Weyl anomalies and quantum cosmology

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Résumé

Nous étudions les conséquences cosmologiques des anomalies de Weyl qui émergent de la renormalisation des opérateurs composés des champs fondamentaux, y compris la métrique. Ces anomalies sont codifiées dans les habillements gravitationnels des opérateurs dans une action effective quantique qui est non-locale. Nous obtenons les équations d'évolution qui découlent de cette action et nous en cherchons des solutions cosmologiques. Par simplicité on se limite à la gravité d'Einstein-Hilbert avec une constante cosmologique.

Nous initions par considérer la gravité en deux dimensions, où des résultats bien établis de la théorie de Liouville nous permettent de calculer explicitement la dimension anormale exacte de la constant cosmologique. En utilisant une formulation invariante de Weyl de la gravité, nous déterminons l'action effective qui est invariante de jauge et non-locale, et nous calculons le tenseur de moment correspondant, qui est aussi non-locale. Les anomalies de Weyl modifient le tenseur entier, pas seulement sa trace, menant à des conséquences intéressantes pour la dynamique cosmologique. En particulier, nous trouvons une énergie du vide qui décline avec le temps et un ralentissement de l'expansion de de Sitter à une de quasi-de Sitter.

En quatre dimensions, motivés par nos résultats en deux dimensions, nous paramétrisons l'action effective avec des habillements gravitationnels qui dépendent de l'échelle, et on obtient les équations d'évolution générales. Dans l'approximation des dimensions anormales constantes, le tenseur de moment conduit encore à une énergie du vide qui décline et une expansion de quasi-de Sitter de roulement lent, comme en deux dimensions. Les dimensions anormales sont calculables à priori dans une certaine théorie microscopique avec des méthodes semi-classiques, mais les calculs sont plus compliqués que dans la théorie de Liouville.

Même si les dimensions anormales sont petites en théorie des perturbations, leur contribution intégrée le long des plusieurs *e-folds* pourrait mener à des effets significatifs. Nous examinons les situations possibles où ces effets pourraient avoir étés pertinents pour la cosmologie primordiale. Nous finissons par décrire les travaux en cours et tracer les directions futures.

Mots clés : Gravité, cosmologie quantique, constante cosmologique, anomalie de Weyl, non-locale, Liouville, inflation.

Abstract

In this thesis we study the cosmological consequences of Weyl anomalies arising from the renormalization of composite operators of the fundamental fields, including the metric. These anomalies are encoded in the gravitational dressings of the operators in a non-local quantum effective action. We derive the evolution equations that follow from this action and look for cosmological solutions. For simplicity, we focus on Einstein-Hilbert gravity with a cosmological constant.

We first consider two-dimensional gravity, where results from Liouville theory allow us to explicitly compute the exact anomalous dimension of the cosmological constant operator. Using a Weyl-invariant formulation of gravity, we determine the manifestly gauge-invariant but non-local effective action, and compute the corresponding non-local momentum tensor. The Weyl anomalies modify the full quantum momentum tensor, not only its trace, and hence lead to interesting effects in the cosmological dynamics. In particular, we find a decaying vacuum energy and a slow-down of the de Sitter expansion to a power-law quasi-de Sitter one.

In four dimensions, motivated by our results in two dimensions, we parametrize the effective action with scale-dependent gravitational dressings, and compute the general evolution equations. In the approximation of constant anomalous dimensions, the momentum tensor leads to a decaying vacuum energy and a slow-roll quasi-de Sitter expansion, just as in two dimensions. The anomalous dimensions are in principle computable in a given microscopic theory using semiclassical methods, but require more elaborate computations than in Liouville theory.

Even though the anomalous dimensions are small in perturbation theory, their integrated effect over several e-folds could add up to something significant. We discuss possible situations where these effects could be relevant in primordial cosmology. We conclude by outlining ongoing research and future directions.

Keywords : Gravity, quantum cosmology, cosmological constant, Weyl anomaly, non-locality, Liouville, inflation.

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Résumé de la Thèse

La théorie de la gravité quantique est une théorie non-renormalizable. Malgré tout on peut la définir perturbativement avec succès comme une théorie effective de champs qui décrit correctement la physique au dessous d'une échelle de coupure M_0 de l'ordre de la masse de Planck M_p [1–3]. Le succès de la théorie effective de la gravité met en évidence qu'il n'y a pas d'incohérence entre la relativité générale et la mécanique quantique: même si une nouvelle théorie est requise au delà de l'échelle de coupure, aux basses énergies on peut calculer les corrections quantiques d'une manière cohérente et qui ne dépend pas de la théorie dans le régime ultraviolet. Le développement de cette théorie est donc très important, car elle devrait nous permettre de calculer des corrections à la relativité générale aux distances longues.

Selon la procédure des théories effectives de champs, l'action effective doit inclure tous les termes compatibles avec les symétries. Le principe de jauge de la relativité générale est la covariance générale. En conséquence, à l'action d'Einstein-Hilbert qui a deux dérivées, on ajoute une constante cosmologique, avec zéro dérivées, et successivement des termes avec de plus en plus de dérivées. Les premières corrections, avec quatre dérivées, sont accompagnées d'un facteur $1/M_p^2$, et sont donc supprimées par rapport à Einstein-Hilbert. Le fait que la masse de Planck soit largement supérieure aux énergies atteintes expérimentalement explique que la théorie effective de la gravité puisse faire des prédictions précises.

Jusqu'à présent, la plupart de résultats ont étés obtenus avec la théorie de la perturbation autour d'un fond de Minkowski en utilisant des techniques diagrammatiques. Le résultat principal de cette approche est le calcul des corrections quantiques dominantes aux potentiel de Newton [4]. Même s'il est remarquable que ces corrections soient bien définies, finies et calculables, elles sont sensées être très faibles à cause de la suppression de Planck. Vu que ces corrections consistent en des termes avec des dérivées, elles sont plus larges aux hautes énergies. Plus concrètement, pour obtenir quelque effet qui puisse être observé, on devrait tester des énergies suffisamment proche à l'échelle de Planck. Un contexte où ces énergies devraient avoir été atteintes est la cosmologie primordiale. La théorie la plus acceptée sur les premiers instants de notre univers c'est la théorie de l'inflation [5–7], d'après laquelle l'univers a expérimenté une période de très forte expansion quasi-exponentielle autour des 10^{-34} secondes après la singularité initiale. L'inflation est probablement un des rares scénarios théoriques où ces corrections pourraient être suffisamment larges. Aussi, les fluctuations quantiques présentes à ce moment devraient avoir été magnifiées du fait de l'expansion de l'espace, en laissant des traces observables sur le ciel d'aujourd'hui.

Pour appliquer la théorie effective de la gravité à la cosmologie, on doit développer perturbativement autour d'un fond de Robertson-Walker. Pour cette théorie effective donc, il faut un formalisme qui soit indépendant du fond et complètement covariant. Un des ingrédients le plus important de ce formalisme est l'action quantique effective. Pour l'écrire, il est très pratique d'utiliser la méthode du champ de fond, qui permet de travailler sur un fond qui n'est pas forcement une solution des équations du mouvement. Une caractéristique fondamentale de l'action effective est qu'elle est non-locale. En effet, elle n'est pas l'action Wilsonienne, mais l'action d'une particule irréductible. Les gravitons sont intégrés depuis l'échelle de coupure jusqu'à zéro énergie, ce qui donne

lieu à des termes non-locaux dans l'action.

Une approche très pratique pour dériver (une partie de) l'action effective gravitationnelle est l'intégration des anomalies de Weyl. Cette approche permet de profiter des simplifications qui ont lieu quand une théorie est très proche d'un point fixe. Les anomalies de Weyl apparaissent dans la trace du tenseur de moment dû aux effets quantiques [8,9]. Ces termes anormales doivent dépendre de la courbure de l'espace-temps, vu qu'ils devraient s'annuler dans l'espace plat. En plus, ils doivent être déterminés par la trace de la variation de l'action effective quantique par rapport à la métrique, et donc ils sont une bonne source d'information sur cette action. En particulier, vu que les anomalies paramétrisent chaque théorie, elles doivent être codées dans l'action dans des termes non-locaux. Autrement, elles pourraient être éliminées avec des contre-termes locaux, et les anomalies dépendraient du schéma de renormalisation. Ces anomalies de Weyl sont très intéressantes quand la théorie classique est invariante de Weyl, car dans ce cas elles sont la seule contribution à la trace du tenseur de moment, et elles violent l'identité de Ward pour l'invariance de Weyl qui dicte que la trace classique doit être nulle. En plus, si l'invariance de Weyl est une invariance de jauge, la somme totale de ces anomalies doit être nulle, ce qui fourni des critères pour les valeurs que les coefficients des anomalies peuvent prendre.

Pour determiner l'action effective à partir des anomalies, on peut traiter l'expression de la trace du tenseur de moment en termes des anomalies comme une équation pour l'action. Ce là est possible parce que, en fait, la trace est aussi donnée par la variation de l'action par rapport au facteur conforme de la metric $\Sigma_g(x)$, où $g_{\mu\nu} = e^{2\Sigma_g}\bar{\eta}_{\mu\nu}$, et $\bar{\eta}_{\mu\nu}$ est une métrique de référence fixée par quelque condition scalaire. En effet, en écrivant les opérateurs qui apparaissent dans la trace dans cette jauge, on obtient une équation pour l'action effective qui peut être intégrée, c'est-à-dire

$$T = \frac{1}{\sqrt{-g}}\frac{\delta\Gamma[g]}{\delta\Sigma_g(x)} = c_i\,\mathcal{O}_i(e^{2\Sigma_g}\bar{\eta}_{\mu\nu})\,. \tag{0.0.1}$$

L'action qui suit de cette integration est écrite automatiquement dans cette jauge conforme, mais on peut la rendre covariante en utilisant l'expression du facteur conforme $\Sigma_g(x)$ en termes de la métrique physique $g_{\mu\nu}$. Cette expression contient une intégrale du scalaire de Ricci avec la fonction de Green du Laplacien sur tout l'espace-temps, et donc le facteur conforme est une fonctionnelle non-locale de la métrique. L'action donc, devient aussi non-locale, comme on s'y attendait.

Deux types différents d'anomalies peuvent apparaître dans la trace du tenseur de moment. D'abord, il y a les anomalies qui résultent du fait que les mesures des champs dans l'intégrale du chemin ne sont pas invariantes de Weyl. Ces anomalies sont caractérisées par les points fixes où les fonctions-beta des opérateurs dans l'action sont nulles, et elles consistent en des termes de courbure avec des coefficients constants qui paramétrisent chaque théorie. Les exemples les plus connus sont la charge centrale c en deux dimensions, qui accompagne le scalaire de Ricci, ou les charges a et b en quatre dimensions, qui accompagnent la densité d'Euler et le tenseur de Weyl respectivement.

Le deuxième type d'anomalies de Weyl vient de la renormalisation des opérateurs présents dans l'action classique. Typiquement, ces opérateurs sont composés des champs fondamentaux, et donc ils enferment des divergences de contact qui doivent être renormalisées. Dans le cas des théories définies dans un espace-temps plat, la renormalisation des opérateurs requiert l'introduction d'une échelle additionnelle M qu'on utilise pour implementer le groupe de renormalisation. Les objets renormalisées acquièrent une dépendance en M, qui peut briser l'invariance d'échelle au niveau quantique. En effet, vu que l'échelle de renormalisation change sous la transformation d'échelle, la dépendance en M acquise par les opérateurs une fois renormalisés, $\mathcal{O}_i(M)$, leur donne une dimension d'échelle anormale $-\gamma_i(M)$. Cette dimension est encodée dans la fonction-beta de la constante de couplage $\lambda_i(M)$ correspondante à l'opérateur, dont l'expression est donnée par $\beta_i(\lambda_i) = (\Delta_i - d + \gamma_i)\, \lambda_i(M)$, où Δ_i est la dimension classique de l'opérateur.

La généralisation de cette ligne de raisonnement à une théorie gravitationnelle requiert la substitution des transformations d'échelle $\delta \ln M$ par les transformations de Weyl $\delta\Sigma_g$. Les dimensions anormales sous les transformations d'échelle deviennent donc des dimensions de Weyl anormales. Cette généralisation suggère que le facteur conforme de la métrique doit prendre le rôle de l'échelle covariante de renormalisation M, et donc que les opérateurs renormalisés acquièrent une dépendance anormale due facteur conforme $\mathcal{O}_i(\Sigma_g)$. Une dépendance additionnelle à la métrique est une conséquence prévisible d'une renormalisation covariante. En plus, une échelle de renormalisation doit devenir locale, $M(x)$, dans une théorie gravitationnelle, où il n'y a pas des énergies bien définies globalement. L'idée intuitive derrière cette affirmation est que pour une théorie définie sur un espace-temps générale, il n'y a aucune raison pour laquelle deux observateurs placés dans des points très séparés dans l'univers devraient choisir la même échelle de renormalisation arbitraire, et donc celle-ci devrait dépendre de la position. L'échelle d'énergie caractéristique est donc remplacée par l'échelle caractéristique de courbure, et en particulier, cette généralisation dicte qu'elle est donnée par le facteur conforme de la métrique.

Comme dans le cas des théories en l'espace plat, les dimensions anormales peuvent en générale dépendre de l'échelle covariante de renormalisation, c'est à dire $\gamma_i(\Sigma_g)$. Les opérateurs renormalisés deviennent donc

$$\mathcal{O}_i(\Sigma_g) = \mathcal{O}_i^0\, \mathcal{Z}_i^{-1}(\Sigma_g) = \mathcal{O}_i^0\, e^{-\int_0^{\Sigma_g} \gamma_i(\Sigma)\, d\Sigma}\,, \tag{0.0.2}$$

où $\mathcal{Z}_i$ désigne le facteur multiplicatif qui contient la variation de l'opérateur renormalisés par rapport à l'échelle de renormalisation, et $\mathcal{O}_i^0$ est l'opérateur nu. L'action quantique effective s'écrit avec les opérateurs renormalisés, et donc elle hérite de leur dépendance quantique dans la métrique. On dit alors que les opérateurs dans l'action deviennent habillés gravitationnellement. En conséquence, la variation de l'action par rapport à la métrique, ainsi que sa trace, héritent aussi de cette dépendance quantique. De la même manière que l'action acquière un comportement anormale sous une transformation de Weyl, la trace du tenseur de moment acquière une anomalie, qui est donnée par

$$T = \frac{1}{\sqrt{-g}} \frac{\delta\Gamma}{\delta\Sigma_g} = (d - \Delta_i - \gamma_i)\ \lambda_i\, \mathcal{O}_i(\Sigma_g)\,, \tag{0.0.3}$$

où les deux premiers termes viennent de la variation de l'action classique. D'après l'expression de la fonction-beta, il est clair que la trace quantique obéit

$$T = -\beta_i(\lambda_i)\, \mathcal{O}_i(\Sigma_g)\,. \tag{0.0.4}$$

Celle-ci est l'équation de l'anomalie qui inclue les dimensions de Weyl anormales acquis par les opérateurs, et en conséquence par l'action effective, à cause de la renormalisation. Il s'agit d'une équation très utile parce qu'en connaissant les dimensions anormales des opérateurs elle permet de calculer leurs habillements gravitationnels dans l'action effective. Cette équation dicte donc que l'action devrait prendre la forme $\Gamma \sim \int \lambda_i \, O_i \, e^{-\Gamma_i(\Sigma_g)}$. Nous allons utiliser cet argument pour proposer une action effective gravitationnelle en quatre dimensions.

La conclusion des raisonnements expliqués ci-dessus est que les constantes de couplages physiques sont les constantes des opérateurs habillés gravitationnellement, et que les dimensions anormales des opérateurs habillés sont en principe différentes des dimensions anormales des opérateurs sans habillement gravitationnel. Ces considérations devraient être valides pour touts les opérateurs, en particulier pour les opérateurs purement gravitationnels qui déterminent les équations du mouvement de la métrique. Une des idées principales qui suit du travail présenté ici est que les habillements gravitationnels modifient le tenseur de moment entier, pas seulement sa trace, et donc modifient la dynamique quantique de la théorie. Les anomalies de Weyl, donc, peuvent avoir des conséquences intéressantes pour la dynamique cosmologique.

Dans cette thèse nous nous intéressons principalement au deuxième type d'anomalies. En particulier nous sommes intéressés par le calcul des anomalies des opérateurs composés qui sont pertinents pour l'évolution cosmologique. Le premier type d'anomalies est aussi supposé être présent, mais nous allons plutôt l'ignorer, et nous allons supposer que les effets des différent types d'anomalies peuvent être traités séparément.

D'après les considérations présentées il serait pratique de développer un formalisme qui soit invariant de jauge et qui permettrait de prédire des caractéristiques générales des effets des anomalies sur l'évolution de l'univers. Dans cette thèse, nous abordons cet objectif en étudiant les effets cosmologiques des anomalies de Weyl de l'action d'Einstein-Hilbert avec une constante cosmologique. Nous nous intéressons donc à l'action effective correspondante qui inclut les habillements gravitationnels appropriées pour les deux opérateurs dans cette action classique. Notre approche se compose principalement de deux étapes : d'abord, nous déterminons ces habillements gravitationnels pour les deux opérateurs et nous écrivons l'action effective ; ensuite, nous calculons les équations d'évolution cosmologique et nous cherchons des solutions pour un univers homogène et isotrope.

À cette fin il est très pratique d'utiliser un formalisme qui permet de donner un traitement spécial au facteur conforme de la métrique, en restant toujours covariante. Un tel formalisme est donné par la formulation invariante de Weyl de la gravité, laquelle a été fortement mise à profit dans cette thèse. La formulation invariante de Weyl [10] consiste à introduire un champ compensateur de Weyl $\Omega(x)$ et une métrique de référence $h_{\mu\nu}$, qui se transforment d'une manière telle que la métrique physique $g_{\mu\nu} = e^{2\Omega} h_{\mu\nu}$ reste invariante. Le compensateur donc, transforme linéairement sous une transformation de Weyl. La théorie qui résulte de cette reformulation de la métrique physique possède un degré de liberté supplémentaire, mais aussi un principe de jauge élargi, qui inclut cette transformation de Weyl ainsi que les difféomorphismes de la théorie originelle. Le nombre de degrés de liberté reste donc le même après avoir imposé l'invariance de Weyl. Une condition essentielle du principe de jauge élargi est que toutes les anomalies qui apparaissent doivent s'annuler puisque l'invariance de Weyl est une invariance de jauge. Ceci fournit un fil directeur très utile.

Une conclusion du travail présenté dans cette thèse est que la formulation invariante de Weyl devient très utile pour calculer les dimensions anormales et les habillements gravitationnels des opérateurs dans le cas de deux dimensions, où l'invariance de jauge élargie permet de fixer le métrique de référence $h_{\mu\nu}$ complètement. Dans ce cas le seul degré de liberté est le compensateur $\Omega(x)$. Les opérateurs covariants deviennent des opérateurs composés seulement de $\Omega(x)$, qui est un scalaire et dont les divergences ultraviolettes sont plus faciles à régulariser.

Un autre avantage de la formulation invariante de Weyl est d'être très appropriée aux calculs cosmologiques. Dans un univers homogène et isotrope, la seule composante dynamique de la métrique est son facteur d'échelle, tandis que la métrique du fond est complètement fixée par les symétries et la choix de la courbure spatiale. Le compensateur $\Omega(x)$ prend le rôle du facteur d'échelle et la métrique de référence $h_{\mu\nu}$ celui de la métrique du fond. Dans notre traitement en quatre dimensions nous utilisons cette formulation essentiellement pour dériver les équations d'évolution cosmologique d'une manière simple.

Face à un problème difficile, il est toujours sage de commencer par un modèle simplifié qui permet de faire des calculs explicites mais qui saisit malgré tout les caractéristiques essentielles du modèle plus général. En accord avec l'esprit de cette approche notre point de départ est l'action d'Einstein-Hilbert avec une constante cosmologique en deux dimensions

$$I_G[g] = \frac{1}{2\kappa^2} \int d^2x\sqrt{-g}\ (R_g - 2\Lambda)\ . \tag{0.0.5}$$

Cette analyse bidimensionnelle constitue la première partie de cette thèse. Une des raisons pour considérer deux dimensions est que toute la dynamique quantique réside dans le facteur conforme de la métrique, comme argumenté précédemment, et donc les dimensions anormales deviennent plus faciles à calculer. En plus, comme nous le montrons, quand l'action d'Einstein-Hilbert est écrite dans sa forme invariante de Weyl en terme du compensateur $\Omega(x)$ et la métrique de référence $h_{\mu\nu}$, l'action classique devient une continuation analytique de l'action bien connue de Liouville, l'action de Liouville de type temps ('timelike'). Le compensateur de Weyl dans cette théorie prend le rôle du champ de Liouville. La théorie de Liouville est une théorie des champs conforme qui a été très étudiée, et pour laquelle l'approche du bootstrap a permis de dériver toutes les données conformes de la théorie, c'est-à-dire son spectre, ses fonctions à deux points et ses constantes de structure. C'est l'apparition de cette riche théorie de Liouville qui nous permet de déterminer l'action effective exacte, et donc qui justifie complètement l'étude de la cosmologie quantique en deux dimensions.

L'action de Liouville est présente dans beaucoup d'approches de la gravité quantique bidimensionnelle. Puisque la théorie timelike de Liouville résulte d'une continuation analytique du champ de Liouville, elle acquière un terme cinétique négatif, d'où le qualificatif de 'timelike'. Ce signe moins du terme cinétique est attendu pour la dynamique du facteur conforme de la métrique [11], et en fait il s'agit d'une des difficultés standards rencontrées quand on essaye de quantifier la gravité dans n'importe quelle dimension d'espace-temps. Heureusement, la quantification complète de la théorie timelike du Liouville n'est pas nécessaire pour les questions qui nous intéressent, pour lesquelles la continuation analytique des résultats générales de la théorie de Liouville suffit.

Notre première étape consiste à calculer l'habillement gravitationnel de l'opérateur de constante cosmologique. Dans la formulation invariante de Weyl, cet opérateur identité $\Lambda\sqrt{-g}$, devient $\Lambda\, e^{2\Omega}\sqrt{-h}$. Après avoir fixé l'invariance de jauge, seul le champ de Liouville $\Omega(x)$ fluctue. Donc, on transforme les divergences de contact de l'opérateur composé original $\sqrt{-g}$ en des divergences de contact de l'opérateur vertex $e^{2\Omega}$. Voilà un exemple de la simplification fournie par la formulation invariante de Weyl.

La renormalisation de cet opérateur vertex dedans la théorie qui l'inclut dans l'action devrait être très complexe. Malgré tout une des caractéristiques la plus remarquable de la théorie de Liouville est que les divergences ultraviolettes peuvent être renormalisées avec un simple ordre normal [12–17]. Ceci simplifie énormément le calcul de l'habillement gravitationnel de la constante cosmologique, étant donné qu'il permet de renormaliser l'opérateur vertex comme dans le cadre d'une théorie libre.

L'habillement gravitationnel calculé dépend du facteur conforme de la métrique de référence $\Sigma_h(x)$, et elle est donnée par l'expression

$$[e^{2\Omega}]_h = e^{2\,\beta\, q\,\Omega}\, e^{-2\beta^2\Sigma_h}\,, \tag{0.0.6}$$

où $[e^{2\Omega}]_h$ indique l'opérateur renormalisé par rapport à la métrique $h_{\mu\nu}$, q est une redéfinition de la constante gravitationnelle κ^2 pour deux dimensions, et β paramétrise l'anomalie. La dimension anormale peut être lue dans l'expression de l'opérateur renormalisé et vaut $\gamma = 2\beta^2$. En imposant l'annulation des anomalies de Weyl on détermine la dépendance du paramètre anormal β dans la constante fondamentale gravitationnelle q, et on reproduit bien la relation de Liouville $\beta(q+\beta)=1$. L'action effective invariante de jauge et non-locale est finalement donnée par

$$I_\Lambda[g] = -\frac{\Lambda}{\kappa^2}\int d^2x\sqrt{-g}\, e^{-2\beta^2\Sigma_g}\,. \tag{0.0.7}$$

La correction quantique dû à l'anomalie devient évidente avec l'habillement gravitationnel de l'opérateur $\sqrt{-g}$. Cette action effective devient non-locale quand on introduit l'expression covariante du facteur conforme, qui suit de l'inversion de la transformation de Weyl du scalaire de Ricci, et qui est donnée par

$$\Sigma_g(x) = \frac{1}{2}\int d^2y\sqrt{-g}\, G_g(x,y)\, R_g(y)\,, \tag{0.0.8}$$

où $G_g(x,y)$ est la fonction de Green du Laplacien sur la métrique $g_{\mu\nu}$.

La deuxième partie de notre analyse bidimensionnelle concerne l'étude des conséquences cosmologiques de cette constante cosmologique renormalisée. À cette fin, on calcule le tenseur quantique de moment qui suit de l'action ci-dessus, qui devient encore non-locale dû aux anomalies. En imposant les symétries d'un univers de Robertson-Walker, le tenseur non-local se simplifie beaucoup et prend la forme d'un fluide parfait donné par

$$p_\Lambda = w_\Lambda\, \rho_\Lambda\,, \quad \text{avec} \quad w_\Lambda = -1 + 2\beta^2\,, \tag{0.0.9}$$

où l'indice barotrope diffère de sa valeur classique -1, dû visiblement à l'anomalie. En introduisant le tenseur de moment dans les équations d'Einstein en deux dimensions et imposant la forme du fluide parfait, on trouve l'équation de Friedmann avec la densité du vide renormalisée. La solution que l'on trouve est donnée par

$$\rho_\Lambda(t) = \rho_*(\frac{a}{a_*})^{-2\beta^2}\,, \qquad a(t) = a_*(1+\beta^2 H_* t)^{\frac{1}{\beta^2}}\,. \tag{0.0.10}$$

Ces solutions montrent que les corrections quantiques mènent à une énergie du vide qui décline dans un univers homogène et isotrope et à un ralentissement de l'expansion exponentielle de de Sitter. Ceci est un résultat très intéressant parce qu'il offre un mécanisme dynamique pour la décroissance de la densité d'énergie de la constante cosmologique, qui est basé seulement sur la physique gravitationnelle aux basses énergies. En plus, une telle décroissance de l'énergie du vide pourrait provoquer une période inflationnaire basée seulement sur des effets quantiques gravitationnels, sans la nécessité d'introduire des scalaires supplémentaires. Elle offre donc un modèle d'inflation sans inflaton.

Le but final de l'étude de ce modèle bidimensionnelle est de tirer quelques conclusions générales qui peuvent être applicables en quatre dimensions. L'avantage de ce modèle est que les effets quantiques importants peuvent être calculés explicitement avec assez d'aisance et sans ambiguïtés, à tous les ordres dans la théorie des perturbations. On s'attend que l'approximation semi-classique sera fiable aux échelles cosmologiques en quatre dimensions.

L'analyse de la gravité en quatre dimensions comprend la deuxième partie de cette thèse. Comme en deux dimensions, on s'intéresse à l'action quantique effective qui est valide aux distances longues comparées à l'échelle de Planck. Notre ingrédient central est de nouveau l'action effective non-locale qui intègre les anomalies de Weyl de la gravité d'Einstein-Hilbert et de la constante cosmologique. C'est-à-dire, l'action qui inclut les habillements gravitationnels des opérateurs R_g et $\sqrt{-g}$.

En quatre dimensions on ne jouit pas de la simplification amenée par la formulation invariante de Weyl et par la théorie de Liouville dans les calculs des habillements gravitationnels en deux dimensions. En effet l'invariance jauge ne fixe pas complètement la métrique de référence et les opérateurs covariants sont composés non seulement du compensateur $\Omega(x)$, mais aussi des composantes de la métrique $h_{\mu\nu}$, qui ne sont pas fixées par la choix de jauge. Ainsi les divergences de contact sont difficiles à renormaliser d'une manière covariante et on doit recourir aux calculs perturbatifs. Tout de même les résultats de nôtre modèle bidimensionnelle et les considérations sur la dépendance anormale dans l'action effective au facteur conforme présentées dessus permettent de déduire la forme la plus logique pour l'action effective gravitationnelle. Celle-ci est donnée par [10]

$$\Gamma_G[g] = \frac{M_p^2}{16\pi}\int d^4x\sqrt{-g}\,\left(R_g\, e^{-\Gamma_K(\Sigma_g)} - 2\,\Lambda\, e^{-\Gamma_\Lambda(\Sigma_g)}\right)\,, \tag{0.0.11}$$

où les $\Gamma_i(\Sigma_g)$, $i = K, \Lambda$ représentent les habillements gravitationnels intégrées. Vu que cette action dépend de $\Sigma_g = \Omega + \Sigma_h$, elle est invariante de jauge automatiquement. Ces dimensions anormales et les habillements gravitationnels sont calculables dans une certaine théorie microscopique. La renormalisation de la constante de Newton et de la constante cosmologique ont été déjà calculées précédemment [18–20]. On peut extraire les évolutions logarithmiques sous le groupe de renormalisation de ces résultats. Toutefois la stratégie que nous suivons dans cette thèse consiste à considérer les habillements intégrés comme des fonctions générales qui paramétrisent l'action effective, et nous différons la détermination de leur expression exacte à un projet ultérieur.

Le but principal de notre analyse en quatre dimensions est de déterminer la dynamique cosmologique qui suit de l'action effective (5.1.1). À cette fin, on détermine les équations intégro-différentielles qui découlent de cette action et qui décrivent la dynamique effective de l'espace-temps aux distances longues, et on cherche des solutions de Robertson-Walker. Le tenseur de moment a encore la forme d'un fluide parfait, et sous l'approximation des anomalies linéaires – c'est-à-dire $\Gamma_K(\Omega) = \gamma_K\,\Omega(x)$ et $\Gamma_\Lambda(\Omega) = \gamma_\Lambda\,\Omega(x)$ (les γ_i sont des constantes) –, il acquière un indice barotrope constant

$$w_e = -1 + \frac{\gamma}{3}\,, \quad \text{avec} \quad \gamma = \gamma_\Lambda - \gamma_K\,, \tag{0.0.12}$$

qui est de nouveau au dessus de sa valeur classique. Même si l'approximation des anomalies constantes est seulement un cas particulier elle pourrait être valide pour des périodes suffisamment longues pendant l'évolution de l'univers.

La solution obtenue de l'équation de Friedmann est

$$\rho_e(t) = \rho_{e*}\left(\frac{a}{a_*}\right)^{-\gamma}\,, \qquad a(t) = a_*(1 + \frac{\gamma}{2}H_* t)^{\frac{2}{\gamma}}\,. \tag{0.0.13}$$

Cette solution décrit une décroissance lente de la densité d'énergie du vide et une expansion accélérée qui suit une loi de puissance, exactement comme en deux dimensions.

Vu que le facteur conforme n'est pas un vrai scalaire mais seulement une composante de la métrique dans une jauge particulière notre résultat fournit un modèle d'inflation quasi-de Sitter sans scalaire fondamental, poussée entièrement par la densité d'énergie du vide dû à la dynamique quantique non triviale du champs Ω. Ainsi il fournit un modèle d'inflation sans inflaton.

Aussi bien en deux comme en quatre dimensions les paramètres de roulement lent peuvent être calculés. Le premier paramètre ε_H devient proportionnel au paramètre de l'anomalie (β ou $\gamma_\Lambda - \gamma_K$ respectivement). Dans les deux cas il est plus petit que l'unité dans la limite semi-classique. Le deuxième paramètre η_H devient exactement zéro, ce qui indique que les deux modèles permettent une inflation de roulement lent pour une période suffisamment longue.

Les corrections quantiques dûes aux anomalies dans les deux cas transforment l'expansion classique exponentielle exact de de Sitter dans une de quasi-de Sitter. Quand on écrit la solution quantique corrigée comme une expansion autour de la solution classique, on voit que les corrections sont logarithmiques, ce qui est cohérent avec le fait que les habillements gravitationnels intégrés dans l'action effective additionnent les logarithmes principaux de la renormalisation des opérateurs.

À ce point, le modèle mène à un univers vide parce qu'il continue d'enfler. Avec des champs de matière il serait possible de construire des scénarios plus réalistes, avec un mécanisme de réchauffage et un Big Bang chaud, ainsi qu'avec des perturbations primordiales.

Contents

Introduction

Quantum gravity is a non-renormalizable quantum field theory. However, it can be successfully treated as a perturbative effective field theory [1–3]. As such, this theory should correctly describe the relevant physics below a given ultraviolet cutoff M_0 of the order of the Planck mass M_p, despite requiring an ultraviolet completion beyond this scale. The effective field theory point of view makes it clear that there is no fundamental inconsistency between general relativity and quantum mechanics: gravity becomes non-perturbative for energies around M_0, where all of the quantum corrections and higher-order terms in the Lagrangian become important, but at low energies the quantum corrections can be computed consistently as for any other perturbative quantum field theory, and they will not depend on the ultraviolet completion. Understanding this theory is hence of utmost interest: it should allow us to compute quantum gravitational corrections perturbatively, which modify classical general relativity at long distances, and should shed light onto the putative ultraviolet completion, for the latter should reproduce these corrections at sufficiently low energies.

As the effective field theory approach dictates, the quantum action has to include all the terms compatible with the symmetries. The gauge principle of general relativity is general coordinate invariance, which allows for all products of the metric, the Riemann tensor and covariant derivatives thereof. Therefore, to the classical Einstein-Hilbert action, one should add a cosmological constant Λ as the trivial operator, and all higher-derivative terms[1]

$$I_G[g] = \frac{M_p^2}{16\pi} \int d^4x\sqrt{-g} \left(R_g - 2\Lambda + \frac{1}{M_p^2} \left(\Lambda_1 \, R_g^2 + \Lambda_2 \, R_{\mu\nu} R^{\mu\nu} \right) + \dots \right) . \qquad (0.0.14)$$

Since the Riemann tensor $R_{\mu\nu\rho\sigma}$ contains two derivatives, all the terms with Λ_i above contain four derivatives, and therefore need a factor of M_p^{-2}. The dots would include higher derivatives, hence operators ever more irrelevant. Higher-derivative corrections therefore, are Planck mass suppressed compared to the perturbative corrections coming from the first two terms.

The large value of the Planck mass renders any experimentally accessible energy in the very low energy regime, and somehow paradoxically makes effective quantum gravity a very good perturbative quantum field theory. This great separation of scales, between experimental energies and the gravitational cutoff, is the origin of both the successful predictivity of effective quantum gravity, and the difficulty to test any ultraviolet completion from which the coefficients Λ_i should be predicted.

[1] The $R^2_{\mu\nu\rho\sigma}$ term is not included because a linear combination with R_g^2 and $R^2_{\mu\nu}$ can be formed which is a total derivative, the Gauss-Bonnet term, and which hence has no effect on the dynamics.

The perturbative expansion of effective gravity is implemented with a gravitational perturbation $h_{\mu\nu}$ around a fixed background $g_{\mu\nu}$

$$g_{\mu\nu} + \sqrt{8\pi G_N}\, h_{\mu\nu}\,, \tag{0.0.15}$$

where G_N is Newton's constant. Up to date, most of the work has been devoted to perturbation theory around the Minkowski background $g_{\mu\nu} = \eta_{\mu\nu}$, using diagrammatic techniques. The chief result of this approach is the computation of the leading quantum gravitational corrections to the Newtonian potential [4]. With this analytic result, the first corrections on the bending of light around the Sun [21] and black hole metrics have been computed [22, 23].

It is to be appreciated that these corrections are well defined, finite and computable with relative ease, and that they yield precise testable predictions. This makes them all the more interesting, and not only fully justifies but also calls for thorough exploration of this theory. However, these corrections are expected to be very small due to their Planck suppression. Higher-derivative terms are suppressed with negative powers of M_p^2, but even the lowest order h^3 corrections coming from the Einstein-Hilbert term are Planck suppressed, with M_p as (0.0.15) dictates. Since interaction terms are of the form $\sim \partial^m\, h^n$, one either needs very high energies or very large field values to enhance the corrections. As for the latter, large field values are found in nature, for example in the vicinities of stars. But even then, the resulting corrections are very small. For instance, the radiative correction coming from the expansion of the Einstein-Hilbert term predicts a shift in the perihelion of Mercury of one part in 10^{90}. This clearly makes any of these observations unmeasurable in any near future.

The alternative to attain any observational check then, is to explore high energies. These have to be of an order close enough to the Planck mass. Clearly, no human-made experiment can nowadays probe such scales. Cosmology on the other hand, may be able to provide such probes. Indeed, in its very early stages, the universe should have reached energies of a few orders of magnitude below the Planck scale. It is widely believed that around 10^{-34} seconds after the initial singularity, the universe underwent a period of great quasi-exponential expansion, the so-called inflationary period [5–7] (for a pedagogical review see [24]). Not only is inflation one of the few theoretical settings where these corrections could have been large enough, but also any of the quantum fluctuations present at the time should have been considerably magnified due to the expansion of space, making their relics observable nowadays. It is therefore of paramount interest to apply the effective field theory of gravity to the primordial evolution of the universe, and to further understand the signature of these quantum corrections in current sky observations like the cosmic microwave background.

To be able to consistently apply effective quantum gravity in a cosmological setting however, the perturbative expansion should be done on a Robertson-Walker background. Therefore, a background-independent and fully covariant formalism of effective quantum gravity is required. One of the important steps in the construction of such a formalism is to write down the quantum effective action. For this purpose, it is very convenient to use the background field method, which allows to quantize theories without losing explicit gauge invariance and choose a completely arbitrary and unspecified background, that need not be a solution of the equations of motion. A key feature of the effective action is that it is non-local. Indeed, it is not to be understood as Wilsonian, but rather as the one-particle-irreducible action. The integration of the massless gravitons is done all the way from the cutoff to zero energy, and therefore it is expected to lead to non-local terms.

One interesting approach to both constructing the effective action and computing loop diagrams covariantly is based on the heat kernel and its expansions [25,26]. This method represents the state of the art techniques to compute higher-order perturbative corrections systematically [27,28]. Even when only the heat kernel trace is necessary for the computations, exact expressions are hard to compute on an arbitrary background, and asymptotic expansions are typically required. Currently, two main expansion schemes have been developed. The first one is the derivative or early time expansion [26,29], which is local, and the second one is an expansion in the curvatures [25,30–34], which is non-local. The local heat kernel expansion is usually employed to compute ultraviolet divergences and anomalies, since these are directly related to local heat kernel coefficients. The non-local expansion is used instead to directly compute the finite part of the effective action. The two expansions are entangled in a non-trivial way, as becomes apparent from (0.0.14), where at fourth order in derivatives, there are both first and second order in curvature terms. In fact, the non-local expansion re-sums an infinite number of local heat kernel coefficients (from the derivative expansion at each curvature order) in the form of non-local structure functions.

An alternative approach to predicting terms in the effective gravitational action is that of the integration of Weyl anomalies. Even if it does not allow to systematically compute higher-order terms and does not give full knowledge of the effective action, it is a very convenient approach because it does not require re-summation of asymptotic expansions, and because it conveniently exploits the simplifications that come when working onto or very close to a fixed point.

It is well known that due to quantum effects, the so-called Weyl anomaly terms appear in the trace of the quantum momentum tensor [8,9].[2] They ought to depend on the background curvature, for they should vanish on flat spacetime. These quantum terms should follow from the traced metric variation of the quantum effective action, hence they are a precious source of information about the latter. In particular, since the anomalies (or rather their coefficients) parametrize each given theory, they have to be encoded in the action in non-local terms. Otherwise, they could be removed by adding local counterterms, and anomalies would be renormalization scheme-dependent. These anomalies are very interesting when the theory is classically Weyl invariant, since then they represent the entire contribution to the momentum tensor's trace, and they are responsible for spoiling the Weyl Ward identity which dictates its tracelessness. Even more so, if Weyl invariance is a gauge invariance, the total sum of anomalies has to cancel. This provides constraints on the values the anomaly coefficients can take.

The Weyl anomalies of a theory at a fixed point are given by purely geometric terms with constant parametric coefficients. The origin of these anomalies is the metric-dependence of the fields measures in the path integral. These measures are defined by an inner product of field perturbations that requires specification of a background metric. This product needs not be Weyl invariant, hence can lead to Weyl invariance

[2]See [35] for a nice historical review.

violation of the quantum effective action that follows from doing the path integral.

To be able to extract the information about the quantum effective action from the anomalies, one can treat the expression of the anomalous trace of the momentum tensor as an equation for the quantum effective action. This can be done by realizing that the trace can be written as the variation of the action with respect to the conformal factor of the metric $\Sigma_g(x)$, where $g_{\mu\nu} = e^{2\Sigma_g}\bar{\eta}_{\mu\nu}$ and $\bar{\eta}_{\mu\nu}$ is a fiducial metric fixed by some scalar condition. Fixing the metric to this so-called conformal gauge spoils covariance, but allows for the integration of the trace. Indeed, substituting this form of the metric into the geometric operators appearing in the trace and using their Weyl transformation properties, leads to a differential equation for the effective action that can be integrated, namely

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma[g]}{\delta\Sigma_g(x)} = c_i\,\mathcal{O}_i(e^{2\Sigma_g}\bar{\eta}_{\mu\nu})\,. \tag{0.0.16}$$

The resulting action lands up in the conformal gauge, but it can be made covariant by rewriting Σ_g in terms of the physical metric $g_{\mu\nu}$. It turns out that this covariantization renders the effective action non local, as it is expected to be, since the expression for the conformal factor in terms of the full metric requires information of the metric field from all over the spacetime manifold.

The paramount example of this approach is the two-dimensional Polyakov action [36]. In two dimensions, the trace anomaly is proportional to the Ricci scalar with the anomaly coefficient being the so-called central charge c

$$T = \frac{c}{24\pi}\,R_g\,. \tag{0.0.17}$$

The Ricci scalar becomes the Laplacian of the conformal factor in the conformal gauge $R_g = -2\nabla^2\Sigma_g$, hence it can be easily integrated to give an effective action $\Gamma_{Pol} \sim \Sigma_g\,\nabla^2\,\Sigma_g \sim R_g\frac{1}{\Box}R_g$, that is non-local through the Laplacian Green function.

In four dimensions, the trace anomaly is parametrized by the a and b coefficients, and reads

$$T = a\,E_4 + b\,C^2\,, \tag{0.0.18}$$

where $C(g)$ is the Weyl tensor and $E_4(g)$ the Euler density.[3] Integration of this trace gives the so-called Riegert action [37]. The integration requires identifying the right combination of the operators which have nice Weyl transformation properties. The resulting Riegert action is a four-derivative non-local action,[4] which is a combination of the operators E_4 and C^2, and a Green function of a quartic differential operator that appears through the covariant expression of the conformal factor. The fact that this action is a higher-derivative one is eventually a consequence of the anomaly coefficients a, b being dimensionless, which requires four-derivative operators in the trace by dimensional analysis. This assumption follows from the fact that on a fixed point,

[3] Another two terms, namely $\nabla^2 R_g$ and R_g^2 are allowed by dimensional analysis, but they can be disregarded as we will argue in §1.1.4.

[4] Besides the non-local pieces, the Riegert action includes a purely local term $\sim \int R_g^2$, coming from the integral of the first term in (0.0.18) the a-anomaly. This is required to exactly reproduce the trace, because E_4 does not have a simple Weyl transformation.

there is no characteristic scale upon which the coefficients could depend. The result is that this Riegert action is Planck-suppressed.

Our starting point is the classical Einstein-Hilbert action with a cosmological constant, which is not Weyl invariant. We consider it at low energies, where the theory is close to the infrared Gaussian fixed point. Since the Einstein-Hilbert and the cosmological constant operators are relevant, one expects relevant operators in the trace of the quantum momentum tensor as well, in addition to the above a and b anomalies that would already be present at the fixed point of the theory. It would be very interesting to find the gravitational two-derivative quantum effective action from anomaly-integration. This is one of the long-term goals of the work presented in this thesis.

The relevant terms in the trace already appear classically, as the theory is not Weyl invariant, but get modified at the quantum level. Indeed, the operators in the action are composite operators of the fundamental fields, in our case the metric. This compositeness entitles contact divergences, that require renormalization. In a theory of gravity, the renormalization procedure has to be covariant, and this results into the renormalized operators acquiring an additional anomalous metric-dependence. The effective action, then, incorporates this anomalous dependence through the so-called gravitational dressings, which dress each operator with its renormalization factor. This propagates finally to the quantum momentum tensor derived by metric variation, modifying its classical metric dependence. Its trace then, acquires additional anomalies.

The anomaly coefficients of these operators in the trace have to be dimensionful, since they are relevant. Hence they have to depend on G_N and Λ, the coupling constants of the aforementioned classical terms. In fact, these coefficients are the β-functions of these coupling constants. The reason behind this connection is the local renormalization group [38–41], the covariant version of the flat spacetime renormalization group, where the conformal factor of the metric $\Sigma_g(x)$ takes the role of the arbitrary renormalization scale M. Just as the flat one, the local renormalization group dictates the flow of coupling constants with the energy scale, although it accounts for the fact that the latter is a local covariant variable set by the curvature of the spacetime. While the renormalization group in the flat case is that of scale transformations $\delta \ln M$, in the covariant case it is that of Weyl transformations $\delta\Sigma_g(x)$. The β-function encodes the classical and anomalous Weyl dimensions of the renormalized operators, and just as the scale anomalous dimensions depend on M as $\gamma(M)$, the Weyl anomalous dimensions can generally depend on the conformal factor, i.e. $\gamma(\Sigma_g)$.

From the local renormalization group then follows that the anomalous metric-dependence of the renormalized operators is in particular a dependence on the conformal factor. The renormalized operators follow from the bare ones through the exponentiated integral of the anomalous dimension, then[5]

$$\mathcal{O}_i \to \mathcal{O}_i(\Sigma_g) = \mathcal{O}_i \, e^{-\int \gamma_i(\Sigma)\,\delta\Sigma} \equiv \mathcal{O}_i \, e^{-\Gamma_i(\Sigma_g)}\,, \tag{0.0.19}$$

where $\mathcal{O}_i(x)$ denotes the non-integrated bare operator. The gravitational dressings in the effective action hence also depend on the conformal factor. Writing the latter

[5]The Γ_i with sub-index refer to the integrated anomalous dimensions, and is not to be confused with Γ without, which refers to the quantum effective action.

covariantly in terms of the full metric results into a non-local action, as mentioned before. Finally, the local renormalization group equation determines the dependence on the local energy scale Σ_g of renormalized operators. In particular, when applied to the effective action $\Gamma[g] \sim \int \lambda_i \, \mathcal{O}_i(\Sigma_g)$, where λ_i is the coupling constant, it reads

$$\left(\frac{1}{\sqrt{-g}} \frac{\delta}{\delta \Sigma_g} + \sum_i \beta_{\lambda_i}(\Sigma_g) \frac{\delta}{\delta \lambda_i} \right) \Gamma[g] = 0 \,, \tag{0.0.20}$$

where the sum would in principle run over the operators present in the effective action, and $\beta_{\lambda_i}(\Sigma_g)$ is the β-function of each coupling, which can depend on the conformal factor through the anomalous dimension. Because the variation with respect to the conformal factor is the traced metric variation, the local renormalization group equation becomes[6]

$$T = -\sum_i \beta_{\lambda_i}(\Sigma_g) \, \mathcal{O}_i(\Sigma_g) \,. \tag{0.0.21}$$

From this follows that the anomaly coefficients of the renormalized operators in the trace are β-functions, which are dimensionless for marginal operators but dimensionful otherwise. In the simple case where the anomalous dimension is a constant, the exponent of the integrated anomaly is linear in the conformal factor, and the trace is simply proportional to the renormalized operator. Knowledge of the anomalous dimensions therefore, allows to regard the above anomaly equation (0.0.21) as an equation for the effective action. From this equation follows that the latter should take the form $\Gamma \sim \int \lambda_i \, \mathcal{O}_i \, e^{-\Gamma_i(\Sigma_g)}$. Hence, integrating the β-function anomalies to get the gravitationally-dressed operators is the analog of integrating the a, b anomalies to get the Riegert action.

Since the gravitational dressings follow from integrating the anomalous dimensions, they are renormalization group-re-summed, namely they re-sum the leading logarithms. When expanded for small anomalous dimensions, they should reproduce the logarithmic corrections one would get from a perturbative computation. Therefore, the anomalous dimensions can in principle be computed independently, even if only perturbatively, and used as an input in the above anomaly equation. Alternatively, the gravitational dressings should also follow from the non-local heat kernel expansion mentioned above of the corresponding operators. Perturbatively, to first order in the curvature scalar (or rather to the next order of the one of the undressed operator), one should find the first term of the expansion of the dressing $e^{-\Gamma_i(\Sigma_g)}$, namely $-\mathcal{O}_i \, \Gamma_i(\Sigma_g)$.

Recapping, the trace has two sources of Weyl anomalies. First, the a and b anomalies, due to the metric-dependence of the measures in the path integral, and which are characterized at the fixed point where the operators in the action have vanishing β-functions. Second, the β-function type of anomalies, which come from the covariant renormalization of the corresponding composite operators in the action.

The upshot of the above lines is that physical coupling constants are the couplings of the gravitationally dressed operators, and that the anomalous dimensions of the gravitationally dressed operators are in principle different from the anomalous dimensions of the undressed operators. These considerations should hold for any operator,

[6] The quantum trace in fact contains other terms besides the β-functions and besides the a and b anomalies, but they are irrelevant for this discussion.

in particular for the purely gravitational ones that determine the equations of motion for the metric field. One of the main ideas that we want to put forward is that gravitational dressings modify the full quantum momentum tensor, not only its trace, hence they modify the full quantum dynamics of the theory. Therefore, Weyl anomalies can have interesting consequences for the cosmological dynamics.

In view of the above considerations, it would be very interesting to develop a gauge-invariant formalism that allows to predict general features of the signatures of anomalies on the evolution of the universe from local renormalization group considerations. As a step in that direction, in this thesis we study the cosmological effects from the β-function type of Weyl anomalies of the Einstein-Hilbert action with a cosmological constant. We are interested then in the lower-derivative effective action that dominates over the Riegert one at low energies, and which follows simply from appropriately gravitationally-dressing the two classical operators.

Even if this is the simplest scenario and eventually one has to add other (matter) operators to this action to construct a realistic model, computing the renormalization of the cosmological constant of purely gravitational origin in the context of integration of Weyl anomalies could help gain understanding about its current smallness. The cosmological constant problem is a problem of fine-tuning in the context of effective field theory: the cosmological constant being the coupling of the identity operator, it should scale as M_0^4, with M_0 being the ultraviolet cutoff (of at least the TeV scale). However, the measured vacuum energy density of the universe, namely the currently observed Hubble constant, is of the order of meV, implying a disagreement of at least 60 orders of magnitude [42].

A more modern version of this problem is formulated with the 'Why now?' question [43], referring to the coincidence at present time between the vacuum energy density or dark energy, and the critical density that ensures a flat universe. The fine-tuning of these two densities again seems to require a fundamental explanation.

To successfully address this problem we need to properly account for quantum gravitational effects. While the renormalization of the cosmological constant as a field theory coupling is sensitive to the ultraviolet or short-scale physics where curvature may not be relevant, its measurement captures an infrared effect, namely the cosmological expansion of the universe. The cosmological constant problem spans therefore more than a hundred logarithmic scales on a gravitational homogeneous and isotropic background. Hence, it requires a formulation and renormalization that respects gravity's general covariance and that is somehow invariant under changes of scales on this background.

The renormalization of the cosmological constant may not only be relevant for explaining the current origin and magnitude of dark energy, but may also be of crucial importance in understanding the first instants of the universe. During inflation, the spacetime was approximately de Sitter, with an energy density that was slowly 'rolling' with time. This subtle time dependence is normally introduced by means of additional scalar fields, most commonly the well-known inflaton, for which a potential drives the rolling. However, it seems natural to consider the most simple scenario where the cosmological constant itself acquires an explicit time dependence through quantum effects. After all, quantum gravity effects should have been relevant in such an epoch, where the energy scale of the universe was slightly lower than the Planck mass.

From the above considerations it seems that it would be very convenient to use a formalism that, while being covariant, allows for a special treatment of the conformal factor of the metric. One such formalism is the Weyl-invariant formulation of gravity, which we put to use in this thesis. The Weyl-invariant formulation [10] consists of introducing a Weyl compensator field $\Omega(x)$ and a fiducial metric $h_{\mu\nu}$, which scale appropriately under Weyl transformations so that the physical metric $g_{\mu\nu} = e^{2\Omega} h_{\mu\nu}$ is left invariant. The resulting theory has an extra scalar degree of freedom but also an enlarged gauge principle, which includes Weyl symmetry in addition to diffeomorphisms. The number of degrees of freedom remains the same upon imposing Weyl invariance.

This formalism is very convenient for studying the renormalization of the quantum gravity path integral because it separates general coordinate invariance from Weyl invariance. On general grounds, general coordinate invariance is not expected to have anomalies, while Weyl invariance does due to the appearance of an additional scale through renormalization. An essential requirement of the enlarged gauge principle is that all such potential anomalies cancel because Weyl symmetry is upgraded to be a gauge symmetry. General coordinate invariance of the original theory then becomes equivalent to general coordinate invariance plus quantum Weyl invariance of the modified theory. This provides a useful guiding principle. See [35, 44–55] for related earlier work, for instance on the use of this formulation for computations of gravitational and more generic Weyl-invariant effective actions, and for the connection between Weyl invariance and the renormalization group.

The Weyl-invariant formulation becomes extremely useful to compute the anomalous dimensions and the gravitational dressings of operators in two spacetime dimensions, where the metric has only three independent components. The enlarged gauge symmetry allows to completely gauge-fix the fiducial metric $h_{\mu\nu}$. The only fluctuating field is then the compensator $\Omega(x)$. Therefore, by introducing the split of the physical metric into $\Omega(x)$ and $h_{\mu\nu}$, the covariant operators become composites of $\Omega(x)$ only, which is a scalar field and whose ultraviolet divergences are hence easier to regularize. Covariant renormalization of the operators then introduces an anomalous dependence on the conformal factor of the background fiducial metric $\Sigma_h(x)$, which determines their gravitational dressings. Imposing gauge-invariance of the resulting effective action, namely that it depends on the $\Sigma_h(x)$ only through the combination $g_{\mu\nu} = e^{2\Omega}\, h_{\mu\nu} = e^{2(\Omega+\Sigma_h)}\, \bar{\eta}_{\mu\nu}$, determines the anomalous dimensions of the operators in terms of the fundamental constants. This procedure then, has the potential of sparing perturbative diagrammatic computations, while giving exact expressions for the anomalous dimensions.

Another of the advantages of the Weyl-invariant formulation is that it is very suited for computations in the cosmological setting. In a homogeneous and isotropic universe, all the dynamics is encoded in the scale factor of the metric, while the background metric is fixed by the symmetries and the choice of spatial curvature. The compensator $\Omega(x)$ takes the role of the scale factor and the fiducial metric that of the background. This formulation then allows to compute general metric variations of the action while profiting from the simplifications of imposing the afore-mentioned symmetries in the solutions, so it becomes very useful to compute the cosmological evolution equations. Due to the local renormalization group considerations exposed above, gravitational dressings relevant for cosmological dynamics depend only on the scale factor. Hence this formulation is very suited to compute the cosmological effects of Weyl anomalies.

When faced with a complex problem, it is always a good starting point to consider a toy model which allows for explicit simple computations and which yet captures the essential features of the more general setting. In the spirit of this approach, we start our exploration in two-dimensional gravity, to which we devote the main part of this thesis. One of the reasons for this is that, as mentioned above, the enlarged symmetry allows to choose a gauge in which the fiducial metric is completely fixed, with no dynamics. The entire quantum dynamics then resides in the dynamics of the scalar Weyl compensator, and the anomalous dimensions are easier to compute. Another reason is that in two dimensions, conformal field theories are not only richer thanks to the symmetry enhancement to an infinite algebra, but also better studied and understood.

Our starting point is the two-dimensional Einstein-Hilbert action with a cosmological constant term. When written in its Weyl-invariant form in terms of the Weyl compensator $\Omega(x)$ and the fiducial $h_{\mu\nu}$, the classical action for $\Omega(x)$ becomes an analytic continuation of the well-known Liouville action, the so-called timelike Liouville action. The Weyl compensator then takes the role of the timelike Liouville field. Liouville theory is a well-studied conformal field theory, for which bootstrap approaches have successfully provided all of its conformal data, the spectrum, the two-point functions and the structure constants. It is the appearance of this resourceful Liouville action that allows us to compute the anomaly exactly and fully motivates the study of quantum cosmology in two dimensions.

First of all, the Liouville action is ubiquitous in approaches to two-dimensional quantum gravity. It arises naturally as the effective action for conformal field theories such as the free scalar, but more importantly, it also becomes the gravitational effective action, appearing from the Jacobian between the translationally-invariant and the Weyl-invariant measures of the conformal factor of the metric. Second, since timelike Liouville follows from an analytic continuation of the Liouville field, it has a negative kinetic term, hence its name timelike. This minus sign is to be expected for the dynamics of the conformal factor of the metric [11], and is at the heart of the difficulties encountered when quantizing gravity in any dimensions, since it implies that the path integral is unbounded from below and therefore that the theory is non-unitary. Quantization then requires a BRST-type of prescription to decouple negative-norm states. In two dimensions, this implies that the analytic continuation between Liouville, which is a unitary theory and can be quantized with canonical methods, and timelike Liouville is very subtle, and in fact is still not fully understood. As a result, the quantum gravitational dynamics is highly non-trivial even in two dimensions. There is extensive literature on both spacelike and timelike Liouville theory. See [56–61] for reviews that emphasize different aspects of the quantum theory. Timelike Liouville as a two-dimensional model for cosmology has been considered earlier from different perspectives in [62–65].

Fortunately, the full quantization of timelike Liouville theory is not necessary to address the physical questions that we are interested in. For these, the analytic continuation of the analogous results in standard Liouville theory will suffice. Our first aim is to compute the gravitational dressing of the cosmological constant operator. In the Weyl-invariant formulation, this identity operator $\Lambda \sqrt{-g}$ becomes $\Lambda\, e^{2\Omega}\sqrt{-h}$. After the gauge invariance has been fixed, only the Liouville field Ω is fluctuating. Therefore, the contact divergences of the original $\sqrt{-g}$ are now contact divergences of the composite vertex operator $e^{2\Omega}$. This exemplifies the simplification brought about by the Weyl-invariant formulation.

The renormalization of this exponential operator within the theory that includes it in the action should be highly non-trivial. However, one of the remarkable features of Liouville theory is that the ultraviolet divergences can be renormalized by simple normal ordering [12–17]. This greatly simplifies the computation of the gravitational dressing for this cosmological constant operator, which is found to be non-local through the dependence on the conformal factor of the fiducial metric Σ_h. By further following the criteria of Weyl anomaly cancellation, we can determine the anomalous dimension in terms of Newton's constant. We can then write down the gauge-invariant non-local quantum effective action.

Our second aim is to study the cosmological consequences of this renormalized cosmological constant. For this, we compute the quantum momentum tensor of the latter, which is non-local through the anomalies, and leads to non-trivial Friedmann equations. One of the main results we find is that the quantum corrections lead to a slow decay of the vacuum energy in an isotropic and homogeneous universe, and a slowing down of the exponential de Sitter expansion. This is a very interesting result because it offers a dynamical mechanism for vacuum decay which relies purely on the infrared gravitational physics. Such a decaying vacuum energy could drive a period of slow-roll inflation from purely quantum gravitational effects, without the need of additional scalar fields. The exploration of this mechanism and its generalizations to higher dimensions is hence clearly motivated.

The eventual goal of the study of this two-dimensional model is to draw some general lessons that may be applicable in four dimensions. The advantage of this model is that the important quantum effects can be computed explicitly with relative ease and without ambiguities to all orders in perturbation theory. We expect the semiclassical approximation to be reliable on cosmological scales in four dimensions. Any effect in the two-dimensional model that could be relevant for four-dimensional physics must manifest itself in the semiclassical limit and should not depend on special properties of two dimensions. For this reason, it is good enough that our analysis is done in the semiclassical limit of timelike Liouville theory.

The analysis of four-dimensional gravity makes up the second part of this thesis. As in two dimensions, we are interested in the quantum effective action valid at distances large compared to the Planck scale, hence we consider any higher-derivative contributions, such as those in the Riegert action, as Planck suppressed. Our key object is then the four-dimensional non-local effective action that incorporates the Weyl anomalies of the Einstein-Hilbert cosmological gravity. Namely, the action that incorporates the gravitational dressings of both the square-root of the determinant of the metric as well as the Ricci scalar.

The simplification brought about by the Weyl-invariant formulation and Liouville theory in computing the two-dimensional gravitational dressings is not to be encountered in four dimensions. There, the enlarged gauge symmetry does not fix the fiducial metric completely, and covariant operators are composites of both the compensator $\Omega(x)$ and the unfixed $h_{\mu\nu}$ components. Hence contact divergences are difficult to renormalize covariantly, and one has to resort to the usual perturbative computations. However, the results of our two-dimensional model and the afore-mentioned considerations of the local renormalization group allow to infer the most logical form of the four-dimensional gravitational effective action, namely [10]

$$\Gamma_G[g] = \frac{M_p^2}{16\pi} \int d^4x\sqrt{-g}\, \left(R_g\, e^{-\Gamma_K(\Sigma_g)} - 2\,\Lambda\, e^{-\Gamma_\Lambda(\Sigma_g)}\right) , \tag{0.0.22}$$

where the $\Gamma_i(\Sigma_g)$, $i = K, \Lambda$ represent the integrated anomalous gravitational dressings. Since this effective action depends on $\Sigma_g = \Omega + \Sigma_h$, it automatically is gauge invariant. Therefore, as opposed to two dimensions, Weyl invariance can not be used to infer the anomalous dimensions. However, these anomalous dimensions and the anomalous dressings are computable in a given microscopic theory. Renormalization of Newton's constant and the cosmological constant has been considered earlier in the literature [18–20]. One can extract the precise logarithmic running from these results. The approach in this thesis, though will be to regard the integrated dressings as general functions that parametrize the effective action, and postpone their computation.

Modifications of Einstein's general relativity have been generically considered to explain the late-time acceleration, and the dynamics of galaxies and clusters without dark energy and dark matter. Among these are non-local generalizations of Einstein-Hilbert gravity [66–72], which were originally motivated by non-localities appearing in graviton quantum loop corrections [73, 74]. Long-scale modifications of Einstein's gravity have to satisfy three criteria. First, they can not affect solar system dynamics, which are already well accounted for by general relativity. Further, they have to overcome two fine-tuning problems: they have to contain a mass scale much smaller than any mass found in nature, and the modifications must have been irrelevant up to today. Non-local models seem to satisfy all three criteria [66, 75]. Also, they can lead to interesting cosmology and can predict measurable deviations from Einstein's gravity in structure growth rates and patterns [75, 76].

However, the approach of most non-local models is to assume a general form of the non-locality in the action, parametrized with an arbitrary free non-local function that can be chosen at will to fit the expansion. The non-locality that we consider, instead, is of a very specific kind: it is constrained by the requirement that the action should be a solution of the local renormalization group equation, and the non-local function is easily computable at least perturbatively. Also, our non-local model contains the cosmological constant, so it is not regarded as an alternative to the later to explain the expansion of the universe, but as a model which incorporates quantum corrections from anomalies on top of a cosmological constant-driven classical background.

The main goal of our analysis in four dimensions is to determine the cosmological dynamics of the parametrized effective action (5.1.1). For that, we compute the integro-differential equations resulting from it, which describe the effective classical dynamics of the spacetime metric at long distances, and look for Robertson-Walker solutions.

Somewhat surprisingly, the effective dynamics can be solved analytically even in four dimensions using the Weyl-invariant formulation. Our main conclusion is that non-zero anomalous gravitational dressings lead to a slow quantum decay of vacuum energy just as in two dimensions.

The idea of vacuum energy decay due to infrared quantum effects [77–83], and more generally about infrared effects in nearly de Sitter spacetime [84–95], has been thoroughly analyzed in four-dimensional gravity. An interesting related idea explored in the literature concerns possible nontrivial fixed points of gravity in the ultraviolet [96–105] and in the infrared [106–108]. This is a different regime than what we consider. Our interest is in the long distance physics on cosmological scales in weakly coupled gravity near the trivial Gaussian fixed point. Some of the methods developed in these investigations could nevertheless be useful for the computation of Weyl anomalies, especially in the very early universe.

To summarize, Weyl anomalies affect the cosmological dynamics through the modification of the full quantum momentum tensor. Developing a gauge-invariant formalism that allows to encode them in an effective action and analyze systematically their cosmological consequences is hence of great interest. Two important questions are first whether generic model-independent signatures can be predicted from the Weyl anomalies based solely on the form of the effective action, and second whether their quantum effects can be relevant in the different scales of the evolution of the universe. Eventually, it would be important to understand if Weyl anomalies can have a relevant role in some of the open questions in cosmology, such as the source of the primordial dynamics of the universe, or the origin and magnitude of its current expansion.

Outline

This thesis is organized as follows. In chapter §1 we present some of the background material regarding mainly Weyl invariance and Weyl anomalies. We also fix the notation and present the Weyl-invariant formulation, highlighting its intuitive origin and convenience. Finally, d-dimensional gravity and cosmology are presented in this formulation. After this introduction, chapters §2 to §4 are devoted to the two-dimensional model, to understanding how to use the Weyl-invariant formulation to compute anomalies and predict cosmological effects. In chapters §5 and §6 we then explore four dimensions, by first generalizing the lessons and conclusions from the two-dimensional model, and second by doing the cosmological analysis.

In chapter §2, we take the two-dimensional limit of Weyl-invariant gravity and show how Liouville theory appears as a theory of gravity, both classically and quantum-mechanically. Some of the key features of this conformal field theory, which are relevant for the gravitational and cosmological contexts, are also presented. The theoretical set up is completed by describing the coupling to the scalar matter and ghost sectors.

Chapter §3 mainly deals with the renormalization of the cosmological constant operator near two dimensions. We write the quantum effective action for this renormalized cosmological operator and compute the resulting non-local momentum tensor. Further, we compute the effective action and momentum tensor for each of the sectors: for the conformal factor, for the gravitational sector, and finally for the full (with matter and ghosts) theory. Some of the subtleties, such as the choice of the vacuum or the renormalization of the gravitational constant are also touched upon. Finally, the quantum Einstein equations are written down.

Chapter §4 is devoted to the analysis of the cosmological implications of the renormalized cosmological constant. With the quantum equations found in the previous chapter, which mainly include the non-local momentum tensor for Λ, we look for homogeneous and isotropic universes. The solution gives a quasi de-Sitter expansion and a decaying vacuum energy density. We comment on the potential applications and implications of this result.

In chapter §5 we start by proposing the non-local quantum effective action for gravity with Λ that most simply generalizes what we find in two dimensions. With the action at hand, we proceed then to derive the equations of motion for Robertson-Walker universes. We do this in two different conformal gauges, the F-flat and the R-flat gauges. Finally, we derive the Einstein equations for a general background to enlighten the potential of the formalism and the complexity of the quantum generalization.

In chapter §6 we put the cosmological evolution equations to use and, as in two dimensions, we find a quasi-de Sitter expanding universe and a decaying vacuum energy density. We further build on the theoretical understanding of such a solution and comment on its implications.

In chapter §7 we end up with the conclusions and the discussion of open questions and future lines of research.

This thesis is based on the papers

- T. Bautista and A. Dabholkar, *Quantum Cosmology Near Two Dimensions.* (November 2015). [109]
- T. Bautista, A. Benevides, A. Dabholkar, and A. Goel, *Quantum Cosmology in Four Dimensions.* (December 2015). [110]

Chapter 1

Background

1.1 Weyl Anomalies

1.1.1 Weyl Invariance

A Weyl transformation consists of a local rescaling of the fields. By definition, the metric transforms as

$$g_{\mu\nu} \to e^{2\xi(x)}\, g_{\mu\nu}\,, \tag{1.1.1}$$

and any other field transforms (classically) according to its mass dimension. For example, a scalar field in d dimensions transforms as

$$\phi(x) \to e^{-\frac{d-2}{2}\xi(x)}\, \phi(x)\,. \tag{1.1.2}$$

So defined, Weyl transformations are not necessarily an invariance of covariant actions. For example, the Einstein-Hilbert action is only Weyl invariant in two dimensions. Adding a cosmological constant to it breaks the Weyl invariance even in two dimensions. The free scalar action is also invariant only in two dimensions, and is never invariant when a mass term is added. However, a Weyl-invariant action can be written for a massless scalar through an appropriate coupling to the background curvature, a so-called improvement term. Free fermions, instead, are Weyl invariant in any dimension due to the non-trivial variation of the vierbein connection

Constant or global Weyl transformations are those transformations for which ξ is constant. They are related to scale transformations, which are defined in flat spacetime as a global rescaling of the coordinates $x^\nu \to e^\sigma\, x^\nu$. The fields transform again according to their mass dimension. Since the theory is defined on a fixed flat spacetime but the rescaling of the coordinates should transform the metric according to the usual tensor transformation rule, the latter has to be compensated by a constant Weyl transformation so that the Minkowski metric is left invariant. With the combination of both transformations, the line element gets multiplied by $e^{2\sigma}$, so scale transformations change distances or scales. In a flat background then, scale invariance really means that the physics is the same at all scales.

In a covariant theory of gravity, the metric is one of the fluctuating fields so there is no need to keep it fixed to a certain background, it can transform like the other fields. To obtain the same kind of global rescaling of scales, we can then simply perform a global Weyl transformation of the metric, without having to embark in coordinate transformations and Weyl compensations. In other words, scale invariance in flat spacetime field theory becomes global Weyl invariance in a covariant theory of gravity.

Finally, conformal transformations involve a Weyl transformation. The former consist of a conformal isometry of the metric followed by a compensating (infinitesimal) Weyl transformation. More concretely, the conformal isometry is performed by a conformal Killing vector ζ^ν, under which the metric is invariant up to a local pre-factor

$$\delta g_{\mu\nu}(x) = -\left(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu\right) = -\frac{2}{d}\left(\nabla \cdot \zeta\right) g_{\mu\nu} \,. \tag{1.1.3}$$

The infinitesimal Weyl transformation $\delta g_{\mu\nu} = 2\delta\xi(x)\, g_{\mu\nu}$ then has to satisfy

$$\delta\xi(x) = \frac{1}{d}\,\nabla \cdot \zeta \tag{1.1.4}$$

so that the metric $g_{\mu\nu}$ stays invariant.[1] Therefore, conformal transformations, and hence conformal invariance, are only defined in quantum field theories on a fixed background, but not in a theory of gravity, where the metric field is dynamical. Moreover, very few backgrounds admit conformal transformations, i.e. only those for which some ζ^ν can be found to solve (1.1.3), and in fact conformal field theories are normally assumed to be in Minkowski (or Euclidean) spacetime. In this case, conformal isometries include translations, Lorentz, scale, and special conformal transformations. Notice though, that scale transformations $x^\nu \to e^\sigma\, x^\nu$ are not part of conformal transformations for a general background, since the diffeomorphism that implements them $\zeta^\nu = \sigma\, x^\nu$, leads to a variation of the metric as

$$\delta g_{\mu\nu} = -\sigma\left(2 + x^\rho\,\partial_\rho\right) g_{\mu\nu}\,, \tag{1.1.6}$$

which is only a conformal isometry if $x^\rho\,\partial_\rho g_{\mu\nu}$ is proportional to the metric itself (up to a local pre-factor). Finally, if a theory is diffeomorphism and Weyl invariant, then it is conformal invariant when fixed to a background that admits conformal Killing vectors.

1.1.2 Identities from Gauge Invariance

Coordinate invariance is related to the conservation of the total momentum tensor, which follows classically from the Einstein equations due to the Bianchi identity of the Einstein tensor. Rigorously, covariance translates into a Ward identity that takes into account the variation under diffeomorphisms of all the fields present in the action. Let's see this for the case of an action with the Einstein-Hilbert term[2]

$$I_K[g] = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}\,R(g)\,, \tag{1.1.7}$$

[1] Notice that covariance of the metric implies that under the conformal isometry

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x) + \zeta^\sigma\partial_\sigma g_{\mu\nu}(x) = \left(1 - \frac{2}{d}(\nabla\cdot\zeta)\right) g_{\mu\nu} + \zeta^\sigma\partial_\sigma\, g_{\mu\nu}\,, \tag{1.1.5}$$

which generically is not proportional to the metric $g_{\mu\nu}$, i.e. $g'_{\mu\nu}(x') \neq \Omega(x)\, g_{\mu\nu}(x)$, as is often wrongly stated in the literature as the definition of conformal transformations. Only in the case when $\zeta^\sigma\partial_\sigma g_{\mu\nu}$ is proportional to the metric up to a local pre-factor, like in flat spacetime, does this hold.

[2] All along this thesis, we will use both $R(g)$ and R_g to denote the Ricci scalar of the $g_{\mu\nu}$ metric.

and several (matter) scalar fields, so that $I = I_K[g] + I_m[g, \phi_i]$. Consider a general coordinate transformation $x^\mu \to x'^\mu$. Scalar fields stay invariant, i.e. $\phi'(x') = \phi(x)$, and the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \, g_{\rho\sigma}(x) \,. \tag{1.1.8}$$

The infinitesimal variation $x'^\nu = x^\nu + \xi^\nu$ as induced by a diffeomorphism vector field $\xi^\nu(x)$, is then given by the Lie derivative of the corresponding field. For the scalar, the latter corresponds to the directional derivative

$$\delta\phi(x) = \phi'(x) - \phi(x) = \phi(x-\xi) - \phi(x) = -\mathcal{L}_\xi \phi(x) = -\xi^\nu \nabla_\nu \phi(x) \,, \tag{1.1.9}$$

for the metric it becomes

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\mathcal{L}_\xi g_{\mu\nu}(x) = -\left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right) \,, \tag{1.1.10}$$

and for the inverse metric

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \,. \tag{1.1.11}$$

The variation of the action then is expressed in terms of the functional variations of all the fields it depends on, namely

$$\begin{aligned} \delta I &= \int d^d x \left(\frac{\delta(I_K + I_m)}{\delta g^{\mu\nu}} \, \delta g^{\mu\nu} + \frac{\delta I_m}{\delta \phi_i} \, \delta\phi_i \right) \\ &= -\int d^d x \sqrt{-g} \; \xi^\nu(x) \left(\nabla^\mu \left(\frac{1}{\kappa^2} \, E_{\mu\nu} - T_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta I_m}{\delta\phi_i} \nabla_\nu \phi_i \right) \,, \end{aligned} \tag{1.1.12}$$

where $E_{\mu\nu}$ denotes the Einstein tensor $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, R$ (see (B.0.10) for the variation of the Einstein-Hilbert action), and the momentum tensor is defined by[3]

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \, \frac{\delta I}{\delta g^{\mu\nu}} \,. \tag{1.1.13}$$

Invariance of the action $\delta I = 0$ under a general diffeomorphism $\xi^\nu(x)$ implies the vanishing of the integrand. In the case the action is just the Einstein-Hilbert term $I = I_K$, the Ward identity becomes the Bianchi identity for the Einstein tensor

$$\nabla^\mu E_{\mu\nu} = 0 \,. \tag{1.1.14}$$

This identity holds for any metric configuration, it does not rely on equations of motion, so it is a fundamental property of the geometry of spacetime. Implementing it in (1.1.12), the Ward identity becomes

$$\nabla^\mu T_{\mu\nu} - \frac{1}{\sqrt{-g}} \, \frac{\delta I}{\delta \phi_i} \, \nabla_\nu \phi_i(x) = 0 \,. \tag{1.1.15}$$

This is the Ward identity for general coordinate invariance. If the action being varied includes the full Lagrangians of all the fields ϕ_i, then the total momentum tensor

[3] In Euclidean signature, the momentum tensor is defined with a + sign instead.

is indeed conserved when all the fields are on-shell. This identity however can be independently written for any term in a covariant action.

If the Einstein equations are satisfied, then the Bianchi identity implies the vanishing of the sum

$$\frac{\delta I}{\delta\phi_i}\,\nabla_\nu\phi_i(x) = 0\,. \tag{1.1.16}$$

Therefore, in the presence of only one scalar field $\phi(x)$, the Bianchi identity implies that its equation of motion is satisfied (or that the field is a constant, which may or may not be a solution to the equations of motion), and one can assert that the full dynamics follows from a geometrical property of spacetime. However, this is not true in general, when several fields are present. In other words, conservation of the total momentum tensor doesn't imply that the fields are on-shell.

Just as diffeomorphism invariance is related to the conservation of the momentum tensor, Weyl invariance is related to the tracelessness of the momentum tensor. That is, the Ward identity for Weyl invariance involves the trace of the momentum tensor. Going back to the case of the gravitational action with several scalar fields $I = I_K + I_m$, the first thing to notice is that the Einstein-Hilbert action I_K is generically not Weyl-invariant (see (B.0.17)). The Ward identity can then be written for the matter action if this is Weyl invariant. The infinitesimal variation of the metric under a Weyl transformation follows from (1.1.1) as

$$\delta g^{\mu\nu} = -2\,\xi(x)\,g^{\mu\nu}\,, \tag{1.1.17}$$

and that of a scalar field follows from (1.1.2)

$$\delta\phi_i(x) = -\Delta_i\,\xi(x)\,\phi_i(x) \tag{1.1.18}$$

with $\Delta_i = \frac{d-2}{2}$. The Weyl variation of the action is then

$$\begin{aligned}\delta I_m &= \int d^dx\left(\frac{\delta I_m}{\delta g^{\mu\nu}}\,\delta g^{\mu\nu} + \frac{\delta I_m}{\delta\phi_i}\,\delta\phi_i\right)\\ &= \int d^dx\sqrt{-g}\;\xi(x)\left(g^{\mu\nu}\,T_{\mu\nu} - \Delta_i\,\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\phi_i}\phi_i\right)\,.\end{aligned} \tag{1.1.19}$$

Invariance under a general Weyl transformation then implies the identity

$$g^{\mu\nu}\,T_{\mu\nu} - \Delta_i\,\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\phi_i}\phi_i = 0\,. \tag{1.1.20}$$

The total momentum tensor is traceless when the equations of motion are satisfied. If further the Einstein equations are satisfied, the above identity can be replaced by

$$\frac{1}{\kappa^2}\,R(g) + \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\phi_i}\phi_i = 0\,, \tag{1.1.21}$$

which means that a Weyl-invariant scalar theory sources a Ricci-flat background. In the case of two dimensions, scalar fields don't transform under a Weyl transformation, therefore classically the momentum tensor is traceless regardless of the equations of motion.

If an action is only invariant under global Weyl transformations, the resulting identity is the integral of (1.1.20), i.e. all that can be inferred from (1.1.19) is that the integrated trace of the momentum tensor has to vanish, or equivalently that the trace vanishes up to a total derivative $T^{\mu}{}_{\mu} = \nabla^{\mu} J_{\mu}$.[4]

In flat spacetime field theories, one can perform conformal transformations instead of general coordinate and Weyl transformations, as explained in the previous section. In this case, conformal invariance imposes conditions on the momentum tensor of the theory analogous to the above Ward identities. To start with, Noether's theorem assigns a conserved current to translational invariance (a global symmetry), which is by definition the momentum tensor. The remaining conformal transformations lead to additional currents, all defined in terms of the momentum tensor. Invariance under these transformations then, or equivalently conservation of the corresponding currents, imposes further conditions on the momentum tensor. First, Lorentz invariance requires a momentum tensor which is symmetric. This can be achieved because the momentum tensor that follows from the Noether's prescription is not unique, is defined up to the so-called Belinfante ambiguity, which allows addition of a term that neither spoils conservation nor changes the equations of motion. Next, scale invariance requires the trace of the momentum tensor to vanish up to a total derivative of some virial current, in analogy with global Weyl transformations in a theory of gravity. Finally, special conformal transformations require the momentum tensor to be exactly traceless.[5]

If a covariant action is Weyl invariant, then it will have conformal invariance when fixed to a flat background, by virtue of the conservation and tracelessness of the momentum tensor. However, the opposite may not hold true. I.e. starting from a conformally-invariant action in flat spacetime, and coupling it to a metric in a covariant way, may or may not lead to a Weyl invariant action. The reason lies on the Belinfante ambiguity. This ambiguity can be seen to arise from general linear curvature terms that can be added to the covariant action, which vanish when writing the action on flat spacetime, but which leave behind non-trivial terms in the momentum tensor. Since these terms in the action are covariant, they lead to conserved terms in the momentum tensor, and they can be tuned to further satisfy tracelessness in flat spacetime. Covariantizing the flat space action may miss these non-trivial terms, leading to an action that is Weyl invariant only after these terms are introduced. Therefore, different choices of the Belinfante ambiguity in flat spacetime lead to different covariant background theories.

The addition of these terms, linear in the curvature tensors, that make the covariant theory Weyl invariant is called Ricci gauging [50]. Starting from a globally-Weyl invariant theory in a general background, this inner symmetry can be gauged by introducing an appropriate connection that covariantizes the derivatives appropriately such that the action is locally-Weyl invariant. The condition for a Weyl-gauged action

[4] The current J^{ν} is called the Virial current.

[5] Actually, since the momentum tensor is defined up to the Belinfante ambiguity, the minimal condition for conformal invariance is not tracelessness, but rather $T^{\nu}{}_{\nu} = \partial^{\nu}\partial^{\mu} L_{\nu\mu}$ for $d \geq 3$ or $T^{\nu}{}_{\nu} = \partial^{\nu}\partial_{\nu} L$ for $d = 2$ (i.e. that the Virial current is the divergence of a tensor). This condition is enough to ensure a traceless momentum tensor after a (Belinfante) improvement term is added.

to admit Ricci gauging, i.e. to be able to rewrite the gauge connection in terms of couplings to the curvature, is precisely that the flat space theory is not only scale invariant but also conformal [50]. For a good review of the above points, see [111].

1.1.3 Weyl Anomalies

The Ward identities written in the previous section were purely classical since they were capturing an invariance of the classical action. A classical invariance may not carry on to be an invariance of the quantum effective action. When this is the case, we say the theory is anomalous. This is made explicit through the appearance of anomalies in the corresponding Ward identities, i.e. terms that spoil their validity.

The possible quantum failure originates in the regularization and renormalization of the theory. These require the introduction of additional energy scales, namely an ultraviolet (UV) regulator or cutoff M_0, and a renormalization scale M, which have the potential to break the invariances.

These scales can in principle be chosen to be a constant or a Lorentz scalar, so it should always be possible to choose a subtraction procedure that is fully covariant. Therefore, anomalies in the diffeomorphism Ward identity are not to be expected. The Weyl Ward identity, on the other hand, is clearly bound to be anomalous.

Just as the classical one, the quantum Weyl Ward identity involves the trace of the quantum momentum tensor. The latter follows from the metric variation of the quantum one-particle-irreducible (1PI) effective action, which can be obtained by means of the background field method. Given a theory with a generic fluctuating field $\phi(x)$, this method requires a shift of the latter with a non-dynamical background $\phi \to \phi_b + \phi$. The resulting shifted classical action $I_b = I[\phi_b + \phi]$ leads to a quantum effective action $\Gamma_b[\phi_b, \phi]$, that reproduces the original unshifted one $\Gamma[\phi_b]$ when restricted to graphs where no ϕ_b fields run in the loops and no ϕ fields appear as external lines. Equivalently, the effective action follows from restricting the path integral of the shifted classical action to these field configurations. We can schematically write

$$\mathcal{Z}[\phi_b, g] = e^{i\Gamma[\phi_b, g]} = \int_{1PI} \mathcal{D}_g\phi \, e^{i\, I[\phi_b+\phi, g]}. \tag{1.1.22}$$

The main property of the 1PI effective action is that it reproduces all the physics of the full quantum theory even when used merely at tree-level, because it includes the path-integration of all the fields.

The infinitesimal variation of the partition function with respect to the metric defines the quantum momentum tensor as

$$\mathcal{Z}[\phi_b, g+\delta g] = \int \mathcal{D}_{g+\delta g}\phi \, e^{i\, I[\phi_b+\phi, g+\delta g]} = \int \mathcal{D}_g\phi \, e^{i\, I[\phi_b+\phi, g]} \left(1 - \frac{i}{2}\int d^dx\sqrt{-g}\; T_{\mu\nu}\,\delta g^{\mu\nu}\right). \tag{1.1.23}$$

The variation of $\Gamma[\phi_b, g]$ then follows from the above

$$\delta\Gamma = -i\,\delta\,(\,\ln\mathcal{Z}\,) = -\frac{1}{2}\int d^dx\sqrt{-g}\,\delta g^{\mu\nu}\,\langle T_{\mu\nu}\rangle\,, \tag{1.1.24}$$

from which

$$\langle T_{\mu\nu}\rangle = \frac{-2}{\sqrt{-g}}\frac{\delta\Gamma[\phi_b, g]}{\delta g^{\mu\nu}}\,. \tag{1.1.25}$$

This momentum tensor includes the variation of both the classical action and the fields measure $\mathcal{D}_g\phi$, and encodes all the quantum information.

For the full effective action, we can write

$$e^{i\Gamma[\phi_b,g_b]} = \int_{1PI} \mathcal{D}_g g_{\mu\nu}\, \mathcal{D}_g\phi\, e^{i\, I[\phi_b+\phi,g_b+g]}\,. \tag{1.1.26}$$

The subscript referring to the background in the fields will be omitted henceforth. The resulting complete effective action $\Gamma = \Gamma_g[g] + \Gamma_\phi[\phi, g]$ includes the quantum effective action for the gravitational sector as well as the contribution of gravitons running in the loops for the ϕ sector. The variation of the total effective action with respect to the (background) metric gives the full quantum equation of motion for the metric.

If the quantum effective action were to be Weyl invariant, it would obey the quantum Ward identity that follows from its variation, the quantum analog of (1.1.20)

$$g^{\mu\nu}\,\langle T_{\mu\nu}\rangle - \Delta_i\,\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi_i}\phi_i = 0\,. \tag{1.1.27}$$

This identity would imply the vanishing of the trace (when the quantum equations of motion are satisfied). The failure of the action to be Weyl invariant translates then into additional terms that spoil this identity, i.e. terms that preclude the tracelessness of the quantum tensor. That's why the Weyl anomaly is also called the trace anomaly. Henceforth, the brackets of the quantum momentum tensor will be omitted, the expectation value will be assumed.

The anomalous Ward identity can also be regarded as an equation for the quantum effective action. Indeed, the general metric variation of the effective action (1.1.24), can in particular be taken in the direction of a Weyl variation. In the conformal gauge, the metric is split into a conformal factor $\Sigma_g(x)$ and a fiducial metric $\bar\eta_{\mu\nu}$ fixed by a scalar condition[6]

$$g_{\mu\nu} = e^{2\Sigma_g(x)}\,\bar\eta_{\mu\nu}\,. \tag{1.1.28}$$

The Weyl variation of the metric then can be expressed as the variation of the conformal factor as

$$\delta g_{\mu\nu} = 2\,\delta\Sigma_g(x)\,g_{\mu\nu}\,, \qquad \delta g^{\mu\nu} = -2\,\delta\Sigma_g(x)\,g^{\mu\nu}\,. \tag{1.1.29}$$

Introducing this in the general variation of the action (1.1.24), the trace follows as

$$T \equiv g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\,\frac{-2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\Sigma_g(x)}\,. \tag{1.1.30}$$

In other words, tracing the general variation of the action with respect to the metric amounts to taking the directional derivative, in the direction of the Weyl variation. Substituting this in (1.1.27), the anomalous Ward identity becomes a functional differential equation for the effective action, the anomaly equation. In the case of two dimensions, diffeomorphisms allow to fix the two components of the fiducial metric $\bar\eta_{\mu\nu}$, leaving the conformal factor as the only fluctuating field. The Weyl variation of the action then is effectively a total derivative, hence the anomaly equation can be

[6]Note that $\bar\eta_{\mu\nu}$ need not be the Minkowski metric. We will make this definition of the conformal gauge clearer later on.

integrated. This is how the Polyakov action is found to appear as a quantum effective action, as we will review in §2.3.

Anomalies must follow from non-local terms in the quantum action. The reason is that if they were encoded in local terms, they could be removed with counterterms, and therefore they could not be encoding something robust about the theory such as its symmetry violations. The anomaly coefficients are therefore universal, in the sense that they cannot depend on the regularization scheme or renormalization conditions.

Anomalies of gauge invariances have to cancel in a complete theory. The reason is that gauge invariances are not an actual symmetry of the theory, but just a redundancy of notation, and therefore their anomalies cannot have any physical implications. This criteria is called anomaly-cancellation, and plays an important role in determining the content of the theory. Therefore, if Weyl invariance is part of the gauge principle of a theory, its Ward identity has to hold also at the quantum level. The familiar example is world-sheet string theory, where Weyl anomaly cancellation leads to a 26-dimensional target spacetime.

1.1.4 Types of Weyl Anomalies

The definition of the quantum effective action (1.1.22) suggests that its metric dependence comes both from the classical action and the integral measure. Consequently, the quantum momentum tensor acquires a contribution from the variations of both dependences, hence there are two sources of Weyl anomalies.

Weyl anomalies coming from the lack of invariance of the path integral measures typically lead to curvature terms in the trace of the momentum tensor. In two dimensions, the only curvature term allowed by dimensional analysis is the Ricci scalar $R(g)$, and the Weyl anomaly equation reads

$$T = \frac{c}{24\pi}\, R(g). \tag{1.1.31}$$

The parameter c is called the central charge, and comes from the fact that the algebra satisfied by the Fourier modes of the momentum tensor acquires a central extension at the quantum level proportional to this parameter c. The central charge is characteristic of every theory, and at conformal fixed points it counts the number of massless degrees of freedom. For a theory with one free massless scalar $c = 1$, and for a free massless Majorana fermion $c = 1/2$. In flat spacetime, the momentum tensor becomes traceless, but the theory is still anomalous, since the measures still vary under a Weyl transformation of the flat metric.

In four dimensions, the most general Weyl anomaly allowed by dimensional analysis is[7]

$$T = a\, E_4 + b\, C^2 + c\, R^2 + e\, \nabla_g^2 R_g\,, \tag{1.1.32}$$

where $C(g) = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2_g$ is the Weyl tensor, which is invariant under a Weyl transformation of the metric, and $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2_g$ is the Euler density. The last

[7] In fact, there is yet another term allowed by dimensional analysis, the so-called Hirzebruch-Pontryagin term $T \propto \epsilon^{\mu\nu\rho\sigma}\, R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the antisymmetric Levi-Civita tensor. Although allowed, this term breaks CP invariance, and therefore seems to appear only for CP-violating theories .

term in the trace is not regarded as an anomaly because it can be removed by adding a local counterterm in the effective action, namely R_g^2. The third term, is actually forbidden by the Wess-Zumino consistency condition [112], an algebraic equation that restricts the possible forms of the anomaly and which is based on the group structure of the invariance. In the case at hand, since the Weyl group is abelian, the consistency condition requires that consecutive Weyl variations of the effective action commute

$$\left(\frac{\delta}{\delta\xi(x_1)}\frac{\delta}{\delta\xi(x_2)} - \frac{\delta}{\delta\xi(x_2)}\frac{\delta}{\delta\xi(x_1)}\right)\Gamma[g] = 0\,. \tag{1.1.33}$$

The term R_g^2 does not satisfy the above condition and so it cannot follow from any effective action. The two terms remaining are accompanied by the coefficients a and b, also characteristic of every theory.

The second source of Weyl anomalies is the renormalization of the operators present in the classical action. Since operators typically are composites of the fundamental fields, they encode contact divergences that need to be renormalized. Renormalization requires the introduction of an additional scale M, used to run the renormalization group. Renormalized quantities acquire a dependence on M, which hence has the potential of breaking Weyl invariance at the quantum level. Let's denote the action as the sum of terms $I = \int \lambda_i\, \mathcal{O}_i$, where λ_i are the coupling constants and $\mathcal{O}_i$ the operators. In the simpler case of flat spacetime theories, Poincaré invariance is expected to be preserved at the quantum level, but scale (or conformal) invariance typically becomes anomalous. Indeed, since the renormalization scale M transforms under a scale transformation, the M-dependence acquired by the renormalized operators $\mathcal{O}_i(M)$ furnishes them with an anomalous scaling dimension $-\gamma_i$, which can be computed as

$$\gamma_i(M) := \frac{\delta \ln \mathcal{Z}_i(M)}{\delta \ln M}\,, \tag{1.1.34}$$

where $\mathcal{Z}_i$ is the multiplicative factor that encodes the running of the renormalized operator $\mathcal{O}_i(M) = \mathcal{O}_i^0\, \mathcal{Z}_i^{-1}(M)$, $\mathcal{O}_i^0$ being the bare one, the operator at the UV cutoff M_0. Integrating the above then

$$\mathcal{O}_i(M) = \mathcal{O}_i^0\, e^{-\int \gamma_i(M)\, d\ln M}\,. \tag{1.1.35}$$

Alternatively, if one considers the renormalization of the coupling constants $\lambda_i(M)$, these develop a non-trivial β-function that encodes the classical dimension Δ_i of the operator and the anomalous one as

$$\beta_i(\lambda_i) = (\Delta_i - d + \gamma_i)\, \lambda_i(M)\,. \tag{1.1.36}$$

How does the above logic generalize to a theory with gravity? As argued, when the metric is dynamical, conformal transformations are better to be replaced by Weyl transformations. The anomalous dimensions under scale transformations $\delta \ln M$ then, should become anomalous dimensions under Weyl transformations $\delta\Sigma_g$. This suggests that the conformal factor of the metric should take the role of the covariant renormalization scale. Namely, that renormalized operators should acquire an anomalous dependence on the conformal factor, so $\mathcal{O}_i(\Sigma_g)$. An additional metric dependence is already expected to appear from a covariant renormalization procedure. In other words, the small-distance regulator requires specification of the background metric in which it is computed. Furthermore, a renormalization energy scale should become local $M(x)$ in a theory of gravity, where in general there are no uniquely and globally-defined energies. The intuitive idea behind is that in a theory on a general background, there is no reason for two far a part observers to choose the same arbitrary renormalization scale, and therefore in general this should depend on position. Specification of the renormalization energy scale should rather be regarded as specification of the characteristic curvature scale, which is local. What this generalization is telling us is that this metric dependence should in particular be on the conformal factor, and that the curvature scale appears only through the latter.

Just as in the flat case, the anomalous dimensions can generically depend on the renormalization scale, i.e. $\gamma_i(\Sigma_g)$. The renormalized operators then become

$$\mathcal{O}_i(\Sigma_g) = \mathcal{O}_i^0\, \mathcal{Z}_i^{-1}(\Sigma_g) = \mathcal{O}_i^0\, e^{-\int_0^{\Sigma_g} \gamma_i(\Sigma)\, d\Sigma}\,. \tag{1.1.37}$$

The quantum effective action is written with the renormalized operators, hence it inherits their quantum metric dependence, and we say that the operators in the action become gravitationally dressed. Consequently, the action also acquires an anomalous behavior under a Weyl transformation. This propagates then to the quantum momentum tensor computed from it. In particular, the contribution to its trace has to be proportional to the anomalous dimension, so that it vanishes in the classical limit.

If the theory is classically Weyl invariant, the above anomalies should then be the only contributions spoiling the tracelessness (up to the anomalies of the path integral measures). If instead the theory has (ir)relevant operators, as it is the case of gravity, then the trace does not vanish already classically. The metric dependence of a generic operator in the classical action is $I \sim \int \sqrt{-g}\, \lambda_i\, \mathcal{O}_i(g)$, whose traced metric variation gives

$$\frac{1}{\sqrt{-g}} \frac{\delta I}{\delta \Sigma_g} = (d - \Delta_i)\, \lambda_i\, \mathcal{O}_i(g)\,, \tag{1.1.38}$$

the first term coming from the variation of $\sqrt{-g}$ and the second one from the classical Weyl transformation of the operator. The trace of the quantum momentum tensor then will acquire the anomalous dimension on top of the two terms above[8]

$$\frac{1}{\sqrt{-g}} \frac{\delta \Gamma}{\delta \Sigma_g} = (d - \Delta_i - \gamma_i)\ \lambda_i\, \mathcal{O}_i(\Sigma_g)\,. \tag{1.1.39}$$

From the expression of the β-function (1.1.36) it is clear then that the quantum trace obeys[9]

$$T = -\beta_i(\lambda_i)\, \mathcal{O}_i(\Sigma_g)\,. \tag{1.1.40}$$

[8] We now drop the classical metric-dependence of the operator and just indicate the anomalous one on the conformal factor.

[9] In fact, the trace of the quantum momentum tensor would include other terms: the anomalies of the path integral measures but also other total derivative terms. This is why vanishing of the β-functions only implies scale invariance but not full conformal invariance in flat spacetime. We ignore these terms in this exposition both to make the main point clear and because they will not play a role in our analysis.

This is the anomaly equation that incorporates the quantum anomalous Weyl dimension acquired by the operators, and consequently by the effective action, through renormalization. It is a very useful equation because knowing the anomalous dimensions of the operators, it allows to compute their gravitational dressings in the effective action.

This anomaly equation is in fact the local renormalization group equation [38–41], a covariant extension of the Callan-Symanzik equations. The local renormalization group is the rigorous framework for the generalization of the flat spacetime renormalization group and anomalous scalings to the gravitational context. We will not require it to the extent of the work in this thesis, hence we will not present it here, but it needs to be mentioned because it eventually is the formal reason why covariant renormalization of contact divergences introduces a dependence on the metric through its conformal factor. Further, it tells us that the gravitational dressings in the effective action, since the latter is a solution of (1.1.40), are re-summing the leading logarithms, and need not be one-loop exact, as is the case for other anomalies. Consequently, it suggests that the effects of these Weyl anomalies in, for example, equations of motion derived from the effective action, should also be re-summing leading logarithms, so it serves as a guidance of what corrections to expect to classical solutions.

Finally, the anomaly equation can include operator mixing, which refers to the appearance of several other operators to the trace anomaly of a single one of them

$$T_i = -Z_{ij}\,\mathcal{O}_j(\Sigma_g)\,. \tag{1.1.41}$$

This does not happen in the regime where perturbation theory is valid, i.e. at weak coupling, since then the operators are 'far apart' from each other and cannot interfere each others renormalization.

The Weyl anomalies of interest in this thesis belong to the second group, i.e. we are interested in computing the anomalies of composite operators, those relevant for the cosmological evolution. The first type of anomalies are generically also present, but we will mostly ignore them, and we will assume that the effects of the different types of anomalies can be analyzed independently.

1.2 Weyl-Invariant Formulation of Gravity

The Einstein-Hilbert action in d spacetime dimensions with a cosmological constant Λ is given by

$$I_G[g] = I_K[g] + I_\Lambda[g] = \frac{M_p^{d-2}}{16\,\pi}\int d^dx\sqrt{-g}\,(R(g)-2\Lambda)\,, \tag{1.2.1}$$

where $R(g)$ is the Ricci scalar and M_p is the Planck mass. The latter is related to the fundamental constants through

$$M_p^{d-2} = \frac{\hbar^{d-3}}{c^{d-5}\,G_N}\,, \tag{1.2.2}$$

and in four dimensions takes value $M_p = 1.22\times 10^{19} GeV/c^2$. One usually thinks of this two-derivative action as a low-energy effective action for scales below the Planck cutoff. At higher energies, one would need to take into account corrections due to higher derivative terms and eventually an ultraviolet completion given by a quantum theory of gravity. In this approach, one introduces a UV cutoff M_0, which typically lays below the Planck scale. One can then write the couplings in terms of dimensionless constants, the ones to acquire Weyl anomalous dimensions through renormalization. For the gravitational and cosmological constants we denote this dimensionless coupling constants κ^2 and λ, defined by

$$\frac{M_p^{d-2}}{16\pi} = \frac{M_0^{d-2}}{2\kappa^2}\,, \qquad \Lambda = \lambda\,\kappa^2 M_0^2\,. \tag{1.2.3}$$

We will often work in the units $M_0 = 1$, with which M_p and Λ become dimensionless.

1.2.1 The Weyl compensator

The Einstein-Hilbert action I_K is not Weyl invariant in dimensions $d > 2$. The cosmological constant spoils the invariance in any dimensions. However, we can generalize this action to a Weyl invariant one at the expense of introducing an additional scalar field φ, a so-called Weyl or conformal compensator. We already mentioned that a free scalar field action can be made Weyl invariant through the addition of an improvement term with a coupling to the background curvature, i.e. through Ricci gauging. This action is

$$I_c[\varphi, g] = -\frac{1}{2}\int d^dx\sqrt{-g}\left(g^{\mu\nu}\,\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{4}\frac{(d-2)}{(d-1)}R(g)\,\varphi^2\right)\,. \tag{1.2.4}$$

For now we will assume that $d > 2$; we will discuss the limit to two dimensions at length in the next section. Again, this action is invariant under the finite Weyl transformation

$$\varphi(x) \to e^{-\frac{d-2}{2}\xi(x)}\,\varphi(x)\,, \qquad g_{\mu\nu} \to e^{2\xi(x)}\,g_{\mu\nu}\,, \tag{1.2.5}$$

since the variation of the improvement term cancels the derivatives on the local parameter generated from the kinetic term. The Weyl invariance can then be used to impose the dilatation gauge

$$\varphi^2(x) = -\varphi_d^2 := -4\,\frac{(d-1)}{(d-2)}\,\frac{M_0^{d-2}}{\kappa^2}\,, \tag{1.2.6}$$

under which we recover the Einstein-Hilbert action. The two actions I_c and I_K are hence gauge-equivalent, we go from the first to the second by gauge-fixing. Notice that to exactly recover Einstein-Hilbert (assuming Newton's constant to be positive), the compensator field has to be imaginary.

An alternative point of view on this generalization, which gives the compensator field φ a more natural gravitational origin, is to disguise the latter inside the metric by doing

$$g_{\mu\nu} \to g'_{\mu\nu} = \left(\frac{\varphi}{\varphi_d}\right)^{\frac{4}{d-2}}\,g_{\mu\nu}\,, \tag{1.2.7}$$

where the φ_d factor is inserted to keep the metric dimensionless. It is clear that the new metric g' is invariant under a Weyl transformation (1.2.5), and so must be the case for the gravitational action written with it

$$I_G[g'] = \frac{M_0^{d-2}}{2\kappa^2}\int d^dx\sqrt{-g'}\,\left(R(g') - 2\Lambda\right)\,. \tag{1.2.8}$$

Expressing the above action in terms of the original metric and compensator field, by using the Weyl transformation of the Ricci scalar (B.0.4), we find

$$I_G[g'] = \frac{1}{2}\int d^dx\sqrt{-g}\left((\nabla\varphi)^2 + \frac{1}{4}\frac{(d-2)}{(d-1)}R(g)\,\varphi^2 - 2\tilde{\Lambda}\,\varphi^{\frac{2d}{d-2}}\right) \equiv I_G[\varphi, g]\,, \quad (1.2.9)$$

where $\tilde{\Lambda}$ absorbs the corresponding numerical factors and constants. The first two terms of this action coincide with the Weyl-coupled scalar action $I_c[\varphi, g]$, but with a relative $-$ sign. This is an important distinction, since now the compensator need not be imaginary in order to recover Einstein-Hilbert, the dilatation gauge becoming $\varphi^2(x) = \varphi_d^2$. However, the price to pay is that the kinetic term has the wrong sign, it is not positive definite. We will come back to this point in due time.

In four dimensions, $I_G[\varphi, g]$ is a polynomial action for the field φ with a quartic interaction term. In other dimensions, the cosmological constant term may be non-polynomial. Due to the dependence on $(d-2)$, it is clear that the limit to two dimensions of this invariant action and the transformations involved is subtle, we will properly define it in the next section. However, formulating the problem in d dimensions exhibits the fact that most considerations depend analytically on the number of spacetime dimensions, and therefore the limit can be taken continuously from higher d.

From the above it becomes clear in which sense the gravitational action is Weyl invariant: only after conveniently dressing the metric with a scalar compensator that makes it effectively invariant under a Weyl transformation. This suggests the definition and distinction of two types of Weyl transformations: physical Weyl and fiducial Weyl transformations. For this, we will denote $g_{\mu\nu}$ as the physical metric, and define a new fiducial metric $h_{\mu\nu}$. The two metrics are related through a Weyl compensator as

$$g_{\mu\nu} = \left(\frac{\varphi}{\varphi_d}\right)^{\frac{4}{d-2}} h_{\mu\nu}\,. \quad (1.2.10)$$

This split is equivalent to the shift (1.2.7), this rewriting though makes clear that now we have to regard the field $\varphi(x)$ as a scalar component of the original gravitational field. The physical Weyl transformation is the familiar one, under which the physical metric gets locally rescaled

$$g_{\mu\nu} \to e^{2\xi(x)}\, g_{\mu\nu}\,. \quad (1.2.11)$$

The fiducial Weyl transformation involves the transformation of the fiducial metric and the Weyl compensator as in (1.2.5)

$$\varphi(x) \to e^{-\frac{d-2}{2}\xi(x)}\,\varphi(x)\,, \qquad h_{\mu\nu} \to e^{2\xi(x)}\, h_{\mu\nu}\,, \quad (1.2.12)$$

in such a way that the physical metric g is left invariant. Any action written in terms of the physical metric g is automatically (classically) invariant under the fiducial Weyl transformation upon rewriting it in terms of the compensator and the fiducial metric h. The Einstein-Hilbert action with the cosmological constant is therefore fiducial Weyl invariant.

Notice that the fiducial metric $h_{\mu\nu}$ is not required to satisfy any scalar conditions, and so has all its $d(d+1)/2$ independent components. Hence, the split (1.2.10) seems to introduce an additional degree of freedom to the gravitational field, namely that of φ. However, the fiducial Weyl invariance that is gained with it should be regarded as a gauge invariance, which can be used to fix the new field introduced and kill its degree of freedom. Therefore, even if the number of fields is increased by one, so is the gauge freedom, and the number of degrees of freedom stays invariant.

To clarify possible confusion, let's notice that the split (1.2.10) does not break the original diffeomorphism invariance. The easiest way to see this is to rather think of this split as the equivalent shift (1.2.7): it simply dresses the original metric with a good Lorentz scalar; clearly this is a fully covariant field-redefinition, therefore in the enlarged field set (φ, g), the gauge principle is accordingly enlarged to diffeomorphisms of both fields times fiducial Weyl invariance.

Finally, we would like the reader to appreciate that there is nothing fundamental about the split (1.2.10) or the fiducial Weyl gauge invariance: it is a redundancy added by hand. However, as we will learn along the way, the separation of this scalar component from a fiducial tensor simplifies greatly the computations of gravitational dressings in two dimensions, since correlation functions and anomalous dimensions of scalar operators are much easier to tackle. Further, it becomes very convenient to model cosmological solutions, for which all the dynamics resides in the scale factor after Robertson-Walker symmetry is imposed.

- **The Ω field**

The fiducial Weyl invariance allows to gauge-fix the Weyl compensator φ, as argued above. Alternatively, it can also be used to fix a degree of freedom of the fiducial metric $h_{\mu\nu}$ and leave the compensator as a fluctuating field. This is actually what needs to be done to go to the conformal gauge, which is defined by a split of the metric into a conformal factor $\Sigma_g(x)$, and a fiducial metric $\bar{\eta}$ constrained by a scalar condition, as

$$g_{\mu\nu} = e^{2\Sigma_g(x)}\,\bar{\eta}_{\mu\nu}\,. \tag{1.2.13}$$

The scalar condition reduces the number of independent components of the fiducial metric by one, so that $g_{\mu\nu}$ maintains its $d(d+1)/2$ components. In two dimensions, the scalar condition can be fixed using the Ricci scalar. The conformal gauge that we will mostly use is the Ricci-flat gauge, defined by $R(\bar{\eta}) = 0$. In two dimensions, given that any metric is locally conformal to the Minkowski metric, and given the Weyl transformation of the Ricci scalar and the Laplacian, the Ricci-flat gauge automatically fixes the fiducial metric to be proportional to the Minkowski metric. Henceforth we will name conformally flat gauge the gauge where the fiducial metric is Minkowski.

In higher dimensions, there are other scalars to be defined given any metric, hence there are different conformal gauges. In particular, in our treatment of four dimensions in chapter §5 we will use the Ricci-flat and the F-flat gauges.

Imposing the scalar condition on the fiducial metric is precisely equivalent to fixing the fiducial Weyl invariance. Namely, $\xi(x)$ in (1.2.12) has to be chosen so that $\bar{\eta}_{\mu\nu} = e^{2\xi(x)}h_{\mu\nu}$. Notice that the resulting transformed compensator, which we can rename $e^{2\Sigma_g(x)}$, is not fixed by this condition.

In the conformal gauge, physical Weyl transformations are implemented by a translation of the conformal factor

$$\Sigma_g(x) \to \Sigma_g(x) + \xi(x)\,, \qquad \bar{\eta}_{\mu\nu} \to \bar{\eta}_{\mu\nu}\,. \tag{1.2.14}$$

This gauge suggests a convenient redefinition of the Weyl compensator

$$\frac{\varphi(x)}{\varphi_d} := e^{\frac{d-2}{2}\Omega(x)}\,, \tag{1.2.15}$$

so that the split (1.2.10) becomes

$$g_{\mu\nu} = e^{2\Omega(x)}\, h_{\mu\nu}\,. \tag{1.2.16}$$

We will call this the Weyl split. In terms of the new compensator, the fiducial Weyl transformation becomes very simple

$$\Omega(x) \to \Omega(x) - \xi(x)\,, \qquad h_{\mu\nu} \to e^{2\xi(x)}\, h_{\mu\nu}\,. \tag{1.2.17}$$

If further the fiducial metric $h_{\mu\nu}$ is fixed to the same conformal gauge as $g_{\mu\nu}$

$$h_{\mu\nu} = e^{2\Sigma_h(x)}\, \bar{\eta}_{\mu\nu}\,, \tag{1.2.18}$$

then

$$\Sigma_g(x) = \Omega(x) + \Sigma_h(x)\,, \tag{1.2.19}$$

and the fiducial Weyl transformation becomes

$$\Omega(x) \to \Omega(x) - \xi(x)\,, \qquad \Sigma_h(x) \to \Sigma_h(x) + \xi(x)\,. \tag{1.2.20}$$

In this formulation with $\Omega(x)$, what we previously called the dilatation gauge $\varphi^2 = \varphi_d^2$ now becomes $\Omega(x) = 0$. We will call this the physical gauge, since then the fiducial metric becomes the physical metric $g_{\mu\nu} = h_{\mu\nu}$. Alternatively, if we choose the conformal gauge $h_{\mu\nu} = \bar{\eta}_{\mu\nu}$, the new Weyl compensator becomes the conformal factor of the metric $\Omega = \Sigma_g$.

The rewriting of $\varphi(x)$ in terms of the Weyl compensator $\Omega(x)$ therefore casts the fiducial Weyl transformation in the most intuitive form: the compensator Ω transforms linearly under a fiducial Weyl just as the conformal factor Σ_g transforms linearly under a physical Weyl. This formulation simplifies computations since it helps make contact with Liouville theory, in which Ω, the Liouville field, transforms linearly under conformal transformations. Also, notice that in terms of Ω, neither the Weyl split nor its transformation depend anymore on the factor $(d-2)$. In fact, it is in this formulation that we are able to properly take the limit to two dimensions of the Weyl invariant gravitational action, as we will see in the next section. Finally, Ω has the interpretation of the Goldstone boson associated to the spontaneous breaking of the (global) fiducial Weyl invariance, which takes place as soon as $\varphi(x)$ takes a non-vanishing expectation value.

Henceforth, we will write Weyl invariance to refer to the fiducial Weyl invariance, and explicitly say physical Weyl invariance when we refer to this one. Also, in what follows all quantities of Riemannian geometry, derivatives and contractions will depend on and refer to the fiducial metric.

1.2.2 Weyl-Invariant Action

The Weyl-invariant formulation of gravity consists of rewriting all metric-dependent quantities in terms of $\Omega(x)$ and $h_{\mu\nu}$ by means of the Weyl split (1.2.16). In the resulting gravitational theory, the gauge principle is general coordinate transformations times Weyl invariance. Even if the gauge Weyl invariance may seem to be given a fundamental status, including it in the gauge principle should rather be regarded as a convenient trick, which helps uncover the anomalous coupling to the conformal factor of composite operators. The defined Weyl transformation (1.2.17), trivially leaves the physical metric invariant and hence, it entitles no statement about the symmetry content of the theory. However, it conveniently makes the fiducial conformal factor transform linearly, effectively performing the renormalization group running and digging out the anomalous dimensions. Consistent with this picture, the Weyl invariant formulation is but a very convenient formulation that makes this invariance manifest and thus helps to compute the gravitational dressings explicitly.

The Weyl-invariant formulation of the gravitational action follows then from introducing the Weyl split in the action $I_G[g] = I_K + I_\Lambda$ (1.2.1). By using the Weyl transformation of the Ricci scalar (B.0.4), it reads

$$I_K[\Omega, h] = \frac{M_0^{d-2}}{2\kappa^2} \int d^dx\sqrt{-h}\; e^{(d-2)\Omega}\left(R(h) + (d-2)(d-1)h^{\mu\nu}\,\nabla_\mu\Omega\nabla_\nu\Omega\right) \qquad (1.2.21)$$

for the purely gravitational action, and

$$I_\Lambda[\Omega, h] = -\frac{M_0^{d-2}}{2\kappa^2} \int d^dx\sqrt{-h}\, e^{d\Omega}\, 2\Lambda \qquad (1.2.22)$$

for the cosmological term. Alternatively, they also follow from introducing the redefinition of the φ compensator in terms of Ω in the Weyl-invariant action $I_G[\varphi, h]$. $I_G[\Omega, h]$ therefore becomes a gravitational action with both fluctuating metric $h_{\mu\nu}$ and scalar field Ω, and non-polynomial couplings and interactions.

- **Ward Identities**

The actions (1.2.21) and (1.2.22) are each independently invariant under diffeomorphisms and Weyl transformations by construction. They satisfy therefore the corresponding Ward identities

$$\nabla^\nu\Big(\frac{-2\,\delta I_a}{\sqrt{-h}\,\delta h^{\mu\nu}}\Big) - \frac{1}{\sqrt{-h}}\frac{\delta I_a}{\delta\Omega}\,\nabla_\mu\Omega \;\equiv 0\,. \qquad (1.2.23)$$

and

$$h^{\mu\nu}\Big(\frac{-2\,\delta I_a}{\sqrt{-h}\,\delta h^{\mu\nu}}\Big) - \frac{1}{\sqrt{-h}}\frac{\delta I_a}{\delta\Omega} \;\equiv 0\,, \qquad (1.2.24)$$

where $a = K, \Lambda$. For the cosmological constant action I_Λ, the first term in the two identities corresponds to the fiducial momentum tensor in the background $h_{\mu\nu}$. Notice though that the purely gravitational action I_K contains the kinetic term for Ω but also that for $h_{\mu\nu}$, hence the first term of the two identities for I_K contains not only the momentum tensor of the free Ω field but also the Einstein tensor of $h_{\mu\nu}$.

• Equations of Motion

The equations of motion of the gravitational action $I_G[\Omega, h]$ comprend the fiducial Einstein equations and the equation of motion for the scalar field Ω. To obtain the former, we perform the variation of this action with respect to the fiducial metric $h_{\mu\nu}$, while treating Ω as a (non-gravitational) scalar field. Formally, this variation leads explicitly to

$$e^{(d-2)\Omega} E_{\mu\nu}(h) = \frac{\kappa^2}{M_0^{d-2}} \left(\hat{T}^{\Omega}_{\mu\nu} + \hat{T}^{\Lambda}_{\mu\nu} \right) \tag{1.2.25}$$

where $E_{\mu\nu}(h)$ is the Einstein tensor of the fiducial metric

$$E_{\mu\nu} = R_{\mu\nu}(h) - \frac{1}{2} h_{\mu\nu} R_h \,, \tag{1.2.26}$$

and

$$\hat{T}_{\mu\nu} = \frac{-2}{\sqrt{-h}} \frac{\delta I}{\delta h^{\mu\nu}} \tag{1.2.27}$$

is the fiducial momentum tensor. However, to look for solutions it is more convenient to express the equations in terms of the physical momentum tensor, the momentum tensor that follows from variation with respect to the physical metric $g_{\mu\nu}$. Since this variation is computed at Ω fixed, the relation between the fiducial and the physical momentum tensors is

$$\hat{T}_{\mu\nu} = \frac{-2}{\sqrt{-h}} \frac{\delta I}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta h^{\mu\nu}} = e^{(d-2)\Omega} T_{\mu\nu} \,. \tag{1.2.28}$$

The fiducial Einstein equations then become

$$E_{\mu\nu}(h) = \frac{\kappa^2}{M_0^{d-2}} \left(T^{\Omega}_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right) (\Omega, h) \,, \tag{1.2.29}$$

with

$$\begin{aligned} \frac{\kappa^2}{M_0^{d-2}} T^{\Omega}_{\mu\nu}(\Omega, h) &= (d-2) \left[\nabla_\mu \nabla_\nu \Omega - (\nabla_\mu \Omega)(\nabla_\nu \Omega) - h_{\mu\nu} \left(\nabla^2 \Omega + \frac{d-3}{2} (\nabla\Omega)^2 \right) \right] , \\ \frac{\kappa^2}{M_0^{d-2}} T^{\Lambda}_{\mu\nu}(\Omega, h) &= -\Lambda \, h_{\mu\nu} \, e^{2\Omega} \,. \end{aligned} \tag{1.2.30}$$

It is easy to check that

$$\frac{\kappa^2}{M_0^{d-2}} T^{\Omega}_{\mu\nu} = -D_{\mu\nu}(\Omega, h) \tag{1.2.31}$$

where $D_{\mu\nu}(\Omega, h)$ is defined in (B.0.3) as the Weyl transformation of the Einstein tensor

$$E_{\mu\nu}(h) + D_{\mu\nu}(h, \Omega) = E_{\mu\nu}(g). \tag{1.2.32}$$

This is simply indicating that the fiducial Einstein equations follow from introducing the Weyl split in the original ones

$$E_{\mu\nu}(g) = \frac{\kappa^2}{M_0^{d-2}} T^{\Lambda}_{\mu\nu}(g) = -\Lambda \, g_{\mu\nu} \,. \tag{1.2.33}$$

The equation of motion for the Ω field is

$$-2\,(d-1)\nabla^2\,\Omega-(d-1)(d-2)(\nabla\Omega)^2+R(h)=\frac{2\,d\,\Lambda}{d-2}e^{2\,\Omega}\,. \tag{1.2.34}$$

As a consequence of (either of the two) Ward identities for $I_G[\Omega,h]$, the equation of motion for Ω is automatically satisfied if the fiducial Einstein equations are satisfied. In fact, (1.2.34) can be recognized as the trace of the physical Einstein equations (1.2.33)

$$R(g)=\frac{2\,d\,\Lambda}{d-2} \tag{1.2.35}$$

again after introducing the Weyl split.

1.3 Weyl-Invariant Formulation of Cosmology

The Weyl-invariant formulation is particularly well-suited to study the cosmological evolution. This is because the physical metric for a homogeneous and isotropic universe is conformally-equivalent to a time-independent metric. This follows purely from symmetry considerations, which further fully fix this time-independent conformal metric after the spatial curvature is chosen. The fiducial metric $h_{\mu\nu}$ can hence be chosen to be this time-independent metric. It then follows that all the dynamics is contained in the scalar field Ω, which leads to a useful simplification of the cosmological evolution equations.

More concretely, a homogeneous and isotropic universe is described by the Robertson-Walker metric

$$ds^2=-dt^2+a^2(t)\,h_{ij}dx^idx^j\,, \tag{1.3.1}$$

where $a(t)$ is the scale factor, which only depends on the comoving cosmological time t. The form and time-dependence of this metric follows from the symmetries considered. The conformal time τ is related to t by

$$d\tau=\frac{dt}{a(t)}\,, \tag{1.3.2}$$

with which the Robertson-Walker metric becomes

$$ds^2=a^2(\tau)\,\left(-d\tau^2+h_{ij}dx^idx^j\right)\,. \tag{1.3.3}$$

For a spatially-flat spacetime, $h_{ij}=\delta_{ij}$, and so one can choose a gauge in which the fiducial metric is just the Minkowski metric, whose line-element is of the form

$$ds^2=-d\tau^2+\delta_{ij}dx^idx^j \tag{1.3.4}$$

on the product space $\mathbb{R}\times\mathbb{R}^{d-1}$. We can then identify the scale factor of the physical metric in conformal time with the Weyl compensator as

$$a(\tau)=e^{\Omega(\tau)}=e^{\Sigma_g(\tau)}\,. \tag{1.3.5}$$

Henceforth, we will denote derivatives with respect to t by a dot and derivatives with respect to τ by a prime.

Consider a universe filled with a perfect fluid of energy density ρ and pressure p. The momentum tensor is given by

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p\, g_{\mu\nu}\,. \tag{1.3.6}$$

The comoving time t can be identified with the proper time of the geodesics of the cosmological fluid elements everywhere, therefore the velocity u of the cosmological fluid is

$$u^\mu = (1,\vec{0})\,, \qquad u_\mu = (-1,\vec{0})\,, \tag{1.3.7}$$

which makes the momentum tensor diagonal as required by isotropy and homogeneity. In conformal coordinates instead,

$$u_c^\mu = (a^{-1}(\tau),\vec{0})\,, \qquad u_\mu^c = (a(\tau),\vec{0})\,, \tag{1.3.8}$$

from which follows that the density and pressure can be read from the momentum tensor in conformal time as

$$\rho = \frac{T_{\tau\tau}}{a^2(\tau)}\,, \qquad p = \frac{T_{ii}}{a^2(\tau)}\,. \tag{1.3.9}$$

The momentum tensor for a classical cosmological constant (1.2.30) is of the perfect fluid form, with density and pressure

$$\rho_\Lambda = \frac{M_0^{d-2}}{\kappa^2}\,\Lambda \qquad p_\Lambda = -\frac{M_0^{d-2}}{\kappa^2}\,\Lambda\,. \tag{1.3.10}$$

The classical evolution of the universe, and hence of the conformal factor, is governed by the first Friedmann-Lemaître equation, which follows from Einstein equations for a metric of the Robertson-Walker form. We derive it now in the Weyl-invariant formulation.

We start with the fiducial Einstein equations (1.2.29) with a momentum tensor for a generic perfect fluid, which we write again

$$E_{\mu\nu}(h) = \frac{\kappa^2}{M_0^{d-2}}\left(T^\Omega_{\mu\nu} + T_{\mu\nu}\right)(\Omega, h)\,. \tag{1.3.11}$$

Since we choose the fiducial metric to be Minkowski, the Einstein tensor $E_{\mu\nu}(h)$ vanishes. The $(\tau\tau)$ component of the above equations, as follows from the expression (1.2.30) for $T^\Omega_{\mu\nu}$, reads

$$\frac{(d-1)(d-2)}{2}\,\Omega'^2 = \frac{\kappa^2}{M_0^{d-2}}\,\rho\, e^{2\Omega}\,. \tag{1.3.12}$$

We define the Hubble scale as usual by

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\Omega'}{e^\Omega}\,. \tag{1.3.13}$$

Then (1.3.12) takes the usual form of the first Friedmann-Lemaître equation

$$H^2 = \frac{2\kappa^2\rho}{(d-2)(d-1)M_0^{d-2}}\,. \tag{1.3.14}$$

The (ii) components of (1.3.11) lead of course to the same equation, since there is only one unknown variable. The conservation of the momentum tensor (1.3.6) implies

$$\rho' = -(d-1)(p+\rho)\,\Omega'\,, \tag{1.3.15}$$

which when written in comoving coordinates leads to the familiar continuity equation

$$\dot{\rho} = -(d-1)(p+\rho)\,H\,. \tag{1.3.16}$$

If the fluid satisfies the barotropic equation of state $p = w\rho$ for some constant barotropic index w, then the solutions to (1.3.14) and (1.3.16) are given by

$$\rho(t) = \rho_*(\frac{a}{a_*})^{-\gamma}\,, \qquad a(t) = a_*(1+\frac{\gamma}{2}H_*t)^{\frac{2}{\gamma}}\,, \tag{1.3.17}$$

where ρ_*, H_*, a_* are the initial values of various quantities at $t = 0$, and

$$\gamma := (d-1)(1+w)\,. \tag{1.3.18}$$

For the classical momentum tensor of the cosmological constant we have $w = -1$ and $\gamma = 0$ in any dimensions, which leads to de Sitter spacetime

$$a^{dS}(t) = a_*^{dS}\,e^{H_*\,t}\,. \tag{1.3.19}$$

Conformal time as a function of comoving time is given by

$$\tau = \tau_*\,(1+\frac{\gamma}{2}\,H_*\,t)^{\frac{\gamma-2}{\gamma}}\,, \quad \text{where} \quad \tau_* := \frac{2}{(\gamma-2)\,H_*\,a_*}\,. \tag{1.3.20}$$

The range of t is $0 \le t < \infty$ with the universe starting with scale factor a_*. The range of τ is

$$|\tau_*| \le \tau < \infty \qquad \text{for} \quad \gamma > 2\,, \tag{1.3.21}$$

$$-|\tau_*| \le \tau < 0 \qquad \text{for} \quad \gamma < 2\,. \tag{1.3.22}$$

As a function of τ, the scale factor and the density are given by

$$a(\tau) = a_*\,(\frac{\tau}{\tau_*})^{\frac{2}{\gamma-2}}\,, \qquad \rho(\tau) = \rho_*\,(\frac{\tau}{\tau_*})^{-\frac{2\gamma}{\gamma-2}}\,. \tag{1.3.23}$$

Chapter 2

Two-Dimensional Quantum Gravity

2.1 Classical Gravity and Cosmology near Two Dimensions

Consider the gravitational action $I_G[\Omega, h]$ in the Weyl-invariant formulation, which is the sum of the Einstein-Hilbert action (1.2.21) and the cosmological term (1.2.22). We would like to analyze the renormalization of this action near two dimensions. For this purpose, we first consider the classical action in $d = 2 + \epsilon$. Henceforth we use R_h instead of $R(h)$ for the Ricci scalar associated with the metric $h_{\mu\nu}$. Keeping only terms at most linear in ϵ and using the rescaling (1.2.3) for the constants, we find

$$I_G = \frac{M_0^\epsilon}{2\kappa^2} \int d^{2+\epsilon}x \sqrt{-h} \left(R_h + \epsilon \left((\nabla\Omega)^2 + R_h \, \Omega \right) \right) - \lambda \, M_0^{2+\epsilon} \int d^{2+\epsilon}x \sqrt{-h} \, e^{2\Omega} \, (1+\epsilon \, \Omega) \, . \tag{2.1.1}$$

To make contact with Liouville theory in the next subsection, we define another two constants q^2 and μ by

$$\kappa^2 = \frac{2\pi\epsilon}{q^2}, \qquad \lambda \, M_0^2 = \mu. \tag{2.1.2}$$

The action then takes the form

$$I_G[\Omega, h] = \frac{q^2}{4\pi} \int d^2x \sqrt{-h} \left(\frac{R_h}{\epsilon} + (\nabla\Omega)^2 + R_h \, \Omega - \frac{4\pi\mu}{q^2} \, e^{2\Omega} \right) . \tag{2.1.3}$$

This action is manifestly coordinate invariant, and also Weyl invariant to this order in ϵ. As a result it satisfies both Ward identities (1.2.23) and (1.2.24). The integral of the Ricci scalar in two dimensions is a topological invariant, given by the Euler characteristic of the spacetime. Therefore, it is normally regarded as trivial, since it contributes only with a finite constant to the action and gives no local dynamics. However, in the above action $I_G[\Omega, h]$, the Einstein-Hilbert term comes with the $1/\epsilon$ pole, which precludes us from dropping this term altogether, and seems to be calling for some kind of renormalization, after which a finite piece should remain. We will address this point in the next section. Note further that to first order in ϵ, the expansion of the factor M_0^ϵ in front of the action would only contribute with an additional Einstein-Hilbert term of order ϵ^0, and can therefore be disregarded according to the argument just given.

In the limit to two dimensions $\epsilon \to 0$, the constants q and μ stay finite. For convenience of the reader, we repeat the original definition of the dimensionless parameter κ^2

$$\frac{M_p^{d-2}}{16\pi} = \frac{M_0^{d-2}}{2\kappa^2}\,. \tag{2.1.4}$$

Since $\kappa^2 = 8\pi\, G_N\, M_0^{d-2}$, this two-dimensional limit (2.1.2) implies $G_N = \epsilon/4q^2$, which means that we are taking Newton's constant to zero. This seems to indicate that, once the limit is taken, we cannot get any quantum perturbative corrections. However, using Newton's constant as the loop-counting parameter is not the best approach, as it generically is not dimensionless. A better one, is to regard the dimensionless κ^2 as the parameter to run the expansion. To define a dimensionless parameter, we had to introduce an arbitrary cutoff by hand, below the Planck scale, which determines the extent of validity of the effective field theory. The dimensionless parameter is as arbitrary as our choice of the cutoff, but we define it such that very small κ corresponds to a cutoff way below the Planck scale. $\kappa \to 0$ then implements the perturbative regime at low energies where gravity is weakly coupled.

Since this definition is arbitrary though, we might as well define the dimensionless ratio as

$$\frac{M_p^{d-2}}{16\pi} = \frac{M_0^{d-2}}{2}\,\frac{q^2}{2\pi\,(d-2)}\,, \tag{2.1.5}$$

in terms of a new dimensionless parameter q^2. With this definition, the limit to two dimensions automatically forces the cutoff to be much below the Planck scale if q^2 stays finite. Therefore, introducing this redefinition in terms of q in the action, we can only reliably explore the low energy perturbative regime. In exact two dimensions, the dependence on the Planck mass and the cutoff drop out, and we are left with a finite arbitrary dimensionless parameter, that can be used as the new fundamental dimensionless constant to perform the perturbative expansion. The semiclassical limit is therefore implemented by $q \to \infty$.

Just to summarize, in higher dimensions we introduce a dimensionless parameter κ^2 by hand, that we can tune to very small values when we want to go to the perturbative regime, thanks to introducing an arbitrary cutoff. In two dimensions, the freedom of tuning this parameter with a cutoff disappears due to the dimensionless Einstein-Hilbert action, so what we do is use the limit to two dimensions to effectively implement the limit to the weak coupling regime, while keeping a dimensionless arbitrary parameter to perform the perturbation theory.

Taking the two-dimensional limit in this way, with the scaling (2.1.2) and keeping q and μ finite, is actually the fruitful way to get non-trivial two-dimensional gravity and cosmology [113,114].[1] First of all, consider again the Weyl-invariant action for a scalar field $\varphi(x)$ in d dimensions (1.2.9). From that point of view, namely that of a scalar field theory on a fixed background, the limit to two dimensions has to be taken such that the resulting theory is still Weyl invariant and still has a canonically-normalized kinetic term. It turns out that this requires a redefinition of the scalar field $\varphi(x)$ precisely as

[1] See [48] for an entropy-based argument.

the one we did in terms of the compensator $\Omega(x)$ (1.2.15), and a further rescaling of the constant φ_d defined in (1.2.6) such that κ^2 is proportional to $(d-2)(d-1)$ [51].

Second, it is the way to take the limit so as to get non-trivial constraints for gravity in two dimensions. Indeed, in two dimensions the Einstein tensor vanishes identically, another way to see why there are no gravitational propagating degrees of freedom. Diffeomorphisms allow to fix two components of the metric. Even if we start with $\Lambda \neq 0$, the vanishing Einstein tensor implies the constraints $T^{\Lambda}_{\mu\nu} = 0$, which enforce $\Lambda = 0$, in which case physical Weyl invariance is reestablished, and the metric can be fully fixed at will. So there seems to be no room for something interesting to happen.

However, we can start with Einstein's equations in $d = 2 + \epsilon$, do the redefinition (2.1.2), and check for the resulting lowest order equations. The momentum tensor for Λ at order ϵ^0 becomes[2]

$$T^{\Lambda}_{\mu\nu}(g) = -\mu\, g_{\mu\nu}\,, \tag{2.1.6}$$

which stays finite in the limit $\epsilon \to 0$. Further, the Einstein constraints read

$$E_{\mu\nu} = \frac{2\pi\epsilon}{q^2}\, T^{\Lambda}_{\mu\nu}\,. \tag{2.1.7}$$

If we take the trace of the above equation, the left-hand side becomes also proportional to ϵ, which then drops from the equation, giving at lower order the non-trivial constraint

$$R_g = \frac{8\pi\mu}{q^2}\,. \tag{2.1.8}$$

This constraint forces a constant curvature solution. The limit $\epsilon \to 0$ can now be taken trivially, since the above constraint is already finite. Using the Weyl transformation of the Ricci scalar in two dimensions

$$R_g = e^{-2\Sigma_g}\left(R_{\bar\eta} - 2\,\nabla^2_{\bar\eta}\Sigma_g(x)\right)\,, \tag{2.1.9}$$

and that of the Laplacian $\nabla^2_g = e^{-2\Sigma_g}\nabla^2_{\bar\eta}$, in the conformally flat gauge $\bar\eta_{\mu\nu} = \eta_{\mu\nu}$ the constraint becomes the Poisson equation

$$-\,2\,\nabla^2_g\,\Sigma_g(x) = \frac{8\pi\mu}{q^2}\,. \tag{2.1.10}$$

So taking the limit in this way not only allows for a classical non-vanishing cosmological constant in exact two dimensions, but also relates the conformal factor of the metric to the cosmological constant, which can now no longer be chosen at will.

To solve this Poisson equation, we have to use the Laplacian's Green function $G_g(x, y)$ and add two independent solutions of the homogeneous equation

$$\Sigma_g(x) = \int d^2y\sqrt{-g}\, G_g(x,y)\,\frac{4\pi\mu}{q^2} \;+\; \Sigma^1_g(x) \;+\; \Sigma^2_g(x)\,. \tag{2.1.11}$$

These two homogeneous solutions encode the on-shell degree of freedom of the field, since in principle they allow for a two-functions-worth choice of initial conditions. However, given the Weyl transformation of the Laplacian, the two homogeneous solutions

[2] Higher-order terms would come from the expansion of $T^{\Lambda}_{\mu\nu} = M_0^{\epsilon}\,\mu\, g_{\mu\nu}$, but we can ignore them since in the Einstein equations, the momentum tensor comes multiplied by $\kappa^2 \sim \epsilon$.

have to satisfy the flat Laplacian equation

$$- \nabla_\eta^2 \Sigma_g^i(x) = -4\, \partial_+ \partial_- \Sigma_g^i(x) = 0\,, \tag{2.1.12}$$

where in the second step we have chosen light-cone coordinates. This equation forces the two functions to be a constant, killing thereby the degree of freedom of Σ_g, since the only freedom left is a constant initial condition. We conclude therefore that the field is not dynamical.

Another way to see this is by remembering that when gauge invariance is used to fix some components of the metric, the equations for these components become constraints, which must be applied to the initial data for those components that are not gauge-fixed. In our case, the unfixed component is the conformal factor, and the only non-trivial constraint is the above Poisson equation (2.1.10). The initial data for the Σ_g field are the functions $\Sigma_g^i(x)$. Putting the solution (2.1.11) into the constraint, leads again to the Laplace equation for the two initial functions, which forces them to be a constant. The conclusion that follows from this reasoning is that Σ_g has no dynamics because it's a scalar component of the gravitational field in a theory gravity, where it is required to impose the constraints of its gauge invariance.

The Poisson equation can be easily solved by noticing that the solution has to be a maximally symmetric space, as it is always the case for constant Ricci scalar metrics in two dimensions. We can then assume that the solution depends only on one of the two coordinates. The Poisson equation becomes thus a total differential equation, with solutions the two-dimensional de Sitter or Anti-de Sitter spacetimes, for $\mu > 0$ or $\mu < 0$ respectively.

Adding the Weyl-invariant formalism doesn't change this conclusion. Variation of the Weyl-invariant action (2.1.3) with respect to $h_{\mu\nu}$ gives the $2+\epsilon$-dimensional fiducial Einstein equations (1.2.29), which with the redefined constants read

$$E_{\mu\nu}(h) = \frac{2\pi\epsilon}{q^2} \left(T^{\Omega}_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right) (\Omega, h)\,, \tag{2.1.13}$$

with

$$T^{\Omega}_{\mu\nu}(\Omega, h) = \frac{q^2}{2\pi} \left[\nabla_\mu \nabla_\nu \Omega - (\nabla_\mu \Omega)(\nabla_\nu \Omega) - h_{\mu\nu} \left(\nabla^2 \Omega - \frac{1}{2} (\nabla \Omega)^2 \right) \right] \tag{2.1.14}$$

$$T^{\Lambda}_{\mu\nu}(\Omega, h) = -\mu\, h_{\mu\nu}\, e^{2\Omega}\,. \tag{2.1.15}$$

These momentum tensors become finite. Taking the trace of the equations again, allows to drop the ϵ on both sides of the equations, and to lowest order we find

$$R_h = 2\, \nabla_h^2 \Omega + \frac{8\pi\mu}{q^2}\, e^{2\Omega}\,. \tag{2.1.16}$$

Thus we get a finite non-trivial constraint for Ω and $h_{\mu\nu}$, which by using the Weyl transformation of the fiducial Ricci scalar is the same as (2.1.8). Using Weyl and diffeomorphism invariance, we can completely fix the fiducial metric to Minkowski $h_{\mu\nu} = \eta_{\mu\nu}$, in which case we are left with

$$- 2\nabla_\eta^2 \Omega = \frac{8\pi\mu}{q^2}\, e^{2\Omega}\,, \tag{2.1.17}$$

which is the same as (2.1.10) since in this gauge $\Omega = \Sigma_g$. Notice that with this gauge-fixing, the fiducial Einstein equations in $d = 2 + \epsilon$ read

$$T^{\Omega}_{\mu\nu} + T^{\Lambda}_{\mu\nu} = 0\,, \tag{2.1.18}$$

which are the Virasoro plus the trace constraints, and which still hold after doing $\epsilon \to 0$ because they are of order $\mathcal{O}(\epsilon^0)$. Even if they are three constraints, they all lead to the same equation (2.1.17) for Ω. Notice further, that this constraint coincides with the equation of motion for Ω, as the latter follows from the Bianchi identity because Ω is the only scalar field present.

Without the Weyl invariant formulation there are no Virasoro constraints because the $\epsilon \to 0$ limit of the Einstein equation (2.1.7) guarantees the vanishing of the Einstein tensor without requiring the vanishing of the momentum tensor. However, in the conformally flat gauge $T^{\Lambda}_{\mu\nu} \propto \eta_{\mu\nu}$, and hence the Virasoro constraints $T^{\Lambda}_{++} = T^{\Lambda}_{--} = 0$ are satisfied trivially.

The cosmological evolution equations depend analytically on d, so we can also dimensionally-continue them. Near two dimensions, they can be derived either by rewriting the Friedmann-Lemaître and continuity equations (1.3.14) and (1.3.16) in terms of the new gravitational constant q^2 and then taking the limit, or by imposing the Robertson-Walker symmetry on the constraint (2.1.16). They become

$$H^2 = \frac{4\pi}{q^2}\,\rho_\Lambda\,, \tag{2.1.19}$$

and

$$\dot{\rho}_\Lambda = -(1 + w_\Lambda)\,H\,\rho_\Lambda\,. \tag{2.1.20}$$

Again, we see how the re-absorption of ϵ in κ^2 and Λ allows for finite equations. As explained above though, this Friedmann-Lemaître equation contains no dynamics, since it actually is a constraint.

For the classical cosmological fluid $w_\Lambda = -1$, the energy density is constant $\rho_\Lambda(t) = \mu$, and the Hubble scale is given by $H^2 = 4\pi\mu/q^2$. The physical metric corresponds to two-dimensional de Sitter in cosmological coordinates with scale factor $a(t) = a_* e^{H_* t}$.

2.1.1 Classical Polyakov action

The first term in the action (2.1.3) may look a bit worry-some because of the $1/\epsilon$ pole, which comes from the redefinition of the gravitational constant κ^2 (2.1.2). In the previous section, we argued that the proper definition of the two-dimensional limit required this constant to be proportional to $(d-2)(d-1)$. According to our redefinition (2.1.2), we chose the proportionality constant to be 2π as

$$\kappa^2 = \frac{2\pi}{q^2}(d-2)(d-1)\,, \tag{2.1.21}$$

which correctly reproduces (2.1.2) at lowest order in the two-dimensional limit. Introducing this redefinition in the Einstein-Hilbert term, we get

$$\frac{1}{2\kappa^2}\int d^d x\sqrt{-h}\,R_h = \frac{q^2}{4\pi}\int d^d x\sqrt{-h}\,\frac{R_h}{(d-2)(d-1)}\,. \tag{2.1.22}$$

This term entitles some indeterminacy since in two dimensions, $\sqrt{-h}\,R_h$ is the Euler density, which effectively has a vanishing bulk integral, so it gives 0/0. It would be good to have a L'Hôpital's rule to get a finite limit as far as the bulk properties are concerned. Such a rule was given in [51], where the Weyl gauging of the d-dimensional free scalar action was considered. Since it provided some guidance, we reproduce it in the following.

We have seen that one way to make the free scalar Weyl invariant is through the addition of the Ricci scalar-dependent improvement term, leading to the action (1.2.4). However, as mentioned already at the end of section §1.1.2, another way of turning a scale-invariant theory into a Weyl-invariant one is by gauging the Weyl invariance, i.e. introducing the gauge connection of the wanted symmetry, such that the covariant derivative is Weyl invariant. In other words, the covariant action

$$I_W[\varphi, W, h] = -\frac{1}{2}\int d^dx\sqrt{-h}\,h^{\mu\nu}\left(\partial_\mu\varphi - \frac{d-2}{2}W_\mu\,\varphi\right)\left(\partial_\nu\varphi - \frac{d-2}{2}W_\nu\,\varphi\right) \tag{2.1.23}$$

is Weyl invariant provided the gauge potential W_μ transforms as

$$W_\mu \to W_\mu - \partial_\mu\,\xi(x)\,. \tag{2.1.24}$$

By demanding that the improved action (1.2.4) coincides with the Weyl-covariant action above, we find the relation between the Weyl gauge connection and the background geometry

$$\frac{R_h}{(d-1)} = 2\,\nabla_h^\mu\,W_\mu + (d-2)\,h^{\mu\nu}\,W_\mu W_\nu\,. \tag{2.1.25}$$

Just as a curiosity, the reason why we can write the gauge connection in terms of the curvature, i.e. that the Weyl-gauged action admits Ricci gauging, is because the free scalar in flat spacetime is not only scale invariant, but fully conformal invariant. Giving the gauge connection a geometric meaning is therefore not possible for all theories.

The above relation (2.1.25) makes manifest that in two dimensions the Ricci scalar is a total derivative. If the Ricci scalar satisfies such a relation with an arbitrary vector field, then it should generically have such an expression. And indeed, from its Weyl transformation (B.0.4) follows its expression in the Ricci-flat conformal gauge of the metric

$$R_h = e^{-2\Sigma_h}\left(-2\,(d-1)\nabla_\eta^2\Sigma_h - (d-2)(d-1)\,\eta^{\mu\nu}\,\nabla_\mu\Sigma_h\nabla_\nu\Sigma_h\right)\,. \tag{2.1.26}$$

Writing the above derivatives in terms of the $h_{\mu\nu}$ metric by means of the Weyl transformation of the Laplacian (E.1.4), and diving the expression by $(d-1)$, we obtain

$$\frac{R_h}{(d-1)} = -2\,\nabla_h^2\Sigma_h + (d-2)\,h^{\mu\nu}\nabla_\mu\Sigma_h\nabla_\nu\Sigma_h\,. \tag{2.1.27}$$

This expression reproduces (2.1.25) with the identification $W_\mu = -\nabla_\mu\Sigma_h$.

Substituting the above relation in the undetermined term in the action (2.1.22), the first term is a total derivative and hence can be dropped, effectively removing the $1/\epsilon$ pole. The limit to two dimensions then gives

$$\lim_{d\to2}\frac{q^2}{4\pi}\int d^dx\sqrt{-h}\,\frac{R_h}{(d-2)(d-1)} = \frac{q^2}{4\pi}\int d^2x\sqrt{-h}\,h^{\mu\nu}\,\nabla_\mu\Sigma_h\nabla_\nu\Sigma_h\,. \tag{2.1.28}$$

This is not yet completely covariant because $\Sigma_h(x)$ is the conformal factor of the metric only in the conformal frame. One might be tempted to write $\Sigma_h(x)$ in terms of the determinant of the metric, but this cannot be correct because it must be a scalar, whereas the determinant of the metric is a scalar density. We can obtain a manifestly coordinate-invariant scalar expression by inverting the Weyl transformation of the Ricci scalar (2.1.26), which in two dimensions becomes

$$R_h = -2\,\nabla_h^2 \Sigma_h(x)\,. \tag{2.1.29}$$

By solving this Poisson equation, the conformal factor can be written as[3]

$$\Sigma_h(x) = \frac{1}{2}\int d^2y\,\sqrt{h}\,G_h(x,y)\,R_h(y)\,, \tag{2.1.30}$$

where $G_h(x,y)$ is the Laplacian's Green function on the $h_{\mu\nu}$ background. This is the fully covariant expression of the conformal factor, but it makes the latter a non-local functional of the metric. This is going to be the origin of the non-localities of the quantum effective actions we will write down.

The finite limit of the gravitational action then becomes

$$\frac{q^2}{4\pi}\int d^2x\sqrt{-h}\,\left(\nabla\Sigma_h\right)^2 = \frac{q^2}{16\pi}\int dx\,dy\,R_h(x)\,G_h(x,y)\,R_h(y) = \frac{q^2}{4}\,I_{Pol}[h]\,, \tag{2.1.31}$$

where we have introduced the short-hand notation $dx \equiv d^2x\sqrt{-h}$. This is proportional to the well-known Polyakov action [36], which for simplicity we will henceforth denote

$$I_{Pol}[g] = \frac{1}{4\pi}\int R_g \frac{1}{\Box} R_g\,, \tag{2.1.32}$$

with $\Box = -\nabla_g^2$, and which inherits the non-locality of the conformal factor through the Green function. The Polyakov action thus arises as the two-dimensional finite limit of the classical Einstein-Hilbert action. The above-presented limit appropriately gives the $1/\epsilon$ pole to the total-derivative piece of the Ricci scalar in the action, so it serves as some kind of regularization mechanism.

The Weyl transformation of this action is

$$\frac{q^2}{16\pi}\int R_g\frac{1}{\Box}R_g = \frac{q^2}{16\pi}\int R_h\frac{1}{\Box}R_h + \frac{q^2}{4\pi}\int\sqrt{-h}\,\left((\nabla\Omega)^2 + R_h\,\Omega\right)\,. \tag{2.1.33}$$

It becomes clear that the (first two terms of the) Ω action (2.1.3) arises independently of the ϵ-expansion: it is the difference between the Polyakov actions in the physical and fiducial metrics.[4] The finite gravitational action in exact two dimensions is then

$$I_G[g] = \frac{q^2}{4\pi}\int dx\,\left(\frac{1}{4}\,R_g\frac{1}{\Box}R_g - \frac{4\pi\mu}{q^2}\right)\,, \tag{2.1.34}$$

[3] Notice that we had already inverted this equation in looking for the cosmological constant solution (2.1.11). The two solutions of the homogeneous equation, which have to be constant as argued there, are further chosen to vanish.

[4] A similar method of taking the two-dimensional limit was presented in [114], where the Liouville action arises from taking the limit of the difference between two Einstein-Hilbert actions corresponding to two different metrics. However, their subtraction criteria is rather ad-hoc, and in fact does not resolve whether the resulting theory is spacelike or timelike Liouville.

which in the Weyl-invariant formulation becomes

$$I_G[\Omega, h] = \frac{q^2}{4\pi} \int dx \left(\frac{1}{4} R_h \frac{1}{\Box} R_h + (\nabla\Omega)^2 + R_h \Omega - \frac{4\pi\mu}{q^2} e^{2\Omega} \right), \tag{2.1.35}$$

the finite version of (2.1.3). With the above Hôpital-type rule, the divergent R_h/ϵ term is replaced by the Polyakov action. However, the Polyakov action is non-local. Non-localities are expected in quantum effective actions, as coming from integrals over massless modes, but not in classical actions. Both terms though, the divergent R_h/ϵ and the finite Polyakov $R_h \frac{1}{\Box} R_h$, have the same transformation, given by (2.1.33), which is finite. We will hence consider both just as valid, and will not further worry about either the divergence of the former or the non-locality of the latter. In the remaining, we will mostly write down the Polyakov finite term but we will still refer to it as the Einstein-Hilbert term. Finally, notice that their transformation is such that it cancels the transformation of the (first two terms of the) Ω-action, in such a way that the total gravitational action $I_G[\Omega, h]$ is Weyl invariant.

2.2 Relation to Timelike Liouville Theory

After using the Weyl invariance to fully fix the fiducial metric $h_{\mu\nu}$, all the dynamics is encoded in the Ω action of (2.1.35). We hence drop its Polyakov term for now. In order to have a canonically-normalized kinetic term we define $\chi := q\,\Omega$. The action then becomes

$$I_{TL}[\chi, h] = \frac{1}{4\pi} \int d^2x \sqrt{-h} \left(|\nabla\chi|^2 + q\, R_h\, \chi - 4\pi\mu\, e^{2\beta\chi} \right). \tag{2.2.1}$$

This is the timelike Liouville action. Note that the kinetic term has a 'wrong sign' because in our conventions the metric has mostly positive signature. For this reason, χ is called 'timelike', by analogy with the field corresponding to the time coordinate of target spacetime on the two-dimensional world-sheet of a string [61, 115–119]. In the classical theory, $\beta = 1/q$, but we keep it as a free parameter in anticipation of quantum corrections.

The timelike nature of the Liouville field in the above action naturally makes the quantization of this theory rather complex. Indeed, many aspects of this theory are still not understood, from its spectrum and correlators to its symmetries and dualities. Luckily though, we want to study semiclassical corrections to the Einstein-Hilbert action, hence we do not expect to require the full power of the quantum theory. Moreover, the timelike Liouville action follows from the analytic continuation of the well-known Liouville action, often called spacelike Liouville to easily distinguish the two phases. Spacelike Liouville theory is the simplest non-trivial conformal field theory (CFT). It has been thoroughly explored and most of its results are well known and understood. It is characterized by a diagonal spectrum, made up of a continuum of unitary Verma modules. It's central charge is $c_L \geq 1$, so it is unitary, although the case $c = 1$ is subtle. Its correlation functions are smooth functions of the central charge and the conformal dimensions of the fields. Its two and three point functions are exactly known, the latter is given by the highly non-trivial DOZZ formula [120, 121], after Dorn-Otto and Zamolodchikov-Zamolodchikov.

The semiclassical results of interest to us, in the timelike regime, can be obtained from the analytic continuation of the analogous results of spacelike Liouville. The analytic continuation relating timelike and spacelike Liouville is

$$Q = iq\,, \qquad \varphi = i\chi\,, \qquad b = -i\beta\,. \tag{2.2.2}$$

The resulting action for spacelike Liouville is

$$I_L[\varphi, h] = -\frac{1}{4\pi}\int d^2x\,\sqrt{-h}\,\left(|\nabla\varphi|^2 + Q\,R_h\,\varphi + 4\pi\mu\,e^{2b\varphi}\right). \tag{2.2.3}$$

This action is real and has the right-sign kinetic term, this is why this theory is much better understood than its timelike sister. The Weyl transformations are given by

$$h_{\mu\nu} \to e^{2\xi(x)}\,h_{\mu\nu}\,, \quad \text{and} \quad \chi \to \chi - q\,\xi(x) \quad \text{or} \quad \varphi \to \varphi - Q\,\xi(x)\,. \tag{2.2.4}$$

Although the linear transformation of the Liouville field has a very natural interpretation when thought of as the Weyl transformation of the conformal factor, it is nevertheless very peculiar from the point of view of a conformal field theory. The charges of these transformations are determined by requiring Weyl invariance of the first two terms of the actions (2.2.1) and (2.2.3). Note though, that the these two terms are not strictly Weyl-invariant under (2.2.4) (as they are missing the variation of the Polyakov term), but their Weyl variation is field independent with this charge assignment. Hence the equations of motion are Weyl-invariant. This is the origin of the conformal invariance of Liouville theory. Since we are interested in the analogy to the higher-dimensional Weyl compensator, it is preferable to include the Polyakov term in (2.1.35), so that not just the equations of motion but the action itself is manifestly invariant under (2.2.4).

In the bootstrap approach to Liouville theory, where there is no action or gravitational intuition, the parameter b (or β) of the cosmological operator is not a priori related to the background charge Q (or q). They become related by imposing Weyl invariance, which classically it consistently requires that

$$Q = 1/b \qquad \text{or} \qquad q = 1/\beta\,. \tag{2.2.5}$$

The semiclassical limit $q \to \infty$ becomes also implemented by $\beta \to 0$. As we discuss in detail in chapter §3, the above relation is modified in the quantum theory because of the anomalous Weyl dimension of the cosmological constant operator.

The central charge of Liouville theory is given by

$$c_L = 1 + 6Q^2 \qquad \text{or} \qquad c_L = 1 - 6q^2\,. \tag{2.2.6}$$

The unit factor is the usual from a scalar field and is purely quantum, while the $6Q^2$ (or $6q^2$) is purely classical, coming form the lack of invariance of the first two terms of the action under the Weyl transformation (2.2.4). Since gravity requires the timelike regime, the semiclassical limit leads to a negative central charge. This indicates that the theory for the conformal factor is non-unitary, consistent with the negative kinetic term in (2.2.1). The appearance of a wrong-sign scalar field theory is to be expected taking into account that this is a theory for the gravitational field. Indeed, it is well known that the conformal factor of the metric has a 'wrong-sign' kinetic term [11]. For this reason, timelike Liouville theory is a better toy model [62–65] of four-dimensional gravity than the much-studied spacelike Liouville theory.

To discuss renormalization in the quantum theory, it is convenient to work in Euclidean space, obtained by doing a Wick rotation.[5] We denote the Lorentzian actions by I and the Euclidean actions by S. The Euclidean action for timelike Liouville is

$$S_{TL}[\chi, h] = \frac{1}{4\pi} \int d^2x \sqrt{h} \left(-|\nabla\chi|^2 - q\, R_h\, \chi + 4\pi\mu\, e^{2\beta\chi} \right) . \tag{2.2.7}$$

For spacelike Liouville it is

$$S_L[\varphi, h] = \frac{1}{4\pi} \int d^2x \sqrt{h} \left(|\nabla\varphi|^2 + Q\, R_h\, \varphi + 4\pi\mu\, e^{2b\varphi} \right) . \tag{2.2.8}$$

2.3 Liouville Gravity

In §2.1 and §2.2 we show how Liouville appears as a classical action for gravity in two dimensions. However, what is known as Liouville gravity [36, 122] is a quantum theory of gravity. In the following, we summarize how it shows up as a quantum effective action, good reviews include [57, 59], and adapt it later to our scenario.

Consider the Euclidean partition function for some matter CFT, with action $S_m[X^i]$, minimally-coupled to gravity, with a (bare) cosmological constant μ_0

$$\mathcal{Z} = \int \frac{\mathcal{D}_g g\, \mathcal{D}_g X^i}{V_{diff}}\, e^{-S_m[X^i] - \mu_0 \int d^2x \sqrt{g}} . \tag{2.3.1}$$

Since the matter sector is a CFT, the corresponding quantum effective action should yield the Weyl anomaly

$$T_m = \frac{c_m}{24\pi} R_g , \tag{2.3.2}$$

where c_m is the matter central charge. As explained in §1.1.3, in two dimensions the Weyl anomaly can be integrated to give the effective action. To perform this integration, we first go to conformally flat gauge, where the anomaly equation reads

$$-\frac{\delta S_{ef}}{\delta \Sigma_g} = \frac{c_m}{24\pi} \left(-2\, \nabla_\delta^2\, \Sigma_g \right) . \tag{2.3.3}$$

This integration can be readily performed, the details can be found in appendix §C, and yields the Euclidean Polyakov action

$$S_{ef}[g] = -\frac{c_m}{96\pi} \int \sqrt{g}\, R_g \frac{1}{\Box} R_g . \tag{2.3.4}$$

[5] To perform a Wick rotation in curved spacetime, it is convenient to regard Euclidean space and Lorentzian spacetime as different real slices of a complexified spacetime. Wick rotation is then a complex coordinate transformation $t = -it_E$ under which all tensors transform as usual. In Lorentzian spacetime, the path integral measure is e^{iI} and the spacetime measure is $\sqrt{-h}$. In Euclidean space the path integral measure is e^{-S} and the spacetime measure is $\sqrt{h_E}$. Using the fact that $\sqrt{-h_E} = -i\sqrt{h_E}$, we obtain $I \to -S$ with all tensors the same except $\sqrt{-h}$ replaced by $\sqrt{h_E}$.

The total matter effective action using the background field method is then

$$S_{ef,m}[X^i, g] = S_m[X^i] + \frac{c_m}{24}\, S_{Pol}[g]\,, \tag{2.3.5}$$

where we have named

$$S_{Pol}[g] = -\frac{1}{4\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g\,. \tag{2.3.6}$$

Just as in (2.1.33), under a Weyl transformation $g_{\mu\nu} = e^{2\Omega} h_{\mu\nu}$, the Euclidean Polyakov action transforms as

$$\frac{1}{4}\, S_{Pol}[g] = \frac{1}{4}\, S_{Pol}[h] - \frac{1}{4\pi}\int d^2x\sqrt{h}\, \Big((\nabla\Omega)^2 + R_h\,\Omega\Big)\,. \tag{2.3.7}$$

As mentioned in §1.1.3, the anomaly (2.3.2) comes from the lack of Weyl invariance of the path integral measure due to its metric dependence. Indeed, the measure of a scalar field is defined through the unit normalization of the Gaussian integral

$$\int \mathcal{D}_g \delta X^i\, e^{-\|\delta X^i\|_g^2} = 1\,, \tag{2.3.8}$$

where the norm in the exponent is defined through the inner product

$$\|\delta X^i\|_g^2 = (\delta X^i, \delta X^i)_g := \int d^2y\sqrt{g}\,\delta X^i \delta X^i\,, \tag{2.3.9}$$

which clearly depends on the metric of the background. With the norm so defined, the measure $\mathcal{D}_g X^i$ is invariant under diffeomorphisms and field translations $X^i \to X^i + f^i(x)$. This last property is the infinite-dimensional analog of the invariance of the integration measure under translation by a constant $dx = d(x+a)$, and it's very useful in perturbative field theory, in order to shift the integration variable. However, the measure is not Weyl invariant. From the above transformation of the effective action follows that the Weyl transformation of the measure is [36]

$$\mathcal{D}_g X^i = e^{-\frac{c_m}{6}\tilde{S}_{TL}[\Omega,h]}\, \mathcal{D}_h X^i\,, \tag{2.3.10}$$

where we have defined

$$\tilde{S}_{TL}[\Omega, h] = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\Big(-(\nabla\Omega)^2 - R_h\,\Omega\Big)\,. \tag{2.3.11}$$

We can now tackle the integral $\mathcal{D}_g g_{\mu\nu}$ over the gravitational field. This is an integral for each of the components of the metric, and the measure is normalized so as to satisfy

$$\int \mathcal{D}_g \delta g\, e^{-\frac{1}{2}\|\delta g\|_g^2} = 1\,, \tag{2.3.12}$$

where the norm is defined by

$$\|\delta g\|_g^2 = (\delta g, \delta g)_g := \int d^2y\sqrt{g}\,(g^{\mu\rho}g^{\nu\sigma} + c\, g^{\mu\nu}g^{\rho\sigma})\delta g_{\mu\nu}\delta g_{\rho\sigma}\,. \tag{2.3.13}$$

c is an arbitrary constant whose exact value does not affect the final measures, and whose only requirement is $c > -\frac{1}{2}$ so that the inner product is positive-defined. This inner product has the convenient property that the trace and traceless parts of the metric fluctuation are orthogonal. Indeed, let's decompose the metric fluctuation as

$$\delta g_{\mu\nu} = \delta g^{\perp}_{\mu\nu} + 2\delta\Omega\, g_{\mu\nu}\,, \tag{2.3.14}$$

where the traceless part satisfies $g^{\mu\nu}\delta g^{\perp}_{\mu\nu} = 0$. It follows then that

$$\|\delta g\|^2_g = \|\delta g^{\perp}\|^2_g + 8\,(1+2\,c)\|\delta\Omega\|^2_g\,, \tag{2.3.15}$$

where the inner product for the traceless part is the same as the above for $\delta g_{\mu\nu}$, and that of the Ω field is the same as (2.3.9) for a scalar field. This implies that we can factorize the corresponding measures as[6]

$$\mathcal{D}_g\delta g_{\mu\nu} = \mathcal{D}_g\delta\Omega\;\; \mathcal{D}_g\delta g^{\perp}_{\mu\nu}\,. \tag{2.3.16}$$

Further, since the variation of the traceless sector due to diffeomorphisms is

$$\delta g^{\perp}_{\mu\nu} = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu + g_{\mu\nu}\,\nabla_\sigma\xi^\sigma\,, \tag{2.3.17}$$

the measure can be written in terms of the integral over the diffeomorphism vector

$$\mathcal{D}_g\delta g^{\perp}_{\mu\nu} = \det\left(\frac{\partial\,\delta g^{\perp}_{\mu\nu}}{\partial\,\xi_\sigma}\right)\mathcal{D}_g\xi_\mu \;= \Delta_{FP}(g)\,\mathcal{D}_g\xi_\mu\,, \tag{2.3.18}$$

where Δ_{FP} is the familiar Faddeev-Popov determinant, which can be written in terms of the ghost fields (b,c) as

$$\Delta_{FP}(g) = \int \mathcal{D}_g(b,c)\,e^{-S_{bc}[b,c]}\,, \tag{2.3.19}$$

with $\mathcal{D}(b,c) \equiv \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\bar{b}\,\mathcal{D}\bar{c}$ and

$$S_{bc}[b,c] = \int d^2z\sqrt{g}\,\left(b\bar{\nabla}c + \bar{b}\nabla\bar{c}\right)\,. \tag{2.3.20}$$

The ghost action is Weyl invariant, but the measure is not, transforming analogously to the scalar fields measure as [36]

$$\mathcal{D}_g(b,c) = e^{-\frac{c_{bc}}{6}\tilde{S}_{TL}[\Omega,h]}\,\mathcal{D}_h(b,c)\,, \tag{2.3.21}$$

with the ghost sector central charge being $c_{bc} = -26$.

Back to the partition function, the integral $\mathcal{D}_g\xi_\mu$ cancels the volume of the gauge group in the denominator and gauges out the diffeomorphisms by fixing completely the transversal metric $g^{\perp}_{\mu\nu}$.[7] For later convenience, we will name the latter $h_{\mu\nu}$, so that the gauge-fixed metric reads $g_{\mu\nu} = e^{2\Omega}h_{\mu\nu}$. Using the Weyl transformation properties of the matter and ghost measures we are left with

$$\mathcal{Z} = \int \mathcal{D}_g\Omega\,\mathcal{D}_h(b,c)\,\mathcal{D}_hX^i\,e^{-\frac{c_m-26}{6}\tilde{S}_{TL}[\Omega,h]}\,e^{-S_m[X^i]-S_{bc}[b,c]-\mu_0\int d^2x\sqrt{h}\,e^{2\Omega}}\,. \tag{2.3.22}$$

[6]We ignore the integral over the moduli because of its irrelevance in our discussion.

[7]Notice that when assuming the split of the metric into a trace and a traceless components, we have implicitly assumed a conformal gauge, so we have fixed Weyl invariance. This gauge fixing needs not be accounted for in the path integral volume since its Jacobian is trivial.

In the critical strings scenario, $c_m = 26$, corresponding to a 26-dimensional target space where the string is moving. Further, $\mu_0 = 0$, allowing physical Weyl invariance as part of the gauge group. In the above partition function then, the dependence of the integrand on Ω drops out, and the integral over the conformal factor cancels the volume of the Weyl group with which the measure should be divided. This decoupling of the conformal factor is the consequence of the c-anomaly cancellation

$$c_m + c_{bc} = 0\,, \tag{2.3.23}$$

required because physical Weyl invariance is gauge. Away from criticality, the integral over the conformal factor becomes more involved. Not only because of the presence of the interacting cosmological constant term, but more importantly because the measure of the conformal factor is not Gaussian. Indeed, as it follows from (2.3.15), the latter is

$$\|\delta\Omega\|_g^2 = \int d^2x \sqrt{g}\,(\delta\Omega)^2 = \int d^2x \sqrt{h}\, e^{2\Omega}\,(\delta\Omega)^2\,. \tag{2.3.24}$$

This measure is diffeomorphism invariant, but the factor $e^{2\Omega}$ clearly spoils shift invariance in field space, which in this case means physical Weyl invariance. Instead, the Gaussian measure $\mathcal{D}_h\Omega$ defined with the norm

$$\|\delta\Omega\|_h^2 = \int d^2x \sqrt{h}\,(\delta\Omega)^2\,, \tag{2.3.25}$$

is translationally invariant, and therefore invariant under a physical Weyl transformation, which leaves the fiducial metric $h_{\mu\nu}$ unchanged.

It follows from this observation that the Jacobian of the measure $\mathcal{D}_g\Omega$ needs not be the same as that for the free-field measures above (2.3.10) and (2.3.21). However, it was shown [12, 13] that the Jacobian between the two measures has the same form as the above ones, but with 'renormalized' coefficients. I.e. that the non-Gaussian measure can be replaced by the Gaussian one at the expense of introducing additional local terms in the Lagrangian density, of the same functional form as the ones already present in the 'bare' action, which effectively behave as counterterms and hence renormalize the accompanying coefficients. This was the original DDK guess, from Distler and Kawai [13], and David independently [12]. It was developed further in [62, 123], and was partially proven later perturbatively by explicit heat kernel computation in [124].

To see how this works concretely, we follow DDK in their assumption that the total effective action has to be local, covariant and Weyl invariant, from which they guessed that the action has to be of the same functional form as the above $\tilde{S}_{TL}$ with the cosmological constant term, i.e.

$$\mathcal{Z} = \int \mathcal{D}_h\Omega\,\mathcal{D}_h(b,c)\,\mathcal{D}_h X^i\, e^{-\frac{1}{4\pi}\int d^2x\sqrt{h}\left(a(\nabla\Omega)^2 + d\,R_h\,\Omega + \mu\, e^{2b\Omega}\right)}\, e^{-S_m[X^i] - S_{bc}[b,c]}\,, \tag{2.3.26}$$

with coefficients to be determined by imposing the invariances just mentioned. Locality and covariance are manifest and do not constraint the coefficients. Weyl invariance, on the other hand, requires that the path integral is invariant under a Weyl transformation[8]

$$h_{\mu\nu} \to e^{2\,\sigma(x)}\, h_{\mu\nu} \qquad \Omega \to \Omega - \sigma(x)\,. \tag{2.3.27}$$

[8]Notice that Weyl invariance had already been fixed in assuming the conformal gauge. However,

The coefficients then have to be such that this invariance is preserved. Under this transformation, the measure $\mathcal{D}_h\Omega$ defined with the norm (2.3.25) transforms as the other Gaussian measures, since its invariant under the Ω field translation, and the transformation of the fiducial is like a physical Weyl transformation

$$\mathcal{D}_{e^{2\sigma}h}\Omega = e^{-\frac{1}{6}\tilde{S}_{TL}[\sigma,h]}\,\mathcal{D}_h\Omega\,, \tag{2.3.28}$$

where the effective central charge is 1, counting the conformal factor as a free scalar. Using this Weyl transformation and those of the scalar and ghost measures (2.3.10) and (2.3.21), invariance of the partition function under the above reparametrization requires the first two coefficients to be

$$a = d = \frac{25 - c_m}{6} \equiv Q^2. \tag{2.3.29}$$

Comparing with the Jacobian of the matter and ghosts part (2.3.22), we conclude that the coefficient in front of the effective action becomes renormalized with an added $+1$, from $c_m - 26$ to $c_m - 25$. The factor of 1 shows that the contribution from the conformal factor to the central charge is that of a quantum scalar. To get a canonically-normalized action, we can rescale the field by doing

$$\varphi(x) := Q\,\Omega(x)\,. \tag{2.3.30}$$

The effective classical action for the rescaled conformal factor then becomes

$$S = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\Big((\nabla\varphi)^2 + Q\,R_h\,\varphi + 4\pi\mu\,e^{2b\varphi}\Big)\,, \tag{2.3.31}$$

where we have renamed the parameter in the exponential $b/Q \to b$. This is exactly the Liouville action, the spacelike (2.2.8) or the timelike one (2.2.7) depending on the reality of Q. Since the central charge of this theory is $c_L = 1 + 6Q^2$, and from (2.3.29) follows that $c_m = 25 - 6Q^2$, then

$$c_m + c_{bc} + c_L = 0\,, \tag{2.3.32}$$

which ensures Weyl invariance at the quantum level.

Finally, the parameter b is determined in terms of Q by demanding the exponential operator to have conformal dimension $(1,1)$ on flat space, which guarantees its invariance in the action. Classically, we argued this leads to the relation $b = 1/Q$. The anomalous dimension of a vertex operator is easily calculated on a flat background, and gives

$$\gamma = \bar{\gamma} = -b^2\,. \tag{2.3.33}$$

Taking into account the classical transformation of φ under the Weyl transformation $\varphi \to \varphi - Q\xi$, then Weyl invariance leads to the well-known Liouville relation $b(Q-b) =$

as the metric $g_{\mu\nu}$ is invariant under a Weyl transformation, so has to be the path integral. Hence it cannot depend on the gauge parameter, i.e. on the fiducial metric chosen. So even once gauge fixed, we can write it in any other conformal gauge. This is why we chose to notation $g = e^{2\Omega}h$ for the gauge-fixed metric.

1, from which the classical $b = 1/Q$ gets corrected with negative odd powers of Q. We will come back to this computation and discuss it in detail in the next chapters.

The derivation just reviewed, where no classical gravitational action is included in the path integral, does not determine which of the two regimes, spacelike or timelike, is the one for the Liouville sector. However, the known classical timelike nature of the conformal factor forces the choice of the timelike regime. Therefore in the above results, the analytic continuation (2.2.2) to the timelike Liouville variables q and $\chi(x)$ has to be performed. In that case, Q becomes purely imaginary and $c_m = 25 + 6q^2$. It is worth mentioning though, that a lot of the literature on this topic considers the opposite regime, the one with little matter $c_m < 25$ and spacelike gravity. One of the reasons behind this choice is that, from a historical perspective, the spacelike theory is and has been much better understood for a long time. However, notice that if the spacelike regime were chosen for gravity, then c_m would become very negative in the semiclassical limit, hence requiring non-unitary matter. Also in that case, the Einstein constraint (2.1.17) would acquire a relative minus sign, which would not reproduce our conventional correlation of having positive curvature when having a positive cosmological constant. We will henceforth assume the timelike regime for gravity.

Through the above derivation, the matter central charge becomes dependent on the gravitational constant q^2. In the semiclassical limit, $c_L \to -\infty$ and $c_m \to +\infty$. It is hard to explain why this limit, which makes gravity weakly coupled, would require such a great amount of matter. In the next section we show how c_m becomes q-independent by introducing the Einstein-Hilbert term in the partition function, and even more, how the amount of conformal matter can be arbitrarily chosen while still preserving anomaly cancellation.

2.4 Gravitational Path Integral

The above derivation regards the gravitational action as classically trivial, and hence does not include it in the path integral. However, our point of view is that the limit to two dimensions that allows for a non-trivial constraint, does so by retaining a finite Einstein-Hilbert term in the action. We therefore consider the Euclidean gravitational path integral

$$\mathcal{Z} = \int \frac{\mathcal{D}_g g\, \mathcal{D}_g X^i}{V_{diff}}\, e^{-S_G[g]}\, e^{-S_m[X^i]}\,, \tag{2.4.1}$$

where we introduce the Euclidean gravitational action with a bare cosmological constant

$$S_G[g] = -\frac{q^2}{4\pi}\int d^2x\,\sqrt{g}\,\left(\frac{1}{4}R_g\frac{1}{\Box}R_g - \frac{4\pi\mu_0}{q^2}\right)\,. \tag{2.4.2}$$

Taking into account the variation of the above action and the transformation of all the integral measures, the analogous of the DDK proposal for the effective action is

$$\mathcal{Z} = \int \mathcal{D}_h\Omega\, \mathcal{D}_h(b,c)\, \mathcal{D}_h X^i\, e^{\frac{q^2}{16\pi}\int\sqrt{h}\,R_h\frac{1}{\Box}R_h}\, e^{-\frac{1}{4\pi}\int d^2x\sqrt{h}\,\left(a(\nabla\Omega)^2 + d\,R_h\,\Omega + \mu\, e^{2b\Omega}\right)}\, e^{-S_m - S_{bc}}\,. \tag{2.4.3}$$

In our treatment, q is defined a priori from the rescaling of Newton's constant (2.1.2). Invariance of the above path integral under fiducial Weyl transformations (2.3.27) requires now

$$a = d = \frac{25 - c_m - 6q^2}{6} \,. \tag{2.4.4}$$

As in the usual treatment, the Jacobian of the conformal factor measure renormalizes the central charges by adding a factor of 1. Further in this case, the presence of the classical Einstein-Hilbert action shifts them with the factor $-6q^2$. Analogously to the derivation above, we demand the coefficient in front of the effective action to be $-q^2/4\pi$, so that the effective Liouville action has central charge $c_L = 1 - 6q^2$. Anomaly cancellation is then satisfied with

$$c_m = 25 \,, \quad c_G = 1 \,. \tag{2.4.5}$$

The central charge of the gravitational sector being $c_G = 1$ (and not $1 - 6q^2$) becomes thus a consequence of retaining the Einstein-Hilbert term in the classical action (2.1.35), since only then is the action invariant under Weyl transformations.

This makes c_m independent of the parameter q. This parametrization of the central charges is more sensible when the parameter q is interpreted as the gravitational coupling constant, since the amount of matter should not depend on the latter. The semiclassical limit corresponds to large q, independent of c_m. This is more natural from the point of view of the continuation to d dimensions.

Assuming anomaly cancellation, the resulting partition function is

$$\mathcal{Z} = \mathcal{Z}_m[h]\, \mathcal{Z}_{bc}[h]\; e^{\frac{q^2}{16\pi}\int \sqrt{h}\, R_h \frac{1}{\Box} R_h} \int \mathcal{D}_h\Omega\, e^{\frac{q^2}{4\pi}\int dx\, \left((\nabla\Omega)^2 + R_h\,\Omega - \frac{4\pi\mu}{q^2}\, e^{2\beta\Omega}\right)} \,, \tag{2.4.6}$$

where again $dx = d^2x\,\sqrt{h}$. We can factorize the matter and ghost partitions functions because they no longer depend on the conformal factor. We can factorize also the purely background piece, since diffeomorphisms have been integrated out. We can now integrate the conformal factor. The classical effective action in the exponent is (proportional to) timelike Liouville. The exponential interaction from the cosmological term makes this path integral highly non-trivial. However, it turns out [12–17] that simple normal ordering already removes all the ultraviolet divergences of the theory. In other words, the cosmological constant operator can be renormalized in the much simpler theory of a free boson. This remarkable feature of Liouville theory is going to be key in our computation of the gravitational dressing of the cosmological constant. At this point, we just use it to be able to drop the exponential interaction from the path integral and perform the Gaussian integral, since it tells us that the cosmological operator will renormalize independently, and can therefore be added to the effective action resulting from the Gaussian integration.

To obtain the 1PI effective action, we use the background field method. For this, we decompose the field into a background component and a quantum fluctuation $\hat{\Omega}$; in the classical action we then do the replacement $\Omega \to \Omega + \hat{\Omega}$. The integral of the conformal factor becomes

$$\mathcal{Z}_\Omega[\Omega, h] = \int \mathcal{D}_h\hat{\Omega}\, e^{\frac{q^2}{4\pi}\int dx\, \left((\nabla\Omega)^2 + R_h\,\Omega\right)}\, e^{\frac{q^2}{4\pi}\int dx\, \left((\nabla\hat{\Omega})^2 + R_h\,\hat{\Omega} + 2\nabla\Omega\nabla\hat{\Omega}\right)} \,. \tag{2.4.7}$$

Further, we choose the background such that $R_h\,\hat{\Omega} + 2\nabla\Omega\nabla\hat{\Omega} = 0$, i.e. so that the terms linear in the quantum perturbation, namely the tadpoles, cancel. The integral then becomes a homogeneous Gaussian, which results into

$$\mathcal{Z}_\Omega[\Omega, h] = \frac{e^{\frac{q^2}{4\pi}\int dx\,\left((\nabla\Omega)^2 + R_h\,\Omega\right)}}{\sqrt{\det\left(-\nabla_h^2\right)}} = e^{\frac{1}{96\pi}\int\sqrt{h}\,R_h\frac{1}{\Box}R_h}\; e^{\frac{q^2}{4\pi}\int dx\,\left((\nabla\Omega)^2 + R_h\,\Omega\right)}\,, \qquad (2.4.8)$$

where the determinant of the Laplacian exponentiates to the Polyakov action as shown in the previous section. From the above we conclude that the quantum effective action for the conformal factor of the metric is

$$S_{ef,\Omega}[\Omega, h] = -\frac{1}{96\pi}\int\sqrt{h}\,R_h\frac{1}{\Box}R_h - \frac{q^2}{4\pi}\int dx\,\Big((\nabla\Omega)^2 + R_h\,\Omega\Big)\,. \qquad (2.4.9)$$

Adding the Einstein-Hilbert piece, the gravitational quantum effective action is

$$S_{ef,K}[\Omega, h] = -\frac{1+6q^2}{96\pi}\int\sqrt{h}\,R_h\frac{1}{\Box}R_h - \frac{q^2}{4\pi}\int dx\,\Big((\nabla\Omega)^2 + R_h\,\Omega\Big)\,. \qquad (2.4.10)$$

The total partition function finally is the product of the three sectors involved

$$\mathcal{Z} = \mathcal{Z}_m[X^i, h]\,\mathcal{Z}_{bc}[b, c, h]\mathcal{Z}_K[\Omega, h]\,, \qquad (2.4.11)$$

where the gravitational partition function is

$$\mathcal{Z}_K[\Omega, h] = e^{-S_{ef,K}[\Omega,h]}\,. \qquad (2.4.12)$$

This derivation makes clear that the Liouville action, which already appears classically, is also the quantum effective action for the conformal factor of the metric, up to the renormalization of the cosmological constant operator that we will compute in the next chapter.

We now check explicitly the central charges of the effective actions by computing the trace of the resulting quantum momentum tensors. The momentum tensor for the conformal factor follows from doing the variation of the action $S_{ef,\Omega}$ with respect to the metric $h_{\mu\nu}$. The variation of the Ω-dependent terms gives

$$T^{\Omega,clas}_{\mu\nu}(\Omega, h) = -\frac{q^2}{2\pi}\left(\nabla_\mu\Omega\nabla_\nu\Omega - \frac{1}{2}h_{\mu\nu}\,(\nabla\Omega)^2 - \Big(\nabla_\mu\nabla_\nu - h_{\mu\nu}\,\nabla_h^2\Big)\,\Omega\right)\,, \qquad (2.4.13)$$

which is exactly the classical momentum tensor $T^{\Omega}_{\mu\nu}$ (2.1.14). The quantum correction to this momentum tensor comes from the Polyakov term that the quantum action acquires through the renormalization of Ω. We postpone the full variation of the Polyakov action to section §3.4. For now we are just interested in its trace, which from the integrated trace anomaly (2.3.4) we know is

$$T = \frac{1}{24\pi}\,R_h\,. \qquad (2.4.14)$$

The trace of the momentum tensor for the conformal factor, after use of the equation of motion $2\nabla_h^2\Omega = R_h$, is

$$T^\Omega = \frac{1-6q^2}{24\pi}\, R_h\,. \tag{2.4.15}$$

which confirms that the central charge is that of timelike Liouville theory $c_\Omega = c_L = 1-6q^2$. The variation of the Einstein-Hilbert piece, will further contribute with a factor

$$T = \frac{6q^2}{24\pi}\, R_h\,, \tag{2.4.16}$$

so that the trace of the total gravitational quantum momentum tensor is

$$T^K = \frac{1}{24\pi}\, R_h\,, \tag{2.4.17}$$

which shows that the central charge of the gravitational sector is $c_G = 1$ as advocated previously. The trace of the total momentum tensor is

$$T = T^m + T^{bc} + T^K = \frac{c_m - 26 + 1}{24\pi}\, R_h\,, \tag{2.4.18}$$

confirming that $c_m = 25$ indeed cancels the anomaly.

Since the action and the partition function are gauge invariant, we can go to the physical gauge $\Omega = 0$ and $h_{\mu\nu} = g_{\mu\nu}$, and we find

$$S_{ef,K}[g] = -\frac{1+6q^2}{96\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g\,. \tag{2.4.19}$$

Comparing this quantum effective action to its classical predecessor (2.4.2), it seems that the classical gravitational coupling constant $6q^2$ gets shifted by a unit factor. However, one has to take into account that the matter and ghost sectors also contribute to the renormalization of the gravitational coupling through the anomaly, i.e. their effective actions also contain a purely gravitational Polyakov term[9]

$$\mathcal{Z} = \mathcal{Z}_m[g]\, \mathcal{Z}_{bc}[g]\, e^{\frac{1+6q^2}{96\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g} = \mathcal{Z}_m[\delta]\, \mathcal{Z}_{bc}[\delta]\, e^{\frac{q^2}{16\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g}\,. \tag{2.4.21}$$

where $\mathcal{Z}_m[\delta]$ and $\mathcal{Z}_{bc}[\delta]$ depend only on the matter and ghost background fields. The total gravitational effective action is then the same as the classical one

$$S_{ef,K}[g] = -\frac{q^2}{16\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g\,. \tag{2.4.22}$$

To complete the discussion, we can compute the trace of the total momentum tensor in the $g_{\mu\nu}$ metric. Since $\mathcal{Z}_m[\delta]$ and $\mathcal{Z}_{bc}[\delta]$ only depend on the background fields, they lead to traceless momentum tensors, and we get

$$T = T^K = \frac{6q^2}{24\pi}\, R_g\,. \tag{2.4.23}$$

[9] Alternatively, if we fix the conformally flat gauge $\Omega = \Sigma_g$ and $h_{\mu\nu} = \delta_{\mu\nu}$, we can confirm that we get the same result

$$\mathcal{Z} = \mathcal{Z}_m[\delta]\, \mathcal{Z}_{bc}[\delta]\, e^{\frac{q^2}{4\pi}\int \sqrt{\delta}\,(\nabla\Omega)^2} = \mathcal{Z}_m[\delta]\, \mathcal{Z}_{bc}[\delta]\, e^{\frac{q^2}{16\pi}\int \sqrt{g}\, R_g \frac{1}{\Box} R_g}\,. \tag{2.4.20}$$

This is not traceless because the theory does not satisfy physical Weyl invariance, and it needs not to. This is actually not an anomaly, it simply reflects the variation of the classical gravitational action.

2.4.1 Arbitrary Central Charges

The above derivation allowed to disentangle the amount of matter c_m from the gravitational coupling constant q^2, so that the semiclassical limit does not require an infinite amount of it. However, it seems to force $c_m = 25$. Can we conclude that the amount of matter can generally be predicted from Weyl anomaly cancellation? The answer is no, and this is due to the arbitrariness of the gravitational constant.

Going back to the DDK-inspired classical effective action for the conformal factor (2.4.3), Weyl invariance of the partition function imposes the coefficients of the effective action to satisfy the relation (2.4.4). Instead of choosing to fix the coefficients $a = d$ to $-q^2/4\pi$, as we did to make contact with the classical action and with what is done in the original derivation, we can choose to rename these coefficients with an arbitrary new coupling constant $\tilde{q}^2$, such that

$$a = d = \frac{25 - c_m - 6q^2}{6} \equiv -\tilde{q}^2 \,, \tag{2.4.24}$$

assuming as well though, that $\tilde{q}^2 > 0$ so that the action lands up in the timelike regime. In this case, the effective action for the conformal factor has a central charge $c_L = 1 - 6\tilde{q}^2$, and the total gravitational sector after addition of the Einstein-Hilbert term $c_G = 1 + 6q^2 - 6\tilde{q}^2$, coming from the gravitational effective action

$$S_{ef,K}[\Omega, h] = -\frac{1 + 6q^2}{96\pi} \int \sqrt{h}\, R_h \frac{1}{\Box} R_h - \frac{\tilde{q}^2}{4\pi} \int \sqrt{h}\, \Big((\nabla\Omega)^2 + R_h\, \Omega \Big) \,. \tag{2.4.25}$$

From the above relation (2.4.24) follows that $c_m = 25 - 6q^2 + 6\tilde{q}^2$, and again Weyl anomaly is canceled as $c_m + c_{bc} + c_G = 0$. Adding the matter and ghost effective actions to the above $S_{ef,K}$, with coefficient $c_m + c_{bc} = -1 - 6q^2 + 6\tilde{q}^2$ leads then to the same total effective action as before (2.4.22) but with the new gravitational coupling

$$S_{ef,K}[g] = -\frac{\tilde{q}^2}{16\pi} \int \sqrt{g}\, R_g \frac{1}{\Box} R_g \,. \tag{2.4.26}$$

Since the gravitational constant is the dimensionless parameter, its name is arbitrary, and the above shift has no effect. This ambiguity is the one that allows to have an arbitrary amount of matter, since now c_m depends on two tunable constants q and $\tilde{q}$. We can then take the semiclassical limit $\tilde{q}^2 \to \infty$, while keeping the difference $q^2 - \tilde{q}^2$, and therefore c_m, fixed to an arbitrary finite value.

Chapter 3

Quantum Momentum Tensor from the Anomalous Cosmological Constant

We turn now to the renormalization of the cosmological constant operator, which we temporarily dropped in the last sections to show how the Liouville action was the quantum effective one for the conformal factor of the metric. We now incorporate it back. The aim of this chapter is to compute its gravitational dressing in the effective action and the resulting momentum tensor.

3.1 Renormalization of the Cosmological Constant Operator

The cosmological constant operator $e^{2\beta\chi}$ is a composite operator and must be renormalized in the quantum theory. It is most convenient to carry out this renormalization in spacelike Liouville theory in Euclidean space. The analytic continuation of these results to timelike Liouville and its Lorentzian interpretation will be discussed later.

The Liouville action contains the non-polynomial exponential interaction. The cosmological constant operator should in principle be regularized in the interacting theory defined by this action. However, as already mentioned in §2.4, it is well known [12–17] that normal ordering removes all short-distance divergences of the theory. In other words, the anomalous dimension of the cosmological constant operator in the fully interacting theory is the same as for a much simpler theory of a free boson. Ultimately, this claim is justified by exact results obtained using the conformal bootstrap [58, 121, 125–127] and agrees with the KPZ critical exponents (from Knizhnik, Polyakov and Zamolodchikov) [128] computed using matrix models [129–131], light-cone quantization [122], and canonical quantization [132–135]. Therefore, we can perform the renormalization of the cosmological constant operator in the free theory.

Anomalous dimensions of exponentials of free fields have been studied extensively in string theory and two-dimensional quantum gravity [36]. By the state-operator correspondence, such exponentials correspond to momentum eigenstates. To obtain the anomalous dimension, it is usually adequate to perform renormalization in flat space by normal ordering [136, 137]. However, we are interested here in all three components of the quantum momentum tensor given by metric variation of the quantum effective action. We thus require the metric dependence of the renormalized operator for an arbitrary curved metric.

Renormalization of the cosmological constant operator in curved spacetime has been well studied in the literature [36,136,138–140]. Since it is of crucial importance for our conclusions, we present below a somewhat lengthy derivation taking into account some of the subtleties both in the UV and in the IR. New conceptual questions of interpretation arise in continuing the Euclidean computations to Lorentzian spacetime which we discuss in section §3.2. We then write down the quantum effective action for this renormalized term, compute the quantum momentum tensor and check explicitly that the Ward identities are satisfied.[1]

Consider the correlation function of exponentials in a free theory in a curved background

$$\mathcal{A}^0(x_1,\ldots,x_n) := \langle\prod_{i=1}^n e^{2a_i\varphi(x_i)}\rangle = \int D\varphi\, e^{-S[\varphi,h]}\prod_{i=1}^n e^{2a_i\varphi(x_i)} = \int D\varphi\, e^{-S[\varphi,h]+\int d^2x\sqrt{h}\,J(x)\,\varphi(x)}\,. \tag{3.1.1}$$

The superscript '0' is a reminder that this is a bare correlation function with the action

$$S[\varphi,h] = \frac{1}{4\pi}\int d^2x\sqrt{h}\,|\nabla\varphi|^2\,,\quad \text{and}\quad J(x) = 2\sum_{i=1}^n a_i\,\delta^{(2)}(x,x_i). \tag{3.1.2}$$

We have set $Q=0$ in (2.2.8) so that the Liouville field is neutral under (2.2.4). While the classical dimension depends on Q, the anomalous dimension of our interest is independent of Q. Using Wick's theorem one obtains

$$\mathcal{A}^0(x_1,\ldots,x_n) = \exp\left[4\pi\sum_{i,j}a_ia_j\,G_h(x_i,x_j)\right] \tag{3.1.3}$$

where G_h is the scalar Green function[2]

$$-\nabla_h^2 G_h(x,y) = \delta_h^{(2)}(x,y) = \frac{\delta^{(2)}(x-y)}{\sqrt{h}}\,. \tag{3.1.4}$$

In general, the Green function for an arbitrary metric $h_{\mu\nu}$ is hard to compute. However, in two dimensions, $\nabla_h^2 = e^{-2\Sigma_h}\nabla_\delta^2$, and hence the (non-compact) Green equation is Weyl invariant. In the conformally flat gauge then, the Green function is given by the flat space Green function. The latter is known to be infrared divergent.[3] To regulate this divergence, consider the class of asymptotically flat metrics so that $\Sigma_h(x)\to 0$ as $|x|\to\infty$. Introduce an IR cutoff by restricting $\mathbb{R}^2$ to a disk in the flat metric

$$|x|^2 := \delta_{\mu\nu}x^\mu x^\nu \le R^2 := 1/m^2 \tag{3.1.5}$$

and impose Dirichlet boundary conditions at $|x|=R$. The resulting Green function is

$$G_h(x,y) = G_\delta(x,y) = -\frac{1}{4\pi}\ln\left(m^2|x-y|^2\right)\quad \text{for}\quad x\neq y\,, \tag{3.1.6}$$

[1] In section §3.6 we reverse the logic and compute the stress tensor from the anomalous trace using the Ward identities.

[2] We define the Laplacian as $-\nabla_h^2$ so that it is a positive operator.

[3] On a compact manifold there is no need for an IR regulator but the Laplacian has a zero mode which has to be treated carefully [138, 140].

where we have ignored the contribution from image charges which are negligible in the limit $R \to \infty$. The boundary condition and the Green equation are both invariant under Weyl transformations that asymptote to unity for $|x| \to \infty$. For all metrics related by such Weyl transformations, the Green function is the same as above. The Green function is invariant also under constant Weyl transformations if we scale the IR cut-off at the same time.

Naively, the Weyl invariance of the Green function implies that the IR-regulated n-point function is Weyl invariant. But on general grounds, one expects that regularization of UV divergences will introduce a dependence on the metric that can violate the Weyl symmetry. To compute this anomalous Weyl variation, we rewrite the n-point function as

$$\mathcal{A}^0(x_1, \ldots, x_n) = \prod_i e^{4\pi a_i^2 \, G_h^0(x_i, x_i)} \cdot \exp\left[4\pi \sum_{i \neq j} a_i a_j \, G_h(x_i, x_j)\right]. \tag{3.1.7}$$

As it stands, this is only a formal expression that is not well defined. The pre-factor is a product over exponentials of Green functions evaluated at the same points, which are divergent. We have therefore added a superscript to underscore the fact that the coincident Green functions are bare quantities. The origin of the UV divergence is clear from (3.1.7): each exponential is a composite operator involving products of the fundamental scalar field with divergent self-contractions. This is shown diagrammatically in Fig.3.1 for a two-point function.

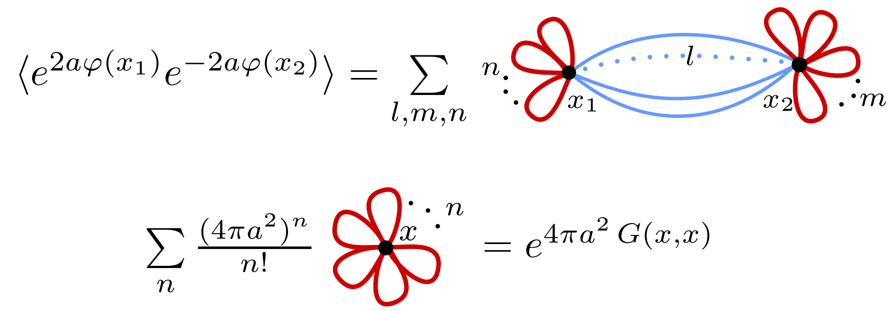

Figure 3.1: *The red daisies at each point come from self-contractions. Each petal of a daisy is a coincident Green function and the sum over these daisies gives a divergent exponential.*

To regulate this divergence we rewrite the coincident Green function as

$$G_h^{\varepsilon}(x,x) = \int d^2y \sqrt{h} \, \delta_h^{(2)}(x,y) \, G_h(y,x) = \int d^2y \sqrt{h} \, K_h(x,y;\varepsilon) \, G_h(y,x) \,, \tag{3.1.8}$$

where we have replaced the delta function by the heat kernel with a short time cutoff[4] ε since

$$K_h(x,y;\varepsilon) \to \delta_h^{(2)}(x,y) \quad \text{as} \quad \varepsilon \to 0 \,. \tag{3.1.9}$$

[4]We use ε for the short-time cutoff and ϵ for the dimensional regulator.

The short-time expansion of the heat kernel can be obtained using standard methods in terms of Seeley-de Witt coefficients. The computations are simpler in the conformally flat gauge. The leading behavior is given by

$$K_h(x, y; \varepsilon) = \frac{1}{4\pi\varepsilon} \exp\left[\frac{e^{2\Sigma_h(x)}|x-y|^2}{4\varepsilon}\right](1+\ldots) \ . \tag{3.1.10}$$

The regularization separates the two points by a distance of order $\sqrt{\varepsilon}$. Using this expansion one obtains

$$G_h^{\varepsilon}(x, x) = \frac{1}{2\pi}\Sigma_h(x) - \frac{1}{4\pi}\ln(4\, e^{-\gamma}\, m^2\varepsilon)\,, \tag{3.1.11}$$

where γ is the Euler-Mascheroni constant. More details and an alternative derivation in dimensional regularization are given in appendix §D.

Using the expression for the conformal factor in terms of the full metric (2.1.30), the manifestly coordinate-invariant and regularized coincident Green function is then

$$G_h^{\varepsilon}(x, x) = \frac{1}{4\pi}\int d^2y\,\sqrt{h}\, G_h(x, y)\, R_h(y) - \frac{1}{4\pi}\ln(4\, e^{-\gamma} m^2\,\varepsilon)\,. \tag{3.1.12}$$

Renormalization now consists in simply adding $\ln(4\, e^{-\gamma} M^2\,\varepsilon)/4\pi$ so the divergent term with ε is removed. Since ε is a pure number[5] independent of coordinates and the metric, this procedure is manifestly coordinate invariant and local. Renormalization has introduced an arbitrary scale M. The renormalized coincident Green function is then given by

$$G_h(x, x) = \frac{1}{4\pi}\int d^2y\,\sqrt{h}\, G_h(x, y)\, R_h(y) + \frac{1}{4\pi}\ln(\frac{M^2}{m^2}) \tag{3.1.13}$$

which however is not a local functional of the metric,[6] the non-locality being introduced by the conformal factor of the metric. The renormalized n-point function can now be obtained simply by replacing the bare coincident Green function $G_h^0(x, x)$ in (3.1.7) by the renormalized coincident Green function (3.1.13). The resulting answer is finite and independent of ε, but one which depends on the renormalization scale. It corresponds to a multiplicative renormalization of each of the bare exponentials

$$[e^{2a\varphi(x)}]_h^{\varepsilon} := e^{-a^2\ln(4\, e^{-\gamma} M^2\varepsilon)}[e^{2a\varphi(x)}]_h := Z_a(M)\,[e^{2a\varphi(x)}]_h \tag{3.1.14}$$

where the notation $[\mathcal{O}]_h^{\varepsilon}$ indicates an operator $\mathcal{O}$ regularized using the metric h and cutoff ε, whereas $[\mathcal{O}]_h$ without superscript indicates the renormalized version of the same operator. We have defined the multiplicative operator renormalization $Z_a(M)$ to make contact with the usual flat-space renormalization. Even though this procedure is manifestly local and coordinate invariant, it is not Weyl invariant because it depends on

[5] It is convenient to regard all quantities including spacetime coordinates and mass scales like m as dimensionless, measured in units of the fundamental UV scale M_0 introduced earlier, which we can set to one.

[6] The non-locality of the conformal factor, and so of models that depend on it, was emphasized earlier in [141].

the background metric. With this renormalization prescription, the n-point function renormalized using the h metric is given by

$$\mathcal{A}_h(x_1, \ldots, x_n) = m^{-2\,(\sum_i a_i)^2} \prod_i (M e^{\Sigma_h(x_i)})^{2a_i^2} \exp\left[-\sum_{i\neq j} a_i a_j \ln|x_i - x_j|^2\right] . \quad (3.1.15)$$

The first factor simply imposes momentum conservation: the correlation function vanishes unless the total momentum is zero.[7] This is to be expected because momentum is the charge corresponding to a continuous global symmetry $\varphi \to \varphi + c$ which cannot be spontaneously broken in two dimensions by the Coleman-Mermin-Wagner theorem. Imposing momentum conservation, the final expression for the renormalized n-point function is given by

$$\mathcal{A}_h(x_1, \ldots, x_n) \quad = \quad \prod_i e^{2a_i^2 \Sigma_h(x_i)} \prod_{i\neq j} \frac{1}{(M|x_i - x_j|)^{2a_i a_j}} \, . \quad (3.1.16)$$

For $\Sigma_h(x) = -\ln M$, we obtain the familiar answer from flat space conformal field theory.

The n-point correlators renormalized in two different metrics are related by

$$\mathcal{A}_{h'}(x_1, \ldots, x_n) = \prod_i e^{2a_i^2 (\Sigma_{h'}(x_i) - \Sigma_h(x_i))} \mathcal{A}_h(x_1, \ldots, x_n) \, . \quad (3.1.17)$$

This follows from (3.1.16) and the fact that the non-coincident Green function given by (3.1.6) is independent of the metric. Interpreting the correlation function in operator language, we conclude that the exponential operator renormalized using the metric h' is related to the one renormalized using the metric h by

$$[e^{2a\hat{\varphi}(x)}]_{h'} = e^{2a^2 (\Sigma_{h'}(x) - \Sigma_h(x))} \, [e^{2a\hat{\varphi}(x)}]_h \quad (3.1.18)$$

where the hatted variable denotes a quantum operator rather than a classical field.

The cosmological constant operator in Liouville theory corresponds to $a = b$. The Weyl transformation of the renormalized cosmological constant operator has an anomalous contribution from (3.1.18) as computed above because of the implicit dependence on the metric through renormalization. In addition, for nonzero Q, there is also a classical contribution because of the explicit dependence on φ which transforms as in (2.2.4). The net Weyl transformation is

$$[e^{2b\hat{\varphi}(x)}]_h \to e^{-(2bQ - 2b^2)\,\xi(x)} \, [e^{2b\hat{\varphi}(x)}]_h \, . \quad (3.1.19)$$

We interpret $2bQ$ as the classical Weyl weight and $-2b^2$ as the anomalous Weyl weight.

[7] Operators with positive Weyl weight are defined only for $a_i = ik_i$ for real k_i. They correspond to normalizable charge eigenstates in the Hilbert space. The prefactor is then a positive power of m which vanishes as $m \to 0$. For operators with negative weight the correlation functions diverge at large separation. The corresponding states are not normalizable and have to be interpreted using an analog of the Gelfand triple [58].

3.2 Lorentzian Interpretation

At a formal level, analytic continuation to timelike Liouville in Lorentzian spacetime is straightforward using (2.2.2) and a Wick rotation. We will use the same covariant expression for Σ_h

$$\Sigma_h(x) = \frac{1}{2}\int d^2y\,\sqrt{-h}\,G_h(x,y)\,R_h(y) \tag{3.2.1}$$

where the Green function[8] is the solution of the Lorentzian Green equation without any i

$$-\nabla^2 G_h(x,y) = \delta_h^{(2)}(x,y) = \frac{\delta^{(2)}(x-y)}{\sqrt{-h}}\,. \tag{3.2.2}$$

Physical interpretation in the Lorentzian signature is subtle. We discuss below some of the puzzles that one encounters in interpreting the Lorentzian action and their resolutions.

- *Choice of the Green function:* The Lorentzian Green function appearing in the expression (3.2.1) for the Σ_h depends on the choice of the boundary condition. The Euclidean Green function in chapter §3 is unique and usually it would continue to the Feynman propagator under a Wick rotation. However, one could equally well choose retarded or advanced boundary conditions, which would lead to very different physics. Which of these Green functions is physically relevant? We are eventually interested in using the quantum effective action to study classical evolution equations. Appearance of Feynman propagators in the effective action would lead to non-causal dynamics because it would involve negative energy modes traveling backward in time. Such an effective action would be unphysical. However, in time-dependent situations as in cosmology, the in-vacuum and the out-vacuum are in general different. A natural object to consider is not the usual in-out effective action, but the in-in effective action in the Schwinger-Keldysh formalism [142, 143]. It is known that one can obtain the in-in effective action from the in-out one by replacing Feynman propagators by retarded Green functions [30, 144–146].
- *Choice of the vacuum:* In canonical formalism in the Lorentzian theory, the choice of the metric used for renormalization corresponds to the choice of the vacuum, as we discuss below. We choose the Minkowski metric $\eta_{\mu\nu}$ as a reference metric, which corresponds to $\delta_{\mu\nu}$ under Euclidean continuation. Continuation of (3.1.18) gives the following equation for the renormalized cosmological constant operator in Lorentzian spacetime

$$[e^{2\beta\hat{\chi}(x)}]_h = e^{-2\beta^2\,\Sigma_h(x)}\,[e^{2\beta\hat{\chi}(x)}]_\eta\,. \tag{3.2.3}$$

As it stands, (3.2.3) is an operator equation with a quantum operator $\hat{\chi}(x)$ in the exponent. In the cosmological term in the quantum effective action, we would like to regard $\chi(x)$ as a classical field. This is achieved using the background field method by replacing $\hat{\chi}(x)$ by $\chi(x)+\hat{\chi}_q(x)$. The un-hatted variable is a classical

[8] A Wick rotation would give a factor of i for the measure and a factor of $-i$ for the Green function. In (3.2.1) and (3.2.2) we drop both factors.

background field and the hatted variable is the fluctuating quantum field. For the free action, $\hat{\chi}_q(x)$ is also a free field[9]. We then have the relation

$$[e^{2\beta\,(\chi(x)+\hat{\chi}_q(x))}]_h = e^{2\beta\chi(x)-2\beta^2\,\Sigma_h(x)}\,[e^{2\beta\hat{\chi}_q(x)}]_\eta\,. \tag{3.2.4}$$

The passage to the in-in effective action still requires a choice of the in-vacuum to compute the in-in matrix element of this operator. Which state should one choose as the in-vacuum? The choice of the vacuum is a deep and unresolved question in cosmology since it concerns the initial state in which the universe 'got prepared'. Even in a free theory, there are many possible Fock vacua that are *a priori* equally valid as initial states. In general, the Fock vacuum depends on the metric used to define the Klein-Gordon inner product. This inner product is essential to obtain the division of the modes of the Klein-Gordon operator (on a globally hyperbolic spacetime) into positive-frequency and negative-frequency modes, and hence to determine the class of annihilation operators that should annihilate the vacuum. A conventional choice is the 'Bunch-Davies' vacuum $|\eta\rangle$, obtained using the Klein-Gordon inner product defined with respect to the flat Minkowski metric η. This would coincide with the conformal or the adiabatic vacuum [149].

In summary, a physically reasonable interpretation of the Lorentzian continuation requires that we consider the in-in quantum effective action and hence use retarded Green functions. The cosmological term can be regarded as the expectation value in the η-vacuum of the operator renormalized using the $h_{\mu\nu}$ metric. We denote this *classical* quantity by $\mathcal{O}_h^\beta$:

$$\mathcal{O}_h^\beta := \langle\eta|[e^{2\beta(\chi(x)+\hat{\chi}_q(x))}]_h|\eta\rangle = e^{2\beta\chi(x)-2\beta^2\,\Sigma_h(x)}\langle\eta|[e^{2\beta\hat{\chi}_q(x)}]_\eta|\eta\rangle = e^{2\beta\chi(x)-2\beta^2\,\Sigma_h(x)}\,, \tag{3.2.5}$$

where in the second equality we have used the fact that, in the Hamiltonian formalism, renormalization in the metric η corresponds to normal ordering with respect to the η-vacuum, and hence the expectation value of the exponential equals one.

With these ingredients, the integrated renormalized cosmological term in the quantum effective action takes the final form

$$\Gamma_\Lambda[\chi, h] = -\mu\int d^2x\,\sqrt{-h}\,\mathcal{O}_h^\beta\,. \tag{3.2.6}$$

The Weyl transformation of $\mathcal{O}_h^\beta$ is given by

$$\mathcal{O}_h^\beta \to e^{-(2\beta q+2\beta^2)\,\xi(x)}\,\mathcal{O}_h^\beta\,. \tag{3.2.7}$$

Since the integration measure $\sqrt{-h}$ has Weyl weight -2, quantum Weyl invariance of the integrated cosmological term implies

$$2\beta q + 2\beta^2 - 2 = 0\,, \tag{3.2.8}$$

[9]In the background field method one chooses an external source as a functional of the background field in such a way as to cancel all tadpoles. See [147, 148] for a concise summary.

which is equivalent to

$$q = \frac{1}{\beta} - \beta\,, \tag{3.2.9}$$

reproducing the well-known (timelike) Liouville relation between the background charge and the coupling [12, 13]. In Liouville literature, $2\beta^2$ is sometimes referred to as the 'anomalous gravitational dressing' of the identity operator. Recall that classically, Weyl invariance required that $\beta = 1/q$. We regard $1/q$ as the coupling constant and interpret our results as quantum corrections to β so that Weyl invariance is maintained at the quantum level

$$\beta = \frac{q}{2}\left(-1 + \sqrt{1 + \frac{4}{q^2}}\right) = \frac{1}{q} - \frac{1}{q^3} + \frac{2}{q^5} + \dots\,. \tag{3.2.10}$$

3.3 Non-local Quantum Effective Action

With this interpretation, the cosmological term in Lorentzian spacetime in terms of Ω becomes

$$\Gamma_\Lambda[\Omega, h] = -\mu \int d^2x\,\sqrt{-h}\,e^{2\,\beta\,q\,\Omega}\,e^{-2\beta^2\,\Sigma_h} = -\mu \int d^2x\,\sqrt{-h}\,e^{2\,\Omega}\,e^{-2\beta^2\,(\Omega+\Sigma_h)}\,. \tag{3.3.1}$$

The complete effective action for the conformal factor is then given by adding this renormalized cosmological operator to the quantum effective action of the free Ω field (2.4.9) (after continuation to Lorentzian signature)

$$\Gamma_{\Omega\Lambda}[\Omega, h] = \frac{1}{96\pi}\int dx\,R_h\frac{1}{\Box}R_h + \frac{q^2}{4\pi}\int dx\,\left(|\nabla\Omega|^2 + R_h\,\Omega - \frac{4\pi\mu}{q^2}\,e^{2\,\Omega}\,e^{-2\beta^2\,(\Omega+\Sigma_h)}\right)\,. \tag{3.3.2}$$

The effective action is non-local and one might worry about possible ghosts. In fact, in the local formulation described in section §4.4, one of the auxiliary fields has a negative kinetic term. Quantization of this degree of freedom would typically lead to a violation of both causality and unitarity. The correct point of view is to regard the quantum effective action as the result of having evaluated a path integral in the presence of a classical background field. Thus, this effective action is not to be quantized further but rather to be used to study the effective dynamics classically. After imposing appropriate initial conditions, one expects a ghost-free causal evolution because the original path integral is well defined.

3.4 Quantum Momentum Tensor

The quantum momentum tensor associated with the effective action (3.3.1) is given by[10]

$$T^{\Lambda}_{\mu\nu}(x) = \frac{-2}{\sqrt{-h}} \frac{\delta\Gamma_\Lambda}{\delta h^{\mu\nu}(x)} \tag{3.4.1}$$

$$= -\mu\, \mathcal{O}^{\beta}_{h}(x)\, h_{\mu\nu} - 4\mu\, \beta^2 \int dy\, \Sigma_{\mu\nu}(x,y)\, \mathcal{O}^{\beta}_{h}(y)\,. \tag{3.4.2}$$

The second term is the variation of the non-local term

$$\Sigma_{\mu\nu}(x,y) := \frac{1}{\sqrt{-h}} \frac{\delta\Sigma_h(y)}{\delta h^{\mu\nu}(x)} = \frac{1}{\sqrt{-h}} \frac{\delta}{\delta h^{\mu\nu}(x)}\, \frac{1}{2} \int dz\, G_{yz}\, R_h(z) \tag{3.4.3}$$

where G_{xy} is a shorthand for $G_h(x,y)$. Using the variation of the integrated Ricci scalar (B.0.10), and the variation of the Green function computed in appendix §E.2, we obtain

$$2\Sigma_{\mu\nu}(x,y) = -\left(\nabla_\mu \nabla_\nu - h_{\mu\nu}\nabla^2\right) G_{xy} - \int dz\, R_h(z) \left(\nabla_{(\mu} G_{yx}\, \nabla_{\nu)} G_{xz} - \frac{1}{2} h_{\mu\nu}\, \nabla_\alpha G_{yx}\, \nabla^\alpha G_{xz}\right)$$

where all derivatives and unspecified arguments of fields such as $h_{\mu\nu}$ correspond to the variable x, and we have used the fact that the Einstein tensor in two dimensions vanishes. The final expression for the quantum momentum tensor can be written as

$$T^{\Lambda}_{\mu\nu}(x) = -\mu\,(1-\beta^2)\, h_{\mu\nu}\, \mathcal{O}^{\beta}_{h}(x) + 2\,\mu\,\beta^2\, S_{\mu\nu}(x) \tag{3.4.4}$$

where $S_{\mu\nu}$ is non-local and traceless and given by

$$S_{\mu\nu}(x) = \int dy \left[\nabla_\mu \nabla_\nu - \frac{1}{2} h_{\mu\nu}\, \nabla^2\right] G_{xy}\, \mathcal{O}^{\beta}_{h}(y) \tag{3.4.5}$$

$$+ \int dy\, dz \Big[\nabla_{(\mu} G_{yx} \nabla_{\nu)} G_{xz} - \frac{1}{2} h_{\mu\nu}\, h^{\alpha\beta}\, \nabla_\alpha\, G_{yx}\, \nabla_\beta\, G_{xz}\Big]\, \mathcal{O}^{\beta}_{h}(y)\, R_h(z)\,.$$

The trace of this tensor is

$$T^{\Lambda} = -2\,\mu\,(1-\beta^2)\, \mathcal{O}^{\beta}_{h}\,. \tag{3.4.6}$$

Comparing with the general form of the anomaly equation $T = -\beta_i(\lambda_i)\, \mathcal{O}_i(\Sigma_g)$, with $\beta_i(\lambda_i) = (\Delta_i - d + \gamma_i)\lambda_i$, the β-function of the cosmological operator is[11]

$$\beta_\mu(\beta) = (-2 + 2\beta^2)\, \mu\,, \tag{3.4.7}$$

confirming that the classical dimension (as any vertex operator) is zero and the anomalous dimension is $\gamma = 2\beta^2$. This highlights one of the peculiar features of Liouville theory: its β-function does not vanish, hence it is a conformal field theory which is not sitting on a fixed point. This is due to the linear transformation of the Liouville field, very unusual for a conformal field, but very natural from the point of view gravity.

[10] In two dimensions, the momentum tensor obtained by varying the fiducial metric $h_{\mu\nu}$ for fixed Ω is the same as the momentum tensor obtained by varying the physical metric $g_{\mu\nu}$.

[11] Notice that our operator has the additional minus sign sitting in front of the action.

Using the same variations as above, the momentum tensor for the Polyakov action is given by

$$T^{Pol}_{\mu\nu} = \frac{-2}{\sqrt{-h}}\frac{\delta I_{Pol}}{\delta\, h^{\mu\nu}} = -\frac{1}{\pi}\left(\hat{R}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\, R_h(x)\right) \equiv -\frac{1}{\pi}\,\hat{E}_{\mu\nu}\,, \tag{3.4.8}$$

where again $\hat{R}_{\mu\nu}$ is a non-local and traceless tensor given by

$$\begin{aligned}\hat{R}_{\mu\nu}(x) \;&=\; -\int dy\left[\nabla_\mu\nabla_\nu - \frac{1}{2}h_{\mu\nu}\nabla^2\right] G_{xy}\, R_h(y) \qquad (3.4.9)\\ &\quad -\frac{1}{2}\int dy\, dz\left[\nabla_{(\mu}G_{yx}\,\nabla_{\nu)}G_{xz} - \frac{1}{2}h_{\mu\nu}\, h^{\alpha\beta}\,\nabla_\alpha\, G_{yx}\,\nabla_\beta\, G_{xz}\right] R_h(y)\, R_h(z)\,.\end{aligned}$$

The total quantum momentum tensor for the conformal factor, as given by the $h_{\mu\nu}$ variation of its effective action $\Gamma_{\Omega\Lambda}$ (3.3.2), reads

$$\begin{aligned}T^{\Omega\Lambda}_{\mu\nu}(x) \;&=\; \frac{1}{24}T^{Pol}_{\mu\nu} + T^{\Omega,clas}_{\mu\nu} + T^{\Lambda}_{\mu\nu}\\ &=\; -\frac{1}{24\pi}\,\hat{E}_{\mu\nu} - \frac{q^2}{2\pi}\left(\nabla_\mu\Omega\nabla_\nu\Omega - \frac{1}{2}h_{\mu\nu}\,(\nabla\Omega)^2 - \left(\nabla_\mu\nabla_\nu - h_{\mu\nu}\,\nabla^2_h\right)\Omega\right)\\ &\quad\; -\mu\,(1-\beta^2)\, h_{\mu\nu}\,\mathcal{O}^\beta_h(x) + 2\,\mu\,\beta^2\, S_{\mu\nu}(x)\,, \qquad (3.4.10)\end{aligned}$$

where $T^{\Omega,clas}_{\mu\nu}$ accounts for the contribution of the kinetic and background charge term, which coincides with the classical Ω momentum tensor (2.1.14). The equation of motion for Ω that follows from $\Gamma_{\Omega\Lambda}[\Omega, h]$ is

$$-\,2\nabla^2\Omega + R_h - \frac{8\pi\mu}{q^2}\,\beta\, q\,\mathcal{O}^\beta_h = 0\,. \tag{3.4.11}$$

Using this equation, the trace of the $T^{\Omega\Lambda}_{\mu\nu}$ momentum tensor is

$$T^{\Omega\Lambda} = \frac{1-6q^2}{24\pi}R_h\,, \tag{3.4.12}$$

where again we recognize the central charge of the Liouville theory $c_\Omega = c_L = 1 - 6q^2$. The factor of 1 comes from the Polyakov term, hence is purely quantum, while the $-6q^2$ comes from the Ω-dependent terms and reflects the classical lack of Weyl invariance of the action. Notice that the presence of the cosmological constant does not change this trace, which is the same for the momentum tensor without cosmological constant (2.4.15). This is because the equations of motion also acquire the cosmological term, and exactly absorb the contribution in the trace.

3.5 Quantum Ward Identities

We first check the Ward identity (1.2.24) for Weyl invariance for the renormalized cosmological term (3.3.1). The left hand side of (1.2.24) evaluates to

$$h^{\mu\nu}\, T^{\Lambda}_{\mu\nu} - \frac{1}{\sqrt{-h}}\frac{\delta\Gamma_\Lambda}{\delta\Omega} = -2\mu\,(1-\beta^2)\,\mathcal{O}^\beta_h + 2\,\mu\,\beta\, q\,\mathcal{O}^\beta_h\,. \tag{3.5.1}$$

It vanishes precisely when β is related to q by (3.2.8). This is to be expected because the Weyl Ward identity is simply the infinitesimal version of invariance under finite Weyl transformations which is what was used to obtain (3.2.8). The important point is that unless we modify β as in (3.2.10) away from its classical value, the full quantum theory would be anomalous. Anomalies in Weyl invariance are unavoidable because of the necessity to regularize the path integral. In the present context, we manage to maintain Weyl invariance at the quantum level by starting with a value of β such that the theory is *not* Weyl invariant at the classical level but it *becomes* Weyl invariant at the quantum level once the anomalous variations are taken into account.

For diffeomorphisms, we do not expect any anomalies because the renormalization procedure is manifestly coordinate invariant. To explicitly check the Ward identity we compute the covariant derivative of (3.4.4). Using the commutator in two dimensions

$$[\nabla_\mu, \nabla_\nu]V^\mu = R_{\mu\nu}V^\mu = \frac{1}{2}R_h\, V_\nu \tag{3.5.2}$$

and the Green equation to cancel terms, we obtain

$$\nabla^\mu T^\Lambda_{\mu\nu} = -\mu\,\nabla_\nu \mathcal{O}^\beta_h - \mu\,\beta^2\,\mathcal{O}^\beta_h(x)\int dz\, R_h(z)\,\nabla_\nu G_{xz} = -2\mu\,\beta\, q\,\mathcal{O}^\beta_h\,\nabla_\nu\Omega\,, \tag{3.5.3}$$

where in the last step we have integrated by parts and used the expression for the renormalized operator (3.2.5). This coincides with the Ω variation of the action

$$\frac{1}{\sqrt{-h}}\frac{\delta\Gamma_\Lambda}{\delta\Omega}\,\nabla_\nu\Omega = -2\mu\,\beta\, q\,\mathcal{O}^\beta_h\,\nabla_\nu\Omega\,, \tag{3.5.4}$$

and then the Ward identity, which subtracts the two, is satisfied.

3.6 Quantum Momentum Tensor from the Weyl Anomaly

One can derive the momentum tensor directly using the Weyl anomaly by reversing the logic of the previous subsection. We *assume* the diffeomorphism Ward identities rather than *verify* them. Our assumption is justified by the fact that our renormalization scheme used for computing the anomalous Weyl dimension is manifestly coordinate invariant and hence there is no possibility of diffeomorphism anomalies. The advantage of this method is that one can avoid the intermediate step of deducing the quantum effective action and directly obtain the quantum momentum tensor required in the equations of motion.

For this purpose it is convenient to use the conformally flat gauge with light-cone coordinates.[12] The only non-vanishing Christoffel symbols are

$$\Gamma^+_{++} = 2\,\partial_+\Sigma_h\,, \quad \Gamma^-_{--} = 2\,\partial_-\Sigma_h\,. \tag{3.6.1}$$

In two dimensions, the momentum tensor has only three independent components. Diffeomorphism Ward identities (1.2.23) give two equations. From the Weyl anomaly one obtains

$$T_{+-} = \frac{1}{2}\,\mu\,(1-\beta^2)\,e^{2\Sigma_h}\,\mathcal{O}^\beta_h(x)\,. \tag{3.6.2}$$

[12] We use the (+++) conventions of Misner, Thorne, and Wheeler. Our light-cone coordinates are $x^\pm := t \pm x$. The flat metric in these coordinates is $\eta_{+-} = -\frac{1}{2}$ with $\sqrt{-\eta} = \frac{1}{2}$ and $\nabla^2_\eta = -4\,\partial_+\partial_-$.

Together we obtain three equations for all three unknowns. The diffeomorphism Ward identity for the $\nu = +$ component gives

$$\partial_- T_{++} + \partial_+ T_{-+} - 2\,\partial_+ \Sigma_h\, T_{-+} = \mu\,(1-\beta^2)\,\partial_+ \Omega\, e^{2\Sigma_h}\, \mathcal{O}_h^\beta \tag{3.6.3}$$

which after use of (3.6.2) becomes

$$\partial_- T_{++} = \frac{1}{2}\mu\,\beta^2\,\partial_+ \left(e^{2\Sigma_h}\,\mathcal{O}_h^\beta\right)\,.$$

Taking a derivative with respect to $+$ on both sides we obtain

$$-\nabla_h^2 T_{++} = 2\mu\,\beta^2 e^{-2\Sigma_h}\,\partial_+^2 \left(e^{2\Sigma_h}\,\mathcal{O}_h^\beta\right) \tag{3.6.4}$$

where $-\nabla_h^2$ is the scalar Laplacian. Solving this Poisson equation we obtain

$$T_{++}(x) = 2\mu\,\beta^2 \int dy\,\partial_+^2 G_{xy}\,\mathcal{O}_h^\beta(y)\,. \tag{3.6.5}$$

We rewrite the partial derivatives as covariant ones and use the expression (2.1.30) for the Σ_h factors in the Christoffel symbols (3.6.1) to obtain a covariant expression

$$T_{++}(x) = 2\mu\,\beta^2 \int dy\,\nabla_+\nabla_+ G_{xy}\,\mathcal{O}_h^\beta(y) \,+\, 2\mu\,\beta^2 \int dy\,dz\,\nabla_+ G_{yx}\nabla_+ G_{xz}\,\mathcal{O}_h^\beta(y)\,R_h(z) \tag{3.6.6}$$

in agreement with (3.4.4). The component T_{--} can be computed similarly.

3.7 Total Effective Action and Quantum Einstein Equations

The full gravitational effective action is given by adding the Einstein-Hilbert piece to the action for the conformal factor (3.3.2)

$$\Gamma_G[\Omega, h] = \frac{q^2}{16\pi} \int dx\, R_h \frac{1}{\Box} R_h + \Gamma_{\Omega\Lambda}[\Omega, h]\,. \tag{3.7.1}$$

As argued in §2.4, the matter and ghost sectors contribute in the effective action with a Polyakov term with coefficient $c_m + c_{bc} = 25 - 26 = -1$, as required by anomaly cancellation. This cancels the same term with coefficient $+1$ from the $\Gamma_{\Omega\Lambda}[\Omega, h]$ action above. The total effective action results in

$$\begin{aligned}\Gamma &= \Gamma_G[\Omega, h] + \Gamma_m[X^i, h] + \Gamma_{bc}[b, c, h] \\ &= \frac{q^2}{4\pi} \int dx\, \left(\frac{1}{4}\, R_h \frac{1}{\Box} R_h + |\nabla\Omega|^2 + R_h\,\Omega - \frac{4\pi\mu}{q^2}\, e^{2\,\Omega}\, e^{-2\beta^2\,(\Omega+\Sigma_h)}\right) + I_m[X^i] + I_{bc}[b, c]\,.\end{aligned} \tag{3.7.2}$$

Upon going to the physical gauge $\Omega = 0$, $h_{\mu\nu} = g_{\mu\nu}$, it takes the form

$$\Gamma[g, X^i, b, c] = \frac{q^2}{4\pi} \int dx\, \left(\frac{1}{4} R_g \frac{1}{\Box} R_g - \frac{4\pi\mu}{q^2}\, e^{-2\beta^2\,\Sigma_g}\right) + I_m[X^i] + I_{bc}[b, c]\,. \tag{3.7.3}$$

We can compare the first two terms to the gravitational classical action (2.1.34). The cosmological term becomes gravitationally dressed through renormalization, while the Einstein-Hilbert term is exactly the same as the classical one. This suggests that this purely gravitational term, or equivalently the gravitational constant q^2 do not get renormalized. However, this is only after the contributions of the renormalization of the matter and ghosts sectors are taken into account, which exactly compensate for the renormalization of the metric field thanks to Weyl anomaly cancellation.

We now derive the field equations that follow from the total effective action. Adding the Einstein-Hilbert term to the action $\Gamma_{\Omega\Lambda}$ adds a term $\frac{q^2}{4} T^{Pol}_{\mu\nu}$ to its momentum tensor $T^{\Omega\Lambda}_{\mu\nu}$, leading to a total gravitational momentum tensor of unit central charge $c_G = 1$, as already argued in §2.4.

Finally, the $h_{\mu\nu}$ variation of the total action $\Gamma[\Omega, h, X^i, b, c]$ (3.7.2) leads to the quantum Einstein equations

$$\hat{E}_{\mu\nu}(h) = \frac{4\pi}{q^2} \left(T^{\Omega,clas}_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right) , \tag{3.7.4}$$

where $I_m[X^i]$ and $I_{bc}[b, c]$ do not contribute because they do not depend on the background metric. The contribution of the matter and ghosts sectors is though encoded on the right-hand side of the above equations, which is purely geometrical, determining the coefficient in front. The trace of these equations is

$$R_h - 2\,\nabla_h^2 \Omega = \frac{8\pi\mu}{q^2}\,(1-\beta^2)\,\mathcal{O}_h^\beta \,. \tag{3.7.5}$$

This trace coincides with the equation of motion for Ω (3.4.11), as it is supposed to be because Ω is effectively the only scalar in the game, hence its equation of motion follows from the Einstein equations due to the Ward identities.

In the physical gauge, the quantum Einstein equations become

$$\hat{E}_{\mu\nu}(g) = \frac{4\pi}{q^2}\, T^{\Lambda}_{\mu\nu}(g)\,. \tag{3.7.6}$$

We can compare these to the classical ones (2.1.7) (in $2+\epsilon$ dimensions)

$$\frac{E_{\mu\nu}(g)}{\epsilon} = \frac{2\pi}{q^2}\, T^{\Lambda,clas}_{\mu\nu}\,. \tag{3.7.7}$$

The $1/\epsilon$ indeterminacy of the left-hand side can be removed using the same l'Hôpital's rule that we used to replace the R_g/ϵ term in the classical Einstein-Hilbert action by the Polyakov one, and the two-dimensional limit gives

$$\lim_{\epsilon\to 0} \frac{E_{\mu\nu}}{\epsilon} = \frac{1}{2}\,\hat{E}_{\mu\nu} = \frac{1}{2}\left(\hat{R}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R_g \right) . \tag{3.7.8}$$

Substituting this on the left-hand side of (3.7.7), gives the classical limit of (3.7.6). This limit allows to have finite tensorial equations in exact two dimensions. However, notice that it does not buy much, since the two tensors (in $2+\epsilon$ and 2 dimensions respectively) have the same trace. Since the only non-trivial constraint is encoded in the trace, both lead to the very same finite constraint.

The classical equations (3.7.7) are derived from a theory without any matter, while the quantum ones are the Einstein equations for a theory with matter and ghosts. Nevertheless notice that classical conformal matter does not couple to the metric, hence does not contribute to the Einstein equations. At the quantum level, the matter and ghost sectors only contribute to the left-hand side of the equations as we have argued, although with a coefficient that exactly cancels that of the renormalization of the metric. We can say that anomaly cancellation allows for a semiclassical treatment where the left-hand side of Einstein equations does not get renormalized, while the momentum tensor of the cosmological constant on the right-hand side does, as we have computed. The only constraint the two-dimensional equations (3.7.6) encode is to be derived from its trace

$$R_g = -\frac{4\pi}{q^2}\, T^{\Lambda}(g) = \frac{8\pi\mu}{q^2}\,(1-\beta^2)\, e^{-2\beta^2}\,. \tag{3.7.9}$$

Indeed, the right-hand side is the same as in the classical constraint (2.1.8), and the left-hand side is reproduced upon taking the $\beta \to 0$ limit.

Chapter 4

Quantum Cosmology in Two Dimensions

In this section we examine the cosmological consequences of the quantum anomalies summarized by the total effective action $\Gamma[\Omega, h]$ (3.7.2), assuming positive cosmological constant $\mu > 0$.

4.1 Quantum Evolution Equations for Cosmology

We look for spatially homogeneous and isotropic solutions, so we impose $h_{\mu\nu} = \eta_{\mu\nu}$ and $\Omega = \Omega(\tau)$ on the fiducial Einstein equations (3.7.4). Since we can treat the cosmological evolution equations semiclassically, the Friedmann-Lemaître equation looks the same as the classical one

$$H^2 = \frac{4\pi}{q^2}\,\rho_\Lambda(\tau)\,, \tag{4.1.1}$$

where the energy density is now derived from the $T^\Lambda_{\mu\nu}$ given by (3.4.4). To identify an energy density function at all, we have to assume that this momentum tensor becomes of the perfect fluid form upon imposing $h_{\mu\nu} = \eta_{\mu\nu}$ and $\Omega = \Omega(\tau)$. But this is in fact a general requirement for homogeneity and isotropy. So if it were not the case, then it could not source solutions with such symmetries, and would trivialize upon imposing these conditions.

Since this momentum tensor is non-local and quite complex, one could still expect the energy density and pressure, even if depending only on time, to be a complicated integral of the $\Omega(\tau)$ field. In that case, the Friedmann-Lemaître equation would become a complicated integro-differential equation. However, not only does the quantum momentum tensor become of the perfect fluid form, but it also simplifies considerably. This is actually to be expected, since the trace of the fiducial Einstein equations becomes $\Omega'' \propto T^\Lambda$. This forces the Hubble parameter, and therefore the energy density, to be simply related to the trace of the momentum tensor, which doesn't keep the integrated non-local terms of $S_{\mu\nu}$.

Let's check this simplification explicitly from the expression of the momentum tensor (3.4.4). The retarded Green's function of the Laplacian in the two-dimensional flat spacetime is given by

$$G_{ret}(x,y) = \frac{1}{2}\Theta(\tau_x - \tau_y - |r_x - r_y|)\,. \tag{4.1.2}$$

For flat fiducial metrics, the second term in the expression of $S_{\mu\nu}$ (3.4.5) vanishes and the first term gives $S_{\mu\nu} = \left(\delta^\tau_\mu \delta^\tau_\nu + \frac{1}{2}\eta_{\mu\nu}\right)\,\mathcal{O}^\beta_h(\tau)$. The total momentum tensor is then given by

$$T^\Lambda_{\mu\nu}(\tau) = -\mu\left(\left(1-2\beta^2\right)\eta_{\mu\nu} - 2\beta^2\,\delta^\tau_\mu\delta^\tau_\nu\right)\mathcal{O}^\beta_h(\tau)\,. \tag{4.1.3}$$

From the components of $T^\Lambda_{\mu\nu}$ we can identify its density and pressure as

$$\rho_\Lambda(\tau) = \mu\,e^{-2\beta^2\Omega(\tau)}\,, \quad p_\Lambda(\tau) = -\mu\,(1-2\beta^2)\,e^{-2\beta^2\Omega(\tau)} \tag{4.1.4}$$

which imply the equation of state

$$p_\Lambda = w_\Lambda \rho_\Lambda \quad \text{with} \quad w_\Lambda = -1 + 2\beta^2\,. \tag{4.1.5}$$

Thus, in the semiclassical limit of small β, the barotropic index is slightly bigger than -1.

Remarkably, the non-local quantum cosmological momentum tensor has reduced to a local one with a particularly simple form corresponding to a barotropic perfect fluid. The net effect of the non-local quantum contribution to the momentum tensor is simply to modify the barotropic index from -1 to $-1+2\beta^2$. With this simplification, the seemingly integro-differential equation reduces to a simple differential one. Applying the formulae from our discussion of classical cosmology, in particular (1.3.17), we see that $\gamma = 2\beta^2$ for the vacuum fluid. We arrive at the conclusion that the quantum cosmological term leads to an expanding universe with decaying vacuum energy density and power law expansion

$$\rho_\Lambda(t) = \rho_*(\frac{a}{a_*})^{-2\beta^2}\,, \qquad a(t) = a_*(1+\beta^2 H_* t)^{\frac{1}{\beta^2}}\,. \tag{4.1.6}$$

Just like the classical cosmological constant, this solution satisfies the null and the weak energy conditions, but not the strong one, since the expansion is accelerated.

4.2 Cosmological Implications of the Quantum Decay of Vacuum Energy

These theoretical conclusions have potentially far-reaching implications for addressing some of the fundamental puzzles in modern cosmology [10]. We briefly comment on some of these consequences that can generalize to higher dimensions in a model-independent way.

The above solution (4.1.6) describes an accelerated power-law expansion. In the semiclassical limit $\beta \to 0$, the power-law becomes the de Sitter exponential expansion $a(t) = a_* e^{H_* t}$. To quantify the expansion, it is convenient to define slow-roll parameters as usual in terms of the fractional change in the Hubble parameter and its derivative

$$\varepsilon_H := -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{H dt}\,, \qquad \eta_H := \frac{\dot{\varepsilon}_H}{H\varepsilon_H} = \frac{d\ln\varepsilon_H}{H dt}\,. \tag{4.2.1}$$

For our model we find

$$\varepsilon_H = \beta^2 \qquad \eta_H = 0\,. \tag{4.2.2}$$

The condition for accelerated expansion ($\ddot{a} > 0$) requires ε_H to be less than one. Slow-roll inflation further requires that $\varepsilon_H \ll 1$. It is also necessary that $\eta_H \ll 1$ so that inflation lasts long enough. Since β is small in the semiclassical approximation, these conditions are satisfied. This quantum dynamics of the Ω field can therefore be a two-dimensional model of slow-roll inflation in the very early universe.

Note that Ω is not really a physical scalar but simply a mode of the metric in a particular gauge. Thus, this is a model of slow-roll inflation without a fundamental scalar, driven entirely by vacuum energy through the non-trivial quantum dynamics of the Ω field. As it stands, the model leads to an empty universe because it simply keeps inflating. With matter fields, it would be possible to construct more realistic scenarios, with graceful exit that can start a hot big bang, and primordial perturbations. It would be interesting to construct a complete two-dimensional model of cosmology with these ingredients.

The quantum decay of vacuum energy can provide a dynamical solution to the cosmological constant problem [42, 43, 150–152]. One can imagine that the universe starts off with a very large cosmological constant. The initial magnitude ρ_* of the vacuum energy density is of the order of M_0^2 for the cutoff scale M_0 which can be of the order of the string scale or the scale of supersymmetry breaking. Classically, one would obtain exactly de Sitter spacetime with exponential expansion and constant energy density. With even a very small value of the anomalous gravitational dressing, the dynamics of the universe is very different and one would obtain instead a slowly rolling, inflating universe. The exponential expansion is slowed down to a power-law expansion. The density is no longer constant but keeps decreasing and can become arbitrarily small compared to its initial value. For an observer at a very late time, the effective vacuum energy density is much smaller than ρ_*.

We have treated the timelike Liouville theory semiclassically. It would be very interesting if one can make sense of the quantum theory as a solvable model and explore the consequences of the full quantization in the context of the above.

In any case, the main lesson that we wish to abstract away is the observation that when gravity is dynamical, various operators coupled to gravity such as the identity operator can have anomalous gravitational dressings. Even small values for these gravitational dressing can have observable effects in the cosmological setting when the universe undergoes exponential expansion with several e-foldings.

4.3 Local Form of the On-Shell Quantum Momentum Tensor

If the fiducial metric is flat then it is possible to obtain a local expression for the quantum momentum tensor upon using the equations of motion for the Ω field

$$\nabla^2\Omega - \frac{1}{2}R_h + \frac{4\pi}{q}\mu\,\beta\,\mathcal{O}_h^\beta = 0 \tag{4.3.1}$$

where we have used $q\beta = 1-\beta^2$. Using this equation with $R_h = 0$, $S_{\mu\nu}$ in (3.4.5) becomes

$$\begin{aligned} S_{\mu\nu} &= -\frac{q}{4\pi\mu\beta}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}\eta_{\mu\nu}\,\nabla\cdot\nabla\right)\int dy\, G_{xy}\,\nabla_y^2\Omega(y) \\ &= \frac{q}{4\pi\mu\beta}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}\eta_{\mu\nu}\nabla^2\right)\Omega(x)\,. \end{aligned} \tag{4.3.2}$$

Substituting it in the total quantum momentum tensor for the conformal factor $T^{\Omega\Lambda}_{\mu\nu}(x)$ (3.4.10) we find[1]

$$T^{\rm q}_{\mu\nu} = \frac{q^2}{2\pi}\left[\frac{1}{\beta q}\left(\nabla_\mu\nabla_\nu - \frac{1}{2}\eta_{\mu\nu}\nabla^2\right)\Omega(x) - \nabla_\mu\Omega\nabla_\nu\Omega + \frac{1}{2}\eta_{\mu\nu}(\nabla\Omega)^2\right]. \tag{4.3.3}$$

Interestingly, the original non-local expression has reduced to a local expression. It is instructive to compare this local expression with the on-shell classical momentum tensor $T^{\Omega,clas}_{\mu\nu} + T^{\Lambda,clas}_{\mu\nu}$, which is already local off-shell,

$$T^{\rm cl}_{\mu\nu} = \frac{q^2}{2\pi}\left[\left(\nabla_\mu\nabla_\nu - \frac{1}{2}\eta_{\mu\nu}\nabla^2\right)\Omega(x) - \nabla_\mu\Omega\nabla_\nu\Omega + \frac{1}{2}\eta_{\mu\nu}(\nabla\Omega)^2\right]. \tag{4.3.4}$$

Both tensors are properly traceless, hence $T_{+-} = 0$. The $(++)$ components are

$$T^{\rm cl}_{++} = -\frac{q^2}{2\pi}\left[(\partial_+\Omega)^2 - \partial_+^2\Omega\right], \quad \beta q = 1\,; \tag{4.3.5}$$

$$T^{\rm q}_{++} = -\frac{q^2}{2\pi}\left[(\partial_+\Omega)^2 - \frac{1}{\beta q}\partial_+^2\Omega\right], \quad \beta q = 1-\beta^2\,. \tag{4.3.6}$$

Imposing the Virasoro constraint corresponds to solving the Einstein equations for spatially flat metrics in two dimensions. The solution is given by

$$e^{\Omega(\tau)} = e^{\Omega_*}\left(\frac{\tau}{\tau_*}\right)^{\frac{-1}{\beta q}} = e^{\Omega_*}\left(\frac{\tau}{\tau_*}\right)^{\frac{2}{\gamma-2}}\,. \tag{4.3.7}$$

In the classical case we have $\beta q = 1$ and $\gamma = 0$ whereas in the quantum case we have $\beta q = 1-\beta^2$ and $\gamma = 2\beta^2$. With $a(\tau) = e^{\Omega(\tau)}$, and after writing the conformal time in terms of the comoving time, the solution is in agreement with (1.3.17).

4.4 Local Formulation with Auxiliary Fields

The non-local action (3.3.1) can be rewritten in a local form [69, 153] by introducing two auxiliary fields $\Sigma(x)$ and $\Psi(x)$ with the action

$$\Gamma_\Lambda = -\mu\int d^2x\sqrt{-h}\,\left[e^{2(1-\beta^2)\Omega}\,e^{-2\beta^2\Sigma} + \Psi(2\,\nabla^2\Sigma + R_h)\right]. \tag{4.4.1}$$

The equations of motion for the auxiliary fields are

$$-\nabla^2\Sigma = \frac{1}{2}R_h\,, \tag{4.4.2}$$

$$-\nabla^2\Psi = -\beta^2\,e^{2\Omega}\,e^{-2\beta^2(\Omega+\Sigma)}\,. \tag{4.4.3}$$

[1] We drop the superscripts of the momentum tensor in the rest of this section, and we just keep q and c to distinguish between the quantum and the classical.

The first equation enforces the field $\Sigma(x)$ to be the conformal factor of the fiducial metric $h_{\mu\nu} = e^{2\Sigma}\eta_{\mu\nu}$. After eliminating the auxiliary fields by using their equations of motion, we recover our non-local action (3.3.1). The action is invariant under the Weyl transformation

$$\Sigma \to \Sigma + \xi, \qquad \Omega \to \Omega - \xi, \qquad h_{\mu\nu} \to e^{2\xi} h_{\mu\nu} \qquad \Psi \to \Psi\,. \tag{4.4.4}$$

The local momentum tensor resulting from this action is

$$T^{\Lambda}_{\mu\nu} = -\mu \left[h_{\mu\nu}(e^{2\Omega}\, e^{-2\beta^2(\Omega+\Sigma)} - 2\nabla\Psi \cdot \nabla\Sigma) + 4\nabla_{(\mu}\Psi\nabla_{\nu)}\Sigma + 2(\nabla_\mu\nabla_\nu - h_{\mu\nu}\nabla^2)\Psi \right] \tag{4.4.5}$$

which again reduces to (3.4.4) after using (4.4.2) and (4.4.3).

Chapter 5

Four-Dimensional Effective Action and Equations

We now move to four dimensions. We are interested in the quantum effective action for the metric obtained by integrating out the quantum fluctuations of various fields, and valid at distances large compared to the Planck distance. Our central object is hence the non-local effective action for Einstein-Hilbert gravity with a cosmological constant. The essential lesson that emerges from the study of the two-dimensional model is that the anomalous dimensions of the dressed operators are in principle different from the anomalous dimensions of the undressed operators. The physical coupling constants are the couplings of the gravitationally-dressed operators. This applies in particular, to the square-root of the determinant of the metric corresponding to the cosmological term, as well as to the Einstein-Hilbert operator.

The quantum effective action then, should take into account the anomalous gravitational dressings of these two operators. The corresponding anomalous dimensions should be calculable in a microscopic theory with perturbative methods. Although fundamental, we postpone these computations for future work. In the next two chapters instead, we simply parametrize the action with the gravitational dressings, which we assume to be non-local functions of the conformal factor. This is the natural generalization of our two-dimensional results, and is consistent with what the local renormalization group dictates. We further compute the evolution equations and analyze the cosmological dynamics.

5.1 A Non-local Action for Gravity

We consider the four-dimensional gravitational effective action[1]

$$I_G[g] = \frac{M_p^2}{16\pi} \int d^4x \sqrt{-g} \, \left(R_g \, e^{-\Gamma_K(\Sigma_g)} - 2\,\Lambda \, e^{-\Gamma_\Lambda(\Sigma_g)} \right) \tag{5.1.1}$$

[1] In the following two sections, we will denote the quantum effective action by I instead of Γ, to avoid confusion with the integrated anomalous dimensions.

where $\Gamma_i(\Sigma_g)$, $i = K, \Lambda$ are the integrated anomalous gravitational dressings. The field $\Sigma_g(x)$ is again a non-local functional of the metric $g_{\mu\nu}$, defined by [37, 154, 155]

$$\Sigma_g(x) = \frac{1}{4} \int d^4y \sqrt{-g}\, G_4(x,y)\, F_4(g)(y)\,, \tag{5.1.2}$$

where[2]

$$F_4(g) = E_4(g) - \frac{2}{3}\nabla_g^2 R_g\,, \qquad E_4(g) = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R_g^2\,; \tag{5.1.3}$$

and $G_4(x,y)$ is the Green function of the Weyl covariant quartic differential operator

$$\Delta_4(g) = \left(\nabla_g^2\right)^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}\left(\nabla^\nu R_g\right)\nabla_\nu - \frac{2}{3}R_g\,\nabla_g^2 \tag{5.1.4}$$

on the $g_{\mu\nu}$ background satisfying

$$\Delta_4^x(g)G_4(x,y) = \delta^{(4)}(x,y) := \frac{\delta^{(4)}(x-y)}{\sqrt{-g}}\,. \tag{5.1.5}$$

For metrics related by a Weyl rescaling

$$g_{\mu\nu} = e^{2\Sigma_g(x)}\,\bar{\eta}_{\mu\nu}\,, \tag{5.1.6}$$

the scalars F_4 are related by

$$F_4(g) = e^{-4\Sigma_g}\left(F_4(\bar{\eta}) + 4\,\Delta_4(\bar{\eta})\,\Sigma_g\right)\,, \tag{5.1.7}$$

and the operators Δ_4 are related by

$$\Delta_4(g) = e^{-4\Sigma_g}\,\Delta_4(\bar{\eta})\,. \tag{5.1.8}$$

One can then choose a conformal gauge in which $\Sigma_g(x)$ becomes the conformal factor of the metric with respect to a reference metric $\bar{\eta}_{\mu\nu}$ which satisfies the F-flatness condition $F_4(\bar{\eta}) = 0$. The expression (5.1.2) is obtained in this F-flat gauge by inverting (5.1.7). Given the transformations (5.1.7) and (5.1.8), it is clear that the F-flat gauge is the four-dimensional analogue of the two-dimensional Ricci-flat gauge. Note that the action (5.1.1) should be regarded as the in-in effective action and hence one must impose retarded boundary conditions. This ensures that the propagation is causal.

We emphasize that the action (5.1.1) is the result of having performed a path integral and is not to be quantized further, but is to be used for studying the effective classical dynamics. For now, we view these functions as a phenomenological parametrization of possible Weyl anomalies.

The variation of (5.1.1) with respect to $g_{\mu\nu}$ is very cumbersome because both Δ_4 and F_4 have a complicated dependence on the metric. The Weyl-invariant formulation

[2] In these two chapters dealing with the four-dimensional Weyl-invariant formulation we will denote the metric-dependence of the different geometric objects in a heterogeneous way. We keep the metric sub-index for R_g and ∇_g^2. Other scalars already exhibiting a sub-index will have the metric dependence as an argument, like $F_4(g)$, $\Delta_4(g)$ and $E_4(g)$. We will omit it all together for most tensors and operators with Lorentz indices, and for Green functions, to avoid clumping the notation. The metric dependence for these should be understood from the metric dependence of neighboring terms.

again leads to considerable simplification by exploiting the fact that the spatially-flat Robertson-Walker metric is Weyl-equivalent to the flat Minkowski metric.

To write the action in the Weyl-invariant form, we introduce the Weyl split

$$g_{\mu\nu} = e^{2\Omega(x)} h_{\mu\nu} \tag{5.1.9}$$

in (5.1.1). The fiducial metric can be further parametrized in terms of the F-flat reference metric $\bar{\eta}_{\mu\nu}$ as

$$h_{\mu\nu} = e^{2\Sigma_h(x)} \bar{\eta}_{\mu\nu} \,. \tag{5.1.10}$$

Then again it holds that

$$\Sigma_g = \Omega + \Sigma_h \,, \tag{5.1.11}$$

where the Weyl factor $\Sigma_h(x)$ is given by (5.1.2) evaluated on $h_{\mu\nu}$, and Σ_g is invariant under a Weyl transformation. The Weyl-invariant quantum effective action becomes[3]

$$I_G[h,\Omega] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-h}\, e^{4\Omega} \left(R_{he^{2\Omega}}\, e^{-\Gamma_K(\Omega+\Sigma_h)} - 2\,\Lambda\, e^{-\Gamma_\Lambda(\Omega+\Sigma_h)} \right) . \tag{5.1.12}$$

Using the Weyl transformation of the Ricci scalar and integrating by parts, we obtain

$$I_G[h,\Omega] = \frac{1}{2\kappa^2} \int dx \, \left[\left(R_h + 6\,(1-\Gamma_K^{(1)})\, |\nabla\Omega|^2 - 6\,\Gamma_K^{(1)}\, \nabla\Omega \cdot \nabla\Sigma_h \right) e^{2\Omega-\Gamma_K} - 2\,\Lambda\, e^{4\Omega-\Gamma_\Lambda} \right] \tag{5.1.13}$$

where $dx \equiv d^4x \sqrt{-h}$ and[4] $\Gamma_i^{(n)}(\Omega+\Sigma_h)$ are the n-th derivatives of the dressing functions.

The action (5.1.13) now has an enlarged gauge symmetry that includes Weyl invariance in addition to diffeomorphisms. As opposed to the two-dimensional case then, we do not gain anything by demanding this gauge invariance, since the effective action is precisely written down so as to satisfy it, namely, it is originally written with the dependence on the conformal factor of the fiducial only through the combination $\Sigma_g = \Omega + \Sigma_h$. The operators in the action are composites of the Weyl compensator and the unfixed components of the fiducial metric. If we were able to explicitly compute the dressings of these operators and hence find the anomalous dependence on the conformal factor Σ_h, then demanding gauge invariance would be useful. In other words, by imposing the Weyl Ward identity to be satisfied we could extract information on the dependence of the anomalous dimensions on the couplings. However, the computations of the gravitational dressings are bound to be highly complex.

As in the two-dimensional case, the physical gauge $\Omega = 0$ brings the above action back to (5.1.1). Alternatively, one can keep Ω arbitrary and impose a scalar gauge condition on the fiducial metric such as $F_4(h) = 0$. In this F-flat gauge $\Sigma_h = 0$ and $\Sigma_g = \Omega$.

[3]Henceforth we choose units so that $M_0 = 1$.

[4]In this and the next two sections, all covariant derivatives and contractions are with respect to the fiducial metric $h_{\mu\nu}$.

5.2 Evolution Equations for Cosmology

We look for the evolution equations that will describe a homogeneous and isotropic universe. We choose the spatial section to be flat, so that the fiducial metric $h_{\mu\nu} = \eta_{\mu\nu}$ and the entire dynamics resides in the Weyl compensator. Since the Minkowski metric is not only F-flat but Riemann-flat, the variation of the non-local terms in the action (5.1.13) simplifies considerably.

Since $F_4(\eta) = 0$, the variation of the Green function does not contribute, and we obtain

$$\delta\Sigma_h(x) = \frac{1}{4}\int dy\, G_4(x,y)\,\delta F_4(h)(y)\,. \tag{5.2.1}$$

Furthermore, the quadratic terms involving the Riemann curvature tensors do not contribute to the variation of $F_4(h)$ when evaluated around $\eta_{\mu\nu}$. The only nonzero contribution comes from the variation of the term linear in the curvature

$$\delta F_4(h)(y) = -\frac{2}{3}\,\nabla^2_\eta\,\delta R_h\,. \tag{5.2.2}$$

The total variation of Σ_h after an integration by parts is then given by

$$\delta\Sigma_h(x) = \frac{1}{6}\int dy\,\delta h^{\mu\nu}(y)\,(\nabla_\mu\nabla_\nu - h_{\mu\nu}\nabla^2_\eta)\,\nabla^2_\eta\,G_4(x,y)\,. \tag{5.2.3}$$

After performing the variation in the fiducial frame, it is convenient to rewrite the equations of motion in terms of the gauge-invariant physical metric using (5.1.9). The Weyl transformation of the Einstein tensor is

$$\begin{aligned} E_{\mu\nu}(g) &= E_{\mu\nu}(h) + D_{\mu\nu}(h,\Omega)\,, \\ D_{\mu\nu}(h,\Omega) &:= -2\left(\nabla_\mu\nabla_\nu - h_{\mu\nu}\nabla^2_h\right)\Omega + 2\left(\nabla_\mu\Omega\nabla_\nu\Omega + \frac{1}{2}h_{\mu\nu}|\nabla\Omega|^2\right)\,. \end{aligned} \tag{5.2.4}$$

Substituting it in the above variation of (5.1.13) yields the equations of motion for the physical metric

$$E_{\mu\nu}(g) = \kappa^2\,(T^K_{\mu\nu} + T^\Lambda_{\mu\nu})\,, \tag{5.2.5}$$

where $T^K_{\mu\nu}$ is the momentum tensor of the 'gravifluid', of purely geometric origin

$$\begin{aligned} \kappa^2\,T^K_{\mu\nu}(x) &= 2\,\Gamma^{(1)}_K\left(\nabla_\mu\Omega\,\nabla_\nu\Omega + \frac{1}{2}\eta_{\mu\nu}|\nabla\Omega|^2\right) \\ &\quad + \left((\Gamma^{(1)}_K)^2 - \Gamma^{(2)}_K\right)\left(\nabla_\mu\Omega\,\nabla_\nu\Omega - \eta_{\mu\nu}|\nabla\Omega|^2\right) - \Gamma^{(1)}_K\left(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\nabla^2_\eta\right)\Omega \\ &\quad - e^{-2\Omega+\Gamma_K}\int dy\,\Gamma^{(1)}_K e^{2\Omega-\Gamma_K}\,(\nabla^2_\eta\Omega + |\nabla\Omega|^2)\left(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\nabla^2_\eta\right)\nabla^2_\eta G_4(x,y)\,; \end{aligned} \tag{5.2.6}$$

and $T^\Lambda_{\mu\nu}$ is the momentum tensor of the 'vacuum fluid'

$$\begin{aligned} \kappa^2\,T^\Lambda_{\mu\nu}(x) &= -\Lambda\,\eta_{\mu\nu}\,e^{2\Omega+\Gamma_K-\Gamma_\Lambda} \\ &\quad - \frac{\Lambda}{3}\,e^{-2\Omega+\Gamma_K}\int dy\,\Gamma^{(1)}_\Lambda\,e^{4\Omega-\Gamma_\Lambda}\left(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\nabla^2_\eta\right)\nabla^2_\eta G_4(x,y)\,. \end{aligned} \tag{5.2.7}$$

We emphasize that the contribution from the 'gravifluid' is purely geometric in origin and in principle belongs to the left hand side of the equation (5.2.5) on the same footing as the Einstein tensor. Since the fiducial metric is flat in this context, the equation (5.2.5) reduces to

$$D_{\mu\nu}(\eta,\Omega) = \kappa^2 \left(T^K_{\mu\nu} + T^\Lambda_{\mu\nu}\right) . \tag{5.2.8}$$

5.3 Cosmological Equations in an Alternative Gauge

It is possible to choose an alternative gauge in which the conformal factor $\tilde{\Sigma}_g(x)$ is defined with respect to an R-flat reference metric $\tilde{\eta}_{\mu\nu}$ [156, 157]. In the R-flat gauge, the expression for the conformal factor follows from the Weyl transformation of the Ricci scalar

$$R_g = e^{-2\tilde{\Sigma}_g} \left(R_{\tilde{\eta}} - 6\nabla^2_{\tilde{\eta}}\tilde{\Sigma}_g - 6|\nabla\tilde{\Sigma}_g|^2\right) . \tag{5.3.1}$$

Imposing $R_{\tilde{\eta}} = 0$, the above equation can be inverted by means of defining the field

$$\Phi_g := 1 - e^{-\tilde{\Sigma}_g} , \tag{5.3.2}$$

in terms of which the above equation becomes

$$\left(-6\,\nabla^2_g + R_g\right)_x \Phi_g = R_g . \tag{5.3.3}$$

This Poisson equation can be inverted by means of the Green function $\tilde{G}(x,y)$ of the differential operator

$$\left(-6\,\nabla^2_g + R_g\right)_x \tilde{G}(x,y) = \delta^{(4)}(x,y) . \tag{5.3.4}$$

The conformal factor then becomes

$$\tilde{\Sigma}_g(x) = -\ \ln\left(1 - \int d^4y\,\sqrt{-g}\ \tilde{G}(x,y)\,R_g(y)\right) . \tag{5.3.5}$$

This gauge can in fact be defined for any dimensions, by starting with the Weyl transformation of the Ricci scalar in any dimensions, defining $\Phi := 1 - e^{-\frac{d-2}{2}\tilde{\Sigma}_g}$, and repeating the above inversion by means of defining the corresponding Green function. The two-dimensional limit can then be taken and shown to reproduce the conformally flat gauge used in section §3.

To use this gauge in the quantum action, we express the fiducial metric as

$$h_{\mu\nu} = e^{2\tilde{\Sigma}_h}\,\tilde{\eta}_{\mu\nu} , \qquad \tilde{\Sigma}_g = \Omega + \tilde{\Sigma}_h . \tag{5.3.6}$$

We introduce the Weyl split into the analog of the action (5.1.1) in the R-flat gauge with gravitational dressing functions[5] $\tilde{\Gamma}_i(\tilde{\Sigma}_g)$. The Weyl-invariant action then becomes

$$I_G[h,\Omega] = \frac{1}{2\kappa^2}\int dx\ \left[\left(R_h + 6\,(1-\tilde{\Gamma}^{(1)}_K)\,|\nabla\Omega|^2 - 6\,\tilde{\Gamma}^{(1)}_K\,\nabla\Omega\cdot\nabla\tilde{\Sigma}_h\right) e^{2\Omega-\tilde{\Gamma}_K} - 2\,\Lambda\,e^{4\Omega-\tilde{\Gamma}_\Lambda}\right] \tag{5.3.7}$$

where now $\tilde{\Gamma}_i = \tilde{\Gamma}_i(\Omega + \tilde{\Sigma}_h)$.

[5] Note that $\tilde{\eta}_{\mu\nu} = e^{2\Sigma_{\tilde{\eta}}}\,\bar{\eta}_{\mu\nu}$ and $\Sigma_g = \tilde{\Sigma}_g + \Sigma_{\tilde{\eta}}$. As a result, the integrated anomalous gravitational dressing functions in the two gauges are related by a shift: $\tilde{\Gamma}_i(\tilde{\Sigma}_g) = \Gamma_i(\tilde{\Sigma}_g + \Sigma_{\tilde{\eta}})$.

The equations of motion for a spatially-flat Robertson-Walker spacetime then follow from the variation of this action around $h_{\mu\nu} = \eta_{\mu\nu}$. On a flat background, the variation of $\tilde{\Sigma}_h$ (5.3.5) receives no contribution from the variation of the Green function and is given by

$$\delta\tilde{\Sigma}_h = \int d^4y\, \tilde{G}(x,y)\, \delta\left(\sqrt{-h}\, R_h(y)\right) = \int d^4y \sqrt{-\eta}\, \delta h^{\mu\nu} \left(-\nabla_\mu\nabla_\nu + h_{\mu\nu}\nabla^2_\eta\right) \tilde{G}(x,y)\,. \tag{5.3.8}$$

Furthermore, since $R_\eta = 0$, the equation (5.3.4) becomes the Green equation for the flat Laplacian. Comparing with the Green equation (5.1.5) of Δ_4 on a flat background, we find that the two Green functions are related through

$$\tilde{G}(x,y) = -\frac{1}{6}\, \nabla^2_\eta G_4(x,y)\,. \tag{5.3.9}$$

Introducing this in (5.3.8) we recover the same variation of $\Sigma_h(x)$ in a flat background (5.2.3)

$$\delta\tilde{\Sigma}_h = \frac{1}{6}\int dy\, \delta h^{\mu\nu}\, (\nabla_\mu\nabla_\nu - h_{\mu\nu}\nabla^2_\eta)\, \nabla^2_\eta\, G_4(x,y)\,. \tag{5.3.10}$$

Since $h_{\mu\nu}$ is taken to be Minkowski, $\Sigma_h = \tilde{\Sigma}_h = 0$, and therefore $\Gamma_i(\Omega) = \tilde{\Gamma}_i(\Omega)$. As a result, the equations of motion obtained in the two gauges are identical.

5.4 General Equations of Motion

The R-flat gauge allows to compute the modified Einstein equations on a general background without the help of the Weyl-invariant formulation. The reason is that in this gauge, the only geometric objects involved are the Ricci scalar and the scalar Laplacian, whose general variations are easily calculable. We therefore compute the $g_{\mu\nu}$ variation of the action

$$I_G[g] = \frac{M_p^2}{16\pi}\int d^4x\sqrt{-g}\,\left(R_g\, e^{-\tilde{\Gamma}_K(\tilde{\Sigma}_g)} - 2\,\Lambda\, e^{-\tilde{\Gamma}_\Lambda(\tilde{\Sigma}_g)}\right)\,. \tag{5.4.1}$$

This requires the general variation of $\tilde{\Sigma}_g(x)$ in terms of $\delta g^{\mu\nu}$, which can be done using again the auxiliary field $\Phi_g(x)$ defined in (5.3.2), from which

$$e^{-\tilde{\Sigma}_g}\,\delta\tilde{\Sigma}_g(x) = \delta\Phi_g(x) = \int d^4y\,\delta\left(\sqrt{-g}\,\tilde{G}(x,y)\,R_g(y)\right)\,. \tag{5.4.2}$$

As opposed to the computations of the equations around the fiducial Minkowski metric, for which the variation of the Green functions did not contribute, the computation of the equations in a general background where R_g does not vanish, does require this variation. The way to compute it is by varying its Green equation (5.3.4), and solving then for the resulting Poisson equation using the general variations of the scalar Laplacian and the Ricci scalar. This computation is presented in appendix §E.3. With the variation $\delta\tilde{G}$ at hand, the variation of the conformal factor is straightforward using the variation of the integrated Ricci scalar (B.0.10). The equations finally read

$$E_{\mu\nu}(g) = \kappa^2\,(T^K_{\mu\nu} + T^\Lambda_{\mu\nu}) \tag{5.4.3}$$

with

$$\begin{aligned}\kappa^2\, T^K_{\mu\nu} \;=\;& \left((\tilde\Gamma^{(1)}_K)^2 - \tilde\Gamma^{(2)}_K\right)\left(\nabla_\mu \tilde\Sigma_g \nabla_\nu \tilde\Sigma_g - g_{\mu\nu}(\nabla\tilde\Sigma_g)^2\right)(x) - \tilde\Gamma^{(1)}_K \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2_g\right)\tilde\Sigma_g(x)\\
&+ e^{\tilde\Gamma_K(x)} \int dy\, e^{\tilde\Sigma_g}\, R_g\, e^{-\tilde\Gamma_K}\, \tilde\Gamma^{(1)}_K(y)\, \Big\{ \left(E_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^2_g\right)_x \tilde G_{y,x}\\
&+ \int dz\, R_g(z) \Big[-E_{\mu\nu}(x)\, \tilde G_{yx}\tilde G_{xz} - 4\left(\nabla_{(\mu}\tilde G_{yx}\,\nabla_{\nu)}\tilde G_{xz} - \frac{1}{4}\, g_{\mu\nu}\,\nabla_\alpha \tilde G_{yx}\nabla^\alpha \tilde G_{xz}\right)_x\\
&+ \tilde G_{yx}\left(\overleftrightarrow{\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2_g}\right)_x \tilde G_{xz}\Big]\Big\}\,, \end{aligned} \tag{5.4.4}$$

$$\begin{aligned}\kappa^2\, T^\Lambda_{\mu\nu} \;=\;& -\Lambda\, e^{\tilde\Gamma_K - \tilde\Gamma_\Lambda}\, g_{\mu\nu}(x) - e^{\tilde\Gamma_K(x)} \int dy\, e^{\tilde\Sigma_g}\; 2\Lambda\, e^{-\tilde\Gamma_\Lambda}\, \tilde\Gamma^{(1)}_\Lambda(y)\;\cdot\\
&\Big\{ \left(E_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^2_g\right)_x \tilde G_{y,x} + \int dz\, R_g(z)\, \Big[-E_{\mu\nu}(x)\, \tilde G_{yx}\tilde G_{xz}\\
&-4\left(\nabla_{(\mu}\tilde G_{yx}\,\nabla_{\nu)}\tilde G_{xz} - \frac{1}{4}\, g_{\mu\nu}\,\nabla_\alpha\tilde G_{yx}\nabla^\alpha\tilde G_{xz}\right)_x + \tilde G_{yx}\left(\overleftrightarrow{\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2_g}\right)_x \tilde G_{xz}\Big]\Big\}\,. \end{aligned} \tag{5.4.5}$$

The over-arrow on the last term indicates the sum of the differential operator acting on the Green function on each side, and the subscripts outside parenthesis indicate which is the independent variable inside.

Even if we do not need these general equations for our cosmological purposes, it is instructive to write them down. First of all, we conclude that even if the gravitational dressings entitle complex and lengthy computations, one can still write down tractable and analytic equations in the general metric, with which solutions other than cosmological can be computed, for example for black holes. This would allow to analytically compute the corrections due to the Weyl anomalous dimensions in solar system or galactic dynamics.

Second, even if the Weyl-invariant formulation is clearly not fundamental but rather a mere computational trick, the above equations enlighten how helpful it becomes, specially for the cosmological solutions. The covariant derivatives, the Laplacian and the Green function all depend on the metric $g_{\mu\nu}$ in a complicated way, and they do not have a simple Weyl transformation (the Laplacian only has a simple Weyl transformation in two dimensions). Therefore, imposing the R-flat gauge in the above equations in order to get the equations in the Weyl-invariant formulation is highly non-trivial. It is much easier to compute them directly from the Weyl-invariant action. Furthermore, the latter computation becomes much shorter, since it does not require the variation of the Green function.

If we were to compute the general equations from the $g_{\mu\nu}$ variation of the action in the F-flat gauge (5.1.1), they would depend instead on $F_4(g)$, $\Delta_4(g)$, which have simple Weyl transformations, and G_4 which does not transform. Therefore the equations would easily be transformed to the Weyl-invariant formulation. The complication however is only traded, since computing the general equations in this gauge requires the general variations of $F_4(g)$, $G_4(g)$, which are clearly very lengthy.

Notice that the great advantage of having the equations in the Weyl-invariant form is that the fiducial metric can be fixed to Minkowski and we can get a total differential equation for the conformal factor, which fully dictates the dynamics of the scale factor of a Robertson-Walker metric. Looking for isotropic and homogeneous solutions with the general equations above is therefore much less practical.

Chapter 6

Quantum Decay of Vacuum Energy

The Einstein equations (5.2.5) are valid generally as long as the Weyl tensor of the physical metric vanishes. In a spatially-flat Robertson-Walker spacetime there is further simplification because the scale factor of the physical metric is a function of only the conformal time τ. With our gauge choice $h_{\mu\nu} = \eta_{\mu\nu}$, we can write $a(\tau) = e^{\Omega(\tau)}$. The momentum tensors (5.2.6) and (5.2.7) now simplify further and the integro-differential equations (5.2.8) reduce to an ordinary differential equation of the usual Friedmann-Lemaître type but for an effective quantum fluid with an unusual equation of state.

6.1 Effective Quantum Fluid

For the Robertson-Walker metric, the explicit form of $G_4(x,y)$ is actually not needed because its contribution to the momentum tensors is of the form

$$\int dy\, F[\Omega]\, \Big(\nabla_\mu \nabla_\nu - h_{\mu\nu}\nabla^2\Big)\, \nabla^2 G_4(x,y)\,. \tag{6.1.1}$$

The differential operator in the parenthesis vanishes when $\mu = \nu = \tau$, so it does not contribute to the energy density. For all other components, the first term in the parenthesis vanishes after integration by parts, and for the components $\mu, \nu = i$, the second term in the parenthesis can be identified with the Green equation of $\Delta_4(\eta)$. It follows that the quantum momentum tensors (5.2.6) and (5.2.7) correspond to perfect fluids, consistent with isotropy and homogeneity, although they are not separately conserved.

The density and pressure of the vacuum fluid are given by

$$\rho_\Lambda(t) = \frac{\Lambda}{\kappa^2}\, e^{\Gamma_K - \Gamma_\Lambda}\,, \quad p_\Lambda(t) = w_\Lambda(t)\, \rho_\Lambda(t)\,, \tag{6.1.2}$$

$$w_\Lambda(t) = \left(-1 + \frac{\Gamma_\Lambda^{(1)}}{3}\right)\,. \tag{6.1.3}$$

The density and pressure of the gravifluid *after using* the equations of motion are given by

$$\rho_K(t) = \frac{\Lambda}{\kappa^2}\frac{\Gamma_K^{(1)}}{1-\Gamma_K^{(1)}}\, e^{\Gamma_K-\Gamma_\Lambda}\,, \quad p_K(t) = w_K(t)\,\rho_K(t)\,, \tag{6.1.4}$$

$$w_K(t) = \left(-1+\frac{\Gamma_\Lambda^{(1)}-1}{3}-\frac{\Gamma_K^{(2)}}{3\,\Gamma_K^{(1)}\,(1-\Gamma_K^{(1)})}\right)\,. \tag{6.1.5}$$

We have written the expressions above in the 'barotropic' form with the effective pressure proportional to the effective density, but the anomalous gravitational dressings $\Gamma_i(\ln a(t))$ are in general non-trivial functions of the comoving time. As a result, the barotropic indices w_Λ and w_K are in general *time-dependent*[1] and should be regarded as a convenient parametrization.

Combining the two contributions one obtains the total momentum tensor on the right-hand side of the equation (5.2.8). It is a perfect fluid with the effective density and pressure given by

$$\rho_e(t) = \frac{\Lambda}{\kappa^2}\frac{1}{1-\Gamma_K^{(1)}}\, e^{\Gamma_K-\Gamma_\Lambda}\,, \quad p_e(t) = w_e(t)\,\rho_e(t) \tag{6.1.6}$$

$$w_e(t) = \left(-1+\frac{\gamma}{3}\right)\,, \quad \gamma = \left(\Gamma_\Lambda^{(1)}-\Gamma_K^{(1)}-\frac{\Gamma_K^{(2)}}{(1-\Gamma_K^{(1)})}\right)\,. \tag{6.1.7}$$

With this effective density, the equation of motion reduces to the first Friedmann equation

$$H^2 = \frac{\kappa^2\rho_e}{3}\,. \tag{6.1.8}$$

Note that our conclusions thus far follow purely from the symmetry considerations of isotropy, homogeneity, and spatial flatness.

The momentum tensor for the gravifluid (6.1.4) is proportional to the cosmological constant after using the equations of motion in a spatially-flat Robertson-Walker spacetime. As a result the total momentum tensor for the effective fluid is proportional to the cosmological constant. This implies that in the absence of the cosmological constant, the Minkowski metric continues to be an exact solution of the new equations (5.2.5) in vacuum. On the other hand, for positive cosmological constant, the classical de Sitter solution is no longer a solution of the quantum equations (5.2.5) as we describe below.

The conservation equation for the effective fluid is

$$\dot\rho_e = -3\,(p_e+\rho_e)\,H\,. \tag{6.1.9}$$

A useful consistency check is that the expressions (6.1.6) and (6.1.7) satisfy the conservation equation. It is of course guaranteed by the fact that the non-local action (5.1.1) is coordinate invariant and hence follows from the Bianchi identity. Note, however, that the gravifluid and the vacuum fluid are not conserved separately for nonzero Γ_K.

[1] Recall that in classical cosmology the commonly encountered fluids have the barotropic index -1 for the cosmological constant, 0 for matter, and $1/3$ for radiation.

6.2 Cosmology of the Decaying Vacuum Energy

The expressions (6.1.6) and (6.1.7) for the effective density and pressure already give their functional dependence on the scale factor. As discussed above, they automatically solve the conservation equation (6.1.9). Our task is then reduced to solving the equation (6.1.8) to obtain the scale factor as a function of the cosmological time. Even though (6.1.8) is much simpler than an integro-differential equation, it is nevertheless a complicated ordinary differential equation. In general, the integrated anomalous dressings Γ_K and Γ_Λ are non-trivial functions of the scale factor and this equation can be solved only numerically.

Analytic solutions are possible when Γ_K and Γ_Λ are both linear functions of Ω

$$\Gamma_K(\Omega) = \gamma_K \, \Omega(x), \qquad \Gamma_\Lambda(\Omega) = \gamma_\Lambda \, \Omega(x)\,, \tag{6.2.1}$$

where γ_i are constants.[2] Even if this is just a particular case, it may be possible to approximate the integrated anomalous gravitational dressings by linear functions for long enough time intervals during the evolution of the universe. In this case, the barotropic index for both the vacuum fluid (6.1.3) and the gravifluid (6.1.5) becomes constant. It is useful to consider this case to gain some understanding of the resulting solutions. From (6.1.3) and (6.1.5) we obtain

$$w_\Lambda = -1 + \frac{\gamma_\Lambda}{3} \qquad w_K = -1 + \frac{\gamma_\Lambda - 1}{3}\,. \tag{6.2.2}$$

More interestingly, the effective fluid appearing on the right-hand side of the Einstein equations becomes also barotropic with index

$$w_e = -1 + \frac{\gamma}{3}\,, \quad \text{with} \quad \gamma = \gamma_\Lambda - \gamma_K\,. \tag{6.2.3}$$

The cosmological solution to (5.2.5) is then given by[3]

$$\rho_e(t) = \rho_{e*}\left(\frac{a}{a_*}\right)^{-\gamma}\,, \qquad a(t) = a_*(1+\frac{\gamma}{2}H_* t)^{\frac{2}{\gamma}}\,, \tag{6.2.4}$$

where ρ_{e*}, H_*, a_* are the initial values at the beginning of universe at time $t = 0$. The densities of the vacuum fluid and the gravifluid are given by

$$\rho_\Lambda(t) = (1-\gamma_K)\,\rho_e(t)\,, \qquad \rho_K(t) = \gamma_K\,\rho_e(t)\,. \tag{6.2.5}$$

In the semiclassical approximation both anomalous dressings are expected to be small. For positive γ, our model describes an expanding universe driven by an effective fluid with a barotropic index that is slightly larger than its classical value -1. In this case, we arrive at the same conclusion as we did in two dimensions: that the vacuum energy density decays from its initial value ρ_{e*} which could be of the order of the string scale or the scale of supersymmetry breaking. The classical exponential expansion of de

[2] In two dimensions, $\Gamma_K(\Omega)$ and $\Gamma_\Lambda(\Omega)$ are indeed linear functions with $\gamma_K = 0$ and $\gamma_\Lambda = 2\beta^2$. This is a consequence of the conformal invariance of the timelike Liouville theory.

[3] A spatially-flat Robertson-Walker solution is only compatible with $\Lambda \geq 0$, just as it happens classically.

Sitter spacetime is slowed down to a power law expansion as a result of the quantum anomalous gravitational dressings. In the limit of vanishing γ, one recovers de Sitter spacetime with constant density.

The slow-roll parameters (4.2.1) for our solution (6.2.4) are

$$\varepsilon_H = \frac{\gamma}{2} \quad \text{and} \quad \eta_H = 0\,. \tag{6.2.6}$$

Slow-roll inflation that lasts long enough requires that $\varepsilon_H \ll 1$ and $\eta_H \ll 1$. Since γ is small in the semiclassical approximation, all these conditions would be satisfied. A generic prediction is that $\eta_H = 0$. Thus, the quantum decay of vacuum energy and the dynamics of the $\Omega(x)$ field provides a new mechanism to drive slow-roll inflation in the early universe. For small γ, the scale factor expands almost exponentially as a power law with a very high exponent. Nonzero ε_H measures the deviation from exact exponential expansion but the parameter η_H vanishes as in exact de Sitter spacetime.

For a general functional form of the anomalous dressings Γ_K and Γ_Λ, the equation (6.1.8) represents a novel generalization of the usual Friedmann equation because the equation of state of the effective fluid is rather unusual. It is conceivable that this has interesting consequences for early cosmology. Numerical integration may be necessary to find the time-dependence of the scale factor. However, we see from (6.1.6) that as long as $\Gamma_\Lambda - \Gamma_K$ is positive during the cosmological history, vacuum energy will decay. For negative γ, the null energy condition would be violated.

Since the proposed four-dimensional action (5.1.1) is simply parametrized by the integrated anomalous gravitational dressings, further analysis of the above results requires specification of the latter. However, from dimensional analysis, the order of magnitude of the anomalous dimensions is expected to be $G_N\Lambda$. Given the UV cutoff M_0, then the vacuum energy is of order M_0^4, Λ is of the order M_0^4/M_p^2, and the Hubble scale is $H = M_0^2/M_p$. The anomalous dimensions would thus be of order H^2/M_p^2. In the very early universe, if for example H is of order $0.1M_p$, these estimates suggest that γ and the slow-roll parameter would be of order 0.01. One can thus obtain slow-roll inflation driven entirely by slowly decaying vacuum energy through the nontrivial effective dynamics of the Ω field. This provides an example of inflation without an inflaton.

Given that the effective action (5.1.1) effectively sums up the leading logarithms, as it is a solution to the local renormalization group equation, we expect our semiclassical solution to encode the same kind of logarithmic corrections to the classical de Sitter solution. To see this explicitly, we expand our solution (6.2.4) for the scale factor for small γ. If the classical exact de Sitter solution is $a^{dS}(t) = a_*^{dS}\, e^{H_* t}$, we find

$$a(t) = a^{dS}(t)\left[1 - \frac{\gamma}{4}\ln^2\left(\frac{a^{dS}}{a_*}\right) + \ldots\right] = a^{dS}(t)\left[1 - \frac{\gamma}{4}N^2(t) + \ldots\right], \tag{6.2.7}$$

where $N(t)$ is the number of e-folds. In terms of the conformal time, we can expand (1.3.23)

$$a(\tau) = a_*\left(\frac{\tau}{\tau_*}\right)^{\frac{2}{\gamma-2}} = a_*\left(\frac{\tau_*}{\tau}\right)\left(\frac{\tau}{\tau_*}\right)^{\frac{-\gamma}{2-\gamma}} = \frac{a_*\tau_*}{\tau}\left[1 - \frac{\gamma}{2}\ln\left(\frac{\tau}{\tau_*}\right) + \ldots\right]. \tag{6.2.8}$$

This makes very apparent that these are infrared quantum effects, as they increase with time. One obtains the usual de Sitter solution when $\gamma = 0$. For nonzero γ, there are logarithmic corrections which add up to a small exponent that slows down the de Sitter expansion.

6.3 Broken Time Translation Symmetry and Stability

This novel mechanism for the decay of the vacuum energy raises the following puzzle. Unlike the classical de Sitter solution, our quantum-corrected slow-roll solution (6.2.4) breaks the global time translation symmetry

$$t \to t + \pi \tag{6.3.1}$$

of the action (5.1.1) for a constant π. If a solution breaks a global symmetry of an action, the symmetry-transform of a given solution generates a new solution. This implies that if one now considers a position dependent symmetry parameter $\pi(x)$, the effective action for $\pi(x)$ must be derivatively coupled so that there is a flat direction, and arbitrary constant π is a solution of the equations of motion that follow from this effective action. Correspondingly, one expects a Nambu-Goldstone like scalar fluctuation mode. In usual inflationary models, this scalar mode can be identified with a gauge-invariant combination of the inflaton and the metric. This idea is the basis of effective field theories of inflation [158, 159] (for a good review see [160]). Where is this additional scalar degree of freedom? One could pose the puzzle slightly differently. Time translation symmetry is part of the diffeomorphism group. How can quantum effects break this symmetry?

The resolution of this puzzle is as follows. The scale factor of our solution has an initial value a_* at the initial value surface $t = 0$. Since we are using semiclassical gravity, a_* can be taken to be of the order of the short-distance cutoff scale a little larger than the Planck length. This means that, unlike the eternal de Sitter solution, one cannot continue this solution to times earlier than $t = 0$. The global time translation symmetry is thus explicitly broken by the fact that one must cutoff the evolution with an initial value surface in the early universe and impose initial conditions. Even though the action is invariant under the time translation symmetry, the initial conditions are not. Thus, one cannot apply the argument above to generate new solutions from a given solution, to deduce the existence of a propagating scalar degree of freedom.

One can state the result slightly differently. The non-local expression (5.1.2) for the Weyl factor follows from inverting (5.1.7) only if one discards all solutions of the homogeneous equation

$$\Delta_4 \Sigma_g = 0 \, . \tag{6.3.2}$$

These solutions correspond precisely to the would-be Nambu-Goldstone scalar fluctuations. The initial conditions on Σ_g on the initial value surface ensure that Σ_g is determined entirely in terms of the metric and is not an additional propagating field.

It is possible to reformulate the argument above using a local action. One can recast the non-local action (5.1.1) in a local form [69, 153] by introducing two auxiliary scalar fields $\Sigma(x)$ and $\Psi(x)$ with the action

$$S[g,\Sigma,\psi]=\frac{M_p^2}{16\pi}\int d^4x\sqrt{-g}\left[R_g e^{-\Gamma_K(\Sigma)}-2\Lambda e^{-\Gamma_\Lambda(\Sigma)}+\Psi\left(\Delta_4\Sigma-\frac{1}{4}F_4(g)\right)\right]. \tag{6.3.3}$$

The equations of motion for the two auxiliary fields are

$$\Delta_4\Psi(x)=R_g\,\Gamma_K^{(1)}\,e^{-\Gamma_K(\Sigma)}-2\Lambda\,\Gamma_\Lambda^{(1)}\,e^{-\Gamma_\Lambda(\Sigma)}\,;\qquad \Delta_4\Sigma(x)=\frac{1}{4}\,F_4(g)\,. \tag{6.3.4}$$

The field $\Psi(x)$ acts therefore as a Lagrange multiplier for the condition $\Sigma=\Sigma_g$, and we recover (5.1.1) upon using its equation of motion in (6.3.3). This local action will reduce to the original non-local action only if the homogeneous solutions of (6.3.4) are eliminated by imposing an initial condition for Ψ and Σ that is similar to the initial condition for Σ_g. This ensures that the only propagating degrees of freedom are the usual tensor fluctuations of the metric and there are no additional scalar fluctuations.

If a Lagrangian depends on higher-time derivatives of the fields, then one should also worry about the possibility of the Ostrogradsky instability [161]. We do not carry out the stability analysis of our action in this paper but refer the reader to the stability analysis for a class of non-local actions [68, 69] similar to the one we consider in this paper in the R-flat gauge of section §5.3.

These conclusions are physically reasonable from the point of view of the original quantum path integral. The action (5.1.1) is the quantum 1PI-effective action for the background metric obtained by a semiclassical evaluation of the path integral at weak coupling. It would be strange if one were to discover an extra scalar degree of freedom or unphysical instability in this infrared effective action if the starting point is a well-defined path integral.

Chapter 7

Discussion

Conclusions

In this thesis we have analyzed the cosmological consequences of the Weyl anomalies arising from the renormalization of the Einstein-Hilbert and cosmological constant operators in two and in four dimensions. Our approach is based on two main steps. The first one consists of writing down the non-local quantum effective action that explicitly incorporates the Weyl anomalies of the two afore-mentioned operators. The second step is to look for the cosmological quantum dynamics this action leads to.

Composite operators of the fundamental fields acquire anomalous Weyl dimensions due to their covariant renormalization. In the effective action, the anomalous dimensions are encoded in the gravitational dressings of the corresponding operators. When gravity is dynamical, the operators acquire an additional anomalous dimension and gravitational dressing, which is a priori different from those of the same operators on a fixed background spacetime. The physical coupling constants are then the couplings of the fully gravitationally-dressed operators.

The gravitational dressings depend on the conformal factor of the metric, which is non-local when written covariantly in terms of the full-metric. The non-locality is to be expected, as this is the action for the low-energy effective theory which should follow from integrating out the fluctuations of various fields all the way down to the characteristic scales of classical gravity. The non-locality of the dressings is further dictated by the local renormalization group, this suggests a specific non-local generalization of Einstein gravity.

The Weyl-invariant formalism turns out to be a very useful tool. This is because it effectively implements a manifestly covariant split of the conformal factor from the fiducial metric. In two dimensions, it greatly facilitates the computation of the gravitational dressing of the cosmological constant. Since it makes the gauge Weyl invariance manifest, it allows to further explicitly compute its anomalous dimension by requiring that the Weyl Ward identity is satisfied, reproducing the well-known Liouville result. In four dimensions, it simplifies the computation of the cosmological quantum evolution equations, by allowing a general metric variation while still profiting from imposing the Robertson-Walker symmetries.

The main lesson that we abstract from our analysis is that the gravitational dressings of the composite operators in the effective action modify, not only the trace, but the full quantum momentum tensor, which in turn can modify the gravitational dynamics. Even small values for these gravitational dressings can have observable effects in the cosmological evolution when the universe undergoes an exponential expansion with several e-foldings.

Our chief result in two dimensions, where the anomalous dimension and gravitational dressing of the cosmological constant can be computed exactly, the renormalized vacuum fluid can source a homogeneous and isotropic universe, and has barotropic index $\gamma = -1 + 2\beta^2$, the anomaly slightly increasing the classical value. This leads to a vacuum energy density that slowly decays with time, which sources a power-law expansion, recovering the de Sitter exponential solution in the classical limit.

In four dimensions, we parametrize the action with the integrated anomalous dimensions γ_K and γ_Λ and assume them constant, both because of simplicity and because this can be a good approximation for long enough time intervals during the evolution of the universe. The main result is that the same vacuum energy decay and quasi de Sitter expansion are found, under the further assumption that $\gamma_\Lambda - \gamma_K > 0$. We conclude therefore that the considered effective action has the potential of being a model for slow-roll inflation, strongly suggesting that Weyl anomalies could have played an important role in the primordial evolution of the universe.

Discussion and Outlook

The appearance of timelike Liouville as the effective action for the conformal factor of the metric is of great advantage to understanding two-dimensional quantum gravity, because of all the well-known results about spacelike Liouville theory. For example, the computation of the gravitational dressing of the cosmological constant was greatly simplified because of the renormalizability of the full interacting theory by simple normal ordering. However, still some question marks remain about the timelike theory, the one relevant for gravity, since not all the spacelike results analytically-continue properly to the timelike regime. The lack of a full understanding of this theory did not affect us because we were interested in the semiclassical regime: we did not require full quantization, identification of the spectrum or its correlation functions. It would be very interesting though, to make sense of the full quantum theory as a solvable model and explore its gravitational consequences, specially in the context of the work presented in this thesis.

Path integral quantization of timelike Liouville theory is complicated because the action is unbounded from below. As opposed to the spacelike regime, the theory is not unitary in the timelike regime, since under the analytic continuation of the background charge $Q = iq$, the timelike central charge becomes $c = 1 - 6q^2 < 1$. This implies that the spectrum need not be the same in the two regimes, which is bound to further modify the correlation functions. In fact, the three point function of spacelike Liouville, given by the highly non-trivial DOZZ formula [120, 121], blows up under the analytic continuation.

Fortunately, in two dimensions, the framework of conformal bootstrap [125] offers an approach to defining the theory based on the exclusive use of the spectrum of operators, together with the two and the three point functions satisfying the bootstrap constraints, namely the crossing symmetry equations. All higher point functions then, can in principle be constructed by gluing. There has been steady progress over the past decade in solving timelike Liouville theory within the conformal bootstrap framework. In particular, an alternative three point function was obtained in [117, 162], which is finite in the timelike regime. This modified DOZZ formula was found to solve the Teschner recursion relations [58], a subset of the bootstrap equations which were originally used to find the DOZZ formula in the spacelike regime. This exact result can be reproduced from semiclassical computations [61] and using Coulomb gas methods [163]. A three point function, though, not only has to solve the Teschner relations, but all the crossing symmetry equations, to lead to a consistent theory. This has in fact recently been shown numerically for this modified DOZZ three point function [119].

Despite these advances, still some subtleties remain to be understood about timelike Liouville, specially in its applicability as a theory of gravity. As shown in §2.3, the natural metric measure of the quantum gravity path integral is the Weyl-invariant one, induced by the norm

$$(\delta\chi, \delta\chi) = \int d^2x\sqrt{-h}\, e^{2\beta\chi}\, \delta\chi(x)\, \delta\chi(x)\,. \tag{7.0.1}$$

This measure on field space suppresses quantum fluctuations from the regions where χ is very negative. In Liouville theory, on the other hand, one uses the shift-invariant measure, as done in (2.3.26), which has no such suppression. This raises the question of whether Liouville theory is the correct model for two-dimensional quantum gravity [16].

If one wishes to suppress the quantum fluctuations by hand, then one should restrict the range of the field χ in the path integral to not reach regions of very negative values. Such regions correspond to very short physical distances, since the latter are computed with the physical metric which depends on the exponential of the field χ. So this is effectively like putting an ultraviolet cutoff on the metric field. Indeed, when the metric is dynamical, putting a physical cutoff such as the Planck length at short distances really requires putting a boundary in field space. This is somewhat natural in a non-renormalizable effective field theory such as gravity in four dimensions, which is well defined only up to some distance scale. However, putting such a cutoff is a generic problem in defining a path integral over metrics, since it automatically spoils covariance as it introduces some background dependence.

Spacelike Liouville theory possesses a nontrivial duality symmetry under $b \to 1/b$, as can be for example checked explicitly from the expression of the DOZZ formula. This suggests that the action should include the dual cosmological constant term $e^{2\,\varphi/b}$, in addition to the usual one $e^{2b\,\varphi}$ [121]. This term in the action grows exponentially for very negative φ, hence can effectively suppress the unwanted quantum fluctuations that the Gaussian measure does not manage to. Timelike Liouville possesses an analogous $\beta \to 1/\beta$ duality symmetry. One can then hope that the dual operator $e^{2/\beta\,\chi}$ implements the same suppression of quantum fluctuations in the ultraviolet regions. However, this is not totally clear, as the dual cosmological term seems to become imaginary along the integration cycle that renders the path integral well defined.

These are nevertheless high energy considerations, related to the ultraviolet regions of the theory, while we are concerned with the low-energy semiclassical physics. This is eventually the reason we do not include the dual cosmological operator in our action. Since its role should be that of precluding the theory to probe small distances, it can be interpreted as the two-dimensional analog of the higher-dimensional non-renormalizability of gravity, which requires an ultraviolet completion to describe the physics at high energies. As long as one is interested in the effective theory then, the effects of such an operator should not come into play. It would be interesting though, to check explicitly the extent of this assertion.

Besides these Liouville theory related subtleties concerning the full quantization of the gravitational theory, there are other directions of further explorations within our two-dimensional model. The immediate one would be the implementation of the techniques developed, based on the use of the Weyl-invariant formulation and the Weyl Ward identity, to compute gravitational dressings of general matter operators. The interaction between gravity and general matter might be complicated, and such that the renormalization of composite operators of both the Liouville field $\chi(x)$ and the matter fields might not be as straightforward as it was for the cosmological operator. Still, Liouville theory should provide simplifications. An example of this is the free massive Majorana fermion. In the Weyl-invariant formulation, the mass term in the classical action would read

$$I_f = m \int d^2x\sqrt{-g}\,\bar{\psi}\,\psi = m \int d^2x\sqrt{-h}\,\bar{\psi}\,\psi\, e^{\Omega}\,, \tag{7.0.2}$$

due to the classical Weyl weight of the bi-linear. The operator is hence a composite of the fermionic and the metric fields, and we write the renormalized one as $[\bar{\psi}\,\psi\, e^{2\alpha\,\chi}]_h$, where the parameter α is introduced to account for the anomaly, just as we did with β for the cosmological operator. Even if the explicit renormalization could be complicated, Liouville theory provides us again with the anomalous dimension of this operator [15, 164], which is $2\alpha^2$. Imposing Weyl invariance then, leads to the equation for this anomalous dimension in terms of the fundamental coupling

$$\alpha\,(q+\alpha) = \frac{1}{2}\,, \tag{7.0.3}$$

which is slightly different from the one for the cosmological operator $\beta\,(q+\beta) = 1$ due to the different classical scalings. Weyl invariance then dictates that the effective action should read

$$\Gamma_f = m \int d^2x\sqrt{-h}\,\bar{\psi}\,\psi\, e^{\Omega}\, e^{-2\alpha^2(\Omega+\Sigma_h)} = m \int d^2x\sqrt{-g}\,\bar{\psi}\,\psi\, e^{-2\alpha^2\Sigma_g}\,, \tag{7.0.4}$$

where in the second step we went to the physical gauge. It seems then that it is feasible to build more general models than the purely gravitational one considered, with composite operators of other matter fields, where the gravitational dressings of the latter are incorporated in the effective action. Such theories would make up for much more realistic cosmological models. Even if two-dimensional, exploring such models is interesting because they allow for explicit computations thanks to the simplicity brought about by the Weyl-invariant formulation and Liouville theory, while they may still lead to non-trivial results that can give guidance on how to do the analogous analysis in four dimensions.

Regarding our four-dimensional model, the obvious next step is the computation of the anomalous dimensions and consequently of the integrated gravitational dressing functions $\Gamma_K(\Sigma_g)$ and $\Gamma_\Lambda(\Sigma_g)$. Renormalization of Newton's constant and the cosmological constant has been considered earlier in the literature [18–20], one can extract the precise logarithmic running from these results. We sketch the logic in the following.

Within the background field method, we consider fluctuations of the metric around a classical background. At one-loop order, the (Euclidean) effective action needs only include up to the quadratic terms in the fluctuations, i.e.

$$e^{-S_{ef}[g]} = e^{-S_G[g]} \int \mathcal{D}h_{\mu\nu}\; e^{-S_G^{(2)}[h]}\,. \tag{7.0.5}$$

Since the integral of the quadratic action is divergent, the classical action should include the counterterms. In renormalized perturbation theory then

$$S_G[g] = -\frac{1}{16\pi G_N\, \mathcal{Z}_G^{-1}} \int d^4x\sqrt{g}\, \left(R_g - 2\Lambda\, \mathcal{Z}_\Lambda^{-1}\right)\,, \tag{7.0.6}$$

where the two couplings are the running ones, and $\mathcal{Z}_i = 1 + \delta_i$ absorb the divergences as usual, their scale variation giving the β-function of each coupling constant. To compute the latter then, we need to regularize the integral of the quadratic action. In particular, if we use dimensional regularization, the β-functions are determined by the coefficient of the $1/\epsilon$ pole.

The dimensional regularization of the quadratic action can be performed using the heat kernel technique, for which one first has to identify the quadratic differential operators Δ_i^Λ of the different sectors in the quadratic action. In our purely gravitational case, these are the traceless symmetric-two-tensor $\bar{h}_{\mu\nu}$, the trace scalar h and the ghost vector field V^ν, which appears from the diffeomorphism gauge fixing. The integral of this quadratic action gives

$$e^{-\Delta S_{ef}} = e^{-\frac{1}{2}\mathrm{Tr}\left(\ln\Delta_2^\Lambda + \ln\Delta_0^\Lambda - 2\ln\Delta_1^\Lambda\right)}\,, \tag{7.0.7}$$

where the operators Δ_i^Λ are quadratic in derivatives and include a non-homogeneous piece proportional to the cosmological constant. The traced logarithms of the operators can be expressed with the Schwinger-time integral of the heat kernel of each operator $K_\Delta(x, y; \tau) = \langle x|e^{-\Delta\tau}|y\rangle$. Further, the trace of the heat kernel admits a short-time expansion $K_\Delta(\tau) = \sum_{m\geq 0} \tau^{m-d/2}\, b_m$, where the $b_m(x)$ are the so-called Seeley-de Witt coefficients. To use dimensional regularization, we do $d+\epsilon$ in this expansion. The $1/\epsilon$ pole then, can only appear in the case of even spacetime dimension, and comes from the term $m = d/2$, since the time integral gives

$$\mathrm{Tr}\ln\Delta = -\int dx \sum_{m\geq 0} b_m(x) \int_0^\infty d\tau\, \tau^{m-d/2-1} = \cdots - \int dx\, b_{d/2}(x)\, \frac{2}{\epsilon} + \ldots \tag{7.0.8}$$

The β-functions therefore, will be determined by the $b_{d/2}$ Seeley-de Witt coefficients of the quadratic operators in the action. In the case of four dimensions, we are interested in the $b_2(x)$ coefficient. General expressions for these coefficients exist for an operator of the kind $\Delta = -\nabla^2 - E$, which is the case of the differential operators in the quadratic gravitational action. Identifying the precise operators E_i from the Δ_i^Λ in (7.0.7), the $b_2(x)$ coefficients for the three operators can be computed. This can be done on-shell, as is allowed within the background field method, then the coefficients become proportional to Λ^2, hence constant [19]. With them, the $1/\epsilon$ pole in the effective action becomes

$$\Delta S_{ef} = -\int d^4x\sqrt{g}\,\frac{1}{\epsilon}\,\left(b_2^{(2)} + b_2^{(0)} - 2\,b_2^{(1)}\right) = \int dx\,\frac{1}{\epsilon}\,\frac{29}{40}\frac{\Lambda^2}{\pi^2}\,. \qquad (7.0.9)$$

This divergence can then be entirely absorbed by the cosmological constant counterterm, since

$$S_{ef}[g] = S_G + \Delta S_{ef} = -\frac{1}{16\pi G_N}\int dx\,\left(R_g - 2\Lambda\,(1-\delta_\Lambda + \frac{M^\epsilon}{\epsilon}\,\frac{29}{5\pi}\Lambda\,G_N)\right)\,. \quad (7.0.10)$$

The anomalous dimensions then follow as

$$\gamma_G = 0 \qquad \gamma_\Lambda = \frac{d\ln\mathcal{Z}_\Lambda}{d\ln M} = \frac{29}{5\pi}\,\Lambda\,G_N\,. \qquad (7.0.11)$$

Hence at one loop, it seems that the dressing Γ_K should vanish, and the cosmological one Γ_Λ would be of order $G_N\Lambda$ as expected from dimensional analysis. In the present era, these effects would indeed be very small. However, they could be relevant during the primordial stages of the universe. In terms of the Hubble scale at the very early universe, we would have $\gamma = (87/5\pi)\,H_*^2/M_p^2$. If again we take the example of the Hubble scale starting of order $0.1M_p$, the anomaly $\gamma = \gamma_\Lambda$ would be around 0.06, and the slow-roll parameter ε_H around 0.03, so the numerical coefficients seem to slightly enhance the effect of the anomalies. The conclusion in any case is that at least at this loop order, our solution does describes a slow-roll inflation, driven entirely by a slowly decaying vacuum energy. A high value of H is ruled out by current bounds on primordial gravitational waves [165], but it is interesting that a mechanism of inflation without any inflaton is possible. It is worth exploring if there are other ways to enhance the anomalous dressings.

As it stands, both our two and four-dimensional models are too simplistic to provide a realistic scenario of primordial inflation, as the cosmological constant would simply keep driving inflation to end up in an empty universe with a vanishing value of the former. It is thus necessary to try to embed them into a realistic cosmology. More concretely, we should add matter fields which could provide a graceful exit to put an end to the inflationary period and start a hot big bang. Also, one should then account for their effects on the running of the cosmological constant, and the effects of the gravitational dressings of their possible interaction terms on their couplings. As explained above, the two-dimensional model makes this program more feasible.

Finally, a realistic model should also reproduce the primordial perturbations observed in the cosmic microwave background. Our gravitational effective action (5.1.1) should lead to a different quadratic action for the tensor perturbations than that of Einstein-Hilbert gravity. Hence it is expected to induce a non-trivial tensor sound speed that depends on the Weyl anomalous dimensions, and which will appear in the tensor power-spectrum. Since this sound speed could depend on time, modifications on both the tensor-to-scalar ratio r and the tensor tilt n_t are generically to be expected. One of the possible consequences of these modifications is a violation of the consistency condition between the previous two $r = -8\,n_t$, typical from standard single-field slow-roll inflation. Such violations have already been found in models with higher-derivative quantum gravitational corrections [166, 167], and they should be observable in future

CMB polarization experiments. This could provide some test on the modifications of gravity due to Weyl anomalies. Regarding scalar perturbations, since our quasi de Sitter solution is entirely driven by the metric, we need to add additional fields to produce them. As a starting point, we can use curvaton-like mechanisms [168–171], where the fields responsible for the perturbations do not drive the inflationary background. High values of r due to the tensor power spectrum put lower bounds on the scalar sound speed or dispersion relation [172], which would act as a consistency condition to impose on our scalar model.

The idea of vacuum energy decay caused by infrared quantum effects has been much explored earlier in four-dimensional gravity. There is considerable divergence in the literature about the final result [77–83, 106, 173] and more generally about infrared effects in nearly de Sitter spacetime [84–95, 174]. One of the new ingredients in the present work is the separation of the computation of the anomalous dimensions from the cosmological evolution they lead to. The anomalous dressings encode the quantum effects in the non-local action in a gauge-invariant way. The semiclassical dynamics then can be computed with a classical approach, by simply deriving equations of motion. This way of organizing the analysis may be useful for future explorations, and gives insight into what kind of quantum effects to expect. For instance, the effects of the anomalies that we find in our quasi de Sitter quantum solution are logarithmic corrections around the classical exact de Sitter solution, as is to be expected from quantum effects coming from an effective action that follows from integration of the renormalization group. Despite the relevance of de Sitter quantum gravity for cosmological purposes, it is much less understood than quantum gravity in flat or anti de Sitter spacetime. It would be interesting to use our new perspective to understand some of its conceptual difficulties, such as the instability of a de Sitter invariant vacuum, or the identification of gauge-invariant observables.

Just as quasi de Sitter is a solution to the equations (5.2.8), so is quasi Anti-de Sitter if we assume a negative cosmological constant. In this case, we impose the solution to be Weyl-equivalent to the Minkowski metric, with a conformal factor that depends only on the radial coordinate $\Omega = \Omega(z)$[1]

$$g_{\mu\nu} = a^2(z)\,\eta_{\mu\nu} = e^{2\Omega(z)}\,(-dt^2 + dz^2 + d\vec{x}^2)\,. \tag{7.0.12}$$

With this assumption, the gravifluid and cosmological momentum tensors simplify considerably just as for the $\Lambda > 0$ case, and the Einstein equations reduce to one total differential equation for the conformal factor. The solution reads

$$a(z) = a_*\,(\frac{z}{z_*})^{\frac{2}{\gamma-2}}\,, \tag{7.0.13}$$

equivalent to the cosmological solution but replacing τ by z. Exact AdS $a(z) \sim 1/z$ is recovered in the semiclassical limit $\gamma \to 0$, and a small value of the anomalous dimension entitles a conformal factor that diverges faster as the boundary is approached at $z \to 0$.

[1] Since $z > 0$, the fiducial is just half of Minkowski spacetime.

This AdS solution is very interesting because it allows us to put the results obtained in the context of the AdS/CFT correspondence. The anomalous dimensions appearing as corrections to the fall-off of the pure AdS solution is consistent with the radial coordinate z effectively running the renormalization group in the dual theory. It would be interesting to figure out how the gravitational effects of Weyl anomalies show up in a possible dual field theory, and to explore whether knowledge of the latter can give input on the gravitational dressings to expect. Finally, since our formalism for incorporating Weyl anomalies in the effective action should not depend on the sign of the cosmological constant, it could be possible to directly translate any knowledge gained about the duals of Weyl anomaly effects within the more familiar AdS/CFT arena, to the more resisting dS/CFT correspondence.

Appendix A

Notation and Conventions

In this appendix we gather the signs and conventions used along this thesis, as well as the main definitions of fields and constants, and the relations between them.

- Our gravitational constant κ^2 is defined to be dimensionless as

$$\frac{M_p^{d-2}}{16\pi} = \frac{M_0^{d-2}}{2\kappa^2}\,, \tag{A.0.1}$$

  where M_p is the Planck mass and M_0 is a UV cutoff below the Planck scale

- We follow the $(+,+,+)$ conventions of Misner, Thorne and Wheeler [175]. Thus, the spacetime signature is $(-,+,...,+)$.

- The Lorentzian action is denoted by I and the Euclidean one by S.

  Euclidean signature is achieved by performing a Wick rotation of the time coordinate. If Lorentzian time is t and Euclidean time is t_e, then the Wick rotation can be thought of the coordinate transformation $t = -i\,t_e$. Under this Wick rotation, all quantities transform according to the tensor transformation rule. However, the Euclidean action is typically defined in the path integral as

$$\mathcal{Z} = \int e^{iI[g,\phi]} \rightarrow \mathcal{Z} = \int e^{-S[g_e,\phi_e]}\,. \tag{A.0.2}$$

  with $S[g_e,\phi_e] = \int d^dx_e\,\sqrt{g_e}\,\mathcal{L}_e(g_e,\phi_e)$. Therefore the Euclidean Lagrangian is defined with an extra minus sign $\mathcal{L}_e(g_e,\phi_e) := -\mathcal{L}(g,\phi)$, even if by coordinate transformation of a scalar they should equal each other.

- The momentum tensor in Lorentzian and Euclidean signatures is defined respectively as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I}{\delta g^{\mu\nu}}\,, \qquad T^e_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\,, \tag{A.0.3}$$

  where in the second formula it is understood that the metric is the Euclidean one. The different minus sign in the two formulas precisely cancels the additional sign in the Euclidean action, therefore the two tensors are the same up to the Wick rotation of the fields.

- Given an operator renormalized at renormalization scale M, such that $\mathcal{O}_0 = \mathcal{Z}(M)\,\mathcal{O}_M$, with $\mathcal{O}_0$ being the operator at the cutoff scale, its anomalous dimension is defined as
$$\gamma := M\frac{\partial \ln \mathcal{Z}(M)}{\partial M}\,. \tag{A.0.4}$$
In the action, the operator appears as $I = \int \sqrt{-\eta}\,\lambda(M)\mathcal{O}_M$, where λ is the (dimensionful) coupling constant. If Δ is the classical scaling dimension of the operator, then the β-function is $\beta_\lambda = (\Delta - d + \gamma)\,\lambda$. The contribution of the anomalous dimension in the trace is
$$\eta^{\mu\nu}T_{\mu\nu} = -\beta_\lambda\,\mathcal{O}_M\,. \tag{A.0.5}$$
In Euclidean signature, both formulas hold with the very same signs, the only sign difference comes into the action, as this one reads $S = -\int\sqrt{\delta}\,\lambda(M)\mathcal{O}_M$.

- Given an operator renormalized with respect to the metric $g_{\mu\nu}$, such that $\mathcal{O}_0 = \mathcal{Z}_g\,\mathcal{O}_g$, its anomalous dimension is defined as
$$\gamma := \frac{\delta \ln \mathcal{Z}_g}{\delta \Sigma_g}\,. \tag{A.0.6}$$
The trace of the resulting momentum tensor then satisfies the same equation (A.0.5).

- The two-dimensional c-anomaly reads $T = \frac{c}{24\pi}R_g$, with the $+$ sign in both Lorentzian and Euclidean signature, even if the definition of the momentum tensor in the two signatures has a relative $-$ sign.

- In spacelike Liouville, the background charge Q, the Liouville coupling b and the central charge are related as
$$b(Q-b) = 1\,, \qquad b_- = \frac{1}{Q} + \frac{1}{Q^3} + \frac{2}{Q^5} + \dots\,, \qquad c = 1 + 6Q^2\,. \tag{A.0.7}$$
In timelike Liouville,
$$\beta(q+\beta) = 1\,, \qquad \beta_+ = \frac{1}{q} - \frac{1}{q^3} + \frac{2}{q^5} + \dots\,, \qquad c = 1 - 6q^2\,. \tag{A.0.8}$$
The analytic continuation between the two is given by
$$Q = iq\,, \qquad \varphi = i\chi\,, \qquad b = -i\beta\,. \tag{A.0.9}$$

Appendix B

Conformal Geometry and Useful Formulas

Conformal geometry concerns the transformation properties of various objects of Riemannian geometry under Weyl rescaling. In the following, we enumerate the transformations that are relevant in this work.

Let's consider the Weyl transformation of metric $h_{\mu\nu}$ with parameter $\Omega(x)$

$$g_{\mu\nu} = e^{2\Omega}\, h_{\mu\nu}\,. \tag{B.0.1}$$

In the following, all covariant derivatives are with respect to the $h_{\mu\nu}$ metric. The transformations of the Ricci and Einstein tensors, and the Ricci scalar are

$$\begin{aligned} R_{\mu\nu}(g) &= R_{\mu\nu}(h) + V_{\mu\nu}(\Omega, h)\,, && \text{(B.0.2)} \\ E_{\mu\nu}(g) &= E_{\mu\nu}(h) + D_{\mu\nu}(\Omega, h)\,, && \text{(B.0.3)} \\ R(g) &= e^{-2\Omega}\left[R(h) + V(\Omega, h)\right]\,, && \text{(B.0.4)} \end{aligned}$$

where

$$\begin{aligned} V_{\mu\nu} &= (d-2)\left[-\nabla_\mu\nabla_\nu\Omega + (\nabla_\mu\Omega)(\nabla_\nu\Omega)\right] - h_{\mu\nu}\left[\nabla_h^2\Omega + (d-2)(\nabla\Omega)^2\right]\,, && \text{(B.0.5)} \\ V &= h^{\mu\nu}V_{\mu\nu} = -2(d-1)\nabla_h^2\Omega - (d-1)(d-2)\,(\nabla\Omega)^2\,, && \text{(B.0.6)} \\ D_{\mu\nu} &= V_{\mu\nu} - \frac{1}{2}h_{\mu\nu}V && \text{(B.0.7)} \\ &= (d-2)\left[-\nabla_\mu\nabla_\nu\Omega + (\nabla_\mu\Omega)(\nabla_\nu\Omega) + h_{\mu\nu}\left(\nabla_h^2\Omega + \frac{d-3}{2}\,(\nabla\Omega)^2\right)\right]\,. && \text{(B.0.8)} \end{aligned}$$

The Einstein tensor for any metric vanishes identically in two dimensions. This can be seen by noting that the Einstein tensor is a functional derivative of the Einstein-Hilbert action. Since in two dimensions the Einstein-Hilbert action is proportional to the Euler character, which is topological invariant, its variation must vanish for any smooth variation of the metric. This implies that the tensor $D_{\mu\nu}$ must also vanish identically for $d = 2$ because it is the difference between two Einstein tensors. This explains why $D_{\mu\nu}$ is proportional to $(d-2)$.

Variation of the Integrated Ricci scalar

Consider a functional of the form[1]

$$I[\Sigma, h] = \int d^d x \sqrt{-h}\; R(h)\, \Sigma(x) \tag{B.0.9}$$

for some scalar function $\Sigma(x)$. Under a general variation of the metric $h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu}$, the variation of this action is given by

$$\delta I = \int d^d x \sqrt{-h}\; \delta h^{\mu\nu} \left[E_{\mu\nu}\, \Sigma(x) - (\nabla_\mu \nabla_\nu - h_{\mu\nu} \nabla^2)\, \Sigma(x) \right] , \tag{B.0.10}$$

where $E_{\mu\nu}$ is the Einstein tensor

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} h_{\mu\nu}\, R\,. \tag{B.0.11}$$

The second term in the square brackets comes from integrating by parts. The Ward identity that follows from the diffeomorphism invariance of this functional

$$-\, 2\nabla^\nu \left(\frac{\delta I}{\delta h^{\mu\nu}} \right) - \frac{\delta I}{\delta \Sigma}\; \nabla_\mu \Sigma = \; 0\,, \tag{B.0.12}$$

can be verified using

$$[\nabla_\lambda, \nabla_\mu]\, \nabla_\nu \Sigma = -R^\sigma{}_{\nu\lambda\mu}\, \nabla_\sigma \Sigma\,, \tag{B.0.13}$$

which leads to

$$\nabla^\nu (\nabla_\mu \nabla_\nu \Sigma - h_{\mu\nu} \nabla^2 \Sigma) = R_{\mu\nu} \nabla^\nu \Sigma\,, \tag{B.0.14}$$

and together with the Bianchi identity for the Einstein tensor, implies that the covariant divergence of the term in the square bracket of (B.0.10) equals

$$-\, \frac{1}{2} R\, \nabla_\mu \Sigma\,. \tag{B.0.15}$$

The variation of the Einstein-Hilbert action[2]

$$I_K[h] = \frac{1}{2\kappa^2} \int d^d x \sqrt{-h}\, R(h)\,, \tag{B.0.16}$$

follows from (B.0.10) by doing $\Sigma(x) = 1$, and gives the Einstein tensor appearing in the left-hand side of Einstein equations. In the case the variation of the metric is a Weyl transformation, $\delta h^{\mu\nu} = -2\, \delta\xi(x)\, h^{\mu\nu}$, it becomes

$$\delta I_K = \frac{(d-2)}{2\kappa^2} \int d^d x \sqrt{-h}\; \delta\xi(x)\, R(h)\,, \tag{B.0.17}$$

which shows that the Einstein-Hilbert action is only Weyl invariant in two dimensions.

[1] All along this thesis, we will use both $R(h)$ and R_h to denote the Ricci scalar of the $h_{\mu\nu}$ metric.
[2] In the following we choose the units $M_0 = 1$ according to our notation (A.0.1).

Appendix C

c-Anomaly and the Polyakov Action

In this appendix we compute the Polyakov action as arising from the integration of the two-dimensional c-anomaly. We do this in Euclidean signature.

The variation of the action $W[h]$ with respect to the metric $h_{\mu\nu}$ is given by

$$W[h+\delta h] - W[h] \ = \delta W[h] = \frac{1}{2}\int d^2x\sqrt{h}\,\delta h^{\mu\nu}\,T_{\mu\nu} \tag{C.0.1}$$

In the conformal gauge, the Weyl variation of the metric can be written in terms of the variation of the conformal factor as $\delta h_{\mu\nu} = 2\,\delta\Sigma_h\,h_{\mu\nu}$. The Weyl variation of the action then follows as

$$\delta W[h] = -\int d^2x\sqrt{h}\,\delta\Sigma_h\,h^{\mu\nu}\,T_{\mu\nu}\,. \tag{C.0.2}$$

The two-dimensional c-anomaly for a CFT with central charge c reads

$$T_h(x) = \frac{c}{24\pi}\,R_h(x)\,. \tag{C.0.3}$$

Inserting it in the variation of the action we find

$$\delta W[h] = -\frac{c}{24\pi}\int d^2x\sqrt{h}\,\delta\Sigma_h\,R_h = -\frac{c}{24\pi}\int d^dx\sqrt{\delta}\,\delta\Sigma_h\,\Big(R_\delta - 2\,{\nabla_\delta}^2\Sigma_h\Big)\,, \tag{C.0.4}$$

where in the second step we have used the Weyl transformation of the Ricci scalar. We can now integrate on both sides

$$W[g] - W[h] = -\frac{c}{24\pi}\int d^2x\sqrt{\delta}\,\Big(R_\delta\,\Sigma - \Sigma\,\nabla_\delta^2\,\Sigma\Big)_{\Sigma_h}^{\Sigma_g} \tag{C.0.5}$$

from which follows

$$W[g] = -\frac{c}{24\pi}\int d^2x\sqrt{\delta}\,\Big((\nabla\Sigma_g)^2 + R_\delta\,\Sigma_g\Big)\,. \tag{C.0.6}$$

To write it fully-covariantly, we can use the expression for the conformal factor

$$\Sigma_g(x) = \frac{1}{2}\int d^2y\sqrt{g}\,G_g(x,y)\,R_g(y)\,, \tag{C.0.7}$$

with which the effective action becomes the Polyakov action

$$W[g] = -\frac{c}{96\pi}\int d^2x\sqrt{g}\, d^2y\sqrt{g}\, R_g(x)\, G_g(x,y)\, R_g(y) = -\frac{c}{96\pi}\int R_g\, \frac{1}{\Box}\, R_g\,, \tag{C.0.8}$$

where $\Box = -\nabla^2$. Further, from (C.0.5) follows that the transformation of the effective or Polyakov action when doing $g_{\mu\nu} = e^{2\Omega}\, h_{\mu\nu}$ is the timelike Liouville action

$$W[g] - W[h] = -\frac{c}{24\pi}\int d^2x\sqrt{h}\,\left((\nabla\Omega)^2 + R_h\Omega\right)\,. \tag{C.0.9}$$

Adding a constant to the trace anomaly equation (C.0.3) would result into an additional cosmological constant term in the effective action, or the cosmological constant operator in the Liouville action.

Appendix D

The Regularized Coincident Green Function

We now compute the coincident Green function, first using short-time cutoff and then using dimensional regularization. Both methods are manifestly local and coordinate invariant.[1] A common basic ingredient is the d-dimensional heat kernel $K_h(x, y; s)$ satisfying the heat equation

$$\left(\partial_s - \nabla_h^2\right) K_h(x, y; s) = \delta(s)\, \delta^{(d)}(x, y) \tag{D.0.1}$$

with the initial condition

$$K_h(x, y; 0) = \delta^{(d)}(x, y)\,. \tag{D.0.2}$$

In flat space, the solution is given by

$$K_\delta(x, y; s) = \frac{e^{-\frac{|y-x|^2}{4s}}}{(4\pi s)^{d/2}}\,. \tag{D.0.3}$$

Since the divergence of the coincident Green function comes from short distances, it suffices to consider the adiabatic expansion of the heat kernel assuming small curvature

$$K_h(x, y; s) = \sqrt{\Delta_h(x, y)}\, \frac{e^{-\sigma(x,y)/2s}}{(4\pi s)^{d/2}} \left[1 + a_1(x, y)s + a_2(x, y)s^2 + ...\right]\,, \tag{D.0.4}$$

where the function $\sigma(x, y)$ is half the square of the geodesic distance between the two points and $\Delta(x, y)$ is the Van Vleck determinant

$$\Delta_h(x, y) = \frac{\det\left[\partial_\mu \partial_\nu\, \sigma(x, y)\right]}{\sqrt{h(y)h(x)}}\,. \tag{D.0.5}$$

The adiabatic expansion parameter is effectively s/L^2, where L^2 is the typical radius of curvature. In two dimensions, in the conformally flat frame, one obtains in this approximation

$$\sigma(x, y) = \frac{1}{2} e^{2\Sigma_h(x)} |x - y|^2, \qquad \Delta_h(x, y) = 1\,, \tag{D.0.6}$$

[1] Other commonly used methods use point-splitting [137] which is not manifestly covariant because of the choice of the direction used for point-splitting. One obtains the correct final answer by averaging over directions.

where the exponential factor in (D.0.4) ensures that corrections are of order $\mathcal{O}(\varepsilon)$. This reproduces the leading behavior of (3.1.10).

For the diagonal heat kernel, the geodesic distance vanishes and the coefficients of the expansion $a_j(x)$ are the so-called Seeley-de Witt coefficients given in terms of local curvature tensors. The Van Vleck determinant can be put to unity in Riemann normal coordinates. Therefore the short-time expansion of the diagonal heat kernel in d dimensions reads

$$K_h(x,x;s) = \frac{1}{(4\pi s)^{d/2}} \left(1 + a_1(x)s + a_2(x)s^2 + ...\right) . \tag{D.0.7}$$

D.1 Short Proper Time Cutoff

As discussed below (3.1.8) the coincident Green function can be regularized as [140]

$$G_h^\varepsilon(x,x) = \int d^2y\sqrt{h}\,\delta_h^{(2)}(x,y)\,G_h(y,x) = \int d^2y\,\sqrt{h}\,K_h(x,y;\varepsilon)\,G_h(y,x) \tag{D.1.1}$$

where the short-time ε effectively puts a cutoff on the distance between the two points. For small ε we need to keep only the leading term of the adiabatic expansion (D.0.4). So in the conformal frame, using (D.0.6) and the explicit expression for the Green function (3.1.6)

$$G_h^\varepsilon(x,x) = -\frac{1}{4\pi} \int d^2y\sqrt{h}\,\frac{1}{4\pi\varepsilon}\exp\left[-\frac{e^{2\Sigma_h(x)}\,|x-y|^2}{4\varepsilon}\right] \ln(m^2|y-x|^2) + \mathcal{O}(\varepsilon)\,. \tag{D.1.2}$$

The $\sqrt{h(y)}$ factor in the integrand is approximated by its value at point x up to terms of higher order in ε. Going to polar coordinates $r = |y-x|$

$$G_h^\varepsilon(x,x) = -\frac{1}{16\pi^2\varepsilon} \int \pi dr^2 e^{2\Sigma_h(x)}\,\ln(m^2\,r^2)\,\exp\left[\frac{-e^{-2\Sigma_h(x)}\,r^2}{4\varepsilon}\right] . \tag{D.1.3}$$

A straightforward integration finally gives

$$G_h^\varepsilon(x,x) = \frac{1}{2\pi}\Sigma_h(x) - \frac{1}{4\pi}\ln(4\,e^{-\gamma}\,m^2\,\varepsilon)\,, \tag{D.1.4}$$

where γ is the Euler-Mascheroni constant.

D.2 Dimensional Regularization

The Green function is related to the heat kernel by

$$G_h(x,y) = \int_0^\infty ds\,K_h(x,y;s)\,. \tag{D.2.1}$$

The coincident Green function is formally obtained by taking $x = y$ as the integral of the diagonal of the heat kernel (D.0.7). Near two dimensions, only the first term of this integral has an ultraviolet logarithmic divergence from the lower end of the integral. This can be regularized by continuing the integral to $d = 2 + \epsilon$ dimensions with ϵ negative and small:

$$G_h^d(x,x) \to G_h^\epsilon(x,x) = \int_0^\infty ds \, \frac{1}{(4\pi s)^{1+\frac{\epsilon}{2}}} \,. \tag{D.2.2}$$

There is an infrared divergence from the upper end of the integral which can be regularized by introducing a mass term. Near two dimensions in the conformal frame the metric can be written as $h_{\mu\nu} = e^{2\Sigma_h}\delta_{\mu\nu}$. Moreover, since only the first term in the adiabatic expansion (D.0.7) matters, we can take Σ_h to be a constant equal to its value at the point x. The infrared divergence can be regulated by considering the massive Green equation in a flat metric $h_{\mu\nu}$ Weyl equivalent to $\delta_{\mu\nu}$ by a *constant* rescaling $e^{2\Sigma_h(x)}$:

$$\left(-e^{-2\Sigma_h(x)}\,\delta^{\mu\nu}\,\partial_\mu\partial_\nu + m_h^2\right) G_h(x,y) = e^{-2\Sigma_h(x)}\,\delta^{(2)}(x-y)\,. \tag{D.2.3}$$

Let $m_h^2 = m^2\, e^{-2\Sigma_h(x)}$. For fixed m the massive Green equation is Weyl invariant and the infrared regulator does no break Weyl invariance. The regulated Green function is given by

$$G_h^\epsilon(x,x) = \frac{1}{(4\pi)^{1+\frac{\epsilon}{2}}} \int_0^\infty \frac{ds}{s^{1+\frac{\epsilon}{2}}}\, e^{-m^2 e^{-2\Sigma_h(x)}\, s} \tag{D.2.4}$$

for fixed m. Following the discussion above (3.1.5) this m can be identified with the IR cutoff $1/R$ introduced in that section.The integral evaluates to

$$\begin{aligned} G_h^\epsilon(x,x) &= \frac{1}{4\pi}\left[1 - \frac{\epsilon}{2}\,\ln(\frac{4\pi e^{2\Sigma_h(x)}}{m^2}) + \mathcal{O}(\epsilon)\right]\Gamma(-\frac{\epsilon}{2}) \\ &= -\frac{1}{2\pi\epsilon} + \frac{\Sigma_h(x)}{2\pi} - \frac{1}{4\pi}\ln(\frac{m^2 e^\gamma}{4\pi}) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{D.2.5}$$

Appendix E

Variation of Green Functions

E.1 Variation of the Scalar Laplacian

The variation of the scalar Laplacian under a general variation of the metric is

$$\begin{aligned}\delta\nabla_h^2 &= \delta\left(\frac{1}{\sqrt{-h}}\,\partial_\mu(\sqrt{-h}\,h^{\mu\nu}\,\partial_\nu)\right)\\ &= -\frac{1}{2}\nabla_\alpha(h_{\mu\nu}\delta h^{\mu\nu})\nabla^\alpha \,+\, (\nabla_\mu\,\delta h^{\mu\nu})\,\nabla_\nu \,+\, \delta h^{\mu\nu}\,\nabla_\mu\nabla_\nu\,. \end{aligned} \tag{E.1.1}$$

In particular, for an infinitesimal Weyl variation $\delta h_{\mu\nu} = 2\,\delta\xi(x)\,h_{\mu\nu}$, the Laplacian transforms as

$$\delta\nabla_h^2 = (d-2)\,h^{\mu\nu}\,(\nabla_\mu\,\delta\xi\,)\,\nabla_\nu \,-2\,\delta\xi\,\nabla_h^2\,. \tag{E.1.2}$$

In the case of two dimensions, the infinitesimal variation can be easily integrated, and under the finite Weyl transformation $g_{\mu\nu} = e^{2\Omega(x)}h_{\mu\nu}$ the Laplacian transforms as

$$\nabla_g^2 = e^{-2\Omega(x)}\,\nabla_h^2\,. \tag{E.1.3}$$

In general dimensions, the finite Weyl transformation of the Laplacian acting on a generic scalar function Σ is

$$\nabla_g^2\Sigma = e^{-2\Omega(x)}\,\left((d-2)\,h^{\mu\nu}\nabla_\mu\Omega\,\nabla_\nu\Sigma + \nabla_h^2\Sigma\right)\,. \tag{E.1.4}$$

E.2 Variation of the Scalar Laplacian Green Function

To compute the metric variation of the scalar Green function we vary the Green equation

$$-\,\nabla_h^2 G(y,z) = \delta_h^{(2)}(y,z) = \frac{\delta^{(2)}(y-z)}{\sqrt{h}} \tag{E.2.1}$$

to obtain

$$\delta\left(-\nabla_h^2\right)\,G_{yz} - \nabla_h^2\,\delta G_{yz} = \frac{1}{2}\,\delta^{(2)}(y,z)\,h_{\mu\nu}\,\delta h^{\mu\nu}\,, \tag{E.2.2}$$

where the right-hand side follows from the variation of the $1/\sqrt{-h}$ factor of the delta function. Using the variation of the Laplacian (E.1.1), we obtain a Poisson equation for δG_{yz} whose solution, after an integration by parts, is given by

$$\delta G_{yz} = \int dw\, \delta h^{\mu\nu}(w) \left(\frac{1}{2} h_{\mu\nu} \nabla_\alpha \left(G_{yw} \nabla^\alpha G_{wz} \right) - \nabla_\mu G_{yw}\, \nabla_\nu G_{wz} \right) + \frac{1}{2} h_{\mu\nu}\, \delta h^{\mu\nu}(z)\, G_{yz} \tag{E.2.3}$$

where all derivatives are taken inside the integral are with respect to the variable w. The final expression for the functional derivative is given by

$$\frac{1}{\sqrt{-h}} \frac{\delta G_{yz}}{\delta h^{\mu\nu}(x)} = -\nabla^x_{(\mu} G_{yx}\, \nabla^x_{\nu)} G_{xz} + \frac{1}{2} h_{\mu\nu}\, h^{\alpha\beta}\, \nabla^x_\alpha G_{yx}\, \nabla^x_\beta G_{xz}\,. \tag{E.2.4}$$

Note that this variation is traceless, as expected from the Weyl invariance of the Laplacian Green equation.

E.3 Variation of the $\tilde{G}$ Green function

The Green function $\tilde{G}(x, y)$ involved in the R-flat gauge treatment obeys the Green equation[1]

$$\left(-6\, \nabla^2_g + R_g \right)_x \tilde{G}(x, y) = \delta^{(4)}(x, y)\,. \tag{E.3.1}$$

Again the variation of the Green function can be computed by first performing the variation of this Green equation. From the resulting Poisson equation we obtain

$$\delta\, \tilde{G}(x, y) = \int dz\; \tilde{G}(x, z) \left[-\, \delta \left(-6\nabla^2_g + R_g \right)_z \tilde{G}(z, y) + \frac{1}{2}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, \delta^{(4)}(z, y) \right] \tag{E.3.2}$$

Using again the Laplacian variation (E.1.1) and the variation of the Ricci scalar

$$\delta R_g = \left(R_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \nabla^2_g \right) \delta g^{\mu\nu}\,, \tag{E.3.3}$$

the variation becomes

$$\begin{aligned} \delta\, \tilde{G}(x, y) \; &= \int dz\; \delta g^{\mu\nu} \Big[-E_{\mu\nu}(z)\, \tilde{G}_{xz} \tilde{G}_{zy} \\ -4 &\left(\nabla_\mu \tilde{G}_{xz}\, \nabla_\nu \tilde{G}_{zy} - \frac{1}{4} g_{\mu\nu}\, \nabla_\alpha \tilde{G}_{xz} \nabla^\alpha \tilde{G}_{zy} \right) + \tilde{G}_{xz} \left(\overleftrightarrow{\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2_g} \right) \tilde{G}_{zy} \Big]\,, \end{aligned} \tag{E.3.4}$$

where the derivatives inside the parenthesis are performed with respect to the variable z, and the over-arrow on the last parenthesis indicates the sum of the operator acting on the Green function on each side.

[1] This computation we perform it in the metric $g_{\mu\nu}$ background to make contact with section §5.4 where it is used.

Appendix F

Nearly Static Coordinates

Static coordinates of de Sitter spacetime are useful for studying the thermodynamic properties of the spacetime [149,176]. Even thought there is no global timelike Killing vector in de Sitter spacetime, the static coordinates provide a timelike future-oriented Killing vector in the static patch. These are also the natural coordinates for a Schwarzschild-de Sitter solution. Is there an analog of the static coordinates for our new solution?

Since our solution violates the de Sitter symmetry, we do not expect exactly static coordinates. Indeed, it can be shown that no such exact static coordinates exist for our cosmological solution (6.2.4). However, since any two Robertson-Walker metrics are Weyl equivalent, our solution admits nearly static coordinates in which it is conformal to static de Sitter:

$$ds^2 = \left(\frac{e^{2hT}}{1 - h^2 R^2}\right)^{\frac{\gamma}{\gamma-2}} \left[-(1 - h^2 R^2)\, dT^2 + \frac{1}{1 - h^2 R^2}\, dR^2 + R^2\, d\Omega_2^2\right] , \tag{F.0.1}$$

where constant h is defined as

$$h := \frac{2 - \gamma}{2}\, H_*\, a_*^{\gamma/2} \, . \tag{F.0.2}$$

In the limit of vanishing γ, one recovers the static de Sitter metric.

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