\cdot,t) = 0\} = \Gamma(t)$, où également $x(t) \in \Gamma(t)$ si et seulement si

$$u(x(t),t) = 0. \tag{1.1}$$

La formulation de la *méthode des surfaces* de niveau permet de calculer directement des quantités intrinsèques comme la normale de $\Gamma(t)$, qui dans le cas de (1.1) est donnée par $n = \frac{\nabla u}{|\nabla u|}$. Si l'on dérive l'équation (1.1) en t on obtient

$$u_t + \nabla u(x(t),t)x'(t) = 0. \tag{1.2}$$

Or, puisque la vitesse normale du point $x(t)$ est $V(x(t),t)$ on a $x'(t) \cdot \frac{\nabla u}{|\nabla u|} = V(x(t),t)$, ce qui nous permet donc d'écrire (1.2) comme

$$u_t(x,t) = -V(x,t)|\nabla u(x,t)|, \tag{1.3}$$

équation à laquelle il faut associer une valeur initiale u_0. On dit que $\{\Gamma_t\}_{t\geq 0}$ est régulier si

$$\overline{\{(x,t):\, u(x,t)>0\}} = \{(x,t):u(x,t)\geq 0\} \text{ et } \{(x,t):\, u(x,t)\geq 0\}^\circ = \{(x,t):\, u(x,t)>0\}, \tag{1.4}$$

où l'on note $\overline{A}$, A° l'adhérence et l'intérieur d'un ensemble A.

A noter que dans (1.3) les trajectoires $x(t)$ ont été éliminées, afin d'étudier seulement la formulation *eulérienne*.

1.2 Conservation de la fonction distance

Plusieurs méthodes d'approximation de la fonction distance sont connues (voir *e.g.* [1] et [12]), mais ces méthodes ne s'appliquent pas dans le cas où l'ensemble de niveau zéro d'une fonction développe un intérieur, c'est-à-dire la relation (1.4) n'est pas satisfaite.

Voici, ce dont il s'agit. D'abord nous étudions la limite pour $\theta\to\infty$ de la solution de *θ-équation*

$$\begin{cases} u_t^\theta = -V(x,t)|\nabla u^\theta| + \theta\beta(u^\theta)h(\nabla u^\theta) & \text{dans } \mathbf{R}^n\times(0,T), & (1.5\text{a})\\ u^\theta(x,0)=u_0(x) & \text{dans } \mathbf{R}^n, & (1.5\text{b})\end{cases}$$

avec u_0 lipschitzienne, $\beta(u) = u/\sqrt{\varepsilon_0^2+u^2}$, $\varepsilon_0>0$ fixe et V continue en (x,t) et lipschitzienne en x et h est une fonction de pénalisation de la distance entre $|\nabla u^\theta|$ et 1, par exemple $h(p)=1-|p|$. La théorie des solutions de viscosité garantit l'unicité de la solution du problème (1.5). En effet nos théorèmes principaux s'appliquent pour des hamiltoniens plus généraux que celui en (1.2) et pour des fonctions plus générales que β et h.

Si nous désignons par d la fonction de distance signée de l'ensemble de niveau zéro de la solution de (1.2), (1.5b), nous obtenons le théorème suivant.

Théorème 1.1. (Convergence de u^θ vers la fonction de distance signée) *On prend u^θ la solution de* (1.5), *avec* $h(p)=1-|p|$. *Alors*

(i)
$$\lim_{\substack{(y,s,\theta)\to(x,t,\infty)\\ s\leq t}} u^\theta(y,s) = d(x,t) \qquad \textit{pour tout } (x,t)\in\mathbf{R}^n\times(0,T),$$

(ii) $u^\theta(\cdot,t)$ converge vers $d(\cdot,t)$ localement de façon uniforme en $x\in\mathbf{R}^n$ si $\theta\to+\infty$ pour tout $t\in(0,T)$,

(iii) si de plus $d(x,t)$ est continue en $(x,t)\in\mathbf{R}^n\times(0,T)$, alors

u^θ converge vers d localement de façon uniforme en $(x,t)\in\mathbf{R}^n\times(0,T)$ si $\theta\to+\infty$.

En général la fonction de distance d n'est pas continue. Les valeurs de la fonction de distance peuvent présenter des sauts quand une partie de l'ensemble de niveau zéro disparaît. En effet, nous pouvons construire un exemple pour lequel on a des parties de l'ensemble de niveau zero qui disparaissent pour presque tous les temps.

Lorsque la fonction de distance est discontinue on ne peut pas s'attendre à ce que les fonctions continues u^θ convergent vers d localement de façon uniforme. En utilisant la propriété de la vitesse finie de propagation pour les équations telles que (1.2), nous montrons que la fonction de distance est continue inférieurement dans le temps, ce qui nous a guidés dans la démonstration du Théorème 1.1 (i).

Les différentes techniques de *ré-initialisation* permettent d'estimer le gradient de la fonction de l'ensemble de niveau par le bas ou par le haut, afin d'obtenir des résultats numériques valides. Si l'on cherche à obtenir une fonction avec une limite inférieure pour le gradient, on peut utiliser comme terme de pénalisation dans (1.5a) la fonction $h(p) = (1 - |p|)_+$, où nous avons noté $(x)_+ = \max\{x, 0\}$ pour $x \in \mathbb{R}$.

Evidemment, dans ce cas la séquence u^θ ne converge pas nécessairement vers la fonction de distance, cependant on obtient une fonction avec les mêmes ensembles de niveau zéro comme la solution de (1.2), mais pour elle nous avons une limite inférieure pour le gradient. Plus exactement, nous obtenons le théorème suivant.

Théorème 1.2. (Une fonction de niveau d'ensemble avec une limite inférieure pour le gradient)
On prend u le solution de (1.3)-(1.5b), *u^θ le solution de* (1.5) *avec $h(p) = (1 - |p|)_+$. Si*

$$\tilde{u} = \begin{cases} \sup\limits_{\theta>0} u^\theta & \textit{dans } \{u > 0\} \\ \inf\limits_{\theta>0} u^\theta & \textit{dans } \{u \le 0\}, \end{cases}$$

On a $\{\tilde{u} = 0\} = \{u = 0\}$ et

$$\begin{cases} |\nabla \tilde{u}| \ge 1 & \textit{dans } \{u > 0\} \\ -|\nabla \tilde{u}| \le -1 & \textit{dans } \{u < 0\}, \end{cases}$$

dans le sens de la viscosité. En particulier, pour tout $t \in (0, T)$, la fonction $\tilde{u}(\cdot, t)$ est lipschitzienne dans $\mathbb{R}^n$ avec $|\nabla \tilde{u}(\cdot, t)| \ge 1$, p.p. dans $\{\pm u > 0\}$.

Observons que le théorème précédent est une conséquence directe du proposition 2.12, théorème 3.1 et la preuve du théorème 2.3 du chapitre 1.

Explication de la θ-équation L'idée de la théorie de la *ré-initialisation*, comme elle a été introduite dans [17] (voir aussi [7]), consiste à arrêter l'évolution de (1.2) régulièrement dans le temps et à trouver plutôt la solution de l'équation de correction

$$u_t = \beta(u)(1 - |\nabla u|) \tag{1.6}$$

avec β comme dans (1.5a). Or, la solution de l'équation (1.6) converge asymptotiquement vers une fonction qui satisfait $|\nabla u| = 1$, qui est une propriété caractéristique de la fonction de distance. Comme telle, la fonction β est une version plus régulière que la fonction signe.

Ensuite, nous pouvons résoudre les équations (1.3) et (1.6) à des intervalles de temps de longueur $k_1 \Delta t$ et $k_2 \Delta t$ respectivement, périodiquement, avec une période qui est terminée à l'étape de temps de longueur $\varepsilon = (k_1 + k_2)\Delta t$.

On parvient à l'algorithme de décomposition suivant

$$u_t^\varepsilon = \begin{cases} -V(x, \frac{t}{1+\frac{k_2}{k_1}})|\nabla u^\varepsilon| & \text{dans } \mathbb{R}^n \times ((i-1)\varepsilon, (i-1)\varepsilon + k_1 \Delta t] \\ \beta(u^\varepsilon)(1 - |\nabla u^\varepsilon|) & \text{dans } \mathbb{R}^n \times ((i-1)\varepsilon + k_1 \Delta t, i\varepsilon], \end{cases} \tag{1.7}$$

pour $i = 1, ..., \lceil \frac{T}{\varepsilon} \rceil$. On note par $\lceil x \rceil$ le plus petit entier qui est non inférieur à $x \in \mathbb{R}$. Le dimensionnement de la vitesse en temps est requis parce que certains intervalles de temps sont réservés pour l'équation de correction.

Enfin, en utilisant les techniques de la théorie de l'homogénéisation, voir [9] et [10], nous arrivons à prouver le théorème suivant.

Théorème 1.3. (Convergence de l'algorithme de décomposition)
Soit h comme dans le théorème 1.1 ou dans le théorème 1.2 et u^θ *la solution de* (1.5)*, alors la solution* u^ε *de l'algorithme de décomposition* (1.7) *avec donnée initiale* u_0*, converge vers* $\bar{u}^\theta$ *(quand* $\varepsilon \to 0$*), localement de façon uniforme, où* $\bar{u}^\theta(x,t) = u^\theta\left(x, \frac{t}{1+\theta}\right)$ *et* $\theta = k_2/k_1$.

Conclusion : Les théorèmes 1.3 et 1.1 donnent une explication rigoureuse de la procédure de *re-initialisation*. Nous ne connaissons pas d'autres résultats, où cette méthode est étudiée de façon précise avec la notion des solutions de viscosité. Nous prévoyons également d'étudier une schéma numérique du problème (1.5) et voir comment les sauts de la fonction distance interviennent dans la convergence de la solution de (1.5).

2 Matériaux non-newtoniens

Comme dans la section précédente, nous allons d'abord présenter le problème plus général et abstrait et les résultats principaux concernant l'existence et l'unicité des solutions associées. Ensuite, nous allons décrire le problème physiquement, donner des estimations explicites pour les grandeurs physiques qui sont impliquées et démontrer la manière dont les paramètres physiques changent la solution du problème.

2.1 Une équation sous-différentielle

On prend $\lambda \geq 0$ et on étudie les solutions non-négatives $u = u(y,z)$ d'inclusion différentielle :

$$\lambda \in -\Delta u - \operatorname{div}(|z|\partial|\nabla u|) \tag{2.1}$$

dans $\Omega = (-1,1) \times (-\infty, 0)$. On note le sous-différentiel d'une fonction $f : \mathbb{R}^N \to \mathbb{R}$ à ce point y, l'ensemble

$$\partial f(y) := \{z \in \mathbb{R}^N : \, f(x) - f(y) \geq z \cdot (x-y) \, \forall x \in \mathbb{R}^N\}.$$

C'est-à-dire il existe $q = q(y,z) \in \mathbb{R}^2$ avec $|q| \leq 1$ et $q \cdot \nabla u = |\nabla u|$ *p.p.* tel que $\operatorname{div}(\nabla u + |z|q) = -\lambda$. Nous supposons des conditions de Dirichlet homogènes sur le bord latéral, c'est-à-dire, $u(\pm 1, \cdot) = 0$. L'équation sous-différentielle (2.1) est la première variation de la fonctionnelle

$$E_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + |z||\nabla u| - \lambda u. \tag{2.2}$$

Deux problèmes principaux se posent lorsque l'on étudie les minimiseurs de E : d'abord la croissance linéaire du terme $|z||\nabla u|$ ne permet pas le calcul direct et rigoureux de la première variation de E ; ensuite, nous étudions l'énergie dans un domaine non borné Ω. Par conséquent, l'existence d'un minimiseur par la méthode directe (et donc d'une solution de (2.1) que nous allons découvrir) n'est plus triviale, car on ne sait pas clairement si le terme $-\int_\Omega \lambda u$ est semi-continu inférieurement.

Dans un premier temps, nous démontrons que la formulation variationnelle est équivalente à l'équation (2.1).

On définit

$$\mathcal{X} := \{v \in W^{1,2}_{0L}(\Omega), \, z\nabla v \in \left(L^1(\Omega)\right)^2\}, \tag{2.3}$$

où

$$W^{1,2}_{0L}(\Omega) = \{v \in W^{1,2}(\Omega) : \, v(\pm 1, \cdot) = 0\}.$$

Nous avons le théorème suivant, dont l'énoncé précis est contenu dans le chapitre 2 de cette thèse.

Théorème 2.1. (Existence et unicité)
Soit E_λ *donnée par* (2.2)*, alors :*

(i) *pour tout* $\lambda \geq 0$ *il existe une unique* $0 \leq u_\lambda \in \mathcal{X}$ *telle que*

$$E_\lambda(u_\lambda) = \inf_{v \in \mathcal{X}} E_\lambda(v).$$

De plus $u_\lambda \equiv 0$ *si* $\lambda \in [0,1]$ *et* $u_\lambda \not\equiv 0$ *si* $\lambda \in (1, +\infty)$.

(ii) $u_\lambda \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ *pour tout* $\alpha \in (0,1)$ *et* $\partial_z u_\lambda(y, 0) = 0$ *pour* $y \in (-1, 1)$.

2.2 Lois constitutives et de conservation

Pour justifier l'interprétation physique de (2.1) nous aurons besoin d'introduire certaines équations. Nous utilisons la description *lagrangienne* afin de suivre la position d'un point matériel $X(t, s; a) \in \mathbb{R}^3$, en temps t, qui occupe la position $a \in \mathbb{R}^3$ en temps s. La vitesse $u(X(t, s; a), t)$ de la particule située à X en temps t est définie par

$$\frac{d}{dt} X(t, s; a) = v(X(t, s; a)).$$

En utilisant la vitesse, on peut éliminer les trajectoires $X(t, s; a)$ en utilisant l'opérateur de transport parallèle

$$\partial_t + v_j(t, x)\partial_{x_j} = \partial_t + v \cdot \nabla \tag{2.4}$$

où $v = (v_j)_{j \in \{1,2,3\}}$ et nous avons utilisé la convention de sommation des indices répétés.

L'opérateur (2.4) est aussi appelé la *dérivée matérielle* et permet de suivre la vitesse $v(x, t)$ d'un point $x \in \mathbb{R}^3$ à un moment donné t au lieu de la position de la particule X. Ceci est la description *eulérienne* du milieu continu. En utilisant cette description nous allons écrire deux lois du mouvement et l'équation constitutive.

1. *Conservation de la masse.* Soit $\rho(x, t)$ la densité d'un matériau qui occupe le domaine $\Omega \subset \mathbb{R}^3$, alors par la conservation de la masse on a $\frac{d}{dt} \int_\Omega \rho \, dx = 0$; de façon équivalente, si la vitesse v est continue donc le taux de variation de la densité dans un volume infinitésimal, $\frac{d}{dt} \rho \, dx$ est égal au flux $-\rho v \cdot n dS = -\mathrm{div}(\rho v) \, dx$, avec dS la mesure de surface infinitésimale et n la normale vers l'extérieur à la limite du volume infinitésimal.

   La conservation de la masse établit que

   $$\partial_t \rho = -\mathrm{div}(\rho v). \tag{2.5}$$

   Dans le cas d'un fluide incompressible, *i.e.* de densité ρ constante, la conservation de la masse (2.5) se transforme en

   $$\mathrm{div}\, v = 0. \tag{2.6}$$

2. *Conservation de la quantité de mouvement.* Selon la conservation de la quantité de mouvement chaque volume de la matière obéit à la deuxième loi de Newton. Soit $\sigma = (\sigma_{ij})_{i,j \in \{1,2,3\}}$, c'est-à-dire une matrice réelle symétrique, qui est telle que la quantité $\int_{\partial\Omega} \sigma_{ij} n$, avec n la normale vers l'extérieur à la limite de $\partial\Omega$, est la force exercée sur le matériau qui est contenu dans Ω. Si nous calculons l'accélération en fonction de la *dérivée matérielle*, les équations de mouvement se transforment en

   $$\partial_t v + (v \cdot \nabla) v = \mathrm{div} \sigma + f, \tag{2.7}$$

   où $f : \mathbb{R}^3 \to \mathbb{R}^3$ est la force extérieure.

3. *L'équation constitutive.* Soit $\sigma_{\text{dev}} := \sigma + pI$ le déviateur de tenseur des contraintes et $D(v) = (\nabla v + (\nabla v)^T)/2$, alors comme dans [13] et [6] nous utilisons l'équation constitutive suivante

$$\begin{cases} \sigma_{\text{dev}} = 2\nu D(v) + k(p)\frac{D(v)}{||D(v)||} & \text{si } D(v) \neq 0, \\ ||\sigma_{\text{dev}}|| \leq k(p) & \text{si } D(v) = 0 \end{cases} \tag{2.8}$$

où nous supposons que la viscosité $\nu > 0$ est constante et $k(p)$ est la limite d'élasticité, qui dépend de la pression. L'équation précédente est le résultat d'une superposition du terme de viscosité $2\nu D(v)$ et du terme lié aux effets de plasticité $k(p)\frac{D(v)}{||D(v)||}$ qui est indépendant de la norme du taux de déformation $||D(v)||$.

Dans le cas de limite d'élasticité $k(p)$ constante, on retrouve le modèle de Bingham ordinaire. Dans le deuxième chapitre de la thèse nous supposerons le critère de plasticité de Drucker-Prager, à savoir

$$k(p) = \mu_s p, \tag{2.9}$$

où $\mu_s = \tan \delta_s$, avec δ_s l'angle de la friction interne (statique).

2.3 Écoulement sur un plan incliné

Nous étudions l'écoulement quasi-statique d'un fluide non-newtonien sur un plan incliné sous l'effet de la pesanteur. Alors, la force de gravité dans (2.7) est donnée par $f = (g_0 \sin\theta, 0, -g_0 \cos\theta)$, avec θ l'angle d'inclinaison du plan et g_0 la constante gravitationnelle. Un matériau non-newtonien se comporte comme un corps rigide dans certaines régions et en tant que fluide dans d'autres.

Afin d'étudier la formation de la partie rigide quand on augmente l'angle d'inclinaison θ, nous supposerons que le plan est situé à l'infini ($z = -\infty$) pour ne pas avoir de frottement avec le plan (voir figure 1). Nous supposons que la vitesse est de la forme $v(x, y, z) = (\tilde{u}(y, z), 0, 0)$ pour $(x, y, z) \in (0, +\infty) \times (-l, l) \times (-\infty, 0)$ où $2l$ est la largeur des parois et le niveau $z = 0$ est la surface sur laquelle le matériau est en contact avec l'atmosphère. La

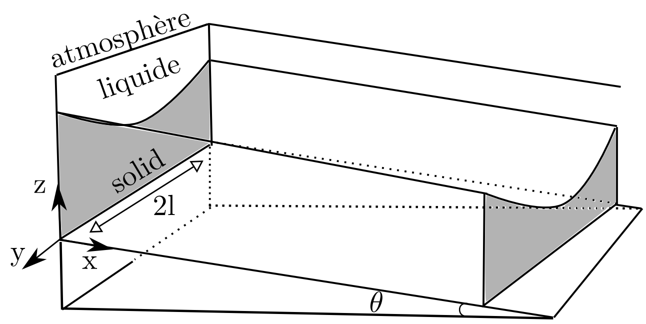

FIGURE 1 – L'écoulement quasi-statique sur un plan incliné

vitesse v satisfait trivialement la condition d'incompressibilité (2.6). On prend les équations de mouvement avec $\partial_t v = 0$ et l'équation constitutive (2.8) pour aboutir à

$$\begin{cases} \nu \operatorname{div}(\nabla\tilde{u}) + \mu_s g_0 \cos\theta \operatorname{div}(|z|q) = -g_0 \sin\theta & \text{in } (-l, l) \times (-\infty, 0), \\ q \in \partial(|\cdot|)(\nabla\tilde{u}) \end{cases} \tag{2.10}$$

Pour plus de détails voir le chapitre 2, section 1. Sur la limite latérale nous supposons que

$$\tilde{u}(\pm l, \cdot) = 0. \tag{2.11}$$

Sur la surface $z = 0$, où le matériau est en contact avec l'atmosphère, nous supposons que le tenseur des contraintes est nul, c'est-à-dire $\sigma_{\mathrm{dev}} \cdot (0,0,1) = 0$, qui, dans notre cas particulier se lit comme

$$\partial_z \tilde{u}(\cdot, 0) = 0. \tag{2.12}$$

Conclusion : Soit $\tilde{u}$ la solution de (2.10)-(2.12), alors la fonction

$$u(y,z) = \frac{\nu}{\mu_s g_0 \cos\theta} \frac{\tilde{u}(ly, lz)}{l^2}, \quad (y,z) \in (-1,1) \times (-\infty, 0)$$

résout (2.1) avec $\lambda = \frac{\tan\theta}{\mu_s}$ et des conditions aux limites (2.11), (2.12). Avec ces notations, le théorème 2.1 (ii) implique que pour $\theta > \operatorname{Arctan} \mu_s$ il existe une solution de (2.10)-(2.12) non triviale, mais pour un angle d'inclination $\theta \leq \operatorname{Arctan} \mu_s$ la solution est nulle, *i.e.* tout le matériau est en phase solide.

En construisant des barrières explicites nous montrons qu'en réalité la fonction u a un support compact et est continue par le Théorème 2.1 (ii) et car $u \geq 0$, ce qui nous permet de définir le frontière libre comme la frontière commune $\partial\{u > 0\} = \partial\{u = 0\}$. Nous appelons phase solide et phase liquide les ensembles $\{u = 0\}$ et $\{u > 0\}$, respectivement.

En général, dans la littérature, le frontière libre est définie par $\partial\{\nabla u \neq 0\}$, mais le rapprochement de cet ensemble nécessiterait des méthodes différentes et plus de régularité de la solution. Les estimations du frontière libre dans le théorème suivant sont explicites. Nous renvoyons le lecteur au chapitre 2 pour une présentation détaillée.

Théorème 2.2. (Propriétés principales)
Soient $\lambda > 1$ *et* u_λ *la solution de* (2.1). *Alors, il existe* $\mathrm{Epi}^{\smile}(\lambda)$ *croissant en* λ, *i.e.* $\mathrm{Epi}^{\smile}(\lambda_1) \subset \mathrm{Epi}^{\smile}(\lambda_2)$ *pour* $1 < \lambda_1 \leq \lambda_2$, *et* $\mathrm{Epi}^{\smile}(\lambda)$ *tel que*

$$\mathrm{Epi}^{\smile}(\lambda) \subset \operatorname{supp} u_\lambda \subset \mathrm{Epi}_{\smile}(\lambda).$$

En fait, l'estimation ci-dessus montre que le frontière libre $\partial\{u_\lambda > 0\}$ *n'atteint jamais l'atmosphère.*

3 Équation de Boltzmann

Nous présentons ici certains modèles d'équations, qui interviennent en théorie cinétique des gaz et qui feront l'objet du troisième chapitre de la thèse et auquel nous renvoyons le lecteur pour une présentation détaillée.

Notons $f : [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ la distribution des particules d'un gaz dans $\mathbb{R}^3$, c'est-à-dire, pout tout $t \in [0,T]$ la quantité $f(t,x,v)dxdv$ est la densité des particules du gaz dans l'élément de volume $dxdv$. L'intervalle de temps peut être fini, $T < +\infty$ ou infini, $T = +\infty$, et la variable temporelle peut prendre aussi des valeurs négatives, $t \in \mathbb{R}$, ce qui correspond aux solutions dites *éternelles*.

On peut établir une corrélation entre la distribution f et les variables macroscopiques (observables) : la densité $\rho = \int f(t,x,v)dv$, la vitesse $\rho u = \int f(t,x,v)vdv$ et la température $\rho|u|^2 + 3\rho T = \int f(t,x,v)|v|^2 dv$. Ce sont les trois premiers moments de f.

3.1 Collisions

En l'absence de collisions entre les particules du gaz, alors la densité f est transportée par la dérivée de matérielle, *i.e.*

$$\partial_t f + v \cdot \nabla_x f = 0. \tag{3.1}$$

Nous supposons que les collisions impliquées sont seulement binaires. Nous supposons également que les collisions sont élastiques, *i.e.* la quantité de mouvement et l'énergie cinétique est conservée. Alors, si v, v_* sont les vitesses de pré-collision de deux particules et v', v'_* leurs vitesses respectives, après leur collision, on a

$$\begin{aligned} v' + v'_* &= v + v_* \\ |v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2. \end{aligned}$$

Ce sont 4 équations à 6 inconnues, on peut donc obtenir une famille à 2 paramètres de solutions, plus précisément pour $\sigma \in \mathbb{S}^2$ la sphère unité dans $\mathbb{R}^3$ on pouvons écrire

$$\begin{aligned} v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \\ v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma. \end{aligned}$$

Lorsque nous prenons en compte les collisions, le zéro sur le côté droit de (3.1) devrait être remplacé par *l'opérateur de collision* $Q = Q^+ - Q^-$, qui est la différence d'un terme de gain et un terme de perte. On suppose que les collisions sont localisées ce qui signifie que $Q^\pm(f, f)(t, x, v) = Q^\pm(f(t, x, \cdot), f(t, x, \cdot))(v)$. Alors pour $v \in \mathbb{R}^3$ le terme de perte $Q^-(f, f)(v)$ est l'intégrale sur tous les v_*, v', v'_* de la probabilité que les vitesses v, v_* passent à v', v'_* après une collision, que nous noterons $p(v, v_* \to v', v'_*)$, pondérée par la densité des particules en commun avec des vitesses v, v_*, $f_2(v, v_*)$, *i.e.*

$$Q^-(f, f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} f_2(v, v_*)p(v, v_* \to v', v'_*)dv'_*dv'dv_*.$$

Nous supposons que les vitesses des particules qui sont sur le point d'entrer en collision sont décorrélées (*l'hypothèse du chaos*), alors $f_2(v, v_*) = f(v)f(v_*)$. Enfin, nous supposons que les collisions sont *micro-réversibles*, c'est-à-dire, pour tout v, v_*, v', v'_* on a $p(v, v_* \to v', v'_*) = p(v', v'_* \to v, v_*)$, alors, nous pouvons écrire le terme de gain

$$Q^+(f, f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} f(v')f(v'_*)p(v, v_* \to v', v'_*)dv'_*dv'dv_*,$$

et l'opérateur de collision

$$Q(f, f)(v) = \int_{v_*}\int_{v'}\int_{v'_*} (f(v')f(v'_*) - f(v)f(v_*))p(v, v_* \to v', v'_*)dv'_*dv'dv_*.$$

3.2 Section efficace

Selon le principe d'invariance galiléenne, la probabilité $p(v, v_* \to v', v'_*)$ ne dépend que de $|v - v_*|$ et du cosinus de l'angle de déviation, *i.e.* $\frac{v-v_*}{|v-v_*|} \cdot \sigma$, ce que nous noterons par $p = \bar{\mathcal{B}}\left(|v - v_*|, \frac{v-v_*}{|v-v_*|} \cdot \sigma\right)$.

Si $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$, lorsque la force interparticulaire est proportionnelle à $r^{-1/\nu}$, avec r qui désigne la distance interparticulaire et $\nu \in (0, 1)$, nous pouvons écrire pour $\theta \in [0, \pi]$

$$\bar{\mathcal{B}}(|v - v_*|, \cos\theta) = |v - v_*|^{1-4\nu}\mathcal{B}(\cos\theta), \tag{3.2}$$

avec $\mathcal{B}$ une fonction régulière, sauf pour $\theta = 0$ pour laquelle nous avons

$$\mathcal{B}(\cos\theta) \overset{\theta\to 0}{\sim} \frac{C}{\theta^{2+2\nu}},$$

avec $C > 0$. A noter que puisque $\nu \in (0,1)$ la singularité de $\mathcal{B}$ est toujours non-intégrable. Pour $\nu \in (0,1/2)$ la singularité est dite *mild* et si $\nu \in [1/2,1)$ est dite *forte*.

Nous disons que $\bar{\mathcal{B}}$ est un *potentiel dur* si $\lim\limits_{|v|\to+\infty} \bar{\mathcal{B}}(v,\cos\theta) = +\infty$, un *potentiel doux* si $\lim\limits_{|v|\to+\infty} \bar{\mathcal{B}}(v,\cos\theta) = 0$ et un *potentiel maxwellien* s'il ne dépend pas de la vitesse relative, *i.e.* il est de la forme $\mathcal{B}(\cos\theta)$.

Avec ces notations, l'équation de Boltzmann homogène pour les molécules maxwelliennes s'écrit

$$\partial_t f(v,t) = Q(f,f)(v,t), \tag{3.3}$$

où la forme bilinéaire Q correspondant à un gaz maxwellien est donnée par

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)(f(v')g(v'_*) - f(v)g(v_*)\, d\sigma dv. \tag{3.4}$$

On notera également qu'en ce qui concerne le changement de variables $\sigma \to -\sigma$ dans (3.4), on peut considérer la place du noyau collisionnel symétrisé

$$\left[\mathcal{B}\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) + \mathcal{B}\left(\frac{v_*-v}{|v-v_*|}\cdot\sigma\right)\right]\chi_{\{(v-v_*)\cdot\sigma>0\}}, \tag{3.5}$$

où χ est la fonction caractéristique de l'ensemble correspondant, en d'autres termes, nous pouvons supposer que $\theta \in [0,\pi/2]$ dans (3.2).

3.3 Solutions éternelles

L'existence d'une solution $f = f(v,t)$, avec $v \in \mathbb{R}^3$ et $t > 0$, de l'équation de Boltzmann homogène pour un gaz de molécules maxwelliennes, éqs. (3.3) – (3.4), est bien connue, voir *e.g.* [19]. Cette solution satisfait les lois de conservation de la masse, la moyenne et la température (énergie) du gaz, *i.e.*

$$\int_{\mathbb{R}^3} f(v,t)\,dv = 1,\ \int_{\mathbb{R}^3} f(v,t)v_i\,dv = 0\,(i=1,2,3),\ \int_{\mathbb{R}^3} f(v,t)|v|^2\,dv = 3, \tag{3.6}$$

et converge asymptotiquement en temps vers un état maxwellien $M(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$.

Un cas particulièrement intéressant est donné lorsque l'inconnue est considérée en fonction de la transformée de Fourier de la variable vitesse, soit $\varphi(\xi,t) = \widehat{f}(\xi,t)$, ce qui conduit, via les formules de Bobylev [2] (voir aussi [8]), à l'équation de Boltzmann simplifiée

$$\partial_t\varphi(\xi,t) = \int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)(\varphi(\xi^+,t)\varphi(\xi^-,t) - \varphi(\xi,t)\varphi(0,t))\,d\sigma, \tag{3.7}$$

avec les vecteurs $\xi^+ = (\xi+|\xi|\sigma)/2$ et $\xi^- = (\xi-|\xi|\sigma)/2$ qui vérifient les relations

$$\xi^+ + \xi^- = \xi \quad \text{et} \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2.$$

Suivant [18], Cannone et Karch ont introduit l'espace $\mathcal{K}^\alpha = \{\varphi : \mathbb{R}^3 \to \mathbb{C}$ des fonctions caractéristiques telles que $\sup_{\xi\in\mathbb{R}^3} |\xi|^{-\alpha}|1-\varphi(\xi)| < \infty\}$ et sous certaines hypothèses assez faibles d'intégrabilité du noyau de collision $\mathcal{B}$, *i.e.*

$$(1-y)^{\alpha_0/4}(1+y)^{\alpha_0/4}\mathcal{B}(y) \in L^1(-1,1) \quad \text{pour un certain } \alpha_0 \in [0,2], \tag{3.8}$$

ils ont montré que si la donnée initiale est dans $\mathcal{K}^\alpha$, alors le problème de Cauchy pour l'équation (3.7) admet une unique solution classique $\varphi \in C([0,\infty),\mathcal{K}^{\alpha_0})$, voir [4, Théorèmes 2.2 et 2.5] pour plus de détails.

Dans cette thèse, nous prouvons de nouveaux résultats d'existence de *solutions éternelles* de l'équation (3.7), autrement dit *définies en tout temps* $t \in \mathbb{R}$.

Nous commençons par introduire la notion de *moment généralisé d'ordre* α, à savoir une fonction $\varphi \in \mathcal{K}^\alpha$, telle que la limite $\lim_{\xi \to 0} |\xi|^{-\alpha}(1 - \varphi(\xi))$ existe. Avec cette notation, nous pouvons annoncer un résultat de propagation des moments généralisés d'ordre α des solutions de l'équation (3.7).

Théorème 3.1. *Soit $\varphi \in C([0,\infty), \mathcal{K}^\alpha)$ avec $\alpha \in (0,2)$ une solution de l'équation (3.7). On suppose que $\lim_{\xi \to 0} |\xi|^{-\alpha}(1 - \varphi(\xi, 0)) = K$ pour une constante $K > 0$. Alors*

$$\lim_{\xi \to 0} \frac{1 - \varphi(\xi, t)}{|\xi|^\alpha} = K e^{\lambda_\alpha t} \qquad \text{pour tout} \quad t \geq 0, \tag{3.9}$$

où la constante $\lambda_\alpha \geq 0$ est définie par l'expression

$$\lambda_\alpha = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma. \tag{3.10}$$

Dans le cas limite $\alpha = 2$, on obtiendrait $\lambda_\alpha = 0$ et la formule (3.9) exprime la conservation du moment d'ordre deux (conservation de l'énergie) contenue dans (3.6). Nous allons utiliser ces moments généralisés d'ordre α pour construire des solutions *éternelles* de l'équation (3.7).

Théorème 3.2. *Supposons que $\mathcal{B}$ vérifie la condition* (3.8) *et fixons $\alpha \in [\alpha_0, 2)$. Pour chaque $K > 0$ il existe une solution éternelle $\overline{\varphi} \in C((-\infty,\infty), \mathcal{K}^\alpha)$, $\bar{w} \not\equiv 1$ de l'équation* (3.7).

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Chapitre 1

A rigorous setting for the reinitialization of first order level set equations

This chapter is a paper [26] (accepted for publication in Interfaces and Free Boundaries) written in collaboration with Nao Hamamuki †.

Abstract

In this paper we set up a rigorous justification for the reinitialization algorithm. Using the theory of viscosity solutions, we propose a well-posed Hamilton-Jacobi equation with a parameter, which is derived from homogenization for a Hamiltonian discontinuous in time which appears in the reinitialization. We prove that, as the parameter tends to infinity, the solution of the initial value problem converges to a signed distance function to the evolving interfaces. A locally uniform convergence is shown when the distance function is continuous, whereas a weaker notion of convergence is introduced to establish a convergence result to a possibly discontinuous distance function. In terms of the geometry of the interfaces, we give a necessary and sufficient condition for the continuity of the distance function. We also propose another simpler equation whose solution has a gradient bound away from zero.

1 Introduction

Setting of the problem In this paper we establish a rigorous setting for the reinitialization algorithm. In the literature "reinitialization" usually refers to the idea of stopping the process of solving an evolution equation regularly in time and changing its solution at the stopping time so that we obtain a function which approximates the (signed) distance function to the zero level set of the solution. A typical example of such evolution equations is

$$u_t = c(x,t)|\nabla u|, \tag{1.1}$$

where $u = u(x,t)$ is the unknown, $u_t = \partial_t u$, $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ and $|\cdot|$ stands for the standard Euclidean norm in $\mathbf{R}^n$. The equation (1.1) describes a motion of an interface Γ_t in $\mathbf{R}^n$ whose normal velocity is equal to $c = c(x,t)$, where at each time the zero level set

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of $u(\cdot,t)$ represents the interface Γ_t. In general, the solution of (1.1) does not preserve the distance function, and its gradient can get very close to zero. For example, the function

$$u(x,t) = 1 - |x|e^{-t}$$

solves the problem

$$\begin{cases} u_t = |x| \cdot |u_x| & \text{in } \mathbf{R} \times (0,T), \\ u(x,0) = 1 - |x| & \text{in } \mathbf{R} \end{cases}$$

in the viscosity sense.

Motivated by the numerical theory of reinitialization we are led to study a new equation which approximates the distance function, namely we study the limit as $\theta \to \infty$ of the solution of

$$\begin{cases} u_t^\theta = H_1(x,t,\nabla u^\theta) + \theta\beta(u^\theta)h(\nabla u^\theta) & \text{in } \mathbf{R}^n \times (0,T), \\ u^\theta(x,0) = u_0(x) & \text{in } \mathbf{R}^n \end{cases} \tag{1.2}$$

with u_0 Lipschitz continuous, $\beta(u) = u/\sqrt{\varepsilon_0^2 + u^2}$, $h(p)$ belongs to a class of functions which are used to penalise the distance of p from 1, see (1.5), and finally $H_1 = H_1(x,t,p)$ is assumed to be geometric (see (H2) in Subsection 2.1) and satisfy suitable continuity conditions which guarantee the uniqueness and existence of viscosity solutions.

For the numerical study of (1.1) several simplifications can be made when the solution is or approximates the distance function. One of the reasons is the fact that the gradient of the distance function is always 1 and thus bounded away from 0. When the gradient degenerates like in the above example, it becomes difficult to compute precisely the zero level sets. The reinitialization is used to overcome such an issue. For a more detailed discussion on the numerical profits of the reinitialization, see [32, 31].

Several reinitialization techniques have been introduced in the literature. In this paper we focus on the one introduced by Sussman, Smereka and Osher ([36]). Their method allows to reinitialize (1.1) without explicitly computing the signed distance function with the advantage that the level set function of their method approximates the signed distance at every time.

We briefly explain the main idea of the method in [36]. Consider the *corrector equation*

$$\phi_t = \operatorname{sign}(\phi)(1 - |\nabla\phi|), \tag{1.3}$$

where $\operatorname{sign}(\cdot)$ is the sign function defined as

$$\operatorname{sign}(r) = \begin{cases} \dfrac{r}{|r|} & \text{if } r \neq 0, \\ 0 & \text{else.} \end{cases}$$

The solution of this equation asymptotically converges to a steady state $|\nabla\phi| = 1$, which is a characteristic property of the distance function ; see Subsection 3.3. The purpose of the sign function in (1.3) is to control the gradient. In the region where ϕ is positive, the equation is $\phi_t = 1 - |\nabla\phi|$. Thus, the monotonicity of ϕ is prescribed by the order of 1 and $|\nabla\phi|$. This forces $|\nabla\phi|$ to be close to 1 as time passes. Also, the relation $\operatorname{sign}(0) = 0$ guarantees that the initial zero level set is not distorted since $\phi_t = 0$ on the zero level. Roughly speaking, the idea of [36] is to stop the evolution of (1.1) periodically in time and solve (1.3) till convergence to the signed distance function is achieved. This method was first applied in [36] for the calculation of the interface of a fluid flow, with the disadvantage that the fluid flow can lose mass, because of the accumulation of numerical errors after many periods are completed. This problem was later fixed in [35].

Up to the authors' knowledge there is no rigorous setting for the reinitialization process described above. In this paper we study an evolution of an interface Γ_t given as the zero level set of the solution u to the initial value problem of the general Hamilton-Jacobi equation

$$u_t = H_1(x, t, \nabla u) \tag{1.4}$$

with a Lipschitz continuous initial datum u_0 and H_1 as in (1.2). As a corrector equation we use a slight modification of (1.3), namely

$$u_t = \beta(u) h(\nabla u), \tag{1.5}$$

where $\varepsilon_0 > 0$ is fixed and the function h can be one of the following :

(1) $h(p) = 1 - |p|$,

(2) The positive part of (1), i.e., $h(p) = (1 - |p|)_+$.

The function $\beta(u)$ is a smoother version of the sign function. Although the function h in (2) does not preserve the distance function in the sense of [36] and in a way that will be made rigorous later in Theorem 2.3 and Example 4.1, it does however prevent the gradient of the solution to approach zero on the zero level set. Moreover, it provides a simple monotone scheme for the numerical solution of the problems which we will encounter. In fact, our result applies for corrector equations which are more general than (1.5), but for the sake of simplicity we present, in this section, the main idea for this model equation.

The idea, as in [36], is to solve (1.4) and (1.5) periodically in time, the first for a period of $k_1 \Delta t$ and the second for $k_2 \Delta t$, where $k_1, k_2, \Delta t > 0$ and one period will be completed at a time step of length $\varepsilon = (k_1 + k_2)\Delta t$. We are thus led to define the following combined Hamiltonian

$$H_{12}(x, t, \tau, r, p) := \begin{cases} H_1(x, \frac{t}{1+\frac{k_2}{k_1}}, p) & \text{if } (i-1) < \tau \leq (i-1) + \frac{k_1 \Delta t}{\varepsilon}, \\ \frac{u}{\sqrt{\varepsilon_0^2 + u^2}} h(\nabla u) & \text{if } (i-1) + \frac{k_1 \Delta t}{\varepsilon} < \tau \leq i \end{cases}$$

for $i = 1, ..., \lceil \frac{T}{\varepsilon} \rceil$. Here by $\lceil x \rceil$ we denote the smallest integer which is not smaller than $x \in \mathbf{R}$. The rescaling of the Hamiltonian H_1 in time is required since certain time intervals are reserved for the corrector equation. More precisely, H_1 is solved in time length $k_1 \Delta t \lceil \frac{T}{\varepsilon} \rceil \sim T \frac{k_1}{k_1 + k_2} = \frac{T}{1 + \frac{k_2}{k_1}}$. One would expect that solving the two equations infinitely often would force the solution of the reinitialization algorithm to converge to the signed distance function to Γ_t ; we denote it by d. Therefore we are led to study the limit as $\varepsilon \to 0$ of the solutions of

$$\begin{cases} u_t^\varepsilon = H_{12}\left(x, t, \dfrac{t}{\varepsilon}, u^\varepsilon, \nabla u^\varepsilon\right) & \text{in } \mathbf{R}^n \times (0, T), \\ u^\varepsilon(x, 0) = u_0(x) & \text{in } \mathbf{R}^n. \end{cases} \tag{1.6}$$

This is a homogenization problem with the Hamiltonian H_{12} being 1-periodic and discontinuous in the fast variable $\tau = t/\varepsilon$. Since the limit above is taken for $\Delta t \to 0$ (and consequently $\varepsilon \to 0$), two free parameters still remain, namely k_1 and k_2. In fact, we show that the solutions of (1.6) converge, as $\varepsilon \to 0$ and after rescaling, to the solution u^θ of (1.2). Here $\theta = k_2 / k_1$ is the ratio of length of the time intervals in which the equations

(1.4) and (1.5) are solved. If we solve the corrector equation (1.5) in a larger interval than the one we solve the original (1.4), we can expect the convergence to a steady state. For this reason we study the limit as $\theta \to \infty$ of the solutions of (1.2).

Let us consider the function h in (1) as the model case. Roughly speaking, the limit $\theta \to \infty$ forces $h(\nabla u^\theta)$ to be close to 0 except on the zero level of u^θ, i.e., $|\nabla u^\theta| \approx 1$ for large value of θ. If we further know that the zero level set of u^θ is the same as that of the solution of (1.4) and hence is equal to Γ_t (we call this property a preservation of the zero level set), then we would get a convergence of u^θ to the signed distance function d, which is known to be a solution of the eikonal equation

$$|\nabla d| = 1 \tag{1.7}$$

with the homogeneous Dirichlet boundary condition on the zero level. The preservation of the zero level set for (1.2) mainly follows from [25].

To justify the convergence to d rigorously, the comparison principle for the eikonal equation (1.7) is used to compare the distance function and a half-relaxed limit of u^θ, which is a weak notion of the limit for a sequence of functions. To do this, we need to know that the limit of u^θ also preserves the zero level set. This is not clear, despite the fact that u^θ always preserves the zero level set for every $\theta > 0$. For the preservation of the zero level set by the limit, continuity of the distance function plays an important role. As is known, if we fix a time, $d(\cdot, t)$ is a Lipschitz continuous function, but d is not continuous in general as a function of (x, t). Indeed, when the interface has an extinction point (Definition 5.3), the distance function can be discontinuous near this point. For our problem, by constructing suitable barrier functions it turns out that, when d is continuous, the zero level set of the half-relaxed limit of u^θ is the same as Γ_t. Consequently, we obtain the locally uniform convergence of u^θ to d ; see Theorem 2.2 (iii).

Concerning the locally uniform convergence, we further consider a condition which guarantees the continuity of d. An important property of first order equations is the finite speed of propagation (Subsection 5.1), which allows us to show that the only way the distance function can be discontinuous is if points at the zero level extinct instantaneously. More precisely, we show that the distance function is continuous at (x, t) if and only if at least one of the nearest points of x to Γ_t is a non extinction point ; see Theorem 5.4 (3). Therefore, if the latter condition is satisfied for every $(x, t) \in \mathbf{R}^n \times (0, T)$, then the solutions u^θ of (1.2) converge locally uniformly to d in $\mathbf{R}^n \times (0, T)$ (Remark 5.5). The converse is also true.

If the signed distance function d is discontinuous, we cannot expect that the continuous solutions u^θ of (1.2) will converge locally uniformly to d. In fact, when d is discontinuous, the zero level sets of the half-relaxed limit of u^θ are not Γ_t, and this prevents us to apply the comparison principle for (1.7). We can however show (Theorem 2.2 (i)) a weaker notion of convergence to d ; namely a convergence to d from below in time as follows :

$$\lim_{\substack{(y,s,\theta)\to(x,t,\infty) \\ s\le t}} u^\theta(y, s) = d(x, t) \quad \text{for all } (x, t) \in \mathbf{R}^n \times (0, T). \tag{1.8}$$

This will be shown by introducing a notion of a half-relaxed limit from below in time and by using the fact that d is continuous from below in time. The result (1.8) also implies a locally uniform convergence at any fixed time, that is, $u^\theta(\cdot, t)$ converges to $d(\cdot, t)$ locally uniformly in $\mathbf{R}^n$ as $\theta \to \infty$; see Theorem 2.2 (ii).

In a future work we plan to introduce a numerical scheme for (1.2), where no reinitialization will be required. We also plan to study numerically and rigorously a similar method for second order equations, including the mean curvature flow, of the form

$$u_t = H_1(x, t, \nabla u, \nabla^2 u) + \theta\beta(u)h(\nabla u).$$

Review of the literature In [11], Chopp used a reinitialization algorithm for the mean curvature flow. His technique does not utilize a corrector equation, instead, at each stopping time he recalculates the signed distance and starts the evolution again with this new initial value. In [32] it is mentioned that another way to reinitialize is to compute the signed distance at each stopping time using the fast marching method. In [35] the problem of the movement of the zero level set which appeared in [36] is fixed by solving an extra variational problem during the iteration of the two equations. Sethian in [32] suggests to use, instead of the reinitialization algorithm, the method of extended velocity described in chapter 11.

In [14] a new nonlinear equation is introduced for the evolution of open sets with thin boundary under a given velocity field. The solution is for every time the signed distance function to the boundary of the open set (called an oriented distance function in [14]). Other numerical methods for preserving the signed distance are presented in [15] and [23].

In [24], the authors use the approximation of the mean curvature flow by the Allen-Cahn equation and they prove the convergence of an equation to the signed distance function. See also [8] for a related theory developed for anisotropic and crystalline mean curvature flow.

Another method for approximating the distance function is presented in [4]. This method as in [24] is motivated by the phase field theory, but is also applied for first order equations. In both papers the assumption of non-fattening of the zero level set is necessary in order to prove convergence. Even for first order equations the zero level set can develop interior when the velocity changes sign or depends on time, see Example 5.2 ; also in [4, Theorem 4.1] it is proved, in fact, that a constant sign or a time independence condition of the velocity is sufficient for the evolution not to develop an interior. We note that for our method we do not need to assume a non-fattening condition or any additional smoothness of the zero level set and we do not impose any restrictions on the sign or the time dependence of the velocity. We also mention that another difference from our method is that in [4] one needs to track the boundaries $\partial\{u > 0\}$ and $\partial\{u \geq 0\}$. This is why the non-fattening condition is needed. Whereas, in our case we only track the set $\{u = 0\}$ which is preserved for all $\theta > 0$ by solutions of (1.2), see Theorem 3.1.

Summary To sum up, the contributions of this paper are the mathematical justification of the reinitialization procedure, the introduction of a new approximate scheme for the distance function of evolving interfaces, i.e., solving (1.2) and taking the limit as $\theta \to \infty$; the formulation of a necessary and sufficient condition for the solution of the scheme (1.2) to converge locally uniformly to the signed distance function, in terms of topological changes of interfaces ; the discovery of a weak notion of a limit which gives the signed distance function even if it is discontinuous.

We also mention that through the rigorous analysis of the reinitialization procedure, we retrieve the correct rescaling in time of the equation (1.4) in order to approximate the signed distance function, and thus we extend the reinitialization procedure to evolutions with time depending velocity fields. Lastly, the equation in (1.2) with h satisfying (2) or more generally (as we will see later) the assumption (2.10) admits a natural numerical scheme with a CFL condition, see also [5] or [33]. We plan to study this last part in a future work.

Organization of the paper In Subsection 2.1 we state the main results, and in Subsection 2.2 we present known results concerning a well-posedness and regularity of viscosity solutions. Section 3 consists of main tools which we use in order to prove our main theorems, namely the preservation of the zero level set (Subsection 3.1), construction of barrier functions (Subsection 3.2) and characterization of the distance function via the eikonal equation (Subsection 3.3). In Section 4 we prove convergence results to the signed distance function d. The proof for continuous d and that for discontinuous d will be given separately. Section 5 is concerned with continuity properties of the distance function, and finally in Section 6 we prove a homogenization result.

2 Main results

2.1 Main theorems

We study the evolution of the zero level set of a function w given by the following problem :

$$\begin{cases} w_t = H_1(x,t,\nabla w) & \text{in } \mathbf{R}^n \times (0,T), \quad (2.1a) \\ w(x,0) = u_0(x) & \text{in } \mathbf{R}^n. \quad (2.1b) \end{cases}$$

Here u_0 is a possibly unbounded Lipschitz continuous function on $\mathbf{R}^n$ and its Lipschitz constant is denoted by L_0. The function $H_1 = H_1(x,t,p) : \mathbf{R}^n \times [0,T] \times \mathbf{R}^n \to \mathbf{R}$ satisfies

(H1) $H_1 \in C(\mathbf{R}^n \times [0,T] \times \mathbf{R}^n)$,

(H2) H_1 is *geometric*, i.e., $H_1(x,t,\lambda p) = \lambda H_1(x,t,p)$ for all $\lambda > 0$, $x \in \mathbf{R}^n$, $t \in [0,T]$ and $p \in \mathbf{R}^n$,

(H3) There is a positive constant L_1 such that

$$|H_1(x,t,p) - H_1(y,t,p)| \le L_1|x-y|$$

for all $x,y \in \mathbf{R}^n$, $t \in [0,T]$ and $p \in \mathbf{R}^n$ with $|p| = 1$,

(H4) There is a positive constant L_2 such that

$$|H_1(x,t,p) - H_1(x,t,q)| \le L_2|p-q|$$

for all $x \in \mathbf{R}^n$, $t \in [0,T]$ and $p,q \in \mathbf{R}^n$,

The condition (H2) is the first order version of *"geometricity"* which was first introduced by Y. G. Chen, Y. Giga, and S. Goto for second order nonlinear operators [10] ; see also a book of Y. Giga [21].

Remark 2.1. *In the literature the assumption* (H3) *is usually given as : There are $L, \bar{L}$ positive, such that*

$$|H(x,t,p) - H(y,t,p)| \le L|x-y||p| + \bar{L}|x-y| \quad (2.2)$$

for all $x,y,p \in \mathbf{R}^n$ and $t \in [0,T]$. However, since the Hamiltonian H is geometric (the assumption (H2)*), it turns out that the conditions* (H3) *and* (2.2) *are equivalent. Indeed, it is clear that* (H3) *implies* (2.2) *with $L = L_1$ and $\bar{L} = 0$. Also, under* (2.2) *we can easily derive* (H3) *with $L_1 = L + \bar{L}$. Here let us also show that, in fact, we can take $L_1 = L$ in* (H3) *when* (2.2) *holds. Let $x,y \in \mathbf{R}^n$, $t \in [0,T]$, $p \in \mathbf{R}^n$ with $|p| = 1$ and $r > 0$. Then we have*

$$|H(x,t,rp) - H(y,t,rp)| \le Lr|x-y| + \bar{L}|x-y|.$$

Dividing both sides by r and using (H2)*, we get*

$$|H(x,t,p) - H(y,t,p)| \le L|x-y| + \bar{L}\frac{|x-y|}{r}.$$

If we now take the limit as $r \to +\infty$, we get (H3) *with $L_1 = L$.*

The assumption (H2) is natural for a geometric evolution problem, while (H3) is used for construction of barriers in Subsection 3.2 and for the proof of Lipschitz continuity of solutions in Appendix A. We call the constant L_2 in assumption (H4) the speed of propagation of the zero level set of the solutions. Existence, uniqueness and other properties of the problem (2.1) can be found in Subsection 2.2. Since the zero level set of the solution w of (2.1) is the main focus of this paper we will use the following notations for $t \in [0,T)$:

$$D_t^{\pm} := \{x \in \mathbf{R}^n \mid \pm w(x,t) \geq 0\}, \tag{2.3}$$

$$\Gamma_t := \{x \in \mathbf{R}^n \mid w(x,t) = 0\} \tag{2.4}$$

and

$$D^+ := \bigcup_{t\in(0,T)} (D_t^+ \times \{t\}), \quad D^- := \bigcup_{t\in(0,T)} (D_t^- \times \{t\}).$$

In what follows we will always suppose that the evolution associated with w is not empty, i.e.,

$$\Gamma_t \neq \emptyset \text{ for all } t \in [0,T). \tag{2.5}$$

Also for $\Omega \subset \mathbf{R}^n$ the distance function $\mathrm{dist}(\cdot,\Omega) : \mathbf{R}^n \to [0,\infty)$ is defined as

$$\mathrm{dist}(x,\Omega) := \inf_{y\in\Omega} |x-y|.$$

It is well-known that the distance function $\mathrm{dist}(\cdot,\Omega)$ is 1-Lipschitz continuous in $\mathbf{R}^n$, i.e.,

$$|\mathrm{dist}(x,\Omega) - \mathrm{dist}(y,\Omega)| \leq |x-y| \quad \text{for all } x,y \in \mathbf{R}^n.$$

For a function $w : \mathbf{R}^n \times [0,T) \to \mathbf{R}$, $\mathrm{Lip}_x[w]$ stands for the Lipschitz constant of w with respect to x, i.e.,

$$\mathrm{Lip}_x[w] := \sup_{\substack{x,y\in\mathbf{R}^n \\ x\neq y}} \sup_{t\in[0,T)} \frac{|w(x,t)-w(y,t)|}{|x-y|} \in [0,\infty].$$

Our first result concerns an equation of the form

$$u_t^\theta = H_1(x,t,\nabla u^\theta) + \theta H_2(u^\theta,\nabla u^\theta) \quad \text{in } \mathbf{R}^n \times [0,T), \tag{2.6}$$

where $\theta > 0$ is a parameter, H_1 is as in (2.1a) and

$$H_2(r,p) = \beta(r)h(p). \tag{2.7}$$

The function β is assumed to satisfy

(B) $\mathrm{Lip}[\beta] =: L_\beta < \infty$ and β is non-decreasing and bounded in $\mathbf{R}$ with $\beta(0) = 0$, $\beta(r) > 0$ if $r > 0$, $\beta(r) < 0$ if $r < 0$,

where by $\mathrm{Lip}[f]$ we denote the Lipschitz constant of a function $f : \mathbf{R}^n \to \mathbf{R}$. Moreover, $h : \mathbf{R}^n \to \mathbf{R}$ is such that

(2.8) there is a modulus ω_h such that $|h(p) - h(q)| \leq \omega_h(|p-q|)$ for all $p,q \in \mathbf{R}^n$.

Here a function $\omega : [0,\infty) \to [0,\infty)$ is called a modulus if ω is non-decreasing and $0 = \omega(0) = \lim_{r\to 0} \omega(r)$. We will also use one of the following assumptions for the function h :

$$\begin{cases} h(p) > 0 \text{ if } |p| < 1, & \text{(2.9a)} \\ h(p) < 0 \text{ if } |p| > 1 & \text{(2.9b)} \end{cases}$$

or

$$
\begin{cases} h(p) > 0 \text{ if } |p| < 1, & (2.10a) \\ h(p) = 0 \text{ if } |p| \geq 1. & (2.10b) \end{cases}
$$

Examples of these functions are

$$H_1(x,t,p) = c(x,t)|p|, \tag{2.11}$$

$$H_2(u,p) = \frac{u^2}{\sqrt{\varepsilon_0^2 + u^2}} h(p) \tag{2.12}$$

for

$$h(p) = 1 - |p| \tag{2.13}$$

or

$$h(p) = (1 - |p|)_+, \tag{2.14}$$

where $\varepsilon_0 > 0$, c is Lipschitz continuous with respect to $x \in \mathbf{R}^n$ uniformly in time, and for $a \in \mathbf{R}$ we denote by

$$a_\pm = \max\{\pm a, 0\}$$

the positive and negative part of a. We see that the function h defined in (2.13) satisfies (2.9) while (2.14) satisfies (2.10).

For a function $w(x,t)$ defined in $\mathbf{R}^n \times [0,T)$, we define the signed distance function $d(x,t)$, from the zero level set of w, as follows :

$$d(x,t) = \begin{cases} \operatorname{dist}(x, \Gamma_t) & \text{if } x \in D_t^+ \cup \Gamma_t, \\ -\operatorname{dist}(x, \Gamma_t) & \text{if } x \in D_t^-. \end{cases} \tag{2.15}$$

Here $D_t^\pm$ and Γ_t are defined in (2.3) and (2.4).

For later use we collect our main assumptions in the following list :

$$\begin{cases} u_0 \text{ is Lipschitz continuous in } \mathbf{R}^n, \;\; H_1 \text{ satisfies (H1)–(H4)}, \\ H_2 \text{ is of the form (2.7)}, \;\; \beta \text{ satisfies (B)}, \;\; h \text{ satisfies (2.8)}. \end{cases} \tag{2.16}$$

For the solution u^θ of (2.6) and (2.1b) we have the following main theorem.

Theorem 2.2 (Convergence of u^θ to the signed distance function)**.** *Assume* (2.16) *and* (2.9)*. Let u^θ be the solution of* (2.6) *and* (2.1b)*. Let d be the signed distance function as in* (2.15)*. Then*

(i)

$$\lim_{\substack{(y,s,\theta)\to(x,t,\infty) \\ s \leq t}} u^\theta(y,s) = d(x,t) \quad \text{for all } (x,t) \in \mathbf{R}^n \times (0,T),$$

(ii) $u^\theta(\cdot,t)$ converges to $d(\cdot,t)$ locally uniformly in $\mathbf{R}^n$ as $\theta \to +\infty$ for all $t \in (0,T)$,

(iii) if in addition $d(x,t)$ is continuous in $\mathbf{R}^n \times (0,T)$, then

$$u^\theta \text{ converges to } d \text{ locally uniformly in } \mathbf{R}^n \times (0,T) \text{ as } \theta \to +\infty.$$

In general, if the signed distance function d is discontinuous, we cannot expect that the continuous functions u^θ will converge to d locally uniformly. The following example shows that the signed distance function can be discontinuous when points of the zero level set disappear instantaneously. We will denote by $B_r(x)$ the open ball of radius $r > 0$ centered at x. Its closure is $\overline{B_r(x)}$. Also, $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product.

Example 2.1 (A single discontinuity)**.** *We study* (2.1) *with*

$$H_1(x,p) = c(x)|p|, \tag{2.17}$$

where $c \in \mathrm{Lip}(\mathbf{R}^n)$ *is bounded and non-negative. Since* H_1 *is written as* $H_1(x,p) = \max_{a \in \overline{B_1(0)}} \langle c(x)a, p\rangle$*, the viscosity solution* w *of* (2.1) *has a representation formula as a value function of the associated optimal control problem ([17, Section 10]), which is of the form*

$$w(x,t) = \sup_{\alpha \in \mathcal{A}} u_0(X^\alpha(t)). \tag{2.18}$$

Here $\mathcal{A} := \{\alpha : [0,T) \to \overline{B_1(0)},\ \text{measurable}\}$ *and* $X^\alpha : [0,T) \to \mathbf{R}^n$ *is the solution of the state equation*

$$(X^\alpha)'(s) = c(X^\alpha(s))\alpha(s) \quad \text{in } (0,T), \quad X^\alpha(0) = x.$$

Each element $\alpha \in \mathcal{A}$ *is called a control.*

We now consider the case where $c(x) = 1$*. This describes a phenomenon where the interface expands at a uniform speed 1. In this case the optimal control forces the state* $X^\alpha(\cdot)$ *to move towards the maximum point of* u_0 *in* $\overline{B_t(x)}$*, and hence*

$$w(x,t) = \max_{|x-y|\le t} u_0(y). \tag{2.19}$$

Take the initial datum as $u_0(x) = \max\{(1-|x-2|)_+,\ (1-|x+2|)_+\}$*. The formula* (2.19) *now implies*

$$w(x,t) = \min\{\max\{(t+1-|x-2|)_+,\ (t+1-|x+2|)_+\}, 1\}.$$

To see this we notice that

$$\max_{|x-y|\le t} (1-|y\pm 2|)_+ = \min\{(t+1-|x\pm 2|)_+, 1\},$$

then using the formula (2.19) *and after changing the order of the maxima, we calculate*

$$\begin{aligned} w(x,t) &= \max\{\min\{(t+1-|x-2|)_+, 1\}, \min\{(t+1-|x+2|)_+, 1\}\} \\ &= \min\{\max\{(t+1-|x-2|)_+,\ (t+1-|x+2|)_+\}, 1\}. \end{aligned}$$

Here we have used the relation $\max\{\min\{a_1,\ b\}, \min\{a_2,\ b\}\} = \min\{\max\{a_1,\ a_2\},\ b\}$ *for* $a_1, a_2, b \in \mathbf{R}$*. We therefore have*

$$\{w = 0\} = \begin{cases} \{|x| \ge t+3\} \cup \{|x| \le 1-t\} & \text{if } t \le 1, \\ \{|x| \ge t+3\} & \text{if } t > 1 \end{cases}$$

and

$$d(x,t) = \begin{cases} \max\{(t+1-|x-2|)_+,\ (t+1-|x+2|)_+\} & \text{if } t \le 1, \\ (t+3-|x|)_+ & \text{if } t > 1. \end{cases}$$

See Figure 1.1. Thus d *is discontinuous on* $\ell := \{(x,1) \mid -2 < |x| < 2\}$*; more precisely,* d *is not upper semicontinuous but lower semicontinuous on* ℓ*.*

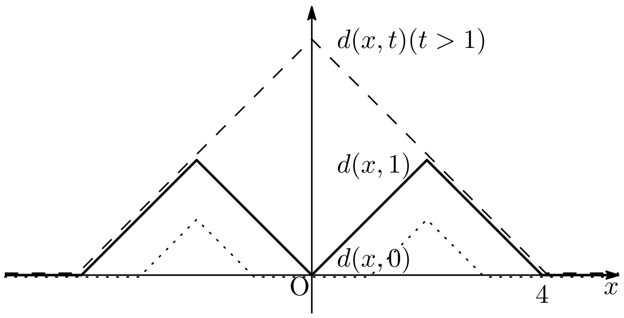

FIGURE 1.1 – The graph of d.

If h satisfies (2.10) we can still estimate from one side the limit with the distance function. More precisely we have the following theorem.

Theorem 2.3. *Assume* (2.16) *and* (2.10) *with* u^θ, d *as in Theorem 2.2. Then*

$$d(x,t) \le \sup_{\theta>0} u^\theta(x,t) < +\infty \qquad \text{for all } x \in D_t^+,$$
$$d(x,t) \ge \inf_{\theta>0} u^\theta(x,t) > -\infty \qquad \text{for all } x \in D_t^-.$$

For the next result we define

$$H_{12}(x,t,\tau,r,p) := \begin{cases} H_1(x, \frac{t}{1+\frac{k_2}{k_1}}, p) & \text{if } (i-1) < \tau \le (i-1) + \frac{k_1 \Delta t}{\varepsilon}, \\ H_2(r,p) & \text{if } (i-1) + \frac{k_1\Delta t}{\varepsilon} < \tau \le i \end{cases} \tag{2.20}$$

for $k_1, k_2 > 0$, $\Delta t > 0$, $\varepsilon = (k_1+k_2)\Delta t$ and $i = 1, ..., \lceil \frac{T}{\varepsilon} \rceil$. By definition H_{12} is 1-periodic in τ, and in general it is discontinuous in τ. In summary, we are led to the following equation :

$$u_t^\varepsilon = H_{12}\left(x, t, \frac{t}{\varepsilon}, u^\varepsilon, \nabla u^\varepsilon\right) \quad \text{in } \mathbf{R}^n \times (0,T). \tag{2.21}$$

Remark 2.4. *A solution of the problem* (2.21), (2.1b) *can be constructed by solving* (2.21) *in the intervals* $[\varepsilon(i-1), \varepsilon(i-1)+k_1\Delta t)$, $[\varepsilon(i-1)+k_1\Delta t, \varepsilon i)$, $i = 1, ..., \lceil \frac{T}{\varepsilon} \rceil$, *iteratively, using as initial condition at each interval, the final value of the solution defined in the previous interval. We call this solution an* iterative *solution.*

Let $\theta = k_2/k_1$. We define

$$\bar{H}(x,t,r,p) = \frac{1}{1+\theta}\left(H_1\left(x, \frac{t}{1+\theta}, p\right) + \theta H_2(r,p)\right)$$

and consider the equation

$$\bar{u}_t^\theta = \bar{H}(x,t,\bar{u}^\theta, \nabla \bar{u}^\theta) \quad \text{in } \mathbf{R}^n \times (0,T). \tag{2.22}$$

Theorem 2.5 (Homogenization)**.** *Assume* (2.16). *Let* $\bar{u}^\theta$ *and* u^ε *be, respectively, the solution of* (2.22), (2.1b) *and the iterative solution of* (2.21), (2.1b). *Then* u^ε *converges to* $\bar{u}^\theta$ *locally uniformly in* $\mathbf{R}^n \times [0,T)$.

Remark 2.6. *If we set* $u^\theta(x,t) = \bar{u}^\theta(x,(1+\theta)t)$ *in Theorem 2.5, then* u^θ *solves the equation* (2.6) *and satisfies the initial data* (2.1b).

Remark 2.7. *All of our main theorems have the same assumptions on u_0 and H_1. For this reason, we will assume* (2.16) *in the rest of the paper except Subsection 3.1, where u_0 will be generalized. For the function h in* (2.7) *we will differentiate the assumptions* (2.9) *and* (2.10)*. Finally we will state clearly whether or not the distance function d is continuous.*

Remark 2.8. *After this work was completed the authors became aware of the article [34]. Using the theory of the Trotter-Kato products for viscosity solutions as in [34] we may justify the reinitialization algorithm by defining an iterative solution of a different Hamiltonian which converges to the solution of* (2.22),(2.1b)*, assuming additionally that the initial data is bounded. Namely, starting with bounded initial data u_0 we solve*

$$u_t = H_1(x, t, \nabla u), \qquad \textit{in } \mathbf{R}^n \times (0, k_1\Delta t], \tag{2.23}$$

using $u(\cdot, k_1\Delta t)$ as new initial data we solve the corrector equation

$$\tilde{u}_t = H_2(\tilde{u}, \nabla\tilde{u}), \quad (\textit{or alternativelly } \tilde{u}_t = H_2(u(\cdot, k_1\Delta t), \nabla\tilde{u})) \qquad \textit{in } \mathbf{R}^n \times (0, k_2\Delta t], \tag{2.24}$$

using $\tilde{u}(\cdot, k_2\Delta t)$ as new initial data we solve again (2.23) *and we iterate this argument. We rescale the time variables $t = \bar{t}/(1+\theta)$, $\bar{t} \in (0, \varepsilon]$ and $t = \tilde{t}\theta/(1+\theta)$, $\tilde{t} \in (0, \varepsilon]$, in equations* (2.23) *and* (2.24) *respectively. Then, using [34, Theorem 4.1] and [34, Remark 4.3] it is not difficult to see that the iterative solution described above and solved in the intervals $((i-1)\varepsilon, i\varepsilon]$, $i = 1, 2, ..., N_\varepsilon = \lceil \frac{T}{\varepsilon} \rceil$, which form the partition $\{0, \varepsilon, ..., N_\varepsilon \varepsilon = T\}$ of $(0, T]$, converges as $\varepsilon \to 0$ to the solution of* (2.22),(2.1b)*, locally uniformly. In other words, in theorem 2.5 we give a different splitting than the one given in [34] and we show the convergence to equation* (2.22) *using the theory of homogenization instead of the theory of difference schemes.*

2.2 Theorems from the literature

In this subsection we will present a comparison principle for general equations of the form

$$u_t = F(x, t, u, \nabla u) \quad \text{ in } \mathbf{R}^n \times (0, T). \tag{2.25}$$

Let us introduce a notion of viscosity solutions. For this purpose, we first define semicontinuous envelopes of functions. Let $K \subset \mathbf{R}^n$. For a function $f : K \to \mathbf{R}$ we denote the upper and lower semicontinuous envelopes by f^* and $f_* : \overline{K} \to \mathbf{R} \cup \{\pm\infty\}$ respectively, which are as follows :

$$f^*(z) := \limsup_{y \to z} f(y) = \lim_{\delta \to 0} \sup\{f(y) \mid y \in B_\delta(z) \cap K\},$$

$$f_*(z) := \liminf_{y \to z} f(y) = \lim_{\delta \to 0} \inf\{f(y) \mid y \in B_\delta(z) \cap K\}.$$

Definition 2.9 (Viscosity Solution)**.** *We say that $u : \mathbf{R}^n \times [0, T) \to \mathbf{R}$ is a* viscosity subsolution *(resp. a* supersolution*) of* (2.25) *if $u^* < +\infty$(resp. $u_* > -\infty$) and if*

$$\phi_t \leq F^*(x_0, t_0, \phi, \nabla\phi) \quad (\textit{resp. } \phi_t \geq F_*(x_0, t_0, \phi, \nabla\phi)) \qquad \textit{at } P_0 = (x_0, t_0)$$

whenever

$$\begin{cases} u^* \leq \phi & \textit{on } B_{r_0}(P_0) \\ u^* = \phi & \textit{at } P_0 \end{cases} \qquad \left(\textit{resp. } \begin{cases} u_* \geq \phi & \textit{on } B_{r_0}(P_0) \\ u_* = \phi & \textit{at } P_0 \end{cases} \right) \tag{2.26}$$

for $\phi \in C^1(\mathbf{R}^n \times (0, T))$, $P_0 \in \mathbf{R}^n \times (0, T)$ and $r > 0$ such that $B_r(P_0) \subset \mathbf{R}^n \times (0, T)$.

Since we already use the notation $D^{\pm}$, we are going to use the symbol $\mathcal{J}^{\pm}$ for the *subdifferential* respectively for the *superdifferential* of a function. More precisely for a function $u : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ we define a *superdifferential* $\mathcal{J}^+u(z,s)$ of u at $(z,s) \in \mathbf{R}^n \times (0,T)$ by

$$\mathcal{J}^+u(z,s) := \left\{ (p,\tau) \in \mathbf{R}^n \times \mathbf{R} \;\middle|\; \begin{array}{c} \exists \phi \in C^1(\mathbf{R}^n \times (0,T)) \text{ such that} \\ (p,\tau) = (\nabla\phi, \partial_t\phi)(z,s) \text{ and} \\ \max\limits_{\mathbf{R}^n \times (0,T)} (u-\phi) = (u-\phi)(z,s) \end{array} \right\}. \tag{2.27}$$

A *subdifferential* $\mathcal{J}^-u(z,s)$ is defined by replacing “max" by “min" in (2.27). Equivalently, we say that a function $u : \mathbf{R}^n \times (0,T) \to \mathbf{R}$ is a *viscosity subsolution* (resp. *viscosity supersolution*) of (2.25) if

$$\tau \leq F^*(z,s,u^*(z,s),p) \text{ (resp. } \tau \geq F_*(z,s,u_*(z,s),p) \text{)}$$

for all $(z,s) \in \mathbf{R}^n \times (0,T)$ and $(p,\tau) \in \mathcal{J}^+u^*(z,s)$ (resp. $(p,\tau) \in \mathcal{J}^-u_*(z,s)$).

In order to guarantee the well-posedness of the problem (2.25) and (2.1b), the following assumptions are usually imposed on the function $F : \mathbf{R}^n \times [0,T] \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$.

(F1) $F \in C(\mathbf{R}^n \times [0,T] \times \mathbf{R} \times \mathbf{R}^n)$,

(F2) There is an $a_0 \in \mathbf{R}$ such that $r \mapsto F(x,t,r,p) - a_0 r$ is non-increasing on $\mathbf{R}$ for all $(x,t,p) \in \mathbf{R}^n \times [0,T] \times \mathbf{R}^n$,

(F3) For $R \geq 0$ there is a modulus ω_R such that

$$|F(x,t,r,p) - F(x,t,r,q)| \leq \omega_R(|p-q|)$$

for all $(x,t,r,p,q) \in \mathbf{R}^n \times [0,T] \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$, with $|p|, |q| \leq R$,

(F4) There is a modulus ω such that

$$|F(x,t,r,p) - F(y,t,r,p)| \leq \omega(|x-y|(1+|p|))$$

for all $(x,y,t,r,p) \in \mathbf{R}^n \times \mathbf{R}^n \times [0,T] \times \mathbf{R} \times \mathbf{R}^n$.

For the convenience of the reader we will state the comparison principle and sketch its proof for the problem (2.25) and (2.1b) in Appendix A. For a detailed proof, see [20, Theorem 4.1].

Proposition 2.10 (Comparison pinciple)**.** *Assume that F satisfies* (F1)-(F4)*. Let u, v be a viscosity subsolution and supersolution respectively of* (2.25) *and assume that they satisfy*

(A1) $u^*(x,0) \leq v_*(x,0)$ for all $x \in \mathbf{R}^n$,

(A2) there is a constant $K > 0$ such that we have on $\mathbf{R}^n \times (0,T)$

$$u(x,t) \leq K(1+|x|), \quad v(x,t) \geq -K(1+|x|),$$

(A3) there is a constant $\tilde{K} > 0$ such that for $x, y \in \mathbf{R}^n$ we have

$$u^*(x,0) - v_*(y,0) \leq \tilde{K}|x-y|.$$

Then

$$u^* \leq v_* \ \text{ in } \mathbf{R}^n \times [0,T).$$

Combining Proposition 2.10 with Perron's method we get the following theorem.

Theorem 2.11 (Existence/Uniqueness). *Assume* (2.16). *Then for all $\theta > 0$, there exists a unique solution $u = u^\theta \in C(\mathbf{R}^n \times [0,T))$ of the problem* (2.6) *and* (2.1b) *with*

$$u^{low} \leq u \leq u^{up} \quad \text{in } \mathbf{R}^n \times [0,T),$$

where $u^{up}(x,t) = u_0(x) + Kt$, $u^{low}(x,t) = u_0(x) - Kt$, for some $K > 0$, are a viscosity supersolution and subsolution respectively of the same problem.

Démonstration. We will only show that $u_0 \pm Kt$ are a subsolution and a supersolution of (2.6) for some $K > 0$, large enough depending on θ, since for the rest of the proof we can use a Perron's argument, see for example [28]. We first suppose that u_0 is smooth. Then by the Lipschitz continuity of u_0 we have $|\nabla u_0| \leq L_0$. By assumptions (H2) and (H4), there is a constant $C > 0$ such that $|H_1(x,t,\nabla u_0)| \leq CL_0$. Also, since h is continuous and β is bounded by (B), there is a constant $M > 0$ such that

$$|H_2(u_0, \nabla u_0)| = \theta|\beta(u_0)h(\nabla u_0)| \leq \theta \max_{|p|\leq L_0} |h(p)|M.$$

Finally, if we choose $K > 0$ such that $K \geq CL_0 + \theta \max_{|p|\leq L_0} |h(p)|M$, we get the desired result. For the case where u_0 is not smooth, we use the same argument for elements of the super- and subdifferential of u_0. □

In order to get a more precise estimate for the Lipschitz constant of solutions considered in this paper, we will use instead of (H3) the following :

(H3-s) There is a function $D \in C([0,T])$ such that

$$|H(x,t,p) - H(y,t,p)| \leq D(t)|x-y|$$

for all $x, y \in \mathbf{R}^n$, $t \in [0,T]$ and $p \in \mathbf{R}^n$ with $|p| = 1$.

Note that the assumption (H3-s) implies the assumption (H3). We define

$$L(t) := \max\{L_0, 1\}e^{\int_0^t D(s)\,ds}.$$

The following proposition is proved in Appendix A.

Proposition 2.12 (Lipschitz continuity of solutions). *Under the assumptions of Proposition 2.10, with H_1 satisfying assumption* (H3-s) *instead of* (H3), *the solution u of the problem* (2.6) *and* (2.1b) *satisfies*

$$|u(x,t) - u(y,t)| \leq L(t)|x-y| \quad \text{for all } x, y \in \mathbf{R}^n \text{ and } t \in [0,T).$$

Remark 2.13. *The Lipschitz continuity of the solution of* (2.1) *will be used in the next section to show that the solution u^θ of* (2.6) *gives the same zero level set as* (2.1) *and that there exist barrier functions of u^θ independent of θ. There, the Lipschitz constant is allowed to depend on the terminal time T. It is well-known that, if H_1 is* coercive, *i.e.,*

$$H_1(x,t,p) \to \infty \quad \text{as } |p| \to \infty \text{ uniformly in } (x,t),$$

then the solution is Lipschitz continuous and its Lipschitz constant does not depend on T. See, e.g., [3]. Since such independence of T is not needed for our study, we do not require H_1 to be coercive in this paper.

One important property of geometric equations (2.1a) is the invariance under the change of dependent variables. This invariance property as well as the comparison principle play a crucial role for the proof of uniqueness of evolutions.

Theorem 2.14 (Invariance)**.** *Let $\zeta : \mathbf{R} \to \mathbf{R}$ be a nondecreasing and upper semicontinuous (resp. lower semicontinuous) function. If w is a viscosity subsolution (resp. supersolution) of* (2.1a)*, then $\zeta \circ w$ is a viscosity subsolution (resp. supersolution) of* (2.1a)*.*

See [21, Theorem 4.2.1] for the proof.

Remark 2.15. *As a simple consequence of the invariance property, we see that, when w is a solution of* (2.1a)*, the characteristic function on D_t^+ (see* (2.3)*) defined as*

$$\chi_{D_t^+}(x) = \begin{cases} 1 & \text{if } x \in D_t^+, \\ 0 & \text{if } x \notin D_t^+ \end{cases}$$

is a supersolution of (2.1a) *since it is written as $\chi_{D_t^+}(x) = \chi_{(0,\infty)}(x) \circ w$. Similarly, $\chi_{D_t^+ \cup \Gamma_t}(x)$ is a subsolution of* (2.1a)*.*

It is known that the evolution of the interface $\{\Gamma_t\}_{t \in (0,T)}$ associated with (2.1a) is independent of a choice of the initial data u_0. In other words, if the zero levels of initial data are the same, then those of the solutions are also the same. See [21, Section 4.2.3 and 4.2.4] for the detailed statement and its proof.

3 Main tools

3.1 Preservation of the zero level set

We believe that the preservation of the zero level set is by itself a useful result. For this reason we present it in a more general framework than the one we are going to apply it for the proof of our main results.

We study a general equation of the form

$$u_t = H_1(x, t, \nabla u) + \beta(u) G(x, \nabla u) \quad \text{in } \mathbf{R}^n \times (0, T). \tag{3.1}$$

Here the second term on the right-hand side is generalized so that it depends on x-variable. (Since we do not assume the particular properties (2.9) and (2.10), in order to distinguish this generalized equation from (2.6), we use a new notation $G(x, p)$.) The function H_1 satisfies (H1)-(H4). For the function β we assume that (B) is true, and for $G : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ we assume

(G) G satisfies (F3), (F4) and is bounded from above in $\mathbf{R}^n \times \mathbf{R}^n$.

Under these assumptions the comparison principle holds for solutions of (3.1). Indeed, the continuity assumptions on H_1, G and β imply that the function $F(x, r, p) := H_1(x, t, p) - \beta(r) G(x, p)$ satisfies (F1), (F3), (F4) while (F2) is fulfilled with $\gamma = L_\beta(\sup_{\mathbf{R}^n \times \mathbf{R}^n} G)$.

To guarantee that solutions of (3.1) preserve the original zero level set, two kinds of sufficient conditions on G are made in our theorem. One is boundedness of G from below, which, unfortunately, excludes the typical case $G(x, p) = 1 - |p|$. The other condition needs only local boundedness of G from below near $p = 0$ but requires solutions of (2.1) to be Lipschitz continuous, which is not true in general if the initial data u_0 is just uniformly continuous.

In the first author's dissertation [25], Theorem 3.1 (i) is established but (ii) is not given. Here we give the proof of both (i) and (ii) not only for the reader's convenience but also in order to show connection with the proof of (ii).

Theorem 3.1 (Preservation of the zero level set)**.** *Let w and u be, respectively, the viscosity solution of* (2.1) *and* (3.1), (2.1b) *with a uniformly continuous u_0. Assume either* (i) *or* (ii) *below :*

(i) G is bounded from below in $\mathbf{R}^n \times \mathbf{R}^n$.

(ii) G is bounded from below in $\mathbf{R}^n \times \overline{B_\rho(0)}$ for some $\rho \in (0,1]$, and $\mathrm{Lip}_x[w] < \infty$.

Then we have $\Gamma_t = \{u(\cdot,t) = 0\}$ and $D_t^\pm = \{\pm u(\cdot,t) > 0\}$ for all $t \in (0,T)$, where $D^\pm$ and Γ_t are defined in (2.3) *and* (2.4) *respectively.*

Remark 3.2. *Assume that G is independent of x and continuous. Then* (i) *is true if G satisfies* (2.10)*, while* (ii) *is true if G satisfies* (2.9).

Démonstration. Assume that (i) is true.

1. Set $G^\star = \max\{\sup_{\mathbf{R}^n\times\mathbf{R}^n} G,\ 0\}$ and $G_\star := \max\{-\inf_{\mathbf{R}^n\times\mathbf{R}^n} G,\ 0\}$. We define

$$v^\star(x,t) := \begin{cases} e^{L_\beta G^\star t} w(x,t) & \text{if } w(x,t) \geq 0, \\ e^{-L_\beta G_\star t} w(x,t) & \text{if } w(x,t) < 0 \end{cases}$$

and

$$v_\star(x,t) := \begin{cases} e^{-L_\beta G_\star t} w(x,t) & \text{if } w(x,t) \geq 0, \\ e^{L_\beta G^\star t} w(x,t) & \text{if } w(x,t) < 0 \end{cases}$$

for $(x,t) \in \mathbf{R}^n \times [0,T)$, where L_β is the Lipschitz constant of β appearing in (B). We claim that $v^\star$ and $v_\star$ are, respectively, a viscosity supersolution and subsolution of (3.1).

2. We shall show that $v^\star$ is a supersolution. If w is smooth and $w(x,t) > 0$, we compute

$$\begin{aligned} v_t^\star - H_1(x,t,\nabla v^\star) &= L_\beta G^\star v^\star + e^{L_\beta G^\star t} w_t - H_1(x,t,e^{L_\beta G^\star t}\nabla w) \\ &= L_\beta G^\star v^\star + e^{L_\beta G^\star t}\{w_t - H_1(x,t,\nabla w)\} \\ &\geq L_\beta G^\star v^\star + 0 \\ &\geq \beta(v^\star) G(x,\nabla v^\star), \end{aligned}$$

which implies that $v^\star$ is a supersolution of (3.1). In the general case where w is not necessarily smooth, taking an element of the subdifferential of w, we see that $v^\star$ is a viscosity supersolution of (3.1). Similar arguments apply to the case when $w(x,t) < 0$, so that $v^\star$ is a supersolution in $\{w > 0\} \cup \{w < 0\}$. It remains to prove that $v^\star$ is a supersolution of (3.1) on $\{w = 0\}$.

Let $(z,s) \in \mathbf{R}^n \times (0,T)$ be a point such that $w(z,s) = 0$, and take $(p,\tau) \in \mathcal{J}^- v^\star(z,s)$. Our goal is to derive

$$\tau \geq H_1(z,s,p)$$

since $\beta(v^\star(z,s)) = 0$. To do this, we consider a characteristic function $g(x,t) = \chi_{D_t}(x)$. We have $v^\star(z,s) = g(z,s) = 0$ and $v^\star \leq g$ near (z,s), and thus $(p,\tau) \in \mathcal{J}^- g(z,s)$. Since g is a supersolution of (2.1a) by Remark 2.15, we have $\tau \geq H_1(z,s,p)$, which is the desired inequality. Summarizing the above arguments, we conclude that $v^\star$ is a supersolution of (3.1). In the same manner we are able to prove that $v_\star$ is a subsolution of (3.1).

3. Since $v^\star(x,0) = v_\star(x,0) = u_0(x)$ for all $x \in \mathbf{R}^n$, the comparison principle (Proposition 2.10) yields

$$v_\star(x,t) \leq u(x,t) \leq v^\star(x,t) \text{ for all } (x,t) \in \mathbf{R}^n \times (0,T).$$

In particular, we have

$$\{v_\star(\cdot,t) > 0\} \subset \{u(\cdot,t) > 0\} \subset \{v^\star(\cdot,t) > 0\}.$$

Since $\{v_\star(\cdot,t)>0\}=\{v^\star(\cdot,t)>0\}=D_t^+$ by the definition of $v_\star$ and $v^\star$, we conclude that $D_t^+=\{u(\cdot,t)>0\}$. Similarly, we obtain $D_t^-=\{u(\cdot,t)<0\}$, and hence $\Gamma_t=\{u(\cdot,t)=0\}$.

Assume that (ii) is true.

1. Set $G_{\star\rho}:=\max\{-\inf_{\mathbf{R}^n\times\overline{B_\rho(0)}}G,\ 0\}$ and $m:=\max\{\mathrm{Lip}_x[w],\ 1\}$. Instead of $v_\star$ and $v^\star$ defined in Step 1 of (i), we consider the functions

$$\tilde{v}^\star(x,t):=\begin{cases} e^{L_\beta G^\star t}w(x,t) & \text{if } w(x,t)\geq 0,\\ (\rho/m)e^{-L_\beta G_{\star\rho}t}w(x,t) & \text{if } w(x,t)<0\end{cases}$$

and

$$\tilde{v}_\star(x,t):=\begin{cases} (\rho/m)e^{-L_\beta G_{\star\rho}t}w(x,t) & \text{if } w(x,t)\geq 0,\\ e^{L_\beta G^\star t}w(x,t) & \text{if } w(x,t)<0.\end{cases}$$

Then $\tilde{v}^\star$ and $\tilde{v}_\star$ are a viscosity supersolution and subsolution of (3.1) respectively.

2. We shall prove that $\tilde{v}_\star$ is a subsolution in $\{w>0\}$. If w is smooth, we have

$$|\nabla\tilde{v}_\star|=(\rho/m)e^{-L_\beta G_{\star\rho}t}|\nabla w|\leq\rho,$$

which implies that $G(x,\nabla\tilde{v}_\star)\geq -G_{\star\rho}$. Similarly to Step 2 of (i), we observe

$$\begin{aligned}(\tilde{v}_\star)_t-H_1(x,t,\nabla\tilde{v}_\star)&=-L_\beta G_{\star\rho}\tilde{v}_\star+(\rho/m)e^{-L_\beta G_{\star\rho}t}\{w_t-H_1(x,t,\nabla w)\}\\ &\leq -L_\beta G_{\star\rho}\tilde{v}_\star+0\\ &\leq\beta(\tilde{v}_\star)G(x,\nabla\tilde{v}_\star),\end{aligned}$$

i.e., $\tilde{v}_\star$ is a subsolution. The rest of the proof runs as before. □

As an immediate consequence of Theorem 3.1, it follows that the evolution which is given as the zero level set of the solution of the non-geometric equation (3.1) does not depend on the choice of its initial data.

3.2 Barrier functions

Throughout this subsection we will assume (2.16). Thanks to Theorem 3.1, for a general h satisfying (2.8) and either one of the assumptions (2.9) or (2.10), the solution u^θ of (2.6) and (2.1b) gives the same zero level set as w, i.e., we have $\Gamma_t=\{u^\theta=0\}$ and $D_t^\pm=\{\pm u^\theta>0\}$ for all $t\in(0,T)$. In order to study the behaviour of u^θ as $\theta\to\infty$ and a relation between Γ_t and the zero level set of the limit of u^θ, we will construct barrier functions independent of θ. More precisely we construct an upper barrier $f^\star$ and a lower barrier $f_\star$ such that

$$\begin{aligned}f_\star\leq u^\theta\leq f^\star,\ \Gamma_t&=\{f^\star=0\}=\{f_\star=0\},\\ D^+&=\{f^\star>0\}=\{f_\star>0\},\\ D^-&=\{f^\star<0\}=\{f_\star<0\}.\end{aligned}$$

In this subsection we often use the fact that, if u is a supersolution (resp. subsolution) of (2.6) in D^+, and if $\Gamma=\{u=0\}$ and $D^\pm=\{\pm u>0\}$, then u_+ is a supersolution (resp. subsolution) of (2.6) in $\mathbf{R}^n\times(0,T)$. This follows from Remark 2.15. Indeed, if $(p,\tau)\in\mathcal{J}^-u_+(z,s)$ and $u_+(z,s)=0$, then we have $(p,\tau)\in\mathcal{J}^-\chi_{D^+}(z,s)$ and this yields the desired viscosity inequality since the characteristic function is a supersolution of (2.1a) by Remark 2.15. The proof for a subsolution is similar.

We first show that the solutions u^θ are monotone with respect to θ when h is non-negative. This gives a lower barrier in D^+ and an upper barrier in D^- in the case of (2.10).

Proposition 3.3 (Monotonicity)**.** *Assume that* $h \geq 0$. *Let* $0 < \theta_1 < \theta_2$ *and* u^{θ_1} *and* u^{θ_2} *be, respectively, the viscosity solution of* (2.6) *with* $\theta = \theta_1$ *and* θ_2. *Then*

$$u^{\theta_1}(x,t) \leq u^{\theta_2}(x,t) \qquad \text{for all } x \in D_t^+, \tag{3.2}$$

$$u^{\theta_1}(x,t) \geq u^{\theta_2}(x,t) \qquad \text{for all } x \in D_t^-. \tag{3.3}$$

Démonstration. In D^+ we observe

$$\begin{aligned} u_t^{\theta_1} &= H_1(x,t,\nabla u^{\theta_1}) + \theta_1 \beta(u^{\theta_1}) h(\nabla u^{\theta_1}) \\ &\leq H_1(x,t,\nabla u^{\theta_1}) + \theta_2 \beta(u^{\theta_1}) h(\nabla u^{\theta_1}) \end{aligned}$$

since h is nonnegative. This implies that u^{θ_1} is a subsolution of (2.6) with $\theta = \theta_2$ in D^+. Applying the comparison principle to a subsolution $(u^{\theta_1})_+$ and a supersolution $(u^{\theta_2})_+$ of (2.6) with $\theta = \theta_2$, we conclude $u^{\theta_1} \leq u^{\theta_2}$ in D^+. By the same argument we see that $u^{\theta_2} \leq u^{\theta_1}$ in D^-. ☐

We show that solutions of (2.1a) with small Lipschitz constants give rise to lower barrier functions in D^+ and upper barrier functions in D^-.

Proposition 3.4. *Assume that* $h(p) \geq 0$ *if* $|p| \leq 1$. *Let* w *be the solution of* (2.1). *Then the viscosity solution* u^θ *of* (2.6) *and* (2.1b) *satisfies*

$$u^\theta(x,t) \geq \varepsilon w(x,t) \qquad \text{for all } x \in D_t^+, \tag{3.4}$$

$$u^\theta(x,t) \leq \varepsilon w(x,t) \qquad \text{for all } x \in D_t^-, \tag{3.5}$$

where $\varepsilon := \min\{1/\mathrm{Lip}_x[w],\ 1\}$.

Démonstration. Set $\tilde{w} := \varepsilon w$. Since $|\nabla \tilde{w}| = \varepsilon |\nabla w| \leq 1$, by the assumption of h we observe

$$\tilde{w}_t = H_1(x,t,\nabla \tilde{w}) \leq H_1(x,t,\nabla \tilde{w}) + \theta \beta(\tilde{w}) h(\nabla \tilde{w})$$

if $\tilde{w} > 0$. In other words, $\tilde{w}$ is a subsolution of (2.6) in $\{\tilde{w} > 0\}$. Applying the comparison principle to a subsolution $(\tilde{w})_+$ and a supersolution $(u^\theta)_+$ of (2.6), we obtain (3.4). The estimate (3.5) is shown in a similar way. ☐

It remains to construct an upper barrier in D^+ and a lower barrier in D^-. In both the cases (2.9) and (2.10), the solutions u^θ are dominated by the signed distance function d with large coefficient. In the proof of Proposition 3.5 below, we use the fact that d is a viscosity supersolution of

$$d_t = H_1(x - d\nabla d, t, \nabla d) \quad \text{in } \{d > 0\}. \tag{3.6}$$

This assertion is more or less known (see, e.g., [16, Proof of Theorem 2.2, Step 1–3]), but we give its proof in Remark 3.6 for the reader's convenience.

Proposition 3.5. *Assume that* $h(p) \leq 0$ *if* $|p| \geq 1$. *Then the viscosity solution* u^θ *of* (2.6) *and* (2.1b) *satisfies*

$$u^\theta(x,t) \leq l e^{L_1 t} d(x,t) \qquad \text{for all } x \in D_t^+, \tag{3.7}$$

$$u^\theta(x,t) \geq l e^{L_1 t} d(x,t) \qquad \text{for all } x \in D_t^-, \tag{3.8}$$

where $l := \max\{L_0,\ 1\}$ *and* L_1 *is the constant in* (H3).

Démonstration. Define $\tilde{d}(x,t) := le^{L_1 t}d(x,t)$. If d is smooth, then

$$\begin{aligned}\tilde{d}_t - H_1(x,t,\nabla\tilde{d}) &= lL_1e^{L_1t}d + le^{L_1t}d_t - H_1(x,t,le^{L_1t}\nabla d)\\ &= le^{L_1t}\{L_1d + d_t - H_1(x,t,\nabla d)\}.\end{aligned}$$

We next apply the fact that d is a supersolution of (3.6) to estimate

$$\begin{aligned}\tilde{d}_t - H_1(x,t,\nabla\tilde{d}) &\geq le^{L_1t}\{L_1d + H_1(x-d\nabla d,t,\nabla d) - H_1(x,t,\nabla d)\}\\ &\geq le^{L_1t}\{L_1d - L_1|d\nabla d||\nabla d|\}\end{aligned}$$

if $d>0$. Noting that $|\nabla d| = 1$, we have

$$\tilde{d}_t - H_1(x,t,\nabla\tilde{d}) \geq le^{L_1t}\{L_1d - L_1d\} = 0.$$

Since $|\nabla\tilde{d}| = le^{L_1t}|\nabla d| \geq 1$, we now have $h(\nabla\tilde{d}) \leq 0$ by assumption. This implies that $\tilde{d}$ is a supersolution of (2.6) in $\{d>0\}$. Even if d is not smooth, the same arguments above work in the viscosity sense.

Finally, since $u_0 \leq ld_+(\cdot,0)$ in $\mathbf{R}^n$, applying the comparison principle to a subsolution u^θ and a supersolution $(\tilde{d})_+$ of (2.6), we conclude (3.7). The proof of (3.8) is similar. ☐

Remark 3.6. *We shall explain why d is a supersolution of* (3.6)*. We first note that d is lower semicontinuous in D^+ (Theorem 5.4* (1)*). Let $(x_0,t_0) \in \mathbf{R}^n \times (0,T)$ be a point satisfying $d(x_0,t_0) > 0$ and take any $(p,\tau) \in \mathcal{J}^-d(x_0,t_0)$. We choose a smooth function $\phi \in C^1$ such that $(p,\tau) = (\nabla\phi,\phi_t)(x_0,t_0)$ and*

$$\min_{\mathbf{R}^n\times(0,T)}(d-\phi) = (d-\phi)(x_0,t_0) = 0.$$

Set $d_0 := d(x_0,t_0)$. Since $p \in \mathcal{J}^-(d|_{t=t_0})(x_0)$, it follows that the closest point of Γ_{t_0} to x_0 is unique and that this point is given by $y_0 := x_0 - d_0p \in \Gamma_{t_0}$; for the proof, refer to [2, Proposition II.2.14] or [7, Corollary 3.4.5 (i), (ii)]. We also remark that $|p| = 1$.

Define $\psi(x,t) := \phi(x+d_0p,t) - d_0$. We now assert

$$\min_{\mathbf{R}^n\times(0,T)}(d_+-\psi) = (d_+-\psi)(y_0,t_0). \tag{3.9}$$

Since $(d_+-\psi)(y_0,t_0) = 0$ and $d_+ \geq 0$, we only need to show $\{\psi>0\} \subset \{d>0\}$. Take a point $(x,t) \in \mathbf{R}^n\times(0,T)$ such that $\psi(x,t) > 0$. We then have $d(x+d_0p,t) \geq \phi(x+d_0p,t) > d_0$. Using the Lipschitz continuity of d, we compute

$$d(x,t) \geq d(x+d_0p,t) - d_0|p| > d_0 - d_0 = 0.$$

Thus (3.9) *is proved. Let $g(x,t) = \chi_{D_t^+}(x)$. Then the relation* (3.9) *implies that $(p,\tau) \in \mathcal{J}^-g(y_0,t_0)$, where we applied $(\nabla\psi,\psi_t)(y_0,t_0) = (\nabla\phi,\phi_t)(x_0,t_0) = (p,\tau)$. Since the characteristic function g is a supersolution of* (2.1a) *(see Remark 2.15), we have*

$$\tau \geq H_1(y_0,t_0,p) = H_1(x_0-d_0p,t_0,p),$$

which is the inequality we need in order to conclude that d is a supersolution of (3.6)*.*

Remark 3.7. *Another way of proving Proposition 3.5 is using the Lipschitz continuity of solutions of* (2.6) *and* (2.1b) *from Proposition 2.12. Using assumption* (H3) *instead of* (H3-s) *in Proposition 2.12, the Lipschitz estimate for u^θ reads as follows :*

$$|u^\theta(x,t) - u^\theta(y,t)| \leq le^{L_1t}|x-y| \quad \textit{for all } x,y \in \mathbf{R}^n,\ t\in[0,T), \tag{3.10}$$

where $l = \max\{L_0,1\}$ ($L_0 = \mathrm{Lip}[u_0]$). If we take the infimum for all $y \in \Gamma_t$ in (3.10) *we get*

$$-le^{L_1t}\mathrm{dist}(x,\Gamma_t) \leq u^\theta(x,t) \leq le^{L_1t}\mathrm{dist}(x,\Gamma_t) \quad \textit{for all } (x,t)\in\mathbf{R}^n\times[0,T),$$

which implies the relations (3.7) *and* (3.8)*.*

3.3 Comparison principle for eikonal equations

We investigate uniqueness of solutions of the eikonal equation $|\nabla u| = 1$ in a possibly unbounded set. To establish a convergence to the signed distance function, we show in the next section that the limit of the solutions u^θ solves the eikonal equation. Since the distance function is a solution of the eikonal equation, the uniqueness result presented below guarantees that the limit is the distance function.

We consider the eikonal equation

$$|\nabla u| = 1 \quad \text{in } \Omega \tag{3.11}$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{3.12}$$

Here $\Omega \subset \mathbf{R}^n$ is a possibly unbounded open set. We denote by d_Ω the distance function to $\partial\Omega$, i.e., $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$. It is well known that d_Ω is a viscosity solution of (3.11) ; see, e.g., [2, Corollary II.2.16] or [7, Corollary 3.4.5 (i), (ii) or Remark 5.6.1]. In other words, the problem (3.11) with (3.12) admits at least one viscosity solution. Comparison principle (and hence uniqueness) of viscosity solutions of (3.11) and (3.12) is established in [29] when Ω is bounded. If Ω is not bounded, the uniqueness of solutions does not hold in general ; for instance, when $\Omega = (0, \infty) \subset \mathbf{R}$, all of the following functions are solutions :

$$d_\Omega(x) = x, \quad -d_\Omega(x) = -x, \quad u_a(x) = \min\{x,\ a - x\}\ (a > 0).$$

However, even if Ω is not bounded, it turns out that nonnegative solutions of (3.11) and (3.12) are unique and equal to d_Ω.

Lemma 3.8. *Let $u : \Omega \to \mathbf{R}$.*

(1) If u is a viscosity subsolution of (3.11) *and $u^* \le 0$ on $\partial\Omega$, then $u^* \le d_\Omega$ in Ω.*

(2) If u is a viscosity supersolution of (3.11) *and $u \ge 0$ in Ω, then $d_\Omega \le u_*$ in Ω.*

Démonstration. (1) It is known that every subsolution of (3.11) is Lipschitz continuous with Lipschitz constant less than or equal to one, that is, $|u^*(x) - u^*(y)| \le |x - y|$ for all $x, y \in \Omega$. (For the proof see, e.g., [22, Lemma 5.6] or [30, Proof of Proposition 2.1, Step 1].) This yields the inequality $u^* \le d_\Omega$.

(2) We consider a bounded set $\Omega_R := \Omega \cap B_R(0)$ with $R > 0$. Define $d_R(x) := \mathrm{dist}(x, \partial\Omega_R)$. We first note that $u_* \ge 0$ on $\overline{\Omega}$ since $u \ge 0$ in Ω, and that $u_* \ge 0 = d_R$ on $\partial\Omega_R$. Thus, by the comparison principle in bounded sets, we see $d_R \le u_*$ in Ω_R. Finally, sending $R \to \infty$, we conclude $d_\Omega \le u_*$ in Ω. □

4 Convergence results

Throughout this section we assume (2.16). We will first prove Theorem 2.2 (iii), it will then be easier for the reader to understand the proof of Theorem 2.2 (i), (ii).

We introduce a notion of the half-relaxed limits ([12, Section 6]), which are weak limits of a sequence of functions and will be used in the proof of the convergence to the distance function. We define an *upper half-relaxed limit* $\overline{u} = \limsup^*_{\theta\to\infty} u^\theta$ and a *lower half-relaxed* *limit* $\underline{u} = \liminf_{*\theta\to\infty} u^\theta$ as

$$\begin{aligned}\overline{u}(x,t) &:= \limsup_{(y,s,\theta)\to(x,t,\infty)} u^\theta(y,s)\\ &= \lim_{\delta\to 0} \sup\{u^\theta(y,s) \mid |x-y|<\delta,\ |t-s|<\delta,\ \theta>1/\delta\},\\ \underline{u}(x,t) &:= \liminf_{(y,s,\theta)\to(x,t,\infty)} u^\theta(y,s)\\ &= \lim_{\delta\to 0} \inf\{u^\theta(y,s) \mid |x-y|<\delta,\ |t-s|<\delta,\ \theta>1/\delta\}.\end{aligned}$$

Thanks to the existence of barrier functions shown in Section 3.2, we see that, in both the cases (2.9) and (2.10), $-\infty < \overline{u} < \infty$ and $-\infty < \underline{u} < \infty$.

The following proposition is true in the general case where the distance function is not necessarily continuous.

Proposition 4.1 (The zero level set of the relaxed limits)**.** *Assume either* (2.9) *or* (2.10)*. Then*

$$\{\underline{u} > 0\} = D^+, \quad \{\underline{u} = 0\} \subset \Gamma, \quad \{\underline{u} < 0\} \supset D^- \tag{4.1}$$

and

$$\{\overline{u} > 0\} \supset D^+, \quad \{\overline{u} = 0\} \subset \Gamma, \quad \{\overline{u} < 0\} = D^-. \tag{4.2}$$

Démonstration. We only show (4.1) since a proof of (4.2) is similar. Let $v := \liminf_{*\theta\to\infty}(u^\theta)_+$. Then it is easily seen that $v = (\underline{u})_+$. From the estimates (3.4) and (3.7) of u^θ by barrier functions we derive

$$\varepsilon w_+ \le (u^\theta)_+ \le L d_+$$

for some $\varepsilon, L > 0$. Taking the lower half-relaxed limit, we obtain

$$\varepsilon w_+ \le v \le L(d_+)_* \le L d_+.$$

Since $\{w_+ > 0\} = \{d_+ > 0\} = D^+$, the above inequalities imply $\{v > 0\} = D^+$, and hence $\{\underline{u} > 0\} = D^+$. We similarly have

$$-Ld_- \le -(u^\theta)_- \le -\varepsilon w_-.$$

In this case, however, taking the lower half-relaxed limit yields only $\{\underline{u} < 0\} \supset \{w < 0\} = D^-$ because $-d_-$ is upper semicontinuous. The inclusion $\{\underline{u} = 0\} \subset \Gamma$ is now clear. $\square$

4.1 Convergence results for continuous distance function

The following general properties of the relaxed limits will be used to prove the convergence of u^θ :

— Assume that each u^θ is a subsolution (resp. supersolution) of the equation $F_\theta = 0$. If F_θ converges to some F locally uniformly and $\overline{u} < \infty$ (resp. $\underline{u} > -\infty$), then $\overline{u}$ is a subsolution (resp. $\underline{u}$ is a supersolution) of $F = 0$.

— If $\overline{u} = \underline{u} =: u$ and $-\infty < u < \infty$, then u^θ converges to u locally uniformly as $\theta \to \infty$.

See [12, Lemma 6.1, Remark 6.4] for the proofs.

Assume that d is continuous in $\mathbf{R}^n \times (0,T)$, in particular, we can now use the additional upper-semicontinuity property of d_+ and d_-. Proceeding in a similar way as in Proposition 4.1 we can show

$$\Gamma = \{\overline{u} = 0\} = \{\underline{u} = 0\}, \quad D^\pm = \{\pm\overline{u} > 0\} = \{\pm\underline{u} > 0\}. \tag{4.3}$$

Proof of Theorem 2.2 (iii). 1. For $\theta > 0$ we define

$$F_\theta(x,t,r,p,\tau) := \frac{1}{\theta}\{\tau - H_1(x,t,p)\} - \beta(r)h(p).$$

Then u^θ is a viscosity solution of the equation $F_\theta(x,t,u,\nabla u,u_t) = 0$ in $\mathbf{R}^n \times (0,T)$. Since F_θ converges to $-\beta(r)h(p)$ locally uniformly as $\theta \to \infty$, it follows that $\overline{u}$ and $\underline{u}$ are, respectively, a viscosity subsolution and a viscosity supersolution of $-\beta(u)h(\nabla u) = 0$ in $\mathbf{R}^n \times (0,T)$.

Recall that h satisfies (2.9). Since $\beta(\overline{u}) > 0$ in D^+ and $\beta(\overline{u}) < 0$ in D^- by (4.3), we see that $\overline{u}$ is a subsolution of

$$|\nabla u(x,t)| = 1 \quad \text{in } D^+ \tag{4.4}$$

and

$$-|\nabla u(x,t)| = -1 \quad \text{in } D^- \tag{4.5}$$

as a function of (x,t). (Note that these two equations are different in the viscosity sense.) Similarly, $\underline{u}$ is a supersolution of both (4.4) and (4.5). Thus, for each fixed $t_0 \in (0,T)$, $\overline{u}|_{t=t_0}$ and $\underline{u}|_{t=t_0}$ are, respectively, a subsolution and a supersolution of

$$|\nabla u(x)| = 1 \quad \text{in } D^+_{t_0} \tag{4.6}$$

as a function of x. (See Remark 4.2 for the details.) By Lemma 3.8 we obtain

$$\overline{u}|_{t=t_0} \le d(\cdot,t_0) \le \underline{u}|_{t=t_0} \quad \text{in } D^+_{t_0}$$

and hence

$$d = \overline{u} = \underline{u} \quad \text{in } D^+.$$

This implies that $u^\theta \to d$ locally uniformly in D^+. For D^- we notice that if $\overline{u}(\cdot,t_0)$ is a subsolution of $-|\nabla u| = -1$ then $-\overline{u}(\cdot,t_0)$ is a supersolution of $|\nabla u| = 1$, hence a comparison with $-d$ this time gives the desired result. $\square$

Remark 4.2. *We claim that, if $u = u(x,t)$ is a subsolution of* (4.4)*, then $u|_{t=t_0}$ is a subsolution of* (4.6) *for a fixed $t_0 \in (0,T)$. To show this, we take a test function $\phi \in C^1(\mathbf{R}^n)$ such that $\max_{\mathbf{R}^n}(u|_{t=t_0} - \phi) = u(x_0,t_0) - \phi(x_0)$ for $x_0 \in D^+_{t_0}$. We may assume that this is a strict maximum. Next define $\psi_M(x,t) := \phi(x) + M(t-t_0)^2$. We then have*

$$\left(\liminf_{M\to\infty}{}_*\psi_M\right)(x,t) = \begin{cases} \phi(x) & \text{if } t = t_0, \\ \infty & \text{if } t \neq t_0, \end{cases}$$

so that $u - (\liminf_* \psi_M)$ has a strict maximum over $\mathbf{R}^n \times (0,T)$ at (x_0,t_0). By [21, Lemma 2.2.5] there exist sequences $\{M_n\}_{n=1}^\infty \subset (0,\infty)$ and $\{(x_n,t_n)\}_{n=1}^\infty \subset \mathbf{R}^n \times (0,T)$ such that $M_n \to \infty$, $(x_n,t_n) \to (x_0,t_0)$ as $n \to \infty$ and $u - \psi_{M_n}$ has a local maximum at (x_n,t_n). Since u is a subsolution of (4.4)*, we have*

$$1 \ge |\nabla \psi_{M_n}(x_n,t_n)| = |\nabla \phi(x_n)|.$$

Sending $n \to \infty$ implies $|\nabla\phi(x_0)| \le 1$; namely, $u|_{t=t_0}$ is a subsolution of (4.6).

4.2 Convergence results for general distance functions

For the case where the distance d is not necessarily continuous we can only compare the half-relaxed limits with the distance function in certain domains due to the fact that only the inclusions in Proposition 4.1 are true.

Lemma 4.3 (Comparison with the distance)**.** *Assume that either* (2.9) *or* (2.10) *hold. Then*

(1)

$$d \le \underline{u} \quad \text{in } D^+, \tag{4.7}$$

$$\overline{u} \le d \quad \text{in } D^-. \tag{4.8}$$

(2) For every $t \in (0, T)$,

$$\underline{u}(\cdot, t) = 0 \quad \text{on } \partial D_t^+, \tag{4.9}$$

$$\overline{u}(\cdot, t) = 0 \quad \text{on } \partial D_t^-. \tag{4.10}$$

Démonstration. We give proofs of (4.7) and (4.9) since (4.8) and (4.10) can be shown in similar ways.

(1) In the same manner as in the proof of Theorem 2.2 (iii), it follows that $\underline{u}(\cdot, t)$ is a viscosity supersolution of (4.6) in D_t^+. Since $\underline{u}(\cdot, t) > 0$ in D_t^+ by (4.1), the comparison principle (Lemma 3.8 (2)) implies that $d(\cdot, t) \le \underline{u}(\cdot, t)$ in D_t^+.

(2) By (3.7) and (3.8) we have

$$-Ld_- \le u^\theta \le Ld_+, \tag{4.11}$$

where $L > 0$ is a constant. Taking the lower half-relaxed limit at (x, t), we obtain

$$-L(d_-)^*(x, t) \le \underline{u}(x, t) \le L(d_+)_*(x, t) \le Ld_+(x, t). \tag{4.12}$$

Let $x \in \partial D_t^+$. Then the right-hand side of (4.12) is 0 since $x \in \Gamma_t$. We next study the limit of $d_-(y, s) = \mathrm{dist}(y, D_s^+ \cup \Gamma_s)$ on the left-hand side. Since $x \in \partial D_t^+ \subset \overline{D_t^+} \subset \overline{\mathrm{int}(D_t^+ \cup \Gamma_t)}$, it is not an extinction point (Definition 5.3) by Proposition 5.6. Therefore Theorem 5.4 (3) ensures that d_- is continuous at (x, t). This implies that the left-hand side of (4.12) is 0, and hence the conclusion follows. □

Proof of Theorem 2.3. Since (2.10) holds, the monotonicity of u^θ (Proposition 3.3) yields the following representations :

$$\underline{u}(x, t) = \sup_{\theta > 0} u^\theta(x, t) \quad \text{for } x \in D_t^+, \quad \overline{u}(x, t) = \inf_{\theta > 0} u^\theta(x, t) \quad \text{for } x \in D_t^-.$$

These relations and Lemma 4.3 (1) conclude the proof. □

For the equation with h satisfying (2.10), Theorem 2.3 guarantees only the one side inequality between the supremum of u^θ and the signed distance function d. However, as the next example shows, the opposite inequality is not true in general even if the initial datum is smaller than the distance function.

Example 4.1. *Let us consider* (2.1) *for the Hamiltonian of the form* (2.17) *with* $c(x) = (1-|x|)_+ + 1$*. We take the initial datum* u_0 *as* $u_0(x) = (1-|x|)_+$*. The unique viscosity solution* w *of this initial value problem is given as the value function* (2.18)*. In this case the optimal control is the one that leads to a straight trajectory with the maximal speed before it comes to the origin and stays there after that moment. Thus direct calculations yield the following simplified representation of* w *:*

$$w(x,t) = \begin{cases} 1 & \text{if } |x| \le 2(1-e^{-t}), \\ (2-|x|)e^t - 1 & \text{if } 2(1-e^{-t}) \le |x| \le 1, \\ e^{t-|x|+1} - 1 & \text{if } 1 \le |x| \le t+1, \\ 0 & \text{if } t+1 \le |x| \end{cases} \quad \text{for } t \le \log 2,$$

and

$$w(x,t) = \begin{cases} 1 & \text{if } |x| \le t+1-\log 2, \\ e^{t-|x|+1} - 1 & \text{if } t+1-\log 2 \le |x| \le t+1, \\ 0 & \text{if } t+1 \le |x| \end{cases} \quad \text{for } t \ge \log 2.$$

See Figure 1.2. In particular, we have $w(x,t) = 1$ *if* $|x| = t+1-\log 2 \ge 1$*. Also,* $\{w = 0\} = \{|x| \ge t+1\}$ *and the signed distance function* d *to the interface is* $d(x,t) = (t+1-|x|)_+$*. We thus have* $d(x,t) = \log 2$ *if* $|x| = t+1-\log 2$*, and so*

$$d(x,t) = \log 2 < 1 = w(x,t) \quad \text{if } |x| = t+1-\log 2 \ge 1. \tag{4.13}$$

Since the solution w *is non-negative, it is a viscosity subsolution of* (2.6) *with* $h \ge 0$ *for every* $\theta > 0$*. Accordingly,* $w \le u^\theta$ *by the comparison principle. From* (4.13) *it follows that*

$$d(x,t) = \log 2 < 1 \le u^\theta(x,t) \quad \text{if } |x| = t+1-\log 2 \ge 1,$$

which implies that the inequality $d \ge \sup_{\theta>0} u^\theta$ *does not hold on the whole space.*

We also remark that, for $\gamma \in (\log 2, 1)$*, the inequality* $d(x,t) < \gamma w(x,t)$ *holds if* $|x| = t+1-\log 2 \ge 1$ *and that* γw *is a solution of* (2.1a) *with the initial datum* γu_0*. From this we see that* u^θ *can be greater than* d *at some point even if we take an initial datum which is strictly less than* $d(x,0)$ *in* $\{d(\cdot,0) > 0\}$*.*

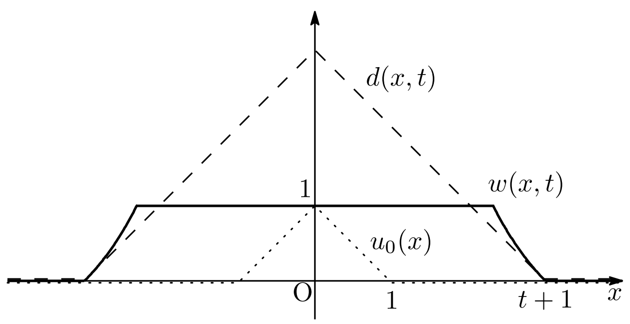

FIGURE 1.2 – The graph of w when $t \ge \log 2$.

In the rest of this subsection we will assume that h satisfies the assumption (2.9). We now introduce several notions of half-relaxed limits. Let $(x,t) \in \mathbf{R}^n \times (0,T)$. We define an *upper* and a *lower half-relaxed limit from below in time* by, respectively,

$$\overline{u}'(x,t) := \limsup_{\substack{(y,s,\theta)\to(x,t,\infty) \\ s \le t}} u^\theta(y,s), \quad \underline{u}'(x,t) := \liminf_{\substack{(y,s,\theta)\to(x,t,\infty) \\ s \le t}} u^\theta(y,s).$$

An *upper* and a *lower half-relaxed limit at a fixed time* are, respectively, given as

$$\overline{u|_t}(x) := \limsup_{(y,\theta)\to(x,\infty)} u^\theta(y,t), \quad \underline{u|_t}(x) := \liminf_{(y,\theta)\to(x,\infty)} u^\theta(y,t).$$

By definitions we have

$$\underline{u}(x,t) \le \underline{u}'(x,t) \le \underline{u|_t}(x) \le \overline{u|_t}(x) \le \overline{u}'(x,t) \le \overline{u}(x,t) \tag{4.14}$$

for all $(x,t) \in \mathbf{R}^n \times (0,T)$.

The next proposition is a crucial step in proving a convergence to the signed distance function in a weak sense.

Proposition 4.4. *Assume either* (2.9) *or* (2.10)*. Then the functions* $\overline{u}'(\cdot,t)$ *and* $\underline{u}'(\cdot,t)$ *are, respectively, a viscosity subsolution of* (4.4) *in* D_t^+ *and a viscosity supersolution of* (4.5) *in* D_t^- *for every* $t \in (0,T)$*.*

Démonstration. Fix $\hat{t} \in (0,T)$ and let us prove that $\overline{u}'(\cdot,\hat{t})$ is a viscosity subsolution of (4.4) in $D_{\hat{t}}^+$.

1. We first introduce an upper half-relaxed limit of u^θ in $\mathbf{R}^n \times (0,\hat{t}]$. For $(x,t) \in \mathbf{R}^n \times (0,\hat{t}]$ we define

$$\overline{v}(x,t) := \limsup_{\substack{(y,s,\theta)\to(x,t,\infty)\\ s\le\hat{t}}} u^\theta(y,s),$$

which is an upper semicontinuous function on $\mathbf{R}^n \times (0,\hat{t}]$. By definition we have

$$\overline{v}(x,t) = \begin{cases} \overline{u}(x,t) & \text{if } t < \hat{t}, \\ \overline{u}'(x,t) & \text{if } t = \hat{t}. \end{cases}$$

2. Take $z \in D_{\hat{t}}^+$ and $\psi \in C^1(\mathbf{R}^n)$ such that $\overline{u}'(\cdot,\hat{t}) - \psi$ attains a maximum at z over $\mathbf{R}^n$. As usual we may assume that this is a strict maximum, and note that, by (4.7) and (4.14),

$$0 < d(z,\hat{t}) \le \underline{u}(z,\hat{t}) \le \overline{u}'(z,\hat{t}). \tag{4.15}$$

We now define $\phi^\theta(x,t) := \psi(x) - \sqrt{\theta}(t-\hat{t})$ and

$$\phi(x,t) := \begin{cases} +\infty & \text{if } t < \hat{t}, \\ \psi(x) & \text{if } t = \hat{t}. \end{cases}$$

Then $\overline{v} - \phi$ attains its strict maximum at $(z,\hat{t})$ over $\mathbf{R}^n \times (0,\hat{t}]$, and $u^\theta - \phi^\theta \to \overline{v} - \phi$ in the sense of the upper half-relaxed limit on $\mathbf{R}^n \times (0,\hat{t}]$. Thus, by [21, Lemma 2.2.5] there exist sequences $\{\theta_j\}_{j=1}^\infty \subset (0,\infty)$ and $\{(x_j,t_j)\}_{j=1}^\infty \subset \mathbf{R}^n \times (0,\hat{t}]$ such that $\theta_j \to \infty$, $(x_j,t_j) \to (z,\hat{t})$ and $(u^{\theta_j} - \phi^{\theta_j})(x_j,t_j) \to (\overline{v} - \phi)(z,\hat{t})$ as $j \to \infty$.

We now claim

$$\overline{u}'(z,\hat{t}) = \lim_{j\to\infty} u^{\theta_j}(x_j,t_j). \tag{4.16}$$

Observe

$$\begin{aligned} u^{\theta_j}(x_j,t_j) &= \{(u^{\theta_j} - \phi^{\theta_j})(x_j,t_j) - (\overline{v} - \phi)(z,\hat{t})\} + \phi^{\theta_j}(x_j,t_j) + (\overline{v} - \phi)(z,\hat{t}) \\ &= \{(u^{\theta_j} - \phi^{\theta_j})(x_j,t_j) - (\overline{v} - \phi)(z,\hat{t})\} + \{\psi(x_j) - \psi(z)\} + \overline{u}'(z,\hat{t}) - \sqrt{\theta}(t_j - \hat{t}) \\ &\ge \{(u^{\theta_j} - \phi^{\theta_j})(x_j,t_j) - (\overline{v} - \phi)(z,\hat{t})\} + \{\psi(x_j) - \psi(z)\} + \overline{u}'(z,\hat{t}). \end{aligned}$$

This implies $\liminf_{j\to\infty} u^{\theta_j}(x_j,t_j) \geq \overline{u}'(z,\hat{t})$. The opposite relation $\limsup_{j\to\infty} u^{\theta_j}(x_j,t_j) \leq \overline{u}'(z,\hat{t})$ follows from the definition of $\overline{u}'$, and therefore (4.16) is proved.

3. Since u^θ is a viscosity solution of (2.6) in $\mathbf{R}^n \times (0,\hat{t})$ and since the viscosity property is extended up to the terminal time $t = \hat{t}$ ([9, Section 7]), we have

$$\phi_t^{\theta_j}(x_j,t_j) \leq H_1(x_j,t_j,\nabla\phi^{\theta_j}(x_j,t_j)) + \theta_j\beta(u^{\theta_j}(x_j,t_j))h(\nabla\phi^{\theta_j}(x_j,t_j)).$$

By the definition of ϕ^θ, this is equivalent to

$$-\sqrt{\theta_j} \leq H_1(x_j,t_j,\nabla\psi(x_j)) + \theta_j\beta(u^{\theta_j}(x_j,t_j))h(\nabla\psi(x_j)).$$

Dividing both the sides by θ_j and sending $\theta_j \to \infty$, we obtain

$$0 \leq \beta(\overline{u}'(z,\hat{t}))h(\nabla\psi(z)),$$

where we have used (4.16). Since $\beta(\overline{u}'(z,\hat{t})) > 0$ by (4.15), using the assumption on h, we conclude that $|\nabla\psi(z)| \leq 1$. □

As a consequence of Proposition 4.4, we obtain

Theorem 4.5. *Assume* (2.9). *Then the following hold.*

(1) $\underline{u}(\cdot,t) = d(\cdot,t)$ *on* $\overline{D_t^+}$ *and* $\overline{u}(\cdot,t) = d(\cdot,t)$ *on* $\overline{D_t^-}$ *for every* $t \in (0,T)$*.*

(2) $\overline{u}' = \underline{u}' = d$ *in* $\mathbf{R}^n \times (0,T)$*, i.e.,*

$$\lim_{\substack{(y,s,\theta)\to(x,t,\infty)\\ s\leq t}} u^\theta(y,s) = d(x,t) \quad \textit{for all } (x,t) \in \mathbf{R}^n \times (0,T).$$

(3) $\overline{u|_t} = \underline{u|_t} = d(\cdot,t)$ *in* $\mathbf{R}^n$ *for every* $t \in (0,T)$*, i.e.,* $u^\theta(\cdot,t)$ *converges to* $d(\cdot,t)$ *locally uniformly in* $\mathbf{R}^n$ *for every* $t \in (0,T)$*.*

Démonstration. 1. We first note that (4.11) yields

$$\overline{u}' = \underline{u}' = 0 \quad \text{on } \Gamma, \tag{4.17}$$

$$\underline{u|_t} = \overline{u|_t} = 0 \quad \text{on } \Gamma. \tag{4.18}$$

Indeed, for $(x,t) \in \Gamma$, taking the upper and lower half-relaxed limit from below in time in (4.11), we see that $\overline{u}'(x,t) = \underline{u}'(x,t) = 0$ since d is continuous from below in time by Theorem 5.4 (2). Similarly, (4.18) follows from the continuity of $d(\cdot,t)$. Thus (2) and (3) were proved on Γ. The equalities in (1) on ∂D_t^+ or ∂D_t^- are consequences of Lemma 4.3 (2).

2. It remains to prove (1)–(3) in D_t^+ and D_t^-. Recall that $\overline{u}'(\cdot,t)$ and $\underline{u}'(\cdot,t)$ are, respectively, a viscosity subsolution of (4.4) in D_t^+ and a viscosity supersolution of (4.5) in D_t^- by Proposition 4.4. Since (4.17) holds, the comparison result (Lemma 3.8 (1)) implies that

$$\overline{u}'(\cdot,t) \leq d(\cdot,t) \quad \text{in } D_t^+, \tag{4.19}$$

$$d(\cdot,t) \leq \underline{u}'(\cdot,t) \quad \text{in } D_t^-. \tag{4.20}$$

Combining (4.7), (4.14) and (4.19), we obtain

$$0 < d(\cdot,t) = \underline{u}(\cdot,t) = \underline{u}'(\cdot,t) = \underline{u|_t} = \overline{u|_t} = \overline{u}'(\cdot,t) \quad \text{in } D_t^+.$$

In the same manner, we see

$$0 > d(\cdot,t) = \underline{u}'(\cdot,t) = \underline{u|_t} = \overline{u|_t} = \overline{u}'(\cdot,t) = \overline{u}(\cdot,t) \quad \text{in } D_t^-.$$

The two relations above conclude the proof. □

This concludes the proof of Theorem 2.2 (i) and (ii).

5 Continuity of distance functions

Throughout this section we study only non-negative distance functions. Namely, we assume $D^- = \emptyset$ so that $d(x,t) = \mathrm{dist}(x,\Gamma_t) \geq 0$ for all $(x,t) \in \mathbf{R}^n \times (0,T)$. Also, we simply write $D_t = D_t^+$ and $D = D^+$. In the general case where d can take negative values, we decompose d as $d = d_+ - d_-$ and apply the following results to d_+ and d_-.

5.1 Finite Propagation

In order to study the continuity of distance functions, we first prepare a property of finite propagation for the Hamilton-Jacobi equation (2.1). For this property, the assumption (H4), the Lipschitz continuity of H_1 in p plays an important role, though we omit the details in this paper.

Let $(x,t) \in \mathbf{R}^n \times (0,T)$ and $r > 0$. We define a cone as

$$\mathcal{C}^r_{(x,t)} := \bigcup_{0<\tau<r} B_{r-\tau}(x) \times \left\{ t + \frac{\tau}{L_2} \right\}.$$

Theorem 5.1 (Local comparison principle). *Let $(x,t) \in \mathbf{R}^n \times (0,T)$, $r > 0$ and set $\mathcal{C} := \mathcal{C}^r_{(x,t)}$. If $u, v \in C(\overline{\mathcal{C}})$ are, respectively, a viscosity sub- and supersolution of* (2.1a) *in $\mathcal{C}$ and $u(\cdot,t) \leq v(\cdot,t)$ in $\overline{B_r(x)}$, then $u \leq v$ in $\overline{\mathcal{C}}$.*

See [2, Theorem III.3.12, (Exercise 3.5)] or in [1, Theorem 5.3] for the proof. As a consequence of Theorem 5.1 we obtain

Proposition 5.2 (Finite propagation). *Let $(x,t) \in \mathbf{R}^n \times (0,T)$ and $r > 0$.*

(1) If $\overline{B_r(x)} \subset D_t$, then $\overline{\mathcal{C}^r_{(x,t)}} \subset D$.

(2) If $\overline{B_r(x)} \subset \Gamma_t$, then $\overline{\mathcal{C}^r_{(x,t)}} \subset \Gamma$.

Démonstration. Let w be the solution of (2.1).

(1) Set $\alpha := \min_{\overline{B_r(x)} \times \{t\}} w > 0$, and define $u(x,t) := \alpha$, which is a constant function satisfying $u(\cdot,t) \leq w(\cdot,t)$ in $\overline{B_r(x)}$. Moreover, u is a solution of (2.1a) by the geometricity of H_1. Therefore Theorem 5.1 implies that $u \leq w$ in $\overline{\mathcal{C}^r_{(x,t)}}$. The positivity of u implies the conclusion.

(2) The proof is similar to (1). We compare w with $u(x,t) := 0$ both from above and from below to conclude that $0 = u \leq w \leq u = 0$ in $\overline{\mathcal{C}^r_{(x,t)}}$. □

5.2 Continuity properties

We first introduce a notion of extinction points.

Definition 5.3 (Extinction point). *Let $x \in \Gamma_t$. We say that x is an* extinction point *if there exist $\varepsilon, \delta > 0$ such that $\overline{B_\varepsilon(x)} \times (t, t+\delta] \subset D$.*

For example the point $0 \in \Gamma_1$ in Example 2.1 is an extinction point. We remark that $x \in \Gamma_t$ is non-extinction point if and only if there exists a sequence $\{(x_j,t_j)\}_{j=1}^\infty$ such that $(x_j,t_j) \to (x,t)$ as $j \to \infty$, $x_j \in \Gamma_{t_j}$ and $t_j > t$ for all j. Define $E_t \subset \mathbf{R}^n$ as the set of all extinction points at time $t \in (0,T)$ and $N_t(x)$ as the set of all the nearest points from $x \in \mathbf{R}^n$ to Γ_t, i.e.,

$$N_t(x) := \{ z \in \Gamma_t \mid d(x,t) = |x-z| \}.$$

Note that we always have $N_t(x) \neq \emptyset$ by (2.5).

Theorem 5.4 (Continuity properties of the distance function). *(1) d is lower semicontinuous in $\mathbf{R}^n \times (0,T)$.*

(2) d is continuous from below in time, i.e.,

$$d(x,t) = \lim_{\substack{(y,s)\to(x,t)\\ s\le t}} d(y,s) \quad \textit{for all } (x,t) \in \mathbf{R}^n \times (0,T).$$

(3) Let $(x,t) \in \mathbf{R}^n \times (0,T)$. Then d is continuous at (x,t) if and only if $N_t(x) \setminus E_t \neq \emptyset$.

Démonstration. (1) The proof can be found in [16, Proposition 2.1].

(2) 1. Suppose by contradiction that d is not continuous at (x,t) from below in time. Since d is lower semicontinuous by (1), we would have a sequence $\{(x_j,t_j)\}_{j=1}^{\infty}$ such that $(x_j,t_j) \to (x,t)$ as $j \to \infty$, $t_j < t$ and

$$\lim_{j\to\infty} d(x_j,t_j) > d(x,t).$$

Set $\alpha := \{\lim_{j\to\infty} d(x_j,t_j) - d(x,t)\}/4 > 0$. Without loss of generality we may assume that $d(x_j,t_j) - d(x,t) \ge 3\alpha$ and $|x_j - x| \le \alpha$ for all $j \ge 1$.

2. Take any $z \in N_t(x)$. We claim that

$$d(y,t_j) \ge \alpha \quad \text{for all } y \in \overline{B_\alpha(z)} \text{ and } j \ge 1. \tag{5.1}$$

Since $d(\cdot,t_j)$ is a Lipschitz continuous function with the Lipschitz constant 1, we calculate

$$\begin{aligned} d(y,t_j) &\ge d(x_j,t_j) - |x_j - y| \\ &\ge \{d(x,t) + 3\alpha\} - (|x_j - x| + |x - z| + |z - y|) \\ &\ge \{d(x,t) + 3\alpha\} - (\alpha + |x - z| + \alpha) \\ &\ge \alpha, \end{aligned}$$

which yields (5.1). By (5.1) we have $\overline{B_\alpha(z)} \times \{t_j\} \subset D$. Thus Proposition 5.2 (1) implies that

$$\overline{\mathcal{C}^{\alpha}_{(z,t_j)}} \subset D. \tag{5.2}$$

Since $t_j \uparrow t$ as $j \to \infty$, we have $(z,t) \in \overline{\mathcal{C}^{\alpha}_{(z,t_j)}}$ for j large, and therefore $z \in D_t$ by (5.2). However, this contradicts the fact that $z \in \Gamma_t$.

(3) 1. We first assume that d is continuous at (x,t). Take any sequence $\{(x_j,t_j)\}_{j=1}^{\infty}$ such that $(x_j,t_j) \to (x,t)$ as $j \to \infty$ and $t_j > t$. By continuity we have $d(x_j,t_j) \to d(x,t)$ as $j \to \infty$. We now take $z_j \in N_{t_j}(x_j)$ for each j. Then $\{z_j\}$ is bounded. Indeed, since $|z_j| \le |x| + |x - x_j| + |x_j - z_j|$ and $|x - x_j| \to 0$, $|x_j - z_j| = d(x_j,t_j) \to d(x,t)$ as $j \to \infty$, we see that $\{z_j\}$ is bounded. From this z_j subsequently converges to some $\bar{z}$ as $j \to \infty$, where we use again the index j. It is easy to see that $\bar{z} \in \Gamma_t$.

Let us show $\bar{z} \in N_t(x) \setminus E_t$. Taking the limit in $d(x_j,t_j) = |x_j - z_j|$, we obtain $d(x,t) = |x - \bar{z}|$, which implies that $\bar{z} \in N_t(x)$. Also, since $z_j \in \Gamma_{t_j}$ and $t_j \downarrow t$ as $j \to \infty$, it follows that $\bar{z}$ is not an extinction point, and hence we conclude that $N_t(x) \setminus E_t \neq \emptyset$.

2. We next assume that d is not continuous at (x,t). By (1) and (2) we have some sequence $\{(x_j,t_j)\}_{j=1}^{\infty}$ such that $(x_j,t_j) \to (x,t)$ as $j \to \infty$, $t_j > t$ and

$$\lim_{j\to\infty} d(x_j,t_j) > d(x,t).$$

We now argue in a similar way to the proof of (2), so that we obtain (5.2) for any $z \in N_t(x)$. Therefore

$$\bigcup_{j=1}^{\infty} \overline{\mathcal{C}^{\alpha}_{(z,t_j)}} \subset D,$$

and it is easily seen that there exist $\varepsilon, \delta > 0$ such that

$$\overline{B_\varepsilon(z)} \times (t, t+\delta] \subset \bigcup_{j=1}^{\infty} \overline{\mathcal{C}^{\alpha}_{(z,t_j)}}.$$

We thus conclude that z is an extinction point, and hence $N_t(x) \setminus E_t = \emptyset$. □

Remark 5.5. *(1) Theorem 5.4* (3) *implies that if every $x \in \Gamma_t$ with $t \in (0,T)$ is a non extinction point then the distance function d is continuous in $\mathbf{R}^n \times (0,T)$ and hence Theorem 2.2* (iii) *holds.*

(2) If d is discontinuous at times $0 < t_1 < t_2 < ... < t_m < T$ (at one or more points in $\mathbf{R}^n$), we can apply Theorem 2.2 in the intervals $(0,t_1), (t_1,t_2), ..., (t_m,T)$. More precisely, under the assumptions of Theorem 2.2 we can show

$$u^\theta \underset{\theta \to +\infty}{\longrightarrow} d \quad \textit{locally uniformly in } \mathbf{R}^n \times ((0,t_1) \cup (t_1,t_2) \cup ... \cup (t_m,T)).$$

Example 5.1 in the next subsection shows that we can construct an evolution $\{\Gamma_t\}_{t\in[0,T)}$ for which the associated distance function has discontinuities for each $t \in \mathbf{Q}$. Therefore, the idea described above cannot be applied.

The next proposition gives a sufficient condition for the non-extinction condition.

Proposition 5.6. *Let $t \in (0,T)$. If $x \in \overline{\mathrm{int}(\Gamma_t)}$, then $x \notin E_t$.*

Démonstration. Let $x \in \overline{\mathrm{int}(\Gamma_t)}$. Then there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathrm{int}(\Gamma_t)$ that converges to x as $j \to \infty$. Set $\varepsilon_j := \mathrm{dist}(x_j, \partial\Gamma_t)$, which converges to 0 as $j \to 0$. Since we have $\overline{B_{\varepsilon_j}(x_j)} \subset \Gamma_t$, Proposition 5.2 (2) implies that $\overline{\mathcal{C}^{\varepsilon_j}_{(x_j,t)}} \subset \Gamma$. In particular $x_j \in \Gamma_{t+(\varepsilon_j/L_2)}$, which is the vertex of the cone, and consequently we see that x is a non-extinction point. □

Remark 5.7. *The converse of the assertion of Proposition 5.6 is not true in general. In fact, it is easy to construct the interface such that $\Gamma_t = \{0\}$ for all $t \in (0,T)$. Any $x \in \Gamma_t$ is a non-extinction point, but* $\mathrm{int}(\Gamma_t) = \emptyset$.

Remark 5.8. *The opposite notion of an extinction point is an emerging point, which is defined as follows : Let $x \in \Gamma_t$. We say that x is an* emerging point *if there exist $\varepsilon, \delta > 0$ such that $\overline{B_\varepsilon(x)} \times [t-\delta, t) \subset D$. However, the property of finite propagation implies that there are no emerging points. Suppose that $x \in \Gamma_t$ is an emerging point, i.e., $\overline{B_\varepsilon(x)} \times [t-\delta, t) \subset D$ for some $\varepsilon, \delta > 0$. Choose $M \geq 1$ large so that $x \in \overline{\mathcal{C}^{\varepsilon}_{(x,t-(\delta/M))}}$. This cone is a subset of D by Proposition 5.2* (1)*. Thus $x \in D$, a contradiction.*

5.3 Some examples

We present an example which shows that the idea presented in Remark 5.5 (2) can not be applied, even if we restrict the evolutions to move inside a bounded domain instead of $\mathbf{R}^n$.

Example 5.1 (A zero level set vanishing for all $t \in \mathbf{Q} \cap (0, T)$)**.** *Let* $\mathbf{Q} \cap (0,T) = \{t_1, t_2, ...\}$.
Case 1. In $\mathbf{R}^n$***.***

Consider disjoint cubes with sides of length at least $2t_n$ *for* $n = 1, 2, ...$ *Then inside every each one of them, we fit a circle* B_{t_n} *of radius* t_n*. The evolution of these circles under the equation*

$$V = -1 \tag{5.3}$$

where V *is the normal velocity (with normal pointing to the exterior of the circles), is given by*

$$\frac{d}{dt} R(t) = -1.$$

Here $R(t)$ *is the radius of the circles. Notice that the evolution of* (5.3) *is the same as the zero level set of the solution* u *of the problem*

$$\begin{cases} u_t = -|\nabla u| & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^n \end{cases}$$

if u_0 *is for example the signed distance function to the circles* B_n*, with positive values in the interior of the circles. For a proof of equivalence of the two evolutions see for example [21, Section 4.2.3 and 4.2.4]. Then* $R(t) = t_n - t$ *and the extinction time of the circles is* $t = t_n$*.*
Case 2. In a bounded domain.

Let Ω *be a bounded open set. For every* $n \in \mathbf{N}$ *we can find points* $x_n \in \Omega$ *and positive numbers* ε_n *such that* $B_{\varepsilon_n}(x_n) \subset \Omega$ *with* $\overline{B_{\varepsilon_n}(x_n)} \cap \overline{B_{\varepsilon_m}(x_m)} = \emptyset$ *for* $n \neq m$*. Then for* $a_n = \varepsilon_n / 2(T + 1/2)$ *we have* $R_n := t_n a_n < \varepsilon_n/2$ *and* $B_{R_n}(x_n) \subset B_{\frac{\varepsilon_n}{2}}(x_n)$*. We then define*

$$c_n(x) = \begin{cases} a_n & \text{in } B_{\varepsilon_n/2}(x_n), \\ 2a_n - \frac{2a_n}{\varepsilon_n}|x - x_n| & \text{in } B_{\varepsilon_n}(x_n) \setminus B_{\varepsilon_n/2}(x_n), \\ 0 & \text{else} \end{cases}$$

and the velocity

$$c(x) = \sup_n c_n(x), \qquad \text{for } x \in \mathbf{R}^n.$$

As in Case 1 we consider the problem

$$\begin{cases} u_t = -c(x)|\nabla u| & \text{in } \mathbf{R}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^n. \end{cases}$$

Here u_0 *is the signed distance function from the set* $\bigcup_{n \in \mathbf{N}} \partial B_{R_n}(x_n)$ *with positive values in each* $B_{R_n}(x_n)$*. The extinction time of* $\partial B_{R_n}(x_n)$ *is as in the first case* $t = t_n$*.*

As we mentioned in the introduction the zero level set may develop an interior even for first order equations, for an example with discontinuous initial data see [4, Proposition 4.4], but since we deal with continuous initial data we present here another example.

Example 5.2 (A zero level set with fattening)**.** *Consider the equation*

$$\begin{cases} u_t = (t-1)|u_x| & \text{in } \mathbf{R} \times (0, \infty), \\ u(x, 0) = \frac{1}{2} - |x| & \text{in } \mathbf{R} \end{cases}$$

then the viscosity solution is

$$u(x,t) = \begin{cases} (t-1)^2/2 - |x| & \text{in } \mathbf{R} \times (0, 1], \\ \min\{(t-1)^2/2 - |x|, 0\} & \text{in } \mathbf{R} \times [1, \infty). \end{cases}$$

Then for $t \geq 1$ *the zero level set is given by* $\overline{B_{(t-1)^2/2}(0)}$*.*

6 Homogenization

We conclude this paper by proving Theorem 2.5. Let us consider

$$u_t = H_1\left(x, \frac{t}{1+\theta}, \nabla u\right) \tag{6.1}$$

and

$$u_t = H_2(u, \nabla u), \tag{6.2}$$

where $\theta = k_2/k_1$ is as in (2.20). By the assumptions on H_1 and H_2, the classical comparison and existence results still hold for the problems (6.1), (2.1b) and (6.2), (2.1b).

To solve the problem (2.21), (2.1b) we use the notion of the iterative solution which was introduced in Remark 2.4. By the comparison and existence results for (6.1) and (6.2), we see that (2.21), (2.1b) admits a unique continuous iterative solution.

6.1 Hamiltonians discontinuous in time

Since the Hamiltonian H_{12} is now discontinuous with respect to time, we have to be careful about the proof of our homogenization result. We do not use the notion of viscosity solutions introduced in Definition 2.9, where the upper- and lower semicontinuous envelopes are used for the equation, because otherwise we could not estimate the difference between $(H_{12})^*$ and $(H_{12})_*$. Thus we first discuss removability of the upper- and lower star of the equation as well as a connection between the iterative solution and the different notions of viscosity solutions of (2.21).

In this section we call u a *viscosity subsolution* (resp. *supersolution*) *with star* if it is a viscosity subsolution (resp. supersolution) in the sense of Definition 2.9. Also, we say that u is a *viscosity subsolution* (resp. *supersolution*) *without star* if it satisfies the viscosity inequality (2.26) with F instead of F^* (resp. F_*). Note that, since $F_* \le F \le F^*$, a viscosity subsolution (resp. supersolution) without star is always a viscosity subsolution (resp. supersolution) with star. Namely, a notion of viscosity solutions without star is stronger than that with star.

Theorem 6.1. *Let u^ε be the iterative solution of* (2.21), (2.1b)*.*

(1) u^ε is a viscosity solution of (2.21), (2.1b) *without star.*

(2) If v is a viscosity solution of (2.21), (2.1b) *with star, then $v = u^\varepsilon$ in $\mathbf{R}^n \times (0,T)$.*

Theorem 6.1 (1) asserts that u^ε is a viscosity solution in $\mathbf{R}^n \times (0,T)$ not only in the sense with star but also in the sense without star. In other words, existence of solutions is established in both the cases. On the other hand, (2) is concerned with uniqueness of solutions since it asserts that any solution should be equal to u^ε. In the sense with star, Perron's method (see Theorem 2.11) gives a viscosity solution u_P of (2.21), (2.1b) which is not necessarily continuous. By (2) we see that $u_P = u^\varepsilon$, and therefore u_P is also a viscosity solution of (2.21), (2.1b) without star and an iterative solution as well.

Démonstration. (1) We apply the fact that the viscosity property is extended up to the terminal time ([9, Section 7]). Since u^ε is a viscosity solution of (6.1) in $\mathbf{R}^n \times (0, k_1\Delta t)$, we see that $u^\varepsilon|_{\mathbf{R}^n \times (0,k_1\Delta t]}$ is a viscosity subsolution of (6.1) in $\mathbf{R}^n \times (0, k_1\Delta t]$. This implies that u^ε is a viscosity subsolution of (2.21) without star on $\mathbf{R}^n \times \{k_1\Delta t\}$. Arguing in the same way on $\mathbf{R}^n \times \{t\}$ with $t = \varepsilon, \varepsilon + k_1\Delta t, 2\varepsilon, \ldots$, we conclude that u^ε is a viscosity subsolution of (2.21) without star. The proof for supersolution is similar.

(2) Since v and u^ε are, respectively, a viscosity subsolution and a supersolution of (6.1) in $\mathbf{R}^n \times (0, k_1\Delta t)$, the comparison principle for (6.1) implies that $v^* \leq u^\varepsilon$ in $\mathbf{R}^n \times (0, k_1\Delta t)$. If we prove $v^* \leq u^\varepsilon$ on $\mathbf{R}^n \times \{k_1\Delta t\}$, we then have $v^* \leq u^\varepsilon$ in $\mathbf{R}^n \times (k_1\Delta t, \varepsilon)$ by the comparison principle for (6.2). Iterating this argument, we finally obtain $v^* \leq u^\varepsilon$ in $\mathbf{R}^n \times (0,T)$. In the same manner, we derive $u^\varepsilon \leq v_*$ in $\mathbf{R}^n \times (0,T)$, and hence $u^\varepsilon = v$ in $\mathbf{R}^n \times (0,T)$.

It remains to prove that $v^*(x, k_1\Delta t) \leq u^\varepsilon(x, k_1\Delta t)$ for $x \in \mathbf{R}^n$. We now use the fact that v^* is left accessible ([9, Section 2, 9]), i.e., there exists a sequence $\{(x_j, t_j)\}_{j=1}^\infty$ such that $t_j < k_1\Delta t$ for all $j \geq 1$, $(x_j, t_j) \to (x, k_1\Delta t)$ and $v^*(x_j, t_j) \to v^*(x, k_1\Delta t)$ as $j \to \infty$. Therefore, taking the limit in $v^*(x_j, t_j) \leq u^\varepsilon(x_j, t_j)$ gives $v^*(x, k_1\Delta t) \leq u^\varepsilon(x, k_1\Delta t)$. □

Remark 6.2. *The same argument in the proof of* (2) *yields the comparison principle for* (2.21). *Namely, if u and v are, respectively, a subsolution and a supersolution of* (2.21) *with star such that $u^*(\cdot, 0) \leq v_*(\cdot, 0)$ in $\mathbf{R}^n$, then $u^* \leq v_*$ in $\mathbf{R}^n \times [0,T)$. Also, similar arguments allow us to prove a local version of the comparison principle. Let $(x,t) \in \mathbf{R}^n \times (0,T)$ and $r > 0$. If u and v are a subsolution and a supersolution of* (2.21) *with star in $B_r(x) \times (t-r, t+r) =: C$, respectively, with $u^* \leq v_*$ on $\partial_P C$, then $u^* \leq v_*$ in C. Here by ∂_P we denote the parabolic boundary, that is, for $\Omega \subset \mathbf{R}^n$ and $a < b$,*

$$\partial_P(\Omega \times (a,b)) := (\partial\Omega \times [a,b)) \cup (\Omega \times \{a\}).$$

Remark 6.3. *See [27, 6] for more results concerning Hamiltonians discontinuous in time.*

6.2 Cell problems

We study an one-dimensional cell problem with discontinuity, whose solution and eigenvalue will be needed in the proof of our homogenization result. Since our problem is periodic in time, the corresponding cell problem is one-dimensional. Consider

$$v'(\tau) + \lambda = H(\tau) \quad \text{in } \mathbf{T}, \tag{6.3}$$

where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the one-dimensional torus, $H \in L^1(\mathbf{T})$ and $\lambda \in \mathbf{R}$. Although we only need to study piecewise continuous H for our homogenization result, we here take it as a L^1-function since the technical aspects of the proof allow us to generalize H without any additional effort. For the special case where H is piecewise continuous, see Remark 6.5. We define

$$H^\#(\tau) := \limsup_{k\downarrow 0}\left(\frac{1}{k}\int_{\tau-k}^{\tau} H(s)ds\right), \quad H_\#(\tau) := \liminf_{k\downarrow 0}\left(\frac{1}{k}\int_{\tau-k}^{\tau} H(s)ds\right).$$

Lemma 6.4 (Solvability of the cell problem)**.** *We set*

$$\lambda := \int_0^1 H(s)ds, \quad v(\tau) := v(0) - \lambda\tau + \int_0^\tau H(s)ds. \tag{6.4}$$

Then v is a viscosity solution of (6.3) *in the following sense : If $\max_{\mathbf{T}}(v-\phi) = (v-\phi)(\tau_0)$ (resp. $\min_{\mathbf{T}}(v-\phi) = (v-\phi)(\tau_0)$) for $\tau_0 \in \mathbf{T}$ and $\phi \in C^1(\mathbf{T})$, then*

$$\phi'(\tau_0) + \lambda \leq H_\#(\tau_0) \quad \textit{(resp. } \phi'(\tau_0) + \lambda \geq H^\#(\tau_0)\textit{)}. \tag{6.5}$$

Démonstration. 1. We first note that v is a periodic function thanks to the choice of λ. Indeed, for all $\tau \in \mathbf{R}$ and $m \in \mathbf{Z}$, we observe

$$\begin{aligned} v(\tau+m) &= v(0) - \lambda(\tau+m) + \int_0^{\tau+m} H(s)ds \\ &= v(0) - \lambda(\tau+m) + \left(\lambda m + \int_0^{\tau} H(s)ds\right) \\ &= v(0) - \lambda\tau + \int_0^{\tau} H(s)ds \\ &= v(\tau). \end{aligned}$$

Thus v is periodic.

2. Take $\tau_0 \in \mathbf{T}$ and $\phi \in C^1(\mathbf{T})$ such that $\max_{\mathbf{T}}(v-\phi) = (v-\phi)(\tau_0)$. For $k>0$ we have

$$\frac{\phi(\tau_0)-\phi(\tau_0-k)}{k} \le \frac{v(\tau_0)-v(\tau_0-k)}{k} = -\lambda + \frac{1}{k}\int_{\tau_0-k}^{\tau_0} H(s)ds.$$

Taking $\liminf_{k\downarrow 0}$ implies the first inequality in (6.5). A similar argument shows that v is a supersolution. □

Remark 6.5. *Let* $0 = \tau_0 < \tau_1 < \cdots < \tau_N = 1$ *be a partition of* $[0,1]$ *and assume that* $H \in L^1(\mathbf{T})$ *is continuous on each* $(\tau_i, \tau_{i+1}]$. *Then we have* $H^\# = H_\# = H$, *and consequently the viscosity inequalities in* (6.5) *become*

$$\phi'(\tau_0) + \lambda \le H(\tau_0) \quad \textit{(resp. } \phi'(\tau_0) + \lambda \ge H(\tau_0)\textit{)}.$$

In other words, v *given by* (6.4) *is a viscosity solution of* (6.3) *without star.*

6.3 Proof of homogenization

We will give a proof of Theorem 2.5. For the proof we employ the perturb test function method by Evans ([18, 19]). The argument is similar to [19, Section 5.2], but we give a full proof in order to show that the method can be extended to discontinuous equations.

Proof of Theorem 2.5. 1. Let u^ε be the iterative solution of (2.21) and (2.1b). We denote by $\overline{u}$ and $\underline{u}$ the upper- and lower half-relaxed limit of u^ε respectively, i.e., $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$ and $\underline{u} = \liminf_{*\varepsilon\to 0} u^\varepsilon$. Since the functions $u_0(x) - Kt$ and $u_0(x) + Kt$ with $K>0$ large are, respectively, a subsolution and a supersolution of (2.21), it follows from comparison that $u_0(x) - Kt \le u^\varepsilon(x,t) \le u_0(x) + Kt$. This implies $-\infty < \underline{u} \le \overline{u} < +\infty$ and $\overline{u}(x,0) = \underline{u}(x,0) = u_0(x)$.

2. Let us show that $\overline{u}$ is a subsolution of (2.22). Let ϕ be a test function for $\overline{u}$ at $(\hat{x},\hat{t})$ from above, i.e,

$$\overline{u} < \phi \quad \text{in } (B_R(\hat{x}) \times (\hat{t}-R, \hat{t}+R)) \setminus \{(\hat{x},\hat{t})\}, \tag{6.6}$$

$$\overline{u}(\hat{x},\hat{t}) = \phi(\hat{x},\hat{t}) \tag{6.7}$$

for some $R>0$ such that $0 < \hat{t} - R < \hat{t} + R < T$. We set

$$H(\tau) := H_{12}(\hat{x}, \hat{t}, \tau, \hat{\phi}, \nabla\hat{\phi}),$$

where $\hat{\phi} = \phi(\hat{x},\hat{t})$ and $\nabla\hat{\phi} = \nabla\phi(\hat{x},\hat{t})$. Then $H \in L^1(\mathbf{T})$ and H is continuous on $(0, k_1\Delta t/\varepsilon]$ and $(k_1\Delta t/\varepsilon, 1]$. Let v and λ be as in (6.4). By Remark 6.5 we see that v is a viscosity solution of (6.3) without star. Noting that $\theta = k_2/k_1$, we observe

$$\begin{aligned}\int_0^1 H_{12}(x,t,\tau,r,p)\,d\tau &= \int_0^{\frac{k_1\Delta t}{\varepsilon}} H_1\left(x, \frac{t}{1+\frac{k_2}{k_1}}, p\right) d\tau + \int_{\frac{k_1\Delta t}{\varepsilon}}^1 H_2(r,p)\,d\tau \\ &= \frac{k_1}{k_1+k_2} H_1\left(x, \frac{t}{1+\frac{k_2}{k_1}}, p\right) + \frac{k_2}{k_1+k_2} H_2(r,p) \\ &= \frac{1}{1+\theta}\left(H_1\left(x, \frac{t}{1+\theta}, p\right) + \theta H_2(r,p)\right) \\ &= \bar{H}(x,t,r,p).\end{aligned}$$

This implies that

$$\lambda = \int_0^1 H(\tau)\,d\tau = \int_0^1 H_{12}(\hat{x},\hat{t},\tau,\hat{\phi},\nabla\hat{\phi})\,d\tau = \bar{H}(\hat{x},\hat{t},\hat{\phi},\nabla\hat{\phi}).$$

Consequently, v solves

$$v'(\tau) + \bar{H}(\hat{x},\hat{t},\hat{\phi},\nabla\hat{\phi}) = H_{12}(\hat{x},\hat{t},\tau,\hat{\phi},\nabla\hat{\phi}) \quad \text{in } \mathbf{T}. \tag{6.8}$$

3. We want to show that

$$\hat{\phi}_t \le \bar{H}(\hat{x},\hat{t},\hat{\phi},\nabla\hat{\phi})$$

with $\hat{\phi}_t = \phi_t(\hat{x},\hat{t})$. Suppose in the contrary that there is $\mu > 0$ such that

$$\hat{\phi}_t \ge \bar{H}(\hat{x},\hat{t},\hat{\phi},\nabla\hat{\phi}) + \mu. \tag{6.9}$$

Let us introduce a perturbed test function. Define

$$\phi^\varepsilon(x,t) = \phi(x,t) + \varepsilon v\left(\frac{t}{\varepsilon}\right).$$

Since v is bounded, we see that ϕ^ε converges to ϕ uniformly as $\varepsilon \to 0$. We will show that ϕ^ε is a supersolution of (2.21) in $B_r(\hat{x}) \times (\hat{t}-r, \hat{t}+r) =: C$, where $r \in (0,R)$ is chosen to be small so that

$$|\phi_t(x,t) - \hat{\phi}_t| \le \frac{\mu}{4}, \tag{6.10}$$

$$\left| H_{12}\left(x,t,\frac{t_0}{\varepsilon},\phi(x,t),\nabla\phi(x,t)\right) - H_{12}\left(\hat{x},\hat{t},\frac{t_0}{\varepsilon},\hat{\phi},\nabla\hat{\phi}\right)\right| \le \frac{\mu}{8} \tag{6.11}$$

for all $(x,t) \in C$. Although $H_{12} = H_{12}(x,t,\tau,r,p)$ is discontinuous in τ, (6.11) is achieved because we fix $\tau = t_0/\varepsilon$. More precisely, (6.11) is satisfied if

$$\begin{aligned}\left|H_1\left(x,t,\nabla\phi(x,t)\right) - H_1(\hat{x},\hat{t},\nabla\hat{\phi})\right| &\le \frac{\mu}{8}, \\ \left|H_2\left(\phi(x,t),\nabla\phi(x,t)\right) - H_2(\hat{\phi},\nabla\hat{\phi})\right| &\le \frac{\mu}{8}\end{aligned}$$

for all $(x,t) \in C$. Allowing a larger error, we are able to replace $\phi(x,t)$ on the left-hand side of (6.11) by $\phi^\varepsilon(x,t)$ with $\varepsilon > 0$ small enough. Namely, we have

$$\left| H_{12}\left(x,t,\frac{t_0}{\varepsilon},\phi^\varepsilon(x,t),\nabla\phi(x,t)\right) - H_{12}\left(\hat{x},\hat{t},\frac{t_0}{\varepsilon},\hat{\phi},\nabla\hat{\phi}\right)\right| \le \frac{\mu}{4}. \tag{6.12}$$

4. Let ψ be a test function for ϕ^ε at $(x_0,t_0) \in C$ from below. Then the function

$$\tau \mapsto v(\tau) - \frac{1}{\varepsilon}\left(\psi(x_0,\varepsilon\tau) - \phi(x_0,\varepsilon\tau)\right)$$

has a local minimum at $\tau_0 := t_0/\varepsilon$. Also, from the smoothness of $\phi^\varepsilon(\cdot, t_0)$, it follows that

$$\nabla\phi^\varepsilon(x_0, t_0) = \nabla\phi(x_0, t_0) = \nabla\psi(x_0, t_0). \tag{6.13}$$

Since v is a viscosity supersolution of (6.8), we have

$$\psi_t(x_0, t_0) - \phi_t(x_0, t_0) + \bar{H}(\hat{x}, \hat{t}, \hat{\phi}, \nabla\hat{\phi}) \geq H_{12}\left(\hat{x}, \hat{t}, \frac{t_0}{\varepsilon}, \hat{\phi}, \nabla\hat{\phi}\right).$$

Let $\phi_0^\varepsilon = \phi^\varepsilon(x_0, t_0)$, $\nabla\phi_0 = \nabla\phi(x_0, t_0)$ and $\nabla\psi_0 = \nabla\psi(x_0, t_0)$. Applying (6.10), (6.12) and (6.9) to the above inequality, we compute

$$\begin{aligned}
\psi_t(x_0, t_0) &\geq \phi_t(x_0, t_0) - \bar{H}(\hat{x}, \hat{t}, \hat{\phi}, \nabla\hat{\phi}) + H_{12}\left(\hat{x}, \hat{t}, \frac{t_0}{\varepsilon}, \hat{\phi}, \nabla\hat{\phi}\right) \\
&\geq \hat{\phi}_t - \frac{\mu}{4} - \bar{H}(\hat{x}, \hat{t}, \hat{\phi}, \nabla\hat{\phi}) + H_{12}\left(x_0, t_0, \frac{t_0}{\varepsilon}, \phi_0^\varepsilon, \nabla\phi_0\right) - \frac{\mu}{4} \\
&\geq H_{12}\left(x_0, t_0, \frac{t_0}{\varepsilon}, \phi_0^\varepsilon, \nabla\psi_0\right) + \frac{\mu}{2}.
\end{aligned} \tag{6.14}$$

For the last inequality we have used (6.13). The above inequality shows that ϕ^ε is a supersolution. Moreover, since $H_{12} = H_{12}(x, t, \tau, r, p)$ is continuous in the r-variable, the estimate (6.14) implies that there is a small $\eta_0 > 0$ such that $\phi^\varepsilon - \eta$ is also a supersolution of (2.21) for every $\eta \in (0, \eta_0]$.

5. Set

$$\delta_0 := -\max_{\partial_P C}(\overline{u} - \phi),$$

which is positive by (6.6). Also, let $\delta := \min\{\delta_0/2,\ \eta_0\}$. We then have

$$\max_{\partial_P C}(u^\varepsilon - \phi^\varepsilon) \leq -\delta,$$

i.e., $u^\varepsilon \leq \phi^\varepsilon - \delta$ on $\partial_P C$ for $\varepsilon > 0$ small enough. We now apply the comparison principle for a subsolution u^ε and a supersolution $\phi^\varepsilon - \delta$ of (2.21) to obtain $u^\varepsilon \leq \phi^\varepsilon - \delta$ in C. Taking $\limsup^*_{\varepsilon\to 0}$ at $(\hat{x}, \hat{t})$, we see $\overline{u}(\hat{x}, \hat{t}) \leq \phi(\hat{x}, \hat{t}) - \delta$. This is a contradiction to (6.7), and hence $\overline{u}$ is a subsolution of (2.22).

6. Similarly we show that $\underline{u}$ is a supersolution of (2.22), and therefore $\overline{u} = \underline{u}$ by comparison. This implies the locally uniform convergence of u^ε to the unique viscosity solution $\bar{u}^\theta$ of (2.22) and (2.1b). □

Remark 6.6. *As long as the comparison principle is true, this homogenization result still holds for more general equations with H_1 and H_2 which are not necessarily of the forms $H_1 = H_1(x, t, p)$ and $H_2 = H_2(r, p)$.*

A Lipschitz continuity of solutions

The properties of the solution of the problem (2.6) and (2.1b) might come in handy when studying numerical results. For this reason we prove here a Lipschitz estimate for the solution, under the assumption that the initial datum is Lipschitz continuous. We also give an explicit representation of the Lipschitz constant in terms of the Lipschitz constant of the initial datum and the Lipschitz constant of the Hamiltonian H_1 denoted by $D(t)$ as in (H3-s). Although there are plenty of results in the literature concerning the Lipschitz continuity of viscosity solutions, a Lipschitz estimate for the Hamiltonians which are being studied in this paper does not exist up to the authors' knowledge. Moreover we are more concerned in a Lipschitz constant that does not depend on the parameter θ.

Proof of Proposition 2.12. 1. Let $\Phi(x,y,t) = u(x,t) - u(y,t) - L(t)|x-y|$ for $x, y \in \mathbf{R}^n$ and $t \in [0,T)$. We proceed by contradiction. Suppose that

$$M = \sup_{x,y\in\mathbf{R}^n, t\in[0,T)} \Phi(x,y,t) > 0.$$

Since u has at most linear growth (Theorem 2.11) and u_0 is Lipschitz continuous with Lipschitz constant L_0, we have

$$\begin{aligned} u(x,t) - u(y,t) &\le u_0(x) + Kt - (u_0(y) - Kt) \\ &\le C_T + L_0|x-y| \end{aligned} \tag{1.1}$$

with $C_T = 2KT$. We define

$$\Phi_\sigma(x,y,t) = u(x,t) - u(y,t) - L_\alpha(t)|x-y| - \frac{\eta}{T-t} - \alpha(|x|^2 + |y|^2),$$

where

$$\begin{aligned} L_\alpha(t) &= \max\left\{L_0,\, 1 + 2\sqrt{\alpha C_T}\right\} e^{\int_0^t (D(s)+\mu(s))\, ds}, \\ \mu(t) &= 2\sqrt{\alpha C_T} D(t). \end{aligned}$$

Set

$$M_\sigma = \sup_{x,y\in\mathbf{R}^n,\, t\in[0,T)} \Phi_\sigma(x,y,t)$$

for $\sigma = (\eta, \alpha)$. Since u has at most linear growth, there are $x_\sigma, y_\sigma \in \mathbf{R}^n$ and $t_\sigma \in [0,T)$ such that

$$M_\sigma = \Phi_\sigma(x_\sigma, y_\sigma, t_\sigma).$$

2. By the definition of M, for every $\delta > 0$ there are x_δ, y_δ and t_δ such that

$$\Phi(x_\delta, y_\delta, t_\delta) \ge M - \delta.$$

Since $L_\alpha(t) \to L(t) = \max\{L_0,\, 1\} e^{\int_0^t D(s)\, ds}$ uniformly in t as $\alpha \to 0$, there is $\varepsilon > 0$ small enough, independent of δ, such that

$$-L_\alpha(t_\delta) > -\varepsilon - L(t_\delta).$$

Then for $\delta = M/4$ we have

$$\begin{aligned} M_\sigma &\ge \Phi_\sigma(x_\delta, y_\delta, t_\delta) \\ &= \Phi(x_\delta, y_\delta, t_\delta) - \frac{\eta}{T-t_\delta} - \alpha(|x_\delta|^2 + |y_\delta|^2) - \varepsilon|x_\delta - y_\delta| \\ &\ge M - \delta - \frac{\eta}{T-t_\delta} - \alpha(|x_\delta|^2 + |y_\delta|^2) - \varepsilon|x_\delta - y_\delta| \\ &\ge \frac{M}{2} > 0 \end{aligned}$$

for $\eta, \alpha, \varepsilon$ small enough, since δ is fixed. From this it follows that

$$\Phi_\sigma(x_\sigma, y_\sigma, t_\sigma) > 0. \tag{1.2}$$

3. We claim

$$\alpha|x_\sigma|,\, \alpha|y_\sigma| \le \sqrt{\alpha C_T}. \tag{1.3}$$

By (1.1) we observe

$$u(x,t) - u(y,t) - L_\alpha(t)|x-y| \le C_T + (L_0 - L_\alpha(t))|x-y| \le C_T$$

for all (x,y,t), and hence we can write

$$u(x_\sigma,t_\sigma) - u(y_\sigma,t_\sigma) - L_\alpha(t_\sigma)|x_\sigma - y_\sigma| - \frac{\eta}{T-t_\sigma} - \alpha|y_\sigma|^2 \le C_T.$$

The left-hand side is equal to $\Phi_\sigma(x_\sigma,y_\sigma,t_\sigma) + \alpha|x_\sigma|^2$. By (1.2) we get

$$\alpha|x_\sigma|^2 \le C_T.$$

Similarly we have $\alpha|y_\sigma|^2 \le C_T$, and these inequalities show (1.3).

4. We prove $t_\sigma > 0$ and $x_\sigma \ne y_\sigma$. Suppose that $t_\sigma = 0$. Then, since $u_0 = u(\cdot,0)$ is Lipschitz continuous with Lipschitz constant $L_0 \le L_\alpha(0)$, we have

$$0 < \Phi_\sigma(x_\sigma,y_\sigma,0) \le u(x_\sigma,0) - u(y_\sigma,0) - L_\alpha(0)|x_\sigma - y_\sigma| \le 0,$$

a contradiction. Since $\Phi_\sigma(x_\sigma,x_\sigma,t_\sigma) < 0$, we have that $x_\sigma \ne y_\sigma$.

5. Since $x_\sigma \ne y_\sigma$, we have $|x-y| > 0$ in a neighbourhood of (x_σ,y_σ). Therefore, we can apply [13, Lemma 2] and get, for $p_\sigma = (x_\sigma - y_\sigma)/|x_\sigma - y_\sigma|$,

$$\begin{aligned}
&\frac{\eta}{T^2} + L'_\alpha(t_\sigma)|x_\sigma - y_\sigma| \\
&\le \{H_1(x_\sigma,t_\sigma,L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) - H_1(y_\sigma,t_\sigma,L_\alpha(t_\sigma)p_\sigma - 2\alpha y_\sigma)\} \\
&\quad + \theta\{\beta(u(x_\sigma,t_\sigma)h(L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) - \beta(u(y_\sigma,t_\sigma))h(L_\alpha(t_\sigma)p_\sigma - 2\alpha y_\sigma)\} \\
&=: I_1 + I_2. \qquad (1.4)
\end{aligned}$$

We can also rewrite I_2 as

$$\begin{aligned}
I_2 = \theta\,\{\beta(u(x_\sigma,t_\sigma)) - \beta(u(y_\sigma,t_\sigma))\}\,h(L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) \\
+ \theta\beta(u(y_\sigma,t_\sigma))\cdot\{h(L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) - h(L_\alpha(t_\sigma)p_\sigma - 2\alpha y_\sigma)\}. \qquad (1.5)
\end{aligned}$$

6. Let us give estimates of I_1 and I_2. Using (H3-s), the Lipschitz continuity in x of H_1, together with (H2), we get

$$\begin{aligned}
I_1 \le D(t_\sigma)|x_\sigma - y_\sigma|\cdot|L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma| \\
+ \{H_1(y_\sigma,t_\sigma,L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) - H_1(y_\sigma,t_\sigma,L_\alpha(t_\sigma)p_\sigma - 2\alpha y_\sigma)\}.
\end{aligned}$$

The first term on the right-hand side can be estimated by (1.3) and $|p_\sigma| = 1$, while we apply (H4), the Lipschitz continuity in p of H_1, to the second term. Then

$$I_1 \le D(t_\sigma)|x_\sigma - y_\sigma|\left(L_\alpha(t_\sigma) + 2\sqrt{\alpha C_T}\right) + 2\alpha L_2|x_\sigma + y_\sigma|.$$

By the definition of μ we have

$$I_1 \le |x_\sigma - y_\sigma|\,(D(t_\sigma)L_\alpha(t_\sigma) + \mu(t_\sigma)) + 2\alpha L_2|x_\sigma + y_\sigma|. \qquad (1.6)$$

We next show that the first term on the right-hand side of (1.5) is not positive. We first note that $u(x_\sigma,t_\sigma) \ge u(y_\sigma,t_\sigma)$ by (1.2). Therefore, we have $\beta(u(x_\sigma,t_\sigma)) \ge \beta(u(y_\sigma,t_\sigma))$ since β is increasing. As a second remark, by the definition of L_α and (1.3), we have

$$|L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma| \ge L_\alpha(t_\sigma) - 2\sqrt{\alpha C_T} \ge 1.$$

Then the assumption (2.10) and the continuity of h imply that $h(L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) \leq 0$. According to these two remarks we have

$$I_2 \leq \theta|\beta(u(y_\sigma, t_\sigma))| \cdot |h(L_\alpha(t_\sigma)p_\sigma + 2\alpha x_\sigma) - h(L_\alpha(t_\sigma)p_\sigma - 2\alpha y_\sigma)|.$$

Using the boundedness of β and the uniform continuity of h, (2.8), we can further estimate the above as follows :

$$I_2 \leq \theta M \omega_h(2\alpha|x_\sigma + y_\sigma|), \tag{1.7}$$

where M is an upper bound for $|\beta|$.

7. Applying (1.6) and (1.7) to (1.4), we get

$$\frac{\eta}{T^2} + L'_\alpha(t_\sigma)|x_\sigma - y_\sigma| \leq |x_\sigma - y_\sigma|(L_\alpha(t_\sigma)D(t_\sigma) + \mu(t_\sigma)) + J, \tag{1.8}$$

where

$$J = 2\alpha L_2|x_\sigma + y_\sigma| + \theta M \omega_h(2\alpha|x_\sigma + y_\sigma|).$$

Note that the function L_α has been chosen so that it solves the differential equation

$$L'_\alpha(t) = (D(t) + \mu(t))L_\alpha(t).$$

According to this, the estimate (1.8) becomes

$$\frac{\eta}{T^2} \leq \mu(t_\sigma)(1 - L_\alpha(t_\sigma))|x_\sigma - y_\sigma| + J.$$

Since $L_\alpha > 1$, we have

$$\frac{\eta}{T^2} \leq J = 2\alpha L_2|x_\sigma + y_\sigma| + \theta M \omega_h(2\alpha|x_\sigma + y_\sigma|).$$

Using (1.3), we can send $\alpha \to 0$ and get a contradiction. □

Proof of Proposition 2.10. Since the comparison principal is more or less classical and similar to the proof of Proposition 2.12, we will only give a sketch of the proof.

1. We may suppose without loss of generality that u, v are upper, respectively, lower semicontinuous. As usual we set

$$\tilde{u}(\cdot, t) = e^{-at}u(\cdot, t),\ \tilde{v}(\cdot, t) = e^{-at}v(\cdot, t),$$

where $a > a_0$ and a_0 is given by (F2). Using the notation u, v instead of $\tilde{u}, \tilde{v}$, we have that u, v are sub- and supersolutions of the equation

$$u_t + (a - a_0)u = \tilde{F}(x, t, u, \nabla u),$$

where $\tilde{F}(x, t, r, p)$ is non-increasing in r and satisfies the conditions (F1)-(F4). As before we denote by F the new $\tilde{F}$.

2. Suppose that

$$M = \sup_{(x,t)\in\mathbf{R}^n\times[0,T)} (u(x, t) - v(x, t)) > 0.$$

We now make the usual doubling of variables trick and define for $\varepsilon, \eta, \alpha > 0$

$$\Phi_\sigma(x, t, y) = u(x, t) - v(y, t) - \frac{|x - y|^2}{\varepsilon} - \frac{\eta}{T - t} - \alpha(|x|^2 + |y|^2)$$

and

$$M_\sigma = \sup_{x,y\in\mathbf{R}^n,\, t\in[0,T)} \Phi_\sigma(x,y,t),$$

where $\sigma = (\varepsilon, \eta, \alpha)$. As usual we have $0 < M_\sigma < +\infty$. In order to proceed we need a priori bounds on the maximum M_σ and to do that we need to be able to control the difference $u(x,t) - v(y,t)$ by the modulus $|x-y|$. One can show (using a doubling of variables trick, see for example [20, Proposition 2.3′, p.464]) that there is a constant $C_T > 0$ such that

$$u(x,t) - v(y,t) \leq C_T(1 + |x-y|).$$

Using this estimate we can show that there is $C > 0$ such that

$$\alpha|x|, \alpha|y| \leq \sqrt{\alpha C}.$$

The above estimate together with (A3) enables us to find $x_\sigma, y_\sigma \in \mathbf{R}^n$ and $t_\sigma \in (0,T)$ such that $\Phi_\sigma(x_\sigma, y_\sigma, t_\sigma) = M_\sigma$.

For the term $|x_\sigma - y_\sigma|^2/\varepsilon$, we will need a more refined estimate than the classical one, namely we need

$$\lim_{\varepsilon\to 0}\left(\limsup_{\eta,\alpha\to 0} \frac{|x_\sigma - y_\sigma|^2}{\varepsilon}\right) = 0. \tag{1.9}$$

A proof of a similar estimate can be found in [20, Proposition 4.4].

3. Doubling the variables again in time or using a similar argument as in [13, Lemma 2], we have for $p_\varepsilon = 2(x_\sigma - y_\sigma)/\varepsilon$

$$\begin{aligned}\frac{\eta}{T^2} + (a - a_0)(u(x_\sigma, t_\sigma) &- v(y_\sigma, t_\sigma)) \\ &\leq F(x_\sigma, t_\sigma, u(x_\sigma, t_\sigma), p_\varepsilon + 2\alpha x_\sigma) - F(y_\sigma, t_\sigma, v(y_\sigma, t_\sigma), p_\varepsilon - 2\alpha y_\sigma).\end{aligned}$$

As in [20, Proposition 2.4] there is a $\delta > 0$ independent of σ, such that $u(x_\sigma, t_\sigma) - v(y_\sigma, t_\sigma) > \delta$. Using properties of F, (F3) and (F4), one gets

$$\frac{\eta}{T^2} + (a - a_0)\delta \leq \omega(|x_\sigma - y_\sigma|(1 + |p_\varepsilon + 2\alpha x_\sigma|)) + \omega_R(2\alpha(|x_\sigma| + |y_\sigma|)),$$

where $R = o(1/\sqrt{\varepsilon}) + o(\sqrt{\alpha})$ as $\varepsilon, \alpha \to 0$. Using (1.9), we can take the limit first as $\alpha, \eta \to 0$ and then as $\varepsilon \to 0$ to get a contradiction. ☐

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Chapitre 2

A solution with free boundary for non-Newtonian fluids with Drucker-Prager plasticity criterion

This chapter is a paper [16] written in collaboration with M Regis [†].

Abstract

We study a free boundary problem which is motivated by a particular case of the flow of a non-Newtonian fluid, with a pressure depending yield stress given by a Drucker-Prager plasticity criterion. We focus on the steady case and reformulate the equation as a variational problem. The resulting energy has a term with linear growth while we study the problem in an unbounded domain. We derive an Euler-Lagrange equation and prove a comparison principle. We are then able to construct a subsolution and a supersolution which quantify the natural and expected properties of the solution ; in particular we show that the solution has in fact compact support, the boundary of which is the free boundary.

The model describes the flow of a non-Newtonian material on an inclined plane with walls, driven by gravity. We show that there is a critical angle for a non-zero solution to exist. Finally, using the sub/supersolutions we give estimates of the free boundary.

1 Introduction

Setting of the problem We study non-negative solutions $u(y,z)$ of the equation

$$\begin{cases} \operatorname{div}(\nabla u + |z|q) = -\lambda & \text{in } (-1,1)\times(-\infty,0), \\ q \in \partial(|\cdot|)(\nabla u), \end{cases} \tag{1.1}$$

with $u(\pm 1, z) = 0$, $q = q(y,z)$, $\lambda \geq 0$ and for a function $f : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$ we define the subdifferential of f at a point $y \in \mathbb{R}^N$ as

$$(\partial f)(y) := \{z \in \mathbb{R}^N : \ f(x) - f(y) \geq z \cdot (x-y) \, \forall x \in \mathbb{R}^N\}. \tag{1.2}$$

The variational formulation of (1.1) consists in minimizing the functional

$$E_\lambda(u) = \int_\Omega \frac{|\nabla u|^2}{2} + |z||\nabla u| - \lambda u, \tag{1.3}$$

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in the space

$$\mathcal{X} = \mathcal{X}(\Omega) := \{u \in W^{1,2}_{0L}(\Omega),\ z\nabla u \in L^1(\Omega, \mathbb{R}^2)\}, \tag{1.4}$$

with

$$W^{1,2}_{0L}(\Omega) := \{u \in W^{1,2}(\Omega) :\ u(\pm 1, \cdot) = 0\},$$

$\Omega = (-1,1) \times (-\infty, 0)$. Note that by Remark 2.1 the functional E_λ is well defined in $\mathcal{X}$. Before we explain the physical interpretation of the mathematical model, we present some of the particularities of the problem.

Since we study the equation (1.1) in an unbounded domain, the variational problem (1.3) is no longer trivial because it is not clear if the linear term $-\int_\Omega \lambda u$ is lower semicontinuous or if the minimizing sequence obtained by the direct method will have a converging subsequence in $\mathcal{X}$. Using Lemma 3.3, we show that the linear term is lower semicontinuous and the well posedness of the problem is established in Theorem 2.2 (i). Also, despite the fact that the energy E includes a term with linear growth (in the gradient variable), a comparison principle still holds for equation (1.1). Using this comparison principle we construct sub/supersolutions and show that in fact the solution of (1.1) is compactly supported.

For the construction of these barriers we use the "curvature like" equation

$$-\mathrm{div}(|z|q) = \lambda, \tag{1.5}$$

which is the first variation of the energy $\int_\Omega |z||\nabla u| - \lambda u$, with $q = \frac{\nabla u}{|\nabla u|}$ when $\nabla u \neq 0$ and $|q| \leq 1$; then the vector $\frac{\nabla u}{|\nabla u|}$ is the normal to the level sets of u. If we suppose that these level sets are given by $-z = \phi(y)$ we are led to study the first variation of the 1-D functional

$$\int_{-1}^{1} -\phi(y)\sqrt{1 + |\phi'(y)|^2} + \lambda\phi. \tag{1.6}$$

Non-Newtonian fluids The model (1.1) is motivated by the motion of non-Newtonian fluids. Let $\Omega \subset \mathbb{R}^3$, open and $v : \Omega \to \mathbb{R}^3$ be the velocity of the fluid, assumed incompressible,

$$\mathrm{div}\, v = 0. \tag{1.7}$$

Let $f : \Omega \to \mathbb{R}^3$ be the external force, then the relevant equation reads as

$$\mathrm{div}\, \sigma + f = (\nabla \cdot v)v + \partial_t v \tag{1.8}$$

where σ is the stress tensor and using the usual summation convention we write $(\nabla \cdot v)v = (v_j \partial_j v_i)_{1 \leq i \leq 3}$. Let σ_{dev} be the stress deviator defined by $\mathrm{tr}(\sigma_{\mathrm{dev}}) = 0$ and

$$\sigma_{\mathrm{dev}} := \sigma + pI, \tag{1.9}$$

where p is the pressure and I is the unit matrix.

We are interested in the flow of rigid visco-plastic fluids, which unlike Newtonian fluids can sustain shear stress. The stress tensor in this case is characterized by a flow/no flow condition, namely when the stress tensor belongs to a certain convex set the fluid behaves like a rigid body, whereas outside this set the material flows like a regular Newtonian fluid.

For a matrix $B = (b_{ij})_{1\leq i,j\leq 3}$ we denote the norm $||B|| = \sqrt{\frac{1}{2}\sum_{i,j=1}^{3} b_{ij}^2}$. Following [10] and [5] we define the stress deviator as

$$\begin{cases} \sigma_{\text{dev}} = 2\nu D(v) + k(p)\frac{D(v)}{||D(v)||} & \text{if } D(v) \neq 0, \\ ||\sigma_{\text{dev}}|| \leq k(p) & \text{if } D(v) = 0 \end{cases} \tag{1.10}$$

where we assume that the viscosity $\nu > 0$ is constant and $k(p)$ is the pressure-dependent yield stress and $D(v) = (\nabla v + (\nabla v)^T)/2$. The above constituent law is a result of a superposition of the viscous contribution $2\nu D(v)$ and a contribution related to plasticity effects $k(p)\frac{D(v)}{||D(v)||}$, which is independent of the norm of the strain rate $||D(v)||$. For constant yield limit $k(p)$ we retrieve the regular Bingham model, which is a generalized Newtonian problem, i.e. the constituent law in this case is described by a dissipative potential, see [6], [8, Chapter 3] and references therein. In this paper we will assume the Drucker-Prager plasticity criterion

$$k(p) = \mu_s p, \tag{1.11}$$

where $\mu_s = \tan\delta_s$, with δ_s the internal friction (static) angle. The existence of a dissipative potential in the case of Bingham flows allows for a variational formulation and in tern the well-posedeness of the problem ; for quasi-static Bingham flows see for example [8]. The case of a Drucker-Prager criterion, however, falls in a wider class of constituent laws called "$\mu(I)-$rheology" which are known to be ill-posed, see [1] and [18]. The strong geophysical interest in the model (1.11) supports however our study. A main result of the present work is that for one-directional steady flows the model is well-posed.

Flow in one direction We study the well-posedeness and certain quantitative properties of quasi-static solutions of (1.7)-(1.8), (1.10)-(1.11), for a material which flows on an inclined plane with sidewalls. We assume that the inclination angle is constant θ and the material moves only in the direction x under the effect of gravity, see Figure 2.1. In what

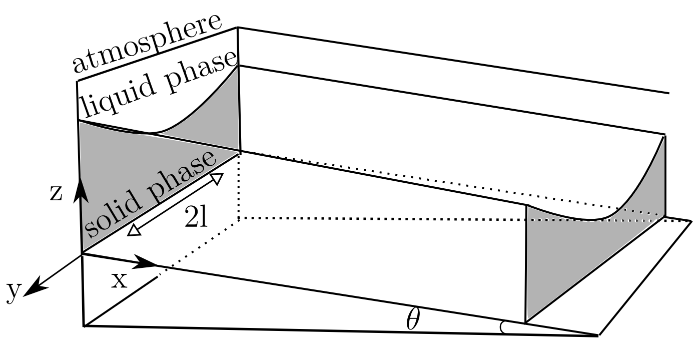


FIGURE 2.1 – Steady flow on an inclined plane.

follows, we will assume that the velocity field is of the form $v(x, y, z) = (u(y, z), 0, 0)$ for $(x, y, z) \in \{(x, y, z) : 0 \leq x, -l \leq y \leq l, b \leq z \leq h(y)\}$ where $h(y)$ is the interface separating the fluid and the air and $z = b$ is the surface of the inclined plane, the width of which is equal to $2l$. Although the well posedness of similar problems have been studied in more generality in a bounded domain, as it will become clear later, in order to study the interface between the solid and the liquid phase, as we increase the inclination angle, we will need to take $b = -\infty$. By the form of v, the incompressibility condition (1.7) is trivially satisfied and equation (1.8) with $\partial_t v = 0$ becomes

$$\operatorname{div}\sigma_{\text{dev}} = -f + \nabla p, \tag{1.12}$$

with σ_{dev} given by (1.10) and

$$f = (g_0 \sin\theta, 0, -g_0 \cos\theta), \tag{1.13}$$

with g_0 the gravitational constant. We also assume that $p = p(y,z)$. We calculate

$$D(v) = \frac{1}{2}\begin{pmatrix} 0 & \partial_y u & \partial_z u \\ \partial_y u & 0 & 0 \\ \partial_z u & 0 & 0 \end{pmatrix} \tag{1.14}$$

and $||Du|| = \frac{1}{2}|\nabla u|$, with $\nabla u = (\partial_y u, \partial_z u)$. If we substitute (1.14) in (1.10), equations (1.12) become, for $D(v) \neq 0$ or equivalently $\nabla u \neq 0$,

$$\begin{cases} \nu \operatorname{div}(\nabla u) + \mu_s \operatorname{div}\left(p \dfrac{\nabla u}{|\nabla u|}\right) = -g_0 \sin\theta & (1.15) \\ 0 = \partial_y p & (1.16) \\ 0 = g_0 \cos\theta + \partial_z p & (1.17) \end{cases}$$

Where the divergence is taken for the coordinates (y,z). If we integrate equation (1.17) from z to $h(y)$ we get $p(y,z) = (h(y) - z) g_0 \cos\theta$, but because of equation (1.16) and because $\theta \in \left[0, \frac{\pi}{2}\right)$ we have $h(y) = h \equiv$ constant. For simplicity we take $h = 0$; then the pressure is given by

$$p(y,z) = |z| g_0 \cos\theta. \tag{1.18}$$

We are lead to study the following equation

$$\begin{cases} \nu \operatorname{div}(\nabla u) + \mu_s g_0 \cos\theta \operatorname{div}(|z| q) = -g_0 \sin\theta \quad \text{in } (-l,l) \times (b,0), \\ q \in \partial(|\cdot|)(\nabla u) \end{cases} \tag{1.19}$$

where $\partial(|\cdot|)$ is the subdifferential of the absolute value. If (u,q) is such that (1.19) holds, with $q = q(y,z) = (q_1(y,z), q_2(y,z))$, then $|q| \leq 1$ and $q = \frac{\nabla u}{|\nabla u|}$ for $\nabla u \neq 0$ and therefore the stress deviator defined by

$$\sigma_{\text{dev}} := \nu \begin{pmatrix} 0 & \partial_y u & \partial_z u \\ \partial_y u & 0 & 0 \\ \partial_z u & 0 & 0 \end{pmatrix} + \mu_s |z| g_0 \cos\theta \begin{pmatrix} 0 & q_1 & q_2 \\ q_1 & 0 & 0 \\ q_2 & 0 & 0 \end{pmatrix} \tag{1.20}$$

is of the form (1.10) with $v(x,y,z) = (u(y,z), 0, 0)$ and solves equations (1.12) with f given by (1.13) and p by (1.18).

Boundary conditions On the surface of the material $z = 0$ we assume a *no stress condition*, i.e. $\sigma \cdot (0,0,1) = 0$; since the pressure is zero on the surface near the atmosphere, this condition becomes $\sigma_{\text{dev}} \cdot (0,0,1) = 0$. Here we assume that the stress deviator is given by (1.20). Then the stress free condition becomes (since $z = 0$)

$$\partial_z u(y,0) = 0. \tag{1.21}$$

On the lateral boundary $y = \pm 1$ we assume the Dirichlet conditions $u = 0$ (no slip), while at the bottom $z = b$, where the material is in contact with the inclined plane, a natural assumption is the friction condition

$$\begin{cases} \sigma n - (\sigma n \cdot n)n = \mu_C v \\ v \cdot n = 0 \end{cases}$$

where v, σ, n, μ_C are the velocity, stress, normal to the plane and a friction coefficient respectively. In our case the friction condition reads as follows

$$\nu \partial_z u + \mu_s |b| g_0 (\cos\theta) q_2 = \mu_C u. \tag{1.22}$$

Variational formulation The variational formulation of equation (1.19) with boundary conditions (1.21), (1.22) and the homogeneous Dirichlet conditions on the lateral boundary constitutes in minimizing the energy

$$\int_{(-l,l)\times(b,0)} \nu \frac{|\nabla u|^2}{2} + \mu_s |z| g_0 \cos\theta |\nabla u| - (g_0 \sin\theta) u + \mu_C \int_{\{z=b\}} \frac{|u|^2}{2} \tag{1.23}$$

with zero lateral boundary conditions, i.e. $u(\pm 1, \cdot) = 0$. Since the energy (1.23) is convex and the domain is bounded we can easily get a non-negative minimizer via the direct method.

We are interested in the properties of the minimizer as we increase the inclination angle θ. We call solid and liquid phases the sets $\{(y, z) : u(y, z) = 0\}$ and $\{(y, z) : u(y, z) > 0\}$ respectively (often abbreviated as $\{u = 0\}$, $\{u > 0\}$ resp.), while their common boundary we call a yield curve. We note that usually in the literature the yield curve is defined, for our setting, as the set $\partial\{\nabla u \neq 0\}$, but approximating this set would require different methods and more regularity of the solution.

For $|b|$ small we expect that for a sufficiently large angle θ all of the material will move due to the gravity, namely there is no solid phase, whereas, if $|b|$ is large enough, even if the inclination is large we expect that there will be a solid phase. In order to study the behaviour and shape of the liquid/solid phases as we increase the inclination angle, we fix $b = -\infty$. However, there is still one more free boundary remaining, the yield curve, i.e. the curve that separates the solid from the liquid phase. Since we study (1.23) in an unbounded domain we drop the friction condition. Let $\tilde{u}$ be a solution of (1.19)-(1.21) with $b = -\infty$, in order to simplify further the equation (1.19) we set

$$u(y, z) = \frac{\nu}{\mu_s g_0 \cos\theta} \frac{\tilde{u}\,(ly, lz)}{l^2}, \quad (y, z) \in (-1, 1) \times (-\infty, 0) \tag{1.24}$$

we also define

$$\lambda := \frac{\tan\theta}{\mu_s}, \tag{1.25}$$

then $\partial|\cdot|(\nabla u(y, z)) = \partial|\cdot|(\nabla \tilde{u}(ly, lz))$ and therefore, u given by (1.24) solves the equation (1.1) if and only if $\tilde{u}$ solves (1.19)

As we will see in Theorem 2.3, the minimizer has compact support, therefore, it trivially satisfies the friction condition (1.22) on the solid phase as long as the level of the plane is taken far enough from the support of the minimizer.

We also show that the critical angle for an non-zero minimizer to exist is $\arctan \mu_s$, namely for $\theta > \arctan \mu_s$ there exists a non-zero solution with a yield curve while for

$0 \leq \theta \leq \arctan \mu_s$ the solution is zero. This angle is known in the literature by experimental study, see for example [17]. The time dependent, one dimensional analogue of our case is studied in [4]; the authors prove that for $\theta > \arctan \mu_s$ there is no solution with solid phase while in our case the solution always has a solid phase. The difference of course lies in our two dimensional setting of the problem in which the existence of the walls where the velocity vanishes is crucial, not just for the physical relevance of the problem. Indeed since we study minimizers of (1.3) in an unbounded domain we will often need to apply Poincaré's inequality, for this reason we need that the projection of the domain in one of the coordinate axes is bounded. In [13] the authors also prove that for $\theta \leq \arctan \mu_s$ the flowing material stops moving in finite time.

Review of the literature For an extensive review of non-Newtonian fluids see [6], also [8] and references therein and [15] for evolutionary problems. The flow of a viscoplastic material with "$\mu(I)-$rheology" is relatively new in the literature, see for example [10]. The inviscid case, i.e. for $\nu = 0$ is similar to another scalar model with applications in image processing, the total variation flow, see for example [19] and [2]. Although the total variation bears more similarities with the Bingham case, many of the tools used to study our problem are similar. In fact the total variation is more difficult to study because of the lack of the quadratic term in the energy which leads to lack of regularity of the solution. For the inviscid case our energy (1.3) falls into a wider class, the "total variation functionals" see [3, Hypothesis 4.1]. We refer to [14] for simulations of a regularized Drucker-Prager model with application to granular collapse. Concerning the case of the inclined plane see [11] and [17].

Organization of the paper In Section 2 we state our main results, Theorems 2.2 and 2.3. In Subsection 3.1 we study the 1-dimensional analogue of (1.3) which we use in Lemma 3.3; this Lemma is the crucial step in order to prove that the linear term $-\lambda \int_\Omega u$ is lower semicontinuous. In Subsection 3.3 we study an approximate problem of the minimizer of (1.3) which helps us to prove certain regularity properties of the solution; we also note that since the minimizer is studied in the half stripe Ω the regularity holds up to the interface seperating the solid from the liquid phase (the support of the minimizer). Using the approximate minimizer we can also calculate the first variation of (1.3). Finally, in Lemma 4.4 we construct a solution of (1.5) which we use together with the comparison principle from Subsection 4.1, in Subsections 4.3 and 4.4 in order to construct a subsolution and supersolution respectively. The Figures 2.2-2.7 as well as the simulations in Table 2.1 have been made with Mathematica.

2 Main results

We begin with a technical remark.

Remark 2.1. *We have $\mathcal{X}(\Omega) \subset W^{1,1}(\Omega)$, which justifies the choice of the space $\mathcal{X}$ as natural functional space for the functional* (1.3)*. Indeed,*

$$\int_{\{|z|\geq 1\}\cap\Omega} |\nabla u| \leq \int_{\{|z|\geq 1\}\cap\Omega} |z||\nabla u| < \infty,$$

we also get that $u \in L^1(\Omega)$ by Poincaré's inequality, see [12, Theorem 12.17]; note also that in our case the proof of Poincaré's inequality requires only that elements of the space $W^{1,2}(\Omega)$ are zero on the lateral boundary of Ω (i.e. on $\{\pm 1\} \times (0, \infty)$). In fact since the width of the walls is 2 we have $\int_\Omega |u|^p \leq \frac{2^p}{p} \int_\Omega |\nabla u|^p$ for $p = 1, 2$.

Let

$$\Lambda := \{q : q \in L^2_{loc}(\Omega, \mathbb{R}^2), \; |q| \leq 1 \text{ a.e.}\}. \tag{2.1}$$

Let $\hat{\Omega} = (-1,1) \times \mathbb{R}$, $u \in W^{1,2}_{0L}(\Omega)$, we denote by $\hat{u} \in W^{1,2}_{0L}(\hat{\Omega})$ the reflection of u with respect the $z = 0$ axes, i.e.

$$\hat{u}(y,z) := \begin{cases} u(y,z) & \text{if } (y,z) \in \Omega, \\ u(y,-z) & \text{if } y,z) \in \hat{\Omega} \setminus \Omega. \end{cases} \tag{2.2}$$

Throughout the paper we will denote the space $\mathcal{X}(\Omega)$ simply by $\mathcal{X}$. Only in Lemma 3.4 we will use the explicit notation, this time for the space $\mathcal{X}(\hat{\Omega})$. The weak formulation of (1.1) is

$$\begin{cases} \int_\Omega \nu \nabla u \cdot \nabla \varphi + |z| q \cdot \nabla \varphi = \lambda \int_\Omega \varphi & \text{for all } \varphi \in \mathcal{X} \\ q \cdot \nabla u = |\nabla u| & \text{a.e.} \end{cases} \tag{2.3}$$

for some $\lambda \geq 0$, $q \in \Lambda$. We can now state our first main Theorem.

Theorem 2.2. (Existence and uniqueness of minimizers of (1.3)**)**
Let $\lambda \geq 0$, E_λ be given by (1.3)*, then the following hold*

(i) there exists a unique $0 \leq u_\lambda \in \mathcal{X}$ such that

$$E_\lambda(u_\lambda) = \inf_{v \in \mathcal{X}} E_\lambda(v), \tag{2.4}$$

moreover, $u_\lambda \equiv 0$ if $\lambda \in [0,1]$ and $u_\lambda \not\equiv 0$ if $\lambda \in (1, +\infty)$,

(ii) there exists $q \in \Lambda$ such that (u_λ, q) solves (2.3)*,*

(iii) $u_\lambda \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for all $\alpha \in (0,1)$, in fact $\hat{u}_\lambda \in W^{2,2}_{\mathrm{loc}}(\hat{\Omega})$ and $\partial_z u_\lambda(y,0) = 0$ for $y \in (-1,1)$,

(iv) if $\lambda > 1$, the pair (u_λ, q) obtained in (ii) *is unique in the sense that if $(\bar{u}_\lambda, \bar{q}) \in \mathcal{X} \times \Lambda$ is another pair satisfying* (2.3) *then*

$$u = \bar{u} \text{ in } \Omega, \quad \text{and } q = \bar{q}, \text{ a.e. in } \{\nabla u \neq 0\}.$$

We set

$$I_m := \inf_{v \in \mathcal{X}} E_\lambda(v).$$

Note that by the continuity of the non-negative function u_λ in Theorem 2.2 we can define the *yield curve* as the common boundary $\partial\{u_\lambda > 0\} = \partial\{u_\lambda = 0\}$. Moreover, the critical value $\lambda = 1$ in the previous Theorem is also a critical value of the physical solution by (1.24), (1.25) and it does not depend on the viscosity constant ν or the width of the walls.

We will give some notations in order to present our second result, the motivation for this notation will become clear in the proofs of the relevant Propositions. Let $\lambda > 1$ for $Z \in [\frac{1}{\lambda}, \frac{1}{\lambda - 1}]$ we define

$$f_\lambda(Z) := \frac{1}{(\lambda^2 - 1)^{3/2}} \left\{ \operatorname{Arcsin}\left[(\lambda^2 - 1)Z - \lambda\right] - \lambda \sqrt{1 - ((\lambda^2 - 1)Z - \lambda)^2} \right\}. \tag{2.5}$$

As we will see in the proof of Lemma 4.4, the function f_λ is strictly increasing in the interval $[\frac{1}{\lambda}, \frac{1}{\lambda-1}]$, i.e. $f_\lambda(Z_1) < f_\lambda(Z_2)$ for $Z_1 < Z_2$, with $Z_1, Z_2 \in [\frac{1}{\lambda}, \frac{1}{\lambda-1}]$; we can therefore define the following function

$$\phi_{K(\lambda)}(y) := K(\lambda) f_\lambda^{-1} \left(f_\lambda \left(\frac{1}{\lambda - 1} \right) + \frac{|y|}{K(\lambda)} \right) \qquad y \in [-1,1], \tag{2.6}$$

where

$$K(\lambda) := \frac{1}{f_\lambda\left(\frac{1}{\lambda}\right) - f_\lambda\left(\frac{1}{\lambda-1}\right)}. \tag{2.7}$$

Note that by the monotonicity of f_λ it is $K(\lambda) < 0$. We also define the half cone

$$\mathcal{C}_\lambda := \{(y,z) \in \mathbb{R}^2 : 0 < |y| < z\frac{\lambda}{K(\lambda)}\} \tag{2.8}$$

and

$$\mathrm{Epi}^{\smile}(\lambda) := \{(y,z) \in \Omega : z > \phi_{K(\lambda)}(y)\}. \tag{2.9}$$

In Lemma 4.4 we show that the sets in (2.9) are increasing in λ in the sense that $\mathrm{Epi}^{\smile}(\lambda) \subsetneq \mathrm{Epi}^{\smile}(\bar{\lambda})$ for $\bar{\lambda} > \lambda$, see Figure 2.3a. For $\lambda_1 > \lambda$ we set

$$\vartheta_{\lambda,\lambda_1} := \frac{\lambda_1 - \lambda}{2\left(1 + \left(\frac{\lambda_1}{K(\lambda_1)}\right)^2\right)}, \tag{2.10}$$

$$b(\lambda, \lambda_1) := 1 + \sqrt{\frac{\lambda_1}{2\vartheta_{\lambda,\lambda_1}}}, \tag{2.11}$$

$$\Pi(\lambda, \lambda_1) := \frac{-K(\lambda_1)}{\lambda_1 - 1} b(\lambda, \lambda_1) + \frac{K(\lambda)}{\lambda - 1}. \tag{2.12}$$

and

$$\mathrm{Epi}_{\smile}(\lambda_1) := \{(y,z) \in \Omega : z > b(\lambda,\lambda_1)\phi_{K(\lambda_1)}\left(\frac{y}{b(\lambda,\lambda_1)}\right)\}. \tag{2.13}$$

In Lemma 4.4 we see that $\min_{|y|\le 1} \phi_{K(\lambda)}(y) = \phi_{K(\lambda)}(0) = \dfrac{K(\lambda)}{\lambda - 1}$ for all $\lambda > 1$, and therefore, the function Π in (2.12) is the distance of the projections on the $z-$axes of the epigraphs $\mathrm{Epi}^{\smile}(\lambda)$ and $\Omega \setminus \mathrm{Epi}_{\smile}(\lambda_1)$. Using (2.7) we calculate

$$\frac{K(\lambda_1)}{\lambda_1 - 1} = \frac{2(\lambda_1+1)\sqrt{\lambda_1^2 - 1}}{2\sqrt{\lambda_1^2-1} + \pi + 2\mathrm{Arcsin}\left(\frac{1}{\lambda_1}\right)},$$

then $\lim_{\lambda_1\to+\infty} \dfrac{K(\lambda_1)}{\lambda_1 - 1} = +\infty$, and similarly one can see that $\lim_{\lambda_1\to+\infty} \dfrac{K(\lambda_1)}{\lambda_1} = +\infty$; if we combine the above two limits, one can check that for all $\lambda > 1$,

$$\lim_{\lambda_1\to+\infty} \Pi(\lambda,\lambda_1) = +\infty. \tag{2.14}$$

We also have

$$\lim_{\lambda_1\to\lambda} \Pi(\lambda,\lambda_1) = +\infty. \tag{2.15}$$

If we combine (2.14), (2.15) and the fact that Π is continuous and we get that for every fixed $\lambda > 1$ the function $\Pi(\lambda,\cdot)$ attains a minimum for some $\lambda_1^\star > \lambda$. In fact numerical simulations (see Figure 2.6) suggest the function $\Pi(\lambda,\cdot)$ attains the minimum at a unique $\lambda_1^\star > \lambda$, but the analytical calculations are too complicated to check.

In the following Theorem we gather the main properties of the solution obtained in Theorem 2.2.

Theorem 2.3. (Main properties)
Let $\lambda > 1$, $\mathcal{X}$ as in (1.4), $(u_\lambda, q) \in \mathcal{X} \times \Lambda$ *be a solution of* (2.3). *Also let $\lambda_1 > \lambda$,* $\mathrm{Epi}_\smile(\lambda_1)$ *be as in* (2.13), *then then function u_λ has compact support and it's support can be estimated as follows*

$$\mathrm{Epi}^\smile(\lambda) \subset \operatorname{supp} u_\lambda \subset \mathrm{Epi}_\smile(\lambda_1). \tag{2.16}$$

Moreover, we can optimize estimate (2.16) *by choosing $\lambda_1 = \lambda_1^\star$.*

Remark 2.4. (Consequences of Theorem 2.3)

1. *In Lemma 4.4 we show that the function $\phi_{K(\lambda)}$ has a strictly negative maximum, therefore estimate* (2.16) *implies that the yield curve $\partial\{u_\lambda > 0\}$ never reaches the surface of the atmosphere $\{z = 0\}$.*
2. *Notice that the sets* $\mathrm{Epi}^\smile(\lambda)$ *and* $\mathrm{Epi}_\smile(\lambda_1)$ *can also estimate the support of the physical solution, by* (1.24) *and they are independent of the viscosity ν.*

3 Existence/Uniqueness

3.1 $1D$-problem

Let $A > 0$ and for $w \in W_0^{1,2}(-1,1)$ we consider the energy

$$\epsilon_A(w) = \int_{-1}^{1} \left(\frac{|w'(y)|^2}{2} + A|w'(y)| \right) dy. \tag{3.1}$$

Using the direct method of calculus of variations it is not difficult to show the following Proposition.

Proposition 3.1. (Minimizer of $1D$-problem)
Let $A > 0$ and $m > 0$. Then there exists a unique function w solving

$$\epsilon_A(w) = \inf_{\substack{\overline{w} \in W_0^{1,2}(-1,1) \\ \int_{-1}^1 \overline{w} = m}} \epsilon_A(\overline{w}).$$

We set

$$I_m^A := \inf_{\substack{\overline{w} \in W_0^{1,2}(-1,1) \\ \int_{-1}^1 \overline{w} = m}} \epsilon_A(\overline{w}). \tag{3.2}$$

The uniqueness of the minimizer of ϵ_A in the above Proposition follows by the strict convexity of the functional or by using similar arguments as in the proof of Step 1 of Theorem 2.2 (i).

We define the set theoretic sign function as

$$\operatorname{sign}(r) := \begin{cases} \left\{ \frac{r}{|r|} \right\} & \text{if } r \neq 0, \\ (-1,1) & \text{else.} \end{cases}$$

Proposition 3.2. (Characterization of the $1D$ minimizer)
Let A, $m > 0$. If $\lambda_A = \lambda_{A,m}$ is the non-negative root of

$$2\lambda_A^3 - 3\lambda_A^2(A+m) + A^3 = 0, \tag{3.3}$$

with $\lambda_A > A + m$,

$$a = \frac{A}{\lambda_A} < 1, \tag{3.4}$$

$$w(y) = \begin{cases} A\left(-\frac{y^2}{2a} + |y| + \frac{1}{2a} - 1\right) & a < |y| < 1, \\ \\ A\frac{(a-1)^2}{2a} & |y| \leq a. \end{cases} \tag{3.5}$$

$$q(y) = \begin{cases} -\frac{y}{|y|} & a < |y| < 1, \\ \\ -\frac{y}{a} & |y| \leq a, \end{cases}$$

then (w, q, λ_A) *solves the equation*

$$-w''(y) - A(q(y))' = \lambda_A, \quad \textit{for a.e. } y \in (-1, 1), \tag{3.6}$$

and $\int_{-1}^{1} w = m$. *In particular* w *is the unique minimizer of* (3.1) *corresponding to the volume constraint* m.

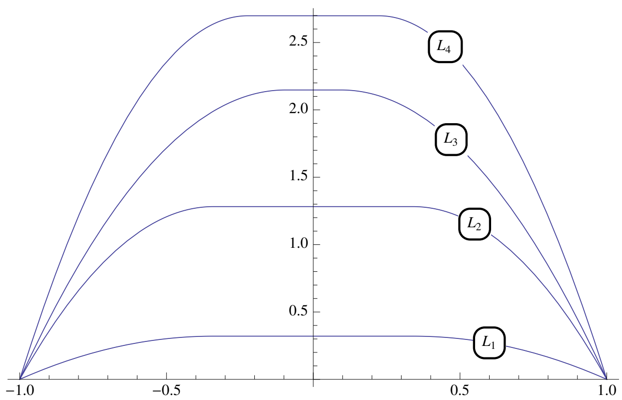


FIGURE 2.2 – L_i, $i = 1, 2, 3, 4$ are the graph of w for $(m, A) = (0.5, 0.5), (2, 2), (3, 0.5), (4, 2)$ respectively.

Proof of Theorem 3.2
Step 1. The trinomial (3.3)

First we will show that the trinomial (3.3) has a unique non-negative root $\lambda_A \geq A+m$. A simple calculation shows that the trinomial (3.3) is increasing in the interval $[A+m, +\infty)$ with values $A^3\left(1 - (1 + \frac{m}{A})^3\right) \leq 0$ and $+\infty$ for $\lambda_A = A + m$ and $\lambda_A = +\infty$ respectively, hence there exist a root for $\lambda \geq A + m$.

Step 2. The equation (3.6)

For a.e. $y \in (-1, 1)$ we have

$$w''(y) = \begin{cases} -\frac{A}{a} & a < |y| < 1, \\ 0 & |y| < a \end{cases} \tag{3.7}$$

and

$$q'(y) = \begin{cases} 0 & a < |y| < 1, \\ -\frac{1}{a} & |y| < a. \end{cases} \tag{3.8}$$

Using (3.7), (3.8) and (3.4) we deduce that (w, q, λ_A) solves (3.6).

Step 3. Volume constraint

It remains to show that $\int_{-1}^{1} w = m$. It is

$$\begin{aligned}
\int_{-1}^{1} w &= 2\left(\frac{A}{2}(1-a)^2 + \int_a^1 w\right) \\
&= 2\left(\frac{A}{2}(1-a)^2 - \frac{A}{2a}\int_a^1 (y-a)^2 - (1-a)^2\,dy\right) \\
&= \frac{A(1-a)^2}{a} - \frac{A}{a}\int_0^{1-a} y^2\,dy \\
&= \frac{A}{3a}(1-a)^2(2+a),
\end{aligned}$$

using equation (3.4) and (3.3) we get

$$\int_{-1}^{1} w = \frac{A}{3a}(1-a)^2(2+a) = \frac{(\lambda_A - A)^2(2\lambda_A + A)}{3\lambda_A^2} = m. \tag{3.9}$$

Step 3. Minimizer

It remains to show that w is the minimizer of ϵ_A in $W_0^{1,2}(-1,1)$ which corresponds to the constraint m. First we notice that $q(x) \in \operatorname{sign}(w'(x)) = \partial(|\cdot|)(w'(x))$ for $x \in (-1,1)$ and the subdifferential is given by (1.2). Let $v \in W_0^{1,2}(-1,1)$ with $\int_{-1}^{1} v = m$, it is

$$\begin{aligned}
\epsilon_A(v) - \epsilon_A(w) &\geq \int_{-1}^{1} w'(v-w)' + Aq(v-w)' \\
&= -\int_{-1}^{1} (w'' + Aq')(v-w) = \lambda_A \int_{-1}^{1} (v-w) = 0.
\end{aligned}$$

□

3.2 A variational problem

The lower semicontinuity of the term $-\int_\Omega \lambda u$ in (1.3) under the weak topology of $W^{1,2}$ is not trivial since the integral is not evaluated in a bounded domain. The following Lemma shows that the L^1-tails of a sequence of functions will converge to zero if the respective values of the functional E_λ are uniformly bounded.

Lemma 3.3. (Compensation of the mass)
Let $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{X}$, suppose that there exists a non-negative constant c independent of k such that $E_\lambda(v_k) < c$ for all $k \in \mathbb{N}$, then

$$\lim_{l\to+\infty}\left(\sup_k\left(\int_l^{+\infty}\int_{-1}^{1} v_k(y,-A)\,dy dA\right)\right) = 0. \tag{3.10}$$

Proof of Lemma 3.3

Step 1 : An estimate for the minimum of ϵ_A

Let $A > 0$ and define

$$m_A = m_A^k = \int_{-1}^{1} v_k(y,-A)\,dy \tag{3.11}$$

Let ϵ_A be given by (3.1) and $I^A_{m_A}$ be the minimum of ϵ_A corresponding to the constraint m_A. Then for λ_A the root of the trinomial in (3.3), it is

$$m_A = \frac{(\lambda_A - A)^2(2\lambda_A + A)}{3\lambda_A^2}.$$

Using (3.5) we calculate

$$I^A_{m_A} = \frac{A^2}{a}\left(\frac{2}{3}\frac{(1-a)^3}{a} + (1-a)^2\right) = \frac{(\lambda_A - A)^2(2\lambda_A + A)}{3\lambda_A} = m_A\lambda_A. \tag{3.12}$$

By Step 1 of the proof of Proposition 3.2 we have $\lambda_A \geq A + m_A$ hence equation (3.12) becomes

$$I^A_{m_A} \geq m_A(A + m_A) \geq Am_A. \tag{3.13}$$

Step 2 : The tails of v_k converge uniformly to 0

We argue by contradiction, suppose that

$$\sup_k\left(\int_l^{+\infty} m^k_A\, dA\right) \nrightarrow 0 \text{ as } l \to +\infty$$

then, there are $\varepsilon > 0$ and a sequence $l_j \to +\infty$ as $j \to +\infty$ such that

$$\sup_k\left(\int_{l_j}^{+\infty} m^k_A\, dA\right) \geq \varepsilon \tag{3.14}$$

By Fubini's Lemma we have for $l_j > \lambda$

$$\begin{aligned} E_\lambda(v_k) &= \int_0^{+\infty}\int_{-1}^{1} \frac{|\nabla v_k|^2}{2} + A|\nabla v_k| - \lambda v_k \,\mathrm{d}y\mathrm{d}A \\ &\geq \int_0^{+\infty}\int_{-1}^{1} \frac{|\partial_y v_k|^2}{2} + A|\partial_y v_k| - \lambda v_k \,\mathrm{d}y\mathrm{d}A \\ &\geq \int_{l_j}^{+\infty} I^A_{m^k_A} - \lambda m^k_A\, dA \geq \int_{l_j}^{+\infty} m^k_A(l_j - \lambda)\, dA, \end{aligned} \tag{3.15}$$

where in the last inequality we used (3.11), (3.2) and (3.13). Taking the supremum over $k \in \mathbb{N}$ we get, using (3.14)

$$c \geq \sup_k E_\lambda(v_k) \geq \sup_k\left(\int_{l_j}^{+\infty} m^k_A(l_j - \lambda)\, dA\right) \geq (l_j - \lambda)\varepsilon \to +\infty \quad \text{as } l_j \to +\infty,$$

a contradiction.

□

We have the following Lemma.

Lemma 3.4. (Approximation by smooth functions)
Let $v \in \mathcal{X}(\hat\Omega)$. Then, there is a sequence $v_A \in W^{1,2}_0(\hat\Omega)$ such that

$$v_A \to v \text{ in } W^{1,2}(\hat\Omega) \cap L^1(\hat\Omega), \tag{3.16}$$

and

$$\lim_{A\to+\infty}\int_{\hat\Omega} |z||\nabla v_A - \nabla v| = 0. \tag{3.17}$$

Proof of Lemma 3.4
First we note that $v \in L^1(\hat{\Omega})$ by Remark 2.1. Let $A > 1$ we define the cut off functions $\eta_A \in W_0^{1,\infty}(\mathbb{R})$ by

$$\eta_A(z) := \begin{cases} 1 & \text{if } |z| \leq A, \\ 1 - \frac{1}{A}(|z| - A) & \text{if } A \leq |z| \leq 2A, \\ 0 & \text{if } 2A \leq |z|. \end{cases}$$

Then

$$|\eta_A'(z)| \leq \frac{2}{|z|} \text{ a.e.} \tag{3.18}$$

The functions $v_A(y, z) := \eta_A(z)v(y, z)$ belong to $W^{1,2}(\hat{\Omega})$, they have compact support in $\hat{\Omega}_A = (-1, 1) \times (-2A, 2A)$ and zero trace on $\partial\hat{\Omega}_A$. Since the boundary of each $\hat{\Omega}_A$ is Lipschitz and bounded we have by [12, Theorem 15.29] that $v_A \in W_0^{1,2}(\hat{\Omega}_A)$. It is not difficult to see that $v_A \to v$ in $W^{1,2}(\hat{\Omega})$, we will show that $\lim_{A\to+\infty} |z||\nabla v_A - \nabla v| = 0$.
We have

$$\begin{aligned} \int_{\hat{\Omega}} |z||\nabla v_A - \nabla v| &\leq \int_{\hat{\Omega}} |z|(|\eta_A' v| + |\eta_A - 1||\nabla v|) \\ &\leq \int_{\hat{\Omega}\cap\{A<|z|<2A\}} 2|v| + \int_{\hat{\Omega}\cap\{A<|z|\}} |z||\nabla v| \end{aligned}$$

then, using (3.18) and the fact that $|z||\nabla v|, |v| \in L^1(\hat{\Omega})$ the right hand side of the above estimate converges to zero as $A \to +\infty$.
The convergence in $L^1(\hat{\Omega})$ in (3.16) follows by Remark 2.1. □

For two sets $U, U' \subset \mathbb{R}^2$, by $U \subset\subset U'$ we mean that U is relatively compact in U', i.e. $\overline{U} \subset U'$ and $\overline{U}$ is compact. Also for a function $u(y, z)$ we define the positive part $u^+(y, z) = \max\{u(y, z), 0\}$.

Proof of Theorem 2.2 (i)
Step 1. Boundedness of E_λ from below
We focus in the cases $\lambda > 0$ since for $\lambda = 0$ the minimizer of E_λ is trivially the zero function. We fix $\lambda > 0$, let $u \in \mathcal{X}$, using Poincaré's inequality in Ω (Remark 2.1) we get

$$\begin{aligned} E_\lambda(u) &= \int_\Omega \frac{|\nabla u|^2}{2} + |z||\nabla u| - \lambda \int_\Omega u \\ &\geq \int_\Omega \frac{|\nabla u|^2}{2} + (|z| - 2\lambda)|\nabla u|. \end{aligned}$$

We split the last integral in the domains $\{|z| \geq 2\lambda\} \cap \Omega$ and $\{|z| \leq 2\lambda\} \cap \Omega$ and get

$$\begin{aligned} E_\lambda(u) &\geq \int_{\{|z|\leq 2\lambda\}\cap\Omega} \frac{|\nabla u|^2}{2} - 2\lambda|\nabla u| \\ &\geq \int_{\{|z|\leq 2\lambda\}\cap\Omega} -2\lambda^2 > -\infty. \end{aligned}$$

Step 2. Minimizing sequence
Let $u_k \in \mathcal{X}$ with $\lim_{k\to+\infty} E_\lambda(u_k) = \inf_{v\in\mathcal{X}} E_\lambda(v)$. We will denote by c a generic positive constant which does not depend on the parameter k. There is a positive constant c such

that $\sup_{k\in\mathbb{N}} E_\lambda(u_k) \le c$, then as in Step 1 we use Poincare's inequality to get

$$\begin{aligned} c &\ge \int_{\{|z|\le 2\lambda\}\cap\Omega} \frac{|\nabla u_k|^2}{2} + (|z|-2\lambda)|\nabla u_k| + \int_{\{|z|\ge 2\lambda\}\cap\Omega} \frac{|\nabla u_k|^2}{2} \\ &\ge \int_{\{|z|\le 2\lambda\}\cap\Omega} \frac{|\nabla u_k|^2}{2} - \frac{|\nabla u_k|^2}{4} - (|z|-2\lambda)^2 + \int_{\{|z|\ge 2\lambda\}\cap\Omega} \frac{|\nabla u_k|^2}{2}, \end{aligned}$$

where in the second inequality we used Young's inequality$\left(|a||b| \le \frac{b^2}{4} + a^2\right)$. Is is easy now to see that

$$\int_\Omega |\nabla u_k|^2 \le c. \tag{3.19}$$

Then by Poincare's inequality and compactness there is $u \in W^{1,2}_{0L}(\Omega)$ such that $u_k \rightharpoonup u$ as $k \to +\infty$.

Using similar arguments we get $c \ge \int_\Omega (|z|-2\lambda)|\nabla u_k|$, or if we split the integral in the domains $\{|z| \ge 4\lambda\} \cap \Omega = \{|z| - 2\lambda \ge |z|/2\} \cap \Omega$ and $\{|z| \le 4\lambda\} \cap \Omega$ we get

(3.20)

$$\frac{1}{2}\int_{\{|z|-2\lambda\ge|z|/2\}\cap\Omega} |z||\nabla u_k| \le \int_{\{|z|\ge 4\lambda\}\cap\Omega} (|z|-2\lambda)|\nabla u_k| \le c - \int_{\{|z|\le 2\lambda\}\cap\Omega} (|z|-2\lambda)|\nabla u_k|,$$

since $\int_{\{2\lambda\le|z|\le 4\lambda\}\cap\Omega} (|z|-2\lambda)|\nabla u_k| \ge 0$. We can now bound the right hand side of (3.20) using Hölders inequality and (3.19) and get eventually that $\int_{\{|z|\ge 4\lambda\}\cap\Omega} |z||\nabla u_k| \le c$. Using Hölders inequality and (3.19), one can also bound the quantity $\int_{\{|z|\le 4\lambda\}\cap\Omega} |z||\nabla u_k|$ uniformly in k, we can therefore conclude that

$$\int_\Omega |z||\nabla u_k| \le c, \tag{3.21}$$

where again c is a positive constant independent of k.

Step 3. Lower semicontinuity

We will show that

$$\int_\Omega \frac{|\nabla u|^2}{2} + |z||\nabla u| \le \liminf_{k\to+\infty} \int_\Omega \frac{|\nabla u_k|^2}{2} + |z||\nabla u_k|, \tag{3.22}$$

and

$$-\lambda \int_\Omega u \le \liminf_{k\to+\infty} \left(-\lambda \int_\Omega u_k\right). \tag{3.23}$$

Equations (3.19), (3.21) and (3.22) imply that $u \in \mathcal{X}$ and then $u \in L^1(\Omega)$ by Remark 2.1. Whereas, equations (3.22) and (3.23) together imply that $E_\lambda(u) \le \liminf_{k\to+\infty} E_\lambda(u_k)$, which shows that u is a minimizer of E_λ in $\mathcal{X}$. Since the integrand in (3.22) is non-negative convex in the gradient variable and measurable in the z variable, the inequality (3.22) follows from [9, Chapter I, Theorem 2.5].

For $l > 0$ fixed we have

$$\begin{aligned} \int_\Omega u_k &= \int_l^{+\infty}\int_{-1}^1 u_k(y,-A)\,dydA + \int_0^l\int_{-1}^1 u_k(y,-A)\,dydA \qquad (3.24) \\ &\le \sup_k \left(\int_l^{+\infty}\int_{-1}^1 u_k(y,-A)\,dydA\right) + \int_0^l\int_{-1}^1 u_k(y,-A)\,dydA. \qquad (3.25) \end{aligned}$$

Since $E_\lambda(u_k)$ is uniformly bounded we can apply Lemma 3.3 and get that (3.10) holds for the sequence u_k. Using (3.10) and the fact that $u \in L^1(\Omega)$, we can take the lim sup in (3.24), as $k \to +\infty$ and then $l \to +\infty$ and get $\limsup_{k\to+\infty} \int_\Omega u_k \leq \int_\Omega u$ or else (3.23), which completes the proof of the lower semi-continuity of E_λ and hence the existence of a minimizer $u \in \mathcal{X}$.

Step 4. Uniqueness

Let $u, \tilde{u} \in \mathcal{X}$ be two minimizers, then using similar arguments as in [6, Section 3.5.4, p.36] one can show that

$$\int_\Omega \nabla u \cdot (\nabla \tilde{u} - \nabla u) + \int_\Omega |z||\nabla \tilde{u}| - \int_\Omega |z||\nabla u| \geq \lambda \int_\Omega \tilde{u} - u, \tag{3.26}$$

$$\int_\Omega \nabla \tilde{u} \cdot (\nabla u - \nabla \tilde{u}) + \int_\Omega |z||\nabla u| - \int_\Omega |z||\nabla \tilde{u}| \geq \lambda \int_\Omega u - \tilde{u}. \tag{3.27}$$

If we add equations (3.26) and (3.27) we get

$$\int_\Omega |\nabla u - \nabla \tilde{u}|^2 \leq 0,$$

hence $u = \tilde{u}$ in Ω since they also have the same lateral boundary conditions.

Step 5. Non-negative minimizer

We have by [20, Corollary 2.1.8, page 47] that $\nabla u^+ = (\nabla u) \cdot \chi_{\{u>0\}}$, where by $\chi_{\{u>0\}}$ we denote the characteristic function of the set $\{(y,z) : u(y,z) > 0\}$. Since also $-\lambda \int_\Omega u^+ \leq -\lambda \int_\Omega u$ we have $E_\lambda(u^+) \leq E_\lambda(u)$, hence $u = u^+$ by the uniqueness of minimizers.

Step 6. $\lambda \in [0,1]$

Our goal is to show that

$$E_\lambda(u) \geq 0, \qquad \text{for all } u \in \mathcal{X}, \tag{3.28}$$

then because $0 \in \mathcal{X}$ and $E_\lambda(0) = 0$ we get that the unique minimizer of E_λ is the zero function. In view of Lemma 3.4, it is enough to prove (3.28) for functions u with $\hat{u} \in W_0^{1,2}(\hat{\Omega})$. Let u be such a function, then as in Step 5 we have

$$E_\lambda(u^+) \leq E_\lambda(u). \tag{3.29}$$

Suppose that the compact support of $\hat{u}^+$ is contained in $[-1,1] \times (-A, A)$ where A is large enough, then we have

$$\begin{aligned}
\int_\Omega |z||\nabla u^+| &\geq \int_{-1}^{1} \int_0^A |z| \left| \frac{\partial u^+}{\partial z} \right| dzdy \\
&\geq \left| \int_{-1}^{1} \int_0^A z \frac{\partial u^+}{\partial z} \, dzdy \right| \\
&= \int_\Omega u^+
\end{aligned}$$

in the last equality we used integration by parts. This estimate together with (3.29) and the fact that $\lambda \leq 1$ gives

$$E_\lambda(u) \geq E_\lambda(u^+) \geq (1 - \lambda) \int_\Omega u^+ \geq 0.$$

Step 7. $\lambda \in (1, +\infty)$

Our goal is to prove that there is $u \in \mathcal{X}$ with $E_\lambda(u) < 0$. Let $\varphi \in C^\infty(-1,1)$, $\varphi \geq 0$ with $\varphi(-1) = 0 = \varphi(1)$ and $\int_{-1}^1 \varphi = 1$ (for example $\varphi(y) = \frac{3}{4}(1-y^2)$). We define

$$u(y,z) := k^{-3} e^{kz} \varphi(y)$$

where $k > 0$ is large enough, to be chosen later. It is $u \in \mathcal{X}$ and

$$\begin{aligned} \int_\Omega \frac{|\nabla u|^2}{2} &= \frac{1}{2} \int_{-1}^1 \left[k^{-2} (\varphi'(y))^2 + \varphi^2(y) \right] dy \int_{-\infty}^0 (k^{-2} e^{kz})^2 \, dz \\ &= \frac{A_k}{4} k^{-5} \end{aligned} \tag{3.30}$$

where we set $A_k = \int_{-1}^1 \left[k^{-2}(\varphi'(y))^2 + \varphi^2(y) \right] \, dy$. Also

$$\begin{aligned} \int_\Omega |z||\nabla u| &= \int_{-1}^1 \sqrt{k^{-2}(\varphi'(y))^2 + \varphi^2(y)} \, dy \int_{-\infty}^0 |z| k^{-2} e^{kz} \, dz \\ &= B_k \int_{-\infty}^0 |z| k^{-2} e^{kz} \, dz \end{aligned} \tag{3.31}$$

where $B_k = \int_{-1}^1 \sqrt{k^{-2}(\varphi'(y))^2 + \varphi^2(y)} \, dy$. If we integrate by parts the second product component of the right hand side of (3.31) we get

$$\int_{-\infty}^0 |z| k^{-2} e^{kz} \, dz = \int_{-\infty}^0 k^{-3} e^{kz} = \int_{-\infty}^0 k^{-3} e^{kz} \int_{-1}^1 \varphi = \int_\Omega u,$$

then (3.31) becomes

$$\int_\Omega |z||\nabla u| = B_k \int_\Omega u. \tag{3.32}$$

We also have $\int_\Omega u = k^{-4}$, then we can write $E_\lambda(u)$ using (3.30) and (3.32) as

$$E_\lambda(u) = \frac{A_k}{4} k^{-5} + (B_k - \lambda) k^{-4}. \tag{3.33}$$

Next we note that $B_k \geq 1$, is decreasing in k (and so is A_k) and $B_k \to 1$ as $k \to +\infty$. Since $\lambda > 1$ we can find k_0 large enough such that $B_{k_0} < \lambda$, then (3.33) becomes

$$E_\lambda(u) \leq \frac{A_{k_0}}{4} k^{-5} + (B_{k_0} - \lambda) k^{-4},$$

for all $k \geq k_0$. We can now conclude if we choose $k \geq k_0$ large enough, since the function k^{-5} decreases faster than k^{-4}, for example $k > \max\{k_0, \frac{A_{k_0}}{4(\lambda - B_{k_0})}\}$. □

3.3 The ε-approximation

Let $\lambda > 0$, u_λ be the minimizer of E_λ given by Theorem 2.2 (i). For $A > 0$ we define $\hat{\Omega}_A = \{(y,z) \in \hat{\Omega} : |z| \leq A\}$, $\Omega_A = \Omega \cap \hat{\Omega}_A$ and

$$\mathcal{H}_A = \{v \in W^{1,2}(\Omega_A),\ v = u_\lambda, \text{ on } \partial\Omega_A \setminus \{z=0\}\},$$

We are interested in approximate minimizers of (1.3), for this we study the minimizers in $\mathcal{H}_A$ of the approximate functional

$$E^A_{\varepsilon,\lambda}(u) = \int_{\Omega_A} \frac{|\nabla u|^2}{2} + |z| \sqrt{\varepsilon^2 + |\nabla u|^2} - \lambda u, \tag{3.34}$$

where $\varepsilon > 0$.
Since we have mixed boundary conditions, an easy way to describe the space of test functions for the weak formulation of the first variation of (3.34) is to use reflection in the domain $\hat{\Omega}_A$. We will simply write $\hat{\phi} \in W_0^{1,2}(\hat{\Omega}_A)$ for the test functions. We have the following Proposition.

Proposition 3.5. **($W^{2,2}_{\text{loc}}$ regularity of approximate problem)**
Let $A, \varepsilon, \lambda > 0$, then there exists a unique minimizer $u_{\varepsilon,A} \in \mathcal{H}_A$ of $E^A_{\varepsilon,\lambda}$. Moreover, $\hat{u}_{\varepsilon,A} \in W^{2,2}_{\text{loc}}(\hat{\Omega}_A)$ and the following equation holds

$$\int_{\Omega_A} \nabla u_{\varepsilon,A} \cdot \nabla \varphi + |z| \frac{\nabla u_{\varepsilon,A} \cdot \nabla \varphi}{\sqrt{\varepsilon^2 + |\nabla u_{\varepsilon,A}|^2}} = \lambda \int_{\Omega_A} \varphi, \quad \textit{for all } \hat{\varphi} \in W_0^{1,2}(\Omega_A), \tag{3.35}$$

and $\partial_z u_{\varepsilon,A}(y, 0) = 0$ for $y \in (-1, 1)$.

The existence of a minimizer is a consequence of the direct method in the bounded domain Ω_A, while the regularity results are standard. We give a sketch of the Proof of Proposition 3.5 in Appendix A.

Proof of Theorem 2.2 (ii)-(iv)
Step 1. Solutions of E-L equation are minimizers of (1.3)
First we will show that for any pair $(u, q) \in \mathcal{X} \times \Lambda$ that satisfies equation (2.3), u is a minimizer of E_λ. Let $v \in \mathcal{X}$, using (2.3) and the fact that $|q| \leq 1$ it is easy to check that $q \in \partial|\cdot|(\nabla u)$ in Ω. By the definition of the subdifferential we have

$$E_\lambda(v) - E_\lambda(u) \geq \int_\Omega \nabla u \cdot \nabla(v - u) + |z| q \cdot \nabla(v - u) - \lambda \int_\Omega (v - u) = 0, \tag{3.36}$$

where we used (2.3) with test function $\varphi = v - u \in \mathcal{X}$.
Step 2. Approximating solutions
As usual we will focus in the case $\lambda > 0$. Let $u = u_\lambda$ be the minimizer of E_λ given by Theorem 2.2 (i). For $\varepsilon > 0$ let $u_{\varepsilon,A}$ be the minimizer of $E^A_{\varepsilon,\lambda}$ given by Proposition 3.5, then for all $A > 0$ we will show that $u_{\varepsilon,A} \to u$ strongly as $\varepsilon \to 0$, in $W^{1,2}(\Omega_A)$ up to a subsequence. Extending $u_{\varepsilon,A}$ by u_λ outside Ω_A, we can write the following variational inequalities as in the Step 1 of the proof of Theorem 2.2 (i)

$$\int_{\Omega_A} \nabla u \cdot (\nabla u_{\varepsilon,A} - \nabla u) + \int_{\Omega_A} |z||\nabla u_{\varepsilon,A}| - \int_{\Omega_A} |z||\nabla u| \geq \lambda \int_{\Omega_A} u_{\varepsilon,A} - u \tag{3.37}$$

and
(3.38)

$$\int_{\Omega_A} \nabla u_{\varepsilon,A} \cdot (\nabla u - \nabla u_{\varepsilon,A}) + \int_{\Omega_A} |z|\sqrt{\varepsilon^2 + |\nabla u|^2} - \int_{\Omega_A} |z|\sqrt{\varepsilon^2 + |\nabla u_{\varepsilon,A}|^2} \geq \lambda \int_{\Omega_A} u - u_{\varepsilon,A}.$$

Adding inequalities (3.37) and (3.38), we get

$$\begin{aligned}
\int_{\Omega_A} |\nabla u_{\varepsilon,A} - \nabla u|^2 &\leq \int_{\Omega_A} |z|(|\nabla u_{\varepsilon,A}| - \sqrt{\varepsilon^2 + |\nabla u_{\varepsilon,A}|^2}) + |z|(\sqrt{\varepsilon^2 + |\nabla u|^2} - |\nabla u|) \\
&\leq \int_{\Omega_A} |z|(\sqrt{\varepsilon^2 + |\nabla u|^2} - |\nabla u|) \\
&= \int_{\Omega_A} |z| \frac{\varepsilon^2}{\sqrt{\varepsilon^2 + |\nabla u|^2} + |\nabla u|} \leq A|\Omega_A|\varepsilon,
\end{aligned}$$

Then using also Poincare's inequality we get for all $A > 0$ and up to a subsequence

$$\nabla u_{\varepsilon,A} \to \nabla u,\ u_{\varepsilon,A} \to u \text{ a.e. in } \Omega_A \text{ as } \varepsilon \to 0. \tag{3.39}$$

Step 3. The function q

For $q_{\varepsilon,A} = \frac{\nabla u_{\varepsilon,A}}{\sqrt{\varepsilon^2+|\nabla u_{\varepsilon,A}|^2}}$, we have $q_{\varepsilon,A} \cdot \nabla u_{\varepsilon,A} \leq |\nabla u_{\varepsilon,A}|$, then using (3.39) it is not difficult to see that

$$q_{\varepsilon,A} \cdot \nabla u_{\varepsilon,A} \to |\nabla u| \text{ a.e. in } \Omega_A \text{ as } \varepsilon \to 0. \tag{3.40}$$

Since $q_{\varepsilon,A} \in L^2_{\text{loc}}(\Omega_A, \mathbb{R}^2)$ with $|q_{\varepsilon,A}| \leq 1$, there exists $q_A \in L^2_{\text{loc}}(\Omega_A, \mathbb{R}^2)$ with $|q_A| \leq 1$ and such that $q_{\varepsilon,A}$ converges weakly to q_A in $L^2(U, \mathbb{R}^2)$, as $\varepsilon \to 0$, for every $U \subset\subset \Omega_A$. Then using also (3.39) we have $\lim_{\varepsilon\to 0} \int_U q_{\varepsilon,A} \cdot \nabla u_{\varepsilon,A} = \int_U q_A \cdot \nabla u$ for all $U \subset\subset \Omega_A$ and by (3.40) we get that $q_A \cdot \nabla u = |\nabla u|$ a.e. in Ω_A. Extending q_A by zero outside Ω_A we may wright $q_A \in L^2_{\text{loc}}(\Omega, \mathbb{R}^2)$ and as before we can find $q \in L^2_{\text{loc}}(\Omega, \mathbb{R}^2)$, with $|q| \leq 1$ and such that q_A converges weakly to q in $L^2(U, \mathbb{R}^2)$, as $A \to +\infty$, for every $U \subset\subset \Omega_A$, and hence $q \cdot \nabla u = |\nabla u|$ a.e.

Step 4. Passing to the limit $\varepsilon \to 0, A \to +\infty$

Let φ with $\hat{\varphi} \in W^{1,2}_0(\hat{\Omega})$, then equation (3.35) with A large enough holds for this test function and since $q_{\varepsilon,A}$ is bounded we can pass to the limit as $\varepsilon \to 0$ and get

$$\int_\Omega \nabla u \cdot \nabla\varphi + |z| q_A \cdot \nabla\varphi = \lambda \int_\Omega \varphi.$$

We can now pass to the limit as $A \to +\infty$ and using also Lemma 3.4 we get (2.3).

Step 5. Uniqueness

Let (u, q), $(\bar{u}, \bar{q})$ be two solutions of (2.3) then by Step 1 we have $u = \bar{u}$, since minimizers of (1.3) in $\mathcal{X}$ are unique by Theorem 2.2 (i). Then in the set $\{\nabla u \neq 0\}$ the vectors $q, \bar{q}$ are parallel to ∇u and so is $q - \bar{q}$, but since $(q - \bar{q}) \cdot \nabla u = 0$ by (2.3) we have $q = \bar{q}$ a.e. in $\{\nabla u \neq 0\}$.

Step 6. Neumann condition

We denote by ∂_{x_i}, $i = 1, 2$ respectively the derivatives ∂_y, ∂_z. Let $i, j \in \{1, 2\}$, $\hat{U} \subset\subset \hat{\Omega}_A$, by Proposition 3.5 we have that $\hat{u}_{\varepsilon,A} \in W^{2,2}_{\text{loc}}(\hat{U})$, by Lemma A.1 the second derivatives of $\hat{u}_{\varepsilon,A}$ are uniformly bounded in $L^2(\hat{U})$, hence for $\varphi \in W^{1,2}_0(\hat{U})$ we have (up to a subsequence)

$$\int_{\hat{U}} \partial_{x_i}\hat{u}\partial_{x_j}\varphi = \lim_{\varepsilon\to 0} \int_{\hat{U}} \partial_{x_i}\hat{u}_{\varepsilon,A}\partial_{x_j}\varphi = -\lim_{\varepsilon\to 0} \int_{\hat{U}} \partial_{x_j}\partial_{x_i}\hat{u}_{\varepsilon,A}\varphi = -\int_{\hat{U}} g\varphi,$$

for some function $g \in L^2(\hat{U})$. We have proved that $\hat{u} \in W^{2,2}_{\text{loc}}(\hat{\Omega})$, then applying a Sobolev embedding Theorem ([7, Section 5.6.3]) we get that $\hat{u} \in C^{0,\alpha}_{\text{loc}}(\hat{\Omega})$ for all $\alpha \in (0, 1)$. As in the proof of Proposition 3.5 we can now define the trace of the derivative of u on $\{z = 0\}$ and $\partial_z u(y, 0) = 0$ for $y \in (-1, 1)$. $\square$

4 Properties of the solution

4.1 Comparison Principle

In view of Theorem 2.2 (i) we will assume that $\lambda > 1$ for the rest of the paper.

Definition 4.1. Sub/supersolution
Let $u \in \mathcal{X}$ be non-negative and $q \in \Lambda$, Λ as in (2.1), we call the pair (u,q) a subsolution (resp. a supersolution) of the equation (2.3) if

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla \varphi + |z| q \cdot \nabla \varphi \leq \lambda \int_\Omega \varphi \ (\textit{resp.} \geq \lambda \int_\Omega \varphi) & \textit{for all } \varphi \in \mathcal{X}, \varphi \geq 0, \\ q \cdot \nabla u = |\nabla u| & \textit{a.e. in } \Omega. \end{cases} \tag{4.1}$$

Proposition 4.2. Comparison principle
Let $u, v \in \mathcal{X}$, $q_u, q_v \in \Lambda$ with $(u, q_u), (v, q_v)$ a subsolution and a supersolution respectively of (2.3), with $0 = u \leq v$ on $\{-1, 1\} \times (-\infty, 0)$ in the sense of traces, then

$$u \leq v, \qquad \textit{in } \Omega.$$

Proof of Proposition 4.2
Let $\varphi = (u-v)_+$, then $\varphi \in \mathcal{X}$. If we write the inequalities (4.1) for u, v with this test function and subtract the one from the other we get

$$\int_\Omega \nabla(u-v) \cdot \nabla(u-v)_+ + |z|(q_u - q_v) \cdot \nabla(u-v)_+ \leq 0,$$

or if we use [20, Corollary 2.1.8, page 47] we can write it as

$$\int_\Omega |\nabla(u-v)|^2 \chi_{\{u-v \geq 0\}} \leq -\int_\Omega |z| \left[(q_u - q_v) \cdot \nabla(u-v)\right] \chi_{\{u-v\geq 0\}}. \tag{4.2}$$

Next we calculate, using the properties of q_u, q_v in Definition 4.1

$$\begin{aligned} (q_u - q_v) \cdot (\nabla u - \nabla v) &= |\nabla u| - q_u \cdot \nabla v - q_v \cdot \nabla u + |\nabla v| \\ &\geq |\nabla u| - |\nabla u| + |\nabla v| - |\nabla v| = 0, \ \text{ a.e.} \end{aligned}$$

then (4.2) implies

$$\nabla(u-v) = 0, \qquad \text{a.e. in } \{u - v \geq 0\}$$

or $\nabla(u-v)_+ = 0$ almost everywhere. Using the boundary conditions we can conclude that $(u-v)_+ = 0$ and hence $u \leq v$ a.e. in Ω. $\square$

Remark 4.3. (Monotonicity in λ)
For u_λ the minimizer of E_λ in $\mathcal{X}$ and $m(\lambda) = \int_\Omega u_\lambda$ the volume rate, using the comparison principle from Proposition 4.2 it is not difficult to see that $m(\lambda)$ is increasing in λ. Unfortunately, the physical volume rate is given, using the rescaling (1.24), by $m_0 = (l^2 \mu_s g_0 \cos\theta) m$, which does not allow us to directly study the monotonicity with respect the inclination angle θ ($\cos\theta$ is decreasing for $\theta \in [0, \pi/2)$ and $\lambda(\theta)$ is increasing by (1.25)).

4.2 Some explicit profiles

As we explained in the introduction, we study the first variation of the functional (1.6), i.e.

$$\frac{\phi\phi''}{(1+|\phi'|^2)^{3/2}} - \frac{1}{\sqrt{1+|\phi'|^2}} + \lambda = 0, \quad y \in (-1, 1). \tag{4.3}$$

Lemma 4.4. (An explicit solution of 4.3)
Let $\lambda > 1$, $K(\lambda)$ be given by (2.7) and $\phi_{K(\lambda)}$ defined in (2.6). Then the function $\phi_{K(\lambda)} \in C^\infty(-1,1) \cap C([-1,1])$ is non-positive and the following properties hold

$$\lim_{y\to -1} \phi'_{K(\lambda)}(y) = -\infty, \ \lim_{y\to 1} \phi'_{K(\lambda)}(y) = +\infty. \tag{4.4}$$

Moreover the function $\phi_{K(\lambda)}$ is convex with minimum $\phi_{K(\lambda)}(0) = \frac{K(\lambda)}{\lambda-1}$ and maximum $\phi_{K(\lambda)}(\pm 1) = \frac{K(\lambda)}{\lambda}$ and if $\bar\lambda > \lambda$ then $\phi_{K(\bar\lambda)}(y) < \phi_{K(\lambda)}(y)$, for $y \in [-1,1]$.

Proof of Lemma 4.4
Step 1. The inverse function
Let $\lambda > 1$ and $Z \in [\frac{1}{\lambda}, \frac{1}{\lambda-1}]$, $f_\lambda(Z)$ be given by (2.5). Notice that f_λ is smooth in $(\frac{1}{\lambda}, \frac{1}{\lambda-1})$ and that it has been chosen so that

$$f'_\lambda(Z) = \frac{(\lambda Z - 1)\sqrt{\lambda^2-1}}{\sqrt{1-((\lambda^2-1)Z-\lambda)^2}}, \tag{4.5}$$

from which we get that f_λ is strictly increasing in $[\frac{1}{\lambda}, \frac{1}{\lambda-1}]$. We set

$$A_\lambda := f_\lambda\left(\frac{1}{\lambda-1}\right) - f_\lambda\left(\frac{1}{\lambda}\right) = \frac{\pi}{2(\lambda^2-1)^{3/2}} + \frac{1}{\lambda^2-1}\left(1 + \frac{\mathrm{Arcsin}\left(\frac{1}{\lambda}\right)}{\sqrt{\lambda^2-1}}\right), \tag{4.6}$$

by the monotonicity of f we can define the positive function ϕ implicitly in the intervals $[-A_\lambda, 0]$ and $[0, A_\lambda]$ as follows

$$f_\lambda(\phi(y)) = f_\lambda\left(\frac{1}{\lambda-1}\right) - |y|, \quad y \in [-A_\lambda, A_\lambda], \tag{4.7}$$

then $f_\lambda(\phi(y)) = f_\lambda(\phi(-y))$ for $y \in [0, A_\lambda]$, which means that ϕ is an even function thanks to the monotonicity of f_λ. Also by (4.7) we have $\phi(0) = 1/(\lambda-1)$ and by (4.5) we can calculate the limit $\lim\limits_{Z\to 1/(\lambda-1)} f'(Z)$ and get $\lim\limits_{y\to 0^+} \phi'(y) = 0$. Since ϕ is even and smooth in the intervals $[-A_\lambda, 0)$ and $(0, A_\lambda]$ we eventually get $\phi'(0) = 0$. We have concluded that $\phi \in C^1(-A_\lambda, A_\lambda)$.

Relation (4.7) gives also for $y \in [-A_\lambda, A_\lambda]$

$$1/\lambda = \phi(\pm A_\lambda) \le \phi(y) \le \phi(0) = 1/(\lambda-1) \tag{4.8}$$

and by (4.5)

$$\phi'(-A_\lambda) = +\infty, \ \phi'(A_\lambda) = -\infty. \tag{4.9}$$

Step 2. ϕ satisfies (4.3)
Using (4.5) we can differentiate (4.7) and taking the squares in both sides of the equation, we get for $y \in (-A_\lambda, A_\lambda)$,

$$|\phi'|^2 \frac{(\lambda\phi-1)^2(\lambda^2-1)}{1-((\lambda^2-1)\phi-\lambda)^2} = 1$$

or after a few simplifications

$$|\phi'|^2 = \frac{1}{(\lambda - \frac{1}{\phi})^2} - 1.$$

Noting that $\phi \geq 1/\lambda > 0$, the above equation can be rewritten as

$$\phi\left(\lambda - \frac{1}{\sqrt{1+|\phi'|^2}}\right) = 1. \tag{4.10}$$

Let $K_0 < 0$, we define

$$\phi_{K_0}(y) := K_0\phi(\frac{y}{K_0}), \quad y \in [A_\lambda K_0, -A_\lambda K_0], \tag{4.11}$$

by (4.10), the negative function ϕ_{K_0} satisfies

$$\phi_{K_0}(y)\left(\lambda - \frac{1}{\sqrt{1+|\phi'_{K_0}(y)|^2}}\right) = K_0, \quad y \in (A_\lambda K_0, -A_\lambda K_0). \tag{4.12}$$

In particular, if $K(\lambda)$ is given by (2.7), differentiating (4.12) with respect to y we get

$$\phi'_{K(\lambda)}\left(\frac{\phi_{K(\lambda)}\phi''_{K(\lambda)}}{(1+|\phi'_{K(\lambda)}|^2)^{3/2}} - \frac{1}{\sqrt{1+|\phi'_{K(\lambda)}|^2}} + \lambda\right) = 0, \quad y \in (-1,0)\cup(0,1). \tag{4.13}$$

Using equation (4.13) we calculate for $y \in (-1,0)\cup(0,1)$

$$\phi''_{K(\lambda)} = \frac{(1+|\phi'_{K(\lambda)}|^2)(\lambda\sqrt{1+|\phi'_{K(\lambda)}|^2} - 1)}{-\phi_{K(\lambda)}} > 0, \tag{4.14}$$

here we have also used equation (4.12) in order to get the sign of the second derivative. Since $\phi_{K(\lambda)} \in C^1(-1,1)$ we get from (4.14) that in fact $\phi_{K(\lambda)} \in C^2((-1,1))$. Differentiating further (4.14) and using (4.8) we get by iteration $\phi_{K(\lambda)} \in C^\infty(-1,1)$.

Step 3. Extrema

By (4.8) and (4.11) we have

$$\begin{cases} \min\limits_{|y|\leq 1} \phi_{K(\lambda)} = \phi_{K(\lambda)}(0) = \dfrac{K(\lambda)}{\lambda - 1} = \dfrac{1}{(\lambda-1)(f_\lambda(\frac{1}{\lambda}) - f_\lambda(\frac{1}{\lambda-1}))}, \\ \max\limits_{|y|\leq 1} \phi_{K(\lambda)} = \phi_{K(\lambda)}(-1) = \phi_{K(\lambda)}(1) = \dfrac{K(\lambda)}{\lambda} = \dfrac{1}{\lambda(f_\lambda(\frac{1}{\lambda}) - f_\lambda(\frac{1}{\lambda-1}))}. \end{cases} \tag{4.15}$$

It is

(4.16)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{K(\lambda)}(1) = -\frac{4(2\lambda^2+1)\mathrm{Arcsin}\left(\frac{1}{\lambda}\right)\sqrt{\lambda^2-1} + 2\pi(2\lambda^2+1)\sqrt{\lambda^2-1} + 4(\lambda^2-1)(\lambda^2+2)}{\lambda^2\left(2\sqrt{\lambda^2-1}+\pi+2\mathrm{Arcsin}\left(\frac{1}{\lambda}\right)\right)^2} < 0$$

and

(4.17)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi_{K(\lambda)}(0) = -\frac{4\left(\lambda-\frac{1}{2}\right)(\lambda-1)\sqrt{\lambda^2-1}\left(\pi+2\mathrm{Arcsin}\left(\frac{1}{\lambda}\right)\right) + 4(\lambda^2-1)\left[(\lambda-1)^2+\frac{\lambda-1}{\lambda}\right]}{(\lambda-1)^2\left(2\sqrt{\lambda^2-1}+\pi+2\mathrm{Arcsin}\left(\frac{1}{\lambda}\right)\right)^2} < 0.$$

Figure 2.3b is the graph of the function $\phi_{K(\lambda)}(1)$ in terms of the variable λ.

Step 4. Monotonicity of the graphs in λ

Let $\bar{\lambda} > \lambda$ we will show that $\phi_{K(\bar{\lambda})}(y) < \phi_{K(\lambda)}(y)$, for $y \in [0,1]$. Since the functions are even and we already have the monotonicity of the boundary points by Step 3, we will focus in the interval $(0,1)$. If we use equation (4.13), we get that the function $w(y) = \phi_{K(\bar{\lambda})}(y) - \phi_{K(\lambda)}(y)$ satisfies the elliptic equation

$$-a_1(y)w''(y) + a_2(y)w'(y) + a_3(y)w(y) = \lambda - \bar{\lambda},$$

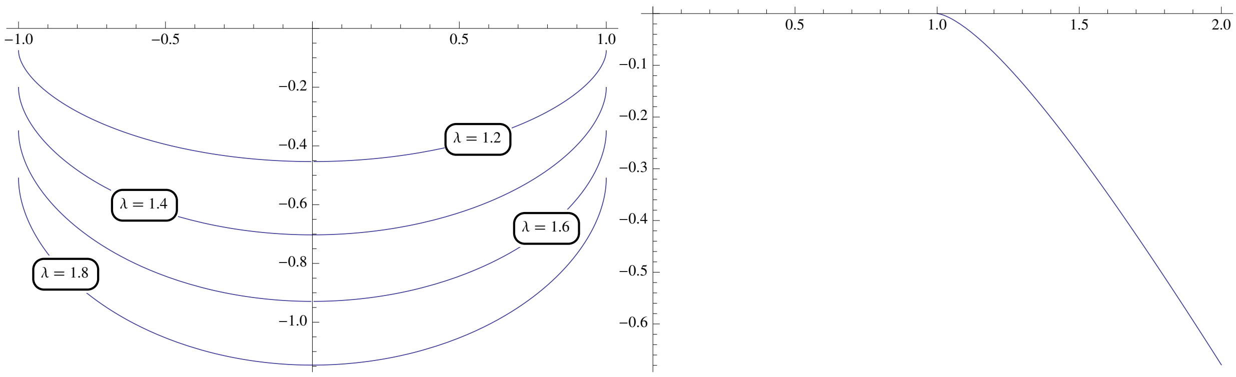
with

$$a_1(y) = \frac{-\phi_{K(\bar{\lambda})}(y)}{(1+|\phi'_{K(\bar{\lambda})}(y)|^2)^{3/2}}, \quad a_3(y) = \frac{\phi''_{K(\lambda)}(y)}{(1+|\phi'_{K(\bar{\lambda})}|^2)^{3/2}},$$

and

$$a_2(y) = \int_0^1 G_1(p(t,y))dt + \phi''_{K(\lambda)}(y)\phi_{K(\lambda)}(y)\int_0^1 G_2(p(t,y))dt,$$

with $p(t,y) = \phi'_{K(\lambda)}(y) + t(\phi'_{K(\bar{\lambda})}(y) - \phi'_{K(\lambda)}(y))$, $G_1(p) = \frac{-p}{(1+|p|^2)^{3/2}}$ and $G_2(p) = \frac{-3p}{(1+|p|^2)^{5/2}}$. It is $a_i \in C(0,1)$, $i = 1,2,3$ with $a_1, a_3 > 0$ in $(0,1)$ and $w \in C^2((0,1)) \cap C([0,1])$ with $w(0), w(1) < 0$ by (4.16), (4.17). We can now conclude that $w < 0$ by a maximum principle. $\square$



(a) $\phi_{K(\lambda)}$ for $\lambda = 1.2, 1.4, 1.6, 1.8$.

(b) $\phi_{K(\lambda)}(1)$

FIGURE 2.3

Using the function $\phi_{K(\lambda)}$ constructed in Lemma 4.4 we can define a diffeomorphism in $\mathcal{C}_\lambda \cap \Omega$, with $\mathcal{C}_\lambda$ as in (2.8). Let $L \in (0,+\infty)$, we define

$$\phi_L(y) := L\phi_{K(\lambda)}\left(\frac{y}{L}\right), \quad y \in [-L, L].$$

We have the following Lemma.

Lemma 4.5. (A diffeomorphism)
Let $\phi_{K(\lambda)}$ be as in (2.6), then for $(y,z) \in \overline{\mathcal{C}_\lambda \cap \Omega} \setminus \{(0,0)\}$ there is a unique $L = L(y,z) \in (0,+\infty)$ implicitly defined by

$$z = L\phi_{K(\lambda)}\left(\frac{y}{L}\right) = \phi_L(y), \tag{4.18}$$

and $L \in C^\infty(\mathcal{C}_\lambda \cap \Omega) \cap C(\overline{\mathcal{C}_\lambda \cap \Omega} \setminus \{(0,0)\})$.

Proof of Lemma 4.5
Since the family of curves $\{(y,\phi_L(y))\}_{L\in(0,+\infty)}$ are obtained as a rescaling of the function $\phi_{K(\lambda)}$ we have that the mapping $(y,L)\mapsto(y,z)$ is a surjection ; it is also an injection since the family of curves $\{(y,\phi_L(y))\}_{L\in(0,+\infty)}$ do not intersect. On the other hand the same bijective correspondence can be established locally by the implicit function theorem since $\frac{y}{L}\phi'_{K(\lambda)}\left(\frac{y}{L}\right)-\phi_{K(\lambda)}\left(\frac{y}{L}\right)>0$ (since $\phi_{K(\lambda)}$ is even and negative), from which we also get the smoothness of $L(y,z)$ in $\mathcal{C}_\lambda\cap\Omega$ because $\phi_{K(\lambda)}$ is smooth. The continuity of L up to the boundary follows from the definition and the continuity of $\phi_{K(\lambda)}$. □

Using the diffeomorphism from Lemma 4.5 we can define $q=q_\lambda(y,z)\in C^\infty(\mathcal{C}_\lambda\cap\Omega,\mathbb{R}^2)\cap C(\overline{\mathcal{C}_\lambda\cap\Omega}\setminus\{(0,0)\},\mathbb{R}^2)$ as follows

$$q(y,z):=\frac{(-\phi'_{L(y,z)}(y),1)}{\sqrt{1+|\phi'_{L(y,z)}(y)|^2}},\quad (y,z)\in\overline{\mathcal{C}_\lambda\cap\Omega}\setminus\{(0,0)\},\tag{4.19}$$

where $\phi'_{L(y,z)}(y)=\phi'_{K(\lambda)}\left(\frac{y}{L(y,z)}\right)$. Note that the boundary values of q make sense because of the boundary values of $\phi'_{K(\lambda)}$ by Lemma (2.6). We have the following Lemma

Lemma 4.6. (An equation for q)
Let $\lambda>1$, q as in (4.19) then

$$-\operatorname{div}(|z|q(y,z))=\lambda,\quad \textit{for } (y,z)\in(\mathcal{C}_\lambda\cap\Omega).\tag{4.20}$$

Proof of Lemma 4.6
All the equations in this proof hold for $(y,z)\in\mathcal{C}_\lambda\cap\Omega$. Having in mind the diffeomorphism $(y,z)\mapsto(\bar{y},L(y,z))$, with $\bar{y}(y)=y$ from Lemma 4.5, we can write $q=q(\bar{y},L(y,z))=(q_1(\bar{y},L(y,z)),q_2(\bar{y},L(y,z)))$. Since $|z|=-z$ in Ω, we have

$$\operatorname{div}_{(y,z)}(|z|q)=|z|\operatorname{div}_{(y,z)}(q)-q_2\tag{4.21}$$

and

$$\begin{aligned}\partial_y q_1&=\partial_{\bar{y}}q_1+\partial_L(q_1)\partial_y L\\ \partial_z q_2&=\partial_L(q_2)\partial_z L.\end{aligned}\tag{4.22}$$

In order to simplify the notation we set $\psi=\phi_{K(\lambda)}$, then using (4.18) we can write $q_1=\frac{-\psi'(y/L)}{\sqrt{1+|\psi'(y/L)|^2}}$ and $q_2=\frac{1}{\sqrt{1+|\psi'(y/L)|^2}}$, from which we can calculate

$$\begin{aligned}\partial_L q_1&=\frac{\psi''\left(\frac{y}{L}\right)\left(\frac{y}{L^2}\right)}{\left(1+|\psi'\left(\frac{y}{L}\right)|^2\right)^{3/2}},\\ \partial_L q_2&=\frac{\psi''\left(\frac{y}{L}\right)\psi'\left(\frac{y}{L}\right)\left(\frac{y}{L^2}\right)}{\left(1+|\psi'\left(\frac{y}{L}\right)|^2\right)^{3/2}}.\end{aligned}\tag{4.23}$$

Differentiating (4.18) in y and z we get

$$\begin{aligned}\partial_z L&=\frac{-1}{\frac{y}{L}\psi'\left(\frac{y}{L}\right)-\psi\left(\frac{y}{L}\right)},\\ \partial_y L&=\frac{\psi'\left(\frac{y}{L}\right)}{\frac{y}{L}\psi'\left(\frac{y}{L}\right)-\psi\left(\frac{y}{L}\right)}.\end{aligned}\tag{4.24}$$

Using (4.23), (4.24) we get $\partial_L(q_1)\partial_y L + \partial_L(q_2)\partial_z L = 0$ and hence we get from (4.22)

$$\operatorname{div}_{(y,z)} q = \partial_{\bar{y}} q_1 = \frac{d}{d\bar{y}} \frac{-\phi_L'(\bar{y})}{\sqrt{1+|\phi_L'(\bar{y})|^2}} = \frac{-\phi_L''(\bar{y})}{(1+|\phi_L'(\bar{y})|^2)^{3/2}}. \tag{4.25}$$

Using the fact that $\bar{y} = y$, $z < 0$, (4.18) and (4.25), equation (4.21) becomes

$$-\operatorname{div}_{(y,z)}(|z|q) = -\frac{\phi_L(y)\phi_L''(y)}{(1+|\phi_L'(y)|^2)^{3/2}} + \frac{1}{\sqrt{1+|\phi_L'(y)|^2}},$$

and finally using the above equation together with (4.13) and the definition of ϕ_L we conclude

$$-\operatorname{div}_{(y,z)}(|z|q) = \lambda, \quad (y,z) \in \mathcal{C}_\lambda \cap \Omega.$$

$\square$

Note also that by (4.24) and the boundary conditions of $\phi'_{K(\lambda)}$ we can extend $L \in C^1(\overline{\mathcal{C}_\lambda \cap \Omega} \setminus \{(0,0)\})$.

Lemma 4.7. (Bound on the Laplacian)
Let L be as in (4.18), then there are positive constants C_1, C_2 such that if

$$C = C(\lambda) := 1 + \left(\frac{\lambda - 1}{K(\lambda)}\right)^2 \tag{4.26}$$

we have

$$(\partial_y L(y,z))^2 + (\partial_z L(y,z))^2 \leq C \quad \textit{for } (y,z) \in \mathcal{C}_\lambda \cap \Omega, \tag{4.27}$$

$$L(y,z)\Delta_{(y,z)} L(y,z) \leq C_1 + C_2 \quad \textit{for } (y,z) \in \mathcal{C}_\lambda \cap \Omega, \tag{4.28}$$

in particular we have

$$0 \leq \Delta_{(y,z)}(L(y,z))^2 \leq 2(C + C_1 + C_2), \quad (y,z) \in \mathcal{C}_\lambda \cap \Omega. \tag{4.29}$$

Proof of Lemma 4.7
As in the proof of Lemma 4.6 we simplify the notation by setting $\psi = \phi_{K(\lambda)}$.
Step 1. Bound on $\partial_z L$ and $\partial_y L$
By (4.14) we have $\psi'' > 0$ in $(-1,1)$, then, using also (4.15) we can estimate by the maximum

$$\frac{1}{y\psi'(y) - \psi(y)} \leq \frac{\lambda - 1}{-K(\lambda)}, \quad \text{for } y \in (-1,1). \tag{4.30}$$

Let $(y,z) \in \mathcal{C}_\lambda \cap \Omega$, by the diffeomorphism in Lemma 4.5 we have $\frac{|y|}{L(y,z)} \leq 1$, hence using (4.30) and the formula of $\partial_z L$ by (4.24) we have $|\partial_z L(y,z)| \leq \frac{\lambda-1}{-K(\lambda)}$.

Similarly for $\partial_y L$ given by the formula (4.24), since

$$\frac{d}{dy}\left(\frac{\psi'(y)}{y\psi'(y) - \psi(y)}\right) = \frac{-\psi(y)\psi''(y)}{(y\psi'(y) - \psi)^2} > 0 \quad \text{for } y \in (-1,1),$$

and $\lim\limits_{y\to 1} \dfrac{\psi'(y)}{y\psi'(y) - \psi(y)} = 1$, we have $|\partial_y L(y,z)| \leq 1$ for $(y,z) \in \mathcal{C}_\lambda \cap \Omega$. Combining the bounds of $\partial_z L$ and $\partial_y L$ we get (4.27).
Step 2. Bound on second derivatives

If we differentiate (4.18) twice in z and y respectively and use (4.24) we get

$$L\partial_{zz}^2 L = \frac{\psi''\left(\frac{y}{L}\right)\left(\frac{y}{L}\right)^2}{\left(\frac{y}{L}\psi'\left(\frac{y}{L}\right) - \psi\left(\frac{y}{L}\right)\right)^3} \tag{4.31}$$

and

$$L\partial_{yy}^2 L = \frac{\psi''\left(\frac{y}{L}\right)\psi^2\left(\frac{y}{L}\right)}{\left(\frac{y}{L}\psi'\left(\frac{y}{L}\right) - \psi\left(\frac{y}{L}\right)\right)^3}. \tag{4.32}$$

We estimate in $\mathcal{C}_\lambda \cap \Omega$

$$L\partial_{zz}^2 L \leq \max\left\{\max_{\frac{|y|}{L}\leq\frac{1}{2}} L\partial_{zz}^2 L, \sup_{\frac{1}{2}<\frac{|y|}{L}<1} L\partial_{zz}^2 L\right\}.$$

Using the fact that $y\psi'(y) \geq 0$ and the maximum of ψ by (4.15) we estimate

$$\max_{\frac{|y|}{L}\leq\frac{1}{2}} L\partial_{zz}^2 L \leq \frac{1}{4}\left(\frac{\lambda}{-K(\lambda)}\right)^3 \max_{|y|\leq\frac{1}{2}} \psi''(y). \tag{4.33}$$

For $1/2 < |y/L| < 1$ it is $\psi' \neq 0$ and we can rewrite (4.31) as

$$L\partial_{zz}^2 L = \frac{\psi''\left(\frac{y}{L}\right)}{\left|\psi'\left(\frac{y}{L}\right)\right|^3} \cdot \frac{\left(\frac{y}{L}\right)^2}{\left(\left|\frac{y}{L}\right| + \frac{-\psi\left(\frac{y}{L}\right)}{\left|\psi'\left(\frac{y}{L}\right)\right|}\right)^3}, \tag{4.34}$$

and by equation (4.14) we calculate in the same interval

$$\frac{\psi''}{|\psi'|^3} = \frac{\lambda\left(\frac{1}{|\psi'|^2}+1\right)^{3/2} - \frac{1}{|\psi'|^3} - \frac{1}{|\psi'|}}{-\psi}. \tag{4.35}$$

Substituting (4.35) in (4.34) and using properties of ψ and the monotonicity of ψ' we get the bound

$$\sup_{\frac{1}{2}<\frac{|y|}{L}<1} L\partial_{zz}^2 L \leq \frac{2\lambda^2}{-K(\lambda)}\left(\frac{1}{\left|\psi'\left(\frac{1}{2}\right)\right|^2}+1\right)^{3/2}. \tag{4.36}$$

Finally by (4.36) and (4.33) we get $\sup_{\mathcal{C}_\lambda\cap\Omega} L\partial_{zz}^2 L \leq C_1$, with C_1 a positive constant. Similarly one can show that $\sup_{\mathcal{C}_\lambda\cap\Omega} L\partial_{yy}^2 L \leq C_2$ with

$$C_2 = \max\left\{\frac{\lambda}{-K(\lambda)}\max_{|y|\leq\frac{1}{2}}\psi''(y), 8\left(\frac{\lambda}{\lambda-1}\right)^2(-K(\lambda))\left(\frac{1}{\left|\psi'\left(\frac{1}{2}\right)\right|^2}+1\right)^{3/2}\right\}.$$

$\square$

4.3 A subsolution

Remark 4.8. *Let $\sigma : \Omega_1 \cup \Omega_2 \to \mathbb{R}^2$, with $\Omega_1, \Omega_2 \subset \mathbb{R}^2$, two bounded domains with Lipschitz boundary and a common smooth boundary $\partial\Omega$, with surface measure dS. Suppose that $\sigma \in \bigcap_{i=1}^{2} (C^1(\Omega_i, \mathbb{R}^2) \cap C(\overline{\Omega_i}, \mathbb{R}^2))$, $\mathrm{div}\sigma \in L^2(\Omega_1) \cap L^2(\Omega_2)$, we denote by $\mathrm{Tr}\,|_{\Omega_i}\,\sigma$, $i = 1, 2$, the limit value of σ from the sides Ω_i respectively. Then for $\phi \in W_0^{1,2}(\Omega_1 \cup \Omega_2)$ with $\mathrm{supp}(\phi) \cap \partial\Omega \neq \emptyset$ it is*

$$\int_{\Omega_1 \cup \Omega_2} \sigma \cdot \nabla\phi = -\int_{\Omega_1 \cup \Omega_2} \mathrm{div}(\sigma)\phi + \int_{\partial\Omega} n \cdot (\mathrm{Tr}\,|_{\Omega_1}\,\sigma - \mathrm{Tr}\,|_{\Omega_2}\,\sigma)\phi\, dS \tag{4.37}$$

where n is the normal to $\partial\Omega$ pointing at the direction of Ω_2.

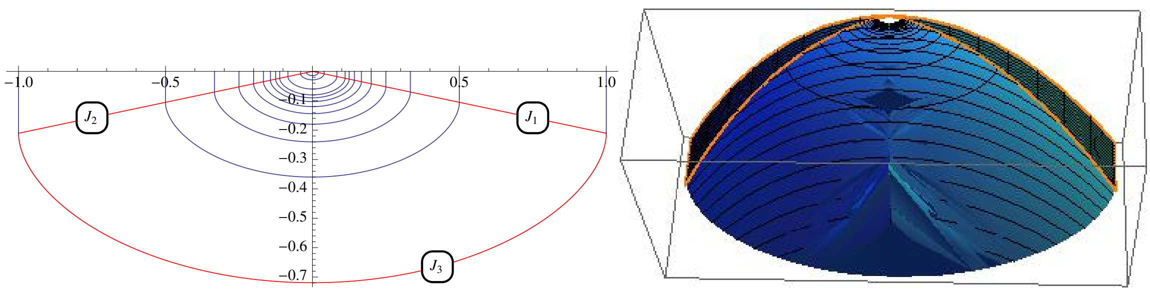
We can now construct a subsolution. In what follows we will favour intuition over mathematical elegance, as far as the notation is concerned, and we will instead denote the set $\mathrm{Epi}^{\smile}(\lambda)$ defined in (2.9), simply by $\{z > \phi_{K(\lambda)}\}$. Let $\zeta > 0$, using the diffeomorfism from Lemma 4.5 we can define the continuous function (see Figure 2.4)

$$u_{\zeta,\lambda}(y, z) := \begin{cases} -\zeta y^2 + \zeta & \text{in } \Omega \setminus \mathcal{C}_\lambda, \\ -\zeta L^2(y, z) + \zeta & \text{in } \mathcal{C}_\lambda \cap \{z \geq \phi_{K(\lambda)}\}, \\ 0 & \text{in } \{z < \phi_{K(\lambda)}\}, \end{cases} \tag{4.38}$$

and for q_λ as in (4.19) we define

$$d_\lambda^{\mathrm{ext}}(y, z) := \begin{cases} \left(-\frac{y}{|y|}, 0\right) & \text{in } \Omega \setminus \mathcal{C}_\lambda, \\ q_\lambda(y, z) & \text{in } \mathcal{C}_\lambda \cap \Omega. \end{cases} \tag{4.39}$$

Then we have that $u_{\zeta,\lambda} \in \mathcal{X}$ with $\partial_z u_{\zeta,\lambda}(y, 0) = 0$ for $y \in (-1, 1)$ and $d_\lambda^{\mathrm{ext}} \in \Lambda$. In the set $\mathcal{C}_\lambda \cap \{z > \phi_{K(\lambda)}\}$ we have $\nabla u_{\zeta,\lambda} = -2\zeta L(\partial_y L, \partial_z L)$, then using also (4.24), (4.38), (4.39), definition (4.19) and the properties of $\phi_{K(\lambda)}$ by Lemma 4.4 we have that $d_\lambda^{\mathrm{ext}} \cdot \nabla u_{\zeta,\lambda} = |\nabla u_{\zeta,\lambda}|$ a.e. in Ω.



(a) Curves with normal d_λ^{ext} (b) $u_{\zeta,\lambda}$

FIGURE 2.4

Proposition 4.9. (Subsolution)
Let $\lambda > 1$, then there is $1 < \lambda_0 < \lambda$ such that for $0 < \zeta_0 \leq \frac{\lambda - \lambda_0}{2(C + C_1 + C_2)}$, with C, C_1, C_2 given by Lemma 4.7, the pair $(u_{\zeta_0,\lambda_0}, d_{\lambda_0}^{\mathrm{ext}})$ given by (4.38)-(4.39), is a subsolution of the equation (2.3).

Proof of Proposition 4.9
Step 1. The subsolution inequalities

We will first show the subsolution inequalities in the set

$$\Omega_1 \cup \Omega_2 \cup \Omega_3 := \left(\Omega \setminus \overline{\mathcal{C}_{\lambda_0}}\right) \cup \left(\mathcal{C}_{\lambda_0} \cap \{z > \phi_{K(\lambda_0)}\}\right) \cup \left(\mathcal{C}_{\lambda_0} \setminus \{z \geq \phi_{K(\lambda_0)}\}\right)$$

where the functions u_{ζ_0,λ_0}, $d^{\mathrm{ext}}_{\lambda_0}$ are smooth. Using (4.20) and (4.39) we calculate

$$-\mathrm{div}(|z| d^{\mathrm{ext}}_{\lambda_0}(y,z)) = \begin{cases} 0 & \text{in } \Omega \setminus \overline{\mathcal{C}_{\lambda_0}}, \\ \lambda_0 & \text{in } \mathcal{C}_{\lambda_0} \cap \Omega. \end{cases} \tag{4.40}$$

Also

$$-\Delta u_{\zeta_0,\lambda_0} = \begin{cases} 2\zeta_0 & \text{in } \Omega \setminus \overline{\mathcal{C}_{\lambda_0}}, \\ 0 & \text{in } \Omega \cap \{z < \phi_{K(\lambda_0)}\}. \end{cases} \tag{4.41}$$

and using Lemma 4.7 we get in $\mathcal{C}_{\lambda_0} \cap \{z > \phi_{K(\lambda_0)}\}$

$$-\Delta u_{\zeta_0,\lambda_0} = \zeta_0 \Delta L^2 \leq 2\zeta_0 (C + C_1 + C_2) \tag{4.42}$$

If we now combine (4.40)-(4.42), use the fact that the positive constant $2(C + C_1 + C_2)$ depends only on λ_0, we can choose $\zeta_0 \leq \frac{\lambda - \lambda_0}{2(C+C_2+C_2)} (< \lambda/2$ since $C + C_1 + C_2 > 1$ by (4.26)) and get

$$-\Delta u_{\zeta_0,\lambda_0} - \mathrm{div}(|z| d^{\mathrm{ext}}_{\lambda_0}) \leq \lambda \quad \text{in } \Omega_1 \cup \Omega_2 \cup \Omega_3. \tag{4.43}$$

It remains to show that inequality (4.43) holds in the rest of Ω. We will use Remark 4.8 for $\sigma = \nabla u_{\zeta_0,\lambda_0} + |z| d^{\mathrm{ext}}_{\lambda_0}$. Note that σ is not defined at $(0,0)$ but we still have that it is bounded near $z = 0$ by Lemma 4.7.

Step 2. The Dirac masses

Note that since $\partial_z u_{\zeta_0,\lambda_0}(y,0) = 0$ and therefore $\sigma(y,0) = 0$, for $y \in (-1,1) \setminus \{(0,0)\}$, in view of (4.37), we do not need to take into account the boundary $\{z = 0\}$. We denote by $J = J_1 \cup J_2 \cup J_3$ the three parts of the boundary of $\Omega_1 \cup \Omega_2 \cup \Omega_3$ as in Figure (2.4a). We will show the subsolution inequalities on J. We need to estimate for $(i,j) \in \{(1,2),(2,3)\}$, the terms

$$n_j \cdot \left(\mathrm{Tr}\,|_{\Omega_i}\, \sigma - \mathrm{Tr}\,|_{\Omega_j}\, \sigma\right), \tag{4.44}$$

where n_j is the normal of the common boundary pointing in the direction of Ω_j. For J_1, the right common boundary of Ω_1 and Ω_2 we have $n_1 = \left(\frac{K(\lambda_0)}{\lambda_0}, -1\right)$, using (4.4) and (4.19) one can see that that $d^{\mathrm{ext}}_{\lambda}$ is continuous in Ω, therefore using (4.38) we get

$$\mathrm{Tr}\,|_{\Omega_1}\, \sigma - \mathrm{Tr}\,|_{\Omega_2}\, \sigma = (-2\zeta_0 y, 0) + 2\zeta_0 L(\partial_y L, \partial_z L) = 0, \quad (y,z) \in J_1, \tag{4.45}$$

where we used the fact that $y = L$ on J_1 and $(\partial_y L, \partial_z L) = (1,0)$ by the Neumann conditions in (4.4). In a similar way we can write (4.44) on J_2 as

$$-n_2 \cdot (\mathrm{Tr}\,|_{\Omega_1}\, \sigma - \mathrm{Tr}\,|_{\Omega_2}\, \sigma) = 0. \tag{4.46}$$

where $n_2 = \left(\frac{-K(\lambda_0)}{\lambda_0}, -1\right)$. On J_3 we simplify the notation and set $\psi = \phi_{K(\lambda_0)}$, then (4.44) becomes

$$n_3 \cdot (\mathrm{Tr}\,|_{\Omega_2}\,\sigma - \mathrm{Tr}\,|_{\Omega_3}\,\sigma) = \left(\frac{\psi'}{\sqrt{1+|\psi'|^2}}, \frac{-1}{\sqrt{1+|\psi'|^2}}\right) \cdot (-2\zeta_0 L(\partial_y L, \partial_z L)) \tag{4.47}$$
$$= -2\zeta_0 \frac{\sqrt{1+|\psi'|^2}}{y\psi' - \psi} \leq 0,$$

where in the last equality we used equations (4.24) and that $L = 1$ on J_3. We can now conclude from estimates (4.45), (4.46) and (4.47). □

Proof of Theorem 2.3 (lower bound)
If we compare the subsolution u_{ζ_0,λ_0} by Proposition 4.9 with the solution u_λ of (2.3) using Proposition 4.2, we get $0 \leq u_{\zeta_0,\lambda_0} \leq u_\lambda$ in Ω for all $\lambda_0 \in (1, \lambda)$, hence by definitions (4.38) and (2.9) we get

$$\{u_{\zeta_0,\lambda_0} > 0\} = \{z > \phi_{K(\lambda_0)}\} = \mathrm{Epi}^{\smile}(\lambda_0) \subset \{u_\lambda > 0\}, \quad \text{for all } \lambda_0 \in (1, \lambda). \tag{4.48}$$

We set $\psi(y, \lambda) = \phi_{K(\lambda)}(y)$ for $(y, \lambda) \in [0, 1] \times (1, +\infty)$. By definition (2.6) we have that ψ satisfies the equation $F(y, \lambda, \psi(y, \lambda)) = 0$ with

$$F : \{(y, \lambda, z) : y \in (0, 1),\ \lambda \in (1, +\infty),\ z \in \left(\frac{K(\lambda)}{\lambda - 1}, \frac{K(\lambda)}{\lambda}\right)\} \to \mathbb{R}$$

given by

$$F(y, \lambda, z) = K(\lambda) f_\lambda\left(\frac{z}{K(\lambda)}\right) - K(\lambda) f_\lambda\left(\frac{1}{\lambda - 1}\right) - y.$$

The using the formulas (2.5), (2.7) and (4.6) one can check that F is smooth in the domain of it's definition. Since $f'_\lambda\left(\frac{\psi(y,\lambda)}{K(\lambda)}\right) > 0$ for $(y, \lambda) \in (0, 1) \times (1, +\infty)$ we have by the implicit function theorem that $\psi \in C^\infty((0, 1) \times (1, +\infty))$. Since $\phi_{K(\lambda)}$ is even, we get that for fixed $y \in (-1, 0) \cup (0, 1)$ the function $\phi_{K(\lambda)}(y)$ is continuous in λ in $(1, +\infty)$. By the formulas of $\phi_{K(\lambda)}(\pm 1)$, $\phi_{K(\lambda)}(0)$ by Lemma (4.4) and the continuity of the function $K(\lambda)$ we get that $\lim\limits_{\lambda_0 \uparrow \lambda} \phi_{K(\lambda_0)}(y) = \phi_{K(\lambda)}(y)$ for all $y \in [-1, 1]$. We can now pass to the limit in (4.48) and conclude. □

4.4 A supersolution

Let $\lambda > 1$, $\lambda_1 > \lambda$ and ϑ, b, Π given by (2.10), (2.11), (2.12) respectively. Using the diffeomorphism from Lemma 4.5 with $\phi_{K(\lambda_1)}$ in (4.18) we can consider sets of the form $\{(y, z) \in \mathcal{C}_\lambda \cap \Omega : 1 \leq L(y, z) \leq b\}$, where the level set $\{(y, z) \in \mathcal{C}_\lambda \cap \Omega : L(y, z) = 1\}$ is the graph $\{z = \phi_{K(\lambda_1)}\}$; we will simply denote by $\{1 \leq L(y, z) \leq b\}$ these sets. We define

$$u_1^{\lambda_1}(y, z) := \frac{\lambda_1}{2}(1 - y^2), \quad (y, z) \in \Omega, \tag{4.49}$$

$$u_2^{\lambda_1,\vartheta}(y, z) := \begin{cases} +\infty & \text{in } \{z > \phi_{K(\lambda_1)}\}, \\ \vartheta(L(y, z) - b)^2 & \text{in } \{1 \leq L(y, z) \leq b\}, \\ 0 & \text{in } \{b \leq L(y, z)\}. \end{cases} \tag{4.50}$$

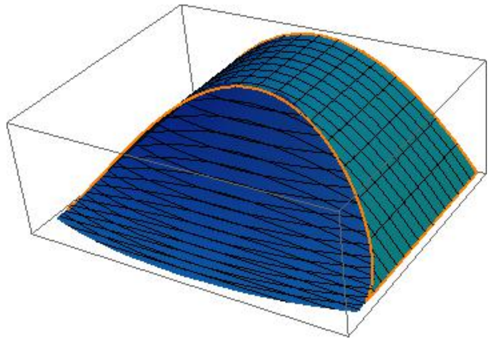


FIGURE 2.5 – $U^{\lambda_1,\vartheta}$

where we simply write ϑ for $\vartheta_{\lambda,\lambda_1}$. Also, we define

$$U^{\lambda_1,\vartheta} = \min\{u_1^{\lambda_1}, u_2^{\lambda_1,\vartheta}\}, \quad \text{in } \Omega. \tag{4.51}$$

We note that the intersection of the graphs of the functions $u_2^{\lambda_1,\vartheta}$ and $u_1^{\lambda_1}$ lies in the domain $\Omega \cap \{L(y,z) < b\}$ and is given by the equation

$$\vartheta(L(y,z) - b)^2 = \frac{\lambda_1}{2}(1-y^2), \quad (y,z) \in \Omega \cap \{L(y,z) < b\}, \tag{4.52}$$

or else since $L < b$

$$L(y,z) = b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)} \geq b - \sqrt{\frac{\lambda_1}{2\vartheta}} = 1,$$

by the definition of b. Also, since $\partial_z L < 0$ in $\mathcal{C}_\lambda \cap \Omega$ the curve defined by the contour (4.52) is the graph of a function which lies in fact in the set $\{1 \leq L(y,z) \leq b\}$, and therefore, the function $U^{\lambda_1,\vartheta}$ is continuous, see Figure 2.5. For q_{λ_1} as in (4.19) we define for a.e. $y \in \Omega$ the vector field

(4.53)

$$q_{\lambda_1}^{\text{ext}}(y,z) := \begin{cases} \left(-\frac{y}{|y|}, 0\right) & \text{in } (\{1 \leq L(y,z) < b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)}\} \cup \{z > \phi_{K(\lambda_1)}\}) \cap \{y \neq 0\} \\ q_{\lambda_1}(y,z) & \text{in } \{b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)} < L(y,z)\}. \end{cases}$$

We have the following Proposition.

Proposition 4.10. (Supersolution)
Let $\lambda > 1$, then the function $U^{\lambda_1,\vartheta}$ defined in (4.51) *is a supersolution of* (2.3).

Proof of Proposition 4.10
A straightforward calculation shows that $\nabla U^{\lambda_1,\vartheta} \cdot q_{\lambda_1}^{\text{ext}} = |\nabla U^{\lambda_1,\vartheta}|$, a.e. in Ω. We also have $\partial_z U^{\lambda_1,\vartheta}(y,0) = \partial_z u_1^{\lambda_1}(y,0) = 0$.
Step 1. Supersolution inequalities

It is

$$-\Delta U^{\lambda_1,\vartheta} = \begin{cases} \lambda_1 & \text{in } \{1 \leq L(y,z) < b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)}\} \cup \{z > \phi_{K(\lambda_1)}\} \\ -2\vartheta((\partial_y L)^2 + (\partial_z L)^2) & \\ \quad + 2\vartheta(b-L)(\partial^2_{yy} L + \partial^2_{zz} L) & \text{in } \{b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)} < L(y,z) < b\} \\ 0 & \text{in } \{b < L(y,z)\}, \end{cases}$$

and as in (4.40) we have

$$-\mathrm{div}(|z| q^{\mathrm{ext}}_{\lambda_1}) = \begin{cases} 0 & \text{in } (\{1 \leq L(y,z) < b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)}\} \cup \{z > \phi_{K(\lambda_1)}\}) \cap \{y \neq 0\} \\ \lambda_1 & \text{in } \{b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)} < L(y,z)\}. \end{cases}$$

Therefore if C is as in (4.26), we have $\vartheta = \frac{\lambda_1 - \lambda}{2C}$ and

$$-\Delta U_{\lambda_1,\vartheta} - \mathrm{div}(|z| q^{\mathrm{ext}}_{\lambda_1}) \geq \lambda, \quad \text{in } \Omega \setminus (\{L(y,z) = b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)}\} \cup \{0\} \times \left(\frac{K(\lambda_1)}{\lambda_1 - 1}, 0\right)).$$

Note that the solution of the equation $L(0,z) = 1$ is $z = \frac{K(\lambda_1)}{\lambda_1 - 1}$. We also note that by (4.31), (4.32) and Step 2 of the proof of Lemma 4.7 we have that $\Delta U_{\theta,\lambda_1}$ is bounded.

Step 2. Dirac masses

The discontinuities of the vector fields $\nabla U_{\lambda_1,\vartheta}$ and $q^{\mathrm{ext}}_{\lambda_1}$ lie on the intersection given by the contour (4.52) and on $\{0\} \times \left(\frac{K(\lambda_1)}{\lambda_1 - 1}, 0\right)$. For the second set only the vector field $q^{\mathrm{ext}}_{\lambda_1}$ is discontinuous and the Dirac mass it creates is

$$|z|(1,0) \cdot ((1,0) - (-1,0)) \geq 0.$$

For the intersection, eq. (4.52), we suppress the indices λ_1, ϑ and we write the Dirac mass as

$$n \cdot \left[(\nabla u_1 - \nabla u_2) + |z| \left(\frac{\nabla u_1}{|\nabla u_1|} - \frac{\nabla u_2}{|\nabla u_2|}\right)\right], \tag{4.54}$$

where n is the normal to the intersection pointing at the direction of $\{L(y,z) > b - \sqrt{\frac{\lambda_1}{2\vartheta}(1-y^2)}\}$. Then the $z-$component of n is negative, and since $L_z < 0$ by (4.24) we have

$$n = \frac{\nabla u_1 - \nabla u_2}{|\nabla u_1 - \nabla u_2|}.$$

Clearly we have $n \cdot (\nabla u_1 - \nabla u_2) \geq 0$. The second term of (4.54) is

$$\frac{|z|}{|\nabla u_1 - \nabla u_2|} \left(|\nabla u_1| + |\nabla u_2| - \nabla u_1 \cdot \nabla u_2 \frac{|\nabla u_1| + |\nabla u_2|}{|\nabla u_1||\nabla u_2|}\right) \geq 0$$

by the Cauchy-Schwartz inequality. This concludes the proof. □

Proof of Theorem 2.3 (upper bound)

We will estimate $\mathrm{supp}\, u$ from above. By Propositions 4.10 and 4.2 we get $0 \leq u_\lambda \leq U_{\lambda_1,\vartheta}$ in Ω and since $\mathrm{supp}\, U_{\lambda_1,\vartheta} = \mathrm{Epi}_\smile(\lambda_1)$ we get the desired estimate. □

Let $\lambda_1^\star = \lambda_1^\star(\lambda) > \lambda$ be a minimizer of $\Pi(\lambda, \cdot)$ (see discussion before Theorem 2.3). In Figure 2.6 we give the graph of $\Pi(\lambda, \lambda_1^\star)$ for different values of λ and in Table 2.1 the corresponding minimizers and minimal values. In fact one notices that the difference $\lambda_1^\star - \lambda$ increases as $\lambda \to +\infty$, see Figure 2.7a.

$\lambda = 1.2$	$\lambda_1^\star = 1.59451$	$\Pi = 3.20584$
$\lambda = 1.4$	$\lambda_1^\star = 1.84198$	$\Pi = 3.66274$
$\lambda = 1.6$	$\lambda_1^\star = 2.09337$	$\Pi = 4.16455$
$\lambda = 1.8$	$\lambda_1^\star = 2.34819$	$\Pi = 4.69225$

TABLE 2.1 – Optimal $\lambda_1^\star$

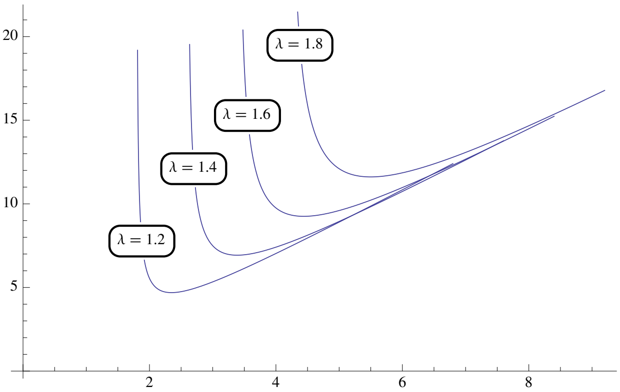
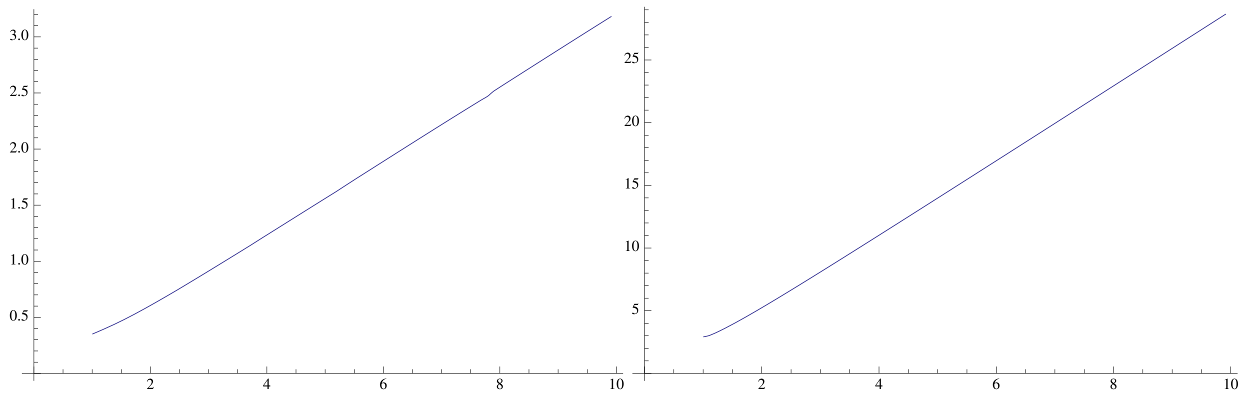


FIGURE 2.6 – $\Pi(\lambda, \lambda_1^\star)$ for $\lambda = 1.2, 1.4, 1.6, 1.8$



(a) $\lambda_1^\star - \lambda$ as $\lambda \to +\infty$

(b) $\Pi(\lambda, \lambda_1^\star)$ as $\lambda \to +\infty$

FIGURE 2.7

A Regularity of ε-minimizers

In what follows we will denote by c a generic constant which does not depend on the ε mentioned in Proposition 3.5.

Proof of Proposition 3.5
Step 1. Existence/Uniqueness

The uniqueness of the minimizer follows by the strict convexity of the functional or using similar arguments as in the proof of Step 1 of Theorem 2.2 (i). The existence is also similar, in fact the lower semicontinuity of the linear term $-\lambda \int_{\Omega_A} u$ is trivial since the domain Ω_A is bounded. We set

$$F(z,p) = \frac{|p|^2}{2} + |z|\sqrt{\varepsilon^2 + |p|^2}, \tag{1.1}$$

for $(z,p) \in \Omega_A \times \mathbb{R}^2$. It is

$$|\xi|^2 \le \frac{\partial^2 F}{\partial p_i \partial p_j}\xi_i\xi_j \tag{1.2}$$

for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, and

$$\left|\frac{\partial^2 F}{\partial p_i \partial p_j}\right| \le c\left(1+\frac{A}{\varepsilon}\right), \text{ for all } i,j \in \{1,2\}, \tag{1.3}$$

we set $c_2 := c\left(1+\frac{A}{\varepsilon}\right)$.

Step 3. Regularity

Since the proof of regularity is standard we are only going to emphasize the particularities of the problem, i.e. the fact that F is only Lipschitz continuous in the z variable. We will simply write $F(z,\nabla u)$ for $F(z,\nabla u(y,z))$. Let φ with $\hat{\varphi} \in W_0^{1,2}(\hat{\Omega}_A)$, then equation (3.35) holds as the first variation of the functional $E^A_{\varepsilon,\lambda}$. Moreover, using a change of variables one can see that the function $w := \hat{u}_{\varepsilon,A}$ satisfies

$$\int_{\hat{\Omega}_A} \nabla w \cdot \nabla\varphi + |z|\frac{\nabla w \cdot \nabla \varphi}{\sqrt{\varepsilon^2 + |\nabla w|^2}} = \lambda \int_{\hat{\Omega}_A} \varphi. \tag{1.4}$$

We study the regularity properties of (1.4). Let $|h| < \operatorname{dist}(\operatorname{supp}\varphi, \partial\hat{\Omega}_A)$, we define $\varphi_{k,h}(y,z) := \varphi((y,z) - he_k)$, $k = 1,2$, with e_k, $k = 1,2$ the unit vectors on the axes y and z respectively. We use $\varphi_{k,h}$ as a test function in (1.4) and estimate the derivative of the difference quotient

$$\Delta_h^k w(y,z) = \frac{w((y,z)+he_k) - w(y,z)}{h}. \tag{1.5}$$

Since the proof is similar we will only present the estimate for e_2. Using $\varphi_{2,h} = \varphi_h$ as a test function in (1.4) and after changing the variables in the integral we get

$$\int_{\hat{\Omega}_A} \partial_{p_i} F(z+h, (\nabla w)_h)\partial_{x_i}\varphi = \lambda \int_{\hat{\Omega}_A} \varphi, \tag{1.6}$$

where $\partial_{p_i} F = \frac{\partial F}{\partial p_i}$, $(\nabla w)_h(y,z) = \nabla w(y, z+h)$ and $\partial_{x_i}\varphi$, $i = 1,2$ is the partial derivative of φ in the directions y, z respectively. As usual subtracting (1.4) from (1.6) we get after a few calculations

$$\int_{\hat{\Omega}_A} \frac{1}{h}(\partial_{p_i}F(z+h,(\nabla w)_h) - \partial_{p_i}F(z+h,\nabla w))\partial_{x_i}\phi = -\int_{\hat{\Omega}_A}\frac{1}{h}(\partial_{p_i}F(z+h,\nabla w) - \partial_{p_i}F(z,\nabla w))\partial_{x_i}\phi. \tag{1.7}$$

The right hand side of (1.7) can be estimated using the Lipschitz continuity of $\nabla_p F$ in the z variable, we have

$$\frac{1}{|h|}|\nabla_p F(z+h,\nabla w) - \nabla_p F(z,\nabla w)| = \frac{|\nabla w|}{\sqrt{\varepsilon^2+|\nabla w|^2}}\frac{||z+h|-|z||}{|h|} \le 1.$$

It is now a standard process to use (1.2) and (1.3) in order to bound the quantity $\int |\nabla \Delta_h w|^2$ uniformly in h, we have

$$\int_{\Omega'} |\nabla\Delta_h w|^2 \le 2c_3(1+2c_2)\int_{\Omega_A} |\nabla w|^2, \tag{1.8}$$

with c_3 a constant independent of h and $\Omega' \subset\subset \Omega'' \subset\subset \Omega_A$. We then have $w \in W^{2,2}(\Omega'')$ by standard arguments.

Step 4. Neumann condition

Since $\hat{u}_\varepsilon \in W^{2,2}_{\mathrm{loc}}(\hat{\Omega}_A)$ we can define $\partial_z u_\varepsilon(y,0)$ for a.e. $y \in (-1,1)$ and since $\hat{u}$ is symmetric with respect to $\{z=0\}$, it is in fact $\partial_z u_\varepsilon(y,z) = -\partial_z u_\varepsilon(y,-z)$ for $(y,z) \in \hat{\Omega}_A$; setting $z=0$ we get the desired result. $\square$

The constant c_2 in the estimate (1.8) depends on ε. Using an argument similar to the proof of [8, Theorem 3.3.4] we can show that the second derivative of u_ε is bounded in L^2, uniformly in ε. We have the following Lemma.

Lemma A.1. (Uniform bound on $|\nabla^2 u_\varepsilon|$)
Let $A>0$, u_ε as in Proposition 3.5 and $\Omega' \subset\subset \Omega'' \subset\subset \hat{\Omega}_A$. Then there exists a positive constant $C = C(A, \mathrm{dist}(\Omega', \partial\hat{\Omega}_A))$ such that

$$\int_{\Omega'} |\nabla \partial_{x_i} \hat{u}_\varepsilon|^2 \leq C(1 + \int_{\hat{\Omega}_A} |\partial_{x_i} \hat{u}_\varepsilon|^2), \quad i = 1,2. \tag{1.9}$$

Proof of Lemma A.1

Since the proof is similar to the proof of Proposition 3.5, we will only give a sketch of it. We will only show the proof of the estimate (1.9) for $|\nabla \partial_z \hat{u}_\varepsilon|$ because the term with the partial derivative in the y variable is easier to estimate, since the integrand F from (1.1) does not depend on y. Let φ be a smooth function with compact support in Ω'' ; using $\partial_z \varphi$ as a test function in (1.4) and integrating by parts we can write, using the usual summation convention and the same notation as in the proof of Proposition 3.5

$$\int_{\Omega''} \partial_{p_i} \partial_z F(z, \nabla \hat{u}_\varepsilon) \partial_{x_i} \varphi = 0. \tag{1.10}$$

Or if we notice that $\partial_z \left(F(z, \nabla \hat{u}_\varepsilon(y,z))\right) = \partial_{\bar{z}} F(\bar{z}, \nabla \hat{u}_\varepsilon(y,z)) \mid_{\bar{z}=z} + \partial_{p_j} F(z, \nabla \hat{u}_\varepsilon(y,z)) \partial_{x_j} \partial_z \hat{u}_\varepsilon(y,z)$ and if $\partial_z |z| = \chi_{(0,+\infty)} - \chi_{(-\infty,0)}$, we may rewrite (1.10) as

$$\int_{\Omega''} \partial_z |z| \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_{x_i} \varphi + \int_{\Omega''} \left(\partial_{p_i} \partial_{p_j} F(z, \nabla \hat{u}_\varepsilon) \partial_{x_j} \partial_z \hat{u}_\varepsilon \right) \partial_{x_i} \varphi = 0. \tag{1.11}$$

As usual we choose a function $\eta \in C^2_0(\Omega'')$ with $\eta = 1$ in Ω' , $0 \leq \eta \leq 1$, $|\nabla \eta| \leq \frac{c}{\mathrm{dist}(\Omega', \partial\Omega_A)}$ and $||\nabla^2 \eta|| \leq \frac{c}{(\mathrm{dist}(\Omega', \partial\Omega_A))^2}$. We set $\varphi = \eta^3 \partial_z \hat{u}_\varepsilon$ in (1.11), use the convexity property (1.2) and the fact that

$$\partial_{p_i} \partial_{p_j} F(z, \nabla \hat{u}_\varepsilon) \partial_{x_j} \partial_z \hat{u}_\varepsilon = \partial_{x_i} \partial_z \hat{u}_\varepsilon + |z| \partial_z \left(\frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \right),$$

we get as in the proof of Proposition 3.5

$$\begin{aligned} \int_{\Omega''} \eta^3 |\nabla \partial_z \hat{u}_\varepsilon|^2 \leq & - \int_{\Omega''} \partial_{x_i} \partial_z \hat{u}_\varepsilon \partial_{x_i}(\eta^3) \partial_z \hat{u}_\varepsilon - \int_{\Omega''} \partial_z |z| \partial_{x_i}(\eta^3) \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_z \hat{u}_\varepsilon \\ & - \int_{\Omega''} \partial_z |z| \eta^3 \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_{x_i} \partial_z \hat{u}_\varepsilon - \int_{\Omega''} |z| \partial_z \left(\frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \right) \partial_{x_i}(\eta^3) \partial_z \hat{u}_\varepsilon. \end{aligned} \tag{1.12}$$

The first three terms of the right hand side of (1.12) can be estimated as in the proof of Proposition 3.5 using Young's inequality, the fact that $\partial_z |z| \leq 1$ and $\frac{|\partial_{x_i} \hat{u}_\varepsilon|}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \leq 1$, for

$i = 1, 2$ uniformly in ε. We will only show the estimate of the last term of (1.12), which we denote by J. Integrating by parts J we get

$$J = \int_{\Omega''} \partial_z |z| \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_{x_i}(\eta^3) \partial_z \hat{u}_\varepsilon + \int_{\Omega''} |z| \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_z \partial_{x_i}(\eta^3) \partial_z \hat{u}_\varepsilon + \int_{\Omega''} |z| \frac{\partial_{x_i} \hat{u}_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \hat{u}_\varepsilon|^2}} \partial_z(\eta^3) \partial_z \partial_z \hat{u}_\varepsilon. \tag{1.13}$$

It is a standard process now to estimate the right hand side of the above equality using Young's inequality with weight $\gamma > 0$, for example the last term of (1.13) can be estimated from above by

$$c \int_{\Omega''} \eta^{1/2} \eta^{3/2} |\nabla \partial_z \hat{u}_\varepsilon| \leq \tilde{c}(\frac{1}{\gamma} + \gamma \int_{\Omega''} \eta^3 |\nabla \partial_z \hat{u}_\varepsilon|^2).$$

Finally, putting all the estimates together and choosing γ small enough we can absorb the terms $\gamma \int_{\Omega''} \eta^3 |\nabla \partial_z \hat{u}_\varepsilon|^2$ on the right hand side of (1.12) by it's left hand side and by noticing that $\eta = 1$ on Ω' we end up with the desired estimate. $\square$

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Chapitre 3

Eternal solutions of the homogeneous Boltzmann equation with infinite energy

This is a work under preparation in collaboration with Marco Cannone † and Grzegorz Karch ‡

Abstract

In the work [7] the authors proposed an approach which allows the study solutions of the initial value problem for the homogeneous Boltzmann equation for Maxwellian molecules in a space of probability measures of infinite second moment (so-called infinite energy solutions). Here, we use the same approach to construct eternal and infinite energy solutions which describe a large time behaviour of other infinite energy solutions.

1 Introduction

Homogeneous Boltzmann equation.

We consider the homogeneous Boltzmann equation in $\mathbb{R}^3$

$$\partial_t f(v,t) = Q(f,f)(v,t) \tag{1.1}$$

with the bilinear form corresponding to a Maxwellian gas

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} \mathcal{B}\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\big(f(v')g(v'_*) - f(v)g(v_*)\big)\, d\sigma\, dv_*. \tag{1.2}$$

Here, the unknown density $f = f(v,t)$ is independent of the space variable, moreover, we denote

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1.3}$$

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with σ varying in the unit sphere S^2. Equation (1.1)–(1.2) is supplemented with a nonnegative initial datum

$$f(v,0) = f_0(v) \tag{1.4}$$

which is assumed to be either a density of a probability distribution or, more generally, a probability measure.

The collision kernel $\mathcal{B}$ in equation (1.2) is a nonnegative function and, in the case of Maxwellian molecules, it depends only on the deviation angle θ, defined by the equation $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma$. It is well-known that the physical collision kernel $\mathcal{B} = \mathcal{B}(y)$ has a nonintegrable singularity as $y \to 1$ of the form $(1-y)^{-5/4}$ (see *e.g.* [5, p. 1043], [16, Ch. 1.1] and references therein).

In the study of the Boltzmann equation, it is natural to assume that the nonnegative initial datum satisfies

$$\int_{\mathbb{R}^3} f_0(v)\,dv = 1, \quad \int_{\mathbb{R}^3} f_0(v)v_i\,dv = 0 \ (i=1,2,3), \quad \int_{\mathbb{R}^3} f_0(v)|v|^2\,dv = 3, \tag{1.5}$$

because these relations are interpreted as the unit mass, the zero mean value, and the unit temperature of the gas, respectively. The existence of a unique solution of the initial value problem (1.1)–(1.4) under assumptions (1.5) and for a large class on nonintegrable collision kernels is well-known, see *e.g.* [4, 15, 16] and the references therein. This solution satisfies $f \in C^1([0,\infty), L^1(\mathbb{R}^3))$ and

$$\int_{\mathbb{R}^3} f(v,t)\,dv = 1, \quad \int_{\mathbb{R}^3} f(v,t)v_i\,dv = 0 \ (i=1,2,3), \quad \int_{\mathbb{R}^3} f(v,t)|v|^2\,dv = 3 \tag{1.6}$$

for all $t > 0$. For more information about the Boltzmann equation and its physical meaning, we refer the reader to the book by Cercignani [9] and to the more recent review article by Villani [16].

Homogeneous Boltzmann equation in Fourier variables.

We study properties of solutions to the initial value problem (1.1)-(1.4) using the Bobylev formulation [4], where the Fourier transform of the unknown variable

$$\varphi(\xi,t) \equiv \widehat{f}(\xi,t) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} f(v,t)\,dv \tag{1.7}$$

satisfies the following simpler equation

$$\partial_t\varphi(\xi,t) = \int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\varphi(\xi^+,t)\varphi(\xi^-,t) - \varphi(\xi,t)\varphi(0,t)\right)d\sigma, \tag{1.8}$$

with

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}. \tag{1.9}$$

We recall that these two vectors ξ^+ and ξ^- satisfy the well-known relations

$$\xi^+ + \xi^- = \xi \quad \text{and} \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2, \tag{1.10}$$

hence,

$$|\xi^+|^2 = |\xi|^2 \frac{1 + \frac{\xi}{|\xi|}\cdot\sigma}{2} \quad \text{and} \quad |\xi^-|^2 = |\xi|^2 \frac{1 - \frac{\xi}{|\xi|}\cdot\sigma}{2}. \tag{1.11}$$

Remark 1.1. *Here, we note that the formula for the Fourier transform of the bilinear operator Q on the right-hand side of equation (1.8) is actually a particular case of a more general one which does not assume the collision kernel to be Maxwellian, see [1, Appendix] for more details.*

In the following, we study properties the solutions of equation (1.8) supplemented with an initial datum

$$\varphi(\xi, 0) = \varphi_0(\xi), \tag{1.12}$$

where φ_0 is a characteristic function *i.e.* the Fourier transform of a probability measure, cf. Section 2.

The spaces $\mathcal{K}^\alpha$.

The authors of [7] proposed to study solutions of problem (1.8)-(1.12) the space

$$\mathcal{K}^\alpha = \Big\{\varphi : \mathbb{R}^3 \to \mathbb{C} \text{ is a characteristic function such that } \|\varphi - 1\|_\alpha < \infty\Big\} \tag{1.13}$$

with $\alpha \in [0, 2]$, where

$$\|\varphi - 1\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \tag{1.14}$$

It was proved in [7] that the set $\mathcal{K}^\alpha$ is nontrivial only for $\alpha \in [0, 2]$, because $\mathcal{K}^\alpha = \{1\}$ for $\alpha > 2$. On the other hand, $\mathcal{K}^0$ coincides with the set of all characteristic functions and the following imbeddings hold true

$$\{1\} \subseteq \mathcal{K}^\alpha \subseteq \mathcal{K}^{\alpha_0} \subseteq \mathcal{K}^0 \quad \text{for all} \quad 2 \geq \alpha \geq \alpha_0 \geq 0. \tag{1.15}$$

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi - \widetilde{\varphi}\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \widetilde{\varphi}(\xi)|}{|\xi|^\alpha}, \tag{1.16}$$

is a complete metric space. We refer the reader to Section 2 for the proofs of all these properties.

Next, for every $\xi \in \mathbb{R}^3 \setminus \{0\}$, we define the quantity which appears systematically in our considerations :

$$\lambda_\alpha \equiv \int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right)\left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma. \tag{1.17}$$

Using the identities $|\xi^\pm|^\alpha = |\xi|^\alpha((1 \pm (\xi/|\xi|) \cdot \sigma)/2)^{\alpha/2}$ we can find $C(\alpha) > 0$ such that

$$\lambda_\alpha = 2\pi \int_{-1}^1 \mathcal{B}(y)\left[\left(\frac{1+y}{2}\right)^{\alpha/2} + \left(\frac{1-y}{2}\right)^{\alpha/2} - 1\right] dy \leq C(\alpha) \int_{-1}^1 \mathcal{B}(y)(1-y^2)^{\alpha/2}\, dy$$

which is finite, independent of ξ, and positive for $0 < \alpha < 2$, under the assumption $(1-y)^{\alpha/2}(1+y)^{\alpha/2}\mathcal{B}(y) \in L^1(-1, 1)$. However, to construct solutions to the initial-value problem (1.8)– (1.12), we have to impose the stronger assumption on the collision kernel, namely,

$$(1-y)^{\alpha_0/4}(1+y)^{\alpha_0/4}\mathcal{B}(y) \in L^1(-1, 1) \quad \text{for some} \quad \alpha_0 \in [0, 2]. \tag{1.18}$$

We are in a position to recall results from [7] on the existence, uniqueness , and stability of solutions to the initial value problem (1.8)–(1.12).

Theorem 1.2 ([7, Theorem 2.2]). *Assume that $\mathcal{B}$ satisfies assumption (1.18) for some $\alpha_0 \in [0,2]$. Then for each $\alpha \in [\alpha_0, 2]$ and every $\varphi_0 \in \mathcal{K}^\alpha$ there exists a classical solution $\varphi \in C([0,+\infty), \mathcal{K}^\alpha)$ of problem (1.8)–(1.12). The solution is unique in the space $C([0,+\infty), \mathcal{K}^{\alpha_0})$.*

Theorem 1.3 ([7, Theorem 2.5]). *Assume that $\mathcal{B}$ satisfies condition (1.18) for some $\alpha_0 \in [0,2]$. Let $\alpha \in [\alpha_0, 2]$. Given $\tau \in \mathbb{R}$, suppose that $\varphi, \widetilde{\varphi} \in C([\tau,+\infty), \mathcal{K}^\alpha)$ solve (1.8)–(1.11). Then for every $\tau \le t$ the following inequality holds true*

$$e^{-\lambda_\alpha t}\|\varphi(t) - \widetilde{\varphi}(t)\|_\alpha \le e^{-\lambda_\alpha \tau}\|\varphi(\tau) - \widetilde{\varphi}(\tau)\|_\alpha, \tag{1.19}$$

where the constant $\lambda_\alpha \ge 0$ is defined by equation (1.17).

Notice that Theorem 1.3 is the stability estimate, proved in [7, Section 4.4] for $\tau = 0$, however, the extension to arbitrary $\tau \in \mathbb{R}$ is straightforward.

Results from Theorem 1.2 were recently improved and generalized by Morimoto and his collaborators. In particular, it was noticed by Morimoto [11] that, to construct global-in-time solutions in the space $\mathcal{K}^\alpha$, it suffices to impose the following weaker condition : $(1-\bar{y})^{\alpha/2}\mathcal{B}(\bar{y}) \in L^1(0,1)$ for some $\alpha \in [0,2]$ (here the symmetrised collisional kernel is assumed). Moreover, Morimoto and Yang [12] proved smoothing effects for solutions to problem (1.1)-(1.4) with a strongly singular kernel satisfying $\mathcal{B}(\cos\theta)\theta^{2+2\nu} \to b_0$ when $\theta \to 0^+$, for some $0 < \nu < 1$ and $b_0 > 0$. Under these assumptions, a global-in-time unique solution to problem (1.1)-(1.4) corresponding to an initial datum f_0 with a finite moment of order α is smooth, see also [14]. Recently, in the preprint [13], the theory from [7] on measure valued infinite energy solutions for the homogeneous Boltzmann equation (1.1)-(1.4) was extended on the case of hard and soft potentials.

2 Continuous positive definite functions

For the convenience of the reader in this section we are going to recall several properties of characteristic and positive definite functions as introduced and described in [7].

Definition 2.1. *A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is called characteristic function if there is a probability measure μ (i.e. a Borel measure with $\int_{\mathbb{R}^N} \mu(dx) = 1$) such that we have the following identity $\varphi(\xi) = \widehat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}\,\mu(dx)$. The set of all characteristic functions $\varphi : \mathbb{R}^N \to \mathbb{C}$ we will be denoted by $\mathcal{K}$.*

Definition 2.2. *A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is called positive definite if for every $k \in \mathbb{N}$ and every vectors $\xi^1, ..., \xi^k \in \mathbb{R}^N$ the matrix $\left(\varphi(\xi^j - \xi^\ell)\right)_{j,\ell=1,...,k}$ is positive Hermitian, i.e. for all $\lambda_1, ..., \lambda_k \in \mathbb{C}$ we have*

$$\sum_{j,\ell=1}^{k} \varphi(\xi^j - \xi^\ell)\lambda_j \overline{\lambda_\ell} \ge 0. \tag{2.1}$$

According to the Bochner theorem a continuous positive definite function is essentially a characteristic function.

Theorem 2.3. *A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is a characteristic function if and only if the following conditions are fulfilled*

(i) φ is a continuous function on $\mathbb{R}^N$

(ii) $\varphi(0) = 1$

(iii) φ *is positive definite.*

We refer the reader to the books either by Berg and Forst [2, Ch. I, §3] or by Jacob [10, Ch. 3] for proofs of properties of positive definite functions which will be listed below. As we will see later the larger set of positive definite functions (instead of simple characteristic functions) allow to deal with the singularity of the collisional kernel in the space of pseudomeasures $\mathcal{K}^\alpha$.

We start with some simple results.

Lemma 2.4. *Every positive definite function* φ *satisfies*

$$\overline{\varphi(\xi)} = \varphi(-\xi) \quad \textit{and} \quad \varphi(0) \geq 0 \tag{2.2}$$

and

$$|\varphi(\xi)| \leq \varphi(0), \quad \textit{hence} \quad \sup_{\xi \in \mathbb{R}^N} |\varphi(\xi)| = \varphi(0). \tag{2.3}$$

Lemma 2.5. *Any linear combination with positive coefficients of positive definite functions is a positive definite function. The set of positive definite functions is closed with respect to the pointwise convergence.*

Lemma 2.6. *The product of two positive definite functions is a positive definite function.*

Proof This is the immediate consequence of Definition 2.2 if we note that for every two positive Hermitian matrices $(a_{jk})_{j,k=1,\dots,N}$ and $(b_{jk})_{j,k=1,\dots,N}$, the matrix $(c_{jk})_{j,k=1,\dots,N}$ with elements $c_{jk} = a_{jk}b_{jk}$ is positive Hermitian, see *e.g.* [10, Lemma 3.5.9]. $\square$

Lemma 2.7. *If* φ *is a positive definite function, so are* $\overline{\varphi}$ *and* $\operatorname{Re}\varphi$.

Proof To show that $\overline{\varphi}$ is a positive definite function it suffices to compute the complex conjugate of inequality (2.1). Using equality $\operatorname{Re}\varphi = (\varphi + \overline{\varphi})/2$ we complete the proof by Lemma 2.5. $\square$

The following inequalities play an important role in dealing with the singularity of the collisional kernel. For the completeness of the exposition, we sketch their proofs, see either [2, Ch. I, §3.4] or [10, Lemma 3.5.10] for more details.

Lemma 2.8. *For any positive definite function* $\varphi = \varphi(\xi)$ *such that* $\varphi(0) = 1$ *we have*

$$|\varphi(\xi) - \varphi(\eta)|^2 \leq 2(1 - \operatorname{Re}\varphi(\xi - \eta)) \tag{2.4}$$

and

$$|\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)|^2 \leq (1 - |\varphi(\xi)|^2)(1 - |\varphi(\eta)|^2) \tag{2.5}$$

for all $\xi, \eta \in \mathbb{R}^N$.

Proof We are going to use inequality (2.1) with suitable chosen vectors ξ^j and constants λ_j. Indeed, for $\xi, \eta \in \mathbb{R}^N$ such that $\varphi(\xi) \neq \varphi(\eta)$ we consider the Hermitian matrix

$$\begin{pmatrix} \varphi(0) & \overline{\varphi(\xi)} & \overline{\varphi(\eta)} \\ \varphi(\xi) & \varphi(0) & \varphi(\xi - \eta) \\ \varphi(\eta) & \overline{\varphi(\xi - \eta)} & \varphi(0) \end{pmatrix}, \tag{2.6}$$

where $\varphi(0) = 1$. Next, with arbitrary and given $s \in \mathbb{R}$, we define

$$\lambda_1 = s, \quad \lambda_2 = \frac{s|\varphi(\xi) - \varphi(\eta)|}{\varphi(\xi) - \varphi(\eta)}, \quad \lambda_3 = -\lambda_2.$$

Hence, applying inequality (2.1), we find by a straightforward calculation

$$1 + 2s^2 + 2s|\varphi(\xi) - \varphi(\eta)| - 2s^2 \mathrm{Re}\, \varphi(\xi - \eta) \geq 0.$$

This means that the discriminate of the quadratic form on the left-hand side (as the function of s) has to be nonpositive, hence,

$$4|\varphi(\xi) - \varphi(\eta)|^2 \leq 4(2 - 2\mathrm{Re}\, \varphi(\xi - \eta)).$$

which completes the proof of (2.4).

On the other hand, inequality (2.5) is equivalent to the fact that the determinant of the Hermitian matrix (2.6) with $\varphi(0) = 1$ is non-negative. $\square$

Let us now recall the definition of the function space

$$\mathcal{K}^\alpha = \Big\{ \varphi : \mathbb{R}^3 \to \mathbb{C} \text{ is a characteristic function such that } \|\varphi - 1\|_\alpha < \infty \Big\},$$

supplemented with the metric

$$\|\varphi - \widetilde{\varphi}\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \widetilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

First, we give some examples of characteristic functions in the space $\mathcal{K}^\alpha$.

Example 2.1. *(1) The function $\varphi = \varphi(\xi)$ satisfying $\varphi(0) = 1$ and $\varphi(\xi) = 0$ for ξ different from zero is a positive definite function, however, it is not a characteristic function (since it is not continuous)*

(2) The function $\varphi(\xi) = e^{-ib\cdot\xi}$, with fixed $b \in \mathbb{R}^3$, is the Fourier transform of the Dirac delta δ_b concentrated at b. It belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 1]$.

(3) Maxwellians in the Fourier variables, $\varphi(\xi) = e^{-A|\xi|^2}$ with fixed $A > 0$, belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 2]$.

(4) The function $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$ is a characteristic function for each $\alpha \in (0, 2]$ because this is the Fourier transform of the probability distribution of an α-stable symmetric Lévy process, see e.g. [10, Examples 3.5.23 and 3.9.17] for more details. Hence, $\varphi_\alpha \in \mathcal{K}^\beta$ for each $\beta \in [0, \alpha]$.

Proposition 2.9. *For every $\alpha \in [0, 2]$, the set $\mathcal{K}^\alpha$ endowed with the distance (1.16) is a complete metric space.*

Proof The proof is immediate because the set of characteristic functions is closed with respect to the pointwise convergence. $\square$

Next, we state without the proof simple properties of the space $\mathcal{K}^\alpha$.

Lemma 2.10. *(i) The space $\mathcal{K}^\alpha$ is not a vector space (e.g. $\varphi(\xi) \equiv 0$ does not belong to $\mathcal{K}^\alpha$).*

(ii) $\varphi \equiv 1 \in \mathcal{K}^\alpha$ for every $\alpha \geq 0$.

(iii) For every $\varphi \in \mathcal{K}^\alpha$ we have $|\varphi(\xi)| \leq \varphi(0) = 1$ (cf. (2.3)).

(iv) *For all* $\varphi, \widetilde{\varphi} \in \mathcal{K}^\alpha$ *their product satisfies* $\varphi\widetilde{\varphi} \in \mathcal{K}^\alpha$.

(v) *Any linear and convex combination of functions from* $\mathcal{K}^\alpha$ *belongs to* $\mathcal{K}^\alpha$ *(cf. Lemma 2.5).*

In the following lemma, we explain why we limit ourselves to $\alpha \in [0,2]$ in the definition of $\mathcal{K}^\alpha$.

Lemma 2.11. *(i)* $\mathcal{K}^0 = \mathcal{K}$

(ii) $\mathcal{K}^{\alpha_1} \subseteq \mathcal{K}^{\alpha_2}$ *if* $\alpha_2 \leq \alpha_1$.

(iii) $\mathcal{K}^\alpha = \{1\}$ *for every* $\alpha > 2$.

Proof In the case of i, it suffices to use (2.3) in order to see that any characteristic function φ is bounded, more precisely, it satisfies $\sup_{\xi\in\mathbb{R}^N} |\varphi(\xi) - 1| \leq \varphi(0) + 1$.

To show ii, for any $\varphi \in \mathcal{K}^{\alpha_1}$, we proceed as follows

$$\begin{aligned}
\|\varphi - 1\|_{\alpha_2} &\leq \sup_{|\xi|\leq 1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_2}} + \sup_{|\xi|>1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_2}} \\
&\leq \sup_{|\xi|\leq 1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_1}} + \sup_{|\xi|>1} |\varphi(\xi) - 1| \\
&\leq \|\varphi - 1\|_{\alpha_1} + \varphi(0) + 1,
\end{aligned}$$

since $\alpha_2 \leq \alpha_1$ and by using (2.3). Hence, $\varphi \in \mathcal{K}^{\alpha_2}$.

Let us show iii. It follows immediately form eq. (1.14) that any $\varphi \in \mathcal{K}^\alpha$ with $\alpha > 2$ satisfies

$$\left|\frac{1 - \varphi(\xi)}{|\xi|^2}\right| \leq |\xi|^{\alpha-2}\|\varphi - 1\|_\alpha \to 0 \quad \text{as} \quad |\xi| \to 0. \tag{2.7}$$

Next, using inequality (2.4) we get for any unit vector $\zeta \in \mathbb{R}^3$ and all $\xi \in \mathbb{R}^3$

$$\left|\frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h}\right|^2 \leq 2\frac{(1 - \operatorname{Re}\varphi(h\zeta))}{h^2} \leq 2\left|\frac{1 - \varphi(h\zeta)}{h^2}\right|,$$

thus, by (2.7), we have

$$\lim_{h\to 0} \frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h} = 0.$$

Hence, for all $\zeta \in \mathbb{R}^3$ the directional derivative $\zeta \cdot \nabla\varphi(\xi)$ exists and is equal to zero, implying that φ is constant. □

Lemma 2.12. *Let* $\alpha \in [0,2]$. *Assume that* $\varphi \in \mathcal{K}^\alpha$. *Then* $\operatorname{Re}\varphi \in \mathcal{K}^\alpha$,

$$\|\operatorname{Re}\varphi - 1\|_\alpha \leq \|\varphi - 1\|_\alpha, \quad \textit{and} \quad \sup_{\xi\in\mathbb{R}^3\setminus\{0\}} \frac{|\operatorname{Im}\varphi(\xi)|}{|\xi|^\alpha} \leq \|\varphi - 1\|_\alpha. \tag{2.8}$$

Proof Let $\varphi \in \mathcal{K}^\alpha$. It is well-known that $\operatorname{Re}\varphi$ is a characteristic function (*e.g.* it suffices to combine Lemma 2.7 with the Bochner Theorem 2.3). Now, by the Pythagorean theorem, we obtain

$$|\varphi(\xi) - 1|^2 = |\operatorname{Im}\varphi(\xi)|^2 + |\operatorname{Re}\varphi(\xi) - 1|^2 \geq |\operatorname{Re}\varphi(\xi) - 1|^2. \tag{2.9}$$

Hence, we complete the proof of the first inequality in (2.8) dividing (2.9) by $|\xi|^\alpha$ and computing the supremum with respect to $\xi \in \mathbb{R}^3$.

To show the second inequality in (2.8), we proceed analogously using the inequality $|\varphi(\xi)-1| \geq |\operatorname{Im}\varphi(\xi)|$ resulting from (2.9). □

The following inequality implies, as we will see later, that the nonlinear term in equation (1.8) is well-defined for functions from $\mathcal{K}^\alpha$ if we impose the condition (1.18) on the collision kernel.

Lemma 2.13. *Let $\alpha \in [0,2]$. Assume that $\varphi \in \mathcal{K}^\alpha$. For every $\xi \in \mathbb{R}^3$ define ξ^+ and ξ^- by equations (1.9) with some fixed $n \in S^2$. Then*

$$|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| \leq 4|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}\|\varphi - 1\|_\alpha. \tag{2.10}$$

Proof First, recall that $\varphi(0) = 1$. We begin the elementary identity

$$1 - |\varphi(\xi^+)|^2 = (1-\varphi(\xi^+))(1+\overline{\varphi(\xi^+)}) + 2\operatorname{Im}\varphi(\xi^+). \tag{2.11}$$

Using the estimate $|1+\overline{\varphi(\xi^+)}| \leq 1 + |\varphi(\xi^+)| \leq 2$ (*cf.* (2.3)) and second inequality in (2.8) we deduce from (2.11)

$$0 \leq 1 - |\varphi(\xi^+)|^2 \leq 4|\xi^+|^\alpha \|\varphi - 1\|_\alpha.$$

Obviously, an analogous inequality holds true if we replace ξ^+ by ξ^-. Now, applying inequality (2.5), we conclude

$$\begin{aligned}|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| &\leq \sqrt{(1-|\varphi(\xi^+)|^2)(1-|\varphi(\xi^-)|^2)} \\ &\leq 4|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}\|\varphi-1\|_\alpha\end{aligned}$$

for all $\xi \in \mathbb{R}^3$. □

Lemma 2.14. *Let $\alpha \in [0,2]$. Assume that μ is a probability measure on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} |v|^\alpha \,\mu(dv)$ is finite. If, moreover, $\alpha \in (1,2]$, assume that $\int_{\mathbb{R}^3} v_i \,\mu(dv) = 0$ for $i \in \{1,2,3\}$. Then $\widehat{\mu} \in \mathcal{K}^\alpha$.*

Proof Consider first $\alpha \in (0,1]$. Using the definition of the Fourier transform of a probability measure $\mu(dv)$ we obtain

$$\frac{|\widehat{\mu}(\xi)-1|}{|\xi|^\alpha} \leq \int_{\mathbb{R}^3} \frac{|e^{-iv\cdot\xi}-1|}{|\xi|^\alpha}\,\mu(dv). \tag{2.12}$$

Note now the by substituting $\xi = \eta/|v|$, we have

$$\sup_{\xi\in\mathbb{R}^3} \frac{|e^{-iv\cdot\xi}-1|}{|\xi|^\alpha} = |v|^\alpha \sup_{\eta\in\mathbb{R}^3} \frac{|e^{-i\eta\cdot v/|v|}-1|}{|\eta|^\alpha} \leq C|v|^\alpha,$$

where, in view of the elementary inequality $|e^{is}-1| \leq |s|$ for all $s \in \mathbb{R}$, the constant $C = \sup_{v,\eta\in\mathbb{R}^3} |e^{-i\eta\cdot v/|v|}-1||\eta|^{-\alpha}$ is finite for $\alpha \in (0,1]$. Hence, we deduce from (2.12) that

$$\|\widehat{\mu}-1\|_\alpha \leq C\int_{\mathbb{R}^3} |v|^\alpha\,\mu(dv),$$

For $\alpha \in (1,2]$, one should proceed analogously using the following counterpart of inequality (2.12)

$$\frac{|\widehat{\mu}(\xi)-1|}{|\xi|^\alpha} \leq \int_{\mathbb{R}^3} \left|\frac{e^{-iv\cdot\xi}+iv\cdot\xi-1}{|\xi|^\alpha}\right|\mu(dv).$$

being the simple consequence of the additional assumption $\int_{\mathbb{R}^3} v_i\,\mu(dv) = 0$, for every $i = \{1,2,3\}$. □

Remark 2.15. *Let us provide a counterexample that the reverse implication in Lemma 2.14 for $\alpha \in (0,2)$ is not true, in other words, we want to show that the space $\mathcal{K}^\alpha$ is bigger than the space of of characteristic functions corresponding to probability measures with finite moments of order α. Is is well-known that the function $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$, with $\alpha \in (0,2)$, is the Fourier transform of the probability density $P_\alpha(x)$ of the α-stable symmetric Lévy process, (see Example 2.1). Obviously, we have $\varphi_\alpha \in \mathcal{K}^\alpha$. On the other hand, it is known that for every $\alpha \in (0,2)$ the function P_α is smooth, nonnegative, and satisfies the estimate $0 < P_\alpha(x) \le C(1+|x|)^{-(\alpha+n)}$ for a constant C and all $x \in \mathbb{R}^n$. Moreover,*

$$\frac{P_\alpha(x)}{|x|^{\alpha+n}} \to c_0 \quad \textit{when} \quad |x| \to \infty, \tag{2.13}$$

where $c_0 = \alpha 2^{\alpha-1}\pi^{-(n+2)/2}\sin(\alpha\pi/2)\Gamma\left(\frac{\alpha+n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$. We refer the reader to [3] for a proof of the formula (2.13) with the explicit constant c_0.

In view of the limit relation (2.13), we have $\int_{\mathbb{R}^3} P_\alpha(x)|x|^\alpha \, dx = \infty$.

3 Propagation of generalized moments

We are in a position to present our new contribution to the existing theory for the homogeneous Boltzmann equation for Maxwellian molecules in the space $\mathcal{K}^\alpha$.

Definition 3.1. *We say that a constant $K > 0$ is an (isotropic) generalized α-moment of a function $\varphi \in \mathcal{K}^\alpha$ if*

$$\lim_{\xi \to 0} \frac{1-\varphi(\xi)}{|\xi|^\alpha} = K$$

provided this limit exists.

Remark 3.2. *If $\varphi \in \mathcal{K}^\alpha$ has a generalized α-moment equal to K than it has to be a positive constant. This is an immediate consequence of the following properties of characteristic functions : $|\varphi(\xi)| \le \varphi(0) = 1$ for all $\xi \in \mathbb{R}^3$, see Lemma 2.4.*

First, we prove a formula for a propagation of generalized α-moment of solutions to equation (1.8), which were constructed in Theorem 1.2.

Theorem 3.3 (Propagation of α-moments)**.** *Assume that a collision kernel $\mathcal{B}$ satisfies condition (1.18) for some $\alpha_0 \in [0,2)$. Consider a solution $\varphi \in C([0,+\infty), \mathcal{K}^\alpha)$ of problem (1.8)–(1.12) with certain $\alpha \in [\alpha_0, 2)$. Suppose that*

$$\lim_{\xi \to 0} \frac{1-\varphi(\xi,0)}{|\xi|^\alpha} = K \qquad \textit{for some} \quad K > 0.$$

Then

$$\lim_{\xi \to 0} \frac{1-\varphi(\xi,t)}{|\xi|^\alpha} = Ke^{\lambda_\alpha t} \qquad \textit{for all} \quad t \ge 0, \tag{3.1}$$

where the constant λ_α is defined in (1.17).

Remark 3.4. *For $\alpha = 2$, we have $\lambda_\alpha = 0$ by relations* (1.10) *applied to equation* (1.17)*. Hence, formula* (3.1) *with $\alpha = 2$, in the case of solutions satisfying* (1.6)*, expresses the conservation of the second moment (i.e. the conservation of energy) recalled in the third equality in* (1.6)*.*

Remark 3.5. *The formula (3.1) for propagation of generalized moments plays a crucial role in our construction of eternal self-similar solutions of equation (1.8), see below in the proof of Theorem 6.1 for more details.*

We need the following auxiliary result in the proof of Theorem 3.3.

Lemma 3.6. *Consider a cut-off collision kernel $\mathcal{B} \in L^1(-1,1)$. Suppose that $\varphi \in \mathcal{K}^\alpha$ satisfies*

$$\lim_{\xi\to 0} \frac{1-\varphi(\xi)}{|\xi|^\alpha} = K \qquad \text{for some} \quad K > 0, \tag{3.2}$$

then

$$\lim_{\xi\to 0} \frac{1}{|\xi|^\alpha} \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left(\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)\right) d\sigma = -K\lambda_\alpha. \tag{3.3}$$

Proof By a direct calculation, we have the following equality

$$\begin{aligned} &\frac{1}{|\xi|^\alpha} \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)\right) d\sigma + K\lambda_\alpha \\ &\qquad = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \left(\frac{|\xi^+|^\alpha}{|\xi|^\alpha} I_1 + \frac{|\xi^-|^\alpha}{|\xi|^\alpha} I_2 + I_3\right) d\sigma, \end{aligned} \tag{3.4}$$

where

$$I_1 = \left(\frac{\varphi(\xi^+)-1}{|\xi^+|^\alpha} + K\right)\varphi(\xi^-) + K(1-\varphi(\xi^-)), \quad I_2 = \frac{\varphi(\xi^-)-1}{|\xi^-|^\alpha} + K, \quad I_3 = \frac{1-\varphi(\xi)}{|\xi|^\alpha} - K$$

Because of inequalities $|\xi^+|, |\xi^-| \le |\xi|$, $|\varphi(\xi)| \le 1$, the equality $\varphi(0) = 1$, as well as relation (3.2), the quantities I_j, for each $j \in \{1,2,3\}$, are bounded and converge to zero as $|\xi| \to 0$. Hence, the existence of the limit on (3.3) is an immediate consequence of the Lebesgue dominated convergence theorem because, here, we consider an integrable collision kernel $\mathcal{B}$. □

Proof of Theorem 3.3
Step 1. Cut-off case

Consider first the cut-off case $\mathcal{B} \in L^1(-1,1)$. Following [7, Proof of Thm 4.5], we use the nonlinear operator

$$\mathcal{G}(\varphi)(\xi) \equiv \int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) \varphi(\xi^+)\varphi(\xi^-)\, d\sigma, \tag{3.5}$$

where ξ^+ and ξ^- are defined in (1.9). Hence, under the cut-off assumption, for the constant

$$\gamma_2 \equiv \int_{S^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right) d\sigma = 2\pi \int_{-1}^{1} \mathcal{B}(s)\, ds,$$

and for φ satisfying $\varphi(0,t) = 1$ for all $t \ge 0$, we may write equation (1.8) in the following form

$$\partial_t\varphi + \gamma_2\varphi = \mathcal{G}(\varphi). \tag{3.6}$$

Next, multiplying equation (3.6) by $e^{\gamma_2 t}$ and integrating with respect to t we obtain the following equivalent formulation of problem (1.8)–(1.12)

$$\varphi(\xi,t) = \varphi_0(\xi)e^{-\gamma_2 t} + \int_0^t e^{-\gamma_2(t-\tau)}\mathcal{G}(\varphi(\cdot,\tau))(\xi)\,d\tau \equiv \mathcal{F}(\varphi)(\xi,t). \tag{3.7}$$

In [7, Thm. 4.5], the authors showed that a solution of equation (3.6) supplemented with an initial datum $\varphi(\cdot,0) = \varphi_0 \in \mathcal{K}^\alpha$ is obtained as a fixed point of equation (3.7) via the Banach contraction principle applied to the nonlinear operator $\mathcal{F}(\varphi)$. More precisely, this operator is a contraction on the metric space $\mathcal{X}_T^\alpha = C([0,T],\mathcal{K}^\alpha)$ supplemented with the metric $\|\varphi - \widetilde{\varphi}\|_{\mathcal{X}_T^\alpha} \equiv \sup_{\tau\in[0,T]} \|\varphi(\cdot,\tau) - \widetilde{\varphi}(\cdot,\tau)\|_\alpha$ provided $T > 0$ is sufficiently small. Moreover, a unique solution of equation (3.6) (or equivalently (3.7)) is obtained as a limit (uniform on each time interval $[0,T]$) of the Picard iterations $\{\varphi_n\}_{n\in\mathbb{N}}$ defined by the recurrence formula $\varphi_{n+1} = \mathcal{F}(\varphi_n)$, which by a direct calculation, we write in the form

$$\frac{1-\varphi_{n+1}(\xi,t)}{|\xi|^\alpha} = \frac{1-\varphi_0(\xi)}{|\xi|^\alpha}e^{-\gamma_2 t} - \int_0^t e^{-\gamma_2(t-\tau)}\left(\frac{\mathcal{G}(\varphi_n(\cdot,\tau))(\xi)-\gamma_2}{|\xi|^\alpha}\right)d\tau. \tag{3.8}$$

By an inductive argument and (3.8) we may define

$$K_n(t) \equiv \lim_{\xi\to 0}\frac{1-\varphi_n(\xi,t)}{|\xi|^\alpha} \qquad \text{for each } n\in\mathbb{N} \text{ and each } t\geq 0, \tag{3.9}$$

and we plan to find a recurrence relation for K_n by passing to the limit as $\xi \to 0$ in recurrence equation (3.8). Using the identity

$$\frac{\mathcal{G}(\varphi)(\xi)-\gamma_2}{|\xi|^\alpha} = \frac{1}{|\xi|^\alpha}\int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\varphi(\xi^+)\varphi(\xi^-)-\varphi(\xi)\right)d\sigma - \frac{\gamma_2(1-\varphi(\xi))}{|\xi|^\alpha},$$

and Lemma 3.6, we may pass to the limit $\xi \to 0$ in (3.8) and get

$$K_{n+1}(t) = Ke^{-\gamma_2 t} + (\lambda_\alpha+\gamma_2)\int_0^t e^{-\gamma_2(t-\tau)}K_n(\tau)\,d\tau \qquad \text{and} \quad K_0(t) = K. \tag{3.10}$$

It is easy to show that the limit $K(t) = \lim\limits_{n\to\infty} K_n(t)$ exists for all $t \geq 0$ because the recurrence equation in (3.10) corresponds to the Picard iterations for the Cauchy problem

$$K'(t) = -\gamma_2 K(t) + (\lambda_\alpha+\gamma_2)K(t), \qquad K(0) = K$$

with the solution $K(t) = Ke^{\lambda_\alpha t}$.

Since the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the metric $\|\cdot\|_\alpha$ towards the solution $\varphi \in C([0,T),\mathcal{K}^\alpha)$, we have

$$\begin{aligned}\limsup_{\xi\to 0}\left|\frac{1-\varphi(\xi,t)}{|\xi|^\alpha} - Ke^{\lambda_\alpha t}\right| &\leq \sup_{\xi\in\mathbb{R}^3}\left|\frac{\varphi_n(\xi,t)-\varphi(\xi,t)}{|\xi|^\alpha}\right| \\ &\quad + \limsup_{\xi\to 0}\left|\frac{1-\varphi_n(\xi,t)}{|\xi|^\alpha} - K_n(t)\right| + |K_n(t) - Ke^{\lambda_\alpha t}|.\end{aligned}$$

Hence using (3.9) and passing to the limit with $n \to \infty$ in the above inequality we obtain relation (3.1) in the cut-off case.

Step 2. Non-cut-off case

To extend this result on the non cut-off case, first, we recall results form [7, Section 5], where a solution φ of problem (1.8)-(1.12) with a singular kernel was obtained as a limit (uniform of compact subsets of $\mathbb{R}^3 \times [0, +\infty)$) of solutions φ_n of corresponding problems with truncated kernels $\mathcal{B}_n(s) = \min\{n, \mathcal{B}\}$, namely

$$\partial_t \varphi_n(\xi, t) = \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi_n(\xi^+, t)\varphi_n(\xi^-, t) - \varphi_n(\xi, t))\, d\sigma. \tag{3.11}$$

Subtracting (3.11) from (3.6) and dividing by $|\xi|^\alpha$ we get (we also suppress the time variable in the integrals)

(3.12)

$$\partial_t h_n(\xi, t) = \int_{\mathbb{S}^2} (\mathcal{B} - \mathcal{B}_n)\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \frac{\varphi_n(\xi^+)\varphi_n(\xi^-) - \varphi_n(\xi)}{|\xi|^\alpha}\, d\sigma$$
$$+ \frac{1}{|\xi|^\alpha} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi(\xi^+) - \varphi_n(\xi^+))\varphi(\xi^-) + (\varphi(\xi^-) - \varphi_n(\xi^-))\varphi_n(\xi^+)\, d\sigma + \gamma_2^n h_n(\xi, t)$$

where

$$h_n(\xi, t) = (\varphi(\xi, t) - \varphi_n(\xi, t))|\xi|^{-\alpha} \quad \text{and} \quad \gamma_2^n = 2\pi \int_{-1}^1 \mathcal{B}_n(y)\, dy.$$

Using inequality (2.10) we can bound the first term of the right hand side of (3.12) by

$$\begin{aligned} a_n &= 4||\varphi - 1||_\alpha \int_{\mathbb{S}^2} (\mathcal{B} - \mathcal{B}_n)\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \frac{|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}}{|\xi|^\alpha}\, d\sigma \\ &= 4||\varphi - 1||_\alpha \int_{\mathbb{S}^2} (\mathcal{B} - \mathcal{B}_n)\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\frac{1 + (\xi/|\xi|) \cdot \sigma}{2}\right)^{\alpha/4} \left(\frac{1 - (\xi/|\xi|) \cdot \sigma}{2}\right)^{\alpha/4} \\ &= 2\pi \int_{-1}^1 (\mathcal{B} - \mathcal{B}_n)(y) \left(\frac{1+y}{2}\right)^{\alpha/4} \left(\frac{1-y}{2}\right)^{\alpha/4}\, dy. \end{aligned} \tag{3.13}$$

Since $\alpha \in [\alpha_0, 2)$ we get by (1.18) and the dominated convergence theorem that $a_n \to 0$. Noting that the maximum of characteristic functions is 1, we may estimate for $|\xi^+|, |\xi^-| \le |\xi| \le R$ the second term of the right hand side of (3.12) by

$$||\varphi(t) - \varphi_n(t)||_{\alpha,R} \int_{\mathbb{S}^2} \mathcal{B}_n\left(\frac{\xi \cdot \sigma}{|\xi}\right) \frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} = ||\varphi(t) - \varphi_n(t)||_{\alpha,R} \cdot \gamma_\alpha^n, \tag{3.14}$$

where

$$||\varphi(t) - \varphi_n(t)||_{\alpha,R} = \sup_{|\xi| \le R} |\varphi(\xi, t) - \varphi_n(\xi, t)||\xi|^{-\alpha}.$$

Combining (3.13) and (3.14) we may write (3.12) as

$$|\partial_t h_n(\xi, t) - \gamma_2^n h_n(\xi, t)| \le \alpha_n + \gamma_\alpha^n ||\varphi(t) - \varphi_n(t)||_{\alpha,R},$$

which we then rewrite as

$$|\partial_t (e^{\gamma_2^n t} h_n(\xi, t))| \le e^{\gamma_2^n t} a_n + \gamma_\alpha^n e^{\gamma_2^n t} ||\varphi(t) - \varphi_n(t)||_{\alpha,R}.$$

Finally noting that $h_n(\xi, 0) = 0$ we get

$$|e^{\gamma_2^n t} h_n(\xi, t)| \le p_n(t) + \gamma_\alpha^n \int_0^t e^{\gamma_2^n \tau} ||\varphi(\tau) - \varphi_n(\tau)||_{\alpha,R}\, d\tau, \tag{3.15}$$

where we set $p_n(t) = a_n \int_0^t e^{\gamma_2^n \tau}\, d\tau = a_n \frac{e^{\gamma_2^n t}-1}{\gamma_2^n}$. Taking the supremum over $|\xi| \le R$ in inequality (3.15), we may apply the Gronwall lemma and get

$$(3.16)\qquad \sup_{|\xi|\le R} \frac{|\varphi(\xi,t)-\varphi_n(\xi,t)|}{|\xi|^\alpha} \le e^{(\gamma_\alpha^n-\gamma_2^n)t}\gamma_\alpha^n \int_0^t e^{-\gamma_\alpha^n \tau} p_n(\tau)\, d\tau + p_n(t) e^{-\gamma_2^n t} = \varepsilon_n(t),$$

and after a few calculations we get

$$(3.17)\qquad \varepsilon_n(t) = a_n \frac{e^{(\gamma_\alpha^n-\gamma_2^n)t}-1}{\gamma_\alpha^n-\gamma_2^n}.$$

By the monotone convergence theorem and (1.17) we obtain

$$\gamma_\alpha^n - \gamma_2^n \nearrow \gamma_\alpha - \gamma_2 = \lambda_\alpha > 0.$$

Hence $\lim_{n\to+\infty} \varepsilon_n(t) = 0$ for all $t \ge 0$. We can now pass to the limit as $R \to 0$ in (3.16) and using also (3.17) we get that for every $t \ge 0$

$$(3.18)\qquad \limsup_{\xi\to 0} \frac{|\varphi(\xi,t)-\varphi_n(\xi,t)|}{|\xi|^\alpha} \le \varepsilon_n(t).$$

By Step 1, we have that the generalized α-moment of $\varphi_n(\xi,t)$ equals $Ke^{\lambda_\alpha t}$, then we have the estimate

$$\begin{aligned}\limsup_{\xi\to 0}\left|\frac{1-\varphi(\xi,t)}{|\xi|^\alpha} - Ke^{\lambda_\alpha t}\right| &\le \limsup_{\xi\to 0}\left|\frac{\varphi(\xi,t)-\varphi_n(\xi,t)}{|\xi|^\alpha}\right| \\ &\quad + \limsup_{\xi\to 0}\left|\frac{1-\varphi_n(\xi,t)}{|\xi|^\alpha} - Ke^{\lambda_\alpha t}\right| \le \varepsilon_n(t),\end{aligned}$$

from which we conclude relation (3.1) if we pass to the limit as $n \to +\infty$. □

4 Self-similar solution by Bobylev and Cercignani

In [5], Bobylev and Cercignani constructed explicit eternal solutions of equation (1.7)-(1.11). Here, we recall their arguments for clarity of the exposition.

The authors of [5] considered isotropic distributions $f(|v|,t)$ with corresponding characteristic function $\hat{f}(|\xi|,t) = \varphi(x,t)$ where $x = |\xi|^2/2$. Then, since f is a probability distribution, we have $\varphi(0,t) = 1$. Moreover,

$$\begin{aligned}\varphi'(0,t) &= 2\int_{\mathbb{R}^3} \lim_{\xi\to 0} \frac{e^{-i\xi\cdot v}-1}{|\xi|^2} f(v,t)\, dv = 4\pi \int_0^{+\infty}\int_0^\pi \lim_{s\to 0}\frac{e^{-isr\cos\theta}-1}{s^2} f(r,t)\sin\theta r^2\, d\theta\, dr \\ &= -4\pi \int_0^\pi \cos^2\theta \sin\theta\, d\theta \int_0^{+\infty} f(r,t) r^4\, dr\end{aligned}$$

or if we notice that $\int_0^\pi \cos^2\theta\sin\theta\, d\theta = 1/3\int_0^\pi \sin\theta\, d\theta$ and passing back to the original variables in the integral we get

$$(4.1)\qquad \varphi'(0,t) = -\frac{1}{3}\int_{\mathbb{R}^3} f(v,t)|v|^2\, dv$$

and the function φ solves

$$(4.2)\qquad \varphi_t = \int_0^1 G(r)(\varphi(rx)\varphi((1-r)x) - \varphi(0)\varphi(x))\, dr$$

with $G(r) = 4\pi\mathcal{B}(1-2r)$, $r \in [0,1]$. The authors of [5] looked for solutions with infinite second moments, which by (4.1) means that $\varphi'(x) \to -\infty$ as $x \to 0^+$. Then in [5], the following power series is considered

$$\varphi(x,t) = \sum_{n=0}^{\infty} \varphi_n(t) \frac{x^{n\alpha/2}}{\Gamma(n\alpha/2+1)}, \tag{4.3}$$

for $\alpha \in (0,2)$ (note that $\alpha \in (0,1)$ in the notation of [5]). Thus, using equation (4.2), one may determine the first to terms of the power series as $\varphi_0(t) = 1$ and $\varphi_1(t) = \varphi_1(0)e^{\lambda_\alpha t}$ with λ_α as in (1.17). Then for small x we have $\varphi(x,t) \simeq 1 - c(xe^{\mu_\alpha t})^\alpha$, with $c > 0$ and $\mu_\alpha = \lambda_\alpha/\alpha$. Bobylev and Cercignani were motivated to represent the functions in (4.3) as $\varphi(x,t) = \psi(xe^{\mu_\alpha t}, t)$, where ψ solves the equation

$$\psi_t + \mu_\alpha x \psi_x = \int_0^1 G(r)(\psi(rx)\psi((1-r)x) - \psi(0)\psi(x))\, dr \tag{4.4}$$

with initial condition (1.12). If one supposes again that

$$\psi(x,t) = \sum_{n=0}^{\infty} \psi_n(t) \frac{x^{n\alpha/2}}{\Gamma(n\alpha/2+1)},$$

one may compute coefficients ψ_n using equation (4.4). Indeed, we have

$$\partial_t \psi_0 = \partial_t \psi_1 = 0 \tag{4.5}$$

and for $n \geq 2$

$$\partial_t \psi_n(t) + \gamma_n(\alpha)\psi_n = \sum_{j=1}^{n-1} E_\alpha(j, n-j)\psi_j \psi_{n-j}, \tag{4.6}$$

with $\gamma_n(\alpha) = n\lambda_\alpha - \lambda(n\alpha)$, $\lambda(p) = \int_0^1 G(r)(r^{p/2} + (1-r)^{p/2} - 1)\, dr$ and

$$E_\alpha(j,l) = \frac{\Gamma(n\alpha/2+1)}{\Gamma(j\alpha/2)\Gamma(l\alpha/2+1)} \int_0^1 G(r) r^{j\alpha/2}(1-r)^{l\alpha/2}.$$

Then for $n \geq 2$ we have

$$\psi_n(t) = \psi_n(0)e^{-\gamma_n(\alpha)t} + \sum_{j=1}^{n-1} E_\alpha(j,n-j) \int_0^t e^{-\gamma_n(\alpha)(t-\tau)} \psi_j(\tau)\psi_{n-j}(\tau). \tag{4.7}$$

It is not difficult to see that $\gamma_n(\alpha) > 0$ for $n \geq 2$, we can therefore pass to the limit as $t \to +\infty$ in (4.7) and get $\lim\limits_{t \to +\infty} \psi_n(t) = u_n$ where u_n is the unique steady solution of (4.5), (4.6) given by $u_0 = 1$, u_1 given and

$$u_n = \frac{1}{\gamma_n(\alpha)} \sum_{j=1}^{n-1} E_\alpha(j,n-j) u_j u_{n-j}, \quad n \geq 2. \tag{4.8}$$

To sum up, setting $x = |\eta|^2/2$, if u_n is as in (4.8) with $u_0 = 1$, $u_1 = -K2^{\alpha/2}\Gamma(\alpha/2+1)$, then the Bobylev-Cercignani functions

$$\varphi_{\alpha,K}(\eta) = \sum_{n=0}^{\infty} \frac{u_n 2^{-n\alpha/2}(|\eta|^\alpha)^n)}{\Gamma(n\alpha/2+1)},$$

are stationary solutions of (4.4), in particular the functions

$$\varphi(\xi,t) = \varphi_{\alpha,K}(\xi e^{\mu_\alpha t}), \tag{4.9}$$

are eternal self-similar solution of (1.7)-(1.11).

5 Boltzmann equation in slow variables

In order to see a scaling property of equation (1.8) we "slow down" the time variable, namely we map $\mathbb{R}$ to $(0, +\infty)$ using the change of variables

$$w(\xi, s) = \varphi(\xi, \log s), \qquad \text{where} \quad t = \log s, \tag{5.1}$$

for which we have

$$\partial_s w(\xi, s) = \frac{1}{s} \int_{S^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(w(\xi^+, s)w(\xi^-, t) - w(\xi, s)w(0, s)\right) d\sigma. \tag{5.2}$$

Remark 5.1. *One can check by a direct calculation that if w be a solution of equation* (5.2)*, then, for all $A, B \in \mathbb{R}\backslash\{0\}$, the function $w_{A,B}(\xi, s) \equiv w(A\xi, Bs)$ is also a solutions of this equation. Our goal is to construct self-similar solutions of equation* (5.2)*, i.e. solutions which are invariant under a particular case of this scaling.*

In order to determine a correct scaling, we rephrase the stability estimate from Theorem 1.3 for solutions of equation (5.2).

Corollary 5.2. *Assume that $\mathcal{B}$ satisfies condition (1.18) for some $\alpha_0 \in [0, 2]$. Let $\alpha \in [\alpha_0, 2]$. If for some $\tau > 0$, we have two solutions $w, \tilde{w} \in C([\tau, +\infty), \mathcal{K}^\alpha)$ of equation* (5.2) *then*

$$s^{-\lambda_\alpha} \|w(s) - \widetilde{w}(s)\|_\alpha \le \tau^{-\lambda_\alpha} \|w(\tau) - \widetilde{w}(\tau)\|_\alpha \qquad \text{for all} \quad \tau \le s. \tag{5.3}$$

Proof It suffices to use the change of variables (5.1) in estimate (1.19) for two solutions φ and $\widetilde{\varphi}$ of equations (1.8)-(1.11). □

Now, for two arbitrary functions $w, \tilde{w} \in C((0, +\infty), \mathcal{K}^\alpha)$ we define their distance

$$|||w - \tilde{w}|||_\alpha = \sup_{s>0} s^{-\lambda_\alpha} \|w(s) - \tilde{w}(s)\|_\alpha \tag{5.4}$$

where the number λ_α is defined in (1.17). This distance appears to be invariant under a particular case of the scaling of equation (5.2) mentioned in Remark 5.1 which we prove in the following lemma.

Lemma 5.3. *Let $w \in C((0, +\infty), \mathcal{K}^\alpha)$ be an arbitrary function such that $|||w - 1|||_\alpha < \infty$. Then for the rescaled functions*

$$w_B(\xi, s) = w(\xi B^{-\lambda_\alpha/\alpha}, sB) \qquad \text{for a constant} \quad B > 0, \tag{5.5}$$

we have

$$|||w_B - 1|||_\alpha = |||w - 1|||_\alpha \qquad \text{for all} \quad B > 0.$$

Proof Here, it suffices to change variables $s \mapsto sB$ in the norm defined in (5.4) and to use a scaling property of the norm (1.14). □

6 Radial eternal self-similar solutions

Our goal of this section is to construct *eternal self-similar solutions* of equation (5.2). Such solutions are called *eternal* because they exist for all $s > 0$, i.e. for all $t \in \mathbb{R}$. They are *self-similar* since they are invariant under scaling (5.5).

Theorem 6.1 (Radial self-similar solutions)**.** *Assume that $\mathcal{B}$ satisfies condition (1.18) for some $\alpha_0 \in [0,2)$. Let $\alpha \in [\alpha_0, 2)$. There exists a solution $\bar{w} \in C((0,+\infty), \mathcal{K}^\alpha)$, $\bar{w} \not\equiv 1$ of* (5.2) *with*

$$|||\bar{w} - 1|||_\alpha < \infty. \tag{6.1}$$

Before we prove Theorem 6.1 we give some remarks.

Remark 6.2. *It follows immediately from the first condition in* (6.1) *that*

$$\lim_{s\to 0} w(\xi, s) = 1 \qquad \textit{in the norm of} \quad \mathcal{K}^\alpha.$$

Thus, coming back to original variables and to the homogeneous Boltzmann equation for Maxwellian molecules (1.1)*, we obtain that the eternal, self-similar, and infinite energy solutions constructed in Theorem 6.1 are concentrated at the origin (i.e. they are a Dirac measure) as $t \to -\infty$.*

Remark 6.3. *According to an analogous theory for Navier-Stokes equation, it is natural to look for self-similar solutions of an equation under considerations in scaling invariant spaces, see e.g. [6]. In this work, the distance defined in* (5.4) *reflects a natural scaling invariant property of eternal solutions constructed by Bobylev and Cercignani in [5]. In fact, we expect that the eternal solution constructed in the previous theorem will be the functions which were constructed in [5], but a uniqueness result for eternal solutions is required to prove this. Moreover we expect a classification of the radial eternal solutions according to the parameters α and their generalized moment at $s = 1$, as defined in definition 3.1.*

Remark 6.4. *Eternal solutions exist also for the Navier-Stokes equations see for example [17].*

We are in a position to prove the main result of this section.

Proof of Theorem 6.1
We divide this proof into a series of steps for a clarity of the exposition.

Step 1. Rescaled solution

We begin with an arbitrary function $w_1 \in \mathcal{K}^\alpha$ satisfying

$$\lim_{\xi\to 0} \frac{1 - w_1(\xi)}{|\xi|^\alpha} = K.$$

For example, we can choose $w_1(\xi) = e^{-K|\xi|^\alpha}$. By Theorem 1.2, there exists a unique solution $w \in C([1,+\infty), \mathcal{K}^\alpha)$ of equation (5.2) with the initial $w(\cdot, 1) = w_1$. Moreover, by the stability estimate from Theorem 1.3

$$s^{-\lambda_\alpha} \|w(s) - 1\|_\alpha \le \|w_1 - 1\|_\alpha \qquad \text{for all} \quad s \ge 1.$$

For every $B > 0$ we consider the rescaled function

$$w_B(\xi, s) = w(\xi B^{-\mu_\alpha}, sB), \tag{6.2}$$

which is defined in $\mathbb{R}^3 \times [\frac{1}{B}, +\infty)$ and, by Remark 5.1, it is a solution of equation (5.2) satisfying the following scale invariant estimate

$$\sup_{s\geq\frac{1}{B}} s^{-\lambda_\alpha}\|w_B(\cdot,s)-1\|_\alpha = \sup_{s\geq 1} s^{-\lambda_\alpha}\|w(\cdot,s)-1\|_\alpha \leq \|w_1-1\|_\alpha. \tag{6.3}$$

Step 2. Compactness

We will show that there exists a function $\bar{w} \in C((0,+\infty),\mathcal{K}^\alpha)$ and a subsequence $B_j \to +\infty$ such that $w_{B_j} \to \bar{w}$, locally uniformly in $\mathbb{R}^3 \times (0,+\infty)$. Extending $w_B(\cdot,s)$ by $w(\xi B^{-\mu_\alpha},1) = e^{-|\xi|^\alpha B^{-\lambda_\alpha}}$ for $s \in (0,1/B)$ and using estimate (6.3), we may write that the family $\{w_B\}_{B>0}$ is uniformly bounded on every compact subset of $\mathbb{R}^3 \times (0,+\infty)$. We obtain the equicontinuity of this family in the space variable combining estimate (6.3) with the reasoning from [7, Step 3 of the proof of Lemma 5.1]. Indeed, we have

$$\begin{aligned}|w_B(\xi,s)-w_B(\eta,s)| &\leq \sqrt{2(1-\operatorname{Re} w_B(\xi-\eta,s))}\\ &\leq \sqrt{2}|\xi-\eta|^{a/2}s^{\lambda_\alpha}\sup_{s\geq 1/B} s^{-\lambda_\alpha}\|w_B(\cdot,s)-1\|_\alpha\\ &\leq \sqrt{2}|\xi-\eta|^{a/2}s^{\lambda_\alpha}\|w_1-1\|_\alpha\end{aligned}$$

for all $\xi,\eta \in \mathbb{R}^3$ and $s>0$. For the equicontinuity in time, we use equation (5.2), again estimate (6.3), and the reasoning from [7, Step 2 of the proof of Lemma 5.1] to obtain for $s > 1/B$

$$\begin{aligned}|\partial_s w_B| &\leq \frac{1}{s}\int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)|w_B(\xi^+,s)w_B(\xi^-,s)-w_B(\xi,s)|d\sigma\\ &\leq 4s^{\lambda_\alpha-1}\beta_\alpha\sup_{s\geq 1/B} s^{-\lambda_\alpha}\|w_B(\cdot,s)-1\|_\alpha\\ &\leq 4s^{\lambda_\alpha-1}\beta_\alpha\|w_1-1\|_\alpha\end{aligned}$$

with

$$\begin{aligned}\beta_\alpha &= \int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\frac{|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}}{|\xi|^\alpha}d\sigma\\ &= 2\pi\int_{-1}^{1}\mathcal{B}(s)\left(\frac{1+s}{2}\right)^{\alpha/4}\left(\frac{1-s}{2}\right)^{\alpha/4}ds < \infty\end{aligned} \tag{6.4}$$

by (1.18). Then by the Arzela-Ascoli theorem and a diagonal argument there exists a function $\bar{w} \in C(\mathbb{R}^3 \times (0,+\infty))$ such that we have up to a subsequence $w_{B_j} \to \bar{w}$ locally uniformly in $\mathbb{R}^3 \times (0,+\infty)$.

Step 3. The limit $B_j \to +\infty$

We have that w_{B_j} satisfies the equation

$$\partial_s w_{B_j}(\xi,s) = \frac{1}{s}\int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)|w_{B_j}(\xi^+,s)w_{B_j}(\xi^-,s)-w_{B_j}(\xi,s)|d\sigma, \tag{6.5}$$

then using (6.4) and the Lebesgue dominated convergence theorem we get that the right hand side of (6.5) converges locally uniformly to a function $\zeta \in C(\mathbb{R}^3 \times (0,+\infty))$ locally uniformly in $\mathbb{R}^3 \times (0,+\infty)$. Then $\partial_s \bar{w} = \zeta$ and $\bar{w}$ is a solution of (5.2) in $\mathbb{R}^3 \times (0,+\infty)$. Moreover, by the stability estimate from Corollary 5.2, we have for $\xi \neq 0$ and $s \geq 1/B_j$

$$s^{-\lambda_\alpha}\frac{|w_{B_j}(\xi,s)-1|}{|\xi|^\alpha} \leq B_j^{-\lambda_\alpha}||w_{B_j}(\cdot,B_j^{-1})-1||_\alpha = ||w(\cdot,1)-1||_\alpha,$$

and passing to the limit as $B_j \to +\infty$ we get $|||\bar{w} - 1|||_\alpha < \infty$.
Step 4. $\bar{w} \not\equiv 1$

It is $w_{B_j}(\xi, 1) = w(\xi B_j^{-\mu_\alpha}, B_j)$, which we may write in fast variables as $\phi(\xi e^{-\mu_\alpha t}, t)$, with ϕ the solution of (1.8) with initial data $w(\xi, 1)$. But by [7, Theorem 2.7] we have that the function ϕ converges as $t \to +\infty$ in self-similar variables to a function with α-moment K at $t = 0$, and therefore, cannot converge to 1.

□

7 Conjectures for the non-radial case

One should notice that a generalized α-moment (discussed in Theorem 3.3) may not exist for a large class of functions from $\mathcal{K}^\alpha$. Indeed, it suffices to consider the following anisotropic case

$$\varphi(\xi) = \exp\big(- K_1|\xi_1|^{\alpha_1} - K_2|\xi_2|^{\alpha_2} - K_3|\xi_3|^{\alpha_3}\big)$$

with different $\alpha_j \in (0, 2)$ and different $K_j \geq 0$ for $j \in \{1, 2, 3\}$. One can show that this is a characteristic function, hence $\varphi \in \mathcal{K}^\alpha$ with $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$, see (1.13). For such functions, we define their generalized moments in the following way.

Definition 7.1. *We call a function $\psi : \mathbb{R}^3 \to \mathbb{R}$ a generalized α-moment of a characteristic function $\varphi \in \mathcal{K}^\alpha$ if*

$$\psi(\xi) = \lim_{\ell \to 0, \ell > 0} \frac{(1 - \varphi(\ell\xi))}{\ell^\alpha}$$

for each $\xi \in \mathbb{R}^3$.

Let us first state elementary properties of such generalized moments. Using the definition of ψ we immediately obtain that it is a homogeneous function of order α : $\psi(\lambda\xi) = \lambda^\alpha\psi(\xi)$ for all $\lambda \in [0, \infty)$ and $\xi \in \mathbb{R}^3$. Moreover, one can show that a generalized moment of a characteristic function has to be negative definite, see *e.g.* [10, Ch. 3.6] for a definition of negative definite functions and their properties.

Now, we are in a position to formulate the following conjecture which is a generalization of Theorem 3.3.

Conjecture 7.2. *Let $\varphi \in C([0, \infty), \mathcal{K}^\alpha)$ be a solution of problem (1.8)–(1.12) constructed in Theorem 1.2. Suppose that there exists ψ_0 the generalized α-moment of $\varphi(\xi, 0) = \varphi_0(\xi)$. Then, the solution $\varphi = \varphi(\xi, t)$ has a generalized α-moment $\psi(\xi, t)$ for all $t \geq 0$. The function $\psi(\xi, t)$ is a solution of the following initial value problem :*

$$\partial_t\psi(\xi, t) = \int_{\mathbb{S}} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right)(\psi(\xi^+, t) + \psi(\xi^-, t) - \psi(\xi, t))d\sigma d\tau, \qquad \psi(\xi, 0) = \psi_0(\xi). \tag{7.1}$$

A formal calculation leads to equation (7.1) immediately. However, it seems that a rigorous proof of this conjecture requires some new ideas (especially in the non cut off case) comparing with the proof of Theorem 3.3.

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Titre

Contribution à la théorie des EDP non linéaires avec applications à la méthode des surfaces de niveau, aux fluides non newtoniens et à l'équation de Boltzmann

Title

A contribution to the theory of non-linear PDEs with applications to the level set method, non-Newtonian fluid flows and the Boltzmann equation