Formal reduction of differential systems : Singularly-perturbed linear differential systems and completely integrable Pfaffian systems with normal crossings

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UNIVERSITÉ DE LIMOGES

ÉCOLE DOCTORALE no 521

FACULTÉ DES SCIENCES ET TECHNIQUES

Année : 2015 Thèse N° X

Thèse

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE LIMOGES

Discipline : Mathématiques et ses Applications

présentée et soutenue par

Suzy S. MADDAH

le 25 Septembre 2015

Formal Reduction of Differential Systems:

Singularly-Perturbed Linear Differential Systems

and Completely Integrable Pfaffian Systems with Normal Crossings

Thèse dirigée par Moulay A. BARKATOU

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Sumayya Suzy MADDAH

Always to my mother, my father,
and my sisters, Layal, Lina, Lara, and Sally

Acknowledgements

Foremost I would like to express my deepest gratitude to Moulay Barkatou for his generous guidance in time and knowledge, and continuous support over the course of my thesis. I was extremely lucky to have him as an advisor. Working with him was extremely enriching on the scientific and the personal levels.

I am greatly indebted to each member in my defense jury for the privilege. I would like to thank the president of the jury Reinhard Schäfke. I deeply appreciate his criticisms and patience which have been indispensable to the improvement of my work. Many thanks are due to George Labahn for his continuous encouragement and support, Carole El Bacha and Jorge Mozo Fernández for their suggestions and attentive reading of the manusript, Nobuki Takayama for references on Pfaffian systems, and Frédéric Chyzak for several discussions over the past two years. I am looking forward to working with him.

This thesis was "en-cotutelle" with the Lebanese University. I would like to thank all my professors in Beirut, especially my co-advisor Hassan Abbas for his continuos support, and Ali Wehbi for participating in my jury. I am also indebted to Ahmed Atiyya at Beirut Arab Univeristy for his encouragement.

I would like to thank all the members of DMI. My special thanks are due to Jacques-Arthur Weil, Marc Rybowicz, Olivier Ruatta, and Thomas Cluzeau for their encouragement and advices. I am also greatly indebted to Karim Tamine, Djamchid Ghazanfarpour, Olivier Terraz, Philippe Gaborit, and Guillaume Gilet for giving me the opportunity to work with them. I learned immensely from them within the ATER experience. I also take the chance to thank Sergei Abramov and Clemens Raab for discussions during their visits to Limoges and encouragements.

I extend my gratitude to Odile Duval, Annie Nicolas, Débora Thomas, and Yolande Vieceli, for their endless administrative help; and to Henri Massias for the technical help.

I would like to thank each of the SpecFun team at INRIA Saclay, IRMA Institute at the University of Strasbourg, and the Graduate Center at the City University of New York for seminar invitations. A special thanks goes to Aline Bostan, Loïc Teyssier, and Alexey Ovchinnikov. I also like to thank the organizers and participants of the numerous conferences in which I have presented parts of this work. And I thank Joris van der Hoeven and Gregoire Lecerf for their help and advices with Mathemagix.

My sincere thanks also goes to all my colleagues in Mathematics and Computer Science: Nora El Amrani, Yoann Weber, Zoé Amblard, Paul Germouty, Jean-Christophe Deneuville, Nicolas Pavie, Mohamad Akil, Gael Thomas, Charlotte Hulek, Amaury Bittmann, Georg Grasegger ... and especially my very dear friends over these three years, Esteban Segura Ugalde, Ekatarina Ilinova, Slim Bettaieb, Sajjad Ahmed, Georges El Nashef, Clement Potier, Karthik Poorna, and Denisse Munante, for the continuous support in the highs and lows.

My deepest appreciation is due to Maximilian Jaroschek for what we share on the personal and professional levels; and also to Sebastien Bouquillon and Guillaume Olivrin for charming lessons in astronomy and artificial intelligence respectively.

I would also like to thank the very beautiful ladies Sina Zietz, Lulu Zhao, Seldag Bankir, Lina Linan, Jasmin Hammon, and Irma Sivileviciute, for all the fun we had in Limoges. And finally, I thank all my friends who attended or helped organize my defense, especially Fatima Saissi and Oussama Mortada.

Contents

Global Notations

This is a list of global notations. Local notations will be set at the beginning of each of the three parts.

- $\mathbb{N}$ is the set of natural integers;
- $\mathbb{Z}$ is the ring of integers;
- $\mathbb{C}$ is the field of complex numbers;
- $\mathcal{K}$ is a computable commutative field of characteristic zero ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$).
- We use upper case letters for algebraic structures, matrices, and vectors (except for easily identified cases); the lower case letters i, j, k, l for indices; and some lower case letters (greek and latin) locally in proofs and sections, such as u, v, σ, ρ, ϱ, etc...
- We use $x,\ t,\ \tau$ as independent variables ($x_i,\ t_i$ in the case of several variables) and ε as a parameter.
- We use n to denote the dimension of the systems considered in this thesis; R to denote the algebraic rank of specified matrices (leading coefficient matrices); p (resp. p_i in case of several variables) to denote the Poincaré rank, i.e. order of pole in x (resp. x_i); h to denote the order of the pole in ε and ξ (also called ε-rank and ξ-rank respectively).
- We use $F,\ G,\ U,\ W,\ Z$ for unknown n-dimensional column vectors (or $n \times n$-matrices). We use $f,\ g$, and w in the case of scalar equations.
- We give the blocks of a matrix M with upper indices, e.g.

$$M = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}.$$

  The size of the different blocks is dropped unless it is unclear from the context.
- $I_{d\times n}$ (resp. I_n) stands for the identity matrix of dimensions $d \times n$ (resp. $n \times n$); and $O_{d\times n}$ (resp. O_n) stands for the zero matrix of dimensions $d \times n$ (resp. $n \times n$); the dimensions are dropped whenever confusion is not likely to arise.
- We say that $A \in \mathcal{M}_n(\mathrm{R})$ whenever the matrix A is a square matrix of size n whose entries lie in a ring R.
- V is used to signify vector spaces of dimension n.
- $GL_n(\mathrm{R})$ (resp. $GL(\mathrm{V})$) is the general linear group of degree n over R (the set of $n \times n$ invertible matrices together with the operation of matrix multiplication).
- We use ∂ to signify a derivation. x dominates this thesis and so we drop it in the derivation. And so we write $\partial = \frac{d}{dx}$, $\partial_t = \frac{d}{dt}$, $\partial_\tau = \frac{d}{d\tau}$ and $\partial_i = \frac{d}{dx_i}$, $\partial_{t_i} = \frac{d}{dt_i}$ in the case of several variables.

- We deal mainly with three systems which are denoted throughout this thesis as follows:
  - $[A]$ represents a singular linear ordinary differential system (ODS, in short) ;
  - $[A_\varepsilon]$ and $[A_\xi]$ represent a singularly-perturbed linear differential system;
  - $[\underline{A}]$ represents a completely integrable Pfaffian system with normal crossings.

How to read this thesis?

- The outline of the thesis as a whole is given at its beginning. Furthermore, each part has its own introduction where the literature review is given, the content of individual chapters is outlined, and the main contributions are identified.
- The list of global notations which will be used within the thesis as a whole was given. However, each of three parts has its own notations as well which is given within their local introduction.
- In the sequel of the thesis, we suppose that the base field is $\mathbb{C}$ for the simplicity of the presentation. However, the base field can be any computable commutative field $\mathcal{K}$ of characteristic zero ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$). The algorithms presented are refined within their implementation to handle efficiently field extensions as explained in Chapter 1 and Section 3.3.
- Chapter 1 is written in a simplified manner and serves as an introduction to the field of study. It describes the local analysis of an ODS $[A]$.
- Chapters 2 discusses the perturbed algebraic eigenvalue problem and can be read independently from the other chapters of the thesis.
- Chapter 3 treats apparent singularities of linear systems with rational function entries and can be read independently from the other chapters of the thesis.
- Part II investigates the construction of formal solutions of singularly-perturbed linear differential systems and can be read independently.
- Part III investigates the construction of formal solutions of completely integrable Pfaffian systems with normal crossings at the origin and can be read independently.
- The users of computer algebra systems come from a multitude of disciplines. Those who are interested in examples of computations solely, can skip to Appendix B where the description of the packages is adjoined.

Gentle Introduction

The study of differential equations stands out as one of the most extensive branches of mathematics, lying at the core of modeling in natural, engineering, and increasingly social sciences. It started off in the year 1675 with the ordinary differential equations arising naturally from the efforts to seek a solution of physical problems. However, the quest of general methods to solve thoroughly differential equations at hand was unfruitful except for very special cases and thus was abandoned after 100 years [117]. Since then, diverse approaches have flourished depending on the classification of the equations.

The local analysis of n^{th}-order linear differential equations and first-order linear differential systems is attained via the computation of their series solutions. At ordinary points, it suffices to consider Taylor series (power series). Any engineering student or scientist is familiar with the resolution procedure and popular computer systems always consider a package for this goal. However, singular points are another story and are the interest of this thesis.

In general, a singularity is a point at which a mathematical object is not defined or not well-behaved. In the words of Harvard Theoretical physicist Andrew Strominger: "We know what a singularity is, it is when we don't know what to do"*. In the second half of the twentieth century, the mathematical knowledge has integrated with electronic computers. Computer algebra, i.e. the application of computers to exact ("almost" exact, in contrast to numeric) mathematical computation, is a prominent outcome of this integration and a dynamic field of contemporary research tackling a multitude of mathematical problems (see, e.g., [62, 56]). While numerical computation involves numbers directly and their approximations, symbolic computation involves symbols and exact values and manipulates algebraic expressions.

*This quote is from a BBC Horizon interview entitled " Who's Afraid of a Big Black Hole?" which addresses the public. This quote is figurative to say that classical methods are not adequate for such problems. It does not really mean that we really do not know what to do. The interview can be accessed at:
$https://www.youtube.com/playlist?list=PLAB9F4AFC3B789C74$. Another way to view singularities is to consider critical points, or black holes: the infinite gravity and infinite density concentrated at an infinitesimally small region. Or alternatively, consider a box of surprises: If we have sufficient information about the event, the people invited, etc..., we might guess the exact content of the box or an approximation to it. However, there remains a chance, inversely proportional to our information and guessing techniques, that we get really surprised. A symbolic solution is the solution to this question: What is happening? What is the "exact" content of the box?

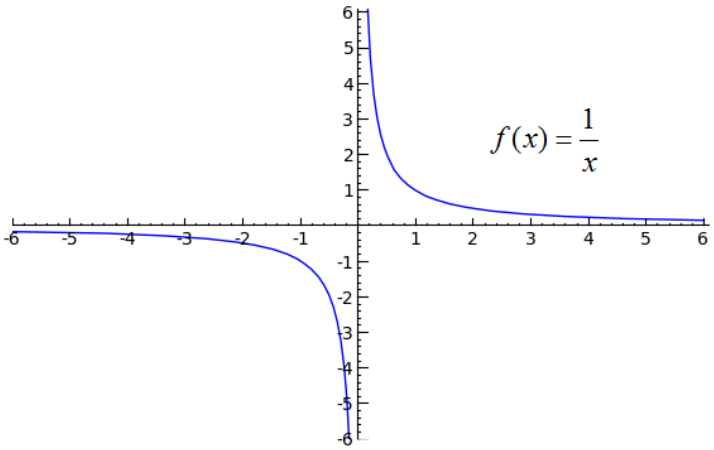


Figure 1: The function $1/x$ is not defined at $x = 0$ and explodes to $\pm\infty$ in a neighborhood of $x = 0$ (Sage calculus tutorial)

Symbolic representation	Numeric representation
$\frac{1}{8}$	0.125
π	$3.14159\ldots$
$tan(24)$	$-2.13489\ldots$

Symbolic computation	Numeric computation
$\int_0^1 xe^{-x}dx = 1 - 2e^{-1}$	$\int_0^1 xe^{-x}dx = 0.26424...$

For this purpose, the language of abstract algebra is used in symbolic computation.

Symbolic Manipulation		
$2 + 3 \;=\; 5$	$\rightarrow$	$x + (x + 1) = 2x + 1,\ x \in \mathbb{Z}$

Such an abstraction serves as an umbrella under which mathematical objects, e.g. equations and differential equations, linear operators and linear differential operators, can be handled analogously. This language is then translated into a code in computer algebra softwares, e.g. MAPLE, MATHEMATICA, and MATHEMAGIX.

Traditionally, a numerical solution is sought whenever the analytical (symbolic) solution cannot be found or have a very high cost in terms of time and memory. And this happens very often. Nevertheless, even in the latter case, a symbolic investigation can still give valuable information about the solution which contributes to the improvement of the numerical methods themselves.

Given the differential equation [11, Preface]

$$x^2 \frac{d}{dx} f \; - \; f \; + \; x = 0,$$

a "traditional" or "true" solution to a differential equation is a solution which satisfies the equation formally, and possesses some smoothness properties. The power series $f(x) = \sum_{k=0}^{\infty} k! x^{k+1}$ can be verified to satisfy the former condition but not the latter since it is divergent. One can explicitly compute solutions of this particular equation in closed form so we do not need to bother with divergent series. However, this is not the case with differential equations and systems in general and the utility of such divergent series stems from this fact. Such divergent series can give valuable information about the asymptotic behavior of "true" solutions.

This thesis is concerned with symbolic resolution, i.e. the explicit computation of such series representations of solutions in a neighborhood of a singularity, rather than their analytic properties.

Thesis Outline

Let K be a commutative field of characteristic zero equipped with a derivation ∂, that is a map $\partial : K \to K$ satisfying

$$\partial(f+g) = \partial f + \partial g \quad \text{and} \quad \partial(fg) = \partial(f)g + f\partial(g) \quad \text{for all} \quad f, g \in K.$$

We denote by $\mathcal{K}$ its field of constants and V an n-dimensional K-vector space. Let A be an element of $\mathcal{M}_n(K)$. Setting $\Delta = \partial - A$, Δ is a ∂-differential operator acting on V, that is, a $\mathcal{K}$-linear endomorphism of V satisfying the Leibniz condition:

$$\forall f \in K,\ v \in V, \quad \Delta(fv) = \partial(f)v + f\Delta(v).$$

By setting $\partial = \frac{d}{dx}$ and $K = \mathcal{K}((x))$, the field of formal Laurent series in x over $\mathcal{K}$, Δ is associated to the widely studied **singular linear ordinary differential system** (ODS). There exist several algorithms, contributing to the **formal reduction** of such systems, i.e. the algorithmic procedure that computes a change of basis w.r.t. which the matrix representation of the differential operator has a normal form. This form allows the construction of a **fundamental matrix of formal solutions** (FMFS).

This thesis is mainly concerned in giving a two-fold generalization of such algorithms to treat a wider class of systems over bivariate and multivariate fields. Namely,

- let $\mathcal{K}[[x]][[\varepsilon]]$ be the ring of formal power series in a parameter ε whose coefficients lie in $\mathcal{K}[[x]]$, the ring of formal power series in x over $\mathcal{K}$. Let K be its field of fractions and set again $\partial = \frac{d}{dx}$. We are interested in the operator Δ_ε associated to a **singularly-perturbed linear differential system**;
- let $\mathcal{K}[[x_1, x_2, \dots x_m]]$, $m \geq 2$, be the ring of formal power series in $x = (x_1, x_2, \dots, x_m)$ over $\mathcal{K}$ and K its field of fractions. Setting $\partial_i = \frac{\partial}{\partial x_i}$, we are interested in **completely integrable Pfaffian systems with normal crossings** associated to

  $$\Delta_i F = 0,\ 1 \leq i \leq m,$$

  for which the following integrability conditions are satisfied (pairwise commutativity of the operators),

  $$\Delta_i \circ \Delta_j = \Delta_j \circ \Delta_i, \quad 1 \leq i, j \leq m.$$

The systems associated to these operators are discussed in three separate parts as explained below.

Part I: Singular Linear differential systems

The first part of this thesis is devoted to topics in local and global analysis over univariate fields. It encompasses three chapters the first of which recalls the local analysis of linear singular differential systems. We describe the formal reduction and its underlying algorithms, leading to the construction of solutions (e.g. uncoupling of the system into systems of lower dimensions, dropping the rank of the singularity to its minimal integer, computing formal invariants).

The perturbed algebraic eigenvalue problem arises naturally within the discussion of the first chapter and so it is discussed in a second chapter (∂ is set to a trivial (zero) derivation and consequently Δ is a linear operator). Applications to this problem arise in numerous fields as well including quantum physics and robust control.

The third chapter is dedicated to removing apparent singularities of systems whose entries are rational functions ($\partial = \frac{d}{dx}$ and $K = \mathcal{K}(x)$).

Main contribution

- The formal reduction algorithm described in the Chapter 1 lead to the package MINIISOLDE [96] written in MAPLE, and the package LINDALG [96] written in MATHEMAGIX, the open source computer algebra and analysis system [135].
- The new algorithm for removing apparent singularities given in Chapter 3 lead to the paper [27] with M.A. Barkatou and a MAPLE package [100] APPSING.

Part II: Singularly-perturbed linear differential systems

The second part of this thesis is dedicated to the study of the singularly-perturbed linear differential systems. Such systems are traced back to the year 1817 [140, Historical Introduction] and are exhibited in a myriad of problems within diverse disciplines including astronomy, hydraulics (namely the work of Prandtl (1904) and his students [112]), and quantum physics [89, 103, 35]. Their study encompasses a vast body of literature as well (see, e.g. [43, 140, 103] and references therein).

It was the hope of Wasow, in his 1985 treatise summing up contemporary research directions and results on such systems, that techniques developed for the treatment of their unperturbed counterparts be generalized to tackle their problems. The bivariate nature of such systems and the phenomenon of turning points renders such a generalization nontrivial. The investigation in this part addresses these obstacles.

Main contribution

- We give an algorithm to compute (outer) formal solutions in a neighborhood of a turning point. The results of this part amount to a paper [2] with A. Barkatou and H. Abbas, a paper under redaction with M. Barkatou [28], and a MAPLE package PARAMINT [98].

Part III: Completely Integrable Pfaffian Systems with Normal Crossings

"Partial differential equations are the basis of all physical theorems" (Bernhard Riemann 1826-1866). In particular, Pfaffian systems arise in many applications including the studies of aerospace and celestial mechanics. By far, the most important for applications are those with normal crossings [111], which are discussed in the third part of this thesis. Although it was not always stated explicitly in physical models that one is simply applying the general theory of Pfaff forms (1-forms), such forms play a broad-ranging role. Fundamental fields of classical physics are defined in terms of forces per unit mass, charge, etc.. And generally forces take that form[†].

A univariate completely integrable Pfaffian system with normal crossings reduces to the ODS discussed in Chapter 1, for which, unlike the multivariate case considered herein, algorithms have been developed. Theoretical studies guarantee the existence of a change of basis which takes the system to Hukuhara-Turrittin's normal form from which the construction of a fundamental matrix of formal solutions is straightforward. However, the formal reduction, that is the algorithmic procedure computing such a change of basis, is a question of another nature and is the interest of this part.

Main contribution

- We give a formal reduction algorithm for bivariate systems which amounts to the paper [1] with H. Abbas and M. Barkatou, and the MAPLE package [99] PFAFFINT. These results are extended under certain conditions to multivariate systems in a submitted paper with M. Barkatou and M. Jaroschek [25].

We conclude each of the three parts separately and give items for further research. There exist several additional notions and approaches to treat an ODS. In this thesis, we only recall notions which correspond to our generalizations. However, we point out in Appendix A some possible future investigations over bivariate and multivariate fields.

A large part of this thesis is code lines. The pseudo code given for some of the algorithms within the chapters serves the purpose of clarification only and does not coincide with the implementation. The description of some of the functionalities of the packages and some examples of computations are adjoined in Appendix B. The packages are available on my web page for download with their manuals.

[†]We refer to [49] where one can find examples from physical systems and references to early notable advances in the understanding of such problems

Part I

Singular Linear Differential Systems

Introduction

> *The theory of linear differential equations is so powerful that one can usually predict the local behavior of the solutions near a point x_0 without knowing how to solve the differential equation. It suffices to examine the coefficient functions of the differential equation in the neighborhood of x_0. [...] Even when the solution to a differential equation can be expressed in terms of the common higher transcendental functions, the techniques of local analysis are still very useful. For example, saying that the solutions to $\partial^2 f = x^4 f$ are expressible in terms of modified Bessel functions of order $1/6$ does not convey much qualitative information to someone who is not an expert on Bessel functions. On the other hand, an easy local analysis of the differential equation shows that solutions behave as linear combinations of $x^{-1} \exp\ (\pm x^3/3)$ as $x \to +\infty$*[‡].

We consider in this first part a linear system of n ordinary differential equations (ODS) in the neighborhood of a singular point:

$$[A] \qquad x^{p+1}\partial F \ = \ A(x)\ F \ = \ (\sum_{k=0}^{\infty} A_k x^k)\ F \tag{1}$$

where p is an integer, and the $A(x)$ is holomorphic in some region (D) of the complex plane. Without loss of generality, we have assumed that the singularity lies at the origin. Otherwise, a simple translation of the independent variable can be performed. Moreover, the change of variable $x \mapsto 1/x$ permits to classify the point $x = \infty$. If $p = 0$ then we say we have a *first kind singularity* at $x = 0$. And if $p > 0$ then we say we have a *second kind singularity*.

It is well-known that in a neighborhood of an *ordinary point* ($p \leq -1$), a solution possesses a power (Taylor) series representation locally (see, e.g., [139, The Existence Theorem, pp 3]). Moreover, the former is holomorphic in the largest disc contained in (D). A classical example is the Airy equation which is given by

$$\frac{d^2}{dx^2} f \ = \ x f,$$

[‡]C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*, Vol. 1. Springer, 1999, quoted from the introduction of Chapter 3

and whose first-order system representation is easily obtained by setting $F = (f, \partial f)^T$:

$$\partial F \;=\; \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} F.$$

The general solution near $x = 0$ is

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{x^{3k}}{9^k\ k!\ \Gamma(k + 2/3)} + c_1 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{9^k\ k!\ \Gamma(k + 4/3)},$$

where Γ stands for the $Gamma$ function and c_0, c_1 depend on the initial conditions[§]. However, if $p \geq 0$ then such a power series representation breaks downs even in simple scalar examples.

| Example 0.1

[35, Exm 1, pp 68] Given $\partial^2 f + \frac{1}{4x^2} f = 0$ whose first-order system representation is:

$$x^2\ \partial F \;=\; \begin{bmatrix} 0 & x^2 \\ -1/4 & 0 \end{bmatrix} F.$$

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. A formal substitution in the given equation yields:

  $$a_0 = a_1 = 0, \text{ and } (4k(k-1) + 1)a_k = 0,\ \forall\ k \geq 2.$$

  Thus, this presentation gives the trivial solution $f(x) = 0$.
- However, considering a Frobenius series representation, $y(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k$ where $\alpha \in \mathbb{Q}$ and $a_0 \neq 0$, it is easy to verify that: $f(x) = a_0\sqrt{x}$.

□

Frobenius series representations may break down as well as illustrated in the following example.

| Example 0.2 ([35], Exm 2, pp 77)

Given

$$x^3\ \partial^2 f \;=\; f$$

whose first-order system representation with $F = [f, \partial f]^T$ is:

$$x^3\ \partial F \;=\; \begin{bmatrix} 0 & x^3 \\ 1 & 0 \end{bmatrix} F.$$

Let $f(x) = x^\alpha \sum_{k=0}^{\infty} a_k x^k$ where $\alpha \in \mathbb{R}$ and $a_0 \neq 0$. Substituting as above we have:

$$(k - 1 - \alpha)(k - 2 + \alpha)a_{k-1} - a_k = 0\ \forall\ k \geq 1$$

[§]One may consult The Encyclopedia of Special Functions, *Microsoft Research - INRIA joint Centre*, at: *http : //algo.inria.fr/esf/*

and $a_0 = 0$ which is an immediate contradiction. Hence, no Frobenius representation exists for this example.

□

For a system $[A]$ with $p \geq 0$, we call p the Poincaré rank. Such systems have been studied extensively (see, e.g., [11, 139] and references therein). It is well-known that a solution is, in general, the product of not only a matrix of formal power series in a root of x (Airy equation at $x = 0$) and a matrix power of x (Example 0.1), but also an exponential of a polynomial in a root of x^{-1} (Example 0.2).
Consider again Airy equation but for large $|x|$ ($x \mapsto 1/x$). It is known to possess two linearly independent solutions, the Airy functions of the first and second kind, $Ai(x)$ and $Bi(x)$, having the following asymptotic representation (see, e.g. [139, Ch. VI]):

$$\begin{aligned} Ai\ (x) &= \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(\frac{-2}{3x^{3/2}}\right) [1 + O(|x|^{-3/2})], \\ Bi\ (x) &= \frac{1}{\sqrt{\pi}} x^{-1/4} \exp\left(\frac{2}{3x^{3/2}}\right) [1 + O(|x|^{-3/2})]. \end{aligned}$$

More generally, it follows from Hukuhara-Levelt-Turrittin normal form, which is an explicitly prescribed form closely resembling Jordan canonical form (JCF) of matrix presentations of linear operators, that a fundamental matrix of formal solutions (FMFS) of $[A]$ is given by (see, e.g., [133, 85, 139])

$$\Phi(x^{1/s})\ x^C\ \exp(Q(x^{-1/s})), \tag{2}$$

where

- s is a positive integer which we refer to as the *ramification index*;
- Φ is a matrix of meromorphic series in $x^{1/s}$ (root-meromorphic in x) over $\mathbb{C}$;
- $Q(x^{-1/s})$ is the *exponential part*. It is a diagonal matrix whose entries are polynomials in $x^{-1/s}$ over $\mathbb{C}$ without constant terms.
- C is a constant matrix which commutes with $Q(x^{-1/s})$.

This gives another classification of the singularity: If $Q(x^{-1/s})$ is the zero matrix then $x = 0$ is a *regular singular point*. In this case, $s = 1$ and the formal series $\Phi(x^{1/s})$ converges whenever the series of $A(x)$ does: All solutions grow at most like a finite power of $|x|$. Otherwise, $x = 0$ is an *irregular singular point* (see, e.g. [139, Chapter 4, Section 2, pp 111]), and the elements of $Q(x^{-1/s})$ determine the main asymptotic behavior of actual solutions as $x \to 0$ in sectors of sufficiently small angular opening (see, e.g. [139, Theorem 19.1, pg 110]). This classification however, based upon the knowledge of a FMFS, is not immediately apparent for a given differential system and will be discussed further in Section 1.5.1 and Chapter 3.
Contrary to the other systems to be considered in this thesis, i.e. the singularly-perturbed linear differential system and the Pfaffian system, algorithms to related problems leading to the construction of formal solutions of system $[A]$ and n^{th}-order scalar differential equations have been developed by various authors (see, e.g., [14, 24, 29, 32, 121, 137, 125, 66, 64, 101] and references therein).

- In Chapter 1, we recall the formal reduction of system $[A]$. We outline the main algorithm given in [14], and its underlying procedures. As we are interested in formal reduction, we drop reference to analytic/asymptotic properties. In the latter direction, one may consult [139, 11] and references therein.
- As we will see in Chapter 1, the one-parameter algebraic eigenvalue problem is central to the theory of differential systems. In Chapter 2, we describe briefly two direct approaches tackling this problem: The approach of [74, 75] which is based on Lidskii's genericity conditions and inspired by the techniques developed within the treatment of system $[A]$; and the approach of [4] within the realm of tropical algebra.
- The study of these system lead to questions of global nature as well. In the scalar univariate case, several desingularization algorithms exist for differential, difference, and more generally, Ore operators. However, the apparent singularity of systems of first-order linear differential equations with rational function coefficients, can be removed by acting directly on it, rather than on the equivalent scalar equation. This is the subject of Chapter 3.

Local Notations

- $\mathcal{K}[[x]]$ is the ring of formal power series in x whose coefficients lie in a commutative field $\mathcal{K}$ of characteristic zero ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$). $\mathcal{K}((x))$ is its fraction field, namely the field of formal meromorphic (or Laurent) series in x with coefficients in $\mathcal{K}$;
- $\mathcal{K}[x]$ is the ring of polynomials in x over $\mathcal{K}$ and $\mathcal{K}(x)$ is the field of rational functions in x over $\mathcal{K}$ (to be used in Chapter 3);
- ∂ denotes the derivation $\frac{d}{dx}$;
- We recall that a valuation of $\mathcal{K}((x))$ indicates the order in x of an element $a(x)$ of this field at zero ($val_x(0) = \infty$). It is defined by the map $val_x : \mathcal{K}((x)) \to \mathbb{Q} \cup \infty$ which satisfies the following properties for all $a(x),\ b(x)$ in $\mathcal{K}((x))$:
  1. $val_x\ (a)\ = \infty$ if, and only if, $a = 0$;
  2. $val_x\ (ab)\ = val_x\ (a) + val_x\ (b)$;
  3. $val_x\ (a + b)\ \geq min\ (val_x\ (a),\ val_x\ (b))$, and equality holds if $val_x\ (a) \neq val_x\ (b)$;

Chapter 1

Formal Reduction (Computing a FFMS)

Contents

Consider again system $[A]$ given by (1):

$$[A] \qquad x^{p+1}\partial F \;=\; A(x)\;F \;=\; (\sum_{k=0}^{\infty} A_k x^k)\;F,$$

where $p \geq 0$ and $A(x) \in \mathcal{M}_n(\mathcal{K}[[x]])$. As mentioned in the introduction, formal reduction is the algorithmic procedure that constructs a fundamental matrix of formal solutions (2) of $[A]$ (see, e.g., [14, 114], and references therein). A recursive algorithmic procedure attaining formal reduction was developed by Barkatou in [14]. As in the classical approach, it consists of computing at every step, depending on the nature of the eigenvalues of the leading matrix coefficient A_0, a *transformation* (change of basis). Accordingly, a block-diagonalized *equivalent* system, an *irreducible* equivalent system, or the *formal exponential order* are computed. These operations will be described individually in the following sections and will be generalized in the later chapters of this thesis. Upon performing this transformation, the resulting (uncoupled) system(s) are of either dimension(s) or Poincaré rank(s) lower than that of $[A]$. Based on the former operations, the packages ISOLDE [31] and LINDALG [96] written respectively in MAPLE and MATHEMAGIX, are dedicated to the symbolic resolution of such systems.

1.1 Gauge transformations

Given system $[A]$, let $T(x) \in GL_n(\bar{\mathcal{K}}((x)))$. A transformation (change of basis) $F = T\ G$ yields

$$[\tilde{A}] \qquad x^{\tilde{p}+1}\ \partial G = \tilde{A}(x)\ G, \tag{1.1}$$

where $\tilde{A} \in \mathcal{M}_n(\bar{\mathcal{K}}[[x]])$, $\tilde{p}$ is an integer, and

$$\frac{\tilde{A}}{x^{\tilde{p}+1}} = T^{-1}\ \frac{A}{x^{p+1}}\ T\ -\ T^{-1}\ \partial T. \tag{1.2}$$

We say that system $[\tilde{A}]$ is *equivalent* to system $[A]$ via $T(x)$ and we write $[\tilde{A}] := T[A]$. One can observe that this transformation deviates from similarity with the term $T^{-1}\ \partial T$. The theory of differential operators henceforth deviates from that of linear operators. We refer to $T^{-1}AT$ as the *similarity term* of (1.2). Two special types are the constant transformation and the shearing transformation. We remark that in literature, T is referred to sometimes as *gauge transformation* or *coordinate transformation*.

Constant Transformations

Let $T \in GL_n(\bar{\mathcal{K}})$. Clearly, $F = T\ G$ yields system $[\tilde{A}]$ such that $\tilde{p} = p$ and $\tilde{A}\ =\ T^{-1}\ A\ T$.

Shearing Transformation

The shearing transformation is a polynomial transformation of the general form:

$$S\ =\ Diag\ (x^{\alpha_1}, \ldots, x^{\alpha_n}), \quad \text{where} \quad \alpha_1, \ldots, \alpha_n \in \mathbb{Z}.$$

Such transformations can change the leading matrix coefficient radically upon altering its eigenvalues. Given system $[A]$, it follows from (1.2) that the transformation $F = T\ G$ yields system $[\tilde{A}]$ such that for $p = \tilde{p}$ we can write

$$\tilde{A} = S^{-1}\ A\ S\ -\ x^p\ Diag\ (\alpha_1, \ldots, \alpha_d).$$

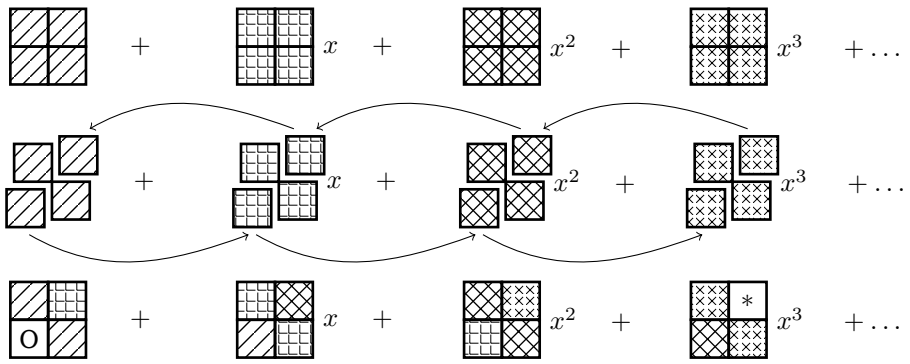
where given $A\ =\ [a_{ij}]_{1 \le i,j \le n}$, we have

$$S^{-1}\ A\ S\ =\ [a_{ij}\ x^{\alpha_j - \alpha_i}]_{1 \le i,j \le n}, \quad \text{or more explicitly,}$$

$$\begin{aligned} \tilde{A} \quad &= \quad S^{-1}\ A\ S\ -\ x^p\ Diag\ (\alpha_1, \ldots, \alpha_d) \\ &= \begin{bmatrix} a_{11} - \alpha_1 x^p & a_{12}\ x^{\alpha_2 - \alpha_1} & \ldots & a_{1n}\ x^{\alpha_n - \alpha_1} \\ a_{21}\ x^{\alpha_1 - \alpha_2} & a_{22} - \alpha_2 x^p & \ldots & a_{2n}\ x^{\alpha_n - \alpha_2} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}\ x^{\alpha_1 - \alpha_n} & a_{n2}\ x^{\alpha_2 - \alpha_n} & \ldots & a_{nn} - \alpha_n x^p \end{bmatrix}. \end{aligned}$$

| Example 1.1

Consider $S = Diag\,(x, \ldots, x, 1, \ldots, 1)$. The following diagram exhibits the shearing ("cropping") effect of S on $[A]$.



□

1.2 Equivalence between a system and an equation

In the introduction, we saw that an n^{th}-order differential equation can be rewritten equivalently as a first-order linear differential system. The other direction holds as well although it is nontrivial. The formal solutions of the latter can then be computed from the former (see, e.g. [136] and references therein). However, it is debatable whether such a treatment is satisfactory. Some experimental results show that the size of the coefficients grows dramatically. In any case, its major drawback is that it overlooks the information that can be derived directly from the system itself. Nevertheless, it plays a key role in the theoretical basis of algorithms treating system $[A]$ and those which we develop later in Part II. And so, it is convenient to recall it in this subsection.

Given system $[A]$, a standard procedure is to consider a nonzero row vector $U(x) = (u_1, \ldots, u_n)$ with entries in $\bar{\mathcal{K}}((x))$ and define inductively the row-vector sequence $\{U_i(x)\}_{0 \le i \le n}$ as follows:

$$U_0 = U, \quad U_i = \partial U_{i-1} + U_{i-1} \frac{A}{x^{p+1}}, \quad 1 \le i \le n.$$

Let

$$f = U_0\, F = u_1\, f_1 + \cdots + u_n f_n\,.$$

By substituting $\partial F = \frac{A}{x^{p+1}} F$ in the successive computations of $\partial f, \ldots, \partial^n f$, we obtain:

$$\partial^i f = U_i F, \quad 0 \le i \le n. \tag{1.3}$$

Now let T (resp. $\tilde{A}$) be the n-square matrices whose i^{th} rows are formed by U_{i-1} (resp. U_i) for $1 \le i \le n$; and let G be the column vector whose i^{th} component is $\partial^{i-1} f$. Then the set of equations in (1.3) can be rewritten as

$$G = T\, F. \tag{1.4}$$

Noting that $\tilde{A} = \partial T + T\frac{A}{x^{p+1}}$, (1.4) can be rewritten as

$$\partial G \;=\; \tilde{A}\; F. \tag{1.5}$$

We say that $U(x)$ is a cyclic vector if the matrix T is invertible. Cyclic vectors always exist (see, e.g. [46] and references therein). We can thus rewrite (1.5) and (1.4) as

$$F \;=\; T^{-1}\; G \quad \text{and} \quad \partial G \;=\; \tilde{\tilde{A}}\; G, \quad \text{where} \quad \tilde{\tilde{A}}(x) = \tilde{A}T^{-1} = T\frac{A}{x^{p+1}}T^{-1} + \partial T T^{-1}$$

is a companion matrix. Denoting the entries in its last row by $(\tilde{\tilde{a}}_i)_{0\leq i\leq n-1}(x) \in \bar{\mathcal{K}}((x))$, the system is obviously equivalent to the n^{th}-order scalar differential equation:

$$\partial^n f \;-\; \tilde{\tilde{a}}_{n-1}(x)\,\partial^{n-1} f \;-\; \ldots \tilde{\tilde{a}}_1(x)\,\partial f - \tilde{\tilde{a}}_0(x)\; f = 0.$$

Another algorithm is that of [13] which computes a companion block diagonal form for system $[A]$. The former can be easily adapted to singularly-perturbed linear differential systems as well to serve theoretical purposes (Chapter 4).
In the sequel, we offer a direct treatment of the system $[A]$, i.e. without resorting to an equivalent n^{th}-order scalar equation. The classical direct treatment of system $[A]$ depends on the nature of the eigenvalues of the leading matrix coefficient A_0. For the clarity of the presentation, we can assume without loss of generality that A_0 is in Jordan canonical form (JCF, in short). This can always be achieved by a constant transformation and efficient algorithms exist for this task [59].

1.3 A_0 has at least two distinct eigenvalues

Whenever A_0 has at least two distinct eigenvalues, system $[A]$ can be uncoupled into systems of lower dimensions via the classical Splitting lemma which we recall here with its constructive proof (see, e.g. [139, Section 12, pp 52-54] or [11, Lemma 3, pp 42-43]).

| Theorem 1.1

Given system $[A]$ with $A(x) \in \mathcal{M}_n(\mathcal{K}[[x]])$. If the leading matrix coefficient is of the form

$$A_0 = \begin{bmatrix} A_0^{11} & O \\ O & A_0^{22} \end{bmatrix} \tag{1.6}$$

where A_0^{11} and A_0^{22} have no eigenvalues in common, then there exists a unique transformation $T(x) \in GL_n(\mathcal{K}[[x]])$ given by

$$T(x) = \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \sum_{k=1}^{\infty} \begin{bmatrix} O & T_k^{12} \\ T_k^{21} & O \end{bmatrix} x^i,$$

such that the transformation $F = TG$ gives

$$x^{p+1}\frac{d}{dx}\,G \;=\; \tilde{A}(x)\;Z \;=\; \begin{bmatrix} \tilde{A}^{11}(x) & O \\ O & \tilde{A}^{22}(x) \end{bmatrix} G$$

where $\tilde{A}_0 = A_0$ and $\tilde{A}(x) \in \mathcal{M}_n(\mathcal{K}[[x]])$.

Proof. We have from (1.2):

$$x^{p+1}\partial T(x) = A(x)T(x) - T(x)\tilde{A}(x). \tag{1.7}$$

Assume the above form for $T(x)$, then we have for $1 \leq \varrho \neq \varsigma \leq 2$:

$$\begin{cases} A^{\varrho\varrho}(x) - \tilde{A}^{\varrho\varrho}(x) + A^{\varrho\varsigma}(x)\;T^{\varsigma\varrho}(x) = O \\ A^{\varsigma\varrho}(x) + A^{\varsigma\varsigma}(x)\;T^{\varsigma\varrho}(x) - T^{\varsigma\varrho}(x)\;\tilde{A}^{\varrho\varrho}(x) = x^{p+1}\partial T^{\varsigma\varrho}(x) \end{cases} \tag{1.8}$$

Inserting the series expansions $A(x) = \sum_{k=0}^{\infty} A_k x^k$ and $\tilde{A}(x) = \sum_{k=0}^{\infty} \tilde{A}_k x^k$ in (1.8), and equating the power-like coefficients, we get recursion formulas of the form:
For $k = 0$ we have

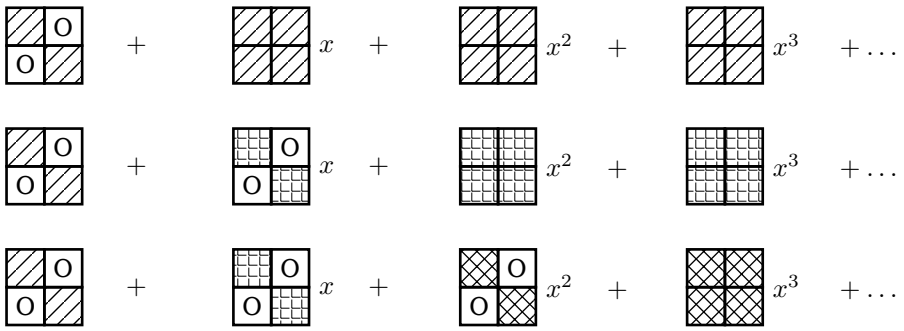
$$\begin{cases} A_0^{\varrho\varrho} = \tilde{A}_0^{\varrho\varrho} \\ A_0^{\varsigma\varsigma}\;T_0^{\varsigma\varrho} - T_0^{\varsigma\varrho}\;\tilde{A}_0^{\varrho\varrho} = O \end{cases}$$

which are satisfied by setting $A_0^{\varsigma\varsigma} = \tilde{A}_0^{\varsigma\varsigma}$ and $T_0^{\varsigma\varsigma} = I,\; T_0^{\varsigma\varrho} = O$.
For $k \geq 1$ we have

$$A_0^{\varsigma\varsigma}T_k^{\varsigma\varrho} - T_k^{\varsigma\varrho}A_0^{\varrho\varrho} \;=\; -A_k^{\varsigma\varrho} - \sum_{j=1}^{k-1}(A_{k-j}^{\varsigma\varsigma}T_j^{\varsigma\varrho} - T_j^{\varsigma\varrho}\tilde{A}_{k-j}^{\varrho\varrho}) + (k-p)\;T_{k-p}^{\varsigma\varrho} \tag{1.9}$$

$$\tilde{A}_k^{\varrho\varrho} \;=\; A_k^{\varrho\varrho} + \sum_{j=1}^{k} A_{k-j}^{\varrho\varsigma}T_j^{\varsigma\varrho} \tag{1.10}$$

where $T_{k-p}^{\varsigma\varrho} = O$ for $k \leq p$.
It's clear that (1.9) is a Sylvester matrix equation that possesses a unique solution due the assumption on the disjoint spectra of $M_0^{\varsigma\varsigma}$ and $M_0^{\varrho\varrho}$ (Lemma 1.1). Remarking that the right hand sides depend solely on the T_j, $\tilde{A}_j$ with $j < k$, the system of equations (1.9) and (1.10) are successively soluble as illustrated by the following diagram:



■

| Lemma 1.1 (see, e.g., [11], Appendix A.1, pp 212-213)

Given $M \in \mathcal{M}_m(\mathcal{K})$ and $N \in \mathcal{M}_n(\mathcal{K})$. If M and N have disjoint spectra, i.e. do not have an eigenvalue in common, then for every $P \in \mathcal{M}_{m\times n}(\mathcal{K})$ the matrix equation

$$M\,X \;-\; X\,N \;=\; P$$

has a unique solution in $\mathcal{M}_{m\times n}(\mathcal{K})$.

Thus the system can be split into two subsystems of lower dimensions and formal reduction proceeds on each of the subsystems in parallel.

1.4 A_0 has a unique nonzero eigenvalue

Supposing that A_0 has a single nonzero eigenvalue $\gamma \in \bar{\mathcal{K}}$, it is easy to verify that the so-called eigenvalue shifting, which is a special type of exponential transformations,

$$F = G\exp(\int^x \gamma z^{-p-1}dz), \tag{1.11}$$

results in a system with a nilpotent leading matrix coefficient (in fact, in (1.2), $\tilde{A}(x) = A(x) - \gamma I_n$ and $\tilde{p} = p$).

1.5 A_0 is nilpotent

This case is the most interesting and requires at least one of the following steps, the first of which is rank reduction. In the following two parts, we will see that the main difficulties arise in the case of nilpotency as well.

1.5.1 Rank reduction

Given a system $[A]$, one would like to determine whether $x = 0$ is a regular or an irregular singularity. It is well known that a singularity of the first kind of system $[A]$, i.e. for which $p = 0$, is a regular singularity (see, e.g. [81, Proposition 3.13, pp 258] and references therein). However the converse is not true: Even when $x = 0$ is a multiple pole (a singularity of second kind), it is still possible for $x = 0$ to be a regular singularity. Horn's theorem [67] states that $[A]$ has a regular singularity if and only if it is equivalent to a system whose singularity is of first kind. Rank reduction is the procedure which constructs a transformation that yields such an equivalent system. More generally, whether the system is regular singular or irregular singular, rank reduction reduces the Poincaré rank to its minimal integer value, called the true Poincaré rank p_{true}. In the case of irregular singularity, p_{true} is the minimal integer which gives an upper bound to the growth order of the solutions in a neighborhood of $x = 0$. There exists several reduction criteria to determine p_{true} and several algorithms to construct an equivalent system whose Poincaré rank is the true Poincaré rank [121, 138, 87, 32]. In this section, and generally in this thesis, we are interested in the Moser-based ones.
Moser-based rank reduction of system $[A]$ is a reduction based on the criterion defined by Moser in [107]. Consider the following two rational numbers called the Moser rank and Moser invariant respectively:

$$\begin{cases} m(A) = \max\,(0, p + \frac{rank(A_0)}{n}) \\ \mu(A) = \min\,\{\, m(T[A]) \text{ for all possible choices of } T \text{ in } GL_n(\bar{\mathcal{K}}((x)))\}, \end{cases} \tag{1.12}$$

If $\mu(A) \leq 1$ then system $[A]$ is regular. For $m(A) > 1$, it is proved in [107, Theorems 1 and 2, pg 381] that $m(A) > \mu(A)$ if and only if the polynomial

$$\theta_A(\lambda) := x^{rank(A_0)}\ \det(\lambda I + \frac{A_0}{x} + A_1)|_{x=0}$$

vanishes identically in λ. If it is the case, we say that system $[A]$ (resp. $A(x)$) is reducible. Constant transformations cannot change the rank of the system unless combined with shearings : $m(A)$ can be diminished by applying the transformation $Y = TZ$ where $T \in GL_n(\mathcal{K}(x))$ is a product of polynomial transformations of the form [15]

$$P\ Diag(x, \ldots, x, 1, \ldots, 1) \quad \text{where} \quad P \in GL_n(\mathcal{K}).$$

Otherwise, the system (resp. $A(x)$) is said to be irreducible*.
Thus, a system has a regular singularity at a point $x = 0$ if and only if it is equivalent to a system $[\tilde{A}]$ with a first-kind singularity at $x = 0$. Such a $T(x) \in GL_n(\mathcal{K}(x))$ can be constructed via the rank reduction algorithms of [32] and references therein. Moser-based rank reduction algorithms result in a system equivalent to $[A]$ which holds not only the true Poincaré rank but also a minimal algebraic rank for the leading matrix coefficient A_0. A minimal algebraic rank of A_0 is a prerequisite for later computations

*In literature, the terminology of *Moser-reducible* and *Moser-irreducible* is used. However, we adopt the simpler terminology of *reducible* and *irreducible* which will be helpful in Parts II and III, where a parameter or several variables are involved. Upon generalizing such reductions to more general systems, we will eventually need to specify the parameter or variable with respect to which the system is reduced, e.g. ε-irreducible, ξ-irreducible, and x_i-irreducible.

(Subsection 1.5.2) and from here stems our interest in such algorithms. Hence, in the rest of this thesis, we generalize this reduction criterion and the Moser-based rank reduction algorithm of [15].
We can now suppose without any loss of generality that $[A]$ is an irreducible system. Three possibilities arise:

- System $[A]$ is regular singular and so it is transformed into an equivalent system whose Poincaré rank is zero. We proceed in the formal reduction as in Section 1.6.
- System $[A]$ is irregular singular and the leading matrix coefficient has at least two distinct eigenvalues (resp. unique nonzero eigenvalue). We thus retreat to Section 1.3 (resp. Section 1.4) .
- System $[A]$ is irregular singular and the leading matrix coefficient is nilpotent. This case demands introducing a ramification in x. This ramification (re-adjustment of the independent variable) is not known from the outset but can be computed as we show in Subsection 1.5.2.

We remark that more recent rank reduction algorithms were given in [32] and references therein. The transformations considered therein were optimal in the following sense: If p can be dropped by one then the similarity transformation computed achieves this goal in one single step. Moreover, $\theta_A(\lambda)$ gives other valuable information about the system leading to a generalized splitting lemma [114]. Roughly speaking, the latter uncouples the system into two subsystems, one of which does not demand a ramification for the retrieval of the leading term of the exponential parts. The former are however out of the scope of this brief description.

1.5.2 Formal exponential order $\omega(A)$

| Definition 1.1

[14, Theorem 1] Given system $[A]$ and its *exponential part* in (2):

$$Q(x^{-1/s}) = Diag(q_1(x^{-1/s}), q_2(x^{-1/s}), \ldots, q_n(x^{-1/s})).$$

Then, the *formal exponential order*, exponential order in short, of $[A]$ (resp. $A(x)$) is the rational number

$$\omega(A) \ = \ -min_{1\leq i\leq n}\ val_x(q_i).$$

In literature, $\omega(A)$ is also referred to as Katz invariant. As mentioned, one cannot retrieve $\omega(A)$ from the outset. However, it is proved in [14, Theorem 1], which we recall here, that $\omega(A)$ can be computed from the characteristic polynomial of $A(x)/x^p$ whenever $A(x)$ is irreducible. Supposing that $\omega(A) = \frac{\ell}{d}$ with ℓ, d relatively prime positive integers, one can then set $t = x^{1/d}$, and perform again rank reduction. The resulting equivalent system has Poincaré rank equal to ℓ and leading matrix coefficient with at least d distinct eigenvalues. Consequently, the Splitting lemma can be reapplied to uncouple the system.

| Theorem 1.2 ([14], Theorem 1)

Given system $[A]$ *with* $p > 1$. *Let*

$$\det\,(\lambda I_n - \frac{A(x)}{x^p}) = \lambda^n + \alpha_{n-1}\lambda_{n-1} + \cdots + \alpha_0.$$

such that $\alpha_n = 1$ *and* $\alpha_i = \sum_{j=val_x(\alpha_i)}^{\infty} \alpha_{i,j}\, x^j$ *for* $0 \leq i < n$. *If* $p > n - \ rank\ (A_0)$, *then we have*

$$\omega(A) = \max_0^{n-1}\,(0,\ \frac{-val_x(\alpha_i)}{n-i}).$$

One can analyze $\det(\lambda I_n - \frac{A(x)}{x^p})$ by associating a Newton polygon (see, e.g. [75, Section 2.1]), and $\omega(A)$ would then be the steepest slope of this polygon. This theorem establishes a relationship between the algebraic Newton polygon, that is the Newton polygon of $A(x)$ on one hand and the differential Newton polygon, that is the Newton polygon of the equivalent scalar n^{th}-order differential equation on the other hand (see also [64, 66, 136]).

| Remark 1.1

The condition $p > n - \ rank\ (A_0)$ *is non-restrictive as it can be always attained by a suitable choice of ramification and a computable transformation [14, Lemma 5].*

The leading term of $Q(x^{-1/s})$ is then given by

$$-\frac{1}{\omega(A)\ x^{\omega(A)}}\ Diag\ (a_1, \ldots a_{deg\ (E)}, 0, \ldots, 0),$$

where the a_k's denote the roots of the (Newton) polynomial given by the algebraic equation

$$E(X) = \sum_{k=0}^{\ell} \alpha_{(i_k, val_x(\alpha_{i_k}))}\ X^{(i_k - i_0)} \tag{1.13}$$

where $0 \leq i_0 < i_1 < \cdots < i_\ell = n$ denote the integers i for which $\omega(A)(n-i) = -val(\alpha_i)$.
Or equivalently $Q(x^{-1/s})$ is given by,

$$-\int^x \frac{1}{(p+1)\ z^{\,p+1}}\ Diag\ (\gamma_1, \ldots, \gamma_n)\ dz,$$

where the γ_i's are the eigenvalues of A_0 in $\bar{\mathcal{K}}$.

1.6 Regular systems

This section is devoted to systems which are regular singular $x = 0$, i.e. system$[A]$ with $p_{true} = 0$. Without loss of generality, we suppose that $[A]$ is irreducible and consequently $p = p_{true} = 0$. Relevant methods of resolution are discussed in [139, Chapter 1], [29, 24] for more general contexts, and references therein.

The discussion is, again, based on the nature of the eigenvalues of A_0:

| Theorem 1.3

Given system $[A]$ *with* $p = 0$. *If the eigenvalues of* A_0 *do not differ by nonzero integers, then there exists*

$$T(x) \;=\; \sum_{i=0}^{\infty} T_k\, x^k \;\in GL_n(\mathcal{K}[[x]]), \quad \textit{where} \quad T_0 = I_n,$$

which yields an equivalent system $[\tilde{A}]$ *for which* $\tilde{A} = A_0$. *Consequently, a formal fundamental matrix of formal solutions of* $[A]$ *is given by* $T(x)\, x^{A_0}$.

Proof. If follows from (1.2) that

$$x\, \partial T(x) \;=\; A(x)\, T(x) \;-\; T(x)\, A_0.$$

Inserting the series representation of $A(x)$ and $T(x)$ yields:

$$\begin{cases} A_0\, T_0 \;- T_0\, A_0 \;=\; O_n \\ (A_0 \;-\; k\, I_n)\, T_k \;-\; T_k\, A_0 \;=\; -\sum_{i=0}^{k-1}\; A_{k-i}\, T_i\;,\; k \geq 1. \end{cases}$$

Choosing $T_0 = I_n$, the T_k's can be successively computed since $A_0 - kI_n$ and A_0 have disjoint spectra. ■

| Proposition 1.1

If the eigenvalues of A_0 *differ by a nonzero integer then there exists* $T \in GL_n(\mathcal{K}((x)))$, *product of shearing (polynomial) transformations and constant transformations such that the eigenvalues of the leading matrix coefficient of the equivalent system do not differ by nonzero integers.*

The proof is constructive and is skipped here since it is very similar to the proof of Proposition 3.4 in Chapter 3. An efficient version of the resulting algorithm is proposed in [84, pp 67 - 68].

| Example 1.2

Given the famous modified Bessel equation of order ν:

$$\partial^2 f + \frac{1}{x}\partial f - (1 + \frac{\nu^2}{x^2})f = 0.$$

We discuss the construction of a fundamental matrix of formal solutions at $x = 0$ according to the values of ν. Let $F = (f, \partial f)^T$. Then the given equation can be rewritten as the following linear first-order differential system of dimension 2:

$$x^2\, \partial F \;=\; A(x)\, F = \begin{bmatrix} 0 & x^2 \\ x^2 + \nu^2 & -x \end{bmatrix} F. \tag{1.14}$$

The leading term $A_0 := A(0)$ is nilpotent and the system is reducible according to Moser's criterion

$(\theta(\lambda) = 0)$. The transformation $F = T\,G$ where $T = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$ yields the equivalent system:

$$x\,\partial G = \tilde{A}(x)G = \begin{bmatrix} -1 & 1 \\ x^2+\nu^2 & -x \end{bmatrix} G.$$

Thus, $p_{true} = 0$ and the system is regular singular at $x = 0$. The new leading matrix $\tilde{A}_0 = \tilde{A}(0) = \begin{bmatrix} -1 & 1 \\ \nu^2 & 0 \end{bmatrix}$ is not nilpotent. The difference between its two eigenvalues is given by $\sqrt{4\nu^2+1}$. We thus distinguish two cases:

- **Case 1** : If $\sqrt{4\nu^2+1} \in \mathbb{N}^*$ then Proposition 1.1 is applied to reduce.
- **Case 2** : Otherwise, we proceed to constructing $T(x)$ of Theorem 1.3 and hence a fundamental matrix of formal solutions is given by $T(x)\,x^{\tilde{A}_0}$.

□

1.7 Formal reduction algorithm

Based on the above, the terms of $Q(x^{-1/s})$ of largest degrees can be found recursively. This process is exhibited in Algorithm 1†, which computes the exponential part of a FMFS (2), and eventually, a full fundamental matrix of formal solutions of system $[A]$. In fact, the transformations performed in the process of computing the former, are endowed in the latter.
Algorithm 1 results in a set of decoupled systems with dimension $n = 1$ (case of first-order linear scalar equations) and/or a set of system(s) whose Poincaré ranks are zeros, and consequently, has(ve) zero exponential part(s) and can be treated as in Section 1.6.

1.8 Remarks about the implementation

In this section, we point out some considerations in the implementation of the packages MINIISOLDE [31] and LINDALG. The implementation is in the former recursive and in the latter iterative although the algorithm itself is recursive.
The polynomial $Q(x^{-1/s})$ of a FMFS given by (2) is determined by at most the first np terms in the Taylor expansion of $A(x)$ in (1) [9, 90]. Thus, finitely many terms are needed for the construction of the exponential part and this number is reduced as the system splits into two or several systems and as the Poincaré rank drops. Moreover, any additional number of terms can be taken into account to increase the precision of $\Phi(x)$ of a FMFS. The base field can be any commutative field $\mathcal{K}$ of characteristic zero. Algorithm 1 and its underlying sub-algorithms can be refined to handle efficiently algebraic extensions of the constant base field, as explained below.

†The pseudo code is provided to give an overview of the main steps of this recursive algorithm. The implementation itself demands different considerations. The source code of LINDALG is accessible with examples of computations within the current release of MATHEMAGIX [135].

Algorithm 1 FMFS_ODS $(p, A(x))$: Computes a fundamental matrix of formal solutions (2) of $[A]$

Input: $p,\ A(x)$ of (1)
Output: A fundamental matrix of formal solutions FMFS (2)

$Q, C, \leftarrow\ O_n$; $\Phi \leftarrow I_n$;
while $p > 1$ and $n \neq 1$ **do**
  if A_0 has at least two distinct eigenvalues **then**
    Split system as in Subsection 1.3; Update Φ;
    FMFS_ODS $(p, \tilde{A}_{11}(x))$; Update Φ, C, Q;
    FMFS_ODS $(p, \tilde{A}_{22}(x))$; Update Φ, C, Q;
  else if A_0 has one non-zero eigenvalue **then**
    Update Q from the eigenvalues of A_0;
    $A(x) \leftarrow$ Follow Subsection 1.4; (A_0 is now nilpotent);
    FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
  else
    Follow Subsection 1.5.1 for rank reduction; Update Φ; Update p; Update A_0;
    if $p > 1$ and A_0 has at least two distinct eigenvalues **then**
      Split system as in Subsection 1.3;
      FMFS_ODS $(p, \tilde{A}_{11}(x))$; Update Φ, C, Q;
      FMFS_ODS $(p, \tilde{A}_{22}(x))$; Update Φ, C, Q;
    else if A_0 has one non-zero eigenvalue **then**
      Update Q from the eigenvalues of A_0;
      $A(x) \leftarrow$ Follow Subsection 1.4; (A_0 is now nilpotent)
      FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
    else
      Follow Subsection 1.5.2;
      $\omega(A) = \frac{\ell}{d}$; $x \leftarrow x^d$;
      $A(x) \leftarrow$ Follow Subsection 1.5.1 for rank reduction; Update Φ; $p \leftarrow \ell$;
      Update Q from eigenvalues of A_0;
      $A(x) \leftarrow$ Follow Subsection 1.4; (A_0 is now nilpotent)
      FMFS_ODS $(p, A(x))$; Update Φ, C, Q;
    end if
  end if
end while
if $n = 1$ **then**
  Proceed by integration up to the first $p - 1$ terms; Update Φ, C, Q;
else if $p = 0$ **then**
  Update Φ, C from Subsection 1.6;
end if
return Φ, C, Q.

The JCF can be avoided in Theorem 1.1, Proposition 1.1, and whenever appropriate, by making use of the following lemma.

Lemma 1.2 (Lemma A.1, [32])

Given a rank deficient matrix $M \in \mathcal{M}_n(\mathcal{K})$. Then there exists a constant matrix $T \in GL_n(\mathcal{K})$ such that

$$\tilde{M} \;=\; T^{-1}\, M\, T \;=\; \begin{bmatrix} D & O_{r\times(n-r)} \\ O_{(n-r)\times r} & N \end{bmatrix} \quad \in \mathcal{M}_n(\mathcal{K}),$$

where $r := \; rank\;(M^n)$, $D \in \mathcal{M}_r(\mathcal{K})$ is nonsingular, and N is a nilpotent matrix.

The proof follows from the rank-nullity theorem. In practice we proceed as in Algorithm 2. Variations of

Algorithm 2 BLOCK_DIAG: Block-diagonalization of M over $\mathcal{K}$

Input: $M \in \mathcal{M}_n(\mathcal{K})$
Output: The transformation of Lemma 1.2

$r = rank(M^n)$;
if $0 < r < n$ **then**
  Compute a basis $\{T_1, \ldots, T_r\}$ of the space generated by the columns of M^n;
  Compute a basis $\{T_{r+1}, \ldots, T_n\}$ of the kernel of M^n (using Gaussian Elimination);
  Form the matrix $T(x)$ whose columns are $\{T_1, \ldots, T_r, T_{r+1}, \ldots, T_n\}$;
else
  $T = I_n$;
end if
return T.

Lemma 1.2 can also be applied to isolate distinct eigenvalues, e.g. to put A_0 in the form 1.6 to prepare $[A]$ for the Splitting lemma or Proposition 1.1. However, despite avoiding JCF, some algebraic extensions might be introduced with the roots of (1.13) and eigenvalue shifting. In fact, let $\omega(A) = \frac{\ell}{d}$ with $gcd(\ell, d) = 1$. If a is a root of the polynomial (1.13) of multiplicity ν then there exists ν polynomial entries on the diagonal of the exponential part $Q(x^{-1/s})$ that have $-\frac{a}{\omega(A)\, x^{\omega(A)}}$ as a leading term, i.e. the leading term of $Q(x^{-1/s})$ is given by

$$-\frac{1}{\omega(A)\; x^{\omega(A)}}\; Diag\;(a, \ldots, a, 0, \ldots, 0).$$

Following Algorithm 1, we apply the ramification $x = t^d$ and eventually carry out the shifting

$$F \;=\; \exp\,(\frac{-a}{\omega(A)\; t^{\ell}})\; G.$$

At this point, we leave the constant field $\mathcal{K}$ and start working in its extension $\mathcal{K}(a)$. These extensions arise naturally but it is possible to restrict their sizes using a trick described in [14, Section 5] which we outline here:

Let u and v be two integers verifying $u\ell \ + \ vd \ = \ 1$. Let $z = a^{-u}\ t$ and $b = a^d$. Then we have:

$$\begin{cases} \frac{-a}{\omega(A)\ t^\ell} \ = \ \frac{-a^{u\ell+vd}}{\omega(A)\ t^\ell} \ = \ \frac{-a^{dv}}{\omega(A)\ (a^{-u}t)^\ell} \ = \ \frac{b^v}{\omega(A)\ z^\ell}, \\ x \ = \ t^d \ = \ a^{du}\ z^d \ = \ b^u\ z^d. \end{cases}$$

One notices that if a is a root of the polynomial (1.13) then $b = a^d$ is a root of a reduced polynomial given by $E_{reduced}\ (X^d) \ = \ E(X)$. Hence, Algorithm 1 can then be modified as follows:

- Choose a root b of $E_{reduced}$;
- compute u and v satisfying $u\ell \ + \ vd \ = \ 1$;
- substitute x by $b^u\ z^d$;
- apply the shifting
$$F = G \exp(\frac{-b^v}{\omega(A)\ z^\ell}).$$

The new constant field is then $\mathcal{K}(b)$ where b is a root of a polynomial of degree equal to the degree of the polynomial (1.13) divided by d. The computations are done up to conjugations.
We illustrate these enhancements with this very simple example.

| Example 1.3 ([14], Example 4)

Let $\mathcal{K} = \mathbb{Q}$ and consider the following system:

$$x^3\ \partial F \ = \ A(x)\ F \ = \ \begin{bmatrix} 2x & x \\ x-1 & 1 \end{bmatrix}.$$

Without applying the above trick, Algorithm 1 computes the exponential part to be:

$$Q(x^{-1/2}) \ = \ Diag\ (q_1(x^{-1/2})\ ,\ q_2(x^{-1/2})),$$

where

$$\begin{cases} q_1(x^{-1/2}) \ = \ -\frac{2\sqrt{-1}}{3x^{3/2}} - \frac{1}{x} + \frac{2\sqrt{-1}}{\sqrt{x}}, \\ q_2(x^{-1/2}) \ = \ +\frac{2\sqrt{-1}}{3x^{3/2}} - \frac{1}{x} - \frac{2\sqrt{-1}}{\sqrt{x}}. \end{cases}$$

However, we proceed instead as follows:

- The leading matrix A_0 is nilpotent and irreducible so we need to compute $\omega(A)$;
- we compute $\omega(A) = 3/2$ and $E(X) = X^2 + 1$ (1.13). Hence we set $E_{reduced} = U + 1$;
- we take $u = 1,\ v = -1$, and $b = -1$ (the unique root of $E_{reduced}$);
- we apply the ramification $x = bt^2 = -t^2$ and rank reduction again to obtain the new system

$$t^4\ \partial_t\ G \ = \ \tilde{A}(t)\ G \ = \ \begin{bmatrix} 2t^3 & -2t^2-2 \\ -2 & -4t-t^3 \end{bmatrix};$$

- The shifting
$$G \;=\; H \exp(\frac{2}{3t^3}),$$
shifts the eigenvalues of the leading matrix coefficient by two which yields
$$t^4\,\partial_t\; H \;=\; \tilde{\tilde{A}}(t)\; H \;=\; \begin{bmatrix} 2t^3+2 & -2t^2-2 \\ -2 & -4t-t^3+2 \end{bmatrix};$$
Now, the leading matrix coefficient $\tilde{\tilde{A}}_0$ of the resulting system has zero as a simple eigenvalue and is given by
$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$
One can then isolate its nilpotent part based on Lemma 1.2 (whose transformation coincides with Jordan transformation in this case): We compute
$$T \;=\; \begin{bmatrix} 8 & 1 \\ -8 & 1 \end{bmatrix} \quad \text{so that} \quad T^{-1}\;\tilde{\tilde{A}}_0\;T \;=\; \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$
- We then split the system to any desired precision by Theorem 1.1. which leads to the scalar equation
$$t^3\;\partial_t f \;=\; n(t)\;f = n(t)\;f = \;(-2-2\,t+\dots)\;f.$$
The latter has $\exp(\frac{1}{t^2}+\frac{2}{t})$ as its exponential part.
- Thus, we obtain a polynomial in $1/t$
$$\mathbf{q}(1/t) \;=\; \frac{2}{3\,t^3} \;+\; \frac{1}{t^2} \;+\; \frac{2}{t}, \quad \text{where} \quad x \;=\; -t^2.$$

Thus, $q_1(x^{-1/2})$ and $q_2(x^{-1/2})$ obtained above by computations in $\mathbb{Q}(i)$ can be both obtained from $\mathbf{q}(t)$ by substituting $t = \pm i x^{1/2}$ in $\mathbf{q}(1/t)$, which is computed in $\mathbb{Q}$.

□

1.9 Conclusion

In this chapter, we revised major components of the formal reduction of linear singular differential systems. For the sake of brevity, we restricted our description to the notions which serve the algorithms that are developed in the later chapters of this thesis. Several other central notions exist and are recalled briefly with prospects of their generalization to wider classes of systems in Appendix A.

LINDALG and MINIISOLDE are currently dedicated to the symbolic local treatment of n^{th}-order linear differential equations and first-order linear differential systems with singularities. Following the track of the MAPLE package ISOLDE, algorithms for computing global solutions will be adjoined as well (e.g. rational, exponential, and liouvillian solutions) in the future.

Chapter 2

Perturbed (Algebraic) Eigenvalue Problem

Contents

As we have seen in Subsection 1.5.2, the formal invariants of linear singular differential system $[A]$ given by (1)

$$x^{p+1}\frac{d}{dx}\,F \;=\; A(x)\;F \;=\; (\sum_{i=0}^{\infty} A_k x^k)\;F,$$

can be retrieved (after certain reductions) from the characteristic polynomial of its matrix $A(x)$. We thus turn our attention in this chapter to the algebraic eigenvalue problem.

We consider the one-parameter $A(x)$ perturbation of an $n \times n$ constant matrix A_0:

$$A(x) = x^{\nu} \sum_{k=0}^{\infty} A_k x^k, \quad \text{for some} \quad \nu \in \mathbb{Z}, \tag{2.1}$$

where the A_k's belong to $\mathcal{M}_n(\mathbb{C})$ and $A_0 \neq O_n$.

It is well-known that the eigenvalues of $A(x)$ can be expressed in some neighborhood of $x = 0$ as a formal Puiseux series (see, e.g., [34, 80]), i.e. each admits a formal expansion in fractional powers of x of the form $x^{\tilde{\nu}} \sum_{k=0}^{\infty} \lambda_k x^{k/s}$, where $\lambda_k \in \mathbb{C}$ (or more specifically some field extension of the base field), $s \geq 1$ is an integer, and $\tilde{\nu} \in \mathbb{Z}$. We suppose without loss of generality that $\lambda_0 \neq 0$ and we refer to it as the leading coefficient and to $\tilde{\nu} + k/s$ as the leading exponent.

These eigenvalue expansions can be computed by applying Newton-Puiseux algorithm to the characteristic polynomial of $A(x)$ (see, e.g. [50]) and there already exist computer algebra packages for this purpose. However, this approach is indirect. A direct approach would recover leading terms from a few low order coefficients of a matrix similar to $A(x)$ rather than an associated scalar equation.

A direct approach to the perturbed eigenvalue problem (and more generally the perturbed eigenvalue-eigenvector problem), is given in the linear case by the perturbation theory of Visik, Ljusternik [136], and Lidskii [88]. For a generic matrix, the leading exponents of the eigenvalues are the inverses of the sizes of the Jordan blocks of A_0 and the leading coefficients can be obtained from certain Schur complements constructed from the entries of A_0 and A_1. However, it might happen that Schur complements do not exist or A_1 has a sparse or structured pattern. The generalization of this theory to cover such cases is the subject of profound research for both matrices and matrix pencils (see, e.g. [4, Introduction] and references therein).

One extension of the former takes advantage of methods of min-plus algebra and is given by Akian-Bapat-Gaubert in [4]. Another extension is given by Jeannerod-Pfluegel in [75] and relies on a differential-like reduction inspired by Moser's reduction criteria (Subsection 1.5.1) for linear singular differential systems. In the latter, a direct algorithm was proposed to find the first terms of the perturbed eigenvalues for which $s = 1$. In the same differential spirit, the case of $s > 1$ was discussed afterwards by Jeannerod [74] based on Lidskii's genericity conditions for perturbed eigenvalues, and their characterization in terms of the Newton diagram of $A(x)$ given by Moro-Burke-Overton in [105]. However, although both approaches generalize Lidskii-Visik-Ljusternik's theory to a wider class of matrices, both leave behind singular cases open to investigation*. This chapter describes briefly the main results of both approaches regarding the computation of the asymptotics of the eigenvalues of (2.1), in Sections 2.1 and 2.2 respectively.

We aim to show the interaction between the studies of differential and linear operators: On one hand, the techniques developed for the former turn out to be useful in the treatment of the latter; and on the other hand, the advancements in the latter contribute directly to the former (see also [138, 108]). Moreover, we give in Subsection 2.2.3 our partial result, in an attempt to explain the singular case in the differential-like approach (Section 2.2), using the results of the tropical approach (Sections 2.1).

2.1 Tropicalization of $A(x)$

Tropical algebra (also min-plus algebra) has been initiated independently by several schools and developed in relation with diverse mathematical fields (see, e.g. [10, 5, 92]). Let $\mathbb{R}$ be the ring of real numbers. The tropical semiring $\mathbb{R}_{min}$ is the semiring $\mathbb{R} \cup \{\infty\}$ equipped with the operations $x \oplus y := \min(x, y)$ and $x \odot y := x + y$.
Tropical analogues of many classical algebraic functions exist. Let $\mathbb{A} = [v_{ij}]_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R}_{min})$. In

*A *singular* matrix herein refers to a Jordan canonical form (JCF) perturbation which cannot be treated by one or none of the two direct approaches discussed in this chapter. This terminology has nothing to do with the terminology of singular regular and singular irregular points used in Chapter 1

particular, the tropical characteristic polynomial of $\mathbb{A}$ is given by:

$$P_{\mathbb{A}}(\lambda) = \bigoplus_{\sigma\in\Sigma_n} \bigodot_{i=1}^{n} (\lambda\, \delta_{i\sigma(i)} \oplus v_{i\sigma(i)}) \quad \in \mathbb{R}_{min}[\lambda],$$

where Σ_n is the set of permutations of $\{1,\dots,n\}$ and $\delta_{i\sigma(i)}$ is Kronecker's delta. A tropical analogue of the fundamental theorem of algebra exists and is due to Cuninhame-Green and Meijer (see, e.g. [4, Theorem 2.9]).
An eigenvalue of $\mathbb{A}$ is a real number λ such that

$$\mathbb{A}\odot v = \lambda \odot v, \quad \text{for some} \quad v \in \mathbb{R}^n_{min}.$$

Although this chapter does not aim to discuss the efficiency of the approaches described, we remark that the eigenvalues of $\mathbb{A}$ can be computed in polynomial time due to a result by Burkard and Betkovič [40]. We recall that $\mathbb{A}$ can be represented by a weighted directed graph $\mathcal{G}(\mathbb{A})$ with n nodes labeled $1, 2, ..., n$ as follows: There is an arc (i, j) from node i to node j if and only if $v_{ij} < \infty$; and the weight v_{ij} is assigned to each arc. The normalized weight of a directed path $i_0, i_1, ..., i_k$ in $\mathcal{G}(\mathbb{A})$ is given by $(\sum_{j=1}^{k} v_{i_{j-1},i_j})\ /k$. If $i_k = i_0$ then the path is a directed cycle (see e.g. [92, pp 127-133]). We say that a directed graph is strongly connected if every node is reachable from every other node, i.e. one can find a directed path connecting both nodes. Such graphs have the following interesting property (see, e.g. [92, Theorem 5.1.1, pp 128], [4, Theorem 2.1], or [10, Theorem 3.23]).

| Theorem 2.1

Let $\mathbb{A}$ be a tropical $n \times n$ matrix whose graph $\mathcal{G}(\mathbb{A})$ is strongly connected. Then $\mathbb{A}$ has a unique eigenvalue $\lambda(\mathbb{A})$, which is equal to the minimal length of any directed cycle in $\mathcal{G}(\mathbb{A})$.

Back to (2.1), let $A(x) = [a_{ij}(x)]_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{C}((x)))$. Then the tropicalization of $A(x)$ is given by the constant matrix $\mathbb{A} = [val_x(a_{ij})]_{1\le i,j\le n}$. Let $P_A(\lambda) = \sum_{i=0}^{n} p_i(x)\lambda^i \in \mathbb{C}[[x]][\lambda]$ be the characteristic polynomial of $A(x)$. For generic entries of $A(x)$, the tropical characteristic polynomial of $\mathbb{A}$ coincides with the tropicalization of the classical characteristic polynomial of $A(x)$ given by $\oplus_{i=0}^{n} val_x(p_i)\odot\lambda^i \in \mathbb{R}_{min}[\lambda]$. Supposing that the underlying digraph of $\mathbb{A}$ is strongly connected, we have:

| Theorem 2.2 ([4], Theorem 3.8)

Let $\alpha = (\alpha_1, ..., \alpha_n)$, $(\alpha_1 \le \cdots \le \alpha_n)$, denote the sequence of the leading exponents in the expansions of the eigenvalues of $A(x)$ counted with their multiplicities and $v = (v_1, \dots, v_n)$, $v_1 \le \cdots \le v_n$) denote the sequence of roots of the tropical characteristic polynomial of $\mathbb{A}$. Then, α is weakly majorized by v, that is([4, Definition 3.5]),

$$\alpha_1 \odot \cdots \odot \alpha_i \ge v_1 \odot \cdots \odot v_i\ , \quad \textit{for all } i = 1, \dots, n.$$

Moreover, for generic values of the a_{ij}'s,

$$\alpha_i = v_i, \quad \textit{for all} \quad i = 1, \dots, n.$$

We illustrate the above theorem by the following example.

| Example 2.1 ([92], pp 132)

Consider the matrix

$$A(x) = \begin{bmatrix} x & 1 & x \\ 1 & x & -x^2 \\ x & x^2 & x \end{bmatrix}$$

The tropicalization of the matrix $A(x)$ is

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

whose underlying graph is strongly connected. The tropical characteristic polynomial of $\mathbb{A}$ is (see [92], pp 132)

$$P_{\mathbb{A}}(\lambda) = \lambda^3 \oplus 1 \odot \lambda^2 \oplus 0 \odot \lambda \oplus 1$$

which factors as follows as a polynomial function:

$$P_{\mathbb{A}}(\lambda) = (\lambda \oplus 0)^2 \odot (\lambda \oplus 1).$$

This reflects that $A(x)$ has two eigenvalues of order 0 and one eigenvalues of order 1. In fact, $A(x)$ has three distinct eigenvalues x, $\sqrt{1+x^2-x^4}+x$, and $-\sqrt{1+x^2-x^4}+x$. We also remark that by Theorem 2.1, $\lambda = 0$ is the only eigenvalue of the matrix $\mathbb{A}$.

□

2.1.1 Application to differential systems

We show in this subsection the relation between the exponential parts of a linear singular differential system (1) and the tropical eigenvalues of the tropicalization $\mathbb{A}$ of $A(x)$. It is investigated in [17] in the light of a refined version of Theorem 1.2. We illustrate this relation in two examples from [17][†].

| Example 2.2

Given the irreducible linear singular differential system $x^2 \frac{d}{dx} F = A(x) F$ where

$$A(x) = \begin{bmatrix} -x & \frac{2}{3}(x-1) & \frac{16}{9}x \\ 0 & -x & \frac{4}{3}x \\ \frac{9}{4} & \frac{3}{4}(1+x) & -x \end{bmatrix}.$$

[†]For the computation of tropical eigenvalues, one may consult the MAPLE package MAXLINEARALGEBRA [116]. And for the computation of the exponential parts using Algorithm 1, one may consult the accompanying examples of LINDALG where this example in particular is treated.

The tropicalization of $x^{-2}\ A(x)$ is given by

$$\mathbb{A} \ = \ \begin{bmatrix} -1 & -2 & -1 \\ \infty & -1 & -1 \\ -2 & -2 & -1 \end{bmatrix}.$$

$\mathcal{G}(\mathbb{A})$ is strongly connected and so it has a unique tropical eigenvalue $-\frac{5}{3}$. This is the root of the tropical characteristic polynomial with multiplicity 3, corresponding to the cycle of minimal weight. $A(x)$ has three eigenvalues of order $x^{-5/3}$. The exponential parts of $A(x)$ are given by (observe the leading term):

$$q_{1,2,3} \ = \ \int^t \ 4\ u^{-5} \ - \ \frac{10}{3}\ u^{-4} \ + \ \frac{8}{3}\ u^{-3}, \quad \text{where} \quad x = -\frac{1}{2}\ t^3.$$

□

We might encounter distinct roots as well:

| Example 2.3

Given the irreducible linear singular differential system $x^3 \frac{d}{dx}\ F \ = \ A(x)\ F$ where

$$A(x) \ = \ \begin{bmatrix} x^2\ (x-1) & -(x-1) & -(3\ +\ x)\ x^2 & x\ (-2x^2+x^4+x^3+1) \\ 0 & -x\ (2\ x\ +\ 1) & 0 & x \\ x^2\ (x+1) & -x & -x^2\ (4+x) & x\ (x^4\ +\ x^3\ +\ 1) \\ 1 & 0 & 0 & -x\ (-1\ +\ 3\ x) \end{bmatrix}.$$

The tropicalization of $x^{-3}\ A(x)$ is given by

$$\mathbb{A} \ = \ \begin{bmatrix} -1 & -3 & -1 & -2 \\ \infty & -2 & \infty & -2 \\ -1 & -2 & -1 & -2 \\ -3 & \infty & \infty & -2 \end{bmatrix}.$$

$\mathcal{G}(\mathbb{A})$ is strongly connected. The roots of the tropical characteristic polynomial are $\frac{-8}{3}$ of multiplicity 3 and -1 of multiplicity 1. Consequently, the eigenvalues of $A(x)$ are of orders $x^{-8/3}$ and x^{-1}. The exponential parts of $A(x)$ are given by:

$$\begin{aligned} q_{1,2,3} \ &= \ \int^t \ u^{-8} \ + \ \frac{1}{3}\ u^{-7} \ - \ \frac{1}{81}\ u^{-5} \ + \ \frac{8}{243}\ u^{-4} \ - \ \frac{7}{3}\ t^{-3}, \quad \text{where} \quad x = \ t^3 \\ q_4 \ &= \ \int^t \ -4\ u^{-1}, \quad \text{where} \quad x = t^3. \end{aligned}$$

□

2.2 Differential-like reduction

Consider again (2.1). The orbit of matrices similar to $A(x)$ is

$$\mathcal{O}_A = \{P^{-1}AP,\ P \in GL_n(\mathbb{C}((x)))\}.$$

This differential-like reduction is based on the observation that the similarity term of (1.2) in Chapter 1 plays a very central role within the formal reduction of the differential system $[A]$ given by (1). Consequently, such reductions are well-suited to treat (2.1) whose orbit consists solely of similar matrices. This algorithm consists of the following main operations which will be described in the sequel:

- Minimizing the rank of the leading matrix A_0 over $\mathcal{O}_A$.
- Recovering the eigenvalues with integer leading exponents ($s = 1$)
  - Generalized splitting lemma.
- Recovering the eigenvalues with fractional leading exponents ($s < 1$)
  - Minimizing the Ségre characteristics of the leading matrix A_0 over $\mathcal{O}_A$.

An application of this approach to a perturbed eigenvalue problem in quantum mechanics is discussed in [76]. The question addressed therein was to determine the stationary states of an electron-like particle in a quantum well of semi-conductors.

2.2.1 Integer leading exponents

The criterion Moser gave in [75] proved to be useful in perturbation theory as well. Based on the former, one can construct the so-called CRV forms, to attain an analog of irreducible systems defined earlier (Section 1.5.1) in the context of differential equations.

Definition 2.1 ([75], Definition 1)

Let $A(x)$ given by (2.1)

$$A(x) = x^{\nu} \sum_{k=0}^{\infty} A_k x^k.$$

$A(x)$ is said to be column reduced by valuation (CRV) if it has both a maximal valuation ν and a minimal number of nonzero columns of valuation ν, among all similar matrices, i.e. over $\mathcal{O}_A$.

The valuation of a CRV matrix is smaller than the smallest leading exponent of the eigenvalues. Following [107, Theorems 1 and 2, pp 381], it can be shown that a matrix $A(x)$ is in CRV form if and only if the polynomial

$$\theta_A(\lambda) := x^{rank(A_0)}\ \det(\lambda I + \frac{A_0}{x} + A_1)|_{x=0} \tag{2.2}$$

does not vanish identically in λ.

An *algebraic rank-reduction algorithm* which puts $A(x)$ in such a form can be obtained by a simple

adaptation of the algorithms of [32]. It suffices to restrict the expression of transformations used therein to their similarity terms. Not only does the polynomial (2.2) give a reduction criterion but also it characterizes the eigenvalues and leads to a non-classical splitting as follows.

| Proposition 2.1 ([75], Proposition 1)

Let $A(x)$ be in CRV form and $\theta_A(\lambda)$ be its associated polynomial. Let μ be a root of $\theta_A(\lambda)$ with multiplicity m. Then $A(x)$ has m eigenvalues of the form
$\lambda(x) = \mu x^{\nu+1} + o(x^{\nu+1})$.

| Example 2.4 ([75], Example 1)

Consider the companion matrix

$$A(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & -2x^{10} \\ 1 & 0 & 0 & 0 & 2x^7 \\ 0 & 1 & 0 & 0 & -3x^5 \\ 0 & 0 & 1 & 0 & x^3 \\ 0 & 0 & 0 & 1 & -x^2 \end{bmatrix}$$

whose five eigenvalues satisfy

$$\begin{cases} \lambda_{1,2}(x) &= \pm x^{3/2} + o(x^{3/2}), \\ \lambda_3(x) &= x^2 + o(x^2), \\ \lambda_4(x) &= 2x^2 + o(x^2), \\ \lambda_5(x) &= o(x^2). \end{cases}$$

One can verify that $\theta_A(\lambda) \equiv 0$. Reducing $A(x)$ yields $\tilde{A}(x) = T^{-1}(x)A(x)T(x)$ whose valuation is 1. Because of $\lambda_{1,2}$, the valuation cannot be increased further over $\mathcal{O}_A$. In fact, $\tilde{A}(x)$ is in CRV form and its associated polynomial is $\theta(\lambda) = -\lambda(\lambda-1)(\lambda-2)$ which is obviously not identical to zero. Thus, the roots of $\theta(\lambda)$ appear in accordance with Proposition 2.1 as the leading (possibly zero) coefficients of the eigenvalues $\lambda_3, \lambda_4,$ and λ_5. Moreover, since the dimension $n = 5$ and $deg\ \theta = 3 > 0$, $\tilde{A}(x)$ can be block-diagonalized into two square matrices of respective dimensions $n - \ deg\ \theta = 2$ and $deg\ \theta = 3$.

□

The CRV form leads to non-classic splitting of $A(x)$ [75, Section 4.2] such that the eigenvalues of each submatrix admit a ramification $s > 1$. It remains thus to deal with the resulting nilpotent submatrices. This is the case relevant to our study of the differential system (1) and it is subject of the next subsection.

2.2.2 Fractional leading exponent

Without loss of generality, we can now assume that $A_0 = J$ is nilpotent in JCF and $\nu = 0$. Moreover, at least one of the eigenvalues of $A(x)$ has its leading exponent between zero and one, i.e. $s > 1$. We

rewrite (2.1) as

$$A(x) = J + A_1 x + O(x^2), \tag{2.3}$$

and we follow the presentation of [74].
J can be completely defined by its *Segré characteristics*, i.e. the sizes of its blocks counted with their multiplicities. We use Arnold's compact notation 0^k to denote a nilpotent Jordan block of size k and $(0^k)^r$ to denote a block-diagonal matrix that consists of r such blocks. Then we write

$$J = (0^{n_1})^{r_1} \dots (0^{n_q})^{r_q}$$

such that $n_1 > \dots > n_q$ where $r_1, \dots, r_q$ are positive integers and obviously $n = \sum_{i=1}^{q} r_i n_i$. The *Segré characteristics* of J may be represented by the following partition of n

$$\kappa(J) = (\underbrace{n_1, \dots, n_1}_{r_1}, \dots, \underbrace{n_q, \dots, n_q}_{r_q}) \tag{2.4}$$

which we may also express as $\kappa(J) = ((r_j, n_j))_{1 \le j \le q}$. Given (2.3), we set $\kappa(A) = \kappa(J)$ (where J is the JCF of A_0) [74, Definition 1].

| Definition 2.2 [91]

Let $\varpi = e^{\frac{2\pi i}{k}}$. A set of k eigenvalues of $A(x)$ that satisfies

$$\lambda_\ell(x) = \mu \varpi^\ell x^{1/k} + o(x^{1/k}),\ \ell = 0, \dots, k-1,\ 0 \neq \mu \in \mathbb{C},$$

is called a *k-ring*. The number μ is the *ring constant*.

If κ is known from the outset, we can then substitute $x = t^{n_j}$ into $A(x)$ and then apply the reduction algorithm of the preceding subsection to $A(t^{n_j})$ (notice the analogy with $o(A)$ of Subsection 1.5.2). This is the case of a Lidskii matrix whose eigenvalue splitting identifies with its leading Jordan structure. Lidskii-Visik-Ljusternik provided in [88, 136] sufficient conditions on A_1, given an arbitrary nilpotent Jordan structure J, for which $A(x)$ is a Lidskii matrix.
Let $u_j = r_1 + \dots + r_j$, $v_j = r_1 n_1 + \dots r_j n_j$, $u = u_q$, $v = v_q = n$. Set $v_0 = u_0 = 0$. We partition A_1 in accordance with the partition of J into u^2 blocks. Lidskii's results assert that the leading coefficients of the eigenvalues of and perturbation $A(x)$ of A_0 depend exclusively on the entries in the lower left corners of these sub-blocks. These entries form the q_j-square constant matrices L_j as illustrated in the following example:

| Example 2.5 ([74], Example 2)

Consider the 13×13 matrix $A(x)$ whose $J = (0^3)^2\ (0^2)^3\ (0^1)^1$. We partition A_1 as follows:

$$A_1 = \left[\begin{array}{ccc|ccc|cc|cc|cc|c} * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ a_{31} & * & * & a_{34} & * & * & a_{37} & * & a_{39} & * & a_{3,11} & * & a_{3,13} \\ \hline * & * & * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * & * & * \\ a_{61} & * & * & a_{64} & * & * & a_{67} & * & a_{69} & * & a_{6,11} & * & a_{6,13} \\ \hline * & * & * & * & * & * & * & * & * & * & * & * & * \\ a_{81} & * & * & a_{84} & * & * & a_{87} & * & a_{89} & * & a_{8,11} & * & a_{8,13} \\ \hline * & * & * & * & * & * & * & * & * & * & * & * & * \\ a_{10,1} & * & * & a_{10,4} & * & * & a_{10,7} & * & a_{10,9} & * & a_{10,11} & * & a_{10,13} \\ \hline * & * & * & * & * & * & * & * & * & * & * & * & * \\ a_{12,1} & * & * & a_{12,4} & * & * & a_{12,7} & * & a_{12,9} & * & a_{12,11} & * & a_{12,13} \\ \hline a_{13,1} & * & * & a_{13,4} & * & * & a_{13,7} & * & a_{13,9} & * & a_{13,11} & * & a_{13,13} \end{array}\right]$$

where $* \in \mathbb{C}$ may be any value. Then the constant matrices L_j, $1 \leq j \leq 3$, are given by:

$$L_1 = \begin{bmatrix} a_{31} & a_{34} \\ a_{61} & a_{64} \end{bmatrix}$$

$$L_2 = \begin{bmatrix} a_{31} & a_{34} & a_{37} & a_{39} & a_{3,11} \\ a_{61} & a_{64} & a_{67} & a_{69} & a_{6,11} \\ a_{81} & a_{84} & a_{87} & a_{89} & a_{8,11} \\ a_{10,1} & a_{10,4} & a_{10,7} & a_{10,9} & a_{10,11} \\ a_{12,1} & a_{12,4} & a_{12,7} & a_{12,9} & a_{12,11} \end{bmatrix}$$

$$L_3 = \begin{bmatrix} a_{31} & a_{34} & a_{37} & a_{39} & a_{3,11} & a_{3,13} \\ a_{61} & a_{64} & a_{67} & a_{69} & a_{6,11} & a_{6,13} \\ a_{81} & a_{84} & a_{87} & a_{89} & a_{8,11} & a_{8,13} \\ a_{10,1} & a_{10,4} & a_{10,7} & a_{10,9} & a_{10,11} & a_{10,13} \\ a_{12,1} & a_{12,4} & a_{12,7} & a_{12,9} & a_{12,11} & a_{12,13} \\ a_{13,1} & a_{13,4} & a_{13,7} & a_{13,9} & a_{13,11} & a_{13,13} \end{bmatrix}$$

□

The matrix L_j is called the *j-Lidskii submatrix* associated with $A(x)$. Let $\Delta_j \equiv \det\ L_j$. We denote by Δ_j^k, for $k = 1, \ldots, r_j$, the sum of the principal minors of L_j with order $t_{j-1} + k$ that contain L_{j-1}. Then we have:

| Theorem 2.3 [88, 105]

We say that a matrix is a Lidskii matrix if and only if $\Delta_j \neq 0$ for $j = 1, \ldots, q$. Additionally, the n_j^{th} power of the ring constants for each j are the roots of the polynomial $\theta_j = \det(L_j - \lambda E_j)$ of degree r_j

where $E_j = \text{diag}\ (0_{t_{j-1}}, I_{r_j})$.

| Example (Continue Example 2.5)

We have,

$$\begin{cases} E_1 &= \text{diag}\ (\lambda, \lambda); \\ E_2 &= \text{diag}\ (0, 0, \lambda, \lambda, \lambda); \\ E_3 &= \text{diag}\ (0, 0, 0, 0, 0, \lambda). \end{cases}$$

□

A generic perturbation of a JCF is a Lidskii matrix. However non-generic perturbations occur and the following is a well-known example.

| Example 2.6 ([74], Example 2)

Consider again $A(x)$ of Example 2.5. In A_1, we set all the entries equal to zero except for

$$a_{31} = a_{67} = a_{89} = a_{10,11} = a_{12,13} = a_{13,4} = 1.$$

One can easily verify that Lidskii conditions are not satisfied in this case since $\Delta_1 = \Delta_2 = 0$ although $\Delta_3 \neq 0$. In fact, the splitting of the eigenvalues comply not with $\kappa(A) = (3, 3, 2, 2, 2, 1)$, but rather with $(3, 2, 2, 2, 2, 2)$.

□

Thus, we need to compare two partitions of n. For this purpose, we recall two of their representations: the Ferrer diagram (Young tableaux) and the Newton envelope.

A Ferrer diagram characterization

The *Ferrer diagram* $\mathcal{R}(J)$ of J represents each Jordan block 0^{n_j} as a stack of n_j coins, ordered by nonincreasing heights from left to right. The move of a coin from a stack to another, from the left to the right, without modifying the monotonicity of the partition is called a rightward coin move, as illustrated by Figure 2.1.

Given two nilpotent Jordan structures J and $\tilde{J}$ such that $rank(J) = rank(\tilde{J})$. We have $\kappa(J) > \kappa(\tilde{J})$ if and only if $\mathcal{R}(\tilde{J})$ can be derived from $\mathcal{R}(J)$ by a rightward coin move.

| Example 2.7

Let $J = (0^3)^2\ (0^2)^3\ 0$ (Example 2.6) and $\tilde{J} = (0^3)\ (0^2)^5$. Then,

$$\kappa(J) = (3, 3, 2, 2, 2, 2, 1)\ >\ \kappa(\tilde{J}) = (3, 2, 2, 2, 2, 2, 1).$$

□

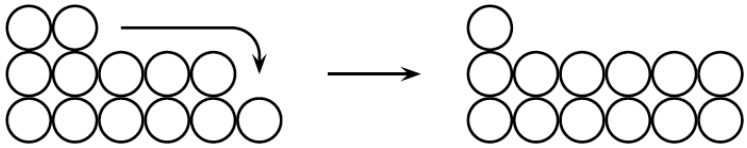


Figure 2.1: Ferrer diagrams for J and $\tilde{J}$ and rightward coin move indicating that $\kappa(J) > \kappa(\tilde{J})$ [74, Fig. 1].

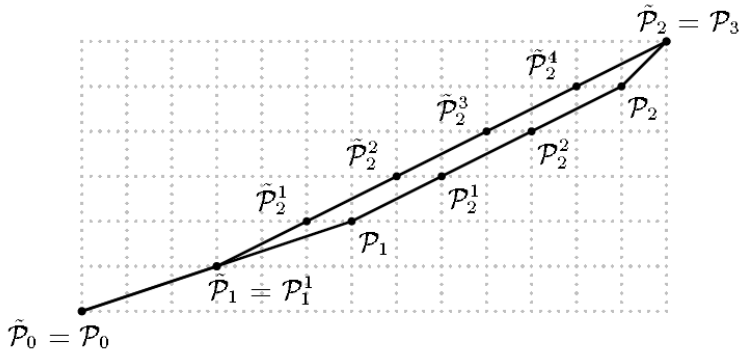


Figure 2.2: Newton envelopes for $J = (0^3)^2\ (0^2)^3\ 0$ and $\tilde{J} = (0^3)\ (0^2)^5$ also indicating that $\kappa(J) > \kappa(\tilde{J})$ [74, Fig. 2].

A Newton diagram characterization

Another possible characterization of Lidskii's conditions is given in terms of a *Newton envelope* associated with J in [105, pg 805-806]. By setting $\mathcal{P}_0 \equiv (0,0)$ and defining for $j = 1,\ldots,q$, $k = 1,\ldots,r_j$

$$\mathcal{P}_j^k \equiv (v_{j-1} + k\ n_j, u_{j-1} + k) \quad \text{and} \quad \mathcal{P}_j \equiv \mathcal{P}_j^{r_j} \equiv (v_j, u_j).$$

The Newton envelope $\mathcal{E}(J)$ associated with J is then the lower convex hull of the set $\{\mathcal{P}_j\}_{0\le j\le q}$.
Given two nilpotent Jordan structures J and $\tilde{J}$ such that $rank(J) = rank(\tilde{J})$. We say that $\mathcal{E}(\tilde{J})$ lies above $\mathcal{E}(J)$ when all the $\tilde{\mathcal{P}}_j^k$ lie on $\mathcal{E}(J)$ with at least one of them lying strictly above $\mathcal{E}(J)$ (see Figure 2.2). We have $\kappa(J) > \kappa(\tilde{J})$ if and only if $\mathcal{E}(\tilde{J})$ lies above $\mathcal{E}(J)$. In fact, $\mathcal{E}(J)$ is the lowest possible Newton diagram that can be obtained by a perturbation of J, which gives rise to the following proposition.

| Proposition 2.2 ([74], Proposition 1)
If $A(x)$ is a Lidskii matrix then $\kappa(A)$ is minimum over $\mathcal{O}_A$.

This proposition suggests two steps:

- Minimizing the rank of J over $\mathcal{O}_A$. This can be achieved as explained in the previous subsection using the algebraic rank-reduction algorithm. In particular, the last Lidskii condition $\Delta_q \neq 0$ is equivalent to $\theta_A(\lambda) \neq 0$ ([74, Proposition 3]). Hence, we can assume without loss of generality that $A(x)$ is in CRV form.

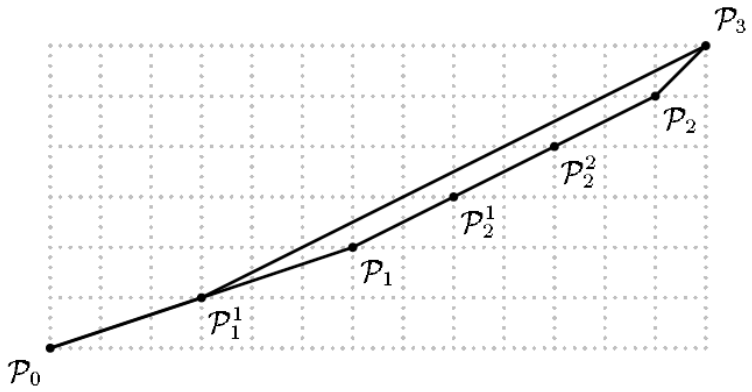


Figure 2.3: Newton diagram and envelope for Example 2.6: $\Delta_1^1 \neq 0$ and $\Delta_3 \neq 0$, whereas $\Delta_1 = \Delta_2^1 = \Delta_2^2 = \Delta_2 = 0$ [74, Fig. 3].

- However, minimizing the rank does not suffice to satisfy all Lidskii conditions as illustrated in Figure 2.3. This suggests minimizing $\kappa(A)$ over the subset of $\mathcal{O}_A$ for which leading matrices have the same minimal rank. This is the subject of the subsection 2.2.2.

Minimizing $\kappa(A)$

Without loss of generality, we can assume that $rank(J)$ is minimal over $\mathcal{O}_A$ and so $\Delta_q \neq 0$. We denote by j the smallest integer of $\{2, \ldots, q\}$ such that $\Delta_{j-1} = 0$ ($n_{j-1} > 1$). Based on an analysis by Newton envelopes, an algorithm was given in [74], which uses shearing transformations and column/row operations to fulfill recursively Lidskii conditions. In the Ferrer diagram, these transformations are illustrated by a rightward coin move (see Figure 2.4). In particular, if $n_j = n_{j-1} - 1$ then the coin has first to roll before falling (see Figure 2.1).

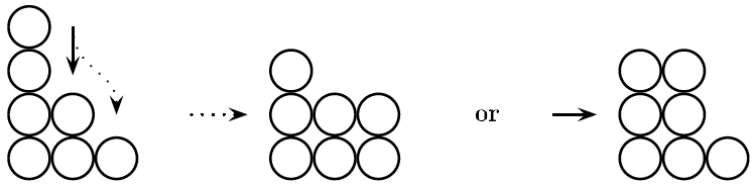


Figure 2.4: Coin move from $0^4 0^2 0$ to $(0^3)^2 0$ or to $0^3 (0^2)^2$ [74, Fig. 4].

The reduction of $A(x)$ in [74] stops either when all Lidskii's conditions $\Delta_j \neq 0$ are satisfied, i.e. either $A(x)$ a Lidskii matrix, or $\kappa(A)$ cannot be decreased strictly any further. The eigenvalues of the output matrix do not split into r_j n_j-rings (see. e.g. [74, Sec. 7]). In such a situation, $A(x)$ has necessarily a leading Jordan form whose block sizes differ at most by one. Otherwise, a rightward movement in the Ferrer diagram can still be attained to decrease $\kappa(A)$ strictly. In fact, Proposition 2.2 is necessary but not sufficient. Thus, there are singular cases which cannot be treated by this algorithm. In the next section, we try to explain these cases using results from the tropical investigation.

2.2.3 Singular case

We illustrate the singular case in the following very simple example.

Example 2.8

Let

$$A(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & x \\ x & 0 & 0 \end{bmatrix} \quad \text{and} \quad B(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x^2 & 0 & 0 \end{bmatrix}$$

which are similar for all $x \neq 0$ and whose eigenvalues are $1^{1/3}\ x^{2/3}$. Clearly, for the matrix $A(x)$ we have

$$L_1 = \begin{bmatrix} 0 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence $\Delta_1 = \det\ L_1 = 0$ and so $A_1(x)$ is not a Lidskii matrix although $\kappa(A)$ is minimum over $\mathcal{O}_A$ (see Figure 2.5).

However, we have observed the following: Consider the tropicalization $\mathbb{A}$ of $A(x)$,

$$\mathbb{A} = \begin{bmatrix} \infty & 0 & \infty \\ \infty & \infty & 1 \\ 1 & \infty & \infty \end{bmatrix}.$$

One can easily verify that its underlying digraph $\mathcal{G}(\mathbb{A})$ is strongly connected (see Figure 2.5). By Theorem 2.1, $\mathbb{A}$ has a unique eigenvalue $\frac{2}{3}$, which coincides with the exponents of the the classical eigenvalues of $A(x)$.

□

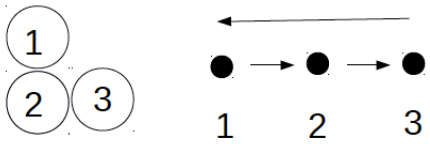


Figure 2.5: Ferrer diagram of $A(x)$ with labeled coins and digraph of $\mathbb{A}$ in Example 2.8.

Example 2.8 is not a particular case regarding the strong connectedness of the underlying digraph. Before proving that this holds true for any singular case of the differential-like reduction, we remark again that, in such a case, $A(x)$ has necessarily a leading Jordan form whose block sizes differ at most by one. We denote this form by

$$J = (0^{n_1})^{r_1}\ (0^{n_2})^{r_2} \quad \text{for some} \quad n_1 > n_2,\ r_1, r_2 \in \mathbb{N}^*.$$

We prove the following:

| Proposition 2.3

Let $A(x)$ of (2.1) be in CRV form and have a minimal $\kappa(A)$ over O_A. Suppose that $A(x)$ is not a Lidskii matrix and denote by $\mathbb{A} \in \mathbb{R}_{min}$ its tropicalization. Then, the underlying digraph $\mathcal{G}(\mathbb{A})$ of $\mathbb{A}$ is strongly connected.

Proof. We recall that the in_degree of a node is the number of arcs leading to that node, and the out_degree of a node is the number of arcs leading away from that node. In order to prove that a digraph is strongly connected, it suffices to prove that every node has each of its in_degree and out_degree strictly greater than zero, upon excluding the arcs from a node to itself (see, e.g. [130]). For this purpose, we label the coins in the Ferrer diagram of $A(x)$ from 1 to n from left to right, up to bottom (see Example 2.9 for an illustration). We divide the nodes of $\mathcal{G}(\mathbb{A})$ to three families with nonempty intersections:

$$\begin{aligned} N_{out} &= \{n_1\, j,\ 0 < j \leq r_1\} \cup \{n_1\, r_1 \,+\, n_2\, j,\ 0 < j \leq r_2\} \\ N_{in} &= \{n_1\, j \,+\, 1,\ 0 \leq j < r_1\} \cup \{n_1\, r_1 \,+\, n_2\, j \,+\, 1,\ 0 \leq j < r_2\} \\ N_J &= \{i,\ 1 \leq i \leq n\} \,-\, N_{out}. \end{aligned}$$

Let (i, j) denote an arc from node i to node j $(1 \leq i, j \leq n)$. For any $i \in N_J$, the structure of J induces the arcs $(i, i+1)$. Remarking that $N_{in} \subset N_J$, we have:

$$\begin{cases} \text{If} \quad i \in N_J & \text{then} \quad out\_degree(i) > 0; \\ \text{If} \quad i \in \{N_J - N_{in}\} \,\cup\, N_{out} & \text{then} \quad in\_degree(i) > 0. \end{cases}$$

Thus, it is left to prove that

$$\text{If} \quad i \in N_{in} \quad \text{then} \quad in\_degree(i) > 0; \tag{2.5}$$

$$\text{If} \quad i \in N_{out} \quad \text{then} \quad out\_degree(i) > 0. \tag{2.6}$$

$$\tag{2.7}$$

Consider the sequences S_{in} and S_{out} formed by ordering the elements of N_{in} and N_{out} respectively with respect to the natural ordering. Then the two 1-Lidskii and 2-Lidskii submatrices can be expressed as follows:

$$\begin{aligned} L_1 &= [a_{ij}]_{\{i \in S_{out},\, j \in S_{in},\, 1 \leq i\, ,j\, \leq n_1\, r_1\}}; \\ L_2 &= [a_{ij}]_{\{i \in S_{out},\, j \in S_{in}\}}. \end{aligned}$$

We recall that for whatever $1 \leq i, j \leq n$, if $a_{ij} \neq 0$ then there exists an arc (i, j). Under the assumptions in the statement of the proposition, $\Delta_1 = \det(L_1) = 0$ and $\Delta_2 = \det(L_2) \neq 0$. It follows from the latter that in every column of L_2 there exists at least one nonzero entry. And in every row there exists at least one nonzero entry. We thus distinguish between two cases:

- If $n_2 > 1$ then (2.5) and (2.6) follow directly from $\Delta_2 \neq 0$.

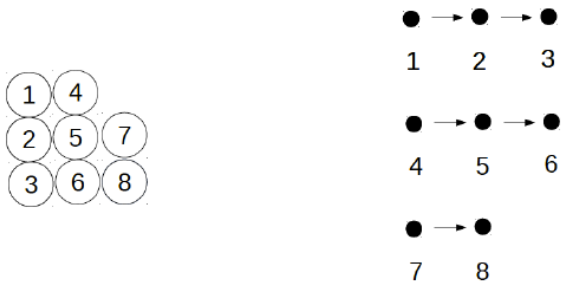


Figure 2.6: Ferrer diagram of $A(x)$ with labeled coins and the arcs of $\mathcal{G}(\mathbb{A})$ induced by J.

- If $n_2 = 1$ then one should consider $\Delta_1 = \det(L_1) = 0$ since $a_{nn} \in S_{in} \cup S_{out}$: Suppose that the only nonzero entry in the last column is a_{nn} (and so the n^{th}-node has an arc to itself and not to another distinct node). By expanding Δ_2 with respect to the last column, $\Delta_2 = a_{nn}\ \Delta_1 = 0$ which is a contradiction. The same argument follows for the last row.

■

| Example 2.9

$$A(x) = \left[\begin{array}{ccc|ccc|cc} * & 1 & * & * & * & * & * & * \\ * & * & 1 & * & * & * & * & * \\ a_{31} & * & * & a_{34} & * & * & a_{37} & * \\ \hline * & * & * & * & 1 & * & * & * \\ * & * & * & * & * & 1 & * & * \\ a_{61} & * & * & a_{64} & * & * & a_{67} & * \\ \hline * & * & * & * & * & * & * & 1 \\ a_{81} & * & * & a_{84} & * & * & a_{87} & * \end{array}\right].$$

We have

$$L_1 = \begin{bmatrix} a_{31} & a_{34} \\ a_{61} & a_{64} \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} a_{31} & a_{34} & a_{37} \\ a_{61} & a_{64} & a_{67} \\ a_{81} & a_{84} & a_{87} \end{bmatrix}$$

Since $J = (0^3)^2\ (0^2)^1$, we have (see Figure 2.6):

$$\begin{aligned} N_{out} &= \{3, 6, 8\} \\ N_{in} &= \{1, 4, 7\} \\ N_J &= \{1, 2, 4, 5, 7\}. \end{aligned}$$

□

One might question whether the digraph is necessarily strongly connected if $\kappa(A)$ is not minimal. The following example gives a negative answer.

| Example 2.10

Let

$$A(x) = \left[\begin{array}{ccc|cc|c} * & 1 & * & * & * & * \\ * & * & 1 & * & * & * \\ a_{31} & * & * & a_{34} & * & a_{36} \\ \hline * & * & * & * & 1 & \\ a_{51} & * & * & a_{54} & * & a_{56} \\ \hline a_{61} & * & * & a_{64} & * & a_{66} \end{array}\right] \quad \text{and} \quad L_3 = \begin{bmatrix} a_{31} & a_{34} & a_{36} \\ a_{51} & a_{54} & a_{56} \\ a_{61} & a_{64} & a_{66} \end{bmatrix}.$$

$$\text{We also have} \quad L_2 = \begin{bmatrix} a_{31} & a_{34} \\ a_{51} & a_{54} \end{bmatrix} \quad \text{and} \quad L_1 = \begin{bmatrix} a_{31} \end{bmatrix}.$$

A quick sketching of the Ferrer diagram shows that $\kappa(A)$ is not minimal. Suppose that Lidskii conditions are not all satisfied: $\Delta_1 = 0$, $\Delta_2 \neq 0$, and $\Delta_3 \neq 0$. The argument of the proof of Proposition 2.3 cannot be replicated here to prove that $in\_degree(6) > 0$: Suppose that a_{66} is the only nonzero entry of the last row of Δ_3. Then, no contradiction arises since in this case $\Delta_3 = a_{66}\ \Delta_2 \neq 0$.

□

We also have the following corollary which explains the singular case of Example 2.8.

| Corollary 2.1

Under the notations and assumptions of Proposition 2.3, the unique eigenvalue of $\mathbb{A}$ is bounded from above by $\frac{r_1+r_2}{n}$.

Proof. Since the underlying graph of $\mathbb{A}$ is strongly connected, it follows from Theorem 2.1 that $\mathbb{A}$ has a unique tropical eigenvalue determined by the minimal normalized weight of cycles. To find the claimed bound for the eigenvalue, it suffices to find a cycle whose normalized weight is given by this bound. We show here the existence of such a cycle. In particular, a Hamiltonian cycle (a cycle that visits each node exactly once) which has only one-weighted arcs and a maximum number of zero-weighted arcs. Such a cycle can be retrieved as follows:
Consider again the sets N_J, N_{out}, and N_{in} in the proof of Proposition 2.3. Since the set N_J corresponds to the arcs induced by J, these arcs are zero-weighted. And their number is equal to the cardinal $card_{N_J}$. In other words, it is equal to the number of "1"s in J. Recalling that $n_2 = n_1 - 1$, it is given by:

$$card_{N_J} = (r_1\, n_1 - r_1) + (r_2\, n_2 - r_2) = (r_1 + r_2)\, n_1 - r_1 - 2\, r_2.$$

To complete our cycle, we choose a minimal number of one-weighted arcs whose number is clearly equal to $n - card_{N_J} = card_{N_{out}}$. Thus, the normalized weight of such cycle is given:

$$\frac{1.\, card_{N_{out}} + 0.\, card_{N_J}}{n} = \frac{r_1 + r_2}{n}.$$

■

In the case the bound is attained, the eigenvalues of $A(x)$ split generically into a $\frac{r_1+r_2}{n}$-ring by Theorem 2.2. This gives an explanation to the exponent $\frac{2}{3}$ of Example 2.8.

2.3 Conclusion

In this chapter, we discussed the interaction between the perturbation theory of JCF, the differential-like reduction, and tropical algebra. We implemented some of the algorithms described in this chapter in the package ALGPARAM in Maple. We conclude with various open questions regarding this interaction, the most prevalent among which are: Do the two approaches presented here superpose so that one can cover the singular case of the other? And what kind of interpretation does this hold for a linear differential system[‡]? For example, can the genericity required in Theorem 2.2 be guaranteed for an irreducible linear singular differential system?

[‡]Miyake introduced in [104] a reduction based on computing a volevič weight (due to L.R.Volevič) of the differential system (1). Roughly speaking, this weight is the minimum of normalized sums of the valuations of the system's matrix $A(x)$. It thus seems plausible to attempt to interpret this weight in terms of an underlying graph of the tropicalization of $A(x)$. Additionally, it is also interesting to study the relation between this weight and notion of irreducibility of the system described in Section (1.5.1)

Chapter 3

Removing Apparent Singularities

Contents

In this chapter we present a new algorithm which, given a system of first-order linear differential system with rational function coefficients, constructs an equivalent system with rational function coefficients, whose finite singularities are exactly the non-apparent singularities of the original system. This chapter is an elaboration [27] and refinement of the algorithm which was first introduced in [19]. We thus consider the following system

$$[A] \qquad \partial F = A(x)F$$

where $A \in \mathcal{M}_n(\mathbb{C}(x))^*$. The finite singularities of system $[A]$ are the poles of $A(x)$ in $\mathbb{C}$. A singular point $x_0 \in \mathbb{C}$ is called an *apparent singularity* if there exists a fundamental matrix solution $\Phi(x)$ which is analytic at $x = x_0$. Consider, for example, the first-order differential system

$$\partial F \;=\; A(x)\; F \;=\; \begin{bmatrix} 0 & 1 \\ \frac{-2}{x} & 1+\frac{2}{x} \end{bmatrix} F \tag{3.1}$$

Clearly, this system is equivalent to the second-order scalar differential equation given by the monic

*We remark that in this chapter we are interested in a global aspect and so we are considering $A \in \mathcal{M}_n(\mathbb{C}(x))$ rather than $A \in \mathcal{M}_n(\mathbb{C}[[x]])$. Obviously, the absence of x^p in the left hand side does not imply that the system is regular because the entries of A(x) are rational functions.

operator L:

$$L := \partial^2 - \frac{x+2}{x}\partial + \frac{2}{x},$$

for which e^x and $1 + x + \frac{x^2}{2}$ form a basis of solutions. Due to this basis, one can deduce that the point $x = 0$ is an apparent singularity of system (6.12). *Desingularization*, i.e. the problem of constructing another operator $\tilde{L}(f)$ of higher order such that the solution space of $\tilde{L}(f) = 0$ contains that of $L(f) = 0$, and for which the factor x is "removed" from the denominator, is an interesting problem of research. For instance, by the method given by Abramov-Barkatou-van Hoeij in [3] (and which we refer to as the ABH method), one can compute a desingularization of order 4 given by the operator

$$\begin{aligned}\tilde{L} \quad = \quad & \partial^4 + (-1 + 1/4\,x)\,\partial^3 + (-1/4 - 3/8\,x)\,\partial^2 \\ & + (1/2 + 1/8\,x)\,\partial - 1/4.\end{aligned}$$

In the scalar case, several desingularization algorithms exist for differential, difference (e.g., [3]), and more generally, Ore operators (see, e.g. [45, 44] and references therein). However, the apparent singularity of system (6.12) (equivalently of L) at $x = 0$, can be also removed by acting directly on it. In fact, by setting [19]

$$F = T(x)\,G, \quad T(x) = \begin{bmatrix} 1 & 0 \\ 1 & x^2 \end{bmatrix},$$

the new variable G satisfies the equivalent first-order differential system of the same size as the order of L, given by

$$[\tilde{A}] \qquad \partial G = \tilde{A}\,G$$

where

$$\tilde{A} := T^{-1}AT - T^{-1}\partial T = \begin{bmatrix} 1 & x^2 \\ 0 & 0 \end{bmatrix}.$$

As in Chapter 1, we say that systems $[A]$ and $T[A] = [\tilde{A}]$ are equivalent over $\mathbb{C}(x)$.

In this chapter, we shall prove that, given any system $[A]$ with rational coefficients, it can be reduced to an equivalent system $[\tilde{A}]$ with rational coefficients, such that the finite singularities of $[\tilde{A}]$ coincide with the non-apparent singularities of $[A]$. Our method can, in particular, be applied to the companion system of any scalar linear differential equation with arbitrary order n. We thus have an alternative method to the standard methods for removing apparent singularities of linear differential operators. However, it is also interesting by its own since first-order linear differential systems with apparent singularities arise naturally in applications (see, e.g., [77, 83, 37] and references therein for applications within Feynman integrals and statistical physics). Moreover, such a desingularization can be very beneficial to numerical methods.

We recall (see Section 1.5.1) that a singular point $x_0 \in \mathbb{C}$ is called a *regular singular point* for the system $[A]$ if in a neighborhood of x_0, there exists a fundamental matrix solution of the form $\Phi(x)(x-x_0)^C$ where $C \in \mathcal{M}_n(\mathbb{C})$ is a constant matrix and $\Phi(x)$ is a matrix which is analytic at x_0; otherwise x_0 is called an *irregular singular point*. The change of variable $x \mapsto 1/x$ permits to classify the point $x = \infty$ as an ordinary, a regular singular, or an irregular singular point of the system $[A]$.

Hence apparent singularities of $[A]$ are among regular singularities of $[A]$. And, as shown in Section 1.5.1, it is proven that in the case where x_0 is a regular singularity there exists a polynomial matrix function T of $x - x_0$ which is nonsingular for $x \neq x_0$ such that the transformation $F = TG$ transforms the system $[A]$ into an equivalent system $[\tilde{A}]$ $\partial G = \tilde{A}(x)G$, for which x_0 is a simple pole. This latter system can be constructed using the so called *Rational Moser algorithm* developed in [15]. This algorithm, establishes *partial desingularization*, as it computes for a given system $[A]$ a polynomial transformation $T(x)$ with $\det(T(x)) \not\equiv 0$ that leads to a system $[\tilde{A}]$ such that:

(i) The finite singularities of $[\tilde{A}]$ are among the finite singularities of $[A]$.

(ii) Every finite singular point of $[\tilde{A}]$ has a minimal pole order among all equivalent systems.

This chapter is organized as follows: In Section 3.1, we give our main result: We show how apparent singularities can be detected, prove the existence of desingularizations, and develop a desingularization algorithm over $\mathbb{C}(x)$. In Section 3.2, we show some examples of treating scalar linear differential equations. In Section 3.3, we give a rational version of our algorithm. And finally, we point out items for further investigation in Section 3.4.

3.1 Detecting and removing apparent singularities

Consider a system $[A]$ $\partial F = A(x)F$ with $A(x) \in \mathcal{M}_n(\mathbb{C}(x))$.

| Definition 3.1

A system $[\tilde{A}]$ $\partial G = \tilde{A}(x)\, G$ is called a desingularization of $[A]$ if:

(i) There exists a polynomial matrix $T(x)$ with $\det(T(x)) \not\equiv 0$ such that $[\tilde{A}] = T[A]$;

(ii) The singularities of $[\tilde{A}]$ are the singularities of $[A]$ that are not apparent.

We first prove that desingularizations do exist:

| Proposition 3.1

If $x = x_0$ is a finite apparent singularity of $[A]$ then there exists a polynomial matrix $T(x)$ with

$$\det\ T(x) = c(x - x_0)^{\alpha}, \quad c \in \mathbb{C}^*, \alpha \in \mathbb{N}$$

such that $[\tilde{A}] := T[A]$ has no pole at $x = x_0$.

Proof. Every fundamental matrix of formal solutions $\Phi(x)$ of $[A]$ is analytic at x_0. Formally, $\Phi(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$. Since $\mathbb{C}[[x]]$ is a principal ideal domain, one can construct unimodular transformations $P(x) \in GL_n(\mathbb{C}[x])$, and $Q(x) \in GL_n(\mathbb{C}[[x]])$ such that (see, e.g.[107, Lemma 1] or Smith normal form)

$$P(x)\Phi(x)Q(x) = \mathrm{diag}((x - x_0)^{\alpha_1}, \ldots, (x - x_0)^{\alpha_n})$$

where $\alpha_1, \dots \alpha_n$ are nonnegative integers. We then set

$$T(x) \;=\; P^{-1}(x)\;\operatorname{diag}((x-x_0)^{\alpha_1}, \dots, (x-x_0)^{\alpha_n}).$$

■

This proof however is not constructive. In the sequel we shall develop an algorithm that produces a desingularization of any system $[A]$ over $\mathbb{C}(x)$. We start by explaining how to remove one apparent singularity.

| Proposition 3.2

If $x = x_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(x)$ with $\det(T(x)) = c(x-x_0)^{\alpha}$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $T[A]$ has at worst a simple pole at $x = x_0$.

Proof. We have:

(i) An apparent singularity is a regular singularity, hence $[A]$ can be reduced to an equivalent system $T[A]$ which has x_0 as a singularity of first kind.

(ii) The transformation T can be constructed by the algorithm in [15] and hence it has the required property (see [15, Theorem 2]).

■

| Proposition 3.3

Suppose that $A(x)$ has simple pole at $x = x_0$ and let

$$A(x) = \frac{A_0}{(x-x_0)} + \sum_{i\geq 1} A_i (x-x_0)^{i-1}, \;\; A_i \in \mathcal{M}_n(\mathbb{C}).$$

If x_0 is an apparent singularity then the eigenvalues of the so-called residue matrix A_0, are nonnegative integers.

Proof. Suppose that A_0 possesses at least one eigenvalue which does not belong to $\mathbb{N}$ and let μ be an eigenvalue of A_0 such that $\mu \in \mathbb{C} \setminus \mathbb{N}$ with a maximal real part. Then the system $[A]$ has a nonzero local vectorial solution of the form:

$$F(x) = (x-x_0)^{\mu} \sum_{k=0}^{+\infty} F_k (x-x_0)^k$$

with $F_k \in \mathbb{C}^n$ and $F_0 \neq 0$, the series being convergent in a disc centered at x_0 (see, e.g. [11, Chapter 2, Theorem 6, pp 32]). Such a solution however is not analytic at $x = x_0$ because $\mu \notin \mathbb{N}$. This implies that x_0 is a singularity which is not apparent. ■

| Proposition 3.4

Suppose that $x = x_0$ is a simple pole of $A(x)$ and that A_0 has only nonnegative integer eigenvalues. Then there exists a polynomial matrix $T(x)$ with $\det(T(x)) = c(x - x_0)^\alpha$ for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $[\tilde{A}] \quad \partial G \;=\; \tilde{A}(x)\, G$ has at most a simple pole at $x = x_0$ and $\tilde{A}_0$ has a single eigenvalue: $\tilde{A}_0 = mI_n + N$ where $m \in \mathbb{N}$ and N is nilpotent. Moreover, x_0 is an apparent singularity iff $N = O$. In this case, the transformation $G = (x - x_0)^m W$ leads to a system for which $x = x_0$ is an ordinary point.

Proof. Let $m_1, \ldots, m_s \in \mathbb{N}$ be the eigenvalues of A_0. For $i = 1, \ldots, s$, denote by ν_i the multiplicity of m_i. Suppose that $m_1 > m_2 > \ldots > m_s$ and put $\ell_i = m_i - m_{i+1} \in \mathbb{N}^*, \;\; i = 1, \ldots, s-1$. By applying a constant transformation we can assume that A_0 is in Jordan form:

$$A_0 = \begin{bmatrix} A_0^{11} & 0 \\ 0 & A_0^{22} \end{bmatrix}, \tag{3.2}$$

where A_0^{11} is an ν_1 by ν_1 matrix having one single eigenvalue m_1:

$$A_0^{11} = m_1 I_{\nu_1} + N_1,$$

N_1 being a nilpotent matrix. The transformation $F = PZ$, where

$$P = \operatorname{diag}((x - x_0)I_{\nu_1}, I_{n-\nu_1}) \tag{3.3}$$

yields the new system:

$$\partial Z = B(x)Z,$$

where

$$B(x) = P^{-1}A(x)P - P^{-1}\partial P.$$

Its residue matrix is given by:

$$B_0 = \left(A_0 + (x - x_0)P^{-1}A_1P - (x - x_0)P^{-1}\partial P\right)_{|x=x_0}.$$

Let A_1 be partitioned as A_0 :

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} \\ A_1^{21} & A_1^{22} \end{bmatrix}, \quad A_1^{11} \in \mathcal{M}_{\nu_1}(\mathbb{C})$$

Then

$$B_0 = \begin{bmatrix} A_0^{11} - I_{\nu_1} & A_1^{12} \\ 0 & A_0^{22} \end{bmatrix}. \tag{3.4}$$

Hence the eigenvalues of B_0 are: $m_1 - 1, m_2, \ldots, m_s$, each with the same initial multiplicities.

By repeating this process ℓ_1 times where $m_1 - \ell_1 = m_2$, the eigenvalues become:

$$m_2, m_2, \ldots, m_s,$$

with the respective multiplicities $\nu_1, \nu_2 \ldots, \nu_s$. Thus, after $\ell_1 + \ldots + \ell_{s-1} = m_1 - m_s$ steps, let $T(x)$ be the product of all the transformations applied, then one gets an equivalent system $[\tilde{A}] := T[A] \quad \partial G = \tilde{A}G$ of the first kind with a residue matrix A_0 with a single eigenvalue m_s of multiplicity $\nu_1 + \ldots + \nu_s = n$. Hence the matrix $N := A_0 - m_s I_n$ is nilpotent. Moreover the matrix T is the product of matrices that are either constant or of the form (3.3). Hence T is a polynomial matrix of degree at most $m_1 - m_s$ and its determinant is of the form $\det(T(x)) = c(x - x_0)^\alpha$ for some $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$.
Due to the form of $\tilde{A}_0$ it follows that the system $[\tilde{A}]$ has, in the neighborhood of x_0, a fundamental matrix solution of the form

$$G(x) = (x - x_0)^{m_s} \Phi(x) (x - x_0)^N$$

where

$$\Phi(x) = I_n + \sum_{k=1}^{+\infty} \Phi_k (x - x_0)^k \quad \text{analytic at } x = x_0.$$

Hence x_0 is an apparent singularity of $[A]$ if and only if N is the zero matrix. Finally, if we put $G = (x - x_0)^{m_s} W$ the resulting system has the above matrix $\Phi(x)$ as a fundamental matrix solution around $x = x_0$. As $\Phi(x_0) = I_n$, the point x_0 is an ordinary point for the latter system. ■

Remark 3.1

One can deduce from the above proof that a necessary (but not sufficient) condition for a first-kind singularity x_0 to be an apparent singularity for $[A]$ is that the residue matrix A_0 be diagonalizable with nonnegative integer eigenvalues. Indeed, if A_0 is not diagonalizable then in (3.2) at least one of the two blocks A_0^{11}, A_0^{22} has a nonzero nilpotent part. It follows from the form of (3.4) that transformation (3.3) cannot annihilate the nilpotent part of the residue matrix. Thus if the residue matrix A_0 of the input system (or one of the intermediate systems) is not diagonalizable then the output system $[\tilde{A}]$ has a residue matrix $\tilde{A}_0 = mI_n + N$ with $N \neq O$.

We thus proved the following theorem:

Theorem 3.1

If $x = x_0$ is a finite apparent singularity of $[A]$ then one can construct a polynomial matrix $T(x)$ with $\det(T(x)) = c(x - x_0)^\alpha$, $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{N}$ such that $\tilde{A}(x) := T[A]$ has no pole at $x = x_0$.

Due to the form of its determinant, the gauge transformation $T(x)$ in the previous theorem does not affect the other finite singularities of $[A]$. This means that the apparent singularity at x_0 is removed without introducing new finite singularities or changing the pole order of the other finite singularities of $[A]$, as illustrated by the following simple example.

| Example 3.1

Given the system $[A]\ \partial F = A(x)F$ where

$$A = \begin{bmatrix} 0 & 1 \\ 2\,\frac{-1+2\,x^2}{x^2+2} & -\frac{3\,x^2-4}{x(x^2+2)} \end{bmatrix}$$

It has a simple pole at $x = 0$ with a residue matrix

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Our algorithm computes the gauge transformation given by

$$T = \begin{bmatrix} 1 & 0 \\ x & -x^2 \end{bmatrix}$$

The matrix of the new equivalent system is

$$\tilde{A} = T^{-1}(AT - \partial T) = \begin{bmatrix} x & -x^2 \\ 1 & -\frac{x\left(x^2+7\right)}{x^2+2} \end{bmatrix}$$

As expected it has $x = 0$ as an ordinary point. Moreover, neither new finite singularities are introduced, nor the pole order of the other non-apparent finite singularities, i.e. the roots of $(x^2 + 2)$, are changed.

□

Thus by successively applying Theorem 3.1 to each finite apparent singularity of $[A]$, we get the following:

| Theorem 3.2

Given a system $[A]$, one can construct a polynomial matrix $T(x)$ which is invertible in $\mathbb{C}(x)$ such that the finite poles of $\tilde{A} := T[A]$ are exactly the poles of A that are not apparent singularities for $[A]$.

| Remark 3.2

If the point at infinity of the original system is singular regular then it will be also singular regular of the computed desingularization. However, the order of the pole at infinity may increase. This follows immediately from the fact that the two systems are gauge equivalent.

Consider a system $[A]\quad \partial F = A(x)F$ and let $\mathcal{P}(A)$ denote the set of finite poles of $A(x)$. We thus have the desingularization Algorithm 3.

Algorithm 3 APP_SING Desingularization Algorithm

Input: $A(x)$;
Output: $T(x) \in GL_n(\mathbb{C}(x))$ and $T[A]$ such that $T[A]$ is a desingularization of the input system $[A]$. And, the two sets $\mathcal{A}pp$ and Σ of apparent singularities (which are "removed") and simple poles (which are not apparent singularities) respectively.

$T \leftarrow$ Use the *Rational Moser Algorithm* of [15] to compute a polynomial matrix $T(x)$ with $\det(T(x)) \not\equiv 0$ such that

- The roots of $\det(T(x)) = 0$ belong to $\mathcal{P}([A])$ (*this implies that the poles of $T^{-1}(x)$ are among the poles of A and hence $\mathcal{P}(T[A]) \subset \mathcal{P}([A])$*)
- The orders of the poles of $T[A]$ are minimal among all equivalent systems

$[A] \leftarrow T[A]$;
$\mathcal{A}pp \leftarrow$ {set of simple poles of A, $x_i : \ 1 \leq i \leq \mu$};

$\Sigma \quad \leftarrow \{\}$;

$i \leftarrow 1$;
while $i \neq \mu + 1$ **do**

- Compute A_{x_i} the residue matrix of A at $x = x_i$

- Compute the SN decomposition of A_{x_i}, namely $A_i = S_{x_i} + N_{x_i}$ where S_{x_i} semi-simple, N_{x_i} nilpotent, and S_{x_i} and N_{x_i} commute. We remark that a rational decomposition exists (see, e.g. [86])

if $N_{x_i} \neq O_n$ or S_{x_i} has at least one eigenvalue in $\mathbb{C} \setminus \mathbb{N}$ **then** $\mathcal{A}pp \leftarrow \mathcal{A}pp \setminus \{x_i\}$; $\Sigma \leftarrow \Sigma \cup \{x_i\}$;
else Use the method presented in the proof of Proposition 3.4 to compute a polynomial matrix T_{x_i} such that $T_{x_i}[A]$ has at worst a simple pole at $x = x_i$ with residue matrix of the form $A_{x_i} = m_{x_i} I_n + N_{x_i}$ where $m_{x_i} \in \mathbb{N}$ and N_{x_i} is nilpotent.
if $N_{x_i} \neq 0_n$ **then** $\mathcal{A}pp \leftarrow \mathcal{A}pp \setminus \{x_i\}$; $\Sigma \leftarrow \Sigma \cup \{x_i\}$;
else $[A] \leftarrow T_{x_i}[A]$; $T \leftarrow T \star (x - x_i)^{m_{x_i}} T_{x_i}$;
end if
end if

- $i \leftarrow i + 1$;

end while
return (T, A, Σ, $\mathcal{A}pp$).

3.2 Application to desingularization of scalar differential equations

The interest in desingularization of scalar differential equations dates back to the 19^{th} century. Since then, several algorithms have been developed for such and more general scalar equations (see, e.g., the introductions of [3, 45] and the references therein). The desingularization algorithms developed specifically for scalar equations are based on computing a least common left multiple of the operator in question and an appropriately chosen operator. This outputs in general an equation whose solution space contains strictly the solution space of the input equation. As we mentioned in the introduction, the algorithm developed in this paper can be used as well for the desingularization of a companion system of any scalar differential equation. This desingularization is based on an adequate choice of a transformation. Thus the desingularized output system is always equivalent to the input system and the dimension of the solution space is preserved. However, a scalar differential equation equivalent to the desingularized system (see, e.g., [13, 46]) would generally feature apparent singularities. Thus, when dealing with scalar differential equations, our algorithm is well-suited to situations where adhering to a scalar representation is insignificant, e.g. reduction prior to computing solutions near singularities via numerical methods.
In this section, we use Algorithm 3 to desingularize companion systems of two equations which are already treated by existing algorithms. But first we recall the definition of desingularization in the scalar case. Let $L \in \mathbb{C}(x)[\partial]$ be a monic differential operator of order n,

$$L = \partial^n + c_{n-1}(x)\partial^{n-1} + \cdots + c_0(x).$$

We denote by $S(L)$ the set of finite singularities of L, i.e. the set of the poles of the c_i's, $0 \leq i \leq n-1$.

| Definition 3.2

An operator $\tilde{L} \in \mathbb{C}[x][\partial]$ is called a desingularization of L if :

(i) $\tilde{L} = RL$ for some $R \in \mathbb{C}(x)[\partial]$,

(ii) $S(\tilde{L}) = \{x_0 \in S(L) \mid x_0 \text{ is not apparent}\}$

An algorithm developed in [3] constructs, for a given a monic operator $L \in \mathbb{C}(x)[\partial]$ of order n, a monic operator $\tilde{L} \in \mathbb{C}(x)[\partial]$ with minimal order $m+1 \geq n$ satisfying (i) and (ii), m being the maximum of the of the set of all local exponent at the different finite apparent singularities of L. This algorithm has been implemented in MAPLE and is referred to in this paper as ABH method. The system of Example 3.1 is in fact the companion system of the following differential equation which we treat below by the ABH method.

| Example 3.2

Consider the operator

$$L = \partial^2 + \frac{(3x^2-4)}{x(x^2+2)}\partial + 2\frac{1-2x^2}{x^2+2}.$$

It has an apparent singularity at $x = 0$ with local exponents 0 and 3. The desingularization

computed by ABH method is the following operator of order 4:

$$\begin{aligned}\tilde{L} &= \partial^4 + \frac{x(24+7x^2)}{2(x^2+2)}\partial^3 + \frac{(58x^2+88+27x^4)}{2(x^2+2)^2}\partial^2 \\ &\quad - \frac{x(-4x^2+4+93x^4+28x^6)}{2(x^2+2)^3}\partial - \frac{4(44x^2+16+42x^4+7x^6)}{(x^2+2)^3}\end{aligned}$$

□

The classical algorithm for differential equations takes the least common left multiple of the given differential operator and a well-chosen auxiliary one (see, e.g., [70]). A "three-fold generalization" of this algorithm to more general operators is given recently in [45]. The following example has been treated therein. However, the removal of one apparent singularity, namely at $x_0 = 0$, introduces new singularities. The latter can then be removed by using a trick introduced in ABH algorithm (see [3, Theorem 2 and Step 6 in Algo "t-desing"] and [45, Sec 3]). As illustrated below, Algorithm 3 removes all apparent singularities at one stroke without introducing any new ones.

| Example 3.3

Let

$$L = \partial^2 - \frac{(x^2-3)(x^2-2x+2)}{(x-1)(x^2-3x+3)x}\partial + \frac{(x-2)(2x^2-3x+3)}{(x-1)(x^2-3x+3)x}.$$

The apparent singularities of L are $x = 0$ and the roots of $x^2 - 3x + 3 = 0$. In what follows, we seek their removal using different algorithms:

(i) A desingularization computed by the classical algorithm [45, Example 1]:

$$\begin{aligned}\tilde{L}_{\text{CLASSICAL}} &= (x-1)(x^4-x^3+3x^2-6x+6)\partial^4 \\ &- (x^5-2x^4+x^3-12x^2+24x-24)\partial^3 \\ &- (3x^3+9x^2)\partial^2 + (6x^2+18x)\partial - (6x+18).\end{aligned}$$

$\tilde{L}_{\text{CLASSICAL}}$ is a desingularization of L at $x = 0$ and $x^2 - 3x + 3 = 0$. However, new apparent singularities, i.e. the roots of $x^4 - x^3 + 3x^2 - 6x + 6 = 0$, are introduced.

(ii) A desingularization computed by the probabilistic of [45], which we refer to as CKS algorithm, (see Example 7(1) therein):

$$\begin{aligned}\tilde{L}_{\text{CKS}} &= (x-1)(x^6-3x^5+3x^4-x^3+6)\partial^4 \\ &- (2x^6-9x^5+15x^4-11x^3+3x^2-24)\partial^3 \\ &- (x^7-4x^6+6x^5-4x^4+x^3+6x-6)\partial \\ &+ (2x^6-9x^5+15x^4-11x^3+3x^2-24).\end{aligned}$$

$\tilde{L}_{\text{CKS}}$ is a desingularization of L at $x = 0$ and $x^2 - 3x + 3 = 0$. However, new apparent singularities, i.e. the roots of $x^6 - 3x^5 + 3x^4 - x^3 + 6 = 0$, are introduced.

(iii) The desingularization computed by ABH method:

$$\begin{aligned}\tilde{L}_{\text{ABH}} &= \partial^4 + \frac{(16x^4-55x^3+63x^2-42x+36)}{9(x-1)}\partial^3 \\ &\quad -\frac{(64x^5-316x^4+591x^3-468x^2+123x+42)}{9(x-1)^2}\partial^2 \\ &\quad +\frac{\beta}{9(x-1)^3}\partial - \frac{96x^5-570x^4+1333x^3-1597x^2+993x-219}{9(x-1)^3},\end{aligned}$$

where

$$\beta = (48x^6-197x^5+148x^4+488x^3-1162x^2+999x-288).$$

(iv) The desingularization computed by algorithm 3:
The companion system of L is given by:

$$[A] \quad \partial F = \begin{bmatrix} 0 & 1 \\ \frac{(x-2)(2x^2-3x+3)}{(x-1)(x^2-3x+3)x} & \frac{(x^2-3)(x^2-2x+2)}{(x-1)(x^2-3x+3)x} \end{bmatrix} F. \tag{3.5}$$

The gauge transformation $F = TG$ where

$$T = \begin{bmatrix} 1 & 0 \\ 1 & (-x^2+3x-3)x^2 \end{bmatrix} \tag{3.6}$$

results in the following equivalent system

$$[\tilde{A}] \qquad \partial G = \begin{bmatrix} 1 & -x^2(x^2-3x+3) \\ 0 & \frac{2}{1-x} \end{bmatrix} G. \tag{3.7}$$

Observe that in (iii) and (iv) no new apparent singularities are introduced while old ones are removed.

□

Note that, as system (3.5) has rational function coefficients, the transformation (3.6) and the equivalent system (3.7), computed by our algorithm, have rational function coefficients as well. In the following section, we describe how such a "rationality" is preserved by our algorithm.

3.3 Rational version of the algorithm

So far, we have presented our algorithm over $\mathbb{C}$ for the sake of clarity. However, in practice, the base field can be taken as any commutative field $\mathcal{K}$ of characteristic zero ($\mathbb{Q} \subseteq \mathcal{K} \subset \bar{\mathcal{K}} \subseteq \mathbb{C}$). Consider now a system

$$[A] \;\; \partial F = A(x)F, \quad \text{with} \quad A(x) \in \mathcal{M}_n(\mathcal{K}(x)). \tag{3.8}$$

Let $\Omega = \{\alpha_1, \dots \alpha_d\} \subset \bar{\mathcal{K}}$ be a set of conjugate simple poles of $A(x)$. We aim to find an equivalent system which is a desingularization of (3.8) at each of the points of Ω. One possible method is the successive

application of Algorithm 3 to each singularity individually. That is, we first compute a transformation T_1 such that the equivalent system $T_1[A]$ is a desingularization in α_1. We then compute a transformation T_2 such that the equivalent system $T_2[T_1[A]] = (T_2T_1)[A]$ is a desingularization in α_2. Eventually, this yields an equivalent system $(\prod_{i=1}^{d} T_i)[A]$ which is a desingularization of $[A]$ at all points of Ω. However, the entries of T_j and $(\prod_{i=1}^{j} T_i)[A]$, $1 \leq j \leq d$, belong to $\mathcal{K}(\alpha_1, \ldots, \alpha_j)[x]$. Thus, this individual treatment of singularities in d steps, requires an algebraic field extension $\mathcal{K}(\alpha_1, \ldots \alpha_d)$.
This section describes our "rational" algorithm, i.e. the algorithm which avoids computations with individual singularities by representing them by the irreducible polynomial $p(x)$. Consequently, it replaces d steps by only one step and the computations of intermediate steps are limited to $\mathcal{K}(x)/(p)$. We remark however that neither the former nor the latter method require a field extension for the final output, i.e. the equivalent system and the transformation.
For this purpose, we work, similar to [15], with the irreducible polynomial $p(x) = \prod_{i=1}^{d}(x - \alpha_i) \in \mathcal{K}[x]$, and consider the p-adic expansions instead of Laurent series expansions at the α_i's.
Let p be an irreducible polynomial in $\mathcal{K}[x]$, i.e. a finite "point". If f is a non-zero element of $\mathcal{K}(x)$, we define $ord_p(f)$ (read order of f at p) to be the unique integer n such that :

$$f = \frac{a}{b}p^n, \text{ with } a, b \in \mathcal{K}[x] \setminus \{0\}, \quad p \not| \, a \text{ and } p \not| \, b.$$

By convention, $ord_p(0) = +\infty$. The *local ring* at p is $\mathcal{O}_p = \{f \in \mathcal{K}(x) \ : \ \mathrm{ord}_p(f) \geq 0\}$. If $f \in \mathcal{O}_p$ then $f = a/b \in \mathcal{K}(x)$, where $\gcd(a, b) = 1$ and $p \not| \, b$. The residue field of $\mathcal{K}(x)$ at p is $\mathcal{O}_p/p\mathcal{O}_p$, which is isomorphic to the field $\mathcal{K}[x]/(p)$.
Let $f \in \mathcal{K}(x)$ then it has a unique $p-$adic expansion given by:

$$f = p^{ord_p f}(f_{0,p} + pf_{1,p} + \cdots)$$

where the $f_{i,p}$'s are polynomials of degree $< \deg p$, and $f_{0,p} \neq 0$ is called the *leading coefficient*.
In analogy, let $A = (a_{i,j})$ be a matrix in $\mathcal{M}_n(\mathcal{K}(x))$. We define the *order* at p of A, notation $\mathrm{ord}_p(A)$, by

$$\underset{p}{\mathrm{ord}}(A) = \min_{i,j}(\underset{p}{\mathrm{ord}}(a_{i,j})).$$

We say that A has a pole at p if $\mathrm{ord}_p(A) < 0$. Similarly, the leading coefficient is $A_{0,p} \neq 0_n$ in the p-adic expansion of A given by:

$$A = p^{\mathrm{ord}_p(A)}(A_{0,p} + pA_{1,p} + \cdots).$$

3.3.1 The residue matrix at p

The following lemma leads to a definition of the *residue matrix at* p.

Lemma 3.1

Consider the system

$$[A] \quad \partial F = A(x)F, \quad \textit{with} \quad A(x) \in \mathcal{M}_n(\mathcal{K}(x)).$$

Let $\Omega = \{\alpha_1, \ldots \alpha_d\} \subset \mathbb{C}$ *be a set of conjugate apparent singularities and* $p(x) = \prod_{i=1}^{d}(x - \alpha_i) \in \mathcal{K}[x]$

be the irreducible polynomial representing them. Consider the p-adic and α_i-Laurent expansions of $A(x)$ given respectively by

$$\begin{aligned} A(x) &= \frac{1}{p}(A_{0,p} + pA_{1,p} + \cdots) \\ A(x) &= \frac{1}{(x-\alpha_i)}(A_{0,\alpha_i} + (x-\alpha_i)A_{1,\alpha_i} + \cdots),\ 1 \le i \le d. \end{aligned}$$

Then we have,

$$\frac{1}{\partial p(\alpha_i)}\ A_{0,p}(\alpha_i) = A_{0,\alpha_i},\ 1 \le i \le d.$$

Proof. From the above expansions, it follows that

$$\begin{aligned} \frac{A_{0,p}}{p}(x) &= \sum_{i=1}^{d} \frac{A_{0,\alpha_i}}{(x-\alpha_i)} \\ &= \sum_{i=1}^{d} \frac{A_{0,\alpha_i}}{p} \prod_{1 \le j \ne i \le d} (x-\alpha_j). \end{aligned}$$

But $\partial p(\alpha_i) = \prod_{1 \le j \ne i \le d}(x-\alpha_j)$, which completes the proof.

■

| *Remark* 3.3

In the following each equivalent g of $\mathcal{K}[x]/(p)$ is represented by the unique polynomial of degree $< \deg p$ belonging to g. The operations of addition and multiplication in $\mathcal{K}[x]/(p)$ are performed on the representatives considered as polynomials and the results are reduced modulo p. For inverting a nonzero element of $\mathcal{K}[x]/(p)$ we use the extended Euclidean algorithm.

Thus, the following definition is well-justified.

| Definition 3.3

The matrix given by

$$\frac{A_{0,p}(x)}{\partial p(x)} \in \mathcal{M}_n((\mathcal{K}[x]/(p)))$$

is called the residue matrix of $A(x)$ at p. We shall denote by $R_{0,p}(x)$ its representative in $\mathcal{M}_n(\mathcal{K}[x])$. The latter is of degree strictly less than d and can be computed as: $(uA_{0,p}\ mod\ p)$, where u denotes the inverse of $(\partial p\ mod\ p)$.

| Example 3.4

Given

$$[A] \qquad \partial F\ =\ A(x)\ F\ =\ \frac{1}{1+x^2}\begin{bmatrix} 1-x & x \\ -x & 1+x \end{bmatrix} F.$$

Let $p := 1 + x^2$, then p is an irreducible polynomial over $\mathbb{Q}[x]$ and its roots are given by $\pm i$ over

$\mathbb{Q}(i)$. Then u=$-\frac{x}{2}$ is the inverse of $\partial p\ mod p$. Thus, $R_{0,p}(x)$ is given by $uA_{0,p}\ mod\ p$ and so we have

$$R_{0,p}(x) = \frac{1}{2}\begin{bmatrix} 1-x & -1 \\ -1 & -1-x \end{bmatrix}.$$

Indeed, one can verify that the residue matrices at $\pm i$, are given by

$$\begin{bmatrix} \frac{1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1+i}{2i} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{-1-i}{2i} & \frac{1}{2} \\ -\frac{1}{2} & \frac{-1+i}{2i} \end{bmatrix}.$$

□

We now proceed to giving a rational algorithm for testing whether the eigenvalues of $R_{0,p}(x)$ are nonnegative integers or not.

3.3.2 Computing the integer eigenvalues of the residue matrix

Given $R_{0,p}(x) \in \mathcal{M}_n(\mathcal{K}[x])$, we wish to compute its integer eigenvalues, in the course of reduction, to identify the nature of the singularity. Let the characteristic polynomial of $R_{0,p}(x)$ be given by

$$\chi_R(x,\lambda) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + \cdots + a_0(x),$$

where $a_i(x) \in \mathcal{K}[x]$ s.t. $deg_x(a_i) < deg_x(p) = d$. Then, $\chi_R(x,\lambda)$ can be rewritten equivalently as

$$\chi_R(x,\lambda) = \sum_{i=0}^{d-1} b_i(\lambda)x^i,$$

where $b_i(\lambda) \in \mathcal{K}[\lambda],\ 0 \le i \le d-1$, are of maximal degree n. Let $h(\lambda) = \ gcd\ \{b_i(\lambda)\ ,\ 0 \le i \le d-1\}$. It follows that the set of integer roots of $h(\lambda)$ coincides with the set of integer roots of $\chi_R(x,\lambda)$.
Additionally, we remark that in order to compute the integer roots of $\chi_R(x,\lambda)$, it suffices to compute those of $\det(\partial p\lambda - A_{0,p})\ mod\ p$. Similar arguments hold true for operations of Proposition 3.4.

3.3.3 Applications

In this subsection, we illustrate the rational version of our algorithm with two examples. We first treat the introductory example of [44], and then an example from [37] arising in statistical physics. We show how the corresponding systems can be desingularized by "rational" transformations at irreducible polynomials of degrees 3 and 4 respectively. A third example with an irreducible polynomial of degree 37 is available within the description of the package [100].

| Example 3.5 Introduction, [44]

Consider the differential operator

$$\begin{aligned} L \quad &= (1+x)(23-20x-x^2+2x^3)\partial^2 \\ &+2(33-9x-3x^2-x^3)\partial - (45+25x-35x^2-x^3+2x^4). \end{aligned}$$

whose companion system is given by:

$$[A] \qquad \partial F = A(x)F$$

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ \frac{(45+25x-35x^2-x^3+2x^4)}{(1+x)(23-20x-x^2+2x^3)} & -\frac{2(33-9x-3x^2-x^3)}{(1+x)(23-20x-x^2+2x^3)} \end{bmatrix}.$$

The gauge transformation $F = TG$ where

$$T = \begin{bmatrix} 1 & \frac{-14}{143}x^2 + \frac{3}{11}x + \frac{153}{143} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2x^3 - x^2 - 20x + 23 & 0 \\ 0 & 1 \end{bmatrix},$$

results in the following system desingularized at
$p = (23 - 20x - x^2 + 2x^3)$:

$$[\tilde{A}] \quad \partial G = \begin{bmatrix} \frac{14x^3-39x^2-258x-175}{143(x+1)} & \frac{98x^2-497x-1385}{-20449(x+1)} \\ \frac{2x^4-x^3-35x^2+25x+45}{x+1} & \frac{14x^3-39x^2-258x+111}{-143(x+1)} \end{bmatrix} G.$$

The algorithm gives as well a negative response for whether $[A]$ can be desingularized at $(1 + x)$. In fact, the eigenvalues of the residue matrix are 0 and -2, which is a negative integer.

□

| Example 3.6 (The Ising Model [37])

Given[†]

$$[A] \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-4\alpha_1}{x^2(-1+16x)^3 q} & \frac{-4\alpha_2}{x^2(-1+16x)^2 q} & \frac{-2\alpha_3}{x(-1+16x)q} \end{bmatrix} F,$$

where

$$\begin{aligned} q &= (4\,x-1)\left(4352\,x^4+3607\,x^3-1678\,x^2+252\,x-8\right) \\ \alpha_1 &= 89128960\,x^7+74981376\,x^6-97687536\,x^5 \\ &\quad 33948640\,x^4-4652220\,x^3+84480\,x^2+9469\,x-294 \\ \alpha_2 &= 17825792\,x^7+13139200\,x^6-16119599\,x^5 \\ &\quad +5128290\,x^4-689440\,x^3+28373\,x^2-185\,x-6 \\ \alpha_3 &= 1183744\,x^6+770128\,x^5-872579\,x^4+252146\,x^3 \\ &\quad -30499\,x^2+1172\,x-12. \end{aligned}$$

[†]We thank J.-A. Weil for pointing out to these large examples.

We are interested in desingularization at

$$p = \left(4352\,x^4 + 3607\,x^3 - 1678\,x^2 + 252\,x - 8\right).$$

Our algorithm computes the gauge transformation
$F = T_1 T_2 G$ where

$$T_1 = \begin{bmatrix} 1 & \gamma_1\gamma_2 & \gamma_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and}$$

$$\begin{aligned}
\gamma_1 &= -6128505692416\,x^3 - 5454831630087\,x^2 + 2041133482952\,x - 215817804724 \\
\gamma_2 &= \frac{10598786\,x^3}{22324211786901375} - \frac{145270123\,x^2}{714374777180844000} - \frac{70951\,x}{71437477718084400} \\
&\quad + \frac{79111}{44648423573802750} \\
\gamma_3 &= \frac{169580576\,x^3}{525308635} - \frac{145270123\,x^2}{1050617270} - \frac{70951\,x}{105061727} + \frac{632888}{525308635}.
\end{aligned}$$

The system is not desingularizable at any of the other polynomials. In fact, the algorithm gives as well the following information:

- $x = 0$ is a simple pole, and a partial desingularization can be computed. However, only two of the eigenvalues of the residue matrix are nonnegative integers.
- The root of $16x - 1 = 0$ is a simple pole, and a partial desingularization can be computed. However, none of the eigenvalues of the residue matrix are nonnegative integers.
- The root of $4x - 1 = 0$ is a simple pole. However, only two of the eigenvalues of the residue matrix are nonnegative integers.

The resulting desingularizaton at p and the partial desingularizations at x and $16x-1$ are available with [100].

□

3.4 Conclusion

In this chapter, we gave an algorithm, implemented as the package APPSING in MAPLE, for detecting and removing the apparent singularities of linear differential systems via a rational algorithm, i.e. an algorithm which avoids the computations with individual conjugate singularities. Our method can be used, in particular, for the desingularization of differential operators in the scalar case.

One field of investigation is the generalization of our algorithm to treat more general systems, e.g. systems with parameters as well as investigating the case of difference systems. First steps in this direction, namely reductions in the parameter and the partial desingularization, are established in [21] and Part II respectively.

Another major field of investigation is the complexity study of the various algorithms existing for the scalar case and comparing their efficiency, as well as studying the complexity of this new algorithm. Partial results in this direction are already obtained in [32].

Part II

Singularly-Perturbed Linear Differential Systems

Introduction

A distinctive feature of the singularly-perturbed equations is that their solution and/or the solution derivatives have intrinsic narrow zones (boundary and interior layers) of large variations in which they jump from one stable state to another or to prescribed boundary values. Such situation occurs in many physical, chemical, biological, and sociological phenomena. In physics, for example, this happens in viscous gas flows in the zones near the boundary layers where the viscous flow jumps from the boundary values prescribed by the condition of adhesion to the inviscid flow or in the zones near the shock wave where the flow jumps from a subsonic to supersonic state. In chemical reactions the rapid transition from one state to another is typical for solution processes. In biology such sharp changes occur in population genetics. Typical examples of rapid transitions in sociology give revolutions and corresponding changes of political institutions[‡].

In this part, we are interested in the symbolic resolution of the first-order singularly-perturbed linear differential system given by

$$[A_\varepsilon] \qquad \varepsilon^h \, \partial \, F \; = A(x,\varepsilon)F, \tag{3.9}$$

where $\partial = \frac{d}{dx}$, h is an integer, and $A(x,\varepsilon)$ is an $n \times n$ matrix, in the variable x and parameter ε, in a region of the (x,ε)-space defined by the inequalities

$$[D] \qquad |x| \leq \alpha_0 \quad , \; 0 < |\varepsilon| \; \leq \; \varepsilon_0, \quad |arg\, \varepsilon| \leq \theta_0 \; , \tag{3.10}$$

where $\alpha_0, \varepsilon_0, \theta_0$ are positive constants. We assume that $A(x,\varepsilon)$ have an asymptotic expansion

$$A(x,\varepsilon) \sim \sum_{k=0}^{\infty} A_k(x)\varepsilon^k$$

which is valid in the given region as ε tends to zero. The $A_k(x)$'s are assumed to be holomorphic for $|x| \leq \alpha_0$ and we write

$$A_0(x) \; = \; A_{00} \; + \; A_{10}\, x \; + \; A_{20}\, x^2 \; + \ldots .$$

In the case of a regular perturbation, i.e. $h \leq 0$, a solution of system $[A_\varepsilon]$ can be sought by presenting

[‡] V. D. Liseĭkin, *Layer resolving grids and transformations for singular perturbation problems*, VSP, 2001, quoted from the preface

it as a power series in ε. The latter can then be inserted into (3.9) and the like powers of ε equated. This reduces the problem of constructing solutions of $[A_\varepsilon]$ to solving recursively a set of (in)homogeneous unperturbed singular linear differential systems over $\mathbb{C}((x))$, the first of which is system $[A]$ given by (1) in Chapter 1 (see Section 5.2). However, singular perturbations cause major complications for their part.

As we have seen in Chapter 1, the classical formal simplification of system $[A]$ given by (1) begins with the reduction of its leading matrix coefficient A_0 to its Jordan form. Hence, in addition to the usual difficulties encountered within the formal reduction itself, additional ones arise for system $[A_\varepsilon]$ since its leading matrix coefficient $A_0(x)$ is a matrix-valued function rather than a constant one. In particular, if it has multiple eigenvalues then they might coalesce (see Section 5.4). In classical textbooks, points where the Jordan form of $A_0(x)$ is unstable, that is to say either the multiplicity of the eigenvalues or the degrees of the elementary divisors are not constant in a neighborhood of such points, are referred to as *turning points*[§]. As illustrated by the following example, the behavior of solutions of the perturbed system $[A_\varepsilon]$ around such points is far more complicated than that of system $[A]$ around an irregular singularity. The difficulties are exhibited by the division of the domain $|x| \leq \alpha_0$ into subdomains in each of which the solutions, which have yet to be constructed, behave quite differently [73].

| Example 3.7 ([110], Introduction, Subsection 1.5)

Consider the decisively simple-looking differential equation

$$\varepsilon^2 \, \partial^2 f^2 \; - \; (x^3 \; - \; \varepsilon) f \; = \; 0$$

over the (x, ε)-region $D : \; |x| \leq \alpha_0 \; , \; 0 < |\varepsilon| \leq \varepsilon_0$. Setting

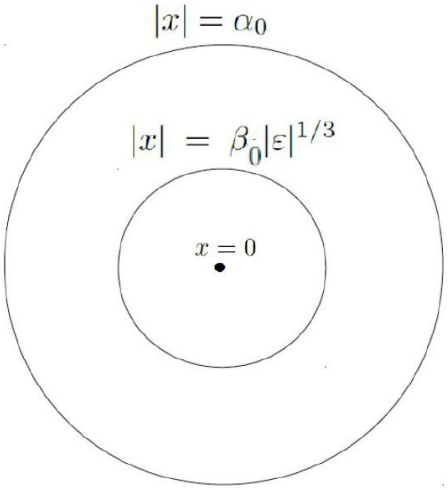
$$F = \begin{bmatrix} f \\ \varepsilon \partial f \end{bmatrix},$$

this equation can be rewritten as the following first-order system:

$$\varepsilon \, \partial F \; = \; A(x, \varepsilon) \; F \; = \; \begin{bmatrix} 0 & 1 \\ x^3 - \varepsilon & 0 \end{bmatrix} F. \tag{3.11}$$

One can observe that the Jordan form of $A_0(x)$ is not stable in any neighborhood around $x = 0$. Thereby, $x = 0$ might be a turning point for (3.11) and our aim is to compute a solution in a full neighborhood of $x = 0$ as the parameter tends to zero.

[§]The definition of such points is subject to debate in literature and is refined in Definition 5.3.



- In the region $\beta_0|\varepsilon|^{1/3} \leq |x| \leq \alpha_0$ where β_0 is a positive constant, the transformation $F = T_1 G$ where

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & x^{3/2} \end{bmatrix}$$

transforms system (3.11) into[¶]

$$(x^{-3}\varepsilon)x^{3/2}\,\partial G \;=\; \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (x^{-3}\varepsilon)\begin{bmatrix} 0 & 0 \\ -1 & \frac{-3}{2}x^{1/2} \end{bmatrix}\}G,$$

which can be rewritten, by a self-explanatory change of notation $x^{-3}\varepsilon = \xi$, as

$$\xi x^{3/2}\partial G \;=\; \{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \xi\begin{bmatrix} 0 & 0 \\ -1 & \frac{-3}{2}x^{1/2} \end{bmatrix}\}G.$$

Then a transformation $G = T_2 U$ where

$$T_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + O(\xi)$$

results in the block-diagonalized system

$$\xi x^{3/2}\partial U \;=\; \{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + O(\xi)\}U.$$

Consequently, the system splits into two first-order linear differential scalar equations and

[¶]See Example 6.8 for the details of computation.

giving rise to the following fundamental matrix of formal solution for (3.11) in this region

$$F_{outer} = T_1\ T_2 \exp(\begin{bmatrix} -\frac{2}{5}x^{5/2}\varepsilon^{-1} + O(x^{-1/2}) & 0 \\ 0 & \frac{2}{5}x^{5/2}\varepsilon^{-1} + O(x^{-1/2}) \end{bmatrix}).$$

- In the region $|x| \leq \beta_0|\varepsilon|^{1/3}$, we shall perform a so-called stretching transformation. Namely, we set $\tau = x\epsilon^{-1/3}$ and $\partial_\tau = \frac{d}{d\tau}$. Then, for all τ such as $|\tau| < \infty$, except possibly for neighborhoods of the roots of $\tau^3 - 1 = 0$[‖], the transformation $F = L_1\ G$ where

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{1/2} \end{bmatrix}$$

reduces the system (3.11) to

$$\varepsilon^{1/6}\partial_\tau\ G\ =\ \begin{bmatrix} 0 & 1 \\ \tau^3 - 1 & 0 \end{bmatrix} G.$$

And the transformation $G = L_2\ U$ where

$$L_2 = \begin{bmatrix} \frac{1}{2} + \frac{1}{8}\tau^3 + O(\tau^6) & \frac{1}{2} + \frac{1}{8}\tau^3 + O(\tau^6) \\ -\frac{i}{2} + \frac{i}{8}\tau^3 + O(\tau^6) & \frac{i}{2} - \frac{i}{8}\tau^3 + O(\tau^6) \end{bmatrix},$$

reduces the above system to

$$\varepsilon^{1/6}\partial_\tau U = \{\begin{bmatrix} -i + \frac{i}{2}\tau^3 + O(\tau^6) & 0 \\ 0 & i - \frac{i}{2}\tau^3 + O(\tau^6) \end{bmatrix} + O(\varepsilon^{1/6})\}U,$$

which has the following fundamental matrix of formal solution in this region:

$$F_{inner} = L_1\ L_2 \exp(\begin{bmatrix} \frac{-i\tau + (i/8)\tau^4 + O(\tau^7)}{\varepsilon^{1/6}} & 0 \\ 0 & \frac{i\tau - (i/8)\tau^4 + O(\tau^7)}{\varepsilon^{1/6}} \end{bmatrix}).$$

We call F_{outer} (resp. F_{inner}) an outer (resp. inner) solution as it is obtained in the outer (resp. inner) region around $x = 0$. For less particular systems, intermediate regions might arise as well.

□

The formal reduction is to take place over a bivariate rather than univariate field. An immediate consequence is the obligation to choose at each step whether to prioritize x or ε, depending on the setting. The treatment of the singularity in x, might complicate dramatically the singularity in ε, and vice-versa, if not handled carefully.

The methods proposed in the literature for the symbolic resolution of system $[A_\varepsilon]$ either exclude turning points (see [43] and references therein), treat systems of dimension two, impose restrictions on the

[‖]The turning points other than $\tau = 0$ do not explicitly correspond to the original turning point. They are referred to as secondary turning points [110].

structure of the leading matrix (see [140] and references therein), or are not fully algorithmic. In [72], the work of [73] on scalar n^{th}-th order differential equations was generalized to treat system $[A_\varepsilon]$, based on results from [71, 72, 73, 122] and references therein. This approach was later simplified in [142]. However, all the former methods make an essential use of the so-called Arnold-Wasow form [7]**. Moreover, there exists no computer algebra package dedicated to the symbolic resolution of neither system $[A_\varepsilon]$ nor the scalar n^{th}-order scalar equation, with the exception of [93] which computes the outer solutions of the latter. In particular, the widely-used softwares MAPLE and MATHEMATICA content themselves to the computation of outer formal solutions for scalar equations.

Within the vast literature on this topic, we list some major results in the direction of symbolic computation of formal solutions which inspire and motivate the approach of the work established in this part. For a survey on numerical techniques, one may refer to [78].

- In [68], Hukuhara established the form of a fundamental matrix of formal solutions of (3.9) under the hypothesis that x does not lie in a neighborhood of a value for which there is an abrupt change in the structure of the Jordan form of the leading matrix coefficient $A_0(x)$.

- In [132], fifteen years later, Turrittin treated independently the same problem at Solomon Lefschetz's suggestion to visit Princeton and work essentially on the same subject of his doctoral dissertation (i.e. solutions of system $[A]$) but in a more general point of view. The same above hypothesis on $A_0(x)$ was made purposely to sidestep the entire turning point problem [134, Sec. 2].

- In [122], Sibuya established a theorem for analytic reduction of $[A_\varepsilon]$ (and equivalently n^{th}-order singularly-perturbed differential equations) under the hypothesis that A_{00} has at least two distinct eigenvalues.

- In [143] and references therein, Wasow attacked the problem of turning points upon subjecting the system to shearing and stretching transformations. A_{00} was assumed to have only one elementary divisor, i.e.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

  A_{10} to have a nonzero left-bottom corner, and h was set to one for the inner expansion. (This is a very particular case of the general turning point treatment we give in Section 5.4.)

- In [73], Iwano and Sibuya studied the reduction of order of singularly-perturbed n^{th}-order differential equations in a neighborhood of a turning point. It turned out that in the absence of the hypothesis of distinct eigenvalues of the leading matrix coefficient, no single theorem can possibly describe all the complexities which might arise. They consequently introduced a convex polygon

** We refer to [38] for the complexity of a form closely-related to this Arnold-Wasow form and its comparison to the cyclic vector method.

similar to the Newton's polygon in the theory of algebraic functions and singular linear differential systems (see Chapter 1). The slopes of this polygon determined the division of the domain $|x| \leq \alpha_0$ into subdomains in each of which the solutions behave differently.

- In [71], Iwano summarized as follows the problems needed to be resolved to obtain the complete knowledge about the asymptotic behavior of the solutions of a singularly-perturbed linear differential system [71, quoted from pp 74]:

  (1) *Divide a domain* $[D]$ *in* (x, ε)*-space into a finite number of subdomains so that the solution behaves quite differently as* ε *tends to zero in each of these subdomains;*

  (2) *Find out a complete set of asymptotic expressions of independent solutions in each of these subdomains*;

  (3) *Determine the so-called connection formula; i.e. a relation connecting two different complete sets of the asymptotic expressions obtained in* (2).

  In order to resolve (1) and (2), Iwano proceeded by associating a convex polygon to first-order systems in analogy with the scalar case, after imposing a precise triangular structure on $A(x, \varepsilon)$. Problem (3) remains generally unresolved and it is out of the scope of this thesis as well (see [69, 140] and references therein).

- The hypotheses on $A(x, \varepsilon)$ were eventually relaxed for special types of systems in a series of papers by Iwano, Sibuya, Wasow, Nakano, Nishimoto, and others (see references in [142]). In [142], Wasow gave a shorter and simpler presentation of the main ideas of these investigations. He used reductions to Arnold's form [7, 94, 13] (see Figure 3.1) to bridge first-order system/ n^{th}-order scalar equation treatments and compute inner and outer solutions.

- Using tools from algebra, the theorem on the existence of a fundamental matrix of formal solutions and its anticipated form was refined and clarified further by Schaefke and Volkmer in [120].

- A fully algorithmic treatment of first-order systems excluding turning points and based on Arnold-Wasow form was given in [43] by Chen.

- An algorithmic treatment of n^{th}-order singularly-perturbed differential equations was given by Macutan in [93] leading to the computation of outer solutions and their Gevrey order.

- A famous and prevalent method among scientists is the WKB method (see, e.g. [82]), which, roughly speaking, reduces the system at hand into one whose asymptotic behavior is known in the literature. However, as pointed out in [141], in view of the great variety and complexity of systems given by $[A_\varepsilon]$, it is rarely expected that a system can be reduced into an already investigated simpler form.

On the other hand, as we have seen in Chapter 1, the research tackling the reduction of system $[A]$ given by (1), advanced profoundly in the last two decades making use of methods of modern algebra and topology. The classical approach has been supported by constructive proofs leading to efficient algorithms that act directly on the system. In addition to their experimental efficiency compared to the classical approach (see, e.g. [14, Introduction] and references therein), these direct algorithms are of a

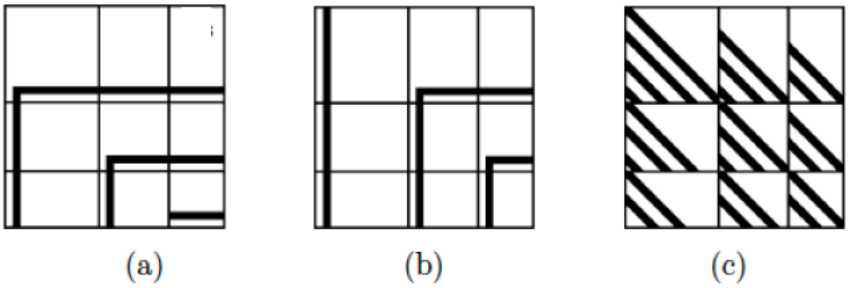


Figure 3.1: Example on Arnold forms. The black stripes correspond to the only possible locations of non-constant entries.

theoretical importance: They allow an understanding of the properties and structure of the system itself as well as its solution.

In this part, we tackle the second problem suggested by Iwano. The difficulty in the symbolic treatment of system $[A_\varepsilon]$ given by (3.9) consists in its reduction to one of the base cases : $h = 0$ or $n = 1$. One can proceed for the former case by integrating a set of inhomogeneous systems successively over a univariate field (see Chapter I) and for the latter case by direct integration (see Section 5.2). We give algorithms to establish this reduction upon generalizing the components of the recursive algorithm described in Chapter 1 for $[A]$, the unperturbed counterpart of system $[A_\varepsilon]$.

We study the invariants of system $[A_\varepsilon]$ and compute explicitly its outer and inner formal solutions to any desired precision over three chapters:

- In Chapter 4, we recall two polygons introduced to study n^{th}-order singularly-perturbed linear differential equations.
- In Chapter 5, we give algorithms for the resolution of turning points and reduction of h to its minimal integer value (ε-rank reduction).
- In Chapter 6, we show how the ε-formal invariants can be retrieved directly from the system without passing through its equivalent companion system (and consequently equivalent scalar equation). We then sum up the work of these chapters to give an algorithm, illustrated by examples, to construct (outer) formal solutions.

In our investigation, we content ourselves with the process of formal reduction. The related problems of matching and connection (corresponding to Iwano's third question) are interesting and difficult problems of open research, that are not discussed in this thesis (see, e.g., [110] for a complete discussion on Example 3.7). Hence, any reference to the asymptotic interpretation of formal solutions will be dropped in the sequel. One may consult in this direction the method of matched asymptotic expansions (see [124, 140] and references therein) or composite asymptotic expansions (see [55, 69] and references therein).

In the sequel, we adopt $\mathbb{C}$ as the base field for the simplicity of the presentation. However, any field $\mathcal{K}$ ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$) can be considered instead. In this case, the restrictions on the extensions of the base field discussed in Section 1.8 apply as well.

Local Notations

- $\partial = \frac{d}{dx}$, $\partial_t = \frac{d}{dt}$, and $\partial_\tau = \frac{d}{d\tau}$;
- $\mathbb{C}[[x]]$ is the ring of formal power series in x whose coefficients lie in $\mathbb{C}$; and $\mathbb{C}((x))$ is its fraction field;
- $\mathbb{C}[[\varepsilon]]$ is the ring of formal power series in ε whose coefficients lie in $\mathbb{C}$; and $\mathbb{C}((\varepsilon))$ is its fraction field;
- $\mathbb{C}[[x]][[\varepsilon]]$ (resp. $\mathbb{C}[[x]]((\varepsilon))$) is the ring of formal (resp. meromorphic) power series in ε whose coefficients lie in $\mathbb{C}[[x]]$;
- $\mathbb{C}((x))[[\varepsilon]]$ (resp. $\mathbb{C}((x))((\varepsilon))$) is the ring (resp. field) of formal (resp. meromorphic) power series in ε whose coefficients lie in $\mathbb{C}((x))$;
- We also recall that the ε-adic valuation of $\mathbb{C}((x))((\varepsilon))$ indicates the ε-order of an element of this field and is defined by a map $val_\varepsilon : \mathbb{C}((x))((\varepsilon)) \to \mathbb{Z} \cup \infty$ which satisfies the following properties for all $a(x,\varepsilon),\ b(x,\varepsilon)$ in $\mathbb{C}((x))((\varepsilon))$:
  - $val_\varepsilon\ (a)\ = \infty$ if, and only if, $a = 0$;
  - $val_\varepsilon\ (ab)\ = val_\varepsilon\ (a) + val_\varepsilon\ (b)$;
  - $val_\varepsilon\ (a+b)\ \geq min\ (val_\varepsilon\ (a),\ val_\varepsilon\ (b))$.

Chapter 4

Singularly-Perturbed n^{th}-order Scalar Differential Equations

Contents

This chapter is mainly concerned with the properties and solutions of an n^{th}-order singularly-perturbed differential equations. We consider the singularly-perturbed linear differential equation given in $[D]$ by

$$[a_\varepsilon] \qquad \partial^n f + a_{n-1}(x,\varepsilon)\,\partial^{n-1} f + \ldots + a_1(x,\varepsilon)\,\partial f + a_0(x,\varepsilon)\,f = 0, \tag{4.1}$$

where $a_n(x,\varepsilon) = 1$ and $a_i(x,\varepsilon) \in \mathbb{C}[[x]]((\varepsilon))$ for all $i \in \{0,\ldots,n-1\}$.

In the literature, two polygons are central to the study of equations of the form $[a_\varepsilon]$: The Newton polygon introduced in [93] and the Iwano-Sibuya characteristic polygon introduced in [73]. We use a rather descriptive terminology and refer to these polygons as ε-polygon (Section 4.1) and (x,ε)-polygon (Section 4.2) respectively.

As mentioned earlier, given a first-order system $[A_\varepsilon]$ of the form (3.9), a transformation $F = TG$ where $T \in GL_n(\mathbb{C}[[x]]((\varepsilon)))$ yields

$$[\tilde{A}_\varepsilon] \qquad \varepsilon^{\tilde{h}}\,\partial G = \tilde{A}(x,\varepsilon)\,G \tag{4.2}$$

where $\tilde{h}$ is an integer and $\tilde{A}(x,\xi) \in \mathcal{M}_n(\mathbb{C}[[x]][[\varepsilon]])$ and

$$\frac{\tilde{A}(x,\varepsilon)}{\varepsilon^{\tilde{h}}} = T^{-1}\,\frac{A(x,\varepsilon)}{\varepsilon^h}\,T - T^{-1}\partial T. \tag{4.3}$$

Systems $[A_\varepsilon]$ and $[\tilde{A}_\varepsilon]$, given by (3.9) and (4.2) respectively, are called *equivalent*.

4.1 ε-Polygon

The definition of the ε-polygon and ε-polynomials is given for the n^{th}-order singularly-perturbed scalar differential equation $[a_\varepsilon]$ in [93] upon generalizing the work of [136], which treats the unperturbed counterpart of these equations. In fact, the definition of the ε-polygon follows the classical definition used in [136] except that it collects the valuations with respect to ε rather than x. We first recall the basic definitions [93, Definition 2.1]:
For $(u_0, v_0) \in \mathbb{R}^2$, we put

$$\begin{aligned} P(u_0, v_0) &= \{(u, v) \in \mathbb{R}^2 |\ u \leq u_0 \text{ and } v \geq v_0\} \\ &= (-\infty, u_0] \times [v_0, \infty). \end{aligned}$$

Consider $[a_\varepsilon]$ given by (4.1) and let $P_\varepsilon(a)$ be the union of $P(i,\ val_\varepsilon\ (a_i(x, \varepsilon))$ for $i \in \{1, \ldots, n\}$. Then the ε-polygon of $[a_\varepsilon]$, denoted by $\mathcal{N}_\varepsilon(a)$, is the intersection of $\mathbb{R}^+ \times \mathbb{R}$ with the convex hull in $\mathbb{R}^2$ of the set $P_\varepsilon(a)$. We denote the slopes of the edges of $\mathcal{N}_\varepsilon(a)$ by $\{e_1, \ldots, e_\ell\}$. These slopes are nonnegative rational numbers.
For every $j \in \{1, \ldots, \ell\}$, we consider the algebraic equation given by

$$(E_j) \qquad \sum_{k=0}^{\ell} a_{i_k,\ val_\varepsilon(a_{i_k})}\ (x)\ X^{(i_k - i_0)} = 0,$$

where $0 \leq i_0 < i_1 < \cdots < i_\ell = n$ denote the integers i for which $(i,\ val_\varepsilon(a_i))$ lie on the edge of slope e_j of the ε-polygon, and $a_{i,\ val_\varepsilon(a_i)}(x) = \varepsilon^{-val_\varepsilon(a_i)}\ a_i(x, \varepsilon)_{|_{\varepsilon=0}}$. We say that (E_j) is the ε-polynomial* associated to the slope e_j. We then have the following proposition:

| Proposition 4.1 ([93], Proposition 4.1)

Consider a nonzero $[a_\varepsilon]$ and its ε-polygon $\mathcal{N}_\varepsilon(a)$ of slopes $\{e_1, \ldots, e_\ell\}$. Let

$$f(x, \varepsilon) = \exp(\int^x q(z, \varepsilon) dz) \quad \textit{with} \quad q(x, \varepsilon) \neq 0 \in \bigcup_{s \in \mathbb{N}^*} \overline{\mathbb{C}((x))}((\varepsilon^{1/s})).$$

If $f(x, \varepsilon)$ satisfies $[a_\varepsilon]$ formally then

$$q(x, \varepsilon) = \frac{1}{\varepsilon^{e_j}}(X(x) + O(\varepsilon))$$

for some $j \in \{1, \ldots, \ell\}$ and $X(x)$ is one of the non zero roots of the ε-polynomial (E_j) associated to e_j.

The full expansion of $q(x, \varepsilon)$ can be obtained with the substitution $f = \exp(\int^x \frac{X(z)}{\varepsilon^{e_j}} dz)\ g$ [93, Proposition

*In the literature, the slopes (resp. ε-polynomials) are also called critical indices (resp. characteristic equations).

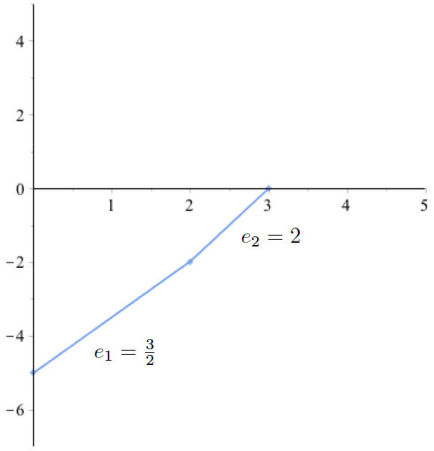


Figure 4.1: The ε-polygon associated to (4.4) in Example 4.1

4.2].

| Example 4.1

Consider the following singularly-perturbed linear differential equation[†]:

$$[a_\varepsilon] \qquad \partial^3 f \; - \; \frac{x}{\varepsilon^2}\, \partial^2 f \; - \; \frac{1}{\varepsilon^5} f \; = \; 0. \tag{4.4}$$

We wish to study this equation according to its ε-polygon (see Figure 4.1). We have two slopes:

- The slope $e_1 = \frac{3}{2}$ for which $(E_1) \quad x\, X^2 + 1 = 0$ whose nonzero solutions are $X = \pm \frac{i}{\sqrt{x}}$.
- The slope $e_2 = 2$ for which $(E_2) \quad X^3 - xX^2 = 0$ whose nonzero solution is $X = x$.

By Proposition 4.1, the leading terms (of the ε-exponential part) of solutions of (4.4) are given by $\exp(\int^x \frac{z}{\varepsilon^2} dz)$ and $\exp(\pm \int^x \frac{i}{\sqrt{z}\varepsilon^{3/2}} dz)$.

□

In analogy to the unperturbed case (see Chapter 1), we give the following definition:

| Definition 4.1

Consider $[a_\varepsilon]$ which is given by (4.1). The largest slope of the ε-polygon $\mathcal{N}_\varepsilon(a)$

$$\omega_\varepsilon(a) = \max\{e_j, 1 \leq j \leq \ell\},$$

is called the ε-formal exponential order (ε-exponential order, in short).

Clearly, we have

$$\omega_\varepsilon(a) \; = \; \max_{i=0}^{n-1} (0, -\frac{val_\varepsilon\,(a_i)}{n-i}).$$

[†]Examples on the harmonic oscillator with damping and Orr-Sommerfeld equation are treated in [93, Section 5].

4.2 (x,ε)-Polygon

The ε-polygon gives valuable information about the ε-formal invariants of $[a_\varepsilon]$, which resolves the second item in Iwano's reformulation of the problem, i.e. the construction of formal solutions. However, it gives no answer to the first item which questions the division of the domain. This is the purpose of the (x,ε)-polygon, introduced by Iwano and Sibuya in [73], and which takes into account the x-adic valuations in addition to the ε-adic valuations considered in the ε-polygon. We follow their aforementioned paper in the presentation of this section.
We consider again $[a_\varepsilon]$ and we make use of the positive rational number $\omega_\varepsilon(a)$ (ω_ε for brevity) to rewrite (4.1) as:

$$[\tilde{a}_\varepsilon] \qquad \partial^n f \ + \ \varepsilon^{-\omega_\varepsilon}\, \tilde{a}_{n-1}(x,\varepsilon)\, \partial^{n-1} f \ + \ \ldots \ + \ \varepsilon^{-n\omega_\varepsilon}\, \tilde{a}_0(x,\varepsilon)\, f \ = \ 0, \tag{4.5}$$

where $\tilde{a}_i(x,\varepsilon) \in \mathbb{C}[[x]][[\varepsilon]]$, for $i \in \{0,\ldots,n-1\}$, and $\tilde{a}_i(x,\varepsilon=0) \ \not\equiv 0$ for at least one index i. We then turn our attention to the algebraic equation

$$\lambda^n \ + \ \tilde{a}_{n-1}(0,0)\, \lambda^{n-1} \ + \ \ldots \ + \ \tilde{a}_0(0,0) \ = \ 0. \tag{4.6}$$

If (4.6) has at least two distinct roots then the reduction of the order of the equation is always possible by an analog of the splitting described in Section 1.3 (see [122, Corollary, pp 529]). We will elaborate further this case in Section 5.3. Thus, we can assume without loss of generality that (4.6) has only one root of order n. The transformation

$$g = f \exp\Big(\int^x \frac{\tilde{a}_{n-1}(z,\varepsilon)}{n\ \varepsilon^{\omega_\varepsilon}}\, dz\Big) \tag{4.7}$$

would then reduce $[\tilde{a}_\varepsilon]$ to

$$[\tilde{\tilde{a}}_\varepsilon] \qquad \partial^n g \ + \ \varepsilon^{-2\omega_\varepsilon}\, \tilde{\tilde{a}}_{n-2}(x,\varepsilon)\, \partial^{n-2} g \ + \ \ldots \ + \ \varepsilon^{-n\omega_\varepsilon}\, \tilde{\tilde{a}}_0(x,\varepsilon) g \ = \ 0, \tag{4.8}$$

which we study under the assumptions:

- $\tilde{\tilde{a}}_i(x,\varepsilon=0) \ \not\equiv 0$ for at least one $i \in \{0,\ldots,n-2\}$;
- and $\tilde{\tilde{a}}_i(x=0,\varepsilon=0) \ = \ 0$ for all $i \in \{0,\ldots,n-2\}$.

Consequently, the algebraic equation

$$\lambda^n \ + \ \tilde{\tilde{a}}_{n-2}(0,0)\, \lambda^{n-2} \ + \ \ldots \ + \ \tilde{\tilde{a}}_0(0,0) \ = \ 0 \tag{4.9}$$

has only one root $\lambda = 0$ of order n for $x = 0$, but it has at least two distinct roots for $x \neq 0$. Hence, $x = 0$ is probably a turning point.

| Example 4.2 (Continue Example 4.1)

Consider again the equation (4.4) of Example 4.1 given by:

$$\partial^3 f \; - \; \frac{x}{\varepsilon^2} \, \partial^2 f \; - \; \frac{1}{\varepsilon^5} f \; = \; 0.$$

We wish to study this equation according to the (x, ε)-polygon. We first compute $\omega_\varepsilon \; = \; max \; \{2, \frac{3}{2}\} = 2$. Thus, (4.4) can be rewritten as

$$\partial^3 f \; + \varepsilon^{-2} \, (-x) \, \partial^2 f \; + \; \varepsilon^{-6} \, (-\varepsilon) \, f \; = \; 0. \tag{4.10}$$

One can verify that $\tilde{a}_2(x, \varepsilon = 0) \; = \; -x \; \not\equiv 0$ and the algebraic equation $\lambda^3 \equiv 0$ has zero as a triple root. The transformation (4.7) yields

$$\partial^3 \, g \; + \varepsilon^{-4} \, (\frac{-1}{3} x^2 \; + \; \varepsilon^2) \, \partial \, g \; + \; \varepsilon^{-6} \, (\frac{2}{27} x^3 \; + \; \varepsilon) \, g \; = \; 0. \tag{4.11}$$

So we have

- $\tilde{\tilde{a}}_1(x, \varepsilon) \; = \; \frac{-1}{3} x^2 \; + \; \varepsilon^2$ and $\tilde{\tilde{a}}_0(x, \varepsilon) \; = \; \frac{2}{27} x^3 \; + \; \varepsilon$;
- The algebraic equation (4.9) is given by

$$\lambda^3 \; - \; \frac{1}{3} x^2 \, \lambda + \; \frac{2}{27} x^3 \; = \; (\lambda + \frac{2}{3} x)(\lambda - \frac{1}{3} x)^2.$$

Clearly, if $x = 0$ then we have $\lambda = 0$ as a triple root otherwise we have two distinct roots. So $x = 0$ is possibly a turning point.

□

Consider again $[\tilde{\tilde{a}}_\varepsilon]$ given by (4.8). The domain $|x| \; \leq \; x_0$ of $[D]$ (3.10) can be divided into a finite number of subdomains in each of which the solution of $[\tilde{\tilde{a}}_\varepsilon]$ behaves quite differently [73, Intro, pp 2]. To construct these subdomains, a sequence of positive rational numbers

$$[\rho] \qquad\qquad 0 = \; \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_m. \tag{4.12}$$

can be employed as follows:

$$[D_1] \qquad \begin{cases} \beta_m \, |\varepsilon|^{\rho_m} \quad \leq \; |x| \; \leq \; \alpha_{m-1} \, |\varepsilon|^{\rho_{m-1}} \, , \\ \beta_{m-1} \, |\varepsilon|^{\rho_{m-1}} \; \leq \; |x| \; \leq \; \alpha_{m-2} \, |\varepsilon|^{\rho_{m-2}} \, , \\ \vdots \\ \beta_2 \, |\varepsilon|^{\rho_2} \; \leq \; |x| \; \leq \; \alpha_1 \, |\varepsilon|^{\rho_1} \, , \\ \beta_1 \, |\varepsilon|^{\rho_1} \; \leq \; |x| \; \leq \; \alpha_0 \, ; \end{cases} \tag{4.13}$$

$$[D_2] \qquad \begin{cases} |x| \; \leq \; \beta_m \, |\varepsilon|^{\rho_m} \, , \\ \alpha_{m-1} \, |\varepsilon|^{\rho_{m-1}} \; \leq \; |x| \; \leq \; \beta_{m-1} \, |\varepsilon|^{\rho_{m-1}} \, , \\ \vdots \\ \alpha_1 \, |\varepsilon|^{\rho_1} \; \leq \; |x| \; \leq \; \beta_1 \, |\varepsilon|^{\rho_1} \, ; \end{cases} \tag{4.14}$$

where the β_i's (resp. α_i's) are sufficiently large (resp. small) numbers.
The sequence $[\rho]$ can be determined with the aid of a convex polygon which we describe in the following subsection.

4.2.1 Construction of (x, ε)-polygon

For the simplicity of the presentation, we re-adjust our notations from $[\tilde{\tilde{a}}_\varepsilon]$ to $[a_\varepsilon]$. Without loss of generality, we consider

$$[a_\varepsilon] \qquad \partial^n f \; + \; \varepsilon^{-2\omega_\varepsilon} \, a_{n-2}(x, \varepsilon) \, \partial^{n-2} f \; + \; \ldots \; + \; \varepsilon^{-n\omega_\varepsilon} \, a_0(x, \varepsilon) f \; = \; 0. \tag{4.15}$$

with the properties of equation $[\tilde{\tilde{a}}_\varepsilon]$. We expand the coefficients with respect to ε:

$$\begin{cases} a_i(x, \varepsilon) \; = \; \sum_{k=0}^{\infty} \, a_{i,k}(x) \, \varepsilon^k, \qquad 0 \leq i \leq n-2, \\ a_{i,k}(x) \; = \; \sum_{j=0}^{\infty} \, a_{i,k,j} \, x^j, \qquad a_{i,j,k} \in \mathbb{C}, \quad k \geq 0. \end{cases} \tag{4.16}$$

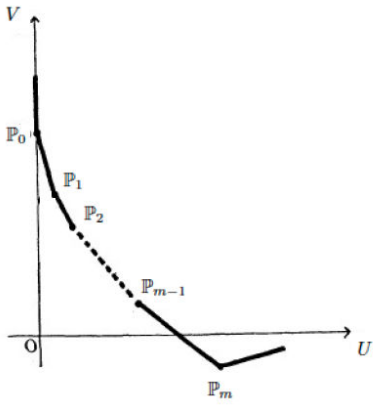


Figure 4.2: (x, ε)-Polygon

We plot this set of points in a cartesian coordinate system (U, V) (Figure 4.2):

$$\begin{aligned} P_{\omega_\varepsilon} &= (\omega_\varepsilon, -1) \\ P_{ik} &= (\frac{val_\varepsilon\,(a_i)}{n-i}\,,\,\frac{val_x\,(a_{i,val_\varepsilon(a_i)})}{n-i}), \quad 0 \le i \le n-2\,,\; k \ge 0, \end{aligned}$$

where $a_{i,\,val_\varepsilon(a_i)}(x) = \varepsilon^{-val_\varepsilon(a_i)}\; a_i(x,\varepsilon)_{|\varepsilon=0}$. Let

$$P_\varepsilon \;=\; \{P_{\omega_\varepsilon}\} \;\cup\; \{P_{ik}\}_{0\le i\le n-2,\; k\ge 0}.$$

Then the (x, ε)-polygon of $[a_\varepsilon]$, which we denote by $\mathcal{N}_{(x,\varepsilon)}(a)$, is the convex downwards polygon formed from finitely many points of P_ε so that none of the points of P_ε is located below it. Clearly, P_{ω_ε} is a vertex of the polygon. By construction, due to the conditions satisfied by the coefficients of $[a_\varepsilon]$, one of the P_{ik}'s on the polygon lies on the V-axis and is not the origin. We shall denote it by $\mathbb{P}_0$. Furthermore, we shall denote P_{ω_ε} by $\mathbb{P}_m$ and the vertices between $\mathbb{P}_0$ and $\mathbb{P}_m$ by $\mathbb{P}_1,\; \mathbb{P}_2,\; \ldots,\; \mathbb{P}_{m-1}$, successively from the left to the right (see Figure 4.2). Set

$$\mathbb{P}_i \;=\; (u_i, v_i), \quad 0 \le i \le m. \tag{4.17}$$

We then define the positive rational numbers ρ_i's of (4.12) as (the negative inverse of the slopes):

$$\rho_i \;= -\,\frac{u_i \;-\; u_{i-1}}{v_i \;-\; v_{i-1}}, \quad 1 \le i \le m.$$

The utility of such a polygon is to clarified upon investigating the effect of the shearing transformations and variable stretching on equation (4.8). This is the goal of the next two subsections.

| Example 4.3 (Continue Example 4.2)

We consider again the equation given by (4.11)

$$\partial^3\, g \;+\varepsilon^{-4}\,(\frac{-1}{3}x^2 \;+\; \varepsilon^2)\;\partial\, g \;+\; \varepsilon^{-6}\,(\frac{2}{27}x^3 \;+\; \varepsilon)\; g \;=\; 0.$$

The (x,ε)-polygon is then defined by the following set of points: $P_{\omega_\varepsilon} = (2,-1)$, $P_{00} = (0,1)$, $P_{01} = (\frac{1}{3},0)$, $P_{10} = (0,1)$, and $P_{12} = (1,0)$. Hence, it has two slopes -3 and -1.

□

4.2.2 Reduction of order in $[D_1]$

Let

$$F \;=\; (f, \varepsilon^{\omega_\varepsilon}\,\partial f, \varepsilon^{2\,\omega_\varepsilon}\,\partial^2\,f, \ldots,\, \varepsilon^{(n-1)\,\omega_\varepsilon}\,\partial^{n-1}\,f)^T.$$

Then the equation $[a_\varepsilon]$ given by (4.15) can be rewritten equivalently as the first-order system

$$[A_\varepsilon] \qquad \varepsilon^{\omega_\varepsilon}\,\partial F \;=\; A(x,\varepsilon)\,F, \tag{4.18}$$

where $A(x,\varepsilon)$ is a companion matrix of the form

$$A(x,\varepsilon) \;=\; \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0(x,\varepsilon) & -a_1(x,\varepsilon) & -a_2(x,\varepsilon) & \ldots & -a_{n-2}(x,\varepsilon) & 0 \end{bmatrix}.$$

In this subsection, we show that in subdomains of $[D_1]$, the system can be uncoupled into at least two subsystems of lower dimensions (and consequently, the order of the scalar equation is reduced).
The shearing transformation $F \;=\; SG$ (see Section 1.1), where S is of the form

$$S \;= \mathrm{diag}(1\,,\; \varepsilon^u x^v\,,\; (\varepsilon^u x^v)^2\,, \ldots,\; (\varepsilon^u x^v)^{n-1}), \tag{4.19}$$

yields the equivalent system

$$[\tilde{A}_\varepsilon] \qquad \varepsilon^{\omega_\varepsilon - u}\,x^{-v}\,\partial G \;=\; \tilde{A}(x,\varepsilon)\,G \;=\; (\,B(x,\varepsilon) \;-\; v\,\varepsilon^{\omega_\varepsilon - u}\,x^{-v-1}\,C)\,G, \tag{4.20}$$

where

$$B(x,\varepsilon) \;=\; \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -(\varepsilon^u x^v)^{-n}\,a_0(x,\varepsilon) & -(\varepsilon^u x^v)^{1-n}\,a_1(x,\varepsilon) & \ldots & -(\varepsilon^u x^v)^{-2}\,a_{n-2}(x,\varepsilon) & 0 \end{bmatrix}$$

and C is a constant diagonal matrix of the form

$$C = \begin{bmatrix} 0 & & & \cdots & \\ & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n-1 \end{bmatrix}.$$

If follows form (4.16) that for $0 \leq i \leq n-2$ we have

$$(\varepsilon^u x^v)^{i-n}\, a_i(x, \varepsilon) \;=\; \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{i,k,j}\; x^{j-v(n-i)}\; \varepsilon^{k-u(n-i)}. \tag{4.21}$$

Thus, the shearing transformation (4.19) is a change of coordinates in the (U, V)-plane (see Figure 4.3). The new coordinate system has its origin at (u, v):

$$U' \;=\; U \;-\; u \quad \text{and} \quad V' \;=\; V \;-\; v. \tag{4.22}$$

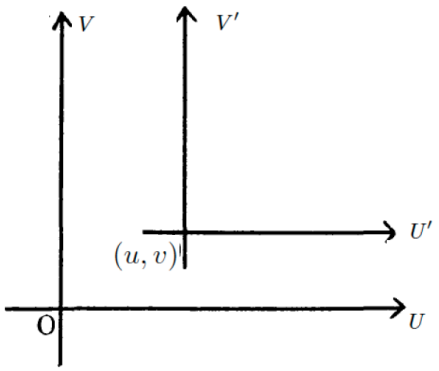


Figure 4.3: Interpretation of a shearing transformation

We seek a subdomain of $[D]$ where $x^{j-v(n-i)}\; \varepsilon^{k-u(n-i)}$ of (4.21) remains bounded. Consider the two straight lines $L = 0$ and $L' = 0$ which pass through the point (u, v) and are given by:

$$L \;=\; \rho\,(V - v) \;+\; (U - u) \tag{4.23}$$

$$L' \;=\; \rho'\,(V - v) \;+\; (U - u), \quad \text{where} \quad 0 < \rho < \rho'. \tag{4.24}$$

Clearly, if for some (u_0, v_0) , $L(u_0, v_0) \geq 0$ (resp. $L'(u_0, v_0) \geq 0$) then the point (u_0, v_0) does not lie below the straight line $L = 0$ (resp. $L' = 0$). Substituting for (U, V)

$$(\frac{k}{n-i}\;,\; \frac{j}{n-i}\;),$$

in (4.23) and (4.24), we obtain

$$\begin{aligned} \rho' L - \rho L' &= \frac{\rho' - \rho}{n-i}(k - u(n-i)) \\ L' - L &= \frac{\rho' - \rho}{n-i}(j - v(n-i)). \end{aligned}$$

Thus, we can rewrite every monomial of (4.21) as

$$x^{j-v(n-i)}\,\varepsilon^{k-u(n-i)} = x^{\frac{(n-i)(L'-L)}{\rho'-\rho}}\,\varepsilon^{\frac{(n-i)(\rho' L - \rho L')}{\rho'-\rho}} \tag{4.25}$$

$$= (x\,\varepsilon^{-\rho})^{\frac{(n-i)L'}{\rho'-\rho}}\;(x\,\varepsilon^{-\rho'})^{\frac{-(n-i)L}{\rho'-\rho}}\,. \tag{4.26}$$

Recalling that we have $L \geq 0$ (resp. $L' \geq 0$) if and only if $(\frac{k}{n-i}\,,\,\frac{j}{n-i})$ does not lie below the straight line defined by $L = 0$ (resp. $L' = 0$), we encounter the following scenarios:

- $L = 0$ and $L' = 0$ which corresponds to $(\frac{k}{n-i}\,,\,\frac{j}{n-i}) \equiv (u, v)$. In (4.26), we have

$$x^{j-v(n-i)}\,\varepsilon^{k-u(n-i)} = 1;$$

- $L = 0$ and $L' > 0$ (resp. $L > 0$ and $L' = 0$), which corresponds to $(\frac{k}{n-i} = u,\ \frac{j}{n-i} > v)$ (resp. $(\frac{k}{n-i} > u,\ \frac{j}{n-i} = v)$). In (4.26), we have

$$x^{j-v(n-i)}\,\varepsilon^{k-u(n-i)} = (x\,\varepsilon^{-\rho})^{\frac{(n-i)L'}{\rho'-\rho}} \quad \text{(resp.} \quad x^{j-v(n-i)}\,\varepsilon^{k-u(n-i)} = (x\,\varepsilon^{-\rho'})^{\frac{-(n-i)L}{\rho'-\rho}}\text{)}.$$

  In this case, (4.26) (and consequently (4.21)) is small when $x\,\varepsilon^{-\rho}$ is small (resp. $x\,\varepsilon^{-\rho'}$ is large).

- $L > 0$ and $L' > 0$ which corresponds to $(\frac{k}{n-i} > u, \frac{j}{n-i} > v)$ for some (u, v) lying above both straight lines (4.23) and (4.24). In this case, (4.26) (and consequently (4.21)) is small when $(x\varepsilon^{-\rho})$ is small and $(x\varepsilon^{-\rho'})$ is large. This corresponds to a domain

$$\beta\,|\varepsilon|^{\rho'} \leq |x| \leq \alpha\,|\varepsilon|^{\rho},$$

  where β and $1/\alpha$ are very large positive constants.

The same can be proved for the substitution

$$U = \omega_\varepsilon \quad \text{and} \quad V = -1.$$

Consider again the subdomains of (4.13) and the vertices of the ε-polygon (4.17). We substitute for ρ and ρ' as follows: For any $i \in \{0, \ldots, m-1\}$, let

$$\begin{aligned} L &= \rho_i\,(V - v_i) + (U - u_i) \\ L' &= \rho_{i+1}\,(V - v_i) + (U - u_i). \end{aligned}$$

then system $[A_\varepsilon]$ can be uncoupled into systems of lower dimensions in each of the subdomains

$$\beta_{i+1}\,|\varepsilon|^{\rho_{i+1}} \;\le\; |x| \;\le\; \alpha_i\,|\varepsilon|^{\rho_i}$$

by applying a shearing transformation (4.19) where $u = u_i$ and $v = v_i$ [73, Fundamental Lemma, pp 12]. In fact, the resulting equivalent system $[\tilde{A}_\varepsilon]$ of (4.20) is given by

$$[\tilde{A}_\varepsilon] \qquad \varepsilon^{\omega_\varepsilon - u_i}\, x^{-v_i}\, \partial G \;=\; \tilde{A}(x,\varepsilon)\, G \;=\; (\tilde{A}_0(x) \;+\; \tilde{A}_1(x)\,\varepsilon \;+\; \tilde{A}_2(x)\,\varepsilon^2 \;+\; \dots)\, G,$$

where $L\,, L' \;\ge 0, \quad \tilde{A}_0(x) \;=\; \tilde{A}_{00} \;+\; \tilde{A}_{10}x \;+\; \tilde{A}_{20}x^2 \;+\; \dots,$ and

$$\tilde{A}_{00} \;=\; \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \tilde{a}_{0,0} & \tilde{a}_{1,0} & \dots & \tilde{a}_{n-2,0} & 0 \end{bmatrix}.$$

At least one of the constants $\tilde{a}_i$'s is different from zero since $L = L' = 0$ for some i. Thus, $\tilde{A}_{00}$ has at least two distinct eigenvalues and consequently, the splitting can be applied to uncouple the system into two subsystems of lower dimensions (see Section 5.3). Consequently, the order of the corresponding scalar equation $[a_\varepsilon]$ drops in such subdomains.

4.2.3 Behavior of solutions in intermediate domains $[D_2]$

Consider again system $[A_\varepsilon]$ given by (4.18):

$$[A_\varepsilon] \qquad \varepsilon^{\omega_\varepsilon}\, \partial F \;=\; A(x,\varepsilon)\, F.$$

We now investigate the behavior of the solution in the subdomains of $[D_2]$ given by (4.14). We study the effect of the so-called stretching which is followed by a shearing (4.19) of the following form:

$$\begin{cases} x \;=\; \varepsilon^\rho \tau & \text{where } \rho \text{ is a positive rational number ,} \\ F \;=\; S\, G & \text{where} \quad S = \operatorname{diag}(1\,,\; \varepsilon^\gamma,\; \varepsilon^{2\gamma}\,, \dots,\; \varepsilon^{(n-1)\gamma}),\; \gamma \in \mathbb{Q}. \end{cases} \tag{4.27}$$

Under (4.27), the resulting system is given by

$$[\tilde{A}_\varepsilon] \qquad \varepsilon^{\omega_\varepsilon - \rho - \gamma}\, \partial_\tau G \;=\; \tilde{A}(\tau,\varepsilon)\, G, \tag{4.28}$$

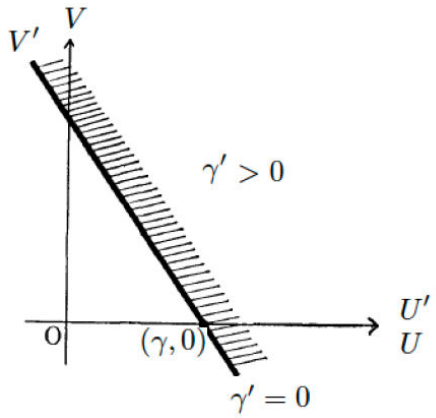


Figure 4.4: Interpretation of a stretching-shearing transformation

where

$$\tilde{A}(\tau,\varepsilon) \;=\; \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\varepsilon^{-n\gamma}\,\tilde{a}_0(\tau,\varepsilon) & -\varepsilon^{(1-n)\gamma}\,\tilde{a}_1(\tau,\varepsilon) & \dots & -\varepsilon^{-2\gamma}\,\tilde{a}_{n-2}(\tau,\varepsilon) & 0 \end{bmatrix},$$

where $\tilde{a}_i(\tau,\varepsilon) \;=\; a_i(x=\varepsilon^{\rho}\tau,\varepsilon)$ for $0 \le i \le n-2$. It follows from (4.16) that for $0 \le i \le n-2$ we have

$$\varepsilon^{\gamma(i-n)}\;\tilde{a}_i(\tau,\varepsilon) \;=\; \varepsilon^{\gamma(i-n)}\;a_i(\varepsilon^{\rho}\tau,\varepsilon) \;=\; \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{i,k,j}\;\tau^{j}\;\varepsilon^{k-\gamma(n-i)+\rho j}. \tag{4.29}$$

Let

$$\gamma' \;=\; \rho\,V \;+\; (U \;-\gamma), \tag{4.30}$$

then $\gamma' = 0$ defines a straight line passing through the point $(\gamma, 0)$ (see Figure 4.4) and

- $\gamma'(u_0, v_0) \ge 0$ implies that the point (u_0, v_0) does not lie below the straight line.
- $\gamma'(u_0, v_0) = 0$ if and only if (u_0, v_0) is on the straight line.

Substituting for (U, V) by

$$(\frac{k}{n-i}\;,\;\frac{j}{n-i}) \quad \text{and} \quad (\omega_{\varepsilon}\;,\;-1),$$

respectively in (4.30) yields respectively

$$\varepsilon^{k-\gamma(n-i)+\rho j} \;=\; \varepsilon^{\gamma'(n-i)} \tag{4.31}$$

$$\varepsilon^{\omega_{\varepsilon}-\rho-\gamma} \;=\; \varepsilon^{\gamma'}. \tag{4.32}$$

Thus, the transformation (4.27) effects a change of coordinates in the (U, V)-plane given by

$$U' \;=\; \rho\, V \;+\; (U - \gamma) \quad \text{and} \quad V' \;=\; V. \tag{4.33}$$

The V'-axis is the straight line defined by $U' = 0$, i.e. $\rho\, V \;+\; (U - \gamma) \;=\; 0$ and the origin of the new coordinate system is the point $(\gamma, 0)$.
Consider again the subdomains of $[D_2]$. For any $i \in \{1, \ldots, m-1\}$, we perform the transformations of (4.27) on $[A_\varepsilon]$ with $\rho = \rho_i$ and γ_i equal to the U-intercept of the straight line $\gamma_i' = 0$ of slope ρ_i and which passes through the points $\mathbb{P}_i$ and $\mathbb{P}_{i-1}$ of the polygon. In a subdomain of (4.14):

$$\alpha_i\, |\varepsilon|^{\rho_i} \;\leq\; |x| \;\leq\; \beta_i\, |\varepsilon|^{\rho_i}\,, \quad (\text{or} \quad \alpha_i \;\leq\; |\omega| \;\leq\; \beta_i)$$

the resulting equivalent system $[\tilde{A}_\varepsilon]$ is given by

$$\varepsilon^{\gamma_i'}\, \partial_\tau G \;=\; \tilde{A}(\tau, \varepsilon)\, G \;=\; (\tilde{A}_0(\tau) \;+\; \tilde{A}_1(\tau)\, \varepsilon \;+\; \tilde{A}_2(\tau)\, \varepsilon^2 \;+\; \ldots)\, G.$$

We have $\gamma_i' > 0$ for $i \in \{1, \ldots, m-1\}$ and $\gamma_m' = 0$. Moreover, since none of the points (4.17) lie below the straight line $\gamma_i' = 0$, we have $\tilde{A}(\tau, \varepsilon) \in \mathbb{C}[[\tau]][[\varepsilon]]$ and in particular

$$\tilde{A}_0(\tau) \;=\; \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ \tilde{a}_0(\tau) & \tilde{a}_1(\tau) & \ldots & \tilde{a}_{n-2}(\tau) & 0 \end{bmatrix},$$

with at least one of the $\tilde{a}_i(\tau)$'s differing from zero. Thus, either $\tilde{A}_0(\tau = 0)$ has at least two distinct eigenvalues and consequently the splitting can be applied (see Section 5.3); or $[\tilde{A}_\varepsilon]$ has possibly turning points[‡]. However, the number of such points is finite and so the subdomain under question can be covered by a finite number of open sets each containing at most one of such points. In each of the open sets, one can make a translation in the variable τ to get back to the situation we started with. And so, a (x, ε)-polygon would be constructed again. It can then be verified that ω_ε of the new system is less than that of the original equation and the point $\mathbb{P}_0$ of the polygon does not lie above the original one. Consequently, after finite repetitions of this process, we arrive at a situation were $\omega_\varepsilon = 0$.

4.3 Conclusion

In this chapter we recalled two polygons which are used to study n^{th}-order singularly-perturbed linear differential equations. In the next chapter, we consider scalar equations and systems of a more general but related form.

[‡]In literature, such points are referred to as *secondary turning points*.

Chapter 5

Two-Fold Rank Reduction of First-Order Systems

Contents

We consider again system (3.9) which is given by

$$[A_\varepsilon] \qquad \varepsilon^h \, \partial F \; = A(x,\varepsilon)F \; = \; (\sum_{k=0}^{\infty} A_k(x)\varepsilon) \; F,$$

where h is an integer and $A(x,\varepsilon)$ has its entries in $\mathbb{C}[[x]][[\varepsilon]]$. We refer to $A_0(x)$ as the leading matrix coefficient and to $A_{00} := A_0(x=0)$ as leading constant matrix.

This chapter is concerned with our contribution in the resolution of turning points and reduction of ε-rank h to its minimal integer value. It is based on [2], where we propose algorithms to treat the *turning points* of system (3.9) and uncouple it into a set of systems of lower dimensions such that each has the

minimal integer value for its ε-rank among all its equivalent counter-parts. This minimization is essential to formal reduction as we will see in the next chapter but is a stand-alone problem as well.

This chapter is organized as follows: In Section 5.1, we introduce more general systems and recall preliminary results on the reduction of system $[A_\varepsilon]$. In Section 5.3, we recall the well-known Splitting lemma and refine the classical proof. In Section 5.4 we treat the turning point. And finally in Section 5.5, we describe the algorithm which reduces h to its minimal integer value.

5.1 Preliminaries

Consider system $[A_\varepsilon]$ given by (3.9). We assume that it has at most one turning point otherwise the region of study can be shrunk. We place this turning point at the origin (otherwise, a translation in the independent variable can be performed.

As illustrated by Example 3.7 in the introduction, the formal reduction of system $[A_\varepsilon]$ leads inevitably to the consideration of more general systems (see, e.g., [142, Chapter 2] and references therein). We thus consider the following two subsets of $\mathbb{C}((x))[[\varepsilon]]$ and $\mathbb{C}((x))((\varepsilon))$ respectively:

$$\begin{aligned} \mathrm{R}_{[[\varepsilon]]} &= \{f=\sum_{k\in\mathbb{N}}^{\infty} f_k(x)\varepsilon^k \in \mathbb{C}((x))[[\varepsilon]] \text{ s.t. } val_x(f_k)\geq \sigma k+p \text{ for all } k\in\mathbb{N}, \text{ for some } \sigma\in\mathbb{Q}^-, p\in\mathbb{Z}\};\\ \mathrm{R}_{((\varepsilon))} &= \{f=\sum_{k\in\mathbb{Z}}^{\infty} f_k(x)\varepsilon^k \in \mathbb{C}((x))((\varepsilon)) \text{ s.t. } val_x(f_k)\geq \sigma k+p \text{ for all } k\in\mathbb{Z}, \text{ for some } \sigma\in\mathbb{Q}^-, p\in\mathbb{Z}\}. \end{aligned}$$

One can verify that $\mathrm{R}_{[[\varepsilon]]}$ is a ring and $\mathrm{R}_{((\varepsilon))}$ is its field of fractions. In these two chapters, we will treat systems of the form $\partial F = M(x,\varepsilon)$ with $M(x,\varepsilon)\in\mathcal{M}_n(\mathrm{R}_{((\varepsilon))})$ or, with a more pleasant notation:

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h\, x^p\, \partial\, F \;=\; A(x,\varepsilon)F=\sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\, F, \tag{5.1}$$

where $A(x,\varepsilon)\in\mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$. Here, p is an integer, σ_A is a nonpositive rational number, and $A_k(x)\in\mathcal{M}_n(\mathbb{C}[[x]])$ for all $k\geq 0$. Without loss of generality, we assume that σ_A is a nonpositive integer, since otherwise, a change of the independent variable $x=t^\alpha$ can be introduced for some appropriate choice of natural number α. We also assume that $A_0(x)$ (resp. A_{00}) is a nonzero matrix otherwise h (resp. p and σ_A) can be re-adjusted. In the sequel, we resort sometimes to the shorthand notation $\xi = x^{\sigma_A}\varepsilon$ for clarity.

Evidently, system $[A_\varepsilon]$ given by (5.1) is a particular case of system $[A_{\varepsilon,\sigma_A}]$ for which $p=\sigma_A=0$. And as for system $[A_\varepsilon]$, h will be assumed to be a positive integer since this is the case where major complications arise. The simpler case of $h\leq 0$ is discussed in Subsection 5.2.1.

5.1.1 Equivalent systems

Consider system $[A_{\varepsilon,\sigma_A}]$ given by (5.1). Let $T \in GL_n(\mathrm{R}_{((\varepsilon))})$ then the transformation $F = TG$ yields

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^{\tilde{h}} x^{\tilde{p}} \partial G \;=\; \tilde{A}(x,\varepsilon)G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_{\tilde{A}}}\varepsilon)^k \; G \tag{5.2}$$

where $\sigma_{\tilde{A}}$ is some nonpositive integer and $\tilde{A}_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ for all $k \geq 0$. One can verify that:

$$\frac{\tilde{A}(x,\varepsilon)}{\varepsilon^{\tilde{h}} x^{\tilde{p}}} = T^{-1} \; \frac{A(x,\varepsilon)}{\varepsilon^h x^p} \; T \;-\; T^{-1}\partial T. \tag{5.3}$$

We say systems $[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}]$ and $[A_{\varepsilon,\sigma_A}]$ are equivalent and we write $[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] = T[A_{\varepsilon,\sigma_A}]$. In the sequel, we seek at each step such a transformation which takes system $[A_{\varepsilon,\sigma_A}]$ to an equivalent system $[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}]$ which either has $\tilde{h} < h$ or can be uncoupled into systems of lower dimensions. Before proceeding, we discuss in the following section the two basic cases $h \leq 0$ and $n = 1$.

5.1.2 Restraining index

As we are lead naturally to the study of system $[A_{\varepsilon,\sigma_A}]$, we are required to investigate the inevitable growth of the order of the poles in x within the reduction. This is the first step towards determining inner and intermediate formal solutions. Such solutions may be constructed by the techniques developed herein for the construction of the (outer) formal solutions, after performing an adequate stretching, i.e. a change of the independent variable of the form $\tau = x\varepsilon^{-\rho}$ for some positive rational number ρ (as discussed for the case of n^{th}-order singularly-perturbed differential equation in Chapter 4).
As described in [142, Chapter 2] and as can be seen from the structure of the ring of coefficients, these poles grow at worst linearly with k. The information on the orders of the poles in x which might be introduced in system $[A_\varepsilon]$ within the process of reduction, is stored in p and σ_A. This allows tracking their growth. More precisely, the first input is system $[A_\varepsilon]$ given by (3.9). We then have $\sigma_A = p = 0$ at the beginning of our treatment. After each transformation T performed (with a careful choice of T), we compute $\sigma_{\tilde{A}}$ and $\tilde{p}$ by adjusting σ_A and p so that for the system $T[A_{\varepsilon,\sigma_A}] := [\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}]$, the entries of the $\tilde{A}_k(x)$'s are again in $\mathbb{C}[[x]]$ and A_{00} is nonzero for a nonzero system. For any such adjustment, $\sigma_{\tilde{A}}$ is chosen to be maximal.

Remark 5.1

Consider system $[A_{\varepsilon,\sigma_A}]$ given by (5.1) with $h > 0$. In the sequel, we might need to assume that $p \geq \sigma_A h + 1$ within certain steps of the reduction. This assumption however is non-restrictive since for a system $[A_{\varepsilon,\sigma_A}]$ with $p < \sigma_A h + 1$, we can set $\sigma_B = \frac{p-1}{h}$. Then, $\sigma_A > \sigma_B$ and we can rewrite x^{σ_A} as $x^{\sigma_B} x^{\sigma_A - \sigma_B}$, and $A(x,\varepsilon)$ as

$$A(x,\varepsilon) = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k = \sum_{k=0}^{\infty} B_k(x)(x^{\sigma_B}\varepsilon)^k$$

with $B_k(x) = A_k(x)x^{(\sigma_A-\sigma_B)k} \in \mathcal{M}_n(\mathbb{C}[[x]])$ for all $k \geq 0$. Hence, the treatment of system $[A_{\varepsilon,\sigma_A}]$

can be substituted by the treatment of the system

$$[B_{\varepsilon,\sigma_B}] \qquad \partial F = B(x,\varepsilon)F = \sum_{k=0}^{\infty} B_k(x)(x^{\sigma_B}\varepsilon)^k F.$$

5.2 Basic cases

A detailed discussion of these basic cases, whenever $\sigma_A = p = 0$, is given in [43, Chapter 3, Section 1, pp 52 -56]. We give here a brief discussion without any condition on σ_A and p.

5.2.1 Case $h \leq 0$

If $h \leq 0$ then the solution of system $[A_{\varepsilon,\sigma_A}]$ can be sought, up to any order μ, upon presenting the solution as a power series in $\xi = x^{\sigma_A}\varepsilon$, i.e.

$$F \;=\; \sum_{k=0}^{\infty} F_k(x)\, \xi^{-h+k}.$$

The latter can then be inserted into $[A_{\varepsilon,\sigma_A}]$ and the like powers of ξ equated. This reduces the symbolic resolution to solving successively a set of inhomogeneous linear singular differential systems in x solely, the first of which is the unperturbed system $[A]$ given by (1).

$$\begin{aligned}
x^p\, \partial F_0 &= (A_0(x) \;+\; h\, \sigma_A\, x^{p-1} I)\, F_0 \\
x^p\, \partial F_1 &= (A_0(x) \;+\; (h-1)\, \sigma_A\, x^{p-1} I)\, F_1 \;+\; A_1(x)\, F_0 \\
&\vdots \\
x^p\;\, \partial F_\mu &= (A_0(x) \;+\; (h-\mu)\, \sigma_A\, x^{p-1} I)\, F_\mu + A_1(x)\, F_{\mu-1} \;+\; \ldots \;+\; A_\mu(x)\, F_0.
\end{aligned}$$

For any $k \geq 0$, the solution of the homogeneous system singular in x can be sought via MINIISOLDE or LINDALG. The solution of the inhomgeneous system can be then obtained by the method of variation of constants (see, e.g. [11, Theorem 3, pp 11]). We remark that a transformation $F_\mu = x^{-\mu\sigma_A} G$ reduces the inhomogeneous system corresponding to F_μ to:

$$x^p\, \partial G \;= (A_0(x) \;+\; h\, \sigma_A\, x^{p-1} I)\, G(x)\;,$$

which coincides with that of $F_0(x)$. Thus we have:

$$F_\mu(x) \;=\; x^{-\mu\sigma_A}\, F_0(x) \int_0^x F_0^{-1}(z)\, z^{\mu\sigma_A}\, [A_1(z)\, F_{\mu-1}(z) \;+\; \ldots \;+\; A_\mu(z)\, F_0(z)]\, dz.$$

5.2.2 The Scalar Case

The resolution of the scalar case of a first-order system $[A_{\varepsilon,\sigma_A}]$ is straightforward (see, e.g., [43, pp 52-55] or [142, page 101]). The n^{th}-order scalar equations with less general entries were discussed in Chapter 4. We discuss here those with entries in $\mathrm{R}_{((\varepsilon))}$.

Equivalence with first-order systems

In Chapter 4, we have seen that a scalar equation $[a_\varepsilon]$ can be expressed trivially as a first-order system $[A_\varepsilon]$. In this section, we are interested in the reverse claim for more general systems: Given a first-order system $[A_{\varepsilon,\sigma_A}]$, can we find an equivalent system which has a companion matrix form? If the answer is positive then by denoting the entries in the last row of this companion matrix by $\{(a_i(x,\varepsilon)), 0 \leq i \leq n-1\} \subset \mathrm{R}_{((\varepsilon))}$, the system is obviously equivalent to the singularly-perturbed scalar differential equation $[a_{\varepsilon,\sigma_a}]$ given by:

$$[a_{\varepsilon,\sigma_a}] \qquad \partial^n f - a_{n-1}(x,\varepsilon)\partial^{n-1} f - \ldots a_1(x,\varepsilon)\partial f - a_0(x,\varepsilon) f = 0.$$

In fact, cyclic vectors in analogy with Section 1.2 exist [46, 120]. Due to this equivalence, the formal solutions of systems can be computed after their reduction to scalar equations. And the results of Subsection 4.1 can be easily generalized to these more general scalar equations. However such a treatment is computationally unsatisfactory (see e.g. [93, Conclusion]) and overlooks the information that we can derive from the system directly. Nevertheless, it plays a key role in the theoretical basis of the algorithm which we will develop in later chapters. In particular, it allows the definition of the ε-formal exponential order of system $[A_{\varepsilon,\sigma_A}]$ as follows:

| Definition 5.1

Consider a singularly-perturbed differential system $[A_{\varepsilon,\sigma_A}]$ given by (5.1). The ε-polygon and ε-polynomials of such a system are those of its equivalent n^{th}-order singularly-perturbed differential equation.

We denote the largest slope of the ε-polygon by $\omega_\varepsilon(A)$. It is called the ε-formal exponential order (ε-exponential order in short) of system $[A_{\varepsilon,\sigma_A}]$. The ε-polygon, ε-polynomials, and $\omega_\varepsilon(A)$ are independent of the chosen cyclic vector (see, e.g., [46, 120]). Additionally, it follows from the equivalence under a cyclic vector, that the ε-exponential order of system $[A_{\varepsilon,\sigma_A}]$ is given by

$$\omega_\varepsilon(A) = \max_{0\leq i\leq n} (0, \ (\frac{-val_\varepsilon\ (a_i)}{n-i})). \tag{5.4}$$

Another possible approach is that of [13] which computes a companion block diagonal form for an unperturbed system $[A]$ given by (1). The corresponding algorithm relies on polynomial (shearing) transformations in x and elementary row/column operations. This algorithm can be easily adapted to the perturbed system $[A_{\varepsilon,\sigma_A}]$ upon using polynomial (shearing) transformations in both x and ε, and elementary row/column operations. Such operations are discussed further in Section 5.5.
As mentioned in the introduction, a classical bridge between scalar equations and systems is Arnold's

form (also called Arnold-Wasow form in this context) which can be attained by elementary row/column operations as well [13, 7, 43, 94]. This form is "sufficiently" sparse, based on the Jordan canonical form of the leading constant matrix coefficient (Figure 3.1), so that, roughly speaking, the techniques used for equations can be adapted to systems. For example, given $A(x,\varepsilon)$ in Arnold's form, the (x,ε)-polygon is introduced for $[A_\varepsilon]$ upon defining its vertices from the nonzero coefficients of $A(x,\varepsilon)$ [142, Chapter 2]. In the following section, we start the process of formal reduction, acting directly on the system, with the well-known Splitting lemma which allows uncoupling certain systems.

5.3 ε-Block diagonalization

A classical tool in perturbation theory is the so-called splitting which separates off the existing distinct coalescence patterns. Whenever the leading constant matrix $A_{00} := A_0(0)$ admits at least two distinct eigenvalues, there exists a transformation which block-diagonalizes system $[A_{\varepsilon,\sigma_A}]$ so that it can be uncoupled into subsystems of lower dimensions whose leading constant matrices have a unique distinct eigenvalue each (see, e.g., [65, Theorem $XII-4-1$, pp 381] and [142, Theorem 8.1, pp 81]).

| Theorem 5.1

Consider system $[A_{\varepsilon,\sigma_A}]$ *given by* (5.1)*:*

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h\, x^p\, \partial\, F \;=\; A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\, F,$$

If $p \geq \sigma_A h + 1$ *and* $A_{00} = \begin{bmatrix} A_{00}^{11} & O \\ O & A_{00}^{22} \end{bmatrix}$ *and* A_{00}^{11} *and* A_{00}^{22} *have no eigenvalues in common, then there exists a unique transformation* $T(x,\varepsilon) = \sum_{k=0}^{\infty} T_k(x)(x^{\sigma_A}\varepsilon)^k$ *given by*

$$T(x,\varepsilon) = \begin{bmatrix} I & T_0^{12}(x) \\ T_0^{21}(x) & I \end{bmatrix} (I + \sum_{k=1}^{\infty} \begin{bmatrix} O & T_k^{12}(x) \\ T_k^{21}(x) & O \end{bmatrix} (x^{\sigma_A}\varepsilon)^k),$$

with $T_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ *for all* $k \geq 0$*, such that the change of basis* $F = TG$ *yields*

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial G \;=\; \tilde{A}(x,\varepsilon)\, G \;=\; \sum_{k=0}^{\infty} \tilde{A}_k (x^{\sigma_A}\varepsilon)^k G \;=\; \begin{bmatrix} \tilde{A}^{11}(x,\varepsilon) & O \\ O & \tilde{A}^{22}(x,\varepsilon) \end{bmatrix} G,$$

where $\tilde{A}_{00} = A_{00}$ *and* $\tilde{A}_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ *for all* $k \geq 0$*.*

Well-known proofs (e.g. [65, pp 382- 385] and [139, pp 138 - 142]) proceed in two steps: Block-diagionalizing $A_0(x)$, followed by the total block-diagonalization of $A(x,\epsilon)$. However, the proofs of the first step are given only for the well-behaved case. This is the case when $A_0(x)$ have at least two distinct eigenvalues and is holomorphically similar to its Jordan form, i.e. the eigenvalues don't coalesce in the region of study (see Example 5.1). We give herein a constructive proof of the first step without imposing any restriction on the eigenvalues and eigenvectors of $A_0(x)$, as we aim to give a general discussion which

does not exclude *turning points*. The second and third steps of the proof are adaptations of [43, Chapter 3, pp 56-59] and [140, Theorem 2.3-1, pp 17], which are based on a result originally due to [122].

Proof. We proceed in three steps:

Step 1: We first block-diagonalize the leading matrix coefficient : We can construct a transformation $F = T_0(x)G$ s.t. the leading matrix coefficient $\tilde{A}_0(x)$ of the equivalent system has the following block-form in accordance with A_{00}:

$$\tilde{A}_0(x) = \begin{bmatrix} \tilde{A}_0^{11}(x) & O \\ O & \tilde{A}_0^{22}(x) \end{bmatrix}.$$

From (5.3) we have

$$O_n \;=\; A_0(x)\,T_0(x) \;-\; T_0(x)\,\tilde{A}_0(x). \tag{5.5}$$

Assuming that $T_0(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ is in the following form

$$T_0(x) = \begin{bmatrix} I & T_0^{12}(x) \\ T_0^{21}(x) & I \end{bmatrix},$$

we have for $1 \leq \varrho \neq \varsigma \leq 2$:

$$\begin{cases} A_0^{\varrho\varrho}(x) \;-\; \tilde{A}_0^{\varrho\varrho}(x) \;+\; A_0^{\varrho\varsigma}(x)\,T_0^{\varsigma\varrho}(x) \;=\; O \\ A_0^{\varsigma\varrho}(x) \;+\; A_0^{\varsigma\varsigma}(x)\,T_0^{\varsigma\varrho}(x) \;-\; T_0^{\varsigma\varrho}(x)\,\tilde{A}_0^{\varrho\varrho}(x) \;=\; O \end{cases} \tag{5.6}$$

We inserting the series expansions $A_0(x) = \sum_{i=0}^{\infty} A_{i0}x^i$ and $\tilde{A}_0(x) = \sum_{i=0}^{\infty} \tilde{A}_{i0}x^i$ in (5.6), and we equate the power-like coefficients ($\tilde{A}_{i0}$ may be written conveniently as $\tilde{A}_{i,0}$):

For $i = 0$ we have

$$\begin{cases} A_{00}^{\varrho\varrho} \;=\; \tilde{A}_{00}^{\varrho\varrho} \\ A_{00}^{\varsigma\varsigma}\,T_{00}^{\varsigma\varrho} - T_{00}^{\varsigma\varrho}\,\tilde{A}_{00}^{\varrho\varrho} = O \end{cases}$$

which are satisfied by setting $\tilde{A}_{00}^{\varsigma\varsigma} = A_{00}^{\varsigma\varsigma}$ and $T_{00}^{\varsigma\varsigma} = I,\; T_{00}^{\varsigma\varrho} = O$.

For $i \geq 1$ we have

$$A_{00}^{\varsigma\varsigma}T_{i0}^{\varsigma\varrho} - T_{i0}^{\varsigma\varrho}A_{00}^{\varrho\varrho} \;=\; -A_{i0}^{\varsigma\varrho} - \sum_{j=1}^{i-1}(A_{i-j,0}^{\varsigma\varsigma}T_{j0}^{\varsigma\varrho} - T_{j0}^{\varsigma\varrho}\tilde{A}_{i-j,0}^{\varrho\varrho}) \tag{5.7}$$

$$\tilde{A}_{i0}^{\varrho\varrho} \;=\; A_{i0}^{\varrho\varrho} + \sum_{j=1}^{i-1} A_{i-j,0}^{\varrho\varsigma}T_{j0}^{\varsigma\varrho}. \tag{5.8}$$

It's clear that (7.13) is a set of Sylvester matrix equations that possess a unique solution due to the assumption on the disjoint spectra of A_{00}^{11} and A_{00}^{22} (see, Lemma 1.1). Remarking that the right hand side depends solely on the A_{j0}, T_{j0}, $\tilde{A}_{j0}$ with $j < i$, equations (7.13) and (7.15) are successively soluble.

Step 2: We can now assume without loss of generality that the leading matrix coefficient $A_0(x)$ of $[A_{\varepsilon,\sigma_A}]$ has a block-diagonal form in accordance with that of A_{00} and we study the effect of the

transformation

$$F = (I + (x^{\sigma_A}\varepsilon)^j T_j(x))\, G, \quad \text{for some } j \geq 1. \tag{5.9}$$

One can verify that (5.9) yields the equivalent system

$$\varepsilon^h x^p \partial G = \tilde{A}(x,\varepsilon) G$$

for which

$$(I + (x^{\sigma_A}\varepsilon)^j T_j(x))\, \tilde{A}(x,\varepsilon) = \; A(x,\varepsilon)\, (I + (x^{\sigma_A}\varepsilon)^j T_j(x)) \; - \; \varepsilon^h x^p (x^{\sigma_A}\varepsilon)^j (\frac{j\sigma_A}{x} + \partial) T_j(x). \tag{5.10}$$

We can rewrite

$$\varepsilon^h x^p (x^{\sigma_A}\varepsilon)^j = \varepsilon^h x^{p+\sigma_A h - \sigma_A h} (x^{\sigma_A}\varepsilon)^j = (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h} (x^{\sigma_A}\varepsilon)^j = (x^{\sigma_A}\varepsilon)^{h+j} x^{p-\sigma_A h}.$$

We then assume that $\tilde{A}(x,\varepsilon) = \sum_{k=0}^{\infty} \tilde{A}(x)(x^{\sigma_A}\varepsilon)^k$. With the help of the shorthand notation $\xi = x^{\sigma_A}\varepsilon$, we insert the series expansions of $A(x,\varepsilon)$ and $\tilde{A}(x,\varepsilon)$ in (5.10), and equating power-like coefficients of ξ. We obtain (x is dropped for clarity):

$$\sum_{k=0}^{\infty} (\tilde{A}_k + T_j \tilde{A}_{k-j})\, \xi^k = \sum_{k=0}^{\infty} (A_k + A_{k-j}\, T_j)\, \xi^k - \xi^{h+j} x^{p-\sigma_A h}\, (\partial + \frac{j\sigma_A}{x})\, T_j$$

where $\tilde{A}_{k-j} = A_{k-j} = O$ if $k < j$.
Equating power-like coefficients, we have:

$$\tilde{A}_k = A_k + A_{k-j}\, T_j - T_j \tilde{A}_{k-j} - \gamma x^{p-\sigma_A h - 1} (x\partial + j\sigma_A\, I) T_j \tag{5.11}$$

where $\gamma = 1$ if $k = h + j$ and $\gamma = 0$ otherwise. In particular, we have:

$$\tilde{A}_k \;\; = \;\; A_k \quad \text{for} \quad 0 \leq k < j \tag{5.12}$$

$$\tilde{A}_j \;\; = \;\; A_j + A_0\, T_j - T_j \tilde{A}_0 \tag{5.13}$$

One can observe that (5.13) is a Sylvester matrix equation of the type discussed in Step 1. Thus, by taking

$$T_j(x) = \begin{bmatrix} I & O \\ O & I \end{bmatrix} + \sum_{i=1}^{\infty} \begin{bmatrix} O & T_{ji}^{12}(x) \\ T_{ji}^{21}(x) & O \end{bmatrix} x^i,$$

and remarking that $A_{00} = \tilde{A}_{00}$, one can construct such a T_j and can compute the $\tilde{A}_k$'s successively.

Step 3: By a product of transformations of type (5.9), namely

$$F = \prod_{j=0}^{\mu} (I + \xi^j T_j(x))\, G$$

the $A_j(x)$'s can be block-diagonalized successively to any desired order $\mu \in \mathbb{N}$ starting with $j = 1$. For

a given $j \geq 1$, the terms of lower degree in ε (equivalently of lower degree in ξ) are not altered, as exhibited by (5.12). ■

| *Remark* 5.2

Under the notations of Theorem 5.1 and the non-restrictive condition $p \geq \sigma_A h + 1$ (see Remark 5.1), one can observe that $\sigma_{\tilde{A}} \geq \sigma_A$.

| *Remark* 5.3

Clearly, Theorem 5.1 is a generalization of Theorem 1.1. The above form of A_{00} is non-restrictive. It suffices that A_{00} has at least two distinct eigenvalues so that Lemma 1.2 be applied. Consequently, in case the base field is $\mathcal{K}$ rather than $\mathbb{C}$, no extension is required.

Consider one of the uncoupled systems and assume its leading constant matrix is in Jordan normal form with a unique eigenvalue $\gamma \in \mathbb{C}$. This can be always attained by a constant transformation. Then, upon applying the so-called eigenvalue shifting, i.e.

$$F = G \exp(\int^x \varepsilon^{-h} z^{-p} \gamma \, dz), \tag{5.14}$$

it is easy to verify from (5.3) that the resulting system has a nilpotent leading constant matrix.

5.4 Resolution of turning points

Without loss of generality, we can now assume that system $[A_{\varepsilon,\sigma_A}]$ is such that $A_0(0)$ is nilpotent. Clearly, it doesn't follow that $A_0(x)$ is nilpotent as well. The origin may be a turning point for system $[A_{\varepsilon,\sigma_A}]$. In fact, if the matrix under reduction has multiple eigenvalues then they might coalesce. We have seen the characterization of such points for the scalar case in Chapter 4 in terms of algebraic equations. The following example gives an illustration of this phenomenon for a system*.

| **Example 5.1 ([140], pp223)**

Let $a(x), b(x)$ be holomorphic for x in a region $\Omega \in \mathbb{C}$ and

$$M(x) = \begin{bmatrix} x & 1 & a(x) \\ 0 & x & b(x) \\ 0 & 0 & 0 \end{bmatrix}.$$

A short calculation shows that most general transition matrix $T(x)$ is given by

$$T(x) = \begin{bmatrix} \alpha(x) & \beta(x) & \gamma(x)(-a(x)x^{-1} + b(x)x^{-2}) \\ 0 & \alpha(x) & -\gamma(x)b(x)x^{-1} \\ 0 & 0 & \gamma(x) \end{bmatrix}.$$

*The terminology *turning point* was coined by physicists as a description for experiments where such differential equations first arose (e.g. total reflection in the propagation of light in inhomogeneous mediums) [140, Historical Introduction].

Thus we have the following in Ω for $x \neq 0$

$$J(x) = T^{-1}M(x)T(x) = \begin{bmatrix} x & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The scalar functions $\alpha(x), \beta(x), \gamma(x)$ are arbitrary, so far, except that α and γ must not vanish anywhere in Ω. If they are holomorphic, so is T, provided zero is not a point of Ω. If $0 \in \Omega$, then $T(x)$ has a pole at $x = 0$ unless the following conditions are satisfied:

$$b(0) = 0 \text{ and } \partial b(0) = a(0).$$

Therefore, $M(x)$ is in general not holomorphically similar to $J(x)$ in regions that contain the origin. Moreover, if the second condition is not satisfied, two possibilities arise:

- if $b(0) = 0$ then there exists a constant invertible matrix S such that

$$S^{-1}M(0)S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

  Then $M(x)$ is pointwise similar to $J(x)$, even in regions containing $x = 0$.

- if $b(0) \neq 0$ then the Jordan form of $M(0)$ is

$$J(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  so that $J(x)$ is not holomorphic at $x = 0$.

Hence, it might occur that the Jordan matrix is not holomorphic in some region or the Jordan matrix is itself holomorphic but, nevertheless, not holomorphically (although perhaps pointwise) similar to $M(x)$.

□

| Definition 5.2 ([142], pp 57)

Consider system $[A_{\varepsilon,\sigma_A}]$ given by (5.1). We refer by turning points to the points where the Jordan form of $A_0(x)$ changes, i.e., either the multiplicity of the eigenvalues or the degrees of the elementary divisors are not constant in a neighborhood of such points.

| Example 5.2 ([142], pp 57)

Consider the following system:

$$\varepsilon^2\partial F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x \end{bmatrix} F \quad \text{where} \quad A_0(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x \end{bmatrix} \quad \text{and} \quad \sigma_A = 0.$$

We remark that this system is obtained from (4.4) by setting $F = [f, \varepsilon^2\partial f, \varepsilon^4\partial^2 f]^T$. It follows from the form of $A_0(x)$ that the origin is a turning point for this system.

□

At this stage of formal reduction, i.e. when A_{00} is nilpotent, our approach diverges from the classical one. As mentioned before, our algorithm requires neither the Arnold-Wasow form for $A(x,\varepsilon)$ nor a cyclic vector to reduce system $[A_{\varepsilon,\sigma_A}]$ to a scalar differential equation. Instead, we proceed by treating the turning point. This treatment transforms the system at hand to a more general system for which $A_0(x)$ is nilpotent as well.

| Proposition 5.1

Consider system $[A_{\varepsilon,\sigma_A}]$ *given by* (5.1)*:*

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p\, \partial F \;=\; A(x,\varepsilon)\; F \;=\; \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\; F.$$

If A_{00} *is a nilpotent matrix and* $A_0(x)$ *has at least one nonzero eigenvalue then there exist an integer* s *and a transformation* $T \in GL_n(\mathbb{C}(x^{1/s}))$ *such that, by setting* $x = t^s$*, and re-adjusting* p *and* σ_A*, the transformation* $F = TG$ *results in the following equivalent system:*

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^h t^{\tilde{p}}\, \partial_t\, G \;=\; \tilde{A}(t,\varepsilon)\; G = \sum_{k=0}^{\infty} \tilde{A}_k(t)(t^{\sigma_{\tilde{A}}}\varepsilon)^k\; G$$

where $\tilde{A}_{00} := \tilde{A}_0(0)$ *has at least one nonzero constant eigenvalue.*

Proof. The eigenvalues[†] of $A_0(x)$ admit a formal expansion in the fractional powers of x in the neighborhood of $x = 0$ (see, e.g., [80]). We are interested only in their first nonzero terms. Let $\mu(x) = \sum_{j=0}^{\infty} \mu_j x^{j/s}$ be a nonzero eigenvalue of $A_0(x)$ whose leading exponent, i.e. smallest j/s for which $\mu_j \neq 0$ and j, s are coprime, is minimal among the other nonzero eigenvalues. Then, without loss of generality, we can assume that $s = 1$; otherwise we set $x = t^s$. Now, let $T \in GL_n(\mathbb{C}(x))$ be the polynomial transformation for which $B_0(x) = T^{-1}A_0(x)T = B_{00} + B_{10}x + \ldots$ is in CRV form, i.e.

$$\theta_{B_0(x)}(\lambda) = x^{rank(B_{00})} \det(\lambda I + \frac{B_{00}}{x} + B_{10})|_{x=0}$$

does not vanish identically in λ (see Section 2.2.1). Let $\nu > 0$ denote the valuation of $B_0(x)$ then, by

[†]For details regarding eigenvalues and perturbed eigenvalue problem, please refer to Chapter 4.

Proposition 2.1, there are $n - deg\theta$ eigenvalues of $B_0(x)$ whose leading exponents lie in $[\nu, \nu+1[$, and deg θ eigenvalues for which the leading exponent is equal to or greater than $\nu+1$. Then, the application of the transformation $T(x)$ to the system $[A_{\varepsilon,\sigma_A}]$ yields

$$[B_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \, \partial G \;=\; B(x,\varepsilon)\, G \;=\; \sum_{k=0}^{\infty} B_k(x)(x^{\sigma_A}\varepsilon)^k \, G$$

where $B_0(x) := x^\nu(B_{00} + B_{10}x + \dots)$ and B_{00} has $n - deg\theta$ eigenvalues whose leading exponents lie in $[0, 1[$. Since $T(x)$ is a polynomial transformation, then for every $k \geq 0$, $val_x(B_k(x)) \geq min(p - \sigma_A h - 1, span_x(T))$ where $span_x(T)$ is the difference between the valuation and the degree of the polynomial transformation $T(x)$ in x. Let q be the maximal nonpositive integer such that

$$val_x(B_k(x)) \geq qk + \nu, \quad \text{for all} \quad k > 0.$$

(Geometrically, q is the maximal nonpositive integer among the slopes of the straight lines which pass through the point $(0, \nu)$ and stays below the points $(k, val_x(B_k))$ for all $k > 0$.) Then upon setting $\sigma_{\tilde{A}} = \sigma_A + q$ and $\tilde{p} = p - \nu$, system $[B_{\varepsilon,\sigma_A}]$ can be rewritten as

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^h x^{\tilde{p}} \, \partial G \;=\; \tilde{A}(x,\varepsilon)\, G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_{\tilde{A}}}\varepsilon)^k \, G$$

where for all $k > 0$, $\tilde{A}_k(x) = x^{-\nu-qk}B_k(x) \in \mathcal{M}_n([[x]])$, $\tilde{A}_0(x) = x^{-\nu}B_0(x)$, and $\tilde{A}_{00} = B_{00}$. Then $\mu(x)$ is an eigenvalue of $\tilde{A}_0(x)$ with a minimal leading exponent and hence it is among those whose leading exponents lie in $[0, 1[$ (see Proposition 2.1). By our assumption $s = 1$, and hence the leading exponent of $\mu(x)$ is zero and $\mu_0 \neq 0$. Since μ_0 is a nonzero eigenvalue of $\tilde{A}_{00}$, it follows that the latter is non-nilpotent. ■

| *Remark* 5.4

The eigenvalues of $A_0(x)$ are the roots of the algebraic scalar equation $f(x,\mu) = \det(A(x) - \mu I_n) = 0$ and can be computed by Newton-Puiseux algorithm or as described in Chapter 2. The linear transformation $T \in GL_n(\mathbb{C}(x^{1/s}))$ can be computed via MINIISOLDE *or* LINDALG.

| Example 5.3

Given the matrix expression of Weber's equation ($\sigma_A = 0$, $p = 0$, $s = 1$):

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon\partial F = \begin{bmatrix} 0 & 1 \\ x^2 & 0 \end{bmatrix} F.$$

To put the leading coefficient in CRV form, we use $T(x) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$. Then $F = TG$ yields the

following equivalent system:

$$[B_{\varepsilon,\sigma_A}] \qquad \varepsilon\partial G = \begin{bmatrix} 0 & x \\ x & -\varepsilon x^{-1} \end{bmatrix} G.$$

Then $\nu = 1$. We compute $\sigma_{\tilde{A}} = -2$ and $\tilde{p} = -1$. We thus have:

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon x^{-1}\partial G = \begin{bmatrix} 0 & 1 \\ 1 & -\varepsilon x^{-2} \end{bmatrix} G.$$

The leading coefficient $\tilde{A}_0(x)$ is a constant matrix now and is clearly in CRV form ($\theta_{\tilde{A}_0(x)}(\lambda) = -1$). The degree of $d := \theta_{\tilde{A}_0}$ is zero and there are $n - d = 2$ eigenvalues of $\tilde{A}_0(x)$ whose leading exponents lie in $[0, 1[$, namely zero.

□

Under the above notations of Proposition 5.1, one can observe that the treatment of a turning point yields $\sigma_{\tilde{A}} \leq \sigma_A$. Proposition 5.1 leads to the following refined definition of a turning point.

| Definition 5.3

Consider system $[A_\varepsilon]$ given by (3.9), and its equivalent expression as system $[A_{\varepsilon,\sigma_A}]$ given by (5.1), with $\sigma_A = 0$ and $p = 0$. We say that system $[A_\varepsilon]$ has a turning point at $x = 0$ if, by the end of the process of formal reduction of system $[A_{\varepsilon,\sigma_A}]$, at least one of its uncoupled subsystems has a nonzero restraining index.

"Actual" solutions of systems of zero restraining index have been studied in [126] and references therein.

| Example 5.4

We consider the following system for which $\sigma_A = 0$:

$$\varepsilon^2\,\partial F \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & x & 0 \end{bmatrix} F \quad \text{where} \quad A_0(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x & 0 \end{bmatrix}.$$

Here, $s = 2$. So we can set $x = t^2$ and let $F = TG$ where $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{bmatrix}$. Or equivalently, let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & x^{1/2} & 0 \\ 0 & 0 & x \end{bmatrix}.$$

This results in:

$$\varepsilon^2\,\partial G \;= x^{1/2}\{\begin{bmatrix}0&1&0\\1&0&0\\0&1&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}x^{-3/2}\varepsilon+\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-2\end{bmatrix}x^{-1}\varepsilon^2\}\,G.$$

Hence, $\nu=1/2$, $\tilde{p}=-1/2$, $\sigma_{\tilde{A}}=-3/2$. Thus the above system can be expressed as:

$$\varepsilon^2x^{-1/2}\,\partial G \;=\{\begin{bmatrix}0&1&0\\1&0&0\\0&1&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}(x^{-3/2}\varepsilon)+\begin{bmatrix}0&0&0\\0&-x^2&0\\0&0&-2x^2\end{bmatrix}(x^{-3/2}\varepsilon)^2\;\}\,G.$$

Now that the leading term is a constant matrix with three distinct eigenvalues, we can proceed by applying the Splitting of Theorem 5.1. Luckily, this system can be uncoupled thoroughly.

□

However, we might encounter different scenarios as well, as illustrated in the following example which was treated initially in [142, pp 88]. We remark however that the techniques proposed in the former start with the reduction of $A(x,\varepsilon)$ to its Arnold-Wasow form followed by the construction of the (x,ε)-polygon. This particular example therein was already given in Arnold-Wasow form for the sake of simplicity and hence, the transformations computed by our algorithm, which doesn't require reduction to this form, do not deviate herein from those in the former.

| Example 5.5 ([142],pp 57)

Consider again the system of Example 5.2 ($\sigma_A=0$, $p=0$) which is equivalent to the scalar equation (4.4):

$$\varepsilon^2\,\partial F \;=\; \begin{bmatrix}0&1&0\\0&0&1\\\varepsilon&0&x\end{bmatrix}F.$$

As mentioned, we have a turning point at $x=0$. A_{00} is nilpotent in Jordan form while the eigenvalues of $A_0(x)$ are $0,0,x$, whence $s=1$. Let $T=\mathrm{diag}(1,x,x^2)$ then upon setting $F=TG$ we get

$$\varepsilon^2\,\partial G \;=\; x\{\begin{bmatrix}0&1&0\\0&0&1\\0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}x^{-3}\varepsilon+\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-2\end{bmatrix}x^{-2}\varepsilon^2\;\}\,G.$$

Hence, $\sigma_{\tilde{A}}=-3$, $\tilde{p}=-1$, and the former can be rewritten equivalently as:

$$\varepsilon^2x^{-1}\,\partial G \;=\{\begin{bmatrix}0&1&0\\0&0&1\\0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}(x^{-3}\varepsilon)+\begin{bmatrix}0&0&0\\0&-x^4&0\\0&0&-2x^4\end{bmatrix}(x^{-3}\varepsilon)^2\;\}\,G.$$

The constant leading matrix coefficient of the resulting matrix is no longer nilpotent. Hence the

system can be uncoupled into two subsystems upon setting $G = TW$ where

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -2 \\ -1 & -1 & 0 \end{bmatrix} (x^{-3}\varepsilon) + O((x^{-3}\varepsilon)^2).$$

The resulting equivalent system then consists of the two uncoupled lower dimension systems where $W = [W_1, W_2]^T$:

$$\begin{aligned} \varepsilon^2 x^{-1}\partial W_1 &= \{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} (x^{-3}\varepsilon) + \begin{bmatrix} 1 & -1 \\ 1 & -1+x^4 \end{bmatrix} (x^{-3}\varepsilon)^2 + O((x^{-3}\varepsilon)^3) \}W_1. \\ \varepsilon^2 x^{-1}\partial W_2 &= \{1 + (x^{-3}\varepsilon) + (1+2x^4)(x^{-3}\varepsilon)^2 + O((x^{-3}\varepsilon)^3) \}W_2. \end{aligned}$$

For the second subsystem, the ε-exponential part is $\int \varepsilon^{-2}x(1 + O(\varepsilon x^{-3}))\ dx = \frac{1}{2}\varepsilon^{-2}x^2 + \ldots$: the eigenvalue x of the leading matrix coefficient of the system we started with is recovered as expected. This is in accordance with the ε-exponential parts obtained for the scalar equation (4.4). As for the first subsystem, $A_0(x)$ and A_{00} are now simultaneously nilpotent.

□

To proceed to the total uncoupling of the above system into scalar equations, we introduce the ε-rank reduction.

5.5 ε-Rank reduction

We consider again system $[A_{\varepsilon,\sigma_A}]$ given by (5.1). We suppose that the turning point of the system is treated and no further reduction can be attained via splitting. We can now assume without loss of generality that $A_0(x)$ and A_{00} are simultaneously nilpotent and we investigate the ε-rank reduction of the system, i.e. we seek to determine the minimal integer value of h among all equivalent systems to $[A_{\varepsilon,\sigma_A}]$ and to construct a transformation for which the equivalent system has this minimal ε-rank.

We first generalize Moser's reduction criterion given in [107] (see Chapter 1). We remark that, as we have seen in Section 1.5.1, the motivation behind [107] was to determine the nature of the singularity (regular or irregular) of the unperturbed system $[A]$ given by (1). Consequently, the reduction of the Poincaré rank p of x to its minimal integer value was investigated. For the perturbed system $[A_{\varepsilon,\sigma_A}]$ (or $[A_\varepsilon]$) however, it seems more plausible to reduce h, rather than p, to its minimal integer value (see Section 5.2 for the resolution of the case $h = 0$).

Moser's criterion was generalized as well to linear functional matrix equations in [21], a particular case of which is the unperturbed system $[A]$. It was also borrowed in [75] from the theory of differential systems to investigate efficient algorithmic resolution of the perturbed algebraic eigenvalue-eigenvector problem (see Chapter 2).However, despite their utility and efficiency for such univariate systems, algorithms based on this criterion are not considered so far over bivariate fields.

Barkatou developed in [15] a Poincaré rank reduction algorithm for the unperturbed differential system $[A]$ with rational coefficients. In this section, we give a rank reduction criterion and generalize this

algorithm to system $[A_{\varepsilon,\sigma_A}]$[‡].

Following the discussion for unperturbed systems $[A]$ of the form (1), we define the ε-Moser rank and ε-Moser invariant of system $[A_{\varepsilon,\sigma_A}]$ respectively as the rational numbers:

$$\begin{aligned} m_\varepsilon(A) &= \max\,(0, h + \frac{rank\ (A_0(x))}{n}) \quad \text{and} \\ \mu_\varepsilon(A) &= \min\,\{m_\varepsilon(T[A_{\varepsilon,\sigma_A}]) \text{ for all possible choices of } T \text{ in } GL_n(\mathrm{R}_{((\varepsilon))})\}. \end{aligned}$$

| Definition 5.4

We say that system $[A_{\varepsilon,\sigma_A}]$ (the matrix $A(x,\varepsilon)$ respectively) is ε-reducible if $m_\varepsilon(A) > \mu_\varepsilon(A)$, otherwise we say that is ε-irreducible.

If $m_\varepsilon(A) \leq 1$, then $h = 0$. Hence, we restrict our attention to the case of $m_\varepsilon(A) > 1$.

| Remark 5.5

This definition is not to be mixed neither with the usual sense of Moser-irreducible unperturbed system $[A]$ given by (1), i.e. the systems whose Poincaré rank p is minimal; nor with the common sense of reduced system in the literature, i.e. the system defined from $[A_{\varepsilon,\sigma_A}]$ by $x^p\partial\mathcal{F} = A_0(x)\mathcal{F}$.

The main result of this section is the following theorem and its sufficient condition whose proof follows after a series of intermediate results. In this section, given a system $[A_{\varepsilon,\sigma_A}]$, we assume, without loss of generality due to Remark 5.1, that $p - \sigma_A h - 1 \geq 0$.

| Theorem 5.2

Consider system $[A_{\varepsilon,\sigma_A}]$ given by (5.1):

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial F = A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\ F.$$

We assume that $h > 1$, $m_\varepsilon(A) > 1$, and $r = rank\ (A_0(x)) < n$. Suppose that there exists a transformation $F = T(x,\varepsilon)G$, where $T = \sum_{k=0}^{\infty} T_k(x)(x^{\sigma_A}\varepsilon)^k$ is invertible, and $T_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$

[‡]Moser himself, within his numerous and diverse interests, was interested in such systems. However, he did not consider his own approach to rank reduction of the unperturbed counterparts of such systems to treat system $[A_{\varepsilon,\sigma_A}]$ as we propose in this section. In an internal technical report [106], he investigates the formal growth order. His treatment resembles essentially that of Iwano in [71] although these works were developed independently. On another rather irrelevant note, Moser had another kind of relationship to researchers interested in such systems: In [112], O'Malley and Robert trace the birth of singular perturbation theory to Göttingen with the seven-page paper of Prandtl entitled *"On fluid motion with small friction"*. This paper, as they quote from Tani (1977), *marked on the epoch in the history of fluid dynamics, opening the way for understanding the motion of real fluids*. Prandtl's life however was characterized by his most famous student Theodore von Kármán as *particularly full of overtones of naïveté*. Indeed, when Prandtl decided he ought to marry in 1909, he wrote to the Föppls asking for the hand of one of their daughters, without specifying one of them in particular. In a discussion whether Prandtl married the appropriate daughter of the Föppls, the authors of [112] remark that, I quote, *the tradition of marrying the daughter of one's professor had been common in academic Germany, sometimes being suggested as the way mathematical talent was transferred between generations. Courant, for example, married Runge's daughter, and his daughters have also married prominent mathematicians, namely Jurgen Moser and Peter Lax, who certainly did not need family assistance to obtain success*.

for all $k \geq 0$, such that the equivalent system

$$[B_{\varepsilon,\sigma_A}] \qquad \varepsilon^{\tilde{h}} x^p \partial G = B(x,\varepsilon)G = \sum_{k=0}^{\infty} B_k(x)(x^{\sigma_A}\varepsilon)^k \; G$$

satisfies either $(\tilde{h} < h)$ or $(\tilde{h} = h$ and $rank(B_0(x)) < r)$. Then, setting $\xi = x^{\sigma_A}\varepsilon$, the polynomial

$$\theta_A(\lambda) := \xi^r \det(\lambda I + \frac{A_0(x)}{\xi} + A_1(x))|_{\xi=0} \tag{5.15}$$

vanishes identically in λ.

Proof. Suppose that such a $T(x,\varepsilon)$ exists, then we can present $T(x,\varepsilon)$ in the form (see [106, Lemma 1]):

$$T(x,\varepsilon) = P(x,\varepsilon)\; \xi^{\alpha}\; Q(x,\varepsilon),$$

where $P(x,\varepsilon), Q(x,\varepsilon)$ are unimodular elements of $\mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$, and $\alpha = \mathrm{diag}(\alpha_1,\ldots,\alpha_n)$ for some integers $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$.
We then define the span $s(T)$ by

$$s := s(T) = \alpha_n - \alpha_1 = \; max_{1\leq i,j\leq n}\,(\alpha_i - \alpha_j).$$

Without loss of generality, one can assume that $s = 1$. Otherwise, T can be written as a product of $s+1$ transformations, the first of which is of span zero and the rest are of span one each, and which can be applied successively to the system. In fact, let $e_{k,l} = 0$ if $k = l$ and $e_{k,l} = 1$ otherwise. Then one can construct[§] $s+1$ sequences

$$\beta^{(i)} = (\beta_1^{(i)},\ldots,\beta_n^{(i)}), \quad i \in \{0,\ldots,s\},$$

[§]We illustrate with the following examples:

- $\alpha = (1,2,4)$ then $s = 3$ and the 4 sequences can be chosen as $(1,1,1)$, $(0,1,1)$, $(0,0,1)$, $(0,0,1)$;
- $\alpha = (2,3,3)$ then $s = 1$ and the 2 sequences can be chosen as $(2,2,2)$, $(0,1,1)$;
- $\alpha = (-2,-1,3)$ then $s = 5$ and the 6 sequences can be chosen as $(-2,-2,-2)$, $(0,1,1)$, $(0,0,1)$, $(0,0,1)$, $(0,0,1)$, $(0,0,1)$.

in the following manner

$$\begin{aligned}
\beta^{(0)} &= (\alpha_1, \alpha_1, \ldots, \alpha_1) \\
\beta^{(1)} &= (0, e_{\alpha_2,\alpha_1}, \ldots, e_{\alpha_n,\alpha_1}) \\
\beta^{(2)} &= (0, e_{\alpha_2,\alpha_1+1}, \ldots, e_{\alpha_n,\alpha_1+1}) \\
&\vdots \\
\beta^{(i)} &= (0, e_{\alpha_2,\alpha_1+i-1}, \ldots, e_{\alpha_n,\alpha_1+i-1}) \\
&\vdots \\
\beta^{(s)} &= (0, e_{\alpha_2,\alpha_1+s-1}, \ldots, e_{\alpha_n,\alpha_1+s-1})
\end{aligned}$$

One can then verify that for any $i \in \{1, \ldots, s\}$ we have

$$\begin{aligned}
\beta_1^{(i)} &\leq \cdots \leq \beta_n^{(i)}, \\
\beta_n^{(i)} &- \beta_1^{(i)} = 1, \quad \text{and} \\
\sum_{i=0}^{s} \beta_j^{(i)} &= \alpha_j,\ j \in \{1, \ldots, n\}
\end{aligned}$$

We then proceed by defining

$$\begin{aligned}
T^{(i)} &= P\xi^{\beta^{(i)}}P^{-1}, \text{ for } i \in \{0, 2, \ldots, s-1\} \\
T^{(s)} &= P\xi^{\beta^{(s)}}Q.
\end{aligned}$$

We then have

$$T = T^{(0)}T^{(1)}T^{(2)}\ldots T^{(s)}$$

where $s(T^{(0)}) = 0$ and $s(T^{(i)}) = 1, i \in \{1, \ldots, s\}$.
We now consider $\tilde{A} = P^{-1}AP - \xi^h x^{p-\sigma_A h}P^{-1}\partial P$ and $\tilde{B} = QBQ^{-1} + \xi^h x^{p-\sigma_A h}(\partial Q)Q^{-1}$. Then we have

$$\tilde{B} = \xi^{-\alpha}\tilde{A}\xi^{\alpha} - \xi^h x^{p-\sigma_A h-1}\,\mathrm{diag}(\sigma\alpha_1, \ldots, \sigma\alpha_n).$$

It then follows from the unimodularity of P and Q with respect to ε (and consequently ξ) that $m_\varepsilon(A) = m_\varepsilon(\tilde{A})$ and $m_\varepsilon(B) = m_\varepsilon(\tilde{B})$. Let $\vartheta(\lambda) = \xi^r \det(\lambda I + \frac{A}{\xi})|_{\xi=0}$, then we have:

$$\begin{aligned}
\vartheta(\lambda) &= \xi^r \det(\lambda I + \frac{A}{\xi})|_{\xi=0} = \xi^r \det(\lambda I + \frac{\tilde{A}}{\xi} + \xi^{h-1}x^{p-\sigma_A h-1}P^{-1}(\partial P))|_{\xi=0} \\
&= \xi^r \det(\lambda I + \frac{\tilde{B}}{\xi} + \tilde{P})\,|_{\xi=0}
\end{aligned}$$

where $\tilde{P} = \xi^{h-1}x^{p-\sigma_A h-1}(\mathrm{diag}(\sigma\alpha_1, \ldots, \sigma\alpha_n) + x\xi^{-\alpha}P^{-1}\partial P\xi^{\alpha})$. Hence, $\tilde{P}(x, \xi)$ has no poles in ξ since $span(T) = 1$. In other words, the pole introduced in $\xi^{-\alpha}P^{-1}\partial P\xi^{\alpha}$ is at worst a simple pole. And so, $\tilde{P}$ has no poles.

Thus the worst pole the determinant of $\vartheta(\lambda)$ can have is of order $\mu \leq rank(\tilde{B}_0(x)) = rank(B_0(x)) < r$, i.e.

$$\vartheta(\lambda) = \xi^{r-\mu}\ E(x,\xi,\lambda)|_{\xi=0}$$

where $E(x,\xi,\lambda)$ is a polynomial in λ whose coefficients lie in $\mathbb{C}((x))[[\xi]]$. Consequently, $\vartheta(\lambda) \equiv 0$. But

$$\vartheta(\lambda) = \xi^r \det(\lambda I + \frac{A}{\xi})|_{\xi=0} \equiv \xi^r \det(\lambda I + \frac{A_0(x)}{\xi} + A_1(x) + \sum_{k=2}^{\infty} A_k(x)\xi^{k-1})|_{\xi=0},$$

and so it suffices to consider A_0 and A_1, which yields $\theta_A(\lambda) \equiv 0$. ■

| *Remark* 5.6

The assumption that $\sigma_A = \sigma_B = \sigma_T$ in the baove theorem does not restrict its generality. In fact, if $\sigma_B \neq \sigma_A$ or $\sigma_T \neq \sigma_A$ then the transformation and both systems can be re-expressed with a common restraining index equals to $\min(\sigma_A, \sigma_B, \sigma_T)$.

In what follows, we prove constructively that the condition on (5.15) is also sufficient whenever $h > 1$. We remark that the distinction between the two cases $h > 1$ and $h = 1$ is technical as explained in the following remark and unlike for the Poincaré rank p of an unperturbed system $[A]$, it does not lead to a classification of regularity.

| *Remark* 5.7

Given system $[A_{\varepsilon,\sigma_A}]$ we study herein the effect of the two transformations that we will perform in the sequel:

- *Let $T(x) = (I_n + Q(x)) \in GL_n(\mathbb{C}[[x]])$ such that*

$$Q(x) = [q_{ij}]_{1\leq i,j\leq n}\ , \quad s.t.\ \begin{cases} q_{nj} \in \mathbb{C}[[x]]\ , & \text{for } r+1 \leq j < n \\ q_{ij} = 0 & \text{elsewhere.} \end{cases}$$

By $F = TG$, the resulting equivalent system is given by:

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial G = \tilde{A}(x,\varepsilon)G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_A}\varepsilon)^k$$

where $\sigma_{\tilde{A}} = \sigma_A$ and

$$\begin{cases} \tilde{A}_0(x) = (I_n + Q(x))^{-1}\ A_0(x)\ (I_n + Q(x)) \\ \tilde{A}_1(x) = (I_n + Q(x))^{-1}\ A_1(x)\ (I_n + Q(x)) - \gamma x^p \partial Q(x); \\ \tilde{A}_2(x) = \dots \end{cases} \tag{5.16}$$

where $\gamma = 1$ if $h = 1$ and $\gamma = 0$ otherwise. But, due to (5.15), we can limit our interest to $A_0(x)$, $A_1(x)$, $\tilde{A}_0(x)$, and $\tilde{A}_1(x)$ solely. Hence, if $h > 1$ then it suffices to investigate $T^{-1}\ A(x,\xi)\ T$, the similarity term of (5.3). However, if $h = 1$ then the term $T^{-1}\partial T(x)$ should be taken into

account.

- *Let* $T = \mathrm{diag}((x_A^\sigma\varepsilon)^{\alpha_1},\ldots,(x_A^\sigma\varepsilon)^{\alpha_n})$ *where* $\alpha_1,\ldots,\alpha_n$ *are integers. Then* $T^{-1}\partial T = \mathrm{diag}\,\frac{1}{x}(\sigma\alpha_1,\ldots,\sigma\alpha_n)$.

The two cases $h>1$ and $h=1$ are discussed separately and respectively in the following two subsections.

5.5.1 ε-Reduction, $h>1$

| Theorem 5.3

Given the system

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial F = A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\ F$$

with $h>1$ *and* $m_\varepsilon(A)>1$. *Set* $rank\ (A_0(x)) = r$ *. Suppose that the polynomial*

$$\theta_A(\lambda) := \xi^r \det(\lambda I + \frac{A_0(x)}{\xi} + A_1(x))|_{\xi=0}$$

vanishes identically in λ. *Then there exists a transformation* $F = R(x,\varepsilon)G$ *such that the equivalent system*

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^{\tilde{h}} x^p \partial G \ = \ \tilde{A}(x,\varepsilon)\ G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_A}\varepsilon)^k\ G$$

satisfies either ($\tilde{h}<h$*) or (*$\tilde{h}=h$ *and* $rank(\tilde{A}_0(x)) < r$*). Moreover, such a* $R(x,\varepsilon)$ *can be always chosen to be a product of unimodular transformations in* $GL_n(\mathbb{C}[[x]])$ *and polynomial transformations in* $(x^{\sigma_A}\varepsilon)$ *of the form* $\mathrm{diag}((x^{\sigma_A}\varepsilon)^{\alpha_1},\ldots,(x^{\sigma_A}\varepsilon)^{\alpha_n})$ *where* $\alpha_1,\ldots,\alpha_n$ *are nonnegative integers.*

The proof of the theorem will be given after a set of intermediate results. We start with the following lemma.

| Lemma 5.1

Given the system

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h\ x^p\ \partial F = A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x_A^\sigma\varepsilon)^k\ F,$$

with $r = rank\ (A_0(x))$. *There exists a unimodular transformation* $U(x)$ *in* $GL_n(\mathbb{C}[[x]])$ *such that the leading matrix coefficient* $\tilde{A}_0(x)$ *of the equivalent system is given by*

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial G = \tilde{A}(x,\varepsilon)G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x_A^\sigma\varepsilon)^k\ G,$$

and $\tilde{A}_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ *for all* $k \geq 0$*, and* $\tilde{A}_0(x)$ *has the form*

$$\tilde{A}_0(x) = \begin{bmatrix} \tilde{A}_0^{11}(x) & O \\ \tilde{A}_0^{21}(x) & O \end{bmatrix}$$

where $\tilde{A}_0^{11}(x)$ *is a square matrix of dimension* r *and* $\begin{bmatrix} \tilde{A}_0^{11}(x) \\ \tilde{A}_0^{21}(x) \end{bmatrix}$ *is a* $n \times r$ *matrix of full column rank* r*.*

We call the transformation of Lemma 5.1 *column reduction*. Remark that the form of $\tilde{A}_0$ is analogous to the CRV form of Definition 2.1.

Proof. By Remark 5.7, $\tilde{A}_0(x) = U^{-1}(x)A_0(x)U(x)$. Hence it suffices to search a similarity transformation $U(x)$. Since $\mathbb{C}[[x]]$ is a principal ideal domain (the ideals of $\mathbb{C}[[x]]$ are of the form $x^k\mathbb{C}[[x]]$), it is well known that one can construct unimodular transformations $Q(x), U(x)$ lying in $GL_n(\mathbb{C}[[x]])$ such that the matrix $Q(x)A_0(x)U(x)$ has the Smith normal form

$$Q(x)\ A_0(x)\ U(x) = \mathrm{diag}(x^{\beta_1}, \ldots, x^{\beta_r}, 0, \ldots, 0))$$

where $\det(U(0)) \neq 0$, $\det(Q(0)) \neq 0$, and $\beta_1, \ldots, \beta_r$ in $\mathbb{Z}$ with $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$.
It follows that we can compute a unimodular matrix $U(x)$ in $GL_n(\mathbb{C}[[x]])$ so that its last $n - r$ columns form a $\mathbb{C}[[x]]$-basis of ker $(A_0(x))$. ■

Remark 5.8

In practice, $U(x)$ *can be obtained, as suggested in* [26]*, by performing Gaussian elimination on the columns of* $A_0(x)$ *taking as pivots the elements of minimum valuation (order) in* x*.*

Hence, we can suppose without loss of generality that $A_0(x)$ is of the form

$$A_0(x) = \begin{bmatrix} A_0^{11}(x) & O \\ A_0^{21}(x) & O \end{bmatrix} \tag{5.17}$$

with r independent columns and $(n-r)$ zero columns. We partition $A_1(x)$ in accordance with $A_0(x)$, i.e. $A_1(x) = \begin{bmatrix} A_1^{11}(x) & A_1^{12}(x) \\ A_1^{21}(x) & A_1^{22}(x) \end{bmatrix}$, and consider

$$G_A(\lambda) = \begin{bmatrix} A_0^{11}(x) & A_1^{12}(x) \\ A_0^{21}(x) & A_1^{22}(x) + \lambda I_{n-r} \end{bmatrix}. \tag{5.18}$$

This consideration of $G_A(\lambda)$ gives an ε-rank reduction criterion equivalent to $\theta_A(\lambda)$ as demonstrated in the following lemma:

Lemma 5.2

Let $\theta_A(\lambda)$ *and* $G_A(\lambda)$ *be given by* (5.15) *and* (5.18) *respectively. Then,* $\det(G_A(\lambda))$ *vanishes identically in* λ *if and only if* $\theta_A(\lambda)$ *does.*

Proof. With ξ denoting $x^{\sigma_A}\varepsilon$, let $D(\xi) = \mathrm{diag}(\xi I_r, I_{n-r})$ where $r = rank\ (A_0(x))$. Then we can write $\xi^{-1}A(x,\xi) = ND^{-1}$ where $N := N(x,\xi) \in \mathcal{M}_n(\mathbb{C}[[x]][[\xi]])$ has no poles in ξ. Set $D_0 = D(0)$ and $N_0 = N(x,0)$. Then we have

$$\begin{aligned}\det(G_A(\lambda)) &= \det(N_0 + \lambda D_0) = \det(N + \lambda D)|_{\xi=0} \\ &= (\det(\frac{A(x,\xi)}{\xi} + \lambda I_n)\det(D))|_{\xi=0} \\ &= (\det(\frac{A_0(x)}{\xi} + A_1(x) + \lambda I_n)\,\xi^r)|_{\xi=0} = \theta_A(\lambda).\end{aligned}$$

■

We illustrate our progress with the following simple example.

| Example 5.6

Consider $\varepsilon^h \partial F = A(x,\varepsilon)F$ where $\sigma_A = 0$, $h > 1$, and

$$A(x,\varepsilon) = \begin{bmatrix} \varepsilon & -x^3\varepsilon & (1+x)\varepsilon & 0 \\ x^2 & x\varepsilon & 0 & -2x\varepsilon \\ -x & 0 & 0 & 2\varepsilon \\ 0 & 2 & 0 & \varepsilon^2 \end{bmatrix}.$$

Clearly, $A_0(x)$ is nilpotent of rank 2 and

$$G_A(\lambda) = \begin{bmatrix} 0 & 0 & x+1 & 0 \\ x^2 & 0 & 0 & -2x \\ -x & 0 & \lambda & 2 \\ 0 & 2 & 0 & \lambda \end{bmatrix}. \tag{5.19}$$

□

We then have the following proposition:

| Proposition 5.2

Consider the system $[A_{\varepsilon,\sigma_A}]$ *given by* (5.1)*:*

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h\, x^p\, \partial F = A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\, F,$$

and set $r = \ rank\ (A_0(x))$*. If* $m_\varepsilon(A) > 1$ *and* $\det(G_A(\lambda)) \equiv 0$ *then there exists a finite product of triangular matrices* $T(x) = P(I_n + Q(x))$ *where* P *is a permutation and* $\det T(x) = \pm 1$*, such that for the equivalent system*

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^h x^p \partial G \ = \ \tilde{A}(x,\varepsilon)\, G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_A}\varepsilon)^k\, G$$

we have $\tilde{A}_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ *for all* $k \geq 0$ *and:*

$$G_{\tilde{A}}(\lambda) = \begin{bmatrix} A_0^{11}(x) & U_1(x) & U_2(x) \\ V_1(x) & W_1(x) + \lambda I_{n-r-\varrho} & W_2(x) \\ M_1(x) & M_2(x) & W_3(x) + \lambda I_\varrho \end{bmatrix}, \tag{5.20}$$

where $0 \leq \varrho \leq n-r$, W_1, W_3 *are square matrices of order* $(n-r-\varrho)$ *and* ϱ *respectively, and*

$$rank\,(\begin{bmatrix} A_0^{11}(x) & U_1(x) \\ M_1(x) & M_2(x) \end{bmatrix}) = rank\,(\begin{bmatrix} A_0^{11}(x) & U_1(x) \end{bmatrix}), \tag{5.21}$$

$$rank\,(\begin{bmatrix} A_0^{11}(x) & U_1(x) \end{bmatrix}) < r. \tag{5.22}$$

Our procedure differs from [15] in the properties of M_1, M_2, and W_3. The nullity of M_1 and M_2 in the former is replaced by the weaker condition (5.21) here. Otherwise, the unimodularity of $T(x)$ cannot be guaranteed. Moreover, this refinement avoids unnecessary computations. We establish Remark 5.9 which we will use in the proof of this Proposition.

| *Remark* 5.9

Suppose that $G_{[A_{\varepsilon,\sigma_A}]}(\lambda)$ *has the form* (5.20) *and there exists a transformation* $T(x) \in GL_n(\mathbb{C}((x)))$ *such that* $G_{T[A_{\varepsilon,\sigma_A}]}(\lambda)$ *has the form*

$$G_{T[A_{\varepsilon,\sigma_A}]}(\lambda) = \begin{bmatrix} A_0^{11} & U_1 & U_2 \\ V_1 & W_1 + \lambda I_{n-r-\varrho} & W_2 \\ O & O & \tilde{W}_3 + \lambda I_\varrho \end{bmatrix},$$

where $0 \leq \varrho \leq n-r$ *and* $\tilde{W}_3$ *is upper triangular with zero diagonal. Then,*

$$\det(G_{T[A_{\varepsilon,\sigma_A}]}(\lambda)) = \lambda^\varrho \det(\begin{bmatrix} A_0^{11} & U_1 \\ V_1 & W_1 + \lambda I_{n-r-\varrho} \end{bmatrix}).$$

If $\det(G_{[A_{\varepsilon,\sigma_A}]}(\lambda)) \equiv 0$ *then we have* $\det(G_{T[A_{\varepsilon,\sigma_A}]}(\lambda)) \equiv 0$ *as well (rank of leading matrix coefficient is unchanged). Hence,*

$$\det(\begin{bmatrix} A_0^{11} & U_1 \\ V_1 & W_1 + \lambda I_{n-r-\varrho} \end{bmatrix}) \equiv 0. \tag{5.23}$$

For a fixed $\varrho \in \{0, \ldots, n-r\}$ *we shall denote by* $G_0^{(\varrho)}$ *the matrix* $\begin{bmatrix} A_0^{11} & U_1 \\ V_1 & W_1 \end{bmatrix}$ *of* (5.20).

Proof. (Proposition 5.2) Since $\det(G_A(\lambda)) \equiv 0$ then in particular, the matrix $G_A(\lambda = 0)$ is singular. Let E_1 (respectively E_2) be the vector space spanned by the first r (resp. last $n-r$) rows of $G_A(\lambda = 0)$. We have

$$\dim(E_1 + E_2) = rank(G_A(\lambda = 0)) < n.$$

If $\dim(E_1) < r$ then setting $\varrho = 0$ suffices to fulfill our claim. Otherwise, since

$$\dim(E_1 + E_2) = \dim(E_1) + \dim(E_2) - \dim(E_1 \cap E_2) < n,$$

it follows that either $\dim(E_2) < n - r$ or $\dim(E_1 \cap E_2) > 0$. In both cases, there exists at least a row vector $\varpi^{(1)}(x) = (\varpi_1^{(1)}(x), \ldots, \varpi_n^{(1)}(x))$ with entries in $\mathbb{C}((x))$ in the left null space of $G_A(\lambda = 0)$, such that $\varpi_j^{(1)}(x) \neq 0$ for some $r + 1 \leq j \leq n$. We can assume without loss of generality that $\varpi^{(1)}(x)$ has its entries in $\mathbb{C}[[x]]$. Indeed, this assumption can be guaranteed by a construction as in Remark 5.8. Let the constant matrix $P^{(1)}$ denote the product of permutation matrices which exchange the rows of $A(x, \varepsilon)$, so that $val_x(\varpi_n^{(1)}(x)) < val_x(\varpi_j^{(1)}(x))$, $r + 1 \leq j \leq n - 1$, where val_x denotes the x-adic valuation (order in x). Let

$$Q^{(1)}(x) = [q_{ij}^{(1)}(x)]_{1 \leq i,j \leq n}\ , \quad s.t. \begin{cases} q_{nj}^{(1)}(x) = -\frac{\varpi_j^{(1)}(x)}{\varpi_n^{(1)}(x)}\ , & \text{for } r + 1 \leq j < n \\ q_{ij}^{(1)} = 0 \quad \text{elsewhere} \end{cases}$$

Thus, $P^{(1)}(I_n + Q^{(1)}(x))$ is unimodular in $GL_n(\mathbb{C}[[x]])$.
Set $F = F^0$, $A(x, \varepsilon) = A^{(0)}(x, \varepsilon)$, and let $A^{(1)}(x, \varepsilon)$ be the matrix of the equivalent system $\varepsilon^h x^p \partial F^{(1)} = A^{(1)}(x, \varepsilon) F^{(1)}$ obtained by the transformation

$$F^{(0)} = P^{(1)}(I_n + Q^{(1)}(x))\ F^{(1)}.$$

Thus, by Remark (5.7), $G_{A^{(1)}}(\lambda)$ has the form (5.20) with (5.21) and $\varrho = 1$.
By Remark 5.9, the matrix $G_{A^{(1)}}(\lambda = 0)$ is singular and the condition (5.22) does not occur, then one can find, by the same argument as above a permutation matrix and a nozero vector $\varpi^{(2)}(x)$ in the left null space of $G^{(1)}(\lambda = 0)$. Let

$$Q^{(2)}(x) = [q_{ij}^{(2)}(x)]_{1 \leq i,j \leq n}\ , \quad s.t. \begin{cases} q_{n-1,j}^{(2)}(x) = -\frac{\varpi_j^{(2)}(x)}{\varpi_{n-1}^{(2)}(x)}\ , & \text{for } r + 1 \leq j < n - 1 \\ q_{ij}^{(2)} = 0 \quad \text{elsewhere} \end{cases}$$

The matrix $G_{A^{(2)}}(\lambda)$ is then of the form (5.20) with (5.21) and $\varrho = 2$.
Consider the finite sequence of equivalent systems obtained by the transformation

$$F^{(s-1)} = P^{(s)}(I_n + Q^{(s)}(x))\ F^{(s)}$$

where $1 \leq s \leq \varrho$ and

$$Q^{(s)}(x) = [q_{ij}^{(s)}(x)]_{1 \leq i,j \leq n}\ , \quad s.t. \begin{cases} q_{n-s+1,j}^{(s)}(x) = -\frac{\varpi_j^{(s)}(x)}{\varpi_{n-s+1}^{(s)}(x)}\ , & \text{for } r + 1 \leq j < n - s + 1 \\ q_{ij}^{(s)} = 0 \quad \text{elsewhere.} \end{cases}$$

Then this process yields an equivalent matrix $\tilde{A}(x, \varepsilon) := A^{(\varrho)}(x, \varepsilon)$ with (5.21) for which either (5.22) occurs or $\varrho = n - r$. But in the latter case one has, again by Remark 5.9, that $\det(A_0^{11}(x)) = 0$, and so (5.22) occurs. It is easy to observe that, under the assumption $p - \sigma_A h - 1 \geq 0$, $\sigma_{\tilde{A}} = \sigma_{A^{(s)}} = \sigma_A$. ■

| Example 5.7 (Continue Example 5.6)

A simple calculation shows that $\det(G_A(\lambda)) \equiv 0$ hence A is ε-reducible. From (5.19), for $\lambda = 0$, we have the singular matrix

$$G_A(\lambda = 0) = \begin{bmatrix} 0 & 0 & x+1 & 0 \\ x^2 & 0 & 0 & -2x \\ -x & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{bmatrix}. \text{ Let } T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

then the transformation $F = TG$ yields the equivalent system $\varepsilon^h \partial G = \tilde{A}(x, \varepsilon)G$ where

$$\tilde{A}(x, \xi) = \begin{bmatrix} \varepsilon & -x^3\varepsilon & 0 & (1+x)\varepsilon \\ x^2 & x\varepsilon & -2x\varepsilon & 0 \\ 0 & 2 & \varepsilon^2 & 0 \\ -x & 0 & 2\varepsilon & 0 \end{bmatrix}.$$

$G_{\tilde{A}}(\lambda)$ has the form (5.20) with $\varrho = 1$ and $r = 2$. In fact,

$$G_{\tilde{A}}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & (1+x) \\ x^2 & 0 & 0 & 0 \\ 0 & 2 & \lambda & 0 \\ -x & 0 & 2 & \lambda \end{bmatrix}.$$

□

| Lemma 5.3

Consider the system $[A_{\varepsilon,\sigma_A}]$ *given by* (5.1)*:*

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h \, x^p \, \partial F = A(x, \varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k \, F.$$

Set $r = rank(A_0(x))$ *and suppose that* $m_\varepsilon(A) > 1$ *and* $G_A(\lambda)$ *has the form* (5.20) *with conditions* (5.21) *and* (5.22) *satisfied. Consider the shearing transformation* $S(x, \varepsilon) = \mathrm{diag}((x^{\sigma_A}\varepsilon)I_r, I_{n-r-\varrho}, (x^{\sigma_A}\varepsilon)I_\varrho)$ *if* $\varrho \neq 0$ *and* $S(x, \varepsilon) = \mathrm{diag}((x^{\sigma_A}\varepsilon)I_r, I_{n-r})$ *otherwise. Then* $F = S(x, \varepsilon)G$ *yields the equivalent system*

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon^h \, x^p \, \partial G = \tilde{A}(x, \varepsilon)G = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_A}\varepsilon)^k \, G,$$

for which $\tilde{A}_k(x) \in \mathcal{M}_n(\mathbb{C}[[x]])$ *for all* $k \geq 0$ *and* $rank(\tilde{A}_0(x)) < r$.

Proof. We partition $A(x,\varepsilon)$ as follows ((x,ε) dropped for clarity)

$$A(x,\varepsilon) = \begin{bmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{bmatrix}$$

where A^{11}, A^{22}, A^{33} are of dimensions $r, n-r-\varrho$, and ϱ respectively. It is easy to verify then that

$$\begin{aligned} \tilde{A}(x,\varepsilon) &= S^{-1}AS - S^{-1}\partial S \\ &= S^{-1}AS - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}\operatorname{diag}(\sigma I_r, O_{n-r-\varrho}, \sigma I_\varrho) \\ &= \begin{bmatrix} A^{11} - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}\sigma I_r & (x^{\sigma_A}\varepsilon)^{-1}A^{12} & A^{13} \\ (x^{\sigma_A}\varepsilon)A^{21} & A^{22} & (x^{\sigma_A}\varepsilon)A^{23} \\ A^{31} & (x^{\sigma_A}\varepsilon)^{-1}A^{32} & A^{33} - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}\sigma I_\varrho \end{bmatrix}. \end{aligned}$$

Hence, since $h, p > 1$, the new leading matrix coefficient is

$$\tilde{A}_0(x) = \begin{bmatrix} A_0^{11} & U_1 & O \\ O & O & O \\ M_1 & M_2 & O \end{bmatrix}$$

and $rank(\tilde{A}_0(x)) = rank(A_0^{11} \quad U_1) < r$. The rest of the properties can be easily verified due to the assumption $p - \sigma_A h - 1 \geq 0$. ■

| Example 5.8 (Continue Example 5.7)

Let $S(\varepsilon) = \operatorname{diag}(\varepsilon, \varepsilon, 1, \varepsilon)$ then $G = S(\varepsilon)U$ yields $\varepsilon^h \partial U = \tilde{\tilde{A}}(x,\varepsilon)U$ where

$$\tilde{\tilde{A}}(x,\varepsilon) = \begin{bmatrix} \varepsilon & -x^3\varepsilon & 0 & (1+x)\varepsilon \\ x^2 & x\varepsilon & -2x & 0 \\ 0 & 2\varepsilon & \varepsilon^2 & 0 \\ -x & 0 & 2 & 0 \end{bmatrix}.$$

It is clear that the leading term $\tilde{\tilde{A}}_0(x)$ has rank $1 < 2 = rank(A_0(x))$.

□

Proof. (Theorem 5.3) By Lemma 5.1, we can assume that $A_0(x)$ is in the form (5.17). Then, $G_A(\lambda)$ is constructed as in (5.18). Since $\det(G_A(\lambda)) \equiv 0$, it suffices to take the change of basis $F = R(x,\varepsilon)G = T(x,\varepsilon)S(x,\varepsilon)G$, where $T(x,\varepsilon)$ and $S(x,\varepsilon)$ are as in Propositions 5.2 and Lemma 5.3 respectively. ■

| Remark 5.10

The ε-reducibility of $[A_{\varepsilon,\sigma_A}]$ implies that the rank of the leading matrix coefficient can be dropped (and consequently the ε-Moser rank) without necessarily reducing the ε-rank h of the system. If the ε-rank reduction criterion is satisfied for a sufficient number of equivalent systems then a repetitive application of such a transformation results in an equivalent system whose leading matrix coefficient

has a zero rank, hence h can be reduced by one (e.g. Example 5.9). At this point, the discussion restarts from the nature of the eigenvalues of the leading constant matrix.

| Example 5.9

Consider the system $\varepsilon^4\partial F = A(x,\varepsilon)F$ where $\sigma_A = 0$ and

$$A(x,\varepsilon) = \begin{bmatrix} 2x\varepsilon^3 & 3x^2\varepsilon^4 & 2x\varepsilon^2 & (2x+1)\varepsilon^5 \\ 0 & \varepsilon^4 & 0 & 0 \\ 0 & 0 & \varepsilon^2 & 0 \\ -2x & 0 & 0 & 0 \end{bmatrix}.$$

The transformation $F = TG$, where

$$T = \begin{bmatrix} 0 & \varepsilon^3 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ \varepsilon^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

computed by our algorithm results in an equivalent ε-irreducible system $\varepsilon^2\partial G = \tilde{A}(x,\varepsilon)G$ where

$$\tilde{A}(x,\varepsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x & 2x\varepsilon & 3x^2 & 2x+1 \\ 0 & 0 & \varepsilon^2 & 0 \\ 0 & -2x\varepsilon & 0 & 0 \end{bmatrix},$$

whose ε-rank is diminished by two .

□

5.5.2 ε-Reduction, $h = 1$

It is left now to treat the case of $h = 1$. Let h_{true} denote the true ε-rank, i.e. the minimal integer value that can be attained for h upon applying an invertible transformation.

| Proposition 5.3

Consider the system $[A_{\varepsilon,\sigma_A}]$ given by (5.1) for which $h = 1$:

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon x^p\partial F = A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\ F$$

such that $m_\varepsilon(A) > 1$. If by a ramification $\varepsilon = \tilde{\varepsilon}^s$ where $s \geq n+1$ and $\tilde{\varepsilon}$-rank reduction of Subsection 5.5.1, the equivalent system is of the form

$$[\tilde{A}_{\tilde{\varepsilon},\sigma_{\tilde{A}}}] \qquad \tilde{\varepsilon}x^p\partial F = \tilde{A}(x,\tilde{\varepsilon})F = \sum_{k=0}^{\infty} \tilde{A}_k(x)(x^{\sigma_{\tilde{A}}}\tilde{\varepsilon})^k\ F,$$

then, the singularity in ε is apparent, i.e. $h_{true} = 0$.

Proof. Let $\omega_\varepsilon(A) = \frac{\ell}{d}$ where ℓ and d are coprime. Then, $d, \ell \geq 0$, $d < n$ and $\omega_\varepsilon(A) \leq h_{true}$ (see Corollary 5.2). Let $\varepsilon = \tilde{\varepsilon}^s$ then $\omega_{\tilde{\varepsilon}}(\tilde{A}) = s\omega_\varepsilon(A)$ and $\omega_{\tilde{\varepsilon}}(\tilde{A}) \leq \tilde{h}_{true}$. Hence, if $s \geq n+1$, we have,

$$\omega_{\tilde{\varepsilon}}(\tilde{A}) = s\omega_\varepsilon(A) = s\frac{\ell}{d} \geq (n+1)\frac{\ell}{n} \geq (1+\frac{1}{n})\ell.$$

Hence, if $\ell \neq 0$ then

$$1 < (1+\frac{1}{n})\ell \leq \omega_{\tilde{\varepsilon}}(\tilde{A}) \leq \tilde{h}_{true}.$$

■

| *Remark* 5.11

Proposition 5.3 lifts the problem from $h = 1$ to $h > 1$. It can be restated as follows: Within the formal reduction, whenever the case $h = 1$ is attained, a ramification $\varepsilon = \tilde{\varepsilon}^s$ where s is an integer greater than $n+1$, is introduced so that the reduction of Subsection 5.5.1 can be applied. After this reduction, two cases arise:

- *If the ε-rank of the resulting system is less than or equal to one then h_{true} of the original system is zero, and we stop.*
- *Otherwise, we proceed with our reduction.*

The case $h_{true} = 0$ can be treated with classical Arnold-Wasow approach. That is, the system is put in Arnold's form first and the (x, ε)-polygon can be constructed to determine the proper shearing[¶].

| **Example 5.10**

Consider the first resulting subsystem of Example 5.5:

$$\varepsilon^2 x^{-1}\partial W_1 = \{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}-1 & 0\\ -1 & 0\end{bmatrix}(x^{-3}\varepsilon) + \begin{bmatrix}1 & -1\\ 1 & -1+x^4\end{bmatrix}(x^{-3}\varepsilon)^2 + O((x^{-3}\varepsilon)^3)\ \}W_1.$$

It's easy to see that the leading coefficient and constant matrices coincide and $\theta(\lambda) = 1$.

□

To proceed in the formal reduction of Example 5.10, one needs to introduce a ramification in ε. This is the objective of Chapter 6. A major constituent of the proofs established in the latter is the ε-Moser invariant of a scalar equations, is introduced in Section 5.6.

[¶] We remark that in the implementation, we first try to find whether there exist constant vectors in the left null space of $G_A(\lambda)$. If such vectors do exist, we use them to construct a transformation which might reduce the rank. If the ε-rank remains one after all the constant vectors are exhausted, we compute a candidate for the ramification in ξ from the characteristic polynomial. If this candidate, after applying again the ε-rank reduction results in a system with non-nilpotent leading coefficient, we keep it. Otherwise, we use Arnold-Wasow approach.

Algorithm 4 ε-RANK_REDUCTION: ε-Rank Reduction of System $[A_{\varepsilon,\sigma_A}]$

Input: $h,\ p,\ A(x,\varepsilon)$ of $[A_{\varepsilon,\sigma_A}]$ (under the non-restrictive assumptions indicated within reduction)
Output: $R(x,\varepsilon) \in GL_n(\mathrm{R}_{((\varepsilon))})$ and an equivalent system given by $\tilde{h}$ and $\tilde{A}(x,\varepsilon)$ which is ε-irreducible.

$R \leftarrow I_n$;
$h \leftarrow \varepsilon$-rank of A;
$U(x) \leftarrow$ Lemma 5.1 so that $U^{-1}A_0(x)U$ has form (5.17);
$R \leftarrow RU$;
$A \leftarrow U^{-1}AU - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}U^{-1}\partial U$;
$d = \det(G_\lambda(A))$;
while $d = 0$ and $h > 0$ **do**
  if $h > 1$ **then**
    $T(x), \varrho \leftarrow$ Proposition 5.2 ;
    $A \leftarrow T^{-1}AT - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}T^{-1}\partial T$;
    $S(x,\varepsilon) \leftarrow$ Lemma 5.3;
    $A \leftarrow S^{-1}AS - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}S^{-1}\partial S$;
    $R \leftarrow RTS$;
  else
    $A \leftarrow$ Proposition 5.3;
  end if
  $U(x) \leftarrow$ Lemma 5.1 ;
  $R \leftarrow RU$;
  $A \leftarrow U^{-1}AU - (x^{\sigma_A}\varepsilon)^h x^{p-\sigma_A h-1}U^{-1}\partial U$;
  $d = \det(G_\lambda(A))$;
  $h \leftarrow \varepsilon$-rank of A;
end while
return (R, A, h).

5.6 ε-Moser invariant of a scalar equation

Due to the equivalence between the scalar equation $[a_\varepsilon]$ and system $[A_\varepsilon]$ (resp. $[A_{\varepsilon,\sigma_A}]$) as discussed in Section 5.2.2, it is natural to define and question the ε-Moser invariant of the former. This is the goal of this section which is fulfilled by generalizing the analogous notion discussed for unperturbed scalar linear differential equations in [107, Part IV]. Hence, we consider again the singularly-perturbed linear differential equations given by

$$[a_{\varepsilon,\sigma_a}] \qquad \partial^n f + a_{n-1}(x,\varepsilon)\,\partial^{n-1} f + \ldots + a_1(x,\varepsilon)\,\partial f + a_0(x,\varepsilon)\,f = 0, \tag{5.24}$$

and, in accordance with the case of systems, we consider the more general entries as follows: $a_n(x,\varepsilon) = 1$ and $a_i(x,\varepsilon) \in \mathrm{R}_{((\varepsilon))}$. Thus, there exists a nonpositive integer σ_a such that for every $i \in \{0,\ldots,n-1\}$, $a_i(x,\varepsilon)$ can be expressed as

$$a_i(x,\varepsilon) = \sum_{k\in\mathbb{Z}}^{\infty} a_{i,k}(x)(x^{\sigma_a}\varepsilon)^k, \quad \text{with} \quad a_{i,k}(x) \in x^p\mathbb{C}[[x]] \quad \text{for all} \quad k \in \mathbb{Z}.$$

| Proposition 5.4

Given the differential equation $[a_{\varepsilon,\sigma_a}]$. Let κ,ν be the smallest integers such that $val_\varepsilon\ (a_i(x,\varepsilon)) \geq (i-n)(\kappa-1)-\nu$. In other words, let

$$\kappa = min\ \{\varrho \in \mathbb{N} |\ val_\varepsilon\ (a_i) + (n-i)\varrho \geq 0,\ 0 \leq i \leq n\} \tag{5.25}$$

$$\nu = max\ \{(i-n)(\kappa-1) - val_\varepsilon\ (a_i),\ 0 \leq i \leq n\} \tag{5.26}$$

Then the ε-Moser invariant of the system given by the associated companion matrix is

$$\mu_\varepsilon(a) = \kappa + \frac{\nu}{n}.$$

| Remark 5.12 *(Geometric interpretation)*

Consider again $\mathcal{N}_\varepsilon(a)$, the ε-polygon (Section 4.1) in a (U,V)-plane. First, one can construct the straight line passing through the point $(n,0)$, with the smallest integer slope κ, which stays below $\mathcal{N}_\varepsilon(a)$. It is the straight line of equation $V = \kappa(U-n)$. Then, one finds among all the parallel lines of slope $(\kappa-1)$, the highest straight line with an integer V-intercept which stays below $\mathcal{N}_\varepsilon(a)$. The V-intercept of the latter is $-\nu - n(\kappa-1)$ where ν is an integer. In other words, the latter has the equation $V = (U-n)(\kappa-1)-\nu$.

Proof. (Proposition 5.4) We define

$$\gamma_i = max\ \{\kappa(i-n), (\kappa-1)(i-n)-\nu\},\ 0 \leq i \leq n-1.$$

Then, by construction, $\gamma_i \leq val_\varepsilon\ (a_i)$ and equality is attained for at least one $i \geq n-\nu$ (otherwise ν and κ can be minimized). Let i_0 represent the smallest integer such that $\gamma_{i_0} = \kappa(i_0 - n)$, then $i_0 = n - \nu$. Geometrically, the γ_i's represent a broken line dominated by the $(i, val_\varepsilon\ (a_i))$. They will aid in the construction of an ε-irreducible system which is equivalent to (5.24). In fact, let

$$w_{i+1} = (x^{\sigma_a}\varepsilon)^{\gamma_i}\ \partial^i f, \quad 0 \leq i \leq n-1.$$

then we have

$$\begin{cases} \partial w_{i+1} = (x^{\sigma_a}\varepsilon)^{-\kappa}\ [\begin{cases} (x^{\sigma_a}\varepsilon)w_{i+2} & 0 \leq i < i_0 \\ w_{i+2} & i_0 \leq i \leq n-2 \end{cases} + \ (x^{\sigma_a}\varepsilon)^{\kappa}\frac{\sigma_a\gamma_i}{x}\ w_{i+1}], \\ w_n = (x^{\sigma_a}\varepsilon)^{\gamma_{n-1}}\ \partial^{n-1}f = (x^{\sigma_a}\varepsilon)^{-\kappa}\ \partial^{n-1}f, \\ \partial w_n = \frac{\sigma_a\gamma_{n-1}}{x}\ w_n + (x^{\sigma_a}\varepsilon)^{-\kappa}\partial^n f = (x^{\sigma_a}\varepsilon)^{-\kappa}[(x^{\sigma_a}\varepsilon)^{\kappa}\ \frac{\sigma_a\gamma_{n-1}}{x}\ w_n + \sum_{i=0}^{n-1}\ \alpha_i(x,\varepsilon)w_{i+1}]. \end{cases}$$

where $\alpha_i(x,\varepsilon) = -a_i(x,\varepsilon)(x^{\sigma_a}\varepsilon)^{-\gamma_i} = \sum_{k\in\mathbb{Z}}^{\infty} a_{i,k}(x)(x^{\sigma_a}\varepsilon)^{k-\gamma_i}$. Let $W = (w_1, \dots w_n)^T$ and $A(x,\varepsilon) =$

$$\begin{bmatrix} \sigma_a\gamma_0(x^{\sigma_a}\varepsilon)^{\kappa} & x(x^{\sigma_a}\varepsilon) & & & & & \\ & \ddots & & & & & \\ & & \sigma_a\gamma_{i_0-1}(x^{\sigma_a}\varepsilon)^{\kappa} & x(x^{\sigma_a}\varepsilon) & & & \\ & & & \sigma_a\gamma_{i_0}(x^{\sigma_a}\varepsilon)^{\kappa} & x & & \\ & & & & \sigma_a\gamma_{i_0+1}(x^{\sigma_a}\varepsilon)^{\kappa} & & \\ & & & & & & x \\ x\alpha_0 & x\alpha_1 & \dots & x\alpha_{i_0} & x\alpha_{i_0+1} & \dots & x\alpha_{n-1} + \sigma_a\gamma_{n-1}(x^{\sigma_a}\varepsilon)^{\kappa} \end{bmatrix}.$$

Then one can verify that,

$$[A_{\varepsilon,\sigma_a}] \qquad x^{\sigma_a\kappa+1}\varepsilon^{\kappa}\partial W = A(x,\varepsilon)W.$$

It remains to prove that this system is ε-irreducible. We first remark that $A(x,0)$ has rank $\nu = n - i_0$ due to the linear independence of its last ν rows. In fact, it's clear that it has $\nu - 1$ linearly independent rows (the rows with the x's). Moreover, $\alpha_i(x,\varepsilon) = -a_i(x,\varepsilon)(x^{\sigma_a}\varepsilon)^{-\gamma_i} = -a_i(x,\varepsilon)(x^{\sigma_a}\varepsilon)^{-val_\varepsilon(a_i)}$, and so $\alpha_i(x,0) \neq 0$ for at least one $i \in \{0,\dots,i_0\}$. Thus the last ν rows are linearly independent. Setting

$\xi = x^{\sigma_a}\varepsilon$ we have:

$$\begin{aligned}
\theta_A \quad (\lambda) &= \xi^{\nu} \det(\lambda I + \frac{A_0(x)}{\xi} + A_1(x))|_{\xi=0} \\
&= \xi^{\nu} \begin{vmatrix}
\lambda+\delta(\sigma_a\gamma_0) & x & & & & & & \\
& \ddots & & & & & & \\
& & \lambda+\delta(\sigma_a\gamma_{i_0-1}) & x & & & & \\
& & & \lambda+\delta(\sigma_a\gamma_{i_0}) & \frac{x}{\xi} & & & \\
& & & & \lambda+\delta(\sigma_a\gamma_{i_0+1}) & & & \\
& & & & & & & \frac{x}{\xi} \\
\frac{x\alpha_0}{\xi} & \frac{x\alpha_1}{\xi} & \dots & \frac{x\alpha_{i_0}}{\xi} & \frac{x\alpha_{i_0+1}}{\xi} & \dots & \lambda+\delta(\sigma_a\gamma_{n-1})+\frac{x\alpha_{n-1}}{\xi}
\end{vmatrix}|_{\xi=0} \\
&= \xi^{\nu-n+i_0} \begin{vmatrix}
\lambda+\delta(\sigma_a\gamma_0) & x & & \\
& \ddots & & \\
& & \lambda+\delta(\sigma_a\gamma_{i_0-1}) & x \\
x\alpha_0 & x\alpha_1 & \dots & x\alpha_{i_0}
\end{vmatrix},
\end{aligned}$$

where $\delta = 1$ if $\kappa = 1$ and is zero otherwise.
Hence, since $\alpha_i(x,0) \neq 0$ for at least one $i \in \{0,\dots,i_0\}$, $\theta_A(\lambda)$ does not vanish identically in λ and its highest possible degree is i_0. It follows from Theorem 5.2 that the system is ε-irreducible and $\mu_\varepsilon(a) = \mu_\varepsilon(A) = \kappa + \frac{\nu}{n}$. ■

| Example 5.11

Let $\sigma_a = -1$ and set $\xi = x^{\sigma_a}\varepsilon$. We consider the scalar differential equation

$$\partial^5 f + (x\xi^{-3} + x)\partial^4 f + 2x^2\xi^{-1}\partial^3 f + (\xi^{-3}+1)\partial^2 f + (-3\xi^{-4} + x^2\xi^{-2})\partial f - \xi^{-2} f = 0.$$

First, we plot the points $(i, val_\varepsilon(a_i)), 0 \leq i \leq n = 5$ and the ε-polygon (see Figure 5.1). Next, we plot the straight line passing through the point $(n,0) = (5,0)$ with smallest integer slope such that it stays below all these point. This yields $V = 3U - 15 = 3(U-5)$ and $\tau = 3$. Then, we plot the straight line of slope $\tau - 1 = 2$ which stays below these points: $V = 2U - 11 = 2(U-5) - 1$. This yields $\nu = 1$. Finally, we consider $\gamma = (\gamma_0,\dots,\gamma_4) = (-11,-9,-7,-5,-3)$. We observe that $i_0 = n - \nu = 4$ and so the equivalent system is given by:

$$\xi^3\partial W = \begin{bmatrix}
\frac{11}{x}\xi^3 & \xi & 0 & 0 & 0 \\
0 & \frac{9}{x}\xi^3 & \xi & 0 & 0 \\
0 & 0 & \frac{7}{x}\xi^3 & \xi & 0 \\
0 & 0 & 0 & \frac{5}{x}\xi^3 & \xi \\
\xi^9 & 3\xi^5 - x^2\xi^7 & -\xi^4 - \xi^7 & -2x^2\xi^4 & -x - (x - \frac{3}{x})\xi^3
\end{bmatrix} W,$$

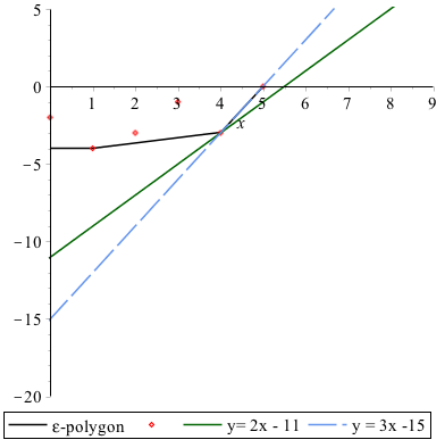


Figure 5.1: Geometric Interpretation

or equivalently

$$x^{-2}\varepsilon^3\partial W = \begin{bmatrix} 11\xi^3 & x\xi & 0 & 0 & 0 \\ 0 & 9\xi^3 & x\xi & 0 & 0 \\ 0 & 0 & 7\xi^3 & x\xi & 0 \\ 0 & 0 & 0 & 5\xi^3 & x\xi \\ x\xi^9 & 3x\xi^5 - x^3\xi^7 & -x\xi^4 - x\xi^7 & -2x^3\xi^4 & -x^2 - (x^2-3)\xi^3 \end{bmatrix} W.$$

And the system is ε-irreducible since $\theta_A(\lambda) = x\lambda^4$.

□

| Corollary 5.1

Consider an ε-irreducible system $[A_{\varepsilon,\sigma_A}]$ $\varepsilon^h x^p \partial F = A(x,\varepsilon)F$. Then, for any choice of a cyclic vector, we have $\kappa = h$ and $\nu = rank(A_0)$, where κ and ν are defined in (5.25) and (5.26) respectively.

Proof. Let $[C_{\varepsilon,\sigma_C}]$ $\varepsilon^{\tilde{h}} x^{\tilde{p}} \partial G = C(x,\varepsilon)G$ denote an equivalent system associated to a cyclic vector (i.e. whose matrix is in companion form, then by the equivalence between both systems and the ε-irreducibility of $A(x,\varepsilon)$, we have:

$$\kappa + \frac{\nu}{n} = \mu_\varepsilon(C) = \mu_\varepsilon(A) = m_\varepsilon(A) = h + \frac{r}{n}$$

Consequently, $\kappa = h$ and $\nu = rank(A_0)$. ■

| Corollary 5.2

Under the given and notations of Corollary 5.1, we have:

$$\kappa - 1 + \frac{\nu}{n} \leq \omega_\varepsilon(A) \leq \kappa \quad \text{and} \quad h - 1 + \frac{rank(A_0)}{n} \leq \omega_\varepsilon(A) \leq h.$$

Proof. Follows from (5.25), (5.26), and (5.4). ■

5.6.1 Example of comparison with Levelt-Barkatou-LeRoux's approach

While Moser defined two rational numbers in [107], Levelt investigated the existence of stationary sequences of free lattices in [87]. Since Moser-based and Levelt's algorithms serve the same utility, i.e. rank reduction of the unperturbed system $[A]$ given by (1), it is natural to question their comparison. These algorithms have different theoretical bases but turned out to have an identical cost in the univariate case, which suggests a further experimental study so that they can be well-compared [84, pp 108]. The latter algorithm was generalized in [26] to a certain class of Pfaffian systems (free of turning points) over bivariate fields. We refer to this generalization as Levelt-Barkatou-LeRoux's algorithm. Following arguments similar to those of [26] whenever the leading matrix coefficient is nilpotent and to Section 5.4 whenever it is not, Levelt-Barkatou-LeRoux algorithm seems adaptable to system $[A_{\varepsilon,\sigma_A}]$ as well. Such an attempt would give rise to Algorithm 5.
For the simplicity of the presentation, we assume that the input system has no turning point and so we give the Algorithm and illustrate by an example for $[A_{\varepsilon,\sigma_A}]$ with $\sigma_A = 0$.

Algorithm 5 PARAM_LATTICE_RANK_REDUCTION: Adaptation of Levelt-Barkatou-LeRoux Rank Reduction to System $[A_\varepsilon]$

Input: $A(x,\varepsilon)$ of system $[A_\varepsilon]$;
Output: $T(x,\varepsilon)$ computed via Levelt's approach and an equivalent system given by $T[A]$ which is ε-irreducible in the sense of Moser.

$T \leftarrow I_n$
$i \leftarrow 0$;
$h \leftarrow \varepsilon$-rank of A;
while $i < n-1$ and $h > 0$ **do** $r \leftarrow rank(A_0)$;
  $U(x) \leftarrow$ unimodular transformation of Lemma 5.1 such that $U^{-1}A_0U$ has the form (5.17);
  $S(x) \leftarrow \mathrm{diag}(\varepsilon I_r, I_{n-r})$;
  $P \leftarrow US$;
  $T \leftarrow TP$;
  $A \leftarrow P^{-1}AP - \varepsilon P^{-1}\partial P$;
  $\tilde{h} \leftarrow \varepsilon$-rank of A;
  if $\tilde{h} < h$ **then** then $i \leftarrow 0$;
  else $i \leftarrow i+1$;
  end if
  $h \leftarrow \tilde{h}$;
end while.
return (T, A).

It is clear that Algorithm 5 coincides with Algorithm 4 for $\varrho = 0$. For $\varrho > 0$, which frequently occurs for matrices of dimension greater than 3, we ran simple examples of systems $[A_{\varepsilon,\sigma_A}]$. Despite the identical cost of both algorithms, these examples exhibit that Algorithm 5 complicates dramatically the matrix coefficients of the system under reduction. One factor in this complication stems from the weak termination criterion of this algorithm. However, even upon adjoining Moser's termination criterion (given by $\theta(\lambda)$) to this algorithm, as suggested in [26, Section 5], the result remains less satisfying than

that of Algorithm 4. Another factor, is that Algorithm 5, contrary to Algorithm 4, demands unnecessary Gaussian eliminations which might explain this growth in the size of coefficients. In fact, one can observe that for $\varrho > 0$, Algorithm 5 applies, for every step in the iteration, Gaussian elimination on the leading matrix coefficient followed by a shearing transformation without any assurance that the rank r will drop. The latter is not guaranteed until $n-1$ steps are performed. On the other hand, for $\varrho > 0$, Algorithm 4 applies a "weak" Gaussian elimination on matrices of decreasing dimensions (see the proof of Proposition 5.2) so that the form (5.20) is attained. With this form, a shearing transformation guarantees a drop in the rank r of the leading matrix coefficient (Proposition 5.3). We exhibit here a selected example over $\mathbb{C}[[x]][[\varepsilon]]$. Additional examples are available for download with the package.

| Example 5.12

Let $\varepsilon^h \partial F = A(x,\varepsilon)F$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & \varepsilon x(\varepsilon x+3) & \varepsilon^3(x+9) & \varepsilon x^2 \\ x & 0 & 0 & 0 & 9\varepsilon^2 x^2 & 0 \\ x^2 & -1 & 0 & \varepsilon x(\varepsilon x^6+1) & 0 & 0 \\ x^3+x & -x & 0 & \varepsilon x^2 & 0 & 0 \\ 0 & 0 & x & -3\varepsilon & 0 & 0 \\ x^3-1 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easily verified that A is ε-reducible. Furthermore, we have $m_\varepsilon(A) = h + \frac{3}{6}$ and $\mu_\varepsilon(A) = h + \frac{2}{6}$. Hence, it suffices to run only one reduction step, for which the rank of the leading matrix coefficient is dropped by one. We give the transformation $T(x,\varepsilon)$ and the matrix of $T[A]$ as computed by Algorithm 4, Algorithm 5 with Moser's criterion adjoined, and Algorithm 5 without Moser's criterion respectively. We illustrate the dramatic growth of their coefficients by listing some entries due to the lack of space. The full computations can be found within the package.

- $T = \begin{bmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \end{bmatrix}$, $T[A] = \begin{bmatrix} 0 & 0 & 0 & \varepsilon^2(x+9) & x^2\varepsilon & \varepsilon x(\varepsilon x+3) \\ x & 0 & 0 & 9x^2\varepsilon & 0 & 0 \\ x^2 & -1 & 0 & 0 & 0 & \varepsilon x(\varepsilon x^6+1) \\ 0 & 0 & \varepsilon x & 0 & 0 & -3\varepsilon^2 \\ x^3-1 & 5 & 0 & 0 & 0 & 0 \\ x(x^2+1) & -x & 0 & 0 & 0 & x^2\varepsilon \end{bmatrix}$;
- $T = \begin{bmatrix} \varepsilon^3 & 0 & 0 & 0 & 0 & -9x\varepsilon^2 \\ 0 & 0 & -1/3x^2\varepsilon^2 & \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon^2 & -1/3\varepsilon x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 \end{bmatrix}$ and $T[A] = [\tilde{a}_{ij}]$ has entries with 2-digit coefficients of degree 8 in x, e.g., $\tilde{a}_{12} = -x(27\varepsilon^2 - \varepsilon x - 3)$ and $\tilde{a}_{53} = -(1/3)\varepsilon^2 x^8$.

- $T = [t_{ij}]$ and $T[A] = [\tilde{a}_{ij}]$ have entries with 4-digit and 10-digit coefficients resp. The degrees in x surpass 8, e.g.,

$$t_{23} = \frac{1458\varepsilon^2 x^5 (x^3 + 3x^2 + 2)^2}{243x^{11} + 1944x^{10} + 3627x^9 + \cdots - 324}$$

$$\begin{aligned} \tilde{a}_{13} &= (39366\varepsilon x^{25} + 1220346\varepsilon x^{24} + 14565420\varepsilon x^{23} \\ &+ 83731482\varepsilon x^{22} + \cdots + 944784)/(243x^{11} \\ &+ 1944x^{10} + 3627x^9 + \cdots + 324)^2. \end{aligned}$$

□

5.7 Conclusion

We proposed in this chapter an algorithm which treats the turning points of a singularly-perturbed linear differential systems in a general setting and reduces its ε-rank to its minimal nonzero integer value. A complementary step to attain the full formal reduction and construct formal solutions is to find the ramification in the parameter which reduces the general case to the case discussed here in a recursive process. This will be the subject of the next chapter. It is also left to reduce a system with $h = 1$ and $h_{true} = 0$ to an equivalent one with $h = 0$, without resorting to Arnold-Wasow form.

Examples comparing this algorithm with a generalization of Levelt's favors the former. However, it suggests that Levelt's algorithm be generalized to system $[A_{\varepsilon,\sigma_A}]$ and that a detailed complexity analysis comparing both algorithms be held alongside. Furthermore, it motivates the consideration of algorithms based on a generalization of Moser's criterion over other differential bivariate fields. This is to be investigated for bivariate completely integrable Pfaffian systems with normal crossings in Chapter 8.

An additional field of investigation is tackling the two-parameter algebraic eigenvalue problem via a differential-like approach. Such investigation is held for the one-parameter case in Chapter 2 and references therein. In fact, by Remark 5.7, the main role in the reduction process is reserved to the similarity term of $T[A_{\varepsilon,\sigma_A}]$, i.e. $T^{-1}AT$ rather than $T^{-1}\partial T$. Hence, for a non-differential operator where $T[A_{\varepsilon,\sigma_A}] = T^{-1}AT$, the discussion is not expected to deviate from the discussion presented here for a differential one.

Chapter 6

Computing the ε-Formal Invariants

Contents

We consider again system $[A_{\varepsilon,\sigma_A}]$ which is given by (5.1) and which we recall here:

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^h\, x^p\, \partial\, F \;=\; A(x,\varepsilon)F = \sum_{k=0}^{\infty} A_k(x)(x^{\sigma_A}\varepsilon)^k\, F. \tag{6.1}$$

In this chapter, we adopt without loss of generality the assumptions of Section 5.1 and the additional assumption that $[A_{\varepsilon,\sigma_A}]$ is ε-irreducible and $h > 0$. The goal of this chapter is to investigate the ε-formal exponential order (ε-exponential order, in short) and ε-polynomials of system (6.1).

As for system $[A]$ given by (1) in the first chapter, one cannot retrieve $\omega_\varepsilon(A)$ of system $[A_{\varepsilon,\sigma_A}]$ from the outset. In Section 6.1, we prove that $\omega_\varepsilon(A)$ can be computed from the characteristic polynomial of $\frac{A(x,\varepsilon)}{\varepsilon^h x^p}$ and an associated ε-polygon whenever the system is ε-irreducible.

The $\omega_\varepsilon(A)$ and the corresponding ε-polynomial give indispensable information for formal reduction. In particular, the former determines the ramification in ε which can lead to a system whose leading matrix is non-nilpotent, so that the process of reduction can be resumed. This leads to the recursive Algorithm 6. Eventually, we can construct fundamental matrices of formal solutions in any given subdomain. We illustrate this algorithm by examples and motivate items for further investigation in Section 6.3.

6.1 Computing ε-exponential order using ε-polygon

Consider the ε-irreducible system $[A_{\varepsilon,\sigma_A}]$ given by (6.1) with $h > 0$ and let

$$\det(\lambda I - \frac{A(x,\varepsilon)}{\varepsilon^h x^p}) = \lambda^n + \alpha_{n-1}(x,\varepsilon)\lambda_{n-1} + \cdots + \alpha_0(x,\varepsilon). \tag{6.2}$$

such that $\alpha_n = 1$ and $\alpha_i(x,\varepsilon) = \sum_{j=val_\varepsilon(\alpha_i)}^{\infty} \alpha_{i,j}(x)\varepsilon^j \in \mathrm{R}_{((\varepsilon))}$ for $i \in \{0,\ldots,n\}$. We define the ε-polygon $\mathcal{N}_\varepsilon(A)$ of $[A_{\varepsilon,\sigma_A}]$ as in Section 4.1, by taking $P_\varepsilon(A)$ to be the union of $P(i,\ val_\varepsilon\ (\alpha_i(x,\varepsilon))$ for $i \in \{1,\ldots,n\}$.

| Theorem 6.1

Consider the ε-irreducible system $[A_{\varepsilon,\sigma_A}]$ given by (6.1) with $h > 0$ and (6.2). If $h > n - rank(A_0(x))$ then the ε-formal exponential order is given by

$$\omega_\varepsilon(A) = \max_{0\le i<n} \left(\frac{-val_\varepsilon(\alpha_i)}{n-i}\right).$$

Additionally, the corresponding ε-polynomial is given by the algebraic equation

$$E_\varepsilon(X) = \sum_{k=0}^{\ell} \alpha_{i_k, val_\varepsilon(\alpha_{i_k})}\ X^{(i_k - i_0)}$$

where $0 \le i_0 < i_1 < \cdots < i_\ell = n$ denote the integers i for which $\omega_\varepsilon(n-i) = -val_\varepsilon(\alpha_i)$ (i.e. lie on the edge of slope ω_ε of the ε-polygon $\mathcal{N}_\varepsilon(A)$ of $[A_{\varepsilon,\sigma_A}]$); and $\alpha_{i,val_\varepsilon(\alpha_i)}(x) = \varepsilon^{-val_\varepsilon\ (\alpha_i)}\ \alpha_i(x,\varepsilon)|_{\varepsilon=0}$.

Not only does this theorem compute these invariants of the system, but also a further reduction of the system as follows: Suppose that $\omega_\varepsilon(A) = \frac{\ell}{d}$ with ℓ, d relatively prime positive integers. One can then set $\tilde{\varepsilon} = \varepsilon^{1/d}$ in $[A_{\varepsilon,\sigma_A}]$ and perform again ε-rank reduction (the true ε-rank is ℓ). This will lead to an equivalent system whose leading matrix coefficient has at least d distinct eigenvalues. The system can thus be uncoupled into subsystems of lower dimensions. By repeating these procedures for each of the resulting subsystems, we can uncouple the initial system into subsystem(s) of lower dimension(s) or zero ε-rank. This leads to the recursive algorithm of Section 6.2.
We remark that the condition $h > n - rank(A_0(x))$ in Theorem 6.1 is non-restrictive. We can always arrive to a system satisfying this condition by a generalization of [14, Lemma 5], which is based on applying the affinity $(U,V) \to (U, d\ V)$ for some integer d to the ε-polygon. In fact, we can prove the following lemma:

| Lemma 6.1

Consider an ε-irreducible system $[A_{\varepsilon,\sigma_A}]$ given by (6.1) with $h > 0$ and $r = rank\ (A_0(x))$. Let d be an integer such that $d \ge \frac{n}{h-1+\frac{r}{n}}$ and let

$$\tilde{\varepsilon}^{\tilde{h}} x^{\tilde{p}} \partial G = \tilde{A}(x,\tilde{\varepsilon}) G.$$

be the $\tilde{\varepsilon}$-irreducible differential system obtained by the ramification $\varepsilon = \tilde{\varepsilon}^d$ and performing $\tilde{\varepsilon}$-rank reduction. Then $\mu_{\tilde{\varepsilon}}(\tilde{A}) = \tilde{h} + \frac{\tilde{r}}{n}$, where $\tilde{r} = rank\ (\tilde{A}_0(x))$ and $\tilde{h} + \tilde{r} > n$.

Proof. By Corollary 5.2 we have, $h - 1 + \frac{r}{n} \le \omega_\varepsilon(A) \le h$. And due to $\varepsilon = \tilde{\varepsilon}^d$, we have $\omega_\varepsilon(A) = \frac{\omega_{\tilde{\varepsilon}}(\tilde{A})}{d}$. Hence, $\omega_{\tilde{\varepsilon}}(\tilde{A}) \ge d(h - 1 + \frac{r}{n}) \ge n$. But $\tilde{h} \ge \omega_{\tilde{\varepsilon}}(\tilde{A})$ and $\tilde{r} \ge 1$, which yields $\tilde{h} + \tilde{r} \ge \omega_{\tilde{\varepsilon}}(\tilde{A}) + 1 \ge d(h - 1 + \frac{r}{n}) + 1 \ge n + 1 > n$. ■

| Example 6.1

Consider the system $[A_{\varepsilon,\sigma_A}] \quad \varepsilon^2\partial F = A(x,\varepsilon)F$ with $\sigma_A = 0$ and

$$A(x,\varepsilon) = \begin{bmatrix} 0 & 0 & \epsilon^2 & 0 & \epsilon & x & 0 \\ \epsilon^2 & \epsilon^3 & x\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 4 \\ 3\,\epsilon\,x^2 & 0 & 0 & 0 & \epsilon^2 & x\epsilon & 0 \\ 0 & x\epsilon & 0 & 0 & 0 & \epsilon^2 & 0 \\ 0 & 0 & x\epsilon & \epsilon^2 & \epsilon^2 & 0 & 0 \\ 0 & \epsilon^3 & 0 & 0 & 0 & 0 & x\epsilon \end{bmatrix}.$$

$A(x,\varepsilon)$ is ε-irreducible, and we have $n = 7$, $h = 2$, and $r := rank(A_0(x)) = 2$. Thus, the condition $h > n - r$ of Theorem 6.1 is not verified. Let us first consider a random ramification in ε regardless of Lemma 6.1. For instance, let us consider $\varepsilon = \tilde{\varepsilon}^3$ and apply $\tilde{\varepsilon}$-rank reduction. This yields system $[C_{\varepsilon,\sigma_C}] \quad \tilde{\varepsilon}^5\partial G = C(x,\tilde{\varepsilon})G$ with $\sigma_C = 0$ and

$$C(x,\tilde{\varepsilon}) = \begin{bmatrix} 0 & 0 & 0 & x\tilde{\varepsilon} & \tilde{\varepsilon}^4 & \tilde{\varepsilon}^4 & 0 \\ 1/4\,\tilde{\varepsilon}^2x^2 & x\tilde{\varepsilon}^2 & \tilde{\varepsilon}^7 & 0 & 1/4\,\tilde{\varepsilon}\,\left(\tilde{\varepsilon}^3 - x^2\right) & 1/4\,\tilde{\varepsilon}^4x & 3/4\,\tilde{\varepsilon}\,x^3 \\ 0 & 0 & \tilde{\varepsilon}^8 & x\tilde{\varepsilon}^2 & 0 & 0 & \tilde{\varepsilon}^5 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ x\tilde{\varepsilon}^3 & 0 & 0 & 0 & 0 & \tilde{\varepsilon}^5 & 3\,\tilde{\varepsilon}^2x^2 \\ \tilde{\varepsilon}^6 & 0 & x\epsilon^2 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & \tilde{\varepsilon}^5 & 0 & \tilde{\varepsilon}^2 & 0 \end{bmatrix},$$

for which, $\tilde{h} = 5$ and $\tilde{r} = 2$. Thus, the condition $\tilde{h} > n - \tilde{r}$ is still not necessarily verified after a random ramification. To guarantee that we will arrive at a system verifying this condition, we make use of Lemma 6.1 and choose an integer d such that $d \geq 7/(1 + 2/7) = 49/9$. Let $d = 7 > 49/9$, perform $\varepsilon = \tilde{\varepsilon}^7$ in $[A_{\varepsilon,\sigma_A}]$, and then $\tilde{\varepsilon}$-rank reduction. This yields the system

$[B_{\tilde{\varepsilon},\sigma_B}]$ $\tilde{\varepsilon}^{11}\partial G = B(x,\tilde{\varepsilon})G$ with $\sigma_B = 0$ and

$$B(x,\tilde{\varepsilon}) = \begin{bmatrix} \tilde{\varepsilon}^4 x & 1/4\,\tilde{\varepsilon}^4 x^2 & \tilde{\varepsilon}^{15} & 0 & 1/4\,\tilde{\varepsilon}\,\left(\tilde{\varepsilon}^7 - x^2\right) & 1/4\,\tilde{\varepsilon}^8 x & 3/4\,\tilde{\varepsilon}\,x^3 \\ 0 & 0 & 0 & \tilde{\varepsilon}\,x & \tilde{\varepsilon}^8 & \tilde{\varepsilon}^8 & 0 \\ 0 & 0 & \tilde{\varepsilon}^{18} & \tilde{\varepsilon}^4 x & 0 & 0 & \tilde{\varepsilon}^{11} \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\varepsilon}^7 x & 0 & 0 & 0 & \tilde{\varepsilon}^{11} & 3\,\tilde{\varepsilon}^4 x^2 \\ 0 & \tilde{\varepsilon}^{14} & \tilde{\varepsilon}^4 x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & \tilde{\varepsilon}^{11} & 0 & \tilde{\varepsilon}^4 & 0 \end{bmatrix}.$$

For system $[B_{\tilde{\varepsilon},\sigma_B}]$, we have: $\tilde{h} = 11$, $\tilde{r} = 2$, and so the condition $\tilde{h} = 11 > n - \tilde{r} = 7 - 2$ holds, and so, Theorem 6.1 can now be applied. From $\det(\lambda I - \frac{B(x,\tilde{\varepsilon})}{\tilde{\varepsilon}^{11}})$, we can compute $\omega_{\tilde{\varepsilon}}(M) = \frac{42}{4}$. And so, we introduce the ramification $\tilde{\varepsilon} = \tilde{\tilde{\varepsilon}}^4$ and apply $\tilde{\tilde{\varepsilon}}$-rank reduction which gives a $\tilde{\tilde{\varepsilon}}$-system of order 42 and whose leading coefficient has 4 nonzero eigenvalues*: In fact, we compute $[\tilde{M}_{\tilde{\tilde{\varepsilon}},\sigma_M}]$ $\tilde{\tilde{\varepsilon}}^{42}\partial W = \tilde{M}(x,\tilde{\tilde{\varepsilon}})W$ with $\sigma_M = 0$ and

$$\tilde{M}(x,\tilde{\tilde{\varepsilon}}) = \begin{bmatrix} \tilde{\tilde{\varepsilon}}^{14} x & 1/4\,\tilde{\tilde{\varepsilon}}^{14} x^2 & 0 & 0 & 1/4\,\tilde{\tilde{\varepsilon}}^{28} - 1/4\,x^2 & 1/4\,\tilde{\tilde{\varepsilon}}^{28} x & 3/4\,x^3 \\ 0 & 0 & 0 & x & \tilde{\tilde{\varepsilon}}^{28} & \tilde{\tilde{\varepsilon}}^{28} & 0 \\ 0 & 0 & 0 & \tilde{\tilde{\varepsilon}}^{14} x & 0 & 0 & \tilde{\tilde{\varepsilon}}^{42} \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\tilde{\varepsilon}}^{28} x & 0 & 0 & 0 & \tilde{\tilde{\varepsilon}}^{42} & 3\,\tilde{\tilde{\varepsilon}}^{14} x^2 \\ 0 & 0 & \tilde{\tilde{\varepsilon}}^{14} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & \tilde{\tilde{\varepsilon}}^{42} & 0 & \tilde{\tilde{\varepsilon}}^{14} & 0 \end{bmatrix}.$$

* One can observe that this procedure was not necessary for such system since $gcd(14, 42, 28) = 14$. Thus a simplification would leave us with a system of order 3. Thus, in the implementation we first try to ignore the condition $h > n - r$ and apply Theorem 6.1 directly to the system $[A_{\varepsilon,\sigma_A}]$. For this particular example we obtain $\omega_\varepsilon(A) = 3/2$. Then by performing $\varepsilon = \tilde{\varepsilon}^2$ and the $\tilde{\varepsilon}$-rank reduction, we arrive at a $\tilde{\varepsilon}$-irreducible system of order 3 whose leading coefficient has 4 nonzero eigenvalues. In fact, we compute $[\tilde{A}_{\tilde{\varepsilon}}]$ $\tilde{\varepsilon}^3\partial G = \tilde{A}(x,\tilde{\varepsilon})G$ where

$$\tilde{A}(x,\tilde{\varepsilon}) = \begin{bmatrix} 0 & 0 & 0 & x & \tilde{\varepsilon}^2 & \tilde{\varepsilon}^2 & 0 \\ 1/4\,\tilde{\varepsilon}\,x^2 & \tilde{\varepsilon}\,x & \tilde{\varepsilon}^4 & 0 & -1/4\,x^2 + 1/4\,\tilde{\varepsilon}^2 & 1/4\,x\tilde{\varepsilon}^2 & 3/4\,x^3 \\ 0 & 0 & \tilde{\varepsilon}^5 & \tilde{\varepsilon}\,x & 0 & 0 & \tilde{\varepsilon}^3 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ x\tilde{\varepsilon}^2 & 0 & 0 & 0 & 0 & \tilde{\varepsilon}^3 & 3\,\tilde{\varepsilon}\,x^2 \\ \tilde{\varepsilon}^4 & 0 & \tilde{\varepsilon}\,x & 0 & 0 & 0 & 0 \\ x & 0 & 0 & \tilde{\varepsilon}^3 & 0 & \tilde{\varepsilon} & 0 \end{bmatrix}.$$

If Theorem 6.1 does not lead to the desired result, we then resort Lemma 6.1. In the case of unperturbed system, it is also a matter of discussion whether this condition is necessary. M. Miyake recently claimed to have such an example (where this condition is necessary).

□

We give the proof of Theorem 6.1 after establishing a series of useful lemmas. The following proofs are an adaptation to the parametrized setting of the proofs in [14, Lemma 3, Lemma 4, Proposition 1, Theorem 1] respectively. For clarity within these intermediate proofs, we will express systems in the equivalent notation $[B_{\varepsilon,\sigma_B}] \quad \partial F = B(x,\varepsilon)F$ where $B(x,\varepsilon \in \mathcal{M}_n(\mathrm{R}_{((\varepsilon))})$, rather than $[B_{\varepsilon,\sigma_B}] \quad \varepsilon^h x^p \partial F = B(x,\varepsilon)F$ where $B(x,\varepsilon) \in \mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$.

| Lemma 6.2

Let $A(x,\varepsilon)$, $W(x,\varepsilon)$ be matrices in $\mathcal{M}_n(\mathrm{R}_{((\varepsilon))})$ and $\mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$ respectively. Put $h = max(0, -val_\varepsilon\ (A(x,\varepsilon)))$ and let

$$\det(\lambda I - A(x,\varepsilon)) - \det(\lambda I - A(x,\varepsilon) + W(x,\varepsilon)) = \alpha_{n-1}(x,\varepsilon)\lambda^{n-1} + \alpha_{n-2}(x,\varepsilon)\lambda^{n-2} + \cdots + \alpha_0(x,\varepsilon),$$

where $\alpha_{n-1}(x,\varepsilon), \alpha_{n-2}(x,\varepsilon), \ldots, \alpha_0(x,\varepsilon)$ lie in $\mathrm{R}_{((\varepsilon))}$). Then

$$val_\varepsilon(\alpha_{n-i}) \geq (1-i)h \quad 1 \leq i \leq n.$$

Proof. For any $i \in \{1, \ldots, n\}$, it follows from Cramer's rule that

$$\alpha_{n-i} = \sum_{l=0}^{\gamma} (w_l \prod_{s=1}^{i-1} a_{l,s})$$

where for $0 \leq l \leq \gamma, 1 \leq s \leq i-1$, w_l are entries in $W(x,\varepsilon)$ and $a_{l,s}$ are entries in $W(x,\varepsilon)$ or $A(x,\varepsilon)$. Consequently,

$$val_\varepsilon(\alpha_{n-i}) \geq (i-1)val_\varepsilon(A(x,\varepsilon)) \geq h.$$

■

| Lemma 6.3

Let $A(x,\varepsilon)$, $B(x,\varepsilon)$ be two matrices in $\mathcal{M}_n(\mathrm{R}_{((\varepsilon))})$ such that $val_\varepsilon(A(x,\varepsilon)) \leq val_\varepsilon(B(x,\varepsilon))$. Consider the two systems $[A_{\varepsilon,\sigma_A}] \quad \partial F = A(x,\varepsilon)F$ and $[B_{\varepsilon,\sigma_B}] \quad \partial G = B(x,\varepsilon)G$. Suppose that there exists $T \in GL_n(\mathrm{R}_{((\varepsilon))})$ such that $[B] = T[A]$. Put $h = max(0, -val_\varepsilon\ (A(x,\varepsilon)))$ and let

$$\begin{cases} \det(\lambda I - A(x,\varepsilon)) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_0 \\ \det(\lambda I - B(x,\varepsilon)) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \beta_{n-2}\lambda^{n-2} + \cdots + \beta_0. \end{cases}$$

Then, we have:

$$val_\varepsilon(\alpha_{n-i}(x,\varepsilon) - \beta_{n-i}(x,\varepsilon)) \geq (1-i)h, \quad 1 \leq i \leq n.$$

Proof. By [106, Lemma 1] we can write $T(x,\varepsilon) = P(x,\varepsilon)\ \varepsilon^\gamma\ Q(x,\varepsilon)$ where P, Q are units of $\mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$ and $\varepsilon^\gamma = \mathrm{diag}(\varepsilon^{\gamma_1}, \ldots, \varepsilon^{\gamma_n})$ for some integers $(\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n)$. Consider $\tilde{A} = P[A]$ and $\tilde{B} = Q^{-1}[B]$.

Then we have

$$\tilde{B} = \varepsilon^{-\gamma}\tilde{A}\varepsilon^{\gamma}, \quad val_\varepsilon(\tilde{A}) = val_\varepsilon(A), \quad \text{and } val_\varepsilon(\tilde{B}) = val_\varepsilon(B).$$

It follows that,

$$\begin{cases} \det(\lambda I - \tilde{A}) = \det(\lambda I - P^{-1}AP + P^{-1}\partial P) = \det(\lambda I - A + (\partial P)P^{-1}) \\ \det(\lambda I - \tilde{B}) = \det(\lambda I - QBQ^{-1} + Q\partial Q^{-1}) = \det(\lambda I - B + (\partial Q^{-1})Q) \\ \det(\lambda I - \tilde{B}) = \det(\lambda I - \varepsilon^{-\gamma}\tilde{A}\varepsilon^{\gamma}) = \det(\lambda I - \tilde{A}). \end{cases}$$

Since P, Q, P^{-1}, Q^{-1}, are units of $\mathcal{M}_n(\mathrm{R}_{[[\varepsilon]]})$, it follows that $(\partial P)P^{-1}$ and $(\partial Q^{-1})Q$ inherit this property as well. The rest of the proof follows as a consequence of Lemma 6.2. ■

| Proposition 6.1

Consider the system $[A_{\varepsilon,\sigma_A}] \quad \partial F = A(x,\varepsilon)F$ *where* $A \in \mathcal{M}_n(\mathrm{R}_{((\varepsilon))})$ *and let:*

$$\det(\lambda I - A(x,\varepsilon)) = \lambda^n + \alpha_{n-1}(x,\varepsilon)\lambda^{n-1} + \alpha_{n-2}(x,\varepsilon)\lambda^{n-2} + \cdots + \alpha_0(x,\varepsilon).$$

Let $[C_{\varepsilon,\sigma_C}] \quad \partial G = C(x,\varepsilon)G$ *where*

$$C(x,\varepsilon) \;=\; Companion\ (c_i(x,\varepsilon))_{0\le i\le n-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{n-1} \end{bmatrix}, \tag{6.3}$$

be a companion system which is equivalent to $[A_{\varepsilon,\sigma_A}]$ *over* $GL_n(\mathrm{R}_{((\varepsilon))})$*. Then we have,*

$$val_\varepsilon\ (\alpha_i - c_i) \ge (1-(n-i))h \quad 0 \le i \le n-1.$$

Proof. Let $[c_{\varepsilon,\sigma_c}]$ with $c_n = 1$ be the scalar equation representing $[C_{\varepsilon,\sigma_C}]$ ($\sigma_c = \sigma_C$). Consider κ of $[c_{\varepsilon,\sigma_c}]$ as defined in Section 5.6. We define $\beta_i = (n-i)\kappa$, for $i \in \{0,\dots,n-1\}$ and $[D_{\varepsilon,\sigma_D}] = \varepsilon^{\beta}[C_{\varepsilon,\sigma_C}]$. Then $D(x,\varepsilon) = \varepsilon^{-\beta}C(x,\varepsilon)\varepsilon^{\beta}$ where $\varepsilon^{\beta} = \mathrm{diag}(\varepsilon^{\beta_0},\dots,\varepsilon^{\beta_{n-1}})$. It follows that $D(x,\varepsilon) = \varepsilon^{-\kappa}\ Companion\ (\varepsilon^{\beta_i}c_i(x,\varepsilon))_{0\le i\le n-1}$. We have, $val_\varepsilon(D(x,\varepsilon)) \ge -\kappa$ since $val_\varepsilon(\varepsilon^{\beta_i}c_i) = (n-i)\kappa + val(c_i) \ge 0$. By the equivalence between $[A_\varepsilon]$ and $[C_\varepsilon]$ (resp. $[D_\varepsilon]$) we have $h \ge \kappa$ (resp. $val_\varepsilon(D(x,\varepsilon)) \ge -\kappa \ge -h$). Let

$$\det(\lambda I - D(x,\varepsilon)) = \lambda^n + d_{n-1}(x,\varepsilon)\lambda^{n-1} + \cdots + d_0(x,\varepsilon).$$

Hence, by Lemma 6.3, we have

$$val_\varepsilon\ (\alpha_i - d_i) \ge (1-(n-i))h,\ 0 \le i \le n.$$

Moreover, by Lemma 6.2

$$val_\varepsilon(d_i - c_i) \geq (1-(n-i))\ max\ (0, -\ val_\varepsilon\ (D(x,\varepsilon))) \geq (1-(n-i))\kappa \geq (1-(n-i))h.$$

It follows that

$$val_\varepsilon(\alpha_i - c_i) \geq\ min\ (val_\varepsilon\ (\alpha_i - d_i),\ val_\varepsilon\ (d_i - c_i)) \geq (1-(n-i))h.$$

■

Proof. (Theorem 6.1) Let $[C_{\varepsilon,\sigma_C}]\quad \partial G = C(x,\varepsilon)G$ be as in Proposition 6.1. Due to their equivalence, $[A_{\varepsilon,\sigma_A}]$ and $[C_{\varepsilon,\sigma_C}]$ have the same ε-exponential order and ε-polynomial. Hence, we have

$$\omega_\varepsilon := \omega_\varepsilon(A) = \omega_\varepsilon(C) = max_{0\leq i<n}\ (\frac{-val_\varepsilon(c_i)}{n-i}) \quad \text{and} \quad E_\varepsilon(X) = \sum_{k=0}^{\ell} c_{i_k, val_\varepsilon(c_{i_k})} X^{(i_k - i_0)}$$

where $0 \leq i_0 < i_1 < \cdots < i_\ell = n$ denote the integer i for which $\omega_\varepsilon(n-i) = -val_\varepsilon(c_i)$. By Corollary 5.2 and Proposition 6.1 one has

$$\begin{aligned} val_\varepsilon\ (\alpha_i - c_i) &\geq (i-n+1)h = (i-n)\omega_\varepsilon + (i-n)(h-\omega_\varepsilon) + h \\ &\geq \omega_\varepsilon(i-n) + (-n)(1-\frac{r}{n}) + h \geq \omega_\varepsilon(i-n) + r + h - n. \end{aligned}$$

It follows from $h + r - n > 0$ that $val_\varepsilon\ (\alpha_i - c_i) > (i-n)\omega_\varepsilon,\ 0 \leq i \leq n$.

- If $i \in \{i_0, \ldots i_\ell\}$ then $\omega_\varepsilon(i-n) = val_\varepsilon\ (c_i)$ and $val_\varepsilon\ (\alpha_i - c_i) >\ \ val_\varepsilon\ (c_i)$. Hence, $val_\varepsilon\ (\alpha_i) = val_\varepsilon\ (c_i)$ and $\alpha_{i,\ val_\varepsilon\ (\alpha_i)} = c_{i,\ val\ (c_i)}$.
- Else $val_\varepsilon\ (\alpha_i - c_i) > \omega_\varepsilon(i-n)$ and $val_\varepsilon\ (c_i) > \omega_\varepsilon(i-n)$. Thus,

$$val_\varepsilon\ (\alpha_i) \geq\ min\ (val_\varepsilon\ (\alpha_i - c_i),\ val_\varepsilon\ (c_i)) > \omega_\varepsilon(i-n).$$

This completes the proof. ■

6.2 Formal reduction algorithm

With the algorithms of uncoupling, turning point resolution, ε-rank reduction, and as can be verified by Algorithm 6[†] below, one can deduce a general form for a fundamental matrix of formal outer solutions for system $[A_\varepsilon]$ given by (3.9). In fact, it is given by

$$F = (\sum_{k=0}^{\infty} \Phi_k(x^{1/s})\ \varepsilon^{k/d}) \exp(\int \mathcal{Q}(x^{1/s}, \varepsilon^{-1/d})), \tag{6.4}$$

[†]The pseudo code of the algorithms is provided to show the sequence of steps. The implementation itself demanded different considerations.

Algorithm 6 EXP_PARAM$(h, p, \sigma_A, A(x,\varepsilon))$: Computes the ε-exponential part of system $[A_\varepsilon]$ or $[A_{\varepsilon,\sigma_A}]$ (performs formal reduction)

. **Input:** $h, p, \sigma_A,\ A(x,\varepsilon)$ of system (5.1) ($p = \sigma_A = 0$ for system (3.9))
Output: $Q(x^{1/s}, \varepsilon^{-1/d})$

$Q \leftarrow \mathrm{diag}(0, \dots 0)$;
while $h > 0$ and $n \neq 1$ **do**
  if A_{00} has at least two distinct eigenvalues **then**
    Apply the ε-block diagonalization of Section 5.3;
    EXP_PARAM$(h, p, \sigma_A, \tilde{A}_{11}(x,\varepsilon))$; Update Q;
    EXP_PARAM$(h, p, \sigma_A, \tilde{A}_{22}(x,\varepsilon))$; Update Q;
  else if A_{00} has one non-zero eigenvalue **then**
    Update Q from the eigenvalues of A_{00};
    $A(x,\varepsilon) \leftarrow$ perform eigenvalue shifting (5.14); (A_{00} is now nilpotent);
    EXP_PARAM$(h, p, \sigma, A(x,\varepsilon))$; Update Q;
  else if $A_0(x)$ is not nilpotent **then**
    $A(x,\varepsilon), p, \sigma_A, \leftarrow$ apply turning point resolution of Subsection 5.4; (A_{00} is now non-nilpotent and σ_A is updated);
    EXP_PARAM$(h, p, \sigma_A, A(x,\varepsilon))$; Update Q;
  else
    $A(x,\varepsilon), h \leftarrow \varepsilon$-rank reduction of Section 5.5;
    if $h > 0$ (i.e. $h_{true} > 0$) **then**
      if A_{00} has at least two distinct eigenvalues **then**
        Apply the ε-block diagonalization of Section 5.3;
        EXP_PARAM$(h, p, \sigma_A, \tilde{A}_{11}(x,\varepsilon))$; Update Q;
        EXP_PARAM$(h, p, \sigma_A, \tilde{A}_{22}(x,\varepsilon))$; Update Q;
      else if A_{00} has one non-zero eigenvalue **then**
        Update Q from the eigenvalues of A_{00};
        $A(x,\varepsilon) \leftarrow$ perform eigenvalue shifting (5.14); (A_{00} is now nilpotent);
        EXP_PARAM$(h, p, \sigma_A, A(x,\varepsilon))$; Update Q;
      else if $A_0(x)$ is not nilpotent **then**
        $A(x,\varepsilon), p, \sigma_A, \leftarrow$ apply turning point resolution of Subsection 5.4; (A_{00} is now non-nilpotent and σ_A is updated);
        Update Q from the eigenvalues of A_{00};
        $A(x,\varepsilon) \leftarrow$ perform Eigenvalue shifting; (A_{00} is now nilpotent);
        EXP_PARAM$(h, p, \sigma_A, A(x,\varepsilon))$; Update Q;
      else
        Use Theorem 6.1 of Section 6.1;
        $\omega_\varepsilon = \frac{\ell}{d}$; $\varepsilon \leftarrow \varepsilon^d$;
        $h, A(x,\varepsilon) \leftarrow$ Apply ε-rank reduction of Section 5.5 ($h \leftarrow \ell$);
        EXP_PARAM$(h, p, \sigma_A, A(x,\varepsilon))$; Update Q;
      end if
    end if
  end if
end while
return (Q).

where s, d are positive integers; $\mathcal{Q}$ is a diagonal matrix whose entries are polynomials in $\varepsilon^{-1/d}$ with coefficients in $\mathbb{C}((x^{1/s}))$. We refer to $Q := \int \mathcal{Q}$ as the *ε-exponential part* (logarithms in a root of x might arise as a result of integration); and the entries of $\Phi_i(x^{1/s})$'s are root-meromorphic in x (see, e.g. [120, Introduction]).

Remark 6.1

Under the notations and statements of Theorem 6.1, the leading term of Q is given by

$$\frac{1}{\varepsilon^{\omega_\varepsilon(A)}} \int^x \operatorname{diag}(X_1(z), \ldots, X_{deg\ (E_\varepsilon)}(z), 0, \ldots, 0)\ dz,$$

where the X_i's denote the roots of the ε-polynomial $E_\varepsilon(X)$. A similar statement can be stated in terms of the eigenvalues of the leading matrix coefficient, x^p, and ε^h.

We sum up our main results in Algorithm 6 which computes the ε-exponential part and consequently a fundamental matrix of formal solutions (6.4) in a given subdomain. We recall that:

- If $n = 1$ then we proceed by integration up to the first h terms;
- If $h \leq 0$ then we follow Subsection 5.2.1 (use MINIISOLDE or LINDALG).

Evidently, computing a FMFS over a commutative field $\mathcal{K}$ of characteristic zero rather than $\mathbb{C}$ may require algebraic extensions of the base field. Chapter 1 describes how one can limit such extensions. One can easily verify that a similar refinement can be applied to Algorithm 6. This is taken into consideration within our implementations in MAPLE.

Remark 6.2 *Current Investigations*

According to Section 4.2, Iwano's first problem can be resolved with the help of a sequence of positive rational numbers $[\rho]$ given by (4.12):

$$[\rho] \qquad 0 = \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_m.$$

In Section 6.3, we try to motivate the following approach to resolve the same problem for a system: Suppose that we start the reduction with system $[A_{\varepsilon,\sigma_A}]$, $\sigma_A = 0$, given by (5.1) as an input. Then, Algorithm 6 can determine the "true" restraining index $\sigma_{A,true}$ of this system at the end of the reduction and its outer solutions. If the restraining index is nonzero then we have a turning point and $\rho_1 = -1/\sigma_{A,true}$. We can then perform the stretching $\tau = x\varepsilon^{-\rho_1}$ for the initial system $[A_{\varepsilon,\sigma_A}]$, and apply to it Algorithm 6. This determines ρ_2. The same process can be repeated with ρ_2 to determine ρ_3, and then with $\rho_3, \ldots$, until we reach a system with a restraining index zero. We are currently investigating the correctness of such a process and questioning the redundancy of certain information by using the equivalence between a system and a scalar equation, and the (x, ε)-polygon recalled in Section 4.2.

6.3 Examples and investigations

We illustrate our algorithm and its flexibility upon applying it to examples in literature. The computations herein are suppressed for clarity and brevity. The computation of the full ε-exponential parts of the following examples is available within the MAPLE package PARAMINT [98].

| Example 6.2 (Continue Example 5.5)

- We continue to compute the outer solution of system $[B_{\varepsilon,\sigma_B}]$ given by:

$$\begin{aligned}\varepsilon^2 x^{-1}\partial W_1 &= B(x,\varepsilon)W_1\\ &= \{\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}-1&0\\-1&0\end{bmatrix}(x^{-3}\varepsilon)+\begin{bmatrix}1&-1\\1&-1+x^4\end{bmatrix}(x^{-3}\varepsilon)^2+O((x^{-3}\varepsilon)^3)\ \}W_1,\end{aligned}$$

with $\sigma_B = -3$. The leading matrix coefficient $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ is nilpotent and ε-irreducible. We compute the characteristic polynomial of $\frac{B(x,\varepsilon)}{\varepsilon^2 x^{-1}}$:

$$\lambda^2-\frac{(\varepsilon x-1)}{\varepsilon x^2}\lambda+\frac{(\varepsilon-x^3)(\varepsilon^2-x^2)}{\varepsilon^3 x^6}.$$

Then the ε-formal exponential order is $\omega_\varepsilon=\frac{3}{2}$ and the ε-polynomial is $X^2+\frac{1}{x}=0$. Thus, by setting $\varepsilon=\tilde{\varepsilon}^2$ (we also adjust σ_B to $\tilde{\sigma_B}=-3/2$) and applying $\tilde{\varepsilon}$-rank reduction of Section 5.5 via $\mathrm{diag}(1,\tilde{\varepsilon})$, we get an $\tilde{\varepsilon}$-irreducible system whose leading matrix coefficient is $\begin{bmatrix}0&1\\-x^{-3}&0\end{bmatrix}$. And so, by the turning point algorithm of Section 5.4, we apply the transformation

$$\begin{bmatrix}1&0\\0&x^{-3/2}\end{bmatrix}$$

which yields (note that we can also express the former using a change of independent variable $x=t^2$)

$$\tilde{\varepsilon}^3 x^{1/2}\partial U=\tilde{A}(x,\tilde{\varepsilon})U \quad \text{where}$$

$$\tilde{A}(x,\tilde{\varepsilon})=\begin{bmatrix}0&1\\-1&0\end{bmatrix}+\begin{bmatrix}-1&0\\0&0\end{bmatrix}(x^{-3/2}\tilde{\varepsilon})+\begin{bmatrix}0&0\\1&0\end{bmatrix}(x^{-3/2}\tilde{\varepsilon})^2+\begin{bmatrix}1&0\\0&-1+x^4\end{bmatrix}(x^{-3/2}\tilde{\varepsilon})^3+O((x^{-3/2}\tilde{\varepsilon})^4).$$

The leading matrix has two distinct eigenvalues. Applying Splitting Lemma we get

$$\tilde{\varepsilon}^3 x^{1/2}\partial R=\tilde{\tilde{B}}(x,\tilde{\varepsilon})R \quad \text{where}$$

$$\tilde{\tilde{B}}(x,\tilde{\varepsilon})=\begin{bmatrix}-i&0\\0&i\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}&0\\0&-\frac{1}{2}\end{bmatrix}(x^{-3/2}\tilde{\varepsilon})+O((x^{-3/2}\tilde{\varepsilon})^2).$$

Thus this system can be uncoupled to any desired precision. The solution follows by straightforward integration. Remark that, as expected, the system has the same ε-formal exponential order and ε-polynomials, as its equivalent scalar equation given in Example 4.1. The leading term of the ε-exponential part Q is then given by: $\exp(\begin{bmatrix} \frac{-2i\sqrt{x}}{\varepsilon^{3/2}} & 0 \\ 0 & \frac{2i\sqrt{x}}{\varepsilon^{3/2}} \end{bmatrix})$. Thus, we have $\rho_1 = 1/3$ which is consistent with Example 4.3. Moreover, further computations yield the full exponential part of outer solutions. In fact, consider again the initial system of Example 5.5:

$$\varepsilon^2 \, \partial F \; = \; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x \end{bmatrix} F.$$

Then the diagonal entries of the ε-part of a fundamental matrix of formal solutions in the outer subdomain are given by:

$$\frac{x^2}{2\varepsilon^2} - \frac{1}{x\varepsilon} \quad \text{and}$$
$$-\frac{1}{20t^5\varepsilon^{1/2}} - \frac{1}{2t^2\varepsilon} + \frac{2t}{\varepsilon^{3/2}} \quad \text{where} \quad x = -t^2.$$

- For the initial system of Example 5.10 given by:

$$\varepsilon^2 \partial F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x \end{bmatrix} F,$$

we have obtained so far outer formal solutions. To compute inner (and probably intermediate) solutions, we set $\tau = x\epsilon^{-\rho_A} = x\epsilon^{-1/3}$ which yields:

$$\varepsilon^{\frac{5}{3}} \partial_\tau F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & \tau\varepsilon^{\frac{1}{3}} \end{bmatrix} F.$$

We can then set $\varepsilon = \tilde{\varepsilon}^3$. And by Algorithm 6, one can compute the transformation $F = TG$ where

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}^2 & 0 & 0 \\ 0 & \tilde{\varepsilon} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resulting system is

$$\tilde{\varepsilon}^4 \partial_\tau G = \tilde{A}(x, \tilde{\varepsilon}) G = \begin{bmatrix} \tau & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} G.$$

One can verify that $\tilde{A}_{00}$ has three distinct roots and so the system can be uncoupled into three scalar equations. The leading term of the ε-exponential part in the inner subdomain is then

given by:

$$\exp(\begin{bmatrix} \frac{\int^\tau dz+\ldots}{\varepsilon^{4/3}} & 0 & 0 \\ 0 & \frac{\int^\tau -1+i/\sqrt{3}+\ldots dz}{2\varepsilon^{4/3}} & 0 \\ 0 & 0 & \frac{\int^\tau -1-i/\sqrt{3}+\ldots dz}{2\varepsilon^{4/3}} \end{bmatrix}).$$

And further computations show that the diagonal entries of the ε-exponential part of a fundamental matrix of formal solutions in the inner subdomain are given by:

$$\frac{\tau + (1/6)\tau^2 + (1/27)\tau^3 + O(\tau^4)}{\varepsilon^{4/3}} \quad \text{and}$$
$$\frac{\tau\,\mathrm{RootOf}(z^2+z+1) + (1/6)\tau^2 - (1/27)(\mathrm{RootOf}(z^2+z+1)+1)\tau^3 + O(\tau^4)}{\varepsilon^{4/3}}.$$

□

| Example 6.3 ([35], Example 1, Section 9.5, pp 446)

Consider the following singularly-perturbed scalar differential equation

$$\varepsilon\partial^3 f - \partial f + xf = 0$$

which can be rewritten as the following differential first order system

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon\partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \\ -x & 1 & 0 \end{bmatrix} F$$

where $F = [f, \partial f, \partial^2 f]^T$ and $\sigma_A = 0$. $A_0(x)$ is nilpotent and consequently no treatment of turning points is required. The transformation $F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} G$ yields the following ε-irreducible equivalent system ($\theta_{\tilde{A}}(\lambda) = -\lambda - x$):

$$[\tilde{A}_{\varepsilon,\sigma_A}] \qquad \varepsilon\partial G = \tilde{A}(x,\varepsilon)G = \begin{bmatrix} -\varepsilon\, x & -x^2\varepsilon - \varepsilon & \varepsilon \\ \varepsilon & \varepsilon\, x & 0 \\ 1 & 0 & 0 \end{bmatrix} G$$

Since the leading matrix coefficient is still nilpotent, we compute the ε-exponential oder. The characteristic polynomial of $\frac{\tilde{A}(x,\varepsilon)}{\varepsilon}$ is given by:

$$\lambda^3 + \frac{\varepsilon - 1}{\varepsilon}\lambda + \frac{x}{\varepsilon}.$$

Consequently, the ε-exponential oder is given by: $\omega_\varepsilon(\tilde{A}) = \ max\ \{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$. Let $\varepsilon = \tilde{\varepsilon}^2$ then we

have:

$$[\tilde{\tilde{A}}_{\varepsilon,\sigma_A}] \qquad \tilde{\varepsilon}^2\partial G = \tilde{\tilde{A}}(x,\tilde{\varepsilon})G = \begin{bmatrix} -\tilde{\varepsilon}^2\, x & -x^2\tilde{\varepsilon}^2 - \tilde{\varepsilon}^2 & \tilde{\varepsilon}^2 \\ \tilde{\varepsilon}^2 & \tilde{\varepsilon}^2\, x & 0 \\ 1 & 0 & 0 \end{bmatrix} G.$$

The transformation $G = \begin{bmatrix} \tilde{\varepsilon} & 0 & 0 \\ 0 & \tilde{\varepsilon} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & -1/2 & -1/2 \end{bmatrix} U$ yields the following $\tilde{\varepsilon}$-irreducible system:

$$[B_{\varepsilon,\sigma_A}] \qquad \tilde{\varepsilon}\partial U = B(x,\tilde{\varepsilon})U = \begin{bmatrix} \tilde{\varepsilon}\, x & \tilde{\varepsilon}/2 & \tilde{\varepsilon}/2 \\ \left(-x^2-1\right)\tilde{\varepsilon} & -1/2\,\tilde{\varepsilon}\, x - 1 & -1/2\,\tilde{\varepsilon}\, x \\ \left(-x^2-1\right)\tilde{\varepsilon} & -1/2\,\tilde{\varepsilon}\, x & -1/2\,\tilde{\varepsilon}\, x + 1 \end{bmatrix} U$$

The leading matrix coefficient B_{00} has three distinct roots and consequently, the system can be uncoupled into three first order scalar differential equations by performing the Splitting lemma. A fundamental matrix of formal solutions is then given by:

$$\Phi(x,\varepsilon^{1/2})\exp\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-x}{\varepsilon^{1/2}} & 0 \\ 0 & 0 & \frac{x}{\varepsilon^{1/2}} \end{bmatrix}\right).$$

where $\Phi(x,\varepsilon^{1/2})$ is the product of all the transformations performed including that of the Splitting lemma. Since the restraining index remains zero, the origin is not a turning point for this system and we do not need stretchings.

□

| Example 6.4 (order 4)

Consider the following scalar equation ($\sigma_a = 0$)

$$[a_{\varepsilon,\sigma_a}] \qquad \partial^4 f \;-\; \varepsilon^{-4}(2x+\varepsilon^3)\partial^2 f \;+\; \varepsilon^{-8}(x^3)f \;=\; 0, \tag{6.5}$$

for which $\omega_\varepsilon = 2$. The (x,ε)-polygon of Section 4.2 is defined by the following set of points: $P_{\omega_\varepsilon(a)} = (2,-1)$, $P_{00} = (0,\frac{3}{4})$, $P_{20} = (0,\frac{1}{2})$, and $P_{23} = (\frac{3}{2},0)$ leading to only one slope given by $\frac{-3}{4}$. Then, according to Section 4.2 the behavior can be investigated with the help of $[\rho] \quad 0 < \rho_1 = 4/3$. Hence, there is only one stretching to consider: $\tau = x\varepsilon^{-4/3}$. We now wish to compute formal solutions and the sequence $[\rho]$ with the matricial representation: Let $F = [f, \varepsilon^2\partial f, \varepsilon^4\partial^2 f, \varepsilon^6\partial^3 f]^T$

then (6.5) can be expressed as the following first order differential system ($\sigma_A = 0$)

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^2 \partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -x^3 & 0 & \varepsilon^3 + 2x & 0 \end{bmatrix} F. \tag{6.6}$$

- For the turning point treatment, we set $x = t^2$ and then apply the transformation

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t^2 & 0 \\ t^3 & 0 & 0 & 0 \\ 0 & t^4 & 0 & 0 \end{bmatrix}$$

which yields the system

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^2 t^{-2} \partial_t G = \tilde{A}(t,\varepsilon)G = \begin{bmatrix} -3\,\frac{\varepsilon^2}{t^3} & 2 & 0 & 0 \\ 4 + 2\,\frac{\varepsilon^3}{t^2} & -4\,\frac{\varepsilon^2}{t^3} & 0 & -2\,t \\ 2 & 0 & -2\,\frac{\varepsilon^2}{t^3} & 0 \\ 0 & 0 & 2\,t & 0 \end{bmatrix} G. \tag{6.7}$$

Hence $\sigma_{\tilde{A}} = -3/2$ (in t and not in x) and we can rewrite (6.7) as

$$\varepsilon^2 t^{-2} \partial_t G = \begin{bmatrix} -3\left(t^{-3/2}\varepsilon\right)^2 & 2 & 0 & 0 \\ 4 + 2\left(t^{-3/2}\varepsilon\right)^3 t^{5/2} & -4\left(t^{-3/2}\varepsilon\right)^2 & 0 & -2\,t \\ 2 & 0 & -2\left(t^{-3/2}\varepsilon\right)^2 & 0 \\ 0 & 0 & 2\,t & 0 \end{bmatrix} G. \tag{6.8}$$

Applying the Splitting lemma uncouples (6.7) into two subsystems (truncated at order 4 in both t and ε)

$$\varepsilon^2 t^{-2} \partial_t U^1 = S^1(t,\varepsilon)U^1 \quad \text{and} \quad \varepsilon^2 t^{-2} \partial_t U^2 = S^2(t,\varepsilon)U^2$$

where

$$S^1(t,\varepsilon) = \begin{bmatrix} -1/2\left(t^{-3/2}\varepsilon\right)^2 \left(\left(t^{-3/2}\varepsilon\right)^4 t^2 + t^2 + 6\right) & -\left(t^{-3/2}\varepsilon\right)^4 t^2 + 2 \\ 2\left(t^{-3/2}\varepsilon\right)^3 t^{5/2} - \left(t^{-3/2}\varepsilon\right)^4 t^2 - t^2 + 4 & -1/2\left(t^{-3/2}\varepsilon\right)^2 \left(3\left(t^{-3/2}\varepsilon\right)^4 t^2 + t^2 + 8\right) \end{bmatrix}$$

whose leading coefficient has two constant nonzero eigenvalues (hence the dimension of the

system /order of the equivalent equation can be reduced). And

$$S^2(t,\varepsilon)=\begin{bmatrix}1/2\,(t^{-3/2}\varepsilon)^2(3\,(t^{-3/2}\varepsilon)^4t^2+t^2-4) & \gamma_1\\ 1/2\,t(2\,(t^{-3/2}\varepsilon)^4t^2+t^2+4) & \gamma_2\end{bmatrix}, \tag{6.9}$$

where $\gamma_1 = -1/8\,t(4\,(t^{-3/2}\varepsilon)^3t^{5/2} - 9\,(t^{-3/2}\varepsilon)^4t^2 - 8\,(t^{-3/2}\varepsilon)^4 - 8)$ and $\gamma_2 = -1/16\,t^2(t^{-3/2}\varepsilon)^2(8\,(t^{-3/2}\varepsilon)^3t^{5/2} - 105\,(t^{-3/2}\varepsilon)^4t^2 - 40\,(t^{-3/2}\varepsilon)^4 - 14\,t^2 - 16$. Since $\varepsilon t^{-3/2} = \varepsilon x^{-3/4}$, we have $\rho_1 = 4/3$.
If we wish to continue the reduction for the second subsystem, then, as usual, we first consider its leading coefficient:

$$S_0^2(t)=t\begin{bmatrix}0 & 1\\ 2+1/2\,t^2 & 0\end{bmatrix}.$$

Hence, by the turning point resolution (here it suffices to factorize t). Thus, the system $[S^2_{\varepsilon,\sigma_{\tilde{A}}}]$ can be rewritten as:

$$[\tilde{S}^2_{\varepsilon,\sigma_{\tilde{S}^2}}]\varepsilon^2t^{-3}t\partial_tR=\tilde{S}^2(t,\varepsilon)R$$

where $\sigma_{\tilde{S}^2}=-2$ (in t) and

$$\tilde{S}^2(t,\varepsilon)=\begin{bmatrix}1/2\,(\varepsilon t^{-2})^2\left(3\,t^4(\varepsilon t^{-2})^4+t^2-4\right) & \tilde{\gamma}_1\\ t^4(\varepsilon t^{-2})^4+1/2\,t^2+2 & \tilde{\gamma}_2\end{bmatrix}$$

where $\tilde{\gamma}_1 = -1/2\,(\varepsilon t^{-2})^3t^4 + \frac{9\,t^4(\varepsilon t^{-2})^4}{8} + (\varepsilon t^{-2})^4t^2 + 1$ and $\tilde{\gamma}_2 = -1/16\,t^2(\varepsilon t^{-2})^2\left(-105\,t^4(\varepsilon t^{-2})^4+8\,(\varepsilon t^{-2})^3t^4-40\,(\varepsilon t^{-2})^4t^2-14\,t^2-16\right)$. The leading constant coefficient has two constant eigenvalues and hence can be uncoupled.

- We consider again system $[A_{\varepsilon,\sigma_A}]$ which is given by (6.6), and we apply the stretching $x=\varepsilon^{4/3}\tau$ which yields

$$\varepsilon^{2/3}\partial_\tau H=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -2\tau^3\varepsilon^4 & 0 & \varepsilon^3+2\tau\varepsilon^{4/3} & 0\end{bmatrix}H. \tag{6.10}$$

Then the ε-rank reduction with

$$T=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & \epsilon^{2/3} & 0\\ 0 & \epsilon^{4/3} & 0 & 0\\ \epsilon^2 & 0 & 0 & 0\end{bmatrix}$$

yields

$$\partial_\tau Q = \begin{bmatrix} 0 & \epsilon^5 + 2\,\tau & 0 & -2\,\tau^3\epsilon^4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} Q$$

whose ε-exponential part is zero.

□

| Example 6.5 (Roo's equation [109])

Given the singularly-perturbed scalar differential equation ($\sigma_a = 0$)

$$[a_{\varepsilon,\sigma_a}] \qquad \varepsilon^4\,\partial^2 f = (x^5 + \varepsilon x^2 + \varepsilon^2) f \tag{6.11}$$

whose (x,ε)-polygon consists of three segments connecting four points [109, pp 607], $P_{00} = (0, 5/2)$, $P_{01} = (1/2, 1)$, $P_{02} = (1, 0)$, and $P_{o_\varepsilon} = (2, -1)$, and thus having the three slopes, $-3, -2$, and -1. Then, according to Section 4.2 the sequence $[\rho]$ of positive rational numbers is given by:

$$[\rho] \qquad 0 < \rho_1 = 1/3 < \rho_2 = 1/2 < \rho_3 = 1.$$

We now wish to compute the sequence $[\rho]$ with the matrix representation of $[a_{\varepsilon,\sigma_a}]$. Let $F = [f, \varepsilon^2\partial f]^T$ then $[a_{\varepsilon,\sigma_a}]$ can be rewritten as the following first order differential system ($\sigma_A = 0$)

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^2\partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & 1 \\ x^5 + \varepsilon x^2 + \varepsilon^2 & 0 \end{bmatrix} F. \tag{6.12}$$

- Let $T = \begin{bmatrix} 1 & 0 \\ 0 & x^{5/2} \end{bmatrix}$ (resolution of turning point) then $F = TG$ yields

$$\varepsilon^2\partial G = \begin{bmatrix} 0 & x^{5/2} \\ x^{5/2} + \varepsilon x^{-1/2} + \varepsilon^2 x^{-5/2} & -\frac{5}{2}\varepsilon^2 x^{-1} \end{bmatrix} G = x^{5/2} \begin{bmatrix} 0 & 1 \\ 1 + \varepsilon x^{-3} + \varepsilon^2 x^{-5} & -\frac{5}{2}\varepsilon^2 x^{-7/2} \end{bmatrix} G$$

which can be rewritten as

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^2 x^2 \partial G = \tilde{A}(x,\varepsilon) = \begin{bmatrix} 0 & 1 \\ 1 + (x^{-3}\varepsilon) + (x^{-3}\varepsilon)x & -\frac{5}{2}(x^{-3}\varepsilon)^2 x^{5/2} \end{bmatrix} G \tag{6.13}$$

with $\sigma_{\tilde{A}} = -3$. The system $[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}]$ can be uncoupled into two scalar equations and we thus have $\rho_1 = 1/3$.

- We consider again system $[A_{\varepsilon,\sigma_A}]$ and perform the stretching $\tau = x\varepsilon^{-1/3}$. We get:

$$[B_{\varepsilon,\sigma_B}] \qquad \varepsilon^{5/3}\partial_\tau U = B(\tau,\varepsilon)U = \begin{bmatrix} 0 & 1 \\ \tau^5\varepsilon^{5/3} + \varepsilon^{5/3}\tau^2 + \varepsilon^2 & 0 \end{bmatrix} U.$$

with $\sigma_B = 0$. Let $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$ (ε-rank reduction) then $U = T_1 G$ yields

$$\varepsilon \partial_\tau G = \begin{bmatrix} 0 & \varepsilon^{1/3} \\ \tau^5 + \tau^2 + \varepsilon^{1/3} & 0 \end{bmatrix} G.$$

The leading coefficient matrix is nilpotent. The computation of the ε-exponential order suggests introducing an additional ramification in ε of order 2. Since we already have a ramification of order 3, we set $\varepsilon = \tilde{\varepsilon}^6$. This yields:

$$\tilde{\varepsilon}^6 \partial_\tau G = \begin{bmatrix} 0 & \tilde{\varepsilon}^2 \\ \tau^5 + \tau^2 + \tilde{\varepsilon}^2 & 0 \end{bmatrix} G.$$

Let $T_2 = \begin{bmatrix} \tilde{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix}$ ($\tilde{\varepsilon}$-rank reduction) then $G = T_2 W$ yields

$$\tilde{\varepsilon}^5 \partial_\tau W = \begin{bmatrix} 0 & 1 \\ \tau^5 + \tau^2 + \tilde{\varepsilon}^2 & 0 \end{bmatrix} W.$$

Let $T_3 = \begin{bmatrix} 1 & 0 \\ 0 & \tau \end{bmatrix}$ (resolution of turning point) then $W = T_3 U$ yields

$$\tilde{\varepsilon}^5 \partial_\tau W = \begin{bmatrix} 0 & \tau \\ \tau^4 + \tau + \tilde{\varepsilon}^2 \tau^{-1} & -\tilde{\varepsilon}^5 \omega^{-1} \end{bmatrix} W$$

which can be rewritten as

$$[\tilde{B}_{\varepsilon,\sigma_{\tilde{B}}}] \qquad \tilde{\varepsilon}^5 \tau^{-1} \partial_\tau W = \tilde{B}(x,\varepsilon) W = \begin{bmatrix} 0 & 1 \\ \tau^3 + 1 + (\tau^{-1}\tilde{\varepsilon})^2 & -(\tau^{-1}\tilde{\varepsilon})^5 \tau^3 \end{bmatrix} W$$

with $\sigma_{\tilde{B}} = -1$(w.r.t. $\tilde{\varepsilon}$). Clealry, system $[\tilde{B}_{\varepsilon,\sigma_{\tilde{B}}}]$ can be uncoupled into two scalar equations. Upon rewriting $\tau^{-1}\tilde{\varepsilon}$ in terms of x and ε,

$$\tau^{-1}\tilde{\varepsilon} = \tau^{-1}\varepsilon^{1/6} = (x^{-1}\varepsilon^{1/3})\varepsilon^{1/6} = x^{-1}\varepsilon^{1/2} = (x^{-2}\varepsilon)^{1/2},$$

we deduce $\rho_2 = 1/2$.

- We consider again system $[A_{\varepsilon,\sigma_A}]$ and perform the stretching $\tau = x\varepsilon^{-1/2}$. We get:

$$[C_{\varepsilon,\sigma_C}] \qquad \varepsilon^{3/2} \partial_\tau Z = C(\tau,\varepsilon) Z = \begin{bmatrix} 0 & 1 \\ \tau^5 \varepsilon^{5/2} + \tau^2 \varepsilon^2 + \varepsilon^2 & 0 \end{bmatrix} Z,$$

with $\sigma_C = 0$. Let $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$ (ε-rank reduction) then $Z = T_1 G$ yields

$$[\tilde{C}_{\varepsilon,\sigma_C}]\varepsilon^{1/2}\partial_\tau G = \begin{bmatrix} 0 & 1 \\ \tau^5\varepsilon^{1/2} + \tau^2 + 1 & 0 \end{bmatrix} G$$

which can be uncoupled into two scalar equations and for which the restraining index remains zero.

And so, by our techniques, we have obtained the two slopes $\rho_1 = 1/3$ and $\rho_2 = 1/2$.

□

| Example 6.6 from [71]

We consider the scalar differential equation

$$[a_{\varepsilon,\sigma_a}] \qquad \varepsilon^3\partial^2 f - (x^3 + \varepsilon)f = 0,$$

with $\sigma_a = 0$. Setting $F = [f, \partial f]^T$, $[a_{\varepsilon,\sigma_a}]$ can be expressed with the following differential first order system

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon^3\partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & \varepsilon^3 \\ x^3 + \varepsilon & 0 \end{bmatrix} F$$

where $\sigma_A = 0$.

- Since A_{00} is nilpotent and $A_0(x)$ is not, we first start with the treatment of the turning point at $x = 0$. It suffices however to factorize x^3 from $A_0(x)$ which modifies the restraining index. We then apply the transformation $F = \begin{bmatrix} (x^{-3}\varepsilon) & 0 \\ 0 & 1 \end{bmatrix} G$ to get the ε-irreducible system:

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad \varepsilon^2\partial G = \tilde{A}(x,\varepsilon)G = \begin{bmatrix} 3x^5(x^{-3}\varepsilon)^2 & (x^{-3}\varepsilon)x^6 \\ 1 + (x^{-3}\varepsilon) & 0 \end{bmatrix} G$$

with $\sigma_{\tilde{A}} = -3$. Since $\tilde{A}_0(x)$ is still nilpotent, we proceed to computing the ε-exponential order from the characteristic polynomial of $\frac{\tilde{A}(x,\varepsilon)}{\varepsilon^2}$ which is given by:

$$\lambda^2 - \frac{3}{x}\lambda - (\frac{x^3}{\varepsilon^3} + \frac{1}{\varepsilon^2}).$$

Hence, $\omega_\varepsilon(\tilde{A}) = \frac{3}{2}$ and the ε-polynomial is given by: $X^2 - x^3 = 0$. Let $\varepsilon = \tilde{\varepsilon}^2$, then we have:

$$[\tilde{\tilde{A}}_{\varepsilon,\sigma_{\tilde{\tilde{A}}}}] \qquad \tilde{\varepsilon}^4\partial G = \tilde{\tilde{A}}(x,\tilde{\varepsilon})G = \begin{bmatrix} 3x^{11}(x^{-3}\tilde{\varepsilon})^4 & (x^{-3}\tilde{\varepsilon})^2x^9 \\ 1 + x^3(x^{-3}\tilde{\varepsilon})^2 & 0 \end{bmatrix} G$$

with $\sigma_{\tilde{\tilde{A}}} = -3$. We then perform $G = \begin{bmatrix} (x^{-3}\tilde{\varepsilon}) & 0 \\ 0 & 1 \end{bmatrix} U$ to get the $\tilde{\varepsilon}$-irreducible system:

$$[B_{\varepsilon,\sigma_{\tilde{\tilde{A}}}}] \qquad x^3\tilde{\varepsilon}^3\partial U = B(x,\tilde{\varepsilon})U = \begin{bmatrix} 3x^{11}(x^{-3}\tilde{\varepsilon})^2((x^{-3}\tilde{\varepsilon})+1) & x^9 \\ 1+x^3(x^{-3}\tilde{\varepsilon})^2 & 0 \end{bmatrix} U.$$

Now that $B_0(x)$ has two distinct eigenvalues and B_{00} is nilpotent, we perform $U = \begin{bmatrix} x^{9/2} & 0 \\ 0 & 1 \end{bmatrix} W$ to treat the turning point. Then we have:

$$[\tilde{B}_{\varepsilon,\sigma_{\tilde{\tilde{A}}}}] \qquad x^{-3/2}\tilde{\varepsilon}^3\partial W = \tilde{B}(x,\tilde{\varepsilon})W = \begin{bmatrix} 3x^{13/2}(x^{-3}\tilde{\varepsilon})^2(1-(1/2)(x^{-3}\tilde{\varepsilon})) & 1 \\ 1+x^3(x^{-3}\tilde{\varepsilon})^2 & 0 \end{bmatrix} W.$$

Since $\tilde{B}_{00}$ has two distinct eigenvalues, ± 1, one can proceed to the Splitting lemma which uncouples the system thoroughly. Hence, from the ε-polynomial or the eigenvalues of $\tilde{B}_{00}$, one can read the ε-exponential part of this system in an outer subdomain

$$\exp\left(\begin{bmatrix} \frac{\int^x z^{3/2}dz}{\varepsilon^{3/2}} & 0 \\ 0 & -\frac{\int^x z^{3/2}dz}{\varepsilon^{3/2}} \end{bmatrix}\right),$$

and deduce that $\rho_1 = 1/3$.

- We consider again system $[A_{\varepsilon,\sigma_A}]$ and perform the stretching $\tau = x\epsilon^{-1/3}$. Then, we have:

$$[B_{\varepsilon,\sigma_B}] \qquad \varepsilon^{5/3}\partial_\tau U = B(\tau,\varepsilon) = \begin{bmatrix} 0 & \varepsilon^2 \\ \tau^3+1 & 0 \end{bmatrix} U$$

with $\sigma_B = 0$. The transformation $U = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} G$ yields

$$[\tilde{B}_{\varepsilon,\sigma_B}] \qquad \varepsilon^{2/3}\partial_\tau G = \tilde{B}(\tau,\varepsilon)G = \begin{bmatrix} 0 & 1 \\ \tau^3+1 & 0 \end{bmatrix} G.$$

which, due to the nature of the eigenvalues of the leading matrix coefficient, can be uncoupled into two scalar equations. The restraining index remains zero which suggests that there are no more stretchings to perform.

- However, we can consider again system $[A_{\varepsilon,\sigma_A}]$ and experiment with the stretching $\tau = x\epsilon^{-1}$. We get:

$$[B_{\varepsilon,\sigma_B}] \qquad \varepsilon\partial_\tau U = B(\tau,\varepsilon)U = \begin{bmatrix} 0 & \varepsilon^2 \\ \tau^3\varepsilon^2+1 & 0 \end{bmatrix} U$$

with $\sigma_B = 0$. By the transformation $U = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} G$ and some simplification we get

$$[\tilde{B}_{\varepsilon,\sigma_B}] \qquad \partial_\tau G = \tilde{B}(\tau,\varepsilon)G = \begin{bmatrix} 0 & 1 \\ \tau^3\varepsilon^2 + 1 & 0 \end{bmatrix} G,$$

which can be uncoupled into two scalar equations as well and has restraining index zero. However, this system has a zero ε-exponential part.

Our approach can thus determine the stretching $\tau = x\varepsilon^{-1/3}$ but not the stretching $\tau = x\varepsilon^{-1}$. However, both stretchings can be obtained by the treatment of this system as a scalar equation [71, Last section].

□

| Example 6.7 ([35], Example 1, Section 9.5, pp 453)

In this example, the input system has $p = 1$. We run our algorithm to compute formal solutions. However, the regions of validity of these solutions is still to be determined. Given

$$\varepsilon^3 x\partial^2 f + x^2\partial f - (x^3 + \varepsilon)f = 0$$

which can be rewritten as the following first order differential system

$$[A_{\varepsilon,\sigma_A}] \qquad x\varepsilon^3\partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & \varepsilon^3 x \\ x^3 + \varepsilon & -x^2 \end{bmatrix} F$$

where $\sigma_A = 0$ and $F = [f, \partial f]^T$. A_{00} is nilpotent while $A_0(x)$ is not. Thus, we first start with the treatment of the turning point at $x = 0$. It suffices here to factorize x^2 which yields

$$[\tilde{A}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad x^{-1}\varepsilon^3\partial F = \tilde{A}(x,\varepsilon)F = \begin{bmatrix} 0 & (x^{-2}\varepsilon)^3 x^5 \\ x + (x^{-2}\varepsilon) & -1 \end{bmatrix} F$$

where $\sigma_{\tilde{A}} = -2$. Now that $\tilde{A}_{00}$ has two distinct eigenvalues, we can proceed to the Splitting lemma. We first block-diagonalize $\tilde{A}_0(x)$ by the transformation $F = \begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} G$ which yields

$$[\tilde{\tilde{A}}_{\varepsilon,\sigma_{\tilde{A}}}] \qquad x^{-1}\varepsilon^3\partial G = \tilde{\tilde{A}}(x,\varepsilon)G = \begin{bmatrix} -(x^{-2}\varepsilon)^3 x^6 - 1 & -(x^{-2}\varepsilon)^3 x^7 - (x^{-2}\varepsilon)^3 x^5 + (x^{-2}\varepsilon) \\ (x^{-2}\varepsilon)^3 x^5 & (x^{-2}\varepsilon)^3 x^6 \end{bmatrix} G,$$

from which we can deduce that the ε-exponential part is given by:

$$\exp\left(\begin{bmatrix} \frac{-x^2}{2\varepsilon^3} + O(ln(x)\varepsilon^{-2}) & 0 \\ 0 & O(ln(x)\varepsilon^{-2}) \end{bmatrix}\right).$$

Since the restraining index is nonzero, we can deduce that $\rho_1 = 1/2$ and apply a stretching in system $[A_{\varepsilon,\sigma_A}]$ and repeat the formal reduction.

□

| Example 6.8

In this example, we use our algorithm to explain some of the computations in the introductory example in the light of the eigenvalues of the matrix of the system. Thus, we consider again the system $[A_{\varepsilon,\sigma_A}]$ given by (3.11):

$$[A_{\varepsilon,\sigma_A}] \qquad \varepsilon\,\partial F = A(x,\varepsilon)F = \begin{bmatrix} 0 & 1 \\ x^3-\varepsilon & 0 \end{bmatrix} F.$$

We have seen that in the region $|x| \leq M|\varepsilon^{1/3}|$ ($\rho_1 = 1/3$ is obtained from the reduction in the outer subdomain), we set $\tau = x\epsilon^{-1/3}$. Then, by a change of parameter $\varepsilon = \tilde{\varepsilon}^3$, we have:

$$\tilde{\varepsilon}^2\partial_\tau F = \tilde{A}(\tau,\tilde{\varepsilon})F = \begin{bmatrix} 0 & 1 \\ \tau^3\tilde{\varepsilon}^3-\tilde{\varepsilon}^3 & 0 \end{bmatrix} F.$$

$\tilde{A}_{00} = \tilde{A}_0(\tau = 0)$ and thus no treatment of turning points is required. We proceed to $\tilde{\varepsilon}$-rank reduction: The transformation $F = T_1 G$ where $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\varepsilon} \end{bmatrix}$ reduces the system to

$$\tilde{\varepsilon}\partial_\tau G = B(\tau,\tilde{\varepsilon})G = \begin{bmatrix} 0 & 1 \\ \tau^3\tilde{\varepsilon}-\tilde{\varepsilon} & 0 \end{bmatrix} G.$$

The system is $\tilde{\varepsilon}$-irreducible and so we proceed now to compute the $\tilde{\varepsilon}$-formal exponential order. From the characteristic polynomial of $\frac{B(\tau,\tilde{\varepsilon})}{\tilde{\varepsilon}}$ which is given by

$$\lambda^2 - \frac{\tau^3-1}{\tilde{\varepsilon}},$$

it is clear that $\omega_{\tilde{\varepsilon}} = \frac{1}{2}$. Thus upon setting $\tilde{\varepsilon} = \check{\varepsilon}^2$, the transformation $G = T_2 U$ where $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & \check{\varepsilon} \end{bmatrix}$, reduces the above system to

$$\check{\varepsilon}\partial_\tau U = \check{B}(\tau,\check{\varepsilon})U = \begin{bmatrix} 0 & 1 \\ \tau^3-1 & 0 \end{bmatrix} U.$$

Now that $\check{B}_{00}$ has two distinct eigenvalues, the Splitting lemma can be applied with L_2. We have: $\varepsilon = \tilde{\varepsilon}^3 = \check{\varepsilon}^6$ and $L_1 = T_1 T_2$. The restraining index remains zero and hence no further stretchings are required.

We remark that the eigenvalues of $A(x,\varepsilon)$ are given by $\lambda_{1,2} = \pm(x^3-\varepsilon)^{1/2}$. We thus have the two possible expansions which correspond to the two subdomains above:

- $\lambda_{1,2} = \pm(x^3-\varepsilon)^{1/2} = \pm i\varepsilon^{1/2}(1-\frac{1}{2}\varepsilon^{-1}x^3-\frac{1}{8}\varepsilon^{-2}x^6+\dots)$, valid for $|\varepsilon^{-1}x^3|<1$.
- $\lambda_{1,2} = \pm(x^3-\varepsilon)^{1/2} = \pm x^{3/2}(1-\frac{1}{2}\varepsilon x^{-3}-\frac{1}{8}\varepsilon^2 x^{-6}+\dots)$, valid for $|\varepsilon x^{-3}|<1$.

□

6.4 Conclusion

In this chapter, we gave a method to compute the ε-formal invariants of the system. This result allowed us to give an algorithm for formal reduction which resolves the second item in Iwano's reformulation of the problems of singularly-perturbed systems, and motivates an approach to resolve the first item in this reformulation.

Conclusion

In this part, we give an algorithm, implemented in MAPLE [98], which computes a formal fundamental matrix of formal solutions for singularly-perturbed linear differential systems, in a given subdomain, in a neighborhood of a turning point. The sub-algorithms are stand-alone as well (e.g. ε-rank reduction, resolution of turning points) and reply interesting questions. The algorithm is based on the generalization of the algorithm given in [14] for the unperturbed counterparts of such systems.

Our results are presented in the formal setting. However, the growth of the order of poles in x is tracked within the formal reduction to compute an outer solution and the restraining index is computed. Consequently, an adequate stretching transformation can be determined and applied to construct a inner and intermediate solutions as well. This approach is promising as we show with some examples and its correctness is under current investigation upon investigating $[A_{\varepsilon,\sigma_A}]$ in the light of the valuations of both x and ε. Another field of investigation is the relation between the algebraic eigenvalues of the matrix $A(x,\varepsilon)$ of system $[A_{\varepsilon,\sigma_A}]$ and the ε-exponential part of its solutions (see Example 6.8 and Chapter 2). Such an investigation would allow us to benefit from the existing work on fractional power series expansions of solutions of bivariate algebraic equations, to compute $[\rho]$ (4.12) (see, e.g. [36, 8] and references therein). On the other hand, it would be interesting to investigate a differential-like reduction for a two-parameter perturbation of a JCF (see Chapter 2 for the one-parameter case).

The generalization of other notions (e.g. notion of simple systems [29, 24]) and efficient algorithms can be investigated as well, making use of the ideas presented in this part, as discussed in Appendix A.

Our algorithm can be extended to treat formally other types of bivariate systems, e.g. systems meromorphic in both x and ε (see, e.g., Examples 6.3 and 6.7), systems with essential singularity [23], and perturbed difference systems. This is motivated by the success of generalizations of the algorithms of the unperturbed counterparts of such systems (see, e.g. [1, 21]).

The multi-parameter case [119] may be investigated using tools from both, this part, and Part III due to the multivariate nature of the problem.

There remains as well the questions which fall under the complexity, e.g. studying the complexity of this algorithm; the bounds on number of coefficients needed in x in computations; developing efficient algorithms for computing a cyclic vector and comparing it to the results of our direct algorithm.

And finally, to give a full answer on the behavior of the solutions, the related problems of connection, matching, and secondary turning points, are to be studied as well (see, e.g. [69, 55] and references therein).

Part III

Completely Integrable Pfaffian Systems with Normal Crossings

Introduction

> *A differential equation- whether ordinary or partial- admits of an infinite number of solutions. The older and the classical point of view, concerning its integration, consisted in finding the so-called "general integral", i.e. a solution of the equation containing as many arbitrary elements (arbitrary parameters or arbitrary functions) as are necessary to represent any solution, save some exceptional ones. But in more recent research, especially as concerns partial differential equations, this point of view had to be given up, not only because of the difficulty or impossibility of obtaining this "general integral", but above all, because the question does not by any means consist merely in its determination. The question, as set by most applications, does not consist in finding "any" solution of the differential equation, but in choosing, amongst all those possible solutions, a particular one defined by properly given accessory conditions. (...) If we have the general integral, there remains the question of choosing the arbitrary elements in its expression so as to satisfy accessory conditions. In the case of ordinary differential equations, so that, at least theoretically, the question may be considered as solved, being reduced to ordinary algebra; but for partial differential equations, the arbitrary elements consist of functions, and the problem of their determination may be the chief difficulty in this question*[‡].

The systems considered in this part are the completely integrable Pfaffian systems with normal crossings. More explicitly, let $R = \mathbb{C}[[x_1, \ldots, x_m]]$ be the ring of formal power series in $x = (x_1, x_2, \ldots, x_m)$ over the field of complex numbers and $\mathrm{Frac}(\mathrm{R})$ be its fraction field. We are interested in the symbolic resolution of the class of linear systems of partial differential equations in m variables and of the dimension n given by

$$[\underline{A}] \qquad x_i^{p_i+1}\, \partial_i F \;=\; A_{(i)}(x)\; F, \quad 1 \leq i \leq m \tag{6.14}$$

where, for all $i \in \{1, \ldots, m\}$, $\partial_i = \frac{\partial}{\partial x_i}$, $A_{(i)}$ is a $n \times n$ matrix whose entries lie in $\mathrm{Frac}(\mathrm{R})$, and p_i is an integer. We therefore explain the terminology:

- System (6.14) is said to have normal crossings, if for each $i \in \{1, \ldots, m\}$ the integer p_i can be

[‡] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations,* Yale University Press 1923, quoted from the introductory paragraph of Chapter 1

re-adjusted so that

$$A_{(i)} \in \mathbb{C}[[x_1, \ldots, x_m]].$$

- The differential form associated to (6.14) is a 1-form given by

$$dF \;=\; \sum_{i=1}^{m} \frac{A_{(i)}}{x_i^{p_i+1}}\, dx_i.$$

  1-forms go back to the 1815 work of Johann Friedrich Pfaff[§], and are consequently referred to as Pfaffian forms.

- A Pfaffian system is called completely integrable[¶], if the following commutation rule holds for all $i, j \in \{1, \ldots, m\}$:

$$x_i^{p_i+1}\, \partial_i A_{(j)}(x) \;+\; A_{(j)}(x)\, A_{(i)}(x) \;=\; x_j^{p_j+1}\, \partial_j A_{(i)}(x) \;+\; A_{(i)}(x)\, A_{(j)}(x). \tag{6.15}$$

Without loss of generality, the singularity is placed at the origin $(x_1 = 0, \ldots, x_m = 0)$. Otherwise, translations in the independent variables can be performed. The singular locus of this system is the union of hyperplanes of coordinates $x_1 x_2 \ldots x_m = 0$. Subsequently, whenever we refer to a Pfaffian system, we assume it is a completely integrable system with normal crossings, unless explicitly stated otherwise.

Clearly, due to (6.15), one can assume that $A_{(i)}$, for any $i \in \{1, \ldots, m\}$, is not a zero matrix (otherwise $\partial_i\, A_{(j)} = 0$ for all $j \in \{1, \ldots, m\}$ and the number of subsystems of $[\underline{A}]$ can be reduced to $m - 1$).

In a series of articles in the seventies and eighties by Levelt, van den Essen, Charrière, Gérard, Sibuya, and others (see, e.g. [41, 42, 54, 58, 53] and references therein), the language of singular linear connections and stable modules over the ring of formal power series was used to establish important theoretical results regarding the existence of the solution, the classification and the nature of singularities (see [84] and references therein). These investigations provided neither constructive proofs nor techniques for the construction of solutions. Nevertheless they provided the indispensable theoretical base for the formal reduction and its sub-algorithms which will be deveoped in this part.

Obviously, the univariate case reduces to singular linear ordinary differential systems studied in Chapter I. A rank reduction algorithm, i.e. an algorithm which, roughly speaking, minimizes the p_i's individually and simultaneously (see Chapter 8), is given in [26] by Barkatou and LeRoux for the bivariate case upon generalizing Levelt's algorithm given in [87] for univariate systems (1). LeRoux investigated as well the construction of formal solutions for systems whose $(p_1, \ldots, p_n) = (0, \ldots, 0)$ in his thesis [84, Chapter 3]. He gave effective algorithms based on the works of Gérard and Levelt [57], Takano and Yoshida [127]. In analogy to the univariate case, we say that such systems have a singularity of the first kind. System $[\underline{A}]$ is said to have a singularity of second kind otherwise. Thus, in our investigation, we will be tackling the general multivariate case with no restriction to the values of p_i's. We also mention [22] for computing global solutions and [63] for numeric evaluations.

[§] J. F. Pfaff. Methodus generalis, aequationes differentiarum partialium, nec non aequationes differentiales vulgares, utrasque primi ordinis inter quotcunque variabiles, complete integrandi, 1815.

[¶] Roughly speaking, to get some intuition, one may think about complete integrability in the context of conservation of the total energy in a physical system (see [49, Part II, Section 5] for an elaborated discussion)

In the hope of constructing solutions, one may wonder whether bivariate Pfaffian system can be treated with the same approach and techniques developed for singularly-perturbed linear differential system in Part II, holding one of the independent variables as a parameter. We show that although both systems are treated over bivariate fields, they manifest distinct difficulties and obstacles. Namely, Pfaffian systems do not share the phenomenon of turning points with singularly-perturbed systems (see Proposition 7.2). Their interesting feature is rather the interdependence between the m subsystems (Section 7.1.1) and, in general, their multivariate nature. Thus, our approach for the symbolic resolution of bivariate and more generally the multivariate Pfaffian systems take a different turn which makes use of a set of m associated univariate subsystems of form (1) discussed in Chapter 1 (Definition 7.4).

To get an intuition of our approach in this part, we show (informally) our proposed steps of reduction with the following simple bivariate Pfaffian system.

| Example 6.9

Given the following bivariate system over the ring of formal power series in (x_1, x_2) with complex coefficients:

$$\begin{cases} x_1^4\,\partial_1 F \;=\; A_{(1)}\;F \;=\; \begin{bmatrix} x_1^3+x_1^2+x_2 & x_2^2 \\ -1 & x_1^3+x_1^2-x_2 \end{bmatrix} F \\ x_2^3\,\partial_2 F \;=\; A_{(2)}\;F \;=\; \begin{bmatrix} x_2^2-2x_2-6 & x_2^3 \\ -2x_2 & -3x_2^2-2x_2-6 \end{bmatrix} F \end{cases} \tag{6.16}$$

Following theoretical results (see Corollary 7.1), a fundamental matrix of solutions of (6.16) is given by

$$F(x_1,x_2) = \Phi(x_1,x_2)\;\; x_1^{C_1}\;\; x_2^{C_2}\;\; \exp(Q(x_1^{-1/s_1}))\;\; \exp(Q(x_2^{-1/s_2})).$$

For this simple example, we show that it can be constructed in four steps:

- Firstly, we associate a system of two ODS subsystems of the form (1) discussed in Chapter 1.

$$\begin{cases} x_1^4\,\partial_1 F \;=\; A^{(1)}(x_1, x_2=0)\;F \;=\; \begin{bmatrix} x_1^3+x_1^2 & 0 \\ -1 & x_1^3+x_1^2 \end{bmatrix} F, \\ x_2^3\,\partial_2 F \;=\; A^{(2)}(x_1=0, x_2)\;F \;=\; \begin{bmatrix} x_2^2-2x_2-6 & x_2^3 \\ -2x_2 & -3x_2^2-2x_2-6 \end{bmatrix} F. \end{cases}$$

  We show in Section 7.2 that the formal invariants Q_1 and Q_2 of (6.16) can be computed from this associated ODS system. Thus, we compute

$$Q_1 = \frac{-1}{x_1} I_2 \quad \text{and} \quad Q_2 = (\frac{3}{x_2^2} + \frac{2}{x_2}) I_2$$

  via MINIISOLDE or LINDALG. Thus we have $s_1 = s_2 = 1$ and $F(x_1, x_2)$ is given by

$$\Phi(x_1,x_2)\; x_1^{C_1}\; x_2^{C_2} \exp(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix})\; \exp(\begin{bmatrix} \frac{3}{x_2{}^2} + \frac{2}{x_2} & 0 \\ 0 & \frac{3}{x_2{}^2} + \frac{2}{x_2} \end{bmatrix}).$$

- Secondly, we apply the so-called eigenvalue shifting (Subsection 7.1.5)

$$F \;=\; G\exp(\frac{-1}{x_1})\;\exp(\frac{3}{x_2^2}+\frac{2}{x_2}),$$

which yields

$$\begin{cases} x_1^4\;\partial_1 G \;=\; \begin{bmatrix} x_1^3+x_2 & x_2^2 \\ -1 & x_1^3-x_2 \end{bmatrix} G \\ x_2^2\;\partial_2 G \;=\; \begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} G \end{cases}$$

And so, it remains to compute:

$$G(x_1,x_2)=\Phi(x_1,x_2)\;x_1^{C_1}\;x_2^{C_2}.$$

- Thirdly, we reduce the orders of the singularity in x_1 and x_2 to their minimal integer values. This is the rank reduction discussed in Chapter 8. We apply $G=T_1H$ where

$$T_1=\begin{bmatrix} x_2x_1^3 & -x_2 \\ 0 & 1 \end{bmatrix}$$

which yields:

$$\begin{cases} x_1\;\partial_1 H \;=\; \begin{bmatrix} -2 & 0 \\ -x_2 & 1 \end{bmatrix} H, \\ x_2\;\partial_2 H \;=\; \begin{bmatrix} -2 & 0 \\ -2x_1^3 & -1 \end{bmatrix} H. \end{cases}$$

- Finally, by a simple calculation (see [84, Chapter 3] for general cases), we compute

$$T_2=\begin{bmatrix} 1 & 0 \\ \frac{x_2}{3}+2x_1^3 & -1 \end{bmatrix}.$$

Then $H \;=\; T_2\;U$ yields

$$\begin{cases} x_1\;\partial_1 U \;=\; C_1\;U \;=\; \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} U, \\ x_2\;\partial_2 U \;=\; C_2\;U \;=\; \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} U. \end{cases}$$

Thus, a fundamental matrix of solutions is given by

$$T_1\;T_2\;x_1^{C_1}\;x_2^{C_2}\;\exp(\begin{bmatrix} \frac{-1}{x_1} & 0 \\ 0 & \frac{-1}{x_1} \end{bmatrix})\;\exp(\begin{bmatrix} \frac{3}{{x_2}^2}+\frac{2}{x_2} & 0 \\ 0 & \frac{3}{{x_2}^2}+\frac{2}{x_2} \end{bmatrix}).$$

□

Unlike this simple example, the steps of computation can be far more evolved and demand multiple levels of recursion. Nevertheless, we give in this part the formal reduction of such systems over three chapters as follows:

- In Chapter 7, we show that the formal invariants of the system (6.14) can be retrieved from an associated linear ordinary differential subsystems (ODS), each having the form $[A]$ given by (1) and discussed in Chapter 1. We then give a formal reduction algorithm, under the assumption that we have at hand a rank reduction algorithm.
- In Chapter 8, we distinguish between bivariate and multivariate systems and propose rank reduction algorithms.

We then discuss two examples which arise from applications and have different structure from the systems considered in this thesis. Chapters 7 and 8 are based on our published paper [1] for the bivariate case and the ongoing investigation for the multivariate case. A Maple package is also given [100].

In the sequel, we adopt $\mathbb{C}$ as the base field for the simplicity of the presentation. However, any other field $\mathcal{K}$ ($\mathbb{Q} \subseteq \mathcal{K} \subseteq \mathbb{C}$) can be considered instead. In this case, the restrictions on the extensions of the base field discussed in Section 1.8 apply as well and are taken into consideration within the implementation.

Local Notations

- We use n for the dimension of each of the subsystems in accordance with the other parts. And we use the integer m where $m \leq 1$ for the number of variables. As mentioned above, for $m = 1$, system (6.14) is simply an ODS.
- As mentioned in the introduction, we denote by x (resp. $\bar{x}_i$) the collection of variables $x_1, \ldots, x_m$ (resp. $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$). We thus refer by R (resp. $\mathrm{R}_{\bar{\mathrm{x}}_\mathrm{i}}$) to $\mathbb{C}[[x_1, \ldots, x_m]]$ (resp. $\mathbb{C}[[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m]]$), the ring of formal power series in x (resp. $\bar{x}_i$) over the field of complex numbers $\mathbb{C}$. And we denote its field of fractions by $\mathrm{Frac}(\mathrm{R})$ (resp. $\mathrm{Frac}(\mathrm{R}_{\bar{\mathrm{x}}_\mathrm{i}})$).
- Aside from the rings R and $\mathrm{R}_{\bar{\mathrm{x}}_\mathrm{i}}$, we will frequently have to work in other algebraic structures. Let L be the set of monomials given by,

$$\mathrm{L} = \{x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_m^{\beta_m} \quad \text{for} \quad \beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}^m\}.$$

  Clealry, L is closed under multiplication and contains the unit element. Then, one can define $\mathrm{R}_\mathrm{L} := \mathrm{L}^{-1}\mathrm{R}$, the localization of R at L, i.e. the ring of series with only finitely many terms having monomials of strictly negative exponents. Unlike in the univariate case, $\mathrm{Frac}(\mathrm{R})$ and R_L do not refer to the same algebraic structure, e.g. $(x_1 + x_2)^{-1}$ is an element of $\mathrm{Frac}(\mathrm{R})$ which is not an element of R_L. In fact, there exists no $\beta \in \mathbb{N}^2$ such that $x^\beta (x_1 + x_2)^{-1} \in \mathrm{R}_\mathrm{L}$ as $(x_1 + x_2)^{-1} = \frac{1/x_1}{1-(-x_2/x_1)}$ whose formal expansion with respect to x_2 is $\sum_{i \geq 0} (-1)^i x_1^{-i-1} x_2^i$, which has infinitely many poles in x_1. For a further characterization of $\mathrm{Frac}(\mathrm{R})$, one may refer to [6].

- In the sequel, it will be necessary to introduce ramifications of the form $x_i = t_i^{\alpha_i}$ for new variables t_i and positive integers α_i ($i \in \{1, \ldots, m\}$). And so we denote by $\mathrm{R_t} = \mathbb{C}[[\mathrm{t_1}, \ldots, \mathrm{t_m}]]$ the ring of formal power series in $t = (t_1, t_2, \ldots, t_m)$ over the field of complex numbers $\mathbb{C}$ and by $\mathrm{Frac}(\mathrm{R_t})$ its field of fractions.

- We set $\partial_i = \frac{\partial^i}{\partial x^i}$ and $\partial_{t_i} = \frac{\partial^i}{\partial t^i}$ for all $i \in \{1, \ldots, m\}$.

- We will have to work with some m-tuples and so we underline them so that they can be easily identified. Let $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_m)$ belong to $\mathbb{Q}^m$. We set $\underline{1} = (1, \ldots, 1)$, $\underline{0} = (0, \ldots, 0)$, and define the order relation $\preceq$ over $\mathbb{Q}^m$ by

$$\underline{a} \preceq \underline{b} \quad \text{if and only if} \quad a_i \leq b_i \; \forall \; i \in \{1, \ldots, m\}.$$

  We also have

$$\underline{a} = \underline{b} \quad \text{if and only if} \quad a_i = b_i \; \forall \; i \in \{1, \ldots, m\},$$

$$\underline{a} \prec \underline{b} \quad \text{if and only if } \underline{a} \preceq \underline{b}, \text{ and } \exists \text{ at least one index } i \in \{1, \ldots, m\} \text{ such that } a_i < b_i,$$

$$\text{and} \quad \underline{a} - \underline{b} \; = \; (a_1 - b_1, \ldots, a_m - b_m).$$

- In the sequel we make use of the following series representations:

  – The expansion of $A_{(i)}(x)$ with respect to x_i:

$$x_i^{p_i+1} \; \partial_i F = (A_{(i),0} + A_{(i),1} x_i + A_{(i),2} x_i^2 + + \cdots + A_{(i),k} x_i^k + \ldots) \; F,$$

  where the entries of $A_{(i),k}$ lie in the ring $\mathrm{R_{\bar{x}_i}}$; we refer to $A_{(i),0} := A_{(i)}(x_i = 0)$ as the leading matrix coefficient of the i^{th} component $[A_{(i)}]$ of system $[\underline{A}]$;

  – Occasionally, we need to refer to the leading matrix coefficient in the expansion of $A_{(i)}(x)$ with respect to x_j where $i \neq j$. In this case, we simply write $A_{(i)}(x_j = 0) \in \mathrm{R_{\bar{x}_j}}$ to avoid any confusion with $A_{(i),0}$, which we use to denote the leading matrix coefficient in the expansion of $A_{(i)}$ with respect to x_i.

  – We define $\mathbb{A}_{(i)}(x_i) := A_{(i)}(x_1 = 0, \ldots, x_{i-1} = 0, x_i, x_{i+1} = 0, \ldots, x_m = 0)$ which has its entries in $\mathbb{C}[[x_i]]$. In other words, $\mathbb{A}_{(i)}(x_i)$ is obtained from $A_{(i)}(x)$ by setting to zero all the variables x_j for which $j \neq i$ where $j \in \{1, \ldots, m\}$.

  – For the monomial expansion of $M(x) \in \mathcal{M}_n(\mathrm{R})$, we use the lexicographical ordering (see, e.g. [6, Section 4]) and the following short-hand notations for $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$:

$$\begin{cases} |\underline{k}| \; = \; k_1 + \cdots + k_m \\ x^{\underline{k}} \; = \; x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m} \end{cases} \qquad \text{and so we write:}$$

$$M(x) \; = \; \sum_{|\underline{k}|=0}^{\infty} M_{\underline{k}} \; x^{\underline{k}} \quad \text{where} \quad M_{\underline{k}} \in \mathcal{M}_n(\mathbb{C}).$$

- A family of matrices is given with lower indices, e.g. $(M_{(i),k})_{1\leq i\leq m,k\geq 0}$, and for a matrix $M_{(i),k}$, blocks are given with upper indices where the sizes of the different blocks are clear from the context. For example, we may partition $A_{(i),k}$, the k^{th} coefficient of the matrix of the i^{th} component, as follows
$$A_{(i),k} = \begin{pmatrix} A^{11}_{(i),k} & A^{12}_{(i),k} \\ A^{21}_{(i),k} & A^{22}_{(i),k} \end{pmatrix}.$$
- We denote by p_i the Poincaré rank of the i^{th} component $[A_{(i)}]$ and by $\underline{p} = (p_1, \ldots, p_m)$ the Poincaré rank of the system $[\underline{A}]$.
- The leading matrix coefficient can be regarded as non-zero without any loss of generality, otherwise p_i can be readjusted. It will play a crucial role in our considerations. We thus adopt the following notations:
  - $A_{(i),0}$ is the leading matrix coefficient of the i^{th} component of $[\underline{A}]$ as explained above;
  - $\underline{A}_0 = (A_{(1),0}, \ldots, A_{(m),0})$ is m-tuple of the leading matrix coefficients of the system $[\underline{A}]$;
  - $A_{(i),\underline{0}} = A_{(i)}(x = \underline{0})$ and $\underline{A}_{\underline{0}} = (A_{(1),\underline{0}}, \ldots, A_{(m),\underline{0}})$ is m-tuple of the constant matrices of the system $[\underline{A}]$.
  - We also use $\underline{\mathbb{A}} = (\mathbb{A}_{(1)}, \ldots, \mathbb{A}_{(m)})$.

Chapter 7

Computing Formal Invariants

Contents

In this chapter, we are interested in the formal reduction of system $[\underline{A}]$ given by (6.14) and which we recall here

$$[\underline{A}] \qquad x_i^{p_i+1}\,\partial_i F \;=\; A_{(i)}(x)\,F, \quad 1 \le i \le m,$$

where $A_{(i)} \in \mathcal{M}_n(\mathrm{R})$ and $p_i \ge 0$ for every $i \in \{1,\dots,m\}$. Let $T(x) \in GL_n(\mathrm{Frac}(\mathrm{R}))$ then the (gauge) transformation $F = TG$ gives rise to

$$[\underline{\tilde{A}}] \qquad x_i^{\tilde{p}_i+1}\,\partial_i G \;=\; \tilde{A}_{(i)}(x)\,G, \quad 1 \le i \le m \tag{7.1}$$

where $\tilde{p}_i$ is an integer and

$$\frac{\tilde{A}_{(i)}}{x_i^{\tilde{p}_i+1}} = T^{-1}\,\Big(\frac{A_{(i)}}{x_i^{p_i+1}}\,T - \partial_i T\Big), \quad 1 \le i \le m. \tag{7.2}$$

We say that system (7.1) is *equivalent* to system (6.14) and we write $[\underline{\tilde{A}}] := T[\underline{A}]$. It can be easily verified that complete integrability is inherited by an equivalent system*. Subsequently, to stay in the same class of systems under study, special care will be taken so that the transformations used in our considerations

*In fact, such a transformation can be viewed as change of basis in the language of operators.

do not alter the normal crossings. In fact, In addition to computing over multivariate fields, a major difficulty within the symbolic manipulation of system (6.14) arises from (7.2) and will be discussed in Subsection 7.1.1.

This chapter is thus organized as follows: In section 7.1, we give some preliminaries regarding some results on the existence of solutions, the classification of singularities, the properties of the leading matrix coefficients, and the uncoupling of the system to systems of lower dimensions (splitting) in this multivariate setting. In Section 7, we describe our proposed algorithm which relies on two major ingredients: The first ingredient is associating to our system a set of ODS's from which its formal invariants can be efficiently derived. This is detailed in Section 7.2. And the second ingredient is the rank reduction which we discuss in Chapter 8.

7.1 Preliminaries

7.1.1 Compatible transformations

It is evident from (7.2) that any transformation alters all the components of $[\underline{A}]$ simultaneously. In particular, the equivalent system does not necessarily inherit the normal crossings even for very simple examples, as exhibited by the bivariate example which we recall here.

| Example 7.1 ([26], Section 4)

Consider the following completely integrable Pfaffian system with normal crossings of Poincaé rank $(3,1)$:

$$[A] \qquad \begin{cases} x_1^4\,\partial_1 F \;=\; A_{(1)}(x_1,x_2)\;F \;=\; \begin{bmatrix} x_1^3+x_2 & x_2^2 \\ -1 & -x_2+x_1^3 \end{bmatrix} F \\ \\ x_2^2\,\partial_2 F \;=\; A_{(2)}(x_1,x_2)\;F \;=\; \begin{bmatrix} x_2 & x_2^2 \\ -2 & -3x_2 \end{bmatrix} F. \end{cases} . \tag{7.3}$$

This system appears within the reduction of the system of Example 6.9 in the introduction. As we have seen, we computed (by the algorithm which we will propose in Chapter 8) a transformation which dropped p_1 to zero. However, this can also be attained by the transformation $F = \begin{bmatrix} x_1^3 & -x_2^2 \\ 0 & x_2 \end{bmatrix} G$ which is computed by the univariate-case Moser-based rank reduction algorithm, upon regarding the first component as an ODS in x_1 with x_2 as a transcendental constant. This transformation would result in the following equivalent system

$$[\tilde{A}] \qquad \begin{cases} x_1\,x_2\,\partial_1 G \;=\; \tilde{A}_{(1)}(x_1,x_2)\;G = \begin{bmatrix} -2x_2 & 0 \\ -1 & x_2 \end{bmatrix} G \\ \\ x_2^3\,\partial_2 G \;=\; \tilde{A}_{(2)}(x_1,x_2)\;G = \begin{bmatrix} -x_2^2 & 0 \\ -2x_1^3 & -2x_2^2 \end{bmatrix} G. \end{cases}$$

We can see that such a transformation achieves the goal of diminishing the Poincaré rank of the first component, considered as an ODS, to its minimum (zero). However, unlike the transformation which we used in the introduction, it alters the normal crossings by introducing the factor x_2 in the denominator of an entry in $\tilde{A}_{(1)}$. Moreover, it increases the Poincaré rank of the second component of $[\underline{A}]$.

□

The urge to preserve the normal crossings motivates the following definition:

| Definition 7.1

Let $T \in GL_n(\mathrm{Frac(R)})$. We say that the transformation $F = TG$ (respectively T) is *weakly compatible* with system $[\underline{A}]$ given by (6.14) if $T[\underline{A}] := [\underline{\tilde{A}}]$ is again a completely integrable Pfaffian system with normal crossings, i.e. $[\underline{\tilde{A}}]$ can be expressed by (7.2) with $\tilde{A}_i \in \mathcal{M}_{\mathrm{n}}(\mathrm{R})$ for every $i \in \{1, \ldots, m\}$.

Clearly, any constant or unimodular invertible matrix is an example of such transformations. We also give a further characterization of weak compatibility in Proposition 7.1.
In the sequel, we will also need to resort to transformations with stronger properties:

| Definition 7.2

Let $T \in GL_n(\mathrm{Frac(R)})$. We say that a transformation $F = TG$ (respectively T) is *compatible* with system $[\underline{A}]$ if it is weakly compatible with $[\underline{A}]$ and the Poincaré rank of each individual component of $T[\underline{A}]$ do not exceed that of the corresponding component of $[\underline{A}]$. In other words, T does not increase $\underline{p}$ of $[\underline{A}]$.

7.1.2 Fundamental matrix of formal solutions FMFS

Before studying how to construct formal solutions to a given system, the question arises if and how many solutions exist. The language of stable modules over the ring of power series is used in [42, Theorem 1] and [54, Main Theorem] independently to establish the following theorem which gives an answer to this question.

| Theorem 7.1

Consider the completely integrable system $[\underline{A}]$ given by (6.14). There exist strictly positive integers α_i, $1 \leq i \leq m$, and an invertible matrix $T(t)$ whose entries lie in R_t such that, upon setting $x_i = t_i^{\alpha_i}$, the transformation $F = TG$ yields the equivalent system:

$$t_i^{\alpha_i \tilde{p}_i + 1}\, \partial_{t_i} G = \tilde{A}_{(i)}(t_i)\, G, \quad 1 \leq i \leq m, \quad \textit{where}$$

$$\tilde{A}_{(i)}(t_i) = \mathrm{diag}\,(\tilde{A}_{(i)}^{11}(t_i), \tilde{A}_{(i)}^{22}(t_i), \ldots, \tilde{A}_{(i)}^{jj}(t_i)),$$

and for every $\ell \in \{1,\dots,j\}$, $\tilde{A}^{\ell\ell}_{(i)}(t_i)$ is a square matrix of dimension n_ℓ of the form

$$\tilde{A}^{\ell\ell}_{(i)} \;=\; w^{\ell\ell}_{(i)}(t_i)\; I_{n_\ell} + \; t_i^{\alpha_i \tilde{p}_i}\; (c^{\ell\ell}_{(i)} I_{n_\ell} + N^{\ell\ell}_{(i)});$$

where

- $n_1 + n_2 + \cdots + n_j = n$*;*
- $w^{\ell\ell}_{(i)}(t_i) = \sum_{j=0}^{\alpha_i \tilde{p}_i - 1}\; \lambda_{ij\ell}\; t_i^j$ *is a polynomial in t_i, with coefficients in $\mathbb{C}$;*
- $c^{\ell\ell}_{(i)} \in \mathbb{C}$ *and $N^{\ell\ell}_{(i)}$ is a constant (with respect to all derivations ∂_{t_i}) d_ℓ-square matrix having nilpotent upper triangular form;*
- *for any fixed $\ell \in \{1,\dots,j\}$, the matrices $\{N^{\ell\ell}_{(i)}\}_{i=1,\dots,m}$ are permutable;*
- *for every $\ell \in \{1,\dots,j-1\}$, there exists $i \in \{1,\dots,m\}$ such that*

$$\begin{aligned} & w^{\ell\ell}_{(i)}(t_i) \;\neq\; w^{(\ell+1)(\ell+1)}_{(i)}(t_i) \\ \text{or} \quad & c^{\ell\ell}_{(i)} - c^{(\ell+1)(\ell+1)}_{(i)} \;\notin\; \mathbb{Z}. \end{aligned}$$

The argument in both papers relies on induction on the number of variables and the dimension of the system. Theorem 7.1 guarantees the existence of a transformation which takes system (6.14) to Hukuhara-Turrittin's normal form from which the construction of a fundamental matrix of formal solutions FMFS (7.4) is straightforward. In fact, we have:

| Corollary 7.1

Consider system $[\underline{A}]$ given by (6.14). A fundamental matrix of formal solutions exists and is of the form

$$\Phi(x_1^{1/s_1},\dots,x_m^{1/s_m}) \prod_{i=1}^{m} x_i^{C_i} \prod_{i=1}^{m} \exp(Q_i(x_i^{-1/s_i})), \tag{7.4}$$

where[†] Φ is an invertible matrix whose entries lie in R_t and for each $i \in \{1,\dots,m\}$ we have:

- *s_i is a positive integer;*
- *the diagonal matrix*

$$Q_i(x_i^{-1/s_i}) = \operatorname{diag}\left(q_{i,1}(x_i^{-1/s_i}), q_{i,2}(x_i^{-1/s_i}), \dots, q_{i,n}(x_i^{-1/s_i})\right)$$

contains polynomials in x_i^{-1/s_i} over $\mathbb{C}$ without constant terms. We refer to $Q_i(x_i^{-1/s_i})$ as the x_i-exponential part. Under the notations of Theorem 7.1, it is obtained by formally integrating $\frac{w^{\ell\ell}_{(i)}}{t_i^{\alpha_i \tilde{p}_i + 1}}$;

- *C_i is a constant matrix which commutes with $Q_i(x_i^{-1/s_i})$.*

[†]Notice the analogy with the univariate system $[A]$ given by (1)

System $[\underline{A}]$ is said to be *regular singular* whenever, for every $i \in \{1, \ldots, m\}$, $Q_i(x_i^{-1/s_i})$ is a zero matrix. In this case $s_i = 1,\ 1 \leq i \leq m$, and the formal series Φ of (7.4) converges whenever one of the leading matrix coefficients has n distinct eigenvalues and the expansions of the $A_{(i)}(x)$'s converge. Otherwise, system (6.14) is said to be *irregular singular* and the entries of $Q_i(x_i^{-1/s_i})$, $1 \leq i \leq m$, determine the main asymptotic behavior of the actual solutions as $x_i \to 0$ in appropriately small sectorial regions [58, Proposition 5.2, pp 232, and Section 4]. We also have the following definition:

| Definition 7.3

Given system $[\underline{A}]$. We denote by $d_{(i),j}$ the minimum order in x_i within the terms of $q_{i,j}(x_i^{-1/s_i})$ in $Q_i(x_i^{-1/s_i})$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Then, the *x_i-formal exponential growth order* (x_i-exponential order, in short) of $[A_{(i)}]$, the i^{th} component of $[\underline{A}]$, is the rational number

$$\omega_i = -\min_{1 \leq j \leq n} d_{i,j}.$$

The m-tuple of rational numbers $\underline{\omega}(\underline{A}) = (\omega_1, \ldots, \omega_m)$ then defines the exponential order of system $[\underline{A}]$.

If two systems are equivalent then they have the same x_i-exponential parts and x_i-exponential order for all $i \in \{1, \ldots, m\}$, under any gauge transformation. And so, we refer to these quantities as the formal invariants.

The above theoretical results on existence do not establish the formal reduction itself, that is the algorithmic procedure which computes explicitly the α_i's of Theorem 7.1 and a transformation which takes the system to a normal form that allows the construction of such solutions. This will be our task in the following sections. But before proceeding to describe the algorithms we propose, we characterize the transformations to be used.

| Proposition 7.1

Consider a system $[\underline{A}]$ given by (6.14). Let $T \in GL_n(\mathrm{Frac(R)})$ and set $T[A] = [\tilde{\underline{A}}]$. If T is a transformation which is weakly compatible with system $[\underline{A}]$ then $T \in GL_n(\mathrm{R_L})$.

Proof. It follows from (7.2) that

$$\partial_i T = \frac{A_i}{x_i^{p_i+1}} T - T \frac{\tilde{A}_i}{x_i^{\tilde{p}_i+1}}, \quad 1 \leq i \leq m.$$

Using the Kronecker tensor product and the vectorization of $T = [t_{j\ell}]_{1 \leq j,\ell \leq n}$, this set of m equations can be expressed equivalently by

$$\partial_i vec(T) = M_i(x)\ vec(T), \quad 1 \leq i \leq m. \tag{7.5}$$

where $vec(T)$ is the column vector given by the transpose of

$$[t_{11}, \ldots, t_{n1}, t_{12}, \ldots, t_{n2}, \ldots, t_{1n}, \ldots, t_{nn}]$$

and

$$M_i = I_n \otimes \frac{A_i}{x_i^{p_i+1}} - \frac{Transpose\ (\tilde{A}_i)}{x_i^{\tilde{p}_i+1}} \otimes I_n.$$

Thus, T itself is a solution of a completely integrable Pfaffian system with normal crossings. By Corollary 7.1, $vec(T)$ has the form (7.4). But $T \in GL_n(\mathrm{Frac(R)})$ and consequently T is free of logarithmic and exponential terms. Hence, T corresponds to a log-free regular solution of (7.5). Thus, $T \in \mathcal{M}_n(\mathrm{R_L})$. The same argument serves to prove that $T^{-1} \in \mathcal{M}_n(\mathrm{R_L})$ as well, upon remarking that $T^{-1}[\underline{\tilde{A}}] = [\underline{A}]$. And thus, $T \in GL_n(\mathrm{R_L})$. ■

The proposition above can also be established by a consequence of the generalization of Abel's formula for the Wronskian (see, e.g. [11, Proposition 1, proof, pp 6]):

$$\partial_i \det T = (\frac{trace\ (A_i)}{x^{p_i+1}} - \frac{trace\ (\tilde{A}_i)}{x^{\tilde{p}_i+1}})\ \det T, \quad 1 \leq i \leq m.$$

7.1.3 Properties of the leading coefficients

Classically, as we have seen in Chapter 1, the starting point in the formal reduction is the nature of eigenvalues of the leading matrix coefficients. The integrability conditions and the property of normal crossings play a major role in establishing Theorem 7.1 and its corollary. In particular, they give rise to interesting and useful aspects of the leading matrix coefficient $\underline{A}_0$ of system $[\underline{A}]$. We first recall that the set of integrability conditions of system $[\underline{A}]$ are given by the pairwise rules of (6.15):

$$x_i^{p_i+1}\ \partial_i A_{(j)}(x)\ -\ x_j^{p_j+1}\ \partial_j A_{(i)}(x)\ =\ A_{(i)}(x)\ A_{(j)}(x)\ -\ A_{(j)}(x)\ A_{(i)}(x), \quad i,j \in \{1,\ldots,m\}.$$

Setting $x_i = 0$ in the above yields

$$x_j^{p_j+1}\ \partial_j A_{(i),0}\ =\ A_{(j)}(x_i=0)\ A_{(i),0}\ -\ A_{(i),0}\ A_{(j)}(x_i=0), \quad i,j \in \{1,\ldots,m\}, \tag{7.6}$$

which can be restated as

$$A_{(i),0}\ \in\ \mathcal{E}(A_{(j)}(x_i=0)) \tag{7.7}$$

where $\mathcal{E}(A_{(j)}(x_i=0))$ is the eigenring of $A_{(j)}(x_i=0)$ which is defined in this context as follows: The set of matrices T with entries in $Frac(\mathrm{R}_{\bar{x}_i})$ satisfying

$$x_j^{p_j+1}\ \partial_j T\ =\ A_{(j)}(x_i=0)\ T\ -\ T\ A_{(j)}(x_i=0).$$

The idea of using eigenrings to investigate factorization of linear differential operators over univariate fields, i.e., computing a change of basis under which the matrix representation has a triangular or block-diagonal form, was introduced by Singer [123], and have been used for factoring systems of linear functional equations over base fields of positive or zero characteristic (see, e.g. [16] and references therein) and for computing closed-form solutions of integrable connections in [22]. Results on eigenrings characterize the interdependence within the m components of $[\underline{A}]$. In particular, we have:

| Proposition 7.2

For any $i \in \{1, \ldots, m\}$, the eigenvalues of $A_{(i),0}$ belong to $\mathbb{C}$.

For a proof in the context of eigenrings, one can refer to [16, Proposition 2.2] and otherwise to [54, Proposition 1, pp 8] and [42, Lemma 4.1, pp 93].
Due to established results on eigenrings, one can relate the structures of $A_{(i),0}$ and $A_{(j)}(x_i = 0)$, $i, j \in \{1, \ldots, m\}$, which will prove to be very beneficial to test and guarantee the compatibility of the transformations used within our treatment. In particular, the following fact can be proved easily in our particular situation but it can also be seen as an adaptation of [16, Theorem 3.1] to our settings:

Lemma 7.1

Let $T, B \in \mathcal{M}_n(\mathrm{K})$ where K is a commutative field of characteristic zero equipped with a derivation ∂. Suppose that $T \in \mathcal{E}(B)$ and set r to be the rank of T. If $T = \begin{bmatrix} T^{11} & O_{r,n-r} \\ T^{21} & O_{n-r,n-r} \end{bmatrix}$ where $\begin{bmatrix} T^{11} \\ T^{21} \end{bmatrix}$ is of full rank r then $B = \begin{bmatrix} B^{11} & O_{r,n-r} \\ B^{21} & B^{22} \end{bmatrix}$.

Proof. Since $T \in \mathcal{E}(B)$ then we have

$$\partial T = BT - TB. \tag{7.8}$$

Let B be partitioned in analogy with the block stucture of T as $B = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix}$. Inserting this form in (7.8) and equating the entries of same positions, we get: $\begin{bmatrix} T^{11} \\ T^{21} \end{bmatrix} B^{12} = O_{n,n-r}$. Since $\begin{bmatrix} T^{11} \\ T^{21} \end{bmatrix}$ is of full rank, we then have $B^{12} = O_{r,n-r}$. ■

7.1.4 Block-diagonalization

Consider system $[\underline{A}]$ given by (6.14) and its m-tuple of leading matrix coefficients $\underline{A}_0 = (A_{(1),0}, \ldots, A_{(m),0})$. Whenever there exists an index $i \in \{1, \ldots, m\}$ such that $A_{(i),0}$ has at least two distinct eigenvalues, system $[\underline{A}]$ can be uncoupled into subsystems of lower dimensions as we demonstrate in Theorem 7.2. Theorem 7.2 generalizes the uncoupling algorithm of the univariate case (see Section 1.3) to this multivariate setting. The proof we give in this subsection is constructive and is based on that of [58, Section 5.2, pp 233]. For a statement and a proof in the language of modules, one can refer to [42, Section 4] and [54, Corollary, pp 9].

Theorem 7.2

Consider system $[\underline{A}]$ given by (6.14) and its m-tuple of leading matrix coefficients $\underline{A}_0 = (A_{(1),0}, \ldots, A_{(m),0})$. Suppose that for some $i \in \{1, \ldots, m\}$, the leading matrix coefficient $A_{(i),0}$ has at least two distinct eigenvalues. Then there exists a unique transformation $T(x) \in GL_n(R)$ of the form

$$T(x) = \begin{pmatrix} 1 & T^{11}(x) \\ T^{22}(x) & 1 \end{pmatrix},$$

which is compatible with $[\underline{A}]$ *and such that the transformation* $F = TG$ *yields the equivalent system*

$$x_j^{p_j+1}\,\partial_j G \;=\; \begin{pmatrix} \tilde{A}^{11}_{(j)} & O \\ O & \tilde{A}^{22}_{(j)} \end{pmatrix} G, \quad 1 \le j \le m,$$

where the entries of $\tilde{A}^{11}_{(j)}$ *and* $\tilde{A}^{22}_{(j)}$ *lie in* R. *If furthermore the constant matrix has the form*

$$A_{(i),\underline{0}} = \begin{bmatrix} A^{11}_{(i),\underline{0}} & O \\ O & A^{22}_{(i),\underline{0}} \end{bmatrix} \tag{7.9}$$

where $A^{11}_{(i),\underline{0}}$ *and* $A^{22}_{(i),\underline{0}}$ *do not have an eigenvalue in common, then* $\tilde{A}_{(i),\underline{0}} = A_{(i),\underline{0}}$.

The theorem can be restated by saying that if one of the m components of $[\underline{A}]$ has a leading matrix with at least two distinct eigenvalues then it can be uncoupled to subsystems of lower dimensions. Moreover, all of the other components will be uncoupled simultaneously. And by Proposition (7.2), it suffices that the constant matrix of one of the components has at least two distinct eigenvalues. The following lemma (see [58, Proposition 5.3, pp 238] for $m = 2$) will be useful in the proof of this splitting.

| Lemma 7.2

Given $P_{(i)}, Q_{(i)}, C_{(i)} \in \mathcal{M}_n(\mathbb{C})$ *such that* $P_{(i)}P_{(j)} = P_{(j)}P_{(i)}$ *and* $Q_{(i)}Q_{(j)} = Q_{(j)}Q_{(i)}$ *for all* $i, j \in \{1, \ldots, m\}$. *If for some* $i \in \{1, \ldots, m\}$, $P_{(i)}$ *and* $Q_{(i)}$ *do not have any eigenvalue in common then the system*

$$P_{(i)}\,X \;-\; X\,Q_{(i)} \;=\; C_{(i)}, \quad 1 \le i \le m$$

has a unique solution if and only if

$$P_{(j)}\,C_{(i)} \;-\; C_{(i)}\,Q_{(j)} \;=\; P_{(i)}\,C_{(j)} \;-\; C_{(j)}\,Q_{(i)}, \quad 1 \le i, j \le m.$$

Proof. (Theorem 7.2) We suppose that for some $i \in \{1, \ldots, m\}$, the leading matrix coefficient $A_{(i),0}$ has at least two distinct eigenvalues. By Proposition 7.2, these are also eigenvalues of the constant matrix $A_{(i),\underline{0}}$. And without any loss of generality, we can assume that $A_{(i),\underline{0}}$ has the block-diagonal structure (7.9), otherwise Lemma 1.2 can be applied. We fix this i and proceed in two steps:

First Step: We show that the block-diagonal structure of $A_{(i),\underline{0}}$ induces the same structure on $A_{(j),\underline{0}}$ for $j \in \{1, \ldots, m\}$. For this purpose, we partition the former in accordance with the partition of the latter as follows:

$$A_{(j),\underline{0}} = \begin{bmatrix} A^{11}_{(j),\underline{0}} & A^{12}_{(j),\underline{0}} \\ A^{21}_{(j),\underline{0}} & A^{22}_{(j),\underline{0}} \end{bmatrix}.$$

We then observe that substituting $x = 0$ in (6.15) yields

$$A_{(i),\underline{0}}\,A_{(j),\underline{0}} \;=\; A_{(j),\underline{0}}\,A_{(i),\underline{0}}. \tag{7.10}$$

And (7.10) yields

$$\begin{aligned} A^{11}_{(i),\underline{0}} A^{12}_{(j),\underline{0}} \; - \; A^{12}_{(j),\underline{0}} A^{22}_{(i),\underline{0}} &= O \\ A^{22}_{(i),\underline{0}} A^{21}_{(j),\underline{0}} \; - \; A^{21}_{(j),\underline{0}} A^{11}_{(i),\underline{0}} &= O \end{aligned}$$

The above equations are Sylvester matrix equations with a unique solution due to the assumption on the disjoint spectra of $A^{11}_{(i),\underline{0}}$ and $A^{22}_{(i),\underline{0}}$ (see Lemma 1.1). Thus $A^{12}_{(j),\underline{0}} = O$ and $A^{21}_{(j),\underline{0}} = O$ for all $j \in \{1, \ldots, m\}$.

Second Step: We construct the monomial expansions of $T(x)$ and $\tilde{A}(x)$ up to any desired precision: Let $T(x) \in GL(V)$ then it follows from (7.2) that for any $j \in \{1, \ldots, m\}$ we have

$$x_j^{p_j+1} \, \partial_j T(x) \quad = \quad A_{(j)}(x) \, T(x) \; - \; T(x) \, \tilde{A}_{(j)}(x) \tag{7.11}$$

Assume the desired forms for $T(x)$ and $\tilde{A}(x)$, then we have for $1 \leq \varrho \neq \varsigma \leq 2$:

$$\begin{cases} A^{\varrho\varrho}_{(j)}(x) - \tilde{A}^{\varrho\varrho}_{(j)}(x) + A^{\varrho\varsigma}_{(j)}(x) \, T^{\varsigma\varrho}(x) = O \\ A^{\varsigma\varrho}_{(j)}(x) + A^{\varsigma\varsigma}_{(j)}(x) \, T^{\varsigma\varrho}(x) - T^{\varsigma\varrho}(x) \, \tilde{A}^{\varrho\varrho}_{(j)}(x) = x_j^{p_j+1} \partial_j T^{\varsigma\varrho}(x) \end{cases} \tag{7.12}$$

Inserting the monomial expansions

$$T(x) = \sum_{|\underline{k}|=0}^{\infty} T_{\underline{k}} \, x^{\underline{k}} \quad A_{(j)}(x) = \sum_{|\underline{k}|=0}^{\infty} A_{(j),\underline{k}} \, x^{\underline{k}}, \quad \tilde{A}_{(j)}(x) \; = \; \sum_{|\underline{k}|=0}^{\infty} \tilde{A}_{(j),\underline{k}},$$

in (7.12) and equating power-like coefficients yields for $|\underline{k}| = 0$ and for all $j \in \{1, \ldots, m\}$

$$\begin{aligned} A^{\varrho\varrho}_{(j),\underline{0}} &= \tilde{A}^{\varrho\varrho}_{(j),\underline{0}} \\ A^{\varsigma\varsigma}_{(j),\underline{0}} \, T^{\varsigma\varrho}_{\underline{0}} - T^{\varsigma\varrho}_{\underline{0}} \, \tilde{A}^{\varrho\varrho}_{(j),\underline{0}} &= O \end{aligned}$$

which are obviously satisfied by setting $A_{(j),\underline{0}} = \tilde{A}_{(j),\underline{0}}$ and $T^{\varsigma\varrho}_{\underline{0}} = O$ and $T^{\varrho\varrho}_{\underline{0}} = I$.
As for $|\underline{k}| > 0$, we have

$$\begin{aligned} A^{\varsigma\varsigma}_{(j),\underline{0}} T^{\varsigma\varrho}_{|\underline{k}|} - T^{\varsigma\varrho}_{|\underline{k}|} A^{\varrho\varrho}_{(j),\underline{0}} &= -A^{\varsigma\varrho}_{(j),|\underline{k}|} + (k_j - p_j) \, T^{\varsigma\varrho}_{|\tilde{\underline{k}}|} && (7.13) \\ &\quad - \sum_{0<|\underline{l}|<|\underline{k}|} (A^{\varsigma\varsigma}_{(j),|\underline{k}-\underline{l}|} T^{\varsigma\varrho}_{|\underline{l}|} - T^{\varsigma\varrho}_{|\underline{l}|} \tilde{A}^{\varrho\varrho}_{(j),|\underline{k}-\underline{l}|}) && (7.14) \end{aligned}$$

$$\tilde{A}^{\varrho\varrho}_{(j),|\underline{k}|} \quad = \quad A^{\varrho\varrho}_{(j),|\underline{k}|} + \sum_{0<|\underline{l}|\leq|\underline{k}|} A^{\varrho\varsigma}_{(j),|\underline{k}-\underline{l}|} T^{\varsigma\varrho}_{|\underline{l}|} \tag{7.15}$$

where

$$\begin{cases} \tilde{\underline{k}} \quad \text{is such that} \quad \begin{cases} \tilde{k}_j = k_j, \quad \text{if} \quad j \neq i \\ \tilde{k}_j = k_j + p_j \quad \text{and} \end{cases} \\ T^{\varsigma\varrho}_{|\tilde{\underline{k}}|} = O \quad \text{if} \quad k_j \leq p_j \end{cases}$$

Hence, for any $\underline{k} \in \mathbb{N}^n$, $j \in \{1, \dots, m\}$, $1 \leq \varrho, \varsigma \leq 2$, the computation of $T^{\varsigma\varrho}_{|\underline{k}|}$, and $\tilde{A}^{\varsigma\varsigma}_{(j),|\underline{k}|}$, depends solely on terms of lower order. Moreover, it can be verified by that the conditions of Lemma 7.2 are guaranteed by the monomial expansion of the integrability conditions and the disjoint spectra of $A^{11}_{(i),\underline{0}}$ and $A^{22}_{(i),\underline{0}}$. Thus, (7.15) and (7.13) can be determined uniquely and successively to any desired precision.

We finally remark that $T(x)$ is clearly compatible with system (6.14) since $T(x = 0) = I_n$. ■

Henceforth, we can assume now without loss of generality that there exists some index $j \in \{1, \dots, m+1\}$ such that for the m-tuple of the leading matrix coefficients $\underline{A}_0 = (A_{(1),0}, \dots, A_{(m),0})$ we have:

- For $1 \leq i < j$, $A_{(i),0}$ is nilpotent.
- For $j \leq i \leq m$, $A_{(i),0}$ has a nonzero unique eigenvalue $\lambda_i \in \mathbb{C}$,

which leads us to the next subsection.

7.1.5 Eigenvalue shifting

Let $i \in \{1, \dots, m\}$ and suppose that $A_{(i),0}$ of system $[\underline{A}]$ has a nonzero unique eigenvalue $\gamma_i \in \mathbb{C}$. We can then apply the so-called eigenvalue shifting

$$F = G \exp(\int^{x_i} \gamma_i z_i^{-p_i-1} dz_i), \tag{7.16}$$

to arrive at a system $[\underline{\tilde{A}}]$ whose i^{th} component has a nilpotent leading matrix coefficient. In fact we have

$$\begin{cases} \tilde{A}_{(i)}(x) = A_{(i)}(x) - \gamma_i I_n \\ \tilde{A}_{(j)}(x) = A_{(j)}(x) \quad \text{for all} \quad 1 \leq j \neq i \leq m \end{cases}.$$

Such a transformation is clearly compatible with system $[\underline{A}]$. Henceforth, by applying to system $[\underline{A}]$

$$F = G \prod_{i=j}^{m} \exp(\int^{x_i} \gamma_i z_i^{-p_i-1} dz_i),$$

we arrive at a system $[\underline{\tilde{A}}]$ with a m-tuple of nilpotent leading coefficients.
Similar to Parts I and II, with this nilpotency arises the main difficulties in formal reduction. In the next section, we show how to compute the formal invariants of such systems.

7.2 Computing the formal invariants

One cannot retrieve $\underline{\omega}(\underline{A})$ (see Definition 7.3) from the outset. In the univariate case, $\underline{\omega}(\underline{A})$ reduces to $\omega(A)$ which, as we have seen in Chapter 1, can be computed from the characteristic polynomial of $A(x)$, i.e. $\det(\lambda I - A(x))$ [14, Theorem 1], based on an analysis by a Newton's polygon. In this section we show that, unlike Part II, one need not search for a generalization of [14, Theorem 1] and its underlying

set of reductions. We show that instead, the formal invariants of $[\underline{A}]$ can be obtained from a system of associated linear singular ordinary differential systems (ODS). And so, not only we give a method to retrieve these invariants but also we reduce computations to univariate fields. We first define these ODS components:

| Definition 7.4

Consider system $[\underline{A}]$ given by (6.14):

$$[\underline{A}] \qquad x_i^{p_i+1}\,\partial_i\,F = A_{(i)}(x)\,F, \quad 1 \leq i \leq m.$$

We call the following the *associated ODS* with $[\underline{A}]$:

$$[\underline{\mathbb{A}}] \qquad x_i^{p_i+1}\,\partial_i\,\mathbb{F} = \mathbb{A}_{(i)}(x_i)\,\mathbb{F}, \quad 1 \leq i \leq m, \tag{7.17}$$

where $\mathbb{A}_{(i)}(x_i) := A(x_1 = 0, \ldots, x_{i-1} = 0, x_i, x_{i+1} = 0, \ldots, x_m = 0)$.

We then introduce the following notations:

- $\underline{\omega}(\underline{\mathbb{A}}) = (\omega(\mathbb{A}_{(1)}), \ldots, \omega(\mathbb{A}_{(m)}))$ the m-tuple of the exponential orders of the components of the associated ODS, and
- $\underline{p}(\underline{\mathbb{A}}) = (p(\mathbb{A}_{(1)}), \ldots, p(\mathbb{A}_{(m)}))$ the m-tuple of the Poincaré ranks of the components of the associated ODS.
- $\underline{p}_{true}(\underline{\mathbb{A}}) = (p_{true}(\mathbb{A}_{(1)}), \ldots, p_{true}(\mathbb{A}_{(m)}))$ the m-tuple of the true Poincaré ranks of the components of the associated ODS, as defined in Section 1.5.1.

We can then establish the following theorem:

| Theorem 7.3

Given the completely integrable system with normal crossings $[\underline{A}]$ and its associated ODS system $[\underline{\mathbb{A}}]$. For every $i \in \{1, \ldots, m\}$, the x_i-exponential parts of the former are the exponential parts of the i^{th} component of the latter (see Figure 7.2). And consequently we have,

$$\underline{\omega}(\underline{A}) \;=\; \underline{\omega}(\underline{\mathbb{A}}).$$

To prove this result, we rely on a triangular form weaker than the Hukuhara-Turrittin's normal form of Theorem 7.1. This weaker form suffices to give an insight into the computation of x_i-exponential parts. The following proposition was first given for the bivariate case in [41, Proposition 3, pp 654] and then generalized to the multivariate general case in [42, Theorem 2.3].

| Theorem 7.4

Consider system $[\underline{A}]$ given by (6.14). There exists a positive integer α_1, and a transformation $T \in GL_n(\mathrm{Frac}(\mathrm{R_t}))$ (where $x_1 = t_1^{\alpha_1}$ and $x_i = t_i,\ 2 \leq i \leq m$), such that the transformation $F = TG$

yields the equivalent system:

$$[\hat{A}] \qquad \begin{cases} t_1^{\alpha_1\hat{p}_1+1}\,\partial_{t_1}\,G = \hat{A}_{(1)}(t_1,x_2,\ldots,x_m)\,G, \\ x_i^{\hat{p}_i+1}\,\partial_{x_i}\,G = \hat{A}_{(i)}(x_2,\ldots,x_m)\,G, \quad 2\le i\le m, \quad \textit{where} \end{cases} \tag{7.18}$$

$$\begin{aligned} \hat{A}_{(1)}(t_1,x_2,\ldots,x_m) &= \operatorname{diag}(\hat{A}^{11}_{(1)},\hat{A}^{22}_{(1)},\ldots,\hat{A}^{jj}_{(1)}),\\ \hat{A}_{(i)}(x_2,\ldots,x_m) &= \operatorname{diag}(\hat{A}^{11}_{(i)},\hat{A}^{22}_{(i)},\ldots,\hat{A}^{jj}_{(i)}), \quad 2\le i\le m, \end{aligned}$$

and for all $\ell\in\{1,\ldots,j\}$ and $i\in\{2,\ldots,m\}$ the entries of $\hat{A}^{\ell\ell}_{(i)}$ lie in $\mathrm{R}_{\bar{\mathrm{x}}_1}$. The $\hat{A}^{\ell\ell}_{(1)}$'s are of the form

$$\hat{A}^{\ell\ell}_{(1)}(x_2,\ldots,x_m) = w^{\ell\ell}_{(1)}(t_1)I_{d_\ell} + t_1^{\alpha_1\hat{p}_1}(\hat{N}^{\ell\ell}_{(1)}(x_2,\ldots,x_m) + c^{\ell\ell}_{(1)}I_{n_\ell}),$$

where

- $n_1+n_2+\cdots+n_j=n$*;*
- $w^{\ell\ell}_{(1)}(t_1)$ *and* $c^{\ell\ell}_{(1)}$ *are as in Theorem 7.1;*
- *If* $\ell,\ell'\in\{1,\ldots,j-1\}$ *and* $\ell\neq\ell'$*, then* $w^{\ell\ell}_{(1)}(t_1)\neq w^{\ell'\ell'}_{(1)}(t_1)$ *or* $c^{\ell\ell}_{(1)}-c^{\ell'\ell'}_{(1)}\notin\mathbb{Z}$*;*
- $\hat{N}^{\ell\ell}_1(x_2,\ldots,x_m)$ *is a nilpotent* n_ℓ*-square matrix whose entries lie in* $\mathrm{R}_{\bar{\mathrm{x}}_1}$*.*

Moreover, T can be chosen as a product of transformations in $GL_n(\mathrm{R_t})$ and transformations of the form $\operatorname{diag}(t_1^{\beta_1},\ldots,t_1^{\beta_n})$*, where $\beta_1,\ldots,\beta_n$ are nonnegative integers.*

Proof. (Theorem 7.3) Upon the change of independent variable $x_1=t_1^{\alpha_1}$, the transformation $F=TG$ yields system (7.18) for which the first component is given by

$$t_1^{\alpha_1\hat{p}_1+1}\,\partial_{t_1}\,G = \hat{A}_{(1)}(t_1,x_2,\ldots,x_m)\,G. \tag{7.19}$$

with the notations and properties as in Theorem 7.4. It then follows from (7.2) that

$$t_1^{\alpha_1\hat{p}_1+1}\,\partial_{t_1}T \;=\; \alpha_1\,A_{(1)}(x_1=t_1^{\alpha_1})\,T \;-\; T\,\hat{A}_{(1)}. \tag{7.20}$$

Due to the particular choice of $T(x)$ characterized in Theorem 7.4, setting $x_i=0,\ 2\le i\le m$ in (7.20) is legitimate. In particular, we are interested in the relation between the leading terms

$$\begin{cases} \mathbb{A}_{(1)}(t_1) \;:=\; A_{(1)}(x_1=t_1^{\alpha_1},x_2=0,\ldots,x_m=0)\,, \\ \hat{\mathbb{A}}_{(1)}(t_1) \;:=\; \hat{A}_{(1)}(t_1,t_2=0,\ldots,t_m=0)\,, \\ \mathbb{T}(t_1) \;=\; T\,(t_1,t_2=0,\ldots,t_m=0)\,, \end{cases}$$

which is clearly given by

$$t_1^{\alpha_1\hat{p}_1+1}\,\partial_{t_1}\mathbb{T} \;=\; \alpha_1\,\mathbb{A}_{(1)}(x_1=t_1^{\alpha_1})\,\mathbb{T} \;-\; \mathbb{T}\,\hat{\mathbb{A}}_{(1)}. \tag{7.21}$$

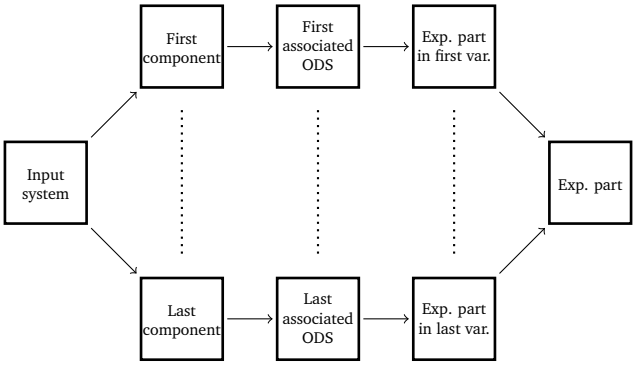


Figure 7.1: Computing the exponential part from associated ODS's

Hence, the systems given by $[\alpha_1\ \mathbb{A}_{(1)}(x_1 = t_1^{\alpha_1})]$ (resp. $[\mathbb{A}_{(1)}(x_1)]$) and $[\hat{\mathbb{A}}_{(1)}(t_1)]$ are equivalent. It follows that they have the same formal invariants.
Clearly, the same result can be obtained for any i^{th} component of the system $[\underline{A}]$, $i \in \{1, \ldots, m\}$, by permuting it with the first component. ■

| Lemma 7.3

The x_i-exponential parts of system $[\underline{A}]$ can be determined by a truncation of total order $\max(\{np_i \mid 1 \leq i \leq m\})$ of each of the components $A_{(i)}$, $i \in \{1, \ldots, m\}$.

Proof. It suffices to remark that for any associated ODS, the exponential part is determined by, at most, the first np_i coefficients [9, 90]. ■

Let $\underline{p}_{true}(\underline{A}) = (p_{1,true}, \ldots, p_{m,true})$ denote the true Poincaré rank of system $[\underline{A}]$, i.e., the m-tuple of the minimal integers bounding the maximal growth orders in the m variables. In other words, it is the m-tuple of the minimal integers satisfying the following inequalities:

$$p_{i,true} - 1 \ < \ \omega_i \ \leq \ p_{i,true}, \quad i \in \{1, \ldots, m\}.$$

Hence, the above theorem gives an insight to this true Poincaré rank of system $[\underline{A}]$ as well and allows its characterization in terms of the invariants of the associated ODS.
It is shown by Deligne and van den Essen separately in [48, 53], for the multivariate case in geometric and algebraic settings respectively, that a necessary and sufficient condition for system $[\underline{A}]$ to be regular singular is that each individual i^{th} component $[A_{(i)}]$, $1 \leq i \leq m$, considered as a system of ordinary differential equations in x_i, with the remaining variables held as transcendental constants, is regular singular. As a consequence, system $[\underline{A}]$ is regular singular if and only if its true Poincaré rank is $(0, 0, \ldots, 0)$. And to test this regularity, algorithms available for the univariate case of $m = 1$ (e.g. [32, 87]) can be applied separately to each of the individual components[‡]. The following corollary

[‡]Transformations computed with this approach might alter the normal crossings of such systems. Hence, although they serve a theoretical purpose here, rank reduction algorithm will be constructed differently in Chapter 8.

follows directly from Theorem 7.3, giving a characterization of the true Poincaré rank in both the regular and irregular cases.

| Corollary 7.2

Under the notations of this section, we have

$$\underline{p}_{true}(\underline{A}) \;=\; \underline{p}_{true}(\underline{\mathbb{A}}).$$

For the proof, it suffices to remark that[§]

$$p_{true}(\mathbb{A}_{(i)}) \;-\; 1 \;<\; \omega_i \;=\; \omega(\mathbb{A}_{(i)}) \;\leq p_{true}(\mathbb{A}_{(i)}).$$

Hence, the formal exponential order, the true Poincaré rank, and most importantly the Q_i's in (7.4), can be computed efficiently over a univariate rather than a multivariate field using the existing packages (see Chapter 1). As mentioned in the introduction, this exponential part is of central importance in applications since it determines the asymptotic behaviour of the solution in the neighborhood of a singularity. However, one might also be interested in computing a full fundamental matrix of formal solutions FMFS as given by (7.4). As we have seen in Chapter 1, the two pillars of formal reduction in case of nilpotent leading coefficients are the formal order growth and rank reduction. So, in order to give an algorithm to compute a fundamental matrix of formal solutions, it remains to give a compatible rank reduction for system $[\underline{A}]$.
We assume in the next section, that we have at hand such an algorithm and we give an algorithm of formal reduction. A rank reduction algorithm will then be investigated in Chapter 8.

7.3 Formal reduction algorithm

Given system $[\underline{A}]$, we discuss the eigenvalues of the leading matrix coefficients of its m components. If for all of these components uncoupling is unattainable, then we fix $i \in \{1, \ldots, m\}$ and proceed to compute the exponential order ω_i from the associated ODS. Suppose that for some i, we compute $\omega_i = \frac{\ell}{d}$ with ℓ, d coprime positive integers (see Figure 7.3 and Algorithm 7). We can then set $t_i = x_i^{1/d}$ and perform again rank reduction to get an equivalent system whose i^{th} component has Poincaré rank equal to ℓ and leading matrix coefficient with at least d distinct eigenvalues. Consequently, block-diagonalization can be re-applied to uncouple the i^{th} component. By Subsection 7.1.4, this uncoupling results in an uncoupling for the whole system. This procedure can be repeated until we attain either a scalar system, i.e. a system whose n components are scalar equations, or a system whose Poincaré rank is given by $(0, \ldots, 0)$. The former is trivial and effective algorithms are given for the latter in [84, Chapter 3].

[§]Stronger bounds are given in [14, Remark 3]

Algorithm 7 FMFS_PFAFF Computes a fundamental matrix of formal solutions (7.4) of a completely integrable Pfaffian system with normal crossings given by (6.14)

Input: $\underline{p} = (p_1, \dots p_m)$, $\underline{A}(x) = (A_{(1)}, \dots, A_{(m)})$ of (6.14)
Output: A fundamental matrix of formal solutions (FMFS) (7.4)

$\{C_i\}_{1\leq i\leq m} \leftarrow O_n$; $\{Q_i\}_{1\leq i\leq m} \leftarrow O_n$; $\Phi \leftarrow I_n$;
while $n \neq 1$ or $p_i > 0$ for some $i \in \{1, \dots, m\}$ **do**
  if $A_{(i),0}$ has at least two distinct eigenvalues **then**
    Split system $[\underline{A}]$ as in Section 7.1.4; Update Φ;
    FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}^{11}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
    FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}^{22}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
  else if $A_{(i),0}$ has one non-zero eigenvalue **then**
    Update Q_i from the eigenvalues of $A_{(i),0}$;
    $\underline{A}(x) \leftarrow$ Follow Subsection 7.1.5; ($A_{(i),0}$ is now nilpotent);
    FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
  else
    Apply rank reduction of Chapter 8; Update Φ;Update p; Update $A_{(i),0}$;
    if $p_i > 0$ and $A_{(i),0}$ has at least two distinct eigenvalues **then**
      Split system as in Section 7.1.4; Update Φ;
      FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}^{11}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
      FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}^{22}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
    else if $A_{(i),0}$ has one non-zero eigenvalue **then**
      Update Q_i from the eigenvalues of $A_{(i),0}$;
      $\underline{A}(x) \leftarrow$ Follow Subsection 7.1.5; ($A_{(i),0}$ is now nilpotent);
      FMFS_PFAFF $(\underline{p}, \tilde{\underline{A}}(x))$; Update Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
    else
      Follow Section 7.2 to compute $\omega_i = \frac{\ell}{d}$;
      $x_i \leftarrow {x_i}^d$;
      Apply rank reduction of Chapter 8; Update Φ; $p_i \leftarrow \ell$; Update $A_{(i),0}$;
      Update Q_i from the eigenvalues of $A_{(i),0}$;
      $\underline{A}(x) \leftarrow$ Follow Subsection 7.1.5; ($A_{(i),0}$ is now nilpotent);
      FMFS_PFAFF $(\underline{p}, \underline{A}(x))$; Update$\Phi$, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$;
    end if
  end if
end while
return $\underline{p}$, $\underline{A}$, Φ, $\{C_i\}_{1\leq i\leq m}$, $\{Q_i\}_{1\leq i\leq m}$.

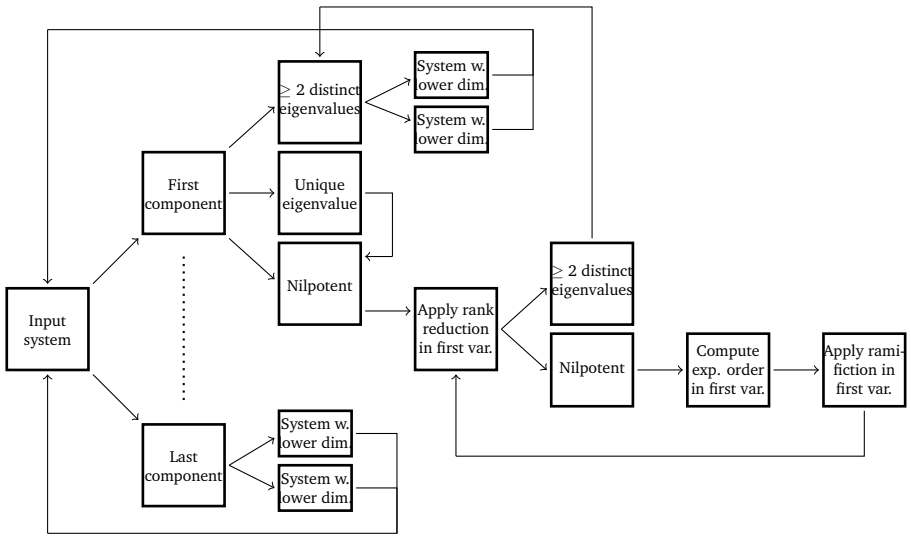


Figure 7.2: Computing a fundamental matrix of formal solutions by studying one of the components, e.g. the first component. The other components would follow the chosen component in the uncoupling.

7.4 Conclusion

In this chapter, we gave an explicit method to compute the exponential parts of completely integrable Pfaffian systems with normal crossings in several variables. This gives the main information about the asymptotic behavior of its solutions. Moreover, this new approach limits the computations to a finite number of constant matrices instead of matrix-valued functions, limits the number of coefficients we need in the expansions in computations to get the exponential part, and constitutes an eminent portion of the process of formal reduction. To complement our work and compute a fundamental matrix of formal solutions, we propose Algorithm 7 which demand a compatible rank reduction algorithm. Such an algorithm is investigated in the following chapter.

Chapter 8

Rank Reduction

Contents

Consider again system $[\underline{A}]$ given by (6.14):

$$[\underline{A}] \qquad x_i^{p_i+1}\,\partial_i\,F = A_{(i)}(x)\,F, \quad 1 \leq i \leq m.$$

The Poincaré rank of the system given by $\underline{p} = (p_1, \ldots, p_m)$ and its true Poincaré rank is given by $\underline{p}_{true}(\underline{A}) = (p_{1,true}, \ldots, p_{m,true})$.

In this chapter, we are interested in the rank reduction of Pfaffian systems, i.e. the explicit computation of a transformation which, given system $[\underline{A}]$, yields an equivalent system whose Poincaré rank is the true Poincaré rank. In other words, we show that the true Poincaré ranks of the m components of $[\underline{A}]$ can be attained simultaneously via a transformation compatible with $[\underline{A}]$ in the bivariate case and under certain conditions in the multivariate case.

As mentioned in the introduction, such an algorithm is developed for bivariate Pfaffian systems in [26] upon generalizing the univariate algorithm of Levelt. Efficiency considerations aside (see Subsection 5.6.1), the former algorithm results in an equivalent system whose Poincaré rank is minimal but does not guarantee that the leading matrix coefficients have minimal algebraic ranks, unlike algorithms based on Moser's criterion. In the univariate case, the latter condition is indispensable for the formal reduction of Algorithm 1.

This chapter is organized as follows: We first discuss two reduction criteria in Section 8.1 within the attempt to generalize Moser's criterion (see Section 1.5.1) and guarantee minimizing both ranks

(the Poincaré's and the algebraic). We then generalize Moser-Barkatou's algorithm of [15] to treat bivariate and, under certain prescribed conditions, multivariate systems. We state our main theorem in Section 8.2. Not only is the generalization of this algorithm to the bivariate case nontrivial, but also the transition from the bivariate setting to the multivariate setting is not straightforward and the obstacle is common in literature. For instance, Theorem 7.1 was first given and proved in [41] for bivariate systems. In the theory developed there, one operator (component) was considered and facts about principal ideal domains (which arise once eliminating one variable) were used in many places to prove that certain modules introduced are free modules. This does not allow an immediate generalization to the case of more than two variables since one can no more rely on properties of such domains. The same obstacle arises in Barkatou-LeRoux-Levelt rank reduction in [26] and in the analogous theory of formal decomposition of m commuting partial linear difference operators studied by Praagman [115]. This problem manifests itself within our algorithm in the column reduction which we discuss in Section 8.3. We then give a constructive proof of our main theorem in Section 8.4.

8.1 Reduction criterion

As mentioned in Section 7.1.1, algorithms available for the univariate case can be adapted to the multivariate setting by considering one of the variables (whose Poincaré rank is to be reduced) as the "principal" variable and regarding the others as transcendental constants. However, we have also seen that the compatibility of the transformations computed via such algorithms with the Pfaffian system is not guaranteed (see Example 7.1). Moreover, computations over bivariate and multivariate fields have a different nature. In order to develop an algorithm suitable for the multivariate setting, we suggest the following two reduction criteria.

8.1.1 Moser's criterion for the associated ODS systems

Consider system $[\underline{A}]$ given by (6.14), we recall that its associated ODS system is defined by (7.17) as follows (see Definition 7.4):

$$[\mathbb{A}] \qquad x_i^{p_i+1}\,\partial_i\,F = \mathbb{A}_{(i)}(x_i)\,F, \quad 1 \leq i \leq m,$$

where $\mathbb{A}_{(i)}(x_i) := A_{(i)}(x_1 = 0, \ldots, x_{i-1} = 0, x_i, x_{i+1} = 0, \ldots, x_m = 0)$. Based on Corollary 7.2, a possible reduction criterion is that of the associated ODS system itself. Hence, we can define the m-tuples of rational numbers as follows*:

$$\underline{m}(\underline{\mathbb{A}}) = (m(\mathbb{A}_{(1)}), \ldots, m(\mathbb{A}_{(m)})) \quad \text{where} \quad m(\mathbb{A}_{(i)}) = \max\,(0, p(\mathbb{A}_{(i)}) + \frac{rank\,(A_{(i),\underline{0}})}{n})$$

$$\underline{\mu}(\underline{\mathbb{A}}) = (\mu(\mathbb{A}_{(1)}), \ldots, \mu(\mathbb{A}_{(m)})) \quad \text{where} \quad \mu(\mathbb{A}_{(i)}) = \min\,\{m(T[\mathbb{A}_{(i)}]) \mid T \in GL_n(\mathrm{R_L})\}.$$

*We are adopting the same notations for the Moser rank and invariant in the literature in the hope that no confusion would arise between $m(A_{(i)})$ which denotes Moser's rank and m which denotes the number of variables.

It then follows from the above definitions that if $\underline{\mu}(\underline{\mathbb{A}}) = \underline{m}(\underline{\mathbb{A}})$ then the Poincaré rank of system (6.14) is its true Poincaré rank. In particular, it is regular singular if and only if $\underline{\mu}(\underline{\mathbb{A}}) \preceq \underline{1}$, i.e. the true Poincaré rank is the zero m-tuple. However, there is no guarantee that the converse holds: It might happen that the true Poincaré rank of system (7.17) (and consequently for system (6.14)) is attained but the minimal algebraic ranks of its constant leading matrices are not attained yet. This is a consequence of $rank\ (A_{(i),\underline{0}}) \leq rank\ (A_{(i),0}), 1 \leq i \leq m$. And so we might have $\underline{\mu}\ (\underline{\mathbb{A}}) \preceq \underline{m}\ (\underline{\mathbb{A}})$ although system (6.14) attained its true Poincaré rank and its leading matrix coefficients attained their minimal algebraic ranks. Hence, we propose a more precise criterion in the next subsection.

8.1.2 Generalized Moser's criterion

We consider again a multivariate Pfaffian system $[\underline{A}]$ given by (6.14). We define the generalized Moser rank and invariant of this system respectively as the following m-tuples of rational numbers (see [15, 107] for univariate systems):

$$\begin{aligned} \underline{m}(\underline{A}) &= (m(A_{(1)}), \ldots, m(A_{(m)})) \qquad \text{where} \qquad m(A_{(i)}) = \max\,(0, p_i + \frac{rank\ (A_{(i),0})}{n}) \\ \underline{\mu}(\underline{A}) &= (\mu(A_{(1)}), \ldots, \mu(A_{(m)})) \qquad \text{where} \qquad \mu(A_{(i)}) = \min\,\{m(T[A]) | T \in GL_n(\mathrm{R_L})\}. \end{aligned}$$

We remark that $\underline{\mu}(\underline{A})$ is well-defined due to Corollary 7.2 and the implication that the true Poincaré rank of the system is attained whenever the true Poincaré ranks of its m components are individually and simultaneously attained.

Definition 8.1

The system $[\underline{A}]$ (the matrix $A(x)$ respectively) is called reducible if $\underline{\mu}(\underline{A}) \prec \underline{m}(\underline{A})$, otherwise it is said to be irreducible.

In other words, System $[\underline{A}]$ is irreducible whenever each of its components is. In particular, it is easy to see from this definition that the system is regular singular if and only if $\underline{\mu}(\underline{A}) \preceq \underline{1}$, i.e. the true Poincaré rank is the zero m-tuple, which coincides with Deligne's and van den Essen's criterion.

8.2 Rank reduction algorithm

By the aid of compatible transformations and the criterion established in Subsection 8.1.2, we study the rank reduction of some component $A_{(i)}(x)$ of $[A]$ which is given by (6.14). We will see that rank reduction can be carried out for each component independently without affecting the individual Poincaré ranks of the other components. We fix $i \in \{1, \ldots, m\}$ and we recall that one can expand the components of $[A]$ w.r.t. x_i. In particular, for the fixed index i we have,

$$x_i^{p_i+1}\partial_i F = A_{(i)}(x)F = (A_{(i),0}(\bar{x}_i) + A_{(i),1}(\bar{x}_i)x_i + A_{(i),2}(\bar{x}_i)x_i^2 + \ldots)F.$$

We set

$$r_i := \text{rank}(A_{(i),0}).$$

For all i we can assume without loss of generality that $A_{(i),0}$ is not the zero matrix and thus the reducibility of system $[A]$ coincides with the existence of an equivalent system such that for some i the rank of the leading matrix coefficient $A_{(i),0}$ is less than r_i. We establish the following theorem which clearly is the multivariate generalization of [107, Theorems 1 and 2, pp 381] for the univariate ODS, and analog of Theorems 5.2 and 5.3 for singularly-perturbed systems discussed in Part II.

| Theorem 8.1

Given a Pfaffian system $[A]$ with $m(A_{(i)}) > 1$ for some $i \in \{1, \ldots, m\}$. If $\mu(A_{(i)}) < m(A_{(i)})$ then the polynomial

$$\theta_i(\lambda) := {x_i}^{r_i} \det(\lambda I + \frac{A_{(i),0}}{x_i} + A_{(i),1})|_{x_i=0} \tag{8.1}$$

vanishes identically in λ. Conversely, if $\theta_i(\lambda)$ vanishes, then under prescribed conditions we can construct a compatible transformation $T \in GL_n(\mathrm{R}_L)$ which reduces $m(A_{(i)})$. Moreover, T can be chosen to be a product of transformations in $GL_n(\mathrm{R}_{\bar{x}_i})$ and polynomial transformations of the form $\text{diag}(x_i^{\beta_1}, \ldots, x_i^{\beta_n})$ where $\beta_1, \ldots, \beta_n$ are non-negative integers.

We first prove that the statement is necessary: We give here the proof of the necessary condition of Theorem 8.1. The proof of the sufficiency follows after a series of intermediate results that will also describe the intermediate steps in the algorithm. We will need two kinds of transformations, shearing transformations and column reductions, which will be explained in the next section.

Proof of Theorem 8.1 (Necessary condition). Suppose that there exists a transformation $T(x) \in GL_n(\mathrm{R_L})$ which reduces $m(A_{(i)})$ for some $i \in \{1, \ldots, m\}$, i.e. , setting $T[A] := \tilde{A}$, we have:

$$m(\tilde{A}_{(i)}) < m(A_{(i)}).$$

We can then view the i^{th} component of system $[A]$ as a system of ordinary differential equations (ODS) in x_i upon considering the x_j's for $j \neq i, j \in \{1, \ldots, m\}$, to be transcendental constants. Hence, by [107, Theorem 1], $\theta_i(\lambda) = 0$. ■

For a given index $i \in \{1, \ldots, m\}$, the algebraic rank of the leading coefficient matrix can be decreased as long as $\theta_i(\lambda)$ vanishes identically in λ. In case the leading matrix coefficient eventually reduces to a zero matrix, the Poincaré rank p_i drops at least by one. This process can be repeated until the Moser rank of system $[A]$ equate its Moser invariant. Due to the compatibility of $T(x)$ in Theorem 8.1 with system $[\underline{A}]$, rank reduction can be applied to any of the components of $[\underline{A}]$ without altering the Moser rank of the others. Hence, by Corollary 7.2, the true Poincaré rank of system $[\underline{A}]$ can be attained by a successive application of the rank reduction algorithm to each of its components. We remark that this process terminates because the Poincaré rank cannot be dropped further than zero. Hence, we have the following corollary.

| Corollary 8.1

Given System (6.14). A necessary and sufficient condition for $[\underline{A}]$ to be reducible, i.e. for the existence of a $T(x) \in Gl(V)$ such that the algebraic rank of at least one of the leading matrix coefficients can be decreased strictly, is that at least one of the polynomials

$$\theta_i(\lambda) := {x_i}^{r_i)} \ \det(\lambda I + \frac{A_{(i),0}}{x_i} + A_{(i),1})|_{x_i=0}, \quad 1 \leq i \leq m,$$

vanishes identically in λ. Moreover, under prescribed conditions identified within its construction, $T(x)$ can be chosen to be in $GL_n(\mathrm{R_L})$ and to be compatible with system $[\underline{A}]$.

By Corollary 7.2, the true Poincaré rank of system (6.14) can be deduced from its associated ODS. If rank reduction is applied within a formal reduction algorithm (Algorithm 7), then one does not need a criterion to detect the true Poincaré rank, as it can be deduced from the associated ODS. Nevertheless this criterion is essential as it furnishes the construction of the change of basis $T(x)$ as will be demonstrated in later sections. Moreover, it allows the rank reduction algorithm to be implemented as a stand-alone procedure which is independent of the formal reduction itself.
The proof of the sufficient condition of Theorem 8.1 follows after a series of intermediary results. As we have already experienced with rank reduction algorithms, this algorithm is based on two kinds of transformations: shearing and column reduction (as in Lemma 5.1). We first explain the conditions required in the following section.

8.3 Column reduction

To enable rank reduction, we alternate between the shearing transformation and transformations which reduce some columns of a leading matrix coefficient to zero:

(P) Given a square matrix $A(x) = [v_1, \ldots, v_n] \in \mathcal{M}_n(\mathrm{R})$ (where v_i denotes the ith column of A) of rank $r < n$ when considered as an element of $\mathcal{M}_n(\mathrm{Frac}(\mathrm{R}))$. It is required to find a *unimodular* transformation $T \in GL_n(\mathrm{R})$ such that the last $n - r$ columns of TAT^{-1} are zero columns.

Before considering the algorithmic aspects, we first discuss the existence of such a transformation. As the next example shows, the desired transformation does not necessarily exist for any matrix A.

| Example 8.1

The matrix

$$\begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is obviously of rank 1 over $\mathrm{Frac}(\mathrm{R})$. There is, however, no unimodular transformation $T \in GL_3(\mathrm{R})$ such that TAT^{-1} contains only one non-zero column.

□

Consider the finitely generated R^n-submodule $M := \langle v_1, \ldots, v_n \rangle$. We call it the column module of A. In order to construct a suitable transformation for bivariate Pfaffian systems (for which the leading matrix coefficients are univariate), one can use the fact that $\mathbb{C}[[x_1]]$ and $\mathbb{C}[[x_2]]$ are principal ideal domains and hence that every finitely generated submodule of a free module over this ring is free. We generalize this for the multivariate case by showing in Corollary 8.2 that the freeness of the column module M is a necessary and sufficient condition for the existence of a transformation that meets our requirements. This is a direct consequence of Nakayama's Lemma for local rings.

| Theorem 8.2

Let R be a local ring, $\mathcal{M}$ its maximal ideal and let M be a finitely generated R-module. Then $v_1, \ldots, v_r \in M$ form a minimal set of generators for M if and only if their images $\bar{v}_1, \ldots, \bar{v}_r$ under the canonical homomorphism $M \to M/\mathcal{M}M$ form a basis of the vector space $M/\mathcal{M}M$ over the field $\mathrm{R}/\mathcal{M}$.

Proof. See [102, Theorem 2.3, pp 8]. ■

The central consequence of Theorem 8.2 for us is that if M is free, a module basis of M can be chosen among the columns of A. We adapt the theorem to our situation to show that we can bring A into a column-reduced form if and only if its column module is free.

| Corollary 8.2

Let $A(x) \in \mathcal{M}_n(\mathrm{R})$ be of rank r over $\mathrm{Frac}(\mathrm{R})$ and let M be the module generated by the columns of A. If M is free, then there exists a subset B of the columns in A with r elements such that B is a module basis of M. Furthermore, B is also a $\mathrm{Frac}(\mathrm{R})$-vector space basis of the column space of A.

Proof. By Theorem 8.2 we can find a basis B of M among the columns of A. By definition, the elements of B are linearly independent over R, so they are also linearly independent over $\mathrm{Frac}(\mathrm{R})$ (otherwise, multiplying a linear relation in $\mathrm{Frac}(\mathrm{R})$ with a common denominator yields a relation in R). Since B is a basis of the column module, it also contains a generating set of the $\mathrm{Frac}(\mathrm{R})$-vector space generated by the columns of A. ■

In theory, Corollary 8.2 would allow the computation of a unimodular column reduction transformation as required in (P) simply via Gaussian elimination. Assume we are given a matrix A and already know a subset $B = \{b_1, \ldots, b_r\}$ of the columns of A which forms a basis of the column module. Let v be a column vector of A which is not in B. Then, since B is a vector space basis, there exist $c_1, \ldots, c_r \in \mathrm{Frac}(\mathrm{R})$ such that

$$c_1 b_1 + \cdots + c_r b_r = v.$$

By assumption, B is also a module basis, so there also exist $d_1, \ldots, d_r \in \mathrm{R}$ with

$$d_1 b_1 + \cdots + d_r b_r = v.$$

The b_i are linearly independent, and therefore the cofactors of v with respect to B are unique. It follows that $c_i = d_i$ for all $1 \leq i \leq r$.
The main algorithmic difficulty stems from the fact that not all formal power series admit a finite representation, as in the introductory example, and even if the initial system is given in a finite form, the splitting transformation as in Theorem 7.2 does not preserve finiteness. In particular, we face two main problems when working with truncated power series:

$(P1)$ Detecting the correct rank and the linear independent columns of A

$(P2)$ If we know the independent columns, a column reduction transformation computed after truncation (in total order) is not uniquely determined.

These computational problems arise for general multivariate and for bivariate systems, but were not addressed in previous algorithmic works on this topic [26, 84]. Before we propose our resolution, we illustrate both problems in the following example:

| Example 8.2

Consider the matrix

$$\begin{pmatrix} x & 0 & x^2 & x^2+x \\ 0 & x & x & x \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Then it the first three columns v_1, v_2, v_3 are linearly independent and generate the column module. A linear combination of the fourth column v_4 is given by

$$1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 = v_4.$$

When truncating at order 1, the system is given as

$$\begin{pmatrix} x & 0 & 0 & x \\ 0 & x & x & x \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The original rank cannot be determined from the truncated matrix. Furthermore, even if we know that v_1, v_2, v_3 are linearly independent, there are several linear combinations of the fourth column after truncation:

$$1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 = v_4.$$

$$1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 = v_4.$$

The cofactors of the second linear combination are not the truncated cofactors of the first. It can not be extended with higher order terms to a linear combination over the formal power series ring without truncation.

□

We can solve both $(P1)$ and $(P2)$ with the help of minors of the original system. Let r be the rank of A. Then there exists a nonzero $r \times r$ submatrix B of A whose determinant is nonzero. Let k be the order of

the determinant. If we take the truncated system $\tilde{A} = A \operatorname{rem} x^{k+1}$, the same submatrix $\tilde{B}$ in $\tilde{A}$ will have a non-zero determinant modulo x^{k+1} and we can therefore identify in $\tilde{A}$ which columns in A are linearly independent. This resolves $(P1)$ as long as the truncation order k is chosen to be big enough.
Now assume that for instance the first r columns of A are linearly independent, i.e. we can choose B such that its columns correspond to $v_1, \ldots, v_r$. Let k be as above, ℓ be a positive integer and let v be a column vector that is linearly dependent on the columns of B. Then there exist $c_1, \ldots, c_r \in \mathrm{R}$ such that

$$B \cdot (c_1, \ldots, c_r) = v.$$

By Cramer's rule, we know that the c_i are given by

$$c_i = \frac{\det(B_i)}{\det(B)}, \tag{8.2}$$

where B_i is the matrix obtained by replacing the i^{th} column of B by v. Rewriting Equation (8.2) gives

$$\det(B)c_i - \det(B_i) = 0, \tag{8.3}$$

and this equation allows the computation of c_i by coefficient comparison. In particular, we are guaranteed to obtain the correct c_i up to order ℓ if in (8.3) we replace B by $\tilde{B}$, its truncation at order $\ell + k + 1$, and B_i by $\tilde{B}_i$, the truncation of B_i at order $\ell + k + 1$. This resolves $(P2)$.
This approach is based on the fact that there is a truncation order such that we can find a submatrix of maximal dimension with non-zero determinant. We have to remark, however, that by the nature of formal power series, it is in general not possible to tell a priori if a given truncation is high enough. Only after computing a wrong result (in comparison to what we can compute with the associated ODS) can we determine that the truncation order has to be increased. This procedure necessarily terminates, since there exists a suitable truncation order. Furthermore, not every $\mathrm{Frac}(\mathrm{R})$-vector space basis of the column space of A is also a module basis, so, in the worst case, $\binom{n}{r}$ submatrices have to be tested.
Subsequently, we will refer to the following conditions whenever necessary:

> We say that $(\mathcal{C})$ (resp. $(\mathcal{R})$) is satisfied if the column (resp. row) module of the matrix under consideration is free.

Evidently, $(\mathcal{C})$ and $(\mathcal{R})$ are always satisfied in the any of the following cases:

- $m = 2$, since $\mathrm{R}_{\bar{x}_1}$ and $\mathrm{R}_{\bar{x}_2}$ would be principal ideal domains. Gaussian elimination can be thus performed by taking as pivots the entries with least valuation in x within the row or column, depending on the context.
- $n = 2$, since one can perform elementary row and/or column operations taking as pivot an entry with a non-zero constant term (see, e.g., Arnold's form [7]).

8.4 Proof of Theorem 8.1

We consider again the multivariate system $[\underline{A}]$ given by (6.14). We fix $i \in \{1, \ldots, m\}$ and investigate the rank reduction of the i^{th} component of (6.14) given by

$$x_i^{p_i+1}\, \partial_i F \;=\; A_{(i)}(x)\; F \;=\; (A_{(i),0} \;+\; A_{(i),1}\, x_i \;+\; A_{(i),2}\, x_i^2 \;+\; A_{(i),3}\, x_i^3 \;+\; \ldots)\; F, \tag{8.4}$$

where the matrices $A_{(i),k}$, $k \geq 0$, have their entries in $\mathrm{R}_{\bar{\mathrm{x}}_\mathrm{i}}$ and the algebraic rank of $A_{(i),0}$ is denoted by r_i.

The proof of Theorem 8.1 for the reduction in x_i follows essentially the steps of that of Section 5.5 in ξ but requires additional structure on the leading matrix coefficient to guarantee the compatibility of the shearing transformations with system $[\underline{A}]$. Hence, whenever adequate, we omit the details which are already described in Chapter 5.

The construction requires successive applications of unimodular and shearing transformations. We remark that in particular, to study the effect of a transformation $T \in GL_n(\mathrm{R}_{\bar{\mathrm{x}}_\mathrm{i}})$ on the i^{th} component of system $[\underline{A}]$, it suffices to study the similarity part since (7.2) reduces to

$$\tilde{A}_{(i)} \;=\; T^{-1}\, A_{(i)}\, T.$$

| Lemma 8.1

Under certain conditions which can be verified constructively, there exists a unimodular transformation $U(\bar{x}_i) \in GL_n(\mathrm{R}_{\bar{x}_i})$ such that for $\tilde{A}_{(i)} = U[A_{(i)}]$ we have

$$\tilde{A}_{(i),0} \;=\; \begin{pmatrix} \tilde{A}^{11}_{(i),0} & O & O \\ \tilde{A}^{21}_{(i),0} & O_{r-d} & O \\ \tilde{A}^{31}_{(i),0} & \tilde{A}^{32}_{(i),0} & O_{n-r} \end{pmatrix} \tag{8.5}$$

with diagonal blocks of sizes $d \times d$, $(r-d) \times (r-d)$ and $(n-r) \times (n-r)$ respectively for some $0 \leq d < r$ and where

$$\begin{pmatrix} \tilde{A}^{11}_{(i),0} \\ \tilde{A}^{21}_{(i),0} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \tilde{A}^{11}_{(i),0} & O \\ \tilde{A}^{21}_{(i),0} & O \\ \tilde{A}^{31}_{(i),0} & \tilde{A}^{32}_{(i),0} \end{pmatrix}$$

are $r \times d$ and $n \times r$ matrices of full column ranks d and r respectively.

Proof. If $A_{(i),0}$ satisfies the condition $(\mathcal{C})$ of Subsection 8.3, then one can compute a unimodular transformation $U_1 \in GL_n(\mathrm{R}_{\bar{x}_i})$ such that

$$U_1^{-1} A_{(i),0} U_1 = \begin{pmatrix} B^{11} & O \\ B^{21} & O \end{pmatrix}$$

has rank r, entries in $\mathrm{R}_{\bar{x}_i}$ and with diagonal blocks of sizes $r \times r$ and $(n-r) \times (n-r)$ respectively. Let d be the rank of B^{11}. If B^{11} also satisfies the condition $(\mathcal{C})$, then one can compute a unimodular transformation $U_2 \in GL_r(\mathrm{R}_{\bar{x}_i})$ such that

$$U_2^{-1} B^{11} U_2 = \begin{pmatrix} E^{11} & O \\ E^{21} & O \end{pmatrix}$$

has properties analogous to $U_1^{-1} A_{(i),0} U_1$. We set $U := \mathrm{diag}(U_2, I_{n-r}) \cdot U_1$. Then the leading coefficient $\tilde{A}_{(i),0}$ of the equivalent system $U[A_{(i)}]$ has the form (8.5). Clearly, U is compatible with system A since it is unimodular. ■

Hence we assume that the leading coefficient $A_{(i),0}(\bar{x}_i)$ of (8.4) is in form (8.5) and we partition $A_{(i),1}(\bar{x}_i)$ in accordance with $A_{(i),0}(\bar{x}_i)$. Let

$$G_{A_{(i)}}(\lambda) \;=\; \begin{bmatrix} A^{11}_{(i),0} & O & A^{13}_{(i),1} \\ A^{21}_{(i),0} & O & A^{23}_{(i),1} \\ A^{31}_{(i),0} & A^{32}_{(i),0} & A^{33}_{(i),1} + \lambda\, I_{n-r} \end{bmatrix}. \tag{8.6}$$

The λ-polynomial $\det(G_{A_{(i)}}(\lambda))$ vanishes identically in λ if and only if $\theta_i(\lambda)$ given by (8.1) does. In fact, let $D(x_i) = \mathrm{diag}(x_i I_r, I_{n-r})$. Then we can write ${x_i}^{-1} A_{(i)}(x) = N(x) D^{-1}(x_i)$ where $N(x) \in \mathcal{M}_n(\mathrm{R})$, and set $D_0 = D(x_i = 0)$, $N_0 = N(x_i = 0)$. Then we have

$$\begin{aligned} \det(G_{A_{(i)}}(\lambda)) &= \det(N_0 + \lambda D_0) = \det(N + \lambda D)|_{x_i=0} \\ &= (\det(\frac{A}{x_i} + \lambda I_n)\ \det(D))|_{x_i=0} \\ &= (\det(\frac{A_{(i),0}}{x_i} + A_{(i),1} + \lambda I_n)\ {x_i}^r)|_{x_i=0}, \end{aligned}$$

and hence

$$\det(G_{A_{(i)}}(\lambda)) = \theta_i(\lambda). \tag{8.7}$$

$G_{A_i}(\lambda)$ has an additional important utility within the construction of a desired transformation as we show in the following proposition.

| Proposition 8.1

Suppose that $m(A_{(i)}) > 1$ and $\det\ (G_{A_{(i)}}(\lambda))$ is identical to zero, where $G_{A_{(i)}}(\lambda)$ is given by (8.6). Then, under explicitly described conditions, there exists a unimodular matrix $Q(\bar{x}_i)$ in $GL_n(\mathrm{R}_{\bar{x}_i})$ with $\det\ Q(\bar{x}_i) = \pm 1$, compatible with system $[\underline{A}]$, such that the matrix $G_{\tilde{A}_{(i)}}(\lambda))$ has the form

$$G_{\tilde{A}_{(i)}}(\lambda)) \;=\; \begin{bmatrix} A^{11}_{(i),0} & O & U_1 & U_2 \\ A^{21}_{(i),0} & O & U_3 & U_4 \\ V_1 & V_2 & W_1 + \lambda\, I_{d-r-\varrho} & W_2 \\ M_1 & M_2 & M_3 & W_3 + \lambda\, I_\varrho \end{bmatrix}, \tag{8.8}$$

where $0 \leq \varrho \leq n-r$, M_2 *is a null matrix, and*

$$rank \begin{bmatrix} A^{11}_{(i),0} & U_1 \\ A^{21}_{(i),0} & U_3 \\ M_1 & M_3 \end{bmatrix} = rank \begin{bmatrix} A^{11}_{(i),0} & U_1 \\ A^{21}_{(i),0} & U_3 \end{bmatrix}, \tag{8.9}$$

$$rank \begin{bmatrix} A^{11}_{(i),0} & U_1 \\ A^{21}_{(i),0} & U_3 \end{bmatrix} < r_i. \tag{8.10}$$

Proof. The transformation $Q(\bar{x}_i)$ can be constructed as in the proof of Proposition 5.2 under the condition (R) of Section 8.3 and its weaker forms whenever necessary for intermediate matrices whose row vector space is inspected. Since $Q(\bar{x}_i)$ is unimodular, it is compatible with system $[\underline{A}]$. It remains to remark however, that each row of $[M_1 \quad M_2 \quad M_3]$ is a linear combination of the rows of

$$\begin{bmatrix} A^{11}_{(i),0} & O & U_1 \\ A^{21}_{(i),0} & O & U_3 \end{bmatrix}.$$

Hence, by construction, M_2 is a zero matrix. ■

We remark that in the particular case of $d = 0$ (8.5) is given by

$$\tilde{A}_{(i),0}(\bar{x}_i) = \begin{pmatrix} O_r & O \\ \tilde{A}^{32}_{(i),0} & O_{n-r} \end{pmatrix}, \quad \text{with} \quad rank(\tilde{A}^{32}_{(i),0}) = r.$$

Consequently, it can be easily verified that (8.8) is given by

$$G_{\tilde{A}_{(i)}}(\lambda) = \begin{pmatrix} O_r & U_3 \\ V_2 & W_1 + \lambda I_{n-r} \end{pmatrix}, \quad \text{and} \quad \varrho = 0.$$

| Proposition 8.2

Consider system $[\underline{A}]$ *given by* (6.14). *If* $m(A_{(i)}) > 1$ *and* $\det\ (G_{A_{(i)}}(\lambda)) \equiv 0$ *is as in* (8.8) *with conditions* (8.9) *and* (8.10) *satisfied, then the* i^{th} *component* $A_{(i)}$ *of* $[\underline{A}]$ *is reducible and reduction can be carried out with the shearing* $F = S(x_i)\, G$ *where*

$$\begin{cases} S(x_i) = diag\ (x_i I_r, I_{n-r-\varrho}, x_i I_\varrho) & \text{if } \varrho \neq 0 \\ S(x_i) = diag(x_i I_r, I_{n-r}) & \text{otherwise.} \end{cases}$$

Furthermore, this shearing is compatible with system $[\underline{A}]$.

Proof. For any $j \in \{1, \dots, m\}$ we partition $A_{(j)}(x)$ in accordance with (8.8)

$$A_{(j)}(x) \;=\; \begin{bmatrix} A^{11}_{(j)} & A^{12}_{(j)} & A^{13}_{(j)} & A^{14}_{(j)} \\ A^{21}_{(j)} & A^{22}_{(j)} & A^{23}_{(j)} & A^{24}_{(j)} \\ A^{31}_{(j)} & A^{32}_{(j)} & A^{33}_{(j)} & A^{34}_{(j)} \\ A^{41}_{(j)} & A^{42}_{(j)} & A^{43}_{(j)} & A^{44}_{(j)} \end{bmatrix}, \quad 1 \le j \le m,$$

where $A^{11}_{(j)}, A^{22}_{(j)}, A^{33}_{(j)}, A^{44}_{(j)}$ are square matrices of dimensions $d, r-d, n-r-\varrho$, and ϱ respectively. It is easy to verify that the equivalent system $S[\underline{A}] := [\underline{\tilde{A}}]$ given by (7.1) admits the form

$$\tilde{A}_{(i)}(x) \;=\; \begin{bmatrix} A^{11}_{(i)} & A^{12}_{(i)} & x_i^{-1}A^{13}_{(i)} & A^{14}_{(i)} \\ A^{21}_{(i)} & A^{22}_{(i)} & x_i^{-1}A^{23}_{(i)} & A^{24}_{(i)} \\ x_iA^{31}_{(i)} & x_iA^{32}_{(i)} & A^{33}_{(i)} & x_iA^{34}_{(i)} \\ A^{41}_{(i)} & A^{42}_{(i)} & x_i^{-1}A^{43}_{(i)} & A^{44}_{(i)} \end{bmatrix} \;-\; x_i^{p_i}\; diag\,(I_r, O_{n-r-\varrho}, I_\varrho)$$

$$\tilde{A}_{(j)}(x) \;=\; \begin{bmatrix} A^{11}_{(j)} & A^{12}_{(j)} & x_j^{-1}A^{13}_{(j)} & A^{14}_{(j)} \\ A^{21}_{(j)} & A^{22}_{(j)} & x_i^{-1}A^{23}_{(j)} & A^{24}_{(j)} \\ x_jA^{31}_{(j)} & x_jA^{32}_{(j)} & A^{33}_{(j)} & x_jA^{34}_{(j)} \\ A^{41}_{(j)} & A^{42}_{(j)} & x_j^{-1}A^{43}_{(j)} & A^{44}_{(j)} \end{bmatrix}, \quad 1 \le j \ne i \le m.$$

Hence, the leading matrix coefficient of the equivalent i^{th} component is given by

$$\tilde{A}_{(i),0}(\bar{x}_i) \;=\; \begin{bmatrix} A^{11}_{(i),0} & O & U_1 & O \\ A^{21}_{(i),0} & O & U_3 & O \\ O & O & O & O \\ M_1 & O & M_3 & O \end{bmatrix}$$

where $rank(\tilde{A}_{(i),0}) < r_i$ since (8.9) and (8.10) are satisfied.
It remains to prove the compatibility of $S(x_i)$ with the system $[\underline{A}]$, in particular, that the normal crossings is preserved. It suffices to prove that the entries of $A_j(x), 1 \le j \ne i \le m$, which are multiplied by x_i^{-1} upon applying $S(x_i)$, namely, the entries of $A^{13}_{(j)}, A^{23}_{(j)}$, and $A^{43}_{(j)}$ are zero matrices modulo x_i otherwise poles in x_i would be introduced. This can be restated as requiring $A^{13}_{(j)}(\bar{x}_i = 0), A^{23}_{(j)}(\bar{x}_i = 0)$, and $A^{43}_{(j)}(\bar{x}_i = 0)$ to be zero submatrices. This requirement is always satisfied due to the integrability condition and the resulting equality given by (7.6) which we restate here

$$x_j^{p_j+1}\, \partial_j A_{(i),0} \;=\; A_{(j)}(x_i = 0)\; A_{(i),0} \;-\; A_{(i),0}\; A_{(j)}(x_i = 0), \quad 1 \le j \ne i \le m.$$

and which induces a structure of $A_{(j)}(x_i = 0)$ depending on that of $A_{(i),0}$. Since $G_{A_{(i)}}(\lambda)$ is as in (8.8) then, before applying the shearing transformation, $A_{(i),0}(\bar{x}_i)$ have the form (8.11) and $A_{(j),0}(x_i = 0)$ can

be partitioned accordingly. And so we have for $1 \leq j \neq i \leq m$:

$$A_{(i),0}(\bar{x}_i) = \begin{bmatrix} A^{11}_{(i),0} & O & O & O \\ A^{21}_{(i),0} & O_{r-d} & O & O \\ V_1 & V_2 & O_{n-r-\varrho} & O \\ M_1 & O & O & O_{\varrho} \end{bmatrix}, \tag{8.11}$$

$$A_{(j)}(x_i = 0) = \begin{bmatrix} A^{11}_{(j)}(x_i = 0) & A^{12}_{(j)}(x_i = 0) & A^{13}_{(j)}(x_i = 0) & A^{14}_{(j)}(x_i = 0) \\ A^{21}_{(j)}(x_i = 0) & A^{22}_{(j)}(x_i = 0) & A^{23}_{(j)}(x_i = 0) & A^{24}_{(j)}(x_i = 0) \\ A^{31}_{(j)}(x_i = 0) & A^{32}_{(j)}(x_i = 0) & A^{33}_{(j)}(x_i = 0) & A^{34}_{(j)}(x_i = 0) \\ A^{41}_{(j)}(x_i = 0) & A^{42}_{(j)}(x_i = 0) & A^{43}_{(j)}(x_i = 0) & A^{44}_{(j)}(x_i = 0) \end{bmatrix}. \tag{8.12}$$

Inserting (8.11) and (8.12) in (7.6), one can obtain the desired results by equating the entries of (7.6) and argument similar to that in the proof of Lemma 7.1. More explicitly, upon investigating the entries in (Column 3), (Rows 1 and 2, Column 2), and (Row 4, Column 2), we observe the following respectively:

- $\begin{bmatrix} A^{11}_{(i),0} & O \\ A^{21}_{(i),0} & O \\ V_1 & V_2 \\ M_1 & O \end{bmatrix} \begin{bmatrix} A^{13}_{(j)}(x_i = 0) \\ A^{23}_{(j)}(x_i = 0) \end{bmatrix} = O_{n,n-r-\varrho}$. The former matrix is of full rank r_i by construction thus $\begin{bmatrix} A^{13}_{(j)}(x_i = 0) \\ A^{23}_{(j)}(x_i = 0) \end{bmatrix}$ is a zero matrix.

- $\begin{bmatrix} A^{11}_{(i),0} \\ A^{21}_{(i),0} \end{bmatrix} A^{12}_{(j)}(x_i = 0) = O_{r,r-d}$. The former is of full rank d by construction thus $A^{12}_{(j)}(x_i = 0)$ is a zero matrix.

- Finally, $A^{43}_{(j)}(x_i = 0)\ V_2 - M_1\ A^{12}_{(j)}(x_i = 0) = O_{\varrho,(r-d)}$. But $A^{12}_{(j)}(x_i = 0)$ is null and V_2 is of full column rank $r - d$ by construction and so $A^{43}_{(j)}(x_i = 0)$ is a zero matrix as well.

This completes the proof. ■

We give hereby the proof of sufficiency of Theorem 8.1.

Proof. (Theorem 8.1) For the sufficiency, under the notations of this section and the conditions described above, we can assume that $A_{(i),0}(\bar{x}_i)$ has the form (8.5). Let $G_{A_{(i)}}(\lambda)$ be given by (8.6). Then, by 8.7, $\det((G_{A_{(i)}}(\lambda)))$ vanishes identically in λ if and only if $\theta_i(\lambda)$ does. Then the system $S[Q[\underline{A}]]$ where S, Q are as in Propositions 8.1 and 8.2 respectively, has the desired property. ■

Algorithm 8 PFAFF_RANK_REDUCTION: Rank Reduction of System (6.14)

Input: $\{p_i,\ A_{(i)}(x)\}_{1\le i\le m}$ of (6.14)
Output: $T(x) \in GL(V)$ a transformation and an irreducible equivalent system $T[\underline{A}]$. In particular, the Poincaré rank of this system is its true Poincaré rank.

T $\leftarrow I_n$
For every i from 1 to m **do**
$T \leftarrow I_n$; $p_i \leftarrow$ Poincaré rank of $A_{(i)}$;
while the described conditions are satisfied **do**
  $U(\bar{x}_i) \leftarrow$ Lemma 8.1;
  $A_{(i)} \leftarrow U(\bar{x}_i)\ A_{(i)}\ U(\bar{x}_i)$; $T \leftarrow TU$;
  while $\det(G_{A_{(i)}}(\lambda)\ =\ 0)$ and $p_i > 0$ **do**
    $Q(\bar{x}_i), \rho \leftarrow$ Proposition 8.1;
    $S(x_i) \leftarrow$ Proposition 8.2;
    $P \leftarrow QS$; $T \leftarrow TP$;
    $A_{(i)} \leftarrow P^{-1}A_{(i)}P - x_i^{p_i}S^{-1}\frac{\partial S}{\partial x_i}$;
    $p_i \leftarrow$ Poincaré rank of $A_{(i)}$;
    $U(\bar{x}_i) \leftarrow$ Lemma 8.1;
    $A_{(i)} \leftarrow U(\bar{x}_i)\ A_{(i)}\ U(\bar{x}_i)$; $T \leftarrow TU$;
  end while
end while
For every $j \neq i$ from 1 to m **do** $A^{(j)} \leftarrow T^{-1}A_{(j)}T - x_j^{p_j+1}T^{-1}\frac{\partial T}{\partial x_j}$;
end for.
T $\leftarrow$ **T** T;
end for.
return (**T**, $A_{(i)}\ i \in \{1 \le i \le m\}$).

| Example 8.3

Examples of dimensions two and three do not give insight to the richness of the techniques presented. So we treat an example of dimension six. Due to the size of the system and the number of necessary computation steps, we are not able to include it directly in this thesis. It is available in several formats at my web page.

□

8.5 Conclusion

In this chapter, we gave a rank reduction criterion and an algorithm which reduces the Poincaré rank of system $[\underline{A}]$ to its true Poincaré rank via compatible transformations of $GL_n(\mathrm{R_L})$. Unlike the bivariate case, the column reduction which is required in the algorithm demands imposing certain conditions in the multivariate case. Although these conditions seem severe, the proposed algorithm gives an insight to the obstacles encountered and resolves the problem on the linear algebra level. The same obstacle of freeness, described in Section 8.3, arise in the attempt to generalize Levelt-Barkatou-LeRoux's algorithm given in [26] for bivariate systems, and in generalizing the univariate case algorithm described in Section 1.6 to construct solutions for regular-singular systems [84, Section 3.2]. An item of further investigation is thus to investigate whether the conditions can be eliminated or weakened. We remark however that the resolution of the linear algebra problem in Section 8.3 allows a generalization of Barkatou-LeRoux-Levelt's to the multivariate case under the condition $(\mathcal{C})$. Hence whenever a required condition of the form $(\mathcal{R})$ do not hold anymore in Algorithm 8, such a generalization can be used as long as condition $(\mathcal{C})$ holds. There remains however, the question on the equivalence of these conditions.

Conclusion and Applications

In this part, we studied completely integrable Pfaffian systems with normal crossings. We showed that one can associate a set of univariate linear singular differential systems from which the formal invariants of the former can be retrieved. This reduces computations to computations over a univariate field via MINIISOLDE or LINDALG, and limits the numbers of coefficients necessary for the computations. We then complemented our work with a rank reduction algorithm based on generalizing Moser's criteria and the algorithm given by Barkatou in [15]. The former is applicable to any bivariate system. However, for multivariate systems, it demands the satisfaction of explicitly described conditions.

We obtained our results via different types of expansions:

- We used a monomial expansion to uncouple the system to systems of lower dimension in case the leading matrix of at least one of the components has at least two distinct eigenvalues.
- To compute the formal invariants of the i^{th} component of system $[\underline{A}]$ for any $i \in \{1, \ldots, m\}$, we used expansions modulo $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$.
- For rank reduction of an i^{th} component of system $[\underline{A}]$, we used an expansion in the variable x_i.
- To study the compatibility of a transformation applied to the i^{th} component of system $[\underline{A}]$, we expanded the other components with respect to x_i.

These different series representations allowed us to retrieve information which was then patched up together to construct formal solutions or study their behavior. In spite of the obstacles met within this treatment, the possible approaches as well as possible series representations to such systems are not exhausted yet due to their rich structure. For instance, one possible approach to construct a FMFS without passing through rank reduction is to represent $T(x)$ in the proof of Theorem 7.3 in its monomial expansion in the light of the information about the formal invariants and construct it term by term.

We mentioned the work of [22] which computed closed form solutions of integrable connections in m variables recursively by adapting existing algorithms handling ordinary linear differential systems. The approach therein consists in considering $m-1$ variables as transcendental constants and running algorithms for ODS. It would be thus interesting to try instead to follow the approach developed herein of considering the associated ODS and studying what helpful information can be derived.

Another prominent field of investigation is the analytic (asymptotic) interpretation of the formal solutions and the recovery of "true" solutions. We mention [58, 95] in this direction and [57, 127] for therein consists regular singularity.

One field of investigation is the formal reduction in the difference case using the approaches given proposed herein. Praagman established in [115] a formal decomposition of m commuting partial linear difference operators. This study was intended as an analog to that established by Levelt, van den Essen, Gérard, Charrière, and others.

Unlike the univariate case, the algorithmic generation of examples on completely integrable Pfaffian systems (with random or special structure) does not seem to be trivial and would be interesting to investigate it. However, as mentioned in the introduction, Pfaffian systems arise naturally in many applications. Systems arising from applications do not necessarily or directly fall into the class of completely integrable Pfaffian systems with normal crossings. Our study in this thesis was restricted to Pfaffian systems with normal crossings at the origin. But what would happen in the general case or different locus? Investigations in this direction can be found in [111, 63, 128, 129] and references therein.

We exhibit herein two examples to point out the obstacles encountered in adapting the techniques developed herein but nevertheless their flexibility whenever the system at hand have a particular structure that can be exploited. The techniques developed so far can then be modified to deal with such a system on an individual basis. Namely, we discuss two bivariate Pfaffian systems, the first of which is perturbed by a parameter ε, and the second lacks the normal crossings discussed in this thesis.

Since both of the systems discussed here are bivariate systems, for the clarity of the presentation, we set $m = n = 2$, $x = x_1$, $y = x_2$, $A_{(1)}(x_1, x_2) = A(x, y)$, $A_{(2)}(x_1, x_2) = B(x, y)$, $\partial_1 = \partial_x$, and $\partial_2 = \partial_y$.

Perturbative quantum chromodynamics

The following system is a simplified version of systems arising in the field of perturbative quantum chromodynamics[†]. Generally the aim is to compute certain aspects of the interaction of elementary particles as precise as possible in order to compare it to experimental data from precision measurements. Interactions of elementary particles can be arbitrarily complicated, involving arbitrarily many intermediate particles, which are then called virtual particles. So one and the same interaction can happen in many different ways internally. The parameter y below is the square of the ratio of the masses of two of the particles involved and ranges from 0 to infinity while x ranges from 0 to 1.
Let $F = (f_1, \ldots, f_7)^T$ denote an unknown vector and ε be a parameter. We have the following Pfaffian system

$$\begin{cases} \partial_x F &= \frac{1}{\varepsilon\, x\, (x-1)\, (xy+\varepsilon-y+1)}\, A(x,y,\varepsilon)\, F \\ \partial_y F &= \frac{1}{y\, (y-1)\, (xy+\varepsilon-y+1)}\, B(x,y,\varepsilon)\, F \end{cases} \tag{8.13}$$

where

$$A(x,y,\varepsilon) = \begin{bmatrix} O_{4\times 4} & O_{4\times 3} \\ A^{21} & A^{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B^{11} & O_{4\times 3} \\ B^{21} & B^{22} \end{bmatrix}.$$

[†]We thank C. Raab and J.A. Weil for communicating to us this example and for discussions. At that time, C. Raab was a postdoctoral researcher at DESY, which is a research centre whose main site lies in Hamburg. The main focus of DESY is experimental physics in the areas of particle physics, photon science, and accelerator technology.

$A(x,y,\varepsilon)$ and $B(x,y,\varepsilon)$ have polynomial entries in $\varepsilon, x,$ and y over $\mathbb{Q}$. We do not show the explicit entries and the uninteresting lengthy computations in the sequel. Our aim is to show how the system can be uncoupled into systems whose form had been already discussed in this thesis. One can easily verify that system (8.13) is completely integrable.
The question here is to compute a solution around $y=1$ as $\varepsilon \to 0$.
Seemingly, the first component is singularly-perturbed in ε. However, we have:

$$A^{21}(x,y,\varepsilon=0) \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{12}(x,y,\varepsilon=0) = O_{4\times 3}.$$

Thus, the transformation $F=TG$ where $T=\operatorname{diag}(\varepsilon,\varepsilon,\varepsilon,1,1,1)$ yields an equivalent system regularly-perturbed in ε. This transformation is compatible with the second component since $B^{12}=O_{4\times 3}$. The singular perturbation was thus apparent. We now consider

$$\begin{cases} \partial_x G \;=\; \frac{1}{x\,(x-1)\,(xy+\varepsilon-y+1)}\;\tilde{A}(x,y,\varepsilon)\;G \;=\; \frac{1}{(x-1)\,x\,(xy+\varepsilon-y+1)} \begin{bmatrix} O_{4\times 4} & O_{4\times 3} \\ \tilde{A}^{21} & A^{22} \end{bmatrix} G \\ \partial_y G \;=\; \frac{1}{y\,(y-1)\,(xy+\varepsilon-y+1)}\;\tilde{B}(x,y,\varepsilon)\;G \;=\; \frac{1}{y\,(y-1)\,(xy+\varepsilon-y+1)} \begin{bmatrix} B^{11} & O_{4\times 3} \\ \tilde{B}^{21} & B^{22} \end{bmatrix} G \end{cases} \tag{8.14}$$

For fixed values of ε, one may test the existence of closed form solutions via the MAPLE package for Integrable Connections developed in [22]. In the sequel, we will be interested in a series expansion in ε. In the view of the denominators of system (8.13), one can expect from the outset the following as ε tends to zero:

- In the neighborhood of $(x=0, y=1)$ the system lacks normal crossings.
- In the neighborhood of $(1,1)$, the system is at worst regular singular in both x and y and a FMFS (7.4) takes the form:
$$\Phi(x,y,\varepsilon)\;(x-1)^{C_1(\varepsilon)}(y-1)^{C_2(\varepsilon)}$$
- In the neighborhood of $x \neq \{0,1\}$ and $y=1$, the system is at worst regular singular in $(y-1)$ and regular in x. Hence we get the following FMFS
$$\Phi(x,y,\varepsilon)\;(y-1)^{C_2(\varepsilon)}$$
An analogous claim is true for ($x \neq \{0,1\}$ and $y=1$) and ($y \neq \{0,1\}$ and $x=0$ or $x=1$).
- In the neighborhood of $(0,0)$ the system is regular singular. Thus we know that a FMFS takes the form
$$\Phi(x,y,\varepsilon)\;x^{C_1(\varepsilon)}y^{C_2(\varepsilon)}.$$

Clearly, due to the sparsity of $A(x, y, \varepsilon)$ and $B(x, y, \varepsilon)$, we can split the 7×7 system into systems of lower dimensions. Let $G = \begin{bmatrix} W \\ U \end{bmatrix}$. With some simplifications related to the explicit values of the entries we have

$$\begin{cases} \partial_x W &= O_{4\times 4} \\ \partial_y W &= \frac{1}{y(y-1)}\, B^{11}(y,\varepsilon)\, W \end{cases} \tag{8.15}$$

and

$$\begin{cases} \partial_x U &= \frac{1}{x\,(-1+x)\,(xy+\varepsilon-y+1)}\, [A^{22}\, U \;+\; \tilde{A}^{21}\, W] \\ \partial_y U &= \frac{1}{y\,(y-1)\,(xy+\varepsilon-y+1)}\, [B^{22}\, U \;+\; \tilde{B}^{21}\, W] \end{cases} \tag{8.16}$$

For a solution in a neighborhood of $(x \neq \{0, 1\}, y = 1)$, one can perform a simple translation $z = y - 1$ and replace rational functions by their series expansions to get

$$\Big\{ \partial W \;=\; \tfrac{1}{z}\, E(z)\, W \tag{8.17}$$

and

$$\begin{cases} \partial_x U &= \tilde{R}(x, z, \varepsilon)\, U \;+\; \tilde{S}(x, z, \varepsilon)\, W \\ \partial_z U &= \tfrac{1}{z}\, [M(x, z, \varepsilon)\, U \;+\; N(x, z, \varepsilon)\, W] \end{cases} \tag{8.18}$$

Thus, the initial system can be broken down to system (8.17) which can be resolved by expanding the solution in ε and solving a set of inhomogeneous systems (See Subsection 5.2.1). And to system (8.18) for which the solution can be expanded analogously and inserted. The problem is then reduced to solving a set of inhomogeneous bivariate Pfaffian systems with normal crossings. For a solution in a neighborhood of $(x = 1, y = 1)$, a translation can be introduced to x as well. And the case $(x = 0, y = 0)$ can be treated analogously.

Statistics

Another interesting example[‡] arises from statistics. The following system is obtained from the Muirhead system in [63, Theorem 1]. Takayama et al. are working on numerical evaluations of solutions of such systems upon introducing the holonomic gradient method [63].

$$\partial_x F \;=\; A(x, y)\, F \tag{8.19}$$

$$\partial_y F \;=\; B(x, y)\, F \tag{8.20}$$

[‡]We thank N. Takayama for pointing out this system to us.

where a, b, c are constants and

$$A = \begin{bmatrix} 0 & -x^{-2} & 0 & 0 \\ -\frac{a}{x} & \frac{(cx-1)y+(-c+1/2)x^2+x}{x^2(y-x)} & -(2\,y-2\,x)^{-1} & 0 \\ 0 & 0 & 0 & -x^{-2} \\ -1/2\,\frac{ay}{y-x} & 3/4\,\frac{y^2}{(y-x)^2} & \alpha_1 & \frac{(cx-1)y+(-c+1/2)x^2+x}{x^2(y-x)} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -y^{-2} & 0 \\ 0 & 0 & 0 & -y^{-2} \\ -\frac{a}{y} & (2\,y-2\,x)^{-1} & \frac{(c-1/2)y^2+(-cx-1)y+x}{y^2(y-x)} & 0 \\ 1/2\,\frac{ax}{y-x} & \alpha_2 & 3/4\,\frac{x^2}{(y-x)^2} & \frac{(c-1/2)y^2+(-cx-1)y+x}{y^2(y-x)} \end{bmatrix}$$

where

$$\begin{cases} \alpha_1 = \frac{((c/2-3/4)x-a)y^2+(-1/2\,cx+2\,a-1/2)xy+(-a+1/2)x^2}{x(y-x)^2}; \\ \alpha_2 = \frac{(-1/2\,cx-a+1/2)y^2+((c/2-3/4)x+2\,a-1/2)xy-ax^2}{y(y-x)^2}. \end{cases}$$

It can be easily verified that this system is completely integrable. However, obviously, it does not have normal crossings. Nevertheless, setting $\varepsilon = (y-x)$ and adapting the ξ-rank reduction algorithm of the singularly-perturbed differential system (5.1) , one can compute

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3/2\,\frac{x^2}{cx-1} & -(cx-1)^{-1} & \frac{ax}{cx-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that the transformation $F = TG$ yields a system for which the order of the pole in $(y-x)$ drops by one as follows:

$$\partial_x G = \tilde{A}(x,y)\,G \tag{8.21}$$

$$\partial_y G = \tilde{B}(x,y)\,G \tag{8.22}$$

where

$$\tilde{A} = \begin{bmatrix} -3/2\,(-y+x)^{-1} & \frac{cx-1}{(-y+x)x^2} & \frac{1}{(-y+x)x^2} & -\frac{a}{(-y+x)x} \\ -1/2 & \frac{cx-1}{x^2} & 0 & -\frac{a}{x} \\ \alpha_3 & -1/2\,\frac{cxy+2\,ax-2\,ay-x}{(-y+x)x} & \alpha_4 & 1/2\,\frac{ay}{-y+x} \\ 0 & -x^{-2} & 0 & 0 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 1/2\,\frac{2\,cxy-2\,cy^2+3\,y^2-2\,x+2\,y}{(-y+x)y^2} & -\frac{yc-1}{(-y+x)y^2} & -\frac{1}{(-y+x)y^2} & \frac{a}{(-y+x)y} \\ 0 & 0 & -y^{-2} & 0 \\ -3/4\,\frac{x^2}{-y+x} & -1/2\,\frac{-cxy+2\,ax-2\,ay+y}{(-y+x)y} & \alpha_5 & -1/2\,\frac{ax}{-y+x} \\ \frac{-y+x}{y^2} & -y^{-2} & 0 & 0 \end{bmatrix}$$

where

$$\begin{cases} \alpha_3 = 1/4\,\frac{2\,cx^2y-2\,cxy^2+4\,ax^2-8\,axy+4\,ay^2+3\,xy^2-2\,x^2+2\,xy}{(-y+x)x}; \\ \alpha_4 = 1/2\,\frac{2\,cx^2-2\,cxy-x^2-2\,x+2\,y}{(-y+x)x^2}; \\ \alpha_5 = 1/2\,\frac{2\,cxy-2\,cy^2+y^2-2\,x+2\,y}{(-y+x)y^2}. \end{cases}$$

Several questions arise:

- Can the rank reduction algorithm be generalized to such systems? And can a criterion be given to determine whether the minimum of the singularity's order is attained?
- A close inspection of the system reveals a kind of symmetry for this system. In particular, Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  then $P[A](y,x) = P^{-1}AP = B(x,y)$. Consequently, one expects some kind of symmetry in the solution which can be exploited to get the formal invariants.

One of the first consequences of the loss of the normal crossings is the instability of the JCF of the leading constant matrix. In the case of singularly-perturbed linear differential systems, this instability gave rise to the turning point phenomenon discussed in Part II. Thus, for a bivariate Pfaffian with singularities of the monomial form $x^\alpha y^\beta$ for some $\alpha, \beta \in \mathbb{N}$ (studied e.g. by H. Majima), would it be plausible to investigate the techniques developed in Part II to uncouple the system into systems of lower dimensions and systems with normal crossings by holding one of the independent variables as a parameter?

Appendix A

Additional notions for Symbolic Resolution

In chapter 1, we recalled the formal reduction algorithm of [14] and its components which are relevant to the work in Parts II and III. However, several other notions of irreducible systems are central to the local analysis of linear systems of ordinary differential equations. Such notions refine Algorithm 1 and give as well theoretical results and estimations of the formal invariants.
Given again system (1) of Chapter 1

$$[A] \qquad x^{p+1}\frac{d}{dx}\,F \;=\; A(x)\,F \;=\; (\sum_{k=0}^{\infty} A_k x^k)\,F,$$

we have the following notions:

Regular solutions

In the case of regular singularity, we have seen that regular solutions can be constructed by the transformations of Section 1.6. However, one might still encounter regular solutions in the case of an irregular singularity. Such solutions correspond to the nullity of at least one of the diagonal entries of the exponential part in (2). A method which computes such regular solutions is proposed in [29]. This method relies on matrix pencil theory and in particular on simple forms.

Super-irreducible systems

The notion of super-irreducible systems introduced in [64] is a generalization of Moser-irreducible systems described in Subsection 1.5.1. The former notion takes into account additional coefficients of $A(x)$ by defining:

$$\begin{cases} m_k(A) \;=\; \max\,(0, p + \frac{\nu_0(A)}{n} + \frac{\nu_1(A)}{n^2} + \ldots \frac{\nu_{k-1}(A)}{n^k}) \\ \mu_k(A) \;=\; \min\,\{\; m_k(\tilde{A}) \text{ for all possible choices of } T \text{ in (1.1)})\} \quad \text{for} 1 \leq k \leq p, \end{cases}$$

where ν_i denotes the number of columns of A having valuation $i-p-1$. Then we have:

| Definition A.1

The system (1) (and A) is called *k-reducible* if $m_k(A) > \mu_k(A)$, otherwise it is said to be *k-irreducible*. The system is called super-irreducible if it is *k-irreducible* for all $1 \leq k \leq p$.

Analogously, a criteria to test the k-irreducibility of a system is defined by:

$$\theta_{(k,A)}(\lambda) := x^{p_k(A)}\ det(x^{p+1-k}A - \lambda I_n)|_{x=0},$$

where $p_k(A) = k\nu_0(A) + (k-1)\nu_1(A) + \cdots + \nu_{k-1}(A)$. Then we have

| Proposition A.1

[64] The system (1) (and A) is k-irreducible for $1 \leq k \leq p$ if and only if the polynomials $\theta_{(i,A)}(\lambda)$, $1 \leq i \leq k$, do not vanish identically in λ.

Inspired by Lidskii matrices, new versions of Moser-based rank reduction algorithm of [15] were introduced in [30, 32]. Moreover, the task of computing super-irreducible form of the system (1) was reduced to that of computing one or several Moser-irreducible forms, using a block-reduction algorithm. However, these algorithms rely on putting the leading matrix coefficient A_0 in JCF. This prerequisite is a first symptom of the non-triviality of generalizing such algorithms.

Simple systems

Certain information derived from k-irreducible systems, such as computing regular solutions, integer slopes of the Newton polygon, computation of global solutions (rational solutions [20], certain exponential solutions [113]), can be derived using weaker forms called k-simple forms (see [24] and references therein). This notion was first introduced in [29] where a more general form of (1) was considered:

$$D(x)\ \nu_k\ (F(x))\ +\ N(x)\ F(x)\ = O_n,$$

where $k \in \mathbb{N}$, $\nu_k = x^{k+1}\frac{d}{dx}$, and $D(x)$ and $N(x)$ belong to $\mathbb{C}[[x]]$. Such a system is said to be k-simple if the matrix pencil defined by

$$L_k(\lambda)\ =\ D(0)\lambda + N(0)$$

is regular, i.e., $det\ (L_k(\lambda)) \neq 0$. Any super-irreducible system is simple [29, Appendix A]. However, the algorithm of [24] computes a k-simple form of the system without requiring super-reduction. The former relies on matrix manipulations over a univariate field and shearing transformations. For system (5.1), the required matrix manipulations are to be performed over a bivariate field. For system (6.14), the required matrix manipulations are to be performed over a multivariate field and the compatibility of the shearing transformation is under question as well. Some tricks employed in this thesis for giving rank reduction algorithms can be employed herein as well. As remarked in [24], corresponding algorithms have implications in the symbolic integration of ordinary differential systems with hyper-exponential

Table A.1 Summary of some non-trivialities encountered whence extending notions and algorithms given for singular linear differential systems (1), to singularly-perturbed linear differential systems (3.9) and completely integrable Pfaffian systems with normal crossings (6.14)

System (1)	System (5.1)	System (6.14)
Reduction algorithms [32]	JCF of leading matrix	JCF of leading matrix
Super-reduction	JCF of leading matrix	JCF of leading matrix
Simple systems	Bivariate field	Multivariate field Transformations' compatibility
Generalized splitting lemma	Simple forms	Simple forms Series representation
Regular solutions	Simple systems	Simple systems Series representation

extensions [33] as well as in the computation of closed form solutions of integrable connections [22].

Generalized Splitting lemma

The generalized splitting lemma given in [114, Proposition 3.1] generalizes the classical splitting lemma described in Section 1.3. In addition to refining the formal reduction algorithm, this generalized splitting allows establishing estimates on former exponential orders [114]. The proof of this lemma requires simple forms.

Increasing the dimension

In Parts I and II, we have discussed the equivalence between a n^{th}-order scalar differential equation and first-order linear differential systems, whether perturbed or unperturbed. Although this equivalence is not employed in the algorithms developed, associating to a first-order system an n^{th}-order scalar differential equation served for the underlying proofs of corresponding theorems.
However, other approaches in literature associate to a system of a given dimension, a sequence of matrices of higher dimensions. Such approaches proved their utility in rank reduction and, more generally, formal reduction of first-order differential systems (see, e.g. [137, 121, 138]). Thus, it would be interesting to study these approaches algorithmically and their extension to more general systems.

Complexity

The complexity estimates of the algorithms proposed herein were not investigated. In the light of the different possible treatments of the systems at hand and experimental results (see, e.g. Subsection 5.6.1), it would be a very interesting item of further research. For univariate systems, first steps are set in [38] where the cyclic vector method was compared to the algorithm which computes a companion block diagonal form [13], in [32] for rank reduction algorithms, and in [52] for simple systems.

Appendix B

Software

The examples of computation and the description of the packages are too large to be included here. Please refer to my web page which is, at the moment of writing this thesis, hosted by the University of Limoges at:

$$http://www.unilim.fr/pages\_perso/suzy.maddah/$$

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Réduction Formelle des Systèmes Différentiels Linéaires Singuliers: Systèmes différentiels linéaires singulièrement perturbés et systèmes de Pfaff complètement intégrables à croisements normaux

Résumé : Dans cette thèse, nous nous sommes intéressés à l'analyse locale de systèmes différentiels linéaires singulièrement perturbés et de systèmes de Pfaff complètement intégrables et multivariés à croisements normaux. De tels systèmes ont une vaste littérature et se retrouvent dans de nombreuses applications. Cependant, leur résolution symbolique est toujours à l'étude. Nos approches reposent sur l'état de l'art de la réduction formelle des systèmes linéaires singuliers d'équations différentielles ordinaires univariées (ODS).

Dans le cas des systèmes différentiels linéaires singulièrement perturbés, les complications surviennent essentiellement à cause du phénomène des points tournants. Nous généralisons les notions et les algorithmes introduits pour le traitement des ODS afin de construire des solutions formelles. Les algorithmes sous-jacents sont également autonomes (par exemple la réduction de rang, la classification de la singularité, le calcul de l'indice de restriction).

Dans le cas des systèmes de Pfaff, les complications proviennent de l'interdépendance des multiples sous-systèmes et de leur nature multivariée. Néanmoins, nous montrons que les invariants formels de ces systèmes peuvent être récupérés à partir d'un ODS associé, ce qui limite donc le calcul à des corps univariés. De plus, nous donnons un algorithme de réduction de rang et nous discutons des obstacles rencontrés.

Outre ces deux systèmes, nous parlons des singularités apparentes des systèmes différentiels univariés dont les coefficients sont des fonctions rationnelles et du problème des valeurs propres perturbées.

Les techniques développées au sein de cette thèse facilitent les généralisations d'autres algorithmes disponibles pour les systèmes différentiels univariés aux cas des systèmes bivariés ou multivariés, et aussi aux systèmes d'équations fonctionnelles.

Mots clés : Calcul formel, réduction formelle, reduction de rang, singularités, points tournants, systèmes différentiels linéaires, systèmes de Pfaff complètement intégrables à croisements normaux, systèmes différentiels linéaires singulièrement perturbés, les algorithmes.

Formal Reduction of Linear Singular Differential Systems:
Singularly-perturbed linear differential systems and completely integrable Pfaffian systems with normal crossings

Abstract: In this thesis, we are interested in the local analysis of singularly-perturbed linear differential systems and completely integrable Pfaffian systems in several variables. Such systems have a vast literature and arise profoundly in applications. However, their symbolic resolution is still open to investigation. Our approaches rely on the state of art of formal reduction of singular linear systems of ordinary differential equations (ODS) over univariate fields.

In the case of singularly-perturbed linear differential systems, the complications arise mainly from the phenomenon of turning points. We extend notions introduced for the treatment of ODS to such systems and generalize corresponding algorithms to construct formal solutions in a neighborhood of a singularity. The underlying components of the formal reduction proposed are stand-alone algorithms as well and serve different purposes (e.g. rank reduction, classification of singularities, computing restraining index).

In the case of Pfaffian systems, the complications arise from the interdependence of the multiple components which constitute the former and the multivariate nature of the field within which reduction occurs. However, we show that the formal invariants of such systems can be retrieved from an associated ODS, which limits computations to univariate fields. Furthermore, we complement our work with a rank reduction algorithm and discuss the obstacles encountered.

The techniques developed herein paves the way for further generalizations of algorithms available for univariate differential systems to bivariate and multivariate ones, for different types of systems of functional equations.

In addition to proposing algorithms for these two systems, we report on apparent singularities of differential systems whose coefficients lie in the univariate field of rational functions and perturbed eigenvalue problem. The main results of this thesis are summed up in three published papers [1, 2, 27], a submitted paper [25], a paper under redaction [28], and five computer algebra packages [96, 97, 98, 99, 100].

Keywords: Computer algebra, formal reduction, rank reduction, singularities, turning points, linear differential systems, Pfaffian systems, singularly-perturbed linear differential systems, algorithms.

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