A stochastic equation with censored jumps related to multi-scale Piecewise Deterministic Markov Processes

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Thèse de doctorat

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Une équation stochastique avec sauts censurés liée à des PDMP à plusieurs régimes

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dirigée par Vlad Bally et Eva Löcherbach

Soutenue le 23 juin 2015 devant le jury composé de :

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Résumé

Titre. Une équation stochastique avec sauts censurés liée à des PDMP à plusieurs régimes.

Résumé. L'ensemble de ce travail est dédié à l'étude de certaines propriétés concernant les processus de sauts d-dimensionnels $X = (X_t)$ dont le générateur est donné par

$$L\psi(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2\psi(x)}{\partial X_i \partial X_j} + g(x)\nabla\psi(x) + \int (\psi(x + c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z)$$

où μ est de masse totale infinie.

Si γ ne dépendait pas de x, nous nous trouverions dans une situation classique où le processus X pourrait être représenté comme une solution d'une équation stochastique comportant une mesure ponctuelle de Poisson de mesure d'intensité $\gamma(z)\mu(\mathrm{d}z)$; lorsque γ dépend de x, on peut s'en représenter l'heuristique en imaginant le processus comme la trajectoire d'une particule, la loi des sauts pouvant alors dépendre de la position de la particule.

Dans la première partie, nous donnons des conditions pour obtenir l'existence et l'unicité de tels processus. Ensuite, nous considérons ce type de processus comme une généralisation des PDMP ; nous montrons qu'ils peuvent être vus comme une limite d'une suite $(X_n(t))$ de PDMP standards pour lesquels l'intensité des sauts tend vers l'infini quand n tend vers l'infini, suivant deux régimes : un lent et un rapide qui, en supposant que les processus en question sont centrés et normalisés convenablement, produit une composante de diffusion à la limite.

Finalement, on prouve la récurrence au sens de Harris de X en utilisant un schéma regénératif entièrement basé sur les sauts du processus. De plus, nous dégageons des conditions explicites par rapport aux coefficients du processus qui nous permettent de contrôler la vitesse de convergence vers l'équilibre en terme d'inégalités de déviation pour des fonctionnelles additives intégrables.

Dans la seconde partie, nous considérons à nouveau le même type de processus $X = (X_t(x))$ partant du point x. Utilisant une approche basé sur un Calcul de Malliavin fini-dimensionnel, nous étudions la régularité jointe de ce processus dans le sens suivant : on fixe $q \geq 1$ et $p > 1$, K un ensemble compact de $\mathbb{R}^d$, et nous donnons des conditions suffisantes pour avoir $\mathrm{P}(X_t(x) \in \mathrm{d}y) = p_t(x,y)\,\mathrm{d}y$ avec $(x,y) \mapsto p_t(x,y)$ appartenant à $W^{q,p}(K \times \mathbb{R}^d)$.

Mots-clés. Calcul de Malliavin, processus de sauts, PDMP, récurrent au sens de Harris

Abstract

Title. A stochastic equation with censored jumps related to multi-scale Piecewise Deterministic Markov Processes.

Abstract. This work is dedicated to the study of some properties concerning the d-dimensional jump type diffusion $X = (X_t)$ with infinitesimal generator given by

$$L\psi(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2\psi(x)}{\partial X_i \partial X_j} + g(x)\nabla\psi(x) + \int(\psi(x + c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z)$$

where μ is of infinite total mass.

If γ did not depend on x, we would be in a classical situation where the process X could be represented as the solution of a stochastic equation driven by a Poisson point measure with intensity measure $\gamma(z)\mu(\mathrm{d}z)$; when γ depends on x, we may have the heuristic idea that, if we were to imagine the process as a trajectory of a particle, the law of the jumps may depend on the position of the particle.

In the first part, we give some conditions to obtain existence and uniqueness of such processes. Then, we consider this type of processes as a generalization of Piecewise Deterministic Markov Processes (PDMP) ; we show that they can be seen as a limit of a sequence $(X_n(t))$ of standard PDMP's for which the intensity of the jumps tends to infinity as n tends to infinity, following two regimes: a slow one, which leads to a jump component with finite variation, and a rapid one which, supposing that the processes at hand are centered and renormalized in a convenient way, produces the diffusion component in the limit.

Finally, we prove Harris recurrence of X using a regeneration scheme which is entirely based on the jumps of the process. Moreover we state explicit conditions in terms of the coefficients of the process allowing to control the speed of convergence to equilibrium in terms of deviation inequalities for integrable additive functionals.

In the second part, we consider again the same type of process $X = (X_t(x))$ starting from x. Using an approach based on a finite dimensional Malliavin Calculus, we study the joint regularity of this process in the following sense : we fix $q \geq 1$ and $p > 1$, K a compact set of $\mathbb{R}^d$, and we give sufficient conditions in order to have $\mathrm{P}(X_t(x) \in \mathrm{d}y) = p_t(x,y)\,\mathrm{d}y$ with $(x,y) \mapsto p_t(x,y)$ in $W^{q,p}(K \times \mathbb{R}^d)$.

Keywords : Malliavin calculus, jumps processes, PDMP, Harris recurrent

Contents

Introduction

Ce travail est dédié à l'étude de certaines propriétés concernant la solution de l'équation différentielle stochastique d-dimensionnelle suivante (où X_t est un processus à valeurs dans $\mathbb{R}^d$, W_s un mouvement brownien multidimensionnel et N une mesure aléatoire de Poisson) :

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} c(z, X_{s^-})\mathbb{1}_{\{u\leq\gamma(z,X_{s^-})\}}N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \quad (1)$$

Tout d'abord, on peut remarquer immédiatement que cette équation est "irrégulière" dans le sens où l'un de ses coefficients contient une fonction indicatrice. Ceci permet de modéliser des processus dont le taux de saut dépend du processus lui-même : la mesure d'intensité du générateur infinitésimal L est $\gamma(z,x)\mu(\mathrm{d}z)$ (au lieu de $\mu(\mathrm{d}z)$ dans le cas usuel des équations avec sauts).

Le générateur infinitésimal associé à un processus markovien X vérifiant cette équation est défini par

$$L\psi(x) = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}\psi(x) + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d}(\psi(x + c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z) \quad (2)$$

où $a \overset{\text{def}}{=} \sigma\sigma^*$ et où μ est une mesure (σ-finie) sur E associée à la mesure d'intensité $\hat{N}$ de N : $\hat{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}t \times \mu(\mathrm{d}z) \times \mathbb{1}_{\{0,\infty\}}(u)\,\mathrm{d}u$.

Si, dans cette définition, γ ne dépendait pas de x, nous serions dans une situation classique où le processus X pourrait être représenté comme une solution d'une équation stochastique associée à une mesure ponctuelle de Poisson de mesure d'intensité $\gamma(z)\mu(\mathrm{d}z)$; lorsque γ dépend de x, une heuristique possible est de se figurer, si l'on imagine le processus comme la trajectoire d'une particule, que la loi des sauts peut dépendre de la position de la particule.

Dans le cas particulier où l'équation (1) ne possède pas de composante brownienne (*ie.* $\sigma = 0$; ou, de manière équivalente, $a = 0$ dans (2)), nous somme amené à étudier un processus markovien déterministe par morceau (PDMP) ; dans le cas où la mesure μ est finie, la partie poissonnienne devient simplement un processus de Poisson composé (et nous avons par conséquent un ensemble dénombrable de sauts $(T_n)_{n\in\mathbb{N}}$ sans points d'accumulation) et une solution peut être construite par morceau entre deux sauts successifs (sur ce dernier point, *cf.* par exemple, la remarque 1.6.1 que nous ferons au chapitre 1).

Existence et unicité

Pour arriver à ce résultat, nous utilisons un argument basé sur le lemme de Gronwall, mais, dans notre cas, il n'est pas possible de travailler directement dans l'espace L^2. C'est une difficulté spécifique de notre problème. Suivant une idée de N. Fournier (communication orale), nous travaillerons ainsi dans l'espace L^1 plutôt que dans l'espace L^2.

Nous démontrerons ainsi, dans la première partie de ce travail l'existence et l'unicité de la solution de (1) ; nous présentons ici brièvement la stratégie qui a été suivie.

On définit une équation sur la restriction $F \subset E$ par

$$X_t^F = x + \int_0^t \sigma(X_s^F)\,\mathrm{d}W_s + \int_0^t g(X_s^F)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_F(z)c(z,X_{s-}^F)\mathbb{1}_{\{u\leq\gamma(z,X_{s-}^F)\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \tag{3}$$

et on prouve, sous certaines hypothèses (*cf.* chapitre 1, hypothèse 1.1 ; grosso modo, σ, g sont lipschitziennes, c, γ sont lipschitziennes par rapport à la seconde variable, et $\overline{\gamma}(z) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d}(|\gamma(z,x)|)$ et $\overline{c}(z) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d}(|c(z,x)|)$ vérifient certaines conditions d'intégrabilité par rapport à μ), que l'on a le résultat suivant (*cf.* chapitre 1, lemme 1.4.1) :

- *avec $G \subset F \subset E$, et $Z_t \stackrel{\text{def}}{=} X_t^F - X_t^G$, il existe $K_1, K_2 \in \mathbb{R}_+$ tels que*

$$\forall t \leq T, \qquad \mathrm{E}[|Z_t|] \leq TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2\int_0^t \mathrm{E}\left[|Z_s|\right]\mathrm{d}s.$$

En utilisant le lemme de Gronwall, l'unicité en découle immédiatement et, considérant une suite croissante (E_n) de E telle que $\bigcup E_n = E$ et $\forall n \in \mathbb{N}$, $\mu(E_n) < \infty$, ce résultat permet de prouver que (sous les mêmes hypothèses 1.1) la suite des solutions approchées $X_t^n \stackrel{\text{def}}{=} X_t^{E_n}$ (qui existent pour chaque n, en utilisant la remarque 1.6.1, évoquée précédemment) converge vers la solution càdlàg de (1) dont les trajectoires appartiennent à l'espace des fonctions L^1.

Solution vue comme une limite de PDMP à plusieurs régimes

Dans ce deuxième chapitre nous considérerons ce type de processus comme une généralisation des processus markovien déterministes par morceaux (PDMP) dans deux directions : dans la théorie standard des PDMP, entre deux temps de sauts, le processus suit une courbe déterministe, solution d'une EDO. Dans notre cas cette courbe déterministe est remplacée par une trajectoire d'un processus de diffusion (ainsi le processus n'est plus désormais déterministe par morceau, mais possède une composante diffusive). Ensuite le processus saute avec une intensité qui dépend de la position de la particule. Il faut remarquer cependant que, dans la définition standard des PDMP, l'intensité de la mesure de saut est finie, et par conséquent les temps de sauts forment un ensemble discret. Dans notre cas, nous considérons une intensité de mesure infinie, et par conséquent les temps de sauts forment un ensemble dense dans $\mathbb{R}_+$.

Nous étudierons par conséquent un théorème limite qui motive l'introduction de l'équation qui fait l'objet de ce travail : on considère une suite X_t^n de PDMP standard pour lesquels l'intensité des sauts tend vers l'infini quand $n \to \infty$, suivant deux régimes : un régime lent, qui induit une composante de saut à variation bornée, et un régime rapide, en supposant que les processus considérés sont centrés et renormalisés d'une façon convenable, qui induit une composante de diffusion en passant à la limite.

En plus de l'hypothèse 1.1 du premier chapitre (qui assure l'existence et l'unicité de la solution), nous ajouterons l'hypothèse suivante (hypothèse qui sera de toute manière nécessaire dans la seconde partie (partie concernant le calcul de Malliavin) et sera contenue alors dans les hypothèses 5.2) :

- *On suppose, dans toute la suite, que γ est borné par $\overline{C} \in \mathbb{R}_+^*$.*

Nous pouvons ainsi supposer dans ce chapitre, sans aucune perte de généralité, que $\overline{C} = \frac{1}{2}$ (cette valeur particulière étant uniquement choisie pour simplifier l'écriture de ce qui suit), l'équation pouvant alors être écrite ici

$$\begin{aligned} X_t = x + \sum_{l=1}^m \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t b(X_s)\,\mathrm{d}s \qquad (4)\\ + \int_0^t\int_{E\times(0,1)} c(X_{s-},z)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned}$$

Rappelons que, si μ est une mesure finie, alors la précédente équation admet la représentation suivante. Soit J_t un processus de Poisson de paramètre $\mu(E)\times[0,1]$ et soit $T_k, k\in N$ les temps de sauts. Considérons aussi la suite des variables aléatoires indépendantes Z_k, $U_k, k\in N$ de lois

$$P(Z_k\in \mathrm{d}z)=\frac{1}{\mu(E)}\mu(\mathrm{d}z) \qquad \text{et} \qquad P(U_k\in \mathrm{d}u)=\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.$$

Alors la précédente équation s'écrit

$$X_t=x+\sum_{l=1}^{m}\int_0^t\sigma_l(X_s)\,\mathrm{d}W_s^l+\int_0^t b(X_s)\,\mathrm{d}s+\sum_{k=1}^{J_t}c(X_{T_k-},Z_k)\mathbb{1}_{\{U_k\leq\gamma(X_{T_k-},Z_k\}}. \tag{5}$$

Ainsi, dans le cas $\mu(E)<\infty$, on prouve que la loi de X_t coïncide avec la loi de $\overline{X}_t$ construite de la façon suivante.

On considère un point $z_*\notin E$ et on note $E_*=E\cup\{z_*\}$. Sur E_* on définit la mesure de probabilité

$$\begin{aligned}\eta(x,\mathrm{d}z)&=\theta(x)\delta_{z_*}(\mathrm{d}z)+\frac{1}{\mu(E)}\mathbb{1}_E(z)\gamma(x,z)\mu(\mathrm{d}z), \qquad \text{avec}\\ \theta(x)&=\frac{1}{\mu(E)}\int_E(1-\gamma(x,z))\mu(\mathrm{d}z).\end{aligned}$$

On pose $\overline{X}_0=x$ et on définit par récurrence

$$\begin{aligned}\overline{X}_{T_k}&=\overline{X}_{T_k-}+c(\overline{X}_{T_k-},\overline{Z}_k)\mathbb{1}_E(\overline{Z}_k) \qquad (6)\\ \overline{X}_t&=\overline{X}_{T_k}+\sum_{l=1}^{m}\int_{T_k}^t\sigma_l(\overline{X}_s)\,\mathrm{d}W_s^l+\int_{T_k}^t b(\overline{X}_s)\,\mathrm{d}s, \qquad T_k\leq t<T_{k+1}.\end{aligned}$$

les variables aléatoires $\overline{Z}_k$ suivant la loi conditionnelle

$$P\Big(\overline{Z}_k\in\mathrm{d}z\mid\overline{X}_{T_k-}=x\Big)=\eta(x,\mathrm{d}z).$$

Remarquons que $T_k, k\in\mathbb{N}$ ne sont pas les véritables temps de sauts de $\overline{X}$: en effet, $\mathbb{1}_E(\overline{Z}_k)=0$ avec probabilité $\theta(x)>0$ et dans ce cas $\overline{X}_{T_k}=\overline{X}_{T_k-}$. Remarquons aussi que la loi du saut au temps T_k dépend de la position $\overline{X}_{T_k-}$. Ainsi, si $\sigma=0$, alors $\overline{X}_t$ est un PDMP. Notre modèle apparaît alors comme une généralisation naturelle de ce type de modèles, qui posséderaient une composante de diffusion $\sum_{l=1}^m\int_0^t\sigma_l(\overline{X}_s)\,\mathrm{d}W_s^l$. La motivation de cette composante de diffusion résulte du théorème de convergence qui suit.

Considérant à nouveau une suite croissante (E_n) de sous-ensembles de E telle que $\bigcup E_n=E$ et $\forall n\in\mathbb{N}$, $\mu(E_n)<\infty$ ainsi qu'une famille $(\nu_n)_{n\in\mathbb{N}}$ de mesures finies définies sur E, on considère une suite X_t^n, $n\in\mathbb{N}$, de PDMP à deux régimes :

$$\begin{aligned}X_t^n=x&+\int_0^t b(X_s^n)\,\mathrm{d}s+\int_0^t\int_{E_n}\int_{(0,1)}c(X_{s-}^n,z)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}}N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \qquad (7)\\ &+\int_0^t\int_E\int_{(0,1)}c(X_{s-}^n,z)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}}\widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).\end{aligned}$$

Ici $N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ et $N_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ sont deux mesures ponctuelles de Poisson avec

$$\widehat{N}_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)=\mathrm{d}s\mathbb{1}_{E_n}(z)\,\mathrm{d}\mu(z)\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u \qquad \text{and} \qquad \widehat{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)=\mathrm{d}s\,\mathrm{d}\nu_n(z)\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.$$

Puisque $\mu(E_n)+\nu_n(E)<\infty$, la solution de (7) est un PDMP standard. Le régime $\mathbb{1}_{E_n}(z)\,\mathrm{d}\mu(z)$ est le régime lent et $\mathrm{d}\nu_n$ le régime rapide. On note $\overline{c}(z)=\sup_{x\in\mathbb{R}^d}|c(x,z)|$ et

$$a_n^{i,j}=\int_E c^i(x,a)c^j(x,a)\gamma(x,a)d\nu_n(a).$$

Soit σ le coefficient dans l'équation (4) et $a = \sigma\sigma^*$. On note

$$\varepsilon(n) = \|a - a_n\|_\infty + \int_{E_n^c} \overline{c\gamma}\,\mathrm{d}\mu + \int_E \overline{c}^3\overline{\gamma} d\nu_n$$

et on construit

$$\varepsilon_*(n) = \inf_{\overline{n}\geq n} \Big(\int_{E_{\overline{n}}^c} \overline{c\gamma}\,\mathrm{d}\mu + \mu(E_{\overline{n}})\varepsilon(n)\Big).$$

Alors, sous l'hypothèse $\gamma(x,z) \geq \underline{\gamma} > 0$ on prouve que (*cf.* théorème 2.3.5)

- *pour tout* $f \in \mathcal{C}_b^3(\mathbb{R}^d)$

$$\sup_{x\in\mathbb{R}^d} \left|\mathrm{E}\left[f(X_t(x))\right] - \mathrm{E}\left[f(X_t^n(x))\right]\right| \leq \frac{C_n}{\underline{\gamma}^3} \|f\|_{3,\infty}\, \varepsilon_*(n), \tag{8}$$

et nous donnons une estimation explicite pour C_n (de manière générale $\sup_n C_n < \infty$). En particulier, si

$$\lim_{n\to\infty} C_n \varepsilon_*(n) = 0,$$

- *on obtient la convergence en loi de* X_t^n *vers* X_t.

Remarquons que l'inégalité (8) n'est pas asymptotique, elle est vraie pour tout n fixé. Ceci nous permet de le considérer d'un point de vue différent, essentiellement comme un résultat pouvant être utilisé pour une simulation, par exemple. C'est une idée remontant à Asmussen et Rosiński [2] qui considèrent une équation stochastique avec des sauts (de type classique) et proposent de simuler la solution d'une telle équation en remplaçant les petits sauts (ceux qui produisent une activité infinie) par un mouvement brownien. Nous faisons la même chose ici : nous remplaçons $c(X_{s-}^n, z)\mathbb{1}_{\{u\leq\gamma(X_{s-}^n,z)\}}\widetilde{N}_n^\nu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ par $\sigma(X_s)\,\mathrm{d}W_s$ (nous considérons le résultat dans le sens inverse). Et alors (8) donne une majoration de l'erreur induite par cette approximation. Ainsi, d'une manière générale, nous répondons à la question : quel est le prix à payer pour remplacer un morceau de la mesure ponctuelle de Poisson par un morceau du mouvement brownien ?

Ergodicité pour une EDS multidimensionnelle avec sauts censurés

Dans le troisième chapitre, nous donnons une première application concernant le possible comportement ergodique de cette solution : l'objectif est de donner des conditions aisément vérifiables sur les coefficients g, σ, c et γ sous lesquelles le processus serait récurrent au sens de Harris[1] et satisferait le théorème ergodique en partant de n'importe quel point de départ x, sans imposer aucune condition de non-dégénérescence sur la partie diffusive.

Pour les EDS avec sauts, une littérature abondante commence a se développer sur le sujet : voir e.g. Masuda [40] (2007) qui travaille dans une situation plus simple où le "terme de censure" $\mathbb{1}_{u\leq\gamma(z,X_{s-})}$ n'est pas présent et qui suit l'approche de Meyn et Tweedie développée dans [42] ou [41], Kulik [34] (2009) qui utilise la méthode de stratification pour prouver l'ergodicité exponentielle des diffusions avec sauts, mais le modèle qu'il considère n'inclut également pas la situation avec censure, ou encore, finalement, Duan and Qiao [22] (2014) qui s'intéressent à des solutions d'équations à coefficients non lipschitziens.

[1] Rappelons qu'un processus X est dit récurrent au sens de Harris s'il possède une mesure invariante m telle que tout ensemble A $m-$mesurable avec $m(A) > 0$ soit visité infiniment souvent par le processus presque sûrement (*cf.* Azéma, Duflo et Revuz [4] (1969)): pour tout $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

Notre but n'est pas d'améliorer les conditions de régularités imposées sur les coefficients, mais de nous concentrer sur le mécanisme de saut. Plus précisément, nous montrons que l'on peut utiliser les sauts eux-mêmes pour générer un schéma de "splitting" qui nous permettra de prouver la récurrence du processus. Il est important de remarquer que la présence du terme de censure $\gamma(z, X_{s-})$ dans (1) implique que l'étude de X est techniquement beaucoup plus lourde que dans la situation non censuré ou γ est minorée et strictement positive.

La méthode que l'on utilise est la méthode de régénération que l'on applique sur les grands sauts. Plus précisément, pour un ensemble convenable E tel que $\mu(E) < \infty$, nous coupons la trajectoire de X en parties de solutions de (3.1.1) dirigées par N restreinte à E^c et qui sont stoppées au premier saut arrivant par le "bruit" z appartenant à E. Dans l'esprit de la technique de "splitting" introduite par Nummelin [44] (1978) et Athreya et Ney [3] (1978), on suppose une condition de non-dégénérescence qui garantit que l'opérateur de saut associé aux grands sauts possède une composante Lesbegue-absolue continue. Cela revient à imposer que les dérivées partielles du terme de saut c par rapport au bruit z soient suffisamment non dégénérées (voir (3.2.6) et (3.2.7)). Nous soulignons que nous n'avons besoin d'aucune condition de non-dégénérescence pour le coefficient diffusif σ.

Remarquons que l'on n'applique pas la technique de "splitting" à une chaîne échantillonnée extraite ni à une chaîne résolutive comme dans Meyn et Tweedie [42] (1993) ; la perte de mémoire nécessaire pour la régénération est produite par certains grands sauts. Cette approche est très naturelle dans ce contexte, mais ne semble pas avoir été utilisée jusqu'à présent dans la littérature, à l'exception de Xu [54] (2011), qui travaille dans un cadre très spécifique et dont les sauts ne dépendent pas de la position du processus.

Ainsi, avec des hypothèses adaptées (grosso modo, c doit être de classe $\mathcal{C}^2$, μ doit être absolument continue par rapport à la mesure de Lebesgue et nous supposons de plus des conditions de type Lyapounov), nous établissons dans ce chapitre le théorème suivant (théorème 3.2.8):

- *le processus X est récurrent au sens de Harris avec une unique mesure de probabilité invariante m, et, de plus, pour toute fonction mesurable $f \in \mathrm{L}^1(m)$, on a*

$$\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s \to m(f)$$

  P_x−presque sûrement pour tout $x \in \mathbb{R}^d$.

Le théorème ergodique ci-dessus est un outil important, *e.g.* pour les inférences statistiques basées sur des observations du processus X en temps continu. Dans cette direction, l'inégalité de déviation suivante est particulièrement intéressante :

- *avec $f \in \mathrm{L}^1(m)$ telle que $\|f\|_\infty < \infty$, $x \in \mathbb{R}^d$ et $0 < \varepsilon < \|f\|_\infty$, pour tout $t \geq 1$ il existe une constante $C(\varepsilon, f) > 0$ (dépendant de ε et f) telle que l'on ait la suivante inégalité :*

$$P_x\left[\left|\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s - m(f)\right| > \varepsilon\right] \leq V(x)\frac{C(\varepsilon, f)}{v(t)}$$

  où $x \mapsto V(x)$ est une fonction de Lyapounov et $v : \mathbb{R}_+ \to \mathbb{R}_+$ une fonction croissante qui fournit l'ordre d'ergodicité.

Avec des hypothèses supplémentaires adaptées sur $V(x)$, nous pouvons avoir pour taux $v(t) = t^{p-1}$, pour un certain $p > 1$, et, avec α tel que $p = \frac{1}{1-\alpha}$, on obtient le contrôle quantitatif de la convergence de la moyenne ergodique suivant (proposition 3.2.10) :

- *pour tout $x, y \in \mathbb{R}^d$,*

$$\left\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - \frac{1}{t}\int_0^t P_s(y,\cdot)\right\|_{\mathrm{TV}} \leq C\frac{1}{t}(V(x)^{(1-\alpha)} + V(y)^{(1-\alpha)}),$$

où $C > 0$ est une constante. En particulier, si $\alpha \geq \frac{1}{2}$, alors

$$\left\| \frac{1}{t} \int_0^t P_s(x,\cdot)\,\mathrm{d}s - m \right\|_{\mathrm{TV}} \leq C \frac{1}{t} V(x)^{(1-\alpha)}$$

(où $\|.\|_{\mathrm{TV}}$ norme de la variation totale).

Régularité jointe pour la densité de la solution

Dans cette partie nous considérons à nouveau la solution X_t^x de l'équation (1), partant du point x, et nous étudierons sa régularité "jointe", dans le sens suivant : soient $q \geq 1$, $p > 1$ et K un compact de $\mathbb{R}^d$ fixés, et nous donnerons des conditions suffisantes pour avoir $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$ avec $(x, y) \mapsto p_{X_t^x}(y) \in W^{q,p}(K \times \mathbb{R}^d)$.

Dans le cas où $\sigma = 0$ (c'est-à-dire sans partie brownienne) et avec une condition initiale x fixée, la régularité de $y \mapsto p_{X_t^x}(y)$ a déjà été étudiée dans [8].

La nouveauté ici, outre la présence de la partie brownienne, est que nous prouvons également la régularité par rapport à la condition initiale. Ceci est un problème non trivial, du fait de la présence de la fonction indicatrice dans un coefficient de l'équation (1). Par conséquent, le flot $x \longrightarrow X_t^x$ n'est pas différentiable. Nous utiliserons certaines idées nouvelles provenant de [7] pour parvenir à passer outre ces difficultés.

De la même façon que dans [8], notre approche est basée sur un calcul de Malliavin finidimensionnel (que nous rappellerons dans le chapitre 4). Mais, dans le cadre de notre étude, la présence du terme supplémentaire $\sigma\,\mathrm{d}W$ rend l'estimation des normes de Sobolev beaucoup plus technique et ardue que dans [8].

Nous expliquons à présent brièvement de quelle manière de tels résultats amènent à la régularité recherchée en présentant un rapide survol de la preuve en elle-même.

On pose $B_M = \{z \in \mathbb{R}^d \ : \ |z| < M\}$, $M \in \mathbb{N}^*$, (alors $\mu(B_M) < \infty$), et on construit (pour chaque M) une approximation $X_t^M(x)$ du processus X_t^x basée sur la restriction N_M de la mesure aléatoire N sur le sous-espace B_M. Pour le moment, on omet le point de départ x, pour simplifier les notations.

En utilisant un résultat similaire au lemme 1.4.1, obtenu dans la première partie de ce travail, nous pouvons établir que la distance dans L^1 entre ces deux processus est bornée comme suit :

$$\forall t \leq T, \qquad \mathrm{E}\left[\left|X_t - X_t^M\right|\right] \leq C_T \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z).$$

$$\forall t \leq T, \qquad \mathrm{E}\left[\left|X_t - X_t^M\right|\right] \leq C_T \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z).$$

Puisque $\mu(B_M) < +\infty$, la mesure aléatoire N_M peut être représentée comme un processus de Poisson composé (où les temps de sauts seront notés T_k^M, $k \in \mathbb{N}$) et la partie poissonnienne du processus X_t^M peut alors être exprimée comme une somme ; néanmoins, du fait de la fonction indicatrice présente dans la solution originelle, les coefficients de l'équation vérifiée par X_t^M demeurent (pour la partie poissonienne) discontinus et, ainsi, on ne peut pas encore utiliser directement le calcul différentiel évoqué précédemment. Nous prouvons donc, de manière intermédiaire que X_t^M possède la même loi que le processus $\overline{X}_t^M$ qui vérifie une équation avec des coefficients réguliers.

Arrivé à ce point, au vue de notre stratégie, il serait souhaitable d'obtenir une formule d'intégration par partie pour le processus $\overline{X}_t^M$, mais il subsiste pour cela une dernière difficulté : il est clair que, pour $t < T_1^M$ (le premier saut de N_M), la mesure aléatoire N_M ne produit pas de bruit, et par conséquent il n'y a aucune possibilité de l'utiliser pour arriver à une intégration par partie (la matrice de Malliavin de variance-covariance étant alors bien sûr dégénérée).

C'est la raison pour laquelle un dernier processus est introduit :

$$F_t^M(x) \stackrel{\text{def}}{=} \overline{X}_t^M(x) + \sqrt{U_M(t)} \times \Delta,$$

où Δ est une gaussienne et où $U_M(t)$ est défini par $U_M(t) = t \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)$.

La distance dans L^1 entre F_M et $\overline{X}_t^M$ est alors bornée, pour $t \leq T$, par

$$K_T \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)}.$$

Le calcul de Malliavin fini-dimensionnel développé dans [8], et présenté au chapitre 4, donne une formule d'intégration par partie pour le processus F_M :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M)H_M\right], \qquad (\mathrm{I}_M)$$

où H_M est calculé à partir de F_M : nous pouvons donc donner une borne supérieure pour $\mathrm{E}\left[|H_M|\right]$ (qui dépend bien entendu de M) et contrôler par conséquent la vitesse de son explosion quand M grandit.

La deuxième étape consiste à prouver la régularité de la densité elle-même

Si le point de départ x est fixé, l'idée est d'utiliser une certaine "balance" entre l'erreur $\mathrm{E}\left[|F_M - X_t|\right]$ (qui tend vers 0) et le poids $\mathrm{E}\left[|H_M|\right]$ (qui tend vers ∞), ce qui a été la stratégie utilisée dans [8]. Ici, l'estimation de la quantité $\mathrm{E}\left[|H_M|\right]$ a été cependant plus délicate que celle effectuée dans [8] à cause de la partie brownienne additionnelle $\sigma\,\mathrm{d}W$. Cependant, une fois ce dernier point réglé (ce qui correspond au chapitre 6 de ce travail), il est possible de conclure de la même façon ; pour rappel, la "balance" utilisée dans [8] était basée sur une méthode de transformation de Fourier.

Mais, puisque nous étudions en plus également la régularité par rapport au point de départ x, on utilise une nouvelle méthode, appelée "Méthode d'Interpolation", développée dans [7].

Cette méthode nous permet, pour une suite donnée de mesures (μ_M) absolument continues par rapport à la mesure de Lebesgue, chacune possédant une densité suffisamment régulière f_M, convergeant vers μ_X (sous une certaine norme, *cf.* section 4.6), de conclure à l'absolue continuité de μ_X et à la régularité de sa densité.

Il suffit de définir , à présent, la mesure μ_X par (où $P_{X_t^x}$ est la loi de X_t^x)

$$\mu_X(\mathrm{d}x, \mathrm{d}y) \stackrel{\text{def}}{=} \Psi_K(x) P_{X_t^x}(\mathrm{d}y)\,\mathrm{d}x \qquad (9)$$

où Ψ_K est une approximation de classe $\mathcal{C}^\infty$ à dérivées de tout ordre bornées de la fonction indicatrice $\mathbb{1}_K$.

Une approximation naturelle de $\mu_X(\mathrm{d}x, \mathrm{d}y)$ serait alors $\Psi_K(x) p_{X_t^M}(x, y)\,\mathrm{d}x\,\mathrm{d}y$. Mais, pour pouvoir utiliser le calcul de Malliavin développé précédemment, il est plus pratique d'utiliser, à la place de X_t^M, l'approximation (en loi) F_M^x que l'on a déjà vue, et nous posons

$$f_M \stackrel{\text{def}}{=} \Psi_K(x) p_{F_t^M}(x, y). \qquad (10)$$

Le point principal (*cf.* théorème 4.6.2) est alors de borner cette norme particulière de f_M, ce qui est le travail développé dans le chapitre 7, où nous sommes amenés à utiliser aussi bien la borne supérieure du poids $\mathrm{E}\left[|H_M|\right]$ que la formule d'intégration par partie (I_M).

Introduction

This work is dedicated to the study of some properties concerning the solution of the following d-dimensional stochastic differential equation (where X_t represents an $\mathbb{R}^d$-valued process, W_s a multidimensional Brownian motion and N a Poisson measure) :

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} c(z, X_{s^-})\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \quad (1)$$

First of all, one may immediately notice that this equation is "irregular" in the sense that one of its coefficients contains an indicator function. This allows to modelize processes for which the rate of jumps of X_t depends on X_t itself : the intensity measure in the infinitesimal operator L is $\gamma(z,x)\mu(\mathrm{d}z)$ (instead of $\mu(\mathrm{d}z)$ in usual jump type equation).

The associated infinitesimal generator of a Markovian process X verifying this equation is defined by

$$L\psi(x) = \frac{1}{2}\sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\psi(x) + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d}(\psi(x + c(z,x)) - \psi(x))\gamma(z,x)\mu(\mathrm{d}z) \quad (2)$$

where $a \overset{\text{def}}{=} \sigma\sigma^*$ and where μ is a (σ-finite) measure on E associated to the intensity measure $\hat{N}$ of N : $\hat{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}t \times \mu(\mathrm{d}z) \times \mathbb{1}_{\{0,\infty\}}(u)\,\mathrm{d}u$.

If, in this definition, γ did not depend on x, we would be in a classical situation where the process X could be represented as the solution of a stochastic equation driven by a Poisson point measure with intensity measure $\gamma(z)\mu(\mathrm{d}z)$; when γ depends on x, we may have the heuristic idea that, if we were to imagine the process as a trajectory of a particle, the law of the jumps may depend on the position of the particle.

In the particular case where the equation 1 does not involve a Brownian part (*ie.* $\sigma = 0$; or, equivalently, $a = 0$ in 2), we are brought to study a piecewise-deterministic Markov process (PDMP) ; in the case where the measure μ is finite, the Poisson part becomes merely a compound Poisson process (and thus we have a countable number of jumps $(T_n)_{n\in\mathbb{N}}$ without accumulation points) and a solution can be constructed piecewise between two successive jumps (concerning that point, *cf.* for instance the Remark 1.6.1 that we will make in Chapter 1).

Existence and uniqueness

In order to establish this point, we use a Gronwall's lemma argument, but, in our case, the work in L^2 fails. This is the specific difficulty of our problem. Following an idea by N. Fournier (oral communication), we will work in the L^1 space, instead of the L^2 space.

Hence, in the first part of this work we will show the existence and the uniqueness of a solution of 1 ; we briefly present here the strategy that has been followed.

We define an equation on a restriction $F \subset E$ by

$$X_t^F = x + \int_0^t \sigma(X_s^F)\,\mathrm{d}W_s + \int_0^t g(X_s^F)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_F(z)c(z, X_{s^-}^F)\mathbb{1}_{\{u\le\gamma(z,X_{s^-}^F)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \quad (3)$$

and prove that, under some Hypothesis (*cf.* Chapter 1, Hypothesis 1.1 ; roughly speaking, σ, g are Lipschitz, c, γ are Lipschitz with respect to their second variable, and $\overline{\gamma}(z) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d}(|\gamma(z,x)|)$ and $\overline{c}(z) \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d}(|c(z,x)|)$ verify some integrability conditions with respect to μ) we have the following result (*cf.* Chapter 1, Lemma 1.4.1) :

- *with $G \subset F \subset E$, and $Z_t \stackrel{def}{=} X_t^F - X_t^G$, there exists $K_1, K_2 \in \mathbb{R}_+$ such that*

$$\forall t \leq T, \qquad \mathrm{E}[|Z_t|] \leq TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s|\right]\mathrm{d}s.$$

Using the Gronwall's lemma, the uniqueness follows immediately and, considering a non-decreasing sequence (E_n) of subsets of E such that $\bigcup E_n = E$ and $\forall n \in \mathbb{N}$, $\mu(E_n) < \infty$, this result allows to show that (under the same hypothesis 1.1) the sequence of approximated solutions $X_t^n \stackrel{\text{def}}{=} X_t^{E_n}$ (which exist for each n, using the remark 1.6.1, quoted earlier) converge to a càdlàg solution of 1 with trajectories belonging to the L^1 space.

Solution viewed as a limit of PDMP with regimes

In Chapter two, we will consider this type of process as a generalization of Piecewise Deterministic Markov Processes (in short PDMP) in two senses: in standard PDMP theory, between two jump times, the process follows a deterministic curve, solution of an ODE. In our case this deterministic curve is replaced by the trajectory of a diffusion process (so the process is no more piecewise deterministic, but has a diffusive component). Then the process jumps with an intensity which depends on the position of the particle. Notice however that, in the standard setting of PDMP's, the intensity of the jump measure is finite, so the jump times are discrete. In our case we consider an infinite intensity measure so that the jump times are dense in $\mathbb{R}_+$.

We will therefore study a limit theorem which motivates the equations introduced above : we consider a sequence X_t^n of standard PDMP's for which the intensity of the jumps tends to infinity as $n \to \infty$, following two regimes: a slow one, which leads to a jump component with finite variation, and a rapid one which, supposing that the processes at hand are centered and renormalized in a convenient way, produces the diffusion component in the limit.

In addition to the Hypothesis 1.1 in the first chapter (which ensure existence and uniqueness of the solution), we will add the following one (that hypothesis will be needed in the Malliavin Calculus part anyway : it will be contained in the Hypothesis 5.2) :

- *We assume, in the following, that γ is bounded by $\overline{C} \in \mathbb{R}_+^*$.*

We can consequently assume in that chapter, without any loss of generality, that $\overline{C} = \frac{1}{2}$ (this value is only chosen in order to simplify the following notations), so our equation can also be written here

$$\begin{aligned} X_t = x + \sum_{l=1}^m \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t b(X_s)\,\mathrm{d}s \\ + \int_0^t \int_{E\times(0,1)} c(X_{s-},z)\mathbb{1}_{\{u\leq\gamma(X_{s-},z)\}} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned} \tag{4}$$

We recall that, if μ is a finite measure, then the above equation admits the following representation. Let J_t be a Poisson process of parameter $\mu(E) \times [0,1]$ and let $T_k, k \in N$ be its jump times. Consider also a sequence of independent random variables $Z_k, U_k, k \in N$ with laws

$$P(Z_k \in \mathrm{d}z) = \frac{1}{\mu(E)}\mu(\mathrm{d}z) \qquad \text{and} \qquad P(U_k \in \mathrm{d}u) = \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.$$

Then the above equation reads

$$X_t = x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\, \mathrm{d}W_s^l + \int_0^t b(X_s)\, \mathrm{d}s + \sum_{k=1}^{J_t} c(X_{T_k-}, Z_k) \mathbb{1}_{\{U_k \le \gamma(X_{T_k-}, Z_k\}}. \tag{5}$$

So, in the case $\mu(E) < \infty$, we prove that the law of X_t coincides with the law of $\overline{X}_t$ constructed in the following way.

We consider a point $z_* \notin E$ and we denote $E_* = E \cup \{z_*\}$. On E_* we define the probability measure

$$\eta(x, \mathrm{d}z) = \theta(x)\delta_{z_*}(\mathrm{d}z) + \frac{1}{\mu(E)} \mathbb{1}_E(z)\gamma(x,z)\mu(\mathrm{d}z), \qquad \text{with}$$
$$\theta(x) = \frac{1}{\mu(E)} \int_E (1 - \gamma(x,z))\mu(\mathrm{d}z).$$

We put $\overline{X}_0 = x$ and we define by recurrence

$$\overline{X}_{T_k} = \overline{X}_{T_k-} + c(\overline{X}_{T_k-}, \overline{Z}_k)\mathbb{1}_E(\overline{Z}_k) \tag{6}$$
$$\overline{X}_t = \overline{X}_{T_k} + \sum_{l=1}^{m} \int_{T_k}^t \sigma_l(\overline{X}_s)\, \mathrm{d}W_s^l + \int_{T_k}^t b(\overline{X}_s)\, \mathrm{d}s, \qquad T_k \le t < T_{k+1}.$$

with the random variables $\overline{Z}_k$ having the conditional law

$$P\Big(\overline{Z}_k \in \mathrm{d}z \mid \overline{X}_{T_k-} = x\Big) = \eta(x, \mathrm{d}z).$$

Notice that $T_k, k \in \mathbb{N}$ are not the real jump times of $\overline{X}$: indeed, $\mathbb{1}_E(\overline{Z}_k) = 0$ with probability $\theta(x) > 0$ and in this case $\overline{X}_{T_k} = \overline{X}_{T_k-}$. Notice also that the law of the jump at time T_k depends on the position $\overline{X}_{T_k-}$. So, if $\sigma = 0$, then $\overline{X}_t$ is a PDMP. Then our model appears as a natural generalization of such models, which introduces a supplementary diffusive component $\sum_{l=1}^m \int_0^t \sigma_l(\overline{X}_s)\, \mathrm{d}W_s^l$. The motivation of this diffusive component is now given by means of the following convergence result.

Considering again a non-decreasing sequence (E_n) of subsets of E such that $\bigcup E_n = E$ and $\forall n \in \mathbb{N}$, $\mu(E_n) < \infty$ and a family $(\nu_n)_{n\in\mathbb{N}}$ of finite measures on E, we consider a sequence X_t^n, $n \in \mathbb{N}$, of $PDMP's$ with two regimes:

$$X_t^n = x + \int_0^t b(X_s^n)\, \mathrm{d}s + \int_0^t \int_{E_n} \int_{(0,1)} c(X_{s-}^n, z)\mathbb{1}_{\{u \le \gamma(X_{s-}, z)\}} N_n^\mu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \tag{7}$$
$$+ \int_0^t \int_E \int_{(0,1)} c(X_{s-}^n, z)\mathbb{1}_{\{u \le \gamma(X_{s-}, z)\}} \widetilde{N}_n^\nu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Here $N_n^\mu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N_n^\nu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ are two Poisson point measures with

$$\widehat{N}_n^\mu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s \mathbb{1}_{E_n}(z)\, \mathrm{d}\mu(z)\mathbb{1}_{(0,1)}(u)\, \mathrm{d}u \qquad \text{and} \qquad \widehat{N}_n^\nu(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s\, \mathrm{d}\nu_n(z)\mathbb{1}_{(0,1)}(u)\, \mathrm{d}u.$$

Since $\mu(E_n) + \nu_n(E) < \infty$, the solution of (7) is a standard $PDMP$. The regime $\mathbb{1}_{E_n}(z)\, \mathrm{d}\mu(z)$ is the slow regime and $\mathrm{d}\nu_n$ will be the rapid regime. We denote $\overline{c}(z) = \sup_{x\in\mathbb{R}^d} |c(x,z)|$ and

$$a_n^{i,j} = \int_E c^i(x,a)c^j(x,a)\gamma(x,a)d\nu_n(a).$$

Let σ be the coefficient in the equation (4) and $a = \sigma\sigma^*$. We denote

$$\varepsilon(n) = \|a - a_n\|_\infty + \int_{E_n^c} \overline{c\gamma}\, \mathrm{d}\mu + \int_E \overline{c}^3\overline{\gamma}d\nu_n$$

and we construct

$$\varepsilon_*(n) = \inf_{\overline{n} \ge n} \Big(\int_{E_{\overline{n}}^c} \overline{c\gamma}\, \mathrm{d}\mu + \mu(E_{\overline{n}})\varepsilon(n) \Big).$$

Then, under the hypothesis $\gamma(x,z) \ge \underline{\gamma} > 0$ we prove that (see Theorem 2.3.5)

- *for every $f \in C_b^3(\mathbb{R}^d)$*

$$\sup_{x\in\mathbb{R}^d} \left|\mathrm{E}\left[f(X_t(x))\right] - \mathrm{E}\left[f(X_t^n(x))\right]\right| \leq \frac{C_n}{\underline{\gamma}^3} \|f\|_{3,\infty}\, \varepsilon_*(n), \tag{8}$$

and we give an explicit estimate for C_n (generally $\sup_n C_n < \infty$). In particular, if

$$\lim_{n\to\infty} C_n \varepsilon_*(n) = 0,$$

- *we obtain the convergence in law of X_t^n to X_t.*

Notice that the estimate (8) is not asymptotic, but holds for every fixed n. This allows to consider it from a different point of view, mainly as a result which may be used for simulation, for example. This is an idea going back to Asmussen and Rosiński [2] : they consider a jump type stochastic equation (of classical type) and propose to simulate the solution of such an equation by replacing the small jumps (the ones which produce the infinite activity) by a Brownian motion. We are doing the same here: we replace $c(X_{s-}^n, z)\mathbb{1}_{\{u\leq\gamma(X_{s-}^n,z)\}}\widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ by $\sigma(X_s)\,\mathrm{d}W_s$ (so now we read the result in a converse sense). And then (8) gives an upper bound of the error introduced by this operation. So, in a general way, we answer the question : what is the price to pay in order to replace a piece of Poisson point measure by a piece of Brownian motion?

Ergodicity for multidimensional EDS's with censored jumps

In Chapter three, we give a first application concerning the possible ergodic behaviour of this solution: the purpose is to give easily verifiable conditions on the coefficients g, σ, c and γ under which the process is recurrent in the sense of Harris[2] and satisfies the ergodic theorem starting from any initial point x, without imposing any non-degeneracy condition on the diffusive part.

There starts to be a huge literature on the subject of EDS with jumps, see e.g. Masuda [40] (2007) who works in a simpler situation where the "censure term" $\mathbb{1}_{u\leq\gamma(z,X_{s-})}$ is not present and who follows the Meyn and Tweedie approach developed in [42] or [41], Kulik [34] (2009) uses the stratification method in order to prove exponential ergodicity of jump diffusions, but the models he considers do not include the censored situation neither. Finally, let us mention Duan and Qiao [22] (2014) who are interested in solutions driven by non-Lipschitz coefficients.

Our aim is not to improve on the regularity conditions imposed on the coefficients but to concentrate on the jump mechanism. More precisely we show that we can use the jumps themselves in order to generate a splitting scheme that will allow to prove recurrence of the process. It is important to note that the presence of the censure term $\gamma(z, X_{s-})$ in (1) implies that the study of X is technically much more involved than the non-censored situation when γ is lower-bounded and strictly positive.

The method we use is the so-called regeneration method which we apply to the big jumps. More precisely, for some suitable set E such that $\mu(E) < \infty$, we cut the trajectory of X into parts of solutions of (3.1.1) driven by N in restriction to E^c and which are stopped at the first jump appearing due to "noise" z belonging to E. In spirit of the splitting technique introduced by Nummelin [44] (1978) and Athreya and Ney [3] (1978), we state a non-degeneracy condition which guarantees that the jump operator associated to the big jumps possesses a Lebesgue absolutely continuous component. This amounts to imposing that the partial derivatives of the jump term

[2]Recall that a process X is called recurrent in the sense of Harris if it possesses an invariant measure m such that any set A of positive $m-$measure $m(A) > 0$ is visited infinitely often by the process almost surely (see Azéma, Duflo and Revuz [4] (1969)): for all $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

c with respect to the noise z are sufficiently non-degenerate (see (3.2.6) and (3.2.7)). We stress that we do not need any non-degeneracy condition for the diffusion coefficient σ.

Notice that we do not apply the splitting technique to an extracted sampled chain nor to the resolvent chain as in Meyn and Tweedie [42] (1993); the loss of memory needed for regeneration is produced by certain big jumps. This approach is very natural in this context, but does not seem to be used so far in the literature, except for Xu [54] (2011), who works in a very specific frame and where the jumps do not depend on the position of the process.

As a result, under appropriate additional assumptions (basically, c has to be $\mathcal{C}^2$, μ has to be absolutely continuous with respect to the Lebesgue measure and we assume some Lyapunov-type conditions) we establish in that chapter the following theorem (Theorem 3.2.8) :

- *the process X is recurrent in the sense of Harris having a unique invariant probability measure m, and, moreover, for any measurable function $f \in \mathrm{L}^1(m)$, we have*

$$\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s \to m(f)$$

  P_x-almost surely for any $x \in \mathbb{R}^d$.

The above ergodic theorem is an important tool, *e.g.* for statistical inference based on observations of the process X in continuous time. In this direction, the following deviation inequality is of particular interest.

- *with $f \in \mathrm{L}^1(m)$ such that $\|f\|_\infty < \infty$, $x \in \mathbb{R}^d$ and $0 < \varepsilon < \|f\|_\infty$, for all $t \geq 1$ there exists a constant $C(\varepsilon, f) > 0$ (depending on ε and f) such that the following inequality holds :*

$$P_x\left[\left|\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s - m(f)\right| > \varepsilon\right] \leq V(x)\frac{C(\varepsilon, f)}{v(t)}$$

  where $x \mapsto V(x)$ is the Lyapunov function and $v : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function which gives the order of ergodicity.

Under appropriate additional assumption on $V(x)$, we can achieve the rate $v(t) = t^{p-1}$, for some $p > 1$, and, with α such that $p = \frac{1}{1-\alpha}$, we obtain the following quantitative control of the convergence of ergodic averages (Proposition 3.2.10) :

- *for any $x, y \in \mathbb{R}^d$,*

$$\left\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - \frac{1}{t}\int_0^t P_s(y,\cdot)\right\|_{\mathrm{TV}} \leq C\frac{1}{t}(V(x)^{(1-\alpha)} + V(y)^{(1-\alpha)}),$$

  where $C > 0$ is a constant. In particular, if $\alpha \geq \frac{1}{2}$, then

$$\left\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - m\right\|_{\mathrm{TV}} \leq C\frac{1}{t}V(x)^{(1-\alpha)}$$

  (where $\|.\|_{\mathrm{TV}}$ is the total variation norm).

Joint regularity for the density of the solution

In this part we consider again a solution X_t^x of the equation (1), starting from x, and we study the joint regularity of it in the following sense : we fix $q \geq 1$ and $p > 1$, K a compact set of $\mathbb{R}^d$, and we will give sufficient conditions in order to have $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$ with $(x, y) \mapsto p_{X_t^x}(y) \in W^{q,p}(K \times \mathbb{R}^d)$.

In the case $\sigma = 0$ (no Brownian part) and for a fixed initial condition x, the regularity of $y \mapsto p_{X_t^x}(y)$ has already been studied in [8].

The new point here is that we prove the regularity with respect to the initial condition as well. This is a non trivial problem, because the coefficient of the equation (1.2.3) involves an indicator function. Consequently , the flow $x \longrightarrow X_t^x$ is not differentiable. We will use some new ideas coming from [7] in order to circumvent this difficulty.

As in [8], our approach is based on a finite dimensional Malliavin Calculus (which we recall in Chapter 4). But in our framework, due to the supplementary term $\sigma \, \mathrm{d}W$, the estimates of the Sobolev norms are much more technical and difficult then in [8].

We will now explain briefly how such results lead to the regularity we were looking for by giving a sketch of the whole proof itself.

We set $B_M = \{z \in \mathbb{R}^d \; : \; |z| < M\}$, $M \in \mathbb{N}^*$, (so $\mu(B_M) < \infty$), and we construct (for each M) an approximation $X_t^M(x)$ of the processes X_t^x based on the restriction N_M of the random measure N on the subset B_M. For the moment, we will omit the starting point x, for more convenience.

Using a similar result as the Lemma 1.4.1, given in the first part of this work, we can then tell that the L^1-distance between these two processes is bounded as follows :

$$\forall t \leq T, \qquad \mathrm{E}\left[\left|X_t - X_t^M\right|\right] \leq C_T \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z) \, \mathrm{d}\mu(z).$$

Since $\mu(B_M) < +\infty$, the random measure N_M may be represented as a compound Poisson process (where the jump times will be denoted by T_k^M, $k \in \mathbb{N}$) and the Poisson part of process X_t^M could be expressed as a sum ; nevertheless, because of the indicator function from the original equation, the coefficients of the equation verified by X_t^M are still (for the Poisson part) discontinuous and so, we cannot use directly the differential calculus presented earlier. Instead we prove that X_t^M has the same law as a process $\overline{X}_t^M$ which verifies an equation with smooth coefficients.

At this point, one would like to obtain an integration by part formula for $\overline{X}_t^M$, but there remains one last difficulty : it is clear that, for $t < T_1^M$ (the first jump of N_M), the random measure N_M produces no noise, and consequently there is no chance to use it for an integration by part (the Malliavin covariance matrix being, of course, degenerated).

That is why one last process will be introduced :

$$F_t^M(x) \overset{\text{def}}{=} \overline{X}_t^M(x) + \sqrt{U_M(t)} \times \Delta,$$

where Δ Gaussian and where $U_M(t)$ is defined by $U_M(t) = t \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z) \, \mathrm{d}\mu(z)$.

The L^1-distance between F_M and $\overline{X}_t^M$ is then bounded, for $t \leq T$, by

$$K_T \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z) \, \mathrm{d}\mu(z).}$$

The finite dimensional Malliavin Calculus developed in [8], and presented in Chapter 4, gives an integration by part formula for the process F_M :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M)H_M\right], \qquad (\mathrm{I}_M)$$

where H_M is computed from F_M, so we can give an upper bound for $\mathrm{E}\left[|H_M|\right]$ (which depends on M of course) and consequently control its speed of explosion.

The second step consists in proving the density regularity.

If the starting point x is fixed, the idea is to use a certain balance between the error $\mathrm{E}\left[|F_M - X_t|\right]$ (which tends to 0) and the weight $\mathrm{E}\left[|H_M|\right]$ (which tends to ∞). This was the strategy used in [8] as well. But here the estimates of $\mathrm{E}\left[|H_M|\right]$ have been more delicate then the corresponding one in [8] because of the additional brownian part $\sigma\,\mathrm{d}W$. Nevertheless, with this last point taken care of (which corresponds to Chapter 6 of this work), it is possible to conclude in the same way ; as a reminder, the balance used in [8] was based on a Fourier transform method.

But, since we additionally study regularity with respect to the starting point x too, we use a new method, called Interpolation Method, developed in [7].

This method allows us, given a sequence (μ_M) of measures absolutely continuous with respect to the Lebesgue measure, each with a sufficiently regular density f_M, converging to μ_X (under a certain norm, *cf.* Section 4.6), to conclude to the absolute continuousness of μ_X and to the regularity of its density.

We just have to define now the measure μ_X by (where $P_{X_t^x}$ is the law of X_t^x)

$$\mu_X(\mathrm{d}x, \mathrm{d}y) \stackrel{\text{def}}{=} \Psi_K(x) P_{X_t^x}(\mathrm{d}y)\,\mathrm{d}x \tag{9}$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$.

A natural approximation of $\mu_X(\mathrm{d}x, \mathrm{d}y)$ would be then $\Psi_K(x) p_{X_t^M}(x,y)\,\mathrm{d}x\,\mathrm{d}y$. But in order to use the Malliavin calculus developed in this work, it is more convenient to use, instead of X_t^M, the approximation (in law) F_M^x of it that is already defined, and we set

$$f_M \stackrel{\text{def}}{=} \Psi_K(x) p_{F_t^M}(x,y). \tag{10}$$

The main point (*cf.* Theorem 4.6.2) is to bound this particular norm of f_M, which is the work developed in Chapter 7, where we will be lead to use as well the upper bound of the weight $\mathrm{E}\left[|H_M|\right]$ and the integration by part (I_M).

Part I

Solution of the equation and Ergodic application

Chapter 1

Existence and uniqueness

1.1 Introduction

We shall start by introducing the following class of jump type stochastic equations (with $E = \mathbb{R}^d$):

$$
\begin{aligned}
X_t = & x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t g(X_s)\,\mathrm{d}s \\
& + \int_0^t \int_{E\times\mathbb{R}_+} c(X_{s-}, z)\mathbb{1}_{\{u\le\gamma(X_{s-},z)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),
\end{aligned}
\tag{1.1.1}
$$

To portray the idea behind the existence and uniqueness of a solution of such an equation, we suppose momentarily that $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a homogeneous Poisson point measure on $E \times (0,1)$ with intensity measure $\mu(\mathrm{d}z) \times \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u$ and the coefficients are $\sigma_l, b : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times E \to \mathbb{R}^d, \gamma : \mathbb{R}^d \times E \to \mathbb{R}$ under some reasonable conditions (*cf.* Hypothesis 1.1). Moreover, let us suppose for a moment that μ is a finite measure. Then the above equation admits the following representation. Let J_t be a Poisson process of parameter $\mu(E)$ and let $T_k, k \in N$ be its jump times. Consider also a sequence of independent random variables $Z_k, U_k, k \in N$ with laws

$$
P(Z_k \in \mathrm{d}z) = \frac{1}{\mu(E)}\mu(\mathrm{d}z) \qquad \text{and} \qquad P(U_k \in \mathrm{d}u) = \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.
$$

Then the above equation reads

$$
X_t = x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t g(X_s)\,\mathrm{d}s + \sum_{k=1}^{J_t} c(X_{T_k-}, Z_k)\mathbb{1}_{\{U_k\le\gamma(X_{T_k-},Z_k)\}}. \tag{1.1.2}
$$

Existence and uniqueness of the solution of the above equation may be proved in a constructive way: on each interval of time $[T_k, T_{k+1})$ one just solves a standard diffusion equation with diffusion coefficients σ_l and drift coefficient g. But the problem becomes non trivial if $\mu(E) = \infty$ because the domain of the Poisson measure is dense in $\mathbb{R}_+$. So a first result of this work consists in proving that, under reasonable hypothesis on the coefficients of the equation, we have the existence and uniqueness property. This is done in Theorem 1.2.3. The proof is based on a natural argument: one considers a sequence of sets $E_n \uparrow E$ with $\mu(E_n) < \infty$ and proves that the sequence $X_t^n, n \in N$ of the solutions of the equations corresponding to E_n (which are of type (1.1.2)) is a Cauchy sequence which, at the limit, produces the solution of (1.1.1). However this is not trivial: a subtle estimate of $\|X_t^n - X_t^m\|_1$ is needed (see Lemma 1.4.1 ; the same argument proves uniqueness as well). We notice that a standard approach based on L^p norms, with $p > 1$, instead of L^1 norms fails — we thank N. Fournier who gave us an important hint in this direction (private communication to Vlad Bally). We also mention that our approach fails if we want to add a martingale term driven by $\mathrm{d}\widetilde{N}$ (roughly speaking this is because $\mathbb{1}^2_{\{U_k\le\gamma(X_{T_k-},Z_k)\}} = \mathbb{1}_{\{U_k\le\gamma(X_{T_k-},Z_k)\}}$).

1.2 Notations and hypothesis

We consider a Poisson point process p with state space $(X, \mathcal{B}(X))$, where $X = E \times \mathbb{R}_+$ (in this framework E will be simply $\mathbb{R}^d$). We refer to [30] for the notations.

We denote by N the counting measure associated to p, we have $N([0,t) \times A) = \#\{0 \leq s < t\,;\, p_s \in A\}$ for $t \geq 0$ and $A \in \mathcal{B}(X)$. We assume that the associated intensity measure is given by $\hat{N}(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}t \times \mu(\mathrm{d}z) \times \mathbb{1}_{\{0,\infty\}}(u)\,\mathrm{d}u$ where $(z,u) \in \mathbb{R}^d \times \mathbb{R}_+$ and $\mu(\mathrm{d}z) = h(z)\,\mathrm{d}z$.

We are interested in the solution of the following d-dimensional diffusion equation where $X_t = (X_t^i)_{i=1,\ldots,d}$ represent an $\mathbb{R}^d$-valued process and $W_t = (W_t^l)_{l=1,\ldots,m}$ is a m-dimensional Brownian motion :

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+} \int_{E\times\mathbb{R}_+} c(z, X_{s^-}) \mathbb{1}_{\{u \leq \gamma(z, X_{s^-})\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \quad (1.2.3)$$

Definition 1.1 *A process $\left(X_t\right)_{t\geq 0}$ will be called "L^1-solution" if*

- *$t \longrightarrow X_t$ is an adapted càdlàg process which verifies* (1.2.3) *and moreover,*
- *for every $T > 0$,*
$$\sup_{t\leq T} \mathrm{E}\left[|X_t|\right] < \infty. \quad (\star)$$

Remark 1.2.1 *We stress that the uniqueness of the solution of the equation* (1.2.3) *is proved, in this work, in the class of processes which verify* $(\star)$.

We notice that the infinitesimal generator of the Markov of such a process X_t is given by

$$L\psi(x) = \frac{1}{2} \sum_{1\leq i,j\leq d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \psi(x) + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z,x)) - \psi(x)) K(x, \mathrm{d}z)$$

where $a \stackrel{\text{def}}{=} \sigma\sigma^*$ and $K(x, \mathrm{d}z) \stackrel{\text{def}}{=} \gamma(z,x)h(z)\,\mathrm{d}z$ depends on the variable $x \in \mathbb{R}^d$ (with $\mathcal{C}_b^2 \subset \mathrm{Dom}(L)$).

To establish the existence and uniqueness of the solution of the above equation, we will assume that the functions $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$, $g : \mathbb{R}^d \to \mathbb{R}^d$, $c : E \times \mathbb{R}^d \to \mathbb{R}^d$, $\gamma : E \times \mathbb{R}^d \to \mathbb{R}^+$ satisfy the following hypothesis [1] :

Hypothesis 1.1 *1. **Lipschitz-conditions** : let L_σ, $L_g \in \mathbb{R}_+$ such that*

- $|\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|, \quad \forall x, y \in \mathbb{R}^d$
- $|g(x) - g(y)| \leq L_g |x - y|, \quad \forall x, y \in \mathbb{R}^d$

and let $L_c : E \to \mathbb{R}_+$ and $L_\gamma : E \to \mathbb{R}_+$ such that

- $|c(z,x) - c(z,y)| \leq L_c(z)|x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall z \in E$
- $|\gamma(z,x) - \gamma(z,y)| \leq L_\gamma(z)|x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall z \in E$

*2. **Integrability conditions** : let $\overline{\gamma}(z) \stackrel{def}{=} \sup_{x\in\mathbb{R}^d}(|\gamma(z,x)|)$ and $\overline{c}(z) \stackrel{def}{=} \sup_{x\in\mathbb{R}^d}(|c(z,x)|)$, we then assume that*

- $C_\gamma \stackrel{def}{=} \int_E L_c(z)\overline{\gamma}(z)\,\mu(\mathrm{d}z) < +\infty$

[1] In all this framework we will denote the euclidean norm of $z \in \mathbb{R}^d$ by $|z|$ (*i.e.* $|z| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^d z_i^2}$).

- $C_c \stackrel{def}{=} \int_E L_\gamma(z)\overline{c}(z)\,\mu(\mathrm{d}z) < +\infty$

- $\int_E \overline{c}(z)\overline{\gamma}(z)\mu(\mathrm{d}z) < +\infty$

3. ***Linear growth conditions*** *: We will assume that σ is bounded and g is such that*

$$|g(x)| \leq K(1+|x|).$$

Remark 1.2.2 • *We could have assumed also $|\sigma(x)| \leq K(1+|x|)$, the bounded condition is here in order to simplify a little the following.*

- *The condition $\int_E \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) < +\infty$ implies that the Poisson part of the equation makes sense.*

- *Because of this particular Poisson part in the equation* (1.2.3)*, we will have to work with* L^1*-norm.*

Because of this last reason we will use the F_p^1 function space (*cf.* Ikeda-Watanabe [30] p.62), defined as

$$F_p^1 = \left\{ f(s,z,u,\cdot);\, f \text{ predictable and } \forall t>0, \quad \mathrm{E}\left[\int_0^t \int_{E\times\mathbb{R}_+} |f(s,z,u,\cdot)|\hat{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right] < +\infty \right\}$$

on which we have the following isometry (recalling that here $\hat{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) = \mathrm{d}s \times \mu(\mathrm{d}z) \times \mathbb{1}_{(0,\infty)}(u)\,\mathrm{d}u$)

$$\forall f \in F_p^1, \qquad \mathrm{E}\left[\int_0^{t+} \int_{E\times\mathbb{R}_+} f(s,z,u,\cdot)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right] = \mathrm{E}\left[\int_0^t \int_{E\times\mathbb{R}_+} f(s,z,u,\cdot)\,\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u\right]. \tag{1.2.4}$$

We can now state the main result of this chapter :

Theorem 1.2.3 *Under the Hypothesis 1.1, the diffusion equation*

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+} \int_{E\times\mathbb{R}_+} c(z,X_{s-})\mathbb{1}_{\{u\leq\gamma(z,X_{s-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$$

has a unique L^1*-solution, in the sense of Definition 1.1.*

1.3 Regularisation

Let us define the following real function by

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} \alpha\exp(-\frac{1}{1-x^2}) & \text{if } |x|<1, \\ 0 & \text{if } |x|\geq 1. \end{cases}$$

where α is chosen in order to have $\int_\mathbb{R} \varphi(x)\,\mathrm{d}x = 1$. This function is then a smooth compactly supported one. Let $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined by :

$$\varphi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right), \qquad \forall x \in \mathbb{R}.$$

Proposition 1.3.1 *$\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a smooth compactly supported function and there exists $C>0$ such that :*

$$\forall x \in \mathbb{R}, \qquad |\varphi_\varepsilon'(x)| \leq \frac{C}{\varepsilon^2} \quad \textit{and} \quad |\varphi_\varepsilon''(x)| \leq \frac{C}{\varepsilon^3}.$$

Proof : The proof is postponed and will be made in Appendix A.1. •

Then one defines $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by :

$$h_\varepsilon(x) \stackrel{\text{def}}{=} |x| \vee 2\varepsilon,$$

and $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by setting :

$$\phi_\varepsilon(x) \stackrel{\text{def}}{=} \big(h_\varepsilon * \varphi_\varepsilon\big)(x) = \int_{\mathbb{R}} h_\varepsilon(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y, \qquad \forall x \in \mathbb{R}.$$

Proposition 1.3.2 *1. $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ converges pointwise (as $\varepsilon \to 0$) to the absolute value function $x \mapsto |x|$ and*

$$\begin{aligned} \phi_\varepsilon(x) &= 2\varepsilon, \quad \text{if} \quad |x| \le \varepsilon, & (1.3.5)\\ &= |x|, \quad \text{if} \quad |x| > 3\varepsilon, & (1.3.6) \end{aligned}$$

and

$$0 \le \phi_\varepsilon(x) \le 4\varepsilon, \quad \text{for} \quad |x| \in]\varepsilon, 3\varepsilon].$$

2. There exists $M > 0$ such that

$$\forall x \in \mathbb{R}, \qquad |\phi_\varepsilon'(x)| \le M \quad \text{and} \quad |\phi_\varepsilon''(x)| \le \frac{M}{\varepsilon}\mathbb{1}_{|x|\le 3\varepsilon}. \tag{1.3.7}$$

Proof : The proof is postponed and will be made in Appendix A.1. •

1.4 Preliminary lemma

Let F and G be two subsets of E such that $G \subset F$ (the case $G = F = E$ is include). We let $(X_t) = (X_t^i)_{i=1,\dots,d}$ and $(Y_t) = (Y_t^i)_{i=1,\dots,d}$ be two càdlàg $\mathbb{R}^d$-valued processes satisfying :

$$X_t = x + \sum_{j=1}^m \int_0^t \sigma_j(X_s)\,\mathrm{d}W_s^j + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_F(z)c(z,X_{s^-})\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$$

and

$$Y_t = x + \sum_{j=1}^m \int_0^t \sigma_j(Y_s)\,\mathrm{d}W_s^j + \int_0^t g(Y_s)\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_G(z)c(z,Y_{s^-})\mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Let

$$Z_t \stackrel{\text{def}}{=} X_t - Y_t = \sum_{j=1}^m \int_0^t \Delta_j\sigma_s\,\mathrm{d}W_s^j + \int_0^t \Delta g_s\,\mathrm{d}s + \int_0^{t+}\int_{E\times\mathbb{R}_+} H_{s^-}(z,u)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$$

where

- $\Delta_j\sigma_s \stackrel{\text{def}}{=} \sigma_j(X_s) - \sigma_j(Y_s)$,
- $\Delta g_s \stackrel{\text{def}}{=} g(X_s) - g(Y_s)$,
- and $H_{s^-}(z,u) \stackrel{\text{def}}{=} \mathbb{1}_F(z)c(z,X_{s^-})\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} - \mathbb{1}_G(z)c(z,Y_{s^-})\mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}$.

Lemma 1.4.1 *Suppose that X_t and Y_t are L^1-solutions (in the sense of Definition 1.1) of the above equations. Under the hypothesis 1.1, and with the above notations,*

1. *we have the following inequality :*

$$\int_0^t \int_{E\times\mathbb{R}_+} |H_{s^-}(z,u))|\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u \le (dC_\gamma + dC_c)\int_0^t |Z_s|\,\mathrm{d}s + dT\int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\ ; \tag{1.4.8}$$

2. *there exist $K_1, K_2 \in \mathbb{R}_+$ such that*

$$\forall t\le T, \qquad \mathrm{E}\left[|Z_t|\right] \le TK_1\Big(\int_{F\setminus G}\overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\Big)e^{K_2 t}. \tag{1.4.9}$$

Proof.
1) By decomposing $H^i_{s^-}(z,u)$ we notice that

$$\int_0^t \int_{E\times\mathbb{R}_+} |H_{s^-}(z,u)|\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u \le C_1 + C_2 + C_3$$

with :

$$C_1 \overset{\text{def}}{=} \sum_{i=1}^d \int_0^t \int_{E\times\mathbb{R}_+} \mathbb{1}_F(z)|c^i(z,X_{s^-}) - c^i(z,Y_{s^-})|\mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u,$$

$$C_2 \overset{\text{def}}{=} \sum_{i=1}^d \int_0^t \int_{E\times\mathbb{R}_+} \mathbb{1}_F(z)|c^i(z,X_{s^-})||\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} - \mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}|\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u,$$

$$C_3 \overset{\text{def}}{=} \sum_{i=1}^d \int_0^t \int_{E\times\mathbb{R}_+} \mathbb{1}_{F\setminus G}(z)|c^i(z,Y_{s^-})|\mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u.$$

So, using that c is Lipschitz with respect to the second variable,

$$\begin{aligned}
C_1 &= \sum_{i=1}^d \int_0^t \int_E |c^i(z,X_{s^-}) - c^i(z,Y_{s^-})|\left(\int_{\mathbb{R}_+}\mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}\,\mathrm{d}u\right)\mathrm{d}s\,\mathrm{d}\mu(z)\\
&\le \sum_{i=1}^d \int_0^t \int_E |X_{s^-} - Y_{s^-}||\gamma(z,Y_{s^-})|L_c(z)\,\mathrm{d}s\,\mathrm{d}\mu(z)\\
&\le \sum_{i=1}^d \int_0^t |X_{s^-} - Y_{s^-}|\left(\int_E \overline{\gamma}(z)L_c(z)\,\mathrm{d}\mu(z)\right)\mathrm{d}s \le dC_\gamma\int_0^t |Z_s|\,\mathrm{d}s.
\end{aligned}$$

We also have

$$\begin{aligned}
C_2 &\le \sum_{i=1}^d \int_0^t \int_{E\times\mathbb{R}_+} |c^i(z,X_{s^-})||\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} - \mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}|\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u\\
&\le d\int_0^t\int_E \overline{c}(z)\left(\int_{\mathbb{R}_+}|\mathbb{1}_{\{u\le\gamma(z,X_{s^-})\}} - \mathbb{1}_{\{u\le\gamma(z,Y_{s^-})\}}|\,\mathrm{d}u\right)\mathrm{d}s\,\mathrm{d}\mu(z)\\
&= d\int_0^t\int_E \overline{c}(z)|\gamma(z,X_{s^-}) - \gamma(z,Y_{s^-})|\,\mathrm{d}s\,\mathrm{d}\mu(z)\\
&\le d\int_0^t |X_{s^-} - Y_{s^-}|\left(\int_E L_\gamma(z)\overline{c}(z)\,\mathrm{d}\mu(z)\right)\mathrm{d}s\\
&\le dC_c\int_0^t |Z_s|\,\mathrm{d}s.
\end{aligned}$$

At last

$$\begin{aligned}
C_3 &= \sum_{i=1}^{d} \int_0^t \int_{E\times\mathbb{R}_+} \mathbb{1}_{F\setminus G} |c^i(z, Y_{s^-})| \mathbb{1}_{\{u\leq\gamma(z,Y_{s^-})\}} \,\mathrm{d}s \,\mathrm{d}\mu(z) \,\mathrm{d}u \\
&= \sum_{i=1}^{d} \int_0^t \int_{F\setminus G} |c^i(z, Y_{s^-})| \gamma(z, Y_{s^-}) \,\mathrm{d}\mu(z) \,\mathrm{d}s \\
&\leq dT \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z) \,\mathrm{d}\mu(z).
\end{aligned}$$

Which completes the proof of (1.4.8).

2) Let, for all $z \in \mathbb{R}^d$,

$$f_\varepsilon(z) \overset{\text{def}}{=} \phi_\varepsilon(|z|).$$

Then f_ε is $\mathcal{C}^2$, and we have :

$$\forall z \in \mathbb{R}^d, \qquad \frac{\partial f_\varepsilon}{\partial z_i}(z) = \phi_\varepsilon'(|z|)\, \frac{z_i}{|z|}.$$

Since $\|\phi_\varepsilon'\|_\infty < M$, the norm of the differential of f_ε is less or equal to M, so f_ε is $M-$Lipschitz.

We have also

$$\forall z \in \mathbb{R}^d, \qquad \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(z) = \left(\phi_\varepsilon''(|z|) - \frac{\phi_\varepsilon'(|z|)}{|z|} \right) \frac{z_i z_j}{|z|^2} + \delta_{i,j} \frac{\phi_\varepsilon'(|z|)}{|z|}.$$

Notice that, if $|z| < \varepsilon$, we have $\phi_\varepsilon'(|z|) = \phi_\varepsilon''(|z|) = 0$, which leads to

$$\frac{\partial f_\varepsilon}{\partial z_i}(z) = \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(z) = 0.$$

Let now $T > 0$. For $t \in [0, T]$, the Itô formula leads to :

$$\begin{aligned}
f_\varepsilon(Z_t) - f_\varepsilon(Z_0) &= \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(Z_s) \sum_{l=1}^{m} (\Delta_l^i \sigma_s \Delta_l^j \sigma_s) \,\mathrm{d}s \\
&+ \sum_{i=1}^{d} \int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s) \sum_{l=1}^{m} \Delta_l^i \sigma_s \,\mathrm{d}W_s^l \\
&+ \sum_{i=1}^{d} \int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s) \Delta^i g_s \,\mathrm{d}s \\
&+ \int_0^{t+} \int_{E\times\mathbb{R}_+} f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-}) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).
\end{aligned}$$

Since σ is bounded $\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s) \Delta_j^i \sigma_s \,\mathrm{d}W_s^j$ is a martingale and, by consequence, its expectation is zero.

Moreover $s \mapsto f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-})$ is clearly predictable and belongs to F_p^1 since for all $i \in [\![1, d]\!]$, $((s, u, z) \mapsto H_{s^-}^i(z, u)) \in F_p^1$ (we can use this condition because $|f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-})| \leq M |(Z_{s^-} + H_{s^-}(z,u)) - Z_{s^-}| = M |H_{s^-}(z,u)|$, since f_ε is M-Lipschitz), so, using the isometry (1.2.4), it comes

$$\begin{aligned}
\mathrm{E}[f_\varepsilon(Z_t)] - \mathrm{E}[f_\varepsilon(Z_0)] = \frac{1}{2}\sum_{i,j=1}^{d} \mathrm{E}\left[\int_0^t \frac{\partial^2 f_\varepsilon}{\partial z_i \partial z_j}(Z_s) \sum_{l=1}^{m}(\Delta_l^i \sigma_s \Delta_l^j \sigma_s)\,\mathrm{d}s\right] \\
+ \sum_{i=1}^{d} \mathrm{E}\left[\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s)\Delta^i g_s\,\mathrm{d}s\right] \\
+ \mathrm{E}\left[\int_0^{t+}\int_{E\times\mathbb{R}} f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-})\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{d}\mathrm{E}\left[\int_0^t \frac{\phi_\varepsilon'(|Z_s|)}{|Z_s|}\sum_{l=1}^{m}(\Delta_l^i\sigma_s)^2\,\mathrm{d}s\right] \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d}\mathrm{E}\left[\int_0^t \left(\phi_\varepsilon''(|Z_s|) - \frac{\phi_\varepsilon'(|Z_s|)}{|Z_s|}\right)\frac{Z_s^i Z_s^j}{|Z_s|^2}\sum_{l=1}^{m}(\Delta_l^i\sigma_s\Delta_l^j\sigma_s)\,\mathrm{d}s\right] \\
&\quad + \sum_{i=1}^{d}\mathrm{E}\left[\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s)\Delta^i g_s\,\mathrm{d}s\right] \\
&\quad + \mathrm{E}\left[\int_0^{t+}\int_{E\times\mathbb{R}} f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-})\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u\right] \\
&\overset{\text{def}}{=} A + A' + B + C.
\end{aligned}$$

Since $\Delta_j^i\sigma_s = \sigma_j^i(X_s) - \sigma_j^i(Y_s)$,

$$(\Delta_l^i\sigma_s)^2 \le L_\sigma^2|Z_s|^2$$

and, since ϕ_ε' is bounded,

$$\begin{aligned}
\left|\frac{\phi_\varepsilon'(|Z_s|)}{|Z_s|}\sum_{l=1}^{m}(\Delta_l^i\sigma_s)^2\right| &\le m\frac{L_\sigma^2|Z_s|^2}{|Z_s|}|\phi_\varepsilon'(|Z_s|)| \\
&\le mL_\sigma^2|Z_s|M.
\end{aligned}$$

So

$$A \le \frac{dmML_\sigma^2}{2}\int_0^t \mathrm{E}\left[|Z_s|\right]\mathrm{d}s.$$

Similarly,

$$|\Delta_l^i\sigma_s||\Delta_l^m\sigma_s| \le L_\sigma^2|Z_s|^2$$

and since using (1.3.7), if $|u| \ge 3\varepsilon$, then $\phi_\varepsilon''(u) = 0$, we have

$$\begin{aligned}
\left|\left(\phi_\varepsilon''(|Z_s|) - \frac{\phi_\varepsilon'(|Z_s|)}{|Z_s|}\right)\frac{Z_s^iZ_s^j}{|Z_s|^2}\sum_{l=1}^{m}(\Delta_l^i\sigma_s\Delta_l^j\sigma_s)\right| &\le \left|\phi_\varepsilon''(|Z_s|) - \frac{\phi_\varepsilon'(|Z_s|)}{|Z_s|}\right|\underbrace{\frac{|Z_s^i||Z_s^j|}{|Z_s|^2}}_{\le 1} mL_\sigma^2|Z_s|^2 \\
&\le mL_\sigma^2\left(\left|\phi_\varepsilon''(|Z_s|)\right||Z_s|^2 + |\phi_\varepsilon'(|Z_s|)|Z_s|\right) \\
&\le mL_\sigma^2\left((\varepsilon)^{-1}M(3\varepsilon)^2 + M|Z_s|\right) \\
&= mL_\sigma^2M(9\varepsilon + |Z_s|)
\end{aligned}$$

so

$$A' \le \frac{1}{2}d^2mL_\sigma^2M\Big(9T\varepsilon + \int_0^t \mathrm{E}\left[|Z_s|\right]\mathrm{d}s\Big).$$

Since $|\frac{\partial f_\varepsilon}{\partial z_i}(Z_s)\Delta^i g_s| \leq M|\Delta^i g_s| \leq ML_g|Z_s|$,

$$\begin{aligned} B &= \sum_{i=1}^{d} \mathrm{E}\left[\int_0^t \frac{\partial f_\varepsilon}{\partial z_i}(Z_s)\Delta^i g_s \,\mathrm{d}s\right] \\ &\leq dML_g \int_0^t \mathrm{E}\left[|Z_s|\right] \mathrm{d}s. \end{aligned}$$

Finally, for the quantity C, recalling that f_ε is M-Lipschitz, it comes, straightly from (1.4.8) :

$$\begin{aligned} |C| &\leq \mathrm{E}\left[\int_0^t \int_{E\times\mathbb{R}_+} |f_\varepsilon(Z_{s^-} + H_{s^-}(z,u)) - f_\varepsilon(Z_{s^-})| \,\mathrm{d}s \,\mathrm{d}\mu(z)\,\mathrm{d}u\right] \\ &\leq M(dC_\gamma + dC_c)\,\mathrm{E}\left[\int_0^t |Z_s|\,\mathrm{d}s\right] + MdT\,\mathrm{E}\left[\int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\right]. \end{aligned}$$

Gathering all the bounds obtained for A, A', B, C, we obtain finally (noticing that $z \mapsto \overline{c}(z)\overline{\gamma}(z)$ is a deterministic function),

$$\mathrm{E}[f_\varepsilon(Z_t)] \leq (TC'+1)\varepsilon + TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s|\right] \mathrm{d}s$$

and since $f_\varepsilon(Z_s) \underset{\varepsilon\to 0}{\longrightarrow} |Z_s|$, using Fatou's lemma[2], it comes :

$$\mathrm{E}[|Z_t|] \leq TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s|\right] \mathrm{d}s.$$

Moreover, since, from the hypothesis made on X_t and Y_t, we have

$$\sup_{0\leq t\leq T} \mathrm{E}[|Z_t|] < +\infty,$$

we may use Gronwall's lemma (*cf.* A.2.1), which proves the result.

1.5 Uniqueness

Let X and Y be two L^1-solutions of the equation (1.2.3). Using the above lemma with $F = G = E$, we obtain $\mathrm{E}[|X_t - Y_t|] = 0$, which shows the uniqueness with regard to the L^1 space of function.

1.6 Existence

Let (E_n) be a non-decreasing sequence of subsets of E such that $\bigcup E_n = E$ and $\forall n \in \mathbb{N}$, $\mu(E_n) < \infty$ and let $X_t^n = (X_t^{i,n})_{1\leq i\leq n}$ be a $\mathbb{R}^d$-valued process solution of the equation :

$$X_t^{i,n} = x^i + \sum_{j=1}^{m} \int_0^t \sigma_j^i(X_s^n)\,\mathrm{d}W_s^j + \int_0^t g^i(X_s^n)\,\mathrm{d}s + \int_0^{t+} \int_{E\times\mathbb{R}_+} \mathbb{1}_{E_n}(z) c^i(z, X_{s^-}^n) \mathbb{1}_{\{u\leq\gamma(z,X_{s^-}^n)\}} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

We will prove that X_t^n converge to X_t, a càdlàg solution of (1.2.3) such that

$$\sup_{0\leq t\leq T} \mathrm{E}[|X_t|] < +\infty. \qquad (\star)$$

[2]If $\varepsilon_l \underset{l\to+\infty}{\longrightarrow} 0$ using Fatou's lemma, all the quantities being non-negative, $\mathrm{E}[|Z_s|] = \mathrm{E}[\liminf_l f_{\varepsilon_l}(Z_s)] \leq \liminf_l \mathrm{E}[f_{\varepsilon_l}(Z_s)] \leq \liminf_l \Big((TC'+1)\varepsilon_l + TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s|\right] \mathrm{d}s\Big) = TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s|\right] \mathrm{d}s$.

Remark 1.6.1 *The existence of X_t^n comes from the fact that $\mu(E_n) < \infty$: we may represent the random measure N by a compound Poisson process ; let $\lambda_n \stackrel{def}{=} \bar{C} \times \mu(E_n)$ and let J_t^n a Poisson process of parameter λ_n. We denote by T_k^n, $k \in \mathbb{N}$, the jump times of J_t^n. We also consider two sequences of independent random variables $(Z_k^n)_{k\in\mathbb{N}}$ and $(U_k)_{k\in\mathbb{N}}$, respectively in $\mathbb{R}^d$ and $\mathbb{R}_+$, which are independent of J^n and such that $Z_k^n \sim \frac{1}{\mu(E_n)} \mathbb{1}_{B_{M+1}}(z)\,\mathrm{d}\mu(z)$, and $U_k \sim \frac{1}{\bar{C}} \mathbb{1}_{[0,\bar{C}]}(u)\,\mathrm{d}u$.*

Then, the last equation may be written as

$$X_t^n = x + \int_0^t \sigma(X_s^n)\,\mathrm{d}W_s + \int_0^t g(X_s^n)\,\mathrm{d}s + \sum_{k=1}^{J_t^n} c(Z_k^n, X_{T_k^n-}^n)\mathbb{1}_{(U_k,\infty)}(\gamma(Z_k^n, X_{T_k^n-}^n)).$$

Then, we can construct, under the hypothesis 1.1, a "piecewise" solution by considering on each interval $]T_k^n, T_{k+1}^n[$ a classic SDE, as it is done, for instance, in [Watanabe [30], p.245]. The solution is càdlàg, with trajectories belonging to the L^1 space, and such that

$$\sup_{0\le t\le T} \mathrm{E}[|X_t^n|] < +\infty. \tag{1.6.10}$$

(For this last assertion cf. also [32].)

For $n > n'$, we set $Z_t^{n,n'} \stackrel{\text{def}}{=} X_t^n - X_t^{n'}$ and the previous lemma, leads to

$$\mathrm{E}[|Z_t^{n,n'}|] \le TK_1 \int_{F\setminus G} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t \mathrm{E}\left[|Z_s^{n,n'}|\right] \mathrm{d}s,$$

and since (1.6.10), we have also $\sup_{0\le t\le T} \mathrm{E}[|Z_t^{n,n'}|] < +\infty$, and the Gronwall lemma implies

$$\mathrm{E}[|Z_t^{n,n'}|] \le TK_1\Big(\int_{E_n\setminus E_{n'}} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\Big)e^{K_2 t},$$

therefore

$$\lim_{\substack{n'\to+\infty\\ n\ge n'}} \mathrm{E}[|X_s^n - X_s^{n'}|] = 0. \tag{1.6.11}$$

It is a Cauchy sequence into the L^1 space, so it converges to a limit X_t. Since, moreover, for $t \le T$ we have

$$\begin{aligned}\mathrm{E}\left[|X_t^n|\right] &\le \mathrm{E}\left[|X_t^n - X_t^1|\right] + \mathrm{E}\left[|X_t^1|\right]\\ &\le TK_1\Big(\int_{E_n} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\Big)e^{K_2 T} + \mathrm{E}\left[|X_t^1|\right]\end{aligned}$$

so that

$$\sup_{0\le t\le T} \mathrm{E}[|X_t|] \le \sup_{0\le t\le T} \mathrm{E}[|X_t^1|] + TK_1\Big(\int_E \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)\Big)e^{K_2 T} < \infty,$$

it remains to be proven that this limit can be chosen càdlàg to end the whole proof. Let X_t such that

$$\forall t \ge 0, \qquad X_t^n \xrightarrow{\mathrm{L}^1} X_t.$$

In order to get a càdlàg behaviour for X_t, we will look for a uniform convergence (with respect to t).

We have

$$X_t^n = x + M_t^n + A_t^n$$

with

- $M_t^{i,n} = \sum_{j=1}^m \int_0^t \sigma_j^i(X_s^n)\,\mathrm{d}W_s^j + \int_0^{t+} \int_{E\times\mathbb{R}_+} \mathbb{1}_{E_n} c^i(z, X_{s-}^n)\mathbb{1}_{\{u\le\gamma(z,X_{s-}^n)\}}\,\mathrm{d}\tilde{N}(s,z,u)$

- $A_t^{i,n} = \int_0^t b^i(X_s^n)\,\mathrm{d}s + \int_0^t \int_{E\times\mathbb{R}_+} \mathbb{1}_{E_n}(z) c^i(z, X_{s-}^n) \mathbb{1}_{\{u\leq\gamma(z,X_{s-}^n)\}}\,\mathrm{d}\hat{N}(s,z,u)$
  $= \int_0^t b^i(X_s^n)\,\mathrm{d}s + \int_0^t \int_E \mathbb{1}_{E_n}(z) c^i(z, X_{s-}^n)\gamma(z, X_{s-}^n)\,\mathrm{d}\mu(z)\,\mathrm{d}s$

consequently, $M_t^{i,n}$ is a martingale.

1) Let us prove that $(M_t^{i,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the L^1 space.
We write

$$|M_t^{i,n} - M_t^{i,n'}| \leq |X_t^{i,n} - X_t^{i,n'}| + |A_t^{i,n} - A_t^{i,n'}|. \tag{1.6.12}$$

First

$$|A_t^{i,n} - A_t^{i,n'}| \leq \int_0^t |g^i(X_s^n) - g^i(X_s^{n'})|\,\mathrm{d}s + \int_0^t \int_{E\times\mathbb{R}_+} |H_{s^-}^{i,n,n'}(z,u))|\,\mathrm{d}s\,\mathrm{d}\mu(z)\,\mathrm{d}u$$

so, using the inequality (1.4.8), it comes,

$$|A_t^{i,n} - A_t^{i,n'}| \leq Tk_1 \int_{E_{n'}\setminus E_n} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + k_2 \int_0^t |X_s^n - X_s^{n'}|\,\mathrm{d}s \tag{1.6.13}$$

thus

$$\mathrm{E}\left[|A_t^{i,n} - A_t^{i,n'}|\right] \leq Tk_1 \int_{E_{n'}\setminus E_n} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + k_2 \int_0^t \mathrm{E}\left[|X_s^n - X_s^{n'}|\right]\mathrm{d}s.$$

Since $(X_t^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^1 (uniformly with respect to $t\in[0,T]$) it follows from this last inequality that $(A_t^n)_{n\in\mathbb{N}}$ is also a Cauchy sequence, and consequently, from (1.6.12), $(M_t^n)_{n\in\mathbb{N}}$ is finally itself a Cauchy sequence.

2) Let us characterize now the uniform convergence.
We have, using the inequality (1.6.13),

$$\begin{aligned} |X_t^n - X_t^{n'}| &\leq |M_t^n - M_t^{n'}| + |A_t^n - A_t^{n'}| \\ &\leq |M_t^n - M_t^{n'}| + TK_1 \int_{E_n\setminus E_{n'}} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + K_2 \int_0^t |X_s^n - X_s^{n'}|\,\mathrm{d}s. \end{aligned}$$

Thus (with $\varepsilon_{n,n'} = K_1 \int_{E_n\setminus E_{n'}} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)$), the Gronwall lemma leads then to

$$|X_t^n - X_t^{n'}| \leq \left(\sup_{u\leq T} |M_u^n - M_u^{n'}| + T\varepsilon_{n,n'}\right) e^{K_2 T}$$

and

$$\sup_{t\leq T} |X_t^n - X_t^{n'}| \leq \left(\sup_{u\leq T} |M_u^n - M_u^{n'}| + T\varepsilon_{n,n'}\right) e^{K_2 T}.$$

If a martingale is only integrable, the Doob's inequality becomes (*cf.* [30] p.34):

$$\forall\lambda>0, \qquad \mathrm{P}\left[\sup_{u\leq T} |M_u^{i,n} - M_u^{i,n'}| > \lambda\right] \leq \frac{\mathrm{E}\left[|M_T^{i,n} - M_T^{i,n'}|\right]}{\lambda}, \qquad \forall i \in [\![1,d]\!].$$

Since we have shown, for all $i\in[\![1,d]\!]$, the L^1-convergence of $(M_t^{i,n})_{n\in\mathbb{N}}$, it comes that, in probability, (X_t^n) converges uniformly to X_t. Therefore there exists a subsequence of (X_t^n) which converges a.s uniformly to X_t (on $[0,T]$, for any T). Since the convergence is uniform, this last limit is, a.s, càdlàg.

Finally, let us show that X_t is solution of the equation (1.2.3) : first, $\int g(X_s^n)\,\mathrm{d}s$ converge (in L^1) to $\int g(X_s^n)\,\mathrm{d}s$. Likewise, the isometry (1.2.4) on the F_p^1-space and (1.4.8) leads to the convergence (in L^1) of

$$\int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_{E_n}(z)c^i(z,X_{s-}^n)\mathbb{1}_{\{u\leq\gamma(z,X_{s-}^n)\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$$

to

$$\int_0^{t+}\int_{E\times\mathbb{R}_+} c^i(z,X_{s-})\mathbb{1}_{\{u\leq\gamma(z,X_{s-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Now, this L^1-convergence implies a convergence in probability and it only remains to show that $\int_0^t \sigma(X_s^n)\,\mathrm{d}W_s$ converges in probability to $\int_0^t \sigma(X_s)\,\mathrm{d}W_s$.

Let

$$A_n = \{\sup_{t\leq T}|X_t^n - X_t| \leq 1\}.$$

Then

$$\mathrm{P}\left(\left|\int_0^T \sigma(X_s^n)-\sigma(X_s)\,\mathrm{d}W_s\right|\geq\delta\right) \leq \mathrm{P}({}^cA_n) + \mathrm{P}\left(A_n\cap\left|\int_0^T \sigma(X_s^n)-\sigma(X_s)\,\mathrm{d}W_s\right|\geq\delta\right)$$

and

$$\begin{aligned}
\mathrm{P}\left(A_n\cap\left|\int_0^T \sigma(X_s^n)-\sigma(X_s)\,\mathrm{d}W_s\right|\geq\delta\right) &\leq \mathrm{P}\left(\left|\int_0^T (\sigma(X_s^n)-\sigma(X_s))\mathbb{1}_{|X_s^n-X_s|\leq 1}\,\mathrm{d}W_s\right|\geq\delta\right)\\
&\leq \frac{1}{\delta^2}\,\mathrm{E}\left[\left|\int_0^T (\sigma(X_s^n)-\sigma(X_s))\mathbb{1}_{|X_s^n-X_s|\leq 1}\,\mathrm{d}W_s\right|^2\right]\\
&= \frac{1}{\delta^2}\,\mathrm{E}\left[\int_0^T (\sigma(X_s^n)-\sigma(X_s))^2\mathbb{1}_{|X_s^n-X_s|\leq 1}\,\mathrm{d}s\right]\\
&\leq \frac{L_\sigma^2}{\delta^2}\,\mathrm{E}\left[\int_0^T |X_s^n-X_s|^2\mathbb{1}_{|X_s^n-X_s|\leq 1}\,\mathrm{d}s\right]\\
&\leq \frac{L_\sigma^2}{\delta^2}\,\mathrm{E}\left[\int_0^T |X_s^n-X_s|\mathbb{1}_{|X_s^n-X_s|\leq 1}\,\mathrm{d}s\right]
\end{aligned}$$

Since (X_s^n) converges in L^1 to X_s, this latter term converges, by dominated convergence, to zero, and, since we even have a uniform convergence, $\mathrm{P}({}^cA_n)$ has zero for limit, which ends the proof.

Chapter 2

Solution viewed as a limit of PDMP with two regimes

2.1 Introduction

We consider again the following class of jump type stochastic equations of which we have already proven the existence and uniqueness in Chapter 1 (*cf.* Theorem 1.2.3) :

$$X_t = x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^t \int_{E\times\mathbb{R}_+} c(z, X_{s-}) \mathbb{1}_{\{u\leq\gamma(X_{s-},z\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

were $E = \mathbb{R}^d$, $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a homogeneous Poisson point measure on $E \times (0, \Gamma)$ with intensity measure $\mu(\mathrm{d}z) \times \mathbb{1}_{(0,\Gamma)}(u)\,\mathrm{d}u$ and the coefficients are $\sigma_l, g : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times E \to \mathbb{R}^d, \gamma : \mathbb{R}^d \times E \to \mathbb{R}$.

We will consider this type of process as a generalization of Piecewise Deterministic Markov Processes (in short PDMP) in two senses: in standard PDMP theory, between two jump times, the process follows a deterministic curve, solution of an ODE. In our case this deterministic curve is replaced by the trajectory of a diffusion process (so the process is no more piecewise deterministic, but has a diffusive component). Then the process jumps with an intensity which depends on the position of the particle. Notice however that, in the standard setting of PDMP's, the intensity of the jump measure is finite, so the jump times are discrete. In our case we consider an infinite intensity measure so that the jump times are dense in $\mathbb{R}_+$.

In this chapter, we will study a limit theorem which motivates the equations introduced above: we consider a sequence X_t^n of standard PDMP's for which the intensity of the jumps tends to infinity as $n \to \infty$, following two regimes: a slow one which leads to a jump component with finite variation. And a rapid one which, supposing that the processes at hand are centred and renormalized in a convenient way, produces the diffusion component in the limit.

In addition to the Hypothesis 1.1 (which ensure existence and uniqueness of the solution), we will add the following one (that hypothesis will be needed, in the Malliavin Calculus part anyway : it will be contained in the Hypothesis 5.2) :

Hypothesis 2.1 (Additional on γ) *We suppose in the following that γ is bounded by $\overline{C} \in \mathbb{R}_+^*$.*

Remark 2.1.1 *Throughout this chapter and unless stated otherwise we assume, without loss of generality, that $\Gamma = 1$ and $\overline{C} = \frac{1}{2}$, so the last equation can also be written*

$$\begin{aligned} X_t = x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t g(X_s)\,\mathrm{d}s \\ + \int_0^t \int_{E\times(0,1)} c(z, X_{s-}) \mathbb{1}_{\{u\leq\gamma(X_{s-},z\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{2.1.1}$$

We recall that, if μ is a finite measure then the above equation admits the following representation. Let J_t be a Poisson process of parameter $\mu(E)\times[0,1]$ and let $T_k, k\in N$ be its jump times. Consider also a sequence of independent random variables $Z_k, U_k, k\in N$ with laws

$$P(Z_k\in \mathrm{d}z)=\frac{1}{\mu(E)}\mu(\mathrm{d}z) \qquad \text{and} \qquad P(U_k\in \mathrm{d}u)=\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.$$

Then the above equation reads

$$X_t=x+\sum_{l=1}^{m}\int_0^t\sigma_l(X_s)\,\mathrm{d}W_s^l+\int_0^t g(X_s)\,\mathrm{d}s+\sum_{k=1}^{J_t}c(X_{T_k-},Z_k)\mathbb{1}_{\{U_k\leq\gamma(X_{T_k-},Z_k\}}. \tag{2.1.2}$$

So, in the case $\mu(E)<\infty$, we prove that the law of X_t coincides with the law of $\overline{X}_t$ constructed in the following way.

We consider a point $z_*\notin E$ and we denote $E_*=E\cup\{z_*\}$. On E_* we define the probability measures

$$\eta(x,\mathrm{d}z)=\theta(x)\delta_{z_*}(\mathrm{d}z)+\frac{1}{\mu(E)}\mathbb{1}_E(z)\gamma(z,x)\mu(\mathrm{d}z), \qquad \text{with}$$
$$\theta(x)=\frac{1}{\mu(E)}\int_E(1-\gamma(z,x))\mu(\mathrm{d}z).$$

(we have here, by construction, $\theta(x)\geq\frac{1}{2}$).

We put $\overline{X}_0=x$ and we define by recurrence

$$\begin{aligned}\overline{X}_{T_k}&=\overline{X}_{T_k-}+c(\overline{X}_{T_k-},\overline{Z}_k)\mathbb{1}_E(\overline{Z}_k)\\ \overline{X}_t&=\overline{X}_{T_k}+\sum_{l=1}^{m}\int_{T_k}^t\sigma_l(\overline{X}_s)\,\mathrm{d}W_s^l+\int_{T_k}^t g(\overline{X}_s)\,\mathrm{d}s, \qquad T_k\leq t<T_{k+1}.\end{aligned} \tag{2.1.3}$$

with the random variables $\overline{Z}_k$ having the conditional law

$$P\Big(\overline{Z}_k\in\mathrm{d}z\mid\overline{X}_{T_k-}=x\Big)=\eta(x,\mathrm{d}z).$$

Notice that $T_k, k\in\mathbb{N}$ are not the real jump times of $\overline{X}$: indeed, $\mathbb{1}_E(\overline{Z}_k)=0$ with probability $\theta(x)>0$ and in this case $\overline{X}_{T_k}=\overline{X}_{T_k-}$. Notice also that the law of the jump at time T_k depends on the position $\overline{X}_{T_k-}$. So, if $\sigma=0$, then $\overline{X}_t$ is standard Piecewise Deterministic Markov Process (in short $PDMP$). Then our model appears as a natural generalization of such models, which introduces a supplementary diffusive component $\sum_{l=1}^m\int_0^t\sigma_l(\overline{X}_s)\,\mathrm{d}W_s^l$. The motivation of this diffusive component is given by means of a convergence result. Consider a sequence X_t^n, $n\in\mathbb{N}$, of $PDMP's$ with two regimes:

$$\begin{aligned}X_t^n=x&+\int_0^t g(X_s^n)\,\mathrm{d}s+\int_0^t\int_{E_n}\int_{(0,1)}c(z,X_{s-}^n)\mathbb{1}_{\{u\leq\gamma(z,X_{s-})\}}N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\ &+\int_0^t\int_E\int_{(0,1)}c(z,X_{s-}^n)\mathbb{1}_{\{u\leq\gamma(z,X_{s-})\}}\widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).\end{aligned} \tag{2.1.4}$$

Here $N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is a Poisson point measure with $\widehat{N}_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)=\mathrm{d}s\mathbb{1}_{E_n}(z)\,\mathrm{d}\mu(z)\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u$ and $N_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is a Poisson point measure with $\widehat{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)=\mathrm{d}s\,\mathrm{d}\nu_n(z)\mathbb{1}_{(0,1)}(u)\,\mathrm{d}u$. We assume that $\mu(E_n)+\nu_n(E)<\infty$ so (2.1.4) is a standard $PDMP$. The regime $\mathbb{1}_{E_n}(z)\,\mathrm{d}\mu(z)$ is the slow regime and $d\nu_n$ will be the rapid regime. We denote

$$\overline{c}(z)=\sup_{x\in\mathbb{R}^d}|c(z,x)|\,, \tag{2.1.5}$$

$$\overline{\gamma}(z)=\sup_{x\in\mathbb{R}^d}|\gamma(z,x)| \tag{2.1.6}$$

and

$$a_n^{i,j}(x) = \int_E c^i(z,x)c^j(z,x)\gamma(z,x)d\nu_n(z).$$

Let σ be the coefficient in the equation (2.1.1) and $a = \sigma\sigma^*$. We denote

$$\varepsilon(n) = \|a - a_n\|_\infty + \int_{E_n^c} \overline{c\gamma}\,\mathrm{d}\mu + \int_E \bar{c}^3\overline{\gamma}d\nu_n$$

and we construct

$$\varepsilon_*(n) = \inf_{\overline{n}\geq n}\Big(\int_{E_{\overline{n}}^c} \overline{c\gamma}\,\mathrm{d}\mu + \mu(E_{\overline{n}})\varepsilon(n)\Big).$$

Then, under the hypothesis $\gamma(z,x) \geq \underline{\gamma} > 0$ we prove that (see Theorem 2.3.5) for every $f \in \mathcal{C}_b^3(\mathbb{R}^d)$

$$\sup_{x\in\mathbb{R}^d} |E(f(X_t(x)) - E(f(X_t^n(x))| \leq \frac{C_n}{\underline{\gamma}^3}\|f\|_{3,\infty}\,\varepsilon_*(n). \tag{2.1.7}$$

and give an explicit estimate for C_n (generally $\sup_n C_n < \infty$). In particular, if

$$\lim_{n\to\infty} C_n\varepsilon_*(n) = 0$$

we obtain the convergence in law of X_t^n to X_t.

Notice that the estimate (2.1.7) is not asymptotic, but holds for every fixed n. This allows to consider it from a different point of view, mainly as a result which may be used for simulation, for example. This is an idea going back to Asmussen and Rosiński [2] : they consider a jump type stochastic equation (of classical type) and propose to simulate the solution of such an equation by replacing the small jumps (the ones which produce the infinite activity) by a Brownian motion. We are doing the same here: we replace $c(z, X_{s-}^n)\mathbb{1}_{\{u\leq\gamma(z,X_{s-}^n)\}}\widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ by $\sigma(X_s)\,\mathrm{d}W_s$ (so now we read the result in a converse sense). And then (2.1.7) gives an upper bound of the error introduced by this operation. So, in a general way, we answer the question: what is the price to pay in order to replace a piece of Poisson point measure by a piece of Brownian motion?

2.2 Regularity of the truncated semi-group

In this section we consider a set $G \subset E$, with $\mu(G) < \infty$, and define $X_t^G(x)$ as the solution of the equation

$$\begin{aligned} X_t^G(x) = x &+ \sum_{l=1}^m \int_0^t \sigma_l(X_s^G)\,\mathrm{d}W_s^l + \int_0^t g(X_s^G)\,\mathrm{d}s \\ &+ \int_0^{t+}\int_{E\times\mathbb{R}_+} \mathbb{1}_G(z)c(z,X_{s^-}^G)\mathbb{1}_{\{u\leq\gamma(z,X_{s^-}^G)\}}\,\mathrm{d}N(s,z,u). \end{aligned} \tag{2.2.8}$$

Remark 2.2.1 *We already have encountered this solution at the beginning of section 1.4, under a slightly different notation (in this section we had $X_t = X_t^F$ and $Y_t = X_t^G$, for some fixed set F and G).*

Our aim is to study the regularity of $P_t^G f(x) = \mathrm{E}\left[f(X_t^G(x))\right]$, where $f \in \mathcal{C}_b^3(\mathbb{R}^d)$. Since $\mu(G) < \infty$ we have the representation

$$X_t^G(x) = x + \sum_{l=1}^m \int_0^t \sigma_l(X_s^G)\,\mathrm{d}W_s^l + \int_0^t g\big(X_s^G\big)\,\mathrm{d}s + \sum_{T_k\leq t} c\big(Z_k, X_{T_k-}^G\big)\mathbb{1}_{\{U_k\leq\gamma(Z_k,X_{T_k-}^G)\}} \tag{2.2.9}$$

where $T_k, k \in N$ are the jump times of a Poisson process J_t of parameter $\mu(G)$ and Z_k, U_k are independent random variables of law

$$P(Z_k \in \mathrm{d}z) = \frac{1}{\mu(G)}\mu(\mathrm{d}z)\mathbb{1}_G(z), \qquad P(U_k \in \mathrm{d}u) = \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u.$$

We introduce some further notations. For a function $f : E \times \mathbb{R}^d \to \mathbb{R}$ we denote

$$\begin{aligned}
\overline{f}(z) &= \sup_{x\in\mathbb{R}^d} |f(z,x)| = \|f(z,\cdot)\|_\infty\,, \qquad (2.2.10)\\
\overline{f}_{(k)}(z) &= \sup_{x\in\mathbb{R}^d} \sum_{|\alpha|=k} |\partial_x^\alpha f(z,x)| = \sum_{|\alpha|=k} \|\partial^\alpha f(z,\cdot)\|_\infty\,,\\
\overline{f}_{[k]}(z) &= \sum_{i=0}^{k} \overline{f}_{(i)}(z).
\end{aligned}$$

and, for a function g defined on $\mathbb{R}^d$, we denote

$$\|g\|_{r,\infty} = 1 \vee \sum_{k=0}^{r} \sum_{|\alpha|=k} \|\partial^\alpha g\|_\infty.$$

Theorem 2.2.2 *Let $q \in N$. Suppose that $\sigma \in \mathcal{C}_b^q(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $g \in \mathcal{C}_b^q(\mathbb{R}^d;\mathbb{R}^d)$ and $1 \geq \gamma(z,x) \geq \underline{\gamma} > 0$ for every $(z,x) \in \mathbb{R}^d \times G$. Then, for all $f \in \mathcal{C}_b^q(\mathbb{R}^d)$,*

$$\left\|P_t^G f\right\|_{q,\infty} \leq \frac{C_q t}{\underline{\gamma}^q} \|f\|_{q,\infty} \Big(1 + \int_G \overline{\gamma}(z)\mu(\mathrm{d}z)\Big) \Theta_{q,l_q}(G) e^{tC_q\theta_{q,l_q}(G)} \qquad (2.2.11)$$

where $\overline{\gamma}$ is defined in (2.1.6)*, where C_q, l_q are constants which depend on q and*

$$\begin{aligned}
\Theta_{q,p}(G) &= 1 + \|\sigma\|_{q,\infty}^{2p} + \|g\|_{q,\infty}^{2p} + \int_G \sum_{i=1}^{q} \overline{c}_{(i)}(1 + \sum_{i=1}^{q} \overline{c}_{(i)})^{2p}\overline{\gamma}(z)\mu(\mathrm{d}z), \qquad (2.2.12)\\
\theta_{q,p}(G) &= 1 + (1 \vee \|\nabla\sigma\|_\infty)^{2p} + (1 \vee \|\nabla g\|_\infty)^{2p} + \int_G \sum_{i=1}^{q} \overline{c}_{(i)}(1 + \sum_{i=1}^{q} \overline{c}_{(i)})^{2p}\overline{\gamma}(z)\mu(\mathrm{d}z)
\end{aligned}$$

For $q = 1, 2, 3$ we have $l_q = q$.

Before starting the proof we have to introduce another representation of the solution of the equation (2.2.8). We define $\overline{X}_t^G, t \geq 0$ in the following way. We denote by $\Phi_{u,v}(x)$ the solution of the diffusion equation

$$\Phi_{u,v}(x) = x + \sum_{l=1}^{m} \int_u^v \sigma_l(\Phi_{u,s}(x))\,\mathrm{d}W_s^l + \int_u^v g(\Phi_{u,s}(x))\,\mathrm{d}s, \qquad (2.2.13)$$

We consider a point $z_* \in E \setminus G$ and we define the probability density

$$\begin{aligned}
q_G(z,y) &= \theta_G(x)\delta_{z_*}(z) + \frac{1}{\mu(G)}\mathbb{1}_G(z)\gamma(z,y) \qquad \text{with} \qquad (2.2.14)\\
\theta_G(y) &= \frac{1}{\mu(G)} \int_G (1 - \gamma(z,y))\mu(\mathrm{d}z).
\end{aligned}$$

We put $\overline{X}_0^G(x) = x$ and we define by recurrence

$$\begin{aligned}
\overline{X}_{T_k}^G(x) &= \overline{X}_{T_k-}^G(x) + c(\overline{Z}_k, \overline{X}_{T_k-}^G(x),)\mathbb{1}_G(\overline{Z}_k) \qquad (2.2.15)\\
\overline{X}_t^G(x) &= \Phi_{T_k,t}(\overline{X}_{T_k}^G(x)) \qquad T_k \leq t < T_{k+1}.
\end{aligned}$$

The random variables $\overline{Z}_k$ have the conditional law

$$P(\overline{Z}_k \in \mathrm{d}z \mid \overline{X}_{T_k-}^G(x) = y) = q_G(z,y)\mu(\mathrm{d}z). \qquad (2.2.16)$$

Remark 2.2.3 *In mathematical physics the above equations are known as "transport equations" and the equation (2.2.8) is called the "fictive shock" representation and the (2.2.15) is the "real shock" representation: see [35] p. 49. This book gives a complete overview of the numerical methods used in the Monte Carlo approach to such equations as well as several possible applications.*

Lemma 2.2.4 *The law of $X_t^G(x)$ coincides with the law of $\overline{X}_t^G(x)$. Moreover, for any non negative and measurable function Ψ the law of $S_t = \sum_{k=1}^{J_t} \Psi(Z_k) \mathbb{1}_{\{U_k \leq \gamma(Z_k, X^G_{T_k -})\}}$ coincides with the law of $\overline{S}_t = \sum_{k=1}^{J_t} \Psi(\overline{Z}_k)$.*

Proof : We have (with the notation from (2.2.9))

$$\begin{aligned}
&\mathrm{E}\left[f(X^G_{T_j}) \mid X^G_{T_j -} = y\right] \\
&\qquad = \mathrm{E}\left[f(y + c(y, Z_j)\mathbb{1}_G(Z_j))\mathbb{1}_{\{U_j \leq \gamma(Z_j, y)\}}\right] + \mathrm{E}\left[f(y)\mathbb{1}_{\{U_j > \gamma(y, Z_j)\}}\right] =: I + J.
\end{aligned}$$

A simple computation shows that $P(U_j > \gamma(Z_j, y)) = \theta_G(y)$ and moreover

$$\begin{aligned}
I &= \int_E \int_0^1 f(y + c(z,y)\mathbb{1}_G(z))\mathbb{1}_{\{u \leq \gamma(z,y)\}} \frac{1}{\mu(G)} \,\mathrm{d}u \mu(\mathrm{d}z) \\
&= \int_E f(y + c(z,y)\mathbb{1}_G(z))\gamma(z,y) \frac{1}{\mu(G)} \mu(\mathrm{d}z)
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{E}\left[f(X^G_{T_j}) \mid X^G_{T_j -} = y\right] &= \int_E f(y + c(z,y)\mathbb{1}_G(z))\gamma(z,y) \frac{1}{\mu(G)} \mu(\mathrm{d}z) + \theta_G(y) f(y) \\
&= \int_E f(y + c(z,y)\mathbb{1}_G(z)) q_G(z,y) \mu(\mathrm{d}z) \\
&= \mathrm{E}\left[f(\overline{X}^G_{T_j}) \mid \overline{X}^G_{T_j -} = y\right].
\end{aligned}$$

We conclude that the law of X_t^G coincides with the law of $\overline{X}_t^G$. In order to check that the laws of S_t and of $\overline{S}_t$ are the same, we just use the previous result for the couple (X_t^G, S_t) and $(\overline{X}_t^G, \overline{S}_t)$.
•

Notice that the process $\overline{X}_t^G$ defined in (2.2.15) verifies the equation

$$\begin{aligned}
\overline{X}_t^G(x) = &x + \sum_{l=1}^m \int_0^t \sigma_l(\overline{X}_s^G(x)) \,\mathrm{d}W_s^l + \int_0^t g(\overline{X}_s^G(x)) \,\mathrm{d}s \\
&+ \sum_{k=1}^{J_t} c(\overline{Z}_k, \overline{X}^G_{T_k -}(x)) \mathbb{1}_G(\overline{Z}_k).
\end{aligned} \tag{2.2.17}$$

It is known (see Ikeda-Watanabe [30]) that one may choose a variant of the stochastic flow $\Phi_{u,v}(x)$ which is differentiable with respect to the initial condition x, and we choose this variant. Then $x \to \overline{X}_t^G(x)$ will be also differentiable. Our aim now is to give an upper bound for the moments of the derivatives.

Remark 2.2.5 *Let us be more precise : for each fixed $\overline{Z}_k = z_k$, $k \in \mathbb{N}$, we consider the function $x \to \overline{X}_t^G(x)$, solution of (2.2.17) ; $\partial_{x_i} \overline{X}_t^G(x)$ designates the derivative of this function with respect to x_i. As mentioned before, the law of $\overline{Z}_k$ depends on $\overline{X}^G_{T_k -}(x)$ and consequently depends on x also. But the derivative ∂_{x_i} does not concern the law of $\overline{Z}_k$ but only the flow $\overline{X}_t^G(x)$ for each fixed $\overline{Z}_k = z_k$. The perturbation of the law of $\overline{Z}_k$ when we move x will be treated in a separate way, in the proof of Theorem 2.2.2.*

Proposition 2.2.6 *For every $q,p \in N$ there exist some constants C (depending on q and p) and l_q (depending on q) such that, for every multi-index α with $|\alpha| = q$*

$$\mathrm{E}\left[\left|\partial_x^\alpha \overline{X}_t^G(x)\right|^p\right] \leq C\Theta_{q,l_qp}(G)e^{tC\theta_{q;l_qp}(G)} \tag{2.2.18}$$

and

$$\mathrm{E}\Big[\sum_{k=1}^{J_t} \mathbb{1}_G(\overline{Z}_k)\left|\partial_x^\alpha \overline{X}_t^G(x)\right|^p\Big] \leq Ct\int_G \overline{\gamma}\,\mathrm{d}\mu \times \Theta_{q,l_qp}(G)e^{tC\theta_{q;l_qp}(G)} \tag{2.2.19}$$

with $\Theta_{q,p}(G), \theta_{q,p}(G)$ defined in (2.2.12). For $q = 1,2,3$ we have $l_q = q$, and in general, $l_q \leq 2^q$.

Proof : We treat the first derivatives. We have (with $e_i = (0,...,0,1,0,...,0)$ with 1 on the $i'th$ position)

$$\begin{aligned}\partial_{x^i}\overline{X}_t^G(x) = e_i &+ \sum_{l=1}^m \int_0^t \Big\langle \nabla\sigma_l(\overline{X}_s^G(x)), \partial_{x^i}\overline{X}_s^G(x)\Big\rangle\,\mathrm{d}W_s^l \\ &+ \int_0^t \Big\langle \nabla g(\overline{X}_s^G(x)), \partial_{x^i}\overline{X}_s^G(x)\Big\rangle\,\mathrm{d}s \\ &+ \sum_{k=1}^{J_t} \Big\langle \nabla_x c(\overline{Z}_k, \overline{X}_{T_k-}^G(x)), \partial_{x^i}\overline{X}_{T_k-}^G(x)\Big\rangle\,\mathbb{1}_G(\overline{Z}_k).\end{aligned} \tag{2.2.20}$$

Using the identity of laws given in Lemma 2.2.4 for the system $(\overline{X}_t^G(x), \nabla_x\overline{X}_t^G(x))_{t\geq 0}$ we conclude that the law of this process coincides with the law of the process $(X_t^G(x), V_{(1),t}(x))_{t\geq 0}$ where $X_t^G(x)$ is the solution of the equation (2.2.8) and $V_{(1),t}^i \in \mathbb{R}^d, i = 1,...,d$ solves the equation

$$\begin{aligned}V_{(1),t}^i(x) = e_i &+ \sum_{l=1}^m \int_0^t \Big\langle \nabla\sigma_l(X_s^G(x)), V_{(1),s}^i(x)\Big\rangle\,\mathrm{d}W_s^l \\ &+ \int_0^t \Big\langle \nabla g(X_s^G(x)), V_{(1),s}^i(x)\Big\rangle\,\mathrm{d}s \\ &+ \sum_{k=1}^{J_t} \Big\langle \nabla_x c(Z_k, X_{T_k-}^G(x)), V_{(1),T_k-}^i(x)\Big\rangle\,\mathbb{1}_G(Z_k)\mathbb{1}_{\{U_k\leq\gamma(X_{T_k-}^G(x),Z_k)\}}.\end{aligned} \tag{2.2.21}$$

We will use Proposition A.3.1 in order to estimate the moments of $V_{(1),t}^i(x)$. In order to fix notations we mention that the index set is now $\Lambda = \{1,...,d\}$ and $\alpha = i$. Moreover $V_{(1),0}^i = e_i$ and $H^i = h^i = Q^i = 0$ so, in particular, $\overline{q} = 0$ and $R^i = 0$. It follows that (see (A.3.3) for the notation)

$$\widehat{c}_{(1),2}(p) = \int_G \overline{c}_{(1)}(z)(1+\overline{c}_{(1)}(z))^{2p}\overline{\gamma}(z)\,\mathrm{d}\mu(z).$$

Then, using the identity of law and (A.3.4) we obtain

$$\mathrm{E}\left[\left|\partial_{x^i}\overline{X}_t^G(x)\right|^{2p}\right] = \mathrm{E}\left[\left|V_{(1),t}^i(x)\right|^{2p}\right] \leq \exp(tC_p(1+\|\nabla\sigma\|_\infty^{2p}+\|\nabla g\|_\infty^{2p}+\widehat{c}_{(1),2}(p))) \leq \exp(tC_p\theta_{1,p}(G))$$

so (2.2.18) is proved. And

$$\begin{aligned}\mathrm{E}\left[\sum_{k=1}^{J_t} \mathbb{1}_G(\overline{Z}_k)\left|\partial_{x^i}\overline{X}_{T_k-}^G(x)\right|^p\right] &= \mathrm{E}\left[\sum_{k=1}^{J_t} \mathbb{1}_G(Z_k)\left|V_{(1),T_k-}^i(x)\right|^p \mathbb{1}_{\{U_k\leq\gamma(X_{T_k-}^G(x),Z_k)\}}\right] \\ &\leq \int_G \overline{\gamma}\,\mathrm{d}\mu \int_0^t \mathrm{E}\left[\left|V_{(1),s}^i(x)\right|^p\right]\mathrm{d}s\end{aligned}$$

so (2.2.19) is also proved with $l_1 = 1$).

We estimate now the second order derivatives. We take derivatives in (2.2.20) and we obtain

$$\begin{aligned}\partial_{x^j}\partial_{x^i}\overline{X}_t^G(x) =&\sum_{l=1}^m\int_0^t \overline{H}_l^{i,j}(s)\,\mathrm{d}W_s^l+\int_0^t \overline{h}^{i,j}(s)\,\mathrm{d}s \\ &+\sum_{k=1}^{J_t}\overline{Q}^{i,j}(T_k-,\overline{Z}_k)\mathbb{1}_G(\overline{Z}_k)\\ &+\sum_{l=1}^m\int_0^t\Big\langle\nabla\sigma_l(\overline{X}_s^G(x)),\partial_{x^j}\partial_{x^i}\overline{X}_s^G(x)\Big\rangle\,\mathrm{d}W_s^l+\int_0^t\Big\langle\nabla g(\overline{X}_s^G(x)),\partial_{x^j}\partial_{x^i}\overline{X}_s^G(x)\Big\rangle\,\mathrm{d}s\\ &+\sum_{k=1}^{J_t}\Big\langle\nabla_x c(\overline{Z}_k,\overline{X}_{T_k-}^G(x)),\partial_{x^j}\partial_{x^i}\overline{X}_{T_k-}^G(x)\Big\rangle\mathbb{1}_G(\overline{Z}_k).\end{aligned}\tag{2.2.22}$$

with (where $\overline{X}^G=\big(\overline{X}^{G,r}(x)\big)_{1\leq r\leq d}$)

$$\overline{H}_l^{i,j}(s)=\sum_{r,r'=1}^d\partial^2_{x^r,x^{r'}}\sigma_l(\overline{X}_s^G(x))\partial_{x^i}\overline{X}_s^{G,r}(x)\partial_{x^j}\overline{X}_s^{G,r'}(x),$$

$$\overline{h}^{i,j}(s)=\sum_{r,r'=1}^d\partial^2_{x^r,x^{r'}}g(\overline{X}_s^G(x))\partial_{x^i}\overline{X}_s^{G,r}(x)\partial_{x^j}\overline{X}_s^{G,r'}(x),$$

and

$$\overline{Q}^{i,j}(s,\overline{Z}_k)=\sum_{r,r'=1}^d\partial^2_{x^r,x^{r'}}c(\overline{Z}_k,\overline{X}_s^G(x))\partial_{x^i}\overline{X}_s^{G,r}(x)\partial_{x^j}\overline{X}_s^{G,r'}(x).$$

Using the identity of laws given in Lemma 2.2.4 for the system $(\overline{X}_t^G(x),\nabla_x\overline{X}_t^G(x),\nabla_x^2\overline{X}_t^G(x))_{t\geq0}$ we conclude that the law of this process coincides with the law of the process $(X_t^G(x),V_{(1),t}(x),V_{(2),t}(x))_{t\geq0}$ where $X_t^G(x)$ is the solution of the equation (2.2.8), and $V_{(1),t}^i\in\mathbb{R}^d, i=1,...,d$ solves the equation (2.2.21) and $V_{(2),t}^{i,j}(x)\in\mathbb{R}^d, i,j=1,...,d$ solves the following equation:

$$\begin{aligned}V_{(2),t}^{i,j}(x)=&\sum_{l=1}^m\int_0^t H_l^{i,j}(s)\,\mathrm{d}W_s^l+\int_0^t h^{i,j}(s)\,\mathrm{d}s\\ &+\sum_{k=1}^{J_t}Q^{i,j}(T_k-,Z_k)\mathbb{1}_G(Z_k)\mathbb{1}_{\{U_k\leq\gamma(X_{T_k-}^G(x),Z_k)\}}\\ &+\sum_{l=1}^m\int_0^t\Big\langle\nabla\sigma_l(\overline{X}_s^G(x)),V_{(2),s}^{i,j}\Big\rangle\,\mathrm{d}W_s^l+\int_0^t\Big\langle\nabla g(\overline{X}_s^G(x)),V_{(2),s}^{i,j}\Big\rangle\,\mathrm{d}s\\ &+\sum_{k=1}^{J_t}\Big\langle\nabla_x c(Z_k,\overline{X}_{T_k-}^G(x)),V_{(2),T_k-}^{i,j}\Big\rangle\mathbb{1}_G(Z_k)\mathbb{1}_{\{U_k\leq\gamma(X_{T_k-}^G(x),Z_k)\}}\end{aligned}\tag{2.2.23}$$

with

$$H_l^{i,j}(s)=\sum_{r,r'=1}^d\partial_r\partial_{r'}\sigma_l(X_s^G(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x),$$

$$h^{i,j}(s)=\sum_{r,r'=1}^d\partial_r\partial_{r'}g(X_s^G(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x),$$

and

$$Q^{i,j}(s,Z_k)=\sum_{r,r'=1}^d\partial_{x^r}\partial_{x^{r'}}c(Z_k,\overline{X}_s^G(x))(V_{(1),s}^i)^r(x)(V_{(1),s}^j)^{r'}(x).$$

We will again use Proposition A.3.1 in order to estimate the moments of $V_{(2),t}(x)$. Now the index set is $\Lambda = \{(i,j); i,j = 1,...,d\}$ and $\alpha = (i,j)$. Moreover $V^{i,j}_{(2),0} = 0$ and $H^{i,j}, h^{i,j}, Q^{i,j}$ are given above. In particular we have $\left|Q^{i,j}(s,Z_k)\right| \leq \overline{q}(z)R^{i,j}(s)$ with $\overline{q}(z) = \overline{c}_{(2)}(z)$ and $R^{i,j}_s = \left|V_{(1),s}\right|^2$. So

$$\widehat{c}_{(2),1}(p) = \int_G (\overline{c}_{(2)}(z) + \overline{c}_{(1)}(z))(1+\overline{c}_{(2)}(z))^{2p}\overline{\gamma}(z)\,\mathrm{d}\mu(z),$$
$$\widehat{c}_{(2),2}(p) = \int_G (\overline{c}_{(2)}(z) + \overline{c}_{(1)}(z))(1+\overline{c}_{(1)}(z))^{2p}\overline{\gamma}(z)\,\mathrm{d}\mu(z).$$

We also have

$$\begin{aligned}
&\int_0^t \mathrm{E}\Big[\sum_{l=1}^m |H^\alpha_l(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_{(2),1}(p)\left|R^\alpha_{s-}\right|^{2p}\Big]\,\mathrm{d}s \qquad (2.2.24)\\
&\quad\leq \int_0^t C_p(\|\sigma\|^{2p}_{2,\infty} + \|g\|^{2p}_{2,\infty} + \widehat{c}_{(2),1}(p))\,\mathrm{E}\left[\left|V_{(1),s}\right|^{4p}\right]\mathrm{d}s\\
&\quad\leq tC_p(\|\sigma\|^{2p}_{2,\infty} + \|g\|^{2p}_{2,\infty} + \widehat{c}_{(2),1}(p))\exp(tC_{2p}(1+\|\nabla\sigma\|^{4p}_\infty + \|\nabla g\|^{4p}_\infty + \widehat{c}_{(1),2}(2p))).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{E}\left[\left|\partial_{x^j}\partial_{x^i}\overline{X}^G_t(x)\right|^{2p}\right] =& \mathrm{E}\left[\left|V^{i,j}_{(2),t}\right|^{2p}\right]\\
\leq& tC_p(\|\sigma\|^{2p}_{2,\infty} + \|g\|^{2p}_{2,\infty} + \widehat{c}_{(2),1}(p))\\
&\times\exp(tC_{2p}(1+\|\nabla\sigma\|^{4p}_\infty + \|\nabla g\|^{4p}_\infty + \widehat{c}_{(1),2}(2p)) + \widehat{c}_{(2),2}(p)))\\
\leq& tC_p\Theta_{2,p}(G)\exp(tC_{2p}\theta_{2,2p}(G))
\end{aligned}$$

(since $\widehat{c}_{(2),2}(p)) \leq \widehat{c}_{(2),2}(2p))$).

So the proof of (2.2.18) is finished and then (2.2.19) follows as above. Notice that $l_2 = 2$ here.

We deal with the third derivatives now. Notice that

$$\begin{aligned}
\partial_{x^r}\partial_{x^j}\partial_{x^i}f(g(a)) =& \sum_{k,k',k''=1}^d \frac{\partial^3}{\partial_{y^{k''}}\partial_{y^{k'}}\partial_{y^k}}f(g(a))\frac{\partial}{\partial_{x^r}}g_{k''}(a)\frac{\partial}{\partial_{x^j}}g_{k'}(a)\frac{\partial}{\partial_{x^i}}g_k(a)\\
&+\sum_{k,k'=1}^d \frac{\partial^2}{\partial_{y^{k'}}\partial_{y^k}}f(g(a))\Big(\frac{\partial^2}{\partial_{x^r}\partial_{x^i}}g_k(a)\frac{\partial}{\partial_{x^j}}g_{k'}(a) + \frac{\partial^2}{\partial_{x^r}\partial_{x^j}}g_{k'}(a)\frac{\partial}{\partial_{x^i}}g_k(a)\\
&\qquad+\frac{\partial^2}{\partial_{x^i}\partial_{x^j}}g_k(a)\frac{\partial}{\partial_{x^r}}g_{k'}(a)\Big)\\
&+\langle\nabla f(g(a)), \partial_{x^r}\partial_{x^j}\partial_{x^i}g(a)\rangle,
\end{aligned}$$

Then, the same pattern as before gives

$$\begin{aligned}
V^{i,j,r}_{(3),t}(x) =& \sum_{l=1}^m\int_0^t H^{i,j,r}_l(s)\,\mathrm{d}W^l_s + \int_0^t h^{i,j,r}(s)\,\mathrm{d}s\\
&+\sum_{k=1}^{J_t} Q^{i,j,r}(T_k-, Z_k)\mathbb{1}_G(Z_k)\mathbb{1}_{\{U_k\leq\gamma(X^G_{T_k-}(x),Z_k)\}}\\
&+\sum_{l=1}^m\int_0^t\left\langle\nabla\sigma_l(\overline{X}^G_s(x)), V^{i,j,r}_{(3),s}\right\rangle\mathrm{d}W^l_s + \int_0^t\left\langle\nabla g(\overline{X}^G_s(x)), V^{i,j,r}_{(3),s}\right\rangle\mathrm{d}s\\
&+\sum_{k=1}^{J_t}\left\langle\nabla_x c(Z_k, \overline{X}^G_{T_k-}(x)), V^{i,j,r}_{(3),T_k-}\right\rangle\mathbb{1}_G(Z_k)\mathbb{1}_{\{U_k\leq\gamma(X^G_{T_k-}(x),Z_k)\}}
\end{aligned}$$

where $H_l^{i,j,r}$, $h^{i,j,r}$, $Q^{i,j,r}$ are defined in a similar way as $H_l^{i,j}$, $h^{i,j}$, $Q^{i,j}$ in the second order case. So we have, (with $|\alpha| = 3$),

$$\mathrm{E}\left[\left|H_l^{\alpha}(s)\right|^{2p}\right] \leq C_p(\|\sigma\|_{2,\infty}^{2p} + \|\sigma\|_{3,\infty}^{2p})\Big(\mathrm{E}\left[|V_{(1),s}|^{6p}\right] + \big(\mathrm{E}\left[|V_{(2),s}|^{4p}\right]\mathrm{E}\left[|V_{(1),s}|^{4p}\right]\big)^{\frac{1}{2}}\Big)$$
$$\mathrm{E}\left[\left|h^{\alpha}(s)\right|^{2p}\right] \leq C_p(\|g\|_{2,\infty}^{2p} + \|g\|_{3,\infty}^{2p})\Big(\mathrm{E}\left[|V_{(1),s}|^{6p}\right] + \big(\mathrm{E}\left[|V_{(2),s}|^{4p}\right]\mathrm{E}\left[|V_{(1),s}|^{4p}\right]\big)^{\frac{1}{2}}\Big)$$

and $|Q^{\alpha}(s, Z_k)| \leq \overline{q}(z)R^{\alpha}(s)$ with $\overline{q}(z) = \overline{c}_{(2)}(z) + \overline{c}_{(3)}(z)$ and $R_s^{\alpha} = \left|V_{(1),s}\right|^3 + \left|V_{(2),s}\right|\left|V_{(1),s}\right|$. So the hypothesis of Proposition A.3.1 are verified with

$$\widehat{c}_{(3),1}(p) = \int_G (\overline{c}_{(3)}(z) + \overline{c}_{(2)}(z) + \overline{c}_{(1)}(z))(1 + \overline{c}_{(2)}(z) + \overline{c}_{(3)}(z))^{2p}\overline{\gamma}(z)\,\mathrm{d}\mu(z),$$
$$\widehat{c}_{(3),2}(p) = \int_G (\overline{c}_{(3)}(z) + \overline{c}_{(2)}(z) + \overline{c}_{(1)}(z))(1 + \overline{c}_{(1)}(z))^{2p}\overline{\gamma}(z)\,\mathrm{d}\mu(z).$$

We also have[1]

$$\int_0^t \mathrm{E}\Big[\sum_{l=1}^m |H_l^{\alpha}(s)|^{2p} + |h^{\alpha}(s)|^{2p} + \widehat{c}_{(3),1}(p)\left|R_{s-}^{\alpha}\right|^{2p}\Big]\,\mathrm{d}s \tag{2.2.25}$$
$$\leq \int_0^t C_p(\|\sigma\|_{2,\infty}^{2p} + \|\sigma\|_{3,\infty}^{2p} + \|g\|_{2,\infty}^{2p} + \|g\|_{3,\infty}^{2p} + \widehat{c}_{(3),1}(p))\Big(\mathrm{E}\left[|V_{(1),s}|^{6p}\right] + \big(\mathrm{E}\left[|V_{(2),s}|^{4p}\right]\mathrm{E}\left[|V_{(1),s}|^{4p}\right]\big)^{\frac{1}{2}}\Big)\,\mathrm{d}s$$
$$\leq t\Big(1 + \sqrt{tC_p\Theta_{2,2p}(G)}\Big)\Theta_{3,p}(G)\exp(tC_p''(\theta_{1,2p}(G) + \theta_{1,3p}(G) + \theta_{2,4p}(G))).$$

It follows that (see A.3.4)

$$\begin{aligned}\mathrm{E}\left[\left|\partial_{x^j}\partial_{x^i}\partial_{x^r}\overline{X}_t^G(x)\right|^{2p}\right] = \mathrm{E}\left[\left|V_t^{i,j,r}\right|^{2p}\right] &\leq t\Big(1 + \sqrt{tC_p\Theta_{2,2p}(G)}\Big)\Theta_{2,3p}(G)\\ &\quad\times\exp(tC_p'(1 + \theta_{1,2p}(G) + \theta_{1,3p}(G) + \theta_{2,4p}(G) + \|\nabla\sigma\|_{\infty}^{2p} + \|\nabla g\|_{\infty}^{2p} + \widehat{c}_{(3),2}(p)))\\ &\leq t\Big(1 + \sqrt{tC_p\Theta_{2,2p}(G)}\Big)\Theta_{3,p}(G)\exp(tC_p\theta_{3,3p}(G)).\end{aligned}$$

So we obtain (2.2.18), with $l_3 = 3$.

For higher order derivatives the proof is the same but it is more difficult to give a precise expression of l_q. For example, when using Hölder's inequality in order to estimate $\mathrm{E}\left[\left|V_{(1),t}\right|^{4p}\left|V_{(2),t}\right|^{4p}\right]$ we are not able to keep $4p$. But it is clear that $l_q = 2^q$ will always work. •

We are now ready to give:

Proof of Theorem 2.2.2. Given a sequence $z = \{z_k\}_{k\in N}$ with $z_k \in E$, we construct $x_t(x, z_1, ..., z_{J_t})$ by $x_0(x) = x$ and

$$\begin{aligned} x_{T_k}(x, z_1, ..., z_k) &= x_{T_k-}(x, z_1, ..., z_{k-1}) + c(x_{T_k-}(x, (z_1, ..., z_{k-1})), z_k)\mathbb{1}_G(z_k) \\ x_t(x, z_1, ..., z_k) &= \Phi_{T_k,t}(x_{T_k}(x, z_1, ..., z_k)) \qquad T_k \leq t < T_{k+1}\end{aligned} \tag{2.2.26}$$

so that $\overline{X}_t^G(x) = x_t(x, \overline{Z}_1, ..., \overline{Z}_{J_t})$. Conditionally to $\mathcal{G}_t = \sigma(W_s, J_s, s \leq t)$, $x_t(x, z_1, ..., z_{J_t})$ is a deterministic function of $z_1, ..., z_{J_t}$. We choose a variant of the stochastic flow $\Phi_{t_0,t}(x)$ which is

[1] Since we already know that

$$\mathrm{E}\left[|V_{(1),s}|^{6p}\right] \leq \exp(tC_p\theta_{1,3p}(G))$$
$$\mathrm{E}\left[|V_{(2),s}|^{4p}\right]\mathrm{E}\left[|V_{(1),s}|^{4p}\right] \leq tC_p\Theta_{2,2p}(G)\exp(tC_{2p}\theta_{2,4p}(G))\exp(tC_p\theta_{1,2p}(G)),$$

so

$$\mathrm{E}\left[|V_{(1),s}|^{6p}\right] + \big(\mathrm{E}\left[|V_{(2),s}|^{4p}\right]\mathrm{E}\left[|V_{(1),s}|^{4p}\right]\big)^{\frac{1}{2}} \leq \big(1 + \sqrt{tC_p\Theta_{2,2p}(G)}\big)\exp(tC_p''(\theta_{1,2p}(G) + \theta_{1,3p}(G) + \theta_{2,4p}(G))).$$

infinitely differentiable with respect to x, so $x \to x_t(x, z_1, ..., z_k)$ is infinitely differentiable. Notice also that $\overline{X}_t^G(x) = x_t(x, \overline{Z}_1, ..., \overline{Z}_{J_t})$ so that, with $\mathrm{E}_{\mathcal{G}_t}$ the conditional expectation with respect to $\mathcal{G}_t$,

$$\begin{aligned}\mathrm{E}\left[f(\overline{X}_t^G(x))\right] &= \mathrm{E}\left[\mathrm{E}_{\mathcal{G}_t}\left[f(x_t(x, \overline{Z}_1, \ldots, \overline{Z}_{J_t}))\right]\right] \\ &= \mathrm{E}\left[\int f(x_t(x, z_1, ..., z_{J_t}))p_{J,t}(x, z_1, ..., z_{J_t})\mu(\mathrm{d}z_1), ..., \mu(\mathrm{d}z_{J_t})\right]\end{aligned}$$

where

$$p_{J,t}(x, z_1, ..., z_{J_t}) = \prod_{k=1}^{J_t} q_G(x_{T_k-}(x, z_1, ..., z_{k-1}), z_k).$$

It follows that

$$\partial_{x_i} \mathrm{E}\left[f(\overline{X}_t^G(x))\right] = A + B$$

with

$$\begin{aligned}A &= \sum_{l=1}^{d} \mathrm{E}\left[\int \partial_l f(x_t(x, z_1, ..., z_{J_t}))\partial_{x_i} x_t^l(x, z_1, ..., z_{J_t})p_{J,t}(x, z_1, ..., z_{J_t})\mu(\mathrm{d}z_1), ..., \mu(\mathrm{d}z_{J_t})\right] \\ &= \sum_{l=1}^{d} \mathrm{E}\left[\partial_l f(x_t(x, \overline{Z}_1, ..., \overline{Z}_{J_t}))\partial_{x_i} x_t^l(x, \overline{Z}_1, ..., \overline{Z}_{J_t})\right]\end{aligned}$$

and

$$\begin{aligned}B &= \mathrm{E}\left[\int f(x_t(x, z_1, ..., z_{J_t}))\partial_{x_i} p_{J,t}(x, z_1, ..., z_{J_t})\mu(\mathrm{d}z_1), ..., \mu(\mathrm{d}z_{J_t})\right] \\ &= \mathrm{E}\left[\int f(x_t(x, z_1, ..., z_{J_t}))\partial_{x_i} \ln p_{J,t}(x, z_1, ..., z_{J_t}) \times p_{J,t}(x, z_1, ..., z_{J_t})\mu(\mathrm{d}z_1), ..., \mu(\mathrm{d}z_{J_t})\right] \\ &= \mathrm{E}\left[f(x_t(x, \overline{Z}_1, ..., \overline{Z}_{J_t}))\partial_{x_i} \ln p_{J,t}(x, \overline{Z}_1, ..., \overline{Z}_{J_t})\right].\end{aligned}$$

Let us estimate A. Using (2.2.18)

$$\begin{aligned}|A| &\leq \|f\|_{1,\infty} \sum_{l=1}^{d} \mathrm{E}\left[\left|\partial_{x_i} x_t^l(x, \overline{Z}_1, ..., \overline{Z}_{J_t})\right|\right] = C \|f\|_{1,\infty} \mathrm{E}\left[\left|\nabla_x \overline{X}_t^G(x)\right|\right] \\ &\leq C \|f\|_{1,\infty} \Theta_{1,1}(G)e^{tC\Theta_{1,1}(G)}.\end{aligned}$$

Let us estimate B. We have

$$\begin{aligned}\partial_{x_i} \ln p_{J,t}(x, z_1, ..., z_{J_t}) =& \sum_{k=1}^{J_t} \psi(z_k)\partial_{x_i} \ln \theta_G(x_{T_k-}(x, z_1, ..., z_{k-1})) \\ &+ \sum_{k=1}^{J_t} \mathbb{1}_G(z_k)\partial_{x_i} \ln \gamma(x_{T_k-}(x, z_1, ..., z_{k-1}), z_k).\end{aligned}$$

Notice that $\theta_G(x) \geq \frac{1}{2}$ and $\gamma(x, a) \geq \underline{\gamma}$. We also have $|\nabla_x \theta_G(x)| \leq \|\gamma\|_{1,\infty}$ so that

$$\begin{aligned}|\partial_{x_i} \ln p_{J,t}(x, z_1, ..., z_{J_t})| \leq& 2 \sum_{k=1}^{J_t} \psi(z_k) \left|\nabla_x \theta_G(x_{T_k-}(x, z_1, ..., z_{k-1}))\right| \times \left|\nabla_x x_{T_k-}(x, z_1, ..., z_{k-1})\right| \\ &+ \frac{1}{\underline{\gamma}} \sum_{k=1}^{J_t} \mathbb{1}_G(z_k) \left|\nabla_x \gamma(x_{T_k-}(x, z_1, ..., z_{k-1}))\right| \times \left|\nabla_x x_{T_k-}(x, z_1, ..., z_{k-1})\right| \\ \leq& \left(2 + \frac{1}{\underline{\gamma}}\right) \|\gamma\|_{1,\infty} \times \sum_{k=1}^{J_t} \mathbb{1}_G(z_k) \left|\nabla_x x_{T_k-}(x, z_1, ..., z_{k-1})\right|.\end{aligned}$$

Using (2.2.19)

$$\mathrm{E}\Big[\,\big|\partial_{x_i}\ln p_{J,t}(x,\overline{Z}_1,...,\overline{Z}_{J_t})\big|\,\Big] \leq C\Big(2+\frac{1}{\underline{\gamma}}\Big)\left\|\gamma\right\|_{1,\infty}\mathrm{E}\Big[\sum_{k=1}^{J_t}\mathbb{1}_G(\overline{Z}_k)\nabla_x\overline{X}^G_{T_k-}(x)\Big]$$
$$\leq Ct\Big(2+\frac{1}{\underline{\gamma}}\Big)\left\|\gamma\right\|_{1,\infty}\Theta_{1,1}(G)e^{Ct\Theta_{1,1}(G)}\int_G\overline{\gamma}\,\mathrm{d}\mu.$$

and this gives

$$|B|\leq Ct\left\|f\right\|_\infty\Big(2+\frac{1}{\underline{\gamma}}\Big)\left\|\gamma\right\|_{1,\infty}\Theta_{1,1}(G)e^{Ct\Theta_{1,1}(G)}\int_G\overline{\gamma}\,\mathrm{d}\mu.$$

For higher derivatives the proof is similar so we skip it.

2.3 The convergence result

Our concern in this section is to give a sequence of $PDMP$'s with two regimes which converge in law to the solution of our equation. The main result is Theorem 2.3.5 bellow. We consider a measurable space E and for each $n>0$ we take an increasing family of sets $E_n\uparrow E$. Moreover we consider a σ-finite non-negative measure μ and some finite non-negative measures ν_n on E and we define the measure η_n on E by

$$\eta_n(\mathrm{d}z)=\nu_n(\mathrm{d}z)+\mathbb{1}_{E_n}(z)\mu(\mathrm{d}z).$$

We denote $\lambda_n=\eta_n(E)$.

Moreover we consider a measurable function $\gamma_n:\mathbb{R}^d\times E\rightarrow[0,\frac{1}{2}]$ and a point $z_*\notin E$ and we denote $E_*=E\cup\{z_*\}$. On E_* we define the probability measures

$$\eta_n^*(\mathrm{d}z,y)=\theta_n(y)\delta_{z_*}(\mathrm{d}z)+\frac{1}{\lambda_n}\mathbb{1}_E(z)\gamma_n(z,y)\eta_n(\mathrm{d}z)\qquad\text{with}$$
$$\theta_n(y)=\frac{1}{\lambda_n}\int_E(1-\gamma_n(z,y))\eta_n(\mathrm{d}z).$$

The PDMP that we have in mind is the following. Let J_t^n be a Poisson process of parameter λ_n and let $T_j,j\in N$ be its jump times (they depend on n but we ignore it for simplicity). The jump coefficient will be $c_n:\mathbb{R}^d\times E_*\rightarrow\mathbb{R}^d$ which verify $c_n(z_*,y)=0$. And the drift component is $\phi_n:\mathbb{R}^d\rightarrow\mathbb{R}^d$. Then we put $\overline{X}_0^n=x$ and we define by recurrence

$$\overline{X}^n_{T_k}=\overline{X}^n_{T_k-}+c_n(\overline{Z}_k,\overline{X}^n_{T_k-})\tag{2.3.27}$$
$$\overline{X}^n_t=\overline{X}^n_{T_k}+\int_{T_k}^t\phi_n(\overline{X}^n_s)\,\mathrm{d}s\qquad T_k\leq t<T_{k+1}.$$

The random variables $\overline{Z}_k$ have the conditional law

$$P\Big(\overline{Z}_k\in\mathrm{d}z\mid\overline{X}^n_{T_k-}=y\Big)=\eta_n^*(y,\mathrm{d}z).\tag{2.3.28}$$

Notice that $T_k,k\in N$ are not the real jump times of $\overline{X}^n$: indeed, since $c_n(z_*,y)=0$, one may have $c_n(\overline{Z}_k,\overline{X}^n_{T_k-},)=0$ with probability $\theta_n(y)>0$ and in this case $\overline{X}^n_{T_k}=\overline{X}^n_{T_k-}$.

Example 2.3.1 ***The generic example*** *we have in mind is the following. We take* $E=(0,1)$ *and* $E_n=(n^{-1},1)$ *and* $z_*=0$. *Then we take* $\nu_n(\mathrm{d}z)=n\mathbb{1}_{(n^{-2},4n^{-2})}z^{-5/2}\,\mathrm{d}z$ *and* $\mu(\mathrm{d}z)=z^{-3/2}\,\mathrm{d}z$. *So*

$$\eta_n(\mathrm{d}z)=(n\mathbb{1}_{(n^{-2},4n^{-2})}z^{-5/2}+\mathbb{1}_{(n^{-1},1)}(z)z^{-3/2})\,\mathrm{d}z.$$

And we take (here $d=1$*)*

$$c_n(z,y)=\sigma(y)z,\qquad\gamma_n(z,y)=\gamma(y).$$

We will now give an alternative representation of $\overline{X}_t^n$ as solution of a jump type stochastic equation. We consider a Poisson point measure N_n on $E \times (0,1)$ with compensator

$$\widehat{N}_n(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s \times \eta_n(\mathrm{d}z) \times \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u \tag{2.3.29}$$

and we denote by X_t^n the solution of the SDE

$$\begin{aligned} X_t^n &= x + \int_0^t \int_{E\times\mathbb{R}_+} c_n(z, X_{s-}^n)\mathbb{1}_{\{u\leq\gamma_n(z,X_{s-}^n)\}} N_n(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \phi_n(X_s^n)\,\mathrm{d}s \\ &= x + \sum_{j=1}^{J_t^n} c_n(Z_j, X_{T_j-}^n)\mathbb{1}_{\{U_j\leq\gamma_n(Z_j,X_{T_j-}^n)\}} + \int_0^t \phi_n(X_s^n)\,\mathrm{d}s. \end{aligned} \tag{2.3.30}$$

Here Z_j and $U_j, j \in N$ are independent random variables with $P(Z_j \in \mathrm{d}z) = \lambda_n^{-1}\eta_n(\mathrm{d}z)$ and $P(U_j \in \mathrm{d}u) = \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u$.

The same argument as in the proof of Proposition 2.2.2 (and more precisely the Lemma 2.2.4) proves that $(\overline{X}_t^n)_{t\geq 0}$ has the same law as $(X_t^n)_{t\geq 0}$.

Our aim is to give sufficient conditions in order to obtain the convergence in law of $\overline{X}_t^n$ (or alternatively of X_t^n) to X_t solution of

$$\begin{aligned} X_t =& x + \sum_{l=1}^m \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t g(X_s)\,\mathrm{d}s \\ &+ \int_0^t \int_E \int_{(0,1)} c(z, X_{s-})\mathbb{1}_{\{u\leq\gamma(z,X_{s-})\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \end{aligned} \tag{2.3.31}$$

where W is an m-dimensional Brownian motion and $\sigma_l, g : \mathbb{R}^d \to \mathbb{R}^d$ and $c : \mathbb{R}^d \times E \to \mathbb{R}^d$, $\gamma : \mathbb{R}^d \times E \to \mathbb{R}$ are some coefficients, and $\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s \times \mu(\mathrm{d}z) \times \mathbb{1}_{(0,1)}(u)\,\mathrm{d}u$. Notice that μ is the measure considered in the construction of η_n, but ν_n disappears. The noise corresponding to ν_n represents the "fast regime" (see the example) and the corresponding contribution will converge to the stochastic integral with respect to the Brownian motion. The relation between the coefficients c_n, ϕ_n, γ_n of the approximation equation and σ, g, c, γ will be specified in the hypothesis that we present later on.

We denote

$$g_n(x) = \phi_n(x) + \int_E c_n(z, x)\gamma_n(z, x)\nu_n(\mathrm{d}z).$$

and (with the notation from (2.2.10))

$$\begin{aligned} C_n =& 1 + \|g_n\|_{1,\infty} + \int_{E_n} \overline{c}_{n,[1]}\overline{\gamma}_n + \overline{c}_n(\overline{c}_n + 1)\overline{\gamma}_{n,(1)}\,\mathrm{d}\mu \\ &+ \int_E (\overline{\gamma}_{n,[1]}\overline{c}_n^2 + \overline{\gamma}_n\overline{c}_n\overline{c}_{n,(1)})\,\mathrm{d}\nu_n. \end{aligned} \tag{2.3.32}$$

We also denote

$$L_n f(x) = \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a_n^{i,j}(x) + \sum_{i=1}^d \partial_i f(x) g_n^i(x) + \int_{E_n} (f(x + c_n(z, x)) - f(x))\gamma_n(z, x)\,\mathrm{d}\mu(z) \tag{2.3.33}$$

with

$$a_n^{i,j} = \int_E c_n^i(z, x)c_n^j(z, x)\gamma_n(z, x)\,\mathrm{d}\nu_n(z) \tag{2.3.34}$$

Lemma 2.3.2 *i) There exists a universal constant C such that for every $t \in [0,1)$*

$$\mathrm{E}\left[|X_t^n(x) - x|^2\right] \leq CtC_n^2 \tag{2.3.35}$$

ii) For every $f \in \mathcal{C}_b^3(\mathbb{R}^d)$

$$\|P_t^n f(x) - f(x) - tL_n f\| \le CC_n^2 \|f\|_{3,\infty} (t^{1/2} + \varepsilon_0(n))t \tag{2.3.36}$$

with

$$\varepsilon_0(n) = \int_E \overline{c}_n^3(z)\overline{\gamma}_n \,\mathrm{d}\nu_n. \tag{2.3.37}$$

Proof : We denote $h_n(z,x,u) = c_n(z,x)\mathbb{1}_{\{u\le\gamma_n(z,x)\}}$ and we consider two independent Poisson point measures N_n^ν and N_n^μ with intensity measures $\widehat{N}_n^\nu(\mathrm{d}t,\mathrm{d}z,\mathrm{d}u) = \mathrm{d}t\nu_n(\mathrm{d}z)\,\mathrm{d}u$ respectively $\widehat{N}_n^\mu(\mathrm{d}t,\mathrm{d}z,\mathrm{d}u) = \mathrm{d}t\mathbb{1}_{E_n}(z)\mu(\mathrm{d}z)\,\mathrm{d}u$. Then we write the equation (2.3.30) as (with $\widetilde{N}_n^\nu \stackrel{\text{def}}{=} N_n^\nu - \widehat{N}_n^\nu$)

$$\begin{aligned} X_t^n = x &+ \int_0^t \int_{E\times\mathbb{R}_+} h_n(z, X_{s-}^n, u)\widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ &+ \int_0^t \int_{E_n\times\mathbb{R}_+} h_n(z, X_{s-}^n, u)N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ &+ \int_0^t g_n(X_s^n)\,\mathrm{d}s. \end{aligned} \tag{2.3.38}$$

Let us prove (2.3.35). We have

$$\begin{aligned} \mathrm{E}\left[|X_t^n - x|^2\right] \le & Ct\int_{E_n} \overline{c}_n^2\overline{\gamma}_n\,\mathrm{d}\mu + Ct^2\Big(\int_{E_n} \overline{c}_n\overline{\gamma}_n\,\mathrm{d}\mu + \|g_n\|_\infty\Big)^2 \\ & + Ct\int_E \overline{c}_n^2\overline{\gamma}_n\,\mathrm{d}\nu_n. \end{aligned}$$

Let us prove (2.3.36). Using Itô's formula, for a function $f \in C^2(\mathbb{R})$, we have

$$f(X_t^n) = f(x) + M_t^n(f) + I_t^n(f) + J_t^n(f) + D_t^n(f)$$

with

$$\begin{aligned} M_t^n(f) &= \int_0^t \int_{E\times\mathbb{R}_+} \big\langle \nabla f(X_{s-}^n), h_n(z, X_{s-}^n, u)\big\rangle\, \widetilde{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \\ I_t^n(f) &= \int_0^t \int_{E\times\mathbb{R}_+} f(X_{s-}^n + h_n(z, X_{s-}^n, u)) - f(X_{s-}^n) - \big\langle \nabla f(X_{s-}^n), h_n(z, X_{s-}^n, u)\big\rangle\, \widehat{N}_n^\nu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ H_t^n(f) &= \int_0^t \int_{E_n\times\mathbb{R}_+} f(X_{s-}^n + h_n(z, X_{s-}^n, u)) - f(X_{s-}^n)N_n^\mu(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ D_t^n(f) &= \int_0^t \langle \nabla f(X_s^n), g_n(X_s^n)\rangle\,\mathrm{d}s. \end{aligned}$$

Since $M_t^n(f)$ is a martingale we obtain

$$P_t^n f(x) - f(x) = \mathrm{E}\left[I_t^n(f)\right] + \mathrm{E}\left[H_t^n(f)\right] + \mathrm{E}\left[D_t^n(f)\right].$$

We estimate now each of these terms.

Let us estimate $\mathrm{E}\left[I_t^n(f)\right]$. We denote

$$\begin{aligned} l(z,x,u) &= \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x)h_n^i(z,x,u)h_n^j(z,x,u) \\ g(z,x,u) &= f(x + h_n(z,x,u)) - f(x) - \langle \nabla f(x), h_n(z,x,u)\rangle - l(z,x,u). \end{aligned}$$

Notice that, since $a_n^{i,j} = \int_E c_n^i(z,x)c_n^j(z,x)\gamma_n(z,x)\,\mathrm{d}\nu_n(z)$,

$$\int_{E\times(0,1)} l(z,x,u)\nu_n(\mathrm{d}z)\,\mathrm{d}u = \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a_n^{i,j}(x)$$

so that

$$\mathrm{E}\left[I_t^n(f)\right] = \frac{t}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a_n^{i,j}(x) + r_1(t,x) + r_2(t,x)$$

with

$$\begin{aligned} r_1(t,x) &= \int_0^t \int_{E\times(0,1)} \mathrm{E}\left[g(z,X_{s-}^n,u))\right]\mathrm{d}u\,\mathrm{d}\nu_n(\mathrm{d}z)\,\mathrm{d}s, \\ r_2(t,x) &= \frac{1}{2}\int_0^t \int_{E\times(0,1)} \mathrm{E}\left[l(z,X_{s-}^n,u) - l(z,x,u)\right]\mathrm{d}u\,\mathrm{d}\nu_n(z)\,\mathrm{d}s. \end{aligned}$$

We have

$$|g(z,x,u)| \le C\,\|f\|_{3,\infty}\,|h_n(z,x,u)|^3 = C\,\|f\|_{3,\infty}\,|c_n(z,x)|^3\,\mathbb{1}_{\{u\le\gamma_n(z,x)\}}$$

so that

$$\begin{aligned} |r_1(t,x)| &\le C\,\|f\|_{3,\infty}\int_0^t\int_E\int_0^1 \mathrm{E}\left[\left|c_n(z,X_{s-}^n)\right|^3\mathbb{1}_{\{u\le\gamma_n(z,X_{s-}^n)\}}\right]\mathrm{d}u\,\mathrm{d}\nu_n(z)\,\mathrm{d}s \\ &= C\,\|f\|_{3,\infty}\int_0^t\int_E \mathrm{E}\left[\left|c_n(z,X_{s-}^n)\right|^3\gamma_n(z,X_{s-}^n)\right]\mathrm{d}\nu_n(z)\,\mathrm{d}s \\ &\le C\,\|f\|_{3,\infty}\,t\int_E \overline{c}_n^3\overline{\gamma}_n\,\mathrm{d}\nu_n. \end{aligned}$$

We estimate now r_2. We write

$$\int_0^1 l(z,x,u)\,\mathrm{d}u = \bar{l}(z,x),$$

with

$$\bar{l}(z,x) = \frac{1}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) c_n^i(z,x)c_n^j(z,x)\gamma_n(z,x).$$

Then

$$r_2(z,x) = \frac{1}{2}\int_0^t\int_E \mathrm{E}\left[\bar{l}(z,X_{s-}^n) - \bar{l}(z,x)\right]\mathrm{d}\nu_n(z)\,\mathrm{d}s.$$

One has

$$\left|\nabla_x\bar{l}(z,x)\right| \le C\,\|f\|_{3,\infty}\,((\overline{\gamma}_n(z) + \overline{\gamma}_{n,(1)}(z))\overline{c}_n^2(z) + \overline{\gamma}_n(z)\overline{c}_n(z)\overline{c}_{n,(1)}(z))$$

so

$$\begin{aligned} |r_2(t,x)| &\le C\,\|f\|_{3,\infty}\int_E \overline{\gamma}_{n,[1]}\overline{c}_n^2 + \overline{\gamma}_n\overline{c}_n\overline{c}_{n,(1)}\,\mathrm{d}\nu_n \times \int_0^t \mathrm{E}\left[\left|X_{s-}^n(x) - x\right|\right]\mathrm{d}s \\ &\le Ct^{3/2}\,\|f\|_{3,\infty}\,C_n^2 \end{aligned}$$

the last inequality being a consequence of (2.3.35). We conclude that

$$\left|\mathrm{E}\left[I_t^n(f)\right] - \frac{t}{2}\sum_{i,j=1}^d \partial_i\partial_j f(x) a_n^{i,j}(x)\right| \le CC_n^2 t\,\|f\|_{3,\infty}\,(t^{1/2} + \int_E \overline{c}_n^3\overline{\gamma}_n\,\mathrm{d}\nu_n).$$

We estimate now $H^n_t(f)$. We have

$$\mathrm{E}\left[H^n_t(f)\right] = \int_0^t \int_{E_n} \mathrm{E}\left[g(z, X^n_{s-})\right] \mu(\mathrm{d}z)\,\mathrm{d}s$$

with

$$g(z,x) = (f(x + c_n(z,x)) - f(x))\gamma_n(z,x).$$

Since

$$|\nabla_x g(z,x)| \leq C \left\|f\right\|_{2,\infty} (\overline{c}_{n,[1]}\overline{\gamma}_n + \overline{c}_n\overline{\gamma}_{n,(1)})(z)$$

it follows that

$$\begin{aligned}
\Big| \mathrm{E}\left[H^n_t(f)\right] - &\int_0^t \int_{E_n} g(z,x)\mu(\mathrm{d}z)\,\mathrm{d}s \Big| \\
&\leq \int_0^t \int_{E_n} \left| \mathrm{E}\left[g(z, X^n_{s-})\right] - g(z,x) \right| \mu(\mathrm{d}z)\,\mathrm{d}s \\
&\leq C \left\|f\right\|_{2,\infty} \int_{E_n} \overline{c}_{n,[1]}\overline{\gamma}_n + \overline{c}_n\overline{\gamma}_{n,(1)}\,\mathrm{d}\mu \times \int_0^t \mathrm{E}\left[\,\left|X^n_{s-}(x)) - x\right|\,\right] \mathrm{d}s \\
&\leq C t^{3/2} \left\|f\right\|_{2,\infty} C_n^2.
\end{aligned}$$

Finally

$$\left| \mathrm{E}\left[D^n_t(f)\right] - \int_0^t \langle \nabla f(x), g_n(x) \rangle \,\mathrm{d}s \right| \leq \left\|g\right\|_{1,\infty} \left\|f\right\|_{2,\infty} \int_0^t \mathrm{E}\left[\,\left|X^n_{s-}(x)) - x\right|\,\right] \mathrm{d}s \leq C C_n^2 t^{3/2}.$$

•

We consider now the Poisson point measure N with compensator $\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$ and the stochastic equation

$$\begin{aligned}
X_t =& x + \sum_{l=1}^m \int_0^t \sigma_l(X_s)\,\mathrm{d}W^l_s + \int_0^t g(X_s)\,\mathrm{d}s \\
&+ \int_0^t \int_G \int_{(0,1)} c(z, X_{s-}) \mathbb{1}_{\{u \leq \gamma(z, X_{s-})\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).
\end{aligned} \tag{2.3.39}$$

We denote $P_t f(x) = \mathrm{E}\left[f(X_t(x))\right]$ the semi-group associated to X_t. This is the limit equation in our framework. We will also use the truncated version of the equation (2.3.39) as defined in (2.2.8) for $G = E_n$. We denote $P^{E_n}_t f(x) = \mathrm{E}\left[f(X^{E_n}_t(x))\right]$ and, with $a = \sigma\sigma^*$,

$$L^{E_n} f(x) = \frac{1}{2} \sum_{i,j=1}^d \partial_i\partial_j f(x) a^{i,j}(x) + \sum_{i=1}^d \partial_i f(x) g^i(x) + \int_{E_n} (f(x + c(z,x)) - f(x))\gamma(z,x) d\mu(z)$$

and

$$C_* \stackrel{\text{def}}{=} 1 + \left\|a\right\|_{1,\infty} + \left\|g\right\|_{1,\infty} + \int_E \overline{c}_{[1]}\overline{\gamma}_{[1]} d\mu.$$

Lemma 2.3.3 ***A.*** *For every* $f \in \mathcal{C}^3_b(R^d)$

$$\left\| P^{E_n}_t f - f - t L^{E_n} f \right\|_\infty \leq C C_*^2 t^{3/2} \left\|f\right\|_{3,\infty} \tag{2.3.40}$$

B*. We also have*

$$\left\| P_t f - P^{E_n}_t f \right\|_\infty \leq C t \int_{E_n^c} \overline{c\gamma} d\mu. \tag{2.3.41}$$

Proof : Assertion **A** is analogue to (2.3.36) so we skip its proof. The estimate (2.3.41) is an immediate consequence of Lemma 1.4.1. •

In order to give our main result we have to introduce some additional notations. We assume that $\gamma(z,x) \geq \underline{\gamma} > 0$ and, for $G \subset E$, we denote

$$\underline{\gamma}(G) \overset{\text{def}}{=} \inf\{\gamma(z,x) : x \in \mathbb{R}^d, z \in G\}.$$

Moreover we recall the definition of $\Theta_{q,p}(G)$ and $\theta_{q,p}(G)$ defined in (2.2.12)

$$\Theta_{q,p}(G) = 1 + \|\sigma\|_{q,\infty}^{2p} + \|g\|_{q,\infty}^{2p} + \int_G \sum_{i=1}^q \overline{c}_{(i)}(1 + \sum_{i=1}^q \overline{c}_{(i)})^{2p} \overline{\gamma}\, \mathrm{d}\mu, \tag{2.3.42}$$

$$\theta_{q,p}(G) = 1 + (1 \vee \|\nabla\sigma\|_\infty)^{2p} + (1 \vee \|\nabla g\|_\infty)^{2p} + \int_G \sum_{i=1}^q \overline{c}_{(i)}(1 + \sum_{i=1}^q \overline{c}_{(i)})^{2p} \overline{\gamma}\, \mathrm{d}\mu$$

Finally we denote

$$\alpha_3(G) = \frac{Ct}{\underline{\gamma}^3(G)} \Theta_{3,3}(G) e^{C\theta_{3,3}(G)} (1 + \int_G \overline{\gamma} d\mu). \tag{2.3.43}$$

Then by (2.2.11),

$$\left\|P_t^G f\right\|_{3,\infty} \leq Ct\alpha_3(G) \|f\|_{3,\infty}. \tag{2.3.44}$$

Moreover we denote

$$\begin{aligned}
\varepsilon_0(n) &= \int_E \overline{c}_n^3 \overline{\gamma}_n d\nu_n, \qquad (2.3.45)\\
\varepsilon_1(n) &= \|a - a_n\|_\infty + \|g - g_n\|_\infty \\
\varepsilon_2(n) &= \int_{E_n} \overline{c}(z) \sup_x |\gamma(z,x) - \gamma_n(z,x)| + \overline{\gamma}(z) \sup_x |c(z,x) - c_n(z,x)|\, \mu(\mathrm{d}z) \\
\varepsilon_3(n) &= \int_{E_n^c} \overline{c\gamma} d\mu
\end{aligned}$$

and, for every $n \in N$ we define

$$\varepsilon_*(n) = \inf_{R \geq n} \Big(\int_{E_R^c} \overline{c\gamma} d\mu + \alpha_3(E_R)(C_*^2 + C_n^2) \sum_{i=0}^3 \varepsilon_i(n) \Big). \tag{2.3.46}$$

Remark 2.3.4 *The definition of $\varepsilon_*(n)$ is based on the idea of an equilibrium (when $R \to +\infty$) between $\int_{E_R^c} \overline{c\gamma} d\mu \downarrow 0$ and $\alpha_3(E_R) \uparrow \infty$. Indeed, if $\int_E \overline{\gamma} d\mu = \infty$ (and this is the really interesting example) then $\alpha_3(E_R)$ blows up, so we have to find an efficient equilibrium with $\int_{E_R^c} \overline{c\gamma} d\mu \downarrow 0$. See the example, section 2.4.*

We are now able to give our main result:

Theorem 2.3.5 *There exists a universal constant C such that for every $n \in N$ and every $f \in \mathcal{C}_b^3(\mathbb{R}^d)$*

$$\|P_t f - P_t^n f\|_\infty \leq Ct \|f\|_{3,\infty} \varepsilon_*(n) \tag{2.3.47}$$

In particular, if $\lim_{n\to\infty} \varepsilon_*(n) = 0$, then, for every $x \in \mathbb{R}^d$ and $t > 0$, $X_t^n(x)$ converges in law to $X_t(x)$.

Proof : Step 1. It is easy to check that

$$\left\| L^{E_n} f - L_n f \right\|_\infty \leq C \left\| f \right\|_{2,\infty} \sum_{i=1}^{2} \varepsilon_i(n).$$

This, together with the previous two lemmas gives (for every $R \geq n$ and every $\delta > 0$)

$$\left| P_\delta^{E_R} f(x) - P_\delta^n f(x) \right| \leq C(C_*^2 + C_n^2)\delta(\delta^{1/2} + \sum_{i=0}^{3} \varepsilon_i(n)) \left\| f \right\|_{3,\infty} \tag{2.3.48}$$

Step 2. Using (2.3.41), for every $R \geq n$

$$\left\| P_t f - P_t^n f \right\|_\infty \leq \left\| P_t^{E_R} f - P_t^n f \right\|_\infty + Ct \left\| f \right\|_{1,\infty} \int_{E_R^c} \overline{c\gamma} d\mu.$$

Step 3. Let $\delta > 0, t_k = k\delta$ and $\Delta_\delta f(x) = P_\delta^n f(x) - P_\delta^{E_R} f(x)$. We write

$$\left\| P_t^n f(x) - P_t^{E_R} f \right\|_\infty \leq \sum_{k \leq t/\delta} \left\| P_{t-t_{k+1}}^n \Delta_\delta P_{t_k}^{E_R} f \right\|_\infty \leq \sum_{k \leq t/\delta} \left\| \Delta_\delta P_{t_k}^{E_R} f \right\|_\infty .$$

By (2.3.48) first and by (2.3.44) then

$$\begin{aligned} \left\| \Delta_\delta P_{t_k}^{E_R} f \right\|_\infty &\leq \left\| P_{t_k}^{E_R} f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \sum_{i=0}^{3} \varepsilon_i(n))\delta \\ &\leq Ct_k \alpha_3(E_R) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \sum_{i=0}^{3} \varepsilon_i(n))\delta. \end{aligned}$$

Summing over $k = 1, ..., t/\delta$ we obtain

$$\begin{aligned} \left\| P_t^{E_R} f - P_t^n f \right\|_\infty &\leq Ct\alpha_3(E_R) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2)(\delta^{1/2} + \sum_{i=0}^{3} \varepsilon_i(n)) \\ &= Ct\alpha_3(E_R) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2) \sum_{i=0}^{3} \varepsilon_i(n) \end{aligned}$$

the last inequality being obtained by taking $\delta^{1/2} = \sum_{i=0}^{3} \varepsilon_i(n)$. We conclude that for every $R \geq n$

$$\left\| P_t f - P_t^n f \right\|_\infty \leq Ct \Big(\int_{E_R^c} \overline{c\gamma} d\mu \left\| f \right\|_{1,\infty} + \alpha_3(E_R) \left\| f \right\|_{3,\infty} (C_*^2 + C_n^2) \sum_{i=0}^{3} \varepsilon_i(n) \Big)$$

and taking the infimum over $R \geq n$ we obtain (2.3.47). Using now this last equation with the function $1 - f_R$, where $f_R : \mathbb{R}^d \to \mathbb{R}$ is smooth and such that $\mathbb{1}_{[-R,R]}(|y|) \leq f_R(y) \leq \mathbb{1}_{[-(R+1),R+1]}(|y|)$, for all $y \in \mathbb{R}^d$, we obtain that the sequence $\left(X_t^n(x) \right)_n$ is tight and so the convergence in law follows. •

2.4 Example

We take $d = 1$ and the coefficients $c_n(x, a) = c(x, a) = \sigma(x)a$, $g_n(x) = g(x)$ and $\gamma_n(x, a) = \gamma(x, a) = \gamma(x)$ with

$$\left\| \sigma \right\|_{3,\infty} + \left\| g \right\|_{3,\infty} + \left\| \gamma \right\|_{3,\infty} = c_* < \infty, \qquad \textit{and} \qquad \gamma(x) \geq \underline{\gamma} > 0. \tag{2.4.49}$$

We also consider the measures

$$\nu_n(\mathrm{d}z) = \frac{n}{z^{5/2}} \mathbb{1}_{(\frac{1}{n^2}, \frac{4}{n^2})}(z)\,\mathrm{d}z, \qquad \mu(\mathrm{d}z) = \frac{1}{z^{3/2}} \mathbb{1}_{(0,1)}(z)\,\mathrm{d}z \quad \textit{and}$$
$$\eta_n(\mathrm{d}z) = \nu_n(\mathrm{d}z) + \mathbb{1}_{(\frac{1}{n},1)}(z)\mu(\mathrm{d}z).$$

So, with the notations from the previous sections, we have $E = (0,1)$ and $E_n = (\frac{1}{n}, 1)$. We also recall that given a measure $\kappa(\mathrm{d}z)$ on E we have denoted by $N^{\kappa}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ the Poisson point measure of compensator $\mathrm{d}s\kappa(\mathrm{d}z)\,\mathrm{d}u$. We will work with the following equations. The limit equation is

$$X_t = x + \frac{1}{\sqrt{2}} \int_0^t \sigma(X_s)\sqrt{\gamma(X_s)}\,\mathrm{d}W_s + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^{t+} \int_{E\times(0,1)} \sigma(X_{s-}) z \mathbb{1}_{\{u\leq\gamma(X_{s-})\}} N^{\mu}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \tag{2.4.50}$$

and the approximation equations are

$$\begin{aligned} X_t^n =& x + \int_0^{t+} \int_{E_n\times(0,1)} \sigma(X_{s-}^n) z \mathbb{1}_{\{u\leq\gamma(X_{s-}^n)\}} N^{\mu}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^{t+} \int_{E\times(0,1)} \sigma(X_{s-}^n) z \mathbb{1}_{\{u\leq\gamma(X_{s-}^n)\}} \widetilde{N}^{\nu_n}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g(X_s)\,\mathrm{d}s \\ =& x + \int_0^{t+} \int_{E\times(0,1)} \sigma(X_{s-}^n) z \mathbb{1}_{\{u\leq\gamma(X_{s-}^n)\}} N^{\eta_n}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \phi_n(X_s)\,\mathrm{d}s \end{aligned} \tag{2.4.51}$$

with

$$\begin{aligned} \phi_n(x) &= g(x) - \int_E \sigma(x) z \gamma(x) \nu_n(\mathrm{d}z) \\ &= g(x) - \sigma(x)\gamma(x) n \int_{1/n^2}^{4/n^2} \frac{\mathrm{d}z}{z^{3/2}} = g(x) - n^2\sigma^2(x)\gamma(x). \end{aligned}$$

Since $n^2\sigma(x)\gamma(x)$ generally blows up (*ie.* except in the case $x \in \{\sigma = 0\} \cup \{\gamma = 0\}$), it is clear that we are obliged to put it in the drift coefficient of the equation (2.4.51). So, with the notation from the previous sections, we have $g_n = g$ and also

$$a_n(x) = \int_E c_n^2(z,x)\gamma_n(z,x)\nu_n(\mathrm{d}z) = \sigma^2(x)\gamma(x) n \int_{1/n^2}^{4/n^2} \frac{\mathrm{d}z}{\sqrt{z}} = \frac{1}{2}\sigma^2(x)\gamma(x).$$

Proposition 2.4.1 *Suppose that (2.4.49) holds true. Then*

$$\|P_t f - P_t^n f\|_\infty \leq \frac{Ctc_*^8 e^{Cc_*^7}}{\underline{\gamma}^3} \|f\|_{3,\infty} \frac{1}{n^{1/4}} \tag{2.4.52}$$

Proof : First, one easily checks that $\Theta_{3,3}(E_R) + \theta_{3,3}(E_R) \leq Cc_*^7$ (recall that $\Theta_{3,3}(E_R)$ and $\theta_{3,3}(E_R)$ are defined in (2.3.42) and the integrals involved in these terms are done with respect to μ and not with ν_n). We compute now

$$\int_{E_R} \overline{\gamma}(z)\mu(\mathrm{d}z) = \gamma(x)\mu(E_R) = \gamma(x)\sqrt{R} \leq c_*\sqrt{R}.$$

It follows that

$$\alpha(E_R) \leq \frac{C}{\underline{\gamma}^3} c_*^8 e^{Cc_*^7} R^{1/2}.$$

Notice also that $C_* + C_n \leq Cc_*^7$. We compute now $\varepsilon_i(n)$ $i = 0, ..., 3$ defined in (2.3.45). First

$$\varepsilon_0(n) = \int_E \overline{c}_n^3 \overline{\gamma}_n d\nu_n \leq c_*^4 n \int_{1/n^2}^{4/n^2} \sqrt{z}\,\mathrm{d}z \leq \frac{Cc_*^4}{z^2}$$

and $\varepsilon_2(z) = \varepsilon_1(z) = 0$. Moreover

$$\varepsilon_3(n) = \int_{E_n^c} \overline{c\gamma} d\mu \leq c_*^2 \int_0^{1/n} \frac{da}{z^{1/2}} da \leq \frac{Cc_*^2}{\sqrt{n}}.$$

We conclude that

$$\sum_{i=0}^{4} \varepsilon_i(n) \leq \frac{Cc_*^4}{n^{1/2}}.$$

Finally

$$\int_{E_R^c} \overline{c\gamma} d\mu \leq c_*^2 \int_0^{1/R} \frac{\mathrm{d}z}{\sqrt{z}} \leq \frac{2c_*^2}{\sqrt{R}}$$

It follows that

$$\begin{aligned} \varepsilon_*(n) &= \inf_{R \geq n} (\int_{E_R^c} \overline{c\gamma} d\mu + \alpha_3(E_R)(C_*^2 + C_n^2) \sum_{i=0}^{3} \varepsilon_i(n)) \\ &\leq \frac{C}{\underline{\gamma}^3} c_*^{15} e^{Cc_*^7} \inf_{R \geq n} (\frac{1}{R^{1/2}} + \frac{R^{1/2}}{n^{1/2}}) = \frac{C}{\underline{\gamma}^3} c_*^{15} e^{Cc_*^7} \times \frac{1}{n^{1/4}} \end{aligned}$$

Using (2.3.47) we obtain (2.4.52). •

Chapter 3

Ergodicity for the solution

3.1 Introduction

Now that we have established in the first chapter the existence of a solution of the stochastic equation (1.2.3), we will give here a first application concerning its possible ergodic behaviour.

So, again, let $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, defined on a probability space $(\Omega, \mathcal{A}, P)$ with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$. We consider a process $X = (X_t)_{t\geq 0}, X_t \in \mathbb{R}^d$, solution of

$$X_t = x + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_{[0,t]}\int_{\mathbb{R}^d\times\mathbb{R}_+} c(z, X_{s-})\mathbb{1}_{u\leq\gamma(z,X_{s-})}N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \quad (3.1.1)$$

$x \in \mathbb{R}^d$, where W is an $m-$dimensional Brownian motion. The associated infinitesimal generator is given for smooth test functions by

$$L\psi(x) = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\psi(x) + g(x)\nabla\psi(x) + \int_{\mathbb{R}^d}(\psi(x + c(z, x)) - \psi(x))\gamma(z, x)\mu(\mathrm{d}z) \quad (3.1.2)$$

where $a = \sigma\sigma^*$. Notice that the jump rate at time t of process depends on the position of the process X_t itself, *i.e.* the intensity measure in the infinitesimal operator L is $\gamma(z, x)\mu(\mathrm{d}z)$. Moreover, since μ has infinite total mass, jumps occur with infinite activity, *i.e.* the process possesses infinitely many small jumps during any finite time interval $[0, T]$.

The principal aim of this chapter is to give easily verifiable conditions on the coefficients b, σ, c and γ under which the process is recurrent in the sense of Harris and satisfies the ergodic theorem starting from any initial point x, without imposing any non-degeneracy condition on the diffusive part. Recall that a process X is called recurrent in the sense of Harris if it possesses an invariant measure m such that any set A of positive $m-$measure $m(A) > 0$ is visited infinitely often by the process almost surely (see Azéma, Duflo and Revuz [4] (1969)): for all $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

There starts to be a huge literature on the subject of EDS with jumps, see e.g. Masuda [40] (2007) who works in a simpler situation where the "censure term" $\mathbb{1}_{u\leq\gamma(z,X_{s-})}$ is not present and who follows the Meyn and Tweedie approach developed in [42] or [41], Kulik [34] (2009) uses the stratification method in order to prove exponential ergodicity of jump diffusions, but the models he considers do not include the censored situation neither. Finally, let us mention Duan and Qiao [22] (2014) who are interested in solutions driven by non-Lipschitz coefficients.

Our aim is not to improve on the regularity conditions imposed on the coefficients but to concentrate on the jump mechanism. More precisely we show that we can use the jumps themselves in order to generate a splitting scheme that will allow to prove recurrence of the process.

It is important to notice that the presence of the censure term $\gamma(z, X_{s-})$ in (3.1.1) implies that the study of X is technically much more involved than the non-censored situation when γ is lower-bounded and strictly positive.

The method we use is the so-called regeneration method which we apply to the big jumps. More precisely, for some suitable set E such that $\mu(E) < \infty$, we cut the trajectory of X into parts of solutions of (3.1.1) driven by N in restriction to E^c and which are stopped at the first jump appearing due to "noise" z belonging to E. In spirit of the splitting technique introduced by Nummelin [44] (1978) and Athreya and Ney [3] (1978), we state a non-degeneracy condition which guarantees that the jump operator associated to the big jumps possesses a Lebesgue absolutely continuous component. This amounts to imposing that the partial derivatives of the jump term c with respect to the noise z are sufficiently non-degenerate, see (3.2.6) and (3.2.7) below. We stress that we do not need any non-degeneracy condition for the diffusion coefficient σ.

Notice that we do not apply the splitting technique to an extracted sampled chain nor to the resolvent chain as in Meyn and Tweedie [42] (1993); the loss of memory needed for regeneration is produced by certain big jumps. This approach is very natural in this context, but does not seem to be used so far in the literature, except for Xu [54] (2011), who works in a very specific frame and where the jumps do not depend on the position of the process.

This chapter is organized as follows. In Section 3.2 we state our main assumptions, prove a lower bound which is of a local Doeblin type and state our main results on Harris recurrence and speed of convergence to equilibrium of the process. Section 3.3 introduces the regeneration technique based on big jumps and proves the existence of certain (polynomial) moments of the associated regeneration times. Section 3.4 is devoted to an informal discussion on explicit and easily verifiable conditions stated in terms of the coefficients g, σ, c and γ which imply the Harris recurrence. Finally, we give in Section 3.5 a proof of the local Doeblin condition which is quite involved due to the fact that the jump mechanism depends on the position x of the process just before jumping.

3.2 Notations

As we did in the first chapter of this work, we consider again a Poisson random measure $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u$, where μ is a $\sigma-$finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ of infinite total mass. Let $X = (X_t)_{t\geq 0}, X_t \in \mathbb{R}^d$, be a solution of

$$X_t = x + \int_0^t g(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_{[0,t]}\int_{\mathbb{R}^d\times\mathbb{R}_+} c(z, X_{s-})\mathbb{1}_{u\leq\gamma(z,X_{s-})}N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \quad (3.2.3)$$

$x \in \mathbb{R}^d$, where W is an $m-$dimensional Brownian motion. Write $I\!\!F = (\mathcal{F}_t)_{t\geq 0}$ for the canonical filtration of the process given by

$$\mathcal{F}_t = \sigma\{W_s, N([0, s] \times A \times B), s \leq t, A \in \mathcal{B}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}_+)\}.$$

3.2.1 Assumptions

In order to grant existence and uniqueness of the above equation, throughout this chapter, we impose the following conditions on the coefficients g, σ, c and γ.

Assumption 3.2.1 *1. g and σ are globally Lipschitz continuous; σ is bounded and g has sub-linear growth; i.e. there exists a constant $B > 0$ such that*

$$|g(x)| \leq B(1 + |x|), \qquad \textit{for all } x \in \mathbb{R}^d.$$

2. c and γ are Lipschitz continuous with respect to x, i.e.

$$|c(z, x) - c(z, x')| \leq L_c(z)|x - x'| \textit{ and } |\gamma(z, x) - \gamma(z, x')| \leq L_\gamma(z)|x - x'|,$$

where $L_c,\ L_\gamma : \mathbb{R}^d \to \mathbb{R}_+$.

3. *Putting $\bar\gamma(z) \overset{def}{=} \sup_{x\in\mathbb{R}^d} \gamma(z,x)$ and $\bar c(z) \overset{def}{=} \sup_{\substack{x\in\mathbb{R}^d \\ \alpha+\beta\le 2}} \big(|c(z,x)| + |\partial_z^\alpha \partial_x^\beta c(z,x)|\big)$, we suppose that $\bar\gamma(z) \le \bar\gamma < \infty$, for all $z \in \mathbb{R}^d$, and that the following integrability condition holds:*
$$\int_{\mathbb{R}^d} (L_\gamma(z)\bar\gamma(z) + L_c(z)\bar c(z))\mu(\mathrm{d}z) < \infty.$$

4. $$\int_{\mathbb{R}^d} \bar\gamma(z)\bar c(z))\mu(\mathrm{d}z) < \infty.$$

Hence, the Theorem 1.2.3 ensures us, since Hypothesis 1.1 are in this case verified (notice that this last set of hypothesis are the same, with an additional stronger condition[1] with respect to the regularity of the coefficient c), that (3.2.3) admits a unique non-explosive adapted solution which is Markov, having càdlàg trajectories.

Notice that our assumptions *do not imply* that there exists a finite total jump rate

$$\int_{\mathbb{R}^d} \gamma(z,x)\mu(dz)$$

for any $x \in \mathbb{R}^d$. In other words, the above integral might be equal to $+\infty$, and jumps occur with infinite activity. We also stress that due to the presence of the censure term $\mathbb{1}_{u\le\gamma(z,X_{s-})}$ in equation (3.2.3) we are not in the classical frame of jump diffusions where the jump term depends in a smooth manner on z and x. Hence we are in a much more difficult situation than the one considered e.g. in Kulik [34] (2009) or in Masuda [40] (2007). In the last chapter we studied the regularity of the associated semi group P_t under stronger assumptions than the ones we are considering here. As a consequence, imposing only the above Assumption 3.2.1 does not ensure that our process is Feller.

In this chapter, we are looking for conditions ensuring that the process X is recurrent in the sense of Harris without using additional regularity of the coefficients, based on some minimal non-degeneracy of the jumps and without imposing any non-degeneracy condition on σ.

Recall that a process X is called recurrent in the sense of Harris if it possesses an invariant measure m such that any set A of positive $m-$measure $m(A) > 0$ is visited infinitely often by the process almost surely (see Azéma, Duflo and Revuz [4] (1969)): For all $x \in \mathbb{R}^d$,

$$P_x\Big[\int_0^\infty 1_A(X_s)\,\mathrm{d}s = \infty\Big] = 1.$$

We will prove Harris recurrence by introducing a splitting scheme that is entirely based on the "big" jumps of X. In order to do so, we introduce the following assumption.

Assumption 3.2.2 *We suppose that $\mu(\mathrm{d}z) = h(z)\,\mathrm{d}z$, for some measurable function $h \ge 0 \in \mathrm{L}^1{}_{loc}(\lambda)$, λ the Lebesgue measure on $\mathbb{R}^d$.*

3.2.2 A useful lower bound

Let $(E_n)_n$ be a non-decreasing sequence of subsets of $\mathbb{R}^d$ such that $\bigcup E_n = \mathbb{R}^d$ and such that $\mu(E_n) < \infty$ for all n. Fix some n and let $N_n(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ be the restriction of $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ to $\mathbb{R}_+ \times E_n \times [0,\bar\gamma]$ where we recall that $\bar\gamma = \sup_x \sup_z \gamma(x,z)$ is an upper bound on the jump rate. Since $\mu(E_n) < \infty$, N_n can be represented as compound Poisson process. We denote its jump times by $T_k^n, k \ge 1$, and the associated marks by (Z_k^n, U_k^n); the T_k^n are the jump times of a rate

[1] Indeed, the quantity $\bar c(z) \overset{\mathrm{def}}{=} \sup_{x\in\mathbb{R}^d}(|c(z,x)|)$ defined in the hypothesis 1.1, is replaced here, with the same notation, by $\bar c(z) \overset{\mathrm{def}}{=} \sup_{\substack{x\in\mathbb{R}^d \\ \alpha+\beta\le 2}} \big(|c(z,x)| + |\partial_z^\alpha \partial_x^\beta c(z,x)|\big)$.

$\bar\gamma \times \mu(E_n)-$Poisson process, and the variables Z_k^n are i.i.d. with law $\frac{1}{\mu(E_n)} 1_{E_n}(z)\mu(\mathrm{d}z)$. Moreover, the U_k^n are i.i.d. with uniform law on $[0,\bar\gamma]$.

Let $\Pi(x,dy) = \mathcal{L}(X_{T_k^n} | X_{T_k^n -} = x)(dy)$ be the transition kernel associated to the jumps. Our aim is to obtain a local Doeblin condition of the type

$$\Pi(x,dy) \geq 1_C(x)\beta\nu(dy), \tag{3.2.4}$$

for a suitable measurable set C, some $\beta \in]0,1[$ and a suitable probability measure ν. It is easy to see that the following lower bound holds.

$$\begin{aligned}\Pi(x,V) &\geq \frac{1}{\mu(E_n)}\int_{E_n} \frac{\gamma(z,x)}{\bar\gamma} 1\!\!1_V(x+c(z,x))\mu(\mathrm{d}z)\\ &= \frac{1}{\mu(E_n)}\int_{E_n} \frac{\gamma(z,x)}{\bar\gamma} 1\!\!1_V(x+c(z,x))h(z)dz,\end{aligned} \tag{3.2.5}$$

where h is the Lebesgue density of μ. It is natural to use a change of variables in the r.h.s. of the above lower bound, i.e. to replace, for fixed initial position x, the argument $x + c(x,z))$ by $y = y(z)$, on suitable subsets of $\mathbb{R}^d$ where $z \mapsto x + c(x,z)$ is a diffeomorphism. The difficulty is to control the dependence on the starting point x, since we are looking for uniform lower bounds (3.2.4), uniform in $x \in C$. This uniform control is achieved in the following proposition.

Proposition 3.2.3 *Suppose that there exist $x_0, z_0 \in \mathbb{R}^d$ and $r, R > 0$ such that for all $x \in \overline{B(x_0,r)}$,*
i) there exists $A > 0$ with

$$|\nabla_z c(z_0,x)h| \geq A|h|, \qquad \forall h \in \mathbb{R}^d, \tag{3.2.6}$$

ii) there exists $K > 0$ such that for all $z \in \overline{B(z_0,R)}$,

$$\left|\left(\nabla_z c(z_0,x)\right)^{-1}\right| \sum_{i,j} \left|\frac{\partial^2 c}{\partial z_i \partial z_j}(z,x)\right| \leq \frac{K}{d}, \tag{3.2.7}$$

iii)

$$\inf_{z:|z-z_0|\leq R, x:|x-x_0|\leq r} \gamma(z,x)h(z) = \varepsilon > 0 \qquad \text{and} \qquad S = \sup_{z:|z-z_0|\leq R} \bar c(z) < +\infty, \tag{3.2.8}$$

where $\mu(dz) = h(z)dz$.

Fix n_0 with $\overline{B(z_0,R)} \subset E_{n_0}$. Then there exist $\eta > 0$ and some ball $B \subset \mathbb{R}^d$ such that for all $n \geq n_0$,

$$\inf_{x\in B(x_0,\eta)} P[X_{T_k^n} \in V | X_{T_k^n -} = x] \geq \frac{1}{\bar\gamma S^d \mu(E_n)} \varepsilon\lambda(V\cap B). \tag{3.2.9}$$

As a consequence of Proposition 3.2.3, the local Doeblin condition (3.2.4) holds with $C = B$, $\beta = \frac{\lambda(B)\varepsilon}{\bar\gamma S^d \mu(E_n)} \wedge 1$ and $\nu(dy) = \frac{1}{\lambda(B)} 1_B(y)dy$. Notice that the set C is not a "petite" set in the sense of Meyn and Tweedie (1993) [42].

The main ingredient of the proof Proposition 3.2.3 is the following result.

Lemma 3.2.4 *Let $\Psi_x(z) = x + c(z,x)$, $\mathcal{K} = \overline{B(z_0,R)}, \Psi_x(\mathcal{K}) = \{\Psi_x(z), z \in \mathcal{K}\}$ and $a_x = x + c(z_0,x) = \Psi_x(z_0)$. Put*

$$\rho = \frac{A}{2}\Big(R \wedge \frac{1}{2K}\Big). \tag{3.2.10}$$

Then there exists $\eta > 0$ such that

$$B(a_{x_0}, \frac{\rho}{2}) \subset \bigcap_{x\in B(x_0,\eta)} \Psi_x(\mathcal{K}). \tag{3.2.11}$$

Moreover, for all $x \in \overline{B(x_0,r)}$, $B(a_x,\rho) \subset \Psi_x(\mathcal{K})$ and there exists $\mathcal{K}_x \subset \mathcal{K}$ such that $z \mapsto \Psi_x(z)$ is a $\mathcal{C}^1$-diffeomorphism from $\mathcal{K}_x$ to $B(a_x,\rho)$.

The proof of this Lemma and of Proposition 3.2.3 is given in Section 3.5 below.

Remark 3.2.5 *The ball B can be chosen as $B = B(a_{x_0}, \rho/2)$ with ρ as in (3.2.10) and $a_{x_0} = x_0 + c(z_0, x_0)$. If moreover*

$$L_c = \sup_{z:|z-z_0|\le R} L_c(z) < \infty,$$

then we can choose

$$\eta = \frac{\rho}{2(1+L_c)} \wedge r = \frac{A\Big(R \wedge \frac{1}{2K}\Big)}{4(1+L_c)} \wedge r.$$

We close this section with two examples where the ball B and the radius η are explicitly given.

Example 3.2.6 *We consider the one-dimensional case, with $\mu(\mathrm{d}z) = \mathrm{d}z$. Throughout this example, f will be a bounded $1-$Lipschitz-function such that $|f(x)| \ge \underline{f} > 0$ for all $x \in \overline{B(x_0, r)}$.*
1. Suppose that $c(z, x) = e^{-|z|} f(x)$ for all $z \in \overline{B(z_0, R)}$ and that $|z_0| \ge R + a$, $a > 0$. Then for all $x \in \overline{B(x_0, r)}$, $|\nabla_z c(z_0, x) h| = \big|f(x) e^{-|z_0|} h\big| \ge A|h|$ with $A = \underline{f} e^{-|z_0|}$. Moreover

$$\big|\big(\nabla_z c(z_0, x)\big)^{-1}\big| \left|\frac{\partial^2 c}{\partial z^2}(z, x)\right| = \frac{e^{|z_0|}}{|f(x)|} |f(x)| e^{-|z|} \le e^{|z_0| - a} =: K, \qquad \forall z \in \overline{B(z_0, R)}.$$

Recall that $B = B(a_{x_0}, \rho/2)$ where $a_{x_0} = x_0 + c(z_0, x_0) = x_0 + e^{-|z_0|} f(x_0)$. We have

$$\frac{\rho}{2} = \frac{A}{4}\Big(R \wedge \frac{1}{2K}\Big) = \frac{\underline{f} e^{-|z_0|}}{4}\Big(R \wedge \frac{e^{a-|z_0|}}{2}\Big).$$

Finally, since $L_c = \sup_{z:|z-z_0|\le R} L_c(z) \le e^{-a}$,

$$\eta = \frac{r}{(1+L_c)} \wedge r \ge \frac{\underline{f} e^{-|z_0|}}{4(1+e^{-a})}\Big(R \wedge \frac{e^{a-|z_0|}}{2}\Big) \wedge r.$$

2. Suppose now that $c(z, x) = \dfrac{f(x)}{1+z^2}$ and that $|z_0| \ge R + a$, $a > 0$. Then for all $x \in \overline{B(x_0, r)}$, $|\nabla_z c(z_0, x) h| = \left|f(x) \frac{2 z_0}{(z_0^2+1)^2} h\right| \ge A|h|$ with $A = \frac{2\underline{f}|z_0|}{(z_0^2+1)^2}$. Moreover

$$\begin{aligned}\big|\big(\nabla_z c(z_0, x)\big)^{-1}\big| \left|\frac{\partial^2 c}{\partial z^2}(z, x)\right| &= \frac{(z_0^2+1)^2}{2|f(x)||z_0|} \times 2|f(x)| \frac{|3z^2 - 1|}{(z^2+1)^3} \\ &\le \frac{(z_0^2+1)^2 |3(|z_0| + R)^2 + 1|}{|z_0|(a^2+1)^3} = K, \qquad \forall z \in \overline{B(z_0, R)}.\end{aligned}$$

In this case,

$$\frac{\rho}{2} = \frac{\underline{f}|z_0|}{2(z_0^2+1)^2}\Big(R \wedge \frac{|z_0|(a^2+1)^3}{2(z_0^2+1)^2|3(|z_0|+R)^2+1|}\Big).$$

Since $L_c = \sup_{z:|z-z_0|\le R} L_c(z) \le \frac{1}{1+a^2}$, we have

$$\eta = \frac{r}{(1+L_c)} \wedge r \ge \Big[\frac{\underline{f}|z_0|}{2(1+a^2)(z_0^2+1)^2}\Big(R \wedge \frac{|z_0|(a^2+1)^3}{2(z_0^2+1)^2|3(|z_0|+R)^2+1|}\Big)\Big] \wedge r.$$

3.2.3 Drift criteria

The set $C = B(x_0, \eta)$ appearing in the local Doeblin condition (3.2.4) will play the role of a small set in the sense of Nummelin [44] (1978) and Meyn-Tweedie [41] (1993). In order to be able to profit from the lower bound (3.2.9), we have to show that $(X_{T_k^n-})_k$ comes back to the set C i.o. For that sake, we introduce a drift condition in terms of the continuous time process, inspired by Douc, Fort and Guillin (2009) in [21].

Assumption 3.2.7 *There exists a continuous function $V : \mathbb{R}^d \to [1,\infty[$, an increasing concave positive function $\Phi : [1,\infty[\to (0,\infty)$ and a constant $b < \infty$ such that for any $s \geq 0$ and any $x \in \mathbb{R}^d$,*

$$\mathrm{E}_x[V(X_s)] + \mathrm{E}_x\left[\int_0^s \Phi \circ V(X_u)\,\mathrm{d}u\right] \leq V(x) + b\,\mathrm{E}_x\left[\int_0^s 1\!\!1_{C'}(X_u)\,\mathrm{d}u\right], \tag{3.2.12}$$

where $C' = B(x_0, \frac{\eta}{2})$, η as in Proposition 3.2.3.

If $V \in \mathcal{D}(\mathcal{A})$ belongs to the domain of the extended generator $\mathcal{A}$ of the process X, then Theorem 3.4 of Douc, Fort and Guillin [21] (2009) shows that the following condition

$$\mathcal{A}V(x) \leq -\Phi \circ V(x) + b1\!\!1_B(x) \tag{3.2.13}$$

implies the above Assumption (3.2.12).

We discuss in Section 3.4 examples where (3.2.12) or (3.2.13) are verified.

Under Assumption 3.2.7, Douc, Fort and Guillin [21] (2009) give estimates on modulated moments of hitting times. Modulated moments are expressions of the type

$$\mathrm{E}_x \int_0^\tau r(s) f(X_s)\,\mathrm{d}s,$$

where τ is a certain hitting time, r a rate function and f any positive measurable function. Knowledge of the modulated moments permits to interpolate between the maximal rate of convergence (taking $f \equiv 1$) and the maximal shape of functions f that can be taken in the ergodic theorem (taking $r \equiv 1$). In the present chapter we are interested in the maximal rate of convergence and hence we shall always take $f \equiv 1$.

For the function Φ of (3.2.12) put

$$H_\Phi(u) = \int_1^u \frac{\mathrm{d}s}{\Phi(s)},\ u \geq 1,\ r_\Phi(s) = r(s) = \Phi \circ H_\Phi^{-1}(s). \tag{3.2.14}$$

If for instance $\Phi(v) = cv^\alpha$ with $0 \leq \alpha < 1$, this gives rise to polynomial rate functions

$$r(s) \sim Cs^{\frac{\alpha}{1-\alpha}};$$

$\alpha = 1$ yields $r(s) = ce^{cs}$. In most of the cases, we will deal with the case $\Phi(v) = cv^\alpha, 0 \leq \alpha < 1$ and thus work in the context of polynomial rates of convergence. In this situation, the most important technical feature about the rate function is the following sub-additivity property

$$r(t+s) \leq c(r(t) + r(s)), \tag{3.2.15}$$

for $t, s \geq 0$ and c a positive constant. We shall also use that

$$r(t+s) \leq r(t)r(s),$$

for all $t, s \geq 0$.

3.2.4 Main results

Theorem 3.2.8 *Grant the assumptions of Proposition 3.2.3, Assumptions 3.2.1, 3.2.2 and 3.2.7. Then the process X is recurrent in the sense of Harris having a unique invariant probability measure m satisfying that $\Phi \circ V \in L^1(m)$. Moreover, for any measurable function $f \in L^1(m)$, we have*

$$\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s \to m(f)$$

P_x−almost surely for any $x \in \mathbb{R}^d$.

The above ergodic theorem is an important tool e.g. for statistical inference based on observations of the process X in continuous time. In this direction, the following deviation inequality is of particular interest. Recall that ν is the measure given in the local Doeblin condition (3.2.4).

Theorem 3.2.9 *Grant the assumptions of Proposition 3.2.3, Assumptions 3.2.1, 3.2.2 and 3.2.7 with $\Phi(v) = cv^\alpha, 0 \le \alpha < 1$. Put $p = 1/(1-\alpha)$. Let $f \in L^1(m)$ with $\|f\|_\infty < \infty$, x be any initial point and $0 < \varepsilon < \|f\|_\infty$. Then for all $t \ge 1$ the following inequality holds:*

$$P_x\left(\left|\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s - m(f)\right| > \varepsilon\right) \le K(p,\nu,X)\,V(x)\,t^{-(p-1)} \times$$
$$\times \left\{ \begin{array}{ll} \frac{1}{\varepsilon^{2(p-1)}}\|f\|_\infty^{2(p-1)} & \text{if } p \ge 2 \\ \frac{1}{\varepsilon^p}\|f\|_\infty^p & \text{if } 1 < p < 2 \end{array} \right\}. \qquad (3.2.16)$$

Here $K(p,\nu,X)$ is a positive constant, different in the two cases, which depends on p,ν and on the process X, but which does not depend on f, t, ε.

Finally, we obtain the following quantitative control of the convergence of ergodic averages.

Proposition 3.2.10 *Grant the assumptions of Proposition 3.2.3, Assumptions 3.2.1, 3.2.2 and 3.2.7 with $\Phi(v) = cv^\alpha, 0 \le \alpha < 1$. Then for any $x, y \in \mathbb{R}^d$,*

$$\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - \frac{1}{t}\int_0^t P_s(y,\cdot)ds\|_{TV} \le C\frac{1}{t}(V(x)^{(1-\alpha)} + V(y)^{(1-\alpha)}), \qquad (3.2.17)$$

where $C > 0$ is a constant. In particular, if $\alpha \ge \frac{1}{2}$, then

$$\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - m\|_{TV} \le C\frac{1}{t}V(x)^{(1-\alpha)}. \qquad (3.2.18)$$

The proof of Theorems 3.2.8 and 3.2.9 and of Proposition 3.2.10 relies on the regeneration method that we are going to introduce now.

3.3 Regeneration for the chain of big jumps

3.3.1 Regeneration times

We show how the lower bound on the jump kernel (3.2.4) allows us to introduce regeneration times for the process X.

We start by defining a split kernel $Q((x,u),dy)$. This is a transition kernel $Q((x,u),dy)$ from $\mathbb{R}^d \times [0,1]$ to $\mathbb{R}^d$ defined by

$$Q((x,u),dy) = \left\{ \begin{array}{ll} \nu(dy) & \text{if } (x,u) \in C \times [0,\beta] \\ \frac{1}{1-\beta}\left(\Pi(x,dy) - \beta\nu(dy)\right) & \text{if } (x,u) \in C \times]\beta,1] \\ \Pi(x,dy) & \text{if } x \notin C. \end{array} \right. \qquad (3.3.19)$$

We now show how to construct a version of the process X recursively over time intervals $[T_k^n, T_{k+1}^n[, k \ge 0$. We start at time $t = 0$ with $X_0 = x$ and introduce the process Z_t defined by

$$Z_t = x + \int_0^t g(Z_s)\,\mathrm{d}s + \int_0^t \sigma(Z_s)\,\mathrm{d}W_s + \int_0^t \int_{E_n^c} \int_0^{\bar\gamma} c(z, Z_{s-}) 1_{u \le \gamma(z, Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

For $t < T_1^n$, we clearly have $Z_t = X_t$. Notice also that T_1^n is independent of the rhs of the above equation and exponentially distributed with parameter $\bar\gamma\mu(E_n)$. We put $X_{T_1^n-} := Z_{T_1^n-}$ (notice that $Z_{T_1^n} = Z_{T_1^n-}$, since Z almost surely does not jump at time T_1^n). On $X_{T_1^n-} = x'$, we do the following.

1. We choose a uniform random variable $U_1 \sim U(0,1)$, independently of anything else.
2. On $U_1 = u$, we choose a random variable $V_1 \sim Q((x', u), dy)$ and we put

$$X_{T_1^n} := V_1. \tag{3.3.20}$$

We then restart the above procedure with the new starting point V_1 instead of x.

We will write $\mathbf{X_t}$ for the process with additional color U_k, defined by

$$\mathbf{X_t} = \sum_{k \geq 0} 1_{[T_k^n, T_{k+1}^n[}(t)(X_t, U_k).$$

Remark 3.3.1 *Notice that the above splitting procedure does not even use the strong Markov property of the underlying process. It only uses the independence properties of the driving Poisson random measure.*

This new process is clearly Markov with respect to its filtration, and by abuse of notations we will not distinguish between the original filtration $I\!\!F$ introduced in Section 3.2 and the canonical filtration of $\mathbf{X_t}$. In this richer structure, where we have added the component U_k to the process, we obtain regeneration times for the process $\mathbf{X}$. More precisely, write

$$A := C \times [0, \beta]$$

and put

$$R_0 := 0, \; R_{k+1} := \inf\{T_m^n > R_k : \mathbf{X}_{T_m^n -} \in A\}. \tag{3.3.21}$$

Then we clearly have

Proposition 3.3.2 *a)* $\mathbf{X}_{R_k} \sim \nu(dx)U(\mathrm{d}u)$ *on* $R_k < \infty$, *for all* $k \geq 1$.
b) $\mathbf{X}_{R_k + \cdot}$ *is independent of* $\mathcal{F}_{R_k -}$ *on* $R_k < \infty$, *for all* $k \geq 1$.
c) If $R_k < \infty$ *for all* k, *then the sequence* $(X_{R_k})_{k \geq 1}$ *is i.i.d.*

It is clear that in this way the speed of convergence to equilibrium of the process is determined by the moments of the extended stopping times R_k. In the next section we show that the drift condition of Assumption 3.2.7 ensures in particular that $R_k < \infty$ P_x-almost surely for any x.

3.3.2 Existence of moments of the regeneration times

Recall the local Doeblin condition (3.2.4) , the definition of the set C and of $C' = B(x_0, \eta/2)$. Let $\tau_{C'} = \inf\{t \geq 0 : X_t \in C'\}$ be the first hitting time of C'. It is known (Douc, Fort and Guillin [21] (2009)) that the condition (3.2.12) implies that

$$\mathrm{E}_x \int_0^{\tau_{C'}} r(s)\, \mathrm{d}s \leq V(x), \tag{3.3.22}$$

where r is given as in (3.2.14).

Return times to C

In particular, equation (3.3.22) implies that $\tau_{C'} < \infty$ P_x-surely for all x. We show that this implies that the regeneration times R_k introduced in (3.3.21) above are finite almost surely. Recall that T_k^n are the successive jump times of the Poisson point process N restricted to $(z, u) \in E_n \times [0, \bar{\gamma}]$. The regeneration times R_k are expressed in terms of the jump chain $X_{T_k^n -}, k \geq 0$. We have to ensure that the control of return times to C' for the continuous time process implies analogous moments for the jump chain.

Before stating the first result going into this direction, we have to introduce the following objects. Let $\|\sigma\|_\infty$ be the sup-norm of the diffusion coefficient σ and recall that $|g(x)| \le B(1+|x|)$, $\forall x \in \mathbb{R}^d$. Finally, we choose n sufficiently large such that

$$\bar{\gamma}\mu(E_n) > B \tag{3.3.23}$$

(recall that $\mu(\mathbb{R}^d) = \infty$) and such that

$$\|\sigma\|_\infty \frac{\sqrt{\pi}}{2} \frac{\bar{\gamma}\mu(E_n)}{(\bar{\gamma}\mu(E_n) - B)^{\frac{3}{2}}} + B_\eta \frac{\bar{\gamma}\mu(E_n)}{(\bar{\gamma}\mu(E_n) - B)^2} < \frac{\eta}{4}, \tag{3.3.24}$$

where $B_\eta = \int_{\mathbb{R}^d} \bar{c}(z)\bar{\gamma}(z)\,\mathrm{d}\mu(z) + B\big(1 + |x_0| + \frac{\eta}{2}\big)$.

Proposition 3.3.3 *For any n verifying* (3.3.23) *and* (3.3.24)*,*

$$\inf_{x \in C'} P_x(X_{T_1^n -} \in C) \ge \frac{1}{2}. \tag{3.3.25}$$

Remark 3.3.4 *The choice $\frac{1}{2}$ in the above lower bound is arbitrary, by choosing larger values of n, we could achieve any bound $1 - \varepsilon$ on the right hand side of* (3.3.25)*.*

Proof : Recall the process Z_t defined by

$$Z_t = x + \int_0^t g(Z_s)\,\mathrm{d}s + \int_0^t \sigma(Z_s)\,\mathrm{d}W_s + \int_0^t \int_{E_n^c} \int_0^{\bar{\gamma}} c(z, Z_{s-}) 1_{u \le \gamma(z, Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and recall that for any $t < T_1^n$, $Z_t = X_t$. Recall also that T_1^n is independent of the rhs of the above equation. Now let $x \in C'$ and upper-bound

$$\mathrm{P}_x[X_{T_1^n -} \notin C] = \mathrm{P}_x[Z_{T_1^n -} \notin C].$$

Clearly, $\mathrm{P}_x\left[|Z_t - x| \ge \frac{\eta}{2}\right] \le \frac{2}{\eta}\,\mathrm{E}_x[|Z_t - x|]$.

Let $T > 0$. Then, with $Z_T^* = \sup_{t\in[0,T]} |Z_t|$ and $m \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{E}_x[|Z_t - x| \mathbb{1}_{Z_T^* < m}] &\le \mathrm{E}_x\left[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s| \mathbb{1}_{Z_T^* < m}\right] + \mathrm{E}_x\left[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^* < m}\right] \\
&\quad + \mathrm{E}_x\left[\int_0^t \int_{E_n^c} \int_0^{\bar{\gamma}} |c(z, Z_{s-})| 1_{u \le \gamma(z, Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \mathbb{1}_{Z_T^* < m}\right] \\
&\le \mathrm{E}_x\left[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|\right] + \mathrm{E}_x\left[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^* < m}\right] \\
&\quad + \mathrm{E}_x\left[\int_0^t \int_{E_n^c} \int_0^{\bar{\gamma}} |c(z, Z_{s-})| 1_{u \le \gamma(z, Z_{s-})} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\right]
\end{aligned}$$

with (using the Itô isometry and the fact that σ is bounded)

$$\mathrm{E}_x\left[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|\right] \le \sqrt{\mathrm{E}_x\left[|\int_0^t \sigma(Z_s)\,\mathrm{d}W_s|^2\right]} \le \|\sigma\|_\infty \sqrt{t}.$$

Moreover, for $x \in B(x_0, \frac{\eta}{2}) = C'$,

$$\begin{aligned}
\mathrm{E}_x\left[\int_0^t |g(Z_s)|\,\mathrm{d}s \mathbb{1}_{Z_T^* < m}\right] &\le \mathrm{E}_x\left[\int_0^t B(1 + |Z_s|)\,\mathrm{d}s \mathbb{1}_{Z_T^* < m}\right] \\
&= Bt + B\int_0^t \mathrm{E}_x\left[|Z_s| \mathbb{1}_{Z_T^* < m}\right] \mathrm{d}s \\
&\le Bt(1 + |x|) + B\int_0^t \mathrm{E}_x\left[|Z_s - x| \mathbb{1}_{Z_T^* < m}\right] \mathrm{d}s \\
&\le Bt\big(1 + |x_0| + \frac{\eta}{2}\big) + B\int_0^t \mathrm{E}_x\left[|Z_s - x| \mathbb{1}_{Z_T^* < m}\right] \mathrm{d}s.
\end{aligned}$$

We upper bound

$$\begin{aligned}\mathrm{E}_x\Big[\int_0^t\int_{E_n^c}\int_0^{\bar\gamma}|c(z,Z_{s-})|1_{u\le\gamma(z,Z_{s-})}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big]&\\
=\mathrm{E}_x\Big[\int_0^t\int_{E_n^c}\int_0^{\bar\gamma}|c(z,Z_{s-})|1_{u\le\gamma(z,Z_{s-})}\,\mathrm{d}z\,\mathrm{d}\mu(z)\,\mathrm{d}u\Big]&\\
\le t\int_{E_n^c}\bar c(z)\bar\gamma(z)\,\mathrm{d}\mu(z)\le t\int_{\mathbb{R}^d}\bar c(z)\bar\gamma(z)\,\mathrm{d}\mu(z)&\end{aligned}$$

and put $B_\eta=\int_{\mathbb{R}^d}\bar c(z)\bar\gamma(z)\,\mathrm{d}\mu(z)+B\big(1+|x_0|+\frac{\eta}{2}\big)$. Then

$$\mathrm{E}_x[|Z_t-x|\mathbb{1}_{Z_T^*<m}]\le\|\sigma\|_\infty\sqrt{t}+B_\eta t+B\int_0^t\mathrm{E}_x\left[|Z_s-x|\mathbb{1}_{Z_T^*<m}\right]\mathrm{d}s.$$

Then Gronwall's lemma (see Proposition A.2.1 in the appendix) implies that

$$\mathrm{E}_x[|Z_t-x|\mathbb{1}_{Z_T^*<m}]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},$$

for all $t\le T$.

Since Z_t is a càdlàg process, Z_T^* is finite almost surely. Therefore $|Z_t-x|\mathbb{1}_{Z_T^*<m}$ tends to $|Z_t-x|$ almost surely as $m\to\infty$, and monotone convergence implies that

$$\mathrm{E}_x[|Z_t-x|]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},\tag{3.3.26}$$

for all $t\le T$. In the above rhs, the constants do not depend on T, hence (3.3.26) is actually true for any $t\ge0$.

Furthermore

$$\mathrm{E}_x[|Z_{t-}-x|]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},\tag{3.3.27}$$

which can be seen as follows. Using (3.3.26), we have, for $s<t$,

$$\mathrm{E}_x[|Z_s-x|\mathbb{1}_{Z_t^*<m}]\le\mathrm{E}_x[|Z_s-x|]\le(\|\sigma\|_\infty\sqrt{s}+B_\eta s)e^{Bs}\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}$$

so, using dominated convergence when s tends to t from inferior values,

$$\mathrm{E}_x[|Z_{t-}-x|\mathbb{1}_{Z_t^*<m}]\le(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt},$$

and letting $m\to+\infty$, monotone convergence gives (3.3.27).

Now, T_1^n is independent from $(Z_t)_t$, exponentially distributed with parameter $\lambda=\bar\gamma\mu(E_n)$. By choice of n, $\lambda>B$. Then

$$\begin{aligned}\mathrm{E}_x[|Z_{T_1-}-x|]&\le\int_0^{+\infty}(\|\sigma\|_\infty\sqrt{t}+B_\eta t)e^{Bt}\lambda e^{-\lambda t}\,\mathrm{d}t\\
&=\lambda\Big(\|\sigma\|_\infty\int_0^{+\infty}\sqrt{t}e^{-(\lambda-B)t}\,\mathrm{d}t+B_\eta\int_0^{+\infty}te^{-(\lambda-B)t}\,\mathrm{d}t\Big)\\
&=\lambda\Big(\|\sigma\|_\infty\frac{\Gamma(\frac32)}{(\lambda-B)^{\frac32}}+\frac{B_\eta\Gamma(2)}{(\lambda-B)^2}\Big)=\lambda\Big(\|\sigma\|_\infty\frac{\sqrt{\pi}}{2}\frac{1}{(\lambda-B)^{\frac32}}+\frac{B_\eta}{(\lambda-B)^2}\Big)\end{aligned}$$

for every $x\in B(x_0,\frac{\eta}{2})$. Since n was chosen to have

$$\|\sigma\|_\infty\frac{\sqrt{\pi}}{2}\frac{\bar\gamma\mu(E_n)}{(\bar\gamma\mu(E_n)-B)^{\frac32}}+B_\eta\frac{\bar\gamma\mu(E_n)}{(\bar\gamma\mu(E_n)-B)^2}<\frac{\eta}{4},$$

we obtain

$$\sup_{x\in B(x_0,\frac{\eta}{2})} \mathrm{P}_x[X_{T_1^n-}\notin C] \le \sup_{x\in B(x_0,\frac{\eta}{2})} \mathrm{P}_x[|Z_{T_1^n-}-x|\ge \frac{\eta}{2}]$$
$$\le \sup_{x\in B(x_0,\frac{\eta}{2})} \frac{2}{\eta}\mathrm{E}_x[|Z_{T_1^n-}-x|] < \frac{1}{2}.$$

•

The above arguments imply the following statement.

Corollary 3.3.5 *Let $S_1=\inf\{T_k^n, k\ge 1 : X_{T_k^n-}\in C\}$. Then $P_x(S_1<\infty)=1$ for all x.*

Proof : We introduce the following sequence of stopping times.

$$t_1=\tau_{C'}, s_1=\inf\{T_k^n>t_1\},\dots,t_l=\inf\{s\ge s_{l-1}: X_s\in C'\}, s_l=\inf\{T_k^n>t_l\}.$$

The above stopping times are all finite almost surely. We put

$$\tau_*=\inf\{l: X_{s_l-}\in C\}.$$

Then, using (3.3.25), for any $x\in\mathbb{R}^d$,

$$P_x(\tau_*\ge n_0)\le\left(\frac{1}{2}\right)^{n_0},$$

which shows that $\tau_*<\infty$ P_x-almost surely for all x. In particular,

$$S_1\le s_{\tau_*}<\infty$$

P_x-almost surely for all x. •

The above proof shows in particular that the polynomial control obtained for the first entrance time in C', obtained in (3.3.22) remains true for S_1. Moreover we have the following control on polynomial moments of the regeneration times.

Proposition 3.3.6 *Grant Assumption 3.2.7 with $\Phi(v)=cv^\alpha, 0\le\alpha<1$. Let $p=\frac{1}{1-\alpha}$. Then there exists a constant c such that*

$$\mathrm{E}_x\left[S_1^p\right]\le cV(x). \tag{3.3.28}$$

Proof : We adopt the notation of the proof of Corollary 3.3.5.

1. In what follows, c will denote a constant that might change from line to line. We start by studying $\mathrm{E}_x\int_0^{s_1} r(s)\,\mathrm{d}s$, where r is as in (3.2.14). Let

$$\lambda=\bar\gamma\mu(E_n)$$

be the rate of the Poisson process associated to $T_k^n, k\ge 1$. Then by definition of s_1,

$$\mathrm{E}_x\int_0^{s_1} r(s)\,\mathrm{d}s=\mathrm{E}_x\int_0^{\tau_{C'}} r(s)\,\mathrm{d}s+\mathrm{E}_x\int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s\le V(x)+\mathrm{E}_x\int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s,$$

where we have used (3.3.22).

Now, using that $s_1-\tau_{C'}$ is independent of $\mathcal{F}_{\tau_{C'}}$, exponentially distributed with parameter λ, we upper-bound

$$\mathrm{E}_x\int_{\tau_{C'}}^{s_1} r(s)\,\mathrm{d}s=\mathrm{E}_x\,\mathrm{E}_{X_{\tau_{C'}}}\int_0^{s_1-\tau_{C'}} r(\tau_{C'}+s)\,\mathrm{d}s$$
$$\le \mathrm{E}_x[r(\tau_{C'})]\,\mathrm{E}[r(s_1-\tau_{C'})]=c\,\mathrm{E}_x[r(\tau_{C'})],$$

since $\mathrm{E}_{X_{\tau_{C'}}}[r(s_1-\tau_{C'})] = \int_0^\infty \lambda e^{-\lambda t} r(t)\,\mathrm{d}t < \infty$ does not depend on $X_{\tau_{C'}}$.
Using that

$$r(t) \le c + \int_0^t r(s)\,\mathrm{d}s, \tag{3.3.29}$$

we obtain

$$\mathrm{E}_x[r(\tau_{C'})] \le c + \mathrm{E}_x \int_0^{\tau_{C'}} r(s)\,\mathrm{d}s \le c + V(x).$$

Therefore,

$$\mathrm{E}_x \int_0^{s_1} r(s)\,\mathrm{d}s \le c + cV(x) \le cV(x), \tag{3.3.30}$$

where we have used that $V(x) \ge 1$.

2. We now use $r(t+s) \le r(t)r(s)$ in order to obtain a control of $\mathrm{E}_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s$. We certainly have

$$\begin{aligned}
\mathrm{E}_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s &= \mathrm{E}_x \int_0^{t_1} r(s)\,\mathrm{d}s + \sum_{n\ge 1} \mathrm{E}_x \int_{t_n}^{t_{n+1}} r(s)\,\mathrm{d}s 1_{\{n<\tau_*\}} \\
&\le V(x) + \sum_{n\ge1} \mathrm{E}_x \left[1_{\{n-1<\tau_*\}} r(t_n) \int_0^{t_{n+1}-t_n} r(s)\,\mathrm{d}s \right] \\
&= V(x) + \sum_{n\ge1} \mathrm{E}_x \left[1_{\{n-1<\tau_*\}} r(t_n)\, \mathrm{E}_{X_{t_n}} \int_0^{t_1} r(s)\,\mathrm{d}s \right] \\
&\le V(x) + \sum_{n\ge1} \mathrm{E}_x \left[1_{\{n-1<\tau_*\}} r(t_n) V(X_{t_n}) \right],
\end{aligned} \tag{3.3.31}$$

where we have used (3.3.22) and the fact that $1_{\{n-1<\tau_*\}}$ is $\mathcal{F}_{s_{n-1}}-$measurable. Now, X_{t_n} belonging to C', we can upper-bound $V(X_{t_n}) \le \|V\|_{C'} = c$, and obtain

$$\mathrm{E}_x \int_0^{t_{\tau_*}} r(s)\,\mathrm{d}s \le V(x) + c \sum_{n\ge1} \mathrm{E}_x \left[1_{\{n-1<\tau_*\}} r(t_n) \right]. \tag{3.3.32}$$

We use $r(t+s) \le r(t)r(s)$ and the Markov property with respect to t_1 to obtain

$$\mathrm{E}_x \left[1_{\{n-1<\tau_*\}} r(t_n) \right] \le \mathrm{E}_x\, r(t_1) \sup_{y\in C'} \mathrm{E}_y \left[r(t_{n-1}) 1_{\{n-2<\tau_*\}} \right].$$

Using (3.3.29), the first factor can be treated as follows

$$\mathrm{E}_x\, r(t_1) \le c + \mathrm{E}_x \int_0^{t_1} r(s)\,\mathrm{d}s \le c + V(x) \le cV(x),$$

since $V(x) \ge 1$.

Further, let $p \in]\frac{1-\alpha}{\alpha} \vee 1, \frac{1}{\alpha}[$, and $q \ge 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, using that $P(n-2 < \tau_*) \le (\frac{1}{2})^{n-2}$,

$$\mathrm{E}_y \left[1_{\{n-2<\tau_*\}} r(t_{n-1}) \right] \le \mathrm{E}_y (r^p(t_{n-1}))^{1/p} \left(\frac{1}{2}\right)^{(n-2)/q}.$$

We have, by definition of r that $r^p(t) \le ct^{\frac{\alpha}{1-\alpha}p}$, where $\frac{\alpha}{1-\alpha}p > 1$ by choice of p. Using Jensen's inequality we obtain

$$r^p(t_{n-2}) \le c(n-2)^{\frac{\alpha}{1-\alpha}p-1} (t_1^{\frac{\alpha}{1-\alpha}p} + \ldots + (t_{n_2} - t_{n-3})^{\frac{\alpha}{1-\alpha}p}).$$

We now use, by choice of p, that $\frac{\alpha}{1-\alpha}p - 1 \le \frac{\alpha}{1-\alpha}\frac{1}{\alpha} - 1 = \frac{\alpha}{1-\alpha}$, and therefore

$$t^{\frac{\alpha}{1-\alpha}p} = \frac{1}{\frac{\alpha}{1-\alpha}p - 1} \int_0^t s^{\frac{\alpha}{1-\alpha}p-1}\,\mathrm{d}s \le c \int_0^t r(s)\,\mathrm{d}s.$$

This allows to rewrite

$$r^p(t_{n-2}) \le c(n-2)^{\frac{\alpha}{1-\alpha}p-1} \left(\int_0^{t_1} r(s)\, \mathrm{d}s + \ldots + \int_0^{t_{n_2}-t_{n-3}} r(s)\, \mathrm{d}s \right).$$

Using successively the Markov property at times $t_1, t_2, \ldots, t_{n-3}$, we obtain

$$\mathrm{E}_y\, r^p(t_{n-2}) \le c(n-2)^{\frac{\alpha}{1-\alpha}p-1}(n-2) \sup_{z\in C'} \mathrm{E}_z \int_0^{t_1} r(s)\, \mathrm{d}s.$$

Finally, by (3.3.22), $\sup_{z\in C'} \mathrm{E}_z \int_0^{t_1} r(s)\, \mathrm{d}s \le \sup_{z\in C'} V(z) = c$, and therefore

$$(\mathrm{E}_y\, r^p(t_{n-2}))^{1/p} \le c(n-2)^{\frac{\alpha}{1-\alpha}}.$$

Coming back to (3.3.32) we conclude that

$$\mathrm{E}_x \int_0^{t_{\tau^*}} r(s)\, \mathrm{d}s \le V(x) + cV(x) \sum_{n\ge 1} \left(\frac{1}{2}\right)^{\frac{n-2}{q}} (n-2)^{\frac{\alpha}{1-\alpha}} \le cV(x).$$

3. We now argue as follows.

$$\begin{aligned}
\mathrm{E}_x \int_0^{s_{\tau^*}} r(s)\, \mathrm{d}s &= \mathrm{E}_x \int_0^{t_{\tau^*}} r(s)\, \mathrm{d}s + \mathrm{E}_x \int_{t_{\tau^*}}^{s_{\tau^*}} r(s)\, \mathrm{d}s \\
&\le cV(x) + \sum_{n\ge 1} \mathrm{E}_x\, 1_{\{\tau^*=n\}} \int_{t_n}^{s_n} r(s)\, \mathrm{d}s \\
&\le cV(x) + \sum_{n\ge 1} \mathrm{E}_x\, 1_{\{\tau^*>n-1\}} r(t_n) \int_0^{s_n-t_n} r(s)\, \mathrm{d}s \\
&\le cV(x) + \sum_{n\ge 1} \mathrm{E}_x [1_{\{\tau^*>n-1\}} r(t_n) \sup_{y\in C'} \int_0^{s_1} r(s)\, \mathrm{d}s] \\
&\le cV(x) + c\sum_{n\ge 1} \mathrm{E}_x\, 1_{\{\tau^*>n-1\}} r(t_n),
\end{aligned}$$

where we have used the Markov property with respect to t_n and (3.3.30). The last sum is treated as (3.3.32), which concludes our proof, since $r(s) \ge cs^{\frac{\alpha}{1-\alpha}}$.

•

The above result implies an analogous control for moments of the regeneration times R_k of (3.3.21). More precisely, we can now define

$$S_l = \inf\{T_k^n > S_{l-1} : X_{T_n^k -} \in C\}, l \ge 2,$$

and let

$$R_1 = \inf\{S_l : U_l \le \beta\}, R_{k+1} = \inf\{S_l > R_k : U_l \le \beta\}. \tag{3.3.33}$$

An analogous argument as the one used in the proof of Proposition 3.3.6 then implies

Theorem 3.3.7 *Grant Assumption 3.2.7 with $\Phi(v) = cv^\alpha, 0 \le \alpha < 1$ and let $p = 1/(1-\alpha)$. Then*

$$\mathrm{E}_x\, R_1^p \le cV(x). \tag{3.3.34}$$

We are now ready to prove Theorems 3.2.8 and 3.2.9.

3.3.3 Proof of Theorems 3.2.8 and 3.2.9

Proof of Theorem 3.2.8.
Let

$$\mathbf{m}(O) := \mathrm{E}\int_{R_1}^{R_2} 1_O(\mathbf{X_s})\,\mathrm{d}s,$$

for any measurable set O. By the strong law of large numbers, any set O with $\mathbf{m}(O) > 0$ is visited i.o. P_x−almost surely by the process $\mathbf{X}$, for any starting point $(x, u) \in \mathbb{R}^d \times [0, 1]$. Hence, the process is recurrent in the sense of Harris, and by the Kac occupation time formula, $\mathbf{m}$ is the unique invariant measure of the process (unique up to multiplication with a constant).

Now, recall that ν is of compact support, hence $V \in L^1(\nu)$. Using (3.3.34) in the case $\alpha = 0$, we obtain $\mathbf{m}(\mathbb{R}^d \times [0, 1]) = \mathrm{E}[R_2 - R_1] = \mathrm{E}_\nu R_1 \le c\nu(V) < \infty$. This implies that $\mathbf{X}$ is positive recurrent.

The invariant measure m of the original process X is the projection onto the first coordinate of $\mathbf{m}$. In particular, X is also positive Harris recurrent, and m can be represented as

$$m(f) = \mathrm{E}\int_{R_1}^{R_2} f(X_s)\,\mathrm{d}s.$$

The ergodic theorem is then simply a consequence of the positive Harris property of X. Finally, the fact that $\Phi \circ V \in L^1(m)$ is an almost immediate consequence of (3.2.12), based on Dynkin's formula. •

Proof of Theorem 3.2.9.
Theorem 3.2.9 follows from Theorem 5.2 of Löcherbach and Loukianova (2013) in [39] together with Proposition 3.3.6. •

We finally proceed to the proof of Proposition 3.2.10.

Proof of Proposition 3.2.10.
Let X and Y be copies of the process, issued from x (from y respectively) at time 0. Let R_1 and R_1' be the respective regeneration times. Using the same realization V_k for X and for Y (recall (3.3.20)), it is clear that R_1 and R_1' are shift-coupling epochs for X and for Y, i.e. $X_{R_1+\cdot} = Y_{R_1'+\cdot}$. If follows then from Thorisson [53] (1994), see also Roberts and Rosenthal [51] (1996), Proposition 5, that

$$\left\|\frac{1}{t}\int_0^t P_s(x,\cdot)\,\mathrm{d}s - \frac{1}{t}\int_0^t P_s(y,\cdot)ds\right\|_{TV} \le C\frac{1}{t}(\mathrm{E}_x[R_1] + \mathrm{E}_y[R_1']). \tag{3.3.35}$$

Recall that $p = 1/(1-\alpha)$. Then

$$\mathrm{E}_x[R_1] \le (\mathrm{E}_x R_1^p)^{1/p} \le c(V(x))^{(1-\alpha)}.$$

Now, if $\alpha \ge \frac{1}{2}$, then $1 - \alpha \le \alpha$ and therefore,

$$\mathrm{E}_x[R_1] \le c\,\Phi \circ V(x) \in L^1(m).$$

In this case, we can integrate (3.3.35) against $m(dy)$ and obtain the second part of the assertion. •

3.4 Discussing the drift condition

In this section, we discuss in an informal way several easily verifiable sufficient conditions implying Assumption 3.2.7 with $\Phi(v) = cv^\alpha$. These conditions will involve different coefficients of the process. Recall that the infinitesimal generator L of the process X is given for every $\mathcal{C}^2$-function ψ with compact support on $\mathbb{R}^d$ by

$$L\psi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\psi(x) + b(x)\nabla\psi(x) + \int_{\mathbb{R}^d} \left[\psi(x + c(z,x)) - \psi(x)\right] K(x, \mathrm{d}z),$$

where $a = \sigma\sigma^*$ and $K(x, \mathrm{d}z) = \gamma(z,x)h(z)\,\mathrm{d}z$. In order to grant Assumption 3.2.7, we are looking for conditions implying that

$$LV \leq -AV^{\alpha}(x) + b\mathbb{1}_{C'}(x), \tag{3.4.36}$$

for some $0 \leq \alpha \leq 1$, with $C' = B(x_0, \frac{\eta}{2})$.

Example 3.4.1 *If we choose for instance $V(x) = |x - x_0|^2$ and $\alpha = \frac{1}{2}$ it suffices to impose that for all $x \in \mathbb{R}^d \setminus C'$,*

$$\mathrm{Tr}(\sigma\sigma^*) + 2\langle g(x), x - x_0\rangle + \int_{\mathbb{R}^d} \langle 2(x - x_0) + c(z,x), c(z,x)\rangle \gamma(z,x)h(z)\,\mathrm{d}z \leq -A|x - x_0|. \tag{3.4.37}$$

We now discuss several concrete sufficient conditions implying (3.4.36). In this context, it is interesting to notice that the influence of the different coefficients can be quite different. Some coefficients can work in a favorable way in order to ensure (3.4.36). In that case we will say that they are "pushing" the diffusion into the set C'. Other coefficients might play a neutral role or even work against (3.4.36). Since we have three natural parts of coefficients (diffusion part, drift and the jump part), we will discuss here the following cases: "pushing" with the jumps only, "pushing" with jumps and drift together[2] and "pushing" with the drift only.

Pushing with the jumps

Consider first a pure jump process, i.e. the case when $a = g = 0$. We choose $V(x) = |x - x_0|^2$ and propose the following conditions.
1. Global condition with respect to z. $\forall z \in \mathbb{R}^d$, $\forall x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\langle c(z,x) + 2(x - x_0), c(z,x)\rangle \leq 0. \tag{3.4.38}$$

2. Local conditions with respect to z on some set $\mathcal{K}$. There exists a set $\mathcal{K}$ such that the following holds.

1. there exists $\xi > 0$ such that for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\int_{\mathcal{K}} |c(z,x)|\gamma(z,x)h(z)\,\mathrm{d}z > \xi. \quad ^3 \tag{3.4.39}$$

2. There exists $\zeta \in (0,1]$ such that for all $z \in \mathcal{K}$ and for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$.

$$\langle c(z,x) + 2(x - x_0), c(z,x)\rangle \leq -\zeta|c(z,x) + 2(x - x_0)||c(z,x)|, \tag{3.4.40}$$

3. For all $z \in \mathcal{K}$ and for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$|c(z,x)| \leq |x - x_0|. \tag{3.4.41}$$

[2] this will be the most interesting case
[3] This condition has to be seen in relation with Condition (3.2.8).

Notice that this last condition implies in particular that $|c(z,x)+2(x-x_0)| \geq |x-x_0|$. Then under the above conditions, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\begin{aligned}
LV(x) &= \int_E (V(x+c(z,x)) - V(x))\gamma(z,x)h(z)\,\mathrm{d}z \\
&= \int_E \langle c(z,x)+2(x-x_0), c(z,x)\rangle \gamma(z,x)h(z)\,\mathrm{d}z \\
&\leq -\zeta \int_{\mathcal{K}} |c(z,x)+2(x-x_0)||c(z,x)|\gamma(z,x)h(z)\,\mathrm{d}z \\
&\leq -\zeta|x-x_0| \int_{\mathcal{K}} |c(z,x)|\gamma(z,x)h(z)\,\mathrm{d}z \\
&\leq -\zeta|x-x_0|\xi = -A(V(x))^{\frac{1}{2}}
\end{aligned}$$

with $A = \zeta\xi > 0$.

Remark 3.4.2 *1. Using the Cauchy-Schwarz inequality,* (3.4.38) *implies that for all $z \in \mathbb{R}^d$ and for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, $|c(z,x)| \leq 2|x-x_0|$. In particular for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, $\sup_{z\in\mathbb{R}^d} |c(z,x)| < +\infty$.*
2. The condition (3.4.41) *is a natural condition to force the diffusion to enter into the set $B\big(x_0, \frac{\eta}{2}\big)$.*
3. There is a simple geometric interpretation of the conditions (3.4.40) *and* (3.4.41)*. Indeed, they lead to the (effective) condition*

$$\langle c(z,x)+2(x-x_0), c(z,x)\rangle \leq -\zeta|x-x_0||c(z,x)|$$

or

$$2\langle (x-x_0), c(z,x)\rangle + |c(z,x)|^2 \leq -\zeta|x-x_0||c(z,x)|.$$

On the one hand, this implies that $\langle (x-x_0), c(z,x)\rangle \leq -\frac{\zeta}{2}|x-x_0||c(z,x)|$, which means that $c(z,x)$ belongs to the convex cone of direction $(x-x_0)$ and angle $\arccos\big(-\frac{\zeta}{2}\big)$*. On the other hand, using* (3.4.41)*, the following condition*

$$2\langle (x-x_0), c(z,x)\rangle + |c(z,x)||x-x_0| \leq -\zeta|x-x_0||c(z,x)|$$

is a sufficient (but not necessary!) condition which leads to $\langle (x-x_0), c(z,x)\rangle \leq -\frac{(1+\zeta)}{2}|x-x_0||c(z,x)|$. In other words, it suffices that $c(z,x)$ belongs to the convex cone of direction $(x-x_0)$ and angle $\arccos\big(-\frac{(1+\zeta)}{2}\big)$*.*

The above conditions on the jump mechanism are naturally quite restrictive since they ensure that from everywhere in $\mathbb{R}^d \setminus C'$, the jumps force the process into the set C'. Nevertheless, this example is useful, and we will come back to these arguments later when discussing the influence of the drift coefficient.

In the next step, let us suppose that $\sigma \neq 0$. Then under the above conditions, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\begin{aligned}
LV(x) &= \mathrm{Tr}(\sigma(x)\sigma^*(x)) + \int_E \langle c(z,x)+2(x-x_0), c(z,x)\rangle\gamma(z,x)h(z)\,\mathrm{d}z \\
&\leq \mathrm{Tr}(\sigma(x)\sigma^*(x)) - \zeta|x-x_0|\xi.
\end{aligned}$$

Let $\Sigma = \sup_{x\in B\left(x_0,\frac{\eta}{2}\right)^c} \frac{|\mathrm{Tr}(\sigma(x)\sigma^*(x))|}{|x-x_0|}$ and suppose that $\mathcal{K}$ is such[4] that $\zeta\xi > \Sigma$. Then

$$LV(x) \leq -A(V(x))^{\frac{1}{2}}$$

with $A = \zeta\xi - \Sigma$.

Finally, if $g \neq 0$, the minimal additional condition $\langle x - x_0, g(x)\rangle < 0$, for all $x \in B\big(x_0, \frac{\eta}{2}\big)^c$, ensures that the above result will remain true.

[4] Actually it is always possible to multiply $\gamma(z,x)$ with a sufficiently large constant ensuring that $\xi > \Sigma/\zeta$.

Pushing with both jumps and drift part

The conditions we made on the jump mechanism in the above paragraph are of course very strong. In this paragraph, we will therefore consider that these conditions hold only for x belonging to some set E_1. Moreover, we will suppose that the drift coefficient contributes to force the diffusion into C' when x belongs to another set E_2.

More precisely, we suppose that $E_1 \subset B\big(x_0, \frac{\eta}{2}\big)^c$ and put $E_2 = B\big(x_0, \frac{\eta}{2}\big)^c \setminus E_1$. We will impose the global condition (3.4.38) but aim to weaken the conditions (3.4.39), (3.4.40) and (3.4.41) by replacing $x \in B\big(x_0, \frac{\eta}{2}\big)^c$ by $x \in E_1$. For $x \in E_2$, we assume additionally that

$$\mathrm{Tr}(\sigma\sigma^*) + 2\langle g(x), x - x_0\rangle \le -A|x - x_0|.$$

Such a condition is true for example if

$$g(x) = -\frac{1}{2}(A + \Sigma)\frac{x - x_0}{|x - x_0|},$$

where we recall that $\Sigma = \sup_{x \in B\left(x_0, \frac{\eta}{2}\right)^c} \frac{|\mathrm{Tr}(\sigma(x)\sigma^*(x))|}{|x - x_0|}$.

Example 3.4.3 *We continue Example 3.2.6 item 1. and consider the one-dimensional case with* $\mu(\mathrm{d}z) = \mathrm{d}z$ *and* $c(z, x) = e^{-|z|}f(x)$. *We suppose that* $x_0 = 0$ *and let* $E_1 = [-M - \frac{\eta}{2}] \cup [\frac{\eta}{2} + M]$. *Moreover we choose* $\mathcal{K} = [a, a + 2R]$ *in such a way that* $\int_{\mathcal{K}} e^{-|z|}\,\mathrm{d}z = \frac{1}{2}$. *Finally we will suppose that for all* $(z, x) \in \mathcal{K} \times E_1$

$$\gamma(z, x) \ge \underline{\gamma} > 0 \qquad \textit{and} \qquad f(x) \ge \underline{f} > 0 \tag{3.4.42}$$

with

$$\underline{f} \cdot \underline{\gamma} > 2\frac{\Sigma}{\zeta}. \tag{3.4.43}$$

It is clear that (3.4.39) *is verified for all* $(z, x) \in \mathcal{K} \times E_1$, *and, moreover, that the jumps are strong enough to ensure the drift condition even in presence of the Brownian part.*

If we impose moreover that for all $x \in B\big(0, \frac{\eta}{2}\big)^c$, $|f(x)| \le |x|$, *then* (3.4.41) *is satisfied. Adding finally the condition that for all* $x \in B\big(0, \frac{\eta}{2}\big)^c$, $\mathrm{sgn}(f(x)) = -\mathrm{sgn}(x)$, (3.4.38) *is true as well and* (3.4.40) *follows with* $\zeta = 1$.

Pushing partially with both

In the last paragraph we supposed that on the subset E_2 where the drift is driving the process towards C', the jumps do not act in a contradictory way – this is actually ensured by the condition (3.4.38). Notice that it is a priori not possible to weaken this assumption on E_2. Indeed, without condition (3.4.38) we have the following structural problem: we cannot even be sure that

$$\int_E |c(z, x)|^2 \gamma(z, x) h(z)\,\mathrm{d}z < +\infty. \tag{3.4.44}$$

Moreover if we do not suppose a global condition as (3.4.38), it will be necessary to compensate the possible non-negative part $\int_{E\setminus\mathcal{K}} \langle c(z, x) + 2(x - x_0), c(z, x)\rangle \gamma(z, x) h(z)\,\mathrm{d}z$ due to jumps in order to obtain a suitable control for $LV(x)$.

Pushing only with the drift

If we decide to ensure the Lyapunov condition by means of the drift coefficient g only, in the same spirit as above, we could take $E_1 = \emptyset$, but would have to keep global conditions, like the condition (3.4.38), if we use the same Lyapunov function.

However, if we choose another Lyapunov function, the situation might be more favorable as we are going to explain now. Let for example $V(x) = |x|$ $(= \sqrt{x_1^2 + \cdots + x_d^2})$ for $x \in B\big(x_0, \frac{\eta}{2}\big)^c$. Then

$$\nabla V(x) = \frac{x}{|x|}, \qquad \frac{\partial^2}{\partial_i \partial_j} V(x) = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}.$$

Let D be such that

$$\int_E |c(z,x)|\gamma(z,x)h(z)\,\mathrm{d}z \le D$$

and $|a_{ij}| < D$, where $a = \sigma\sigma^*$. With $\gamma > 0$ such that $\gamma|x|_1 \le |x|$ (where $|x|_1 = |x_1| + \cdots + |x_d|$) and $\tilde{D} \overset{\text{def}}{=} \frac{D}{2}\big(d + \frac{1}{\gamma^2}\big)$ we assume that g verifies, for every $x \in B\big(x_0, \frac{\eta}{2}\big)^c$,

$$\langle x, g(x)\rangle \le -A|x|^2 - D|x| - \tilde{D}. \tag{3.4.45}$$

Then it is immediately clear to see that

$$\begin{aligned} LV(x) &\le \frac{D}{2}\Big(\frac{d}{|x|} + \sum_{1\le i,j\le d} \frac{|x_i||x_j|}{|x|^3}\Big) + \frac{\langle x, g(x)\rangle}{|x|} + \underbrace{\int_E |c(z,x)|\gamma(z,x)h(z)\,\mathrm{d}z}_{\le D} \\ &= \frac{D}{2}\Big(\frac{d}{|x|} + \frac{|x|_1^2}{|x|^3}\Big) + \frac{\langle x, g(x)\rangle}{|x|} + D \\ &\le \frac{\tilde{D}}{|x|} + \frac{\langle x, g(x)\rangle}{|x|} + D \\ &\le \frac{\tilde{D}}{|x|} - \frac{1}{|x|}(A|x|^2 + D|x| + \tilde{D}) + D \le -A|x|. \end{aligned}$$

3.5 Proofs

3.5.1 Proof of Proposition 3.2.3

Proof : We first admit Lemma 3.2.4 and we put $\mathcal{K} = \overline{B(z_0, R)}$. As a consequence, there exists a ball $B(x_0, \eta)$ such that for all $x \in B(x_0, \eta)$, $B(a_{x_0}, \frac{\rho}{2}) \subset \Psi_x(\mathcal{K})$. Choose $\mathcal{K}'' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}'' \to B(a_{x_0}, \frac{\rho}{2})$ is a $\mathcal{C}^1$-diffeomorphism for all $x \in B(x_0, \eta)$.[5] Since for all $(z, x) \in \mathcal{K} \times B(x_0, \eta)$, $\gamma(z,x)h(z) \ge \varepsilon$, we now have

$$\begin{aligned} \int_{E_n} \mathbb{1}_V(\psi_x(z))\gamma(z,x)\,\mathrm{d}\mu(z) &\ge \varepsilon \int_{K''} \mathbb{1}_V(\psi_x(z))\,\mathrm{d}z \\ &= \varepsilon \int_{B(a_{x_0}, \frac{\rho}{2})} \mathbb{1}_V(y)|J_{\psi_x^{-1}}(y)|\,\mathrm{d}y. \end{aligned}$$

Put $z = \psi_x^{-1}(y)$, then

$$|J_{\psi_x^{-1}}(y)| = \frac{1}{|J_{\psi_x}(z)|)} = \frac{1}{|\nabla_z c(z,x)|}$$

and, using Hadamar's Inequality,

$$|\nabla_z c(z,x)| \le \prod_{i=1}^d |\partial_{z_i} c(z,x)|.$$

[5]Indeed, from Lemma 3.2.4, there exists $\mathcal{K}' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}' \to B(a_x, \rho)$ is a $\mathcal{C}^1$-diffeomorphism, and since $B(a_{x_0}, \frac{\rho}{2}) \subset B(a_x, \rho)$, there exists $\mathcal{K}'' \subset \mathcal{K}' \subset \mathcal{K}$ such that $\Psi_x : \mathcal{K}'' \to B(a_{x_0}, \frac{\rho}{2})$ is a $\mathcal{C}^1$-diffeomorphism.

As a consequence, we obtain

$$\int_{E_n} \mathbb{1}_V(\psi_x(z))\gamma(z,x)\,\mathrm{d}\mu(z) \geq \frac{\varepsilon}{S^d}\lambda\Big(V \cap B\big(a_{x_0}, \frac{\rho}{2}\big)\Big) \tag{3.5.46}$$

which, together with (3.2.5), ends the proof. •

It remains to give a proof of Lemma 3.2.4. This proof goes through several intermediate steps which are given now.

Lemma 3.5.1 *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{C}^2$-function such that*

1. $g(0) = 0$,
2. $dg_0 = \mathrm{Id}$,
3. *there exist $R, K > 0$ such that for all $z \in B(0,R)$,*

$$\sum_{i,j,k} \left| \frac{\partial^2 g_k}{\partial z_i \partial z_j}(z) \right| \leq K$$

.

Put $\tilde{R} = R \wedge \frac{1}{2K}$. Then $B\Big(0, \frac{\tilde{R}}{2}\Big) \subset g\big(B(0,\tilde{R})\big)$.

Proof : The third condition allows to apply the Mean Value Inequality to $z \mapsto dg_z$ since

$$\|d(dg)_z\| \leq K, \qquad \forall z \in B(0,R).$$

Therefore, with $\tilde{R} = R \wedge \frac{1}{2K}$,

$$\|dg_z - \mathrm{Id}\| = \|dg_z - dg_0\| \leq K|z| \leq \frac{1}{2}, \qquad \forall z \in B(0,\tilde{R}).$$

Let now $y \in B\Big(0, \frac{\tilde{R}}{2}\Big)$ and set $h : \overline{B(0,\tilde{R})} \to \mathbb{R}^d$, $z \mapsto h(z) := y + z - g(z)$. We have

$$\|dh_z\| = \|\mathrm{Id} - dg_z\| \leq \frac{1}{2}, \qquad \forall z \in B(0,\tilde{R}).$$

Using again the Mean Value Inequality, we obtain for all $z, z' \in \overline{B(0,\tilde{R})}$,

$$|h(z) - h(z')| \leq \frac{1}{2}|z - z'|.$$

In particular $|h(z)| \leq \frac{1}{2}|z - z'| + |h(z')|$, so $|h(z)| \leq \frac{1}{2}|z| + |h(0)| = \frac{1}{2}|z| + |y| < \tilde{R}$, for all $z \in \overline{B(0,\tilde{R})}$.

This last result highlights two facts. First, h is an $\frac{1}{2}$-contraction from the complete space $\overline{B(0,\tilde{R})}$ into itself, so the fixed-point theorem gives us the existence of $z \in \overline{B(0,\tilde{R})}$ such that $h(z) = z$, and, secondly, the range of h defined on $\overline{B(0,\tilde{R})}$ is $B(0,\tilde{R})$, so we have in fact the existence of $z \in B(0,\tilde{R})$ such that $h(z) = z$, or equivalently, $g(z) = y$, which ends the proof. •

Remark 3.5.2 *1. g is in fact a $\mathcal{C}^1$-diffeomorphism from $V = B(0,\tilde{R}) \cap g^{-1}\Big(B\Big(0, \frac{\tilde{R}}{2}\Big)\Big)$ to $B\Big(0, \frac{\tilde{R}}{2}\Big)$.*
2. We could have taken, of course, $\tilde{R} = R \wedge \frac{1-\varepsilon'}{K}$ for any $\varepsilon' \in]0,1[$.

Lemma 3.5.3 *Let A be a $d \times d$ matrix such that*

$$\forall h \in \mathbb{R}^d, \qquad |Ah| \geq K|h|.$$

Then

$$B(Au, K\tilde{R}) \subset A(B(u, \tilde{R})).$$

Proof : Notice first that A is clearly invertible. Let now $y \in B(Au, K\tilde{R})$. Then for $v \in \mathbb{R}^d$,

$$|v| = |A(A^{-1}v)| \geq K|A^{-1}v|,$$

so, with $v = y - Au$,

$$K\tilde{R} \geq |y - Au| \geq K|A^{-1}(y - Au)| = K|A^{-1}y - u|,$$

or, equivalently, $\tilde{R} \geq |A^{-1}y - u|$ implying that $A^{-1}y \in B(u, \tilde{R})$ and $y \in A(B(u, \tilde{R}))$. •

We now have the following extension of Lemma 3.5.1.

Proposition 3.5.4 *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ a $\mathcal{C}^2$-function and $a \in \mathbb{R}^d$ such that*

1. *$|df_a h| \geq A|h|$ for all $h \in \mathbb{R}^d$,*
2. *there exist $R, K > 0$ such that for all $y \in B(a, R)$,*

$$|df_a^{-1}| \sum_{i,j} \left| \frac{\partial^2 f}{\partial z_i \partial z_j}(y) \right| \leq \frac{K}{d}.$$

Then, with $\tilde{R} = R \wedge \frac{1}{2K}$,

$$B\Big(f(a), A\frac{\tilde{R}}{2}\Big) \subset f\big(B(a, \tilde{R})\big).$$

Proof : **1)** We use Lemma 3.5.1 with

$$g(z) = df_a^{-1}(f(a+z) - f(a)).$$

All hypotheses needed in Lemma 3.5.1 are satisfied since

$$\frac{\partial^2 g}{\partial z_i \partial z_j}(z) = df_a^{-1} \frac{\partial^2 f}{\partial z_i \partial z_j}(a+z).$$

Thus

$$B\Big(0, \frac{\tilde{R}}{2}\Big) \subset g\big(B(0, \tilde{R})\big).$$

2) Since $f(y) = df_a g(y-a) + f(a)$, using Lemma 3.5.3,

$$B\Big(0, A\frac{\tilde{R}}{2}\Big) \subset df_a\Big(B\Big(0, \frac{\tilde{R}}{2}\Big)\Big) \subset df_a g\big(B(0, \tilde{R})\big),$$

where we have used the preceding step in order to obtain the last inclusion. Therefore,

$$B\Big(f(a), A\frac{\tilde{R}}{2}\Big) \subset f\big(B(a, \tilde{R})\big).$$

•

We are now able to prove Lemma 3.2.4.

Proof : [of Lemma 3.2.4] **1)** Let $x \in \overline{B(x_0, r)}$. We can apply Proposition 3.5.4 with $a = z_0$, $f = \Psi_x$ which gives $\rho = \frac{A}{2}\Big(R \wedge \frac{1}{2K}\Big)$ such that

$$B(a_x, \rho) \subset \Psi_x\Big(B(z_0, \frac{2\rho}{A})\Big) \subset \Psi_x(\mathcal{K}),$$

where we recall that $\mathcal{K} = \overline{B(z_0, R)}$. Since our conditions are uniform in x, the radius ρ will be the same for all $x \in \overline{B(x_0, r)}$.

2) The previous point implies in particular that

$$B(a_{x_0}, \rho) \subset \Psi_{x_0}(\mathcal{K}).$$

Since $x \mapsto \Psi_x(z_0)$ is continuous, there exists η with $r > \eta > 0$ such that

$$|x - x_0| < \eta \quad \Longrightarrow \quad |\Psi_x(z_0) - \Psi_{x_0}(z_0)| < \frac{\rho}{2}. \tag{3.5.47}$$

Therefore,

$$\bigcap_{y \in B(x_0, \eta)} B(a_y, \rho) \subset \Psi_x(\mathcal{K}),$$

so it is sufficient to prove that

$$B(a_{x_0}, \frac{\rho}{2}) \subset \bigcap_{y \in B(x_0, \eta)} B(a_y, \rho)$$

which can be seen as follows. Let $y \in B(a_{x_0}, \frac{\rho}{2})$ and $x \in B(x_0, \eta)$, then

$$\begin{aligned} |a_x - y| &\leq |a_{x_0} - y| + |a_x - a_{x_0}| \\ &= |a_{x_0} - y| + |\Psi_x(z_0) - \Psi_{x_0}(z_0)| \\ &< \frac{\rho}{2} + \frac{\rho}{2} = \rho, \end{aligned}$$

so $y \in B(a_x, \rho)$, for every $x \in B(x_0, \eta)$ and the statement is proved. •

Proof : [of Remark 3.2.5] Recall that we have imposed the additional hypothesis $L_c = \sup_{z \in \mathcal{K}} L_c(z) < \infty$. Since

$$|c(z, x) - c(z, y)| \leq L_c(z)|x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall z \in E,$$

it is sufficient to set

$$\eta = \frac{\rho}{2(1 + L_c)} \wedge r,$$

in order to grant (3.5.47). •

Part II

Regularity of the density

Chapter 4

Differential calculus and Integration by part

4.1 Introduction

In the next chapters of this work we will study the regularity of the density of the following stochastic equation

$$\begin{aligned}X_t =& x + \sum_{l=1}^{m} \int_0^t \sigma_l(X_s)\,\mathrm{d}W_s^l + \int_0^t b(X_s)\,\mathrm{d}s \\ &+ \int_0^t \int_{E\times\mathbb{R}_+} c(X_{s-}, z)\mathbb{1}_{\{u\leq\gamma(X_{s-},z\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\end{aligned}$$

of which we have already studied some properties in the first part of this work.

The way to do so used here is based on a two step strategy. First, we construct an approximation (F_M) of the process X_t (basically given a non-decreasing sequence of subsets $\left(B_M\right)_{M\in\mathbb{N}^*}$, with $\mu(B_M) < \infty$, recovering E, the approximation F_M will be constructed (for each M) from a restriction of the processes X_t based on the restriction of the random measure N on the subset B_M) verifying an integration by part formula :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M)H_M\right].$$

This integration by part is obtained within a general framework developed in [8] by V. Bally and E. Clément, whose main results used in the sequel are presented in this chapter.

The second step consists in proving the density regularity itself. The idea is to use a certain balance between the error $\mathrm{E}\left[|F_M - X_t|\right]$ (which tends to 0) and the weight $\mathrm{E}\left[|H_M|\right]$ (which tends to ∞). This was the strategy used in [8] as well. But here the estimates of $\mathrm{E}\left[|H_M|\right]$ will appear to be more delicate than the corresponding one in [8] because of the additional Brownian part $\sigma\,\mathrm{d}W$. Moreover, the balance used in [8] was based on a Fourier transform method while here we use the new method developed by V. Bally and L. Caramellino in [7].

This new method allowed us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [8] ; the part of [7] used for our purpose is presented, in this chapter, Section 4.6.

4.2 Notations, tools of differential calculus

4.2.1 Notations and differentials operators

We consider a sequence $(V_i)_{i\in\mathbb{N}^*}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, a σ-algebra $\mathcal{G} \subset \mathcal{F}$ and a $\mathcal{G}$-measurable random variable J, with values in $\mathbb{N}$. We assume that the variables

(V_i) and J satisfy the following integrability condition :

$$\forall p \geq 1, \qquad \mathrm{E}\left[J^p\right] + \mathrm{E}\left[\left(\sum_{i=1}^{J} V_i^2\right)^p\right] < \infty.$$

Following Bally and Clément, we will define a differential calculus based on the variables (V_i), conditionally on $\mathcal{G}$.

First we will define the following set

Definition 4.1 *Let $\mathcal{M}$ be the class of functions $f : \Omega \times \mathbb{R}^{\mathbb{N}^*} \to \mathbb{R}$ such that :*

- *f can be written as*

$$f(\omega, v) = \sum_{j=1}^{\infty} f^j(\omega, v_1, \ldots, v_j) \mathbb{1}_{\{J(\omega)=j\}}$$

where $f^j : \Omega \times \mathbb{R}^j \to \mathbb{R}$ are $\mathcal{G} \times \mathcal{B}(\mathbb{R}^j)$-measurable functions ;

- *there exists a random variable $C \in \bigcap_{q \geq 1} \mathrm{L}^q(\Omega, \mathcal{F}, \mathrm{P})$ and $p \in \mathbb{N}^*$ such that*

$$|f(\omega, v)| \leq C(\omega)\Big(1 + \Big(\sum_{i=1}^{J(\omega)} v_i^2\Big)^p\Big)$$

(in other words, conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ have polynomial growth with respect to the variables (v_i)).

We will define also

- $\mathcal{G}_i$ the σ-algebra generated by $\mathcal{G} \cup \sigma(V_j, 1 \leq j \leq J, j \neq i)$,
- $(a_i(\omega))$ and $(b_i(\omega))$ two sequences of $\mathcal{G}_i$-measurable random variables satisfying

$$-\infty \leq a_i(\omega) < b_i(\omega) \leq +\infty, \qquad \forall i \in \mathbb{N}^*$$

- O_i the open set of $\mathbb{R}^{\mathbb{N}^*}$ defined by $O_i = P_i^{-1}(]a_i, b_i[)$, where P_i is the coordinate map $\mathbb{R}^{\mathbb{N}^*}$ (*ie.* $P_i(v) = v_i$).

We localize the differential calculus on the sets (O_i) by introducing some weights (π_i), satisfying the following hypothesis.

Hypothesis 4.1 *For all $n \in \mathbb{N}^*$, $\pi_i \in \mathcal{M}$ and*

$$\{\pi_i > 0\} \subset O_i.$$

Moreover for all $j \geq 1$, π_i^j is infinitely differentiable with bounded derivatives with respect to the variables $(v_1, \ldots, v_j)$.

At last, we associate to these weights π_i, the spaces $C_\pi^k \subset \mathcal{M}$, $k \in \mathbb{N}^*$, defined recursively as follows.

- For $k = 1$, C_π^1 denotes the space of functions $f \in \mathcal{M}$ such that for each $i \in \mathbb{N}^*$, f admits a partial derivative with respect to the variable v_i on the open set O_i. We then define

$$\partial_i^\pi f(\omega, v) \stackrel{\text{def}}{=} \pi(\omega, v) \frac{\partial}{\partial v_i} f(\omega, v)$$

and we assume that $\partial_i^\pi f \in \mathcal{M}$.

- Suppose now that C_π^k is already defined. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^{*k}$, we define recursively $\partial_\alpha^\pi = \partial_{\alpha_1}^\pi \cdots \partial_{\alpha_k}^\pi$ and C_π^{k+1} is the space of functions $f \in C_\pi^k$ such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^{*k}$ we have $\partial_\alpha^\pi f \in C_\pi^1$. Notice that if $\partial_\alpha^\pi f \in \mathcal{M}$ for each α with $|\alpha| \leq k$.

- Finally we define

$$C_\pi^\infty \stackrel{\text{def}}{=} \bigcap_{k \in \mathbb{N}^*} C_\pi^k.$$

Definition 4.2 (Simple functionals) *A random variable F is called a simple functional if there exist $f \in C_\pi^\infty$ such that $F = f(\omega, V)$, where $V = (V_i)$. We denote by $\mathcal{S}$ the space of the simple functionals (it is an algebra) ; moreover, it is worth to notice that, conditionally on $\mathcal{G}$, $F = f^J(V_1, \ldots, V_J)$.*

Definition 4.3 (Simple processes.) *A simple process is a sequence of random variables $U = (U_i)_{i \in \mathbb{N}^*}$ such that for each $i \in \mathbb{N}^*$, $U_i \in \mathcal{S}$. Consequently, conditionally on $\mathcal{G}$, we have $U_i = u_i^J(V_1, \ldots, V_J)$. We denote by $\mathcal{P}$ the space of the simple processes and we define the scalar product*

$$\langle U, V \rangle_J = \sum_{i=1}^J U_i V_i \qquad (\in \mathcal{S}).$$

We can now define the derivative operator and state the integration by parts formula.

Definition 4.4 (The derivative operator.) *We define $\mathrm{D} : \mathcal{S} \to \mathcal{P}$ by*

$$\mathrm{D}\, F \stackrel{def}{=} (\mathrm{D}_i\, F) \in \mathcal{P} \quad \textit{where} \quad \mathrm{D}_i\, F \stackrel{def}{=} \partial_i^\pi f(\omega, v).$$

Notice that $\mathrm{D}_i\, F = 0$, for $i > J$.

Definition 4.5 (Malliavin covariance matrix) *For $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, the Malliavin covariance matrix is defined by*

$$\sigma^{k,k'}(F) = \langle \mathrm{D}\, F^k, \mathrm{D}\, F^{k'} \rangle_J = \sum_{i=1}^J \mathrm{D}_i\, F^k\, \mathrm{D}_i\, F^{k'}$$

We denote

$$\Lambda(F) = \{\det \sigma(F) \neq 0\} \qquad \text{and} \qquad \gamma(F)(\omega) = \sigma^{-1}(F)(\omega), \quad \omega \in \Lambda(F)$$

In order to derive an integration by parts formula, we need some additional assumptions on the random variables (V_i). The main hypothesis is that conditionally on $\mathcal{G}$, the law of the vector $(V_1, \ldots, V_J)$, admits a locally smooth density with respect to the Lebesgue measure on $\mathbb{R}^J$.

Hypothesis 4.2 *1. Conditionally on $\mathcal{G}$, the vector $(V_1, \ldots, V_J)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^J$ and we denote by p_J the conditional density.*

2. The set $\{p_J > 0\}$ is open in $\mathbb{R}^J$ and on $\{p_J > 0\}$, $\ln p_J \in C_\pi^\infty$.

3. For all $q > 1$, there exists a constant C_q such that

$$(1 + |v|^q) p_J \leq C_q$$

where $|v|$ stands for the euclidean norm of the vector $(v_1, \ldots, v_J)$.

Assumption 3) implies in particular that conditionally on $\mathcal{G}$, the functions of $\mathcal{M}$ are integrable with respect to p_J and that for $f \in \mathcal{M}$:

$$\mathrm{E}_{\mathcal{G}}[f(\omega, V)] = \int_{\mathbb{R}^J} f^J \times p_J(\omega, v_1, \ldots, v_J)\,\mathrm{d}v_1, \ldots, \mathrm{d}v_J.$$

Definition 4.6 (The divergence operator) *Let $U = (U_i)_{i\in\mathbb{N}^*} \in \mathcal{P}$ with $U \in \mathcal{S}$. We define $\delta : \mathcal{P} \to \mathcal{S}$ by*

$$\delta_i(U) \stackrel{def}{=} -(\partial_{v_i}(\pi_i U_i) + U_i \mathbb{1}_{\{p_J>0\}} \partial_i^\pi \ln p_J) \tag{4.2.1}$$

$$\delta(U) = \sum_{i=1}^{J} \delta_i(U) \tag{4.2.2}$$

For $F \in \mathcal{S}$, we then define

$$\mathrm{L}(F) \stackrel{\text{def}}{=} \delta(\mathrm{D}\,F) \tag{4.2.3}$$

4.3 Duality and integration by parts formulae

4.4 IPP

The duality between δ and D is given by the following proposition.

Proposition 4.4.1 *Assuming the two preceding hypothesis, then for all $F \in \mathcal{S}$ and for all $U \in \mathcal{P}$ we have*

$$\mathrm{E}_{\mathcal{G}}[\langle \mathrm{D}\,F, U\rangle_J] = \mathrm{E}_{\mathcal{G}}[F\delta(U)].$$

Lemma 4.4.2 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$. Then $\phi(F) \in \mathcal{S}$ and*

$$\mathrm{D}\,\phi(F) = \sum_{r=1}^{d} \partial_r \phi(F)\,\mathrm{D}\,F^r. \tag{4.4.4}$$

If $F \in \mathcal{S}$ and $U \in \mathcal{P}$, then

$$\delta(FU) = F\delta(U) - \langle \mathrm{D}\,F, U\rangle_J.$$

Moreover, for $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, we have

$$\mathrm{L}\phi(F) = \sum_{r=1}^{d} \partial_r \phi(F) \mathrm{L}F^r - \sum_{r,r'=1}^{d} \partial_{r,r'}\phi(F)\langle \mathrm{D}\,F^r, \mathrm{D}\,F^{r'}\rangle_J.$$

We can now state the main results of this section.

Theorem 4.4.3 *Assuming the two preceding hypothesis, let $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, $G \in \mathcal{S}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth bounded function with bounded derivatives. Let $\Lambda \in \mathcal{G}$, $\Lambda \subset \Lambda(F)$ such that*

$$\mathrm{E}\left[|\det \gamma(F)|^p \mathbb{1}_\Lambda\right] < \infty, \qquad \forall p \geq 1.$$

Then,

1. *for every $r = 1, \ldots, d$,*

$$\mathrm{E}_{\mathcal{G}}[\partial_r \phi(F) G]\mathbb{1}_\Lambda = \mathrm{E}_{\mathcal{G}}[\phi(F) H_r(F, G)]\mathbb{1}_\Lambda$$

with

$$H_r(F,G) = \sum_{r'=1}^{d} \delta(G\gamma^{r',r}(F)\,\mathrm{D}\,F^{r'}) = \sum_{r'=1}^{d} \left(G\delta(\gamma^{r',r}(F)\,\mathrm{D}\,F^{r'}) - \gamma^{r',r}\langle \mathrm{D}\,F^{r'}, \mathrm{D}\,G\rangle_J\right); \tag{4.4.5}$$

2. for every multi-index $\beta = (\beta_1, \dots, \beta_q) \in \{1, \dots, d\}^q$

$$\mathrm{E}_{\mathcal{G}}[\partial_\beta \phi(F) G] \mathbb{1}_\Lambda = \mathrm{E}_{\mathcal{G}}[\phi(F) H^q_\beta(F, G)] \mathbb{1}_\Lambda \tag{4.4.6}$$

where the weights H^q are defined recursively by (4.4.5) *and*

$$H^q_\beta(F, G) = H_{\beta_1}\Big(F, H^{q-1}_{(\beta_2,\dots,\beta_q)}(F, G)\Big). \tag{4.4.7}$$

4.5 Estimations of H^q

In order to estimate the weights H^q appearing in the integration by parts formulae of the previous section, we first need to define iterations of the derivative operator. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a multi-index, with $\alpha_i \in \{1, \dots, J\}$, for $i = 1, \dots, k$ and $|\alpha| = k$.

For $F \in \mathcal{S}$ we define recursively

$$\mathrm{D}^k_{(\alpha_1,\dots,\alpha_k)} F \stackrel{\text{def}}{=} \mathrm{D}_{\alpha_k}\left(\mathrm{D}^{k-1}_{(\alpha_1,\dots,\alpha_{k-1})} F\right) \qquad \text{and} \qquad \mathrm{D}^k F \stackrel{\text{def}}{=} \Big(\mathrm{D}^k_{(\alpha_1,\dots,\alpha_k)} F\Big)_{\alpha_i \in \{1,\dots,J\}}.$$

Notice that $\mathrm{D}^k F \in \mathbb{R}^{J \otimes k}$, and consequently we define the norm of $\mathrm{D}^k F$ as

$$\left|\mathrm{D}^k F\right| \stackrel{\text{def}}{=} \sqrt{\sum_{\alpha_1,\dots,\alpha_k=1}^{J} \left|\mathrm{D}^k_{(\alpha_1,\dots,\alpha_k)} F\right|^2}. \tag{4.5.8}$$

Moreover we introduce the following norms, for $F \in \mathcal{S}$:

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{k=1}^{l} \left|\mathrm{D}^k F\right| \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} |F| + |F|_{1,l} = \sum_{k=0}^{l} \left|\mathrm{D}^k F\right|. \tag{4.5.9}$$

For $F = (F_1, \dots, F_d) \in \mathcal{S}^d$:

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{r=1}^{d} \left|F^r\right|_{1,l} \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} \sum_{r=1}^{d} \left|F^r\right|_l,$$

and, similarly, for $F = \left(F^{r,r'}\right)_{r,r'=1,\dots,d}$

$$|F|_{1,l} \stackrel{\text{def}}{=} \sum_{r,r'=1}^{d} \left|F^{r,r'}\right|_{1,l} \qquad \text{and} \qquad |F|_l \stackrel{\text{def}}{=} \sum_{r,r'=1}^{d} \left|F^{r,r'}\right|_l.$$

Notation 4.5.1
- *In the sequel, we will generally denote simply D^k_α by D_α (where α is a multi-index of length k).*
- *We will also use the following generalisation for $F \in \mathcal{S}^d$ and $G \in \mathcal{S}^{d \times k}$: we will simply set*

$$\mathrm{D}_\alpha F \stackrel{\text{def}}{=} \Big(\mathrm{D}_\alpha F_i\Big)_{1 \le i \le d} \qquad \textit{and} \qquad \mathrm{D}^k G \stackrel{\text{def}}{=} \Big(\mathrm{D}^k G_{i,j}\Big)_{\substack{1 \le i \le d \\ 1 \le j \le k}}.$$

4.5.1 Differentiability lemmas

In this section we will use directly the notations from Chapter 5 defined in 5.5.1 and 5.5.2, where we will apply the previous general differential framework.

In order to express the form of the different multi-derivatives we will use in the next chapter, let us set the following notations :

- if $F \in \mathcal{S}^d$, we will denote the n-th derivative

$$\mathrm{D}_{(k_n,r_n)}\Big(\mathrm{D}_{(k_{n-1},r_{n-1})}\Big(\cdots\Big(\mathrm{D}_{(k_1,r_1)}(F)\Big)\Big)$$

by

$$\mathrm{D}_\alpha(F)$$

with $\alpha = (\alpha_1, \dots, \alpha_n)$ and, for all $i \in \{1, \dots, n\}$, $\alpha_i \stackrel{\text{def}}{=} (k_i, r_i)$;

- for $1 \le l \le n$, we denote by

$$\mathcal{M}_n(l) = \Big\{M = (M_1, \dots, M_l),\ \bigcup_{i \in [\![1,l]\!]} M_i = \{1, \dots, n\} \text{ and } M_i \cap M_j = \varnothing, \text{ for } i \neq j\Big\} ;$$

the set of the partitions of length l of $\{1, \dots, n\}$.

Remark 4.5.2 *The multi-derivatives defined above are not commutative : in general*

$$\mathrm{D}_{(k,r)}\Big(\mathrm{D}_{(m,n)}(F)\Big) \neq \mathrm{D}_{(m,n)}\Big(\mathrm{D}_{(k,r)}(F)\Big).$$

We can now state :

Lemma 4.5.3 *Let $A, B \in \mathcal{S}$, $\phi : \mathbb{R}^d \to \mathbb{R}$ and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be smooth functions and $F = (F^1, \dots, F^d)$, $G = (G^1, \dots, G^d) \in \mathcal{S}^d$. Then*

1. *for every $(k, r) \in [\![1, J]\!] \times [\![1, d]\!]$,*

$$\mathrm{D}_{k,r}(AB) = \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B) \,; \tag{4.5.10}$$

and for every $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \stackrel{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$,

$$\mathrm{D}_\alpha(AB) = \sum_{\substack{\alpha_i \oplus \alpha_j = \alpha \\ \alpha_i, \alpha_j \text{ ordered}}} \mathrm{D}_{\alpha_i} A\, \mathrm{D}_{\alpha_j} B \,; \tag{4.5.11}$$

(by "ordered" we mean that if $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_k})$, then $i_1 < \cdots < i_k$)

2. *for every $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \stackrel{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$,*

$$\begin{aligned}\mathrm{D}_\alpha\,\phi(F) &= \sum_{l=1}^{n} \sum_{\substack{\beta = (\beta_1, \dots, \beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l} & (4.5.12)\\ &= T_\alpha\big(\phi\big)(F) + \nabla\phi(F)\, \mathrm{D}_\alpha F, & (4.5.13)\end{aligned}$$

where

- *for $M = (M_1, \dots, M_l) \in \mathcal{M}_n(l)$, if $M_j = (i_1, \dots, i_r) \subseteq \{1, \dots, n\}$,*

$$M_j(\alpha) \stackrel{def}{=} (\alpha_{i_1}, \dots, \alpha_{i_r}),$$

- *and*

$$T_\alpha\big(\phi\big)(F) \stackrel{def}{=} \sum_{l=2}^{n} \sum_{\substack{\beta = (\beta_1, \dots, \beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F)\, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l} \tag{4.5.14}$$

3. *for every* $\alpha = (\alpha_1, \ldots, \alpha_n)$, *with* $\alpha_i \stackrel{def}{=} (k_i, r_i) \in [\![1, J]\!] \times [\![1, d]\!]$, *and using the same notations,*

$$\mathrm{D}_\alpha c(F,G) = \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_r,\beta'_{r+1},\ldots,\beta'_l) \\ \beta_i \in [\![1,d]\!],\ \beta'_j \in [\![d+1,2d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta c(F,G) \, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_r(\alpha)} F_{\beta_r} \times \mathrm{D}_{M_{r+1}(\alpha)} G_{\beta'_{r+1}} \cdots \mathrm{D}_{M_l(\alpha)} G_{\beta'_l}$$

$$= U_\alpha(c)(F,G) + \nabla_f c(F,G) \, \mathrm{D}_\alpha F + \nabla_g c(F,G) \, \mathrm{D}_\alpha G.$$

Remark 4.5.4 *The non-symmetric form* (4.5.13) *is used in the sequel in recurrence's purpose : all the elements* $M_i(\alpha)$ *from* T_α *are such that* $|M_i(\alpha)| < \alpha$ *so the degree of derivation of* $\mathrm{D}_{M_i(\alpha)}$ *is strictly inferior to the one of* D_α *itself.*

With the same notations :

Lemma 4.5.5 *Let* $\phi : \mathbb{R}^d \to \mathbb{R}$ *a smooth function and* $F = (F^1, \ldots, F^d) \in \mathcal{S}^d$, α *a multi-index and* $n \stackrel{def}{=} |\alpha|$. *Then there exists* $C_{n,p,d} > 0$ *such that*

$$|\,\mathrm{D}_\alpha \phi(F)|^{2p} \leq C_{n,p,d} \big(|\phi|_n(F)\big)^{2p} \sum_{l=0}^{n} \sum_{M \in \mathcal{M}_n(l)} \Big(|\,\mathrm{D}_{M_1(\alpha)} F|^{2pd} + \cdots + |\,\mathrm{D}_{M_l(\alpha)} F|^{2pd} \Big) \tag{4.5.15}$$

with $|\phi|_n(F) \stackrel{def}{=} \sup_{|\beta| \leq n} |\partial_\beta \phi(F)|$.

Proof :

From Lemma 4.5.3 we have

$$\mathrm{D}_\alpha \phi(F) = \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F) \, \mathrm{D}_{M_1(\alpha)} F_{\beta_1} \cdots \mathrm{D}_{M_l(\alpha)} F_{\beta_l}.$$

It follows

$$|\,\mathrm{D}_\alpha \phi(F)| \leq C_n |\phi|_n(F) \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} |\,\mathrm{D}_{M_1(\alpha)} F_{\beta_1}| \cdots |\,\mathrm{D}_{M_d(\alpha)} F_{\beta_l}|$$

$$|\,\mathrm{D}_\alpha \phi(F)|^{2p} \leq C_{n,p} \big(|\phi|_n(F)\big)^{2p} \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} |\,\mathrm{D}_{M_1(\alpha)} F_{\beta_1}|^{2p} \cdots |\,\mathrm{D}_{M_d(\alpha)} F_{\beta_l}|^{2p}$$

Now[1]

$$|\,\mathrm{D}_{M_1(\alpha)} F|^{2p} \cdots |\,\mathrm{D}_{M_l(\alpha)} F|^{2p} \leq \frac{1}{d} \Big(|\,\mathrm{D}_{M_1(\alpha)} F|^{2pd} + \cdots + |\,\mathrm{D}_{M_d(\alpha)} F|^{2pd} \Big),$$

[1]Since, if $a_1, \ldots, a_n \in \mathbb{R}_+^*$, it is well-known that $\sqrt[n]{\prod_{i=1}^n a_i} \leq \frac{1}{n} \sum_{i=1}^n a_i$,

$$\prod_{i=1}^{n} a_i \leq \frac{1}{n^n} \Big(\sum_{i=1}^{n} a_i \Big)^n \leq \frac{1}{n^n} n^{n-1} \sum_{i=1}^{n} a_i^n,$$

so

$$\prod_{i=1}^{n} a_i \leq \frac{1}{n} \sum_{i=1}^{n} a_i^n.$$

so

$$|\,\mathrm{D}_\alpha\,\phi(F)|^{2p} \leq C_{n,p,d}\big(|\phi|_n(F)\big)^{2p}\sum_{l=1}^{n}\sum_{M\in\mathcal{M}_n(l)}\Big(|\,\mathrm{D}_{M_1(\alpha)}\,F|^{2pd}+\cdots+|\,\mathrm{D}_{M_d(\alpha)}\,F|^{2pd}\Big).$$

•

We will also need an extended version of the first item of Lemma 4.5.3 :

Lemma 4.5.6 *Let $A, B \in \mathcal{S}^{d\times d}$. Then for every $(k,r) \in [\![1,J]\!]\times[\![1,d]\!]$,*

$$\mathrm{D}_{k,r}(AB) = \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B)\,; \tag{4.5.16}$$

and for every $\alpha = (\alpha_1,\ldots,\alpha_n)$, with $\alpha_i \overset{def}{=} (k_i,r_i) \in [\![1,J]\!]\times[\![1,d]\!]$,

$$\mathrm{D}_\alpha(AB) = \sum_{\substack{\alpha_i\oplus\alpha_j=\alpha\\ \alpha_i,\alpha_j \text{ ordered}}} \mathrm{D}_{\alpha_i}\,A\,\mathrm{D}_{\alpha_j}\,B\,; \tag{4.5.17}$$

(by "ordered" we mean that if $\alpha_i = (\alpha_{i_1},\ldots,\alpha_{i_k})$, then $i_1 < \cdots < i_k$).

Proof : Let $A = (A_{i,j})_{1\leq i,j\leq d}$ and $B = (B_{i,j})_{1\leq i,j\leq d}$ with $A_{i,j}$, $B_{i,j} \in \mathcal{S}$. Then,

$$\begin{aligned}
\mathrm{D}_{k,r}(AB) &= \mathrm{D}_{k,r}\Big(\big(\sum_{m=1}^{d} A_{i,m}B_{m,j}\big)_{1\leq i,j\leq d}\Big)\\
&= \Big(\big(\sum_{m=1}^{d}\mathrm{D}_{k,r}\,(A_{i,m}B_{m,j})\big)_{1\leq i,j\leq d}\Big)\\
&= \Big(\big(\sum_{m=1}^{d}\mathrm{D}_{k,r}\,(A_{i,m})B_{m,j} + A_{i,m}\,\mathrm{D}_{k,r}\,(B_{m,j})\big)_{1\leq i,j\leq d}\Big) \qquad \text{(using (4.5.10))}\\
&= \mathrm{D}_{k,r}(A)B + A\,\mathrm{D}_{k,r}(B).
\end{aligned}$$

But the proof of (4.5.17) only requires an induction over the formal relation (4.5.16) (and does not need any commutativity in the product of A by B). •

Corollary 4.5.7 *Let $A = (A_{i,j})_{1\leq i,j\leq d}$, $B = (B_{i,j})_{1\leq i,j\leq d}$ with $A_{i,j}$, $B_{i,j} \in \mathcal{S}$ and $l \in \mathbb{N}^*$. Then there exists $C_l > 0$ such that*

$$|AB|_l \leq C_l|A|_l|B|_l. \tag{4.5.18}$$

We also have the following result proven in [8] (Lemma 8) :

Lemma 4.5.8 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and $F \in \mathcal{S}^d$ then for all $l \geq 1$ we have*

$$|\phi(F)|_{1,l} \leq |\nabla\phi(F)||F|_{1,l} + C_l \sup_{2\leq\beta\leq l}|\partial_\beta\phi(F)||F|_{1,l-1}^l.$$

Result that we will essentially use in this work through this corollary :

Corollary 4.5.9 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^\infty$ bounded function with bounded derivatives of any order and $F \in \mathcal{S}^d$ then for all $l \geq 1$ there exists $C_{\phi,l} > 0$ such that*

$$|\phi(F)|_l \leq C_{\phi,l}\big(1 + |F|_l + |F|_{l-1}^l\big). \tag{4.5.19}$$

4.5.2 Some bounds on H^q

The further theorem, proven in [8], gives some estimates for the weights H^q in terms of the derivatives of G, F, $\mathrm{L}F$ and $\gamma(F)$.

Theorem 4.5.10 *For $F \in \mathcal{S}^d$, $G \in \mathcal{S}$ and for all $q \in N^*$ there exists a universal constant $C_{q,d}$ such that for every multi-index $\beta = (\beta_1, \dots, \beta_q)$*

$$\left| H^q_\beta(F,G) \right| \leq \frac{C_{q,d}|G|_q(1+|F|_{q+1})^{(6d+1)q}}{|\det \sigma(F)|^{3q-1}} \left(1 + |\mathrm{L}F|^q_{q-1}\right).$$

Remark 4.5.11 *In the sequel, we will simply denote $H^q_\beta(F,1)$ by $H^q_\beta(F)$.*

4.6 Interpolation method : notations and theoretical result

All this section is directly taken from the article of Bally and Caramelino [7].

4.6.1 Notations and definitions

Let us define for ξ and γ some multi-indexes

$$x^\gamma \stackrel{\text{def}}{=} \prod_{i=1}^d x_i^{\gamma_i} \tag{4.6.20}$$

$$f_{\xi,\gamma}(x) \stackrel{\text{def}}{=} x^\gamma \partial_\xi f(x) \tag{4.6.21}$$

$$\|f\|_{k,l,p} = \sum_{0 \leq |\gamma| \leq l} \sum_{0 \leq |\xi| \leq k} \|f_{\xi,\gamma}\|_p \tag{4.6.22}$$

For all γ such that $|\gamma| \leq l$,

$$|x^\gamma| \stackrel{\text{def}}{=} \prod_{i=1}^d |x_i|^{\gamma_i} \leq \prod_{i=1}^d |x|^{\gamma_i} \leq |x|^{\sum \gamma_i} \leq (1+|x|)^l$$

so

$$\|f\|_{k,l,p} \leq C_d \sum_{0 \leq |\xi| \leq k} \|(1+|x|)^l \partial_\xi f(x)\|_p. \tag{4.6.23}$$

Since in the sequel we will have to bound the quantity $\|f_M\|_{2m+q,2m,p}$, let us notice that we have directly

$$\|f_M\|_{2m+q,2m,p} \leq C_d \sum_{0 \leq |\xi| \leq 2m+q} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \left[(1+|(x,y)|^{2m}) \partial_\xi \big(f_M(x,y) \big) \right]^p \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}. \tag{4.6.24}$$

Moreover, we will define a distance between two measures μ and ν in the following way :

$$d_k(\mu,\nu) \stackrel{\text{def}}{=} \sup \left\{ \left| \int \phi \, \mathrm{d}\mu - \int \phi \, \mathrm{d}\nu \right| : \phi \in \mathcal{C}^\infty(\mathbb{R}^d), \sum_{0 \leq |\xi| \leq k} \|\partial_\xi \phi\|_\infty \leq 1 \right\}. \tag{4.6.25}$$

Remark 4.6.1 *Here, we will only use the case $k = 1$, which is called the bounded variation distance (or, also, the Fortet-Mourier distance).*

4.6.2 Main result

Theorem 4.6.2 *Let $q, k \in \mathbb{N}$, $m \in \mathbb{N}^*$, $p > 1$ and set*

$$\eta > \frac{q+k+d/p^*}{2m}. \tag{4.6.26}$$

We consider a non negative finite measure μ and a family of finite non negative measures

$$\mu_\delta(\mathrm{d}x) = f_\delta(x)\,\mathrm{d}x, \qquad \delta > 0.$$

We assume that there exist C, $r > 0$ such that

$$\lambda_{q,m}(\delta) \stackrel{def}{=} \sup_{\delta \le \delta' \le 1} \|f_{\delta'}\|_{2m+q,2m,p} \le C\delta^{-r}$$

and moreover, with η given in (4.6.26),

$$\lambda_{q,m}(\delta)^\eta d_k(\mu, \mu_\delta) \le C. \tag{4.6.27}$$

Then $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ with $f \in W^{q,p}$.

Chapter 5

Regularity of the Density

5.1 Introduction

As we briefly mentioned in the introduction of the last chapter, the main purpose of this second part is to study the regularity of the law of the random variable X_t solution of the following stochastic equation with jumps :

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^{t+}\int_{E\times\mathbb{R}_+} c(z, X_{s^-})\mathbb{1}_{\{u\leq\gamma(z,X_{s^-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g(X_s)\,\mathrm{d}s \quad (5.1.1)$$

(again, for the existence and the uniqueness of such stochastic equation see Chapter 1).

Our global aim is to give sufficient conditions in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. That was the point of the study made in [8] as well, with an equation of this type but without the Brownian part.

But, here, we will not only consider the existence and the regularity of the density $y \mapsto p_{X_t}(y)$ (defined by $\mathrm{P}_{X_t}(\mathrm{d}y) = p_{X_t}(y)\,\mathrm{d}y$) with a given starting point $x \in \mathbb{R}^d$: we will consider instead the behaviour of $(x, y) \mapsto p_{X_t^x}(y)$ (with $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$ where X_t^x stands for the solution of (5.1.1) starting at x).

This joint density regularity property, in addition to being obviously a stronger result, will allow us (and it was, at first, one of the motivations to extend the result obtained for the regularity of $y \mapsto p_{X_t^x}(y)$) to obtain an interesting application concerning the Harris-recurrence of the process (section 7.4).

5.2 Hypothesis and notations

Let us recall that the associated intensity measure of the counting measure N is given by

$$\hat{N}(\mathrm{d}t,\mathrm{d}z,\mathrm{d}u) = \mathrm{d}t \times \mu(\mathrm{d}z) \times \mathbb{1}_{\{0,\infty\}}(u)\,\mathrm{d}u$$

where $(z, u) \in X = \mathbb{R}^d \times \mathbb{R}_+$ and $\mu(\mathrm{d}z) = h(z)\,\mathrm{d}z$.

In this section we make the following hypothesis on the functions γ, g, h and c.

Hypothesis 5.1 *We assume that γ, g, h and c are infinitely differentiable functions in both variables z and x. Moreover we assume that*

- *g and its derivatives are bounded ;*
- *$\ln h$ has bounded derivatives ;*
- *both γ and $\ln\gamma$ have bounded derivatives.*

Hypothesis 5.2 *We assume that there exist two functions $\overline{\gamma}, \underline{\gamma} : \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$\overline{C} \geq \overline{\gamma}(z) \geq \gamma(z,x) \geq \underline{\gamma}(z) \geq 0, \qquad \forall x \in \mathbb{R}^d$$

where $\overline{C}$ is a constant.

Hypothesis 5.3 *1. We assume that there exists a non negative and bounded function $\overline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \overline{c}(z)\mu(\mathrm{d}z) < \infty$ and*

$$|c(z,x)| + |\partial_z^\beta \partial_x^\alpha c(z,x)| \leq \overline{c}(z), \qquad \forall z, x \in \mathbb{R}^d.$$

We need this hypothesis in order to estimate the Sobolev norms.

2. *There exists a measurable function $\hat{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^d} \hat{c}(z)\mu(\mathrm{d}z) < \infty$ and*

$$\|\nabla_x c \times (\mathrm{Id} + \nabla_x c)^{-1}(z,x)\| \leq \hat{c}(z), \qquad \forall z, x \in \mathbb{R}^d.$$

In order to simplify the notations we assume that $\hat{c}(z) = \overline{c}(z)$.

3. *There exists a non negative function $\underline{c} : \mathbb{R}^d \to \mathbb{R}_+$ such that, for all $z \in \mathbb{R}^d$,*

$$\sum_{r=1}^{d} \langle \partial_{z^r} c(z,x), \xi \rangle^2 \geq \underline{c}^2(z)|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d$$

and we assume that there exists $\theta \in \overline{\mathbb{R}}_+^*$ such that

$$\liminf_{a \to \infty} \frac{1}{\ln a} \int_{\{\underline{c}^2 \geq \frac{1}{a}\}} \underline{\gamma}(z)\mu(\mathrm{d}z) = \theta. \tag{5.2.2}$$

Remark : assumptions 2) and 3) give sufficient conditions to prove the non degeneracy of the Malliavin covariance matrix as defined in the previous chapter.

5.3 Main result

We are now able to state the density property of X_t^x and the joint regularity (in x and y) of it : we fix $q \geq 1$ and $p > 1$, K a compact set of $\mathbb{R}^d$, and we will give sufficient conditions in order to have $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}\,\mathrm{d}y$ with $(x,y) \mapsto p_{X_t^x} \in W^{q,p}(K \times \mathbb{R}^d)$.

Theorem 5.3.1 *Let q, $p \geq 1$. We assume that hypotheses 5.1, 5.2 and 5.3 hold. Let $(B_M)_{M \in \mathbb{N}^*}$ such that $\bigcup_{M \in \mathbb{N}^*} B_M = E$ and, for all $i \in \mathbb{N}^*$*

$$B_i \subset B_{i+1} \qquad \text{and} \qquad \mu(B_i) < +\infty.$$

Let K a compact set of $\mathbb{R}^d$ and (with p^* such that $\frac{1}{p} + \frac{1}{p^*} = 1$)

$$\eta > \frac{q + 1 + d/p^*}{2}. \tag{5.3.3}$$

If there exists C, $r > 0$ such that

$$\mu(B_M)^{6(d+q+3)^3} \leq C M^r \tag{5.3.4}$$

and if

$$\limsup_{M} \left(\mu(B_M)^{6(d+q+3)^3 \eta} \Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)} \Big) \right) < +\infty, \tag{5.3.5}$$

then, for $t > \frac{4d(3q'-1)}{\theta}$, with $q' = d + q + 2$, and for every $x \in K$, the law of X_t^x is absolutely continuous with respect to the Lebesgue measure, ie. $\mathrm{P}_{X_t^x}(\mathrm{d}y) = p_{X_t^x}(y)\,\mathrm{d}y$, and the function $(x,y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.

Remark 5.3.2 *The quantity η and the related condition 5.3.3, come directly from the main theorem of the interpolation method (Theorem 4.6.2), in the particular case $k = 1$, as we stated in Remark 4.6.1, and with $m = 1$ (this last choice is discussed in Remark 5.5.4).*

Remark 5.3.3 *If $\theta = \infty$, then, for all $t > 0$, the law of X_t^x is absolutely continuous with respect to the Lebesgue measure and the density $p_{X_t^x}$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.*

Remark 5.3.4 *Recalling that (cf. Brézis [15], p.168, Corollaire IX.13, for example), with $k \overset{def}{=} \left[q - \frac{d}{p}\right]$, we have (with $O \in \mathbb{R}^n$ an open ball), $W^{q,p}(O) \subset \mathcal{C}^k(O)$, in the sense that each element of $W^{q,p}(O)$ has a $\mathcal{C}^k$ representative, this theorem can also be used to characterize the $\mathcal{C}^k$ behaviour of the function $(x, y) \mapsto p_{X_t^x}(y)$ (as we will briefly see in the examples at 5.3.1).*

Notation 5.3.5 *In the sequel, since x will belong to a fixed compact set, we will often write simply Y_t instead of Y_t^x for any process Y starting at x, if this precision is not strictly needed (that is why the starting point will never explicitly appear within the Chapter 6 but will be always used in Chapter 7).*

Before starting the proof itself, we will first try to give a sketch of the strategy that we will use. The global idea is articulated in two steps :

1. to obtain an integration by part formula on an appropriate approximation of the process X_t ;

2. to use this last result to prove the regularity of the density.

The terms from the condition (5.3.5) are a direct consequence of this pattern.

For the first step, and first of all, given a non-decreasing sequence of subsets $\left(B_M\right)_{M \in \mathbb{N}^*}$, with $\mu(B_M) < \infty$, recovering E, we construct (for each M) an approximation X_t^M of the process X_t based on the restriction N_M of the random measure N on the subset B_M.

Using a similar result as the Lemma 1.4.1, given in the first part of this work, we can then say that the L^1-distance between these two processes is bounded as follows :

$$\forall t \leq T, \qquad \mathrm{E}\left[\left|X_t - X_t^M\right|\right] \leq C_T \int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\mu(\mathrm{d}z),$$

which explains the presence of the term $\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z)$ in the condition (5.3.5).

Since $\mu(B_M) < +\infty$, the random measure N_M may be represented as a compound Poisson process (where the jump times will be denoted by T_k^M, $k \in \mathbb{N}$) and the Poisson part of process X_t^M could be expressed as a sum ; nevertheless, because of the indicator function from the original equation, the coefficients of the equation verified by X_t^M are still (for the Poisson part) discontinuous and therefore, we cannot use directly the differential calculus presented earlier. Instead we prove that X_t^M has the same law as the process $\overline{X}_t^M$ which verifies an equation with smooth coefficients.

At this point, one would like to obtain an integration by part formula for $\overline{X}_t^M$, but there remains one last difficulty : it is clear that, for $t < T_1^M$ (the first jump of N_M), the random measure N_M produces no noise, and consequently there is no chance to use it for an integration by part (the Malliavin covariance matrix being, of course, degenerated).

That is why one last process will be introduced :

$$F_M \overset{\mathrm{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta,$$

where Δ Gaussian and where $U_M(t)$ is defined by $U_M(t) = t \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)$.

The L^1-distance between F_M and $\overline{X}_t^M$ is then bounded, for $t \le T$, by

$$K_T \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)},$$

which gives a natural interpretation for the last term of the condition (5.3.5).

We are now able to obtain an integration by part formula for the process F_M :

$$\mathrm{E}\left[\varphi'(F_M)\right] = \mathrm{E}\left[\varphi(F_M)H_M\right]. \qquad (\mathrm{I}_M)$$

The second step consists in proving the density regularity. The idea is to use a certain balance between the error $\mathrm{E}\left[|F_M - X_t|\right]$ (which tends to 0) and the weight $\mathrm{E}\left[|H_M|\right]$ (which tends to ∞). This was the strategy used in [8] as well. But here the estimates of $\mathrm{E}\left[|H_M|\right]$ have been more delicate then the corresponding one in [8] because of the additional brownian part $\sigma\,\mathrm{d}W$. Moreover, the balance used in [8] was based on a Fourier transform method while here we use the new method developed in [7].

This new method allows us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [8] ; finally, we give an application of this improvement since we can then consider a regenerative scheme to obtain an interesting result concerning the Harris-recurrence of the process (section 7.4).

5.3.1 Examples

In this example we assume that $h = 1$ so $\mu(\mathrm{d}z) = \mathrm{d}z$ and $\underline{\gamma}(z)$ is equal to a constant $\underline{\gamma} > 0$. We then have

$$\mu(B_M) = r_d M^d$$

where r_d is the volume of the unit ball in $\mathbb{R}^d$. We will also assume that x is in some compact set $K \overset{\mathrm{def}}{=} \overline{B(0,R)}$, $R > 0$.

We will consider two types of behaviour for c.

i) Exponential decay : we assume that $\overline{c}(z) = e^{-a|z|^c}$ for some constants $0 < b \le a$ and $c > 0$.

We then have

$$\int_{\{\overline{c}^2 > \frac{1}{u}\}} \underline{\gamma}(z)\,\mathrm{d}\mu(\mathrm{d}z) = \underline{\gamma}\frac{r_d}{(2a)^{\frac{d}{c}}} \times (\ln u)^{\frac{d}{c}}.$$

we then deduce for the constant θ (definied in (5.2.2))

$$\begin{aligned} \theta &= 0 && \text{if} \quad c > d, \\ \theta &= \infty && \text{if} \quad 0 < c < d, \\ \theta &= \frac{\underline{\gamma} r_d}{2a} && \text{if} \quad c = d. \end{aligned} \qquad (5.3.6)$$

If $c > d$, hypothesis 5.3.3 fails, which is coherent with the result of Bichteler, Gravereaux and Jacod in [12]. Now observe that

$$\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)} \le K e^{-\xi M^c}$$

for some $\xi > 0$, so the condition (5.3.5) is always well verified.

When $0 < c < d$, since $\theta = \infty$, for every $t > 0$, $(x,y) \mapsto p_{X_t^x}(y)$ belongs to $W^{\infty,p}(K \times \mathbb{R}^d)$ ($\forall p \ge 1$), which implies, according to the Remark 5.3.4, that $(x,y) \mapsto p_{X_t^x}(y)$ can be considered as an element of $\mathcal{C}^\infty(K \times \mathbb{R}^d)$.

If $c = d$, then appears a more particular behaviour and it is interesting to compare the result obtained here with the example from [8] (recalling that, in this last case, X_t does not possess a Brownian part, though), assuming here, for that sake, that x is fixed, $k \in \mathbb{N}$ and $p > 1$:

	[8]	Present work
Domain	$t > \frac{8da}{\underline{\gamma} r_d}(3d+2)$	$t > \frac{8da}{\underline{\gamma} r_d}\Big(3d\Big(1+\frac{1}{p}\Big)+3k+8\Big)$
Regularity of p_{X_t}	$\mathcal{C}^k$ with $k \leq \frac{1}{3}\Big(1+\frac{\underline{\gamma} r_d}{8da}t\Big)-d$	$\mathcal{C}^k$ with $k \leq \frac{1}{3}\Big(1+\frac{\underline{\gamma} r_d}{8da}t\Big)-d\Big(1+\frac{1}{p}\Big)-2$

Remark 5.3.6 *In fact, in this work, $(x,y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$ ($\forall p \geq 1$) with*

$$q \leq \frac{1}{3}\Big(1+\frac{\underline{\gamma} r_d}{8da}t\Big)-d-2,$$

so, using again the Remark 5.3.4, $(x,y) \mapsto p_{X_t^x}(y)$ can be considered as an element of $\mathcal{C}^k(K \times \mathbb{R}^d)$, with

$$k \overset{def}{=} \Big\lfloor q-\frac{d}{p}\Big\rfloor \leq q-\frac{d}{p} \leq \frac{1}{3}\Big(1+\frac{\underline{\gamma} r_d}{8da}t\Big)-d\Big(1+\frac{1}{p}\Big)-2.$$

In particular it requires, at least, $q \geq \frac{d}{p}$ to obtain some regularity.

ii) Polynomial decay : We assume now that $\overline{c}(z) = \frac{b}{1+|z|^v}$ and $\underline{c}(z) = \frac{a}{1+|z|^v}$ for some constants $0 < a \leq b$ and $v > d$. We have here

$$\int_{\{\overline{c}^2 > \frac{1}{u}\}} \underline{\gamma}(z)\,\mathrm{d}\mu(\mathrm{d}z) = \underline{\gamma} r_d \times (a\sqrt{u}-1)^{\frac{d}{v}},$$

so $\theta = \limsup_{u\to\infty} \frac{1}{\ln u}\underline{\gamma} r_d(a\sqrt{u}-1)^{\frac{d}{v}} = \infty$ and then, in this case, the regularity result stands for every $t > 0$.

A simple computation gives us the following bounds :

$$\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) \leq \frac{C}{M^{v-d}} \qquad \text{and} \qquad \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z) \leq \frac{C}{M^{2v-d}}.$$

So with C and $r > 0$ such that $\mu(B_M)^{6(d+q+3)^3} \leq CM^r$ (condition (5.3.4)), we have

$$\begin{aligned}\mu(B_M)^{6(d+q+3)^3\eta}\Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)}\Big)&\\ \leq C'M^{r\eta}\Big(\frac{1}{2M^{v-d}}+\frac{1}{2M^{2v-d}}\Big)&\\ \leq C'M^{r\eta-v+d}.&\end{aligned}$$

Hence, the condition (5.3.5) is true if

$$\eta \leq \frac{v-d}{r}$$

and, since here $\mu(B_M) = r_d M^d$, (5.3.4) gives $r = 6d(d+q+3)^3$ and, with (5.3.3), we have the following condition :

$$\frac{q+1+d/p^*}{2} \leq \frac{v-d}{r}.$$

Finally, with v such that

$$v > 6d(d+q+3)^3\frac{q+1+d}{2}, \tag{5.3.7}$$

$(x,y) \mapsto p_{X_t^x}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$, for all $p \geq 1$.

5.4 Approximation of X_t

In order to prove that the process X_t, solution of (5.1.1), has a smooth density, we will apply the differential calculus and the integration by parts formula from Chapter 4. But since the random variable X_t cannot be viewed as a simple functional, the first step consists in approximating it. We describe in this section our approximation procedure. We consider a non-negative and smooth function φ such that $\varphi(z) = 0$ for $|z| > 1$ and $\int_{\mathbb{R}^d} \varphi(z)\,\mathrm{d}z = 1$. And for $M \in \mathbb{N}$, we denote

$$\Phi_M(z) = \varphi * \mathbb{1}_{B_M}$$

with $B_M = \{z \in \mathbb{R}^d \ : \ |z| < M\}$. Then $\Phi_M \in C_b^\infty$ and we have $\mathbb{1}_{B_{M-1}} < \Phi_M < \mathbb{1}_{B_{M+1}}$. We denote by X_t^M the solution of the equation

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t \int_E c_M(z, X_{s-}^M)\mathbb{1}_{\{u \le \gamma(z, X_{s-}^M)\}} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g(X_s^M)\,\mathrm{d}s. \quad (5.4.8)$$

where

$$c_M(z,x) \stackrel{\text{def}}{=} c(z,x)\Phi_M(z).$$

If we set

$$N_M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \stackrel{\text{def}}{=} \mathbb{1}_{B_{M+1}}(z) \times \mathbb{1}_{[0,2\overline{C}]}(u) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

since $\{u < \gamma(z, X_{s-}^M)\} \subset \{u < 2\bar{C}\}$ and $\Phi_M(z) = 0$ if $|z| > M+1$,we may replace N by N_M in the above equation and consequently X_t^M is solution of the equation

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t \int_E c_M(z, X_{s-}^M)\mathbb{1}_{\{u \le \gamma(z, X_{s-}^M)\}}\,\mathrm{d}N_M(s,z,u) + \int_0^t g(X_s^M)\,\mathrm{d}s.$$

Since the intensity measure $\hat{N}_M$ is finite, we may represent the random measure N_M by a compound Poisson process. Let

$$\lambda_M \stackrel{\text{def}}{=} 2\bar{C} \times \mu(B_{M+1}) = t^{-1}\,\mathrm{E}\left[N_M(t, E)\right]$$

and let J_t^M a Poisson process of parameter λ_M. We denote by T_k^M, $k \in \mathbb{N}$, the jump times of J_t^M. We also consider two sequences of independent random variables $(Z_k^M)_{k\in\mathbb{N}}$ and $(U_k)_{k\in\mathbb{N}}$, respectively in $\mathbb{R}^d$ and $\mathbb{R}_+$, which are independent of J_t^M and such that

$$Z_k^M \sim \frac{1}{\mu(B_{M+1})}\mathbb{1}_{B_{M+1}}(z)\mu(\mathrm{d}z), \qquad \text{and} \qquad U_k \sim \frac{1}{2\bar{C}}\mathbb{1}_{[0,2\bar{C}]}(u)\,\mathrm{d}u.$$

Then, the last equation may be written as

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \sum_{k=1}^{J_t^M} c_M(Z_k^M, X_{T_k^M-}^M)\mathbb{1}_{(U_k,\infty)}(\gamma(Z_k^M, X_{T_k^M-}^M)) + \int_0^t g(X_s^M)\,\mathrm{d}s. \quad (5.4.9)$$

The random variable X_t^M solution of (5.4.9) is a function of $(Z_1, \ldots, Z_{J_t^M})$ but it is not a simple functional, as defined in Chapter 4, because the coefficient $c_M(z,x)\mathbb{1}_{\{u \le \gamma(z,x)\}}$ is not differentiable with respect to z. In order to avoid this difficulty we use the following alternative representation. Let $z_M^* \in \mathbb{R}^d$ such that $|z_M^*| = M+3$. We define

$$q_M(z,x) \stackrel{\text{def}}{=} \varphi(z - z_M^*)\theta_{M,\gamma}(x) + \frac{1}{2\overline{C}\mu(B_{M+1})}\mathbb{1}_{B_{M+1}}(z)\gamma(z,x)h(z) \quad (5.4.10)$$

$$\theta_{M,\gamma}(x) \stackrel{\text{def}}{=} \frac{1}{\mu(B_{M+1})}\int_{\{|z| \le M+1\}} \left(1 - \frac{1}{2\overline{C}}\gamma(z,x)\right)\mu(\mathrm{d}z). \quad (5.4.11)$$

We recall that φ is the function defined at the beginning of this subsection : a non-negative and smooth function with $\int \varphi = 1$ and which is null outside the unit ball. Moreover from Hypothesis 5.2, $0 \leq \gamma(z,x) \leq \overline{C}$ and then

$$1 \geq \theta_{M,\gamma}(x) \geq \frac{1}{2}. \tag{5.4.12}$$

From this last inequality it is easy to deduce the following result :

Lemma 5.4.1 *Let q_M defined as in* (5.4.10). *Then $\ln q_M$ has bounded derivatives of any order.*

By construction the function q_M satisfies $\int q_M(z,x)\,\mathrm{d}z = 1$. Hence we can check that (*cf.* appendix A.4 for a proof)

$$\mathrm{E}\left[f(X^M_{T^M_k-}) \mid X^M_{T^M_k-} = x\right] = \int_{\mathbb{R}^d} f(x + c_M(z,x)) q_M(z,x)\,\mathrm{d}z. \tag{5.4.13}$$

From the relation (5.4.13) we construct a process $(\overline{X}^M_t)$ equal in law to (X^M_t) in the following way.

Let $0 \leq u \leq v$ and $y \in \mathbb{R}^d$, we denote by $\Psi_{u,v}(y)$ the solution of

$$\Psi_{u,v}(x) = y + \int_u^v \sigma(\Psi_{u,s}(y))\,\mathrm{d}W_s + \int_u^v g(\Psi_{u,s}(y))\,\mathrm{d}s.$$

We assume that the times T_k, $k \in \mathbb{N}$ are fixed and we consider a sequence $(z_k)_{k\in\mathbb{N}}$ with $z_k \in \mathbb{R}^d$. Then we define x_t, $t \geq 0$ by $x_0 = x$ and, if x_{T_k} is given, then

$$\begin{aligned} x_t &= \Psi_{T_k,t}(x_{T_k}), \qquad\qquad T_k \leq t < T_{k+1}, \\ x_{T_{k+1}} &= x_{T^-_{k+1}} + c_M\big(z_{k+1}, x_{T^-_{k+1}}\big). \end{aligned} \tag{5.4.14}$$

We note that for $T_k \leq t < T_{k+1}$, x_t is a function of x, z_1, ..., z_k. Notice also that x_t solves the equation

$$x_t = x + \int_0^t \sigma(x_s)\,\mathrm{d}W_s + \sum_{k=1}^{J^M_t} c_M(z_k, x_{T^-_k}) + \int_0^t g(x_s)\,\mathrm{d}s.$$

We consider now a sequence of random variables $(\overline{Z}_k)$, $k \in \mathbb{N}^*$, independent of the Brownian motion W_t, and we denote $\mathcal{G}_k = \sigma(T_p, p \in \mathbb{N}) \vee \sigma(\overline{Z}_p, p \leq k)$ and

$$\overline{X}^M_t = x_t(\overline{Z}_1, \ldots \overline{Z}_{J^M_t}). \tag{5.4.15}$$

We assume that the law of $\overline{Z}_{k+1}$ conditionally on $\mathcal{G}_k$ is given by

$$\mathrm{P}(\overline{Z}_{k+1} \in \mathrm{d}z \mid \mathcal{G}_k) = q_M(z, x_{T^-_{k+1}}(\overline{Z}_1, \ldots \overline{Z}_k))\,\mathrm{d}z = q_M(z, \overline{X}^M_{T^-_{k+1}})\,\mathrm{d}z. \tag{5.4.16}$$

Then $\overline{X}^M_t$ satisfies the equation

$$\overline{X}^M_t = x + \int_0^t \sigma(\overline{X}^M_s)\,\mathrm{d}W_s + \sum_{k=1}^{J^M_t} c_M(\overline{Z}_k, \overline{X}^M_{T^-_k}) + \int_0^t g(\overline{X}^M_s)\,\mathrm{d}s \tag{5.4.17}$$

and $\overline{X}^M_t$ has the same law as X^M_t. Moreover we can prove a bit more.

Lemma 5.4.2 *For a locally bounded and measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$ let*

$$\overline{S}_t(\psi) = \sum_{k=1}^{J^M_t} (\Phi_M\psi)(\overline{Z}_k), \qquad S_t(\psi) = \sum_{k=1}^{J^M_t} (\Phi_M\psi)(Z_k) 1\!\!1_{\{\gamma(Z_k, X^M_{T^-_k}) > U_k\}},$$

then $(\overline{X}^M_t, \overline{S}_t(\psi))_{t\geq 0}$ has the same law as $(X^M_t, S_t(\psi))_{t\geq 0}$.

Proof : Observing that $(\overline{X}^M_t, \overline{S}_t(\psi))_{t\geq 0}$ solves a system of equations similar to (5.4.17), but in dimension $d+1$, it suffices to prove that $(\overline{X}^M_t)_{t\geq 0}$ has the same law as $(X^M_t)_{t\geq 0}$, which is done in detail in the appendix A.5. •

5.5 The integration by part formula

The random variable $\overline{X}_t^M$ constructed previously is a simple functional but unfortunately its Malliavin covariance matrix is degenerated. To avoid this problem we use a classical regularization procedure. Instead of the variable $\overline{X}_t^M$, we consider the regularized one F_M defined by

$$F_M \stackrel{\text{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta, \tag{5.5.18}$$

where Δ is a d-dimensional standard Gaussian variable independent of the variables $(\overline{Z}_k)_{k\geq 1}$ and $(T_k)_{k\geq 1}$ and $U_M(t)$ is defined by

$$U_M(t) = t \int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z). \tag{5.5.19}$$

Notation 5.5.1 *We observe that $F_M \in \mathcal{S}^d$ where $\mathcal{S}$ is the space of simple functionals for the differential calculus based on the variables $(\overline{Z}_k)_{k\geq 1}$ with $\overline{Z}_0 = (\Delta^r)_{1\leq r\leq d}$ and $\overline{Z}_k = (\overline{Z}_k^r)_{1\leq r\leq d}$ and we are now in the framework of the previous chapter (Section 4.2) by taking $\mathcal{G} \stackrel{def}{=} \sigma(T_k, k \in \mathbb{N})$ and defining the weights (π_k) by setting $\pi_0^r = 1$ and*

$$\pi_k^r \stackrel{def}{=} \phi_M(\bar{Z}_k) \tag{5.5.20}$$

for $1 \leq r \leq d$.

Conditionally on $\mathcal{G}$, the density of the law of $(\overline{Z}_1, \ldots, \overline{Z}_{J_t^M})$ is given by

$$p_M(\omega, z_1, \ldots, z_{J_t^M}) = \prod_{j=1}^{J_t^M} q_M\big(z_j, \Psi_{T_{j-1},T_j}(\overline{X}_{T_{j-1}}^M)\big) \tag{5.5.21}$$

where $\overline{X}_{T_{j-1}}^M$ is a function of z_i, $1 \leq i \leq j-1$ (moreover, we can notice that $\Psi_{T_{j-1},T_j}(\overline{X}_{T_{j-1}}^M) = \overline{X}_{T_j-}^M)$; we can check that p_M satisfies the Hypothesis 4.2 of Chapter 4.

Notation 5.5.2 *To clarify the notation, the derivative operator can be written in this framework for $F \in \mathcal{S}$ by $\mathrm{D}\,F = (\mathrm{D}_{k,r}\,F)$ where $\mathrm{D}_{k,r} = \pi_k^r \partial_{\overline{Z}_k^r}$ for $k \geq 0$ and $1 \leq r \leq d$. Consequently we deduce that $\mathrm{D}_{k,r}\,F_M^{r'} = \mathrm{D}_{k,r}\,\overline{X}_t^{M,r'}$, for $k \geq 1$ and $\mathrm{D}_{0,r}\,F_M^{r'} = \sqrt{U_M(t)}\delta_{r,r'}$ with $\delta_{r,r'} = 0$ if $r \neq r'$, $\delta_{r,r'} = 1$ otherwise.*

The Malliavin covariance matrix of $\overline{X}_t^M$ is equal to

$$\sigma\big(\overline{X}_t^M\big)_{i,j} = \sum_{k=1}^{J_t^M} \sum_{r=1}^{d} \mathrm{D}_{k,r}\,\mathrm{D}_{k,r}\,\overline{X}_t^{M,i}\overline{X}_t^{M,j}$$

for $1 \leq i, j \leq d$ and finally the Malliavin covariance matrix of F_M is given by

$$\sigma\big(F_M\big) = \sigma\big(\overline{X}_t^M\big) + U_M(t) \times \mathrm{Id}\,.$$

Using the results of Chapter 4, we can state an integration by part formula and give a bound for the weight $H^q(F_M, 1)$ in terms of the Sobolev norms of F_M, the divergence $\mathrm{L}\,F_M$ and the determinant of the inverse of the Malliavin covariance matrix $\det\sigma\big(F_M\big)$.

The control of these last three quantities is rather technical and is studied in detail in the next chapter.

Since we are looking here also for the regularity with respect to the starting point x, and in order to use the Interpolation method (*cf.* 4.6) we will have to look a little bit further. It is clear, from its definition, that the law $P_{F_M^x}$ of F_M^x possesses a smooth density : $P_{F_M^x}(\mathrm{d}y) = p_{F_M^x}(y)\,\mathrm{d}y$. We will then define

$$f_M(x,y) \stackrel{\text{def}}{=} \Psi_K(x) p_{F_M^x}(y) \tag{5.5.22}$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$, and study its behaviour with respect to the norm defined by (4.6.22). More precisely, we will admit for the moment the following result (for the proof, see section 7.3) :

Lemma 5.5.3 *Let $q \in \mathbb{N}$, $m \in \mathbb{N}^*$, $p > 1$. Then*

$$\|f_M\|_{2m+q,2m,p} \le C\mu(B_M)^{6(d+2m+q+1)^3} \tag{5.5.23}$$

where C does not depend on M.

5.5.1 Proof of the main result

To do so, as we said earlier, we will use a more powerful method then the usual "balance" that can be made, with some reasonable conditions, when an integration by part formula is available for a convergent sequence of processes (for a more detailed explanation, see [7], from which this new tool is taken) : here, we will use the Theorem 4.6.2, taken directly from this last cited article.

Proof : Let $t > \frac{4d(3q'-1)}{\theta}$, with $q' = d+2+q$, and let us define the measure μ_X defined by (where $P_{X_t^x}$ is the law of X_t^x)

$$\mu_X(\mathrm{d}x, \mathrm{d}y) \stackrel{\text{def}}{=} \Psi_K(x) P_{X_t^x}(\mathrm{d}y)\,\mathrm{d}x \tag{5.5.24}$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$. A natural approximation of $\mu_X(\mathrm{d}x, \mathrm{d}y)$ would then be $\Psi_K(x) p_{X_t^M}(x,y)\,\mathrm{d}x\,\mathrm{d}y$. But in order to use the Malliavin calculus developed in this work, it is more convenient to use, instead of X_t^M, the approximation (in law) F_M of it. Let us recall that

- $F_M \stackrel{\text{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta$, where Δ is Gaussian and where $U_M(t)$ is defined by $U_M(t) = t\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)$
- $f_M \stackrel{\text{def}}{=} \Psi_K(x) p_{F_t^M}(x,y)$.

We will then use the Theorem 4.6.2, with $\delta \stackrel{\text{def}}{=} M^{-1}$ (to be rigorous with the notations, we should define and work with $\tilde{f}_\delta \stackrel{\text{def}}{=} f_M$, but we will simply use f_M).

On one hand, using Lemma 5.5.3 with $m = 1$, we find that

$$\|f_M\|_{2+q,2,p} \le C\mu(B_M)^{6(d+q+3)^3} \tag{5.5.25}$$

where C does not depend on M.

On the other hand, using the definition (4.6.25) of the distance d_k in the case $k = 1$:

$$\begin{aligned}
d_1(\mu_X, f_M) &\stackrel{\text{def}}{=} \sup_{\substack{g \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d) \\ \|g\|_\infty, \|\nabla g\|_\infty \le 1}} \Big| \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x,y)\mu_X(\mathrm{d}x,\mathrm{d}y) - \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x,y) f_M(x,y)\,\mathrm{d}x\,\mathrm{d}y \Big| \\
&= \int_{\mathbb{R}^d} \Psi_K(x) \Big(\mathrm{E}\left[g(x, X_t(x))\right] - \mathrm{E}\left[g(x, F_t^M(x))\right] \Big)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \Psi_K(x)\,\mathrm{E}\left[\left|X_t(x) - F_t^M(x)\right|\right]\mathrm{d}x
\end{aligned}$$

so

$$d_1(\mu_X, f_M) \le C_K \Big(\int_{B_M^c} \overline{c}(z)\overline{\gamma}(z)\,\mathrm{d}\mu(z) + \sqrt{\int_{B_{M+1}^c} \underline{c}^2(z)\underline{\gamma}(z)\,\mathrm{d}\mu(z)} \Big). \tag{5.5.26}$$

It follows that, with the conditions (5.3.4) and (5.3.5), the conditions of the Theorem 4.6.2 are well verified and we can directly conclude. •

Remark 5.5.4 *Let us give a quick explanation why we only considered the case where the parameter* m *is equal to* 1. *This parameter was made to loosen up the lower bound condition for* η *in Theorem 4.6.2 (roughly speaking this lower bound is a* $O\left(\frac{1}{m}\right)$*) which could help to obtain a better condition from* (4.6.27). *But, the upper bound obtained for* $\|f_M\|_{2m+q,2m,p}$ *with respect to* m*, is a* $O(m^3)$*, so we lose here completely the possible advantage of taking* $m > 1$.

Chapter 6

Bounding of the weights $H^q_\beta(F_M)$

6.1 Introduction

In this chapter we consider that the starting point x is fixed.

The final result of that part is to bound the quantity $H^q_\beta(F_M)$; to do so, we will use the bounding given by the Theorem 4.5.10, which implies :

$$\left|H^q_\beta(F_M)\right| \leq C_{q,d}\frac{1}{|\det\sigma(F_M)|^{3q-1}}\big(1+|F_M|_{q+1})^{(6d+1)q}\big)\big(1+|\mathrm{L}F_M|^q_{q-1}\big),$$

and so will be brought to bound, in particular, on one hand $\|\frac{1}{|\det\ \sigma(F_M)|}\|_p$, which will be done at Lemma 6.6.5 and, on the other hand $\||F_M|_n\|_p$ and $\||\mathrm{L}F_M|_n\|_p$ (where $\|.\|_p$ is the L^p-norm). To do this last thing, because of the similar structure of the linear equations verified by the different processes involved here, we will develop in the first place a way to bound this type of processes, in a recursive way (which is natural, since we want, in particular, to bound successive derivatives of our process). Moreover, this theoretical result will be helpful in the Chapter 7, when we will study further the density continuity of the process X_t.

The upper bound of this quantity allows us to prove, under some similar conditions to (5.3.5), the existence of a regular density for X_t : (with $q \geq 1$ and $p > 1$ fixed) we have $\mathrm{P}_{X_t}(\mathrm{d}y) = p_{X_t}\,\mathrm{d}y$ with $p_{X_t} \in W^{q,p}(\mathbb{R}^d)$ (using a Fourier transform method as in [8], or some weaker version of the interpolation method (*cf.* [7]) quoted here). In this sense, this chapter is "self-contained" ; that is one of the reasons why we give the Lemma 6.7.1. The other reason (and it is globally true for the whole chapter) is to show a pattern of the proof, in a simpler case, which will be used again in the more general Lemma 5.5.3 (proved in the section 7.3).

Even though, to conclude in the general case (joint regularity), we need some further results, made in the next chapter, the main part of the needed techniques is presented in this one, with less heavier notations, since the starting point x is momentarily put aside.

Notations

- In all the sequel we will denote by E_W the expectation with respect to the Brownian motion ; *i.e.* conditionally with respect to the Poisson measure.
- As we have already pointed out in 5.3.5, in all this chapter, since x will belong to a fixed compact set, we will always write Y_t instead of Y^x_t for any process Y starting at x.

6.2 An upper bound lemma for a family of linear SDE's

In this section we give L^p bounds for the solution of a family of linear equations which represent the general framework in which the Malliavin derivatives fit.

Hypothesis

We fix a finite set I and we consider the multi-indexes of the type $\alpha \stackrel{\text{def}}{=} (\alpha_1, \dots, \alpha_n)$, with $\alpha_i \in I$. We define and denote the length of α by $|\alpha| \stackrel{\text{def}}{=} n$. We also consider the void multi-index $\alpha = \emptyset$ and in this case we put $|\alpha| \stackrel{\text{def}}{=} 0$.

Then we denote

$$A_n \stackrel{\text{def}}{=} \{\alpha \stackrel{\text{def}}{=} (\alpha_1, \dots, \alpha_n) \ : \ \alpha_i \in I\}$$

the set of multi-indexes of length n and define then

$$A \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} A_n$$

and

$$n_k \stackrel{\text{def}}{=} \#\{\alpha \in A, \ |\alpha| \le k\}$$

(since I is finite, n_k is a defined finite number).

We define a family of process $(\overline{V}_t^\alpha)_{t \ge 0}$, $\alpha \in A$ in the following way.

- If $|\alpha| = 0$ we put $\overline{V}_t^0 \stackrel{\text{def}}{=} \overline{X}_t^M$ with $\overline{X}_t^M$ solution of the equation (5.4.17) :

$$\overline{X}_t^M = x + \int_0^t \sigma(\overline{X}_s^M)\,\mathrm{d}W_s + \sum_{k=1}^{J_t^M} c_M(\overline{Z}_k, \overline{X}_{T_k^-}^M) + \int_0^t g(\overline{X}_s^M)\,\mathrm{d}s.$$

- Suppose now that we have already defined $\overline{V}^\alpha$ for $|\alpha| < n-1$. We denote by

$$\overline{V}_{(k-1)}(t) \stackrel{\text{def}}{=} \left(\overline{V}_t^\beta\right)_{|\beta| \le k-1}$$

(so $\overline{V}_{(0)}(t) = (\overline{V}_t^0) = (\overline{X}_t^M)$, a family of d-dimensional one element). Then let $\overline{V}^\alpha$ be, for $|\alpha| = n$, the solution of

$$\begin{aligned}\overline{V}_t^\alpha = \overline{V}_0^\alpha &+ \int_0^t G^\alpha\big(\overline{V}_{(k-1)}(s)\big)\,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} d_j^\alpha\big(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-)\big) + \int_0^t g^\alpha\big(\overline{V}_{(k-1)}(s)\big)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(\overline{V}_s^0)\overline{V}_s^\alpha\,\mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \beta^\alpha\big(\overline{Z}_j, \overline{V}_{T_j^-}^0\big)\overline{V}_{T_j^-}^\alpha + \int_0^t b^\alpha(\overline{V}_s^0)\overline{V}_s^\alpha\,\mathrm{d}s, \quad (6.2.1)\end{aligned}$$

with the functions $G^\alpha : \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^{d \times m}$, $d_j^\alpha : \mathbb{R}^d \times \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^d$, $g^\alpha : \mathbb{R}^{d \times n_{k-1}} \to \mathbb{R}^d$, $\rho_l^\alpha : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $\beta^\alpha : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $b^\alpha : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and the following Hypothesis :

Hypothesis 6.1 *1. There exist $w \in \mathbb{N}$ and $K \in \mathbb{R}_+$ such that, for all $v \in \mathbb{R}^{n_{k-1}}$ and $z \in \mathbb{R}^d$,*

$$|G^\alpha(v)| \le K(1 + |v|)^w, \qquad |g^\alpha(v)| \le K(1 + |v|)^w, \qquad (6.2.2)$$

and

$$|\beta^\alpha(z, v)| \le \overline{\beta}(z), \qquad |d_j^\alpha(z, v)| \le K\overline{c}(z)(1 + |v|)^w \quad (\forall j \in \mathbb{N}), \qquad (6.2.3)$$

with, for all $n \in \mathbb{N}$,

- $\int_{\mathbb{R}^d} \overline{c}(z)^n \mu(\mathrm{d}z) < \infty$;
- $\int_{\mathbb{R}^d} \overline{\beta}(z)^n \mu(\mathrm{d}z) < \infty$

2. bounding conditions :

- $\overline{\rho} \stackrel{def}{=} \sup_{s \in \mathbb{R}^d} \sup_{\alpha \in A} \sup_{1 \le l \le m} |\rho_l^\alpha(s)| < \infty$;

- $\overline{b} \overset{def}{=} \sup_{s\in\mathbb{R}^d}\sup_{\alpha\in A}|b^\alpha(s)| < \infty$.

Lemma 6.2.1 *Let $p \in \mathbb{N}^*$. We assume that the Hypothesis 6.1 holds and we set*

$$\Theta_{p,k}(t) = \sup_{\substack{\alpha\in A\\|\alpha|\le k}} \mathrm{E}_W\left[|\overline{V}_t^\alpha|^{2p}\right].$$

For all $T > 0$, there exists a constant $C_{T,p,k}$ (which does not *depend on M* nor *on the set A, and, in particular, does not depend on the size of I) such that*

$$\mathrm{E}\left[\Theta_{p,k}(t)\right] \le C_{T,p,k}. \tag{6.2.4}$$

Proof : In order to use stochastic calculus, we will come back to the process X_t^M, so, with the same notations as before, and

$$V_{(k)}(t) \overset{\mathrm{def}}{=} \left(V_t^\beta\right)_{|\beta|\le k}$$

with the convention $V_t^0 \overset{\mathrm{def}}{=} X_t^M$ (so $V_{(0)}(t) = X_t^M$), V_t^α is then defined as a solution of the following SDE (where $k = |\alpha|$):

$$\begin{aligned}V_t^\alpha = V_0^\alpha &+ \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \sum_{j=0}^{J_t^M} d_j^\alpha\big(Z_j, V_{(k-1)}(T_j^-)\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s\\ &+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \sum_{j=0}^{J_t^M}\beta^\alpha\big(Z_j, V^0_{T_j^-}\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}}V^\alpha_{T_j^-} + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s.\end{aligned} \tag{6.2.5}$$

In order to express the first compound Poisson process with an integral with respect to the Poisson measure, we put (with the convention $T_0 = 0$)

$$e_s^\alpha(z,v) \overset{\mathrm{def}}{=} \sum_{j=1}^{\infty}\mathbb{1}_{]T_{j-1},T_j]}(s)d_j^\alpha(z,v)$$

from (6.2.3) it is clear that,

$$|e_s^\alpha(z,v)| \le K\overline{c}(z)(1+|v|)^w. \tag{6.2.6}$$

Therefore

$$\begin{aligned}V_t^\alpha &= V_0^\alpha + \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \sum_{j=0}^{J_t^M} e^\alpha_{T_j^-}\big(Z_j, V_{(k-1)}(T_j^-)\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}} + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s\\ &\quad+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \sum_{j=0}^{J_t^M}\beta^\alpha\big(Z_j, V^0_{T_j^-}\big)\mathbb{1}_{\{U_j\le\gamma(Z_j,X^M_{T_j-})\}}V^\alpha_{T_j^-} + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s\\ &= V_0^\alpha + \int_0^t G^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}W_s + \int_0^t\int_E e^\alpha_{s^-}\big(z, V_{(k-1)}(s^-)\big)\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g^\alpha\big(V_{(k-1)}(s)\big)\,\mathrm{d}s\\ &\quad+ \sum_{l=1}^m \int_0^t \rho_l^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}W_s^l + \int_0^t\int_E \beta^\alpha\big(z, V^0_{s-}\big)\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}V^\alpha_{s-}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t b^\alpha(V_s^0)V_s^\alpha\,\mathrm{d}s.\end{aligned}$$

From Lemma 5.4.2, V_t^α and $\overline{V}_t^\alpha$ are sharing the same law, so we will prove that, for all $k \ge 0$,

$$\mathrm{E}\Big[\sup_{\substack{\alpha\in A\\|\alpha|\le k}} \mathrm{E}_W\left[|V_t^\alpha|^{2p}\right]\Big] \le C_{T,p}, \tag{6.2.7}$$

which will be shown by recurrence on k.

- For $k=0$, $V^0_t = X^M_t$, it is the Proposition B.0.1.
- For $k \geq 1$, we first simplify the notations by writing :

$$\begin{aligned} G^\alpha(s) &\overset{\text{def}}{=} G^\alpha\big(V_{(k-1)}(s)\big), & g^\alpha(s) &\overset{\text{def}}{=} g^\alpha\big(V_{(k-1)}(s)\big), \\ \rho^\alpha_l(s) &\overset{\text{def}}{=} \rho^\alpha_l(V^0_s), & b^\alpha(s) &\overset{\text{def}}{=} b^\alpha(V^0_s), \\ h^\alpha(s^-,z,u) &\overset{\text{def}}{=} e^\alpha_{s^-}\big(z, V_{(k-1)}(s^-)\big)\mathbb{1}_{\{u\leq\gamma(z,X^M_{s^-})\}}, & \beta^\alpha(s^-,z,u) &\overset{\text{def}}{=} \beta^\alpha\big(z, V^0_{s^-}\big)\mathbb{1}_{\{u\leq\gamma(z,X^M_{s^-})\}}, \end{aligned}$$

which gives

$$\begin{aligned} V^\alpha_t = V^\alpha_0 &+ \int_0^t G^\alpha(s)\,\mathrm{d}W_s + \int_0^t\int_E h^\alpha(s^-,z,u)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g^\alpha(s)\,\mathrm{d}s \\ &+ \sum_{l=1}^m \int_0^t \rho^\alpha_l(s)V^\alpha_s\,\mathrm{d}W^l_s + \int_0^t\int_E \beta^\alpha(s^-,z,u)V^\alpha_{s^-}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t b^\alpha(s)V^\alpha_s\,\mathrm{d}s. \end{aligned}$$

In order to use the recurrence hypothesis, we bound the coefficients of this last equation in the following way : with

$$\overline{h}_k(s) \overset{\text{def}}{=} K(1+|V_{(k-1)}(s)|)^w$$

and

$$\overline{\xi}(z,u) \overset{\text{def}}{=} (\overline{c}(z)+\overline{\beta}(z))\mathbb{1}_{\{u\leq\overline{\gamma}(z)\}}$$

we have (according to the hypothesis 6.1)

$$\sup_{\substack{\alpha\in A\\|\alpha|\leq k}} G^\alpha(s) \leq \overline{h}_k(s) \qquad \text{and} \qquad \overline{h}_k(s) \overset{\text{def}}{=} \sup_{\substack{\alpha\in A\\|\alpha|\leq k}} g^\alpha(s) \leq \overline{h}_k(s)$$

and (using (6.2.6) and (6.2.3))

$$\forall s,z,u, \qquad |h^\alpha(s^-,z,u)| \leq \overline{h}_k(s^-)\overline{\xi}(z,u) \qquad \text{and} \qquad |\beta^\alpha(s^-,z,u)| \leq \overline{\xi}(z,u). \tag{6.2.8}$$

At last, we have to notice, using the recurrence hypothesis for $k-1$, that, for all $n\in\mathbb{N}^*$,

$$\sup_{0\leq s\leq T} \mathrm{E}\left[|\overline{h}_k(s)|^n\right] < \infty; \tag{6.2.9}$$

this last result comes directly from the following bounding :

$$\begin{aligned} \mathrm{E}\left[|V_{(k-1)}(s)|^{2m}\right] = \mathrm{E}\left[\mathrm{E}_W\left[|V_{(k-1)}(s)|^{2m}\right]\right] &\leq K\,\mathrm{E}\left[\sum_{\beta\in A_{k-1}} \mathrm{E}_W\left[|V^\beta_s|^{2m}\right]\right] \\ &\leq K'\sqrt{\mathrm{E}\left[\Big(\sup_{\beta\in A_{k-1}} \mathrm{E}_W\left[|V^\beta_t|^{2m}\right]\Big)^2\right]} \\ &\leq K'\sqrt{C_{T,m,k-1}} < \infty. \end{aligned}$$

Step 1

In order to use the Itô's formula, we will first have to localize our problem by using the sequence $(\tau^M_K)_{K\in\mathbb{N}^*}$ of stopping times defined by

$$\tau^M_K(k) \overset{\text{def}}{=} \inf\{t>0 \;:\; \sup_{s\leq t}\sum_{|\alpha|\leq k}|V^\alpha_s| \geq K\}. \tag{6.2.10}$$

Let us prove that a.s. $\lim_{K\to\infty}\tau^M_K = \infty$.

From the hypothesis made on the coefficients of V_t^α, it is clear that, for all $t \geq 0$,

$$\sum_{|\alpha| \leq k} \mathrm{E}\left[\sup_{s \leq t} |V_s^\alpha|\right] \leq \infty. \tag{6.2.11}$$

We have, for $t \geq 0$

$$\begin{aligned}\lim_{K \to \infty} \mathrm{P}(\tau_K^M < t) &= \lim_{K \to \infty} \mathrm{P}(\sup_{s \leq t} \sum_{|\alpha| \leq k} |V_s^\alpha| > K) \\ &\leq \lim_{K \to \infty} \frac{1}{K} \mathrm{E}\left[\sup_{s \leq t} \sum_{|\alpha| \leq k} |V_s^\alpha|\right] = 0.\end{aligned}$$

$(\tau_K^M)_{K \in \mathbb{N}^*}$ tends to ∞ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau_K^M)_{K \in \mathbb{N}^*}$) which tends to ∞ a.s.

In this case we have

$$|V_t^\alpha|^{2p} \mathbb{1}_{\tau_K^M > t} \uparrow |V_t^\alpha|^{2p} \qquad \text{a.s.}$$

so

$$\mathrm{E}_W \left[|V_t^\alpha|^{2p} \mathbb{1}_{\tau_K^M > t}\right] \uparrow \mathrm{E}_W \left[|V_t^\alpha|^{2p}\right] \qquad \text{a.s.}$$

and

$$\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_t^\alpha|^{2p} \mathbb{1}_{\tau_K^M > t}\right] \uparrow \sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_t^\alpha|^{2p}\right] \qquad \text{a.s.}$$

If we admit for the moment that there exists a constant $C_{p,T,k}$ which does not depend on K and M and such that, for all $0 \leq t \leq T$,

$$\mathrm{E}\left[\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_{t \wedge \tau_K^M}^\alpha|^{2p}\right]\right] \leq C_{T,p,k} \tag{6.2.12}$$

The monotone convergence theorem implies then

$$\mathrm{E}\left[\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_t^\alpha|^{2p}\right]\right] = \sup_K \mathrm{E}\left[\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_t^\alpha|^{2p} \mathbb{1}_{\tau_K^M > t}\right]\right] = \sup_K \mathrm{E}\left[\sup_{\substack{\alpha \in A \\ |\alpha| \leq k}} \mathrm{E}_W \left[|V_{t \wedge \tau_K^M}^\alpha|^{2p}\right]\right] \leq C_{T,p,k}.$$

Step 2

We have to establish now (6.2.12).

For a single component we have (omitting for a moment the parameter α in order to simplify the notations)

$$\begin{aligned}V_t^i = V_0^i &+ \sum_{l=1}^m \int_0^t G_{il}(s)\, \mathrm{d}W_s^l + \int_0^t \int_E h^i(s^-, z, u) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t g^i(s)\, \mathrm{d}s \\ &+ \sum_{l=1}^m \sum_{h=1}^d \int_0^t \rho_{ih}^l(s) V_s^h\, \mathrm{d}W_s^l + \sum_{h=1}^d \int_0^t \int_E \beta_{ih}(s^-, z, u) V_{s^-}^h N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \sum_{h=1}^d \int_0^t b_{ih}(s) V_s^h\, \mathrm{d}s\end{aligned}$$

Then, applying Itô's formula with $f(x) = x^{2p}$

$$\begin{aligned}(V_{t \wedge \tau_K^M}^i)^{2p} = (V_0^i)^{2p} &+ \sum_{l=1}^m \int_0^{t \wedge \tau_K^M} 2p (V_s^i)^{2p-1} \Big(G_{il}(s) + \sum_{h=1}^d \rho_{ih}^l(s) V_s^h\Big)\, \mathrm{d}W_s^l \\ &+ 2p \int_0^{t \wedge \tau_K^M} (V_s^i)^{2p-1} \Big(g_i(s) + \sum_{h=1}^d b_{ih}(s) V_s^h\Big)\, \mathrm{d}s \\ &+ p(2p-1) \sum_{l=1}^m \int_0^{t \wedge \tau_K^M} (V_s^i)^{2p-2} \Big(G_{il}(s) + \sum_{h=1}^d \rho_{ih}^l(s) V_s^h\Big)^2 \mathrm{d}s \\ &+ \int_0^{t \wedge \tau_K^M} \int_E \Big(V_{s^-}^i + h_i(s^-, z, u) + \sum_{h=1}^d \beta_{ih}(s^-, z, u) V_{s^-}^h\Big)^{2p} - (V_{s^-}^i)^{2p} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\end{aligned}$$

We take now the expectation with respect to the Brownian motion (*i.e.* conditionally with respect to all the others random quantities) :

$$\begin{aligned}
\mathrm{E}_W\left[(V^i_{t\wedge\tau_K^M})^{2p}\right] =& \mathrm{E}_W\left[(V^i_0)^{2p}\right] + 2p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W\left[(V^i_s)^{2p-1}\Big(g_i(s)+\sum_{h=1}^d b_{ih}(s)V^h_s\Big)\right]\mathrm{d}s \\
&+ p(2p-1)\sum_{l=1}^m \int_0^{t\wedge\tau_K^M} \mathrm{E}_W\left[(V^i_s)^{2p-2}\Big(G_{il}(s)+\sum_{h=1}^d \rho^l_{ih}(s)V^h_s\Big)^2\right]\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_K^M}\int_E \mathrm{E}_W\left[\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d \beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p} - (V^i_{s^-})^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Since $s\le t\wedge\tau_K^M$, we have $X_s = X_{s\wedge\tau_K^M}$, and obviously $t\le t\wedge\tau_K^M$, so we have

$$\begin{aligned}
\mathrm{E}_W\left[|V^i_{t\wedge\tau_K^M}|^{2p}\right] =& \mathrm{E}_W\left[|V^i_0|^{2p}\right] + 2p\int_0^{t} \mathrm{E}_W\left[|V^i_{s\wedge\tau_K^M}|^{2p-1}\Big(|g_i(s)|+\sum_{h=1}^d |b_{ih}(s)||V^h_{s\wedge\tau_K^M}|\Big)\right]\mathrm{d}s \\
&+ p(2p-1)\sum_{l=1}^m \int_0^{t} \mathrm{E}_W\left[|V^i_{s\wedge\tau_K^M}|^{2p-2}\Big(|G_{il}(s)|+\sum_{h=1}^d |\rho^l_{ih}(s)||V^h_{s\wedge\tau_K^M}|\Big)^2\right]\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_K^M}\int_E \mathrm{E}_W\left[\left|\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d \beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p} - (V^i_{s^-})^{2p}\right|\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\Big(V^i_{s^-}+h_i(s^-,z,u)+\sum_{h=1}^d \beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^{2p} - (V^i_{s^-})^{2p}\right| \\
&\qquad = \left|\sum_{k=1}^{2p}\binom{2p}{k}\Big(h_i(s^-,z,u)+\sum_{h=1}^d \beta_{ih}(s^-,z,u)V^h_{s^-}\Big)^k {V^i_{s^-}}^{2p-k}\right| \\
&\qquad \le \sum_{k=1}^{2p}\binom{2p}{k}\big(|h|(s^-,z,u)+|\beta|(s^-,z,u)|V|_{s^-}\big)^k|V|_{s^-})^{2p-k} \\
&\qquad = (|V|_{s^-}+|h|(s^-,z,u)+|\beta|(s^-,z,u)|V|_{s^-})^{2p} - |V|^{2p}_{s^-},
\end{aligned}$$

it follows (with $\rho \overset{\text{def}}{=} \sup_l|\rho_l|$)

$$\begin{aligned}
\mathrm{E}_W[(V^i_{t\wedge\tau_K^M})^{2p}] \le& \mathrm{E}_W[|V_0|^{2p}] + 2p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s|^{2p-1}(|g|(s)+|b|(s)|V_s|)]\,\mathrm{d}s \\
&+ p(2p-1)\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s|^{2p-2}(|G|(s)+\rho(s)|V_s|)^2]\,\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_K^M}\int_E \mathrm{E}_W[(|V_{s^-}|+|h|(s^-,z,u)+|\beta|(s^-,z,u)|V_{s^-}|)^{2p} - |V_{s^-}|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

Then (writing again from now on the parameter α, in order to see clearly which components depend of it or not), we have (using, among others things, the inequality (6.2.8))

$$\begin{aligned}
\mathrm{E}_W[|V^\alpha_{t\wedge\tau_K^M}|^{2p}] \le& \mathrm{E}_W[|V^\alpha_0|^{2p}] + 2p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V^\alpha_s|^{2p-1}(\overline{h}_k(s)+\overline{b}|V^\alpha_s|)]\,\mathrm{d}s \\
&+ p(2p-1)\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V^\alpha_s|^{2p-2}(\overline{h}_k(s)+\overline{\rho}|V^\alpha_s|)^2]\,\mathrm{d}s \\
&+ \int_0^{t\wedge\tau_K^M}\int_E \mathrm{E}_W[(|V^\alpha_{s^-}|+\overline{h}_k(s^-)\overline{\xi}(z,u)+\overline{\xi}(z,u)|V^\alpha_{s^-}|)^{2p} - |V^\alpha_{s^-}|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).
\end{aligned}$$

(Notice that, since V_t^α is an adapted process, $|V_0^\alpha|$ is a constant, so $\mathrm{E}_W[|V_0^\alpha|^{2p}] = |V_0^\alpha|^{2p}$.)

To bound the first integral, using the elementary inequality

$$\forall x, y \geq 0,\ \forall u, v > 0 \qquad x^u y^v \leq x^{u+v} + y^{u+v}, \tag{6.2.13}$$

we notice that :

$$\begin{aligned} ||V_s^\alpha|^{2p-1}\overline{h}_k(s) + \overline{b}|V_s^\alpha|^{2p}| &\leq \overline{h}_k(s)^{2p} + |V_s^\alpha|^{2p} + \overline{b}|V_s^\alpha|^{2p} \\ &= \overline{h}_k(s)^{2p} + (1+\overline{b})|V_s^\alpha|^{2p}, \end{aligned}$$

and, similarly, for the second one,

$$\begin{aligned} |V_s^\alpha|^{2p-2}(\overline{h}_k(s) + \overline{\rho}|V_s^\alpha|)^2 &\leq 2\overline{h}_k(s)^{2p} + 2|V_s^\alpha|^{2p} + 2\overline{\rho}^2|V_s^\alpha|^{2p} \\ &= 2\overline{h}_k(s)^{2p} + 2(1+\overline{\rho}^2)|V_s^\alpha|^{2p}. \end{aligned}$$

It follows that,

$$\begin{aligned} 2p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p-1}(\overline{h}_k(s) + \overline{b}|V_s^\alpha|)]\,\mathrm{d}s + p(2p-1)\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p-2}(\overline{h}_k(s) + \overline{\rho}|V_s|)^2]\,\mathrm{d}s \\ \leq C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p}]\,\mathrm{d}s. \end{aligned}$$

For the third integral, we will see that

$$(|V_{s^-}^\alpha| + \overline{h}_k(s^-)\overline{\xi}(z,u) + \overline{\xi}(z,u)|V_{s^-}^\alpha|)^{2p} - |V_{s^-}^\alpha|^{2p} \leq \overline{\xi}(z,u)P(\overline{\xi}(z,u))\big(|V_{s^-}^\alpha|^{2p} + (\overline{h}_k(s^-))^{2p}\big) \tag{6.2.14}$$

where P is a polynomial function.

Let us prove now (6.2.14) : if $u, v \geq 0$,

$$\left|u^{2p} - v^{2p}\right| \leq |u-v|(u+v)^{2p-1}, \tag{6.2.15}$$

so, it follows that, for a, b, $c \geq 0$:

$$\begin{aligned} (a + c(b+a))^{2p} - a^{2p} &\leq c(b+a)(a(2+c) + cb)^{2p-1} \\ &\leq 2^{2p-1}c(b+a)(a^{2p-1}(2+c)^{2p-1} + (cb)^{2p-1}) \\ &\leq 2^{2p-1}c(a^{2p}(2+c)^{2p-1} + a(cb)^{2p-1} + ba^{2p-1}(2+c)^{2p-1} + c^{2p-1}b^{2p}) \end{aligned}$$

using (6.2.13), we have

$$a(cb)^{2p-1} \leq a^{2p} + (cb)^{2p} \qquad \text{and} \qquad ba^{2p-1}(2+c)^{2p-1} \leq a^{2p} + b^{2p}(2+c)^{2p(2p-1)}$$

which brings to

$$(a + c(b+a))^{2p} - a^{2p} \leq 2^{2p-1}c\big[a^{2p}(2 + (2+c)^{2p-1}) + b^{2p}(c^{2p-1} + c^{2p} + (2+c)^{2p(2p-1)})\big],$$

or, more generally, to

$$(a + c(b+a))^{2p} - a^{2p} \leq cP(c)\big[a^{2p} + b^{2p})\big], \tag{6.2.16}$$

where $P \in \mathbb{R}[X]$, which proves (6.2.14).

Gathering all those results,

$$\begin{aligned} \mathrm{E}_W[|V_{t\wedge\tau_K^M}^\alpha|^{2p}] \leq \mathrm{E}_W[|V_0^\alpha|^{2p}] + C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[|V_s^\alpha|^{2p}]\,\mathrm{d}s \\ + \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[\bar{h}_k^{2p}(s^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{t\wedge\tau_K^M}\int_E \bar{\xi}P(\bar{\xi})\,\mathrm{E}_W[|V_{s^-}^\alpha|^{2p}]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned}$$

We then have, directly, with $\Theta_{p,k}(t\wedge\tau_K^M) \overset{\text{def}}{=} \sup_{\substack{\alpha\in A\\|\alpha|\le k}} \mathrm{E}_W\left[|V^\alpha_{t\wedge\tau_K^M}|^{2p}\right]$,

$$\begin{aligned}\Theta_{p,k}(t\wedge\tau_K^M) \le& \Theta_{p,k}(0) + C_p\int_0^{t\wedge\tau_K^M} \mathrm{E}_W[\overline{h}_k(s)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t\wedge\tau_K^M}\Theta_{p,k}(s)\,\mathrm{d}s\\ &+\int_0^{t\wedge\tau_K^M}\int_E \bar\xi P(\bar\xi)\,\mathrm{E}_W[\bar h_k^{2p}(s^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{t\wedge\tau_K^M}\int_E \bar\xi P(\bar\xi)\Theta_{p,k}(s-)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\end{aligned} \tag{6.2.17}$$

Since $s\le t\wedge\tau_K^M$, we have, for any process Y, $Y_s = Y_{s\wedge\tau_K^M}$, and obviously $t\le t\wedge\tau_K^M$, so we have

$$\begin{aligned}\Theta_{p,k}(t\wedge\tau_K^M) \le& \Theta_{p,k}(0) + C_p\int_0^{t} \mathrm{E}_W[\overline{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + C_p'\int_0^{t}\Theta_{p,k}(s\wedge\tau_K^M)\,\mathrm{d}s\\ &+\int_0^{t}\int_E \bar\xi P(\bar\xi)\,\mathrm{E}_W[\bar h_k^{2p}((s\wedge\tau_K^M)^-)]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^{t}\int_E \bar\xi P(\bar\xi)\Theta_{p,k}((s\wedge\tau_K^M)-)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\end{aligned} \tag{6.2.18}$$

Step 2

The last step is to bound

$$\mathrm{E}\left[\Theta_{p,k}(t)\right].$$

By setting

$$R_1 \overset{\text{def}}{=} \int \bar\xi P(\bar\xi)\,\mathrm{d}u\mu(\mathrm{d}z)$$

we have (using the isometry in F_p^1, *cf.* (1.2.4))

$$\begin{aligned}&\mathrm{E}\left[\Theta_{p,k}(t\wedge\tau_K^M)\right]\\
&\quad\le \mathrm{E}\left[\Theta_{p,k}(0)\right] + C_p\,\mathrm{E}\left[\int_0^t \mathrm{E}_W[\overline{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s\right] + C_p'\,\mathrm{E}\left[\int_0^t\Theta_{p,k}(s\wedge\tau_K^M)\,\mathrm{d}s\right]\\
&\qquad+\mathrm{E}\left[\int_0^t\int_E \bar\xi P(\bar\xi)\,\mathrm{E}_W[\bar h_k^{2p}((s\wedge\tau_K^M)^-))]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right] + \mathrm{E}\left[\int_0^t\int_E \bar\xi P(\bar\xi)\Theta_{p,k}((s\wedge\tau_K^M)^-))N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\right]\\
&\quad= \mathrm{E}\left[\Theta_{p,k}(0)\right] + C_p\int_0^t \mathrm{E}[\overline{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + C_p'\int_0^t \mathrm{E}[\Theta_{p,k}(s\wedge\tau_K^M)]\,\mathrm{d}s\\
&\qquad+R_1\int_0^t \mathrm{E}[\bar h_k^{2p}((s\wedge\tau_K^M)^-))]\,\mathrm{d}s + \int_0^t \mathrm{E}[\Theta_{p,k}((s\wedge\tau_K^M)^-))]\,\mathrm{d}s\\
&\quad= \mathrm{E}\left[\Theta_{p,k}(0)\right] + (C_p+R_1)\int_0^t \mathrm{E}[\overline{h}_k(s\wedge\tau_K^M)^{2p}]\,\mathrm{d}s + (C_p'+R_1)\int_0^t \mathrm{E}[\Theta_{p,k}(s\wedge\tau_K^M)]\,\mathrm{d}s\end{aligned}$$

With $A_p(T) \overset{\text{def}}{=} \mathrm{E}\left[\Theta_{p,k}(0)\right] + (C_p+R_1)\int_0^T \mathrm{E}[\overline{h}_k(s)^{2p}]\,\mathrm{d}s$, which, by virtue of (6.2.9), is a finite quantity, the Gronwall's lemma gives here :

$$\mathrm{E}\left[\Theta_{p,k}(t\wedge\tau_K^M)\right] \le A_p(T)\exp[(C_p'+R_1)t]$$

which proves the assertion (6.2.12).

•

To bound (in L^p, $p\ge 1$) the Sobolev norm $\left|\bar X_t^M\right|_l$, we will proceed by recurrence on $l\in\mathbb{N}^*$, and we will show in detail the case corresponding to the first order norm, since in this particular case, the structure of the general method already appears with lesser notations than used in the general case.

In all the following we will set

$$A_k(L) \stackrel{\text{def}}{=} ([\![1, L]\!] \times [\![1, d]\!])^k$$

and

$$A(L) \stackrel{\text{def}}{=} \bigcup_{k \in \mathbb{N}^*} A_k(L).$$

We will also need the following lemma's :

Lemma 6.2.2 *Let $n, p \geq 1$ and $F \in \mathcal{S}^d$. Then*

$$\mathrm{E}\left[|D^n F|^{2p}\right] \leq d^{np} \sqrt{\mathrm{E}\left[(J_t^M)^{2np}\right]} \sup_L \sqrt{\mathrm{E}\left[\max_{\alpha \in A_n(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha F|^{4p}\right]\right]}.$$

Proof : By definition

$$\mathrm{E}\left[|D^n F|^{2p}\right] = \mathrm{E}\left[\Big(\sum_{\alpha \in ([\![1, J_t^M]\!] \times [\![1,d]\!])^n} |\mathrm{D}_\alpha F|^2\Big)^p\right];$$

on the other hand

$$\mathrm{E}\left[|D^n F|^{2p}\right] = \sup_L \mathrm{E}\left[|D^n F|^{2p}\, \mathbb{1}_{J_t^M \leq L}\right]$$

and

$$\begin{aligned}
\mathrm{E}\left[|D^n F|^{2p}\, \mathbb{1}_{J_t^M \leq L}\right] &= \mathrm{E}\left[\Big(\sum_{\alpha \in ([\![1, J_t^M]\!] \times [\![1,d]\!])^n} |\mathrm{D}_\alpha F|^2\Big)^p \mathbb{1}_{J_t^M \leq L}\right] \\
&\leq \mathrm{E}\left[(dJ_t)^{n(p-1)} \sum_{\alpha = ((k_n, r_n), \ldots, (k_1, r_1)) \in A_n(L)} |\mathrm{D}_\alpha F|^{2p} \prod_{i=1}^n \mathbb{1}_{k_i \leq J_t^M \leq L}\right] \\
&\leq \mathrm{E}\left[(dJ_t)^{n(p-1)} \sum_{\alpha \in A_n(L)} \mathrm{E}_W\left[|\mathrm{D}_\alpha F|^{2p}\right]\right] \\
&\leq \mathrm{E}\left[d^{np} J_t^{np} \max_{\alpha \in A_n(L)} \mathrm{E}_W\left[|\mathrm{D}_\alpha F|^{2p}\right]\right] \\
&\leq d^{np} \sqrt{\mathrm{E}\left[(J_t^M)^{2np}\right]} \sqrt{\mathrm{E}\left[\max_{\alpha \in A_n(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha F|^{4p}\right]\right]}.
\end{aligned}$$

•

Lemma 6.2.3 *Let $j \geq 1$. Then there exists $C_{|\alpha|} > 0$ such that*

$$|\,\mathrm{D}_\alpha \bar{Z}_j| \leq C_{|\alpha|}.$$

Proof : For $|\alpha| = 1$, $\alpha = (k, r)$ and (recalling that $\pi_k^r = \phi_M(\bar{Z}_k)$)

$$\mathrm{D}_{k,r} \bar{Z}_j = \begin{pmatrix} \pi_k^r \partial_{\bar{Z}_k^r} \bar{Z}_j^1 \\ \vdots \\ \pi_k^r \partial_{\bar{Z}_k^r} \bar{Z}_j^r \\ \vdots \\ \pi_k^r \partial_{\bar{Z}_k^r} \bar{Z}_j^d \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \phi_M(\bar{Z}_k)\delta_{k,j} \\ \vdots \\ 0 \end{pmatrix} \tag{6.2.19}$$

so $|\,\mathrm{D}_{k,r} \bar{Z}_j| \leq \|\phi\|_\infty = 1$.

Since ϕ has bounded derivatives of any order, the recursive differentiation of (6.2.19) gives the general bounding property ; although this recursive differentiation is rather clear, we show the case $|\alpha| = 2$, to highlight the mechanism of it :

let $\alpha = ((m,n)(k,r))$, then (using (6.2.19)),

$$\mathrm{D}_\alpha \bar{Z}_j = \mathrm{D}_{m,n} \mathrm{D}_{k,r} \bar{Z}_j = \begin{pmatrix} 0 \\ \vdots \\ \pi^n_m \partial_{\bar{Z}^n_m} \big(\phi_M(\bar{Z}_k)\big)\delta_{k,j} \\ \vdots \\ 0 \end{pmatrix}$$

with

$$\pi^n_m \partial_{\bar{Z}^n_m} \big(\phi_M(\bar{Z}_k)\big)\delta_{k,j} = \phi_M(\bar{Z}_m)\partial_n \phi_M(\bar{Z}_k)\delta_{m,k}\delta_{k,j},$$

and a derivative of higher order will be of the following form :

$$\mathrm{D}_\alpha \bar{Z}_j = \begin{pmatrix} 0 \\ \vdots \\ \sum c \prod \partial_\beta \phi_M(\bar{Z}_l)) \\ \vdots \\ 0 \end{pmatrix}.$$

so (since the sum and the product are finite),

$$|\mathrm{D}_\alpha \bar{Z}_j| \leq \sum c \prod \|\partial_\beta \phi\|_\infty \overset{\text{def}}{=} C_{|\alpha|}.$$

•

Recalling the following notation (*cf.* Section 4.5.1) : for $1 \leq l \leq n$,

$$\mathcal{M}_n(l) \overset{\text{def}}{=} \Big\{ M = (M_1, \ldots, M_l), \bigcup_{i \in [\![1,l]\!]} M_i = \{1, \ldots, n\} \text{ and } M_i \cap M_j = \varnothing, \text{ for } i \neq j \Big\},$$

we have, in fact, a more precise result :

Lemma 6.2.4 *Let $k \geq 1$ and $\alpha = ((k_n, r_n), \ldots, (k_1, r_1))$.*

$$\mathrm{D}_\alpha \bar{Z}^r_k = \delta_{k_n,k} \cdots \delta_{k_1,k}\delta_{r_1,r} f_\alpha(\bar{Z}_k) \tag{6.2.20}$$

where (with $r \overset{def}{=} (r_n, \ldots, r_2)$)

$$f_\alpha(\bar{Z}_k) \overset{def}{=} \sum_{\beta = (\beta_1, \cdots, \beta_n) \in \mathcal{M}_{n-1}(n)} c_\beta \prod_{i=1}^n \partial_{\beta_i(r)} \phi_M(\bar{Z}_k) \tag{6.2.21}$$

with

- $c_\beta \in \mathbb{N}$ *;*
- *we denote again $\mathcal{M}_{n-1}(n)$, but, here, we allow β_i to be empty.*

Proof : By induction over the length k of α. •

6.3 First order norm

Proposition 6.3.1 *If $p \geq 1$, for all $T > 0$, exists a constant $C_{T,p} > 0$ such that*

$$\forall t \in [0,T], \qquad \|D^1 \bar{X}^M_t\|_{2p} \leq C_{T,p} \sqrt{\|J^M_T\|_{2p}}. \tag{6.3.22}$$

Proof :

First, from Lemma 6.2.2, we have

$$\mathrm{E}\left[\left|D^1\bar{X}_t^M\right|^{2p}\right] \le d^p\sqrt{\mathrm{E}\left[(J_T^M)^{2p}\right]}\sup_L\sqrt{\mathrm{E}\left[\max_{\alpha\in A_1(L)}\mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}_t^M|^{4p}\right]\right]}$$

Hence, to conclude, it remains to bound, independently of L, the quantity $\mathrm{E}\left[\max_{\alpha\in A_1(L)}\mathrm{E}_W\left[|\,\mathrm{D}_\alpha\,\bar{X}_t^M|^{4p}\right]\right]$. Recalling that $\bar{X}_t^M$ solves the following diffusion equation :

$$\bar{X}_t^M = x + \int_0^t \sigma(\bar{X}_s^M)\,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) + \int_0^t g(\bar{X}_s^M)\,\mathrm{d}s,$$

we have

$$\begin{aligned}\mathrm{D}_{k,r}\,\bar{X}_t = \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r}\,\bar{Z}_k + \sum_{l=1}^m \int_{T_k}^t \nabla\sigma_l(\bar{X}_s^M)\,\mathrm{D}_{k,r}\,\bar{X}_s^M\,\mathrm{d}W_s^l \\ + \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{k,r}\,\bar{X}_{T_j-}^M \\ + \int_{T_k}^t \nabla_x g(\bar{X}_s^M)\,\mathrm{D}_{k,r}\,\bar{X}_s^M\,\mathrm{d}s.\end{aligned}$$

We can then apply the bounding Lemma 6.2.1 with $\alpha = (k,r)$ and $\overline{V}_t^\alpha \stackrel{\text{def}}{=} \mathrm{D}_{k,r}\,\bar{X}_t$, since the Hypothesis 6.1 are well verified, for we have ($\alpha = (k,r)$)

$$G^\alpha = 0, \quad d_j^\alpha\big(\overline{Z}_j, \overline{V}_{(0)}(T_j^-)\big) = \partial_{z_r} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\Phi_M(\bar{Z}_j)\delta_{k,j}, \quad g^\alpha = 0$$

$$\rho_l^\alpha(\overline{V}_s^0)\overline{V}_s^\alpha = \nabla\sigma_l(\bar{X}_s^M), \quad \beta^\alpha\big(\overline{Z}_j, \overline{V}_{T_j^-}^0\big)\overline{V}_{T_j^-}^\alpha = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{k,r}\,\bar{X}_{T_j-}^M,$$

$$b^\alpha(\overline{V}_s^0)\overline{V}_s^\alpha = \nabla_x g(\bar{X}_s^M)\,\mathrm{D}_{k,r}\,\bar{X}_s^M$$

with (using Lemma 6.2.3)

$$|\partial_{z_r} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\Phi_M(\bar{Z}_j)\delta_{k,j}| \le C_1\bar{c}\big(\bar{Z}_j\big) \qquad \text{and} \qquad |\nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)| \le \bar{c}\big(\bar{Z}_j\big),$$

which completes the proof.

•

6.4 Norm of higher order

Following the very same path as before, we find, recalling that $\lambda_M \stackrel{\text{def}}{=} 2\bar{c}\mu(E_M)$ (where E_1 is chosen in order to have $\mu(E_1) > 0$)

Proposition 6.4.1 *If $p \ge 1$ and $l \in \mathbb{N}^*$, there exists a constant $C_{p,l,T} > 0$ such that*

$$\forall t \in [0,T], \qquad \big\|\,\big|\bar{X}_t^M\big|_l\,\big\|_{2p} \le C_{p,l,T}(1 + \sqrt{\|(J_T^M)^l\|_{2p}}), \tag{6.4.23}$$

and, consequently, there exists a constant $C'_{p,l,T} > 0$ such that[1]

$$\forall t \in [0,T], \qquad \big\|\,\big|\bar{X}_t^M\big|_l\,\big\|_{2p} \le C'_{p,l,T}\sqrt{(\lambda_M)^l}. \tag{6.4.24}$$

[1]We note, indeed, that $J_t^M \sim \mathcal{P}(t\lambda_M)$ which implies $\mathrm{E}\left[{J_t^M}^n\right] = P(t\lambda_M)$, where P is polynomial of degree n : when M is growing, (for $t \le T$), we have

$$\mathrm{E}\left[{J_t^M}^n\right] = O(\lambda_M^n).$$

Proof : We have

$$\mathrm{E}\left[\left|\bar X^M_t\right|^{2p}_l\right] \le C_p \sum_{k=0}^{l} \mathrm{E}\left[\left|D^k \bar X^M_t\right|^{2p}\right]$$

and, using Lemma 6.2.2,

$$C_p \sum_{k=0}^{l} \mathrm{E}\left[\left|D^k \bar X^M_t\right|^{2p}\right] \le C_p \sum_{k=0}^{l} d^{kp}\sqrt{\mathrm{E}\left[(J^M_T)^{2kp}\right]}\sup_L \sqrt{\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha \bar X^M_t|^{4p}\right]\right]}.$$

So, if we admit for a moment that, for $k \in [\![0,l]\!]$, $\sup_L \sqrt{\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha \bar X^M_t|^{4p}\right]\right]} \le C_{k,p}$, then

$$\mathrm{E}\left[\left|\bar X^M_t\right|^{2p}_l\right] \le C_p \sum_{k=0}^{l} d^{kp}\sqrt{\mathrm{E}\left[(J^M_T)^{2kp}\right]}C_{k,p} \le C_{p,l,T}(1+\sqrt{\mathrm{E}\left[(J^M_T)^{2lp}\right]}) \tag{6.4.25}$$

which proves the proposition.

Hence, to conclude, it remains to bound, independently of L, the quantity $\mathrm{E}\left[\max_{\alpha\in A_k(L)} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha \bar X^M_t|^{4p}\right]\right]$, which will be done by recurrence on $|\alpha|$ in the next lemma.

•

Lemma 6.4.2 *Let $p \ge 1$, and $n \in \mathbb{N}$ and ; there exists $C_{n,p}$ such that*

$$\sup_L \mathrm{E}\left[\max_{\substack{\alpha\in A(L)\\ |\alpha|\le n}} \mathrm{E}_W\left[|\,\mathrm{D}_\alpha \bar X^M_t|^{2p}\right]\right] \le C_{n,p}. \tag{6.4.26}$$

Proof :

The case $|\alpha| = 1$ is corresponding to the first order norm case.

Else, starting again from

$$\bar X^M_t = x + \int_0^t \sigma(\bar X^M_s)\,\mathrm{d}W_s + \sum_{j=1}^{J^M_t} c_M(\bar Z_j, \bar X^M_{T_j-}) + \int_0^t g(\bar X^M_s)\,\mathrm{d}s,$$

we have (using Lemma 4.5.3, with

$$\sum_{(1)} \overset{\text{def}}{=} \sum_{l=2}^{k} \sum_{\substack{\beta=(\beta_1,\dots,\beta_l)\\ \beta_i\in[\![1,d]\!]}} \sum_{M\in\mathcal{M}_k(l)} \qquad \text{and} \qquad \sum_{(2)} \overset{\text{def}}{=} \sum_{l=1}^{k} \sum_{\substack{\beta=(\beta_1,\dots,\beta_r,\beta'_{r+1},\dots,\beta'_l)\\ \beta_i\in[\![1,d]\!],\ \beta'_j\in[\![d+1,2d]\!]}} \sum_{M\in\mathcal{M}_k(l)},$$

where $k \overset{\text{def}}{=} |\alpha|$, in order to shorten the equation)

$$\begin{aligned}
\mathrm{D}_\alpha \bar X^M_t =& \sum_{l=1}^{m}\int_0^t \mathrm{D}_\alpha(\sigma_l(\bar X^M_s))\,\mathrm{d}W^l_s + \sum_{j=1}^{J^M_t} \mathrm{D}_\alpha(c_M(\bar Z_j, \bar X^M_{T_j-})) + \int_0^t \mathrm{D}_\alpha(g(\bar X^M_s))\,\mathrm{d}s\\
=& \sum_{l=1}^{m}\int_0^t \sum_{(1)} \partial_\beta\sigma_l(\bar X^M_s)\,\mathrm{D}_{M_1(\alpha)}(\bar X^M_s)_{\beta_1}\cdots \mathrm{D}_{M_l(\alpha)}(\bar X^M_s)_{\beta_l} + \nabla\sigma_l(\bar X^M_s)\,\mathrm{D}_\alpha(\bar X^M_s)\,\mathrm{d}W^l_s\\
&+ \sum_{j=1}^{J^M_t}\sum_{(2)} \partial_\beta c_M(\bar Z_j, \bar X^M_{T_j-})\,\mathrm{D}_{M_1(\alpha)}\,\bar Z_j^{\beta_1}\cdots \mathrm{D}_{M_r(\alpha)}\,\bar Z_j^{\beta_r} \times \mathrm{D}_{M_{r+1}(\alpha)}(\bar X^M_{T_j-})_{d-\beta'_{r+1}}\cdots \mathrm{D}_{M_l(\alpha)}(\bar X^M_{T_j-})_{d-\beta'_l}\\
&\qquad + \nabla_z c_M(\bar Z_j, \bar X^M_{T_j-})\,\mathrm{D}_\alpha(\bar Z_j) + \nabla_x c_M(\bar Z_j, \bar X^M_{T_j-})\,\mathrm{D}_\alpha(\bar X^M_{T_j-})\\
&+ \int_0^t \sum_{(1)} \partial_\beta(g(\bar X^M_s))\,\mathrm{D}_{M_1(\alpha)}(\bar X^M_s)_{\beta_1}\cdots \mathrm{D}_{M_l(\alpha)}(\bar X^M_s)_{\beta_l} + \nabla g(\bar X^M_s)\,\mathrm{D}_\alpha(\bar X^M_s)\,\mathrm{d}s
\end{aligned} \tag{6.4.27}$$

Then we apply the upper bound Lemma 6.2.1 with $\overline{V}_t^\alpha \overset{\text{def}}{=} \mathrm{D}_\alpha \bar{X}_t$ (and consequently $\overline{V}_{(k-1)}(t) = \Big(\mathrm{D}_\beta \bar{X}_t\Big)_{|\beta|<|\alpha|}$). Using Lemma 6.2.3, it follows that the Hypothesis 6.1 are well verified ; for example :

$$G_l^\alpha\big(\overline{V}_{(k-1)}(s)\big) = G_l^\alpha\Big(\Big(\mathrm{D}_\beta \bar{X}_t\Big)_{|\beta|<|\alpha|}\Big)$$
$$\overset{\text{def}}{=} \sum_{(1)} \partial_\beta(\sigma_l(\bar{X}_s^M))\,\mathrm{D}_{M_1(\alpha)}(\bar{X}_s^M)_{\beta_1}\cdots \mathrm{D}_{M_l(\alpha)}(\bar{X}_s^M)_{\beta_l}$$

so there exists $w \in \mathbb{N}$ such that
$$|G^\alpha(v)| \leq K(1+|v|)^w;$$

and

$$d_j^\alpha\big(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-)\big)$$
$$= d_j^\alpha\big(\overline{Z}_j, \Big(\mathrm{D}_\beta \bar{X}_{T_j^-}\Big)_{|\beta|<|\alpha|}\big)$$
$$\overset{\text{def}}{=} \sum_{(2)} \partial_\beta c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{M_1(\alpha)}\,\bar{Z}_j^{\beta_1}\cdots \mathrm{D}_{M_r(\alpha)}\,\bar{Z}_j^{\beta_r} \times \mathrm{D}_{M_{r+1}(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_{r+1}}\cdots \mathrm{D}_{M_l(\alpha)}(\bar{X}_{T_j-}^M)_{d-\beta'_l}.$$

Notice that it is legitimate to consider each $\mathrm{D}_{M_u(\alpha)}\,\bar{Z}_j^{\beta_u}$ as a function of j and $\bar{Z}_j$ since, using directly the Lemma 6.2.4

$$\mathrm{D}_{M_u(\alpha)}\,\bar{Z}_j^{\beta_u} = \delta_{(M_u(\alpha))_n^1,j}\cdots\delta_{(M_u(\alpha))_1^1,j}\delta_{(M_u(\alpha))_1^2,\beta_u} f_\alpha(\bar{Z}_j)$$

where f_α is defined in (6.2.21).

Since, from the Lemma 6.2.3, every $|\,\mathrm{D}_{M_i(\alpha)}\,\bar{Z}_j^{\gamma_i}|$ is bounded, there comes an inequality of the form
$$|d_j^\alpha(\overline{Z}_j, \overline{V}_{(k-1)}(T_j^-))| \leq K\bar{c}(\overline{Z}_j)(1+|\overline{V}_{(k-1)}(T_j^-)|)^w,$$
and likewise to the other quantities.

•

6.5 Operator L

Proposition 6.5.1 *For $p \leq 1$, and all $l \in \mathbb{N}^*$, it exists $N_{T,l,p} > 0$ such that*

$$\big\||\,\mathrm{L}\,\bar{X}_t^M\big|_l\big\|_{2p} \leq N_{T,l,p}\big(\lambda_M\big)^{(l+2)^2}, \tag{6.5.28}$$

with $\lambda_M = \mu(E_M)$.

Proof :

In the following C_p, C_l, $C_{p,l}$ are "flying constants" which may change during the calculation ; we have

$$\mathrm{L}(F) = -\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} \partial_{k,r}(\pi_{k,r})\,\mathrm{D}_{k,r}\,F + \mathrm{D}_{k,r}(\mathrm{D}_{k,r}\,F) + \mathrm{D}_{k,r}\ln p_J\,\mathrm{D}_{k,r}\,F, \tag{6.5.29}$$

hence, using the fact that $|.|_l$ is a norm (and with $|AB|_l \le C_l|A|_l|B|_l$: *cf.* 4.5.7)

$$\begin{aligned}
|\operatorname{L}(F)|_l &\le \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r}) \operatorname{D}_{k,r} F|_l + |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}\ln p_J \operatorname{D}_{k,r} F|_l \\
&\le C_l \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r})|_l |\operatorname{D}_{k,r} F|_l + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l + C_l \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}\ln p_J||\operatorname{D}_{k,r} F|_l \\
&\le C_l \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\partial_{k,r}(\pi_{k,r})|_l^2 + |\operatorname{D}_{k,r} F|_l^2 + \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l + C_l \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}\ln p_J|_l^2 + |\operatorname{D}_{k,r} F|_l^2 \\
&\le \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l + C_l(|\pi_{k,r}|_{l+1}^2 + |\ln p_J|_{l+1}^2 + 2|F|_{l+1}^2)
\end{aligned}$$

(the last inequality follows from the fact that $\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r} G|_l^2 = \sum_{k=1}^{J_t^M}\sum_{r=1}^{d}\sum_{|\alpha|\le l} |\operatorname{D}_\alpha \operatorname{D}_{k,r} G|^2 \le \sum_{|\alpha|\le l+1} |\operatorname{D}_\alpha G|^2 = |G|_{l+1}^2$), which implies :

$$\begin{aligned}
|\operatorname{L}(F)|_l^{2p} &\le C_p, l\Big(\Big(\sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l\Big)^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big) \\
&\le C_p, l\Big(\Big(J_t^M \sum_{k=1}^{J_t^M}\sum_{r=1}^{d} |\operatorname{D}_{k,r}(\operatorname{D}_{k,r} F)|_l^2\Big)^{p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big) \\
&\le C_p, l\Big((J_t^M)^p |F|_{l+2}^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_J|_{l+1}^{4p}\Big).
\end{aligned}$$

And, finally (noticing that, as a consequence of Lemma 6.2.3, $|\pi_{k,r}|_{l+1}^{4p} \le C_{l,p}$),

$$\operatorname{E}\Big[|\operatorname{L}(F)|_l^{2p}\Big] \le C_p, l\Big(1 + \sqrt{\operatorname{E}\big[(J_t^M)^{2p}\big]}\sqrt{\operatorname{E}\big[|F|_{l+2}^{4p}\big]} + \operatorname{E}\big[|F|_{l+1}^{4p}\big] + \operatorname{E}\big[|\ln p_J|_{l+1}^{4p}\big]\Big) \qquad (6.5.30)$$

We put $F = \bar{X}_t^M$. Let us recall that, from (6.4.25)

$$\operatorname{E}\Big[|\bar{X}_t^M|_{l+1}^{4n}\Big] \le C_{p,l,T}(1 + \sqrt{\operatorname{E}\big[(J_T^M)^{4n(l+1)}\big]}) \le C'_{p,l,T}\sqrt{\lambda_M)^{4n(l+1)}}.$$

Then, if we admit for the moment Lemma 6.5.2, we obtain the following upper bound :

$$\operatorname{E}\Big[|\operatorname{L}(\bar{X}_t^M)|_l^{2p}\Big] \le N_{T,l,p}(\lambda_M)^{2p(l+2)^2},$$

which ends the proof. •

Hence, all that remains is to prove the following result :

Lemma 6.5.2 *Let $l, n \in \mathbb{N}$.*

$$\operatorname{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \le C_{q_M,l,n,T}\big(\mu(E_M)\big)^{n(l+2)^2}.$$

Proof :

We recall that

$$\ln p_J = \sum_{j=1}^{J_T^M} \ln q_M(\bar{Z}_j, \bar{X}_{T_j-}^M),$$

and recalling that $\ln q_M$ has bounded derivatives of any order, using Lemma 5.4.1, which implies, by corollary 4.5.9, the existence of $C_{q_M,l} > 0$ such that

$$|q_M(F)|_{l+1} \le C_{q_M,l}\big(1 + |F|_{l+1} + |F|_l^{l+1}\big)$$

we have,

$$
\begin{aligned}
|\ln p_J|_{l+1} &\leq \sum_{j=1}^{J_T^M} |\ln q_M(\bar{Z}_j, \bar{X}^M_{T_j-})|_{l+1} \\
&\leq C_{q_M,l} \sum_{j=1}^{J_T^M} 1 + |(\bar{Z}_j, \bar{X}^M_{T_j-})|_{l+1} + |(\bar{Z}_j, \bar{X}^M_{T_j-})|_l^{l+1} \\
&\leq C_{q_M,l} \Big(J_T^M + \sum_{j=1}^{J_T^M} |\bar{Z}_j|_{l+1} + |\bar{X}^M_{T_j-}|_{l+1} + 2^l (|\bar{Z}_j|_l^{l+1} + |\bar{X}^M_{T_j-}|_l^{l+1}) \Big) \\
&\leq C'_{q_M,l} \Big(J_T^M + \sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l+1} + |\bar{X}^M_{T_j-}|_l^{l+1} \Big)
\end{aligned}
$$

it directly follows

$$
\begin{aligned}
\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n} \Big] &\leq C_{q_M,l,n} \Big(\mathrm{E}\big[(J_T^M)^{2n} \big] + \mathrm{E}\Big[\Big(J_T^M \sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l+1}^2 + (|\bar{X}^M_{T_j-}|_l^{l+1})^2 \Big)^n \Big] \Big) \\
&\leq C_{q_M,l,n} \Big(\mathrm{E}\big[(J_T^M)^{2n} \big] + \mathrm{E}\Big[\Big(J_T^M \Big)^n \Big(\sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l+1}^2 + |\bar{X}^M_{T_j-}|_l^{2(l+1)} \Big)^n \Big] \Big) \\
&\leq C_{q_M,l,n} \Big(\mathrm{E}\big[(J_T^M)^{2n} \big] + \sqrt{\mathrm{E}\Big[\Big(J_T^M \Big)^{2n} \Big]} \sqrt{\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l+1}^2 + |\bar{X}^M_{T_j-}|_l^{2(l+1)} \Big)^{2n} \Big]} \Big)
\end{aligned}
$$

Our aim is then to bound the second term of the rhs of this last inequality.

We have

$$
|\bar{X}^M_{T_j-}|_{l+1}^2 = \sum_{|\alpha| \leq l+1} |\, \mathrm{D}_\alpha \bar{X}^M_{T_j-}|^2
$$

and from (6.4.27) we know that we can put $\overline{V}_t^\alpha \overset{\text{def}}{=} \mathrm{D}_\alpha \bar{X}_t$ with $\overline{V}_t^\alpha$ defined in (6.2.1) ; then, there exists a process V_t^α with the same law and verifying (6.2.5). So :

$$
\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l+1}^2 \Big)^{2n} \Big] &= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} \sum_{|\alpha| \leq l+1} |\, \mathrm{D}_\alpha \bar{X}^M_{T_j-}|^2 \Big)^{2n} \Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} \sum_{|\alpha| \leq l+1} |\, \mathrm{D}_\alpha \bar{X}^M_{T_j-}|^2 \Big)^{2n} \Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} \sum_{|\alpha| \leq l+1} |\overline{V}^\alpha_{T_j-}|^2 \Big)^{2n} \Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} \sum_{|\alpha| \leq l+1} |V^\alpha_{T_j-}|^2 \Big)^{2n} \Big] \\
&\leq C_n \sqrt{\mathrm{E}\Big[(J_T^M)^{4n-2} \Big]} \sqrt{\mathrm{E}\Big[\sum_{j=1}^{J_T^M} \Big(\sum_{|\alpha| \leq l+1} |V^\alpha_{T_j-}|^2 \Big)^{2n} \Big]}.
\end{aligned}
$$

In the same way,

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\bar{X}^M_{T_j-}|_l^{2(l+1)}\Big)^{2n}\Big] \leq C'_n\sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{4n-2}\Big]}\sqrt{\mathrm{E}\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq l}|V^\alpha_{T_j-}|^2\Big)^{2n(l+1)}\Big]}.$$

But, using the F_p^1 isometry, with $v,\, w \in \mathbb{N}^*$,

$$\begin{aligned}
\mathrm{E}\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\leq v}|V^\alpha_{T_j-}|^2\Big)^{w}\Big] &= \mathrm{E}\Big[\int_0^t\int_{E_M\times[0,2\bar{C}]}\Big(\sum_{|\alpha|\leq v}|V^\alpha_{s-}|^2\Big)^{w}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\Big]\\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\leq v}|V^\alpha_{s-}|^2\Big)^{w}\,\mathrm{d}s\Big]\\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\leq v}|\overline{V}^\alpha_{s-}|^2\Big)^{w}\,\mathrm{d}s\Big]\\
&= 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t|\bar{X}^M_{s-}|_v^{2w}\,\mathrm{d}s\Big]\\
&= 2\bar{C}\mu(E_M)\int_0^t\mathrm{E}\big[|\bar{X}^M_{s}|_v^{2w}\big]\,\mathrm{d}s.
\end{aligned}$$

Now from (6.4.25) we have

$$\mathrm{E}\Big[\big|\bar{X}^M_t\big|_v^w\Big] \leq C_{p,l,T}(1+\sqrt{\mathrm{E}\big[(J_T^M)^{vw}\big]})$$

Gathering these results, we obtain :

$$\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \leq C_{q_M,l,n,T}\Big(\,\mathrm{E}\big[(J_T^M)^{2n}\big]+\sqrt{\mathrm{E}\Big[\Big(J_T^M\Big)^{2n}\Big]}\Big(\mu(E_M)\,\mathrm{E}\Big[\big(J_T^M\big)^{4n-2}\Big]\sqrt{\mathrm{E}\big[(J_T^M)^{4n(l+1)l}\big]}\Big)^{\frac{1}{4}}\Big)$$

Since (for $t\leq T$),

$$\mathrm{E}\Big[J_t^{M\,n}\Big] = \underset{M\to+\infty}{O}(\lambda_M^n),$$

we have

$$\mathrm{E}\Big[|\ln p_J|_{l+1}^{2n}\Big] \leq C_{q_M,l,n,T}\big(\mu(E_M)\big)^{\frac{1}{4}}\big(\mu(E_M)\big)^{2n-1+\frac{n(l+1)l}{2}} \leq C_{q_M,l,n,T}\big(\mu(E_M)\big)^{n(l+2)^2}$$

•

6.6 The covariance matrix

6.6.1 Preliminaries

We consider a Poisson point measure $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ on $\mathbb{R}^d\times\mathbb{R}$; with compensator $\mu(\mathrm{d}z)\times\mathbb{1}_{(0,\infty)}(u)\,\mathrm{d}u$ and two non-negative measurable functions $f,g:\mathbb{R}^d\to\mathbb{R}_+$. For a measurable set $B\subset\mathbb{R}^d$ we denote $B_g=\{(z,u):z\in B,u<g(z)\}\subset\mathbb{R}^d\times\mathbb{R}_+$, and we consider the process

$$N_t(\mathbb{1}_{B_g}f) \stackrel{\text{def}}{=} \int_0^t\int_{B_g}f(z)N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Moreover we note $\nu_g(\mathrm{d}z)=g(z)\mu(\mathrm{d}z)$ and

$$\alpha_{g,f}(s)=\int_{\mathbb{R}^d}(1-e^{-sf(z)})\,\mathrm{d}\nu_g(\mathrm{d}z),\qquad \beta_{B,g,f}(s)=\int_{B^c}(1-e^{-sf(z)})\,\mathrm{d}\nu_g(\mathrm{d}z)$$

We have the following result.

Lemma 6.6.1 *Let* $\phi(s) = \mathrm{E}\left[e^{-sN_t(\mathbb{1}_{B_g}f)}\right]$ *the Laplace transform of the random variable* $N_t(\mathbb{1}_{B_g}f)$ *then we have*

$$\phi(s) = e^{-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))}.$$

Proof : From Itô's formula we have

$$\exp(N_t(\mathbb{1}_{B_g}f)) = 1 - \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}_+} \exp(-s(N_{r-}(\mathbb{1}_{B_g}f)))(1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\,\mathrm{d}N(r,z,u)$$

and consequently

$$\mathrm{E}\left[\exp(N_t(\mathbb{1}_{B_g}f))\right] = 1 - \int_0^t \mathrm{E}\left[\exp(-s(N_{r-}(\mathbb{1}_{B_g}f)))\right]\mathrm{d}r \int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\mu(\mathrm{d}z)\,\mathrm{d}u.$$

But

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}_+} (1-\exp(-sf(z)\mathbb{1}_{B_g}(z,u)))\mu(\mathrm{d}z)\,\mathrm{d}u &= \int_{\mathbb{R}^d\times\mathbb{R}_+} \mathbb{1}_{B_g}(z,u)(1-\exp(-sf(z)))\mu(\mathrm{d}z)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} \mathbb{1}_B(z)(1-\exp(-sf(z)))\int_{\mathbb{R}_+} \mathbb{1}_{\{u<g(z)\}}\mu(\mathrm{d}z)\,\mathrm{d}u \\ &= \int_B (1-\exp(-sf(z)))g(z)\mu(\mathrm{d}z) = \alpha_{g,f}(s) - \beta_{B,g,f}(s),\end{aligned}$$

It follows that

$$\mathrm{E}\left[\exp(N_t(\mathbb{1}_{B_g}f))\right] = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

□

We consider an abstract measurable space E, a measure ν on this space and a non-negative measurable function $f : E \to \mathbb{R}_+$, such that $\int f\,\mathrm{d}\nu < \infty$. For $t > 0$ and $p \geq 1$ we notice

$$\alpha_f(t) = \int_E (1 - e^{-tf(a)})\,\mathrm{d}\nu(a), \qquad \text{and} \qquad I_t^p(f) = \int_0^{+\infty} s^{p-1}e^{-t\alpha_f(s)}\,\mathrm{d}s.$$

Lemma 6.6.2 1. *Suppose that for* $p \geq 1$ *and* $t > 0$

$$\liminf_{u\to\infty} \frac{1}{\ln u}\alpha_f(u) > \frac{p}{t} \tag{6.6.31}$$

then

$$I_t^p(f) < \infty.$$

2. *A sufficient condition for* (6.6.31) *is*

$$\liminf_{u\to\infty} \frac{1}{\ln u}\nu\Big(f \geq \frac{1}{u}\Big) > \frac{p}{t}. \tag{6.6.32}$$

In particular, if $\liminf_{u\to\infty} \frac{1}{\ln u}\nu\Big(f \geq \frac{1}{u}\Big) = \infty$, *then* $\forall p \geq 1$ *and* $\forall t > 0$,

$$I_t^p(f) < \infty.$$

We notice that if ν is finite then (6.6.32) cannot be satisfied.

Proof : 1) From (6.6.31) one can find $\varepsilon > 0$ such that as s goes to infinity $s^{p-1}e^{-t\alpha_f(s)} \leq \frac{1}{s^{1+\varepsilon}}$ and consequently $I_t^p(f) < \infty$.

2) With the notation $n(\mathrm{d}z) = \nu \circ f^{-1}(\mathrm{d}z)$, we have

$$\alpha_f(u) = \int_0^{+\infty} (1 - e^{-uz}) \, \mathrm{d}n(z) = \int_0^{+\infty} e^{-y} n\Big(\frac{y}{u}, \infty\Big) \, \mathrm{d}y.$$

Using Fatou's lemma and (6.6.32), we obtain

$$\liminf_{u\to\infty} \frac{1}{\ln u} \int_0^{+\infty} e^{-y} n\Big(\frac{y}{u}, \infty\Big) \, \mathrm{d}y \geq \int_0^{+\infty} e^{-y} \liminf_{u\to\infty} \frac{1}{\ln u} n\Big(\frac{y}{u}, \infty\Big) \, \mathrm{d}y > \frac{p}{t}.$$

□

We consider the Poisson point measure $N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $\mathbb{R}^d \times \mathbb{R}_+$ with compensator $\mu(\mathrm{d}z) \times \mathbb{1}_{(0,\infty)}(u) \, \mathrm{d}u$. We recall that

$$N_t(\mathbb{1}_{B_g} f) \stackrel{\text{def}}{=} \int_0^t \int_{B_g} f(z) N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

for $f, \, g : \mathbb{R}^d \to \mathbb{R}_+$ and $B_g = \{(z,u) : z \in B, u < g(z)\} \subset \mathbb{R}^d \times \mathbb{R}_+$ and that (with $\nu_g(\mathrm{d}z) \stackrel{\text{def}}{=} g(z)\mu(\mathrm{d}z)$)

$$\alpha_{g,f}(s) = \int_{\mathbb{R}^d} \big(1 - e^{-sf(z)}\big) \, \mathrm{d}\nu_g(\mathrm{d}z), \qquad \beta_{B,g,f}(s) = \int_{B^c} \big(1 - e^{-sf(z)}\big) \, \mathrm{d}\nu_g(\mathrm{d}z).$$

We have the following result (with $\Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-s} \, \mathrm{d}s$).

Lemma 6.6.3 *Let $U_t = t \int_{B^c} f(z) \, \mathrm{d}\nu_g(z)$, then, for all $p \geq 1$,*

$$\mathrm{E}\left[\frac{1}{\big(N_t(\mathbb{1}_{B_g} f) + U_t\big)^p}\right] \leq \frac{1}{\Gamma(p)} \int_0^{\infty} s^{p-1} \exp(-t\alpha_{g,f}(s)) \, \mathrm{d}s. \tag{6.6.33}$$

If it is supposed that, for some $0 < \theta \leq \infty$,

$$\liminf_{a\to\infty} \frac{1}{\ln a} \nu_g\Big(f \geq \frac{1}{a}\Big) = \theta, \tag{6.6.34}$$

then for every $t > 0$ and $p \geq 1$ such that $\frac{p}{t} < \theta$

$$\mathrm{E}\left[\frac{1}{\big(N_t(\mathbb{1}_{B_g} f) + U_t\big)^p}\right] < \infty.$$

Proof : By a change of variables we obtain for every $\lambda > 0$,

$$\lambda^{-p}\Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-\lambda s} \, \mathrm{d}s.$$

Taking the expectation in the previous equality with $\lambda = N_t(\mathbb{1}_{B_g} f) + U_t$ we obtain

$$\mathrm{E}\left[\frac{1}{\big(N_t(\mathbb{1}_{B_g} f) + U_t\big)^p}\right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} s^{p-1} \, \mathrm{E}\left[\exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t))\right] \mathrm{d}s.$$

Now from Lemma 6.6.1 we have

$$\mathrm{E}\left[\exp(-sN_t(\mathbb{1}_{B_g} f))\right] = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

Moreover, from the definition of U_t one can easily verify that $\exp(-sU_t) \leq \exp(-t\beta_{B,g,f}(s))$ and then

$$\mathrm{E}\left[\exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t))\right] \leq \exp(-t\alpha_{g,f}(s))$$

this completes the proof of (6.6.33). The second part of the lemma follows directly from Lemma 6.6.2.

□

6.6.2 The Malliavin covariance matrix

In this section, we prove, under some additional assumptions on p and t, that $\mathrm{E}\left[\frac{1}{|\det\ \sigma(F_M)|^p}\right]$ is bounded (uniformly on M), for the Malliavin matrix $\sigma(F_M)$ defined at the definition 4.5.

From the diffusion equation (5.4.17)

$$\bar{X}_t^M = x + \int_0^t \sigma(\bar{X}_s^M)\,\mathrm{d}W_s + \sum_{j=1}^{J_t^M} c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) + \int_0^t g(\bar{X}_s^M)\,\mathrm{d}s,$$

let us consider the tangent flow

$$Y_t^M = \mathrm{Id} + \sum_{l=1}^m \int_0^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M\,\mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M + \int_0^t \nabla_x g(\bar{X}_s^M) Y_s^M\,\mathrm{d}s.$$

We then define the following process (with $\nabla_x c_j = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)$) :

$$\hat{Y}_t^M \stackrel{\text{def}}{=} \mathrm{Id} - \sum_{l=1}^m \int_0^t \hat{Y}_s^M \nabla\sigma_l(\bar{X}_s^M)\,\mathrm{d}W_s^l - \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M \nabla_x c_j (\mathrm{Id} + \nabla_x c_j)^{-1} + \int_0^t \hat{Y}_s^M \Big(\frac{1}{2}\sum_{l=1}^m \nabla\sigma_l(\bar{X}_s^M)^2 - \nabla_x g(\bar{X}_s^M)\Big)\,\mathrm{d}s.$$

Lemma 6.6.4 *We have, for all $t \geq 0$,*

$$Y_t^M \hat{Y}_t^M = \mathrm{Id}\,. \tag{6.6.35}$$

Proof : The proof is postponed in the Appendix C. •

Lemma 6.6.5 *Assuming hypothesis 5.1, 5.2, 5.3 we have, for $p \geq 1$, $T > t > 0$ such that $\frac{2dp}{t} < \theta$*

$$\mathrm{E}\left[\frac{1}{|\det\ \sigma(F_M)|^p}\right] \leq C_p, \tag{6.6.36}$$

where the constant C_p does not depend on M.

Proof :

Since

$$\mathrm{D}_{k,r}\,\bar{X}_t = \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r}\,\bar{Z}_k + \sum_{l=1}^m \int_0^t \nabla\sigma_l(\bar{X}_s^M)\,\mathrm{D}_{k,r}\,\bar{X}_s^M\,\mathrm{d}W_s^l$$
$$+ \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)\,\mathrm{D}_{k,r}\,\bar{X}_{T_j-}^M + \int_0^t \nabla_x g(\bar{X}_s^M)\,\mathrm{D}_{k,r}\,\bar{X}_s^M\,\mathrm{d}s.$$

We have

$$Y_t^M = Y_{T_k}^M + \sum_{l=1}^m \int_{T_k}^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M\,\mathrm{d}W_s^l + \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M + \int_{T_k}^t \nabla_x g(\bar{X}_s^M) Y_s^M\,\mathrm{d}s$$

and, with $A_k = \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r}\,\bar{Z}_k$,

$$Y_t^M \hat{Y}_{T_k}^M A_k = \underbrace{Y_{T_k}^M \hat{Y}_{T_k}^M}_{=\mathrm{Id}} A_k + \sum_{l=1}^m \int_{T_k}^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M \hat{Y}_{T_k}^M A_k\,\mathrm{d}W_s^l$$
$$+ \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M \hat{Y}_{T_k}^M A_k + \int_{T_k}^t \nabla_x g(\bar{X}_s^M) Y_s^M \hat{Y}_{T_k}^M A_k\,\mathrm{d}s.$$

Then

$$\mathrm{D}_{k,r}\,\bar{X}_t = Y_t^M \hat{Y}_{T_k}^M A_k = Y_t^M \hat{Y}_{T_k}^M \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r}\,\bar{Z}_k.$$

Therefore

$$\begin{aligned}
\sum_{r=1}^{d} \langle \mathrm{D}_{k,r}\,\bar{X}_t, \xi\rangle^2 &= \sum_{r=1}^{d} \langle Y_t^M \hat{Y}_{T_k}^M \nabla_z c_M(\bar{Z}_k, \bar{X}_{T_k-}^M)\,\mathrm{D}_{k,r}\,\bar{Z}_k, \xi\rangle^2 \\
&= \sum_{r=1}^{d} \pi_k^2 \langle \partial_{z^r} c_M(\bar{Z}_k, \bar{X}_{T_k-}^M), (Y_t^M \hat{Y}_{T_k}^M)^* \xi\rangle^2 \\
&\geq \sum_{r=1}^{d} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \langle \partial_{z^r} c(\bar{Z}_k, \bar{X}_{T_k-}^M), (Y_t^M \hat{Y}_{T_k}^M)^* \xi\rangle^2
\end{aligned}$$

since $\pi_k \geq \mathbb{1}_{B_{M-1}}(\bar{Z}_k)$ and $c_M = c$ on B_{M-1} ; using Hypothesis 5.3 item 3., it follows that

$$\rho_t \geq \inf_{|\xi|=1} \sum_{r=1}^{J_t^M} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \underline{c}^2(\bar{Z}_k) |(Y_t^M \hat{Y}_{T_k}^M)^* \xi|^2 \geq (\|Y_{T_k}^M \hat{Y}_t^M\|)^{-2} \sum_{r=1}^{J_t^M} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \underline{c}^2(\bar{Z}_k).$$

With $\sigma(F_M) = \sigma(\bar{X}_t^M) + U_M(t)$, we have[2]

$$\mathrm{E}\left[\left|\frac{1}{\det\,\sigma(F_M)}\right|^p\right] \leq \mathrm{E}\left[\left|\frac{1}{\rho_t + U_M(t)}\right|^{dp}\right] \leq \mathrm{E}\left[\left(\frac{1 + (\|Y_{T_k}^M \hat{Y}_t^M\|)^2}{\sum_{r=1}^{J_t^M} \mathbb{1}_{B_{M-1}}(\bar{Z}_k) \underline{c}^2(\bar{Z}_k) + U_M(t)}\right)^{dp}\right]$$

Now observe that the denominator of the last fraction is equal in law to

$$\sum_{r=1}^{J_t^M} \mathbb{1}_{B_{M-1}}(Z_k) \underline{c}^2(Z_k) \mathbb{1}_{U_k < \gamma\left(Z_k, X_{T_k-}^M\right)} + U_M(t) \geq N_t(\mathbb{1}_{B_{\underline{\gamma}}^M} \underline{c}^2) + U_M(t),$$

with $B_{\underline{\gamma}}^M = \{(z,u) : z \in B_{M-1}, u < \underline{\gamma}(z)\}$. Assuming Hypothesis 5.3 item 3., we can apply Lemma 6.6.3 with $f = \underline{c}^2$ and $\mathrm{d}\nu(z) = \underline{\gamma}(z)\mu(\mathrm{d}z)$. This gives $p' \geq 1$ such that $\frac{p'}{t} < \theta$

$$\mathrm{E}\left[\left(\frac{1}{N_t(\mathbb{1}_{B_{\underline{\gamma}}^M} \underline{c}^2) + U_M(t)}\right)^{p'}\right] \leq C_{p'}.$$

Finally, since the moments of $||\hat{Y}_t^M||$ are bounded uniformly on M, the result follows from Cauchy-Schwarz inequality :

$$\mathrm{E}\left[\frac{1}{|\det\,\sigma(F_M)|^p}\right] \leq C_p.$$

•

6.7 Bounding the weights

Lemma 6.7.1 *Let $q, p \in \mathbb{N}^*$ and $T > 0$. For $T > t > 0$ with $\frac{4d(3q-1)}{t} < \theta$, there exists a constant $C_{p,q,T}$ such that*

$$\|H_\beta^q(F_M)\|_p \leq C_{p,q,T} \lambda_M^{q^2(4q+6d+9)}. \tag{6.7.37}$$

[2] If a, b, c and d are non-negative real numbers,

$$\frac{1}{\frac{a}{c} + d} \leq \frac{1+c}{a+d}.$$

Proof : Let $q \in N^*$ and $\beta = (\beta_1, \ldots, \beta_q)$ a multi-index. We have to bound $H^q_\beta(F_M) \stackrel{\text{def}}{=} H^q_\beta(F_M, 1)$; from Theorem 4.5.10 there exists a universal constant $C_{q,d}$ such that (recalling that $F_M \in \mathbb{R}^d$)

$$\left|H^q_\beta(F_M)\right| \leq C_{q,d} \frac{1}{|\det \sigma(F_M)|^{3q-1}} \big(1 + |F_M|_{q+1})^{(6d+1)q}\big) \big(1 + |\mathrm{L}F_M|^q_{q-1}\big).$$

So (for $T > t > 0$ such that $\frac{2d(6q-2)}{t} < \theta$; C_q will be, in the following lines, a " flying constant")

$$\begin{aligned}
\mathrm{E}\left[H^q_\beta(F_M)\right] &\leq C_{q,d} \sqrt{\mathrm{E}\left[\frac{1}{|\det \sigma(F_M)|^{6q-2}}\right]} \sqrt{\mathrm{E}\left[\big(1 + |F_M|_{q+1})^{(6d+1)q}\big)^2 \big(1 + |\mathrm{L}F_M|^q_{q-1}\big)^2\right]} \\
&\leq C_q \Big(\mathrm{E}\left[\big(1 + |F_M|_{q+1})^{(6d+1)q}\big)^4\right] \mathrm{E}\left[\big(1 + |\mathrm{L}F_M|^q_{q-1}\big)^4\right] \Big)^{\frac{1}{4}}
\end{aligned}$$

since we know that, from Lemma 6.6.5, for $p \geq 1$ and $T > t > 0$ such that $\frac{2dp}{t} < \theta$,

$$\mathrm{E}\left[\frac{1}{|\det\, \sigma(F_M)|^p}\right] \leq C_p.$$

But, from (6.4.23) and (6.5.28), there exists $C_{q,T} > 0$ such that

$$\mathrm{E}\left[(|F_M|_{q+1})^{4q(6d+1)}\right] \leq C_{q,T} \lambda_M^{2(q+1)q(6d+1)} \qquad \text{and} \qquad \mathrm{E}\left[|\mathrm{L}F_M|^{4q}_{q-1}\right] \leq C_{q,T} \lambda_M^{4q(q+1)^2},$$

so (since $q \geq 1$, $q^2 \geq \frac{q(q+1)}{2}$),

$$\mathrm{E}\left[H^q_\beta(F_M)\right] \leq C_{p,q,T} \lambda_M^{\frac{(q+1)q}{2}(6d+1)} \lambda_M^{q(q+1)^2}.$$

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Chapter 7

Joint density regularity

7.1 Introduction

We recall that we made an approximation in law F_M^x of our process X_t^x. It is clear, from its definition (*cf.* (5.5.18)), that the law $P_{F_M^x}$ of F_M^x possesses a smooth density : $P_{F_M^x}(\mathrm{d}y) = p_{F_M^x}(y)\,\mathrm{d}y$. Then, we have defined

$$f_M(x,y) \stackrel{\text{def}}{=} \Psi_K(x) p_{F_M^x}(y)$$

where Ψ_K is a smooth version with bounded derivatives of any order of the indicator function $\mathbb{1}_K$.

In this chapter we will highlight the behaviour of $f_M(x,y)$ with respect to the norm defined by (4.6.22), which will prove the Lemma 5.5.3 and, consequently, will end the proof of our main result.

7.2 Bounds for the Sobolev norms of the tangent flow and its derivatives

A simple generalisation (a little bit heavier with respect to the notations, but using the very same ideas and methods) of Proposition 6.4.1 gives straightforwardly the following result :

Proposition 7.2.1 *Let $l, q \in \mathbb{N}^*$, $p \geq 1$ and $t < T$. For every multi-index $\beta \in \{1,\ldots,d\}^q$, there exists $C_{l,p,q,T} > 0$ such that*

$$\| |\partial_\beta \overline{X}_t^M|_l \|_p \leq C_{l,p,q,T} \sqrt{(\lambda_M)^l}. \tag{7.2.1}$$

7.3 Proof of 5.5.3 : an upper bound for $\|f_M\|_{2m+q,2m,p}$

We already have all the tools to prove the Proposition 5.5.3, which was the key for proving our main joint density result 5.3.1. To bound the quantity $\|f_M\|_{2m+q,2m,p}$, it is sufficient, by (4.6.24), to bound the quantities

$$\partial_\xi\big(f_M(x,y)\big), \qquad \xi \leq 2m+q,$$

which is the exact point of the Proposition 7.3.3 that we will prove now ; we will deal first with the case $\xi = 0$ (which is the point of the next proposition), to show more conveniently the method that we used.

Notation 7.3.1 *In all the sequel, φ_ε will represent a mollifier converging weakly, as ε tends to 0, to the Dirac distribution. We will also define*

$$\Phi_\varepsilon(x_1,\ldots,x_d) \stackrel{def}{=} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \varphi_\varepsilon(t_1,\ldots,t_d)\,\mathrm{d}t_1 \ldots \mathrm{d}t_d. \tag{7.3.2}$$

Proposition 7.3.2 *Let $T > 0$. For every $t \in]\frac{4d(3d-1)}{\theta}, T[$*

$$f_M(x,y) \le C_{d,T}\Psi_K(x)\Big(1 \wedge \frac{1}{(|y|-3)^{d+1}}\Big)\lambda_M^{5(d+1)^3} \tag{7.3.3}$$

Proof : Let us note that, formally, $p_{F_t^M}(x,y) = \mathrm{E}\left[\delta_0(F_t^M(x) - y)\right]$, where δ_0 is the Dirac distribution.

In order to work in the direction of this last representation, we will therefore consider the following approximation of f_M :

$$f_{M,\varepsilon} \overset{\text{def}}{=} \Psi_K(x)\,\mathrm{E}\left[\varphi_\varepsilon(F_t^M(x) - y)\Psi_2(F_t^M(x) - y)\right], \tag{7.3.4}$$

where φ_ε is, as we said before, a mollifier and where Ψ_2 is a smooth version (with bounded derivatives of any order) of the indicator function with respect to the ball centred at 0 with radius 2.

We will consequently look, in the first place, for an upper-bound of $f_{M,\varepsilon} = \Psi_K(x)\,\mathrm{E}\left[\varphi_\varepsilon(F_t^M(x) - y)\Psi_2(F_t^M(x) - y)\right]$; using Theorem 4.4.3,

$$\mathrm{E}\left[\varphi_\varepsilon(F^M - y)\Psi_2(F_t^M(x) - y)\right] = \mathrm{E}\left[\Phi_\varepsilon(F^M - y)H_M^d(F_M, \Psi_2(F_t^M(x) - y))\right]$$

It directly follows, since $\|\Phi_\varepsilon\|_\infty \le 1$ (which is clear from the definition (7.3.2)) and the weight H_M does not depend on ε, the pointwise convergence of $f_{M,\varepsilon}(x,y)$ when ε tends to 0.

Using Theorem 4.5.10 and denoting temporarily $G_M \overset{\text{def}}{=} \Psi_2(F_t^M(x) - y)$,

$$\Big|H_M^d(F_M, G_M)\Big| \le C_d|G_M|_d\frac{(1 + |F_M|_{d+1})^{(6d+1)d}}{|\det\sigma(F_M)|^{3d-1}}\big(1 + |\mathrm{L}F_M|_{d-1}^d\big).$$

Following the same pattern as we did in the proof of Lemma 6.7.1, for $T > t > 0$ such that $\frac{2d(6q-2)}{t} < \theta$ (here $q = d$), we have

$$\begin{aligned}\mathrm{E}\Big[|H_M^d(F_M, G_M)|\Big] &\le C_d\Big(\mathrm{E}\Big[\frac{1}{|\det\sigma(F_M)|^{6d-2}}\Big]\,\mathrm{E}\Big[\big(1 + |F_M|_{d+1}\big)^{4(6d+1)d}\Big]\,\mathrm{E}\Big[\big(1 + |\mathrm{L}F_M|_{d-1}^d\big)^4\Big]\Big)^{\frac14}\||G_M|_d\|_4\\ &\le C_d'\Big(\mathrm{E}\Big[|F_M|_{d+1}^{4(6d+1)d}\Big]\,\mathrm{E}\Big[|\mathrm{L}F_M|_{d-1}^{4d}\Big]\Big)^{\frac14}\||G_M|_d\|_4\end{aligned}$$

and, for M big enough (provided that $\lambda_M \to +\infty$, when $M \to +\infty$) we found that[1]

$$\mathrm{E}\Big[|H_M^d(F_M, G_M)|\Big] \le C_{d,T}\lambda_M^{4(d+1)^3}\||G_M|_d\|_4.$$

The Lemma 4.5.8, with $\phi(\cdot) \overset{\text{def}}{=} \Psi_2(\cdot - y)$ (and with $|\phi|_n(F) \overset{\text{def}}{=} \sup_{|\beta|\le n}|\partial_\beta\phi(F)|$; it is clear then that $|\phi|_n(F_t^M(x)) = |\Psi_2|_n(F_t^M(x) - y))$ implies :

$$\begin{aligned}|\phi(F_t^M)|_d &\le C_d|\phi|_d(F_t^M)\big(1 + |F_t^M|_{1,d} + |F_t^M|_{1,d-1}^d\big)\\ &\le C_d|\phi|_d(F_t^M)\big(1 + |\bar{X}_t^M|_d + |\bar{X}_t^M|_{d-1}^d\big).\end{aligned}$$

So, noting that $|\Psi_2|_n(u) \le C_n\mathbb{1}_{\{|u|\le 3\}}$,

$$\begin{aligned}\||G_M|_d\|_4 &= \||\Psi_2(F_t^M(x) - y)|_d\|_4\\ &\le K_d\big(\mathrm{E}\left[\mathbb{1}_{\{|F_t^M(x)-y|\le 3\}}\right]\big)^{\frac18}\big(\mathrm{E}\left[\big(1 + |\bar{X}_t^M|_d + |\bar{X}_t^M|_{d-1}^d\big)^8\right]\big)^{\frac18}\\ &= C_{d,T}\big(\mathrm{P}\left[|F_t^M(x) - y| \le 3\right]\big)^{\frac18}(\lambda_M)^{\frac{d^2}{2}}\end{aligned}$$

[1] using the non-optimal inequality :

$$d((d-1)^2 + 2) + \frac{d}{2}(6d+1)(d+1) < 4(d+1)^3$$

and, since

$$\begin{aligned}\mathrm{P}\big[|F_t^M(x)-y|\le 3)\big] &\le \mathrm{P}\big[|F_t^M(x)|\ge |y|-3\big]\\ &\le \frac{\mathrm{E}\left[|F_t^M(x)|^{8(d+1)}\right]}{(|y|-3)^{8(d+1)}}\\ &\le \frac{K'_d}{(|y|-3)^{8(d+1)}}\end{aligned}$$

(we used the fact that $\mathrm{E}\left[|F_t^M(x)|^{8(d+1)}\right]=\||F_t^M(x)|_0\|^{8(d+1)}$ which is bounded from Proposition 6.4.1) so

$$\||G_M|_d\|_4 \le \frac{C'_{d,T}(\lambda_M)^{\frac{d^2}{2}}}{(|y|-3)^{d+1}}.$$

So, since by definition $\|\Phi_\varepsilon\|_\infty\le 1$,

$$\mathrm{E}\left[\varphi_\varepsilon(F^M-y)\Psi_2(F_t^M(x)-y)\right]\le K_{d,T}\frac{\lambda_M^{4(d+1)^3}(\lambda_M)^{\frac{d^2}{2}}}{(|y|-3)^{d+1}}\le K_{d,T}\frac{\lambda_M^{5(d+1)^3}}{(|y|-3)^{d+1}}.$$

And

$$f_{M,\varepsilon}(x,y)\le C_{d,T}\Psi_K(x)\Big(1\wedge\frac{1}{(|y|-3)^{d+1}}\Big)\lambda_M^{5(d+1)^3}. \tag{7.3.5}$$

•

We are now ready to deal with the general case $|\xi|>1$. We have to be aware that the density, conditionally on $\mathcal{G}=\sigma(T_k,k\in\mathbb{N})$, of the law of $(\overline{Z}_1,\dots,\overline{Z}_{J_t^M})$, density given by

$$p_{J_t^M,x}(\omega,z_1,\dots,z_{J_t^M})=\prod_{j=1}^{J_t^M}q_M\big(z_j,\Psi_{T_j-T_{j-1}}(\overline{X}^M_{T_{j-1}})\big)$$

depends on x, which makes the differentiation more complicated.

Proposition 7.3.3 *Let $T>0$, m, $q\in\mathbb{N}$. For every $t\in]\frac{4d(3q'-1)}{\theta},T[$, with $q'=d+2m+q$, and every multi-index ξ such that $|\xi|\le 2m+q$,*

$$|\partial_\xi f_M(x,y)|\le C_{d,m,q,T}\mathbb{1}_{K+1}(x)\Big(1\wedge\frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{6(d+2m+q+1)^3}. \tag{7.3.6}$$

Proof :

Because of what we said concerning the density, we will separate the differentiation with respect to x and to y, and hence define two multi-indexes α and β such that

$$\partial_x^\alpha\partial_y^\beta \overset{\text{def}}{=} \partial_\xi.$$

Then, we will start to work on the quantity $\partial_\xi f_{M,\varepsilon}(x,y)=\partial_x^\alpha\partial_y^\beta\big(f_{M,\varepsilon}(x,y)\big)$.

It is clear that $\partial_y^\beta\big(\varphi_\varepsilon(F^M-y)\big)=(-1)^\beta\partial^\beta\varphi_\varepsilon(F^M-y)$, so

$$\begin{aligned}\partial_x^\alpha\partial_y^\beta\big(f_{M,\varepsilon}(x,y)\big) &= \partial_x^\alpha\partial_y^\beta\Big(\Psi_K(x)\,\mathrm{E}\left[\varphi_\varepsilon(\overline{X}_t^M(x)-y)\Psi_2(\overline{X}_t^M(x)-y)\right]\Big)\\ &= \partial_x^\alpha\Big(\Psi_K(x)\,\mathrm{E}\big[\sum_{\beta'\oplus\beta''=\beta}\partial_y^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x)-y)\partial_y^{\beta''}\Psi_2(\overline{X}_t^M(x)-y)\big]\Big)\\ &= (-1)^\beta\sum_{\beta'\oplus\beta''=\beta}\partial_x^\alpha\Big(\Psi_K(x)\,\mathrm{E}\left[\partial^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x)-y)\partial^{\beta''}\Psi_2(\overline{X}_t^M(x)-y)\Big)\right]\\ &= (-1)^\beta\sum_{\beta'\oplus\beta''=\beta}\sum_{\alpha'\oplus\alpha''=\alpha}\partial_x^{\alpha'}\Psi_K(x)\partial_x^{\alpha''}\Big(\mathrm{E}\left[\partial^{\beta'}\varphi_\varepsilon(\overline{X}_t^M(x)-y)\partial^{\beta''}\Psi_2(\overline{X}_t^M(x)-y)\right]\Big).\end{aligned}$$

Lemma 7.3.7 implies then that $\partial_x^\alpha \partial_y^\beta \big(f_{M,\varepsilon}(x,y)\big)$ converges (when ε tends to 0) and the existence of $C_{d,m,q,T} > 0$ such that

$$\partial_\xi f_{M,\varepsilon}(x,y) \leq C_{d,m,q,T} \mathbb{1}_{K+1}(x) \Big(1 \wedge \frac{1}{(|y|-3)^{2m+d+1}}\Big) \lambda_M^{6(d+2m+q+1)^3} \tag{7.3.7}$$

which allows us to state that $\lim_{\varepsilon \to 0} \partial_\xi f_{M,\varepsilon}(x,y) = \partial_\xi f_M(x,y)$, and letting $\varepsilon \to 0$ in (7.3.7), we obtain (7.3.6). •

In this last proof, we used the Lemma 7.3.7 ; to prove it, we will first need two preliminary lemmas :

Lemma 7.3.4 *Let $f : \mathbb{R}^d \to \mathbb{R}_+^*$ and β a multi-index, then (with the notations used in Lemma 4.5.3)*

$$\partial^\beta f(x) = f(x) \sum_{l=1}^{|\beta|} \sum_{M \in \mathcal{M}_{|\beta|}(l)} c_M \prod_{i=1}^{l} \partial^{M_i(\beta)} \ln f(x) \tag{7.3.8}$$

with $c_M \in \mathbb{N}$.

Proof : By induction on $|\beta|$.

We will just show the mechanism, which will then be rather clear for a higher range, for $|\beta| = 1,\ 2$.

For $|\beta| = 1$, let $\beta = (x_i)$; then it is clear that

$$\partial_{x_i} f(x) = f(x) \partial_{x_i} \ln f(x)$$

(so $c_M = 1$ for $M = \{\{1\}\}$).

For $|\beta| = 2$, let $\beta = (x_j, x_i)$ then

$$\begin{aligned} \partial_{x_j}\partial_{x_i} f(x) &= \partial_{x_j}\big(f(x)\partial_{x_i} \ln f(x)\big) \\ &= \partial_{x_j} f(x) \partial_{x_i} \ln f(x) + f(x) \partial_{x_j}\partial_{x_i} \ln f(x) \\ &= \big(f(x)\partial_{x_j} \ln f(x)\big)\partial_{x_i} \ln f(x) + f(x)\partial_{x_j}\partial_{x_i} \ln f(x) \\ &= f(x)\big(\partial_{x_j} \ln f(x) \partial_{x_i} \ln f(x) + \partial_{x_j}\partial_{x_i} \ln f(x)\big) \end{aligned}$$

(so $c_M = 1$ for $M = \{\{1,2\}\}$ or $M = \{\{1\},\{2\}\}$).

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Lemma 7.3.5 *Let $l, q \in \mathbb{N}$, $p \geq 1$ and $t < T$. For every multi-index*[2] *$\beta \in \{1, \ldots, d\}^q$, there exists $C_{l,p,q,T} > 0$ such that*

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\partial_x^\beta \bar{X}_{T_j-}^M|_v^{2w}\Big)^u\Big] \leq C_{u,v,w,|\beta|,T} (\lambda_M)^{u(vw+1)}. \tag{7.3.9}$$

Proof : We have

$$|\partial_x^\beta \bar{X}_{T_j-}^M|_v^2 = \sum_{|\alpha| \leq v} |\,\mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_{T_j-}^M|^2.$$

In the very same way as we did in (6.4.27) (with an obvious generalisation[3] of Lemma 4.5.3) we can set

$$\overline{V}_t^{\alpha,\beta} \overset{\text{def}}{=} \mathrm{D}_\alpha\, \partial_x^\beta \bar{X}_t$$

[2] With the convention $\{1, \ldots, d\}^0 = \{0\}$ and $\partial_x^\beta F = F$.

[3] Indeed, in this lemma, the derivatives were purely formal, therefore, we can use it directly with an operator $\mathrm{D}'_{\alpha'}$ where $\alpha' = (\alpha, \beta)$ and $\mathrm{D}'_{\alpha'} \overset{\text{def}}{=} \mathrm{D}_\alpha\, \partial_x^\beta$.

with $\overline{V}_t^{\alpha,\beta}$ defined as in (6.2.1) ; then, there exists a process $V_t^{\alpha,\beta}$ with the same law and verifying (6.2.5), so we have :

$$\begin{aligned}
\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\partial_x^\beta \bar{X}^M_{T_j-}|_v^{2w}\Big)^u\Big] &= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|\,\mathrm{D}_\alpha\,\partial_x^\beta \bar{X}^M_{T_j-}|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|\,\mathrm{D}_\alpha\,\partial_x^\beta \bar{X}^M_{T_j-}|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|\overline{V}^{\alpha,\beta}_{T_j-}|^2\Big)^w\Big)^u\Big] \\
&= \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|V^{\alpha,\beta}_{T_j-}|^2\Big)^w\Big)^u\Big] \\
&\le \sqrt{\mathrm{E}\Big[\big(J_T^M\big)^{2u-2}\Big]}\sqrt{\mathrm{E}\Big(\Big[\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|V^{\alpha,\beta}_{T_j-}|^2\Big)^{wu}\Big)^2\Big]}.
\end{aligned}$$

But, using the F_p^2 isometry, with $w' \in \mathbb{N}^*$, and setting $f(y) \stackrel{\text{def}}{=} \Big(\sum_{|\alpha|\le v}|V_y^{\alpha,\beta}|^2\Big)^{w'}$,

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}\Big(\sum_{|\alpha|\le v}|V^{\alpha,\beta}_{T_j-}|^2\Big)^{w'}\Big)^2\Big] \\
&= \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\tilde{N}(s,z,u) + \int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\hat{N}(s,z,u)\Big)^2\Big] \\
&\le \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)^2\,\mathrm{d}\hat{N}(s,z,u)\Big] + \mathrm{E}\Big[\Big(\int_0^t\int_{E_M\times[0,2\bar{C}]} f(s^-)\,\mathrm{d}\hat{N}(s,z,u)\Big)^2\Big] \\
&\le 2\bar{C}\mu(E_M)\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big] + \big(2\bar{C}\mu(E_M)\big)^2 T\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big] \\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t f(s^-)^2\,\mathrm{d}s\Big] \\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\le v}|V^{\alpha,\beta}_{s^-}|^2\Big)^{2w'}\,\mathrm{d}s\Big] \\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t\Big(\sum_{|\alpha|\le v}|\overline{V}^{\alpha,\beta}_{s^-}|^2\Big)^{2w'}\,\mathrm{d}s\Big] \\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\,\mathrm{E}\Big[\int_0^t |\partial_x^\beta \bar{X}^M_{s^-}|_v^{4w'}\,\mathrm{d}s\Big] \\
&= (2\bar{C}\mu(E_M) + \big(2\bar{C}\mu(E_M)\big)^2 T)\int_0^t \mathrm{E}\big[|\partial_x^\beta \bar{X}^M_s|_v^{4w'}\big]\,\mathrm{d}s.
\end{aligned}$$

Now from (7.2.1) we have (with $\lambda_M = \mu(E_M)$),

$$\mathrm{E}\Big[\Big|\partial_x^\beta \bar{X}_t^M\Big|_v^{4w'}\Big] \le C_{v,w',|\beta|,T}(\lambda_M)^{2vw'}$$

Gathering these results, we obtain :

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M}|\partial_x^\beta \bar{X}^M_{T_j-}|_v^{2w}\Big)^u\Big] \le C_{u,v,w,|\beta|,T}(\lambda_M)^{u-1}(\lambda_M)^{uvw+1}.$$

•

Lemma 7.3.6 *Let $l, q \in \mathbb{N}^*$, $p \geq 1$ and $t < T$. For every multi-index $\alpha \in \{1, \ldots, d\}^q$, there exists $C_{l,p,q,T} > 0$ such that*

$$\| |\partial_x^\beta \ln p_{J_t^M,x}(\overline{Z}_1, \ldots, \overline{Z}_{J_t^M})|_l \|_{2p} \leq C_{l,p,q,T}(\lambda_M)^{l(q+l)+1}. \tag{7.3.10}$$

Proof : In this proof we will denote J_t^M simply by J (and sometimes $\bar{X}^M_{T_j-}(x)$ simply by $\bar{X}^M_{T_j-}$).

Using a similar formula as (4.5.12) (the Malliavin derivatives were used in a formal way, so it is in particular true with the usual differential operator), we have (with $\alpha \in \{1, \ldots, d\}^q$),

$$\partial_x^\alpha \phi(F) = \sum_{l=1}^{n} \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)} \partial_\beta \phi(F) \partial_x^{M_1(\alpha)} F_{\beta_1} \cdots \partial_x^{M_l(\alpha)} F_{\beta_l},$$

so, with $\alpha \overset{\text{def}}{=} \sum_{l=1}^n \sum_{\substack{\beta=(\beta_1,\ldots,\beta_l) \\ \beta_i \in [\![1,d]\!]}} \sum_{M \in \mathcal{M}_n(l)}$,

$$\partial_x^\alpha (\ln q_M(\bar{X}^M_{T_j-}(x), \bar{Z}_j)) = \sum_{(\alpha)} \partial_\beta (\ln q_M)(\bar{X}^M_{T_j-}(x), \bar{Z}_j) \partial_x^{M_1(\alpha)} (\bar{X}^M_{T_j-}(x))_{\beta_1} \cdots \partial_x^{M_l(\alpha)} (\bar{X}^M_{T_j-}(x))_{\beta_l}.$$

By corollary 4.5.9, we have the existence of $C'_{q_M,l} > 0$ such that

$$|\ln q_M(F)|_l \leq C_{q_M,l}\big(1 + |F|_l + |F|_{l-1}^l\big)$$

which leads to (recalling $q = |\alpha|$)

$$|\partial_x^\alpha (\ln q_M(\bar{X}^M_{T_j-}(x), \bar{Z}_j))|_l \leq C_{q_M,l}\Big(1 + |\bar{X}^M_{T_j-}|_l^2 + |\bar{X}^M_{T_j-}|_{l-1}^{2l} + \sum_{\beta \subset \alpha} |\partial_x^\beta (\bar{X}^M_{T_j-}(x))|_l^{2q}\Big).$$

Then

$$\begin{aligned}
|\partial_x^\alpha \ln p_J|_l &\leq \sum_{j=1}^{J_T^M} |\partial_x^\alpha (\ln q_M(\bar{X}^M_{T_j-}(x), \bar{Z}_j))|_l \\
&\leq C_{q_M,l}\Big(\sum_{j=1}^{J_T^M} 1 + |\bar{X}^M_{T_j-}|_l^2 + |\bar{X}^M_{T_j-}|_{l-1}^{2l} + \sum_{\beta \subset \alpha} |\partial_x^\beta (\bar{X}^M_{T_j-}(x))|_l^{2q}\Big) \\
&\leq C'_{q_M,l}\Big(J_T^M + \sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_l^2 + |\bar{X}^M_{T_j-}|_{l-1}^{2l} + \sum_{\beta \subset \alpha} \sum_{j=1}^{J_T^M} |\partial_x^\beta (\bar{X}^M_{T_j-}(x))|_l^{2q}\Big)
\end{aligned}$$

it directly follows, using the Lemma 7.3.5,

$$\begin{aligned}
\mathrm{E}\Big[|\partial_x^\alpha \ln p_J|_l^{2n}\Big] &\leq C_{q_M,l,n,|\alpha|}\Big(\mathrm{E}\big[(J_T^M)^{2n}\big] + \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_l^2\Big)^{2n}\Big] \\
&\qquad + \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\bar{X}^M_{T_j-}|_{l-1}^{2l}\Big)^{2n}\Big] + \sum_{\beta \subset \alpha} \mathrm{E}\Big[\Big(\sum_{j=1}^{J_T^M} |\partial_x^\beta (\bar{X}^M_{T_j-}(x))|_l^{2q}\Big)^{2n}\Big]\Big) \\
&\leq C_{q_M,l,n,|\alpha|}\Big((\lambda_M)^{2n} + C_{n,l,T}(\lambda_M)^{2n(l+1)} + C'_{n,l,T}(\lambda_M)^{2n(l(l-1)+1)} + \sum_{\beta \subset \alpha} C_{n,l,q,|\beta|,T}(\lambda_M)^{2n(ql+1)}\Big) \\
&\leq C_{n,l,\alpha,T}(\lambda_M)^{2n(l(q+l)+1)}.
\end{aligned}$$

•

Now we can prove the wanted result :

Lemma 7.3.7 *Let $T>0$. Let α, β and γ multi-indexes such that $|\alpha|+|\beta|+|\gamma|\le 2m+q$. For every $t\in]\frac{4d(3q'-1)}{\theta},T[$, with $q'=d+|\beta|$, the quantity $\partial_x^\alpha \mathrm{E}\left[\partial^\beta\varphi_\varepsilon(F^M-y)\partial^\gamma\Psi_2(F_t^M(x)-y)\right]$ converges (when ε tends to 0) and*

$$\partial_x^\alpha \mathrm{E}\left[\partial^\beta\varphi_\varepsilon(F^M-y)\partial^\gamma\Psi_2(F_t^M(x)-y)\right]\le K_{d,m,q,T}\Big(1\wedge\frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{6(d+2m+q+1)^3}. \quad (7.3.11)$$

Proof : (in this proof we will denote simply J_t^M by J ; let us set $\delta_M\overset{\text{def}}{=}\sqrt{U_M(t)}$ and temporarily $\Psi\overset{\text{def}}{=}\partial^\gamma\Psi_2$ and $f(u,z)\overset{\text{def}}{=}\delta_M u+x_t(x,z_1,\dots,z_J))$

$$\begin{aligned}
&\partial_x^\alpha \mathrm{E}\left[\partial^\beta\varphi_\varepsilon(F^M-y)\Psi(F_t^M(x)-y)\right]\\
=&\,\mathrm{E}\left[\partial_x^\alpha\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial^\beta\varphi_\varepsilon(\delta_M u+x_t(x,z_1,\dots,z_J)\Psi(\delta_M u+x_t(x,z_1,\dots,z_J)P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\right]\\
=&\sum_{\alpha_1\oplus\alpha_2}\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
&+\sum_{\substack{\alpha_1\oplus\alpha_2\oplus\alpha_3\\ \alpha_3\neq\varnothing}}\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)\partial_x^{\alpha_3}P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
=&\big(1\big)+\big(2\big).
\end{aligned}$$

For the part $\big(1\big)$, since

$$\begin{aligned}
&\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
=&\,\mathrm{E}\left[\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(F^M-y)\big)\partial_x^{\alpha_2}\big(\Psi(F^M-y)\big)\right]\\
=&\,\mathrm{E}\Big[\Big(\sum_{(\alpha_1)}\partial_\beta\varphi_M(F_t^M(x)-y)\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\Big)\\
&\qquad\times\Big(\sum_{(\alpha_2)}\partial_{\beta'}\Psi(F_t^M(x)-y)\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}\Big)\Big],
\end{aligned}$$

we are brought, on one hand, to prove the convergence (when ε tends to 0) and afterwards to bound the quantity

$$\mathrm{E}\left[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y_{N,N'}\right] \quad (7.3.12)$$

with $Y_{N,N'}\overset{\text{def}}{=}\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}$.

On the other hand, for the part $\big(2\big)$, Lemma 7.3.4 leads to

$$\partial_x^{\alpha_3}P_{J,x}=P_{J,x}\sum_{l=1}^{|\alpha_3|}\sum_{M\in\mathcal{M}_{|\alpha_3|}(l)}c_M\prod_{i=1}^{l}\partial^{M_i(\alpha_3)}\ln P_{J,x} \quad (7.3.13)$$

so, letting $\tilde{p}_{J,x}^{|\alpha_3|}\overset{\text{def}}{=}\sum_{l=1}^{|\alpha_3|}\sum_{M\in\mathcal{M}_{|\alpha_3|}(l)}c_M\prod_{i=1}^{l}\partial^{M_i(\alpha_3)}\ln P_{J,x}$, we have :

$$\begin{aligned}
&\mathrm{E}\Big[\int_{\mathbb{R}^d}\nu(\mathrm{d}u)\int_{\mathbb{R}^J}\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(f(u,z))\big)\partial_x^{\alpha_2}\big(\Psi(f(u,z))\big)\partial_x^{\alpha_3}P_{J,x}(z_1,\dots,z_J)\,\mathrm{d}z_1\cdots\mathrm{d}z_J\Big]\\
=&\,\mathrm{E}\left[\partial_x^{\alpha_1}\big(\partial^\beta\varphi_\varepsilon(F^M-y)\big)\partial_x^{\alpha_2}\big(\Psi(F^M-y)\tilde{p}_{J,x}^{|\alpha_3|}(\overline{Z}_1,\dots,\overline{Z}_n)\right]\\
=&\,\mathrm{E}\Big[\Big(\sum_{(\alpha_1)}\partial_\beta\varphi_M(F_t^M(x)-y)\partial_{N_1(\alpha_1)}\bar{X}_{\beta_1}^{t,M}\cdots\partial_{N_l(\alpha_1)}\bar{X}_{\beta_l}^{t,M}\Big)\\
&\times\Big(\sum_{(\alpha_2)}\partial_{\beta'}\Psi(F_t^M(x)-y)\partial_{N_1'(\alpha_2)}\bar{X}_{\beta_1'}^{t,M}\cdots\partial_{N_l'(\alpha_2)}\bar{X}_{\beta_l'}^{t,M}\Big)\tilde{p}_{J,x}^{|\alpha_3|}(\overline{Z}_1,\dots,\overline{Z}_n)\Big],
\end{aligned}$$

so we are brought, again, to prove the convergence (when ε tends to 0) and afterwards to bound the quantity

$$\mathrm{E}\left[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)\tilde{Y}_{N,N'}\right] \tag{7.3.14}$$

with $\tilde{Y}_{N,N'} \overset{\text{def}}{=} \partial_{N_1(\alpha_1)}\bar{X}^{t,M}_{\beta_1}\cdots\partial_{N_l(\alpha_1)}\bar{X}^{t,M}_{\beta_l}\partial_{N'_1(\alpha_2)}\bar{X}^{t,M}_{\beta'_1}\cdots\partial_{N'_l(\alpha_2)}\bar{X}^{t,M}_{\beta'_l}\tilde{p}^{|\alpha_3|}_{J,x}(\overline{Z}_1,\ldots,\overline{Z}_n)$.

We can see a similar structure between (7.3.12) and (7.3.14) : for the moment we will treat them at the same time ; we will temporarily denote by Y either $Y_{N,N'}$ or $\tilde{Y}_{N,N'}$, and keep working on

$$\mathrm{E}\left[\partial_\beta\varphi_\varepsilon(F_t^M(x)-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y\right]. \tag{7.3.15}$$

Letting $G_M \overset{\text{def}}{=} \partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y$, using Theorem 4.4.3, we have

$$\mathrm{E}\left[\partial_\beta\varphi_\varepsilon(F^M-y)\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)Y\right] = \mathrm{E}\left[\Phi_\varepsilon(F^M-y)H_M^{d+|\beta|}(F_M,G_M)\right]. \tag{7.3.16}$$

It directly follows, since $\|\Phi_\varepsilon\|_\infty \le 1$ and the weight H_M does not depend on ε, the pointwise convergence of $\partial_\alpha\big(f_{M,\varepsilon}(x,y)\big)$ when ε tends to 0.

Now, with Theorem 4.5.10, and following the same pattern as we did in the proof of Lemma 6.7.1, for $T>t>0$ such that $\frac{4d(3q'-1)}{t}<\theta$ (recalling $q'=d+|\beta|$),

$$\begin{aligned}
&\mathrm{E}\left[H_M^{d+|\beta|}(F_M,G_M)\right]\\
&\quad\le C_d\Big(\mathrm{E}\left[\left(1+|\mathrm{L}F_M|^{q'}_{q'-1}\right)^4\right]\Big)^{\frac14}\Big(\mathrm{E}\left[\left(1+|F_M|_{q'+1})^{(6d+1)q'}\right)^4\right]\Big)^{\frac14}\||G_M|_{q'}\|_4.
\end{aligned}$$

So, for M big enough (provided that $\lambda_M\to+\infty$, when $M\to+\infty$) we find that

$$\begin{aligned}
\mathrm{E}\left[H_M^{d+|\beta|}(F_M,G_M)\right] &\le C_{d,q',T}\lambda_M^{((q'-1)^2+2)q'+\frac{d}{2}(6d+1)(d+1)}\||G_M|_{q'}\|_4\\
&\le C_{d,q',T}\lambda_M^{(q'+1)^3+3(d+1)^3}\||G_M|_{q'}\|_4
\end{aligned}$$

Moreover (with Cauchy-Swartz and 4.5.7)

$$\begin{aligned}
\||G_M|_{q'}\|_8 &\le C_{q'}\||\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)|_{q'}|Y|_{q'}\|_4\\
&\le C_{q'}\||\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)|_{q'}\|_8\||Y|_{q'}\|_8.
\end{aligned}$$

With the fact that, for all multi-index τ

$$\partial_\tau\Psi_2(u)\le C\mathbb{1}_{|u|\le 3} \tag{7.3.17}$$

We can show that, as we already did in the proof of Proposition 7.3.2, for all β, β', γ of length less than $2m+q$,

$$\||\partial_{\beta'}\partial_\gamma\Psi_2(F_t^M(x)-y)|_{q'}\|_8 \le \frac{K'_{d,m,q}(\lambda_M)^{\frac{q'^2}{2}}}{(|y|-3)^{2m+d+1}}. \tag{7.3.18}$$

using 7.3.6, 7.2.1 (and also 4.5.7), it appears that

$$\||Y_{N,N'}|_{q'}\|_p \le C(\lambda_M)^{\frac{q'}{2}|\alpha|}\le C(\lambda_M)^{\frac{q'}{2}(2m+q)}$$

and

$$\||\tilde{Y}_{N,N'}|_{q'}\|_p \le C_{l,p,q,T}(\lambda_M)^{\frac{q'}{2}|\alpha|}(\lambda_M)^{|\alpha|(q'(q'+|\alpha|)+1)}\le C_{l,p,q,T}(\lambda_M)^{(2m+q)(q'^2+(2m+q+1)q'+1)}$$

Since $\lambda_M\to+\infty$, we have finally

$$\partial_x^\alpha\,\mathrm{E}\left[\partial^\beta\varphi_\varepsilon(F^M-y)\partial^\gamma\Psi_2(F_t^M(x)-y)\right]\le K_{d,m,q,T}\Big(1\wedge\frac{1}{(|y|-3)^{2m+d+1}}\Big)\lambda_M^{A(d,m,q,q')},$$

with

$$A(d, m, q, q') \stackrel{\text{def}}{=} (q' + 1)^3 + 3(d + 1)^3 + \frac{q'^2}{2} + (2m + q)(q'^2 + (2m + q + 1)q' + 1).$$

Now, setting $\overline{q} \stackrel{\text{def}}{=} d + 2m + q \geq q'$,

$$A(d, m, q, q') \leq 4(\overline{q} + 1)^3 + 2\overline{q}^3 + \frac{3}{2}\overline{q}^2 + \overline{q} \leq 6(\overline{q} + 1)^3.$$

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7.4 Regenerative scheme and Harris-recurrence

We assume that the process X_t, solution of (1.2.3), admits a transition density for any time $t > 0$, denoted $p_t(x, y)$, which is strictly positive and continuous in x and y ; a criteria for such a situation is given by Theorem 5.3.1.

As a consequence, for any $t > 0$ and any compact set C, there exists a probability measure ν and a constant $\alpha > 0$ such that the local Doeblin condition is verified :

$$P_t(x, \mathrm{d}y) \geq \alpha \mathbb{1}_C(x)\nu(\mathrm{d}y). \tag{7.4.19}$$

In order to obtain some ergodic result over the stochastic process X_t, the main heuristic idea is to approximate, when $t \to +\infty$, the quantity $\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s$ by $\frac{1}{n}\sum_{i=1}^n \int_{R_i}^{R_{i+1}} f(X_s)\,\mathrm{d}s$, where the r.v. R_i are to be defined and where the r.v. $\int_{R_i}^{R_{i+1}} f(X_s)\,\mathrm{d}s$ would be i.i.d which will allow to conclude by applying the strong law of large numbers.

To do so in a rigorous way, we will follow the path developed by Eva Löcherbach in [37], *Ergodicity and speed of convergence to equilibrium for diffusion processes*, 2013 (*cf.* also Ikeda, Nagasawa and Watanabe (1966) [29]). First, given a càdlàg Markov process Y_t, we will define the notion of regeneration times :

Definition 7.1 *A sequence $(R_n)_{n\geq 1}$ is called generalized sequence of* ***regeneration times****, if*

1. *$R_n \uparrow \infty$ as $n \to +\infty$.*
2. *$R_{n+1} = R_n + R_1 \circ \vartheta_{R_n}$ (ϑ is the shift operator defined by $\vartheta_t f = f(t + .)$, where $f : \mathbb{R}_+ \to E$ is càdlàg).*
3. *$Y_{R_n+.}$ is independent of $\mathcal{F}^Y_{S_r-}$.*
4. *At regeneration times, the process starts afresh from $Y_{R_n} \sim \nu(\mathrm{d}y)$.*
5. *The trajectories $(Y_{R_n+s}, 0 \leq s \leq R_{n+1} - R_n)_n$ are 2-independent , i.e. $(Y_{R_n+s}, 0 \leq s \leq R_{n+1} - R_n)$ and $(Y_{R_m+s}, 0 \leq s \leq R_{m+1} - R_m)$ are independent if and only if $|m - n| \geq 2$.*

These regeneration times do not exist for the original solution X_t, but they exist for a version of the process on an extended probability space, rich enough to support the driving Brownian motion, the Poisson measure and an i.i.d sequence of uniform random variable $(U_n)_{n\geq 1}$.

We will construct, then, a stochastic process $(Y_t)_{t\geq 0}$ on this richer probability space, equal in law to $(X_t)_{t\geq 0}$.

First we will fix a compact C and a time parameter $t_* > 0$ such that (7.4.19) is true :

$$P_{t_*}(x, \mathrm{d}y) \geq \alpha \mathbb{1}_C(x)\nu(\mathrm{d}y).$$

Then we set $Y_t = X_t$ for all $0 \leq t \leq \tilde{S}_1$, where

$$\tilde{S}_1 \stackrel{\text{def}}{=} \inf\{t \geq t_* : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_1 \stackrel{\text{def}}{=} \tilde{S}_1 + t_*.$$

At time $\tilde{S}_1$, we choose U_1, the first of the uniform random variables. If $U_1 \leq \alpha$, we choose

$$Y_{\tilde{R}_1} \sim \nu(\mathrm{d}y). \tag{7.4.20}$$

Else, if $U_1 > \alpha$, given $Y_{\tilde{S}_1} = x$, we choose

$$Y_{\tilde{R}_1} \sim \frac{P_{t_*}(x, \mathrm{d}y) - \alpha\nu(\mathrm{d}y)}{1 - \alpha}. \tag{7.4.21}$$

Finally, given $Y_{\tilde{R}_1} = y$, we fill in the missing trajectory $(Y_t)_{t\in]\tilde{S}_1,\tilde{R}_1[}$ between time $\tilde{S}_1$ and time $\tilde{R}_1$ according to the diffusion bridge law

$$\frac{p_{t-\tilde{S}_1}(x,z)p_{\tilde{R}_1-t}(z,y)}{p_{t_*}(x,y)}\,\mathrm{d}z. \tag{7.4.22}$$

Notice that by construction, if we do not care about the exact choice of the auxiliary random variable U_1, then we have that $(Y_t)_{t\leq\tilde{R}_1} \overset{\mathcal{L}}{=} (X_t)_{t\leq\tilde{R}_1}$.

We continue this construction after time $\tilde{R}_1$: choose Y_t equal to X_t for all $t\in]\tilde{R}_1,\tilde{S}_2]$ where

$$\tilde{S}_2 \overset{\text{def}}{=} \inf\{t > \tilde{R}_1 : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_2 \overset{\text{def}}{=} \tilde{S}_2 + t_*.$$

At time $\tilde{S}_2$, we choose U_2 in order to realize the choice of $Y_{\tilde{R}_2}$ according to the splitting of the transition kernel P_{t_*}, as in (7.4.20) and (7.4.21). More generally, the construction is therefore achieved along the sequence of stopping times

$$\tilde{S}_{n+1} \overset{\text{def}}{=} \inf\{t > \tilde{R}_n : X_t \in C\} \qquad \text{and} \qquad \tilde{R}_{n+1} \overset{\text{def}}{=} \tilde{S}_{n+1} + t_*, \qquad n\geq 1,$$

where during each $]\tilde{R}_n,\tilde{S}_{n+1}]$, Y follows the original solution of the SDE, whereas the intervals $[\tilde{S}_{n+1},\tilde{R}_{n+1}]$ are used to construct the splitting. In particular, every time that we may choose a transition according to (7.4.20), we introduce a regeneration event for the process Y, and therefore the following two sequences of generalized stopping times will play a role. Firstly,

$$S_1 = \inf\{\tilde{S}_n : U_n \leq \alpha\}, \quad \dots, \quad S_n = \inf\{\tilde{S}_m > S_{n-1} : U_m \leq \alpha\}, \qquad n\geq 2,$$

and secondly,

$$R_n \overset{\text{def}}{=} S_n + t_*, \qquad n\geq 1.$$

The above construction of the process X, since at each time $\tilde{S}_n$, a projection into the future is made.

Let $\tilde{N}_t \overset{\text{def}}{=} \sup\{n : U_n \leq t\}$ and

$$\mathcal{F}_t^Y = \sigma\{Y_s, s\leq t, U_n, Y_{\tilde{R}_n}, n\leq \tilde{N}_t\}, \qquad t\geq 0,$$

be the canonical filtration of the process Y. The sequence of $(\mathcal{F}_t^Y)_{t\geq 0}$-stopping times $(R_n)_{n\geq 1}$ is a generalized sequence of regeneration times as it was defined in Definition 7.1.

Remark 7.4.1 *The trajectories of Y are not the same as those of the original solution X of the SDE. However, by definition, the Harris-recurrence is only a property in law. As a consequence, if, for a given set A, we succeed to show that almost surely, Y visits it infinitely often, the same is automatically true for X as well.*

We can now state the theorem we were looking for (the demonstration of it is directly taken from Eva Löcherbach's lecture, *Ergodicity and speed of convergence to equilibrium for diffusion processes*, and we give it here only for the convenience of the reader) :

Theorem 7.4.2 *If for all $x\in\mathbb{R}^d$ we have $\mathrm{P}_x[R_1<\infty]=1$, then the process X is recurrent in the sense of Harris.*

Proof :

Define a measure π on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ by

$$\pi(A) \overset{\text{def}}{=} \mathrm{E}\Big[\int_{R_1}^{R_2} \mathbb{1}(Y_s)\,\mathrm{d}s\Big], \qquad A\in\mathcal{B}(\mathbb{R}^d).$$

For any $n \geq 2$, put $\xi_n \stackrel{\text{def}}{=} \int_{R_{n-1}}^{R_n} \mathbb{1}(Y_s)\,\mathrm{d}s$. By construction, the random variables ξ_{2n}, $n \geq 1$, are i.i.d. and so are, on the other hand, as well, the random variables ξ_{2n+1}. Put

$$N_t = \sup\{n : R_n \leq t\}$$

and observe that $N_t \to \infty$ as $t \to \infty$. Hence, applying the strong law of large numbers separately to the sequence $(\xi_{2n})_{n\geq 1}$ and the sequence $(\xi_{2n+1})_{n\geq 1}$, we have that

$$\lim_{t\to\infty} \frac{\int_0^t \mathbb{1}(Y_s)\,\mathrm{d}s}{N_t} = \pi(A)$$

P_x-almost surely, for any $x \in \mathbb{R}^d$. This implies that any set A such that $\pi(A) > 0$ is visited infinitely often by the process Y almost surely. Thus, we have the recurrence property also for the process X, for any set A such that $\pi(A) > 0$. Then, by a deep theorem of Azéma, Duflo and Revuz (1969) [4], see also Theorem 1.2. of Höpfner and Löcherbach (2003) [28], the process is indeed Harris.

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Appendix A

Miscellaneous

A.1 Regularisation

Let us define on $\mathbb{R}$ the following function:

$$\varphi(x) \stackrel{\text{def}}{=} \begin{cases} \alpha \exp(-\frac{1}{1-x^2}) & \text{si } |x| < 1 \\ 0 & \text{si } |x| \geq 1 \end{cases}$$

where α is chosen in order to have $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$. Numerically, $\alpha \approx (0,44399)^{-1}$.

Proposition A.1.1 *$\varphi : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^\infty$ compact support function.*

We then define $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\varphi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right), \qquad \forall x \in \mathbb{R}.$$

Thus defined, function φ_ε converges weakly, as ε tends to 0, to the Dirac distribution ; it is a mollifier: for every continuous function f,

1. $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = f(x)$
2. $f * \varphi_\varepsilon$ is $\mathcal{C}^\infty$

(with $f * \varphi_\varepsilon(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(y)\varphi_\varepsilon(x-y)\,\mathrm{d}y$)).

Proposition A.1.2 *$\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ remains a $\mathcal{C}^\infty$ compact support function and there exists $M > 0$ such that:*

$$\forall x \in \mathbb{R}, \qquad |\varphi'_\varepsilon(x)| \leq \frac{M}{\varepsilon^2} \quad \textit{and} \quad |\varphi''_\varepsilon(x)| \leq \frac{M}{\varepsilon^3}.$$

Proof : If $|x| \geq \varepsilon$,

$$\varphi'_\varepsilon(x) = \varphi''_\varepsilon(x) = 0.$$

Elsewhere, $\varphi_\varepsilon(x) = \frac{\alpha}{\varepsilon}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)$ and

$$\begin{aligned}\varphi'_\varepsilon(x) &= \frac{\alpha}{\varepsilon}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \times \frac{-2x}{\varepsilon^2} \times \frac{1}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2} \\ &= \frac{-2\alpha}{\varepsilon^2}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\frac{\frac{x}{\varepsilon}}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}.\end{aligned}$$

The function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right)\frac{y}{(1-y^2)^2}$ is defined and continuous (considering a continuous extension for $y=\pm 1$) on $\mathbb{R}$ and bounded, as we can easily prove by considering its limits when y tends respectively to $\pm\infty$, -1 and 1.
Denoting now by M_1 an upperbound and letting $C_1 = 2\alpha M_1$, we obtain the first assumption.

Now,

$$\begin{aligned}
\varphi_\varepsilon''(x) &= \frac{\alpha}{\varepsilon^3}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\left(\frac{-2x}{\varepsilon^2\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}\times\frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}+\frac{\partial}{\partial x}\left(\frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}\right)\right)\\
&= \frac{\alpha}{\varepsilon^3}\exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)\left(\frac{4\left(\frac{x}{\varepsilon}\right)^2}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^4}-\frac{2}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2}+\frac{4x}{\varepsilon^2}\times\frac{-2x}{\left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^3}\right).
\end{aligned}$$

Similarly, the function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right)\left(\frac{4y^2}{(1-y^2)^4}-\frac{2}{(1-y^2)^2}-\frac{8y^2}{(1-y^2)^3}\right)$ is *bounded* on $\mathbb{R}$ by a constant M_2. With $C_2 = \alpha M_2$ and $M = \max(C_1, C_2)$ we then obtain the last property. •

We then define $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$h_\varepsilon(x) \stackrel{\text{def}}{=} |x| \vee 2\varepsilon$$

and $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\phi_\varepsilon(x) = h_\varepsilon * \varphi_\varepsilon(x).$$

Proposition A.1.3 *1. ϕ_ε converges pointwise to the absolute value function $x \mapsto |x|$ and*

$$\phi_\varepsilon(x) = \begin{cases} 2\varepsilon & \text{if} \quad |x| \le \varepsilon \\ |x| & \text{if} \quad |x| > 3\varepsilon \end{cases}$$

and

$$0 \le \phi_\varepsilon(x) \le 4\varepsilon \quad \text{if} \quad |x| \in]\varepsilon, 3\varepsilon].$$

2. There exists $C > 0$ such that

$$\forall x \in \mathbb{R}, \qquad |\phi_\varepsilon'(x)| \le C \qquad \text{and} \qquad |\phi_\varepsilon''(x)| \le \frac{C}{\varepsilon}\mathbb{1}_{|x|\le 3\varepsilon}.$$

Proof : First, let us remark that, since ϕ_ε is defined by a convolution, it is a smooth function and, for all $n \in \mathbb{N}^*$, $\phi_\varepsilon^{(n)}(x) = h_\varepsilon * \varphi_\varepsilon^{(n)}(x)$.

We will now prove the two items by dividing the problem into the following three cases.

- **Case $|x| \le \varepsilon$:**

  We have

$$\begin{aligned}
\phi_\varepsilon(x) = h_\varepsilon * \varphi_\varepsilon(x) &= \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x-y) h_\varepsilon(y)\,\mathrm{d}y \\
&= 2\varepsilon \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x-y)\,\mathrm{d}y \\
&= 2\varepsilon \underbrace{\int_{\mathbb{R}} \varphi_\varepsilon(z)\,\mathrm{d}z}_{=1} = 2\varepsilon.
\end{aligned}$$

  Hence, for all $|x| < \varepsilon$,

$$\phi_\varepsilon'(x) = \phi_\varepsilon''(x) = 0,$$

  (which remains true when $|x| = \varepsilon$, since ϕ_ε is a smooth function).

- **Case** $|x| > 3\varepsilon$**:**

  noticing that $z \mapsto z\varphi_\varepsilon(z)$ is an odd function (with compact support),

$$\begin{aligned}\phi_\varepsilon(x) = \int_{\mathbb{R}} h_\varepsilon(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z &= \int_{|z|\le\varepsilon} h_\varepsilon(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \int_{|z|\le\varepsilon} |x-z|\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \int_{|z|\le\varepsilon} \mathrm{sign}(x)(x-z)\varphi_\varepsilon(z)\,\mathrm{d}z \\ &= \mathrm{sign}(x) \times \Big(x - \underbrace{\int_{\mathbb{R}} \varphi_\varepsilon(z) z\,\mathrm{d}z}_{=0}\Big) = |x|.\end{aligned}$$

  Hence, for all $|x| > 3\varepsilon$,

$$\phi_\varepsilon'(x) = \pm 1 \qquad \text{and} \qquad \phi_\varepsilon''(x) = 0,$$

  (and obviously again, it remains true when $|x| = 3\varepsilon$).

- **Case** $|x| \in]\varepsilon, 3\varepsilon]$**:**

  Let g be a continuous function (with compact support), then

$$\begin{aligned}h_\varepsilon * g(x) = \int_{\mathbb{R}} g(x-y)h_\varepsilon(y)\,\mathrm{d}y &= \int_{-4\varepsilon}^{4\varepsilon} g(x-y)h_\varepsilon(y)\,\mathrm{d}y \\ &\le 4\varepsilon \int_{\mathbb{R}} |g(z)|\,\mathrm{d}z.\end{aligned}$$

  With $g \stackrel{\text{def}}{=} \varphi_\varepsilon$, $\int_{\mathbb{R}} |g(z)|\,\mathrm{d}z = 1$ and it follows that

$$\phi_\varepsilon(x) \le 4\varepsilon\ ;$$

  with respectively $g \stackrel{\text{def}}{=} \varphi_\varepsilon'$ and $g \stackrel{\text{def}}{=} \varphi_\varepsilon''$, and using the Proposition A.1.2, it follows that

$$\phi_\varepsilon'(x) = h_\varepsilon * \varphi_\varepsilon' \le 4\varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{M}{\varepsilon^2}\,\mathrm{d}y = 8M\ ;$$

  and

$$\phi_\varepsilon''(x) = h_\varepsilon * \varphi_\varepsilon'' \le 4\varepsilon \int_{-\varepsilon}^{\varepsilon} \frac{M}{\varepsilon^3}\,\mathrm{d}y = \frac{8M}{\varepsilon}.$$

Gathering the three cases, we may conclude. •

A.2 Gronwall's lemma

In all this work we used the following version of the Gronwall's lemma:

Proposition A.2.1 *If a measurable function* $g : [0,T] \to \mathbb{R}^+$ *is such that*

1. $G = \sup_{t\in[0,T]} g(t) < +\infty$ *;*
2. *for all* $t \in [0,T]$,

$$g(t) \le A + B\int_0^t g(s)\,\mathrm{d}s$$

then, for all $t \in [0,T]$,

$$g(t) \le A\exp(Bt).$$

Proof : It is easy to obtain by induction that, for every $n \in \mathbb{N}^*$,

$$g(t) \le A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + B^n \int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} g(t_n)\, \mathrm{d}t_n \cdots \mathrm{d}t_1\, \mathrm{d}t\Big),$$

which implies

$$g(t) \le A\Big(1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + G\frac{(Bt)^n}{n!}\Big).$$

Since $\lim_{n\to+\infty} G\frac{(Bt)^n}{n!} = 0$, the assertion follows. •

A.3 Moment inequalities

In this section of the appendix we prove some moment inequalities which we used in Section 2.2, so the notations introduced in that section prevail.

We assume in this section that $\mu(E) < \infty$. This is just to simplify the notations — in concrete applications we will replace μ by $\mathbb{1}_G \mu$. Then we consider an index set Λ and we denote by α the elements of Λ. Moreover we consider a family of processes $V_t^\alpha \in \mathbb{R}^d, \alpha \in \Lambda$ which verify the following equation

$$\begin{aligned} V_t^\alpha &= V_0^\alpha + \sum_{l=1}^m \int_0^t (H_l^\alpha(s) + \langle \nabla\sigma_l(X_s), V_s^\alpha\rangle)\, \mathrm{d}W_s^l \\ &\quad + \int_0^t (h^\alpha(s) + \langle \nabla b(X_s), V_s^\alpha\rangle)\, \mathrm{d}s \\ &\quad + \int_0^t \int_{E\times(0,1)} (Q^\alpha(s-,z) + \left\langle \nabla_x c(X_{s-},z), V_{s-}^\alpha\right\rangle)\mathbb{1}_{\{u\le\gamma(X_{s-},z)\}}\, \mathrm{d}N(s,u,z). \end{aligned} \tag{A.3.1}$$

Here $H_l^\alpha, h_l^\alpha$and Q^α are adapted càdlàg processes which verify

$$\int_0^T (|H_l^\alpha(s)|^2 + |h^\alpha(s)| + \int_E |Q^\alpha(s,z)|\, \overline{\gamma}(z) d\mu(z))\, \mathrm{d}s < \infty.$$

(Where the functions σ, b, c and γ are the ones introduced in Section 2.2.) So the corresponding stochastic integrals in (A.3.1) make sense.

Proposition A.3.1 *We suppose that*

$$|Q^\alpha(s,z)| \le \overline{q}(z)\, |R_s^\alpha| \tag{A.3.2}$$

for some adapted càdlàg process R^α and some measurable function $\overline{q} : E \to \mathbb{R}_+$ and we denote

$$\begin{aligned} \widehat{c}_1(p) &= \int_E (\overline{q}(z) + \overline{c}_{(1)}(z))(1 + \overline{q}(z))^{2p} \overline{\gamma}(z) d\mu(z), \\ \widehat{c}_2(p) &= \int_E (\overline{q}(z) + \overline{c}_{(1)}(z))(1 + \overline{c}_{(1)}(z))^{2p} \overline{\gamma}(z) d\mu(z). \end{aligned} \tag{A.3.3}$$

For every $p \in \mathbb{N}$ there exists a universal constant C_p such that $0 \le t \le T$

$$\begin{aligned} \mathrm{E}\left[\, |V_t^\alpha|^{2p}\,\right] &\le \exp\left(C_p t(1 + \|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}_2(p))\right) \\ &\quad \times \Big(|V_0^\alpha|^{2p} + C_p \int_0^t \mathrm{E}\Big[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_1(p)\, \big|R_{s-}^\alpha\big|^{2p}\,\Big]\, \mathrm{d}s\Big) \end{aligned} \tag{A.3.4}$$

Proof : Using Itô's formula for $f(x) = x^{2p}$, we obtain[1]

$$(V_t^\alpha)^{2p} = (V_0^\alpha)^{2p} + M_t^\alpha + I_t^\alpha + J_t^\alpha$$

with

$$M_t^\alpha = \sum_{l=1}^m \int_0^t 2p(V_s^\alpha)^{2p-1}(H_l^\alpha(s) + \langle \nabla\sigma_l(X_s), V_s^\alpha\rangle)\,\mathrm{d}W_s^l,$$

$$\begin{aligned} I_t^\alpha =& \sum_{l=1}^m \int_0^t p(2p-1)(V_s^\alpha)^{2p-2}\sum_{l=1}^m (H_l^\alpha(s) + \langle \nabla\sigma_l(X_s), V_s^\alpha\rangle)^2\,\mathrm{d}s \\ &+ 2p\int_0^t (V_s^\alpha)^{2p-1}(h^\alpha(s) + \langle \nabla b(X_s), V_s^\alpha\rangle)\,\mathrm{d}s \end{aligned}$$

and

$$J_t^\alpha = \int_0^t \int_{E\times(0,1)} \Big(\big(V_{s-}^\alpha + Q^\alpha(s-,z) + \big\langle \nabla_x c(X_{s-},z), V_{s-}^\alpha\big\rangle\big)^{2p} - \big(V_{s-}^\alpha\big)^{2p}\Big)\,\mathbb{1}_{\{u\le\gamma(X_{s-},z)\}}\,\mathrm{d}N(s,u,z).$$

Using the trivial inequality $a^u b^v \le a^{u+v} + b^{u+v}$ we obtain

$$\begin{aligned} \mathrm{E}\big[\,|I_t^\alpha|\,\big] \le& C_p\int_0^t \mathrm{E}\Big[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p}\Big]\,\mathrm{d}s \\ &+ C_p(1+\|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p})\int_0^t \mathrm{E}\big[\left|V_s^\alpha\right|^{2p}\big]\,\mathrm{d}s. \end{aligned}$$

We estimate now J_t^α. Using the elementary inequality

$$(a+b)^{2p} - a^{2p} \le C_p\,|b|\,(|a|^{2p-1} + |b|^{2p-1})$$

we obtain

$$\begin{aligned} &\left|V_{s-}^\alpha + Q^\alpha(s-,z) + \big\langle \nabla_x c(X_{s-},z), V_{s-}^\alpha\big\rangle\right|^{2p} - \left|V_{s-}^\alpha\right|^{2p} \\ \le\;& C_p(|Q^\alpha(s-,z)| + \bar{c}_{(1)}(z)\left|V_{s-}^\alpha\right|)(\left|V_{s-}^\alpha\right|^{2p-1}(1+\bar{c}_{(1)}^{2p-1}(z)) + |Q^\alpha(s-,z)|^{2p-1}). \end{aligned}$$

Recall that $|Q^\alpha(s-,z)| \le \bar{q}(z)\left|R_{s-}^\alpha\right|$ so the above term is upper bounded by

$$\begin{aligned} & C_p(\bar{q}(z)\left|R_{s-}^\alpha\right| + \bar{c}_{(1)}(z)\left|V_{s-}^\alpha\right|)(\left|V_{s-}^\alpha\right|^{2p-1}(1+\bar{c}_{(1)}(z))^{2p-1} + \left|\bar{q}(z)R_{s-}^\alpha\right|^{2p-1}) \\ \le\;& C_p(\bar{q}(z) + \bar{c}_{(1)}(z))(\left|R_{s-}^\alpha\right| + \left|V_{s-}^\alpha\right|)(((1+\bar{c}_{(1)}(z))^{2p-1}\left|V_{s-}^\alpha\right|^{2p-1} + \left|\bar{q}(z)R_{s-}^\alpha\right|^{2p-1}) \end{aligned}$$

We use once again the inequality $a^u b^v \le a^{u+v} + b^{u+v}$ and we upper bound the above term by

$$C_p(\bar{q}(z) + \bar{c}_{(1)}(z))(\left|R_{s-}^\alpha\right|^{2p}(1+\bar{q}(z))^{2p} + (1+\bar{c}_{(1)}(z))^{2p}\left|V_{s-}^\alpha\right|^{2p}).$$

It follows that

$$\begin{aligned} \mathrm{E}\big[\,|J_t^\alpha|\,\big] \le& C_p\int_E (\bar{q}(z) + \bar{c}_{(1)}(z))(1+\bar{q}(z))^{2p}\bar{\gamma}(z)d\mu(z)\int_0^t \mathrm{E}\big[\left|R_{s-}^\alpha\right|^{2p}\big]\,\mathrm{d}s \\ &+ C_p\int_E (\bar{q}(z) + \bar{c}_{(1)}(z))(1+\bar{c}_{(1)}(z))^{2p}\bar{\gamma}(z)d\mu(z)\int_0^t \mathrm{E}\big[\left|V_{s-}^\alpha\right|^{2p}\big]\,\mathrm{d}s. \end{aligned}$$

Since M_t^α is a martingale we obtain

$$\mathrm{E}\big[\,(V_t^\alpha)^{2p}\,\big] = \mathrm{E}\big[\,(V_0^\alpha)^{2p}\,\big] + \mathrm{E}\big[I_t^\alpha\big] + \mathrm{E}\big[J_t^\alpha\big]$$

[1] For $x = (x_i)_{1\le i\le d} \in \mathbb{R}^d$ we simply denote $(x_i^{2p})_{1\le i\le d}$ by x^{2p}.

and (we recall the notation in (A.3.3))

$$\begin{aligned}
\mathrm{E}\left[\,|V_t^\alpha|^{2p}\,\right] &\le |V_0^\alpha|^{2p} + \mathrm{E}\left[\,|I_t^\alpha|\,\right] + \mathrm{E}\left[\,|J_t^\alpha|\,\right] \\
&\le |V_0^\alpha|^{2p} + C_p \int_0^t \mathrm{E}\left[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_1(p)\left|R_{s-}^\alpha\right|^{2p}\right] \mathrm{d}s \\
&\quad + C_p(1 + \|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}_2(p)) \int_0^t \mathrm{E}\left[\left|V_s^\alpha\right|^{2p}\right] \mathrm{d}s.
\end{aligned}$$

The Gronwall's lemma then gives

$$\begin{aligned}
\mathrm{E}\left[\,|V_t^\alpha|^{2p}\,\right] &\le \exp\left(C_p t(1 + \|\nabla\sigma\|_\infty^{2p} + \|\nabla b\|_\infty^{2p} + \widehat{c}_2(p))\right) \\
&\quad \times \left(|V_0^\alpha|^{2p} + C_p \int_0^t \mathrm{E}\left[\sum_{l=1}^m |H_l^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \widehat{c}_1(p)\left|R_{s-}^\alpha\right|^{2p}\right] \mathrm{d}s\right).
\end{aligned}$$

•

A.4 Proof of (5.4.13)

$$\begin{aligned}
I &= \mathrm{E}\left[f(X_{T_k}^M)\mathbb{1}_{\{U_k \ge \gamma(Z_k, X_{T_k-}^M)\}} | X_{T_k-}^M = x\right] \\
&= \int_{\langle X_{T_k-}^M = x\rangle} f(X_{T_k}^M)\mathbb{1}_{\{U_k \ge \gamma(Z_k, X_{T_k-}^M)\}} \frac{\mathrm{dP}}{\mathrm{P}(X_{T_k-}^M = x)} \\
&= \int_{\langle X_{T_k-}^M = x\rangle \cap \langle U_k \ge \gamma(Z_k, X_{T_k-}^M)\rangle} f(X_{T_k}^M) \frac{\mathrm{dP}}{\mathrm{P}(X_{T_k-}^M = x)}
\end{aligned}$$

On the event $\langle X_{T_k-}^M = x\rangle \cap \langle U_k \ge \gamma(Z_k, X_{T_k-}^M)\rangle$ we have $X_{T_k}^M = x$, so I becomes

$$\begin{aligned}
I &= f(x) \frac{\mathrm{P}(\langle X_{T_k-}^M = x\rangle \cap \langle U_k \ge \gamma(Z_k, X_{T_k-}^M)\rangle)}{\mathrm{P}(X_{T_k-}^M = x)} \\
&= f(x) \frac{\mathrm{P}(\langle X_{T_k-}^M = x\rangle \cap \langle U_k \ge \gamma(Z_k, x)\rangle)}{\mathrm{P}(X_{T_k-}^M = x)} \\
&= f(x)\,\mathrm{P}(U_k \ge \gamma(Z_k, x)) && \text{(since } U_k \text{ and } X_{T_k-}^M \text{ are independent)} \\
&= f(x) \frac{1}{\mu(B_{M+1})2\overline{c}} \int_{B_{M+1}} \int_0^{2\overline{c}} \mathbb{1}_{\{u \ge \gamma(z,x)\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u \\
&= f(x) \frac{1}{\mu(B_{M+1})} \int_{B_{M+1}} \left(1 - \frac{\gamma(z,x)}{2\overline{c}}\right) \mathrm{d}\mu(z) \\
&= f(x)\theta_{M,\gamma}(x) = f(x)\theta_{M,\gamma}(x) \int_{\mathbb{R}^d} \phi(z - z_M^*)\,\mathrm{d}z && \textstyle(\int \phi = 1 \text{ and } \phi(B(0,1)^c) = 0) \\
&= \int_{|z|>M+1} f(x + c_M(z,x))\phi(z - z_M^*)\theta_{M,\gamma}(x)\,\mathrm{d}z && \text{(since } c_M(z,x) = 0 \text{ if } |z| > M+1) \\
&= \int_{|z|>M+1} f(x + c_M(z,x)) q_M(z,x)\,\mathrm{d}z.
\end{aligned}$$

On the other hand:

$$
\begin{aligned}
J &= \mathrm{E}\left[f(X^M_{T_k})\mathbb{1}_{\{U_k<\gamma(Z_k,X^M_{T_k-})\}}|X^M_{T_k-}=x\right]\\
&= \int_{\langle X^M_{T_k-}=x\rangle} f(X^M_{T_k})\mathbb{1}_{\{U_k<\gamma(Z_k,X^M_{T_k-})\}}\frac{\mathrm{dP}}{\mathrm{P}(X^M_{T_k-}=x)}\\
&= \int_{\langle X^M_{T_k-}=x\rangle\cap\langle U_k\geq\gamma(Z_k,X^M_{T_k-})\rangle} f(x+c_M(Z_k,x))\frac{\mathrm{dP}}{\mathrm{P}(X^M_{T_k-}=x)}\\
&= \int_\Omega f(x+c_M(Z_k,x))\mathbb{1}_{\{U_k<\gamma(Z_k,x)\}}\,\mathrm{dP} \qquad \text{(since } (U_k,Z_k) \text{ and } X^M_{T_k-} \text{ are independent)}\\
&= \frac{1}{\mu(B_{M+1})2\overline{c}}\int_{B_{M+1}}\int_0^{2\overline{c}} f(x+c_M(z,x))\mathbb{1}_{\{u<\gamma(z,x)\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u\\
&= \frac{1}{\mu(B_{M+1})}\int_{B_{M+1}} f(x+c_M(z,x))\frac{\gamma(z,x)}{2\overline{c}}\,\mathrm{d}\mu(z) \qquad \text{(with } \mathrm{d}\mu(z)=h(z)\,\mathrm{d}z)
\end{aligned}
$$

We finally have

$$
\mathrm{E}\left[f(X^M_{T_k})|X^M_{T_k-}=x\right]=\int_{\mathbb{R}^d} f(x+c_M(z,x))q_M(z,x)\,\mathrm{d}z.
$$

Now $\sigma(\overline{X}_{T_{k+1}-})\subset\mathcal{G}_k$, so

$$
\begin{aligned}
&\mathrm{E}\left[f(\overline{X}^M_{T_{k+1}})|\overline{X}^M_{T_{k+1}-}=x\right]\\
&\qquad= \mathrm{E}\left[f(\overline{X}^M_{T_{k+1}-}+c_M(\overline{Z}_{k+1},\overline{X}^M_{T_{k+1}-}))|\overline{X}^M_{T_{k+1}-}=x\right]\\
&\qquad= \mathrm{E}\left[\,\mathrm{E}\left[f(\overline{X}^M_{T_{k+1}-}+c_M(\overline{Z}_{k+1},\overline{X}^M_{T_{k+1}-}))|\mathcal{G}_k\right]|\overline{X}^M_{T_{k+1}-}=x\right]\\
&\qquad= \mathrm{E}\left[\int_{\mathbb{R}^d} f(\overline{X}^M_{T_{k+1}-}+c_M(z,\overline{X}^M_{T_{k+1}-}))q_M(\overline{X}^M_{T_{k+1}-},z)\,\mathrm{d}z|\overline{X}^M_{T_{k+1}-}=x\right]\\
&\qquad= \int_{\mathbb{R}^d}\int_{\langle\overline{X}^M_{T_{k+1}-}=x\rangle} f(\overline{X}^M_{T_{k+1}-}+c_M(z,\overline{X}^M_{T_{k+1}-}))q_M(\overline{X}^M_{T_{k+1}-},z)\frac{\mathrm{dP}}{\mathrm{P}(\overline{X}^M_{T_{k+1}-}=x)}\,\mathrm{d}z\\
&\qquad= \int_{\mathbb{R}^d} f(x+c_M(z,x))q_M(z,x)\,\mathrm{d}z
\end{aligned}
$$

So we have:

$$
\mathrm{E}\left[f(X^M_{T_k})|X^M_{T_k-}=x\right]=\mathrm{E}\left[f(\overline{X}^M_{T_k})|\overline{X}^M_{T_k-}=x\right] \tag{A.4.5}
$$

We can prove now that the processes X^M_t and $\overline{X}^M_t$ are sharing the same law.

A.5 X^M_t and $\overline{X}^M_t$ share the same law

- First, if $0\leq t<T_1$, $x_t=\Psi_t(x)$ and then $\overline{X}^M_t=\Psi_t(x)\sim X^M_t$.
- Moreover, if $\overline{X}^M_{T_k-}\sim X^M_{T_k-}$ we have $\overline{X}^M_{T_k}\sim X^M_{T_k}$: recalling that, if $\phi(x):=\mathrm{E}\left[Y|X=x\right]$, we have $\int_A\phi(x)\,\mathrm{dP}_X(x)=\int_{\langle X\in A\rangle}Y\,\mathrm{d}P$, since $\mathrm{P}_{\overline{X}^M_{T_k-}}=\mathrm{P}_{X^M_{T_k-}}$ the relation (A.4.5) leads to

$$
\mathrm{E}\left[f(X^M_{T_k})\right]=\mathrm{E}\left[f(\overline{X}^M_{T_k})\right]
$$

- Finally, if $T_k\leq t<T_{k+1}$, $\overline{X}^M_t=\Psi_{t-T_k}(\overline{X}^M_{T_k})\sim\Psi_{t-T_k}(X^M_{T_k})=X^M_t$.

Appendix B

Sobolev norms of X_t^M and its derivatives.

We have used, within the proof of the Lemma 6.2.1 that X_t^M has moments of any order. The proof of that result follows the same pattern of this same lemma, although a bit simpler. The following result is the equation (6.2.7) in the special case $k = 0$.

Proposition B.0.1 *Let $M \in \mathbb{N}^*$. For all $T > 0$ and $p \geq 1$, there exists a constant $C_{T,p} > 0$ (which does not depend on M) such that*

$$\mathrm{E}\left[|X_t^M|^{2p}\right] \leq C_{T,p}. \tag{B.0.1}$$

Proof :

We localize our problem by using the sequence $(\tau_K^M)_{K\in\mathbb{N}^*}$ of stopping times defined by

$$\tau_K^M \stackrel{\text{def}}{=} \inf\{t > 0 \ : \ |X_t^M| \geq K\}. \tag{B.0.2}$$

We can prove that a.s. $\lim_{K\to\infty} \tau_K^M = \infty$:
From the hypothesis made on the coefficients of X_t^M, it is clear that, for all $t \geq 0$,

$$\mathrm{E}\left[\sup_{s\leq t} |X_s^M|\right] < \infty. \tag{B.0.3}$$

We have, for $t \geq 0$

$$\begin{aligned}
\lim_{K\to\infty} \mathrm{P}(\tau_K^M < t) &= \lim_{K\to\infty} \mathrm{P}(\sup_{s\leq t} |X_s^M| > K) \\
&\leq \lim_{K\to\infty} \frac{1}{K} \mathrm{E}\left[\sup_{s\leq t} |X_s^M|\right] = 0.
\end{aligned}$$

$(\tau_K^M)_{K\in\mathbb{N}^*}$ tends to ∞ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau_K^M)_{K\in\mathbb{N}^*}$) which tends to ∞ a.s.

If we admit for the moment the Lemma B.0.2, we know that there exists a constant $C_{p,T}$ which does not depend on K and M and such that, for all $0 \leq t \leq T$,

$$\mathrm{E}\left[|X_{t\wedge\tau_K^M}^M|^{2p}\right] \leq C_{T,p}.$$

The monotone convergence theorem implies then

$$\mathrm{E}\left[|X_t^M|^{2p}\right] = \sup_K \mathrm{E}\left[|X_t^M|^{2p} \mathbb{1}_{\tau_K^M > t}\right]$$

and

$$\sup_K \mathrm{E}\left[|X_t^M|^{2p} \mathbb{1}_{\tau_K^M > t}\right] = \sup_K \mathrm{E}\left[|X_{t\wedge\tau_K^M}^M|^{2p}\right] \leq C_{T,p}.$$

•

Lemma B.0.2 *Let $M \in \mathbb{N}^*$ and a sequence $(\tau_K^M)_{K\in\mathbb{N}^*}$ of stopping times defined by* (B.0.2). *There exists a constant $C_{p,T}$, which does not depend on K and M, and such that, for all $0 \le t \le T$,*

$$\mathrm{E}\left[|X^M_{t\wedge\tau_K^M}|^{2p}\right] \le C_{T,p}.$$

Proof :

Recalling the definition (5.4.8) of X_t^M:

$$X_t^M = x + \int_0^t \sigma(X_s^M)\,\mathrm{d}W_s + \int_0^t\int_E c_M(z,X_{s-}^M)\mathbb{1}_{\{u\le\gamma(z,X_{s-}^M)\}}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t g(X_s^M)\,\mathrm{d}s,$$

we have, for a single component (omitting for a moment the parameter M in order to simplify the notations), applying Itô's formula with $f(x) = x^{2p}$ with respect to every component of the process $X^M_{t\wedge\tau_K^M}$,

$$\begin{aligned}(X^i_{t\wedge\tau_K^M})^{2p} = (X_0^i)^{2p} &+ \sum_{l=1}^m \int_0^{t\wedge\tau_K^M} 2p(X_s^i)^{2p-1}\sigma_{il}(X_s)\,\mathrm{d}W_s^l\\ &+ 2p\int_0^{t\wedge\tau_K^M}(X_s^i)^{2p-1}g_i(X_s)\,\mathrm{d}s\\ &+ p(2p-1)\sum_{l=1}^m\int_0^{t\wedge\tau_K^M}(X_s^i)^{2p-2}\big(\sigma_{il}(X_s)\big)^2\,\mathrm{d}s\\ &+ \int_0^{t\wedge\tau_K^M}\int_E\Big(X^i_{s^-} + c_M(z,X^M_{s-})\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}\Big)^{2p} - (X^i_{s^-})^{2p}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\end{aligned}$$

We now take the expectation with respect to the Brownian motion (*i.e.* conditionally with respect to all the other random quantities):

$$\begin{aligned}\mathrm{E}_W\left[(X^i_{t\wedge\tau_K^M})^{2p}\right] =& \mathrm{E}_W\left[(X_0^i)^{2p}\right] + 2p\int_0^{t\wedge\tau_K^M}\mathrm{E}_W\left[(X_s^i)^{2p-1}g_i(X_s)\right]\mathrm{d}s\\ &+ p(2p-1)\sum_{l=1}^m\int_0^{t\wedge\tau_K^M}\mathrm{E}_W\left[(X_s^i)^{2p-2}\big(\sigma_{il}(X_s)\big)^2\right]\mathrm{d}s\\ &+ \int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W\left[\Big(X^i_{s^-} + c_M(z,X^M_{s-})\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}\Big)^{2p} - (X^i_{s^-})^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\end{aligned}$$

Since $s \le t\wedge\tau_K^M$, we have $X_s = X_{s^M}$, and obviously $t \ge t\wedge\tau_K^M$, so we have

$$\begin{aligned}\mathrm{E}_W\left[|X^i_{t\wedge\tau_K^M}|^{2p}\right] \le& \mathrm{E}_W\left[|X_0^i|^{2p}\right] + 2p\int_0^t\mathrm{E}_W\left[|X^i_{s\wedge\tau_K^M}|^{2p-1}|g_i(X_s)|\right]\mathrm{d}s\\ &+ p(2p-1)\sum_{l=1}^m\int_0^t\mathrm{E}_W\left[|X^i_{s\wedge\tau_K^M}|^{2p-2}\big|\sigma_{il}(X_s)\big|^2\right]\mathrm{d}s\\ &+ \int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W\left[\Big|\Big(X^i_{s^-} + c_M(z,X^M_{s-})\mathbb{1}_{\{u\le\gamma(z,X^M_{s-})\}}\Big)^{2p} - (X^i_{s^-})^{2p}\Big|\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\end{aligned}$$

It follows (since $(a+b)^{2n} - a^{2n} \le (|a|+|b|)^{2n} - a^{2n}$, for all $a,b\in\mathbb{R}$ and using the inequality $x^uy^v \le x^{u+v} + y^{u+v}$)

$$\begin{aligned}\mathrm{E}_W\left[|X_{t\wedge\tau_K^M}|^{2p}\right] \le& \mathrm{E}_W\left[|X_0|^{2p}\right] + 2p\int_0^t\mathrm{E}_W\left[|X_{s\wedge\tau_K^M}|^{2p}\right]\mathrm{d}s + 2pT\|g^{2p}\|_\infty\\ &+ p(2p-1)m\int_0^t\mathrm{E}_W\left[|X_{s\wedge\tau_K^M}|^{2p}\right]\mathrm{d}s + p(2p-1)mT\|\sigma^{2p}\|_\infty\\ &+ \int_0^{t\wedge\tau_K^M}\int_E\mathrm{E}_W\left[\Big(|X_{s^-}| + \overline{c}(z)\mathbb{1}_{\{u\le\overline{\gamma}\}}\Big)^{2p} - |X_{s^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u);\end{aligned}$$

using now $|a^{2p}-b^{2p}| \leq |a-b|(a+b)^{2p-1}$, we have $\Big(|X_{s^-}|+\overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\Big)^{2p} - |X_{s^-}|^{2p} \leq 2^{2p}\overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}|X_{s^-}|^{2p}$, so,

$$\begin{aligned}\mathrm{E}_W\left[|X_{t\wedge\tau_K^M}|^{2p}\right] \leq & C_T + A_p\int_0^t \mathrm{E}_W\left[|X_{s\wedge\tau_K^M}|^{2p}\right]\mathrm{d}s + B_p\int_0^{t\wedge\tau_K^M}\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\,\mathrm{E}_W\left[|X_{s^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\ \leq & C_T + A_p\int_0^t \mathrm{E}_W\left[|X_{s\wedge\tau_K^M}|^{2p}\right]\mathrm{d}s + B_p\int_0^t\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\,\mathrm{E}_W\left[|X_{(s\wedge\tau_K^M)^-}|^{2p}\right]N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).\end{aligned}$$

With

$$\theta_t^K \stackrel{\text{def}}{=} \sup_{0\leq u\leq t}\mathrm{E}_W\left[|X^M_{u\wedge\tau_K^M}|^{2p}\right],$$

we then have

$$\theta_t^K \leq C_T + A_p\int_0^t \theta_s^K\,\mathrm{d}s + B_p\int_0^t\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\theta_{s^-}^K N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

With

$$R_1 \stackrel{\text{def}}{=} \int_0^t\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\,\mathrm{d}\mu(z)\,\mathrm{d}u$$

we have

$$\begin{aligned}\mathrm{E}\left[\theta_t^K\right] \leq & C_T + A_p\,\mathrm{E}\Big[\int_0^t \theta_s^K\,\mathrm{d}s\Big] + B_p\,\mathrm{E}\Big[\int_0^t\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\theta_{s^-}^K\,\mathrm{d}N(s,z,u)\Big]\\ \leq & C_T + A_p\int_0^t \mathrm{E}\left[\theta_s^K\right]\mathrm{d}s + B_p\int_0^t\int_E \overline{c}(z)\mathbb{1}_{\{u\leq\overline{\gamma}\}}\,\mathrm{E}\left[\theta_{s^-}^K\right]\mathrm{d}s\mu(\mathrm{d}z)\,\mathrm{d}u\\ \leq & C_T + (A_p+R_1)\int_0^t \mathrm{E}\left[\theta_s^K\right]\mathrm{d}s.\end{aligned}$$

The Gronwall's lemma ends then the proof.

•

Appendix C

Tangent flow

Let us recall the expression of the tangent flow Y_t^M:

$$Y_t^M = \mathrm{Id} + \sum_{l=1}^{m} \int_0^t \nabla\sigma_l(\bar{X}_s^M) Y_s^M \, \mathrm{d}W_s^l + \sum_{j=1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M) Y_{T_j-}^M + \int_0^t \nabla_x g(\bar{X}_s^M) Y_s^M \, \mathrm{d}s.$$

We have then defined the following process (with $\nabla_x c_j = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j-}^M)$):

$$\hat{Y}_t^M \stackrel{\text{def}}{=} \mathrm{Id} - \sum_{l=1}^{m} \int_0^t \hat{Y}_s^M \nabla\sigma_l(\bar{X}_s^M) \, \mathrm{d}W_s^l - \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M \nabla_x c_j (\mathrm{Id} + \nabla_x c_j)^{-1} \\ + \int_0^t \hat{Y}_s^M \Big(\frac{1}{2} \sum_{l=1}^{m} \nabla\sigma_l(\bar{X}_s^M)^2 - \nabla_x g(\bar{X}_s^M) \Big) \, \mathrm{d}s,$$

and stated that:

Lemma C.0.3 *For all $t \geq 0$,*

$$Y_t^M \hat{Y}_t^M = \mathrm{Id}\,. \tag{C.0.1}$$

Proof :

Step 1

Let us consider (for the moment $m = 1$) the stochastic process sharing the same law as Y_t (that we will continue to denote Y_t, in order to simplify the notations) defined by

$$Y_t = \mathrm{Id} + \int_0^t \Sigma Y_s \, \mathrm{d}W_s + \sum_{j=1}^{J_t^M} C_j Y_{T_j-}^M + \int_0^t G Y_s \, \mathrm{d}s$$

with

$$C_j = \nabla_x c_M(Z_j, X_{T_j-}^M) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j-}^M)\}}$$

and let us set

$$\hat{Y}_t = \mathrm{Id} + \int_0^t \hat{Y}_s A \, \mathrm{d}W_s + \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M H_j + \int_0^t \hat{Y}_s B \, \mathrm{d}s$$

that is

$$Y_t^{i,j} = \delta_{i,j} + \sum_{h=1}^{d} \int_0^t \Sigma_{i,h} Y_s^{h,j} \, \mathrm{d}W_s + \sum_{k'=1}^{J_t^M} \sum_{h=1}^{d} C_{k'}^{i,h} Y_{T_{k'}-}^{h,j} + \sum_{h=1}^{d} \int_0^t G_{i,h} Y_s^{h,j} \, \mathrm{d}s$$

and

$$\hat{Y}_t^{i,j} = \delta_{i,j} + \sum_{h=1}^{d} \int_0^t \hat{Y}_s^{i,h} A_{h,j} \, \mathrm{d}W_s + \sum_{k'=1}^{J_t^M} \sum_{h=1}^{d} \hat{Y}_{T_{k'}-}^{i,h} H_{k'}^{h,j} + \sum_{h=1}^{d} \int_0^t \hat{Y}_s^{i,h} B_{h,j} \, \mathrm{d}s$$

We have $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q} = \sum_{n=1}^{d} \hat{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q}$. Using Itô's formula it follows:

$$\begin{aligned}
&\hat{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q,n} \\
&\quad = \int_0^t Y_s^{n,q} \sum_{h=1}^{d} \hat{Y}_s^{p,h} A_{h,n} \, \mathrm{d}W_s + \int_0^t \hat{Y}_s^{p,n} \sum_{h=1}^{d} \Sigma_{n,h} Y_s^{h,q} \, \mathrm{d}W_s \\
&\quad + \frac{1}{2} \int_0^t \left(\sum_{g=1}^{d} \hat{Y}_s^{p,g} A_{g,n} \right) \left(\sum_{h=1}^{d} \Sigma_{n,h} Y_s^{h,q} \right) \\
&\quad + \int_0^t Y_s^{n,q} \sum_{h=1}^{d} \hat{Y}_s^{p,h} B_{h,n} \, \mathrm{d}s + \int_0^t \hat{Y}_s^{p,n} \sum_{h=1}^{d} G_{n,h} Y_s^{h,q} \, \mathrm{d}s \\
&\quad + \int_0^t \int_E \left(Y_{s-}^{n,q} + \sum_{h=1}^{d} C_{s-}^{n,h}(z,u) Y_{s-}^{h,q} \right) \left(\hat{Y}_{s-}^{p,n} \sum_{h=1}^{d} \hat{Y}_{s-}^{p,h} H_{s-}^{h,n}(z,u) \right) - Y_{s-}^{n,q} \hat{Y}_{s-}^{p,n} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)
\end{aligned}$$

The integrated term with respect to $\mathrm{d}W_s$ of $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q}$ has the following form:

$$\begin{aligned}
\sum_{n=1}^{d} \sum_{h=1}^{d} Y_s^{n,q} \hat{Y}_s^{p,h} A_{h,n} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_s^{p,n} Y_s^{h,q} \Sigma_{n,h} &= \sum_{n=1}^{d} \sum_{h=1}^{d} Y_s^{h,q} \hat{Y}_s^{p,n} A_{n,h} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_s^{p,n} Y_s^{h,q} \Sigma_{n,h} \\
&= \sum_{n=1}^{d} \sum_{h=1}^{d} Y_s^{h,q} \hat{Y}_s^{p,n} (A_{n,h} + \Sigma_{n,h}).
\end{aligned}$$

This term is null for

$$A = -\Sigma$$

which we will suppose in the following of this proof.

For the third term

$$\begin{aligned}
\sum_{n=1}^{d} \left(\sum_{g=1}^{d} \hat{Y}_s^{p,g} A_{g,n} \right) \left(\sum_{h=1}^{d} \Sigma_{n,h} Y_s^{h,q} \right) &= \sum_{n=1}^{d} \sum_{g=1}^{d} \left(\sum_{h=1}^{d} \hat{Y}_s^{p,g} A_{g,n} \Sigma_{n,h} Y_s^{h,q} \right) \\
&= \sum_{g=1}^{d} \sum_{h=1}^{d} \left(\sum_{n=1}^{d} A_{g,n} \Sigma_{n,h} \right) \hat{Y}_s^{p,g} Y_s^{h,q} \\
&= \sum_{n=1}^{d} \sum_{h=1}^{d} \left(\sum_{g=1}^{d} A_{n,g} \Sigma_{g,h} \right) \hat{Y}_s^{p,n} Y_s^{h,q} \\
&= \sum_{n=1}^{d} \sum_{h=1}^{d} (A\Sigma)_{n,h} \, \hat{Y}_s^{p,n} Y_s^{h,q}
\end{aligned}$$

The integrated term with respect to ds of $(\hat{Y}_t Y_t - \mathrm{Id})_{p,q}$ has the following form:

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{d}\sum_{h=1}^{d}(A\Sigma)_{n,h}\,\hat{Y}_s^{p,n}Y_s^{h,q} + \sum_{n=1}^{d}\sum_{h=1}^{d}Y_s^{n,q}\hat{Y}_s^{p,h}B_{h,n} + \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}G_{n,h}Y_s^{h,q}& \\
= \frac{1}{2}\sum_{n=1}^{d}\sum_{h=1}^{d}(A\Sigma)_{n,h}\,\hat{Y}_s^{p,n}Y_s^{h,q} + \sum_{n=1}^{d}\sum_{h=1}^{d}Y_s^{h,q}\hat{Y}_s^{p,n}B_{n,h} &+ \sum_{n=1}^{d}\sum_{h=1}^{d}\hat{Y}_s^{p,n}G_{n,h}Y_s^{h,q} \\
= \sum_{n=1}^{d}\sum_{h=1}^{d}\left(\frac{1}{2}(A\Sigma)_{n,h} + B_{n,h} + G_{n,h}\right)\hat{Y}_s^{p,n}Y_s^{h,q}& \\
= \sum_{n=1}^{d}\sum_{h=1}^{d}\left(\frac{1}{2}A\Sigma + B + G\right)_{n,h}\hat{Y}_s^{p,n}Y_s^{h,q}.&
\end{aligned}$$

This term is null if (with $A = -\Sigma$)

$$B = \frac{1}{2}\Sigma^2 - G.$$

Step 2

Multidimensional Brownian case:

$$Y_t = \mathrm{Id} + \int_0^t \Sigma_1 Y_s\,\mathrm{d}W_s^1 + \cdots + \int_0^t \Sigma_m Y_s\,\mathrm{d}W_s^n + \sum_{j=1}^{J_t^M} C_j Y_{T_j-}^M + \int_0^t GY_s\,\mathrm{d}s$$

and let us set

$$\hat{Y}_t = \mathrm{Id} + \int_0^t \hat{Y}_s A_1\,\mathrm{d}W_s^1 + \cdots + \int_0^t \hat{Y}_s A_m\,\mathrm{d}W_s^n + \sum_{j=1}^{J_t^M} \hat{Y}_{T_j-}^M H_j + \int_0^t \hat{Y}_s B\,\mathrm{d}s$$

that is

$$Y_t^{i,j} = \delta_{i,j} + \sum_{l=1}^{m}\sum_{h=1}^{d}\int_0^t \Sigma_{i,h}^l Y_s^{h,j}\,\mathrm{d}W_s^l + \sum_{k'=1}^{J_t^M}\sum_{h=1}^{d} C_{k'}^{i,h}Y_{T_{k'}-}^{h,j} + \sum_{h=1}^{d}\int_0^t G_{i,h}Y_s^{h,j}\,\mathrm{d}s$$

and

$$\hat{Y}_t^{i,j} = \delta_{i,j} + \sum_{l=1}^{m}\sum_{h=1}^{d}\int_0^t \hat{Y}_s^{i,h}A_{h,j}^l\,\mathrm{d}W_s^l + \sum_{k'=1}^{J_t^M}\sum_{h=1}^{d} \hat{Y}_{T_{k'}-}^{i,h}H_{k'}^{h,j} + \sum_{h=1}^{d}\int_0^t \hat{Y}_s^{i,h}B_{h,j}\,\mathrm{d}s$$

and

$$\begin{aligned}
\hat{Y}_t^{p,n}Y_t^{n,q} - \delta_{p,q,n} = \int_0^t Y_s^{n,q}\sum_{l=1}^{m}\sum_{h=1}^{d}\hat{Y}_s^{p,h}A_{h,n}^l\,\mathrm{d}W_s^l + \int_0^t \hat{Y}_s^{p,n}\sum_{l=1}^{m}\sum_{h=1}^{d}\Sigma_{n,h}^l Y_s^{h,q}\,\mathrm{d}W_s^l \\
+ \sum_{l=1}^{m}\frac{1}{2}\int_0^t\left(\sum_{g=1}^{d}\hat{Y}_s^{p,g}A_{g,n}^l\right)\left(\sum_{h=1}^{d}\Sigma_{n,h}^l Y_s^{h,q}\right) \\
+ \int_0^t Y_s^{n,q}\sum_{h=1}^{d}\hat{Y}_s^{p,h}B_{h,n}\,\mathrm{d}s + \int_0^t \hat{Y}_s^{p,n}\sum_{h=1}^{d}G_{n,h}Y_s^{h,q}\,\mathrm{d}s \\
+ \int_0^t\int_E\left(Y_{s-}^{n,q} + \sum_{h=1}^{d}C_{s-}^{n,h}(z,u)Y_{s-}^{h,q}\right)\left(\hat{Y}_{s-}^{p,n}\sum_{h=1}^{d}\hat{Y}_{s-}^{p,h}H_{s-}^{h,n}(z,u)\right) - Y_{s-}^{n,q}\hat{Y}_{s-}^{p,n}N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)
\end{aligned}$$

The same computations, by a straight superposition, give us then, for all $l \in [\![0, m]\!]$,

$$A_i = -\Sigma_i$$

and

$$B = \frac{1}{2}\left(\sum_{i=1} \Sigma_i^2\right) - G.$$

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