Sovereign risk modelling and applications

Jean-Francois, Shanqiu Li

Université Pierre et Marie Curie - Paris VI
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Shanqiu LI

Modélisation des risques souverains et applications

Sovereign risk modelling and applications

Co-dirigée par Pr. **Ying JIAO** et Pr. **Huyên PHAM**

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Stéphane	Crépey	Rapporteur
Caroline	Hillairet	Examinatrice
Monique	Jeanblanc	Examinatrice
Ying	Jiao	Directrice de thèse
Idris	Kharroubi	Examinateur
Gilles	Pagès	Président du jury
Huyên	Pham	Directeur de thèse

LPMA - Université Paris Diderot
Bâtiment Sophie Germain, 5ème étage
8, place Aurélie Nemours
75205 PARIS CEDEX 13
FRANCE
Case courrier 7012

LPMA - UPMC
Couloir 16 - 26, 1er étage
4, place Jussieu
75252 PARIS CEDEX 05
FRANCE
Case courrier 188

À ma famille

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Résumé

La présente thèse traite la modélisation mathématique des risques souverains et ses applications.

Dans le premier chapitre, motivé par la crise de la dette souveraine de la zone euro, nous proposons un modèle de risque de défaut souverain. Ce modèle prend en compte aussi bien le mouvement de la solvabilité souveraine que l'impact des événements politiques critiques, en y additionnant un risque de crédit idiosyncratique. Nous nous intéressons aux probabilités que le défaut survienne aux dates d'événements politiques critiques, pour lesquelles nous obtenons des formules analytiques dans un cadre markovien, où nous traitons minutieusement quelques particularités inhabituelles, entre autres le modèle CEV lorsque le paramètre d'élasticité $\beta > 1$. Nous déterminons de manière explicite le processus compensateur du défaut et montrons que le processus d'intensité n'existe pas, ce qui oppose notre modèle aux approches classiques.

Dans le deuxième chapitre, en examinant certains modèles hybrides issus de la littérature, nous considérons une classe de temps aléatoires dont la loi conditionnelle est discontinue et pour lesquels les hypothèses classiques du grossissement de filtrations ne sont pas satisfaites. Nous étendons l'approche de densité à un cadre plus général, où l'hypothèse de Jacod s'assouplit, afin de traiter de tels temps aléatoires dans l'univers du grossissement progressif de filtrations. Nous étudions également des problèmes classiques : le calcul du compensateur, la décomposition de la surmartingale d'Azéma, ainsi que la caractérisation des martingales. La décomposition des martingales et des semimartingales dans la filtration élargie affirme que l'hypothèse H' demeure valable dans ce cadre généralisé.

Dans le troisième chapitre, nous présentons des applications des modèles proposés dans les chapitres précédents. L'application la plus importante du modèle de défaut souverain et de l'approche de densité généralisée est l'évaluation des titres soumis au risque de défaut. Les résultats expliquent les sauts négatifs importants dans le rendement actuariel de l'obligation à long terme de la Grèce pendant la crise de la dette souveraine. La solvabilité de la Grèce a tendance à s'empirer au fil des années et le rendement de l'obligation a des sauts négatifs lors des événements politiques critiques. En particulier, la taille d'un saut dépend de la gravité d'un choc exogène, du temps écoulé depuis le dernier événement politique, et de la valeur du recouvrement. L'approche de densité généralisée rend aussi possible la modélisation des défauts simultanés qui, bien que rares, ont un impact grave sur le marché.

Mots-clefs

Risque souverain, solvabilité souveraine, risque de crédit idiosyncratique, décomposition de temps d'arrêt, hypothèse de densité généralisée, grossissement progressif de filtrations, caractérisation de martingale, décomposition de semimartingale, propriété d'immersion, défauts simultanés, obligation souveraine à long terme.

Abstract

This dissertation deals with the mathematical modelling of sovereign credit risk and its applications.

In Chapter 1, motivated by the European sovereign debt crisis, we propose a hybrid sovereign risk model which takes into account both the movement of the sovereign solvency and the impact of critical political events besides the idiosyncratic credit risk. We are interested in the probability that the default occurs at critical political dates, for which we obtain closed-form formulae in a Markovian setting, where we deal with some unusual features, such as a treatment of the CEV model when the elasticity parameter $\beta > 1$. We compute explicitly the compensator process of default and show that the intensity process does not exist.

In Chapter 2, by studying certain hybrid models in literature on credit risks, we consider a type of random times whose conditional probability distribution is not continuous and by which standard intensity and density hypotheses in the enlargement of filtrations are not satisfied. We propose a generalised density approach, where the hypothesis of Jacod is relaxed, in order to deal with such random times in the framework of progressive enlargement of filtrations We also study classic problems such as the computation of the compensator process of the random time, the decomposition of the Azéma supermartingale, as well as the martingale characterisation. The martingale and semimartingale decompositions in the enlarged filtration show that the H'-hypothesis holds in this generalised framework.

In Chapter 3, we display several applications of the models proposed in the previous chapters. The most important application of the hybrid default model and the generalised density approach is the valuation of default claims. The results explain the significant negative jumps in the long-term Greek government bond yield during the sovereign debt crisis. The solvency of Greece tends to fall gradually through time and the bond yield has negative jumps when critical political events are held. In particular, the size of a jump depends on the seriousness of an exogenous shock, the elapsed time since the last political event, and the value of the recovery payment. The generalised density approach also makes possible the modelling of simultaneous defaults, which are rare but may have an important impact.

Keywords

Sovereign risk, sovereign solvency, idiosyncratic credit risk, decomposition of stopping times, generalised density hypothesis, progressive enlargement of filtrations, martingale characterisation, semimartingale decomposition, immersion property, simultaneous defaults, long-term government bond.

Notations

Chapter 1

- $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space equiped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.
- τ is the default time.
- $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the progressive enlargement of the filtration $\mathbb{F}$ by τ.
- $W = (W_t, t \geq 0)$ is an $\mathbb{F}$-adapted standard Brownian motion.
- $S = (S_t, t \geq 0)$ is the solvency process of a sovereign.
- τ_0 is the first hitting time of the barrier $L > 0$ by the process S.
- $(\tau_i)_{i=1}^n$ is a sequence of first hitting times of the decreasing barrier sequence $L_1, \ldots, L_n$ by the process S.
- ζ and ζ^* are accessible stopping times, and ξ is a totally inaccessible stopping time.
- $N = (N_t, t \geq 0)$ is a Poisson process.
- λ^N is the constant Poisson intensity, and $\lambda^N(t)$ is the time-dependent Poisson intensity function.
- $\lambda = (\lambda_t, t \geq 0)$ is the idiosyncratic default intensity process, and $\Lambda = (\Lambda_t, t \geq 0) = (\int_0^t \lambda_s \, ds, t \geq 0)$.
- η is an $\mathcal{A}$-measurable random variable.
- $\sigma_1 \wedge \sigma_2$ is the minimum of two random times σ_1 and σ_2.
- $p^i = (p^i_t, t \geq 0)$ is the $\mathbb{F}$-conditional probability that τ coincides with τ_i, i.e.,

$$p^i_t = \mathbb{P}(\tau = \tau_i | \mathcal{F}_t), \quad i \in \{1, \ldots, n\}.$$

- $G = (G_t, t \geq 0)$ is the Azéma $\mathbb{F}$-supermartingale of τ.
- $\Lambda^{\mathbb{F}} = (\Lambda^{\mathbb{F}}_t, t \geq 0)$ is the $\mathbb{F}$-compensator process such that $(\mathbb{1}_{\{\tau \leq t\}} - \Lambda^{\mathbb{F}}_{t\wedge\tau}, t \geq 0)$ is a

$\mathbb{G}$-martingale.

- $\Lambda^{\mathbb{G}} = (\Lambda_t^{\mathbb{G}}, t \geq 0)$ is the $\mathbb{G}$-compensator process such that $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{F}}$.
- $\Gamma = (\Gamma_t, t \geq 0)$ is the hazard process of τ such that $\Gamma_t = -\ln G_t$.
- $\mathcal{E}(X)$ is the Doléans-Dade exponential of any process X.
- $\mathcal{L}$ is the infinitesimal generator of a diffusion.
- $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions of the first and second kind.
- $M_{n,m}(x)$ and $W_{n,m}(x)$ are Whittaker functions of the first kind and second kind.
- $F_1(a,b,x)$ and $F_2(a,b,x)$ are Kummer confluent hypergeometric functions of the first kind and second kind.

Chapter 2

- η is a non-atomic σ-finite Borel measure on $\mathbb{R}_+$.
- $\mathcal{B}$ is Borel σ-algebra.
- $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ is a filtered probability space where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the reference filtration satisfying the usual conditions.
- $\mathcal{O}(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ are the optional and predictable σ-algebra associated to the filtration $\mathbb{F}$.
- τ is an arbitrary random time which is not an $\mathbb{F}$-stopping time.
- $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the enlargement of the filtration $\mathbb{F}$ by τ.
- $(\tau_i)_{i=1}^N$ is a finite family of $\mathbb{F}$-stopping times.
- $\alpha(\cdots) = (\alpha_t(\cdots), t \geq 0)$ is the generalised density process.
- $\sigma_1 \vee \sigma_2$ is the maximum of two random times σ_1 and σ_2.
- $D^i = (D_t^i, t \geq 0)$ is the indicator process associated to τ_i, i.e., $D_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$, $i \in \{1, \ldots, N\}$.
- $\Lambda^i = (\Lambda_t^i, t \geq 0)$ is the compensator of D^i such that $M^i = D^i - \Lambda^i$ is an $\mathbb{F}$-martingale, $i \in \{1, \ldots, N\}$.
- $A = (A_t, t \geq 0)$ is the $\mathbb{F}$-dual predictable projection of the indicator process $(\mathbb{1}_{\{\tau \leq t\}}, t \geq 0)$.

Chapter 3

— $\mathbb{Q}$ is an equivalent probability measure.
— $(r_t, t \geq 0)$ is default-free interest rate process.
— $(R_t, t \geq 0)$ is the recovery paiement process.
— $D(t, T)$ is the value of the defaultable zero-coupon bond.
— $Y^d(t, T)$ and $Y(t, T)$ are respectively the yield to maturity of the defaultable zero-coupon bond and that of a classic default-free zero-coupon bond.
— $S(t, T)$ is the credit spread of the defaultable zero-coupon bond.
— κ is the CDS spread.

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Introduction générale en français

Cette thèse traite la modélisation mathématique du risque souverain ainsi que ses applications en finance. Ces études se basent sur la crise de la dette souveraine dans la zone euro, provoquée par la mise en lumière de la crise de la dette grecque, qui demeure un sujet d'actualité depuis fin 2009.

La crise de la dette dans la zone euro résulte d'importants déficits publics et des effets de contagion financière. Plusieurs états membres de la zone euro (Grèce, Portugal, Irlande, Espagne, et Chypre) furent dans l'incapacité de rembourser ou de refinancer leurs dettes publiques sans intervention d'un tiers, tel que d'autres états membres de la zone euro, la Banque Centrale Européenne (BCE), et le Fond Monétaire International (FMI), etc.

Différent du risque de crédit d'entreprise, le risque souverain a des caractères mixtes dus à une combinaison de facteurs macroéconomiques et géopolitiques complexes, comme c'est le cas pour un état membre de la zone euro. Des études empiriques suggèrent que les facteurs macroéconomiques déterminants peuvent être résumés par un facteur commun nommé solvabilité souveraine (voir e.g. Alogoskoufis [Alo12]), et les impacts des décisions politiques surviennent notamment lors d'un événement politique critique fixé à l'avance. Ces impacts politiques sont caractérisés par une probabilité de défaut élevée antérieurement à l'événement susmentionné, ainsi qu'une chute subite de cette probabilité à la suite de celui-ci. Ce phénomène s'interprète par les équilibres multiples du marché de dette en présence du risque de crédit et se visualise dans les variations importantes du rendement actuariel des obligations souveraines. Précisément, du point de vue économique, l'équilibre dominant dépend des attentes des investisseurs sur la probabilité de défaut (e.g., Calvo [Cal88]). Avant un événement politique critique, les investisseurs s'attendent à un défaut souverain avec une probabilité élevée, et le marché d'obligations souveraines manifeste donc un équilibre de spread [1] large. Peu après cet événement, les attentes des investisseurs sur le défaut souverain diminuent soudainement de sorte que le marché d'obligations d'état se trouve dans un équilibre de spread étroit. Pour cette raison, la probabilité de défaut au moment de l'événement est non nulle, caractérisée par un saut dans le rendement actuariel.

Dans les modèles en temps continu, le temps de défaut est habituellement modélisé comme un temps aléatoire, et en particulier, un temps d'arrêt par rapport à une certaine filtration. En théorie de probabilité, les temps d'arrêt sont classifiés en trois catégories : temps d'arrêt prévisibles, accessibles, et totalement inaccessibles. Intuitivement, un temps d'arrêt prévisible est connu juste avant l'arrivée d'un événement puisqu'il est annoncé par une suite croissante de temps d'arrêt ; un temps d'arrêt accessible peut être parfaitement couvert par une suite de temps d'arrêt prévisibles ; et un temps d'arrêt totalement inaccessible est l'instant même d'une surprise totale qui ne coïncide jamais avec un temps d'arrêt prévisible.

Dans la littérature sur les modélisations de risque de crédit, il existe déjà deux approches classiques (voir les livres de Bielecki et Rutkowski [BR02], de Duffie et Singleton [DS03] et de Schönbucher [Sch03], et aussi les ouvrages de Bielecki, Jeanblanc et Rutkowski [BJR04b] et de Schmidt et Stute [SS03] pour une description détaillée) : l'approche structurelle (Black et Scholes [BS73], Merton [Mer74], Black et Cox [BC76]) où le temps de défaut est souvent un temps d'arrêt prévisible, et l'approche à forme réduite (aussi communément appelé approche d'intensité, Jarrow et Turnbull [JT92, JT95], Lando [Lan98], Duffie et Singleton [DS99]), où le temps de défaut est un temps d'arrêt totalement inaccessible. D'ailleurs, dans certains modèles structurels avec rapports de comptabilité imparfaits (e.g., Duffie et Lando [DL01b], Giesecke [Gie06]), le temps de défaut est également totalement inaccessible. Récemment, Jarrow and Protter ([JP04]) montrent d'un point de vue informationnel que l'on peut transformer un temps d'arrêt en modifiant l'ensemble d'information à la disposition du modélisateur.

1. Spread est un anglicisme qui signifie l'écart entre le rendement actuariel d'une obligation et un taux de référence.

Pour l'analyse de risque de crédit, le grossissement progressif de filtrations (e.g., Barlow, Jacod, Yor, et Jeulin [Bar78, Jac85, Jeu80, JY78, Yor78]) a été systématiquement adopté pour modéliser les événements de défaut qu'un modélisateur ne peut pas observer du flux informationel du marché hors défaut, ou de manière mathématique, le temps de défaut n'est pas modélisé comme un temps d'arrêt par rapport à la filtration de référence (voir également Mansuy et Yor [MY06], Protter [Pro05], Dellacherie, Maisonneuve et Meyer [DMM92], Yor [Yor12], Brémaud et Yor [BY78], Nikeghbali [Nik06], Ankirchner [Ank05], Song [Son87], Ankirchner, Dereich et Imkeller [ADI07], Yœurp [Yœu85]). Dans les travaux de Elliott, Jeanblanc et Yor [EJY00] et de Bielecki et Rutkowski [BR02], les auteurs ont proposé d'utiliser le grossissement progressif de filtrations pour décrire les informations du marché qui incluent tant l'information hors défaut que celle à l'égard du défaut. Précisément, les informations globales du marché sont modélisées comme la plus petite filtration contenant toute l'information hors défaut telle que le temps de défaut est un temps d'arrêt. Plus récemment, afin d'étudier l'impact des événements de défaut, une nouvelle approche a été développée par El Karoui, Jeanblanc and Jiao [EKJJ10, EKJJ15] où l'on suppose l'hypothèse de densité. En particulier, l'approche de densité nous permet d'analyser ce qui arrive postérieurement à un événement de défaut et fait naître des applications intéressantes dans les études de risque de défaut de contrepartie. Remarquons que, dans l'approche d'intensité et l'approche de densité, le temps de défaut est un temps d'arrêt totalement inaccessible.

La théorie de la décomposition de temps d'arrêt stipule que chaque temps d'arrêt peut être décomposé de façon unique en une partie accessible et une autre totalement inaccessible (Dellacherie [Del72]). La décomposition de temps aléatoire apparaît également dans la littérature sur la théorie du grossissement de filtrations (e.g., Coculescu [Coc09], Aksamit, Choulli et Jeanblanc [ACJ16]). Pour le cas du défaut souverain, du point de vue d'une telle décomposition, si le temps de défaut peut coïncider avec une date critique arrêtée d'avance dont la probabilité est strictement positive, alors cela signifie que le temps de défaut possède une partie accessible outre qu'une partie totalement inaccessible. D'une part, ni l'approche à forme réduite ni l'approche de densité ne sont réalistes car le temps de défaut ainsi modélisé est totalement inaccessible et évite tous les temps d'arrêt prévisibles. D'autre part, un modèle structurel n'est pas adapté au défaut souverain comme révèle Matsumura dans [Mat06], parce qu'il n'est pas clair quelle est la valeur d'actif à prendre pour référence et l'impact des décisions politiques ne se reflète pas sur la définition structurelle dans la littérature. Pour cette raison, nous proposons un modèle hybride qui se base sur les deux approches classiques et en même temps prend en compte aussi bien le niveau de la solvabilité souveraine que l'impact des événements politiques critiques.

Andreasen [And03] est l'un des premiers à étudier ce type de modèle hybride. Il existe dans la littérature sur le risque de crédit d'autres modèles hybrides tels que le modèle généralisé à processus de Cox de Bélanger, Shreve et Wong [BSW04], le modèle hybride de migration de crédit de Chen et Filipović [CF05], les modèles CEV [2] de défaut ponctuel de Carr et Linetsky [CL06] et de Campi, Polbennikov et Sbuelz [CPS09], ainsi que le cadre général sans intensité de Gehmlich et Schmidt [GS16] et Fontana et Schmidt [FS16]. Plus précisément, dans [BSW04], le processus de compensateur peut posséder des sauts ; dans [CF05], le défaut de la firme est provoqué soit par des dégradations successives de sa note de crédit, soit par un saut imprévisible d'un simple processus ponctuel ; dans [CL06], la valeur d'action est une diffusion CEV ponctuée par un possible saut à zéro qui correspond à un défaut, et le temps de défaut est décomposé en une partie prévisible, qui est le premier temps de passage à zéro par le processus de la valeur d'action, et une partie totalement inaccessible donnée par un modèle à processus de Cox ; dans [CPS09], la valeur d'action est un processus CEV, et le temps de défaut est le minimum du premier instant de saut du processus de Poisson et le premier temps d'absorption du processus de la valeur d'action par zéro en l'absence de sauts ; dans [GS16], la surmartingale d'Azéma du temps de défaut contient des sauts de telle sorte que l'intensité n'existe pas, et [FS16] généralise l'approche de [GS16].

Dans le Chapitre 1, nous proposons un modèle hybride de défaut souverain où le temps de défaut combine une partie accessible prenant en compte le mouvement de la solvabilité souveraine et l'impact des événements politiques critiques, et une partie totalement inaccessible pour le risque de crédit idiosyncratique. Les caractéristiques principales de

2. Modèles à élasticité de variance constante.

notre modèle comprennent : le temps de défaut peut coïncider avec une famille de temps d'arrêt prévisibles, la modélisation de l'impact des facteurs macroéconomiques et des événements politiques est séparée de celle du risque idiosyncratique, la solvabilité peut être corrélée avec le risque idiosyncratique, la propriété d'immersion est satisfaite et peut être facilement relaxée.

Nous nous inspirons des modèles CEV de défaut ponctuel dans [CL06] et [CPS09], qui furent initialement proposés pour évaluer les risques de crédit d'entreprise. Ce qui fait la différence, c'est que le défaut dans notre modèle peut coïncider avec de multiples événements politiques et peut arriver à la suite de n'importe lequel de ces événements, alors que le temps de défaut dans [CL06] et [CPS09] est borné par son seul composant prévisible.

La notion de solvabilité souveraine est importante dans notre modèle. la solvabilité est une variable unifiée qui reflète l'impact des facteurs macroéconomiques sur la capacité d'un souverain de remplir ses obligations à long terme. Nous utilisons la définition existante de la solvabilité souveraine en temps discret ([Alo12]) à un modèle en temps continu et considérons un souverain (e.g., Grèce) avec processus de solvabilité $(S_t, t \geq 0)$, adapté par rapport à la filtration de référence $\mathbb{F}$. Notre modèle est basé sur l'abstraction mathématique du scénario suivant : les autorités (BCE, Commission Européenne, FMI, etc.) établissent une condition de budget pour le souverain. Si le processus de solvabilité S tombe en-dessous du niveau requis, nous considérons que le souverain devient sévèrement insolvable, et un événement politique critique doit être organisé, et lors de celui-ci des décisions politiques doivent être prises à l'égard du souverain. Le résultat des décisions peut être la faillite immédiate du souverain ou un plan d'aide financière pour celui-ci dans le but d'améliorer sa situation budgétaire. Dans ce dernier cas, si la situation aggravée de la dette et du déficit est excessive et ne permet pas d'amélioration satisfaisante, les autorités peuvent relaxer la condition budgétaire et anticipent de nouveaux événements politiques critiques.

Le temps de défaut τ dans notre modèle se décompose de manière unique en une partie accessible ζ^* et une partie totalement inaccessible ξ sur une unique partition de l'ensemble d'échantillons Ω :

$$\tau(\omega) = \zeta^*(\omega) \wedge \xi(\omega).$$

La partie accessible ζ^* est recouverte par une suite croissante de premiers temps de passage ($\mathbb{F}$-temps d'arrêt) à n seuils de solvabilité $\tau_1, \ldots, \tau_n$:

$$\tau_i = \inf\{t \geq 0 : S_t < L_i\}.$$

Le fait de savoir si le défaut coïncide ou non avec l'un des $\mathbb{F}$-temps d'arrêt τ_i dépend d'un facteur exogène (e.g., l'ampleur d'un choc financier externe qui précède τ_i), modélisé pour simplicité par un processus de Poisson inhomogène $(N_t, t \geq 0)$ avec fonction d'intensité $\lambda^N(t)$. La partie totalement inaccessible ξ admet un processus d'intensité $(\lambda_t, t \geq 0)$, qui peut être corrélé avec la solvabilité S. La structure globale d'information $\mathbb{G}$ est le grossissement progressif de la filtration $\mathbb{F}$ par le temps de défaut τ.

Nous calculons les probabilités que le défaut souverain ait lieu à des dates spécifiques d'événements politiques critiques :

$$\mathbb{P}(\tau = \tau_i | \mathcal{F}_t) = \mathbb{E}\left[\left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\right) e^{-\int_0^{\tau_i} \lambda_s\, ds} \middle| \mathcal{F}_t\right].$$

Tout comme ce que nous voudrions montrer, ces probabilités de défaut sont strictement positives dans notre modèle hybride, ce qui implique la présence des singularités dans la loi du temps de défaut τ :

$$\mathbb{P}(\tau > u | \mathcal{F}_t) = \mathbb{E}\left[\exp\left(-\sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \int_0^u \lambda_s\, ds\right) \middle| \mathcal{F}_t\right].$$

Par conséquent, la surmartingale d'Azéma n'est pas continue à $\tau_1, \ldots, \tau_n$, et le processus de hasard [3]

$$\left(\int_0^t \lambda_s\, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds, \quad t \geq 0\right).$$

n'est pas égal au processus $\mathbb{F}$-compensateur

$$\left(\int_0^t \lambda_s\, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} \left(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds}\right), \quad t \geq 0\right)$$

bien qu'ils aient la même partie continue.

3. La traduction littérale du terme en anglais *hazard process*.

Dans un cadre markovien spécifique, où le processus de solvabilité et le processus de Poisson sont homogènes, les probabilités $\mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$ peuvent être déduites de la transformée de Laplace du temps d'arrêt $\rho = \inf\{t \geq 0 : S_t \leq L\}$:

$$\mathbb{E}\left[\exp\left(-k\rho - \int_0^\rho \lambda(S_u)\,du\right)\right],$$

qui est la répresentation de la solution au problème de Dirichlet suivant

$$\mu(z)u'(z) + \frac{1}{2}\sigma^2(z)u''(z) = (\lambda(z)+k)u(z) \quad \text{sur } \{z > L\};$$
$$u(L) = 1.$$

Nous obtenons les formules analytiques en résolvant des équations de Sturm-Liouville dans les cas de mouvement brownien géométrique et de processus CEV. Plus précisément, quand le processus de solvabilité est modélisé par un mouvement brownien géométrique

$$dS_t = S_t(\mu\,dt + \sigma\,dW_t),$$

il s'agit des expressions en termes des solutions à une équation de Bessel modifée

$$(xy')' - c_1x^{-1}y' = c_2xy$$

où $c_1, c_2 > 0$, alors que dans le cas du processus de solvabilité modélisé par une diffusion CEV

$$dS_t = S_t(\mu\,dt + \delta S_t^\beta dW_t),$$

nous devons résoudre l'équation différentielle suivante

$$\frac{1}{2}\delta^2x^{2+2\beta}u'' + \mu xu' - (ax^{-2|\beta|} + b + k)u = 0,$$

où nous discutons le signe du paramètre d'élasticité β et obtenons les solutions fondamentales en employant les fonctions de Whittaker et les fonctions de Bessel modifiées.

La propriété d'immersion possède l'avantage d'impliquer la complétude du marché (e.g. [JLC09a]). Cependant, il est généralement impossible de supposer l'immersion dans les cas de marché incomplet, de multi-defauts non-ordonnés, et de corrélation entre différents temps de défaut. Dans notre modèle hybride, la propriété d'immersion est controlée par la barrière aléatoire, ou plus précisément, la propriété d'immersion est satisfaite si la barière aléatoire est indépendante de $\mathcal{F}_\infty$. En relaxant l'hypothèse d'indépendance susmentionnée, nous pouvons étendre le modèle au delà du paradigme d'immersion. Par conséquent, la surmartingale d'Azéma n'est plus un processus décroissant, et nous devons faire appel à la décomposition multiplicative pour obtenir le processus compensateur.

D'un point de vue probabiliste appuyé par notre calcul, les probabilités de défaut non nulle aux dates d'événements politiques critiques signifient que la loi du temps de défaut τ admet des singularités. Afin d'étendre notre modèle de risque souverain à un cadre plus général, nous considérons un type de temps aléatoires qui peuvent être soit accessibles soit totalement inaccessibles et proposons de généraliser l'approche de densité dans [EKJJ10]. Plus précisément, nous supposons que la loi conditionnelle de τ contient une partie singulière outre que la partie absolument continue qui a une densité.

Dans le Chapitre 2, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ est la filtration de référence et $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ est le grossissement progressif de $\mathbb{F}$ par τ, et nous supposons l'*hypothèse de densité généralisée* que la loi $\mathbb{F}$-conditionnelle de τ évitant une famille de $\mathbb{F}$-temps d'arrêt $(\tau_i)_{i=1}^n$ a une densité (nommée densité généralisée) par rapport à une mesure borélienne σ-finie non-atomique η sur $\mathbb{R}_+$:

$$\mathbb{E}[\mathbb{1}_H h(\tau) \,|\, \mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-p.s.},$$

où H désigne l'ensemble aléatoire

$$\{\tau < \infty\} \cap \bigcap_{i=1}^n \{\tau \neq \tau_i\}.$$

Notre hypothèse clef est satisfaite par d'autres modèles hybrides précédemment cités tels que [BSW04, CF05, CL06, CPS09, GS16], ainsi que notre modèle de risque souverain du Chapitre 1 dans les deux versions avec et sans propriété d'immersion, et nous pouvons expliciter le processus de densité généralisée pour chacun de ces modèles. Sous l'hypothèse de densité généralisée, τ n'a que la possibilité de rencontrer les $\mathbb{F}$-temps d'arrêt qui peuvent coïncider avec $(\tau_i)_{i=1}^n$. Nous prouvons que le processus de densité généralisée $\alpha(\cdot)$ est une $\mathbb{F}$-martingale càdlàg paramétrée. Notons $(p_t^i, t \geq 0)$ les probabilités $\mathbb{F}$-conditionnelle que τ rencontre τ_i, $i \in \{1, \dots, n\}$, alors toute espérance $\mathcal{G}_t$-conditionnelle peut être calculée sous la forme décomposée en termes de $\alpha(\cdot)$ et $(p^i)_{i=1}^n$. En outre, nous pouvons montrer que la formule de décomposition suivante pour les processus $\mathbb{G}$-optionnels est valable, ce qui n'est pas le cas en général

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(\tau), \quad \mathbb{P}\text{-p.s.}$$

La validité de la formule ci-dessus a été vérifiée à maintes reprises pour les modèles existants, entre autres sous l'hypothèse de densité, et on peut en trouver les conditions dans Song [Son14].

Le processus compensateur sous l'hypothèse de densité généralisée n'est pas en général un processus continu et donc le processus d'intensité n'existe pas toujours. Nous traitons la décomposition de Doob-Meyer de la surmartingale d'Azéma G et nous nous concentrons sur sa partie discontinue afin d'obtenir la forme générale du compensateur

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \mathbb{1}_{\{G_{s-}>0\}} \frac{\alpha_s(s)\eta(ds)}{G_{s-}} + \sum_{i=1}^n \int_{(0,t\wedge\tau]} \mathbb{1}_{\{G_{s-}>0\}} \frac{p_{s-}^i d\Lambda_s^i + d\langle M^i, p^i\rangle_s}{G_{s-}}, \quad t \geq 0,$$

où $\Lambda^i = (\Lambda_t^i, t \geq 0)$ est le compensateur de $(\mathbb{1}_{\{\tau_i \leq t\}}, t \geq 0)$ et

$$M^i = \Big(\mathbb{1}_{\{\tau_i \leq t\}} - \Lambda_t^i, \quad t \geq 0\Big)$$

pour tout $i = \{1, \dots, n\}$. Si $(\tau_i)_{i=1}^n$ sont des $\mathbb{F}$-temps d'arrêt prévisibles, alors τ est un $\mathbb{G}$-temps d'arrêt accessible et le processus de compensateur de τ possède une partie absolument continue et une partie avec des sauts à $(\tau_i)_{i=1}^n$; s'ils sont par ailleurs $\mathbb{F}$-temps d'arrêt totalement inaccessibles, alors τ est un $\mathbb{G}$-temps d'arrêt totalement inaccessible et le processus compensateur de τ est continu.

Nous caractérisons également les martingales dans la filtration $\mathbb{G}$ en vérifiant trois conditions sur des $\mathbb{F}$-martingales. Différent de [EKJJ10], les conditions nécessaires et les conditions suffisantes sont subtilement distinguées. La stabilité des semimartingales est aussi un problème classique à étudier quand la filtration de référence est élargie. Nous obtenons la décomposition d'une $\mathbb{F}$-martingale locale en tant que $\mathbb{G}$-semimartingale

$$\begin{aligned} U_t^{\mathbb{F}} = U_t^{\mathbb{G}} &+ \int_{(0,t\wedge\tau]} \frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}} \\ &+ \mathbb{1}_{\cap_{i=1}^N \{\tau \neq \tau_i\}} \int_{(\tau, t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, \alpha(u)\rangle_s}{\alpha_{s-}(u)}\bigg|_{u=\tau} + \sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}} \int_{(\tau,t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p_{s-}^i}, \end{aligned}$$

où $U^{\mathbb{G}}$ est une $\mathbb{G}$-martingale locale et $\bar{M}$ est une $\mathbb{F}$-martingale BMO, calculée par

$$\bar{M}_t = \mathbb{E}\left[\int_0^\infty \alpha_u(u)\eta(du)\middle|\mathcal{F}_t\right] + \sum_{i=1}^n p^i_{t\wedge\tau_i} + p^\infty_t, \quad t \geq 0,$$

qui est en général différente de celle dans la décompostion de Doob-Meyer de G. La formule de décomposition ci-dessus signifie que l'hypothèse (H') de Jacod est satisfaite sous l'hypothèse de densité généralisée.

Dans le Chapitre 3, nous appliquons les résultats des chapitres précédents à l'évaluation et à la couverture des risques de crédit. L'une des applications les plus importantes du modèle de risque souverain est l'évaluation des titres soumis au risque souverain tels que les obligations souveraines. Nous nous intéressons particulièrement au comportement du rendement actuariel à long terme durant la crise de la dette souveraine, et nous montrons que le modèle hybride fournit une interprétation au comportement des sauts autour des dates d'événements politiques critiques. Précisément, le saut dans la valeur d'obligation $D(t,T)$ à la date τ_i est caractérisé en termes du montant de recouvrement et du processus $\mathbb{F}$-compensateur, donné par la formule suivante

$$\Delta D(\tau_i, T) = (D(\tau_i, T) - R_{\tau_i})\Lambda^{\mathbb{F}}_{\tau_i} \quad \text{on } \{\tau_i < T\}.$$

Le spread de crédit se comporte dans le sens inverse du prix d'obligation, qui est en général plus élevé que le recouvrement, il est donc vraisemblable que le spread de crédit ait des sauts négatifs.

Nous donnons un exemple de valorisation à l'aide de l'optimisation de portefeuille dans un modèle hybride, où un investisseur s'investit dans une action soumise au risque de banqueroute. Hodges et Neuberger [HN89] ont introduit l'évaluation par indifférence dans les marchés incomplets (voir Henderson et Hobson [HH04] pour un survol général de littérature sur le sujet, et aussi les travaux de Bielecki, Jeanblanc et Rutkowski [BJR04a] et la thèse de Sigloch [Sig09] pour un aperçu dans le contexte de risque de défaut). Le temps de défaut peut être, soit le premier temps de passage à zero du prix d'action modélisé par une diffusion CEV, soit l'instant même d'une banqueroute ponctuelle qui entraîne la disparition de l'action. Alors, la richesse totale $\hat{X}$ dépend de deux régimes de défaut. Étant donnée la fonction d'utilité U qui remplit certaines conditions générales, nous formulons le problème de contrôle suivant :

$$v(x) = \sup_{\varphi \in \mathcal{A}} \mathbb{E}\left[U(\hat{X}_T^{x,\varphi})\right],$$

où la fonction de gain a un horizon fini avec gain terminal dépendant d'un régime d'arrêt.

Nous étudions également l'évaluation des titres soumis au risque de défaut dans le cadre général, où un titre peut procurer des paiements de coupons. D'ailleurs, aucune propriété d'immersion n'est a priori satisfaite, et aucune probabilité risque-neutre n'est a priori connue. Dans Coculescu, Jeanblanc et Nikeghbali [CJN12], les auteurs ont montré de quelle manière la propriété d'immersion a été modifiée à travers d'un changement de probabilité. Dans le cadre de densité généralisée étudié dans le Chapitre 2, nous montrons que l'hypothèse de densité généralisée est invariable sous un changement de probabilité, et que nous pouvons toujours trouver une mesure de probabilité équivalente sous laquelle la propriété d'immersion est satisfaite, ce qui implique donc la condition non-arbitrage. Nous donnons l'expression d'un $\mathbb{G}$-mouvement brownien en appliquant la formule de décomposition canonique des semimartingales et procédons à un changement de probabilité.

Dans la littérature de modélisation de multi-défauts, on suppose souvent que deux événements de défaut n'arrivent jamais simultanément, notamment dans les modèles d'intensité et de densité classiques. Très peu d'ouvrages traitent les modèles explicites des double-défauts (e.g., Bielecki et al. [BCCH12, BCCH14], Brigo and Capponi [BC10], Crépey and Song [CS14], Giesecke [Gie03]). Comme la crise de la dette souveraine s'est aggravée à cause d'un effet de contagion, nous étudions également les risques extrêmes tels que celui de deux défauts simultanés qui, bien que rares, peuvent avoir un impact désastreux sur le marché financier. L'approche de densité généralisée fournit des outils mathématiques puissants pour ce genre de problèmes en faisant appel à la méthode de récurrence. Dans le modèle que nous étudions, le temps de défaut σ_2 satisfait l'hypothèse de densité généralisée et peut donc coïncider avec un autre temps de défaut σ_1, qui est un temps d'arrêt par rapport à la filtration de référence. Différent des autres exemples précédemment évoqués, σ_1 est totalement inaccessible, ce qui implique que σ_2 est également totalement inaccessible.

Introduction

Based on the European sovereign debt crisis, which remains a worldwide topical issue since the end of 2009, this dissertation deals with the mathematical modelling of sovereign credit risk and its applications in the financial industry.

Briefly speaking, the European sovereign debt crisis is an on-going multi-year debt crisis arising from important sovereign risks and strong financial contagion effects. Several euro area member states (Greece, Portugal, Ireland, Spain, and Cyprus) were unable to repay or refinance their government debt without the assistance of third parties such as other Eurozone countries, the European Central Bank, and the International Monetary Fund, etc.

Different from corporate credit risk, sovereign risk has mixed features which result from a combination of complex macroeconomic and political factors, which is especially true for a euro area member state. Empirical studies show that the determinant macroeconomic factors can be summarised by a common one known as sovereign solvency (see e.g. Alogoskoufis [Alo12]), and the impact of political decisions arises notably when a predetermined critical political event happens. The political impact is described by a high probability of default before a critical political event and a sharp fall slightly after it. This phenomenon is pictured in the large variations of long-term government bond yield and interpreted by the multiple equilibria in debt markets in presence of credit risk. More precisely, the prevailing equilibrium depends on the expectations of investors about the probability of default (e.g., Calvo [Cal88]). Before a critical political event, investors expect a sovereign default with a high probability, in which case the government bond market is in a large-spread equilibrium. Shortly after the critical political event, the expectation of investors about the sovereign default is suddenly reduced to keep the government bond market in a narrow-spread equilibrium. Therefore, the probability of default at the time of a critical political event is nonzero, characterised by a jump in long-term government bond yield.

In continuous-time models, the time of default is usually modelled as a random time, in particular, a stopping time with respect to a proper filtration. In probability theory, the stopping times can be classified into three categories : predictable stopping times, accessible stopping times, and totally inaccessible stopping times. Intuitively, a predictable stopping time is known just before the event happens since it is annouced by an increasing sequence of stopping times ; an accessible stopping time can be perfectly covered by a sequence of predictable stopping times ; and a totally inaccessible stopping time is the instance of a total surprise which can never coincide with a predictable stopping time.

In the literature on credit risk modelling, two classic approaches already exist (see the books of Bielecki and Rutkowski [BR02], of Duffie and Singleton [DS03], and of Schönbucher [Sch03], and also the surveys of Bielecki, Jeanblanc and Rutkowski [BJR04b] and of Schmidt and Stute [SS03] for a detailed description) : structural approach (Black and Scholes [BS73], Merton [Mer74], Black and Cox [BC76]), where the default time is usually predictable, and reduced-form approach (also called intensity approach, Jarrow and Turnbull [JT92, JT95], Lando [Lan98], Duffie and Singleton [DS99]), where the default time is totally inaccessible. Besides, in a structural model with imperfect accounting reports (e.g., Duffie and Lando [DL01b], Giesecke [Gie06]), the default time is also a totally inaccessible stopping time. Recently, Jarrow and Protter ([JP04]) point out from an informational perspective that the difference between classes of stopping times can be characterised in terms of the information known to the modeller and thus one can transform a predictable stopping time into a totally inaccessible stopping time by modifying the information set used by the modeller.

In credit risk analysis, the progressive enlargement of filtrations (e.g., Barlow, Jacod, Yor, and Jeulin [Bar78, Jac85, Jeu80, JY78, Yor78]) has been systematically adopted to model the default event that a modeller cannot observe from the default-free market information flow, or mathematically speaking, the default time is not modelled as a stopping time with respect to the reference filtration (see also Mansuy and Yor [MY06], Protter [Pro05], Dellacherie, Maisonneuve and Meyer [DMM92], Yor [Yor12], Brémaud and Yor [BY78], Nikeghbali [Nik06], Ankirchner [Ank05], Song [Son87], Ankirchner, Dereich and Imkeller [ADI07], Yœurp [Yœu85]). In the work of Elliot, Jeanblanc and Yor [EJY00] and Bielecki and Rutkowski [BR02], the authors have proposed to use the progressive enlargement of filtrations to describe the market information which includes both the default-free information and the default information. Precisely, the global market information is modelled as the smallest filtration containing all the default-free information such that the default time is a stopping time. More recently, in order to study the impact of default events, a new approach has been developed by El Karoui, Jeanblanc and Jiao [EKJJ10, EKJJ15] where we assume the density hypothesis. In particular, the density approach allows us to analyse what happens after a default event and has interesting applications in the study of counterparty default risks. We note that, in both intensity and density approaches, the default time is a totally inaccessible stopping time.

The theory of stopping time decomposition postulates that every stopping time can be uniquely decomposed into an accessible stopping time and a totally accessible stopping time (Dellacherie [Del72]). The decomposition of random times also appears in literature on the theory of enlargement of filtrations such as Coculescu [Coc09] and Aksamit, Choulli and Jeanblanc [ACJ16]. For the case of sovereign default, from the point of view of decomposition, if the default time can coincide with a predetermined critical date with a positive probability, then it means that the default time has an accessible part in addition to a totally inaccessible one. On the one hand, neither the reduced-form approach nor the density approach is realistic since the default time modelled by these approaches is only totally inaccessible and avoids any predictable stopping time. On the other hand, a structural model is not appropriate as revealed by Matsumura in [Mat06], because it is not obvious which asset value could be taken as reference, and the impact of political decisions are not reflected in the structural definition of default in literature. For this reason, we propose a hybrid model which is based on both approaches of the classic credit risk models and in the meantime takes into account the level of the sovereign solvency and the impact of critical political events.

Andreasen [And03] was one of the first to study this type of hybrid model. There exist in the credit risk literature other hybrid models such as the generalised Cox process model in Bélanger, Shreve and Wong [BSW04], the credit migration hybrid model in Chen and Filipović [CF05], the jump to default CEV models in Carr and Linetsky [CL06] and Campi, Polbennikov and Sbuelz [CPS09], and the generalised framework without intensity in Gehmlich and Schmidt [GS16] and Fontana and Schmidt [FS16]. More precisely, in [BSW04], the hazard process can have jumps ; in [CF05], the default of the firm is triggered either by successive downgradings of the credit notes or an unpredictable jump of a simple point process ; in [CL06], the equity value is a CEV diffusion punctuated by a possible jump to zero which corresponds to default. The default time is decomposed into a predictable part, which is the first hitting time of zero by the equity value process, and a totally inaccessible part, given by a Cox process model ; in [CPS09], the equity value is a CEV process, and the default time is the minimum of the first Poisson jump and the first absorption time of the equity value process by zero in absence of jumps ; in [GS16], the Azéma supermartingale of the default time contains jumps so that the intensity does not exist, and [FS16] generalises the approach in [GS16].

In Chapter 1, we propose a hybrid sovereign default model which combines an accessible part which takes into account the movement of the sovereign solvency and the impact of critical political events, and a totally inaccessible part for the idiosyncratic credit risk. The principal features of our model include that the default time can coincide with a family of predictable stopping times, the modelling of the impacts of macroeconomic facteurs and political events are separated from that of the idiosyncratic risk, the solvency can be correlated with the idiosyncratic risk, and the immersion property holds but can be easily relaxed. We are inspired by the jump to default CEV models in [CL06] and [CPS09], which were originally proposed for assessing corporate credit risks. The main differences are that the default in our model can coincide with multiple political events and can occur after any of these events, while the default time in [CL06] and [CPS09] is bounded by its single predictable part.

The notion of sovereign solvency is important in our sovereign risk model. The solvency is a unified variable which reflets the impacts of macroeconomic factors on the ability of a sovereign to meet its long-term obligations. We use the existing definition of the sovereign solvency in discrete-time case ([Alo12]) to the continuous-time case and consider a sovereign (e.g. Greece) with solvency process $(S_t, t \geq 0)$, included in the reference information $\mathbb{F}$. Our model is based on the mathematical abstraction of the following scenario : the authorities (European Central Bank, European Commission, International Monetary Fund, etc.) set a re-adjustable budget requirement for the sovereign. If the solvency process S falls below the required level, we consider that the sovereign becomes seriously insolvent, and a critial political event should be organised at which political decisions need to be made concerning the sovereign. The result of the decisions can be an immediate bankruptcy of the sovereign or a financial aid package for the sovereign in the aim of improving its financial situation. In the latter case, if the debt and deficit situation of the sovereign is excessive and has no improvement, the authorities may relax the budget requirement and anticipate other critical political events. Then, the default time τ in our model can be (uniquely) decomposed into an accessible part ζ^* and a totally inaccessible part ξ with a unique partition of the sample set Ω :

$$\tau = \zeta^* \wedge \xi.$$

The accessible part ζ^* is covered by an increasing sequence of n solvency barrier hitting times ($\mathbb{F}$-stopping times) $\tau_1, \ldots, \tau_n$:

$$\tau_i = \inf\{t \geq 0 : S_t < L_i\},$$

and whether ζ^* meets one of them depends on an exogenous facteur (e.g. the occurrence of an external financial shock) modelled for simplicity by an inhomogenous Poisson process $(N_t, t \geq 0)$ with intensity function $\lambda^N(t)$. The totally inaccessible part ξ admits an intensity process $(\lambda_t, t \geq 0)$, which can be correlated with the solvency process S. The global information structure $\mathbb{G}$ is the progressive enlargement of the filtration $\mathbb{F}$ by the sovereign default time τ.

We compute the probabilities that the sovereign default occurs on specific critical political dates :

$$\mathbb{P}(\tau = \tau_i | \mathcal{F}_t) = \mathbb{E}\left[\left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds} \right) e^{-\int_0^{\tau_i} \lambda_s \, ds} \middle| \mathcal{F}_t \right].$$

As we show, such default probabilities are nonzero in the hybrid model, which implies singularities in the probability distribution of the default time τ :

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}\left[\exp\Big(-\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i\le u\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds - \int_0^u \lambda_s\, ds\Big)\Big|\mathcal{F}_t\right].$$

Consequently, the Azéma supermartingale is discontinuous at $\tau_1,\dots,\tau_n$, and the hazard process

$$\left(\int_0^t \lambda_s\, ds + \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i\le t\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds, \quad t\ge 0\right).$$

is not equal to the $\mathbb{F}$-compensator process

$$\left(\int_0^t \lambda_s\, ds + \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i\le t\}}\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds}\Big), \quad t\ge 0\right)$$

although they have an identical countinuous part.

In specific Markovian settings, where the solvency process and the Poisson process are homogeneous, the probabilities $\mathbb{P}(\tau=\tau_i|\mathcal{F}_t)$ can be derived from the Laplace transform of the stopping time $\rho_x = \inf\{t\ge 0 : S_t^x \le L\}$:

$$\mathbb{E}\left[\exp\Big(-k\rho_x - \int_0^{\rho_x}\lambda(S_u^x)\, du\Big)\right],$$

which is the representation of the solution to the following Dirichlet problem

$$\mu(z)u'(z) + \frac{1}{2}\sigma^2(z)u''(z) = (\lambda(z)+k)u(z) \quad \text{on } \{z>L\};$$
$$u(L)=1.$$

We obtain closed-form formulae by solving Sturm-Liouville equations in the cases of geometric brownian motion and CEV process. More precisely, when the solvency process is modelled by a geometric Brownian motion

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t),$$

it turns out to be expressed in terms of the solution to a modified Bessel equation

$$(xy')' - c_1x^{-1}y' = c_2xy$$

where $c_1, c_2 > 0$, while in the case of the solvency process modelled by a CEV diffusion

$$dS_t = S_t(\mu\, dt + \delta S_t^{\beta} dW_t),$$

we need to solve the following differential equation

$$\frac{1}{2}\delta^2 x^{2+2\beta}u'' + \mu x u' - (ax^{-2|\beta|} + b + k)u = 0,$$

where we discuss the sign of the elasticity parameter β and obtain the fundamental solutions by using Whittaker and modified Bessel functions.

The immersion property has the advantage to imply the market completeness (e.g. [JLC09a]). However, it is usually impossible to assume the immersion property in the cases of incomplete market, non-ordered multi-defaults, and correlation between different default times. In our hybrid model, the immersion property is controlled by the random barrier η, or more precisely, the immersion property holds when η is independent of $\mathcal{F}_\infty$. By relaxing the last assumption, we can extend the model beyond the immersion paradigm. Consequently, the Azéma supermartingale is no more a decreasing process, and we compute its multiplicative decomposition to obtain the compensator process.

From a probabilistic point of view, the nonzero probabilities of default on critical political event dates means that the probability distribution of the default time admits singularities. In order to extend our sovereign risk model to a general framework, we consider a type of random times which can be either accessible or totally inaccessible and propose to generalise the density approach in [EKJJ10]. More precisely, we assume that the conditional probability distribution of τ contains a discontinuous part, besides the absolutely continuous part which has a density.

In Chapter 2, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the reference filtration and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the progressive enlargement of $\mathbb{F}$ by τ, and we assume the *generalised density hypothesis* that the $\mathbb{F}$-conditional probability distribution of τ avoiding a family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$ has a density (called generalised density) with respect to a non-atomic σ-finite Borel measure η on $\mathbb{R}_+$:

$$\mathbb{E}[\mathbb{1}_H h(\tau)\,|\,\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.},$$

where H denotes the random set

$$\{\tau < \infty\} \cap \bigcap_{i=1}^{n} \{\tau \neq \tau_i\}.$$

Our key assumption is satisfied by other previously cited hybrid models such as the generalised reduced-form model in [BSW04], the credit migration model in [CF05], the jump to default CEV models in [CL06] and [CPS09], the extension of the HJM-approach in [GS16], as well as our own hybrid sovereign risk model in Chapter 1 in both the cases with and without immersion, and we can compute the generalised density process for each of these models. Under the generalised density hypothesis, τ has only the possibility to meet the $\mathbb{F}$-stopping times that can coincide with $(\tau_i)_{i=1}^n$. We prove that the generalised density process $\alpha(\cdot)$ is a parametered càdlàg $\mathbb{F}$-martingale. We denote by $(p_t^i, t \geq 0)$ the $\mathbb{F}$-conditional probability that τ meets τ_i, $i \in \{1, \dots, n\}$, and then any $\mathcal{G}_t$-conditional expection can be computed in a decomposed form in terms of $\alpha(\cdot)$ and $(p^i)_{i=1}^n$. Furthermore, we can prove that the following decomposition formula for $\mathbb{G}$-optional process is true, which is not the case in general

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} Y_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(\tau), \quad \mathbb{P}\text{-a.s.}$$

The last formula has been proved to be valid for the existing models, in particular under the density hypothesis, and one can find the conditions in Song [Son14].

The compensator process under the generalised density hypothesis is not a continuous process in general and so the intensity process does not always exist. We deal with the Doob-Mayer decomposition of the Azéma supermartingale G and focus on its discontinuous part, and we obtain the general form of the compensator

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \mathbb{1}_{\{G_{s-}>0\}} \frac{\alpha_s(s)\eta(ds)}{G_{s-}} + \sum_{i=1}^n \int_{(0,t\wedge\tau]} \mathbb{1}_{\{G_{s-}>0\}} \frac{p_{s-}^i d\Lambda_s^i + d\langle M^i, p^i\rangle_s}{G_{s-}}, \quad t \geq 0,$$

where $\Lambda^i = (\Lambda_t^i, t \geq 0)$ is the compensator process of $(\mathbb{1}_{\{\tau_i \leq t\}}, t \geq 0)$ and

$$M^i = \left(\mathbb{1}_{\{\tau_i \leq t\}} - \Lambda_t^i, t \geq 0\right)$$

for any $i = \{1, \dots, n\}$. If $(\tau_i)_{i=1}^n$ are predictable $\mathbb{F}$-stopping times, then τ is an accessible $\mathbb{G}$-stopping time and the compensator process of τ has an absolutely continuous part and a jump part ; if they are totally inaccessible $\mathbb{F}$-stopping times, then τ is a totally inaccessible $\mathbb{G}$-stopping time and the compensator process of τ is continuous.

We also characterise the martingale processes in the filtration $\mathbb{G}$ by using three $\mathbb{F}$-martingale conditions. Different from [EKJJ10], the necessary conditions and the sufficient ones are subtly distinguished. It is also a classic problem to investigate the stability of semimartingales when the reference filtration is enlarged. We obtain the canonical decomposition of an $\mathbb{F}$-local martingale as a $\mathbb{G}$-semimartingale

$$\begin{aligned}U_t^{\mathbb{F}} = U_t^{\mathbb{G}} + \int_{(0,t\wedge\tau]} \frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}} \\ + \mathbb{1}_{\cap_{i=1}^N\{\tau\neq\tau_i\}} \int_{(\tau,t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, \alpha(u)\rangle_s}{\alpha_{s-}(u)}\bigg|_{u=\tau} + \sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}} \int_{(\tau,t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p^i_{s-}},\end{aligned}$$

where $U^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale and $\bar{M}$ is an BMO $\mathbb{F}$-martingale computed by

$$\bar{M}_t = \mathbb{E}\left[\int_0^\infty \alpha_u(u)\eta(du)\Big|\mathcal{F}_t\right] + \sum_{i=1}^n p^i_{t\wedge\tau_i} + p_t^\infty, \quad t \geq 0.$$

The decompostion formula above means that the (H')-hypothesis of Jacod is satisfied under generalised density hypothesis.

In Chapter 3, we apply the results of the previous chapters to the valuation and hedging of credit risks. [JLC09b] and Coculescu, Jeanblanc and Nikeghbali [CJN12]. Precisely, if the immersion property is satisfied under the risk-neutral probability measure, then the market model is arbitrage-free. In [JLC09b], the credit event is modelled by a random time satisfying the density hypothesis, which insures that the semimartingales in the reference filtration remain semimartingales in the enlarged filtration, which gives way to a change of probability measure by using Girsanov theorem.

One of the most important applications of the sovereign default model is the valuation of sovereign defaultable claims such as sovereign bonds. We are particularly interested in the behaviour of long-term bond yield during the sovereign debt crisis and we show that the hybrid model provides an explanation to the jump behaviours of the bond yield around critical political event dates. Precisely, the jump in the bond value at a critial political event date τ_i is characterised in terms of the recovery payment and the $\mathbb{F}$-compensator by the following equality

$$\Delta D(\tau_i, T) = (D(\tau_i, T) - R_{\tau_i})\Lambda^{\mathbb{F}}_{\tau_i} \quad \text{on } \{\tau_i < T\}.$$

The credit spread behaves in the opposite direction of the bond price. Since the bond price is in general higher than the recovery payment, the credit spread is likely to have negative jumps.

We give an example of indifference pricing with portfolio optimisation in a CEV credit risk model, where an investor trades a stock with a bankruptcy risk. Hodges and Neuberger [HN89] were the first to introduce utility indifference pricing in incomplete markets (see Henderson and Hobson [HH04] for a general survey of literature on this topic, and the research paper of Bielecki, Jeanblanc and Rutkowski [BJR04a] and the thesis of Sigloch [Sig09] for an overview in the context of default risk). The default time can be either the first hitting time of zero by the stock price modelled by a CEV diffusion or the time of a jump to default which forces the stock price to zero. Then, the total wealth $\hat{X}$ depends on two different default regimes. Given a utility function U satisfying some general conditions, we study the following control problem :

$$v(x) = \sup_{\varphi \in \mathcal{A}} \mathbb{E}\left[U(\hat{X}_T^{x,\varphi})\right],$$

where the gain function has a finite horizon with terminal payoff depending on a stopping regime.

We also study the pricing of defaultable claims in a general framework, where no immersion property holds and no risk-neutral probability is given. In Coculescu, Jeanblanc and Nikeghbali [CJN12], the authors have also pointed out how the immersion property is modified under an equivalent change of probability measure. In the generalised density framework studied in Chapter 2, we prove that the generalised density hypothesis holds under an equivalent change of probability measure, and that we can always find an equivalent probability measure under which the immersion property holds true, which implies no-arbitrage conditions. We compute the $\mathbb{G}$-Brownian motion by using the canonical decomposition formula and conduct a change of probability.

In the literature of multi-default modelling, one often supposes that there are no simultaneous defaults, notably in the classic intensity and density models. Only few papers consider explicit models of double defaults (e.g. Bielecki et al. [BCCH12], Giesecke [Gie03]). Since the sovereign debt crisis is contagious, we also study extremal risks such as simultaneous defaults whose occurrence is rare but will have significant impact on financial market. The generalised density approach provides mathematical tools to study multi-default models with simultaneous defaults by using a recurrence method. In the model that we study, the default time σ_2 satisfies the generalised density hypothesis and can coincide with another default time σ_1, which is a stopping time in the reference filtration. Different from other examples, σ_1 is totally inaccessible, which implies that σ_2 is also totally inaccessible.

Chapter 1

A hybrid model for sovereign risk

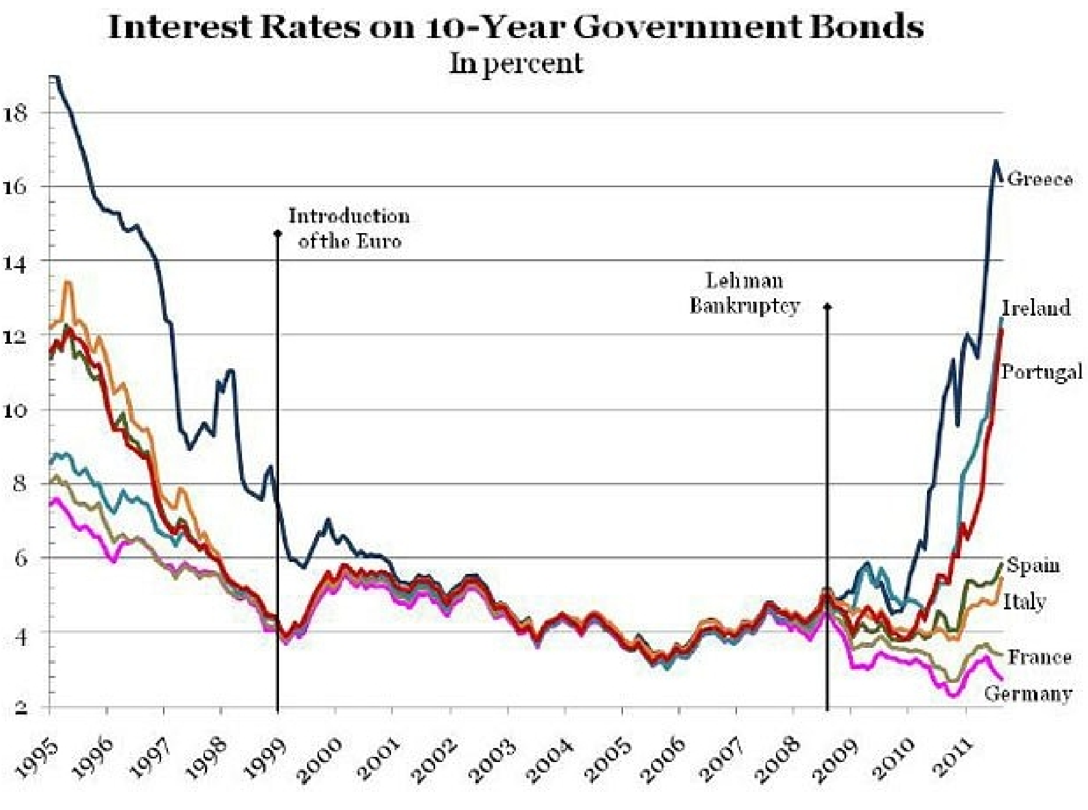
Motivated by the European sovereign debt crisis, we propose a hybrid sovereign default model which combines an accessible part which takes into account the movement of the sovereign solvency and the impact of critical political events, and a totally inaccessible part for the idiosyncratic credit risk. As a consequence, the probability distribution of the default time admits singularities. We are interested in the the probability that the default occurs at critical political dates, for which we obtain closed-form formulae in a Markovian setting. We compute the compensator process of default and show that the intensity process does not exist.

Keywords : Sovereign risk, sovereign solvency, idiosyncratic credit risk, decomposition of stopping times.

1.1 Introduction

The European sovereign debt crisis (often also referred to as the Eurozone crisis or the European debt crisis) is an on-going multi-year debt crisis which took place at the end of 2009, when the long-term interest rates of euro area countries began to diverge significantly. Figure 1.1 is an interesting graph from Vicky Price of FTI Consulting which shows the main story behind the Eurozone crisis [1]. Several European countries (e.g., Greece, Ireland, Portugal, Cyprus) faced the collapse of financial institutions, high government debt and rapidly rising bond yield spreads in government securities, which has made it difficult for them to refinance their public debts without aid of third parties. The crisis has also led to a crisis of confidence for European businesses and economies and a growing amount of attention to sovereign risks from governments and financial markets.

Figure 1.1 – Historical interest rates on 10-year government bonds before 2012



1. http://www.economonitor.com/blog/2011/12/which-graph-best-summarizes-the-eurozone-crisis/

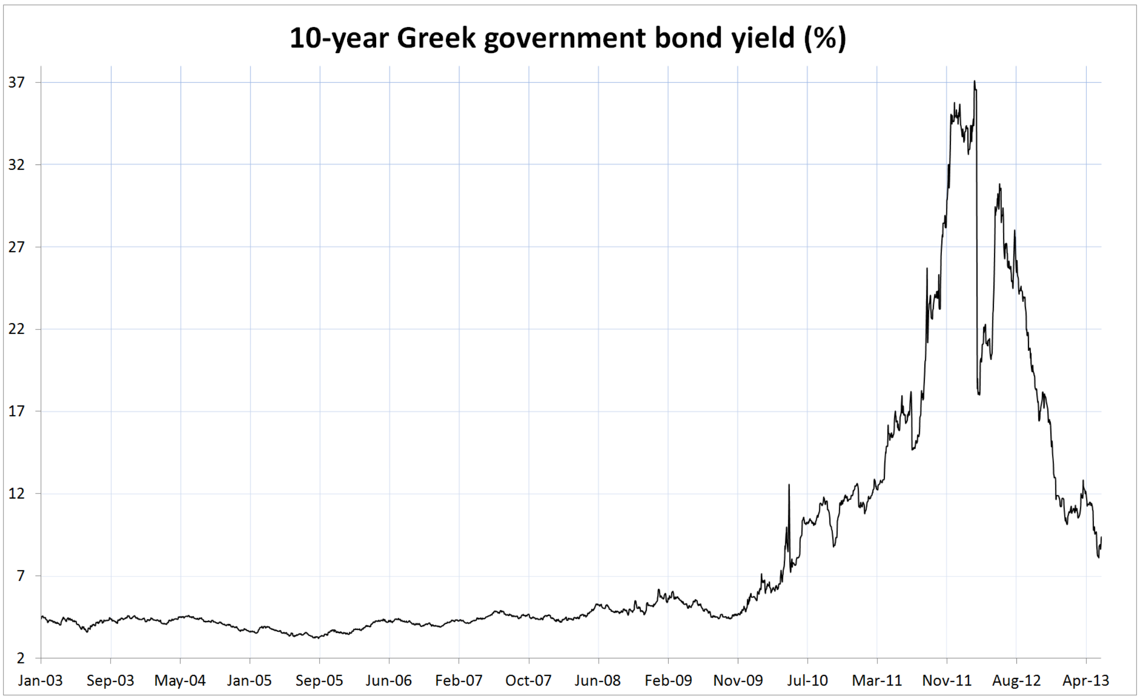
Sovereign risk is the possibility that the government of a country could default on its debt or other obligations (definition from Financial Times Lexicon). It belongs to the family of credit risks, and is a fundamental component of risks in government bond yield curves. However, the modelling of sovereign risks is a challenging subject and may differ from the corporate credit risks. Firstly, sovereign default is usually influenced by macroeconomic factors such as GDP, total public debt, government revenue and expenditure, and inflation, etc.. Secondly, political events and decisions have important impacts on the sovereign default, especially for a European Union member state: the government can opt for defaulting on internal or external debt, and the same bond can be renegotiated many times, etc..

In literature (see e.g. Alogoskoufis [Alo12]), the determinant macroeconomic variables can be summarised by a single one known as sovereign solvency, which can be measured and monitored easily. We are also interested in the impact of political decisions, notably when a critical political event happens. In practice, we observe that on a critical date when important political events are held, the probability of sovereign default can become significant. This point may be justified by the following intuitive argument: when a sovereign is unable to repay its public debt, it solicits an international financial aid as a last resort; if the sovereign is not able to receive the financial support, it can end up in bankruptcy. We have chosen as example three critical dates, noted as T_1, T_2 and T_3, all of which concern the financial aid packages for Greece:

1. on 2 May 2010 (T_1), the euro area member states and International Monetary Fund (IMF) agree on a 110-billion-euro financial aid package for Greece;
2. on 21 July 2011 (T_2), the government heads of the euro area agree to support a new financial aid program of 109-billion-euro for Greece;
3. on 8 March 2012 (T_3), the European Central Bank (ECB) governing council acknowledges the activation of the buy-back scheme for Greece and decides that debt instruments issued or fully guaranteed by Greece will be again accepted as collateral in European credit operations, without applying the minimum credit rating threshold for collateral eligibility, until further notice.

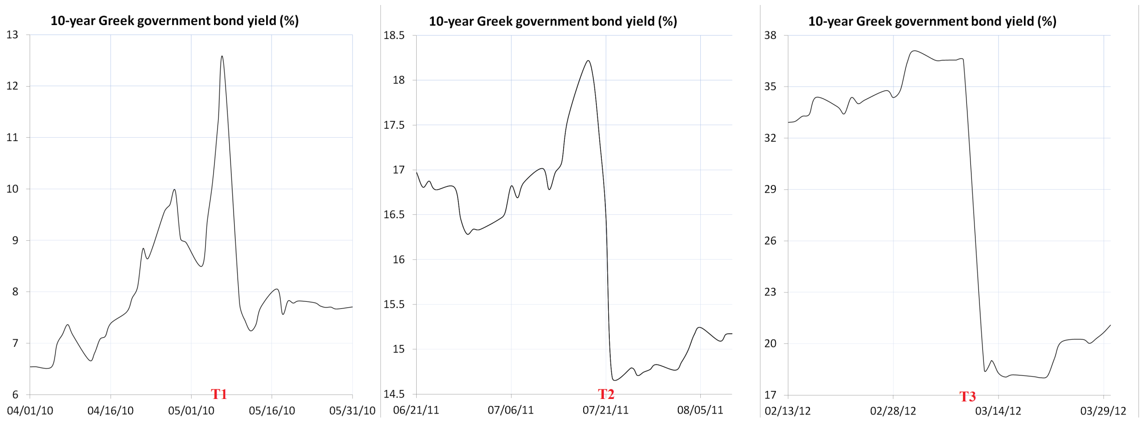
We notice that T_1, T_2 and T_3 are predetermined dates publicly known to investors since political events are in general arranged in advance and they can be found on the official website of European Central Bank. The impact of these political events can be observed in the long-term Greek government bond yield. Figure 1.2 shows the historical 10-year bond yield of the Greek government from 2003 to 2013 (data from Bloomberg), where we observe significant movements of the bond yield during the sovereign crisis. In particular, as illustrated by Figure 1.3, where the three pictures are extracted from Figure 1 around the three dates T_1, T_2 and T_3 respectively, the bond yield has large variations with very high levels of the yield before and negative jumps at or slightly after these critical dates.

Figure 1.2 – Historical 10-year Greek bond yield from 2003 to 2013



Macroeconomic models of debt crisis emphasise such phenomena by the multiple equilibria in debt markets in presence of credit risk. The prevailing equilibrium depends on the expectations of investors about the probability of default (e.g., Calvo [Cal88]). Before a critical political event, investors expect a sovereign default with a high probability, in which case the spread of the government bond is very large, and the debt market is in a large-spread equilibrium. Shortly after the critical political event, the expectation

Figure 1.3 – Greek bond yield around critical dates T_1, T_2 and T_3 (extracted from Figure 1.2)



of investors about the sovereign default is suddenly discharged to keep a narrow-spread equilibrium. Therefore, one can say that the probability of default at the time of a critical political event is nonzero, characterised by a jump in long-term government bond yield.

From a mathematical point of view, the nonzero probability of default on a predetermined date means that the random time of default has a predictable component. In the literature on credit risk modelling, two classic approaches exist: structural approach and reduced-form approach. In a standard structural model, the default time is often a predictable stopping time defined as the first hitting time of a certain default barrier by the asset value process of a firm; while in a reduced-form model, it is usually made a totally inaccessible (probabilistic jargon for an unexpected surprise) stopping time modelled as the first jump time of a point process with stochastic intensity. Besides, in a structural model with incomplete information (e.g., [DL01b]) where the accounting reports are imperfect, and the default time is a totally inaccessible stopping time. Both approaches have been widely used to model corporate credit risks (see the books of Bielecki and Rutkowski [BR02] and Duffie and Singleton [DS03] for a detailed description). For the case of sovereign default, on the one hand, the reduced-form approach is not realistic since the default time modelled by this approach is totally inaccessible and avoids any predictable stopping time. On the other hand, a structural model is not appropriate as revealed by Matsumura in [Mat06], because it is not obvious which asset value could be taken as reference, and the impact of political decisions are not reflected in the structural definition of default in literature.

In this chapter, we propose a hybrid model which is based on both approaches of the classic credit risk models and in the meantime takes into account the level of the sovereign solvency and the impact of critical political events. We intend to explain in Chapter 3 the significant movements of the sovereign bond yield during the sovereign debt crisis by the mixed characteristic of the hybrid model. We are inspired by the jump to default CEV (constant elasticity of variance) models in Carr and Linetsky [CL06] and Campi, Polbennikov and Sbuelz [CPS09], which were originally proposed for assessing corporate credit risks. In [CL06], the equity value is a CEV diffusion punctuated by a possible jump to zero which corresponds to default. The default time is decomposed into a predictable part, which is the first hitting time of zero by the equity value process, and a totally inaccessible part, given by a Cox process model. In [CPS09], the equity value is a CEV process, and the default time is the minimum of the first Poisson jump and the first absorption time of the equity value process by zero in absence of jumps. So the default time can be either predictable, according to the CEV process, or unpredictable, according to a Poisson jump. In these models, we note that the default time is bounded by its predictable part, that is, a default never occurs after a predictable stopping time, and as a consequence, the Azéma supermatingale jumps to zero at the predictable stopping time. In the credit risk literature, there exist other hybrid models such as the generalised Cox process model in Bélanger, Shreve and Wong [BSW04] where the hazard process admits jumps, and the credit migration hybrid model in Chen and Filipović [CF05] where the default of the firm is triggered either by successive downgradings of the credit notes or an unpredictable jump of a simple point process. Recently, Gehmlich and Schmidt [GS16] consider models where the Azéma supermartingale of the default time contains jumps (so that the intensity does not exist) and develop the associated HJM credit term structures and no-arbitrage conditions. The decomposition of random times also appears in literature on the theory of enlargement of filtrations such as Coculescu [Coc09] and Aksamit, Choulli and Jeanblanc [ACJ16] in a more theoretical and general setting.

The hybrid sovereign default model that we propose in this chapter also combines the structural and the reduced-form approaches. On the one hand, the accessible part of the sovereign default time, which depends on solvency process and exogenous macroeconomic factors, describes the critical political dates. On the other hand, the totally inaccessible part, which represents the idiosyncratic credit risk, is given by the standard Cox process model. In this model, the probability distribution of the default time can have singularities and the critical political dates and decisions have important impacts on the sovereign default probability. In order to analyse the political impact, we compute the probability that the sovereign default occurs at critical dates and we obtain closed-form formulae in a Markovian setting by solving Sturm-Liouville equations. More precisely, when the solvency process is modelled by a geometric Brownian motion, the default probability is given in terms of the solution to a modified Bessel equation. In the case of the solvency process modelled by a CEV diffusion, we discuss the sign of the elasticity parameter and obtain the default probability by extending CEV ordinary differential equations in [DL01a, Lin04] and using Whittaker and modified Bessel functions. Numerical tests show that the political decisions have an important impact on the probability of sovereign default. Different from [CPS09] and [CL06], the default time in this model can go beyond its predictable components, while under suitable conditions we can recover the jump to default CEV model. We will also revisit in Chapter 2 this sovereign default model in a general setting which extends the default density approach introduced in El Karoui, Jeanblanc and Jiao [EKJJ10].

1.2 Hybrid model of sovereign default time

In this section, we present a hybrid model for the sovereign default which takes into account the macroeconomic situations of the country, the impact of critical political events and the idiosyncratic default risk.

1.2.1 Sovereign solvency: a structural model

We start by introducing the notion of solvency. Sovereign default is tightly related to the macroeconomic factors of the country. Notably, the sovereign solvency is an important indicator since it includes several determinant macroeconomic variables of sovereign default, such as, public debt, primary surplus, GDP growth rates, etc.. For example, in January 2010, the Greek Ministry of Finance published the *Stability and Growth Program 2010*. The report listed main causes for eruption of the government-debt crisis among which: low GDP growth rates, huge fiscal imbalances, and high government debt level. Here, we borrow the definition used in [Alo12] for discrete-time case and derive the continuous-time version.

Definition 1.1. The sovereign solvency at time t is defined by

$$\ln S_t = \pi_t - \frac{b_{t-1}(r_t - g_t)}{1 + g_t}, \tag{1.1}$$

where b_{t-1} denotes the public debt to GDP ratio of the previous observation year, π_t is the primary surplus to GDP ratio, r_t is the real interest rate on government bonds, and g_t is the GDP growth rate. In particular, we say that the government is fiscally sustainable if $S_t \geq 1$, and is insolvent if $S_t < 1$.

The definition above can be derived by starting from the government budget constraint:

$$B_t - B_{t-1} = r_t B_{t-1} - P_t, \tag{1.2}$$

where B_t and B_{t-1} represent respectively the amount of public debt of the current observation year and that of the previous observation year, r_t is the real interest rate on government bonds, and P_t is the amount of the primary surplus of the government budget of the year. Dividing through by GDP, one has

$$b_t - b_{t-1} = \frac{b_{t-1}(r_t - g_t)}{1 + g_t} - \pi_t. \tag{1.3}$$

The right-hand side of the equality above quantifies the government deficit, the negation of which reflects thus the solvency. Slightly different from the initial definition, we use the exponential form such that the solvency takes only positive values. By definition, four factors determine whether a government is solvent. The predetermined historical debt is known from the government's balance sheet of the preceding year. The real interest rate on government bonds, which is the cost of debt refinancing, can be deduced from bond yield curves and consumer price indices (or break-even indices). The GDP growth rate is observable directly from the economic cycle. The primary surplus, which is the measurement of government deficit, can be computed from the government revenue and expenditure as well as the fiscal dynamics. In practice, these data are available for discrete-time observations.

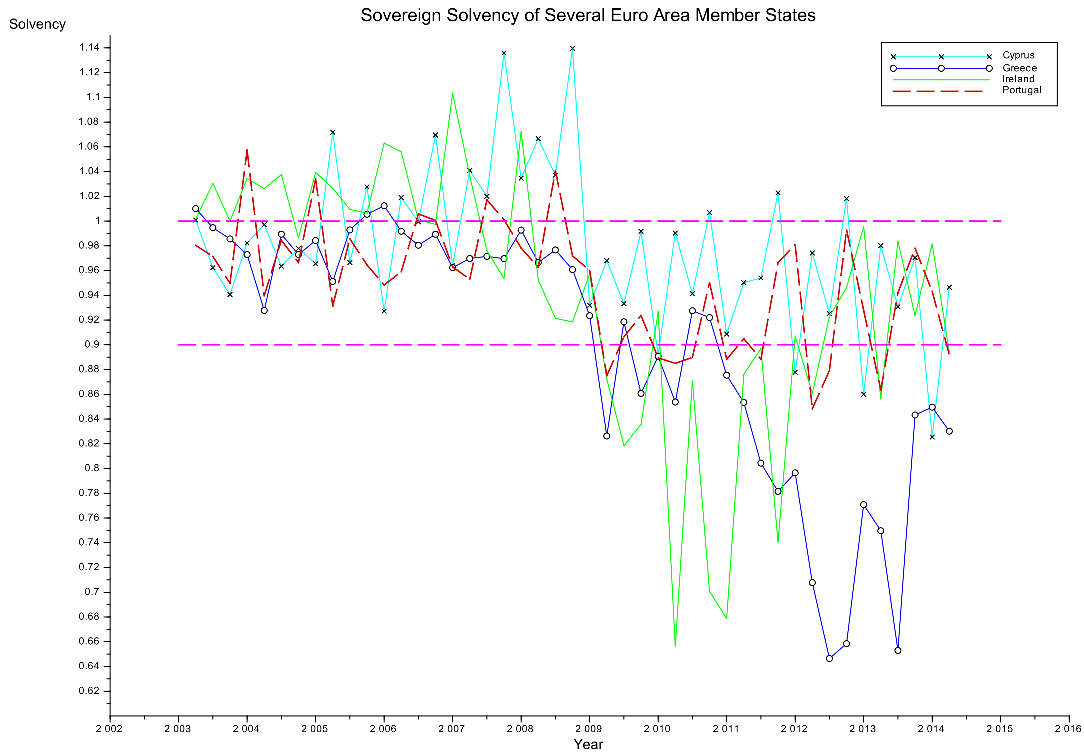
We illustrate in Figure 1.4 the solvency values from the first quater of 2003 to the third quater of 2015 computed by using (1.5) for the following four member states of the euro area: Cyprus, Greece, Ireland, and Portugal. The data for the interest rates comes from the official website of European Central Bank (sdw.ecb.europa.eu) and that for the other factors from the official website of European Commission (ec.europa.eu/eurostat). We notice that all these countries are insolvent during the crisis (solvency lower than 1 from 2009), with Greece and Ireland in the worst situation. This observation is rather coherent with the reality since Greece and Ireland are the first countries that solicit aid from third parties (Greece on 23/04/2010 and Ireland on 21/11/2010). Furthermore, the crisis starts at the end of 2009 when the solvency of several countries falls below 0.9, so we can consider 0.9 as an approximate threshold of the debt crisis. Indeed, fears about a debt crisis begin to spread when the solvency of a country hits down a certain threshold.

In a long-term time scale, we model the sovereign solvency by a continuous-time process. For this reason, we derive at first the continuous-time version of the solvency. From the flow equation (1.2) we obtain the continuous-time equivalent

$$dB(t) = r(t)B(t)dt - dP(t),$$

where $B(t)$ (respectively $P(t)$) is the instantaneous amount of public debt (respectively primary surplus) and $r(t)$ is the instantaneous real interest rate on government bonds at time t. Let us now consider $G(t)$ the amount of GDP at time t, with the additional assumption that the GDP grows following the countinous compounding, characterised

Figure 1.4 – Solvency of four countries of euro area.



by the differential equation $dG(t) = G(t)g(t)dt$, where $g(t)$ is the instantaneous GDP growth rate at time t. We assume that all the processes are continuous. Denoting by $b(t)$ (respectively $\pi(t)$) the instantaneous debt-to-GDP (respectively surplus-to-GDP) ratio at time t, i.e., $b(t) = B(t)/G(t)$ (respectively $\pi(t) = P(t)/G(t)$), one has

$$\begin{aligned} d\left(\frac{B(t)}{G(t)}\right) &= \frac{dB(t)}{G(t)} - \frac{B(t)\,dG(t)}{G^2(t)} \\ &= \frac{r(t)B(t) - dP(t)}{G(t)} - \frac{B(t)g(t)\,dt}{G(t)} \\ &= (r(t) - g(t))\,b(t)\,dt - d\left(\frac{P(t)}{G(t)}\right) - \frac{P(t)g(t)\,dt}{G(t)} \\ &= [r(t)b(t) - g(t)b(t) - g(t)\pi(t)]\,dt - d\pi(t). \end{aligned}$$

Finally, the continuous-time version of (1.3) is written as

$$db(t) = [r(t)b(t) - g(t)b(t) - g(t)\pi(t)]\, dt - d\pi(t). \tag{1.4}$$

Similar to the discrete-time case, the negation of the right-hand side of the equality above defines the dynamic of the soveregn solvency, and we give the following definition before the modelling in a probability space.

Definition 1.2. The continuous-time sovereign solvency at time t is defined by

$$S(t) = S(0) + \pi(t) - \pi(0) + \int_0^t [g(s)\pi(s) + g(s)b(s) - r(s)b(s)]\, ds, \tag{1.5}$$

where $b(t)$ denotes the debt-to-GDP ratio, $\pi(t)$ is the surplus-to-GDP ratio, $r(t)$ is the instantaneous real interest rate on government bonds, and $g(t)$ is the instantaneous GDP growth rate.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, i.e., $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets and $\mathbb{F}$ is right-continuous: $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$. We recall the definitions of three types of stopping times in the Appendix A, Definition A.1–A.3 (see e.g. the book of Protter [Pro05, Chapter III.2] or the survey of Nikeghbali [Nik06, Chapter 2.3] for details).

Let $W = (W_t, t \geq 0)$ be a standard Brownian motion which is $\mathbb{F}$-adapted. For a given country, we assume that the solvency is governed by a process $S = (S_t, t \geq 0)$ satisfying the following diffusion:

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_0 = x, \tag{1.6}$$

where μ and σ are Lipschitz continuous functions such that

$$\int_0^T |\mu(t, S_t)|dt + \int_0^T \sigma^2(t, S_t)dt < \infty$$

for any $T > 0$. Let L be a real positive constant with $L < S_0$ which represents a threshold of the debt crisis. More precisely, if S falls below L, we consider that the sovereign becomes seriously insolvent, i.e., the risk of default is extremely high. Then, we define a random time τ_0 as the first hitting time of the barrier L by the solvency process S, i.e.,

$$\tau_0 = \inf\{t \geq 0 : S_t < L\}, \tag{1.7}$$

with the convention $\inf \emptyset = \infty$. Note that τ_0 is a predictable $\mathbb{F}$-stopping time. In general, the début ("entering time") of a predictable random set is a predictable stopping time (Jacod and Shiryaev [JS13, Proposition 2.13]).

1.2.2 Critical political event

Generally speaking, when a sovereign becomes fiscally vulnerable, a political meeting will be organised at which political decisions need to be made concerning the relevant sovereign country. In reality, a multitude of political events are involved during these meetings. In this section, we propose a simplified model to explore the political impact, while real situations tend to be more complicated and less transparent. For the concerned sovereign, the meeting date is a critical date and often comes shortly after the solvency barrier hitting time τ_0. In our model, we assume that the critical date coincides with the time τ_0. We also assume that the result of political decisions depends on some exogenous factor, such as an economic or financial shock: if the shock has occurred before the critical date, then the sovereign can possibly end up in default at τ_0; otherwise, it receives a financial aid package without immediate default at τ_0. Indeed, the term financial aid package is perceived as any assistance from third parties with the aim of improving the solvency of the country in debt crisis, including a bailout loan, a quantitative easing policy, etc., in return for which the beneficiary should take austerity measures to ameliorate its financial situation. From an economic point of view, when the solvency is below the threshold, an exogenous shock can make things worse so that the aids from third parties will be too costly (for example, austerity policies can do harms to the economy and so people vote against it) and the political decisions are in favour of a sovereign default.

We model the exogenous shock by the jump of a Poisson process $N = (N_t, t \geq 0)$ with intensity $\lambda^N > 0$ which is independent of the filtration $\mathbb{F}$. Then, the result of political decisions depends on the value of N at the critical date τ_0. More precisely, we define

$$\zeta = \begin{cases} \tau_0, & \text{on } \{N_{\tau_0} \geq 1\}, \\ \infty, & \text{on } \{N_{\tau_0} = 0\}. \end{cases} \tag{1.8}$$

The random time ζ takes into account both the sovereign solvency and the political decisions. Obviously, ζ is not an $\mathbb{F}$-stopping time. However, ζ is an honest time (e.g. Barlow [Bar78]), which by definition is $\mathcal{F}_t$-measurable on $\{\zeta \le t\}$. Definitely, the event $\{\zeta \le t\}$ implies $\{\tau_0 \le t\}$, then for any $t \ge 0$,

$$\mathbb{1}_{\{\zeta \le t\}}\zeta = \mathbb{1}_{\{\zeta \le t\}}\tau_0 = \mathbb{1}_{\{\zeta \le t\}}(t \wedge \tau_0),$$

where $(t \wedge \tau_0)$ is $\mathcal{F}_t$-measurable. We can make ζ a stopping time with respect to a larger filtration: the progressive enlargement of $\mathbb{F}$ by ζ, namely the filtration $\mathbb{F}^\zeta = (\mathcal{F}_t^\zeta)_{t \ge 0}$, where

$$\mathcal{F}_t^\zeta := \cap_{s>t}(\sigma(\{\zeta \le u\} : u \le s) \vee \mathcal{F}_t), \quad t \ge 0.$$

We note that

$$\mathbb{P}\{\omega : \tau_0(\omega) = \zeta(\omega) < \infty\} = \mathbb{P}(\zeta < \infty),$$

then ζ is an accessible stopping time with respect to $\mathbb{F}^\zeta$.

1.2.3 Idiosyncratic credit risk: a Cox process model

In the book of Lando [Lan09], the author decomposes the structure of the intensity of an individual firm into two independent components, one coming from a common factor and one being idiosyncratic. In our model, besides the macroeconomic and political impacts, we also consider the credit risk related to the idiosyncratic financial circumstance of the sovereign, and we adopt the widely-used Cox process model.

Let $\lambda = (\lambda_t, t \ge 0)$ be a nonnegative $\mathbb{F}$-adapted process, and η be an $\mathcal{A}$-measurable exponentially distributed random variable of unit parameter representing the idiosyncratic factor, independent of both $\mathcal{F}_\infty$ and the Poisson process N. In the literature on the corporate credit risks, the default intensity can depend on the pre-default equity price process. In our case, λ can depend on the solvency S, which in turn depends on common factors, i.e., the macroeconomic factors. For example, let $\lambda_t = \lambda(t, S_t)$, where $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function whose form differs from country to country. In general, the intensity λ is decreasing with respect to the solvency S and remains bounded when S tends to $+\infty$, which implies that in a healthy situation of solvency, the idiosyncratic default risk is limited. We call the process λ the idiosyncratic default intensity.

We introduce the default hazard process $\Lambda = (\Lambda_t, t \geq 0)$ as $\Lambda_t = \int_0^t \lambda_s \, ds$. Let ξ be the time of default due to the idiosyncratic credit risk, given by a Cox process model, i.e.,

$$\xi := \inf \{t \geq 0 : \Lambda_t > \eta\}. \tag{1.9}$$

By definition, for any $t \geq 0$, one has $\{\xi \geq t\} = \{\Lambda_t \leq \eta\}$. As usual, the random time ξ is a totally inaccessible stopping time with respect to the progressive enlargement of the filtration $\mathbb{F}$ by ξ, namely the filtration $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t\geq 0}$, where

$$\mathcal{F}_t^\xi = \cap_{s>t}(\sigma(\{\xi \leq u\} : u \leq s) \vee \mathcal{F}_t), \quad t \geq 0.$$

1.2.4 Sovereign default time: a hybrid model

We now model the sovereign default by combining the economic and political influences described by ζ and the idiosyncratic credit risk described by ξ. We recall the partition theorem of any stopping time T. For any $\mathcal{A}$-measurable set A, we denote by T_A the restriction of T on A, defined by

$$T_A(\omega) = \begin{cases} T(\omega), & \omega \in \mathrm{A}, \\ \infty, & \omega \notin \mathrm{A}. \end{cases}$$

Theorem 1.3 ([DM75],Theorem 81)**.** *Let T be an $\mathbb{F}$-stopping time. There exists a unique (up to a $\mathbb{P}$-null set) partition of the set $\{T < \infty\}$ into two sets A and B which belong to $\mathcal{F}_{T-}$ such that T_A is accessible and T_B is totally inaccessible. The stopping time T_A is called the accessible part of T while T_B is called the totally inaccessible part of T.*

Conversely, given two stopping times, one accessible, the other totally inaccessible, by taking the minimum, one can consider to construct a new stopping time (by making precise the filtration of course). In this way, let the sovereign default time be

$$\tau := \zeta \wedge \xi, \tag{1.10}$$

which means that the sovereign default can result either from macroeconomic and political events, or from its own idiosyncratic financial situations. This sovereign default model has a hybrid nature of both structural and reduced-form approaches.

We make some comparisons with the jump to default extended CEV credit risk model:

1. We note that the default time τ in our model is not bounded by its predictable component τ_0. In fact, on the set $\{\tau_0 < \xi\} \cap \{N_{\tau_0} = 0\}$, $\tau = \xi > \tau_0$. If $(S_t, t \geq 0)$ follows a CEV diffusion, then the default time τ defined in (1.10) is an extension of the jump to default extended CEV model in [CL06]. We refer the readers to [DL01b, DS02] for the background about the CEV process and we shall discuss the CEV case in detail in Section 1.3.3.2.

2. If the Poisson process intensity of the exogenous shock $\lambda^N \to 0$, i.e., the external environment of the sovereign is relatively stable, then the default never occurs at τ_0, and our model converges to a Cox process model. On the contrary, when $\lambda^N \to \infty$, we have $\tau = \tau_0 \wedge \xi$. In this case, we recover the jump to default extended CEV model for corporate credit risk.

1.2.5 Extension to re-adjusted solvency thresholds

In practice, if the debt and deficit situation of the sovereign is excessive and has no improvement, the authorities may be less confident and consequently relax the requirements on the solvency barrier. For example, after the European Union and the International Monetary Fund formally agreed a first bailout package for Greece, the European Central Bank has been forced to relax the lending rules by suspending the minimum credit rating required on collateral, which lowers indirectly the requirements on the greek solvency, since the solvency is reflected by credit rating on the market. In this case, other critical political events may be gradually anticipated. This observation motivates us to extend the hybrid model to the case of multiple critical dates where solvency thresholds can be re-adjusted.

Let $n \in \mathbb{N}$, and $L_1, L_2, \ldots, L_n \in \mathbb{R}_+$ such that $S_0 > L_1 > L_2 > \ldots > L_n$, representing different levels of solvency requirements. We define a sequence of solvency barrier hitting times $\{\tau_i\}_{i=1}^n$ as

$$\tau_i = \inf\{t \geq 0 : S_t < L_i\}, \quad i \in \{1, \ldots, n\}. \tag{1.11}$$

The sequence $\{\tau_i\}_{i=1}^n$ is increasing since the solvency requirements are decreasing. When the solvency falls below a certain requirement L_i, for $i \in \{1, \ldots, n-1\}$, we assume that a critical political event is organised immediately at τ_i. If an exogenous shock has already arrived before, then the sovereign can possibly default at τ_i and no more critical political events will be planned; if no exogenous shock arrives before τ_i, the sovereign may obtain a financial aid to avoid an immediate default, which makes it possible to predetermine another critical political event when the solvency falls below L_{i+1}, and so on and so forth, until the requirements on the solvency are exhausted.

The exogenous factor is modelled by an inhomogeneous Poisson process N with intensity function $\lambda^N(t)$, and we define a random time ζ^* as

$$\zeta^* = \tau_i, \qquad \text{on } \{N_{\tau_{i-1}} = 0\} \cap \{N_{\tau_i} \geq 1\}, \quad i \in \{1, \ldots, n+1\} \tag{1.12}$$

with convention $\tau_0 = 0$ and $\tau_{n+1} = \infty$. Note that for $u \geq 0$, one has

$$\mathbb{1}_{\{\zeta^* > u\}} = \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_i > u\}} (\mathbb{1}_{\{N_{\tau_{i-1}}=0\}} - \mathbb{1}_{\{N_{\tau_i}=0\}}) = \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{1}_{\{N_{\tau_{i-1}}=0\}}. \tag{1.13}$$

We can check that

$$\mathbb{P}(\cup_{i=1}^n \{\omega : \tau_i(\omega) = \zeta^*(\omega) < \infty\}) = \mathbb{P}(\zeta^* < \infty),$$

then ζ^* is an accessible stopping time with respect to the progressive enlargement of $\mathbb{F}$ by ζ^*, namely the filtration $\mathbb{F}^{\zeta^*} = (\mathcal{F}_t^{\zeta^*})_{t \geq 0}$, where

$$\mathcal{F}_t^{\zeta^*} := \cap_{s>t} (\sigma(\{\zeta^* \leq u\} : u \leq s) \vee \mathcal{F}_s), \quad t \geq 0.$$

Similar to the case of single critical date, the sovereign can be caused either by the successive downgrade of solvency or by the idiosyncratic credit risk. Then, the sovereign default time is defined as

$$\tau = \zeta^* \wedge \xi, \tag{1.14}$$

where ξ is still given by (1.9). Let the global information structure be given as usual by the progressive enlargement of the filtration $\mathbb{F}$ by the sovereign default time τ, denoted by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ and defined as

$$\mathcal{G}_t = \bigcap_{s>t} \Big(\sigma(\{\tau \leq u\} : u \leq s) \vee \mathcal{F}_s\Big), \quad t \geq 0.$$

Then, τ is a $\mathbb{G}$-stopping time. However, ζ^* and ξ are not necessarily $\mathbb{G}$-stopping time. In order to investigate the decomposition $\tau = \zeta^* \wedge \xi$ where all the three random times are stopping times, we need a still larger filtration. We denote by $\mathbb{H} = (\mathcal{H})_{t\geq 0}$ the progressive enlargement of $\mathbb{F}$ by both ζ^* and ξ. To be more precise, we set

$$\mathcal{H}_t = \cap_{s>t}(\sigma(\{\zeta^* \leq u\} : u \leq s) \vee \sigma(\{\xi \leq u\} : u \leq s) \vee \mathcal{F}_t), \quad t \geq 0.$$

Obviously, $\mathbb{F}^{\zeta^*}$ and $\mathbb{F}^{\xi}$ are included in $\mathbb{H}$.

Lemma 1.4. *The following inclusion relations hold:* $\mathbb{F} \subsetneq \mathbb{G} \subsetneq \mathbb{H}$.

PROOF: The first inclusion is obvious since τ is not an $\mathbb{F}$-stopping time. For the second inclusion, the σ-algebra

$$\sigma(\{\tau \leq s\} : s \leq t) = \sigma(\{\zeta^* \leq s\} \cup \{\xi \leq s\} : s \leq t),$$

which is generated by the sets of the form $\{\zeta^* \leq s\} \cup \{\xi \leq s\}$ for any $s \leq t$. On the one hand, one has $\mathbb{G} \subseteq \mathbb{H}$ since

$$\{\zeta^* \leq s\} \cup \{\xi \leq s\} \in \sigma(\{\zeta^* \leq s\} : s \leq t) \vee \sigma(\{\xi \leq s\} : s \leq t);$$

on the other hand, the sets of the form $\{\zeta^* \leq s\} \cap \{\xi \leq s\}$ for any $s \leq t$ do not belong to $\sigma(\{\tau \leq s\} : s \leq t)$, which implies $\mathbb{G} \neq \mathbb{H}$. The lemma is thus proved. □

The following proposition makes precise the unique partition of the set $\{\tau < \infty\}$ on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{H}, \mathbb{P})$.

Proposition 1.5. *Let τ be defined by* (1.14) *and let*

$$A = \cup_{i=1}^{n} \Big(\{\tau_i < \xi\} \cap \{N_{\tau_{i-1}} = 0\} \cap \{N_{\tau_i} \geq 1\}\Big),$$
$$B = \{\tau < \infty\}/A.$$

Then, $A, B \in \mathcal{H}_{\tau-}$ and T_A is accessible and T_B is totally inaccessible, and $\tau = T_A \wedge T_B$ is the unique decomposition a.s..

PROOF: Firstly, the set $\{\tau < \infty\}$ belongs to $\mathcal{H}_{\tau-}$ since τ is measurable with respect to $\mathcal{H}_{\tau-}$. Secondly, since τ_i is a predictable stopping time for any $i = 1, \ldots, n$, by [Nik06, Theorem 2.23] one has

$$\{\tau_i < \xi\} \cap \{N_{\tau_{i-1}} = 0\} \cap \{N_{\tau_i} \geq 1\} = \{\tau = \tau_i\} \in \mathcal{H}_{\tau-},$$

which implies that $A, B \in \mathcal{H}_{\tau-}$. Furthermore, one can check that $\tau_A = \zeta^*$ and $\tau_B = \xi$. Then, by Theorem 1.3, $\tau = \tau_A \wedge \tau_B$ is the unique decomposition up to a $\mathbb{P}$-null set. $\square$

In this case, the stopping time τ is decomposed into an accessible part, which has n predictable components, and a totally inaccessible part.

1.2.6 Immersion property under minimum

In literature on the theory of enlargements of filtrations, we say that the couple $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property or the so-called (H)-hypothesis (or $\mathbb{F}$ is immersed in $\mathbb{G}$) if any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. In the previous section, the sovereign default time τ is defined as the minimum of two random times ζ^* and ξ. In [Li12, Chapter 6], the author has studied the problem of stability for the immersion property under minimum of two random times. More precisely, given two random times σ_i for $i = 1, 2$, if the immersion property holds between the reference filtration $\mathbb{F}$ and the respective progressive enlargements of $\mathbb{F}$ by σ_i for $i = 1, 2$, the author gives conditions under which the immersion property is satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ by $\sigma_1 \wedge \sigma_2$. By using this result, we investigate separately the two parts of τ and prove, without computing explicitly the conditional probability distribution, that $\mathbb{F}$ is immersed in $\mathbb{H}$ and then by filtration shrinkage that $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property.

It is well known (e.g., Bielecki and Rutkowski [BR02]) that if the random time ξ is constructed by Cox process model, then the couple $(\mathbb{F}, \mathbb{F}^{\xi})$ satisfies the immersion property. We prove that it is also the case for $(\mathbb{F}, \mathbb{F}^{\zeta^*})$.

Lemma 1.6. *The immersion property holds for the couple* $(\mathbb{F}, \mathbb{F}^{\zeta^*})$.

PROOF: For any $s, t \in \mathbb{R}_+$ and $s \leq t$, by (1.13) one has

$$\mathbb{P}(\zeta^* \leq s|\mathcal{F}_\infty) = 1 - \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq s < \tau_i\}} \mathbb{P}(N_{\tau_{i-1}} = 0|\mathcal{F}_\infty).$$

The independence of the Poisson process and the filtration $\mathbb{F}$ implies that

$$\mathbb{P}(N_{\tau_{i-1}} = 0|\mathcal{F}_\infty) = \mathbb{P}(N_{\tau_{i-1}} = 0|\mathcal{F}_t)$$

on the set $\{\tau_{i-1} \leq s\}$, which in turn implies

$$\mathbb{P}(\zeta^* \leq s|\mathcal{F}_\infty) = \mathbb{P}(\zeta^* \leq s|\mathcal{F}_t).$$

This last equality is (H5)-condition in Elliott, Jeanblanc and Yor [EJY00], which is equivalent to (H)-hypothesis. □

Proposition 1.7. *The filtration* $\mathbb{F}$ *is immersed in* $\mathbb{H}$.

PROOF: For any $c, k \in \mathbb{R}_+$, by (1.13) one has

$$\begin{aligned}
\mathbb{P}(\zeta^* > c, \xi > k|\mathcal{F}_\infty) &= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq c < \tau_i\}} \mathbb{P}(N_{\tau_{i-1}} = 0, \xi > k|\mathcal{F}_\infty) \\
&= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq c < \tau_i\}} \mathbb{P}(N_{\tau_{i-1}} = 0, \eta > \Lambda_k|\mathcal{F}_\infty).
\end{aligned}$$

Since both η and the Poisson process N are independent of $\mathcal{F}_\infty$, we deduce that

$$\mathbb{P}(N_{\tau_{i-1}} = 0, \eta > \Lambda_k|\mathcal{F}_\infty) = \mathbb{P}(N_{\tau_{i-1}} = 0, \eta > \Lambda_k|\mathcal{F}_t)$$

on the set $\{\tau_{i-1} \leq c\}$, which yields

$$\mathbb{P}(\zeta^* > c, \xi > k|\mathcal{F}_\infty) = \mathbb{P}(\zeta^* > c, \xi > k|\mathcal{F}_t).$$

By [Li12, Chapter 6, Lemma 6.2.4], the immersion property holds for the couple $(\mathbb{F}, \mathbb{H})$.
□

Since any $\mathbb{F}$-martingale is $\mathbb{G}$-adapted, Proposition 1.7 and [FP11, Theorem 2] imply that $\mathbb{F}$ is immersed in $\mathbb{G}$ (see also [Li12, Chapter 6, Lemma 6.2.3]).

1.3 Probability of default on multiple critical dates

In this section, we are interested in the probability that the sovereign default occurs on specific critical political dates. As we show, such default probabilities are nonzero in the hybrid model, which implies singularities in the probability distribution of default.

1.3.1 Conditional default and survival probabilities

We consider the sovereign default given by the hybrid model (1.14). For any $i \in \{1, \ldots, n\}$, let the $\mathbb{F}$-conditional probability that the sovereign default time τ coincides with τ_i be denoted by $p_t^i := \mathbb{P}(\tau = \tau_i | \mathcal{F}_t)$, $t \geq 0$.

Proposition 1.8. *The process $(p_t^i, t \geq 0)$ is an $\mathbb{F}$-martingale stopped at τ_i and is given by*

$$p_t^i = \mathbb{E}\Big[\Big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\Big)e^{-\Lambda_{\tau_i}} | \mathcal{F}_t\Big], \quad i = 1, \ldots n, \tag{1.15}$$

where we recall that λ^N is the time-dependent Poisson intensity.

PROOF: The event $\{\tau = \tau_i\}$ equals $\{\tau_i \leq \xi, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1\}$. Since τ_i is an $\mathbb{F}$-stopping time, the Poisson process N and the random variable η are mutually independent and in addition independent of $\mathbb{F}$, one has

$$\begin{aligned}
\mathbb{P}(\tau = \tau_i | \mathcal{F}_\infty) &= \mathbb{P}(\tau_i \leq \xi, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1 | \mathcal{F}_\infty) \\
&= \mathbb{P}(\Lambda_{\tau_i} \leq \eta, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1 | \mathcal{F}_\infty) \\
&= \mathbb{P}(\Lambda_{\tau_i} \leq \eta | \mathcal{F}_\infty)\mathbb{P}(N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1 | \mathcal{F}_\infty) \\
&= e^{-\Lambda_{\tau_i}} \left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\right),
\end{aligned} \tag{1.16}$$

which implies (1.15) and that p_t^i is stopped at τ_i, i.e., $p_{t \wedge \tau_i}^i = p_t^i$. □

We notice from the above proposition that on the set $\{\tau_i \leq t\}$, p_t^i does not depend on t, which means that the information concerning the impact of a political decision is neutralised after the event. In particular, we have

$$\mathbb{P}(\tau = \tau_i) = p_0^i = \mathbb{E}\left[\Big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\Big)e^{-\Lambda_{\tau_i}}\right]. \tag{1.17}$$

We now compute the $\mathbb{F}$-conditional survival probability of the sovereign and re-visit the immersion property.

Proposition 1.9. *For all $u, t \in \mathbb{R}_+$, the $\mathbb{F}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \le u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\Big)\Big|\mathcal{F}_t\Big]. \tag{1.18}$$

PROOF: For all $u, t \in \mathbb{R}_+$, by (1.13) one has

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= \mathbb{P}(\zeta^* > u, \xi > u|\mathcal{F}_t) \\ &= \mathbb{E}\Big[\Big(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le u < \tau_i\}} \mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\Big)\mathbb{1}_{\{\xi > u\}}\Big|\mathcal{F}_t\Big].\end{aligned}$$

If $u \le t$, then

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le u < \tau_i\}} \mathbb{E}\Big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\xi > u\}}\Big|\mathcal{F}_t\Big] \\ &= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le u < \tau_i\}} \mathbb{E}\Big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\eta > \Lambda_u\}}\Big|\mathcal{F}_t\Big] \\ &= \Big(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le u < \tau_i\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds}\Big) e^{-\Lambda_u}. \\ &= \exp\Big(-\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le u < \tau_i\}} \int_0^{\tau_{i-1}} \lambda^N(s)ds\Big) e^{-\Lambda_u} \\ &= \exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \le u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\Big).\end{aligned}$$

If $u > t$, we compute $\mathbb{P}(\tau > u|\mathcal{F}_t)$ as the $\mathcal{F}_t$-conditional expectation of $\mathbb{P}(\tau > u|\mathcal{F}_u)$, which implies (1.18). □

We can easily check that the couple $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property. Definitely, by Proposition 1.9, when $u \le t$, the $\mathbb{F}$-conditional probability does not depend on t, i.e.,

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{F}_u), \quad u \le t.$$

This last equality is equivalent to (H)-hypothesis (see Elliott, Jeanblanc and Yor [EJY00]).

Corollary 1.10. *For all $t, T \in \mathbb{R}_+$ such that $t \le T$, the $\mathbb{G}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{t < \tau_i \le T\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \int_t^T \lambda_s\, ds\Big)\Big|\mathcal{F}_t\Big]. \tag{1.19}$$

PROOF: This is the direct result of the key lemma in Elliott, Jeanblanc and Yor (c.f. [EJY00, Lemma 3.1]). Precisely,

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{P}(\tau > T|\mathcal{F}_t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left[e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i\le T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds-\int_0^T\lambda_s\,ds}\middle|\mathcal{F}_t\right]}{e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i\le t\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds-\int_0^t\lambda_s\,ds}},$$

which finishes the proof. □

1.3.2 Compensator process

The compensator and the intensity processes of default play an important role in the reduced-form approach of credit risk modelling. In a hybrid model, however, the compensator process of τ is in general discontinuous and the intensity does not necessarily exist.

Recall that an increasing càdlàg $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$ is called $\mathbb{F}$-compensator process of a random time τ if the process $(\mathbb{1}_{\{\tau\le t\}} - \Lambda^{\mathbb{F}}_{t\wedge\tau}, t \ge 0)$ is a $\mathbb{G}$-martingale. The process $\Lambda^{\mathbb{G}} = (\Lambda^{\mathbb{F}}_{t\wedge\tau}, t \ge 0)$ is called the $\mathbb{G}$-compensator of τ. The general method for computing the compensator is given in [EJY00] (see also [JY78, Proposition 2]) by using the Doob-Meyer decomposition of the Azéma supermartingale. This theorem is recalled in Appendix A.

In the case where the immersion property holds, the Azéma supermartingale becomes a decreasing process and then $A = 1 - G$. In this case, G is the unique solution of the following stochastic differential equation

$$dG_t = -G_{t-}d\Lambda^{\mathbb{F}}_t, \quad t > 0, \quad G_0 = 0.$$

The solution to the equation above is well-known: $G = \mathcal{E}(-\Lambda^{\mathbb{F}})$, where $\mathcal{E}$ denotes the Doléan-Dade exponential ([Pro05, Chapter II, page 85]). More precisely,

$$G_t = \mathcal{E}(-\Lambda^{\mathbb{F}})_t = \exp\left(-\Lambda^{\mathbb{F}}_t - \frac{1}{2}[\Lambda^{\mathbb{F}}, \Lambda^{\mathbb{F}}]_t\right).$$

In the sovereign default model (1.14), the Azéma supermartingale has an explicit form given by Proposition 1.9 as

$$G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \exp\Big(- \int_0^t \lambda_s \, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds \Big), \quad t \in \mathbb{R}_+, \tag{1.20}$$

which is a decreasing process. We can also compute the hazard process Γ ([BR02, Chapter 5]) of the sovereign default time τ as

$$\Gamma_t = - \ln G_t = \int_0^t \lambda_s \, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds, \quad t \in \mathbb{R}_+. \tag{1.21}$$

By applying Itō's formula for semimartingales to $\exp(-\Gamma)$, one has

$$e^{-\Gamma_t} = - \int_0^t e^{-\Gamma_{s-}} d\Gamma_s + \sum_{0<s\le t} (e^{-\Gamma_s} - e^{-\Gamma_{s-}} + e^{-\Gamma_{s-}}) \Delta\Gamma_s,$$

where $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$. By consequence, we obtain

$$\Lambda_t^{\mathbb{F}} = - \int_0^t \frac{dG_s}{G_{s-}} = - \int_0^t \frac{de^{-\Gamma_s}}{e^{-\Gamma_{s-}}} = \Gamma_t + \sum_{0<s\le t} \Big(1 - e^{-\Delta\Gamma_s} - \Delta\Gamma_s\Big),$$

which yields the following relationship between the processes $\Lambda^{\mathbb{F}}$ and Γ

$$\Lambda_t^{\mathbb{F}} = \Gamma_t^c + \sum_{0<s\le t} \Big(1 - e^{-\Delta\Gamma_s}\Big), \quad t \in \mathbb{R}_+,$$

where Γ^c is the continuous part of Γ. Thus, the $\mathbb{F}$-compensator is

$$\Lambda_t^{\mathbb{F}} = \int_0^t \lambda_s \, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \Big(1 - e^{- \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds}\Big), \quad t \in \mathbb{R}_+, \tag{1.22}$$

and the $\mathbb{G}$-compensator is

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \lambda_s \, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\wedge\tau\}} \Big(1 - e^{- \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds}\Big), \quad t \in \mathbb{R}_+. \tag{1.23}$$

We underline that the intensity process of sovereign default does not exist because of the discontinuity of the compensator process at the $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$. In literature, Gehmlich and Schmidt [GS16] provide a class of examples where the Azéma supermartingale contains jumps and propose a generalisation of this class with an additional stochastic integral containing singularities at predictable stopping times.

We observe that the absolutely continuous part of $\Lambda^{\mathbb{F}}$ and that of Γ are identical and depend on the idiosyncratic default intensity λ, while their jump parts are different and depend on the solvency (through the political critical dates) and the exogenous shock. When the external shock is small ($\lambda^N(t)$ is small), we can approximate $\Lambda^{\mathbb{F}}$ by Γ.

Remark 1.11. It is known that if the compensator process is continuous, then so is the hazard process and the two processes coincide with each other ([BR02, Proposition 6.2.2]). Bélanger, Shreve and Wong ([BSW04]) has pointed out that when there are jumps in the Azéma supermartingale, the hazard process defined in their paper (which is technically the $\mathbb{F}$-compensator) differs from the one defined in [BR02]. In the sovereign default model previously mentionned in this section, we provide an example where the hazard process is not equal to the $\mathbb{F}$-compensator.

1.3.3 Default probability in a Markovian setting

The general form of the sovereign default probability at critical dates p_t^i, $i = 1, \ldots, n$, is given by Proposition 1.8. We now consider several specific settings and provide explicit formulae when the solvency process is a geometric Brownian motion or a Constant Elasiticity of Variance (CEV) process.

We first make some simplified assumptions. We assume that the equation (1.6) is homogeneous and that the solvency process is given by

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t, \quad S_0 = x,$$

where $\mu(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy regular enough conditions for the existence and the pathwise uniqueness of a strong solution $\{S_t^x, t \geq 0\}$. Recall that the existence and pathwise uniqueness holds in each of the following cases (see e.g. Oksendal [Oks03, Theorem 5.2.1] and Revuz and Yor [RY99, Theorem 3.5] for details):

(a) μ and σ are Lipschitz continuous, and for any $x \in \mathbb{R}_+$ there exists a constant C such that $|\mu(x)| + |\sigma(x)| \leq C(1 + |x|)$;

(b) μ is Lipschitz continuous, and for any $x, y \in \mathbb{R}_+$, $|\sigma(x) - \sigma(y)|^2 \leq \varphi(|x - y|)$;

(c) μ and σ are bounded, and for any $x, y \in \mathbb{R}_+$, $|\sigma(x)-\sigma(y)|^2 \leq \varphi(|x-y|)$, $\sigma \geq \varepsilon > 0$;

(d) μ is bounded, and for any $x, y \in \mathbb{R}_+$, $|\sigma(x) - \sigma(y)|^2 \leq |g(x) - g(y)|$ where the function $g : \mathbb{R}_+ \to \mathbb{R}$ is increasing and bounded, $\sigma \geq \varepsilon > 0$.

Let $\mathcal{L}$ denote the generator of S, i.e., for any function $f \in C^2 : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathcal{L}f(z) = \mu(z)f'(z) + \frac{1}{2}\sigma^2(z)f''(z).$$

The technical hypotheses in [CL06] remain valid here with suitable financial interpretation. We specify the idiosyncratic intensity as a decreasing function of the solvency, i.e., $\lambda_t = \lambda(S_t)$ with $\lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ being decreasing. When the solvency $S \to \infty$, the sovereign has almost no chance to default, then λ should remain bounded. When $S \to 0$, the solvency is bad enough to trigger a default, so that the idiosyncratic default intensity can explode to infinity in this case.

Suppose in addition that the intensity of the exogenous shock is constant, so in the inhomogeneous Poisson process, the intensity function is $\lambda^N(t) = \lambda^N > 0$ for any $t \geq 0$. Furthermore, we specify the idiosyncratic default intensity process λ as a decreasing function of the solvency $\lambda_t = \lambda(S_t)$ with $\lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ being decreasing.

We consider the Laplace transform for the $\mathbb{F}$-stopping time

$$\rho_x := \inf\{t \geq 0 : S_t^x \leq L\} \quad \text{with} \quad S_0^x = x.$$

For any $k \geq 0$, let

$$Q(x; k, L) := \mathbb{E}\Big[\exp\Big(-k\rho_x - \int_0^{\rho_x} \lambda(S_u^x)\, du\Big)\Big]. \tag{1.24}$$

Intuitively, if the process S starts from a higher level than x, then it takes on average more time to hit the threshold L, and so the function in (1.24) should be decreasing. However, this is not necessarily true when the process S is inhomogeneous.

Proposition 1.12. *The parameterised function $Q(\cdot; k, L)$ defined in* (1.24) *is decreasing.*

Proof: For any $\omega \in \Omega$, any $x, x' \in \mathbb{R}_+$ such that $x \leq x'$, define the $\mathbb{F}$-stopping times

$$\rho_{x',x}(\omega) := \inf\{t \geq 0 : S_t^{x'}(\omega) \leq x\} \quad \text{and}$$
$$\rho_{x'}(\omega) := \inf\{t \geq 0 : S_t^{x'}(\omega) \leq L\} \quad \text{with} \quad S_0^{x'}(\omega) = x'.$$

The random time ρ_x is in distribution equal to $\rho_{x'} - \rho_{x',x}$, then by pathwise uniqueness and markovian properties one has

$$\begin{aligned}
Q(x';k,L) &= \mathbb{E}\left[\exp\Big(-k(\rho_{x'}-\rho_{x',x}+\rho_{x',x}) - \int_{\rho_{x',x}}^{\rho_{x'}} \lambda(S_u^{x'})du - \int_0^{\rho_{x',x}} \lambda(S_u^{x'})du\Big)\right] \\
&\leq \mathbb{E}\left[\exp\Big(-k(\rho_{x'}-\rho_{x',x}) - \int_{\rho_{x',x}}^{\rho_{x'}} \lambda(S_u^{x'})du\Big)\right] \\
&= \mathbb{E}\left[\exp\Big(-k\rho_x - \int_0^{\rho_x} \lambda(S_u^{x})du\Big)\right] \\
&= Q(x;k,L)
\end{aligned}$$

since $k\rho_{x',x} + \int_0^{\rho_{x',x}} \lambda(S_u^{x'})du \geq 0$. The proposition is thus proved. □

It is difficult to compute directly the right-hand side of (1.24). However, one can prove that (1.24) is the representation of the solution to a differential equation. Then one can compute the Laplace transform by solving an ODE.

Theorem 1.13. *The right-hand side of* (1.24) *is the representation of the solution to the following Dirichlet problem*

$$\begin{aligned}
\mathcal{L}u(z) - (\lambda(z)+k)u(z) = 0 \quad &on\ \{z > L\}; \\
u(L) = 1. & \qquad (1.25)
\end{aligned}$$

PROOF: Indeed, since ρ_x is a predictable stopping time, there exists an increasing sequence of stopping times $(\rho_m)_{m\geq 1}$ such that $\rho_m < \rho_x$ and $\lim_{m\to\infty}\rho_m = \rho_x$ $\mathbb{P}$-a.s.. Let

$$\beta_t^x = \exp\Big(-\int_0^t (k+\lambda(S_s^x))ds\Big)$$

for any $t \geq 0$. By Itō's formula, on the set $\{t < \rho_x\}$ one has

$$\begin{aligned}
d(\beta_t^x u(S_t^x)) &= -u(S_t^x)\beta_t^x(k+\lambda(S_t^x))dt + \beta_t^x \mathcal{L}u(S_t^x)dt + \beta_t^x u'(S_t^x)\sigma(S_t^x)dW_t \\
&= \beta_t^x u'(S_t^x)\sigma(S_t^x)dW_t, \qquad (1.26)
\end{aligned}$$

where u is a solution to the Dirichlet problem (1.25). We then have

$$\mathbb{E}[\beta_{\rho_m}^x u(S_{\rho_m}^x)] - u(x) = \mathbb{E}\left[\int_0^{\rho_m} \beta_s^x u'(S_s^x)\sigma(S_s^x)dW_s\right],$$

where the right-hand side vanishes thanks to the boundedness of β, the smoothness of u. Thus, when m tends to $+\infty$,

$$u(x) = \mathbb{E}[\beta_{\rho_x}^x u(L)] = \mathbb{E}\left[\exp\Big(-\int_0^{\rho_x} (\lambda(S_s^x)+k)ds\Big)\right].$$

The theorem is thus proved. □

We refer the reader to Karatzas and Shreve [KS02, Chapter 5, Proposition 7.2] and Touzi [Tou13, Theorem 2.8] for a general representation of this kind of Dirichlet problem.

The probability that the sovereign bankruptcy occurs on a critical political date can be derived from $Q(\cdot; k, L)$.

Proposition 1.14. *The $\mathbb{F}$-martingale $(p_t^i, t \geq 0)$, $i \in \{1, \dots, n\}$, is computed as*

$$p_t^i = e^{-\int_0^{t\wedge\tau_i} \lambda(S_u)du} Q(S_{t\wedge\tau_{i-1}}; \lambda^N, L_{i-1}) \\ \cdot \left[e^{-\lambda^N(t\wedge\tau_{i-1})} Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}}; 0, L_i) - e^{-\lambda^N(t\wedge\tau_i)} Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}}; \lambda^N, L_i)\right], \quad t \geq 0, \tag{1.27}$$

with $L_0 = S_0 = x$.

PROOF: By Proposition 1.8 and the section assumptions, we have for any $i \in \{1, \dots, n\}$ and $t \in \mathbb{R}_+$ that

$$p_t^i = \mathbb{E}\left[(e^{-\lambda^N \tau_{i-1}} - e^{-\lambda^N \tau_i}) e^{-\int_0^{\tau_i} \lambda(S_u)du} \middle| \mathcal{F}_t\right].$$

Since p^i is a martingale stopped at τ_i, it suffices to compute p_t^i on the set $\{t < \tau_i\}$. On the set $\{\tau_{i-1} \leq t < \tau_i\}$, by the Markovian property of the process S, we obtain

$$\begin{aligned} p_t^i &= e^{-\lambda^N \tau_{i-1}} \mathbb{E}\left[e^{-\int_0^{\tau_i} \lambda(S_u)du} \middle| \mathcal{F}_t\right] - \mathbb{E}\left[e^{-\lambda^N \tau_i - \int_0^{\tau_i} \lambda(S_u)du} \middle| \mathcal{F}_t\right] \\ &= e^{-\int_0^t \lambda(S_u)du} \left[e^{-\lambda^N \tau_{i-1}} Q(S_t; 0, L_i) - e^{-\lambda^N t} Q(S_t; \lambda^N, L_i)\right]. \end{aligned}$$

In particular, one has

$$p_{\tau_{i-1}}^i = e^{-\lambda^N \tau_{i-1} - \int_0^{\tau_{i-1}} \lambda(S_u)du} \left[Q(L_{i-1}; 0, L_i) - Q(L_{i-1}; \lambda^N, L_i)\right],$$

which yields that on the set $\{t < \tau_{i-1}\}$,

$$p_t^i = e^{-\lambda^N t - \int_0^t \lambda(S_u)du} Q(S_t; \lambda^N, L_{i-1}) \left[Q(L_{i-1}; 0, L_i) - Q(L_{i-1}; \lambda^N, L_i)\right].$$

Finally, we note that $Q(S_{\tau_{i-1}}; k, L_{i-1}) = Q(L_{i-1}; k, L_{i-1}) = 1$ for any k, which implies (1.27). □

The proposition above shows that it is essential to compute the quantity $Q(x; k, L)$ to obtain explicit form of the probabilities of default at critical political events. We present below explicit formulae for two widely used cases.

1.3.3.1 Case of geometric Brownian motion

Let the solvency process S be a geometric Brownian motion which is the solution to

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad t \geq 0,$$

where W is a standard Brownian motion, $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$ and $S_0 = x$. Similar as in [CL06], we suppose that the idiosyncratic default intensity λ is a decreasing function of the solvency S:

$$\lambda_t = \lambda(S_t) = \frac{a}{S_t^{2\beta}} + b, \tag{1.28}$$

where $a > 0$, $b, \beta \geq 0$ represent respectively the scale parameter governing the sensitivity of λ to S, the constant lower bound and the elasticity parameter. Then, by (1.25), $u(x) = Q(x; k, L)$ is the solution to the following Sturm-Liouville equation (see [Eve05]):

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - (ax^{-2\beta} + b + k)u(x) = 0 \quad \text{on } (L, +\infty); \\ u(L) = 1. \end{aligned} \tag{1.29}$$

Case $\beta = 0$: Let $\hat{k} = a + b + k$. Then, the equation (1.29) becomes

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - \hat{k} u(x) = 0, \quad (L, +\infty); \\ u(L) = 1, \end{aligned} \tag{1.30}$$

to which the fundamental solution has the form x^γ, where $\gamma \in \mathbb{R}$ is a constant to be identified. Then, one can compute that $\gamma_{1,2} = -\nu \pm \sqrt{\nu^2 + \frac{2\hat{k}}{\sigma^2}}$, where $\nu = \mu/\sigma^2 - 1/2$. The general solution to the equation (1.30) has the form

$$u(x) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2},$$

where $A_1, A_2 \in \mathbb{R}$. Since $u(x)$ is bounded when $x \to \infty$, we deduce that $A_1 = 0$, and the boundary condition $u(L) = 1$ yields $A_2 = L^{-\gamma_2}$.

Apart from solving the equation (1.30), one can also compute directly the expectation $\mathbb{E}[e^{-\hat{k}\rho_x}]$. Precisely,

$$S_t = xe^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} = xe^{\sigma \tilde{W}_t}, \quad t \geq 0,$$

where $\tilde{W}_t = \nu\sigma t + W_t$, and the first hitting time ρ_x can be defined equivalently as

$$\rho_x = \inf\{t \geq 0 : \tilde{W}_t \leq \frac{1}{\sigma}\ln(L/x)\}.$$

By Girsanov's theorem, we can define a probability $\tilde{\mathbb{P}}$ under which $\tilde{W}$ is a Brownian motion. The Radon-Nikodym density of $\mathbb{P}$ with respect to $\tilde{\mathbb{P}}$ restricted on $\mathcal{F}_t$ is given by

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\bigg|_{\mathcal{F}_t} = e^{\nu\sigma\tilde{W}_t - \frac{1}{2}\nu^2\sigma^2 t}, \quad t \geq 0.$$

Then, by a change of probability one has

$$\begin{aligned}\mathbb{E}\left[e^{-\hat{k}\rho_x}\right] &= \tilde{\mathbb{E}}\left[\exp\{\nu\sigma\tilde{W}_{\rho_x} - \frac{1}{2}\nu^2\sigma^2\rho_x - \hat{k}\rho_x\}\right] \\ &= \left(\frac{L}{x}\right)^{\nu}\tilde{\mathbb{E}}\left[\exp\left\{-\left(\frac{1}{2}\nu^2\sigma^2 + \hat{k}\right)\rho_x\right\}\right],\end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the probability $\tilde{\mathbb{P}}$. Since the process

$$\left(\exp\left\{-\sqrt{\nu^2\sigma^2 + 2\hat{k}}\tilde{W}_{t\wedge\rho_x} - (\nu^2\sigma^2/2 + \hat{k})(t \wedge \rho_x)\right\}, \quad t \geq 0\right)$$

is bounded by $e^{-\sqrt{\nu^2\sigma^2+2\hat{k}}(\ln L - \ln x)/\sigma}$, it is a uniformly integrable $\tilde{\mathbb{P}}$-martingale. Doob's optional sampling theorem (e.g. [Pro05, Theorem 16]) yields the same result (see c.f. [BS02, Part II, Chapter 9, 2.0.1])

$$Q(x; k, L) = \left(\frac{L}{x}\right)^{\sqrt{\nu^2 + 2\hat{k}/\sigma^2} + \nu}.$$

Case $\beta > 0$: We let $w(z) = u(z^{-\frac{1}{\beta}})z^{-\frac{\nu}{\beta}}$. Then, w satisfies the following Bessel equation in a modified form (e.g. [Eve05, Chapter 17])

$$(zw'(z))' - \frac{1}{\beta^2}\left(\nu^2 + 2(k+b)/\sigma^2\right)z^{-1}w'(z) = \frac{2az}{\beta^2\sigma^2}w(z), \quad z \in (0, L^{-\beta}). \tag{1.31}$$

Let $\psi = \frac{1}{\beta}\sqrt{\nu^2 + 2(k+b)/\sigma^2}$, then the equation above admits two linearly independent fundamental solutions $I_\psi(z\sqrt{2a}/\sigma\beta)$ and $K_\psi(z\sqrt{2a}/\sigma\beta)$, where I and K are modified Bessel functions of the first and second kind, and are defined by

$$\begin{aligned}I_\psi(x) &:= \sum_{i=0}^{\infty}\frac{(x/2)^{\psi+2i}}{i!\Gamma(\psi+i+1)}, \\ K_\psi(x) &:= \frac{\pi}{2}\frac{I_{-\psi}(x) - I_\psi(x)}{\sin(\psi\pi)},\end{aligned}$$

with Γ being the gamma function. Then, the general solution to the equation (1.31) has the form

$$w(z) = B_1 I_\psi(z\sqrt{2a}/\sigma\beta) + B_2 K_\psi(z\sqrt{2a}/\sigma\beta),$$

where $B_1, B_2 \in \mathbb{R}$. The modified Bessel functions have the following asympthotic behaviours when $z \to 0$ (see e.g. [BS02, Appendix 2.4]):

$$\begin{aligned} I_\psi(z) &\simeq \frac{1}{\Gamma(\psi)}\left(\frac{z}{2}\right)^{\psi}, \\ K_\psi(z) &\simeq \frac{\Gamma(\psi)}{2}\left(\frac{z}{2}\right)^{-\psi}, \quad \psi > 0, \quad K_0(z) \simeq -\ln z. \end{aligned}$$

The boundedness of $u(z^{-\frac{1}{\beta}})$ as $z \to 0$ implies that $B_2 = 0$ and the boundary condition $u(L) = 1$ yields $B_1 = L^\nu / I_\psi(\sqrt{2a}/\sigma\beta L^\beta)$. Thus, we have (see also [Ken78, Theorem 3.1] and [BS02, Part II, Chapter 9, 2.8.3])

$$Q(x; k, L) = \frac{w(x^{-\beta})}{x^\nu} = \left(\frac{L}{x}\right)^\nu \frac{I_\psi(\sqrt{2a}/\sigma\beta x^\beta)}{I_\psi(\sqrt{2a}/\sigma\beta L^\beta)}.$$

We can notice that, in the both cases, when the threshold L tends to 0, $Q(x; k, L) \to 0$, which implies that the sovereign bankrupt occurs on a critical political date with zero probability. This is because the solvency process modelled by a geometric Brownian motion never hits 0.

1.3.3.2 Case of the CEV process

In practice, for sovereigns in severe financial crise (e.g. Greece), the volatility of the solvency fluctuates notably over time. We now consider another widely used process in the financial industry, the CEV model in which the volatility is a monotonic function of the solvency. Let the solvency process be a CEV process driven by the following diffusion

$$dS_t = \mu S_t\, dt + \delta S_t^{\beta+1} dW_t, \qquad S_0 = x, \tag{1.32}$$

where $\beta \in \mathbb{R}$ and $\delta > 0$ are respectively the elasticity parameter and the scale parameter of the volatility. In particular, the process S is a geometric Brownian motion if $\beta = 0$. We distinguish two cases according to the sign of β. For $\beta < 0$, the volatility $\sigma(S) = \delta S^\beta$

is a decreasing function of S. From a financial point of view, when the solvency decreases, lower solvency (higher deficit) indicates higher level of government borrowing, leading to lower growth rate, as well as smaller future expenditure to improve the budgetary situation. All these add more uncertainty to the solvency. For $\beta > 0$, the volatility is an increasing function of S. Namely, when the solvency increases, besides higher growth rate, higher solvency (surplus) may imply higher fiscal revenue which the government is under pressure to disburse for social welfare, also making the solvency become more uncertain. In other words, the volatility has the possibility to be either an increasing or a decreasing function of the solvency. We note in addition that when $\beta > 0$, the conditions for the existence and the pathwise uniqueness of a strong solution are not satisfied and the process S is a strictly local martingale (c.f. [EM82]) , which describes situations where bubbles may exist on financial market. To remedy the situation, one needs to regularise the process for large values (see Davydov and Linetsky [DL01b, Appendix C] for details).

Recall that the CEV diffusion has the following boundary characterisation. For $\beta < 0$, $+\infty$ is a natural boundary and zero is reached almost surely. For $-1/2 \leq \beta < 0$, zero is an absorbing boundary. For $\beta < -1/2$, zero is a reflecting boundary. For $\beta = 0$, the CEV process reduces to a geometric Brownian motion, and both zero and $+\infty$ are natural boundaries. For $\beta > 0$, zero is a natural boundary and $+\infty$ is an entrance boundary.

The specification of the idiosyncratic default intensity $\lambda(S)$ depends on the sign of the parameter β. More precisely, when $\beta > 0$ (respectively $\beta < 0$), $\lambda(S)$ is an affine function of $\frac{1}{\sigma^2(S)}$ (respectively $\sigma^2(S)$), i.e.,

$$\lambda(S) = \frac{a}{S^{2|\beta|}} + b, \quad a > 0, \quad b \geq 0, \quad \beta \in \mathbb{R}. \tag{1.33}$$

Then, $u(x) = Q(x; k, L)$ is the decreasing solution of the following equation:

$$\begin{aligned} \frac{1}{2}\delta^2 x^{2+2\beta} u'' + \mu x u' - (a x^{-2|\beta|} + b + k)u = 0, \quad &\text{on } (L, +\infty); \\ u(L) = 1.& \end{aligned} \tag{1.34}$$

The fundamental solutions to this last equation are different according to the sign of β.

In the following, we consider separately the two cases according to the sign of β. In literature, the case $\beta < 0$ has been studied for the valuation of path-dependent options in [Lin04] and the jump to default CEV model in [MACL10, MAL11]. The case $\beta > 0$ is more unusual. We shall use another similar equation, called CEV ordinary differential equation (ODE), which has been studied in [DL01b] where the coefficient of u is a negative constant. We make use of the knowledge of the solutions to the CEV ODE to solve the equation (1.34).

Case $\beta > 0$: We let $v(x) = u(x)e^{\kappa x^{-2\beta}}$, where $\kappa = \frac{1}{2\beta\delta^2}(\sqrt{\mu^2 + 2a\delta^2} - \mu) > 0$. Then, v satisfies the following CEV ODE:

$$\frac{1}{2}\delta^2 x^{2+2\beta} v'' + \sqrt{\mu^2 + 2a\delta^2} x v' - \left(\kappa\beta(2\beta+1)\delta^2 + b + k\right) v = 0, \quad x \in (L, \infty). \tag{1.35}$$

The equation above corresponds to another CEV diffusion:

$$dx_t = x_t(\sqrt{\mu^2 + 2a\delta^2}\, dt + \delta x_t^{\beta} dW_t), \quad t \geq 0,$$

for which $+\infty$ is an entrance boundary. Then, the equation (1.35) admits two linearly independent fundamental solutions v_+ and v_-, respectively increasing and decreasing, satisfying the following boundary conditions at $+\infty$ ([BS02, pages 18-19]):

$$\lim_{x\to+\infty} v_+(x) = +\infty, \quad \lim_{x\to+\infty} v_-(x) > 0,$$

and the generaly solution to the equation (1.35) has the form

$$v(x) = C_1 v_+(x) + C_2 v_-(x),$$

where $C_1, C_2 \in \mathbb{R}$. Since $u(x)$ is bounded as $x \to \infty$, so is $v(x)$, and we deduce that $C_1 = 0$, $C_2 = e^{-\kappa/L^{2\beta}}/v_-(L)$. The decreasing fundamental solution v_- is given by [DL01b, Proposition 5] as

$$v_-(x) = x^{\beta+\frac{1}{2}} e^{\frac{\sqrt{\mu^2+2a\delta^2}}{2\beta\delta^2} x^{-2\beta}} M_{n,m}\left(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2} x^{-2\beta}\right),$$

where $n = \frac{1}{2} + \frac{1}{4\beta} - \frac{\kappa\beta(2\beta+1)\delta^2+b+k}{2\beta\sqrt{\mu^2+2a\sigma^2}} = \frac{\mu(2\beta+1)+2b+2k}{4\beta\sqrt{\mu^2+2a\delta^2}}$, $m = \frac{1}{4\beta}$ and $M_{n,m}(z) := z^{m+1/2}e^{-z/2}F_1(m-n+1/2, 2m+1, z)$ is Whittaker function of the first kind with

$$F_1(a, b, z) := 1 + \sum_{j=1}^{\infty} \frac{a(a+1)\dots(a+j-1)z^j}{b(b+1)\dots(b+j-1)j!}$$

being Kummer confluent hypergeometric function of the first kind. This fundamental solution implies that

$$Q(x;k,L) = C_2 v_-(x) e^{-\kappa/x^{2\beta}} = \frac{x^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2} x^{-2\beta}} M_{n,m}\left(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2} x^{-2\beta}\right)}{L^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2} L^{-2\beta}} M_{n,m}\left(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2} L^{-2\beta}\right)},$$

which is valid for any $\mu \in \mathbb{R}$.

Case $\beta < 0$: We let $y(z) = u(z^{\frac{1}{\gamma}}) z^{\frac{1}{2}-\frac{1}{2\gamma}}$, where

$$\gamma = \begin{cases} \sqrt{1+8a/\delta^2}, & \mu > 0, \\ -\sqrt{1+8a/\delta^2}, & \mu \leq 0. \end{cases}$$

If $\gamma < -1$, y is an increasing function on $(0, L^\gamma)$ and $\lim_{z\to 0+} y(z) = 0$; if $\gamma > 1$, y is a decreasing function on $(L^\gamma, +\infty)$ and $\lim_{z\to+\infty} y(z) = 0$. Moreover, y satisfies another CEV ODE as follows:

$$\frac{1}{2}\delta^2\gamma^2 z^{2+2\hat{\beta}} y'' + \mu\gamma z y' - \left(b + k + \frac{\mu\gamma - \mu}{2}\right) y = 0, \tag{1.36}$$

where $\hat{\beta} = \frac{\beta}{\gamma}$, the sign of which depends on the sign of μ, and we note that $b+k+\frac{\mu\gamma-\mu}{2} > 0$ for any $\mu \in \mathbb{R}$. The equation above corresponds to another CEV diffusion:

$$dz_t = \gamma z_t(\mu\, dt + \delta z_t^{\hat{\beta}} dW_t), \quad t \geq 0.$$

The equation (1.36) has two linearly independent solutions y_+ (increasing) and y_- (decreasing), and the general solution is the linear combination of y_+ and y_-.

If $\hat{\beta} < 0$ (namely $\mu > 0$ and $\gamma > 1$), then $+\infty$ is a natural boundary, and one has

$$\lim_{z\to+\infty} y_-(z) = 0 \quad \text{and} \quad \lim_{z\to+\infty} y_+(z) = +\infty$$

and hence y is proportional to y_-. If $\hat{\beta} > 0$ (namely $\mu \geq 0$ and $\gamma < -1$), then 0 is a natural boundary, and one has

$$\lim_{z\to 0+} y_-(z) = +\infty \quad \text{and} \quad \lim_{z\to 0+} y_+(z) = 0$$

and hence y is proportional to y_+. Therefore, by [DL01b, Proposition 5], when $\mu \neq 0$ there exists a constant $D > 0$ such that

$$y(z) = D z^{\frac{\beta}{\gamma}+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2} z^{-2\beta/\gamma}} W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2} z^{-2\beta/\gamma}\right),$$

where $n' = \mathrm{sgn}(\mu\beta)(\frac{1}{2} + \frac{\gamma}{4\beta}) - \frac{2b+2k+\mu\gamma-\mu}{4|\mu\beta|} = \frac{2b+2k-\mu(2\beta+1)}{4|\mu|\beta}$, $m' = -\frac{|\gamma|}{4\beta} = -\frac{\sqrt{1+8a/\delta^2}}{4\beta}$ and the function $W_{n,m}(x) := x^{m+1/2}e^{-x/2}F_2(m-n+1/2, 2m+1, x)$ is Whittaker function of the second kind with

$$F_2(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)}F_1(a,b,x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}F_1(1+a-b, 2-b, x)$$

being Kummer confluent hypergeometric function of the second kind. One can compute the constant D by using the relation $y(L^\gamma) = L^{\gamma/2-1/2}$. If $\mu = 0$ and thus $\gamma < -1$ and $\hat{\beta} > 0$, then

$$y(z) = Dy_+(z) = D\sqrt{z}K_{2m'}\left(-\frac{z^{-\beta/\gamma}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right),$$

where $K_\psi(x)$ is modified Bessel function of the second kind, defined as

$$K_\psi(x) := \frac{\pi}{2\sin(\psi\pi)}\left(I_{-\psi}(x) - I_\psi(x)\right).$$

Therefore we obtain

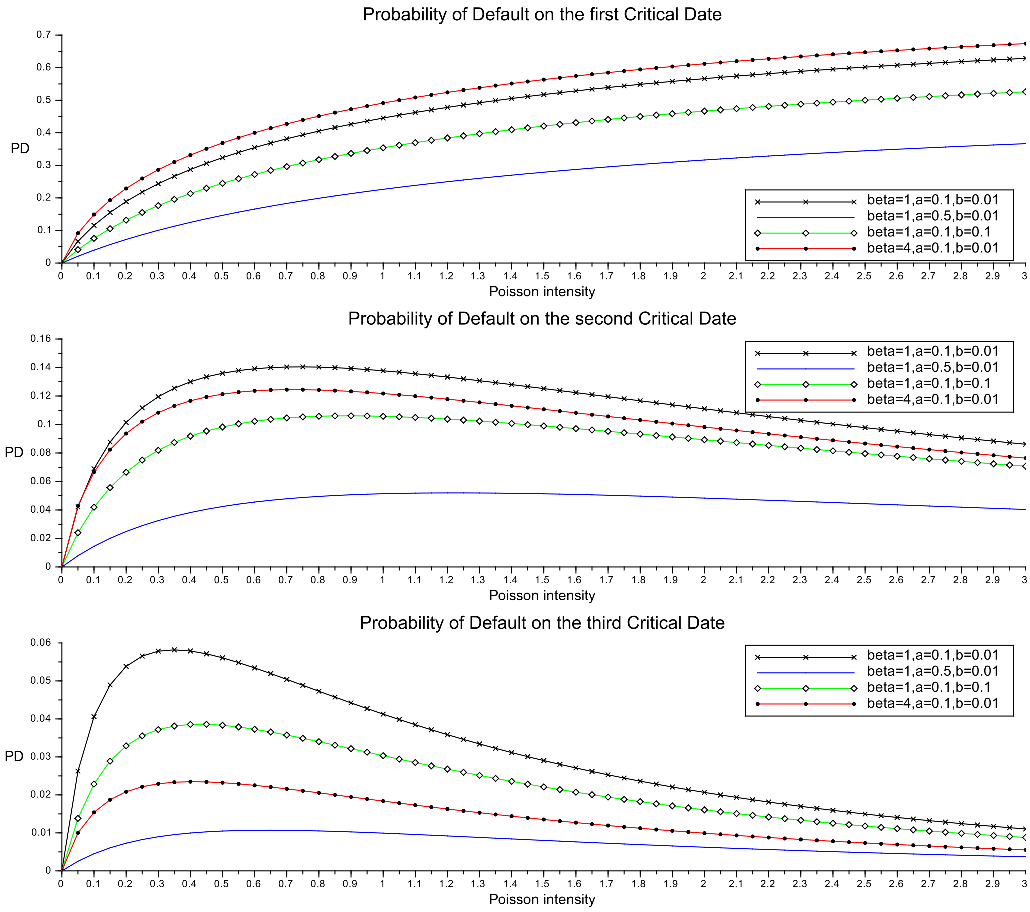
$$Q(x;k,L) = y(x^\gamma)x^{\frac{1}{2}-\frac{\gamma}{2}} = \begin{cases} \dfrac{x^{\beta+\frac{1}{2}}e^{\frac{\mu}{2\beta\delta^2}x^{-2\beta}}W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2}x^{-2\beta}\right)}{L^{\beta+\frac{1}{2}}e^{\frac{\mu}{2\beta\delta^2}L^{-2\beta}}W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2}L^{-2\beta}\right)}, & \mu \neq 0, \\[2ex] \dfrac{\sqrt{x}K_{2m'}\left(-\frac{x^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right)}{\sqrt{L}K_{2m'}\left(-\frac{L^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right)}, & \mu = 0. \end{cases}$$

1.3.4 Numerical illustrations

We now present numerical examples to illustrate the results obtained previously concerning the sovereign default probability and the defaultable bond yield. s

In the first example, we are interested in the default probability p_0^i on a political critical date τ_i, $(i = 1, 2, 3)$, given by (1.17). We assume that the solvency process S is modelled by a geometric Brownian motion as in Section 1.3.3.1, and we use the solvency data of Greece during the period from 2003 to 2013 to estimate the parameters and obtain $S_0 = 1.01$, $\mu = -0.01$ and $\sigma = 0.14$. Let the idiosyncratic default intensity process λ be specified by $\lambda(S) = \frac{a}{S^{2\beta}} + b$ as in (1.28) and the Poisson intensity be a constant λ^N. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$. Figure 1.5

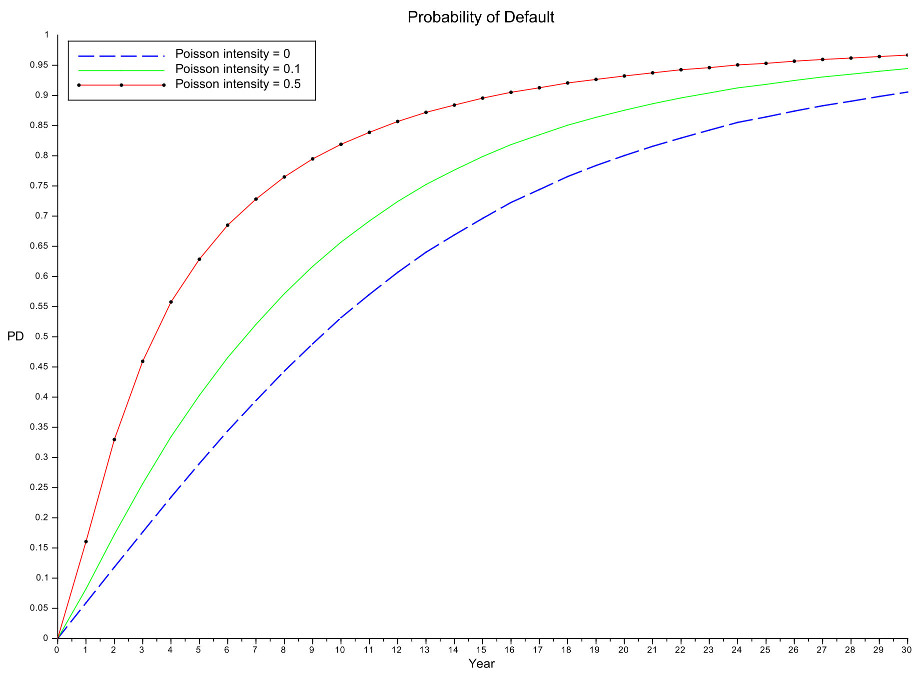
Figure 1.5 – Probability of sovereign default on τ_1, τ_2 and τ_3 respectively.



plots the probabilities that the sovereign default occurs on τ_1, τ_2 and τ_3 respectively as functions of the Poisson intensity λ^N for different parameters a, b and β, and we show in particular the impact of the of the exogenous shock intensity λ^N on political decisions and sovereign default. We observe that the probability of default on τ_1 is an increasing function of λ^N since it is more probable for the exogenous shock to occur when λ^N is larger, in which case the sovereign has higher possibility to default at the first critical date τ_1 due to unfavorable political decisions. However, when λ^N is large, the probability of default on other critical dates after τ_1 is reduced because the exogenous shock has more chance to occur before τ_1. As a result, the probabilities of default on τ_2 and τ_3 are increasing functions of λ^N for small λ^N and decreasing for large λ^N. For comparison concerning the parameters of the idiosyncratic intensity process, we set the parameters

$a = 0.1$, $b = 0.01$ and $\beta = 1$ and examine the impact of each parameter by considering values $a = 0.5$, $b = 0.1$ and $\beta = 4$ respectively. Other things being equal, the probabilities of default on τ_1, τ_2 and τ_3 are smaller for bigger a (respectively bigger b) because it is more probable for the sovereign to default due to the idiosyncratic credit risk when $\lambda(S)$ is bigger. The impact of the elasticity parameter β depends on the level of solvency, more precisely, $\lambda(S)$ is decreasing (respectively increasing) when $S \geq 1$ (respectively $S < 1$). Consequently, the probability of default on τ_1 (respectively τ_2, τ_3) is smaller for smaller β (respectively bigger β).

Figure 1.6 – Sovereign default probability.



In the second example, we consider the sovereign default probability $\mathbb{P}(\tau \leq T)$, which can be computed by Proposition 1.9. The solvency process S is given as a geometric Brownian motion with the same parameters as in the previous example. We fix the values of $a = 0.1$, $b = 0.01$ and $\beta = 1$ for idiosyncratic default intensity. Figure 1.6 plots the probability of default from 1 to 30 years for different values $\lambda^N = 0, 0.05$ and 0.2

respectively. We note that an exogenous shock with larger intensity value increases the sovereign default probability.

1.4 Hybrid model beyond immersion paradigm

The links between the market completeness and the immersion property have been studied in the credit risk literature (e.g. [JLC09a]). The advantage of the immersion is that the market risk premium takes into account the jump risk premium and so the risk-neutral probability does not need to be changed. However, it is usually impossible to assume the immersion property in the cases of incomplete market, non-ordered multi-defaults, and correlation between different default times. For this reason, we tend to adapt our model to a more general level.

The discussions in Subsection 1.2.6 show that the immersion property of the couple $(\mathbb{F}, \mathbb{G})$ preserved by the model results principally from the independence between the random barrier η and $\mathcal{F}_\infty$ in the Cox process model when we define the totally inaccessible part of the default time. In order to obtain a model without immersion, similar to [EKJJZ14, Section 4.2], we relax the assumption that the random variable η is independent of $\mathcal{F}_\infty$ by simply assuming that the $\mathbb{F}$-conditional probability distribution of η is absolutely continuous. We make the following density hypothesis.

Assumption 1.15. *We assume that η is a positive random variable whose conditional probability distribution with respect to $\mathbb{F}$ admits a density process, i.e., there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\beta_t(u)$ such that*

$$\mathbb{P}(\eta > \theta|\mathcal{F}_t) = \int_\theta^\infty \beta_t(u)du.$$

This assumption made on a random barrier is especially useful in studies on insider trading (see e.g. [GP98]). By the fact that Λ is increasing and absolutely continuous

$$\mathbb{P}(\eta > \Lambda_u|\mathcal{F}_t) = \int_{\Lambda_u}^\infty \beta_t(s)ds = \int_u^\infty \beta_t(\Lambda_s)\lambda_s\, ds.$$

One can re-compute the conditional default and survival probabilities in the new setting.

On the set $\{\tau_i \leq t\}$, one has

$$\begin{aligned}
\mathbb{P}(\tau = \tau_i|\mathcal{F}_t) &= \mathbb{P}(\tau_i < \xi, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1|\mathcal{F}_t) \\
&= \mathbb{P}(\Lambda_{\tau_i} < \eta, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1|\mathcal{F}_t) \\
&= \mathbb{P}(\Lambda_{\tau_i} < \eta|\mathcal{F}_t)\mathbb{P}(N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1|\mathcal{F}_t) \\
&= \left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\right) \int_{\tau_i}^{\infty} \beta_t(\Lambda_s)\lambda_s \, ds,
\end{aligned}$$

and on $\{\tau_i > t\}$, we compute $\mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$ as the $\mathcal{F}_t$-conditional expectation of $\mathbb{P}(\tau = \tau_i|\mathcal{F}_{\tau_i})$. Furthermore, if $u \leq t$, then

$$\begin{aligned}
\mathbb{P}(\tau > u|\mathcal{F}_t) &= \mathbb{E}\Big[\Big(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\Big)\mathbb{1}_{\{\xi > u\}}\Big|\mathcal{F}_t\Big] \\
&= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{E}\Big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\xi > u\}}\Big|\mathcal{F}_t\Big] \\
&= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{E}\Big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\eta > \Lambda_u\}}\Big|\mathcal{F}_t\Big] \\
&= \exp\Big(- \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq u < \tau_i\}} \int_0^{\tau_{i-1}} \lambda^N(s)ds\Big) \int_u^{\infty} \beta_t(\Lambda_s)\lambda_s \, ds \\
&= \exp\Big(- \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \leq u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\Big) \int_u^{\infty} \beta_t(\Lambda_s)\lambda_s \, ds.
\end{aligned}$$

If $u > t$, we compute $\mathbb{P}(\tau > u|\mathcal{F}_t)$ as the $\mathcal{F}_t$-conditional expectation of $\mathbb{P}(\tau > u|\mathcal{F}_u)$.

Let $Z_t = \int_t^{\infty} \beta_t(\Lambda_s)\lambda_s \, ds$, which is an $\mathbb{F}$-supermartingale. Then the Azéma supermartingale of τ is

$$G_t = Z_t \exp\left(-\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \leq t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\right).$$

It is much more difficult to compute the compensator process because the additive form of the Doob-Meyer decomposition of G is difficult to be performed when G has a multiplicative form. It is well known that a nonnegative càdlàg supermartingale can be uniquely factorised as a positive local martingale multiplied by a decreasing process (see e.g. Itō and Watanabe [IW65]). By [EKJJ10, Proposition 4.1], the Doob-Meyer decomposition of Z is given by

$$Z_t = 1 + M_t - \int_0^t \beta_s(\Lambda_s)\lambda_s \, ds,$$

where M is the càdlàg square integrable $\mathbb{F}$-martingale defined by

$$M_t = \int_0^t \lambda_s[\beta_s(\Lambda_s) - \beta_t(\Lambda_s)]ds, \quad \text{a.s..}$$

Let $\theta = \inf\{t \geq 0 : Z_{t-} = 0\}$ and define $\gamma_t = \frac{\beta_t(\Lambda_t)\lambda_t}{Z_{t-}}$ on $\{t < \theta\}$, then the factorisation of Z is given by

$$Z_t = L_t \exp\left(-\int_0^t \gamma_{s\wedge\theta} ds\right),$$

where L is a positive $\mathbb{F}$-local martingale solution to the SDE

$$dL_t = e^{\int_0^t \gamma_{s\wedge\theta}ds} dM_t, \quad L_0 = 1,$$

which implies the multiplicative decomposition of G:

$$G_t = L_t \exp\left(-\int_0^t \gamma_{s\wedge\theta} ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\right). \tag{1.37}$$

Again, we refer the readers to Gehmlich and Schmidt [GS16], where a class of examples of Azéma supermartingale containing jumps are provided.

On the other hand, by [Jac79, Corollaire 6.35] and the uniqueness of the factorisation of G, one has

$$G_t = L_t \mathcal{E}(-\Lambda_t^{\mathbb{F}}) = L_t e^{-\Lambda_t^{\mathbb{F},c}} \prod_{0<s\leq t} (1 - \Delta\Lambda_s^{\mathbb{F}}),$$

where $\Lambda^{\mathbb{F},c}$ is the continuous part of $\Lambda^{\mathbb{F}}$, and by comparison, we obtain the $\mathbb{F}$-compensator

$$\Lambda_t^{\mathbb{F}} = \int_0^t \gamma_{s\wedge\theta} ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} \left(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds}\right), \quad t \in \mathbb{R}_+, \tag{1.38}$$

and the $\mathbb{G}$-compensator

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \gamma_{s\wedge\theta} ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\wedge\tau\}} \left(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds}\right), \quad t \in \mathbb{R}_+. \tag{1.39}$$

Since the Azéma supmartingale G is no more a decreasing process, the classic definition of the hazard process $\Gamma = -\ln G$ loses its significance. Let us give the following new definition.

Definition 1.16. The hazard process of the default time τ, noted Γ, is the unique increasing $\mathbb{F}$-predictable process such that $G\exp(\Gamma)$ is a positive $\mathbb{F}$-local martingale.

By Definition 1.16 and (1.37), the hazard process is

$$\Gamma_t = -\ln\left(\frac{G_t}{L_t}\right) = \int_0^t \gamma_{s\wedge\theta} ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds.$$

We note that in the case of generalised random barrier, the continuous part shared by the hazard process and the $\mathbb{F}$-compensator has changed while the respective discontinuous parts and the link between the two processes remain unchanged.

In a generic manner, we work on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$. Let $(\tau_i)_{i=1}^n$ be a sequence of $\mathbb{F}$-predictable stopping times, representing the dates on which political decisions are made concerning the sovereign default. Let the result of political decisions depend on an $\mathcal{A}$-measurable event (set) E_i for any $i = 1, \ldots, n$. More precisely,

— if the event E_i occurs ($\omega \in E_i$), then the sovereign can possibly default at τ_i;

— otherwise, it goes without immediate default at τ_i (but the bankruptcy can still occur at another time).

Then, we define a random time ζ which coincides only with τ_i when it is finite. Precisely,

$$\zeta = \tau_i, \quad \text{if } \omega \in E_i, \quad i \in \{1, \ldots, n\}, \quad \text{and } \zeta = \infty, \quad \text{if } \omega \in \Omega/(\cup_{i=1}^n E_i).$$

Furthermore, let ξ be another possible default time, of which the conditional survival probability is denoted by $Z_t(u) = \mathbb{P}(\xi > u|\mathcal{F}_t)$, $u, t \ge 0$, and in particular, the Azéma supermatingale of ξ is $Z_t := Z_t(t)$. We assume that, for any $i = 1, \ldots, n$, ξ and the events E_i are independent. This assumption means that the occurrence of the events (E_i) have no impact on the possible default time ξ. Let the sovereign default time be the minimum of ζ and ξ,

$$\tau = \zeta \wedge \xi.$$

Then, the default can occur either at $(\tau_i)_{i=1}^n$ or at ξ.

We denote by Q^i a càdlàg martingale such that $Q_t^i = \mathbb{P}(E_i|\mathcal{F}_t)$, $t \in \mathbb{R}_+$. Then, for any

$u, t \in \mathbb{R}_+$, the $\mathbb{F}$-conditional survival probability is given by

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \begin{cases} Z_t(u) \sum_{i=1}^n \mathbb{1}_{\{\tau_i > u\}} Q_u^i, & u \leq t, \\ \mathbb{E}\left[Z_u \sum_{i=1}^n \mathbb{1}_{\{\tau_i > u\}} Q_u^i \middle| \mathcal{F}_t \right], & u > t. \end{cases}$$

which yields in particular the Azéma supermatingale of τ:

$$G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = Z_t \sum_{i=1}^n \mathbb{1}_{\{\tau_i > t\}} Q_t^i, \quad t \in \mathbb{R}_+.$$

Finally, for any $i \in \{1, \ldots, n\}$, the $\mathbb{F}$-conditional probability that the sovereign default time τ coincides with τ_i is computed as

$$p_t^i = \mathbb{P}(\tau = \tau_i|\mathcal{F}_t) = \mathbb{1}_{\{\tau_i \leq t\}} Q_t^i Z_t(\tau_i) + \mathbb{1}_{\{\tau_i > t\}} \mathbb{E}[Q_{\tau_i}^i Z_{\tau_i}|\mathcal{F}_t], \quad t \in \mathbb{R}_+.$$

In the next chapter, we will extend this modelling to a more general framework where we are interested in the structure of the probability distribution of the random time.

Chapter 2

Generalised density approach for sovereign risk

By studying certain hybrid models in literature on credit risks, in particular the sovereign default risk model in the previous chapter, we consider in the present chapter a type of random times whose probability distribution can have singularities, where standard intensity and density hypotheses in the enlargement of filtrations are not satisfied. We propose a generalised density approach in order to deal with such random times in the framework of progressive enlargement of filtrations, and we study classic problems such as the computation of the compensator process of the random time, the factorisation of the Azéma supermartingale, as well as the martingale characterisation and the semimartingale decomposition in the enlarged filtration.

Keywords : Generalised density hypothesis, progressive enlargement of filtrations, martingale characterisation, semimartingale decomposition, sovereign default modelling.

2.1 Introduction

The theory of enlargement of filtrations has been developed since the 1970s by the French school of probability (see e.g. Barlow [Bar78], Jacod [Jac85], Jeulin [Jeu80], Jeulin and Yor [JY78] and Yor [Yor78]). There are two types of enlargement of filtrations: initial enlargement and progressive enlargement. In the credit risk analysis, the progressive enlargement of filtrations has been systematically adopted to model the default event that cannot be observed from the default-free market information flow, or mathematically speaking, the default time is not modelled as a stopping time with respect to the reference filtration (see also Mansuy and Yor [MY06], Protter [Pro05], Dellacherie, Maisonneuve and Meyer [DMM92], Yor [Yor12], Brémaud and Yor [BY78], Nikeghbali [Nik06], Ankirchner [Ank05], Song [Son87], Ankirchner, Dereich and Imkeller [ADI07], Yœurp [Yœu85]). In the work of Elliot, Jeanblanc and Yor [EJY00] and Bielecki and Rutkowski [BR02], the authors have proposed to use the progressive enlargement of filtrations to describe the market information which includes both the default-free information and the default information. Precisely, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a càdlàg reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ representing the default-free market information, and let τ be a nonnegative random variable which represents the default time. Then, the global market information is modelled by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, which is the smallest càdlàg filtration containing $\mathbb{F}$ such that τ is a $\mathbb{G}$-stopping time and $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ by τ. In this framework, the reduced-form modelling approach has been widely used, where one often assumes the existence of the $\mathbb{G}$-intensity of τ, i.e. the $\mathbb{G}$-adapted process $(\lambda_t, t \geq 0)$ such that the process

$$\left(\mathbb{1}_{\{\tau\leq t\}} - \int_0^{\tau\wedge t} \lambda_s ds, t \geq 0\right)$$

is a $\mathbb{G}$-martingale. The process λ, also called default intensity process, plays an important role in the default event modelling. The intuitive meaning of the intensity, interpreted by Duffie and Lando [DL01b], is that it gives a local default rate, in that

$$\mathbb{P}(\tau \in (t, t+dt]|\mathcal{G}_t) = \lambda_t \mathbb{1}_{\{\tau>t\}}\, dt.$$

More recently, in order to study the impact of default events, a new approach has been developed by El Karoui, Jeanblanc and Jiao [EKJJ10, EKJJ15] where they assume the density hypothesis: the $\mathbb{F}$-conditional distribution of τ admits a density with respect to a non-atomic measure η, i.e., for all $\theta, t \geq 0$,

$$\mathbb{P}(\tau \in d\theta | \mathcal{F}_t) = \alpha_t(\theta)\eta(d\theta),$$

where $\alpha_t(\cdot)$ is an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function. The density hypothesis has been firstly introduced by Jacod [Jac85] in a theoretical setting of initial enlargement of filtrations and is essential to ensure that an $\mathbb{F}$-semimartingale remains a semimartingale in the initially enlarged filtration. For this reason, the random time that satisfies the density hypothesis is called initial time in some literature (e.g., Jeanblanc and Le Cam [JLC09b]). There exist explicit links between the intensity and density processes of the default time τ, which establish a relationship between the two approaches of default modelling. In particular, the density approach allows us to analyse what happens after a default event, i.e., on the set $\{\tau \leq t\}$, and has interesting applications in the study of counterparty default risks. We note that, in both intensity and density approaches, the random time τ is a totally inaccessible $\mathbb{G}$-stopping time which usually avoids $\mathbb{F}$-stopping times.

The stability of the class of semimartingales with respect to the enlargement of filtrations is an important subject, which is in particular useful for change of probabilities and implies no-arbitrage conditions in several applications in finance. We say that the couple $(\mathbb{F}, \mathbb{G})$ satisfies the (H')-hypothesis of Jacod (c.f. [Jac85]) if any $\mathbb{F}$-semimartingale remains a $\mathbb{G}$-semimartingale, which, unfortunately, does not always hold. In the framework of the progressive enlargement, no general theorem guarantees the (H')-hypothesis, and it deeply depends on the properties of the random time. Precisely, any $\mathbb{F}$-semimartingale stopped at τ is a $\mathbb{G}$-semimartingale (e.g. [Yor78]), and the (H')-hypothesis holds if τ is an honest time, i.e., for any $t \geq 0$, τ is equal to an $\mathcal{F}_t$-measurable random variable $\{\tau \leq t\}$ (e.g., Barlow [Bar78]). It is also well known that the (H')-hypothesis holds if the density hypothesis is satisfied, and any $\mathbb{F}$-martingale $X = (X_t)_{t \geq 0}$ admits a canonical decomposition ([JLC09b, Theorem 3.1]):

$$X_t = M_t + \int_{]0,t\wedge\tau]} \frac{d\langle X, G\rangle_u + \check{X}^p_u}{G_{u-}} + \int_{]t\wedge\tau,t]} \frac{d\langle X, \alpha.(\theta)\rangle_u}{\alpha_{u-}(\theta)}\bigg|_{\theta=\tau},$$

where $M = (M_t)_{t\geq 0}$ is a $\mathbb{G}$-martingale, $G = (G_t)_{t\geq 0}$ is the Azéma supermartingale, the process $\breve{X}^p = (\breve{X}^p_t)_{t\geq 0}$ is the dual $\mathbb{F}$-predictable projection of the process $(\mathbb{1}_{\{\tau\leq t\}}\Delta X_\tau, t \geq 0)$.

In this chapter, we consider a type of random times which can be either accessible or totally inaccessible. The motivation comes from the recent European sovereign debt crisis. As pointed out in Chapter 1, compared with the classic corporate credit risk, the sovereign default is often influenced by political events. During the European sovereign debt crisis, the decisions and interventions of European Central Bank are crucial. For example, the euro area members and IMF agree on a 110-billion-euro financial aid package for Greece on 02/05/2010 and another financial aid program of 109-billion-euro on 21/07/2011 (see Chapter 1 for details). The eventuality of default-or-not of the Greek government depends on the decisions made at the political meetings held at these dates. From the point of view of a market investor, there are important risks that the Greek government may default at such critical dates. Similar cases have inspired Gehmlich and Schmidt [GS16] where a promised payment cannot be made, which leads to default at pre-specified times, such as coupon dates, e.g., the recently missed coupon payment by Argentina as well as the default of Greece on the 1st of July regarding the failure of 1.5 Billon euros on a scheduled debt repayment to the International Monetary Fund.

From a mathematical point of view, the existence of these political events and critical dates means that the probability distribution of the random time τ admits singularities. Hence, the sovereign default time can coincide with some predetermined dates, modelled as predictable $\mathbb{F}$-stopping times (see Chapter 1 for details). In this case, the classic default modelling approaches, in particular, both intensity and density models are no longer adapted. To overcome this difficulty, we propose to generalise the density approach in [EKJJ10] to add singularities to the probability distribution of τ. More precisely, we assume that the $\mathbb{F}$-conditional probability distribution of τ contains a discontinuous part, besides the absolutely continuous part which has a density. This generalised density approach allows us to consider a random time τ which has positive probability to meet a finite family of $\mathbb{F}$-stopping times.

There are related works in literature on credit risk modelling. In Bélanger, Shreve and Wong [BSW04], a general framework is proposed where reduced-form models, in particular the widely-used Cox process model, can be extended to the case where default can occur at specific dates. In Gehmlich and Schmidt [GS16], the authors consider models where the Azéma supermartingale of τ, i.e., the process

$$(\mathbb{P}(\tau > t|\mathcal{F}_t)\,, t \geq 0)$$

contains jumps (so that the intensity process does not exist) and develop the associated HJM credit term structures and no-arbitrage conditions. Chen and Filipović [CF05], Carr and Linetsky [CL06], and Campi et al. [CPS09] have studied the hybrid credit models where the default time depends on both a first hitting time in the structural approach and an intensity-based random time in the reduced-form approach. The generalised density approach that we propose can also be viewed as such hybrid credit risk models and all the hybrid models cited above belong to this generalised framework.

We shall investigate, under the generalised density hypothesis, some classic problems in the enlargement of filtrations from a theoretical point of view. In particular, we deduce the compensator process of the random time τ, which can be discontinuous in this case. This means that the intensity process does not necessarily exist. We also characterise the martingale processes in the enlarged filtration $\mathbb{G}$ and obtain the canonical decomposition of an $\mathbb{F}$-martingale as an $\mathbb{G}$-semimartingale, which shows that in the generalised density setting, the (H')-hypothesis of Jacod is satisfied. The main contribution of our work is to focus on the impact of the discontinuous part of the $\mathbb{F}$-conditional probability distribution of τ and study the impact of the critical dates on the random time.

2.2 Generalised density hypothesis

In this section, we introduce our key assumption, the generalised density hypothesis, and some basic properties. Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a reference filtration satisfying the usual conditions, namely the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ is a $\mathbb{P}$-complete σ-algebra. We use the expressions $\mathcal{O}(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$

to denote respectively the optional and predictable σ-algebra associated to the filtration $\mathbb{F}$. Let τ be a random time on the probability space valued in $[0, +\infty]$. Denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the progressive enlargement of $\mathbb{F}$ by τ, defined as

$$\mathcal{G}_t = \bigcap_{s>t} \sigma(\{\tau \leq u\} : u \leq s) \vee \mathcal{F}_t, \quad t \geq 0.$$

Let $(\tau_i)_{i=1}^n$ be a finite family of $\mathbb{F}$-stopping times.

2.2.1 Key assumption

We assume that the $\mathbb{F}$-conditional probability distribution of τ avoiding $(\tau_i)_{i=1}^n$ has a density with respect to a non-atomic σ-finite Borel measure η on $\mathbb{R}_+$. Namely, for any $t \geq 0$, there exists a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable

$$(\omega, u) \mapsto \alpha_t(\omega, u)$$

such that, for any bounded Borel function h on $\mathbb{R}_+$, one has

$$\mathbb{E}[\mathbb{1}_H h(\tau) \,|\, \mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}, \tag{2.1}$$

where H denotes the event

$$\{\tau < \infty\} \cap \bigcap_{i=1}^n \{\tau \neq \tau_i\}.$$

The family $\alpha_t(\cdot)$ is called the $\mathbb{F}$-density of τ avoiding the family $(\tau_i)_{i=1}^n$ (generalised density for short). In particular, the case where the function h is constant and takes the value 1 leads to the relation

$$\mathbb{P}\left(\{\tau < \infty\} \cap \bigcap_{i=1}^n \{\tau \neq \tau_i\} \,\middle|\, \mathcal{F}_t\right) = \int_{\mathbb{R}_+} \alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}$$

Furthermore, some of the $\mathbb{F}$-stopping times can be deterministic and known, which would call of an extension of this framework.

Remark 2.1. The assumption above implies that the random time τ avoids any $\mathbb{F}$-stopping time σ such that $\mathbb{P}(\sigma = \tau_i < \infty) = 0$ for all $i \in \{1, \cdots, n\}$. Namely for such

$\mathbb{F}$-stopping time σ, one has $\mathbb{P}(\tau = \sigma < \infty) = 0$. Indeed, for any $T \in \mathbb{R}_+$ such that $\sigma \le T$, we have

$$\begin{aligned}\mathbb{P}\left(\{\tau = \sigma\} \cap \bigcap_{i=1}^{n}\{\tau \neq \tau_i\}\right) &= \mathbb{E}\left[\mathbb{P}\left(\{\tau = \sigma\} \cap \bigcap_{i=1}^{n}\{\tau \neq \tau_i\}\Big|\mathcal{F}_T\right)\right] \\ &= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{u=\sigma\}}\alpha_T(u)\eta(du)\right] \\ &= 0,\end{aligned}$$

which implies that $\mathbb{P}(\tau = \sigma) = 0$. However, the random time τ is allowed to coincide with some of the stopping times in the family $(\tau_i)_{i=1}^n$ with a nonzero probability. Moreover, without loss of generality, we may assume that the family $(\tau_i)_{i=1}^n$ is increasing. In fact, if we denote by $(\tau^{(i)})_{i=1}^n$ the order statistics of $(\tau_i)_{i=1}^n$, then

$$\{\tau < \infty\} \cap \bigcap_{i=1}^{n}\{\tau \neq \tau_i\} = \{\tau < \infty\} \cap \bigcap_{i=1}^{n}\{\tau \neq \tau^{(i)}\}.$$

The following proposition shows that we can even assume that the family $(\tau_i)_{i=1}^n$ is strictly increasing until reaching infinity.

Proposition 2.2. *Let $(\tau_i)_{i=1}^n$ be an increasing family of $\mathbb{F}$-stopping times. Then, there exists a family of $\mathbb{F}$-stopping times $(\sigma_i)_{i=1}^n$ which verify the following conditions:*

(a) For any $\omega \in \Omega$ and $i, j \in \{1, \cdots, n\}$, $i < j$, if $\sigma_i(\omega) < \infty$, then $\sigma_i(\omega) < \sigma_j(\omega)$; otherwise, $\sigma_j(\omega) = \infty$.

(b) For any $\omega \in \Omega$, one has $\{\sigma_1(\omega), \cdots, \sigma_N(\omega), \infty\} = \{\tau_1(\omega), \cdots, \tau_N(\omega), \infty\}$, which implies

$$\{\tau < \infty\} \cap \bigcap_{i=1}^{n}\{\tau \neq \tau_i\} = \{\tau < \infty\} \cap \bigcap_{i=1}^{n}\{\tau \neq \sigma_i\}.$$

Proof: The case where $n = 1$ is trivial. We prove the result by induction and assume $n \geqslant 2$. Let $\tau_{n+1} = \infty$ by convention. For each $k \in \{2, \cdots, n\}$, let

$$E_k = \{\tau_1 = \cdots = \tau_k < \infty\}.$$

Moreover, for $k \in \{2, \ldots, n\}$, we define

$$\tau_k' = \mathbb{1}_{E_k^c}\tau_k + \sum_{i=k}^{n}\mathbb{1}_{E_i \setminus E_{i+1}}\tau_{i+1}.$$

Note that for each $i \geqslant k$, the set E_i is $\mathcal{F}_{\tau_k}$-measurable. Therefore

$$\forall\, t \geq 0, \quad \{\tau'_k \leq t\} = \Big(E_k^c \cap \{\tau_k \leq t\}\Big) \cup \bigcup_{i=k}^{n} \Big((E_i \setminus E_{i+1}) \cap \{\tau_{i+1} \leq t\}\Big) \in \mathcal{F}_t,$$

so τ'_k is an $\mathbb{F}$-stopping time. By definition one has $\tau_1 \leq \tau'_2 \leq \cdots \leq \tau'_N \leq \tau'_{n+1}$, where $\tau'_{n+1} = \infty$. One also has, for any ω,

$$\{\tau_1(\omega), \tau_2(\omega) \cdots, \tau_{n+1}(\omega)\} = \{\tau_1(\omega), \tau'_2(\omega), \cdots, \tau'_{n+1}(\omega)\}.$$

Moreover, the strict inequality $\tau_1 < \tau'_2$ holds on $\{\tau_1 < \infty\}$. Then by the induction hypothesis on $(\tau'_2, \cdots, \tau'_{n+1})$, we obtain the required result. □

For the purpose of the dynamic study of the random time τ, we need the following result which is analogous to [Jac85, Lemme 1.8].

Proposition 2.3. *There exists a nonnegative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\widetilde{\alpha}(\cdot)$ such that $\widetilde{\alpha}(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ and that*

$$\mathbb{E}[\mathbb{1}_H h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\widetilde{\alpha}_t(u)\,\eta(du) \qquad \mathbb{P}\text{-}a.s. \tag{2.2}$$

for any bounded Borel function h.

PROOF: Let $(\alpha_t(\cdot))_{t\geqslant 0}$ be a family of random functions such that the relation (2.1) holds for any $t \geqslant 0$. We fix a countable dense subset D in $\mathbb{R}_+$ such as the set of all non-negative rational numbers. If s and t are two elements in D, $s < t$, there exists a positive $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\alpha_{t|s}(\cdot)$ such that

$$\forall\, \theta \in \mathbb{R}_+, \quad \alpha_{t|s}(\theta) = \mathbb{E}[\alpha_t(\theta)\,|\,\mathcal{F}_s] \qquad \mathbb{P}\text{-a.s.}.$$

Note that for any bounded Borel function h, one has

$$\mathbb{E}[\mathbb{1}_H h(\tau)|\mathcal{F}_s] = \mathbb{E}\left[\int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du)\,\middle|\,\mathcal{F}_s\right] = \int_{\mathbb{R}_+} h(u)\alpha_{t|s}(u)\,\eta(du) \quad \mathbb{P}\text{-a.s.}.$$

Hence there exists an η-negligeable set $B_{t,s}$ such that $\alpha_s(u) = \alpha_{t|s}(u)$ $\mathbb{P}$-a.s. for any $u \in \mathbb{R}_+ \setminus B_{t,s}$. Let $B = \bigcup_{(s,t)\in D^2, s<t} B_{t,s}$ and let $\widehat{\alpha}_t(\cdot) = \mathbb{1}_B(\cdot)\alpha_t(\cdot)$ for any $t \in D$. We then obtain that

$$\widehat{\alpha}_s(u) = \mathbb{E}[\widehat{\alpha}_t(u)|\mathcal{F}_s]$$

for any $u \in \mathbb{R}_+$ and all elements s, t in D such that $s < t$. Moreover, since B is still η-negligeable, for any $t \in D$,

$$\mathbb{E}[\mathbb{1}_H h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\widehat{\alpha}_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}. \tag{2.3}$$

By [DM80, Theorem VI.1.2], for any $\theta \in \mathbb{R}_+$, there exists a $\mathbb{P}$-negligeable subset E_θ of Ω such that, for any $\omega \in \Omega \setminus E_\theta$, the following limits exist

$$\widehat{\alpha}_{t+}(\omega, \theta) := \lim_{s \in D,\, s \downarrow t} \widehat{\alpha}_t(\omega, \theta),$$
$$\widehat{\alpha}_{t-}(\omega, \theta) := \lim_{s \in D,\, s \uparrow t} \widehat{\alpha}_t(\omega, \theta).$$

Moreover, we define

$$\widetilde{\alpha}_t(\omega, \theta) = \begin{cases} \widehat{\alpha}_t(\omega, \theta), & \text{if } \omega \notin E_\theta, \\ 0, & \text{if } \omega \in E_\theta. \end{cases}$$

Then $\widetilde{\alpha}(\theta)$ is a càdlàg $\mathbb{F}$-martingale, and therefore the random function $\tilde{\alpha}(\cdot)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. We then deduce the proposition from (2.3). □

We summarise the *generalised density hypothesis* as below. In what follows, we always assume this hypothesis.

Assumption 2.4. *We assume that there exist a non-atomic σ-finite Borel measure η on $\mathbb{R}_+$, a finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$, together with an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\alpha(\cdot)$ such that $\alpha(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ and that*

$$\mathbb{E}\left[\mathbb{1}_{\{\tau < \infty\}} h(\tau) \prod_{i=1}^n \mathbb{1}_{\{\tau \neq \tau_i\}} \,\middle|\, \mathcal{F}_t\right] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}$$

for any bounded Borel function h.

Remark 2.5. 1) The condition $\mathbb{P}(\tau_i = \tau_j < \infty) = 0$ is not assumed in Assumption 2.4 because it is not essential. In fact, for an arbitrary finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$, if we suppose that the random time τ has an $\mathbb{F}$-density $\alpha(\cdot)$ with respect to η avoiding $(\tau_i)_{i=1}^n$, then by Remark 2.1 and Proposition 2.2, we can always obtain another family of $\mathbb{F}$-stopping times $(\sigma_i)_{i=1}^n$ such that $\mathbb{P}(\sigma_i = \sigma_j < \infty) = 0$ for $i \neq j$ and that τ has an $\mathbb{F}$-density avoiding the family $(\sigma_i)_{i=1}^n$. Moreover, the $\mathbb{F}$-density of τ avoiding $(\sigma_i)_{i=1}^n$

coincides with $\alpha(\cdot)$.
2) For each $i \in \{1, \cdots, n\}$, by Theorem 1.3, there exists a subset $\Omega_i \in \mathcal{F}_{\tau_i}$ such that

$$\tau_i' := \tau_i \mathbb{1}_{\Omega_i} + \infty \mathbb{1}_{\Omega_i^c}$$

is an accesible $\mathbb{F}$-stopping time and

$$\tau_i'' := \tau_i \mathbb{1}_{\Omega_i^c} + \infty \mathbb{1}_{\Omega_i}$$

is a totally inaccessible $\mathbb{F}$-stopping time, and $\tau_i = \tau_i' \wedge \tau_i''$. Then, we have

$$\{\tau \neq \tau_i\} = \{\tau \neq \tau_i'\} \cap \{\tau \neq \tau_i''\},$$

which implies that τ also admits an $\mathbb{F}$-density avoiding the family $(\tau_i', \tau_i'')_{i=1}^n$ and the $\mathbb{F}$-density is still $\alpha(\cdot)$. Therefore, without loss of generality, we may assume in addition that each $\mathbb{F}$-stopping time τ_i is either accessible or totally inaccessible.

We present a simple example as below, where Assumption 2.4 is satisfied and the generalised density can be explicitly computed.

Example 2.6. Let $W = (W_t, t \geq 0)$ be a standard Brownian motion and $\mathbb{F}$ be the canonical Brownian filtration. Let $N = (N_t, t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. We denote by τ_1 the first hitting time of a negative level by the Brownian motion

$$\tau_1 = \inf\{t \geq 0 : W_t = a < 0\},$$

and ξ the first jump time of N

$$\xi = \inf\{t \geq 0 : N_t \geq 1\},$$

with the convention $\inf \emptyset = \infty$. Define a random time τ as the minimum of τ_1 and ξ

$$\tau = \tau_1 \wedge \xi. \tag{2.4}$$

We compute firstly the $\mathbb{F}$-conditional distribution of τ_1. For any $0 \leq t < \theta$, one has

$$\mathbb{P}(\tau_1 > \theta | \mathcal{F}_t) = \mathbb{P}\left(\min_{0 \leq s \leq \theta} W_s > a \middle| \mathcal{F}_t\right) = \mathbb{1}_{\{\tau_1 > t\}} \mathbb{P}\left(\min_{t \leq s \leq \theta} W_s > a \middle| W_t\right).$$

By reflection principle (see c.f. [BS02, 1.2.4, page 126]),

$$\mathbb{P}\left(\min_{t\le s\le\theta} W_s > a \middle| W_t = x\right) = 1 - \mathbb{P}_x\left(\min_{0\le s\le\theta-t} W_s \le a\right) = \mathrm{erf}\left(\frac{x-a}{\sqrt{2(\theta-t)}}\right),$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}dv$ is the Gauss error function. Then,

$$\mathbb{P}(\tau_1 > \theta|\mathcal{F}_t) = \mathbb{1}_{\{\tau_1>t\}}\mathrm{erf}\left(\frac{W_t-a}{\sqrt{2(\theta-t)}}\right), \quad \theta > t. \tag{2.5}$$

Next, for any $t \in \mathbb{R}_+$,

$$\mathbb{P}(\tau = \tau_1|\mathcal{F}_t) = \mathbb{P}(\tau_1 \le \xi|\mathcal{F}_t) = \mathbb{1}_{\{\tau_1\le t\}}e^{-\lambda\tau_1} + \mathbb{1}_{\{\tau_1>t\}}\mathbb{E}[e^{-\lambda\tau_1}|\mathcal{F}_t].$$

Recall that, for any $l \in \mathbb{R}_+$, the process

$$\left(e^{-lW_t-\frac{1}{2}l^2t}, t \ge 0\right)$$

is a martingale. For $a < 0$, the stopped martingale

$$\left(e^{-lW_{t\wedge\tau_1}-\frac{l^2}{2}(t\wedge\tau_1)}, t \ge 0\right)$$

is uniformly integrable, bounded by e^{-la}. Then, on the set $\{\tau_1 > t\}$, the optional sampling theorem (see e.g. [Pro05, Theorem 16]) yields

$$\mathbb{E}\left[e^{-lW_{\tau_1}-\frac{1}{2}l^2\tau_1}\middle|\mathcal{F}_t\right] = e^{-lW_t-\frac{1}{2}l^2t},$$

which implies that

$$\mathbb{1}_{\{\tau_1>t\}}\mathbb{E}\left[e^{-\frac{1}{2}l^2\tau_1}\middle|\mathcal{F}_t\right] = e^{l(a-W_t)-\frac{1}{2}l^2t}.$$

Then, one has

$$\mathbb{P}(\tau = \tau_1|\mathcal{F}_t) = \mathbb{1}_{\{\tau_1\le t\}}e^{-\lambda\tau_1} + \mathbb{1}_{\{\tau_1>t\}}e^{\sqrt{2\lambda}(a-W_t)-\lambda t}, \quad t \ge 0. \tag{2.6}$$

Furthermore, τ satisfies Assumption 2.4 with generalised density

$$\alpha_t(\theta) = \lambda e^{-\lambda\theta}\left[\mathbb{1}_{\{\theta\le t\}}\mathbb{1}_{\{\tau_1>\theta\}} + \mathbb{1}_{\{\theta>t\}}\mathbb{1}_{\{\tau_1>t\}}\mathrm{erf}\left(\frac{W_t-a}{\sqrt{2(\theta-t)}}\right)\right], \quad t \ge 0. \tag{2.7}$$

Indeed, for any $0 \leq \theta \leq u \leq t$,

$$\begin{aligned}\mathbb{P}(\tau \leq u, \tau \neq \tau_1 | \mathcal{F}_t) &= \mathbb{P}(\xi \leq u, \xi \leq \tau_1 | \mathcal{F}_t) \\ &= \mathbb{P}(\xi \leq u \wedge \tau_1 | \mathcal{F}_t) \\ &= 1 - e^{-\lambda(u \wedge \tau_1)} \\ &= \int_0^u \lambda e^{-\lambda\theta} \mathbb{1}_{\{\theta < \tau_1\}} d\theta,\end{aligned}$$

which implies that $\alpha_t(\theta) = \lambda e^{-\lambda\theta} \mathbb{1}_{\{\theta<\tau_1\}}$ if $\theta \leq t$. If $\theta > t$, by martingale property and the equality (2.5), one has

$$\alpha_t(\theta) = \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t] = \lambda e^{-\lambda\theta}\mathbb{P}(\tau_1 > \theta|\mathcal{F}_t) = \lambda e^{-\lambda\theta}\mathbb{1}_{\{\tau_1>t\}}\mathrm{erf}\left(\frac{W_t - a}{\sqrt{2(\theta - t)}}\right).$$

The following example is similar to the extension of Heath-Jarrow-Merton model (see [GS16]), which gives the martingale representation of the generalised density process.

Example 2.7. Let $W = (W_t, t \geq 0)$ be a standard Brownian motion and $\mathbb{F}$ be the canonical Brownian filtration, and τ a random time that is not an $\mathbb{F}$-stopping time. Denote by $G_t(\theta)$ the $\mathbb{F}$-conditional probability distribution of τ, namely

$$G_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t), \quad t, \theta \in \mathbb{R}_+,$$

It is clear that $G(\theta)$ is an $\mathbb{F}$-martingale for any $\theta \geq 0$. By predictable representation theorem, there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $Z(\cdot)$, satisfying $Z_t(0) = 0$ for any $t \geq 0$, such that

$$dG_t(\theta) = Z_t(\theta)dW_t, \quad t \in \mathbb{R}_+.$$

We assume that there exist $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $x(\cdot)$ and $y(\cdot)$ such that

$$Z_t(\theta) = \int_0^\theta [x_t(u)\eta(du) + y_t(u)\delta_A(du)]$$

with the constraint $\int_0^\infty [x_t(u)\eta(du) + y_t(u)\delta_A(du)] = 0$, where A is the graph union of a family of $\mathbb{F}$-stopping times $\{\tau_i\}_{i=1}^n$, namely $A = \bigcup_{i=1}^n [\![\tau_i]\!]$, and δ_A is a Dirac measure defined as

$$\delta_A(u) = \begin{cases} 1, & u \in A, \\ 0, & \text{otherwise.} \end{cases}$$

By Fubini theorem, one has

$$\begin{aligned}
G_t(\theta) &= G_0(\theta) + \int_0^t Z_s(\theta)dW_s \\
&= G_0(\theta) + \int_0^t dW_s \int_0^\theta [x_s(u)\eta(du) + y_s(u)\delta_A(du)] \\
&= G_0(\theta) + \int_0^\theta \left[\eta(du)\int_0^t x_s(u)dW_s + \delta_A(du)\int_0^t y_s(u)dW_s\right] \\
&= G_0(\theta) - \int_\theta^\infty \left[\eta(du)\int_0^t x_s(u)dW_s + \delta_A(du)\int_0^t y_s(u)dW_s\right].
\end{aligned}$$

If τ satisfies Assumption 2.4, we can deduce that the generalised density $\alpha(\cdot)$ has the following martingale representation

$$\alpha_t(u) = \alpha_0(\theta) - \int_0^t x_s(u)dW_s, \quad t \in \mathbb{R}_+.$$

For each $i \in \{1, \cdots, n\}$, let p^i be a càdlàg version of the $\mathbb{F}$-martingale

$$\left(\mathbb{E}[\mathbb{1}_{\{\tau=\tau_i<\infty\}}|\mathcal{F}_t], t \geq 0\right),$$

which is closed by

$$p^i_\infty = \mathbb{E}[\mathbb{1}_{\{\tau=\tau_i<\infty\}}|\mathcal{F}_\infty].$$

We also consider the case where τ may reach infinity and denote by p^∞ a càdlàg version of the $\mathbb{F}$-martingale

$$\left(\mathbb{E}[\mathbb{1}_{\{\tau=\infty\}}|\mathcal{F}_t], t \geq 0\right),$$

which is closed by

$$p^\infty_\infty = \mathbb{E}[\mathbb{1}_{\{\tau=\infty\}}|\mathcal{F}_\infty].$$

Note that Assumption 2.4 implies that, for any $t \geq 0$,

$$\int_{\mathbb{R}_+} \alpha_t(u)\,\eta(du) + \sum_{i=1}^n p^i_t + p^\infty_t = 1 \qquad \mathbb{P}\text{-a.s.} \tag{2.8}$$

We define

$$G_t := \int_t^\infty \alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i>t\}}p^i_t + p^\infty_t. \tag{2.9}$$

Note that $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$, $\mathbb{P}$-a.s. The process G is a càdlàg $\mathbb{F}$-supermartingale and is called Azéma supermatingale of the random time τ. Moreover, for any bounded Borel function h, one has

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\tau=\tau_i<\infty\}}h(\tau)|\mathcal{F}_t] &= \mathbb{1}_{\{\tau_i\leq t\}}h(\tau_i)\mathbb{E}[\mathbb{1}_{\{\tau=\tau_i\}}|\mathcal{F}_t] + \mathbb{1}_{\{\tau_i>t\}}\mathbb{E}\left[\mathbb{1}_{\{\tau_i<\infty\}}h(\tau_i)\mathbb{E}[\mathbb{1}_{\{\tau=\tau_i\}}|\mathcal{F}_{\tau_i}]\Big|\mathcal{F}_t\right].\\ &= \mathbb{1}_{\{\tau_i\leq t\}}h(\tau_i)p_t^i + \mathbb{1}_{\{\tau_i>t\}}\mathbb{E}\left[\mathbb{1}_{\{\tau_i<\infty\}}h(\tau_i)p_{\tau_i}^i|\mathcal{F}_t\right].\end{aligned}$$

Then,

$$\mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\eta(du) + \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{\tau_i<\infty\}}h(\tau_i)p_{\tau_i\vee t}^i|\mathcal{F}_t]. \tag{2.10}$$

The following proposition shows that any $\mathcal{G}_t$-conditional expectation can be computed in a decomposed form, which can be viewed as a direct extension to [EKJJ10, Theorem 3.1].

Proposition 2.8. *Let $Y_T(\cdot)$ be an $\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_+)$-measurable random variable such that*

1) $\mathbb{1}_{\cap_{i=1}^n\{\tau_i\neq\theta\}}Y_T(\theta)\alpha_T(\theta)$ is integrable for any $\theta\in\mathbb{R}_+$ and

$$\int_{\mathbb{R}_+}\Big|\mathbb{E}[Y_T(\theta)\alpha_T(\theta)]\Big|\eta(d\theta) < +\infty,$$

2) $\mathbb{1}_{\{\tau_i<\infty\}}Y_T(\tau_i)p_{\tau_i\vee T}^i$ is integrable for any $i\in\{1,\cdots,n\}$.

Then the random variable $\mathbb{1}_{\{\tau<\infty\}}Y_T(\tau)$ is integrable, and for any $t\leq T$,

$$\mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}Y_T(\tau)|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}\tilde{Y}_t + \mathbb{1}_{\{\tau\leq t\}}\hat{Y}_t(\tau)\quad \mathbb{P}\text{-}a.s. \tag{2.11}$$

where

$$\tilde{Y}_t = \frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\left[\int_t^{+\infty}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\eta(d\theta) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i>t\}}\mathbb{E}[\mathbb{1}_{\{\tau_i<\infty\}}Y_T(\tau_i)p_{\tau_i\vee T}^i|\mathcal{F}_t]\right] \tag{2.12}$$

and

$$\hat{Y}_t(\theta) = \mathbb{1}_{\cap_{i=1}^n\{\theta\neq\tau_i\}}\frac{\mathbb{1}_{\{\alpha_t(\theta)>0\}}}{\alpha_t(\theta)}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t] + \sum_{i=1}^n \mathbb{1}_{\{\theta=\tau_i\}}\frac{\mathbb{1}_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}[Y_T(\tau_i)p_T^i|\mathcal{F}_t],\quad \theta\leq t. \tag{2.13}$$

PROOF: We may assume that $Y_T(\cdot)$ is non-negative without loss of generality so that the following proof works without discussing the integrability (as a byproduct, we can prove the case where $Y_T(\cdot)$ is non-negative without any integrability condition). The integrability of $Y_T(\tau)$ results from the finiteness of each term in the following formulae.

The first term on the right-hand side of (2.11) is obtained as a consequence of the so-called key lemma in the progressive enlargement of filtrations ([EJY00, Lemma 3.1]):

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}Y_T(\tau)|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\mathbb{E}[\mathbb{1}_{\{t<\tau<\infty\}}Y_T(\tau)|\mathcal{F}_t].$$

Note that

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{t<\tau<\infty\}}Y_T(\tau)|\mathcal{F}_T] &= \int_t^{+\infty} Y_T(u)\alpha_T(u)\eta(du) + \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{t<\tau=\tau_i<\infty\}}Y_T(\tau_i)|\mathcal{F}_T] \\ &= \int_t^{+\infty} Y_T(u)\alpha_T(u)\eta(du) + \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{t<\tau_i<\infty\}}Y_T(\tau_i)p^i_{\tau_i\vee T}|\mathcal{F}_T]\end{aligned}$$

which implies (2.12). For the second term in (2.11), we shall prove by verification. Let $Z_t(\cdot)$ be a bounded $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}_+)$-measurable random variable, one has

$$\begin{aligned}\mathbb{E}[\hat{Y}_t(\tau)Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}] &= \mathbb{E}\Big[\mathbb{1}_{H\cap\{\tau\le t\}}\frac{\mathbb{1}_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]_{\theta=\tau}\Big] \\ &\quad + \sum_{i=1}^n \mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_i\le t\}}\frac{\mathbb{1}_{\{p^i_t>0\}}}{p^i_t}\mathbb{E}[Y_T(\tau_i)Z_t(\theta)p^i_T|\mathcal{F}_t]_{\theta=\tau}\Big].\end{aligned}$$

Note that

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{H\cap\{\tau\le t\}}\frac{\mathbb{1}_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]_{\theta=\tau}\Big] = \mathbb{E}\Big[\int_0^t \mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]\eta(d\theta)\Big] \\ &\quad = \int_0^t \mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)]\eta(d\theta) \\ &\quad = \mathbb{E}\Big[\mathbb{1}_{H\cap\{\tau\le t\}}Y_T(\tau)Z_t(\tau)\Big].\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_{\{\tau=\tau_i\le t\}}\frac{\mathbb{1}_{\{p^i_t>0\}}}{p^i_t}\mathbb{E}[Y_T(\tau_i)Z_t(\theta)p^i_T|\mathcal{F}_t]_{\theta=\tau}\Big] &= \mathbb{E}[\mathbb{1}_{\{\tau_i\le t\}}Y_T(\tau_i)Z_t(\tau_i)p^i_T] \\ &= \mathbb{E}[\mathbb{1}_{\{\tau=\tau_i\le t\}}Y_T(\tau)Z_t(\tau)].\end{aligned}$$

Therefore we obtain

$$\mathbb{E}[\mathbb{1}_{\{\tau\le t\}}Y_T(\tau)|\mathcal{G}_t] = \mathbb{1}_{\{\tau\le t\}}\hat{Y}_t(\tau)\quad \mathbb{P}\text{-a.s.}$$

since $\hat{Y}_t(\cdot)$ is $\mathcal{F}_t\otimes\mathcal{B}([0,t])$-measurable. The proposition is thus proved. □

Remark 2.9. (1) For any integrable $\mathcal{G}_T$-measurable random variable Z, one can always find an $\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_+)$-measurable function $Y_T(\cdot)$ such that

$$\mathbb{1}_{\{\tau<\infty\}}Z = \mathbb{1}_{\{\tau<\infty\}}Y_T(\tau),\quad \mathbb{P}\text{-a.s.}$$

which verifies the conditions of integrability in the previous proposition. Without loss of generality, we can assume that Z is nonnegative. We begin with an arbitrary $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable nonnegative random function $Z_T(\cdot)$ such that

$$\mathbb{1}_{\{\tau<\infty\}} Z = \mathbb{1}_{\{\tau<\infty\}} Z_T(\tau).$$

Then by Proposition 2.8 in the nonnegative case (where the integrability conditions are not necessary), one has

$$\int_0^\infty \mathbb{E}[Z_T(\theta)\alpha_T(\theta)]\eta(d\theta) < \infty.$$

Therefore, the set K of $\theta \in \mathbb{R}_+$ such that $\mathbb{E}[Z_T(\theta)\alpha_T(\theta)] = +\infty$ is η-negligeable. By replacing $Z_T(\cdot)$ by zero on the set

$$(\Omega \times K) \cap \bigcap_{i=1}^n \{(\omega,\theta) \in \Omega \times \mathbb{R}_+ \mid \tau_i(\omega) \neq \theta\},$$

we find another random function $Y_T(\cdot)$ such that $Y_T(\tau) = Z_T(\tau)$ $\mathbb{P}$-a.s.. Moreover, $Y_T(\cdot)$ satisfies the integrability conditions as in the proposition.

(2) As a direct consequence, for any $t \leq T$, one has

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \left[\int_T^\infty \alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{\tau_i>T\}} p_T^i | \mathcal{F}_t] + p_t^\infty \right] \quad \mathbb{P}\text{-a.s.} \tag{2.14}$$

In the literature on progressive enlargement of filtrations, the following formula ([Jeu80], Lemme 4.4) has been widely used: for any $\mathbb{G}$-predictable process $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}}, t \geq 0)$, there exist an $\mathbb{F}$-predictable process $Y^{\mathbb{F}} = (Y_t^{\mathbb{F}}, t \geq 0)$ and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $Y(\cdot)$ such that

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} Y_t^{\mathbb{F}} + \mathbb{1}_{\{\tau<t\}} Y_t(\tau), \quad t \in \mathbb{R}_+. \tag{2.15}$$

The equality (2.15) has been proved by monotone class theorem and is valid in general. An optional version of the formula (2.15) has also been used: for any $\mathbb{G}$-optional process $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}}, t \geq 0)$, there exist an $\mathbb{F}$-optional process $Y^{\mathbb{F}} = (Y_t^{\mathbb{F}}, t \geq 0)$ and a $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $Y(\cdot)$ such that

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}} Y_t^{\mathbb{F}} + \mathbb{1}_{\{\tau \leq t\}} Y_t(\tau), \quad t \in \mathbb{R}_+. \tag{2.16}$$

However, as pointed out by Barlow [Bar78] and later by Song [Son14], the equality (2.16) does not hold in general. We affirm that the formula (2.16) can be used under generalised density hypothesis without doubt. Indeed, for any bounded $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function h, Proposition 2.8 yields a decomposed form for the $\mathbb{G}$-martingale $h^{\mathbb{G}} = (\mathbb{E}[h(\tau)|\mathcal{G}_t], t \geq 0)$. The $\mathbb{F}$-optional projections in (2.12) and (2.13) have right-continuous paths, and by taking right limit we prove that the martingale $h^{\mathbb{G}}$ satisfies the equality (2.16). When we send T to $+\infty$, by [Son14, Lemma 3.6], the equality (2.16) is valid for any $\mathbb{G}$-optional process.

2.2.2 Examples in literature

In literature on credit risk, there are other hybrid models such as the generalised reduced-form model in [BSW04], the credit migration model in [CF05], the jump to default CEV models in [CL06] and [CPS09], as well as the extension of the HJM-approach in [GS16]. In this section, we prove that these models can be included in the generalised density framework. To avoid ambiguity, we keep the same notations as in the original papers.

To make it precise, in [BSW04], the reference filtration $\mathbb{F}$ is the standard augmented Brownian filtration and the default time τ is defined by a generalised Cox process model

$$\tau = \inf\{0 \leq t \leq \bar{T} : \Gamma_t \geq \Theta\},$$

where the fixed positive number $\bar{T}$ is the horizon, and $(\Gamma_t, 0 \leq t \leq \bar{T})$ is a nondecreasing $\mathbb{F}$-predictable process which satisfies $\Gamma_0 = 0$ a.s., and Θ is a strictly positive random variable independent of $\mathcal{F}_{\bar{T}}$. Denote $F(\cdot)$ the right-continuous cumulative distribution function of Θ, then for any $0 \leq u \leq t$, the $\mathbb{F}$-conditional survival probability is

$$\begin{aligned}
S_u &:= \mathbb{P}(\tau > u|\mathcal{F}_t) \\
&= 1 - \mathbb{P}(\Theta \leq \Gamma_u|\mathcal{F}_t) \\
&= 1 - \mathbb{P}(\Theta \leq x|x = \Gamma_u) \\
&= 1 - F(\Gamma_u),
\end{aligned}$$

where S is the survival process satisfying the stochastic differential equation

$$S_t = 1 - \int_{(0,t]} S_{s-} d\Gamma_s, \quad 0 \le t \le \bar{T},$$

where $\Gamma = (\Gamma_t, 0 \le t \le \bar{T})$ is the $\mathbb{F}$-compensator (the authors call it hazard process) with discontinuous part $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$ and continuous part $\Gamma_t^c = \Gamma_t - \sum_{0<u\le t} \Delta\Gamma_u$. Then one has

$$S_t = 1 - \int_0^t S_{s-} d\Gamma_s^c - \sum_{0<s\le t} S_{s-} \Delta\Gamma_s.$$

The probability that the default occurs at u, $0 \le u \le t$, is

$$\mathbb{P}(\tau = u | \mathcal{F}_t) = S_{u-} - S_u = S_{u-} \Delta\Gamma_u. \tag{2.17}$$

If we assume in addition that the continuous part of Γ is absolutely continuous, namely

$$\Gamma_t^c = \int_0^t \lambda_t \, dt,$$

and Γ has N jumps at $(\tau_i)_{i=1}^n$, then, on the set $\{\tau_i \le t\}$, the $\mathbb{F}$-conditional probability that τ coincides with τ_i is

$$p_t^i := \mathbb{P}(\tau = \tau_i | \mathcal{F}_t) = S_{\tau_i -} \Delta\Gamma_{\tau_i}.$$

On the set $\{\tau_i > t\}$, $p_t^i = \mathbb{E}[p_{\tau_i}^i | \mathcal{F}_t]$. Furthermore, for any $0 \le u \le t$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\tau \le u\}} \prod_{i=1}^n \mathbb{1}_{\{\tau \ne \tau_i\}} \middle| \mathcal{F}_t\right] &= \mathbb{P}(\tau \le u | \mathcal{F}_t) - \mathbb{E}\left[\mathbb{1}_{\{\tau \le u\}} \sum_{i=1}^n \mathbb{1}_{\{\tau = \tau_i\}} \middle| \mathcal{F}_t\right] \\
&= 1 - S_u - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le u\}} p_t^i \\
&= 1 - S_u - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le u\}} S_{\tau_i -} \Delta\Gamma_{\tau_i} \\
&= \int_0^u S_{s-} \lambda_s \, ds,
\end{aligned}$$

which yields that the generalised density is $\alpha_t(u) = S_{u-}\lambda_u$ for any $0 \le u \le t$. If $u > t$, by martingale property we have $\alpha_t(u) = \mathbb{E}[\alpha_u(u) | \mathcal{F}_t]$.

In [CF05], $\mathbb{F}$ represents the global filtration. To relate the results in this paper to our framework, we let $\mathbb{F}^\Delta = (\mathcal{F}_t^\Delta)_{t \ge 0}$ be the filtration generated by the short rate process Y^1

as well as the credit index Y^2. Then, $\mathbb{F}^\Delta$ is the reference filtration ($\mathbb{F}$ in our framework) and $\mathbb{F}$ is the enlarged filtration ($\mathbb{G}$ in our framework). The firm can default due to credit downgradings at T_Δ, which is an $\mathbb{F}^\Delta$-stopping time, or an unpredictable jump of an associated point process Y^3 at T_J. We denote by $T_\mathcal{D}$ the default time and $G = (G_t, t \geq 0)$ the Azéma supermartingale associated to $T_\mathcal{D}$, namely $G_t = \mathbb{P}(T_\mathcal{D} > t|\mathcal{F}_t^\Delta)$ for any $t \in \mathbb{R}_+$. We compute firstly the $\mathbb{F}$-conditional survival probability of T_Δ. For any $0 \leq t < u$, by [CF05, Remark 2.1], one has

$$\begin{aligned} H_t(u) := \mathbb{P}(T_\Delta > u|\mathcal{F}_t) &= \mathbb{E}[\mathbb{1}_{\{Y_u \neq \Delta\}}|\mathcal{F}_t] \\ &= e^{\phi(u-t,0)+\psi_1(u-t,0)Y_t^1+\psi_2(u-t,0)Y_t^2+\psi_3(u-t,0)Y_t^3} \\ &= \int_u^\infty g_t(\theta)d\theta, \end{aligned} \tag{2.18}$$

where the functions $\phi = \phi(t,v)$ and $\psi_i = \psi_i(t,v)$, $i = 1,2,3$, solve the generalised Riccati equations [CF05, (7)], and $g(\cdot)$ is an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function such that, for any $0 \leq t < \theta$

$$g_t(\theta) = -H_t(\theta)(\partial_t\phi(\theta-t,0) + \partial_t\psi_1(\theta-t,0)Y_t^1 + \partial_t\psi_2(\theta-t,0)Y_t^2 + \partial_t\psi_3(\theta-t,0)Y_t^3).$$

Let $\alpha_t^\Delta(\theta) = \mathbb{E}[g_t(\theta)|\mathcal{F}_t^\Delta]$, then the $\mathbb{F}^\Delta$-conditional survival probability of T_Δ is given by

$$\mathbb{P}(T_\Delta > u|\mathcal{F}_t^\Delta) = \mathbb{E}[\mathbb{P}(T_\Delta > u|\mathcal{F}_t)|\mathcal{F}_t^\Delta] = \mathbb{E}\left[\int_u^\infty g_t(\theta)d\theta|\mathcal{F}_t^\Delta\right] = \int_u^\infty \alpha_t^\Delta(\theta)d\theta, \quad t \leq u.$$

The default time coincides with T_Δ if T_Δ occurs before the first jump time of the point process Y^3, then we have

$$\mathbb{1}_{\{T_\mathcal{D}=T_\Delta\}} = \mathbb{1}_{\{Y^3_{T_\Delta}=0\}} = \lim_{k\to\infty} e^{-kY^3_{T_\Delta}}.$$

Then, on the set $\{T_\Delta > t\}$, the $\mathbb{F}^\Delta$-conditional probability that $T_\mathcal{D}$ coincides with T_Δ is

$$\begin{aligned} p_t^\Delta &:= \mathbb{P}(T_\mathcal{D} = T_\Delta|\mathcal{F}_t^\Delta) \\ &= \mathbb{E}[\mathbb{P}(T_\mathcal{D} = T_\Delta|\mathcal{F}_t)|\mathcal{F}_t^\Delta] \\ &= \mathbb{E}\left[\mathbb{E}[\lim_{k\to\infty}\exp(-kY_T^3)|\mathcal{F}_t, T = T_\Delta]\Big|\mathcal{F}_t^\Delta\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{T_J>t\}}e^{\tilde\phi(T_\Delta-t,0)+\tilde\psi_1(T_\Delta-t,0)Y_t^1+\tilde\psi_2(T_\Delta-t,0)Y_t^2}\Big|\mathcal{F}_t^\Delta\right]. \end{aligned}$$

Furthermore, by [CF05, (13)], for any $0 \leq t < u$, the $\mathbb{F}^\Delta$-conditional survival probability of $T_\mathcal{D}$ is computed as

$$\begin{aligned} G_t^\mathcal{D}(u) &:= \mathbb{P}(T_\mathcal{D} > u | \mathcal{F}_t^\Delta) \\ &= \mathbb{E}[\mathbb{P}(T_\mathcal{D} > u | \mathcal{F}_t) | \mathcal{F}_t^\Delta] \\ &= G_t e^{\tilde{\phi}(u-t,0) + \tilde{\psi}_1(u-t,0)Y_t^1 + \tilde{\psi}_2(u-t,0)Y_t^2}, \end{aligned}$$

where $\tilde{\phi}$, $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are solutions to generalised Riccati equations [CF05, (14)]. Define an $\mathcal{O}(\mathbb{F}^\Delta) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\alpha^\mathcal{D}(\cdot)$ such that, for any $0 \leq t < \theta$,

$$\alpha_t^\mathcal{D}(\theta) = -G_t^\mathcal{D}(\theta) \left(\partial_t \tilde{\phi}(\theta - t, 0) + \partial_t \tilde{\psi}_1(\theta - t, 0) Y_t^1 + \partial_t \tilde{\psi}_2(\theta - t, 0) Y_t^2 \right).$$

Then, for any $0 \leq t < u$,

$$\begin{aligned} \mathbb{P}(T_\mathcal{D} > u, T_\mathcal{D} \neq T_\Delta | \mathcal{F}_t^\Delta) &= \mathbb{P}(T_\mathcal{D} > u | \mathcal{F}_t^\Delta) - \mathbb{P}(T_\mathcal{D} > u, T_\mathcal{D} = T_\Delta | \mathcal{F}_t^\Delta \\ &= \mathbb{P}(T_\mathcal{D} > u | \mathcal{F}_t^\Delta) - \mathbb{E}[\mathbb{1}_{\{T_\mathcal{D} > u\}} p_{T_\Delta}^\Delta | \mathcal{F}_t^\Delta] \\ &= \int_u^\infty \left(\alpha_t^\mathcal{D}(\theta) - p_\theta^\Delta \alpha_t^\Delta(\theta) \right) d\theta, \end{aligned}$$

which implies that the $\mathbb{F}^\Delta$-generalised density is $\alpha_t(\theta) := \alpha_t^\mathcal{D}(\theta) - p_\theta^\Delta \alpha_t^\Delta(\theta)$ for any $0 \leq t < \theta$. However, we cannot obtain the full knowledge of $\alpha(\cdot)$ without specific conditions (e.g. under the immersion property).

In [CL06], default can occur either at T_0 via equity price diffusion S to zero or prior to T_0 via a jump to bankruptcy at $\tilde{\zeta}$, intensity of which $\lambda(S, t)$ is time-and-state-dependent. The reference filtration $\mathbb{F}$ is generated by S. The $\mathbb{F}$-conditional probability that the default time ξ coincides with T_0 is

$$p_t^0 := \mathbb{P}(\xi = T_0 | \mathcal{F}_t) = \mathbb{P}(T_0 < \tilde{\zeta} | \mathcal{F}_t) = \mathbb{E}\left[e^{-\int_0^{T_0} \lambda(S_s, s) ds} \middle| \mathcal{F}_t \right].$$

Then, for any $0 \leq u \leq t$,

$$\begin{aligned} \mathbb{P}(\xi \leq u, \xi \neq T_0 | \mathcal{F}_t) &= \mathbb{P}(\xi \leq u | \mathcal{F}_t) - \mathbb{P}(\xi \leq u, \xi = T_0 | \mathcal{F}_t) \\ &= 1 - \mathbb{P}(\xi > u | \mathcal{F}_t) - \mathbb{1}_{\{T_0 \leq u\}} \mathbb{P}(\xi = T_0 | \mathcal{F}_t) \\ &= 1 - \mathbb{1}_{\{T_0 > u\}} e^{-\int_0^u \lambda(S_s, s) ds} - \mathbb{1}_{\{T_0 \leq u\}} e^{-\int_0^{T_0} \lambda(S_s, s) ds} \\ &= 1 - e^{-\int_0^{u \wedge T_0} \lambda(S_s, s) ds} \\ &= \int_0^u \mathbb{1}_{\{\theta \leq T_0\}} \lambda(S_\theta, \theta) e^{-\int_0^\theta \lambda(S_\theta, \theta)} d\theta, \end{aligned}$$

which implies that the generalised density is

$$\alpha_t(\theta) = \mathbb{1}_{\{\theta \leq T_0\}} \lambda(S_\theta, \theta) e^{-\int_0^\theta \lambda(S_\theta, \theta)}$$

for any $0 \leq \theta \leq t$. If $\theta > t$, by martingale property we have $\alpha_t(\theta) = \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]$.

In [CPS09], the default time is defined as the time of absorption at zero of a jump diffusion S, noted $\tau \wedge \xi$, where τ is the jump time and ξ is the time of absorption at zero without jump. We denote by $\mathbb{F}^z$ the canonical Brownian filtration, which corresponds to the reference filtration in our framework (our $\mathbb{F}$). This model is similar to that in [CL06] with constant λ. Let ξ^c be the time of absorption at zero of the continuous part of S, which is an $\mathbb{F}^z$-stopping time. Then, by similar computation as in the previous paragraph, the $\mathbb{F}^z$-conditional probability that $\tau \wedge \xi$ coincides with ξ^c is

$$\mathbb{P}(\tau \wedge \xi = \xi^c | \mathcal{F}_t^z) = \mathbb{E}[\exp(-\lambda \xi^c) | \mathcal{F}_t^z],$$

where $\lambda > 0$ is the intensity of the Poisson jump, and the default time satisfies Assumption 2.4 with the $\mathbb{F}^z$-generalised density computed as

$$\alpha_t(\theta) = \lambda \exp(-\lambda\theta) \mathbb{P}(\xi^c \geq \theta | \mathcal{F}_t^z), \quad t \in \mathbb{R}_+$$

The Example [GS16, Example 2.1] is a particular case of the generalised reduced-form model in [BSW04], where the random time τ defined by a generalised Cox process model satisfies Assumption 2.4 with

$$\alpha_t(u) = \mathbb{E}\left[\lambda(u) e^{-\int_0^u \lambda(s)ds - \sum_{u_i < u} \lambda_i'} \middle| \mathcal{F}_t\right], \quad t \in \mathbb{R}_+,$$

where $\lambda = (\lambda(t), t \geq 0)$ is a nonnegative process, $0 < u_1 < \ldots < u_N$ are constants, and $\lambda_1', \ldots, \lambda_N'$ are positive random variables. Still, τ can coincide with the constants $(u_i)_{i=1}^n$, with

$$p_t^i := \mathbb{P}(\tau = u_i | \mathcal{F}_t) = \mathbb{E}[e^{-\int_0^{u_i} \lambda(s)ds - \sum \lambda_{i-1}'}(1 - e^{-\lambda_i'}) | \mathcal{F}_t].$$

2.3 Compensator process

As we have mentioned in the previous chapter, the compensator and the intensity processes of τ play an important role in the default event modelling. The general method for computing the compensator is Theorem A.11. Particularly, the compensator in the Cox process model is just the stopped hazard process $(\int_0^{t\wedge\tau} \lambda_s \, ds, t \geq 0)$, where $\lambda = (\lambda_t, t \geq 0)$ is the default intensity process. In [EKJJ10], an explicit result is obtained under the density hypothesis (see also [GJLR10] and [Li12])

$$\left(\int_0^t \frac{\alpha_u(u)du}{\int_u^\infty \alpha_u(s)ds}, t \geq 0 \right),$$

where $\alpha(\cdot) = (\alpha_t(\cdot), t \geq 0)$ is the density process, and the compensator is absolutely continuous, which implies the existence of the intensity process.

In this section, we focus on the compensator process under the generalised density hypothesis, when the intensity process does not always exist because of the singularities in the probability distribution of τ. We introduce the following notations. For any $i \in \{1, \cdots, n\}$, denote by D^i the process $(\mathbb{1}_{\{\tau_i \leq t\}}, t \geq 0)$. We use the expression Λ^i to denote the $\mathbb{F}$-compensator process of D^i, that is, Λ^i is an increasing $\mathbb{F}$-predictable process such that $M^i := D^i - \Lambda^i$ is an $\mathbb{F}$-martingale with $M_0^i = 0$. Note that, if τ_i is a predictable $\mathbb{F}$-stopping time, then $\Lambda^i = D^i$ and $M^i = 0$. The following result generalises [EKJJ10, Proposition 4.1(1)]. Here, the Azéma supermartingale G is a process with jumps and needs to be treated with care.

Proposition 2.10. *The Doob-Meyer decomposition of the Azéma supermartingale G is given by $G_t = G_0 + M_t - A_t$, where A is an $\mathbb{F}$-predictable increasing process given by*

$$A_t = \int_0^t \alpha_\theta(\theta)\eta(d\theta) + \sum_{i=1}^n \int_{(0,t]} p_{s-}^i d\Lambda_s^i + \sum_{i=1}^n \langle M^i, p^i \rangle_t. \tag{2.19}$$

PROOF: For any $t \geq 0$, let

$$C_t = \int_0^t \alpha_\theta(\theta)\eta(d\theta).$$

The process $C = (C_t, t \geq 0)$ is $\mathbb{F}$-adapted and increasing. It is moreover continuous since η is assumed to be non-atomic. Note that by (2.9),

$$G_t = \mathbb{E}\bigg[\int_t^\infty \alpha_\theta(\theta)\eta(d\theta) \,\bigg|\, \mathcal{F}_t \bigg] + \sum_{i=1}^n \mathbb{1}_{\{\tau_i > t\}} p_t^i + p_t^\infty.$$

The process

$$C_t + \int_t^\infty \alpha_t(\theta)\eta(d\theta) = \mathbb{E}\bigg[\int_0^\infty \alpha_\theta(\theta)\eta(d\theta) \,\bigg|\, \mathcal{F}_t \bigg], \quad t \geq 0$$

is a square integrable $\mathbb{F}$-martingale since

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\int_0^\infty \alpha_\theta(\theta)\eta(d\theta)\bigg)^2\bigg] \\
&= 2\mathbb{E}\bigg[\int_0^\infty \eta(d\theta)\alpha_\theta(\theta)\int_\theta^\infty \eta(du)\alpha_u(u)\bigg] \\
&= 2\mathbb{E}\bigg[\int_0^\infty \alpha_\theta(\theta)\mathbb{E}[\mathbb{1}_{H\cap\{\tau>\theta\}}|\mathcal{F}_\theta]\eta(d\theta)\bigg] \\
&\le 2.
\end{aligned}$$

Moreover, one has

$$\begin{aligned}
\mathbb{1}_{\{\tau_i>t\}}p_t^i &= \mathbb{1}_{\{\tau_i>0\}}p_0^i + \int_{(0,t]}\mathbb{1}_{\{\tau_i\ge s\}}dp_s^i - \int_{(0,t]}p_{s-}^i dD_s^i - [D^i,p^i]_t, \\
&= \mathbb{1}_{\{\tau_i>0\}}p_0^i + \int_{(0,t]}\mathbb{1}_{\{\tau_i\ge s\}}dp_s^i - \int_{(0,t]}p_{s-}^i dM_s^i - \int_{(0,t]}p_{s-}^i d\Lambda_s^i - [D^i,p^i]_t,
\end{aligned}$$

where

$$[D^i,p^i]_t = \sum_{0<s\le t}\Delta D_s^i\Delta p_s^i = \mathbb{1}_{\{\tau_i\le t\}}\Delta p_{\tau_i}^i.$$

One can also rewrite $[D^i,p^i]$ as

$$\begin{aligned}
[D^i,p^i] &= [\Lambda^i,p^i] + [M^i,p^i] \\
&= [\Lambda^i,p^i] + ([M^i,p^i] - \langle M^i,p^i\rangle) + \langle M^i,p^i\rangle.
\end{aligned}$$

Note that $[\Lambda^i,p^i]$ is an $\mathbb{F}$-martingale since Λ^i is $\mathbb{F}$-predictible and p^i is an $\mathbb{F}$-martingale (see [DM80, VIII.19]). Moreover $\langle M^i,p^i\rangle$ is an $\mathbb{F}$-predictable process such that $[M^i,p^i] - \langle M^i,p^i\rangle$ is an $\mathbb{F}$-martingale. Therefore we obtain that

$$A_t = C_t + \int_{(0,t]} p_{s-}^i d\Lambda_s^i + \langle M^i,p^i\rangle_t, \quad t\ge 0$$

is a predictable process, and $G+A$ is an $\mathbb{F}$-martingale. □

In the following, we denote by $\Lambda^{\mathbb{F}}$ the process

$$\bigg(\Lambda_t^{\mathbb{F}} := \int_{(0,t]}\frac{\mathbb{1}_{\{G_{s-}>0\}}}{G_{s-}}dA_s, \quad t\ge 0\bigg) \tag{2.20}$$

which is an $\mathbb{F}$-predictable process. By Theorem A.11, the $\mathbb{G}$-compensator of τ is

$$\Lambda^{\mathbb{G}} = (\Lambda_t^{\mathbb{G}}, t\ge 0) = (\Lambda_{\tau\wedge t}^{\mathbb{F}}, t\ge 0).$$

More precisely,

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \mathbb{1}_{\{G_{s-}>0\}} \frac{\alpha_s(s)\eta(ds)}{G_{s-}} + \sum_{i=1}^{n} \int_{(0,t\wedge\tau]} \mathbb{1}_{\{G_{s-}>0\}} \frac{p_{s-}^i d\Lambda_s^i + d\langle M^i, p^i\rangle_s}{G_{s-}}. \tag{2.21}$$

We observe from (2.21) that the compensator $\Lambda^{\mathbb{G}}$ is in general a discontinuous process and the intensity does not exist in this case. Remember that a general model where the Azéma supermartingale is discontinuous has also been studied in [BSW04, GS16].

We can deal with general $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$(see Remark 2.5). On the one hand, if they are predictable $\mathbb{F}$-stopping times, then $\Lambda_t^i = \mathbb{1}_{\{\tau_i \le t\}}$ and $M_t^i = 0$, so the last term on the right-hand side of (2.19) vanishes and we obtain

$$\Lambda_t^{\mathbb{G}} = \int_0^{t\wedge\tau} \mathbb{1}_{\{G_{s-}>0\}} \frac{\alpha_s(s)\eta(ds)}{G_{s-}} + \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \le t\wedge\tau, G_{\tau_i -}>0\}} \frac{p_{\tau_i -}^i}{G_{\tau_i -}}. \tag{2.22}$$

On the other hand, if $\{\tau_i\}_{i=1}^N$ are totally inaccessible $\mathbb{F}$-stopping times, then τ is a totally inaccessible $\mathbb{G}$-stopping time. In this case, the compensator process of τ is continuous. A similar result can be found in Coculescu [Coc09].

Proposition 2.11. *If $(\tau_i)_{i=1}^n$ is a family of totally inaccessible $\mathbb{F}$-stopping times, then τ is a totally inaccessible $\mathbb{G}$-stopping time.*

PROOF: Since τ_i is totally inaccessible, the $\mathbb{F}$-compensator process Λ^i is continuous. Moreover, $\langle M^i, p^i\rangle$ is the compensator of the process

$$[D^i, p^i] = \left(\mathbb{1}_{\{\tau_i \le t\}} \Delta p_{\tau_i}^i, t \ge 0\right)$$

and hence is continuous (see [DM80, VI.78] and the second part of its proof for details). Therefore the process A in the Doob-Meyer decomposition of G is continuous since η is non-atomic. This implies that the $\mathbb{F}$-compensator $\Lambda^{\mathbb{F}}$ is continuous. Thus the process

$$\left(\mathbb{1}_{\{\tau > t\}} + \Lambda_{\tau\wedge t}^{\mathbb{F}}, t \ge 0\right)$$

is a uniformly integrable $\mathbb{G}$-martingale, which is continuous outside the graph of τ, and has jump of size 1 at τ. Still by [DM80, VI.78], τ is a totally inaccessible $\mathbb{G}$-stopping time. □

Recall that there exists a unique multiplicative decomposition of the Azéma supermartingale (e.g. [IW65]). In the following, we give the explicit multiplicative decomposition under the generalised density hypothesis as a general case of [EKJJ10, Proposition 4.1 (2)].

Proposition 2.12. *Let $\xi := \inf\{t > 0 : G_t = 0\}$ and denote by $\Lambda^{\mathbb{F},c}$ the continuous part of $\Lambda^{\mathbb{F}}$. The multiplicative decomposition of the Azéma supermartingale G is given by*

$$G_t = L_t e^{-\Lambda_t^{\mathbb{F},c}} \prod_{0<u\leq t} (1 - \Delta\Lambda_u^{\mathbb{F}}), \quad t \geq 0, \tag{2.23}$$

where L is an $\mathbb{F}$-martingale solution of the stochastic differential equation

$$L_t = 1 + \int_{(0,t\wedge\xi]} \frac{L_{s-}}{(1 - \Delta\Lambda_s^{\mathbb{F}})G_{s-}} dM_s, \quad t \geq 0. \tag{2.24}$$

Proof. For any process X on $(\Omega, \mathcal{A}, \mathbb{P})$, we denote by ${}^oX$ and ${}^pX$ respectively the $\mathbb{F}$-optional projection and $\mathbb{F}$-predictable projection of X. We have the relations

$$\Delta A = {}^o(\mathbb{1}_{[\![0,\tau]\!]}) - {}^o(\mathbb{1}_{[\![0,\tau[\![})$$

and $G_- = {}^p(\mathbb{1}_{]\!]0,\tau]\!]})$ on the set $\{0 < \tau < \infty\}$ (Jeulin [Jeu80, page 63]). Since $\Delta\Lambda^{\mathbb{F}}$ is $\mathbb{F}$-predictable, we have

$$\Delta\Lambda^{\mathbb{F}} = {}^p(\Delta\Lambda^{\mathbb{F}}) = {}^p\left(\frac{\Delta A}{G_-}\right) = {}^p\left(\frac{{}^o(\mathbb{1}_{[\![0,\tau]\!]}) - {}^o(\mathbb{1}_{[\![0,\tau[\![})}{G_-}\right) = \frac{{}^p(\mathbb{1}_{[\![\tau]\!]})}{{}^p(\mathbb{1}_{[\![0,\tau]\!]})} \leq 1.$$

On the one hand, for any $t \geq 0$, if there exists $u \in (0, t]$ such that $\Delta\Lambda_u^{\mathbb{F}} = 1$, making the right-hand side of (2.23) vanish, then we have ${}^p(\mathbb{1}_{[\![0,\tau[\![})_u = 0$, which implies that $G_u = 0$. It is a classic result that G is a nonnegative supermartingale which sticks at 0 (c.f. [Pro05, page 379]), then $G_t = 0$. On the other hand, if $\Delta\Lambda^{\mathbb{F}} < 1$, we denote by $M^{\mathbb{F}}$ the $\mathbb{F}$-martingale defined as

$$dM_t^{\mathbb{F}} = \frac{\mathbb{1}_{\{G_{t-}>0\}}}{G_{t-}} dM_t.$$

Let $S = M^{\mathbb{F}} - \Lambda^{\mathbb{F}}$. Then one has

$$G_t = G_0 + \int_{(0,t]} G_{u-} dS_u$$

for all $t \in \mathbb{R}_+$. By [Jac79, Corollaire 6.35], $G = \mathcal{E}(S) = L\mathcal{E}(-\Lambda^{\mathbb{F}})$, where $L = \mathcal{E}(\tilde{M}^{\mathbb{F}})$ such that

$$d\tilde{M}_t^{\mathbb{F}} = \frac{\mathbb{1}_{\{0<t\leq\xi\}}}{1-\Delta\Lambda_t^{\mathbb{F}}} dM_t^{\mathbb{F}}$$

(here we use the fact that $\xi = \inf\{t > 0 : \Delta S_t = -1\}$ and $-\Delta\Lambda^{\mathbb{F}} \neq -1$ on $]\!]0, \xi]\!]$). Then, L is the solution of

$$L_t = 1 + \int_{(0,t]} L_{s-} d\tilde{M}_s^{\mathbb{F}}, \quad t \geq 0.$$

The proposition is thus proved. □

2.4 Sovereign default model revisited

The generalised density approach provides a general setting for hybrid default models. In particular, the sovereign default model that we have developed in the previous chapter is also a special case which satisfies the generalised density hypothesis.

Proposition 2.13. *The random time τ defined in* (1.14) *satisfies Assumption 2.4, and for all $u, t \in \mathbb{R}_+$, the generalised $\mathbb{F}$-density $\alpha(\cdot)$ is given by*

$$\alpha_t(u) = \mathbb{E}\Big[\lambda_u \exp\Big(-\int_0^u \lambda_s\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds\Big)\Big|\mathcal{F}_t\Big]. \tag{2.25}$$

The proof of the proposition above is given in Appendix B.

We can check that the compensator process satisfies (2.22), which gives the same result as in (1.23).

By using the same argument, one can also prove that if the random threshold η in the Cox process model (1.9) satisfies Assumption 1.15, the sovereign default time still satisfies the generalised density hypothesis, and for all $u, t \in \mathbb{R}_+$, the generalised $\mathbb{F}$-density $\alpha(\cdot)$ is given by

$$\alpha_t(u) = \begin{cases} \lambda_u \beta_t(\Lambda_u) e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds}, & \text{if } u \leq t, \\ \mathbb{E}\left[\lambda_u \beta_u(\Lambda_u) e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds} \Big| \mathcal{F}_t\right], & \text{if } u > t. \end{cases} \tag{2.26}$$

In this case, the compensator process also satisfies (2.22), which yields the same result as in (1.39).

2.5 Martingales and semimartingales in $\mathbb{G}$

In this section, we are interested in the $\mathbb{G}$-martingales and semimartingales. We characterise firstly the $\mathbb{G}$-martingales by using $\mathbb{F}$-martingale conditions, as done in [EKJJ10, Proposition 5.6]. However, under the generalised density hypothesis, we shall distinguish the necessary and sufficient conditions although they have similar forms at the first sight. As a matter of fact, the decomposition of a $\mathbb{G}$-adapted process is not unique, and the martingale property can not hold true for all modifications. This makes the necessary and sufficient conditions subtily different.

Proposition 2.14. *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-adapted process, which is written in the decomposed form $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\leq t\}}Y_t(\tau), t \geq 0$, $\mathbb{P}$-a.s., where Y is an $\mathbb{F}$-adapted process and $Y(\cdot)$ is an $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted process. Then $Y^{\mathbb{G}}$ is a $\mathbb{G}$-(local) martingale if the following conditions are verified:*

(a) $\mathbb{1}_{\cap_{i=1}^n\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ is an $\mathbb{F}$-(local) martingale on $[\theta,\infty)$ for any $\theta\in\mathbb{R}_+$;

(b) $Y(\tau_i)p^i$ is an $\mathbb{F}$-(local) martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,n\}$;

(c) the process

$$\left(Y_tG_t + \int_0^t Y_u(u)\alpha_u(u)\eta(du) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\leq t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i},\, t\geq 0\right)$$

is an $\mathbb{F}$-(local) martingale.

Proof. We treat firstly the martingale case. By Proposition 2.8, the conditional expectation $\mathbb{E}[Y_T^{\mathbb{G}}|\mathcal{G}_t]$ can be written as the sum of

$$\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\mathbb{E}[\mathbb{1}_{\{\tau>T\}}Y_T + \mathbb{1}_{\{t<\tau\leq T\}}Y_T(\tau)|\mathcal{F}_t] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\Bigg(\mathbb{E}[Y_TG_T|\mathcal{F}_t] + \int_t^T \mathbb{E}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\eta(du) + \sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{t<\tau_i\leq T\}}Y_T(\tau_i)p^i_T|\mathcal{F}_t]\Bigg)$$

and

$$\mathbb{1}_{\{\tau\leq t\}}\left(\mathbb{1}_{\cap_{i=1}^n\{\tau\neq\tau_i\}}\frac{\mathbb{1}_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]_{\theta=\tau} + \sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}}\frac{\mathbb{1}_{\{p^i_t>0\}}}{p^i_t}\mathbb{E}[Y_T(\tau_i)p^i_T|\mathcal{F}_t]\right).$$

Hence, $\mathbb{E}[Y_T^{\mathbb{G}}|\mathcal{G}_t] - Y_t^{\mathbb{G}}$ equals the sum of the following terms

$$\begin{aligned}\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\bigg(&\mathbb{E}\left[Y_T G_T - Y_t G_t|\mathcal{F}_t\right] + \int_t^T \mathbb{E}\left[Y_T(u)\alpha_T(u)|\mathcal{F}_t\right]\eta(du)\\ &+\sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{t<\tau_i\leq T\}}Y_T(\tau_i)p_T^i|\mathcal{F}_t]\bigg)\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\mathbb{1}_{\{\tau\leq t\}}\bigg(&-Y_t(\tau) + \mathbb{1}_{\cap_{i=1}^n\{\tau\neq\tau_i\}}\frac{\mathbb{1}_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]_{\theta=\tau}\\ &+\sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}}\frac{\mathbb{1}_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}[Y_T(\tau_i)p_T^i|\mathcal{F}_t]\bigg).\end{aligned} \tag{2.28}$$

Since the measure η is non-atomic, one has

$$\int_t^T \mathbb{E}[Y_T(u)\alpha_T(u)|\mathcal{F}_t]\eta(du) = \mathbb{E}\left[\int_t^T \mathbb{1}_{\cap_{i=1}^n\{\tau_i\neq u\}}Y_T(u)\alpha_T(u)\eta(du)\bigg|\mathcal{F}_t\right].$$

By condition (a), it is equal to

$$\left[\int_t^T \mathbb{1}_{\cap_{i=1}^n\{\tau_i\neq u\}}Y_u(u)\alpha_u(u)\eta(du)\bigg|\mathcal{F}_t\right] = \left[\int_t^T Y_u(u)\alpha_u(u)\eta(du)\bigg|\mathcal{F}_t\right],$$

where we use again the fact that η is non-atomic. Therefore, by condition (b), one can rewrite the term (2.27) as

$$\begin{aligned}\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\bigg(&\mathbb{E}\left[Y_T G_T - Y_t G_t|\mathcal{F}_t\right] + \mathbb{E}\left[\int_t^T Y_u(u)\alpha_u(u)\eta(du)\bigg|\mathcal{F}_t\right]\\ &+\sum_{i=1}^n \mathbb{E}[\mathbb{1}_{\{t<\tau_i\leq T\}}Y_{\tau_i}(\tau_i)p_{\tau_i}^i|\mathcal{F}_t]\bigg),\end{aligned} \tag{2.29}$$

which vanishes thanks to condition (c). Moreover, by condition (a) and (b), we can rewrite (2.28) as

$$\mathbb{1}_{\{\tau\leq t\}}\left(-Y_t(\tau) + \mathbb{1}_{\cap_{i=1}^n\{\tau\neq\tau_i\}}\frac{\mathbb{1}_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}Y_t(\tau)\alpha_t(\tau) + \sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}}\frac{\mathbb{1}_{\{p_t^i>0\}}}{p_t^i}Y_t(\tau_i)p_t^i\right),$$

which also vanishes.

In the following, we treat the local martingale case. Assume that the processes in conditions (a)-(c) are $\mathbb{F}$-local martingales, then there exists a common sequence of $\mathbb{F}$-stopping times which localises the processes (a)-(c) simultaneously. Thus, it remains to prove the following claim: assume that σ is an $\mathbb{F}$-stopping time such that

(1) $1\!\!1_{\cap_{i=1}^n\{\tau_i\neq\theta\}}1\!\!1_{\{\sigma>0\}}Y^\sigma(\theta)\alpha^\sigma(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty)$ for any $\theta\in\mathbb{R}_+$,

(2) $1\!\!1_{\{\sigma>0\}}Y^\sigma(\tau_i)p^{i,\sigma}$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,n\}$,

(3) the process $1\!\!1_{\{\sigma>0\}}(Y_t^\sigma G_t^\sigma+\int_0^{t\wedge\sigma}Y_u(u)\alpha_u(u)\eta(du)+\sum_{i=1}^n 1\!\!1_{\{\tau_i\leq t\wedge\sigma\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i},\ t\geq 0)$ is an $\mathbb{F}$-martingale,

then the process $1\!\!1_{\{\sigma>0\}}Y^{\mathbb{G},\sigma}$ is a $\mathbb{G}$-martingale.

Note that the process $\alpha(\theta)$ and p^i are all $\mathbb{F}$-martingales for $\theta\in\mathbb{R}_+$, $i\in\{1,\dots,n\}$. Therefore, the conditions (1) and (2) imply the corresponding conditions if we replace $\alpha^\sigma(\theta)$ and $p^{i,\sigma}$ by $\alpha(\theta)$ and p^i respectively. We then deduce the following conditions

(1') $1\!\!1_{\cap_{i=1}^n\{\tau_i\neq\theta\}}1\!\!1_{\{\sigma>0\}}\left(1\!\!1_{\{\sigma<\theta\}}Y^\sigma+1\!\!1_{\{\sigma\geq\theta\}}Y^\sigma(\theta)\right)\alpha(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty)$ for any $\theta\in\mathbb{R}_+$,

(2') $1\!\!1_{\{\sigma>0\}}\left(1\!\!1_{\{\sigma<\tau_i\}}Y^\sigma+1\!\!1_{\{\sigma\geq\tau_i\}}Y^\sigma(\tau_i)\right)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,n\}$,

(3') the process

$$1\!\!1_{\{\sigma>0\}}\bigg(Y_t^\sigma G_t^\sigma+\int_0^{t\wedge\sigma}(1\!\!1_{\{\sigma<u\}}Y_u^\sigma+1\!\!1_{\{\sigma\geq u\}}Y_u^\sigma(u))\alpha_u(u)\eta(du)\\ +\sum_{i=1}^n 1\!\!1_{\{\tau_i\leq t\wedge\sigma\}}(1\!\!1_{\{\sigma<\tau_i\}}Y_{\tau_i}^\sigma+1\!\!1_{\{\sigma\geq\tau_i\}}Y_{\tau_i}^\sigma(\tau_i))p^i_{\tau_i}\bigg),\quad t\geq 0$$

is an $\mathbb{F}$-martingale.

One has $1\!\!1_{\{\sigma<\theta\}}Y_t^\sigma=1\!\!1_{\{\sigma<\theta\}}Y_\sigma$ on $[\theta,\infty)$ and hence

$$\begin{aligned}&\Big(1\!\!1_{\{\sigma<\theta\}}Y_t^\sigma+1\!\!1_{\{\sigma\geq\theta\}}Y_t^\sigma(\theta)\Big)\alpha_t(\theta)-Y_t^\sigma(\theta)\alpha_t^\sigma(\theta)\\ &=1\!\!1_{\{\sigma<\theta\}}\Big(Y_\sigma\alpha_t(\theta)-Y_\sigma(\theta)\alpha_\sigma(\theta)\Big)+1\!\!1_{\{\sigma\geq\theta\}}Y_{t\wedge\sigma}(\theta)\Big(\alpha_t(\theta)-\alpha_{t\wedge\sigma}(\theta)\Big)\\ &=1\!\!1_{\{\sigma<\theta\}}\Big(Y_\sigma\alpha_t(\theta)-Y_\sigma(\theta)\alpha_\sigma(\theta)\Big)+1\!\!1_{\{\sigma\geq\theta\}}Y_\sigma(\theta)\Big(\alpha_t(\theta)-\alpha_{t\wedge\sigma}(\theta)\Big),\quad t\geq\theta\end{aligned}$$

is an $\mathbb{F}$-martingale, which implies that (1) leads to (1'). Similarly, one has $1\!\!1_{\tau_i>\sigma}Y_t^\sigma=1\!\!1_{\{\tau_i>\sigma\}}Y_\sigma$ on $[\![\tau_i,\infty[\![$ and hence (2) leads to (2'). Finally, by (2.9) we obtain that the process

$$\left(G_t+\int_0^t\alpha_u(u)\eta(du)+\sum_{i=1}^n 1\!\!1_{\{\tau_i\leq t\}}p^i_{\tau_i},t\geq 0\right)$$

is an $\mathbb{F}$-martingale and hence the process

$$1\!\!1_{\{\sigma>0\}}Y_\sigma\left(G_t-G_t^\sigma+\int_{t\wedge\sigma}^t\alpha_u(u)\eta(du)+\sum_{i=1}^n 1\!\!1_{\{t\wedge\sigma<\tau_i\leq t\}}p^i_{\tau_i}\right),\quad t\geq 0$$

is also an $\mathbb{F}$-martingale. Hence the conditions (3) leads to (3'). By the martingale case of the proposition proved above, applied to the process

$$\mathbb{1}_{\{\sigma>0\}}Y_t^{\mathbb{G},\sigma} = \mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\sigma>0\}}Y_t^{\sigma} + \mathbb{1}_{\{\tau\le t\}}\mathbb{1}_{\{\sigma>0\}}\left(\mathbb{1}_{\{\tau>\sigma\}}Y_t^{\sigma} + \mathbb{1}_{\{\tau\le\sigma\}}Y_t^{\sigma}(\tau)\right),$$

we obtain that $\mathbb{1}_{\{\sigma>0\}}Y^{\mathbb{G},\sigma}$ is a $\mathbb{G}$-martingale. In fact, if we replace in the conditions (a)-(c) the process Y by $\mathbb{1}_{\{\sigma>0\}}Y^{\sigma}$, and $Y_t(\theta)$ by $\mathbb{1}_{\{\sigma>0\}}\left(\mathbb{1}_{\{\sigma<\theta\}}Y^{\sigma} + \mathbb{1}_{\{\sigma\ge\theta\}}Y^{\sigma}(\theta)\right)$, then the conditions (a)-(c) become (1')-(3'). The proposition is thus proved. □

In view of Proposition 2.14, it is natural to examine whether the converse is true. However, given a $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$, the decomposition $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\le t\}}Y_t(\tau)$, $\mathbb{P}$-a.s. is not unique. For example, if one modifies arbitrarily the value of $Y(\theta)$ on $\bigcap_{i=1}^{N}\{\tau_i \neq \theta\}$ for θ in an η-negligiable set, the decomposition equality remains valid. However, the $\mathbb{F}$-martingale property of $\mathbb{1}_{\cap_{i=1}^{n}\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ cannot hold for all such modifications. In the following, we prove that, if $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, then one can find at least one decomposition of $Y^{\mathbb{G}}$ such that Y and $Y(\cdot)$ satisfy the $\mathbb{F}$-martingale conditions in Proposition 2.14.

Proposition 2.15. *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-martingale. There exist a càdlàg $\mathbb{F}$-adapted process Y and an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable processes $Y(\cdot)$ which verify the following conditions :*

(a) $\mathbb{1}_{\cap_{i=1}^{n}\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty[$ for any $\theta\in\mathbb{R}_+$;

(b) $Y(\tau_i)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,n\}$;

(c) the process

$$\left(Y_tG_t + \int_0^t Y_u(u)\alpha_u(u)\eta(du) + \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i\le t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}, t\ge 0\right)$$

is an $\mathbb{F}$-martingale;

and such that, for any $t\ge 0$, one has $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\le t\}}Y_t(\tau), t\ge 0$, $\mathbb{P}$-a.s..

Proof. The process $Y^{\mathbb{G}}$ can be written in the following decomposition form

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\tilde{Y}_t + \mathbb{1}_{\{\tau\le t\}}\hat{Y}_t(\tau), \tag{2.30}$$

where $\tilde{Y}$ and $\hat{Y}(\cdot)$ are respectively $\mathbb{F}$-adpated and $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted processes. Since $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, for $i \in \{1, \cdots, n\}$ and $0 \leq t \leq T$, one has

$$\mathbb{E}[Y_T^{\mathbb{G}} \mathbb{1}_{\{\tau=\tau_i \leq t\}} | \mathcal{F}_t] = \mathbb{E}[Y_t^{\mathbb{G}} \mathbb{1}_{\{\tau=\tau_i \leq t\}} | \mathcal{F}_t],$$

which implies

$$\mathbb{1}_{\{\tau_i \leq t\}} \mathbb{E}[\hat{Y}_T(\tau_i) p_T^i | \mathcal{F}_t] = \mathbb{1}_{\{\tau_i \leq t\}} \hat{Y}_t(\tau_i) p_t^i.$$

This equality shows that $\hat{Y}(\tau_i) p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i, \infty[\![$. We take a càdlàg version of this martingale and replace $\hat{Y}(\tau_i)$ on $[\![\tau_i, \infty[\![$ by the càdlàg version of this martingale multiplied by $\mathbb{1}_{\{p^i > 0\}} (p^i)^{-1}$. This gives an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable version of $\hat{Y}(\cdot)$ and the equality (2.30) remains true $\mathbb{P}$-almost surely.

Similarly, for $0 \leq t \leq T$, one has

$$\mathbb{E}[Y_T^{\mathbb{G}} \mathbb{1}_{\{\tau \leq t\}} \mathbb{1}_{\cap_{i=1}^n \{\tau \neq \tau_i\}} | \mathcal{F}_t] = \mathbb{E}[Y_t^{\mathbb{G}} \mathbb{1}_{\{\tau \leq t\}} \mathbb{1}_{\cap_{i=1}^n \{\tau \neq \tau_i\}} | \mathcal{F}_t],$$

which implies

$$\int_0^t \mathbb{E}[\hat{Y}_T(\theta) \alpha_T(\theta) | \mathcal{F}_t] \, \eta(d\theta) = \int_0^t \hat{Y}_t(\theta) \alpha_t(\theta) \eta(d\theta). \tag{2.31}$$

Let D be a countable dense subset of $\mathbb{R}_+$. For any $\theta \in \mathbb{R}_+$ and all $s, t \in D$ such that $\theta \leq s \leq t$, let

$$\hat{Y}_{t|s}(\theta) = \frac{\mathbb{1}_{\{\alpha_s(\theta) > 0\}}}{\alpha_s(\theta)} \mathbb{E}[\hat{Y}_t(\theta) \alpha_t(\theta) | \mathcal{F}_s].$$

The equality (2.31) shows that there exists an η-negligeable Borel subset B of $\mathbb{R}_+$ such that $\hat{Y}_{t|s}(\theta) \alpha_s(\theta) = \mathbb{E}[\hat{Y}_t(\theta) \alpha_t(\theta) | \mathcal{F}_s]$ provided that $\theta \notin B$. By the same arguments as in the proof of Proposition 2.3, we obtain a càdlàg $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted process $Y(\cdot)$ verifying the conditions (a) and (b), and such that $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} \tilde{Y}_t + \mathbb{1}_{\{\tau \leq t\}} Y_t(\tau)$, $\mathbb{P}$-a.s..

For the last condition (c), for any $t \geq 0$, let

$$Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}} | \mathcal{F}_t] = \tilde{Y}_t G_t + \int_0^t Y_t(\theta) \alpha_t(\theta) \, \eta(d\theta) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} Y_t(\tau_i) p_t^i.$$

The process $Y^{\mathbb{F}}$ is an $\mathbb{F}$-martingale. Since the conditions (a) and (b) hold for $Y(\cdot)$, we obtain that the process

$$\tilde{Y}_t G_t + \int_0^t Y_\theta(\theta) \alpha_\theta(\theta) \, \eta(d\theta) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} Y_{\tau_i}(\tau_i) p_{\tau_i}^i, \quad t \geq 0$$

is also an $\mathbb{F}$-martingale. Let Z be a càdlàg version of this $\mathbb{F}$-martingale and let

$$Y_t = \frac{\mathbb{1}_{\{G_t>0\}}}{G_t}\left(Z_t - \int_0^t Y_\theta(\theta)\alpha_\theta(\theta)\,\eta(d\theta) - \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}\right), \quad t\ge 0$$

which is a càdlàg version of the process $\tilde{Y}$. The equality $Y^{\mathbb{G}}_t = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\le t\}}Y_t(\tau)$, $\mathbb{P}$-a.s. still holds. The result is thus proved. □

In the theory of enlargement of filtrations, it is a classic problem to study whether an $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale. The standard assumption under which this assertion holds true is the density hypothesis (c.f. [Jac85, Section 2] in the initial enlargement of filtrations and [EKJJ10, Proposition 5.9] and [JLC09b, Theorem 3.1] in the progressive enlargement of filtrations). Aksamit, Choulli and Jeanblanc [ACJ16, Theorem 3.5] proves this stability in a general framework. We now give an affirmative answer under the generalised density hypothesis, which provides a weaker condition.

Proposition 2.16. *Any $\mathbb{F}$-local martingale $U^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale which has the following decomposition:*

$$\begin{aligned} U^{\mathbb{F}}_t = U^{\mathbb{G}}_t &+ \int_{(0,t\wedge\tau]} \frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}} \\ &+ \mathbb{1}_{\cap_{i=1}^n\{\tau\neq\tau_i\}} \int_{(\tau,t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, \alpha(u)\rangle_s}{\alpha_{s-}(u)}\bigg|_{u=\tau} + \sum_{i=1}^n \mathbb{1}_{\{\tau=\tau_i\}} \int_{(\tau,t\vee\tau]} \frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p^i_{s-}}, \end{aligned} \tag{2.32}$$

where $U^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale and $\bar{M}$ is an $\mathbb{F}$-martingale defined as

$$\bar{M}_t = \mathbb{E}\left[\int_0^\infty \alpha_u(u)\eta(du)\bigg|\mathcal{F}_t\right] + \sum_{i=1}^n p^i_{t\wedge\tau_i} + p^\infty_t, \quad t\ge 0. \tag{2.33}$$

Proof. Let

$$\bar{A}_t = \int_0^t \alpha_u(u)\eta(u) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le t\}}p^i_{\tau_i}.$$

One has $G = \bar{M} - \bar{A}$. We denote by

$$K_t = \int_{(0,t]} \frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}}$$

and for $\theta\le t$,

$$K_t(\theta) = \mathbb{1}_{\cap_{i=1}^n\{\theta\neq\tau_i\}} \int_{(\theta,t]} \frac{d\langle U^{\mathbb{F}}, \alpha(\theta)\rangle_s}{\alpha_{s-}(\theta)} + \sum_{i=1}^n \mathbb{1}_{\{\theta=\tau_i\}} \int_{(\theta,t]} \frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p^i_{s-}}.$$

We define the process $U^{\mathbb{G}}$ as

$$\begin{aligned} U_t^{\mathbb{G}} &= \mathbb{1}_{\{\tau>t\}}\left(U_t^{\mathbb{F}} - K_t\right) + \mathbb{1}_{\{\tau\le t\}}\left(U_t^{\mathbb{F}} - K_\tau - K_t(\tau)\right) \\ &= \mathbb{1}_{\{\tau>t\}}\tilde{U}_t + \mathbb{1}_{\{\tau\le t\}}\hat{U}_t(\tau), \end{aligned}$$

where $\tilde{U}_t = U_t^{\mathbb{F}} - K_t$ and $\hat{U}_t(\theta) = U_t^{\mathbb{F}} - K_\theta - K_t(\theta)$. We check firstly that $\tilde{U}$ and $\hat{U}(\cdot)$ verify the condition (c) in Proposition 2.14. Let $Z = (Z_t, t \ge 0)$ be a process defined as

$$Z_t = \tilde{U}_t G_t + \int_0^t \hat{U}_u(u) d\bar{A}_u, \quad t \ge 0.$$

Then

$$\begin{aligned} dZ_t &= d(\tilde{U}_t G_t) + \hat{U}_t(t) d\bar{A}_t \\ &= d(U_t^{\mathbb{F}} G_t) - d(K_t G_t) + (U_t^{\mathbb{F}} - K_t) d\bar{A}_t \\ &= U_{t-}^{\mathbb{F}} dG_t + G_{t-} dU_t^{\mathbb{F}} + d[U^{\mathbb{F}}, G]_t - K_t dG_t - G_{t-} dK_t + U_{t-}^{\mathbb{F}} d\bar{A}_t - K_t d\bar{A}_t + d[U^{\mathbb{F}}, \bar{A}]_t \\ &= (U_{t-}^{\mathbb{F}} - K_t) d\bar{M}_t + G_{t-} dU_t^{\mathbb{F}} + d[U^{\mathbb{F}}, \bar{M}]_t - d\langle U^{\mathbb{F}}, \bar{M}\rangle_t. \end{aligned}$$

Therefore Z is an $\mathbb{F}$-local martingale.

We check now the conditions (a) and (b) in Proposition 2.14. On the set

$$\{\alpha_t(\theta) > 0\} \cap \bigcap_{i=1}^n \{\theta \ne \tau_i\}$$

one has

$$d\left(\hat{U}_t(\theta)\alpha_t(\theta)\right) = \left(U_{t-}^{\mathbb{F}} - K_\theta - K_t(\theta)\right) d\alpha_t(\theta) + \alpha_{t-}(\theta) dU_t^{\mathbb{F}} + d[U^{\mathbb{F}}, \alpha(\theta)]_t - d\langle U^{\mathbb{F}}, \alpha(\theta)\rangle_t, \quad \theta \le t$$

and on the set $\{\tau_i \le t\} \cap \{p_t^i > 0\}$ for all $i \in \{1, \dots, n\}$,

$$d\left(\hat{U}_t(\tau_i) p_t^i\right) = \left(U_{t-}^{\mathbb{F}} - K_{\tau_i} - K_t(\tau_i)\right) dp_t^i + p_{t-}^i dU_t^{\mathbb{F}} + d[U^{\mathbb{F}}, p^i]_t - d\langle U^{\mathbb{F}}, p^i\rangle_t.$$

Therefore the process $\mathbb{1}_{\cap_{i=1}^n\{\theta\ne\tau_i\}}\hat{U}(\theta)\alpha(\theta)$ is an $\mathbb{F}$-local martingale on $[\theta, \infty)$, and the process $\hat{U}(\tau_i)p^i$ is an $\mathbb{F}$-local martingale on $[\![\tau_i, \infty[\![$ for all $i \in \{1, \dots, n\}$. By Proposition 2.14, we obtain that $U^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. □

Remark 2.17. We note that the decomposition $G = \bar{M} - \bar{A}$ in the proof of the above proposition is different from the Doob-Meyer decomposition of G since $\bar{A}$ is an $\mathbb{F}$-optional process. However, if $\mathbb{F}$ is quasi left continuous, this decomposition coincides with the Doob-Meyer decomposition.

Chapter 3

Applications of sovereign default risk models in finance

As applications of the generalised density approach, we discuss about the sufficient condition for the immersion property in the generalised density framework, which is widely adopted in the credit risk models, and study a two-name default model with simultaneous defaults. Finally we apply the hybrid model and the generalised density to the valuation of sovereign default claims and explain the significant jumps in the long-term government bond yield during the sovereign crisis.

Keywords : Immersion property, simultaneous defaults, long-term government bond.

3.1 Introduction

In this chapter, we apply the results of the previous chapters to the valuation and hedgeing of credit risks. We first discuss about the immersion property (H-hypothesis) which is widely adopted in the credit risk models in the framework of the progressive enlargement of filtrations to guarantee the no-arbitrage conditions while we valuate some defaultable claims by using a risk-neutral probability measure. Elliott, Jeanblanc and Yor [EJY00] survey classic sufficient and necessary conditions for immersion property, which is satisfied by, among others, the Cox process model. El Karoui, Jeanblanc and Jiao [EKJJ10] propose another equivalent condition for immersion property in the density framework, which postulates that the immersion property holds if and only if the density process $\alpha(\cdot)$ satisfies the condition $\alpha_t(\theta) = \alpha_\theta(\theta)$ for any $t > \theta$. We extend this last condition to the generalised density framework and underline the converse is not ture in general.

The financial justification of the links between the immersion property and no-arbitrage conditions has been studied by Jeanblanc and Le Cam [JLC09b] and Coculescu, Jeanblanc and Nikeghbali [CJN12]. Precisely, if the immersion property is satisfied under the risk-neutral probability measure, then the market model is arbitrage-free. In [JLC09b], the credit event is modelled by a random time satisfying the density hypothesis, which insures that the semimartingales in the reference filtration remain semimartingales in the enlarged filtration, which opens a door to a change of probability measure by using Girsanov theorem. In [CJN12], the authors have also pointed out how the immersion property is modified under an equivalent change of probability measure. In the generalised density framework studied in Chapter 2, we prove that the generalised density hypothesis holds under an equivalent change of probability measure, and that we can always find an equivalent probability measure under which the immersion property holds true, which means that the generalised density hypothesis implies no-arbitrage conditions.

As soon as the no-arbitrage condition and the change of probability are clear in the generalised density framework, we apply the sovereign default model and the generalised density approach to the valuation of defaultable claims and particularly sovereign zero-coupon bonds and credit default swaps. The pre-default value of a defaultable zero-coupon bond is composed of two parts: one related to the face value promised at maturity and the other related to the recovery payment. We show that the bond yield deduced in the model can have large jumps at the critical dates, which allows to explain the significant movements of the sovereign bond yield during the sovereign debt crisis.

Since the sovereign debt crisis is contagious, we also study extremal risks such as simultaneous defaults whose occurrence is rare but will have significant impact on financial market.

Finally, we study an portofolio maximisation problem under a hybrid model.

3.2 Valuation of defaultable claims

In this section, we apply the sovereign default model in Chapter 1 and results of the generalised density approach in Chapter 2 to sovereign defaultable claims such as sovereign bonds and sovereign credit default swaps. We are particularly interested in the behaviour of long-term bond yield during the sovereign debt crisis and we show that the hybrid model provides an explanation to the jump behaviours of the bond yield around critical political event dates.

3.2.1 Sovereign bond and credit spread

We consider a defaultable sovereign zero-coupon bond of maturity $T \in \mathbb{R}_+$ with face value 1. The recovery payment at default is represented by an $\mathbb{F}$-predictable process R valued in $[0, 1)$ if the sovereign default occurs prior to the maturity. The default time τ is defined by the hybrid sovereign default model (1.14), where we recall that the immersion property and the generalised density hypothesis are satisfied (see Subsection 1.2.6). In a financial market with credit risk, when the immersion property holds, the risk-neutral probability in $\mathbb{F}$ is also a risk-neutral probability in $\mathbb{G}$ (c.f. Coculescu, Jeanblanc and Nikeghbali [CJN12]). By Proposition 3.5, the generalised density hypothesis remains valid under an equivalent probability change. For this reason, let $\mathbb{Q}$ be a risk-neutral probability and we assume for the moment that all the dynamics of the sovereign default model are defined under $\mathbb{Q}$. We will see in the next subsection how a change of probability happens when the sovereign default model is not defined under $\mathbb{Q}$. We denote by $r = (r_t, t \geq 0)$ the default-free interest rate process and by $D(t,T)$ the value at $t < T$ of the zero-coupon bond.

Proposition 3.1. *The value of the defaultable zero-coupon bond is given by*

$$D(t,T) = D^0(t,T) + D^1(t,T), \tag{3.1}$$

where D^0 is the pre-default price related to the payment at maturity, computed as

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T (r_s+\lambda_s)ds - \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds\Big)\Big|\mathcal{F}_t\Big], \tag{3.2}$$

and D^1 is related to the recovery payment, given by

$$D^1(t,T) = \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\,\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s\,ds}R_u\alpha_u(u)du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}e^{-\int_t^{\tau_i} r_s\,ds}R_{\tau_i}p^i_{\tau_i}\Big|\mathcal{F}_t\Big], \tag{3.3}$$

where $G_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$.

Proof: The pre-default value of the bond is given by

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_t^T r_s\,ds}\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t\Big] + \mathbb{E}_{\mathbb{Q}}\Big[e^{\int_t^\tau r_s\,ds}\mathbb{1}_{\{t<\tau\leq T\}}R_\tau|\mathcal{G}_t\Big] \\ &=: D^0(t,T) + D^1(t,T). \end{aligned}$$

The first term $D^0(t,T)$ is obtained by using the key-lemma of the enlargement of filtration:

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\frac{G_T}{G_t}e^{-\int_t^T r_s\,ds}\Big|\mathcal{F}_t\Big]$$

together with (2.9), and the second term results from [BR02, Proposition 5.1.1] as

$$\begin{aligned} D^1(t,T) &= \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{t<\tau\leq T\}}\exp(-\int_t^\tau r_s\,ds)R_\tau|\mathcal{F}_T\Big]\Big|\mathcal{F}_t\Big] \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s\,ds}R_u\alpha_T(u)du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}e^{-\int_t^{\tau_i} r_s\,ds}R_{\tau_i}p^i_T\Big|\mathcal{F}_t\Big]. \end{aligned}$$

We complete the proof by using the equality (2.9) and the following properties:

$$\alpha_T(u) = \alpha_u(u) \text{ for } t \le u \le T \text{ on } \bigcap_{i=1}^{n} \{\tau_i \neq u\},$$
$$p_T^i = p_{\tau_i}^i \text{ on } \{t < \tau_i \le T\} \text{ for any } i \in \{1, \cdots, n\}$$

(see Remark 3.4). □

We are interested in the bond prices on the political critical dates $(\tau_i)_{i=1}^n$ and in particular the jump behavior. Let

$$\Delta D(t,T) := D(t,T) - D(t-,T), \quad t \le T,$$

which is the sum of

$$\Delta D^0(t,T) := D^0(t,T) - D^0(t-,T)$$

and

$$\Delta D^1(t,T) := D^1(t,T) - D^1(t-,T)$$

that we compute respectively in the following. In order to determine the jumps $\Delta D^0(t,T)$ and $\Delta D^1(t,T)$, we assume that the filtration $\mathbb{F}$ only supports continuous martingales. On the one hand,

$$\begin{aligned} D^0(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_0^T (r_s + \lambda_s)\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le T\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\, ds \Big) \Big| \mathcal{F}_t \Big] \\ \cdot \exp\Big(\int_0^t (r_s + \lambda_s)\, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\, ds \Big), \quad t \le T, \end{aligned}$$

where the conditional expectation term on the right-hand side is a continuous process on $t \in [0,T]$ due to the above assumption. Hence

$$\Delta D^0(t,T) = D^0(t,T) \sum_{i=1}^n \mathbb{1}_{\{\tau_i = t\}} \Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\, ds} \Big), \quad \text{on } \{\tau > t\}. \tag{3.4}$$

On the other hand, for the similar reason, we deduce from (3.3) the following formula

$$\begin{aligned} \Delta D^1(t,T) = \Delta(G_t^{-1}) \mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s ds} R_u \alpha_T(u) du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i \le T\}} e^{-\int_t^{\tau_i} r_s\, ds} R_{\tau_i} p_{\tau_i}^i \Big| \mathcal{F}_t \Big] \\ - \frac{1}{G_{t-}} \sum_{i=1}^n \mathbb{1}_{\{\tau_i = t\}} R_{\tau_i} p_{\tau_i}^i, \quad \text{on } \{\tau > t\}. \end{aligned}$$

Moreover, by (2.9) one has

$$\Delta(G_t^{-1}) = \frac{1}{G_t}\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big).$$

We then deduce

$$\Delta D^1(t,T) = D^1(t,T)\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) - \frac{1}{G_{t^-}}\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau > t\}, \tag{3.5}$$

which implies, combining (3.4) and (3.5), that

$$\Delta D(t,T) = \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}D(\tau_i,T)\Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) - \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\frac{1}{G_{\tau_i-}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau > t\}.$$

By using the relation

$$\frac{p^i_{\tau_i}}{G_{\tau_i-}} = 1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds},$$

we obtain finally

$$\Delta D(t,T) = \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(D(\tau_i,T) - R_{\tau_i}\Big)\Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big), \tag{3.6}$$

and in particular,

$$\Delta D(\tau_i,T) = \Big(D(\tau_i,T) - R_{\tau_i}\Big)\Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) \quad \text{on} \quad \{\tau_i \le T\}. \tag{3.7}$$

Let the pre-default yield to maturity of the defaultable bond on $\{t < \tau\}$ be

$$Y^d(t,T) = -\frac{\ln D(t,T)}{T-t}. \tag{3.8}$$

Similarly, the yield to maturity of a classic default-free zero coupon bond is given as

$$Y(t,T) = -\frac{\ln B(t,T)}{T-t}.$$

where $B(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s\,ds}|\mathcal{F}_t]$ denotes the price at $t \le T$ of the default-free zero-coupon bond of maturity T. Let the pre-default credit spread, noted $S(t,T)$, be defined as the difference between the two yields to maturity, i.e.,

$$S(t,T) := Y^d(t,T) - Y(t,T) = -\frac{1}{T-t}\ln\frac{D(t,T)}{B(t,T)}.$$

Then,

$$\Delta S(t,T) = S(t,T) - S(t-,T) = -\frac{\Delta \ln D(t,T)}{T-t} = -\frac{1}{T-t}\ln\left(1+\frac{\Delta D(t,T)}{D(t-,T)}\right). \quad (3.9)$$

which implies by (3.7) that the jump of the bond yield at a critical date τ_i is negative if and only if $\Delta D(\tau_i,T)$ is positive. More precisely, $\Delta S(\tau_i,T) < 0$ on $\{\tau_i < T \wedge \tau\}$ if and only if

$$D(\tau_i,T) > R_{\tau_i} \quad \text{a.s..} \quad (3.10)$$

In practice, the recovery rate is often assumed to be of expectation 0.48 according to Moody's service. Since the bond price is in general higher than this value, the inequality (3.10) is often satisfied, which means that at critical dates, the credit spread is likely to have negative jumps. In particular, if $R \equiv 0$, one has $D^1 = 0$ and

$$\Delta \ln D^0(t,T) = \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i=t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds.$$

Then, we can compute explicitly the jump in the credit spread at τ_i as

$$\Delta S(\tau_i,T) = -\mathbb{1}_{\{\tau_i<T\}}\frac{\Delta \ln D^0(\tau_i,T)}{T-\tau_i} = -\mathbb{1}_{\{\tau_i<T\}}\frac{\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds}{T-\tau_i}. \quad (3.11)$$

We notice that whether the jump of sovereign bond yield at a critical date τ_i is negative or not depends on the intensity of the exogenous shock, the elapsed time between τ_{i-1} and τ_i (the solvency indirectly), and the value of the recovery payment at τ_i. When the recovery payment is small enough, the condition in (3.10) can be satisfied. Moreover, if no recovery payment is made, the size of the jumps only depends on the solvency and the exogenous shock.

3.2.2 Sovereign CDS

We can also apply the sovereign default model to the credit default swap (CDS). Let κ be the spread of the CDS, which is payed by the protection buyer throughout the life of CDS until $T \wedge \tau$. The value at $t < T \wedge \tau$ of the cash flow received by the protection buyer is given by

$$O_t = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{\tau \le T\}} e^{-\int_t^\tau r_s\,ds}(1-R_\tau)|\mathcal{G}_t]$$

and that payed is given by

$$I_t(\kappa) = \mathbb{E}_{\mathbb{Q}}[\int_t^{T\wedge\tau} \kappa e^{-\int_t^u r_s\,ds} du|\mathcal{G}_t].$$

Then, the pre-default value of the CDS is given by

$$C_t(\kappa) = O_t - I_t(\kappa),$$

where O and I are respectively computed as

$$O_t = \mathbb{1}_{\{\tau>t\}} G_t^{-1} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T e^{-\int_t^u r_s\,ds}(1-R_u)\alpha_u(u)du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\le T\}} e^{-\int_t^{\tau_i} r_s\,ds}(1-R_{\tau_i})q^i_{\tau_i} \middle| \mathcal{F}_t \right],$$

and

$$I_t(\kappa) = \mathbb{1}_{\{\tau>t\}} G_t^{-1} \mathbb{E}_{\mathbb{Q}} \left[\kappa \int_t^T e^{-\int_t^u r_s\,ds} G_u\,du \middle| \mathcal{F}_t \right].$$

The value of the CDS spread is obtained such that $C_0(\kappa) = 0$ and is given by

$$\kappa = \frac{\mathbb{E}_{\mathbb{Q}}\left[\int_0^T e^{-\int_0^u r_s\,ds}(1-R_u)\alpha_u(u)du + \sum_{i=1}^n \mathbb{1}_{\{0<\tau_i\le T\}} e^{-\int_0^{\tau_i} r_s\,ds}(1-R_{\tau_i})q^i_{\tau_i}\right]}{\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{\alpha_u(u)}{\lambda_u} \exp\left(-\int_0^u r_s\,ds - \sum_{i=1}^n \mathbb{1}_{\{u=\tau_i\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\right) du\right]},$$

where we use the relationship $\alpha_t(t) = \lambda_t G_{t-}$, $t \ge 0$, implied by (1.22) and (2.20).

3.2.3 Example of indifference pricing in a hybrid model

In the previous subsection, the valuation of defaultable claim has been done in a complete market model. When the market is incomplete, the indifference pricing approach is often used. The indifference price is the price at which an agent would have the same expected utility level by exercising a financial transaction as by not doing so. Hodges and Neuberger [HN89] were the first to introduce utility indifference pricing in incomplete markets (see Henderson and Hobson [HH04] for a general survey of literature on this topic). As emphasised by El Karoui and Rouge [REK00], prices should depend on the wealth of the underlying firm. So, we consider here a corporate instead of a sovereign country, because it is difficult to value the total wealth of a country. The fundamental element of finding the indifference price lies in solving two portfolio optimisation problems, and the standard methods are dynamic programming, martingale methods and backward stochastic differential equation (BSDE) approach (see, e.g., Pham [Pha09] for Brownian diffusions and Øksendal and Sulem [ØS05] for jump diffusions). In the context of optimisation and pricing under default risk, see the research paper of Bielecki, Jeanblanc and Rutkowski [BJR04a] and the thesis of Sigloch [Sig09] for an overview, as well as Pham [Pha10] in the case of multi-defaults. Besides, Lim and Quenez [LQ11] uses a BSDE approach in which the default time has an intensity and the immersion property holds, and Jiao and Pham [JP11] decompose a default-sensitive equity trading problem into two optimisation problems according to the default regime under density hypothesis, which is generalised in the case of multi-defaults under density hypothesis in Jiao, Kharroubi and Pham [JKP13]. Recently, Kharroubi, Lim and Ngoupeyou [KLN13] study a problem of mean-variance hedging with BSDE approach where the horizon is uncertain because of a possible jump of the asset price process, and the jump time admits an intensity.

We introduce an example of indifference pricing of defaultable claim under a hybrid model without density or intensity. We fix a constant horizon $T > 0$ and start with a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a Brownian motion $W = (W_t)_{0 \le t \le T}$ and an exponentially distributed random variable η of unit parameter independent of W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the canonical filtration of W, which represents all the information accessible to the investors.

We consider a defaultable stock whose pre-default price dynamic follows a Jump to Default extended Constant Elasticity of Variance (JDCEV) diffusion

$$dP_t = [\mu(t) + \lambda(t, P_t)]P_t\, dt + \delta(t)P_t^{\beta+1} dW_t, \quad P_0 = p, \tag{3.12}$$

where $\mu(t)$ is the bounded $\mathbb{R}$-valued time-dependent return rate function, $\delta(t) > 0$ is the bounded time-dependent volatility scale function, $\beta < 0$ is the volatility elasticity parameter, and $\lambda(t, P) \ge 0$ is the default intensity of the firm, which can depend on both the time and the pre-default stock price process P. We assume that $\lambda(t, P)$ remains uniformly bounded as $P \to +\infty$ and explodes to $+\infty$ when $P \to 0$. For example, as in [CL06], we can let

$$\lambda_t = \lambda(t, P_t) = a\delta^2(t)P_t^{2\beta} + b(t), \tag{3.13}$$

where $a \ge 0$ is a positive constant parameter governing the sensitivity of λ to the local volatility, and $b(t) \geq 0$ is a nonnegative deterministic function of t, in which case $\lambda(t, P)$ tends to infinity when $P \to 0$.

We assume that the firm goes bankrupt when the stock price hits zero, and the price process is killed at the first hitting time of zero $\tau_0 = \inf\{t \geq 0 : P_t = 0\}$ with the convention that $\tau_0 = 0$ if the price does not hit zero. Besides, the bankruptcy of the firm can also be triggered by an unexpected jump to default at time ξ, modelled by the first jump time of a Cox process with intensity process λ, precisely,

$$\xi = \inf\left\{t \geq 0 : \int_0^t \lambda_s \, ds > \eta\right\}. \tag{3.14}$$

We assume that the stock price P is killed from a positive value when a jump to default happens, so the stock process is actually $(P_t \mathbb{1}_{\{\xi > t\}}, 0 \leq t \leq T)$. Denote by $D = (D_t)_{0 \leq t \leq T}$ the indicator process associated to ξ, namely $D_t = \mathbb{1}_{\{\xi \leq t\}}$. Since D is a process of pure jump, one has

$$[P, D]_t = \sum_{u \leq t} \Delta P_u \Delta D_u = \mathbb{1}_{\{\xi \leq t\}} \Delta P_\xi.$$

Then, the dynamic of the stock process is

$$d(P_t \mathbb{1}_{\{\xi > t\}}) = \mathbb{1}_{\{\xi \geq t\}} P_t \left[(\mu(t) + \lambda(t, P_t))dt + \delta(t) P_t^\beta dW_t\right] - P_t \, dD_t.$$

The default of the firm in this model can occur either at time τ_0 via diffusion to zero or at time ξ via a jump to default, whichever comes first. The time of default is then decomposed into a predictable part and a totally inaccessible part. We define the time of default τ as

$$\tau = \min\{\tau_0, \xi\}. \tag{3.15}$$

We note that τ_0 is an $\mathbb{F}$-stopping time but it is not the case for ξ. Let $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ be the progressive enlargement of $\mathbb{F}$ by τ, defined as

$$\mathcal{G}_t = \bigcap_{s > t} \Big(\sigma(\{\tau \leq u\} : u \leq s) \vee \mathcal{F}_s\Big), \quad t \in [0, T],$$

representing the global information available for the investors over $[0, T]$.

Consider now an investor who trades continuously the stock on the financial market by holding a positive total wealth at any time. The current account interest rate is set to 0 for simplicity. We denote by $X = (X_t)_{0\le t\le T}$ the wealth of the investor. The trading strategy of the investor is a $\mathbb{G}$-predictable process $\hat{\varphi} = (\hat{\varphi}_t)_{t\in[0,T]}$ representing the proportion of wealth invested in the stock. The stock disappears after the bankruptcy of the firm and so the investor's position on the stock becomes worthless. Then, for any $t \in [0,T]$, $\hat{\varphi}_t = \mathbb{1}_{\{\tau\ge t\}}\varphi_t$, where φ is an $\mathbb{F}$-predictable process, and the dynamic of the wealth process on $\{t < \tau\}$ is

$$dX_t = X_t\varphi_t\left[(\mu(t) + \lambda(t,P_t))dt + \delta(t)P_t^{\beta}dW_t\right], \quad X_0 = x, \tag{3.16}$$

where $X = (X_t)_{0\le t\le T}$ is an $\mathbb{F}$-adapted process. We denote by $\hat{X}$ the $\mathbb{G}$-adapted wealth process. If the default occurs before t, the investor loses the wealth invested in the stock at the moment of the default τ and holds the current account till t, i.e.,

$$\mathbb{1}_{\{\tau\le t\}}\hat{X}_t = \mathbb{1}_{\{\tau\le t\}}\hat{X}_\tau = \mathbb{1}_{\{\tau\le t\}}\hat{X}_{\tau-}(1-\varphi_\tau).$$

Then, $\hat{X}$ has the following decomposed form

$$\hat{X}_t = \mathbb{1}_{\{\tau>t\}}X_t + \mathbb{1}_{\{\tau\le t\}}\hat{X}_{\tau-}(1-\varphi_\tau).$$

The preference of the investor is described by a utility function $U : (0,\infty) \to \mathbb{R}$ which is strictly increasing, strictly concave, C^1 on $(0,\infty)$ with $U(0) = 0$, and satisfies the usual Inada conditions: $U'(0^+) := \lim_{x\to 0} U'(x) = \infty$, $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$. The aim of the investor is to maximize the expectation of the utility over the finite horizon T. We denote by $\mathcal{A}$ the set of admissible trading strategies. Then, the value function of the optimal investment problem is

$$v(x) = \sup_{\varphi\in\mathcal{A}} \mathbb{E}\left[U(\hat{X}_T^{x,\varphi})\right]. \tag{3.17}$$

By law of iterated conditional expectations, one has

$$\begin{aligned}\mathbb{E}\left[U(\hat{X}_T)\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}U(X_T) + \mathbb{1}_{\{\tau\le T\}}U(\hat{X}_\tau)\right]\\ &= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau>T\}}U(X_T)|\mathcal{F}_T] + \mathbb{E}[\mathbb{1}_{\{\tau\le T\}}U\Big(\hat{X}_{\tau-}(1-\varphi_\tau)\Big)|\mathcal{F}_T]\right].\end{aligned}$$

Furthermore, for all $0 < u \le T$,

$$\mathbb{P}(\tau > u|\mathcal{F}_T) = \mathbb{1}_{\{\tau_0>u\}}\mathbb{P}(\xi > u|\mathcal{F}_T) = \mathbb{1}_{\{\tau_0>u\}}\exp\left(-\int_0^u \lambda(s,P_s)ds\right),$$

and let $F_T(u) = \mathbb{P}(\tau \leq u|\mathcal{F}_T)$. Then,

$$\begin{aligned}\mathbb{E}\left[U(\hat{X}_T)\right] &= \mathbb{E}\left[U(X_T)\mathbb{P}(\tau > T|\mathcal{F}_T) + \int_{(0,T]} U\Big(X_u(1-\varphi_u)\Big)dF_T(u)\right] \\ &= \mathbb{E}\left[\int_0^{T\wedge\tau_0} e^{-\int_0^u \lambda(s,P_s)ds}\lambda(u,P_u)U\Big(X_u(1-\varphi_u)\Big)du \right. \\ &\qquad \left. + \mathbb{1}_{\{\tau_0>T\}}e^{-\int_0^T \lambda(s,P_s)ds}U(X_T)\right].\end{aligned}$$

Let $f : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ be a function defined as

$$f: \quad (t,x,p,\varphi) \quad \mapsto \quad \lambda(t,p)U(x(1-\varphi)).$$

For all $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$, we denote by $\mathcal{A}(t,x,p)$ the subset of the controls φ in $\mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T \left|f(s,X_s^{t,x},P_s^{t,p},\varphi_s)\right| ds\right] < \infty, \tag{3.18}$$

and we assume that $\mathcal{A}(t,x,p)$ is not empty for all $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$.

Let $\tau_t = \inf\{s \in [t,+\infty) : P_s = 0\}$, $\gamma(t,s) = \exp(-\int_t^s \lambda(r,P_r)dr)$, and define the gain function as

$$J(t,x,p,\varphi) = \mathbb{E}\left[\int_t^{T\wedge\tau_t} \gamma(t,s)f(s,X_s^{t,x},P_s^{t,p},\varphi_s)ds + \mathbb{1}_{\{\tau_t>T\}}\gamma(t,T)U(X_T^{t,x})\right] \tag{3.19}$$

for all $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$ and all $\varphi \in \mathcal{A}(t,x,p)$. To maximize over control processes the gain function J, we introduce the associated value function

$$v(t,x,p) = \sup_{\varphi\in\mathcal{A}(t,x,p)} J(t,x,p,\varphi). \tag{3.20}$$

The gain function (3.19) has a finite horizon with terminal payoff depending on a stopping regime. We formulate the dynamic programming principle in the following theorem, the proof of which is given in Appendix B.

Theorem 3.2. *For any $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$ and any $\mathbb{F}$-stopping time θ valued in $[t,T]$, the value function* (3.20) *is*

$$\begin{aligned}v(t,x,p) = \sup_{\varphi\in\mathcal{A}(t,x,p)} \mathbb{E}&\left[\int_t^{\theta\wedge\tau_t} \gamma(t,s)f(s,X_s^{t,x},P_s^{t,p},\varphi_s)ds \right. \\ &\left. + \mathbb{1}_{\{\theta\leq\tau_t\}}\gamma(t,\theta)v(\theta,X_\theta^{t,x},P_\theta^{t,p})\right].\end{aligned} \tag{3.21}$$

The Hamilton-Jacobi-Bellman (HJB) equation describes the infinitesimal behaviour of the value function (3.21) when the stopping time θ tends to t. We assume that the value function v is smooth enough, then v satisfies the following HJB equation with Cauchy and Dirichlet conditions:

$$\begin{aligned} -\frac{\partial v}{\partial t} + \lambda(t,p)v - \sup_{\phi\in A}\left[\mathcal{L}^{\phi}v + f(t,x,p,\phi)\right] = 0, \quad (t,x,p) \in [0,T)\times\mathbb{R}_+\times(0,+\infty), \\ v(T,x,p) = U(x), \\ v(t,x,0) = 0, \end{aligned} \tag{3.22}$$

where $\mathcal{L}^{\phi}$ is the infinitesimal generator associated to the diffusions (3.12) and (3.16) for the constant control ϕ, defined by

$$\mathcal{L}^{\phi}v = [\mu(t) + \lambda(t,p)]\left(x\phi\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial p}\right) + \frac{1}{2}\delta^2(t)p^{2\beta}\left(x^2\phi^2\frac{\partial^2 v}{\partial x^2} + p^2\frac{\partial^2 v}{\partial p^2}\right).$$

Let us consider a defaultable zero-coupon bond issued by the firm, whose price process is a $\mathbb{G}$-adapted process, noted $(H_t, 0 \le t \le T)$, where $H_0 = h$. The payoff of the zero-coupon bond is

$$H_T = \mathbb{1}_{\{\tau>T\}} + \mathbb{1}_{\{\tau\le T\}}R(\tau),$$

where $R(\cdot)$ is a $[0,1)$-valued function. In order to compute the initial price h, we consider another agent investing in this defaultable claim besides the stock with the same initial endowment x. Then, the second investor is in face of the following optimal investment problem:

$$\nu(x-h) = \sup_{\varphi\in\mathcal{A}}\mathbb{E}\left[U(\hat{X}_T^{x-h,\varphi} + H_T)\right]. \tag{3.23}$$

By indifference pricing principle, the value functions of the two investors should be equal if the price h is fair, i.e., $v(x) = \nu(x-h)$. If we write the dynamic version of the value function above as $\nu(t,x,p)$, then ν satisfies the following HJB equation with Cauchy and Dirichlet conditions:

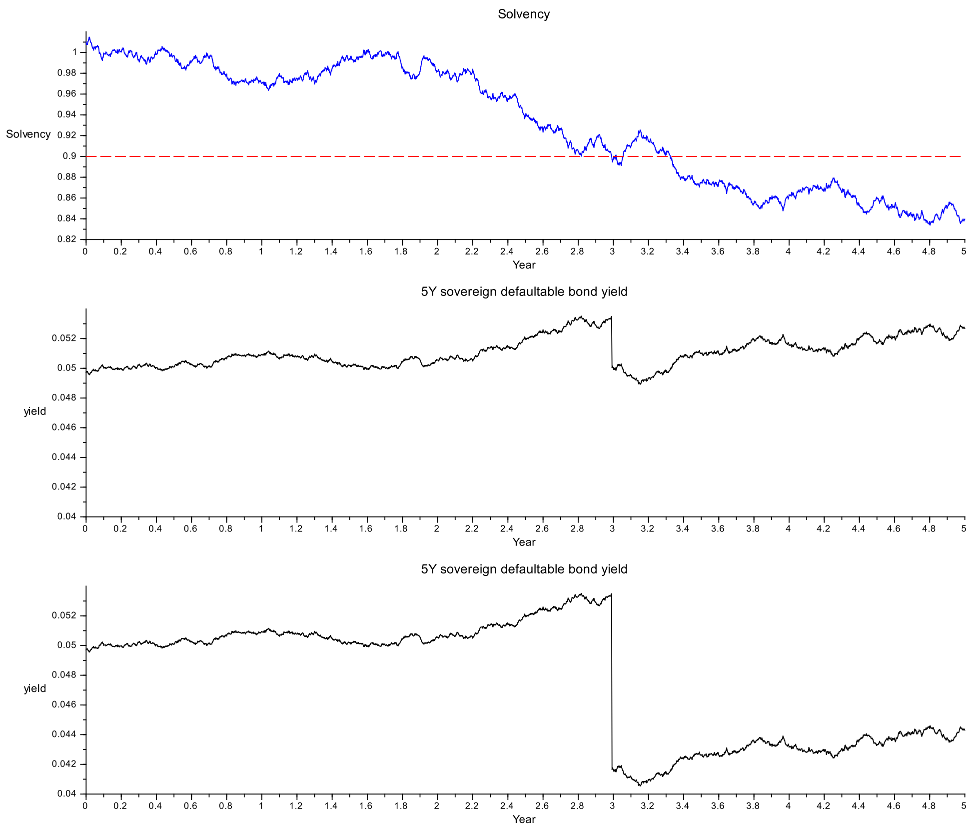
$$\begin{aligned} -\frac{\partial \nu}{\partial t} + \lambda(t,p)\nu - \sup_{\phi\in A}\left[\mathcal{L}^{\phi}\nu + g(t,x,p,\phi)\right] = 0, \quad (t,x,p) \in [0,T)\times\mathbb{R}_+\times(0,+\infty), \\ \nu(T,x,p) = U(x+1), \\ \nu(t,x,0) = 0, \end{aligned} \tag{3.24}$$

where $g(t, x, p, \phi) = \lambda(t, p)U(x(1 - \phi) + R(t))$.

3.2.4 Numerical illustrations

We now present numerical examples to illustrate the results obtained previously concerning the defaultable bond yield. We use the solvency data of Greece during the period from 2003 to 2013 to estimate the parameters.

Figure 3.1 – Jump at a critical date in the sovereign defaultable bond yield (with the corresponding solvency sample path): $\lambda^N = 0, 0.05$ and 0.2 respectively.



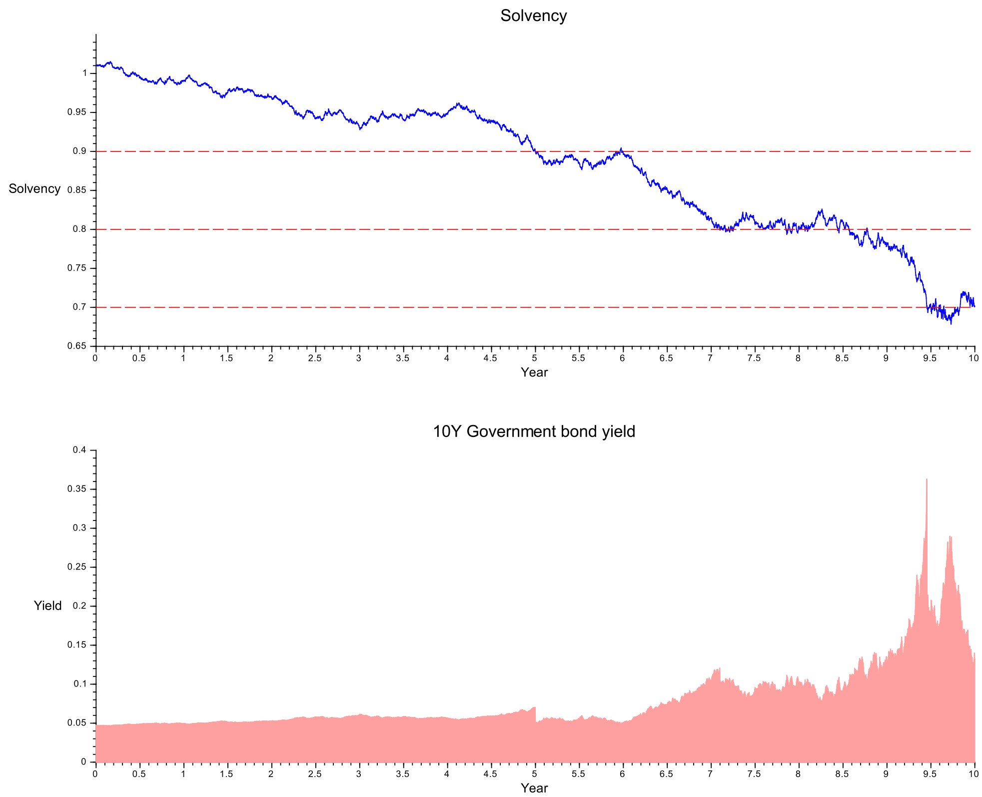
In the first example, we illustrate the bond yield and its jump at a critical date for a sovereign defaultable bond of maturity 5 years. The solvency is described by a CEV process as in (1.32) and we set the parameters to be $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and

$\beta = 1$. The idiosyncratic default intensity process λ is specified by $\lambda(S) = \frac{a}{S^{2|\beta|}} + b$ as in (3.13) with coefficients $a = 0.005$ and $b = 0.01$. We assume that there is only one critical date with the solvency barrier $L = 0.9$ and that the risk-free interest rate is 0. Figure 3.1 plots the time-varying bond yield (3.8) of a defaultable zero-coupon bond without recovery payment, as well as the corresponding simulation scenario of the solvency process. We present two different exogenous shock intensities: $\lambda^N = 0.005$ and $\lambda^N = 0.02$. Note that in this test, when the solvency hits 0.9, the bond yield has a negative jump, the size of which depends on the value of λ^N. A larger λ^N results in a larger jump.

In the second example, we consider the long term Greece government bond yield of maturity 10 years. The solvency of Greece is described by a CEV process. We estimate the parameters by using the solvency data as in Figure 1.4 where δ and β are jointly calibrated (c.f. [CEMY93] and [YYC01]) and obtain $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and $\beta = -4.92$. The coefficients of the idiosyncratic default intensity (as in (3.13)) are $a = 0.013$ and $b = 0.035$, estimated from the 3-month Greek bond yield. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$.

We suppose that the exogenous shock intensity of the inhomogeneous Poisson process is a piecewise constant function which change its value at each critical date. By Figure 1.2, given the sizes of the three jumps, we let $\lambda^N(t) = 0.07$ for $t \in [0, \tau_1]$, $\lambda^N(t) = 0.16$ for $t \in (\tau_1, \tau_2]$ and $\lambda^N(t) = 3.15$ for $t \in (\tau_2, \tau_3]$, which are computed using (3.11). Figure 3.2 plots the time-varying bond yield of a 10-year Greek government zero-coupon bond, as well as a sample path of the solvency of Greece which corresponds to the period of 2003-2013. We observe that the solvency of Greece tends to fall gradually through time. The bond yield has three negative jumps at the barrier hitting times: there is a large negative jump when the solvency falls below 0.7 since the exogenous shock intensity is at a high level; while the first two are relatively small. This looks like the full view of the historical data in Figure 1.2 where the three jumps correspond respectively to T_1, T_2 and T_3 in Figure 1.3.

Figure 3.2 – Simulated 10-year Greek government bond yield with re-adjustable Poisson intensity and the corresponding solvency sample path.



3.2.5 Pricing credit risk in generalised density framework

In this section, we provide a general framework for pricing credit risk. Recall that in the theory of the enlargement of filtrations, given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$, the pair of filtrations $(\mathbb{F}, \mathbb{G})$ is said to verify the immersion property (or H-hypothesis) if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. In the context of credit risk, $\mathbb{F}$ represents the default-free market information and $\mathbb{G}$ is the global market information obtained through the progressive enlargement of $\mathbb{F}$ by the default time τ. In the literature of default modelling, the immersion property is often assumed for the pricing of credit derivatives before default since the immersion property under a risk-neutral probability seems to be a suitable no-arbitrage condition (see e.g. [JLC09a, CJN12]), and the risk-neutral probability in $\mathbb{F}$ is also that in $\mathbb{G}$. We give below a criterion under the generalised density hypothesis for the immersion property to hold true. In this section, we assume that Assumption 2.4 holds for τ.

Proposition 3.3. *The immersion property holds for* $(\mathbb{F},\mathbb{G})$ *if the following conditions are satisfied:*

(a) $\alpha_t(\theta)=\alpha_\theta(\theta)$ *for* $0\le\theta\le t$ *on* $\bigcap_{i=1}^n\{\tau_i\neq\theta\}$*;*

(b) $p_t^i=p_{\tau_i\wedge t}^i$ *for any* $i\in\{1,\cdots,n\}$*,*

where p^i *is a càdlàg version of the* $\mathbb{F}$*-martingale* $\mathbb{E}[\tau=\tau_i<\infty|\mathcal{F}_t]$.

Proof. Let Y be an $\mathbb{F}$-martingale. It can be considered as a $\mathbb{G}$-adapted process and admits the following decomposition

$$Y_t=\mathbb{1}_{\{\tau>t\}}Y_t+\mathbb{1}_{\{\tau\le t\}}Y_t,\quad t\ge 0.$$

We intend to prove that Y is a $\mathbb{G}$-martingale by using the characterisation in Proposition 2.14. The condition (a) implies that the process $\mathbb{1}_{\bigcap_{i=1}^n\{\tau_i\neq\theta\}}\alpha(\theta)Y$ is an $\mathbb{F}$-martingale on $[\theta,\infty)$ for any $\theta\ge 0$. The condition (b) implies that Yp^i is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,n\}$. For the last condition in Proposition 2.14, we have

$$\begin{aligned}
&Y_tG_t+\int_0^t Y_u\alpha_u(u)\eta(du)+\sum_{i=1}^n\mathbb{1}_{\{\tau_i\le t\}}Y_{\tau_i}p_{\tau_i}^i\\
&=Y_t\int_0^\infty\alpha_t(u)\eta(du)+\sum_{i=1}^n Y_{\tau_i\wedge t}p_{\tau_i\wedge t}^i+Y_tp_t^\infty\\
&=Y_t\bigg(\int_0^\infty\alpha_t(u)\eta(du)+\sum_{i=1}^n p_t^i+p_t^\infty\bigg)+\sum_{i=1}^n(Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i\\
&=Y_t+\sum_{i=1}^n(Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i,
\end{aligned}$$

where the second equality comes from the fact $p_{\tau_i\wedge t}^i=p_t^i$ and the third equality comes from (2.8). Since Y is an $\mathbb{F}$-martingale, $\Big((Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i,t\ge 0\Big)$ is an $\mathbb{F}$-martingale for any $i\in\{1,\cdots,n\}$. Hence we obtain the result. □

Remark 3.4. We note that in the sovereign default model (1.14), the following equalities are satisfied: $\alpha_t(u)=\alpha_u(u)$ for $0\le u\le t$ on $\bigcap_{i=1}^n\{\tau_i\neq u\}$ (see Proposition 2.13) and

$p_t^i = p_{\tau_i\wedge t}^i$ for any $i \in \{1, \cdots, n\}$ (see Proposition 1.8), which implies the immersion property. This implication is coherent with our analysis in Chapter 1. By the same observation, among the examples in literature cited in Chapter 2, the immersion property holds in the hybrid models [BSW04, CL06, CPS09, GS16].

Conversely, if the immersion property holds, then

(a') we can choose suitable conditional density process $\alpha(\cdot)$ such that $\alpha_t(\theta) = \alpha_\theta(\theta)$ for $0 \leq \theta \leq t$ on $\bigcap_{i=1}^n \{\tau_i \neq \theta\}$

(b') for any $i \in \{1, \ldots, n\}$, the $\mathbb{F}$-martingale p^i is stopped at τ_i.

However, the condition (a') may not hold in general since we are allowed to change the value of $\alpha_t(\theta)$ for θ on an η-negligible set without changing the $\mathbb{F}$-conditional probability distribution of τ.

The immersion property depends strongly on the probability measure that we have chosen and is in general not preserved under a change of probability measure. For this reason, the reduced-form models are usually given under a risk-neutral probability. In the following propositions, we study the change of probability measures based on the results of the $\mathbb{G}$-martingale characterisation in Chapter 2. Firstly, we prove that the generalised density hypothesis still holds after a change of probability measure and we deduce relevant processes under the new probability measure. Secondly, we show that by starting from an arbitrary probability measure (under which the immersion is not necessarily satisfied), we can always find a change of probability which is invariant on $\mathbb{F}$, and the immersion property holds under the new probability measure.

Proposition 3.5. *Let $Y^{\mathbb{G}}$ be a positive $\mathbb{G}$-martingale of expectation 1, which is written in the decomposed form as $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}} Y_t + \mathbb{1}_{\{\tau\leq t\}} Y_t(\tau)$ where Y and $Y(\cdot)$ are positive processes which are respectively $\mathbb{F}$-adapted and $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted. Let $\mathbb{Q}$ be the probability measure such that $d\mathbb{Q}/d\mathbb{P} = Y_t^{\mathbb{G}}$ on $\mathcal{G}_t$ for any $t \geq 0$. Then the random time τ satisfies Assumption 2.4 under the probability $\mathbb{Q}$, and the $(\mathbb{F}, \mathbb{Q})$-conditional density avoiding $(\tau_i)_{i=1}^n$ and the $(\mathbb{F}, \mathbb{Q})$-conditional probability of $\{\tau = \tau_i < \infty\}$ can be written in the following form*

$$\alpha_t^{\mathbb{Q}}(\theta) = \mathbb{1}_{\{\theta\leq t\}} \frac{Y_t(\theta)}{Y_t^{\mathbb{F}}} \alpha_t(\theta) + \mathbb{1}_{\{\theta>t\}} \frac{\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]}{Y_t^{\mathbb{F}}}, \quad p_t^{i,\mathbb{Q}} = \frac{Y_t(\tau_i)p_t^i}{Y_t^{\mathbb{F}}} \tag{3.25}$$

where

$$Y_t^{\mathbb{F}} := \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_tY_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^n 1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^i.$$

Proof. Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-martingale defined as $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}Y_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau)$, $t\ge 0$. Let h be a bounded Borel function, then

$$\mathbb{E}^{\mathbb{Q}}[1\!\!1_{\{\tau<\infty\}}h(\tau)|\mathcal{F}_t] = \lim_{m\to+\infty}\mathbb{E}^{\mathbb{Q}}[1\!\!1_{\{\tau\le m\}}h(\tau)|\mathcal{F}_t] = \lim_{m\to+\infty}\frac{\mathbb{E}[1\!\!1_{\{\tau\le m\}}Y_{\tau\vee t}^{\mathbb{G}}h(\tau)|\mathcal{F}_t]}{\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t]} \tag{3.26}$$

where we use the optional stopping theorem of Doob for the second equality. Note that

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_tY_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^n 1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^i$$

and for any $m\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}[1\!\!1_{\{\tau\le m\}}Y_{\tau\vee t}^{\mathbb{G}}h(\tau)|\mathcal{F}_t] = &\int_0^m \Big(1\!\!1_{\{\theta\le t\}}Y_t(\theta)\alpha_t(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]\Big)h(\theta)\,\eta(d\theta)\\ &+\sum_{i=1}^n\Big(1\!\!1_{\{\tau_i\le t\wedge m\}}Y_t(\tau_i)p_t^ih(\tau_i) + 1\!\!1_{\{\tau_i>t\}}\mathbb{E}[Y_{\tau_i}(\tau_i)p_{\tau_i}^ih(\tau_i)1\!\!1_{\{\tau_i\le m\}}|\mathcal{F}_t]\Big).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{m\to+\infty}\mathbb{E}[1\!\!1_{\{\tau\le m\}}Y_{\tau\vee t}^{\mathbb{G}}h(\tau)|\mathcal{F}_t] = &\int_0^\infty \Big(1\!\!1_{\{\theta\le t\}}Y_t(\theta)\alpha_t(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]\Big)h(\theta)\,\eta(d\theta)\\ &+\sum_{i=1}^n\Big(1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^ih(\tau_i) + 1\!\!1_{\{\tau_i>t\}}\mathbb{E}[Y_{\tau_i}(\tau_i)p_{\tau_i}^ih(\tau_i)1\!\!1_{\{\tau_i<\infty\}}|\mathcal{F}_t]\Big),\end{aligned}$$

which implies the required result together with (3.26). □

Proposition 3.6. *We assume that the processes $\alpha(\cdot)$ and p^i, $i\in\{1,\cdots,n\}$, are positive. Let Y and $Y(\cdot)$ be respectively $\mathbb{F}$-adapted and $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted processes such that*

$$Y_t = \frac{1}{G_t}\Big(1 - \int_0^t \alpha_\theta(\theta)\,\eta(d\theta) - \sum_{i=1}^n 1\!\!1_{\{\tau_i\le t\}}p_{\tau_i}^i\Big), \tag{3.27}$$

$$Y_t(\theta) = 1\!\!1_{\cap_{i=1}^n\{\tau_i\neq\theta\}}\frac{\alpha_\theta(\theta)}{\alpha_t(\theta)} + \sum_{i=1}^n 1\!\!1_{\{\tau_i=\theta\}}\frac{p_\theta^i}{p_t^i},\quad 0\le\theta\le t. \tag{3.28}$$

Then, the $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$ defined by $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}Y_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau)$ is a positive $\mathbb{G}$-martingale with expectation 1. Moreover, if we denote by $\mathbb{Q}$ the probability measure such that $d\mathbb{Q}/d\mathbb{P} = Y_t^{\mathbb{G}}$ on $\mathcal{G}_t$, then the restriction of $\mathbb{Q}$ on $\mathcal{F}_\infty$ coincides with $\mathbb{P}$ and $(\mathbb{F},\mathbb{G})$ verifies the immersion property under the probability $\mathbb{Q}$. Moreover, one has $\alpha_\theta^{\mathbb{Q}}(\theta) = \alpha_\theta(\theta)$ on $\bigcap_{i=1}^n\{\tau_i\neq\theta\}$ and $p_{\tau_i}^{i,\mathbb{Q}} = p_{\tau_i}^i$.

Proof. The assertion that $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale results from Proposition 2.14. Moreover, one has

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_t Y_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} Y_t(\tau_i) p_t^i = 1.$$

Therefore the expectation of $Y_t^{\mathbb{G}}$ is 1, and the restriction of $\mathbb{Q}$ to $\mathcal{F}_\infty$ coincides with $\mathbb{P}$. It remains to verify that $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property under the probability $\mathbb{Q}$ and the invariance of the values of $\alpha_\theta(\theta)$ and $p_{\tau_i}^i$. By the previous proposition, on $\bigcap_{i=1}^n \{\tau_i \neq \theta\}$ one has

$$\begin{aligned}
\alpha_t^{\mathbb{Q}}(\theta) &= \mathbb{1}_{\{\theta \le t\}} Y_t(\theta)\alpha_t(\theta) + \mathbb{1}_{\{\theta > t\}} \mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t] \\
&= \mathbb{1}_{\{\theta \le t\}} \alpha_\theta(\theta) + \mathbb{1}_{\{\theta > t\}} \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]
\end{aligned}$$

and

$$p_t^{i,\mathbb{Q}} = Y_t(\tau_i) p_t^i = p_{\tau_i}^i \text{ on } \{\tau_i \le t\}.$$

In particular, one has $\alpha_\theta^{\mathbb{Q}}(\theta) = \alpha_\theta(\theta)$ on $\bigcap_{i=1}^n \{\tau_i \neq \theta\}$ and $p_{\tau_i}^{i,\mathbb{Q}} = p_{\tau_i}^i$. By Proposition 3.3, we obtain that $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property under the probability $\mathbb{Q}$. The result is thus proved. □

We fix a constant horizon $T < \infty$ and start with a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with an n-dimensional standard Brownian motion $W = (W^1, \dots, W^n)'$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the usual augmented Brownian filtration: $\mathcal{F}_t := \sigma(W_s, s \le t) \vee \mathcal{A}_0$ where $\mathcal{A}_0$ is the collection of all $\mathbb{P}$-null sets. $\mathbb{F}$ represents the default-free market information. We consider a market model which consists of one riskless asset (money market instrument or short-rate bond) and n continuously-traded risky assets. The price per unit of the riskless asset S^0 is governed by the equation

$$dS_t^0 = S_t^0 r_t \, dt, \quad t \ge 0,$$

where $r = (r_t, 0 \le t \le T)$ is a nonnegative predictable process representing the short-rate, and the price process for one share of i-th risky asset S^i is modelled by the linear stochastic differential equation

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \right), \quad t \in (0, T],$$

where $\mu = (\mu^1, \dots, \mu^n)'$ is an $\mathbb{R}^n$-valued $\mathbb{F}$-predictable vector process, $\sigma = (\sigma^{i,j})_{1\le i,j\le n}$ is an $\mathbb{F}$-predictable $n \times n$ matrix with full rank a.s.. We assume that there exists a bounded $\mathbb{R}^n$-valued $\mathbb{F}$-predictable risk premium process $\theta = (\theta^1, \dots, \theta^n)'$ such that

$$\mu_t - r_t \mathbb{1} = \sigma_t \theta_t, \quad d\mathbb{P} \otimes dt \text{ a.s.}, \tag{3.29}$$

where $\mathbb{1}$ is a n-dimensional vector each component of which is 1.

Let us consider furthermore a defaultable claim (T, F, C, τ, R), where T is the maturity, F is an $\mathcal{F}_T$-measurable random variable representing the nominal (face) value due at the maturity, $C = (C_t, 0 \le t \le T)$ is an nonnegative $\mathbb{F}$-predictable process of finite variation with $C_0 = 0$ representing the accrued interest payment (coupon) [1], τ is a nonnegative random variable valued on $[0, \infty]$ representing the date when the counterpart of the claim defaults, and R is an $[0, 1)$-valued $\mathbb{F}$-predictable process representing the recovery payment at τ if a default occurs prior to or at the maturity. We assume that $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, T]$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T}$, defined as $\mathcal{G}_t = \cap_{s>t}\sigma(\{\tau \le u\} : u \le s) \vee \mathcal{F}_t$, represents the global market information. Let Assumption 2.4 hold for τ under Lebesgue measure. Since the augmented Brownian filtration is quasi left continuous, the $\mathbb{F}$-stopping times τ_1, …, τ_n are predictable.

By predictable representation theorem, there exists an $\mathbb{R}^n$-valued $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process φ such that

$$\mathbb{E}\left[\int_0^T |\varphi_s(\theta)|^2 ds\right] < \infty \quad \text{and} \quad \alpha_t(\theta) = \alpha_0(\theta) + \int_0^t \alpha_t(\theta)\varphi_s(\theta) dW_s, \quad t \in [0, T], \quad \theta \in \mathbb{R}_+,$$

and there exists a family of $\mathbb{R}^n$-valued $\mathbb{F}$-predictable processes $\{\psi^i\}_{i=1}^n$ such that

$$\mathbb{E}\left[\int_0^T |\psi_s^i|^2 ds\right] < \infty \quad \text{and} \quad p_t^i = \int_0^t p_t^i \psi_s^i dW_s, \quad t \in [0, T], \quad i \in \{1, \dots, n\}.$$

The following lemma gives the canonical decomposition of the $\mathbb{F}$-Brownian motion as a $\mathbb{G}$-martingale.

1. Depending on the type of claim, the coupon rate can be fixed at the very beginning or readjusted every 3 months at the beginning of each 3-month period. So, we consider C to be an $\mathbb{F}$-predictable process.

Lemma 3.7. *The Brownian motion W is a continuous $\mathbb{G}$-semimartingale, which can be decomposed in the following form:*

$$W_t = \tilde{W}_t + \int_0^t ds \left[\mathbb{1}_{\{\tau \geq s\}} \frac{\Phi_s}{G_{s-}} + \mathbb{1}_{\{\tau < s\}} \left(\sum_{i=1}^n \mathbb{1}_{\{\tau = \tau_i\}} \psi_s^i + \prod_{i=1}^n \mathbb{1}_{\{\tau \neq \tau_i\}} \varphi_s(\tau) \right) \right], \quad t \in [0, T], \tag{3.30}$$

where $\tilde{W}$ is an n-dimensional $\mathbb{G}$-Brownian motion, and Φ is an $\mathbb{R}^n$-valued $\mathbb{F}$-predictable process given by

$$\Phi_t = -\int_0^t \alpha_t(u) \varphi_t(u) du - \sum_{i=1}^n \mathbb{1}_{\{\tau_i < t\}} p_t^i \psi_t^i.$$

PROOF: By Proposition 2.16, the n-dimensional Brownian motion W admits the following decomposition:

$$W_t = \tilde{W}_t + \int_0^{t \wedge \tau} \frac{d\langle W, \bar{M} \rangle_s}{G_{s-}} + \sum_{i=1}^n \mathbb{1}_{\{\tau = \tau_i\}} \int_\tau^{t \vee \tau} \frac{d\langle W, p^i \rangle_s}{p_s^i} + \prod_{i=1}^n \mathbb{1}_{\{\tau \neq \tau_i\}} \int_\tau^{t \vee \tau} \frac{d\langle W, \alpha(u) \rangle_s}{\alpha_s(u)} \Big|_{u=\tau},$$

where $\bar{M}$ is the BMO martingale defined as

$$\bar{M}_t = \mathbb{E}\left[\int_0^\infty \alpha_u(u) du \Big| \mathcal{F}_t \right] + \sum_{i=1}^n p_{t \wedge \tau_i}^i, \quad t \in [0, T].$$

The quasi left continuous filtration $\mathbb{F}$ makes $\bar{M}$ coincide with the martingale part of the Azéma supermartingale in the Doob-Meyer decomposition. Then, one has

$$\begin{aligned}
\bar{M}_t &= 1 + \int_0^t (\alpha_u(u) - \alpha_t(u)) du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \leq t\}} (p_{\tau_i}^i - p_t^i) \\
&= 1 - \int_0^t du \int_u^t \alpha_s(u) \varphi_s(u) dW_s - \sum_{i=1}^n \int_{t \wedge \tau_i}^t p_s^i \psi_s^i dW_s \\
&= 1 - \int_0^t dW_s \int_0^s \alpha_s(u) \varphi_s(u) du - \sum_{i=1}^n \int_0^t \mathbb{1}_{\{\tau_i < s\}} p_t^i \psi_s^i dW_s,
\end{aligned}$$

and $\langle W, \bar{M} \rangle$ is n-dimensional with

$$d\langle W, \bar{M} \rangle_t = -\left(\int_0^t \alpha_t(u) \varphi_t(u) du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i < t\}} p_t^i \psi_t^i \right) dt.$$

Similarly, $\langle W, p^i \rangle$ and $\langle W, \alpha(u) \rangle$ are n-dimensional with

$$d\langle W, p^i \rangle_t = p_t^i \psi_t^i \, dt, \quad d\langle W, \alpha(u) \rangle_t = \alpha_t(u) \varphi_t(u) dt.$$

The lemma is thus proved. □

By Lemma 3.7, the $\mathbb{F}$-Brownian motion can be written as $W_t = \tilde{W}_t + \int_0^t b_s\,ds$, $t \in [0,T]$, where $b = (b_t, 0 \le t \le T)$ is an $\mathbb{R}^n$-valued $\mathbb{G}$-predictable process given by

$$b_t = \mathbb{1}_{\{\tau \ge t\}} \frac{\Phi_t}{G_{t-}} + \mathbb{1}_{\{\tau < t\}} \left(\sum_{i=1}^n \mathbb{1}_{\{\tau = \tau_i\}} \psi_t^i + \prod_{i=1}^n \mathbb{1}_{\{\tau \neq \tau_i\}} \varphi_t(\tau) \right), \quad t \in [0,T].$$

Then the dynamics of the prices of the risky assets $S = (S^1, \ldots, S^n)'$ can be rewritten in terms of the $\mathbb{G}$-Brownian motion $\tilde{W}$ as:

$$dS_t = S_t[(\mu_t + \sigma_t b_t)dt + \sigma_t d\tilde{W}_t], \tag{3.31}$$

The condition (3.29) implies that

$$\mu_t + \sigma_t b_t - r_t \mathbb{1} = \sigma_t(\theta_t + b_t), \quad d\mathbb{P} \otimes dt \text{ a.s..}$$

Then, by Girsanov theorem, there exists an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$ such that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \exp\left(-\int_0^t (\theta_s + b_s) d\tilde{W}_s - \frac{1}{2}\int_0^t |\theta_s + b_s|^2 ds \right),$$

and that S is a $(\mathbb{Q}, \mathbb{G})$-martingale. In particular, if the immersion property holds, then we have $b \equiv 0$, $\tilde{W} = W$, and $\mathbb{Q}$ is the risk-neutral probability in $\mathbb{F}$. By Proposition 3.5, the default time τ still satisfies Assumption 2.4 under the probability $\mathbb{Q}$, and the $(\mathbb{F}, \mathbb{Q})$-conditional density avoiding $(\tau_i)_{i=1}^n$ and the $(\mathbb{F}, \mathbb{Q})$-conditional probability of $\{\tau = \tau_i < \infty\}$ are noted $\alpha^{\mathbb{Q}}(\cdot)$ and $p^{i,\mathbb{Q}}$, computed by Proposition 3.5.

Recall that the $\mathbb{F}$-compensator is given by

$$\Lambda_t^{\mathbb{F}} = \int_0^t \frac{\alpha_s^{\mathbb{Q}}(s)\eta(ds)}{G_{s-}} + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \frac{p_{\tau_i -}^{i,\mathbb{Q}}}{G_{\tau_i -}}.$$

Let $\lambda_t = \frac{\alpha_t^{\mathbb{Q}}(t)}{G_{t-}}$ for any $t \in [0,T]$. Then, by Proposition 2.12, the Azéma supermartingale G can be factorised as

$$G_t = L_t e^{-\int_0^t \lambda_u\,du} \prod_{i=1}^n \left(1 - \mathbb{1}_{\{\tau_i \le t\}} \frac{p_{\tau_i -}^{i,\mathbb{Q}}}{G_{\tau_i -}} \right), \quad 0 \le t \le T,$$

where L is a positive $\mathbb{F}$-martingale of expectation 1, solution to the following stochastic differential equation

$$dL_t = G_{t-}^{-1}(1 - \Delta\Lambda_t^{\mathbb{F}})^{-1} dM_t, \quad t \in [0,T],$$

where $M_t = \int_0^t (\alpha_u^{\mathbb{Q}}(u) - \alpha_t^{\mathbb{Q}}(u))du + \sum_{i=1}^n (p_{t\wedge\tau_i}^{i,\mathbb{Q}} - p_t^{i,\mathbb{Q}})$. By Girsanov theorem, there exists a probability $\mathbb{Q}^*$ on $(\Omega, \mathcal{A}, \mathbb{F})$ equivalent of $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\Big|_{\mathcal{F}_T} = L_T = \exp\left(\int_0^T \frac{dM_s}{G_{s-}(1-\Delta\Lambda_s^{\mathbb{F}})} - \frac{1}{2}\int_0^T \frac{d\langle M, M\rangle_s}{G_{s-}^2(1-\Delta\Lambda_s^{\mathbb{F}})^2}\right).$$

Let $B_t = \exp(\int_0^t r_s\, ds)$ denote the current account value at t, then the discount factor between t and T is given by $B_t B_T^{-1}$. The pre-default value of the claim at a certain date $t \leq \tau \wedge T$ is the conditional expectation of the discounted value of all the future cash flows between t and T given $\mathcal{G}_t$:

$$\begin{aligned} V(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[B_t B_T^{-1}\left(\mathbb{1}_{\{\tau>T\}}F + \int_{]t,T]} \mathbb{1}_{\{\tau>u\}} B_u^{-1} B_T dC_u + \mathbb{1}_{\{t<\tau\leq T\}} B_\tau^{-1} B_T R_\tau\right)\Big|\mathcal{G}_t\right] \\ &= B_t \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}} B_T^{-1} F + \int_{]t\wedge\tau, T\wedge\tau]} B_u^{-1} dC_u + \mathbb{1}_{\{t<\tau\leq T\}} B_\tau^{-1} R_\tau \Big|\mathcal{G}_t\right] \\ &= B_t \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}\left(B_T^{-1}F + \int_{]t,T]} B_u^{-1} dC_u\right) + \mathbb{1}_{\{t<\tau\leq T\}}\left(B_\tau^{-1} R_\tau + \int_{]t,\tau]} B_u^{-1} dC_u\right)\Big|\mathcal{G}_t\right]. \end{aligned}$$

We note $X(t,\theta) = \int_{(t,\theta]} B_u^{-1} dC_u$ and

$$\begin{aligned} I(t,T) &= B_t \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}} B_T^{-1} F \Big|\mathcal{G}_t\right], \\ J(t,T) &= B_t \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}} X(t,T) + \mathbb{1}_{\{t<\tau\leq T\}} X(t,\tau)\Big|\mathcal{G}_t\right], \\ K(t,T) &= B_t \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{t<\tau\leq T\}} B_\tau^{-1} R_\tau \Big|\mathcal{G}_t\right]. \end{aligned}$$

Then, $V(t,T) = I(t,T) + J(t,T) + K(t,T)$. By using classic properties of enlargement of filtrations, we get

$$\begin{aligned} I(t,T) &= \mathbb{1}_{\{\tau>t\}} B_t G_t^{-1} \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} F G_T \Big|\mathcal{F}_t\right], \\ J(t,T) &= \mathbb{1}_{\{\tau>t\}} B_t G_t^{-1} \mathbb{E}_{\mathbb{Q}}\left[\int_{(t,T]} B_u^{-1} G_{u-} dC_u \Big|\mathcal{F}_t\right], \\ K(t,T) &= \mathbb{1}_{\{\tau>t\}} B_t G_t^{-1} \mathbb{E}_{\mathbb{Q}}\left[\int_t^T B_u^{-1} R_u \alpha_u^{\mathbb{Q}}(u) du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}} B_{\tau_i}^{-1} R_{\tau_i} p_{\tau_i}^{i,\mathbb{Q}} \Big|\mathcal{F}_t\right]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} I(t,T) &= \mathbb{1}_{\{\tau>t\}} B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{L_T}{L_t} F B_T^{-1} e^{-\int_t^T \lambda_u\, du} \prod_{i=1}^n \left(1 - \mathbb{1}_{\{t<\tau_i\leq T\}} \frac{p_{\tau_i-}^{i,\mathbb{Q}}}{G_{\tau_i-}}\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}^*}\left[F e^{-\int_t^T (r_u+\lambda_u) du} \prod_{i=1}^n \left(1 - \mathbb{1}_{\{t<\tau_i\leq T\}} \frac{p_{\tau_i-}^{i,\mathbb{Q}}}{G_{\tau_i-}}\right)\Big|\mathcal{F}_t\right]. \end{aligned}$$

3.3 A two-name model with simultaneous defaults

In the literature of multi-default modelling, one often supposes that there are no simultaneous defaults, notably in the classic intensity and density models. For example, if we suppose that the conditional joint $\mathbb{F}$-density exists for two default times, then the probability that the two defaults coincide equals to zero (see [EKJJ15]). Only few papers consider explicit models of double defaults (e.g. Bielecki et al. [BCCH12], Giesecke [Gie03]). During the crisis, it is important to study extremal risks such as simultaneous defaults whose occurrence is rare but will have significant impact on financial market. The generalised density approach provides mathematical tools to study multi-default models with simultaneous defaults. The idea consists of using a recurrence method.

In the following, we consider two random times σ_1 and σ_2 defined on the probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and we assume that

$$\mathbb{P}(\sigma_1 \in d\theta_1, \sigma_2 \in d\theta_2 | \mathcal{F}_t) = \beta_t(\theta_1, \theta_2) d\theta_1 d\theta_2 + \Delta_*(q_t(\theta)d\theta), \tag{3.32}$$

where $\beta(\cdot)$ and $q(\cdot)$ are respectively càdlàg $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+^2)$ and $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted process, and $\Delta : \mathbb{R}_+ \to \mathbb{R}_+^2$ denotes the diagonal embedding which sends $x \in \mathbb{R}_+$ to $(x, x) \in \mathbb{R}^2$, and $\Delta_*(q_t(\theta)d\theta)$ is the direct image of the Borel measure $q_t(\theta)d\theta$ by the map Δ, namely for any bounded Borel function $h(\cdot)$ on $\mathbb{R}_+^2$, one has

$$\mathbb{E}[h(\sigma_1, \sigma_2)|\mathcal{F}_t] = \int_{\mathbb{R}_+^2} \beta_t(\theta_1, \theta_2) h(\theta_1, \theta_2) d\theta_1 d\theta_2 + \int_{\mathbb{R}_+} q_t(\theta) h(\theta, \theta) d\theta.$$

In particular

$$\mathbb{P}[\sigma_1 = \sigma_2 | \mathcal{F}_t] = \int_{\mathbb{R}_+} q_t(\theta)\, d\theta.$$

We shall apply the results obtained in the previous sections to this two-name model. More precisely, let $\mathbb{F}^1$ be the progressive enlargement of $\mathbb{F}$ by the random time σ_1. Then σ_1 is an $\mathbb{F}^1$-stopping time. The filtration $\mathbb{F}^1$ will play the role of the reference filtration in the previous sections.

Proposition 3.8. *The random time σ_2 satisfies the generalized density hypothesis with respect to the filtration $\mathbb{F}^1$ with the $\mathbb{F}^1$-conditional density of σ_2 exceeding σ_1 given as*

$$\alpha_t^{2|1}(\theta) = \mathbb{1}_{\{\sigma_1 > t\}} \frac{\int_t^\infty \beta_t(s, \theta) ds}{G_t^1} + \mathbb{1}_{\{\sigma_1 \le t\}} \frac{\beta_t(\sigma_1, \theta)}{\alpha_t^1(\sigma_1)}, \quad t \ge 0 \tag{3.33}$$

and

$$p_t := \mathbb{P}(\sigma_2 = \sigma_1 | \mathcal{F}_t^1) = \mathbb{1}_{\{\sigma_1 > t\}} \frac{\int_t^\infty q_t(\theta) d\theta}{G_t^1} + \mathbb{1}_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}. \tag{3.34}$$

Proof. The hypothesis (3.32) implies that

$$\mathbb{P}(\sigma_1 \in d\theta | \mathcal{F}_t) = \left(\int_{\mathbb{R}_+} \beta_t(\theta, \theta_2) d\theta_2 + q_t(\theta) \right) d\theta.$$

So the random time σ_1 admits an $\mathbb{F}$-conditional density which is given by

$$\alpha_t^1(\theta) := \int_{\mathbb{R}_+} \beta_t(\theta, \theta_2) d\theta_2 + q_t(\theta). \tag{3.35}$$

Let $G_t^1 = \mathbb{P}(\sigma_1 > t | \mathcal{F}_t) = \int_t^\infty \alpha_t^1(\theta) d\theta$. Direct computations yield

$$\mathbb{P}(\sigma_2 = \sigma_1 | \mathcal{F}_t^1) = \mathbb{1}_{\{\sigma_1 > t\}} \frac{\int_t^\infty q_t(\theta) d\theta}{G_t^1} + \mathbb{1}_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}.$$

In fact, the term on the set $\{\sigma_1 > t\}$ is classic. For the term on the set $\{\sigma_1 \le t\}$, consider a bounded test function $Y_t(\cdot)$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, by (3.32) one has

$$\mathbb{E}[\mathbb{1}_{\{\sigma_1 = \sigma_2 \le t\}} Y_t(\sigma_1)] = \int_0^t \mathbb{E}[q_t(\theta) Y_t(\theta)] d\theta.$$

Since

$$\mathbb{E}\Big[\mathbb{1}_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)} Y_t(\sigma_1)\Big] = \int_0^t \mathbb{E}\Big[\frac{q_t(\theta)}{\alpha_t^1(\theta)} Y_t(\theta) \alpha_t^1(\theta)\Big] d\theta = \int_0^t \mathbb{E}[q_t(\theta) Y_t(\theta)] d\theta,$$

then

$$\mathbb{1}_{\{\sigma_1 \le t\}} \mathbb{P}(\sigma_2 = \sigma_1 | \mathcal{F}_t^1) = \mathbb{1}_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}.$$

In a similar way, we obtain (3.33). □

Since the random time σ_1 admits an $\mathbb{F}$-density, it is a totally inaccessible $\mathbb{F}^1$-stopping time. We are interested in the compensator process of σ_2 in the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \ge 0}$ which is the progressive enlargement of $\mathbb{F}^1$ by the random time σ_2, i.e. $\mathcal{G}_t = \bigcap_{s>t} \Big(\sigma(\{\sigma_1 \le u\}, \{\sigma_2 \le u\} : u \le s) \vee \mathcal{F}_t\Big)$. By Proposition 2.11, we know that σ_2 is a totally inaccessible $\mathbb{G}$-stopping time and the intensity exists.

Proposition 3.9. *The random time σ_2 has a $\mathbb{G}$-intensity given by*

$$\lambda_t^{2,\mathbb{G}} = \mathbb{1}_{\{\sigma_2>t\}}\bigg(\mathbb{1}_{\{\sigma_1>t\}}\frac{\int_t^\infty \beta_t(\theta_1,t)d\theta_1+q_t(t)}{\int_t^\infty\int_t^\infty \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2+\int_t^\infty q_t(\theta)d\theta}+\mathbb{1}_{\{\sigma_1\le t\}}\frac{\beta_t(\sigma_1,t)}{\int_t^{+\infty}\beta_t(\sigma_1,\theta)d\theta}\bigg).$$

Similarly, the $\mathbb{G}$-intensity of σ_1 is given by

$$\lambda_t^{1,\mathbb{G}} = \mathbb{1}_{\{\sigma_1>t\}}\bigg(\mathbb{1}_{\{\sigma_2>t\}}\frac{\int_t^\infty \beta_t(t,\theta_2)d\theta_2+q_t(t)}{\int_t^\infty\int_t^\infty \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2+\int_t^\infty q_t(\theta)d\theta}+\mathbb{1}_{\{\sigma_2\le t\}}\frac{\beta_t(t,\sigma_2)}{\int_t^{+\infty}\beta_t(\theta,\sigma_2)d\theta}\bigg).$$

Proof. By the symmetry between σ_1 and σ_2, it suffices to prove the first assertion. By [EKJJ10, Proposition 4.4], we obtain from (3.35) that the $\mathbb{F}^1$-compensator of the process $(\mathbb{1}_{\{\tau_1\le t\}})_{t\ge 0}$ is

$$\Lambda_t^1 := \int_0^{\sigma_1\wedge t}\frac{\alpha_s^1(s)}{G_s^1}ds, \quad t\ge 0.$$

By Lemma 2.10, the compensator of the $\mathbb{F}^1$-conditional survival process of σ_2 is $-A^{2|1}$ with

$$A_t^{2|1} = \int_0^t \alpha_\theta^{2|1}(\theta)d\theta + \int_0^t p_{s-}d\Lambda_s^1 + \langle M^1,p\rangle_t, \quad t\ge 0,$$

where $M_t^1 = \mathbb{1}_{\{\tau_1\le t\}} - \Lambda_t^1$. Note that $\langle M^1,p\rangle$ is the compensator of the process

$$\mathbb{1}_{\{\sigma_1\le t\}}\Delta p_{\sigma_1} = \mathbb{1}_{\{\sigma_1\le t\}}\bigg(\frac{q_{\sigma_1}(\sigma_1)}{\alpha_{\sigma_1}^1(\sigma_1)} - \frac{\int_{\sigma_1}^\infty q_{\sigma_1}(\theta)d\theta}{G_{\sigma_1}^1}\bigg), \quad t\ge 0,$$

which is

$$\int_0^{\sigma_1\wedge t}\frac{\alpha_s^1(s)H_s}{G_s^1}ds, \quad t\ge 0$$

by [EKJJ10, Corollary 4.6], where

$$H_t = \frac{q_t(t)}{\alpha_t^1(t)} - \frac{\int_t^\infty q_t(\theta)d\theta}{G_t^1}.$$

Hence we obtain that

$$A_t^{2|1} = \int_0^t \alpha_s^{2|1}(s)ds + \int_0^{\sigma_1\wedge t}\frac{q_s(s)}{G_s^1}ds.$$

Let

$$G_t^{2|1} = \mathbb{P}(\sigma_2>t|\mathcal{F}_t^1) = \frac{\mathbb{1}_{\{\sigma_1>t\}}}{G_t^1}\bigg(\int_t^\infty\int_t^\infty \beta_t(s,\theta)dsd\theta+\int_t^\infty q_t(\theta)d\theta\bigg)+\frac{\mathbb{1}_{\{\sigma_1\le t\}}}{\alpha_t^1(\sigma_1)}\int_t^{+\infty}\beta_t(\sigma_1,\theta)d\theta.$$

By Proposition 2.10, the $\mathbb{G}$-compensator of $(\mathbb{1}_{\{\sigma_2\le t\}})_{t\ge 0}$ is then

$$\Lambda_t^{2,\mathbb{G}} = \int_0^{\tau_2\wedge t}\frac{dA_s^{2|1}}{G_s^{2|1}}, \quad t\ge 0.$$

Therefore, the random time σ_2 has a $\mathbb{G}$-intensity

$$\lambda_t^{2,\mathbb{G}} = \mathbb{1}_{\{\sigma_2>t\}}\left(\mathbb{1}_{\{\sigma_1>t\}}\frac{\int_t^\infty \beta_t(\theta_1,t)d\theta_1 + q_t(t)}{\int_t^\infty\int_t^\infty \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_t^\infty q_t(\theta)d\theta} + \mathbb{1}_{\{\sigma_1\leq t\}}\frac{\beta_t(\sigma_1,t)}{\int_t^{+\infty}\beta_t(\sigma_1,\theta)d\theta}\right).$$

□

Remark 3.10. The relation

$$\mathbb{P}(\sigma_1\wedge\sigma_2 > t|\mathcal{F}_t) = \int_t^\infty\int_t^\infty \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_t^{+\infty} q_t(\theta)d\theta$$

shows that $\mathbb{F}$-intensity process of $\sigma_1\wedge\sigma_2$ is

$$\lambda_t^{\min} := \frac{\int_t^\infty \beta_t(\theta,t)+\beta_t(t,\theta)d\theta + q_t(t)}{\int_t^\infty\int_t^\infty \beta_t(\theta,\theta_2)d\theta_1 d\theta_2 + \int_t^\infty q_t(\theta)d\theta}.$$

Note that the relation

$$\mathbb{1}_{\{\sigma_1\wedge\sigma_2>t\}}\lambda^{\min} = \mathbb{1}_{\{\sigma_1\wedge\sigma_2>t\}}(\lambda_t^{1,\mathbb{G}} + \lambda_t^{2,\mathbb{G}})$$

does not hold in general.

In the model that we study, the default time σ_2 satisfies the generalised density hypothesis and can coincide with another default time σ_1, which is a stopping time in the reference filtration. Different from other examples, σ_1 is totally inaccessible, which implies that σ_2 is also totally inaccessible.

Appendix A

Some classic results

Definition A.1. A stopping time T is predictable if there exists an increasing sequence of stopping times $(T_n)_{n\geq 1}$ such that $T_n < T$ on $\{T > 0\}$ for all n and $\lim_{n\to\infty} T_n = T$ a.s.. Such a sequence (T_n) is said to announce T.

Definition A.2. A stopping time T is accessible if there exists a sequence of predictable stopping times $(T_n)_{n\geq 1}$ such that

$$\mathbb{P}\left(\bigcup_n \{\omega : T_n(\omega) = T(\omega) < \infty\}\right) = \mathbb{P}(T < \infty).$$

Definition A.3. A stopping time T is totally inaccessible if for every predictable stopping time $\bar{T}$,

$$\mathbb{P}\left\{\omega : T(\omega) = \bar{T}(\omega) < \infty\right\} = 0.$$

Definition A.4. The optional σ-algebra $\mathcal{O}$ is the σ-algebra, defined on $\mathbb{R}_+ \times \Omega$, generated by all adapted processes with càdlàg paths.

Definition A.5. The predictable σ-algebra $\mathcal{P}$ is the σ-algebra, defined on $\mathbb{R}_+ \times \Omega$, generated by all adapted processes with left-continuous paths on $(0, \infty)$.

Theorem A.6. *Let X be a measurable process either positive or bounded. There exists a unique (up to indistinguishability) optional ${}^oX$ such that*

$$\mathbb{E}[X_\sigma \mathbb{1}_{\{\sigma<\infty\}} | \mathcal{F}_\sigma] = {}^oX_\sigma \mathbb{1}_{\{\sigma<\infty\}} \quad a.s.$$

for any stopping time σ. The process ${}^oX$ is called the optional projection of X.

Theorem A.7. *Let X be a measurable process either positive or bounded. There exists a unique (up to indistinguishability) predictable ${}^pX$ such that*

$$\mathbb{E}[X_\sigma \mathbb{1}_{\{\sigma<\infty\}} | \mathcal{F}_{\sigma-}] = {}^pX_\sigma \mathbb{1}_{\{\sigma<\infty\}} \quad a.s.$$

for any predictable stopping time σ. The process ${}^pX$ is called the predictable projection of X.

Definition A.8. Let (A_t) be an integrable submartingale. We call dual optional projection of A the optional increasing process (A_t^o) defined by

$$\mathbb{E}\left[\int_{[0,\infty)} X_s \, dA_s^o\right] = \mathbb{E}\left[\int_{[0,\infty)} {}^oX_s \, dA_s\right],$$

for any bounded measurable process X. We call dual predictable projection of A the predictable increasing process (A_t^p) defined by

$$\mathbb{E}\left[\int_{[0,\infty)} X_s \, dA_s^p\right] = \mathbb{E}\left[\int_{[0,\infty)} {}^pX_s \, dA_s\right],$$

for any bounded measurable process X.

Theorem A.9 (Doob-Meyer decomposition)**.** *An adapted càdlàg process S is a submartingale of class (D) null at 0 if and only if S may be written as*

$$S = M + A$$

where M is a uniformly integrable martingale null at 0 and A a predictable integrable increasing process null at 0. Moreover, the decomposition above is unique.

Theorem A.10 (Itō and Watanabe, [IW65])**.** *Let (Z_t) be a nonnegative càdlàg supermartingale such that $Z_0 > 0$ a.s. Then, Z can be factorised as*

$$Z_t = N_t D_t$$

where N is a positive local martingale and B is a decreasing process. The factorisation above is unique on the random interval $\{t \geq 0 : Z_t(\omega) > 0\}$.

Theorem A.11 (Jeulin and Yor, [JY78]). *Let $\mathbb{G}$ the progressive enlargement of filtration $\mathbb{F}$ by τ. Then, the process*

$$\left(\mathbb{1}_{\{\tau\leq t\}} - \int_{(0,t\wedge\tau]} \mathbb{1}_{\{G_{s-}>0\}} \frac{dA_s}{G_{s-}}, \quad t\geq 0\right)$$

is a $\mathbb{G}$-martingale, where the process G is the Azéma supermartingale $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ and the process A is the $\mathbb{F}$-dual predictable projection of the submartingale $(\mathbb{1}_{\{\tau\leq t\}}, t\geq 0)$, or in other words, the predictable increasing process in the Doob-Meyer decomposition of $1-G$.

Theorem A.12 (Itō's forumla). *Let X be a semimartingale and f be a $\mathcal{C}^2$ function. Then,*

$$f(X_t) = f(X_0)+\int_0^t f'(X_{s-})dX_s+\frac{1}{2}\int_0^t f''(X_s)d\langle X^c, X^c\rangle_s+\sum_{0<s\leq t}[f(X_s)-f(X_{s-})-f'(X_{s-})\Delta X_s],$$

where X^c is the continuous part of X and $\Delta X = X - X_-$.

Let $\mathbb{F}$ be the reference filtration and τ a random time. Let $\mathbb{G}$ be the progressive enlargement of $\mathbb{F}$ by τ. Denote by Z the Azéma supermartingale of τ. The supermartingale Z can be decomposed as $Z = U - A$ where U is a BMO martingale and A is an optional increasing process.

Theorem A.13. *Every $\mathbb{F}$-martingale M stopped at τ is a $\mathbb{G}$-semimartingale with canonical decomposition*

$$M_{t\wedge\tau} = \tilde{M}_t + \int_{0^t\wedge\tau} \frac{d\langle M, U\rangle_s}{Z_{s-}},$$

where $\tilde{M}$ is a $\mathbb{G}$-local martingale.

Definition A.14. A random time τ is honest if τ is equal to an $\mathcal{F}_t$-measurable random variable on $\{\tau < t\}$.

Theorem A.15. *Let τ be an honest time. We assume that τ avoids stopping times. Then, evey $\mathbb{F}$-local martingale M is a $\mathbb{G}$-semimartingale with canonical decomposition*

$$M_{t\wedge\tau} = \tilde{M}_t + \int_{0^t\wedge\tau} \frac{d\langle M, U\rangle_s}{Z_{s-}} - \int_\tau^{t\vee\tau} \frac{d\langle M, U\rangle_s}{1-Z_{s-}},$$

where $\tilde{M}$ is a $\mathbb{G}$-local martingale.

Appendix B

Some proofs

B.1 Proof of Proposition 2.13

Proof: When $u \le t$, we only need to verify that, for any $w \le t$,

$$J(w) := \mathbb{P}(\tau \le w|\mathcal{F}_t) = \int_0^w \alpha_t(u)du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i$$

where we recall that $p_t^i = \mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$ are given by Proposition 1.8 as

$$p_t^i = \left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\right) e^{-\int_0^{\tau_i} \lambda_s\, ds}, \quad \text{on } \{\tau_i \le t\}, \quad i \in \{1, \dots, n\}.$$

Indeed, for any $w \le t$,

$$\begin{aligned}
J_t(w) &= \int_0^w \lambda_u e^{-\int_0^u \lambda_s\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i < u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i \\
&= -\int_0^w e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i < u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i \\
&= -\int_0^w \left(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} < u \le \tau_i\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds}\right) d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i \\
&= -\sum_{i=1}^{n+1} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} \int_0^w \mathbb{1}_{\{\tau_{i-1} < u \le \tau_i\}} d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i \\
&= -\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} \left(e^{-\int_0^{w \wedge \tau_i} \lambda_s\, ds} - e^{-\int_0^{\tau_{i-1}} \lambda_s\, ds}\right) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} p_t^i.
\end{aligned}$$

By rewriting explicitly p^i, one has

$$
\begin{aligned}
J_t(w) &= 1 - \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{w\wedge\tau_i} \lambda_s\, ds} + \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i\le w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{\tau_i} \lambda_s\, ds} \\
&= 1 - \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le w<\tau_i\}} e^{-\int_0^{w} \lambda_s\, ds - \int_0^{\tau_{i-1}} \lambda^N(s)ds} \\
&= 1 - e^{-\int_0^{w} \lambda_s\, ds - \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i\le w\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} \\
&= 1 - \mathbb{P}(\tau > w|\mathcal{F}_t) \\
&= \mathbb{P}(\tau \le w|\mathcal{F}_t).
\end{aligned}
$$

When $u > t$, we have by martingale property $\alpha_t(u) = \mathbb{E}[\alpha_u(u)|\mathcal{F}_t]$, which finishes the proof. □

B.2 Proof of Theorem 3.2

PROOF: 1. By pathwise uniqueness of the solution to the stochastic differential equation (3.16), one has

$$
\begin{aligned}
J(t,x,p,\varphi) &= \mathbb{E}\left[\int_t^{\theta\wedge\tau_t} \gamma(t,s) f(s, X_s^{t,x}, P_s^{t,p}, \varphi_s) ds + \gamma(t,\theta\wedge\tau_t) J(\theta\wedge\tau_t, X_{\theta\wedge\tau_t}^{t,x}, P_{\theta\wedge\tau_t}^{t,p}, \varphi)\right] \\
&= \mathbb{E}\left[\int_t^{\theta\wedge\tau_t} \gamma(t,s) f(s, X_s^{t,x}, P_s^{t,p}, \varphi_s) ds + \mathbb{1}_{\{\theta\le\tau_t\}} \gamma(t,\theta) J(\theta, X_\theta^{t,x}, P_\theta^{t,p}, \varphi)\right].
\end{aligned}
$$

Since $J(\theta, X_\theta, P_\theta, \varphi) \le v(\theta, X_\theta, P_\theta)$, we have proved the inequality

$$
\begin{aligned}
v(t,x,p) \le \sup_{\varphi\in\mathcal{A}(t,x,p)} \mathbb{E}\Bigg[&\int_t^{\theta\wedge\tau_t} \gamma(t,s) f(s, X_s^{t,x}, P_s^{t,p}, \varphi_s) ds \\
&+ \mathbb{1}_{\{\theta\le\tau_t\}} \gamma(t,\theta) v(t, X_\theta^{t,x}, P_\theta^{t,p})\Bigg]. \qquad \text{(B.1)}
\end{aligned}
$$

2. We fix some arbitrary control $\varphi \in \mathcal{A}(t,x,p)$ and θ valued in $[t,T]$. By definition of the value function, for any $\varepsilon > 0$ and $\omega \in \Omega$, there exists $\varphi^\varepsilon(\omega) \in \mathcal{A}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), P_{\theta(\omega)}^{t,p}(\omega))$, which is an ε-optimal control for the problem $v(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), P_{\theta(\omega)}^{t,p}(\omega))$, i.e.,

$$
v(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), P_{\theta(\omega)}^{t,p}(\omega)) - \varepsilon \le J(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), P_{\theta(\omega)}^{t,p}(\omega), \varphi^\varepsilon(\omega)).
$$

Define the process

$$\tilde{\varphi}_s(\omega) = \begin{cases} \varphi_s(\omega), & s \in [t, \theta(\omega)], \\ \varphi^\varepsilon_s(\omega), & s \in [\theta(\omega), T]. \end{cases}$$

By measurable selection theorem, the process $\tilde{\varphi} = (\tilde{\varphi}_t)_{0 \le t \le T}$ is progressively measurable and lies in $\mathcal{A}(t,x,p)$. Then we have

$$\begin{aligned} J(t,x,p,\tilde{\varphi}) &= \mathbb{E}\left[\int_t^{\theta \wedge \tau_t} \gamma(t,s) f(s, X^{t,x}_s, P^{t,p}_s, \varphi_s) ds + \mathbb{1}_{\{\theta \le \tau_t\}} \gamma(t,\theta) J(\theta, X^{t,x}_\theta, P^{t,p}_\theta, \varphi^\varepsilon)\right] \\ &\ge \mathbb{E}\left[\int_t^{\theta \wedge \tau_t} \gamma(s,t) f(s, X^{t,x}_s, P^{t,p}_s, \varphi_s) ds \right. \\ &\qquad \left. + \mathbb{1}_{\{\theta \le \tau_t\}} \gamma(t,\theta) v(\theta, X^{t,x}_\theta, P^{t,p}_\theta)\right] - \varepsilon \mathbb{E}[\mathbb{1}_{\{\theta \le \tau_t\}} \gamma(t,\theta)] \\ &\ge \mathbb{E}\left[\int_t^{\theta \wedge \tau_t} \gamma(s,t) f(s, X^{t,x}_s, P^{t,p}_s, \varphi_s) ds + \mathbb{1}_{\{\theta \le \tau_t\}} \gamma(t,\theta) v(\theta, X^{t,x}_\theta, P^{t,p}_\theta)\right] - \varepsilon. \end{aligned}$$

By arbitrariness of $\varphi \in \mathcal{A}(t,x,p)$ and $\varepsilon > 0$, we get the inequality

$$\begin{aligned} v(t,x,p) \ge \sup_{\varphi \in \mathcal{A}(t,x,p)} \mathbb{E}&\left[\int_t^{\theta \wedge \tau_t} \gamma(t,s) f(s, X^{t,x}_s, P^{t,p}_s, \varphi_s) ds \right. \\ &\left. + \mathbb{1}_{\{\theta \le \tau_t\}} \gamma(t,\theta) v(t, X^{t,x}_\theta, P^{t,p}_\theta)\right]. \end{aligned} \tag{B.2}$$

The theorem is thus proved by combining the two inequalities (B.1) and (B.2). ∎

List of Figures

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