Minimal lipschitz extension

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Extensions lipschitziennes minimales

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Summary

The thesis is concerned to some mathematical problems on minimal Lipschitz extensions.

Chapter 1: We introduce some basic background about minimal Lipschitz extension (MLE) problems.

Chapter 2: We study the relationship between the Lipschitz constant of 1-field and the Lipschitz constant of the gradient associated with this 1-field. We produce two Sup-Inf explicit formulas which are two extremal minimal Lipschitz extensions for 1-fields. We explain how to use the Sup-Inf explicit minimal Lipschitz extensions for 1-fields to construct minimal Lipschitz extension of mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. Moreover, we show that Wells's extensions of 1-fields are absolutely minimal Lipschitz extensions (AMLE) when the domain of 1-field to expand is finite. We provide a counter-example showing that this result is false in general.

Chapter 3: We study the discrete version of the existence and uniqueness of AMLE. We prove that the tight function introduced by Sheffield and Smart is a Kirszbraun extension. In the real-valued case, we prove that the Kirszbraun extension is unique. Moreover, we produce a simple algorithm which calculates efficiently the value of the Kirszbraun extension in polynomial time.

Chapter 4: We describe some problems for future research, which are related to the subject represented in the thesis.

Résumé

Cette thèse est consacrée aux quelques problèmes mathématiques concernant les extensions minimales de Lipschitz. Elle est organisée de manière suivante.

Le chapitre 1 est dédié à l'introduction des extensions minimales de Lipschitz.

Dans le chapitre 2, nous étudions la relation entre la constante de Lipschitz d' 1-field et la constante de Lipschitz du gradient associée à ce 1-field. Nous proposons deux formules explicites Sup-Inf, qui sont des extensions extrêmes minimales de Lipschitz d'1-field. Nous expliquons comment les utiliser pour construire les extensions minimales de Lipschitz pour les applications de $\mathbb{R}^m$ à $\mathbb{R}^n$. Par ailleurs, nous montrons que les extensions de Wells d'1-fields sont les extensions absolument minimales de Lipschitz (AMLE) lorsque le domaine d'expansion d'1-field est infini. Un contre-exemple est présenté afin de montrer que ce résultat n'est pas vrai en général.

Dans le chapitre 3, nous étudions la version discrète de l'existence et l'unicité de l'AMLE. Nous montrons que la fonction tight introduite par Sheffield and Smart est l'extension de Kirszbraun. Dans le cas réel, nous pouvons montrer que cette extension

est unique. De plus, nous proposons un algorithme qui permet de calculer efficacement la valeur de l'extension de Kirszbraun en complexité polynomiale. Pour conclure, nous décrivons quelques pistes pour la future recherche, qui sont liées au sujet présenté dans ce manuscrit.

Chapter 1

Introduction générale

1.1 Le problème classique d'extension lipschitzienne

Nous considérons une paire d'espaces métriques (X, d_X) et (Y, d_Y). Soit Ω un sous-ensemble de X et $f : \Omega \to Y$ une fonction lipschitzienne. Nous noterons

$$\mathrm{Lip}(f, \Omega) := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x,y)}$$

la constante de Lipschitz de f sur Ω.

Le problème de l'extension lipschitzienne classique demande des conditions sur les d'espaces métriques (X, d_X) et (Y, d_Y) de sorte que pour tout $\Omega \subset X$ et pour toute fonction lipschitzienne $f : \Omega \to Y$, nous avons une fonction $g : X \to Y$ qui étend f et avec la même constante de Lipschitz que f, c'est à dire, $\mathrm{Lip}(f, \Omega) = \mathrm{Lip}(g, X)$. Cela signifie que nous pouvons toujours étendre les fonctions tout en préservant leur constante de Lipschitz. La paire (X,Y) est dit avoir la *propriété d'extension isométrique.* Il est rare pour une paire d'espaces métriques (X,Y) d'avoir la propriété d'extension isométrique. Dans cette section, nous présentons quelques exemples célèbres pour la paire (X,Y) qui ont la propriété d'extension isométrique.

1.1.1 Théorème de Kirszbraun

Kirszbraun trouvé un exemple très célèbre de paire d'espaces métriques (X,Y) ayant la propriété d'extension isométrique.

Théorème 1.1.1. *(Théorème de Kirszbraun) Soient H_1 et H_2 deux espaces de Hilbert. Si Ω est un sous-ensemble de H_1, et $f : \Omega \to H_2$ est une fonction lipschitzienne, alors il existe une fonction $g : H_1 \to H_2$ satisfaisant*

$$g = f \textit{ dans } \Omega \;\; \textit{et} \;\; \mathrm{Lip}(g, H_1) = \mathrm{Lip}(f, \Omega).$$

Kirszbraun a prouvé ce théorème en 1934 [32] pour les paires d'espaces euclidiens. Plus tard, il a été prouvé de façon indépendante par Valentine en 1943 [52]. Valentine a aussi généralisé les résultats de Kirszbraun à des paires d'espaces de Hilbert de dimension arbitraire. Ce théorème est appelé théorème de Kirszbraun, il est parfois aussi appelé théorème de Kirszbraun-Valentine. Ce théorème affirme que si X et Y sont des espaces de Hilbert, alors (X,Y) a la propriété d'extension isométrique.

Parce que la preuve de ce théorème pour le cas $H_1 = \mathbb{R}^m$ et $H_2 = \mathbb{R}^n$ (tous deux équipés de la norme euclidienne) est très simple et élégante, nous la reproduisons ci-dessous. Tout d'abord, nous rappelons le résultat intéressant utilisé dans la preuve du théorème de Kirszbraun:

Lemme 1.1.2. *[19, Lemma 2.10.40] Soit P un compact de $\mathbb{R}^n \times \{r : 0 < r < \infty\}$ et*

$$Y_t = \{y : \|y-a\| \leq rt \text{ chaque fois que } (a,r) \in P\}$$

pour $0 \leq t < +\infty$. Alors $c = \inf\{t : Y_t \neq \emptyset\} < +\infty$, Y_c se compose d'un seul point b et b appartient à l'enveloppe convexe de

$$A = \{a : \text{ pour certains } r, (a,r) \in P, \text{ et } \|b-a\| = rc\}.$$

La preuve du lemme ci-dessus peut voir dans le livre de Federer *Geometric Measure Theory* [19, Lemma 2.10.40].

Preuve du théorème de Kirszbraun . (Pour le cas $H_1 = \mathbb{R}^m$ et $H_2 = \mathbb{R}^n$ tous deux équipés de la norme euclidienne)

Sans perte de généralité, nous pouvons supposer $\mathrm{Lip}(f,\Omega) = 1$.

***Étape 1:** Dans cette étape, nous étendons f en un point supplémentaire , c'est-à-dire, pour $x \in H_1 \backslash \Omega$, nous devons trouver $f_x \in H_2$ telle que

$$\|f_x - f(a)\| \leq \|x-a\|, \ \forall a \in \Omega.$$

Ceci est équivalent à

$$\bigcap_{a \in \Omega} B(f(a), \|x-a\|) \neq \emptyset.$$

Comme ces boules sont compactes, il suffira de vérifier que

$$\bigcap_{a \in F} B(f(a), \|x-a\|) \neq \emptyset. \tag{1.1}$$

pour chaque sous-ensemble fini F de Ω.

Appliquant le Lemme 1.1.2 (en utilisant la même notation) avec

$$P = \{(f(a), \|x-a\|) : a \in F\},$$

nous pouvons trouver $x_1, \ldots, x_k \in A$ et b appartenant à l'enveloppe convexe $\{f(x_i)\}_{i \in \{1,\ldots,k\}}$ telle que

$$\|b - f(x_i)\| = c\|x - x_i\|.$$

Notre tâche est de montrer que $c \leq 1$.

Nous écrivons $b = \sum_{i=1}^{k} \lambda_i f(x_i)$ pour $\lambda_i \in [0,1]$ et $\sum_{i=1}^{k} \lambda_i = 1$. En utilisant la formule

$$2\langle u,v\rangle = \|u\|^2 + \|v\|^2 - \|u-v\|^2,$$

nous avons

$$
\begin{aligned}
0 &= 2\|\sum_i \lambda_i (f(x_i)-b)\|^2 \\
&= 2\sum_{i,j} \lambda_i\lambda_j \langle f(x_i)-b, f(x_j)-b\rangle \\
&= \sum_{i,j} \lambda_i\lambda_j \left[\|f(x_i)-b\|^2 + \|f(x_j)-b\|^2 - \|f(x_i)-f(x_j)\|^2\right] \\
&\geq \sum_{i,j} \lambda_i\lambda_j \left[c^2\|x_i-x\|^2 + c^2\|x_j-x\|^2 - \|x_i-x_j\|^2\right] \\
&= \sum_{i,j} \lambda_i\lambda_j \left[2\langle c(x_i-x), c(x_j-x)\rangle + (c^2-1)\|x_i-x_j\|^2\right] \\
&= 2\|c\sum_i \lambda_i (x_i-x)\|^2 + (c^2-1)\sum_{i,j} \lambda_i\lambda_j \|x_i-x_j\|^2.
\end{aligned}
$$

Donc , $c \leq 1$.

***Etape 2:** Nous considérons la classe

$$\mathscr{L} = \{h : \Omega \subset \mathrm{dom}(h), h = f \text{ dans } \Omega \text{ et } \mathrm{Lip}(h, \mathrm{dom}(h)) = \mathrm{Lip}(f, \Omega)\}.$$

Pour $h_1, h_2 \in \mathscr{L}$, nous définissons la relation d'ordre:

$$(h_1 \leq h_2) \Leftrightarrow (\mathrm{dom}(h_1) \subset \mathrm{dom}(h_2) \text{ et } h_2 = h_1 \text{ dans } \mathrm{dom}(h_1))$$

En utilisant le lemme de Zorn, $\mathscr{L}$ possède un élément maximal $g : \Omega_1 \to H_2$. La preuve de ce théorème est complète si $\Omega_1 = H_1$. Supposons, par l'absurde que $\Omega_1 \neq H_1$. Alors il existe $\xi \in H_1 \backslash \Omega_1$. En utilisant l'étape 1, il existe $\eta \in H_2$ telle que

$$\|\eta - g(a)\| \leq \|\xi - a\|, \ \forall a \in \Omega_1.$$

Par conséquent, si nous définissons $g_1 = g$ dans Ω_1 et $g_1(\xi) = \eta$, alors $g_1 \in \mathscr{L}$, $g \leq g_1$ et $g \neq g_1$. Ainsi g ne serait pas maximale dans $\mathscr{L}$. Nous obtenons une contradiction. $\square$

L'idée principale dans la preuve ci-dessus est que : dans l'étape 1, nous utilisons des caractéristiques géométriques des espaces de Hilbert pour étendre f en un point supplémentaire, et dans l'étape 2 nous utilisons une certaine forme de l'axiome de choix pour étendre f à tout l'espace. Cette idée est la même que la preuve du classique théorème de classique Hahn-Banach, et les caractéristiques des espaces de Hilbert comme l'existence d'un produit scalaire sont très importants dans cette démonstration. La résultat correspondant pour les espaces de Banach est pas vrai en général, pas même pour les espaces de Banach de dimension finie. Nous pouvons construire des contre-exemples où le domaine est un sous-ensemble de $\mathbb{R}^n$ avec la norme sup et l'application est à valeurs dans $\mathbb{R}^m$ avec la norme Euclidienne. Un contre-exemple simple est la suivante:

$$
\begin{aligned}
&X = \mathbb{R}^2 \text{ avec } d_X(x,y) = \sup\{|x_1 - y_1|, |x_2 - y_2|\},\\
&\text{ où } x = (x_1,x_2), y = (y_1,y_2) \in X,\\
&Y = \mathbb{R}^2 \text{ avec } d_Y(x,y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2},\\
&\text{ où } x = (x_1,x_2), y = (y_1,y_2) \in Y,\\
&\Omega = \{(1,-1),(-1,1),(1,1)\} \subset X,\ f : \Omega \to Y,\\
&f(1,-1) = (1,0),\ f(-1,1) = (-1,0),\ f(1,1) = (0,\sqrt{3}).
\end{aligned}
$$

Nous avons $d_X(x,y) = 2 = d_Y[f(x), f(y)], \forall x,y \in \Omega$ et $d_X(x,0) = 1, \forall x \in \Omega$, mais il n'existe aucun $\xi \in \mathbb{R}^2$ tel que $d_Y(\xi, f(x)) \leq 1, \forall x \in \Omega$.

Plus généralement, ce théorème n'est pas vrai pour $\mathbb{R}^m$ équipé de la norme ℓ_p ($p \neq 2$) (voir Schwartz 1969 [50, p. 20]).

1.1.2 Extensions extrémales de McShane-Whitney

Si $Y = \mathbb{R}$, alors pour tout espace métrique X arbitraire et tout sous-ensemble Ω de X, chaque fonction lipschitzienne $f : \Omega \to \mathbb{R}$ a une extension g lipschitzienne satisfaisant

$$
g = f \text{ dans } \Omega, \text{ et } \mathrm{Lip}(g,X) = \mathrm{Lip}(f,\Omega). \tag{1.2}
$$

En fait, McShane [39] et Whitney [56] en 1934 produisent deux solutions explicites de (1.2)

$$
\begin{aligned}
,m^+(f,\Omega)(\xi) &= \inf\{f(x) + Lip(f,\Omega)d_X(x,\xi) : x \in \Omega\} \text{ pour } \xi \in X, &(1.3)\\
m^-(f,\Omega)(\xi) &= \sup\{f(x) - Lip(f,\Omega)d_Y(x,\xi) : x \in \Omega\} \text{ pour } \xi \in X. &(1.4)
\end{aligned}
$$

De plus, $m^{\pm}$ sont extrémales: la première est maximale et la seconde est minimale, c'est-à-dire

$$
m^-(f,\Omega)(x) \leq g(x) \leq m^+(f,\Omega)(x), \forall x \in X,
$$

pour toute g autre solution de (1.2).

Remarque 1.1.3. Il est clair que les solutions de (1.2) sont uniques si et seulement si $m^+(f,\Omega) = m^-(f,\Omega)$ sur $\mathbb{R}^n$. Cela arrive rarement.

Exemple 1.1.4. Soit $X = \mathbb{R}$, $\Omega = \{-1,0,1\}$, $f(-1) = f(0) = 0, f(1) = 1$. Alors $\mathrm{Lip}(f,\Omega) = 1$. Les fonctions $m^+(f,\Omega)$ et $m^-(f,\Omega)$ sont tracées ci-dessous (voir Figure 1.1 et Figure 1.2).

1.1.3 Rétraction absolument 1-Lipschitz

Nous pouvons demander des conditions sur l'espace métrique Z : pour chaque espace métrique X, la paire (X,Z) a la propriété d'extension isométrique. Pour répondre à cette question, nous introduisons le concept de *rétraction absolument 1-Lipschitz*

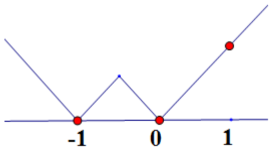

Figure 1.1: Illustration $m^+(f,\Omega)$

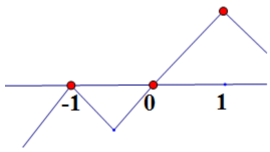

Figure 1.2: Illustration $m^-(f,\Omega)$

Définition 1.1.5. Soit (L,d_L) un espace métrique et X une sous-ensemble de L.

Une fonction Lipschitz $r : L \to X$ est appelée une *rétraction 1-Lipschitz* si elle est l'application identique dans X et $\mathrm{Lip}(r,L) = \mathrm{Lip}(r,X)$.

Quand un tel rétraction 1-Lipschitz existe nous disons que X est une rétracter $1-$ Lipschitz de L.

Un espace métrique X est appelé un *rétraction absolument 1-Lipschitz* si elle est un $1-$ rétracter Lipschitz de l'espace métrique Y chaque fois que Y contient X.

Exemple 1.1.6. L'ensemble $\mathbb{R}$, les arbres métriques et $l_\infty = \{(x_n)_n : x_n \in \mathbb{R}\}$ (par rapport à la norme $\|x\|_\infty = \max_n |x_n|$ où $x = (x_n)_n$) sont rétraction absolument 1-Lipschitz (see [10], [29]).

Théorème 1.1.7. *[10, Proposition 1.2, Proposition 1.4] Soit (Z,d_Z) un espace métrique. Les conditions suivantes sont équivalentes*

(i) Pour chaque espace métrique X, la paire (X,Z) a la propriété d'extension isométrique.

(ii) Z est rétraction absolument 1-Lipschitz.

(iii) (Z,d_Z) est métriquement convexe [1] et a la propriété d'intersection binaire [2].

1.2 Extensions lipschitziennes absolument minimales

1.2.1 Introduction à la notion d'extensions lipschitziennes absolument minimales

Nous considérons une paire de espace métrique (X,d_X) et (Y,d_Y) qui a la propriété d'extension isométrique.

[1] L'espace métrique (Z,d_Z) est appelé *métrique convexe* si pour tout $x,y \in Z$ et $\lambda \in [0,1]$, il existe un $z \in Z$ telle que $d_Z(x,z) = \lambda d_Z(x,y)$ et $d_Z(y,z) = (1-\lambda)d_Z(x,y)$.

[2] L'espace métriquement (Z,d_Z) est dit d'avoir la propriété d'*intersection binaire* si chaque collection de boules fermées ayant une intersection deux à deux non vide, a un point commun.

Définition 1.2.1. Soit Ω un sous-ensemble de X et $f : \Omega \to Y$ une fonction lipschitzienne. Si g étend f et $\mathrm{Lip}(g,X) = \mathrm{Lip}(f,\Omega)$ alors nous disons que g est une extension lipschitzienne minimale (MLE) de f.

Dans le cas $X \subset \mathbb{R}^n$ et $Y = \mathbb{R}$, tous deux équipés de la norme euclidienne, les formules de McShane- Whitney (1.3) et (1.4) nous donnent deux MLEs extrémales m^+ et m^- de f. Ainsi, à moins que $m^+ \equiv m^-$, nous avons pas unicité d'une MLE de f.

Ces MLE extrémales m^+ et m^- ne satisfont pas de principe de comparaison et ne sont pas stables. Plus précisément, la relation $f_1 \leq f_2$, en général, ne signifie pas

$$m^+(f_1,\Omega) \leq m^+(f_2,\Omega), \ \text{dans } X,$$

ou

$$m^-(f_1,\Omega) \leq m^-(f_2,\Omega), \ \text{dans } X,$$

et $m^+(m^+(f,\Omega),\partial V)$ peut être différente de $m^+(f,\Omega)$ sur un ensemble ouvert $V \subset\subset X\backslash\Omega$. De plus, pour une MLE g de f, $\mathrm{Lip}(g,V)$ est probablement strictement supérieur à $\mathrm{Lip}(g,\partial V)$ pour certains $V \subset\subset X\backslash\Omega$. Donc, si nous remplaçons g par la nouvelle fonction

$$g_1(x) = g(x) \text{ pour } x \in X\backslash V, \ \text{ et } g_1(x) = m^+(g,\partial V)(x) \text{ pour } x \in V,$$

alors g_1 est aussi une MLE de f et

$$\mathrm{Lip}(g_1,V) = \mathrm{Lip}(g,\partial V) < \mathrm{Lip}(g,V).$$

Ceci veut dire cela nous pouvons réduire la constante de Lipschitz locale en répétant l'application de l'opérateur m^+ ou m^-.

De la discussion ci-dessus, la question suivante se pose naturellement: Est-il possible de trouver une MLE u avec des propriétés supplémentaires de sorte que u satisfasse un principe de comparaison et soit stable ? Cette extension peut-elle être unique ?

Évidemment, si ces fonctions existent, elles doivent satisfaire

$$\mathrm{Lip}(u,V) = \mathrm{Lip}(u,\partial V), \ \text{ pour tout ouvert } V \subset\subset X\backslash\Omega, \tag{1.5}$$

parce que sinon elles ne seraient pas stables.

Cette question a d'abord été discutée paru dans une série de papiers de Aronsson dans les années 1960 [3, 4, 5]. Aronsson a proposé le concept d'*extensions lipschitziennes absolument minimales* (AMLE):

Definition 1.2.2. Une fonction $u : X \to \mathbb{R}$ est appelé AMLE de f si u est une MLE de f et u satisfait (1.5).

Cela signifie que u a une constante de Lipschitz minimale dans chaque ensemble ouvert $V \subset\subset X\backslash\Omega$.

Les opérateurs McShane-Whitney fournissent une idée naturelle pour construire AMLE en réduisant la constante de Lipschitz minimale dans des domaines où elle n'est pas optimale. Aronsson (1967) [5] a utilisé cette idée pour prouver l'existence d'AMLE.

Après les travaux de Aronsson [3, 4, 5], il y a eu beaucoup de recherches sur les AMLE et les problèmes liés, voir [13, 15, 17, 28, 30, 46].

L'approche la plus populaire pour traiter cette question est d'interpréter le AMLE comme une limite formelle de u_p quand $p \to \infty$, où u_p minimise la fonctionnelle

$$I_p[u] := \int_U \|\nabla u\|^p dx, \tag{1.6}$$

où $U \subset \mathbb{R}^n$ est ouvert, $u \in W^{1,p}(U;\mathbb{R})$ avec la condition $u = f$ dans ∂U, et $\nabla u = (u_{x_1}, ..., u_{x_n})$ est le gradient.

Cette approche conduit aussi à réécrire le problème initial: (1.5) est remplacé par

$$u = v \text{ sur } \partial V \text{ entraîner } \|\nabla u\|_{L^\infty(V)} \leq \|\nabla v\|_{L^\infty(V)}, \tag{1.7}$$

pour chaque $V \subset U$, et pour chaque $v \in C(\overline{V})$.

Ce problème est un problème de calcul des variations en norme sup. Dans le cas $p < +\infty$, il était bien connu à l'époque que ce problème conduisait à l'équation d'Euler-Lagrange $\Delta_p u$=0 pour un minimisateur u mais on ne connaissait pas l'equation d'Euler-Lagrange pour le problème (1.7).

Aronsson en 1967 [5] a découvert l'équation d'Euler-Lagrange pour le problème (1.7), qui est

$$\Delta_\infty u = 0, \tag{1.8}$$

où, pour $u \in C^2$,

$$\Delta_\infty u := \sum_{j,k=1}^{n} u_{x_j} u_{x_k} u_{x_j x_k}.$$

L'équation ci-dessus est aujourd'hui connu sous le nom *d'infini laplacien.* Aronsson découvert cette équation par approximation. Expliquons l'idée au moins formellement. Il a considéré l'opérateur non-linéaire p-laplacien (p fini)

$$\Delta_p u = div(\|\nabla u\|^{p-2}\nabla u) = (p-2)\|\nabla u\|^{p-4}\Delta_\infty u + \|\nabla u\|^{p-2}\Delta u.$$

L'équation $\Delta_p u = 0$ est l'équation d'Euler-Lagrange pour le problème de minimisation (1.6).

Divisant l'équation $\Delta_p u = 0$ par $(p-2)\|\nabla u\|^{p-4}$ conduit à

$$\Delta_\infty u + \frac{1}{p-2}\|\nabla u\|^2 \Delta u = 0.$$

En faisant $p \to +\infty$. On aboutir à l'équation

$$\Delta_\infty u = 0.$$

Aronsson a prouvé en 1967 [5] que si U est la fermeture d'un domaine borné dans $\mathbb{R}^n$ et f est une fonction donnée sur ∂U alors une fonction $u \in C^2$ est une AMLE de f si et seulement si elle est solution de l' équation du laplacien infini (1.8) avec la condition au bord $u = f$ sur ∂U.

Toutefois, les AMLE ne sont pas C^2 en général. Les solutions classiques de l'équation eikonale $|Du| = constante$ produisent des exemples d'AMLE non C^2. Une AMLE non-régulière explicite a été donnée par Aronsson en 1984 [6] $u(x,y) = x^{4/3} - y^{4/3}$. Les derivées première de u sont höldériennes avec un exposant $1/3$ et les dérivées secondes n'existent pas sur les lignes $x = 0$ et $y = 0$. À l'époque, l'existence et l'unicité des AMLE était inconnue en général, et la notion de solutions pour l'équation du laplacien infini (1.8) n'était pas claire.

Une percée importante dans ce sens a été faite par Jensen en 1993 en utilisant les solutions de viscosité introduites par Crandall et Lions dans un papier très célèbre en 1983 [16]:

Définition 1.2.3. (i) Une fonction semi-continue supérieurement $u : \overline{U} \to \mathbb{R}$ est une *sous-solution* de (1.8) si $\Delta_\infty \phi(x) \geq 0$, pour chaque $(x, \phi) \in U \times C^2(U)$ telles que $(u - \phi)(x) \geq (u - \phi)(y)$ pour tous $y \in U$.

(ii) Une fonction semi-continue inférieurement $u : \overline{U} \to \mathbb{R}$ est une *sur-solution* de (1.8) si $\Delta_\infty \phi(x) \leq 0$, pour chaque $(x, \phi) \in U \times C^2(\Omega)$ telles que $(u - \phi)(x) \leq (u - \phi)(y)$ pour tous $y \in U$.

(iii) Une fonction continue $u : \overline{U} \to \mathbb{R}$ est une *solution de viscosité* de (1.8) si elle est à la fois une sous-solution et sur-solution

Le concept de solution de viscosité est une généralisation de la notion classique pour des fonctions non-lisse. En utilisant cette définition, en 1993 Jensen a prouvé l'existence et l'unicité de la solution de viscosité de l'infini laplacien (1.8) avec la condition au bord $u = f$ dans ∂U [28].

Jensen a montré aussi que cette solution est également solution du problème de Aronsson : pour chaque sous-ensemble ouvert borné V de U et pour chaque $v \in C(\overline{V})$, si u est la solution de viscosité de l'infini laplacien alors

$$u = v \ \text{ sur } \ \partial V \ \text{ implique } \ \|\nabla u\|_{L^\infty(V)} \leq \|\nabla v\|_{L^\infty(V)}. \tag{1.9}$$

Le travail de Jensen a suscité un intérêt considérable dans les longs développements de la théorie des AMLE en particulier en ce qui concerne l'existence et l'unicité.

1.2.2 Comparaison avec les cônes

Nous introduisons la propriété de "Comparaison avec les cônes". On note que toute solution d'une équation eikonale $|Dv| = constante$ est une solution classique de $-\Delta_\infty v = 0$ partout où elle est lisse. Ainsi, le cône

$$C(x) = a + b\|x - x_0\|$$

est une solution classique pour $x \neq x_0$.

Si b est positif alors le cône C est une sous-solution de viscosité globale sur $\mathbb{R}^n$, mais elle n'est pas une solution de viscosité globale. La comparaison avec les cônes est l'outil de base de la théorie.

Définition 1.2.4. (i) Une fonction $u \in C(U)$ est dite *comparable avec des cônes par au-dessus* dans U si pour tout ouvert borné V sous-ensemble de U et chaque $x_0 \in \mathbb{R}^n, a, b \in \mathbb{R}$, tels que

$$u(x) \leq C(x) = a + b\|x - x_0\|, \ x \in \partial(V \backslash \{x_0\}),$$

alors nous avons $u \leq C$ dans V.

(ii) Une fonction $u \in C(U)$ est dite *comparable avec des cônes par au-dessous* dans U si pour tout ouvert borné V sous-ensemble de U et chaque $x_0 \in \mathbb{R}^n, a, b \in \mathbb{R}$, tels que

$$u(x) \geq C(x) = a + b\|x - x_0\|,\; x \in \partial(V \backslash \{x_0\}),$$

alors nous avons $u \geq C$ dans V.

(iii) Une fonction $u \in C(U)$ est dite *comparable avec des cônes* dans U si elle est à la fois comparable avec des cônes par au-dessus et comparable avec des cônes par au-dessous.

En 2001, Crandall, Evans, et Gariepy [15] ont prouvé que si u est une fonction continue dans un ouvert borné $U \subset \mathbb{R}^n$, alors

$$\begin{aligned} & u \text{ est un } \textit{AMLE} \text{ qui satisfait (2.7)} \\ \Leftrightarrow\ & u \text{ est solution de viscosité de l'infini laplacien (2.8)} \\ \Leftrightarrow\ & u \text{ est comparable avec les cônes.} \end{aligned}$$

En 2010, Armstrong et Smart [2] ont présenté une preuve élégante et élémentaire de l'unicité des fonctions comparables avec les cônes. Cette preuve ne fait aucune utilisation des équations aux dérivées partielles et ils n'ont pas besoin d'utiliser les solutions de viscosité développées pour les équations elliptiques du deuxième ordre. D'après les résultats de Crandall, Evans, et Gariepy [15] sur l'équivalence des concepts d'AMLE, des solutions de viscosité de l'infini laplacien et des fonctions comparables avec les cônes, le résultat d'Armstrong et Smart produit une preuve nouvelle et facile pour l'unicité de l'AMLE et l'unicité de la solution de viscosité de l'équation de l'infini laplacien.

1.2.3 Généralisation dans les espaces métriques et jeux

La définition de la comparaison avec des cônes (Definition 1.2.4) s'étend facilement à d'autres espaces métriques. Champion et De Pascale [14] adapté cette définition à des espaces de longueur [3], où les cônes sont remplacés par des fonctions de la forme $\phi(x) = bd(x,z) + c$ pour $b > 0$.

Juutinen [30] en 2002 a utilisé la méthode de Perron pour établir l'existence d'AMLE pour des espace séparables.

Peres, Schramm, Sheffield et Wilson [46] en 2009 ont utilisé des techniques de la théorie des jeux pour prouver l'existence et l'unicité d'AMLE pour les espaces généraux de longueur . Cette preuve établit un joli lien entre le laplacien infini et le jeu "tug of war".

L'équation fonctionnelle intéressante

$$u^{\varepsilon}(x) = \frac{1}{2}\left(\max_{|y| \leq \varepsilon} u^{\varepsilon}(x + \varepsilon y) + \min_{|z| \leq \varepsilon} u^{\varepsilon}(x + \varepsilon z)\right) \tag{1.10}$$

qui apparaît dans le travail de Peres, Schramm, Sheffield et Wilson [46] joue un rôle important dans le lien avec les AMLE (voir Le Gruyer [37](2007) et Oberman [45](2004)).

[3]Un espace métrique (X, d_X) est un *espaces de longueur* si pour tout $x, y \in X$, la distance $d_X(x,y)$ est la borne inférieure des longueurs des courbes dans X qui relie x et y.

1.2.4 Régularité

Nous précisons les propriétés derégularité des solutions de viscosité de

$$\Delta_\infty u = 0, \text{ dans } U, \tag{1.11}$$

où $U \subset \mathbb{R}^n$ est un ensemble ouvert.

Une solution de viscosité u est appelé *fonction ∞-harmonique* .

Considérons l'exemple du Aronsson: $u(x,y) = x^{4/3} - y^{4/3}$. Les dériveés premières de u sont höldériennes avec un exposant $1/3$; les dériveés secondes n'existent pas sur les lignes $x = 0$ et $y = 0$. Par conséquent, dans l'exemple de Aronsson, $u \in C^1$ mais $u \notin C^2$. La question est la suivante : les fonctions ∞-harmoniques sont elles nécessairement continûment différentiable ?

Supposons que u soit une solution de viscosité de l'équation (1.11) et $B(x,r) \subset U$. Nous définissons

$$L_r^+(x) := \frac{\max_{y \in \partial B(x,r)} u(y) - u(x)}{r},$$

et

$$L_r^-(x) := \frac{u(x) - \min_{y \in \partial B(x,r)} u(y)}{r}.$$

Crandall, Evans et Gariepy [15] (2001) ont prouvé que les limites

$$L(x) := \lim_{r \to 0} L_r^+(x) = \lim_{r \to 0} L_r^-(x)$$

existent et sont égales en chaque point $x \in U$.

Evans et Smart [18] (2011) établi que:

$$\lim_{r \to 0} \frac{u(ry + x) - u(x)}{r} = \langle a, y \rangle$$

existe localement uniformément pour certains $a \in \mathbb{R}^n$ qui satisfont $\|a\| = L(x)$.

En conséquence, les fonctions ∞-harmoniques sont partout dérivables et $L(x) = \|Du(x)\|$.

Savin en 2006 [48] a montré que les ∞-harmoniques pour $n = 2$ sont en fait continûment différentiables. Savin a utilisé très fortement la topologie de $\mathbb{R}^2$, et il est difficile de généralises les arguments de Savin pour le cas $n > 2$. La question de savoir si les fonctions ∞-harmoniques sont nécessairement C^1 en général reste le problème plus important ouvert dans ce cadre.

1.2.5 L'extension tight

Présentons une version discrète des AMLE pour les fonctions à valeurs vectorielles. Soit $G = (V, E, \Omega)$ un graphe fini connexe de sommets $V \subset \mathbb{R}^n$, d'arêtes E et $\Omega \subset V$ un ensemble non-vide. Soit $f : \Omega \to \mathbb{R}^m$.

Nous noterons $E(f)$ l'ensemble de toutes les extensions de f sur G. La *constante de lipschitz locale de* v au sommet $x \in V \backslash \Omega$ est définie par

$$Lv(x) := \sup_{y \in S(x)} \frac{\|v(y) - v(x)\|}{\|y - x\|},$$

où

$$S(x) := \{y \in V : (x, y) \in E\} \tag{1.12}$$

est un voisinage de x dans G.

Définition 1.2.5. [4] Si $u, v \in E(f)$ satisfont

$$\max\{Lu(x) : Lu(x) > Lv(x), x \in V \backslash \Omega\} > \max\{Lv(x) : Lv(x) > Lu(x), x \in V \backslash \Omega\},$$

alors nous disons que v est *plus tight* que u dans G. Nous disons que u est une *extension tight* de f dans G si il n'y a pas v plus tight que u .

Théorème 1.2.6. *[51, Theorem 1.2] Il existe une unique extension u de f qui est tight sur G. De plus, u est plus tight que tout autre extension v de f.*

L'extension tight est la limite des $p-$ extensions harmoniques discrètes.

Théorème 1.2.7. *[51, Theorem 1.3] En plus de les hypothèses du théorème 2.2.6, supposons que pour chaque $p > 0$, la fonction $u_p : V \to \mathbb{R}^m$ minimise*

$$I_p[u] = \sum_{x \in V \backslash \Omega} (Lu(x))^p,$$

où u_p est une extension de f. Alors la suite (u_p) convege vers l'extension tight de f.

1.3 Le problème de l'extension de Whitney

1.3.1 Théorème de Withney

Le problème a pour origine de Hassler Whitney: Soit Ω un sous-ensemble de $\mathbb{R}^n$, et soit f un fonction continue à valeurs réelles dans le domaine Ω. Sous quelles conditions la fonction f s'étend-elle en une fonction C^m sur $\mathbb{R}$? Si la fonction f est dans un certain sens différentiable dans Ω, l'extension F de f peut-être différentiable du même ordre sur $\mathbb{R}^n$?

Dans les papiers [56, 57, 58] en 1934 Whitney a développé d'importantes techniques d'analyse et de géométrie pour résoudre le problème. La difficulté est que les sous-ensembles d'espaces euclidiens manquent en géneral de structure différentiable. Nous avons donc besoin de comprendre ce que signifie exactement la dérivée d'une fonction sur un tel ensemble. Le point de départ est un examen du théorème de developement de Taylor. Compte tenu d'une fonction de valeurs de réelles $f \in C^m(\mathbb{R}^n)$,

[4] Par convention, si $C = \emptyset$ alors $\max_C = 0$.

théorème de Taylor affirme que pour chaque $a,x,y \in \mathbb{R}^n$, il existe une fonction $R_\alpha(x,y) \to 0$ uniformément quand $x,y \to a$ telle que

$$f(x) = \sum_{|\alpha| \leq m} \frac{D^\alpha f(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=m} \frac{R_\alpha(x,y)}{\alpha!}(x-y)^\alpha, \tag{1.13}$$

où α est multi-indices entier.

Soit $f_\alpha = D^\alpha f$ pour chaque multi-indice α. En différenciant (1.13) par rapport à x, on obtient

$$f_\alpha(x) = \sum_{|\beta| \leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y), \tag{1.14}$$

où $R_\alpha = o(|x-y|^{m-\alpha}) \to 0$ uniformément quand $x,y \to a$.

La définition des dérivées dans d'une fonction un ensemble général donnée par (1.14) se pose naturellement à partir d'un examen de la formule de Taylor (1.13). L'extension du théorème de Whitney est une réciproque partielle au théorème de Taylor. Dans le papier original de Whitney en 1934 [56] on a :

Théorème 1.3.1. *[56, Whitney extension theorem] Supposons que $(f_\alpha)_\alpha$ sont une collection de fonctions sur un sous-ensemble fermé Ω de $\mathbb{R}^n$ pour tout multi-indice α avec $|\alpha| \leq m$ satisfaisant à la condition de compatibilité (1.14). Alors il existe une fonction $F(x)$ de classe C^m telle que:*
(i) $F = f_0$ dans Ω.
(ii) $D^\alpha F = f_\alpha$ dans Ω.
(iii) F est un réelle analytique en chaque point de $\mathbb{R}^n \backslash \Omega$.

Depuis le travail de Whitney, des progrès fondamentaux ont été faits par Georges Glaeser, Yuri Brudnyi, Pavel Shvartsman, Edward Bierstone, Pierre Milman, Erwan Le Gruyer... Dans une série d'articles récents, Charles Fefferman a résolu le problème initial de Whitney en toute généralité. Ses méthodes ont conduit à des développements très importants dans le domaine (voir [20, 21, 22, 23]).

1.3.2 MLE pour les champs d'ordre 1

Soit Ω un sous-ensemble de l'espace euclidien $\mathbb{R}^n$.

Nous définissons

$$\mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) \triangleq \{P : a \in \mathbb{R}^n \mapsto P(a) = p + \langle v,a \rangle\,,\ p \in \mathbb{R}, v \in \mathbb{R}^n\}.$$

Prenons un champ F d'order 1 (dit aussi 1-champ) sur Ω défini par

$$\begin{aligned} F : \Omega &\to \mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) \\ x &\mapsto F(x)(a) := f_x + \langle D_x f ; a - x \rangle, \end{aligned} \tag{1.15}$$

où $a \in \mathbb{R}^n$ est la variable du polynôme $F(x)$ de degré 1 et $f : x \in \Omega \mapsto f_x \in \mathbb{R}$, $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$ sont des applications associées à F.

En conséquence du Théorème 1.1.1, nous avons $\mathrm{Lip}^*(f) = \mathrm{Lip}(f,\Omega)$ où

$$\mathrm{Lip}^*(f) := \inf\{\mathrm{Lip}(g,\mathbb{R}^n) : g \text{ extension lipschitzienne totale de } f\}.$$

Soit F un 1-champ. Nous définissons que le 1-champ G est appelé *l'extension* de F si $dom(G) = \mathbb{R}^n$ et $G(x) = F(x)$ sur Ω. La question naturelle qui se pose est de savoir si

$$\mathrm{Lip}^*(F) = \inf\{\mathrm{Lip}(D_x g,\mathbb{R}^n) : G \text{ est un extension de } F\}.$$

Erwan Le Gruyer [36] (2009) a introduit la constante lipschitzienne du 1-champ F définie par:

$$\Gamma^1(F;\Omega) \triangleq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \Gamma^1(F;x,y), \tag{1.16}$$

où

$$\Gamma^1(F;x,y) \triangleq 2 \sup_{a\in\mathbb{R}^n} \frac{|F(x)(a) - F(y)(a)|}{\|x-a\|^2 + \|y-a\|^2}. \tag{1.17}$$

Le Gruyer prouvé que la constante Γ^1 du 1-champ joue un rôle dans le problème de l'extension de façon similaire à la constante Lip de fonction continue:

Théorème 1.3.2. *[36] Soit F un 1-champ. La fonctionnelle $\Gamma^1 : F \to \Gamma^1(F,\mathrm{dom}(F)) \in \mathbb{R}^+ \cup \{+\infty\}$ est l'unique satisfaisant*

$(P0)$ Γ^1 *est croissante, c'est-à-dire que si U étend F, alors*

$$\Gamma^1(U,\mathrm{dom}(U)) \geq \Gamma^1(F,\mathrm{dom}(F)).$$

$(P1)$ *Si $dom(U) = \mathbb{R}^n$, et $\Gamma^1(U,\mathrm{dom}(U)) < +\infty$, alors la fonction u définie par $u(x) := U(x)(x)$ est différentiable et sa dérivée ∇u est lipschitzienne.*

$(P2)$ *Si u un fonction différentiable telle que $dom(u) = \mathbb{R}^n$ et ∇u lipschitzienne, alors*

$$\Gamma^1(U) = \mathrm{Lip}(\nabla u),$$

où U est le 1-champ associé à u, c'est-à-dire, $U(x)(a) := u(x) + \langle \nabla u(x); a-x\rangle, \forall x,a \in \mathbb{R}^n$.

$(P3)$ *Pour chaque F telle que $\Gamma^1(F,\mathrm{dom}(F)) \leq +\infty$, F s'étend en un 1-champ U satisfaisant $dom(U) = \mathbb{R}^n$ et*

$$\Gamma^1(U,\mathbb{R}^n) = \Gamma^1(F,\mathrm{dom}(F)).$$

Comme conséquence immédiate de ce théorème, pour tout 1-champ F, nous avons $\mathrm{Lip}^*(F) = \Gamma^1(F,\mathrm{dom}(F))$.

Ce théorème est vrai non seulement dans $\mathbb{R}^n$, mais aussi dans tout espace de Hilbert, séparable ou non. Par conséquent, ce théorème est une extension du théorème de Whitney classique. La calcul de Γ^m qui généralise la fonctionnelle Γ^1 introduite par Le Gruyer au ces des m-champs est une question très difficile.

1.4 Résultats de la thèse

Dans cette section, nous présentons les principaux résultats de cette thèse.

1.4.1 Les relations entre Γ^1 et Lip

Soit $F : \Omega \to \mathscr{P}^1(\mathbb{R}^n, \mathbb{R})$ un 1-champ. Il est intéressant de se demander quelle est la relation entre $\Gamma^1(F;\Omega)$ et $\mathrm{Lip}(Df;\Omega)$.

De [36], nous savons que

$$\mathrm{Lip}(Df;\Omega) \leq \Gamma^1(F;\Omega)$$

et si $\Omega = \mathbb{R}^n$ alors

$$\mathrm{Lip}(Df;\mathbb{R}^n) = \Gamma^1(F;\mathbb{R}^n).$$

Mais en général, la fonction $\Gamma^1(F;\Omega)$ peut être strictement plus grande que $\mathrm{Lip}(Df;\Omega)$.

Exemple 1.4.1. Nous donnons un exemple très simple pour lequel est $\Gamma^1(F;\Omega) > \mathrm{Lip}(Df;\Omega)$. Soit A et B deux ensembles ouverts dans $\mathbb{R}^n$ telles que $A \cap B = \emptyset$. Soit $\Omega = A \cup B$ et $F \in \mathscr{F}^1(\Omega)$ telles que $f_x = 0$ si $x \in A$, $f_x = 1$ si $x \in B$, et $D_x f = 0$,$\forall x \in \Omega$. Nous avons

$$\mathrm{Lip}(Df;\Omega) = 0$$

et

$$\Gamma^1(F;\Omega) = \sup_{x\in A}\sup_{y\in B} \frac{4}{\|x-y\|^2} > 0.$$

Nous donnons maintenant deux résultats où nous avons $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$.

Proposition 1.4.2. *Soit $F \in \mathscr{F}^1(\Omega)$. Supposons qu'il existe $a, b \in \Omega$, $a \neq b$ tels que $\Gamma^1(F;a,b) = \Gamma^1(F;\Omega)$. Nous avons $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

Proposition 1.4.3. *Soit $F \in \mathscr{F}^1(\Omega)$. Supposons qu'il existe $\Omega' \subset\subset \Omega$ telle que $\Gamma^1(F;\Omega') = \Gamma^1(F;\Omega)$. Nous avons $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

Nous voyons que, dans certains cas, nous avons $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$. De plus, dans l'exemple 1.4.1, Ω est ouvert mais pas convexe. Ainsi, nous pouvons espérer que si Ω est convexe alors nous avons $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$. Malheureusement, cela est encore faux en général pour un ensemble Ω convexe et ouvert. Nous donnons un contre-exemple où $\mathrm{Lip}(\nabla f;\Omega) < \Gamma^1(F;\Omega)$ pour Ω convexe ouverte et $F \in \mathscr{F}^1(\Omega)$ dans la Proposition 3.3.6 de la thèse.

Notre résultat principal dans cette section est

Theorem 1.4.4. *Soit Ω un sous-ensemble ouvert de $\mathbb{R}^n$. Nous avons*

$$\Gamma^1(F;\Omega) = \max\{\Gamma^1(F;\partial\Omega), \mathrm{Lip}(Df;\Omega)\}, \tag{1.18}$$

où $\partial\Omega$ est la frontière de Ω.

De plus, si Ω est un sous-ensemble convexe de $\mathbb{R}^n$, alors

$$\Gamma^1(F;\Omega) \leq 2\mathrm{Lip}(Df;\Omega). \tag{1.19}$$

Pour comprendre le lien entre $\Gamma^1(F;\Omega)$ et $\mathrm{Lip}(Df;\Omega)$, il est important de connaître l'ensemble d'unicité des extensions minimales de F lorsque Ω est constitué de deux points (cette étude a été réalisée dans [27]).

1.4.2 MLE de 1-champs données explicitement par des formules en sup-inf

Soit $F : \Omega \to \mathscr{P}^1(\mathbb{R}^n,\mathbb{R})$ un 1-champ. Par le théorème 2.3.2, il existe un 1-champ G sur $\mathbb{R}^n$ *extension lipschitzienne minimale* (MLE) de F, c'est à dire, $G = F$ dans Ω et $\Gamma^1(G,\mathbb{R}^n) = \Gamma^1(F,\Omega)$.

Nous présentons deux MLEs U^+ et U^- de F de la forme

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) := u^+(x) + \langle D_x u^+; y - x\rangle,\ y \in \mathbb{R}^n, \tag{1.20}$$

où

$$u^+(x) := \max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F,x,a,v),\ D_x u^+ := \arg\max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F,x,a,v), \tag{1.21}$$

et

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) := u^-(x) + \langle D_x u^-; y - x\rangle,\ y \in \mathbb{R}^n, \tag{1.22}$$

où

$$u^-(x) := \min_{v\in\Lambda_x} \sup_{a\in\Omega} \Psi^-(F,x,a,v),\ D_x u^- := \arg\min_{v\in\Lambda_x} \sup_{a\in\Omega} \Psi^-(F,x,a,v), \tag{1.23}$$

où Λ_x est un ensemble non vide et convexe de $\mathbb{R}^n$ défini dans la Définition 3.4.7 et $\Psi^\pm$ sont des fonctions définies dans la Définition 3.4.8.

Les MLE $u^\pm$ sont *extrémales* : la première est plus grande possible et la seconde la plus petite possible c'est-à-dire

$$u^-(x) \le g_x \le u^+(x),\ \forall x \in \mathbb{R}^n,$$

pour toute MLE G de F.

Les formules de $u^\pm$ et leurs gradients sont explicites. De plus, elles ne dépendent que de F. Les formules de $u^\pm$ dans le carde des $1-$ champ sont similaires aux formules (1.3) et (1.4) de $m^\pm$ provenant du travail de McShane [39] et Whitney [56].

Soit κ est une constante. Lors de la conférence Whitney en 2011, M. Hirn a remarqué que: $\kappa \ge \Gamma^1(F;\Omega)$ si et seulement si

$$f_y \le f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{\kappa}{4}(x-y)^2 - \frac{1}{4\kappa}(D_x f - D_y f)^2, \forall x, y \in \Omega. \tag{1.24}$$

De plus, si $\kappa = \Gamma^1(F;\Omega)$, alors le travail de Wells (voir [54, Theorem 2]) nous donne que w^+ (voir la définition de la fonction w^+ dans l'annexe 3.8) est une MLE. De plus, dans ce cas w^+ est une extension minimal extrémale de supérieure de F.

La construction de w^+ de Wells est explicite quand Ω est fini. Il est possible d'étendre cette construction au domaine infini par passage à la limite mais il n'y a alors plus de formule explicite. Dans le Chapitre 3 Section 3.4, nous prouvons que si $\kappa = \Gamma^1(F;\Omega)$, alors $u^+ = w^+$. De la même manière que pour w^+, nous construisons une fonction de Wells w^- qui est une extension minimale inférieure de F et $w^- = u^-$.

Dans le chapitre 3 Section 3.6, nous prouvons que si Ω est fini alors $W^\pm$ sont des AMLE de F, où $W^\pm$ sont les 1-champ associés à $w^\pm$ respectivement. Cela signifie que pour tout ouvert borné D satisfaisant $\overline{D} \subset \mathbb{R}^n \backslash \Omega$ nous avons

$$\Gamma^1(W^\pm, D) = \Gamma^1(W^\pm, \partial D). \tag{1.25}$$

Ce résultat donne d'existence d'AMLEs de F lorsque Ω est fini. En général, nous avons pas l'unicité, car on peut avoir $w^- < w^+$. En fait, nous pourrions avoir un nombre infini de AMLE de F.

Lorsque Ω est infini, W^+ et W^- sont des MLE extrémales, mais en général ne sont pas des AMLE de F. Nous donnons un contre-exemple.

Exemple 1.4.5. Soit $\Omega_1 = \partial B(0;1)$, $\Omega_2 = \partial B(0;2)$ et $\Omega = \Omega_1 \cup \Omega_2$. Nous définissons $F \in \mathscr{F}^1(\Omega)$ comme suit

$$f_x = 0 \text{ pour } x \in \Omega_1,\ f_x = 1 \text{ pour } x \in \Omega_2, \text{ et } D_x f = 0 \text{ pour } x \in \Omega.$$

Définissons

$$V = \{x \in \mathbb{R}^2 : \|x\| < 3/4\} \subset\subset \mathbb{R}^2/\Omega.$$

En utilisant de la construction de w^+ nous pouvons calculer directement

$$\Gamma^1(W^+;V) = 4 \text{ and } \Gamma^1(W^+;\partial V) = \frac{4}{3}.$$

Ainsi W^+ n'est pas un AMLE de F.

Dans l'exemple ci-dessus, W^+ et W^- ne sont pas des AMLE de F. Nous pouvons vérifier que $\frac{1}{2}(W^+ + W^-)$ est l'unique AMLE de F. De plus, cette fonction n'est pas C^2. La question de l'existence d'une AMLE pour les 1-champs est un problème ouvert et difficile quand Ω est infini.

1.4.3 Formules explicites de MLE pour des applications de $\mathbb{R}^m$ dans $\mathbb{R}^n$

Dans la preuve du théorème 1.1.1 qui est connu comme le problème d'extension de Kirszbraun-Valentine [32, 53], nous avons utilisé une certaine forme de l'axiome du choix. Par conséquent, nous n'avons pas de formules explicites pour les MLE d'applications de $\mathbb{R}^m$ dans $\mathbb{R}^n$.

Expliquons comment utiliser la formule explicite en Sup-Inf des MLE pour des 1-champs pour construire des MLE d'applications de $\mathbb{R}^m$ dans $\mathbb{R}^n$. Appelons $\mathscr{Q}_0$ le problème de MLE pour les applications lipschitziennes et $\mathscr{Q}_1$ le problème de MLE pour les 1-champs. Nous montrons que le problème $\mathscr{Q}_0$ est un sous-problème du problème $\mathscr{Q}_1$. En conséquence, nous obtenons deux formules explicites qui permettent de résoudre le problème $\mathscr{Q}_0$.

Plus précisement, soit $n, m \in \mathbb{N}^*$ et $\omega \subset \mathbb{R}^m$. Soit $u : \omega \to \mathbb{R}^n$. Supposons que $\mathrm{Lip}(u;\omega) < +\infty$. Définissons

$$\Omega := \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^n : x \in \omega\}.$$

Si $x \in \mathbb{R}^{m+n}$, nous noterons $x := (x^{(m)}, x^{(n)}) \in \mathbb{R}^{m+n}$ où $x^{(m)} \in \mathbb{R}^m$ et $x^{(n)} \in \mathbb{R}^n$. Pour chaque fonction u de domaine ω nous associons le 1-champ F de $\Omega \subset \mathbb{R}^{n+m}$ dans $\mathscr{P}^1(\mathbb{R}^{n+m}, \mathbb{R})$ comme suit :

$$f_{(x,0)} := 0 \text{ et } D_{(x,0)} f := (0, u(x)), \text{ pour tous } x \in \omega. \tag{1.26}$$

Soit G une MLE de F. Nous définissons la fonction $\tilde{u} : \mathbb{R}^m \to \mathbb{R}^n$ par

$$\tilde{u}(x) := (D_{(x,0)}g)^{(n)},\ x \in \mathbb{R}^m. \tag{1.27}$$

Theorem 1.4.6. *L'extension $\tilde{u}$ est une MLE de u.*

Si nous remplaçons G par deux 1-champ extrémaux U^- et U^+ de F, alors nous obtenons deux formules explicites qui résolvent le problème $\mathscr{Q}_0$.

Si ω est fini, en utilisant la construction explicite de U^+ ou U^- de Wells, nous pouvons calculer facilement U^+ et U^-. Ainsi le résultat de Wells donne une construction explicite de MLE pour le problème $\mathscr{Q}_0$. De plus, nous pouvons les calculer efficacement.

1.4.4 L'extension de Kirszbraun sur un graphe fini connexe

Nous commençons par étudier la version discrète de l'existence et l'unicité des AMLE. Soit $G = (V, E, \Omega)$ un graphe fini connexe de sommets $V \subset \mathbb{R}^n$, d'arêtes E et soit $\Omega \subset V$ un ensemble non vide.

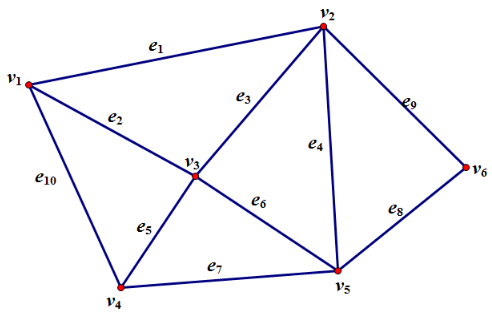

Figure 1.3: Exemple de graphe connexe fini G

Pour $x \in V$, nous définissons

$$S(x) := \{y \in V : (x,y) \in E\} \tag{1.28}$$

un voisinage de x dans G

Exemple 1.4.7. Dans la figure 2.3 nous avons $V = \{v_1, ..., v_6\}, E = \{e_1, ..., e_{10}\}, S(v_3) = \{v_1, v_2, v_4, v_5\}$.

Soit $f : \Omega \to \mathbb{R}^m$. Nous considérons l'équation fonctionnelle suivante avec une condition de Dirichlet:

$$u(x) = \begin{cases} K(u, S(x))(x)\ \forall x \in V \backslash \Omega; \\ f(x)\ \forall x \in \Omega, \end{cases} \tag{1.29}$$

où la fonction $K(u, S(x))(x)$ est définie précisement par (4.5).

Nous disons qu'une fonction u satisfaisant (2.29) est un *extension de Kirszbraun* de f dans le graphe G. Dans le chapitre 4, nous prouvons que l'extension tight introduite par Sheffield et Smart (2012) [51] (voir la définition de l'extension tight dans la Définition 1.2.5) est une extension de Kirszbraun. Donc, nous avons l'existence de l'extension Kirszbraun, mais en général l'extension Kirszbraun peut ne pas être unique. Cette extension est l'extension lipschitzienne optimale de f sur le graphe G puisque pour tout $x \in V \backslash \Omega$, il n'y a aucun moyen de réduire $\operatorname{Lip}(u, S(x))$ en modifiant la valeur de u en x.

Dans le cas $m = 1$, Le Gruyer [37] a obtenu une formule explicite pour $K(u, S(x))(x)$ comme suit

$$K(u,S(x))(x) = \inf_{z \in S(x)} \sup_{q \in S(x)} M(u,z,q)(x), \tag{1.30}$$

où

$$M(u,z,q)(x) := \frac{\|x-z\|u(q) + \|x-q\|u(z)}{\|x-z\| + \|x-q\|}.$$

Le Gruyer a étudié la solution de (1.29) sur un réseau (voir Définition 4.1.2) où $K(u,S(x))(x)$ satisfait (1.30). Cette solution joue un rôle important dans les arguments d'approximation pour les AMLE dans Le Gruyer [37]. L'extension de Kirszbraun u est une généralisation de la solution dans les travaux de Le Gruyer pour le cas $m \geq 2$.

Dans le chapitre 4, nous prouvons que dans le cas $m = 1$ l'extension de Kirszbraun u est unique. De plus, dans le cas $m = 1$, nous produisons un algorithme simple qui calcule efficacement la valeur de l'extension de Kirszbraun avec une complexité polynomiale. Cet algorithme est analogue à l'algorithme produit par Lazarus et et al [34] (1999) quand ils calculent la fonction de coût de Richman. En supposant que les hypothèses de Jensen [28], sont satisfaites cet algorithme calcule exactement la solution de (4.7). En utilisant l'argument de Le Gruyer [37], nous obtenons une nouvelle méthode pour approcher la solution de viscosité de l'équation $\Delta_\infty u = 0$ avec la condition de Dirichlet.

Dans l'algorithme ci-dessus, la formule explicite $K(u,S(x))$ donnée par (1.30) et la structure d'ordre de l'ensemble des nombres réels jouent un rôle important. La généralisation de l'algorithme à $m \geq 2$ est difficile puisque nous ne connaissons pas de formule explicite de $K(u,S(x))$ quand $m \geq 2$ et $\mathbb{R}^2$ n'a pas de structure d'ordre utile. Ces difficultés ont limité le nombre de résultats dans le cas $m \geq 2$.

Chapter 2

Introduction

2.1 The classical Lipschitz extension problem

We consider a pair of metric spaces (X, d_X) and (Y, d_Y). Let Ω be a subset of X and $f : \Omega \to Y$ be a Lipschitz function. We denote

$$\mathrm{Lip}(f, \Omega) := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{d_Y(f(x), f(y))}{d_X(x,y)}$$

to be the Lipschitz constant of f on Ω.

The classical Lipschitz extension problem asks for conditions on pair of metric spaces (X, d_X) and (Y, d_Y) such that for all $\Omega \subset X$ and for all Lipschitz function $f : \Omega \to Y$, then there is a function $g : X \to Y$ that extends f and has the same Lipschitz constant as f, i.e. $\mathrm{Lip}(f, \Omega) = \mathrm{Lip}(g, X)$. It means that we can always extend functions while preserving their Lipschitz constant. The pair (X,Y) is said to have the *isometric extension property*. It is rare for a pair of metric spaces (X,Y) to have the isometric extension property. In this section we introduce some famous examples for the pair (X,Y) that have the isometric extension property.

2.1.1 Kirszbraun theorem

Kirszbraun found a very famous instance for a pair of metric spaces (X,Y) to have the isometric extension property

Theorem 2.1.1. *(Kirszbraun theorem) Let H_1 and H_2 be Hilbert spaces. If Ω is a subset of H_1, and $f : \Omega \to H_2$ is a Lipschitz function, then there is a function $g : H_1 \to H_2$ satisfying*

$$g = f \text{ on } \Omega, \quad \text{and } \mathrm{Lip}(g, H_1) = \mathrm{Lip}(f, \Omega).$$

Kirszbraun first proved this theorem in 1934 [32] for the pairs of Euclidean spaces. Later it was reproved independently by Frederick Valentine in 1943 [52], where he also generalized it to pairs of Hilbert spaces of arbitrary dimension. This theorem is called Kirszbraun theorem, sometimes it is also called Kirszbraun-Valentine theorem.

This theorem asserts that if X and Y are Hilbert spaces, then (X,Y) have the isometric extension property.

Because the proof of this theorem for case $H_1 = \mathbb{R}^m$ and $H_2 = \mathbb{R}^n$ both equipped with the Euclidean norm is very simple and elegant, let us reproduce it below. First of all, we recall the interesting result used in the proof of Kirszbraun theorem:

Lemma 2.1.2. *[19, Lemma 2.10.40] Let P be a compact subset of $\mathbb{R}^n \times \{r : 0 < r < \infty\}$ and*

$$Y_t = \{y : \|y-a\| \le rt \text{ whenever } (a,r) \in P\}$$

for $0 \le t < +\infty$, then $c = \inf\{t : Y_t \neq \emptyset\} < +\infty$, Y_c consists of a single point b and b belongs to the convex hull of

$$A = \{a : \text{ for some } r, (a,r) \in P, \text{ and } \|b-a\| = rc\}.$$

The proof of the above lemma can see in Federer's book: *Geometric Measure Theory*, Springer-Verlag, 1969 [19, Lemma 2.10.40].

Proof of Kirszbraun theorem. (For case $H_1 = \mathbb{R}^m$ and $H_2 = \mathbb{R}^n$ both equipped with the Euclidean norm).

Without loss of generality, we can suppose $\mathrm{Lip}(f,\Omega) = 1$.

***Step 1:** In this step we extend f to one additional point, i.e. let $x \in H_1 \backslash \Omega$, we need to find $f_x \in H_2$ such that

$$\|f_x - f(a)\| \le \|x-a\|, \ \forall a \in \Omega.$$

This is equivalent to

$$\bigcap_{a\in\Omega} B(f(a), \|x-a\|) \neq \emptyset.$$

Since these balls are compact, it will suffice to verify that

$$\bigcap_{a\in F} B(f(a), \|x-a\|) \neq \emptyset. \tag{2.1}$$

for every finite subset F of Ω.

Applying Lemma 2.1.2 (using the same notation) with

$$P = \{(f(a), \|x-a\|) : a \in F\},$$

we can find $x_1, ..., x_k \in A$, and b belongs to the convex hull of $\{f(x_i)\}_{i\in\{1,...,k\}}$ such that

$$\|b - f(x_i)\| = c\|x - x_i\|.$$

Our task is to show that $c \le 1$.

We write $b = \sum_{i=1}^{k} \lambda_i f(x_i)$, where $\lambda_i \in [0,1]$ and $\sum_{i=1}^{k} \lambda_i = 1$. Using the formula

$$2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u-v\|^2,$$

we obtain

$$\begin{aligned}
0 &= 2\|\sum_i \lambda_i(f(x_i)-b)\|^2 \\
&= 2\sum_{i,j}\lambda_i\lambda_j\langle f(x_i)-b, f(x_j)-b\rangle \\
&= \sum_{i,j}\lambda_i\lambda_j\left[\|f(x_i)-b\|^2+\|f(x_j)-b\|^2-\|f(x_i)-f(x_j)\|^2\right] \\
&\geq \sum_{i,j}\lambda_i\lambda_j\left[c^2\|x_i-x\|^2+c^2\|x_j-x\|^2-\|x_i-x_j\|^2\right] \\
&= \sum_{i,j}\lambda_i\lambda_j\left[2\langle c(x_i-x),c(x_j-x)\rangle+(c^2-1)\|x_i-x_j\|^2\right] \\
&= 2\|c\sum_i\lambda_i(x_i-x)\|^2+(c^2-1)\sum_{i,j}\lambda_i\lambda_j\|x_i-x_j\|^2.
\end{aligned}$$

Therefore, $c \leq 1$.

***Step 2:** We consider the class

$$\mathscr{L} = \{h : \Omega \subset \mathrm{dom}(h), h = f \text{ on } \Omega, \text{ and } \mathrm{Lip}(h, \mathrm{dom}(h)) = \mathrm{Lip}(f, \Omega)\}.$$

For $h_1, h_2 \in \mathscr{L}$, we define the order relation:

$$(h_1 \leq h_2) \Leftrightarrow (\mathrm{dom}(h_1) \subset \mathrm{dom}(h_2) \text{ and } h_2 = h_1 \text{ on } \mathrm{dom}(h_1))$$

By Hausdorff's maximal principle $\mathscr{L}$ has a maximal element $g : \Omega_1 \to H_2$. The proof of this theorem is complete if $\Omega_1 = H_1$. Suppose, by contradiction, that $\Omega_1 \neq H_1$. Then there exists $\xi \in H_1 \backslash \Omega_1$. Applying step 1, there exist $\eta \in H_2$ such that

$$\|\eta - g(a)\| \leq \|\xi - a\|, \ \forall a \in \Omega_1.$$

Hence, if we define $g_1 = g$ on Ω_1 and $g_1(\xi) = \eta$, then $g_1 \in \mathscr{L}$, $g \leq g_1$ but $g \neq g_1$. Thus g would not be maximal in $\mathscr{L}$. We get a contradiction. □

The main idea in the above proof is that: In step 1 we use geometric features of Hilbert spaces to extend f to one additional point, and in step 2 we use some form of the axiom of choice to extend f to whole space. This idea is the same as the proof of the classical Hahn-Banach theorem, and the features of Hilbert spaces like inner product are very important in this proof. The corresponding statement for Banach spaces is not true in general, not even for finite-dimensional Banach spaces. We can construct counterexamples where the domain is a subset of $\mathbb{R}^n$ with the maximum norm and the map is valued in $\mathbb{R}^m$ with Euclidean norm. A simple counterexample is the following:

$X = \mathbb{R}^2$ with $d_X(x,y) = \sup\{|x_1 - y_1|, |x_2 - y_2|\}$,
where $x = (x_1, x_2), y = (y_1, y_2) \in X$,
$Y = \mathbb{R}^2$ with $d_Y(x,y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}$,
where $x = (x_1, x_2), y = (y_1, y_2) \in Y$,
$\Omega = \{(1,-1), (-1,1), (1,1)\} \subset X$, $f : \Omega \to Y$,
$f(1,-1) = (1,0)$, $f(-1,1) = (-1,0)$, $f(1,1) = (0,\sqrt{3})$.

Then $d_X(x,y) = 2 = d_Y[f(x), f(y)], \forall x,y \in \Omega$ and $d_X(x,0) = 1, \forall x \in \Omega$, but there exists no $\xi \in \mathbb{R}^2$ such that $d_Y(\xi, f(x)) \leq 1, \forall x \in \Omega$.

More generally, the theorem fails for $\mathbb{R}^m$ equipped with any ℓ_p norm ($p \neq 2$) (see Schwartz 1969 [50, p. 20]).

2.1.2 McShane-Whitney extremal extensions

If $Y = \mathbb{R}$, then for any arbitrary metric space X and any Ω subset of X, every Lipschitz function $f : \Omega \to \mathbb{R}$ has a Lipschitz extension g satisfying

$$g = f \text{ on } \Omega, \text{ and } \mathrm{Lip}(g,X) = \mathrm{Lip}(f,\Omega). \tag{2.2}$$

In fact, McShane [39] and Whitney [56] in 1934 produced two explicit solutions of (2.2)

$$\begin{aligned} m^+(f,\Omega)(\xi) &= \inf\{f(x) + Lip(f,\Omega)d_X(x,\xi) : x \in \Omega\} \text{ for } \xi \in X, & (2.3)\\ m^-(f,\Omega)(\xi) &= \sup\{f(x) - Lip(f,\Omega)d_Y(x,\xi) : x \in \Omega\} \text{ for } \xi \in X. & (2.4)\end{aligned}$$

Moreover, $m^{\pm}$ are extremal: the first is maximal and the second is minimal, that is

$$m^-(f,\Omega)(x) \leq g(x) \leq m^+(f,\Omega)(x), \forall x \in X,$$

for any g other solution of (2.2).

Remark 2.1.3. Clearly solutions of (2.2) are unique if and only if $m^+(f,\Omega) = m^-(f,\Omega)$ on $\mathbb{R}^n$. This rarely happens.

Example 2.1.4. Let $X = \mathbb{R}$, $\Omega = \{-1,0,1\}$, $f(-1) = f(0) = 0, f(1) = 1$. Then $\mathrm{Lip}(f,\Omega) = 1$. The functions $m^+(f,\Omega)$ and $m^-(f,\Omega)$ are drawn below:

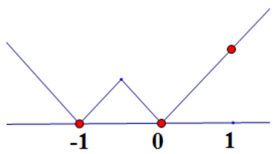

Figure 2.1: Illustration $m^+(f,\Omega)$

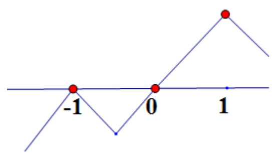

Figure 2.2: Illustration $m^-(f,\Omega)$

2.1.3 Absolute $1-$Lipschitz retract

We may ask for conditions on a metric space Z such that, for every metric space X, the pair (X,Z) has the isometric extension property. To answer this question, we introduce the concept of *absolute* $1-$*Lipschitz retract*.

Definition 2.1.5. Let (L,d_L) be a metric space and X be a subset of L.

A Lipschitz map $r : L \to X$ is called a *1-Lipschitz retraction* if it is the identity on X and $\mathrm{Lip}(r,L) = \mathrm{Lip}(r,X)$.

When such a $1-$Lipschitz retraction exists, we say that X is a *1-Lipschitz retract* of L.

A metric space X is called an absolute $1-$Lipschitz retract if it is a $1-$Lipschitz retract of every metric space containing it.

Example 2.1.6. $\mathbb{R}$, metric trees and $l_\infty = \{(x_n)_n : x_n \in \mathbb{R}\}$ with respect to the norm $\|x\|_\infty = \max_n |x_n|$ (where $x = (x_n)_n$) are absolute $1-$Lipschitz retracts (see [10], [29]).

Theorem 2.1.7. *[10, Proposition 1.2, Proposition 1.4] Let (Z,d_Z) be a metric space. The following statements are equivalent*

(i) *For every metric space X, the pair (X,Z) has the isometric extension property.*

(ii) *Z is an absolute 1-Lipschitz retract.*

(iii) (Z,d_Z) *is metrically convex* [1] *and has the binary intersection property* [2].

2.2 Absolutely minimal Lipschitz extension

2.2.1 The beginning of concept absolutely minimal Lipschitz extension

We consider a pair of metric space (X,d_X) and (Y,d_Y) that has isometric extension property.

Definition 2.2.1. Let Ω be a subset of X and $f : \Omega \to Y$ be a Lipschitz function. If g extends f and $Lip(g,X) = Lip(f,\Omega)$ then we say that g is a *minimal Lipschitz extension* (MLE) of f.

When $X \subset \mathbb{R}^n$ and $Y = \mathbb{R}$, both equipped with the Euclidean norm, from McShane-Whitney formulas (2.3) and (2.4), we have two extremal MLEs m^+ and m^- of f. Thus, unless $m^+ \equiv m^-$, we have no uniqueness of MLE of f.

These extremal MLEs m^+ and m^- fail to obey comparison and stability principle. More precisely, the relation $f_1 \leq f_2$ on Ω does not, in general, imply either

$$m^+(f_1,\Omega) \leq m^+(f_2,\Omega), \text{ on } X,$$

or

$$m^-(f_1,\Omega) \leq m^-(f_2,\Omega), \text{ on } X,$$

[1] The metric space (Z,d_Z) is called *metrically convex* if for every $x,y \in Z$ and $\lambda \in [0,1]$ there is $z \in Z$ such that $d_Z(x,z) = \lambda d_Z(x,y)$ and $d_Z(y,z) = (1-\lambda)d_Z(x,y)$.

[2] The metric space (Z,d_Z) is said to have the *binary intersection property* if every collection of pairwise intersecting closed balls in Z has a common point.

and $m^+(m^+(f,\Omega),\partial V)$ may be different from $m^+(f,\Omega)$ in an open $V \subset\subset X\backslash\Omega$.

Moreover, for g MLE of f, $\mathrm{Lip}(g,V)$ is probably strictly larger than $\mathrm{Lip}(g,\partial V)$ for some $V \subset\subset X\backslash\Omega$.

Therefore, if we replace g by the new function

$$g_1(x) = g(x) \text{ for } x \in X\backslash V, \text{ and } g_1(x) = m^+(g,\partial V)(x) \text{ for } x \in V,$$

then g_1 is also a MLE of f and

$$\mathrm{Lip}(g_1,V) = \mathrm{Lip}(g,\partial V) < \mathrm{Lip}(g,V).$$

This means that we can decrease the local Lipschitz constant by repeating application of the operator m^+ or m^-.

From the above discussion, the following question naturally arises: Is it possible to find a special function u MLE that obeys comparison and stability principle? And furthermore, could this special extension be unique?

Obviously, if such functions exist, then they must satisfy

$$\mathrm{Lip}(u,V) = \mathrm{Lip}(u,\partial V), \text{ for any open } V \subset\subset X\backslash\Omega, \tag{2.5}$$

because otherwise the stability would not hold.

This observation was first appeared in a series of papers of Aronsson in the 1960's [3, 4, 5]. Aronsson proposed the notion of an *absolutely minimal Lipschitz extension* (AMLE):

Definition 2.2.2. A function u defined on X is called AMLE of f if u is a MLE of f and u satisfies (2.5).

It means that u has the least possible Lipschitz constant in every open set whose closure is compact and contained in $X\backslash\Omega$.

The McShane-Whitney operators provide a natural idea to construct AMLE by reducing the Lipschitz constant in domains where it is not optimal. Aronsson (1967) [5] used this idea to prove the existence of AMLE.

After the works of Aronsson [3, 4, 5], there has been many researches devoted to the study of AMLEs and problems related to them, see e.g. [13, 15, 17, 28, 30, 46]. The most popular line of research has arisen from the idea of interpreting the AMLE as a formal limit of u_p, as $p \to \infty$, where u_p is the minimizing of the functional

$$I_p[u] := \int_U \|\nabla u\|^p dx, \tag{2.6}$$

where $U \subset \mathbb{R}^n$ is open, $u \in W^{1,p}(U;\mathbb{R})$ with boundary condition $u = f$ on ∂U, and $\nabla u = (u_{x_1},\ldots,u_{x_n})$ is the gradient.

This approach also leads to rewrite the original problem in which (2.5) is replaced by "calculus of variations problems in the sup-norm":

$$u = v \text{ on } \partial V \text{ implies } \|\nabla u\|_{L^\infty(V)} \le \|\nabla v\|_{L^\infty(V)}, \tag{2.7}$$

for all $V \subset U$, and for all $v \in C(\overline{V})$.

A central question has been to understand minimization problems involving this related functionals. In that time, it was well-know that the problem (2.6) leads as usual to the Euler-Lagrange equation $\Delta_p u = 0$, but it was unclear what is the correct ”Euler-Lagrange” equation for the problem (2.7).

Aronsson (1967)[5] discovered the ”Euler-Lagrange” equation for the problem (2.7), that is

$$\Delta_\infty u = 0, \tag{2.8}$$

defined on smooth real-valued function $u = u(x)$ by

$$\Delta_\infty u := \sum_{j,k=1}^{n} u_{x_j} u_{x_k} u_{x_j x_k}.$$

This nonlinear equation is a highly degenerate elliptic equation. The above ”Euler-Lagrange” equation is nowadays known as the *infinity Laplace equation.*

Aronsson discovered the infinity Laplace equation by approximation procedure. Let us explain the idea at least formally. He considered for finite p the nonlinear $p-$Laplacian operator:

$$\Delta_p u = div(\|\nabla u\|^{p-2}\nabla u) = (p-2)\|\nabla u\|^{p-4}\Delta_\infty u + \|\nabla u\|^{p-2}\Delta u.$$

The equation $\Delta_p u = 0$ arises as the Euler-Lagrange equation for the problem of minimizing (2.6).

Dividing the equation $\Delta_p u = 0$ by $(p-2)\|\nabla u\|^{p-4}$ leads to

$$\Delta_\infty u + \frac{1}{p-2}\|\nabla u\|^2 \Delta u = 0.$$

Let p tend to infinity. This leads to the equation

$$\Delta_\infty u = 0.$$

Aronsson proved in 1967 [5] that if U is the closure of the bounded domain in $\mathbb{R}^n$ and f is a function given on ∂U then a C^2 function $u : U \to \mathbb{R}$ is an AMLE of f if and only if it is a solution of the infinity Laplace equation (2.8) with boundary condition $u = f$ on ∂U.

However, AMLEs are not C^2 in general, the classical solutions of the eikonal equation $|Du| = constant$ is an example of AMLEs which are not C^2. The best known explicit irregular AMLE - outside of the relatively regular solutions of eikonal equations - was exhibited again by Aronsson in 1984 [6]: $u(x,y) = x^{4/3} - y^{4/3}$. The first derivatives of u are Holder continuous with exponent $1/3$ and the second derivatives do not exist on the lines $x = 0$ and $y = 0$. At the time, the existence and uniqueness of AMLEs was unknown in general, and it was also unclear what is correct notion for non-smooth solution of infinity Laplace equation (2.8).

An important breakthrough in this direction was made by Jensen in 1993 using the *viscosity solution* concept introduced by Crandall and Lions in a very famous paper in 1983 [16]:

Definition 2.2.3. (i) An upper semi-continuous function $u : \overline{U} \to \mathbb{R}$ is a *sub-solution* of (2.8) if $\Delta_\infty \phi(x) \geq 0$, for every $(x, \phi) \in U \times C^2(U)$ such that $(u - \phi)(x) \geq (u - \phi)(y)$ for all $y \in U$.

(ii) An lower semi-continuous function $u : \overline{U} \to \mathbb{R}$ is a *super-solution* of (2.8) if $\Delta_\infty \phi(x) \leq 0$, for every $(x, \phi) \in U \times C^2(\Omega)$ such that $(u - \phi)(x) \leq (u - \phi)(y)$ for all $y \in U$.

(iii) A continuous function $u : \overline{U} \to \mathbb{R}$ is a *viscosity solution* of (2.8) if it is both a sub-solution and super-solution.

The viscosity solution concept is a generalization of the classical concept to treat the problem for non-smooth function. Using this definition, in 1993 Jensen proved the existence and uniqueness of viscosity solution of infinity Laplace equation (2.8) with boundary condition $u = f$ on ∂U [28]. Jensen showed as well that this solution is also identified by Aronsson: for every open bounded subset V of U and for each $v \in C(\overline{V})$, if u is the viscosity solution of infinity Laplace equation then

$$u = v \text{ on } \partial V \text{ implies } \|\nabla u\|_{L^\infty(V)} \leq \|\nabla v\|_{L^\infty(V)}. \tag{2.9}$$

Jensen's work generated considerable interest in the long developments in the existence and uniqueness theory of AMLEs.

2.2.2 Comparison with cones

We next introduce the property of "comparison with cones". Notice that any solution of an eikonal equation $|Dv| =$ constant is a classical solution of $-\Delta_\infty v = 0$ wherever it is smooth. Thus the cone

$$C(x) = a + b\|x - x_0\|$$

is a classical solution for $x \neq x_0$.

If b is positive, then the cone C is a global viscosity sub-solution, but it is not a global viscosity solution. Comparison with cones is the basic tool of the theory.

Definition 2.2.4. (i) A function $u \in C(U)$ is said to *enjoys comparison with cones from above* in U if for every bounded open subset V of U and every $x_0 \in \mathbb{R}^n, a, b \in \mathbb{R}$ for which

$$u(x) \leq C(x) = a + b\|x - x_0\|$$

holds on $\partial(V \backslash \{x_0\})$, we have $u \leq C$ in V as well.

(ii) A function $u \in C(U)$ is said to *enjoys comparison with cones from below* in U if for every bounded open subset V of U and every $x_0 \in \mathbb{R}^n, a, b \in \mathbb{R}$ for which

$$u(x) \geq C(x) = a + b\|x - x_0\|$$

holds on $\partial(V \backslash \{x_0\})$, we have $u \geq C$ in V as well.

(iii) A function $u \in C(U)$ is said to *enjoys comparison with cones* in U if it enjoin comparison with cones both above and below.

Roughly speaking, a function enjoys comparison with cones if whenever it is less (greater) than a cone on the boundary of a domain, it is less (greater) than the cone in the interior.

In 2001, Crandall, Evans, and Gariepy [15] proved that if u is a continuous function on a bounded open set $U \subset \mathbb{R}^n$ then

$$\begin{aligned} & u \text{ is } AMLE \text{ that satisfies (2.7)} \\ \Leftrightarrow \quad & u \text{ is viscosity solution of infinity Laplace equation (2.8)} \\ \Leftrightarrow \quad & u \text{ enjoys comparison with cones.} \end{aligned}$$

In 2010, Armstrong and Smart [2] presented an elegant and elementary proof of the uniqueness of the functions enjoying comparison with cones. This proof makes no use of partial differential equations and does not need the viscosity solution machinery developed for second-order elliptic equations. From the results of Crandall, Evans, and Gariepy [15] about the equivalence of the concepts of AMLE, viscosity solution of infinity Laplace equation and function enjoying comparison with cones, the result of Armstrong and Smart implies a new and easy proof for the uniqueness of AMLE and the uniqueness of the viscosity solution of the infinity Laplace equation.

2.2.3 Generalizing in metric spaces and games

The definition of comparison with cones (Definition 2.2.4) easily extends to other metric spaces. Champion and De Pascale [14] adapted this definition to length spaces, where cones are replaced by functions of the form $\phi(x) = bd(x,z) + c$ where $b > 0$.

Juutinen [30] in 2002 used Perron's method to establish the existence of AMLE extensions for separable length space domains.

Peres, Schramm, Sheffield and Wilson [46] in 2009 used game-theoretic techniques to prove the existence and uniqueness of AMLE for general length spaces[3]. It relied on some complicated probabilistic arguments and a beautiful connection between the infinity Laplace equation and random-turn "tug of war" game.

The interesting functional equation

$$u^{\varepsilon}(x) = \frac{1}{2}\left(\max_{|y|\leq\varepsilon} u^{\varepsilon}(x+\varepsilon y) + \min_{|z|\leq\varepsilon} u^{\varepsilon}(x+\varepsilon z)\right) \tag{2.10}$$

appeared in the work of [46] is called Harmonious function. Le Gruyer and Archer [35] (1998) presented a nice proof for the existence of Harmonious function u^{ε} for any $\varepsilon > 0$. The Harmonious function plays an important role in the approximation of AMLE (see Le Gruyer [37](2007) and Oberman [45](2004)).

2.2.4 Regularity

We study the differentiability properties of viscosity solutions of the PDE

$$\Delta_{\infty} u = 0, \ \text{ in } U, \tag{2.11}$$

[3] A metric space (X, d_X) is a *length space* if for all $x, y \in X$, the distance $d_X(x,y)$ is the infimum of the lengths of curves in X that connect x to y.

where $U \subset \mathbb{R}^n$ is an open set.

A viscosity solution u is called an *infinity harmonic function*. After the existence and uniqueness of infinity harmonic function, one wants to know about the regularity of infinity harmonic function.

Consider the Aronsson's example: $u(x,y) = x^{4/3} - y^{4/3}$. The first derivatives of u are Holder continuous with exponent $1/3$; the second derivatives do not exist on the lines $x = 0$ and $y = 0$. Therefore, in Aronsson's example, $u \in C^1$ but $u \notin C^2$. The question is that: Are infinity harmonic functions necessarily continuously differentiable?

Let us assume that u is a viscosity solution of Equation (2.11) and $B(x,r) \subset U$. We then define

$$L_r^+(x) := \frac{\max\limits_{y \in \partial B(x,r)} u(y) - u(x)}{r},$$

and

$$L_r^-(x) := \frac{u(x) - \min\limits_{y \in \partial B(x,r)} u(y)}{r}.$$

Crandall, Evans and Gariepy [15] (2001) proved that the limits

$$L(x) := \lim_{r \to 0} L_r^+(x) = \lim_{r \to 0} L_r^-(x)$$

exist and are equal for each point $x \in U$.

Moreover, any blow-up limit around any point $x \in U$ must be a linear function (see [17] for a fairly simple proof): For each $r_j \to 0$, there exists a subsequence $\{r_{j_k}\}$ such that

$$\frac{u(r_{j_k}y + x) - u(x)}{r_{j_k}} \to \langle a, y \rangle \text{ locally uniformly,}$$

for some $a \in \mathbb{R}^n$ that satisfies $\|a\| = L(x)$.

Evans and Smart [18] (2011) established uniqueness for the blow-up limit, from which it follows that the full limit

$$\lim_{r \to 0} \frac{u(ry + x) - u(x)}{r} = \langle a, y \rangle$$

exist locally uniformly for some $a \in \mathbb{R}^n$ that satisfies $\|a\| = L(x)$.

As a consequence, the infinity harmonic function is everywhere differentiable and $L(x) = \|Du(x)\|$. Savin [48](2006) has shown that infinity harmonic functions in $n = 2$ variables are in fact continuously differentiable. Savin's arguments uses the topology of $\mathbb{R}^2$ very strongly, and it is difficult to general Savin's arguments for case $n > 2$. The question of whether infinity harmonic functions are necessarily C^1 in general remains the most conspicuous open problem in the area.

2.2.5 Tight extension

Let us introduce the discrete version of AMLEs for the vector valued case. Let $G = (V,E,\Omega)$ be a connected finite graph with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$ and let $f : \Omega \to \mathbb{R}^m$.

We denote $E(f)$ to be the set of all extensions of f on G.

Let $v \in E(f)$. The *local Lipschitz constant* of v at vertex $x \in V \backslash \Omega$ is given by

$$Lv(x) := \sup_{y \in S(x)} \frac{\|v(y) - v(x)\|}{\|y - x\|},$$

where

$$S(x) := \{y \in V : (x,y) \in E\} \tag{2.12}$$

is a neighborhood of x on G.

Definition 2.2.5. [4] If $u, v \in E(f)$ satisfy

$$\max\{Lu(x) : Lu(x) > Lv(x), x \in V \backslash \Omega\} > \max\{Lv(x) : Lv(x) > Lu(x), x \in V \backslash \Omega\},$$

then we say that v is *tighter* than u on G. We say that u is a *tight extension* of f on G if there is no v tighter than u .

Theorem 2.2.6. *[51, Theorem 1.2] There exists a unique extension u that is tight of f on G. Moreover, u is tighter than every other extension v of f.*

The unique tight extension is the limit of the discrete $p-$ harmonic extensions.

Theorem 2.2.7. *[51, Theorem 1.3] In addition to the hypotheses of Theorem 2.2.6, suppose that for each $p > 0$, the function $u_p : V \to \mathbb{R}^m$ minimizes*

$$I_p[u] = \sum_{x \in V \backslash \Omega} (Lu(x))^p,$$

where u_p extension of f. The u_p converge to the unique tight extension of f.

2.3 Whitney's extension problem

2.3.1 Withney theorem

The subject is originated from Hassler Whitney who deals with the following problem: Let Ω be a subset of $\mathbb{R}^n$, and let f be a real-valued function defined and continuous in Ω. How can we decide whether f extends to a C^m function on $\mathbb{R}^n$? If the given function $f(x)$ is in some sense differentiable in Ω, can the extension $F(x)$ be made differentiable to the same order on $\mathbb{R}^n$?

In the seminal papers of 1934 (see [56, 57, 58]) Whitney developed important analytic and geometric techniques which allowed him to solve the problem. The difficulty

[4] By convention, if $C = \emptyset$ then $\max_C = 0$.

is that subsets of Euclidean spaces in general lack a differentiable structure. Thus we need careful considerations of what it means to prescribe the derivative of a function on a set. The starting point is an examination of the statement of Taylor's theorem. Given a real-valued C^m function $f(x)$ on $\mathbb{R}^n$, Taylor's theorem asserts that for each $a,x,y \in \mathbb{R}^n$, there is a function $R_\alpha(x,y) \to 0$ uniformly as $x,y \to a$ such that

$$f(x) = \sum_{|\alpha|\leq m} \frac{D^\alpha f(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=m} \frac{R_\alpha(x,y)}{\alpha!}(x-y)^\alpha, \tag{2.13}$$

where the sum is over multi-indices α.

Let $f_\alpha = D^\alpha f$ for each multi-index α. Differentiating (2.13) with respect to x yields

$$f_\alpha(x) = \sum_{|\beta|\leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y), \tag{2.14}$$

where R_α is $o(|x-y|^{m-\alpha})$ uniformly as $x,y \to a$.

The definition of the derivatives of a function in a general set given by (2.14) arises naturally from a consideration of Taylor's formula (2.13). The Whitney extension theorem is a partial converse to Taylor's theorem. It was first proved in the original paper of Whitney (1934)[56]:

Theorem 2.3.1. *[56, Whitney extension theorem] Suppose that $(f_\alpha)_\alpha$ are a collection of functions on a closed subset Ω of $\mathbb{R}^n$ for all multi-indices α with $|\alpha| \leq m$ satisfying the compatibility condition (2.14). Then there exists a function $F(x)$ of class C^m such that:*
(i) $F = f_0$ on Ω.
(ii) $D^\alpha F = f_\alpha$ on Ω.
(iii) F is a real analytic at every point of $\mathbb{R}^n \backslash \Omega$.

Since Whitney's seminal work, a fundamental advances to the problem were made by Georges Glaeser, Yuri Brudnyi, Pavel Shvartsman, Edward Bierstone, Pierre Milman, Erwan Le Gruyer... In a series of recent papers, Charles Fefferman solved the original problem of Whitney in full generality. His methods led to a number of very important developments in the field, including new analytic and geometric methods in the study of Lipschitz structures on finite sets (see [20, 21, 22, 23]).

2.3.2 The minimal Lipschitz extension for 1-field

Let Ω be a subset of the Euclidean space $\mathbb{R}^n$. Let $\mathscr{P}^1(\mathbb{R}^n,\mathbb{R})$ be the set of first degree polynomials mapping $\mathbb{R}^n$ to $\mathbb{R}$, i.e

$$\mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) \triangleq \{P : a \in \mathbb{R}^n \mapsto P(a) = p + \langle v,a\rangle\,,\ p \in \mathbb{R}, v \in \mathbb{R}^n\}.$$

Let us consider a 1-field F on Ω defined by

$$\begin{aligned} F :\ & \Omega \to \mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) \\ & x \mapsto F(x)(a) := f_x + \langle D_x f; a-x\rangle, \end{aligned} \tag{2.15}$$

where $a \in \mathbb{R}^n$ is the evaluation variable of the polynomial $F(x)$ and $f : x \in \Omega \mapsto f_x \in \mathbb{R}$, $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$ are mappings associated with F.

Let us review the Kirszbraun's extension theorem (see Theorem 2.1.1): Let $\Omega \subset \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ be a Lipschitz function. Then there exists a total Lipschitz extension g of f such that $\mathrm{Lip}(g, \mathbb{R}^n) = \mathrm{Lip}(f, \Omega)$. As a consequence we have $\mathrm{Lip}^*(f) = \mathrm{Lip}(f, \Omega)$ where

$$\mathrm{Lip}^*(f) := \inf\{\mathrm{Lip}(g, \mathbb{R}^n) : g \text{ total Lipschitz extension of } f\}.$$

Let F be a 1-field. We define that a 1-field G is called an *extension* of F if $dom(G) = \mathbb{R}^n$ and $G(x) = F(x)$ on Ω. The natural question is that what is

$$\mathrm{Lip}^*(F) = \inf\{\mathrm{Lip}(D_x g, \mathbb{R}^n) : G \text{ is an extension of } F\}.$$

Erwan Le Gruyer [36] (2009) introduced the Lipschitz constant of 1-field F:

$$\Gamma^1(F;\Omega) \triangleq \sup_{\substack{x,y \in \Omega \\ x \neq y}} \Gamma^1(F;x,y), \tag{2.16}$$

where

$$\Gamma^1(F;x,y) \triangleq 2 \sup_{a \in \mathbb{R}^n} \frac{|F(x)(a) - F(y)(a)|}{\|x-a\|^2 + \|y-a\|^2}. \tag{2.17}$$

Le Gruyer proved that the constant Γ^1 of 1-field plays a role in the extension problem similarly to the constant Lip of continuous function:

Theorem 2.3.2. *[36] Let F be a 1-field. The functional $\Gamma^1 : F \to \Gamma^1(F, \mathrm{dom}(F)) \in \mathbb{R}^+ \cup \{+\infty\}$ is the unique one satisfying*

$(P0)$ Γ^1 *is increasing, that is, U extends F implies that*

$$\Gamma^1(U, \mathrm{dom}(U)) \geq \Gamma^1(F, \mathrm{dom}(F)).$$

$(P1)$ *If U has total domain satisfying $\Gamma^1(U, \mathrm{dom}(U)) < +\infty$, then the total function u defined by $u(x) := U(x)(x)$ is differentiable and its derivative ∇u is Lipschitz.*

$(P2)$ *If u is a differentiable function of total domain with ∇u Lipschitz, then*

$$\Gamma^1(U) = \mathrm{Lip}(\nabla u),$$

where U is the 1-field associate to u, i.e. $U(x)(a) := u(x) + \langle \nabla u(x); a - x \rangle, \forall x, a \in \mathbb{R}^n$.

$(P3)$ *For any F such that $\Gamma^1(F, \mathrm{dom}(F)) \leq +\infty$, F extends to a total 1-field U satisfying*

$$\Gamma^1(U, \mathrm{dom}(U)) = \Gamma^1(F, \mathrm{dom}(F)).$$

As an immediate consequence of this theorem, for any 1-field F, we have $\mathrm{Lip}^*(F) = \Gamma^1(F, \mathrm{dom}(F))$.

This main theorem holds not only in $\mathbb{R}^n$ but in fact in any Hilbert space, separable or not. Therefore this theorem generalizes Whitney's extension theorem in the differentiable real valued case.

To compute the norm Γ^m of the minimal extension on C^m which generalizes Le Gruyer's work on minimal C^1 extensions is a very difficult problem and the main thrust is some attempts to guess the natural norm for which one can obtain the minimal extension.

2.4 Results of the thesis

In this section, we introduce the main results in this thesis.

2.4.1 Relationships between Γ^1 and Lip

Let $F : \Omega \to \mathscr{P}^1(\mathbb{R}^n, \mathbb{R})$ be a 1-field. It is worth asking what is it the relationship between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$? From [36], we know that

$$\mathrm{Lip}(Df;\Omega) \leq \Gamma^1(F;\Omega).$$

When $\Omega = \mathbb{R}^n$, we know that (see [36, Proposition 2.4])

$$\mathrm{Lip}(Df;\mathbb{R}^n) = \Gamma^1(F;\mathbb{R}^n),$$

but in general $\Gamma^1(F;\Omega)$ may be strictly bigger than $\mathrm{Lip}(Df;\Omega)$.

Example 2.4.1. We give a very simple example that is $\Gamma^1(F;\Omega) > \mathrm{Lip}(Df;\Omega)$. Let A and B be open sets in $\mathbb{R}^n$ such that $A \cap B = \emptyset$. Let $\Omega = A \cup B$ and $F \in \mathscr{F}^1(\Omega)$ such that $f_x = 0$ if $x \in A$, $f_x = 1$ if $x \in B$, and $D_x f = 0$,$\forall x \in \Omega$. Then

$$\mathrm{Lip}(Df;\Omega) = 0,$$

and we have

$$\Gamma^1(F;\Omega) = \sup_{x\in A}\sup_{y\in B} \frac{4}{\|x-y\|^2} > 0.$$

We now give two results where we have $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$.

Proposition 2.4.2. *Let $F \in \mathscr{F}^1(\Omega)$. Suppose there exist $a, b \in \Omega$, $a \neq b$ such that $\Gamma^1(F;a,b) = \Gamma^1(F;\Omega)$. Then $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

Proposition 2.4.3. *Let $F \in \mathscr{F}^1(\Omega)$. Suppose there exists $\Omega' \subset\subset \Omega$ such that $\Gamma^1(F;\Omega') = \Gamma^1(F;\Omega)$. Then $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

We see that in some cases we have $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$. Moreover, in Example 2.4.1, Ω is open but not convex. Thus we can hope that when Ω is convex then we have $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$. Unfortunately, this is still untrue for open convex sets Ω. We give a counterexample that is $\mathrm{Lip}(\nabla f;\Omega) < \Gamma^1(F;\Omega)$ for open convex Ω and $F \in \mathscr{F}^1(\Omega)$ in Proposition 3.3.6.

Our main result in this section is

Theorem 2.4.4. *Let Ω is an open subset of $\mathbb{R}^n$. We have*

$$\Gamma^1(F;\Omega) = \max\{\Gamma^1(F;\partial\Omega), \mathrm{Lip}(Df;\Omega)\}, \tag{2.18}$$

where $\partial\Omega$ is a boundary of Ω.

Moreover, if Ω is a convex subset of $\mathbb{R}^n$ then

$$\Gamma^1(F;\Omega) \leq 2\mathrm{Lip}(Df;\Omega). \tag{2.19}$$

To make the connection between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$, it is important to know the set of uniqueness of minimal extensions of F when the cardinality of Ω equals 2 (this study was performed in [27]).

2.4.2 Sup-Inf explicit minimal Lipschitz extensions for 1-Fields

Let $F : \Omega \to \mathscr{P}^1(\mathbb{R}^n, \mathbb{R})$ be a 1-field. From Theorem 2.3.2, there exists a total 1-field G *minimal Lipschitz extension* (MLE) of F, i.e. $G = F$ on Ω and $\Gamma^1(G, \mathbb{R}^n) = \Gamma^1(F, \Omega)$.

We present two MLEs U^+ and U^- of F of the form

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) := u^+(x) + \langle D_x u^+; y - x\rangle,\ y \in \mathbb{R}^n, \tag{2.20}$$

where

$$u^+(x) := \max_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v),\ D_x u^+ := \arg\max_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F, x, a, v), \tag{2.21}$$

and

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) := u^-(x) + \langle D_x u^-; y - x\rangle,\ y \in \mathbb{R}^n, \tag{2.22}$$

where

$$u^-(x) := \min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v),\ D_x u^- := \arg\min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v), \tag{2.23}$$

where Λ_x is a non empty and convex set of $\mathbb{R}^n$ defined in Definition 3.4.7, and $\Psi^\pm$ are functions defined in Definition 3.4.8 .

$u^\pm$ are *extremal* : the first is *over* and the second is *under* that is

$$u^-(x) \le g_x \le u^+(x),\ \forall x \in \mathbb{R}^n,$$

for all MLE G of F.

The formulas of $u^\pm$ and their gradients are explicit and they only depend on F. The formulas of $u^\pm$ in the $1-$field case are similar to the formulas (2.3) and (2.4) of $m^\pm$ coming from from the work of McShane [39] and Whitney [56].

During the workshop Whitney problems in 2011, M. Hirn made the link between the constant Γ^1 and the *allowable for F*. We call the real $\kappa \in \mathbb{R}$, with $\kappa > 0$, to be *allowable for F* if κ satisfies the following inequality

$$f_y \le f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{\kappa}{4}(x-y)^2 - \frac{1}{4\kappa}(D_x f - D_y f)^2, \forall x, y \in \Omega. \tag{2.24}$$

M. Hirn proved that κ is allowable for F if and only if $\Gamma^1(F;\Omega) \le \kappa$. Moreover if κ is assigned to be the Lipschitz constant of the field F, then from Wells's works (see [54, Theorem 2]) we have w^+ (see the definition of the function w^+ in Appendix 3.8) is a MLE. Further, in this case w^+ is an over extremal extension of F.

The construction of Wells w^+ is explicit when Ω is finite. It is possible to extend this construction to infinite domain by passing to the limit but there is no explicit formula. In Chapter 3 Section 3.4, we prove that if κ is assigned to be the Lipschitz constant of the field F, then $u^+ = w^+$. In a similar way of the construction of w^+, we construct a Wells function w^- which is an under minimal extension of F and $w^- = u^-$.

In Chapter 3 Section 3.6, we prove that if Ω is finite then $W^\pm$ are AMLEs of F, where $W^\pm$ are 1-fields associated to $w^\pm$ respectively. This means that for any bounded open D satisfying $\overline{D} \subset \mathbb{R}^n \backslash \Omega$ we have

$$\Gamma^1(W^\pm, D) = \Gamma^1(W^\pm, \partial D). \tag{2.25}$$

This result give the existence of AMLEs of F when Ω is finite. In general, we have not uniqueness, since it may happen $w^- < w^+$. In fact, we may even have an infinite number of AMLE of F.

When Ω is infinite, W^+ and W^- are extremal MLEs, but in general are not AMLE of F. We give a counter-example:

Example 2.4.5. Let $\Omega_1 = \partial B(0;1)$, $\Omega_2 = \partial B(0;2)$ and $\Omega = \Omega_1 \cup \Omega_2$. We define $F \in \mathscr{F}^1(\Omega)$ as following

$$f_x = 0 \text{ for } x \in \Omega_1,\ f_x = 1 \text{ for } x \in \Omega_2, \text{ and } D_x f = 0 \text{ for } x \in \Omega.$$

Let us define

$$V = \{x \in \mathbb{R}^2 : \|x\| < 3/4\} \subset\subset \mathbb{R}^2/\Omega.$$

Using the construction of w^+ we can compute directly

$$\Gamma^1(W^+;V) = 4 \text{ and } \Gamma^1(W^+;\partial V) = \frac{4}{3}.$$

Thus W^+ is not AMLE of F.

In the above example, W^+ and W^- are not AMLE of F. We can check that $\frac{1}{2}(W^+ + W^-)$ is the unique AMLE of F. Moreover this function is not C^2 although the domain Ω of F is regular and F is a regular 1-field. The question of the existence of an AMLE remains an open and difficult problem when Ω is infinite, see [27] and the references therein.

2.4.3 The explicit formulas of MLEs for maps from $\mathbb{R}^m$ to $\mathbb{R}^n$

In the proof of Theorem 2.1.1 that is known as Kirszbraun-Valentine extension problem [32, 53], we uses some form of the axiom of choice. Therefore, we have no the explicit formulas of MLE of mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We explain how to use the Sup-Inf explicit minimal Lipschitz extensions for 1-Fields to construct MLE of mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let us define $\mathscr{Q}_0$ as the problem of the minimal Lipschitz extension for Lipschitz continuous functions and $\mathscr{Q}_1$ as the problem of the minimal Lipschitz extension for 1-fields. Curiously, we show that the problem $\mathscr{Q}_0$ is a sub-problem of the problem $\mathscr{Q}_1$. As a consequence, we obtain two explicit formulas that solve the problem $\mathscr{Q}_0$.

More specifically, fix $n, m \in \mathbb{N}^*$ and $\omega \subset \mathbb{R}^m$ with $\#\omega \geq 2$. Let u be a function from ω maps to $\mathbb{R}^n$. Suppose $\mathrm{Lip}(u;\omega) < +\infty$. Let us define

$$\Omega := \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^n : x \in \omega\}.$$

A current element x of $\mathbb{R}^{m+n}$ is denoted by $x := (x^{(m)}, x^{(n)}) \in \mathbb{R}^{m+n}$, with $x^{(m)} \in \mathbb{R}^m$ and $x^{(n)} \in \mathbb{R}^n$. For each function u of domain ω we associate the 1-field F from $\Omega \subset \mathbb{R}^{n+m}$ maps to $\mathscr{P}^1(\mathbb{R}^{n+m}, \mathbb{R})$ as the following

$$f_{(x,0)} := 0, \text{ and } D_{(x,0)} f := (0, u(x)), \text{ for all } x \in \omega. \tag{2.26}$$

Let G be an minimal Lipschitz extension of F. We define the map $\tilde{u}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ as follows

$$\tilde{u}(x) := (D_{(x,0)} g)^{(n)},\ x \in \mathbb{R}^m. \tag{2.27}$$

Theorem 2.4.6. *The extension $\tilde{u}$ is a minimal Lipschitz extension of u.*

If we replace the MLE 1-field G of F by two extremal MLEs 1-fields U^- and U^+ of F, then we obtain two explicit formulas which solve the problem $\mathscr{Q}_0$.

If ω is finite using the previous transformation $u \longrightarrow F$ then the Wells explicit construction of u^+ or u^- allows to compute easily u^+ and u^-. Thus when the domain of the function to extend is finite, the result of Wells gives explicit construction of minimal Lipschitz extensions for problem $\mathscr{Q}_0$. Moreover, we can compute them efficiently.

2.4.4 Kirszbraun extension on a connected finite graph

We begin by studying the discrete version of the existence and uniqueness of AMLE. Let $G = (V, E, \Omega)$ be a connected finite graph with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$. For $x \in V$. We define

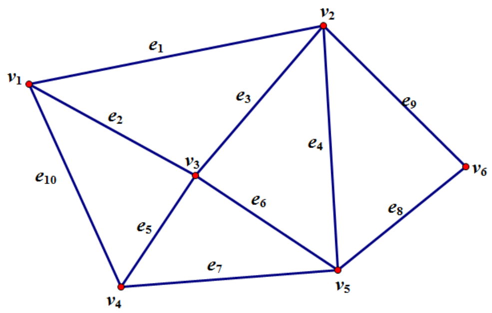

Figure 2.3: A simple picture of graph G

$$S(x) := \{y \in V : (x,y) \in E\} \tag{2.28}$$

to be the neighborhood of x on G.

Example 2.4.7. In Figure 2.3 we have $V = \{v_1, ..., v_6\}$, $E = \{e_1, ..., e_{10}\}$, $S(v_3) = \{v_1, v_2, v_4, v_5\}$.

Let $f : \Omega \to \mathbb{R}^m$. We consider the following functional equation with Dirichlet's condition:

$$u(x) = \begin{cases} K(u, S(x))(x) \ \forall x \in V \backslash \Omega; \\ f(x) \ \forall x \in \Omega, \end{cases} \tag{2.29}$$

where the function $K(u, S(x))(x)$ is defined at (4.5).

We say that the function u satisfying (2.29) is a *Kirszbraun extension* of f on graph G. In Chapter 4, we prove that the tight function introduced by Sheffield and Smart (2012) [51] (see Definition 2.2.5) is a Kirszbraun extension. Therefore, we have the existence of Kirszbraun extension, but in general Kirszbraun extension maybe not unique. This extension is the optimal Lipschitz extension of f on graph G since for any $x \in V \backslash \Omega$, there is no way to decrease $\mathrm{Lip}(u, S(x))$ by changing the value of u at x.

In real valued case $m = 1$, the function $K(u,S(x))(x)$ was considered by Oberman [45] and he used this function to obtain a convergent difference scheme for the AMLE. Le Gruyer [37] showed the explicit formula for $K(u,S(x))(x)$ as follows

$$K(u,S(x))(x) = \inf_{z\in S(x)} \sup_{q\in S(x)} M(u,z,q)(x), \tag{2.30}$$

where

$$M(u,z,q)(x) := \frac{\|x-z\|u(q) + \|x-q\|u(z)}{\|x-z\| + \|x-q\|}.$$

Le Gruyer studied the solution of (2.29) on a network (see Definition 4.1.2) where $K(u,S(x))(x)$ satisfying (2.30). This solution plays an important role in approximation arguments for AMLE in Le Gruyer [37]. The Kirszbraun extension u is a generalization of the solution in the previous works of Le Gruyer for vector valued cases ($m \geq 2$).

In Chapter 4, we prove that in the case $m = 1$ the Kirszbraun extension u is unique. Moreover, in the real-valued case ($m = 1$) we produce a simple algorithm which calculates efficiently the value of the Kirszbraun extension in polynomial time. This algorithm is similar to the algorithm produced by Lazarus el al. [34] (1999) when they calculate the Richman cost function. Assuming Jensen's hypotheses [28], since this algorithm computes exactly solution of (4.7) and by using the argument of Le Gruyer [37], we obtain a new method to approximate the viscosity solution of Equation $\Delta_\infty u = 0$ under Dirichler's condition.

In the above algorithm, the explicit formula of $K(u,S(x))$ in (4.8) and the order structure of real number set play important role. The generalization of the algorithm to vector valued case ($m \geq 2$) is difficult since we do not know the explicit formula of $K(u,S(x))$ when $m \geq 2$ and $\mathbb{R}^2$ does not have an adequate order structure for this problem. The difficulties have limited the number of results in the case $m \geq 2$, see [27] and the references therein.

Chapter 3

Some results of the Lipschitz constant of 1-Field on $\mathbb{R}^n$

Abstract: We study the relationship between the Lipschitz constant of 1-field introduced in [36] and the Lipschitz constant of the gradient canonically associated with this 1-field. Moreover, we produce two explicit formulas which are two extremal minimal Lipschitz extensions for 1-fields. As a consequence of the previous results, for the problem of minimal extension by Lipschitz continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, we produce explicit formulas similar to those of Bauschke and Wang (see [9]). Finally, we show that Wells's extensions (see [54]) of 1-fields are absolutely minimal Lipschitz extension when the domain of 1-field to expand is finite. We provide a counter-example showing that this result is false in general.

Key words: Minimal, Lipschitz, Extension, Differentiable Function, Convex Analysis

AMS Subject Classification: 54C20, 58C25, 46T20, 49

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3.1 Introduction

Let Ω be a subset of Euclidean space $\mathbb{R}^n$, with $\#\Omega \geq 2$. Let $\mathscr{P}^1(\mathbb{R}^n,\mathbb{R})$ be the set of first degree polynomials mapping $\mathbb{R}^n$ to $\mathbb{R}$, i.e.

$$\mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) := \{P : a \in \mathbb{R}^n \mapsto P(a) = p + \langle v,a\rangle\,,\ p \in \mathbb{R}, v \in \mathbb{R}^n\}.$$

Let us consider a 1-field F on domain $\mathrm{dom}(F) := \Omega$ defined by

$$\begin{aligned} F :\Omega &\to \mathscr{P}^1(\mathbb{R}^n,\mathbb{R}) \\ x &\mapsto F(x)(a) := f_x + \langle D_x f; a-x\rangle, \end{aligned} \tag{3.1}$$

where $a \in \mathbb{R}^n$ is the evaluation variable of the polynomial $F(x)$ and $f : x \in \Omega \mapsto f_x \in \mathbb{R}$, $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$ are mappings associated with F.

We will always use capital letters to denote the 1-field and small letters to denote these mappings.

The Lipschitz constant of F introduced in [36] is

$$\Gamma^1(F;\Omega) := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \Gamma^1(F;x,y), \tag{3.2}$$

where

$$\Gamma^1(F;x,y) := 2 \sup_{a\in\mathbb{R}^n} \frac{|F(x)(a) - F(y)(a)|}{\|x-a\|^2 + \|y-a\|^2}. \tag{3.3}$$

If $\Gamma^1(F;\Omega) < +\infty$, then the Whitney's conditions [56], [25] are satisfied and the 1-field F can be extended on $\mathbb{R}^n$: there exists $g \in \mathscr{C}^{1,1}(\mathbb{R}^n,\mathbb{R})$ such that

$$g(x) = f_x, \text{ and } \nabla g(x) = D_x f,\ \forall x \in \Omega,$$

where ∇g is the usual gradient.

Moreover, from [36, Theorem 2.6] we can find g which satisfies

$$\Gamma^1(G;\mathbb{R}^n) = \Gamma^1(F;\Omega),$$

where G is the 1-field associated to g, i.e.

$$G(x)(y) = g(x) + \langle \nabla g(x), y-x\rangle\,,\ x \in \Omega,\ y \in \mathbb{R}^n.$$

It means that the Lipschitz constant does not increase when extending F by G. We say that G is a minimal Lipschitz extension (MLE for short) of F and we have

$$\Gamma^1(G;\mathbb{R}^n) = \inf\{\mathrm{Lip}(\nabla h;\mathbb{R}^n)\ :\ h(x) = f_x, \nabla h(x) = D_x f,\ x \in \Omega,\ h \in \mathscr{C}^{1,1}(\mathbb{R}^n,\mathbb{R})\},$$

where the notation $\mathrm{Lip}(u;.)$ means that

$$\mathrm{Lip}(u;x,y) := \frac{\|u(x)-u(y)\|}{\|x-y\|},\ x,y \in \Omega,\ x \neq y, \text{ and } \mathrm{Lip}(u;\Omega) := \sup_{x\neq y\in\Omega} \mathrm{Lip}(u;x,y) \tag{3.4}$$

It is worth asking what is it the relationship between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$? From [36], we know that

$$\mathrm{Lip}(Df;\Omega) \leq \Gamma^1(F;\Omega).$$

In the special case $\Omega = \mathbb{R}^n$ we have

$$\mathrm{Lip}(Df;\mathbb{R}^n) = \Gamma^1(F;\mathbb{R}^n),$$

but in general the formula

$$\mathrm{Lip}(Df;\Omega) = \Gamma^1(F;\Omega)$$

is untrue.

In this paper we will prove that if Ω is an open subset of $\mathbb{R}^n$ then

$$\Gamma^1(F;\Omega) = \max\{\Gamma^1(F;\partial\Omega), \mathrm{Lip}(Df;\Omega)\}, \tag{3.5}$$

where $\partial\Omega$ is a boundary of Ω.

Moreover, if Ω is a convex subset of $\mathbb{R}^n$ then

$$\Gamma^1(F;\Omega) \leq 2\mathrm{Lip}(Df;\Omega). \tag{3.6}$$

To make the connection between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$, it is important to know the set of uniqueness of minimal extensions of F when $\#\Omega = 2$ (this study was performed in [27]). Indeed, many results of Section 3.3 use this knowlege. For further more details see Section 3.3.

In Section 3.4, we present two MLEs U^+ and U^- of F of the form

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) := u^+(x) + \langle D_x u^+; y - x\rangle,\ y \in \mathbb{R}^n, \tag{3.7}$$

where

$$u^+(x) := \max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F,x,a,v),\ D_x u^+ := \arg\max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F,x,a,v), \tag{3.8}$$

and

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) := u^-(x) + \langle D_x u^-; y - x\rangle,\ y \in \mathbb{R}^n, \tag{3.9}$$

where

$$u^-(x) := \min_{v\in\Lambda_x} \sup_{a\in\Omega} \Psi^-(F,x,a,v),\ D_x u^- := \arg\min_{v\in\Lambda_x} \sup_{a\in\Omega} \Psi^-(F,x,a,v), \tag{3.10}$$

where Λ_x is a non empty and convex set of $\mathbb{R}^n$, defined in Definition 3.4.7.

The point here is that (3.7), (3.8), (3.9), (3.10) give explicit sup inf formulas for $u^\pm$ and their gradients that is to say they only depend on F. In the above formulas, an important remark is that Λ_x is a non-empty convex set of $\mathbb{R}^n$ (this study was performed in [36]), see Definition 3.4.7 for further details.

In addition $u^\pm$ are *extremal* : the first is *over* and the second is *under* that is

$$u^-(x) \leq g_x \leq u^+(x),\ \forall x \in \mathbb{R}^n,$$

for all MLE G of F.

During the workshop Whitney problems in 2011, M. Hirn had make the link between the constant Γ^1 (see Definition (3.2)) and [54, Theorem 2] using Proposition 3.2.4 and Lemma 3.4.2 as following.

We call the real $\kappa \in \mathbb{R}$, with $\kappa > 0$, to be allowable for F if κ satisfies the following inequalities

$$f_y \leq f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{\kappa}{4}(x-y)^2 - \frac{1}{4\kappa}(D_x f - D_y f)^2, \forall x, y \in \Omega, \quad (3.11)$$

From [54, Theorem 2], we know that if $\kappa > 0$ is allowable for F, then there exists $w^+ \in \mathscr{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$ such that

$$w^+(x) = f_x,\ \nabla w^+(x) = D_x f,\ \forall x \in \Omega,\ \text{ and } \mathrm{Lip}(\nabla w^+, \mathbb{R}^n) \leq \kappa.$$

Further, if $g \in \mathscr{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$ with $g(x) = f_x$, $\nabla g(x) = D_x f$ for all $x \in \Omega$ and $\mathrm{Lip}(\nabla g, \mathbb{R}^n) \leq \kappa$, then

$$g(x) \leq w^+(x),\ \forall x \in \mathbb{R}^n.$$

The construction of Wells w^+ is explicit when Ω is finite. It is possible to extend this construction to infinite domain by passing to the limit but there is no explicit formula.

During the workshop Whitney problems in 2011, M. Hirn made the link between the constant Γ^1 (see (3.2)) and the constant $\kappa > 0$ in (3.11) that is the real κ to be allowable for F if and only if $\Gamma^1(F;\Omega) \leq \kappa$. Moreover if κ is assigned the Lipschitz constant of the field F, then w^+ is a MLE. Further, in this case w^+ is an over extremal extension of F.

In section 3.4, we will prove that if κ is assigned the Lipschitz constant of the field F, then $u^+ = w^+$. In a similar way (see Appendix 3.8), we can construct a Wells function w^- which is an under minimal extension of F and $w^- = u^-$.

We pay attention to the case when Ω is finite. In section 3.6, using the explicit constructions of $w^\pm$, we prove that $W^\pm$ are absolutely minimal Lipschitz extensions (AMLEs for short) of F, where $W^\pm$ are 1-fields associated to $w^\pm$ respectively. This means that for any bounded open D satisfying $\overline{D} \subset \mathbb{R}^n \backslash \Omega$ we have

$$\Gamma^1(W^\pm, D) = \Gamma^1(W^\pm, \partial D). \quad (3.12)$$

This result give the existence of AMLEs of F when Ω is finite. In general, we have not uniqueness, since it may happen $w^- < w^+$. In fact, we may even have infinity AMLE of F (see Corollary 3.6.2).

When Ω is infinite, W^+ and W^- are extremal MLEs, but in general are not AMLE of F. To prove this, we present, in section 3.6, an example of mapping F for which W^+ and W^- are not AMLE of F. In this particular example, we can check that $\frac{1}{2}(W^+ + W^-)$ is the unique AMLE of F. Moreover this function is not C^2 although the domain Ω of F is regular and F is a regular 1-field. The question of the existence of an AMLE remains an open and difficult problem when Ω is infinite, see [27] and the references therein.

In Section 3.5, we explain how to use the previous ideas and methods to construct MLE of mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$, i.e. to solve the Kirszbraun-Valentine extension problem [32, 53]. Let us define $\mathscr{Q}_0$ as the problem of the minimum extension for Lipschitz functions, and $\mathscr{Q}_1$ as the problem of the minimum extension for 1-fields. Curiously, we will show that the problem $\mathscr{Q}_0$ is a sub-problem of the problem $\mathscr{Q}_1$. As a consequence, we obtain two explicit formulas see (3.44) and (3.45) that solve the problem $\mathscr{Q}_0$. The Bauschke-Wang result [9] gives an explicit formula for the Kirsbraun-Valentine problem from $\mathbb{R}^m$ to $\mathbb{R}^n$. By our approach, we produce analogous formulas. Moreover, when the domain of the function to extend is finite, the result of Wells gives an explicit construction of minimal Lipschitz extensions that we can compute efficiently.

Let $u : \mathrm{dom}(u) \subset \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Departing from the work of McShane [39] and Whitney [56] in 1934, it is known that the following extensions

$$\begin{aligned} m^+(x) &:= \inf_{y \in \mathrm{dom}(u)} (u(y) + \mathrm{Lip}(u; \mathrm{dom}(u)) \|x - y\|), \\ m^-(x) &:= \sup_{y \in \mathrm{dom}(u)} (u(y) - \mathrm{Lip}(u; \mathrm{dom}(u)) \|x - y\|), \end{aligned}$$

are two extremal minimal Lipschitz extension of u, so that if m is an arbitrary minimal Lipschitz extension of u, then $m^- \leq m \leq m^+$. Among all minimal Lipschitz extensions of u, one can search extensions that have good additional properties. In the 1960s , Aronsson published a series of papers [3, 4, 5, 6], in which the notion of AMLE appeared. An AMLE has a very good stability properties like harmonious extensions (see [35]) which is related to “tug of war” game and the infinity Laplacian (see [46]) . Moreover it is ”locally best” and this notion is also positively correlated with the infinity harmonic functions, we refer the reader to [7] and the references therein. Note that the formulas which define u^+ and u^- for 1-fields in this paper are similar to those of Whitney-McShane in the continuous case. The results of this paper allows to think that the notion of an AMLE of 1-field is not sufficient from all this point of view. Indeed, the minimal extensions u^+ and u^- are extremal like m^+ and m^- in the continuous case, but also they are two AMLE when the domain Ω is finite. Moreover they are not the “locally best” extensions since the Lischitz constant is not local. Despite these disappointing results, one might think that there exists some extensions that have good stability properties and “locally best” like harmonious extensions in the continuous case (for the definition of stability properties for 1-fields see [1]).

3.2 Preliminaries

In this paper all subsets $\Omega \subset \mathbb{R}^n$ satisfy $\#\Omega \geq 2$. If Ω is open, we denote by $\mathscr{C}^{1,1}(\Omega, \mathbb{R})$ the set of all real-valued function f that is differentiable on Ω and the differential ∇f is Lipschitz continuous, that is $\mathrm{Lip}(\nabla f; \Omega) < +\infty$. The 1-field F of domain Ω is defined by (see (3.1))

$$\begin{aligned} F : \Omega &\to \mathscr{P}^1(\mathbb{R}^n, \mathbb{R}) \\ x &\mapsto F(x)(a) = f_x + \langle D_x f; a - x \rangle, a \in \mathbb{R}^n \end{aligned} \tag{3.13}$$

with $f : x \in \Omega \mapsto f_x \in \mathbb{R}$, and $Df : x \in \Omega \mapsto D_x f \in \mathbb{R}^n$.

Definition 3.2.1. We call F to be a Taylorian field on Ω if F is a 1-field on Ω and $\Gamma^1(F,\Omega) < +\infty$. Denote by $\mathscr{F}^1(\Omega)$ the set of all Taylorian fields on Ω.

Information and precision for the reader : Let $F \in \mathscr{F}^1(\Omega)$. Let us define the map

$$f(x) := F(x)(x) = f_x,\ x \in \Omega.$$

Using [36, Theorem 1.1] there exists $\tilde{F} \in \mathscr{F}^1(\mathbb{R}^n)$ which extends F. Moreover $\tilde{f} \in \mathscr{C}^{1,1}(\mathbb{R}^n,\mathbb{R})$ and $\nabla f(x) := \nabla \tilde{f}(x) = D_x f$, $x \in \Omega$. Therefore, we can canonically associate F and f.

Let V be a subset of $\mathbb{R}^n$, $V \subset\subset \Omega$ means that $\overline{V}$ is compact in Ω, and $\overline{V}$ is the closure of V.

Let $x,y \in \mathbb{R}^n$. We define $B(x;r) := \{y \in \mathbb{R}^n : \|y-x\| < r\}$ and $B_{1/2}(x,y)$ is the closed ball of center $\frac{x+y}{2}$ and radius $\frac{\|x-y\|}{2}$.

The line segment joining two points x and y is denote by $[x,y]$, i.e. $[x,y] := \{tx + (1-t)y : 0 \le t \le 1\}$.

The $|$ symbol designates in restriction to.

Definition 3.2.2. Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ and $F \in \mathscr{F}^1(\Omega_1)$.

We call $G \in \mathscr{F}^1(\Omega_2)$ a *extension* of F on Ω_2 if $G(x) = F(x)$ for $x \in \Omega_1$.

We say that $G \in \mathscr{F}^1(\Omega_2)$ is a *minimal Lipschitz extension* (MLE) of F on Ω_2 if G is an extension of F on Ω_2 and

$$\Gamma^1(G;\Omega_2) = \Gamma^1(F;\Omega_1).$$

We say that $G_1 \in \mathscr{F}^1(\Omega_2)$ is an *over extremal Lipschitz extension* (over extremal for short) and G_2 is an *under extremal Lipschitz extension* of F on Ω_2 if G_1 and G_2 are MLEs of F on Ω_2 and

$$g_2(x) \le k(x) \le g_1(x),\ x \in \Omega_2,$$

for all K MLE of F.

We say that $G \in \mathscr{F}^1(\Omega_2)$ is an *absolutely minimal Lipschitz extension* (AMLE) of F on Ω_2 if G is a MLE of F on Ω_2 and

$$\Gamma^1(G;V) = \Gamma^1(G;\partial V),$$

for any bounded open V satisfying $\overline{V} \subset \Omega_2 \backslash \Omega_1$.

Definition 3.2.3. Let Ω be a subset of $\mathbb{R}^n$ and let $F \in \mathscr{F}^1(\Omega)$. For any $a \neq b \in \Omega$, we define

$$\begin{aligned} A_{a,b}(F) &:= \frac{2(f_a - f_b) + \langle D_a f + D_b f, b-a\rangle}{\|a-b\|^2}. \\ B_{a,b}(F) &:= \frac{\|D_a f - D_b f\|}{\|a-b\|}. \end{aligned}$$

We recall some results in [36] that will be useful in sections 3.3 and 3.4:

Proposition 3.2.4. *[36, Proposition 2.2 and remark 2.3] Let Ω be a subset of $\mathbb{R}^n$ and let $F \in \mathscr{F}^1(\Omega)$ then for any $a, b \in \Omega$, $a \neq b$ we have*

$$\Gamma^1(F;a,b) = \sqrt{A_{a,b}(F)^2 + B_{a,b}(F)^2} + \left|A_{a,b}(F)\right| = 2 \sup_{y \in B_{1/2}(a,b)} \frac{F(a)(y) - F(b)(y)}{\|a-y\|^2 + \|b-y\|^2}.$$

Theorem 3.2.5. *[36, Theorem 2.6] Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ and let $F \in \mathscr{F}^1(\Omega_1)$ then there exists a MLE $G \in \mathscr{F}^1(\Omega_2)$ of F on Ω_2.*

3.3 Relationships between $\Gamma^1(F;\Omega)$ and $\mathrm{Lip}(Df;\Omega)$

In this section Ω is an open subset of $\mathbb{R}^n$. Let $F \in \mathscr{F}^1(\Omega)$. From Proposition 3.2.4, we have

$$\Gamma^1(F;\Omega) \geq \mathrm{Lip}(Df;\Omega).$$

When $\Omega = \mathbb{R}^n$, we know that (see [36, Proposition 2.4])

$$\mathrm{Lip}(Df;\mathbb{R}^n) = \Gamma^1(F;\mathbb{R}^n),$$

but in general $\Gamma^1(F;\Omega)$ may be strictly bigger than $\mathrm{Lip}(Df;\Omega)$. For example, let A and B be open sets in $\mathbb{R}^n$ such that $A \cap B = \emptyset$. Let $\Omega = A \cup B$ and $F \in \mathscr{F}^1(\Omega)$ such that $f_x = 0$ if $x \in A$, $f_x = 1$ if $x \in B$, and $D_x f = 0$,$\forall x \in \Omega$. Then

$$\mathrm{Lip}(Df;\Omega) = 0,$$

and from Proposition 3.2.4 we have

$$\Gamma^1(F;\Omega) = \sup_{x \in A} \sup_{y \in B} \frac{4}{\|x-y\|^2} > 0.$$

We now give two new results where we have $\Gamma^1(F,\Omega) = \mathrm{Lip}(Df,\Omega)$.

Proposition 3.3.1. *Let $F \in \mathscr{F}^1(\Omega)$. Suppose there exist $a, b \in \Omega$, $a \neq b$ such that $\Gamma^1(F;a,b) = \Gamma^1(F;\Omega)$. Then $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

Proof. It is enough to prove that $\Gamma^1(F;\Omega) \leq \mathrm{Lip}(Df;\Omega)$.

Let $G = F|_{\{a,b\}}$ be a Taylorian field on $\mathrm{dom}(G) = \{a,b\}$ with $G(a) = F(a)$ and $G(b) = F(b)$.

Let U be a MLE of F on $\mathbb{R}^n$. We have

$$U(a) = F(a) = G(a),\ U(b) = F(b) = G(b),$$

and

$$\Gamma^1(U;\mathbb{R}^n) = \Gamma^1(F;\Omega) = \Gamma^1(F;a,b) = \Gamma^1(G;\mathrm{dom}(G)).$$

Therefore U is a MLE of G on $\mathbb{R}^n$.

Using [27, Lemma 8 and Lemma 10], there exists a point $c \in B_{1/2}(a,b)$ such that

$$\mathrm{Lip}(Du;x,y) = \mathrm{Lip}(Du;s,t) = \Gamma^1(G;a,b) = \Gamma^1(F;a,b), \tag{3.14}$$

for all $x, y \in [a,c]$ $(x \neq y)$ and $s,t \in [b,c]$ $(s \neq t)$.

Since $a \neq b$, we have $c \neq a$ or $c \neq b$. We can assume $c \neq a$. Because Ω is open, there exists $x \neq y \in [a,c] \cap \Omega$, thus from (3.14) we have

$$\mathrm{Lip}(Df;\Omega) \geq \mathrm{Lip}(Df;x,y) = \mathrm{Lip}(Du;x,y) = \Gamma^1(F;a,b) = \Gamma^1(F;\Omega).$$

□

Proposition 3.3.2. *Let $F \in \mathscr{F}^1(\Omega)$. Suppose there exists $\Omega' \subset\subset \Omega$ such that $\Gamma^1(F;\Omega') = \Gamma^1(F;\Omega)$. Then $\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega)$.*

Proof. It is enough to prove that $\Gamma^1(F;\Omega) \leq \mathrm{Lip}(Df;\Omega)$. Let $h > 0$, we define $\Lambda_h = \{(a,b) \in \overline{\Omega'} \times \overline{\Omega'} : |a-b| \geq h\}$ and $\Gamma^1_h(F;\overline{\Omega'}) = \sup\limits_{(a,b)\in\Lambda_h} \Gamma^1(F;a,b)$. Applying Proposition 3.2.4, the mapping $(a,b) \mapsto \Gamma^1(F;a,b)$ is continuous on Λ_h. Moreover, Λ_h is compact, thus there exists $(a_h,b_h) \in \Lambda_h$ such that

$$\Gamma^1(F;a_h,b_h) = \Gamma^1_h(F;\overline{\Omega'}). \tag{3.15}$$

Case 1. There exists $h > 0$ such that

$$\Gamma^1_h(F;\overline{\Omega'}) = \Gamma^1(F;\overline{\Omega'}). \tag{3.16}$$

From (3.15),(3.16) and the condition $\Gamma^1(F;\Omega') = \Gamma^1(F;\Omega)$, we have

$$\Gamma^1(F;a_h,b_h) = \Gamma^1(F;\Omega).$$

Applying Proposition 3.3.1 we have

$$\Gamma^1(F;\Omega) = \mathrm{Lip}(Df;\Omega).$$

Case 2. For all $h > 0$, we always have

$$\Gamma^1_h(F;\overline{\Omega'}) < \Gamma^1(F;\overline{\Omega'}). \tag{3.17}$$

Let $h = 1/n$, then for any $n \in \mathbb{N}$ there exists $(a_n,b_n) \in \Lambda_{1/n}$ such that

$$\Gamma^1(F;a_n,b_n) = \Gamma^1_{1/n}(F;\overline{\Omega'}).$$

Since $(a_n),(b_n) \subset \overline{\Omega'}$ and $\overline{\Omega'}$ is compact, there exist a subsequence (a_{n_k}) of (a_n) and a subsequence (b_{n_k}) of (b_n) such that (a_{n_k}) converges to an element a of $\overline{\Omega'}$ and (b_{n_k}) converges to an element b of $\overline{\Omega'}$.

If $a \neq b$ then

$$\Gamma^1(F;\overline{\Omega'}) = \lim_{k\to\infty} \Gamma^1_{1/n_k}(F;\overline{\Omega'}) = \lim_{k\to\infty} \Gamma^1(F;a_{n_k},b_{n_k}) = \Gamma^1(F;a,b).$$

But this is not possible because for $l = |a-b| > 0$ we deduce from (3.17) that

$$\Gamma^1(F;a,b) \leq \Gamma^1_l(F;\overline{\Omega'}) < \Gamma^1(F;\overline{\Omega'}).$$

Therefore, we must have $a = b$.

From the proof of [36, Proposition 2.4], we see that if $B_{1/2}(x,y) \subset \Omega$ then

$$\Gamma^1(F;x,y) \le \mathrm{Lip}(Df;\Omega).$$

We will use this property for proving in the case $a=b$.

For any $\varepsilon > 0$, since (a_{n_k}) and (b_{n_k}) are both converge to $a \in \overline{\Omega'} \subset \Omega$, there exists $k \in \mathbb{N}$ such that $B_{1/2}(a_{n_k}, b_{n_k}) \subset \Omega$ and

$$\varepsilon + \Gamma^1(F;a_{n_k},b_{n_k}) \ge \Gamma^1(F;\overline{\Omega'}) = \Gamma^1(F;\Omega).$$

Since $B_{1/2}(a_{n_k}, b_{n_k}) \subset \Omega$ we have

$$\Gamma^1(F;a_{n_k},b_{n_k}) \le \mathrm{Lip}(Df;\Omega).$$

Therefore

$$\varepsilon + \mathrm{Lip}(Df;\Omega) \ge \Gamma^1(F;\Omega).$$

This inequality holds for any $\varepsilon > 0$, so that we have $\mathrm{Lip}(Df;\Omega) \ge \Gamma^1(F;\Omega)$. $\square$

Proposition 3.3.3. *Let Ω be an open and convex set in $\mathbb{R}^n$ and let $F \in \mathscr{F}^1(\Omega)$. Then*

$$\Gamma^1(F;\Omega) \le 2\,\mathrm{Lip}(Df;\Omega).$$

Proof. Let f be the canonical associate to F. We can write

$$f(x) - f(y) = \int_0^1 \langle \nabla f(y + t(x-y)), x-y \rangle dt.$$

For any $x, y \in \Omega$ and $z \in \mathbb{R}^n$ we have

$$\begin{aligned}
&F(x)(z) - F(y)(z) \\
&= f(x) - f(y) + \langle \nabla f(x), z-x \rangle - \langle \nabla f(y), z-y \rangle \\
&= \int_0^1 \langle \nabla f(y+t(x-y)) - \nabla f(x), x-z \rangle dt + \int_0^1 \langle \nabla f(y+t(x-y)) - \nabla f(y), z-y \rangle dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&|F(x)(z) - F(y)(z)| \\
&\le \int_0^1 \mathrm{Lip}(\nabla f;\Omega)\|x-y\|\|x-z\|(1-t)dt + \int_0^1 \mathrm{Lip}(\nabla f;\Omega)\|(x-y)\|\|z-y\|t dt \\
&= \frac{1}{2}\mathrm{Lip}(\nabla f;\Omega)\|x-y\|(\|x-z\| + \|z-y\|) \\
&\le \frac{1}{2}\mathrm{Lip}(\nabla f;\Omega)(\|x-z\| + \|z-y\|)^2 \\
&\le \mathrm{Lip}(\nabla f;\Omega)\left(\|x-z\|^2 + \|z-y\|^2\right).
\end{aligned}$$

Therefore $\Gamma^1(F;\Omega) \le 2\,\mathrm{Lip}(\nabla f;\Omega) = 2\,\mathrm{Lip}(Df;\Omega)$. $\square$

From the proof of Proposition 3.3.3, we obtain

Corollary 3.3.4. *Let Ω be an open and convex set in $\mathbb{R}^n$ and $f \in \mathscr{C}^{1,1}(\Omega,\mathbb{R})$. Then $F \in \mathscr{F}^1(\Omega)$ where F is the 1-field associated to f.*

Lemma 3.3.5. *Let $u \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ then*

$$\mathrm{Lip}(\nabla u; x,y) \leq \inf_{z\in[x,y]} \max\{\mathrm{Lip}(\nabla u;x,z), \mathrm{Lip}(\nabla u;z,y)\}, \text{ for all } x \neq y \in \mathbb{R}^n.$$

Proof. Let $x,y \in \Omega$ and $z \in [x,y]$ then we have $\|x-y\| = \|x-z\| + \|z-y\|$. It follows

$$\begin{aligned}
\mathrm{Lip}(\nabla u;x,y) &\leq \frac{\|x-z\|}{\|x-z\|+\|y-z\|}\mathrm{Lip}(\nabla u;x,z) + \frac{\|z-y\|}{\|x-z\|+\|y-z\|}\mathrm{Lip}(\nabla u;z,y) \\
&\leq \max\left\{\mathrm{Lip}(\nabla u;x,z), \mathrm{Lip}(\nabla u;z,y)\right\}.
\end{aligned}$$

□

Proposition 3.3.6. *There exist an open convex Ω and $F \in \mathscr{F}^1(\Omega)$ such that*

$$\mathrm{Lip}(\nabla f;\Omega) < \Gamma^1(F;\Omega).$$

Proof. Suppose $n=2$. Let $a=(-1,0), b=(1,0)$. We define $U \in \mathscr{F}^1(\{a,b\})$ as

$$D_a u = (0,1),\ D_b u = (0,-1),\ u_a = \frac{1}{\sqrt{3}},\ u_b = -\frac{1}{\sqrt{3}}.$$

Let us define

$$\kappa := \Gamma^1(U;\{a,b\}),\ h := \frac{b-a}{2},$$

$$v := \frac{D_a u - D_b u}{2\kappa}, \text{ and } \beta := \|v\| \in (0,1).$$

Using [27, Lemma 7,8], we define

$$\begin{aligned}
&c = \frac{a+b}{2} + v, \\
&\widetilde{u}_c = U(a)(c) - \frac{\kappa}{2}\|a-c\|^2 = U(b)(c) + \frac{\kappa}{2}\|b-c\|^2, \\
&D_c\widetilde{u} = D_a u + \kappa(a-c) = D_b u - \kappa(b-c), \\
&\widetilde{U}(c)(z) = \widetilde{u}_c + \langle D_c\widetilde{u}, z-c\rangle,\ z \in \mathbb{R}^n,
\end{aligned}$$

and

$$f(z) := \begin{cases}
\widetilde{U}(c)(z) - \dfrac{\kappa}{2}\dfrac{\langle z-c, a-c\rangle^2}{\|a-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \leq 0, \\
\widetilde{U}(c)(z) + \dfrac{\kappa}{2}\dfrac{\langle z-c, b-c\rangle^2}{\|b-c\|^2}, & \text{if } p(z) \leq 0 \text{ and } q(z) \geq 0, \\
\widetilde{U}(c)(z), & \text{if } p(z) \leq 0 \text{ and } q(z) \leq 0, \\
\widetilde{U}(c)(z) - \dfrac{\kappa}{2}\dfrac{\langle z-c, a-c\rangle^2}{\|a-c\|^2} + \dfrac{\kappa}{2}\dfrac{\langle z-c, b-c\rangle^2}{\|b-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \geq 0,
\end{cases}$$

where $p(z) = \langle a-c, z-c\rangle$ and $q(z) = \langle b-c, z-c\rangle$.

Then the 1-field F which is associated with f is a MLE of U.

Since $\|h\| = 1$, $\beta = \|v\| \in (0,1)$, $c-a = h+v$, and $c-b = -h+v$, we have

$$\langle c-a, w_a\rangle = 0,\ \langle c-b, w_b\rangle = 0. \tag{3.18}$$

where

$$w_a = h - \frac{1}{\beta^2}v,\ w_b = -h - \frac{1}{\beta^2}v.$$

We choose $\alpha_0 \in (0,+\infty)$ (see Figure 1) such that $x_a, x_b \in$ convex hull$\{a,c,b\}$, where

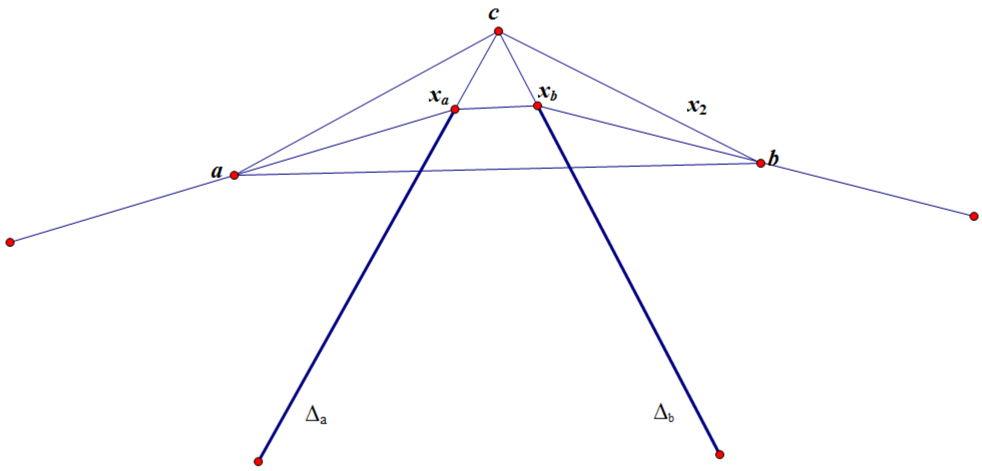

Figure 3.1: Illustration of subsets Δ_a , Δ_b of $\mathbb{R}^2$.

$$x_a = c + \alpha_0 w_b,\ \ x_b = c + \alpha_0 w_a.$$

Let us define (see Figure 1)

$$\begin{aligned} \Delta_a &= \{x \in \mathbb{R}^2 : x = c + \alpha w_b, \alpha \geq \alpha_0\}, \\ \Delta_b &= \{x \in \mathbb{R}^2 : x = c + \alpha w_a, \alpha \geq \alpha_0\}. \end{aligned}$$

Step 1. We will prove there exists $k \in (0,1)$ which depends on β and α_0 such that

$$\max_{i=1,2,3} \Gamma^1(F;\omega_i) \leq k\kappa,$$

where

$$\omega_1 = \Delta_a \cup \{a\},\ \omega_2 = \Delta_a \cup \Delta_b,\ \omega_3 = \Delta_b \cup \{b\}.$$

Using [27, Lemma 9] we have

$$A_{x,x'}(F) = 0,\ \forall i \in \{1,2,3\},\ \forall x \neq x' \in \omega_i. \tag{3.19}$$

Using (3.18) and the definition of F and by noting that

$$\|h\| = 1,\ \beta = \|v\| \in (0,1),$$

$$c-a=h+v, \text{ and } c-b=-h+v,$$

it is easy to calculate the following expressions

If $x,x' \in \Delta_a$, then

$$\frac{1}{\kappa}\mathrm{Lip}(\nabla f;x,x') = \frac{\|\langle x-x',a-c\rangle\|}{\|x-x'\|\|a-c\|} = 1-\frac{(\beta-1)^2}{1+\beta^2}. \tag{3.20}$$

If $x,x' \in \Delta_b$, then

$$\frac{1}{\kappa}\mathrm{Lip}(\nabla f;x,x') = \frac{\|\langle x-x',b-c\rangle\|}{\|x-x'\|\|b-c\|} = 1-\frac{(\beta-1)^2}{1+\beta^2}. \tag{3.21}$$

If $x \in \Delta_a$, $x' \in \Delta_b$, then

$$\begin{aligned}\frac{1}{\kappa}\mathrm{Lip}(\nabla f;x,x') &= \left\|\frac{-\langle x-x',c-a\rangle(c-a)}{\|x-x'\|\|c-a\|^2}+\frac{\langle x-x',c-b\rangle(c-b)}{\|x-x'\|\|c-b\|^2}\right\| \\ &= 1-\frac{(\beta-1)^2}{1+\beta^2}.\end{aligned} \tag{3.22}$$

If $x=c+\alpha w_b \in \Delta_a$, then

$$\begin{aligned}\frac{1}{\kappa}\mathrm{Lip}(\nabla f;x,a) &= \frac{\|\langle x-a,a-c\rangle\|}{\|x-a\|\|a-c\|} \\ &= 1-\frac{\alpha^2(\beta^2-1)^2}{(1+\beta^2)(\beta^2(1-\alpha)^2+(\beta^2-\alpha)^2)}.\end{aligned} \tag{3.23}$$

If $x=c+\alpha w_a \in \Delta_b$, then

$$\begin{aligned}\frac{1}{\kappa}\mathrm{Lip}(\nabla f;x,b) &= \frac{\|\langle x-b,b-c\rangle\|}{\|x-b\|\|b-c\|} \\ &= 1-\frac{\alpha^2(\beta^2-1)^2}{(1+\beta^2)(\beta^2(1-\alpha)^2+(\beta^2-\alpha)^2)}.\end{aligned} \tag{3.24}$$

From (3.19),...,(3.24), Proposition 3.2.4 and noting that

$$1-\frac{\alpha^2(\beta^2-1)^2}{(1+\beta^2)(\beta^2(1-\alpha)^2+(\beta^2-\alpha)^2)} \le 1-\frac{\alpha_0^2(\beta^2-1)^2}{(1+\beta^2)(\beta^2(1-\alpha_0)^2+(\beta^2-\alpha_0)^2)},$$

for all $\alpha \ge \alpha_0$, we obtain

$$\max_{i=1,2,3}\Gamma^1(F;\omega_i) \le k\kappa, \tag{3.25}$$

where

$$k=\sup\{1-\frac{(\beta-1)^2}{1+\beta^2}, 1-\frac{\alpha_0^2(\beta^2-1)^2}{(1+\beta^2)(\beta^2(1-\alpha_0)^2+(\beta^2-\alpha_0)^2)}\}. \tag{3.26}$$

Step 2. We will define an open convex Ω and a 1-field $G \in \mathscr{F}^1(\Omega)$ such that

$$\mathrm{Lip}(\nabla g;\Omega) < \Gamma^1(G;\Omega).$$

Let us use the notation

$$R_{x,y} = \{x + t(y-x) \; : \; t \in \mathbb{R}^+\}.$$

Define (see Figure 2)

$A_1 = \text{convex hull}(\Delta_a \cup R_{x_a,a})$, $A_2 = \text{convex hull}(\Delta_a \cup \Delta_b)$, $A_3 = \text{convex hull}(R_{x_b,b} \cup \Delta_b)$, and Ω be the interior of $A_1 \cup A_2 \cup A_3$. Then Ω open and convex. Then $\overline{\Omega} = A_1 \cup A_2 \cup A_3$ is convex.

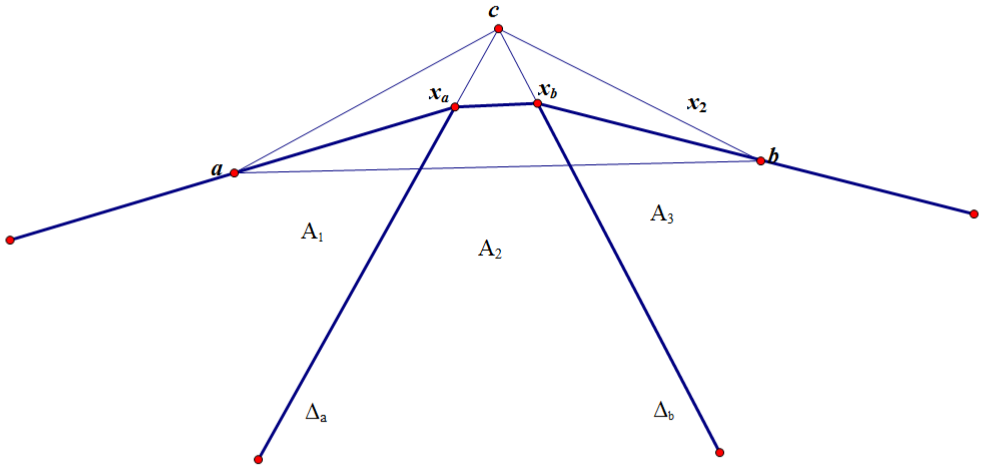

Figure 3.2: Illustration of 1-field G.

For each $i \in \{1,2,3\}$, let us consider G_i be a MLE of $F|_{\omega_i}$. We define a 1-field G on $\overline{\Omega}$ by

$$G(x) = G_i(x), \text{ for } \; x \in A_i, \text{ for } i \in \{1,2,3\}.$$

Let us show that $g \in \mathscr{C}^1(\Omega, \mathbb{R})$. Indeed, for all $x \in \Omega$, there exists $r > 0$ such that $B(x,r) \subset \Omega$. For all $h \in B(x,r)$, we have

$$\begin{aligned} |g_{x+h} - g_x - \langle D_x g; h\rangle| &= |G(x+h)(x+h) - G(x)(x+h)|, \\ &\leq \frac{1}{2} \max_{i=1,2,3} \Gamma^1(G_i; A_i)\|h\|^2. \end{aligned}$$

Hence $g \in \mathscr{C}^1(\Omega, \mathbb{R})$. Thus, by applying Lemma 3.3.5, Proposition 3.2.4 and (3.25) we have

$$\text{Lip}(\nabla g; \Omega) \leq \max_{i=1,2,3} \text{Lip}(\nabla g; A_i) \leq \max_{i=1,2,3} \Gamma^1(F; \omega_i) \leq k\kappa. \tag{3.27}$$

Therefore, by applying Corollary 3.3.4 we have $G \in \mathscr{F}^1(\Omega)$. On the other hand, we have

$$\Gamma^1(G; \Omega) = \Gamma^1(G; \overline{\Omega}) \geq \Gamma^1(G; a, b) = \kappa. \tag{3.28}$$

From (3.27) and (3.28). We have

$$\text{Lip}(\nabla g; \Omega) \leq k\kappa \leq k\Gamma^1(G; \Omega) < \Gamma^1(G; \Omega).$$

$\square$

Remark 3.3.7. With the same notation as the proof of Proposition 3.3.6. There exist an open strictly convex Ω' subset of Ω such that $a,b \in \overline{\Omega'}$, and we have

$$\operatorname{Lip}(\nabla g;\Omega') \leq \operatorname{Lip}(\nabla g;\Omega) \leq k\kappa \leq k\Gamma^1(G;\overline{\Omega'}) = k\Gamma^1(G;\Omega') < \Gamma^1(G;\Omega').$$

Thus in Proposition 3.3.6 we can replace Ω *open convex* by Ω *open strictly convex*.

Moreover, when we let x_a, x_b such that $\operatorname{dist}(x_a,[a,b])$ and $\operatorname{dist}(x_b,[a,b])$ converge to 0. Then the constant k satisfying (3.26) converges to $\frac{\sqrt{3}}{2}$. An interesting question is that what is the optimal constant c that is the largest constant and satisfies $\operatorname{Lip}(\nabla g,\Omega) \geq c\Gamma^1(F,\Omega)$ for all Ω open convex set and for all $F \in \mathscr{F}^1(\Omega)$? We do not exact value of the optimal constant c, but from above consideration and Proposition 3.3.3, we obtain $c \in [\frac{1}{2}, \frac{\sqrt{3}}{2}]$.

Theorem 3.3.8. *Let Ω be an open set in $\mathbb{R}^n$ and let $F \in \mathscr{F}^1(\overline{\Omega})$. We have*

$$\Gamma^1(F;\Omega) = \max\left\{\operatorname{Lip}(Df;\Omega), \Gamma^1(F;\partial\Omega)\right\}.$$

Proof. From [36, Proposition 2.10] We have $\Gamma^1(F;\Omega) = \Gamma^1(F;\overline{\Omega})$. Thus

$$\Gamma^1(F;\Omega) \geq \Gamma^1(F;\partial\Omega). \tag{3.29}$$

Furthermore, we know that $\Gamma^1(F;\Omega) \geq \operatorname{Lip}(Df;\Omega)$. Therefore,

$$\Gamma^1(F;\Omega) \geq \max\left\{\operatorname{Lip}(Df;\Omega), \Gamma^1(F;\partial\Omega)\right\}.$$

Conversely, let us turn to the proof of the opposite inequality:

$$\Gamma^1(F;\Omega) \leq \max\left\{\operatorname{Lip}(Df;\Omega), \Gamma^1(F;\partial\Omega)\right\}. \tag{3.30}$$

Let $F_{|\partial\Omega}$ be the restriction of F to $\partial\Omega$ and let G be a MLE of $F_{|\partial\Omega}$ on $\mathbb{R}^n\backslash\Omega$. We have $G = F$ on $\partial\Omega$ and

$$\Gamma^1(G;\mathbb{R}^n\backslash\Omega) = \Gamma^1(F;\partial\Omega). \tag{3.31}$$

We define

$$H(x) := \begin{cases} F(x), & \text{if } x \in \Omega, \\ G(x), & \text{if } x \in \mathbb{R}^n\backslash\Omega. \end{cases}$$

Step 1. We will prove that $H \in \mathscr{F}^1(\mathbb{R}^n)$. Indeed, let $x,y \in \mathbb{R}^n$ ($x \neq y$). We have three cases:

Case 1. If $x,y \in \Omega$ ($x \neq y$) then

$$\Gamma^1(H;x,y) = \Gamma^1(F;x,y) \leq \Gamma^1(F;\Omega).$$

Case 2. If $x,y \in \mathbb{R}^n\backslash\Omega$ ($x \neq y$) then since (3.29) and (3.31) we have

$$\Gamma^1(H;x,y) = \Gamma^1(G;x,y) \leq \Gamma^1(G,\mathbb{R}^n\backslash\Omega) = \Gamma^1(F;\partial\Omega) \leq \Gamma^1(F,\Omega).$$

Case 3. If $x \in \Omega$ and $y \in \mathbb{R}^n \backslash \Omega$. Let $H_{|\{x,y\}}$ be the restriction of H to $\mathrm{dom}(H_{|\{x,y\}}) = \{x,y\}$. From [27, Proposition 2], there exists $c \in B_{1/2}(x,y)$ such that:

$$\Gamma^1(H;x,y) \leq \max\{\Gamma^1(H;x,z), \Gamma^1(H;z,y)\}, \text{ for all } z \in [x,c] \cup [y,c].$$

Let $z \in ([x,c] \cup [y,c]) \cap \partial\Omega$, we obtain

$$\Gamma^1(H;x,y) \leq \max\{\Gamma^1(H;x,z), \Gamma^1(H;z,y)\}.$$

Moreover, since $x, z \in \overline{\Omega}$ we get

$$\Gamma^1(H;x,z) \quad = \quad \Gamma^1(F;x,z) \leq \Gamma^1(F;\overline{\Omega}) = \Gamma^1(F;\Omega),$$

and since $y, z \in \mathbb{R}^n \backslash \Omega$ we get

$$\Gamma^1(H;z,y) \quad = \quad \Gamma^1(G;z,y) \leq \Gamma^1(G;\mathbb{R}^n \backslash \Omega) = \Gamma^1(F;\partial\Omega) \leq \Gamma^1(F;\Omega).$$

Therefore

$$\Gamma^1(H;x,y) \leq \Gamma^1(F;\Omega).$$

Combining these three cases we have

$$\Gamma^1(H;\mathbb{R}^n) \leq \Gamma^1(F;\Omega) < +\infty.$$

This implies that $H \in \mathscr{F}^1(\mathbb{R}^n)$.

Step 2. We will prove (3.30). Since $H \in \mathscr{F}^1(\mathbb{R}^n)$, we have $\Gamma^1(H;\mathbb{R}^n) = \mathrm{Lip}(\nabla h;\mathbb{R}^n)$ by ([36, Proposition 2.4]). Thus

$$\Gamma^1(F;\Omega) = \Gamma^1(H;\Omega) \leq \Gamma^1(H;\mathbb{R}^n) = \mathrm{Lip}(\nabla h;\mathbb{R}^n).$$

On the other hand, $\mathrm{Lip}(Df;\Omega) = \mathrm{Lip}(\nabla h;\Omega)$ and

$$\Gamma^1(F;\partial\Omega) = \Gamma^1(G;\mathbb{R}^n \backslash \Omega) = \Gamma^1(H;\mathbb{R}^n \backslash \Omega) \geq \mathrm{Lip}(\nabla h;\mathbb{R}^n \backslash \Omega).$$

Therefore, to prove (3.30), it suffices to show that

$$\mathrm{Lip}(\nabla h;\mathbb{R}^n) \leq \max\left\{\mathrm{Lip}(\nabla h;\Omega), \mathrm{Lip}(\nabla h;\mathbb{R}^n \backslash \Omega)\right\}.$$

The final inequality is true from Lemma 3.3.5. □

3.4 Sup-Inf explicit minimal Lipschitz extensions for 1-Fields

In this section let Ω be a subset of $\mathbb{R}^n$ and $F \in \mathscr{F}^1(\Omega)$. To better understand the sections 3.4 and 3.6, let us recall some selected results in [54, Theorem 1,2].

Definition 3.4.1. The real $\kappa \in \mathbb{R}$, with $\kappa > 0$, is allowable for F if κ satisfies the following inequalities

$$f_y \leq f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{\kappa}{4}(x-y)^2 - \frac{1}{4\kappa}(D_x f - D_y f)^2, \forall x, y \in \Omega. \quad (3.32)$$

Lemma 3.4.2. *The real κ is allowable for F iff $\Gamma^1(F;\Omega) \le \kappa$.*

Proof. Applying Proposition 3.2.4, we have $\Gamma^1(F;\Omega) \le \kappa$ if and only if

$$\sqrt{A_{x,y}^2(F) + B_{x,y}^2(F)} + A_{x,y}(F) \le \kappa, \ \forall x, y \in \Omega,$$

This inequality is equivalent to

$$\frac{B_{x,y}^2(F)}{2\kappa} + A_{x,y}(F) \le \frac{\kappa}{2},$$

and hence it is equivalent to

$$f_y \le f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{1}{4}\kappa\|y-x\|^2 - \frac{1}{4\kappa}\|D_x f - D_y f\|^2, \text{ for any } x, y \in \Omega.$$

□

Definition 3.4.3. Let κ be allowable for F. Denote by $w^+(F,\Omega,\kappa)$ the Wells function which is described in [54, Theorem 1] (or see in Appendix 3.8) when Ω is finite and

$$w^+(F,\Omega,\kappa)(x) = \inf_{P \in \mathscr{P}(\Omega)} w^+(F,P,\kappa)(x),$$

when Ω is infinite, where $\mathscr{P}(\Omega)$ is the set of all finite subsets of Ω.

The following corollaries are the direct consequences of the Lemma 3.4.2, [36, Proposition 2.4] and [54, Theorem 1,2].

Corollary 3.4.4. *$W^+(F,\Omega,\kappa)$ is an extension of F and $\Gamma^1(W^+(F,\Omega,\kappa);\mathbb{R}^n) \le \kappa$. Moreover for any extension G of F on $\mathbb{R}^n$ which satisfies $\Gamma^1(G;\mathbb{R}^n) \le \kappa$ we have*

$$g(x) \le w^+(F,\Omega,\kappa)(x), \ x \in \mathbb{R}^n.$$

Corollary 3.4.5. *If $\Omega \subset \Omega_1 \subset \Omega_2$, then*

$$w^+(F,\Omega_2,\kappa)(x) \le w^+(F,\Omega_1,\kappa)(x), \ x \in \mathbb{R}^n.$$

Corollary 3.4.6. *If $\kappa = \Gamma^1(F;\Omega)$ then $w^+(F,\Omega,\kappa)$ is an over minimal Lipschitz extension of F.*

In the remainder of this section, we define $\kappa := \Gamma^1(F;\Omega)$. We will give two explicit formulas for extremal extension problem of F on $\mathbb{R}^n$.

Definition 3.4.7. For any $a, b \in \Omega$ and $x \in \mathbb{R}^n$, we define

$$\begin{aligned}
v_{a,b} &:= \frac{1}{2}(D_a f + D_b f) + \frac{\kappa}{2}(b-a),\\
\alpha_{a,b} &:= 2\kappa(f_a - f_b) + \kappa\langle D_a f + D_b f, b-a\rangle - \frac{1}{2}\|D_a f - D_b f\|^2 + \frac{\kappa^2}{2}\|a-b\|^2\\
&= (\kappa A_{a,b}(F) - \frac{B_{a,b}(F)^2}{2} + \frac{\kappa^2}{2})\|a-b\|^2,\\
\beta_{a,b}(x) &:= \left\|\frac{1}{2}(D_a f - D_b f) + \frac{\kappa}{2}(2x - a - b)\right\|^2.
\end{aligned}$$

Clear from proof of Lemma 3.4.2, we know that $\alpha_{a,b} \geq 0$ thus we can define

$$\begin{aligned} r_{a,b}(x) &:= \sqrt{\alpha_{a,b} + \beta_{a,b}(x)}, \\ \Lambda_x &:= \left\{ v \in \mathbb{R}^n : \|v - v_{a,b}\| \leq r_{a,b}(x), \forall a,b \in \Omega \right\}. \end{aligned}$$

Definition 3.4.8. For any $a \in \Omega$, $x \in \mathbb{R}^n$ and $v \in \Lambda_x$ we define

$$\begin{aligned} \Psi^+(F,x,a,v) &:= f_a + \frac{1}{2}\langle D_a f + v, x - a\rangle + \frac{\kappa}{4}\|a - x\|^2 - \frac{1}{4\kappa}\|D_a f - v\|^2, \\ \Psi^-(F,x,a,v) &:= f_a + \frac{1}{2}\langle D_a f + v, x - a\rangle - \frac{\kappa}{4}\|a - x\|^2 + \frac{1}{4\kappa}\|D_a f - v\|^2, \\ u^+(x) &:= \sup_{v \in \Lambda_x} \inf_{a \in \Omega} \Psi^+(F,x,a,v). \end{aligned}$$

An important part of the proof of [36, Theorem 2.6] shows that Λ_x is non-empty for all $x \in \mathbb{R}^n$. This allows us to define u^+.

The map u^+ is well defined. Indeed, from the proof of [36, Theorem 2.6] we have

$$-\infty < u(x) \leq \inf_{a \in \Omega} \Psi^+(F,x,a,v),$$

for any U MLE of F on $\mathbb{R}^n$.

Thus $-\infty < u^+(x)$.

Moreover since Λ_x is compact, and the map $v \in \mathbb{R}^n \longmapsto \Psi^+(F,x,a,v)$ is continuous for any $a \in \Omega$, we have

$$\sup_{v \in \Lambda_x} \Psi^+(F,x,a,v) < +\infty$$

for any $a \in \Omega$.

Thus $u^+(x) < +\infty$. Therefore u^+ is well defined.

Proposition 3.4.9. *Fix $x \in \mathbb{R}^n$. Then there exists a unique element $v_x^+ \in \Lambda_x$ such that*

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F,x,a,v_x^+).$$

Proof. Since Λ_x is compact and non-empty, there exists $v_x^+ \in \Lambda_x$ such that

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F,x,a,v_x^+).$$

We will prove that v_x^+ is uniquely determined. Indeed, for any $a \in \Omega$ we define

$$g_a(v) = \Psi^+(F,x,a,v), \text{ for } v \in \Lambda_x.$$

Then for any $t \in (0,1)$ and $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$g_a(tv_1 + (1-t)v_2) = tg_a(v_1) + (1-t)g_a(v_2) + \frac{1}{4\kappa}t(1-t)\|v_1 - v_2\|^2.$$

Thus g_a is strictly concave.

If we define $g(v) = \inf_{a\in\Omega} g_a(v)$ for $v \in \Lambda_x$, then for any $t \in (0,1)$ and $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ we have

$$g(tv_1 + (1-t)v_2) \geq tg(v_1) + (1-t)g(v_2) + \frac{1}{4\kappa}t(1-t)\|v_1 - v_2\|^2.$$

Thus g is also strictly concave.
To prove v_x^+ is uniquely determined, we need to prove that if $g(v) = g(v_x^+)$ then $v = v_x^+$, where $v \in \Lambda_x$.

Assume by contradiction there exists $v \in \Lambda_x$ such that $v \neq v_x^+$ and $g(v) = g(v_x^+)$.

Since Λ_x is a convex subset of $\mathbb{R}^n$, we have

$$tv + (1-t)v_x^+ \in \Lambda_x,$$

for $t \in (0,1)$.

Thus

$$g(tv + (1-t)v_x^+) > tg(v) + (1-t)g(v_x^+) = g(v_x^+).$$

which contradicts the equality $g(v_x^+) = \sup_{v\in\Lambda_x} g(v)$. □

The previous proposition allows to define the following 1-field

Definition 3.4.10.

$$U^+ : x \in \mathbb{R}^n \mapsto U^+(x)(y) := u^+(x) + \langle D_x u^+ ; y - x\rangle,\ y \in \mathbb{R}^n, \tag{3.33}$$

with

$$u^+(x) := \max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F, x, a, v),\ D_x u^+ := \arg\max_{v\in\Lambda_x} \inf_{a\in\Omega} \Psi^+(F, x, a, v).$$

Using the proof of [36, Theorem 2.6] , we can easily show the following proposition

Proposition 3.4.11. *Let $x_0 \in \mathbb{R}^n$ and define $\Omega_1 = \Omega \cup \{x_0\}$. Let U an extension of F on Ω_1. Then the following conditions are equivalent*
(i) U is a MLE of F on Ω_1.
(ii)

$$\sup_{a\in\Omega} \Psi^-(F, x, a, D_x u) \leq u(x) \leq \inf_{a\in\Omega} \Psi^+(F, x, a, D_x u),\ \forall x \in \Omega_1.$$

Furthermore

$$\left[\sup_{a\in\Omega} \Psi^-(F, x, a, D_x u) \leq \inf_{a\in\Omega} \Psi^+(F, x, a, D_x u)\right] \Leftrightarrow [D_x u \in \Lambda_x],\ \forall x \in \Omega_1.$$

Corollary 3.4.12. *Let Ω_1 be a subset of $\mathbb{R}^n$ such that $\Omega \subset \Omega_1$. Let G be a MLE of F on Ω_1. For all $x \in \Omega_1$, we have $D_x g \in \Lambda_x$ and*

$$\sup_{a\in\Omega} \Psi^-(F, x, a, D_x g) \leq g(x) \leq \inf_{a\in\Omega} \Psi^+(F, x, a, D_x g) \leq u^+(x),\ \forall x \in \Omega_1.$$

Proof. The proof is immediate from Proposition 3.4.11. □

Theorem 3.4.13. *The* 1*-field* U^+ *is the unique over extremal extension of* F.

Proof. Applying Corollary 3.4.6, $W^+(F;\Omega,\kappa)$ is an over extremal extension of F on $\mathbb{R}^n$. Let w^+ be an over extremal extension of F on $\mathbb{R}^n$. Let W^+ be the 1-field canonical associated to w^+. We will prove $U^+ = W^+$ on $\mathbb{R}^n$.

Step 1. Let $x \in \Omega$. Since W^+ is an extension of F we have $W^+(x) = F(x)$.

Noting that Λ_x has a unique element to be $D_x f$ (since $x \in \Omega$ and from the definition of Λ_x). So that $D_x u^+ = D_x f$ and

$$u^+(x) \quad = \quad \inf_{a\in\Omega} \Psi^+(F,x,a,D_x f).$$

From Proposition 3.4.11 we have

$$\Psi^+(F,x,a,D_x f) \geq f(x), \text{ for any } a \in \Omega.$$

Furthermore, when $a = x$ we have

$$\Psi^+(F,x,x,D_x f) = f(x).$$

Therefore

$$u^+(x) = f(x).$$

Conclusion for all $x \in \Omega$,

$$U^+(x) = W^+(x) = F(x).$$

Step 2. Let $x \in \mathbb{R}^n \backslash \Omega$. We first prove that $u^+(x) \geq w^+(x)$.

Since W^+ is a MLE of F on $\mathbb{R}^n$, we can apply Proposition 3.4.11 to obtain $D_x w \in \Lambda_x$ and

$$w^+(x) \leq \inf_{a\in\Omega} \Psi^+(F,x,a,D_x w) \leq u^+(x).$$

Conversely, we will prove that $u^+(x) \leq w^+(x)$.

Applying Proposition 3.4.9, $D_x u^+$ is the unique element in Λ_x such that

$$u^+(x) = \inf_{a\in\Omega} \Psi^+(F,x,a,D_x u^+).$$

We define the 1-field G of domain $\Omega \cup \{x\}$ as

$$G(y) := F(y),\ y \in \Omega \text{ and } G(x) := U^+(x).$$

Since $D_x g = D_x u^+ \in \Lambda_x$, we can apply Proposition 3.4.11 to have

$$g(x) = u^+(x) = \inf_{a\in\Omega} \Psi^+(F,x,a,D_x g) \geq \sup_{a\in\Omega} \Psi^-(F,x,a,D_x g). \tag{3.34}$$

From (3.34) and Proposition 3.4.11, we have G to be a MLE of F on $\Omega \cup \{x\}$.

By applying [36, Theorem 2.6] there exists $\tilde{G}$ to be a MLE of G on $\mathbb{R}^n$.

Since $\mathrm{dom}(F) \subset \mathrm{dom}(G)$, $\tilde{G}$ is also a MLE of F on $\mathbb{R}^n$.

Since W^+ is over extremal extension of F on $\mathbb{R}^n$, we have

$$\tilde{g}(x) \leq w^+(x),\ \forall x \in \mathbb{R}^n.$$

Thus

$$u^+(x) = g(x) = \tilde{g}(x) \leq w^+(x).$$

Combining this with $w^+(x) \leq u^+(x)$ we have

$$u^+(x) = w^+(x).$$

Finally, using Proposition 3.4.11 and the previous equality we have

$$u^+(x) = w^+(x) \leq \inf_{a\in\Omega} \Psi^+(F,x,a,D_x w^+) \leq u^+(x) = \inf_{a\in\Omega} \Psi^+(F,x,a,D_x u^+).$$

Thus we obtain the following equality

$$\inf_{a\in\Omega} \Psi^+(F,x,a,D_x w^+) = \inf_{a\in\Omega} \Psi^+(F,x,a,D_x u^+).$$

Therefore $D_x w^+ = D_x u^+$ by Proposition 3.4.9. Conclusion for all $x \in \mathbb{R}^n$, $U^+(x) = W^+(x)$.

The uniqueness of over extremal extension of F arises since W^+ is an arbitrary over extremal extension of F. □

Proposition 3.4.14. *Let Ω_1 be a subset of $\mathbb{R}^n$ such that $\Omega \subset \Omega_1$. Let G be an over extremal extension of F on Ω_1. Then $g_x = u^+(x)$ and $D_x g = \nabla u^+(x)$ for all $x \in \Omega_1$.*

Proof. We first prove that $g_x = u^+(x)$ for all $x \in \Omega_1$.

Indeed, since G is an over extremal extension of F on Ω_1 and since $u^+_{|\Omega_1}$ is MLE of F on Ω_1, we have

$$u^+(x) \leq g_x\ \forall x \in \Omega_1.$$

Conversely, by applying Proposition 3.2.5, there exists $\tilde{G}$ is a MLE of G on $\mathbb{R}^n$.

Since G is MLE of F on Ω_1, we have $\tilde{G}$ is a MLE of F on $\mathbb{R}^n$.

By Theorem 3.4.13 we know that u^+ is an over extremal extension of F on $\mathbb{R}^n$, thus

$$\tilde{g}(x) \leq u^+(x),\ \forall x \in \mathbb{R}^n.$$

Thus

$$g_x = \tilde{g}(x) \leq u^+(x), \forall x \in \Omega_1.$$

And thus

$$g_x = u^+(x),\ \forall x \in \Omega_1.$$

We will prove that $D_x g = \nabla u^+(x)$ for all $x \in \Omega_1$.

Fix $x \in \Omega_1$, by applying Proposition 3.4.12 we have $D_x g \in \Lambda_x$ and

$$g_x \leq \inf_{a\in\Omega} \Psi^+(F,x,a,D_x g) \leq u^+(x).$$

Since $u^+(x) = g_x$, we have

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, D_x g).$$

By applying Proposition 3.4.9, to prove $D_x g = \nabla u^+(x)$ we need to prove

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, \nabla u^+(x)).$$

By Proposition 3.4.12, since $U^+_{|\Omega_1}$ is a MLE of F on Ω_1 we have

$$\nabla u^+(x) \in \Lambda_x$$

and

$$u^+(x) \leq \inf_{a \in \Omega} \Psi^+(F, x, a, \nabla u^+(x)) \leq u^+(x).$$

Thus

$$u^+(x) = \inf_{a \in \Omega} \Psi^+(F, x, a, \nabla u^+(x))$$

as desire.

□

Proposition 3.4.15. *Let $\Omega \subset \Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$. Let G be an over extremal extension of F on Ω_1 and let K be an over extremal extension of G on Ω_2 then $k_x = u^+(x)$ and $D_x k = \nabla u^+(x)$ for all $x \in \Omega_2$.*

Proof. We first prove that $k_x = u^+(x)$ for all $x \in \Omega_2$.

Indeed, since G is a MLE of F on Ω_1 and since K is a MLE of G on Ω_2, we have K is a MLE of F on Ω_2.

By applying Proposition 3.2.5, there exists $\widetilde{K}$ is a MLE of K on $\mathbb{R}^n$ and so that $\widetilde{K}$ is also a MLE of F on $\mathbb{R}^n$.

Since u^+ is an over extremal extension of F on $\mathbb{R}^n$, we have

$$\widetilde{k}(x) \leq u^+(x), \ \forall x \in \mathbb{R}^n.$$

Thus

$$k_x = \widetilde{k}(x) \leq u^+(x), \ \forall x \in \Omega_2.$$

Conversely, we have

$$\Gamma^1(U^+; \Omega_2) = \Gamma^1(G; \Omega_1) = M$$

and by Proposition 3.4.14 we have

$$u^+(x) = g_x, \nabla u^+(x) = D_x g,$$

for $x \in \Omega_1$.

So that U^+ is a MLE of G on Ω_2. Thus $u^+(x) \leq k_x$ for all $x \in \Omega_2$. And thus $u^+(x) = k_x$ for all $x \in \Omega_2$.

Since $k(x) = u^+(x)$ for all $x \in \Omega_2$, K is a MLE of F on Ω_2 and $u^+_{|\Omega_2}$ is an over extremal extension of F on Ω_2, we have K to be an over extremal extension of F on Ω_2.

Applying Proposition 3.4.14 we have $D_x k = \nabla u^+(x)$ for all $x \in \Omega_2$. □

Thanks to result of Theorem 3.8.14. Indeed, this result allows to define an under extremal extension of F. That is

Definition 3.4.16. For any $a \in \Omega$, $x \in \mathbb{R}^n$ and $v \in \Lambda_x$ we define

$$u^-(x) \quad := \quad \inf_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v).$$

Using the strict convexity of the map $v \longmapsto \Psi^-(F, x, a, v)$ and the compacity of Λ_x as in the proof of proposition 3.4.9 when concavity is replaced with convexity, we obtain the following proposition

Proposition 3.4.17. *Let $x \in \mathbb{R}^n$. Then there exists a unique element $v_x^- \in \Lambda_x$ such that $u^-(x) = \sup_{a \in \Omega} \Psi^-(F, x, a, v_x^-)$.*

This allows us to define the following 1-field

Definition 3.4.18.

$$U^- : x \in \mathbb{R}^n \mapsto U^-(x)(y) := u^-(x) + \langle D_x u^-; y - x \rangle, \; y \in \mathbb{R}^n, \tag{3.35}$$

with

$$u^-(x) := \min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v), \; D_x u^- := \arg\min_{v \in \Lambda_x} \sup_{a \in \Omega} \Psi^-(F, x, a, v).$$

Theorem 3.4.19. *The 1-field U^- is the unique under extremal extension of F.*

Proof. Using Theorem 3.8.14 and Proposition 3.4.17, the proof uses similar arguments as in the proof of Theorem 3.4.13. □

In conclusion we have the following corollary

Corollary 3.4.20. *For all minimal Lischitz extension G of F we have*

$$u^-(x) \leq g(x) \leq u^+(x), \; \forall x \in \mathbb{R}^n.$$

3.5 Sup-Inf explicit minimal Lipschitz extensions for functions from $\mathbb{R}^m$ maps to $\mathbb{R}^n$

Now, we propose to use the results of the previous section to produce formulas comparable to those Bauschke and Wang have found see [9]. Let us define $\mathscr{Q}_0$ as the problem of the minimum extension for Lipschitz continuous functions and $\mathscr{Q}_1$ as the problem of the minimum extension for 1-fields. Curiously, we will show that the problem $\mathscr{Q}_0$ is a sub-problem of the problem $\mathscr{Q}_1$. As a consequence, we obtain two explicit formulas that solve the problem $\mathscr{Q}_0$.

More specifically, fix $n, m \in \mathbb{N}^*$ and $\omega \subset \mathbb{R}^m$ with $\#\omega \geq 2$. Let u be a function from ω maps to $\mathbb{R}^n$. Suppose $\mathrm{Lip}(u; \omega) < +\infty$ and define $l := \mathrm{Lip}(u; \omega)$. Let us define

$$\Omega := \{(x, 0) \in \mathbb{R}^m \times \mathbb{R}^n : x \in \omega\}.$$

A current element x of $\mathbb{R}^{m+n}$ is denoted by $x := (x^{(m)}, x^{(n)}) \in \mathbb{R}^{m+n}$, with $x^{(m)} \in \mathbb{R}^m$ and $x^{(n)} \in \mathbb{R}^n$. For each function u of domain ω we associate the 1-field F from $\Omega \subset \mathbb{R}^{n+m}$ maps to $\mathscr{P}^1(\mathbb{R}^{n+m}, \mathbb{R})$ as the following

$$f_{(x,0)} := 0, \text{ and } D_{(x,0)}f := (0, u(x)), \text{ for all } x \in \omega. \tag{3.36}$$

Let $a, b \in \omega$, with $a \neq b$. Observing that

$$f_{(a,0)} = f_{(b,0)} = 0, \text{ and } \langle D_{(a,0)}f + D_{(b,0)}f, (b-a, 0)\rangle = 0,$$

and applying Proposition 3.2.4 we have

$$\Gamma^1(F, (a,0), (b,0)) = \frac{\|D_{(a,0)}f - D_{(b,0)}f\|}{\|b-a\|}.$$

Therefore

$$\Gamma^1(F, \Omega) = \mathrm{Lip}(u, \omega). \tag{3.37}$$

Let G be an minimal Lipschitz extension of F. We have $G \in \mathscr{F}^1(\mathbb{R}^{m+n})$ and

$$\Gamma^1(F, \Omega) = \Gamma^1(G, \mathbb{R}^{m+n}). \tag{3.38}$$

Using [36, Proposition 2.4] we have

$$\Gamma^1(G; \mathbb{R}^{m+n}) = \mathrm{Lip}(Dg; \mathbb{R}^{m+n}). \tag{3.39}$$

Now we define the map $\tilde{u}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ as following

$$\tilde{u}(x) := (D_{(x,0)}g)^{(n)}, \; x \in \mathbb{R}^m. \tag{3.40}$$

We will show that $\tilde{u}$ is a minimal Lipschitz extension of u. Let $x \in \omega$. Since G is an extension of F and by construction of F we have

$$\tilde{u}(x) = (D_{(x,0)}g)^{(n)} = u(x).$$

Thus $\tilde{u}$ is an extension of u. Let $x, y \in \mathbb{R}^m$ with $x \neq y$. Using (3.37), (3.38) and (3.39) we have

$$\begin{array}{rcl} \mathrm{Lip}(\tilde{u}; x, y) & = & \frac{\|(D_{(x,0)}g)^{(n)} - (D_{(y,0)}g)^{(n)}\|}{\|x-y\|}, \\ & \leq & \frac{\|D_{(x,0)}g - D_{(y,0)}g\|}{\|x-y\|}, \\ & \leq & \Gamma^1(G; \mathbb{R}^{m+n}), \\ & = & \mathrm{Lip}(u, \omega). \end{array} \tag{3.41}$$

Conclusion, $\tilde{u}$ is an minimal Lipchitz extension of u. Therefore, we obtain another proof of Kirsbraun's theorem (see [32]).

Theorem 3.5.1. *Let u be a function from $\omega \subset \mathbb{R}^m$ to $\mathbb{R}^n$. Suppose that u is a Lipschitz continuous function. Let F be the 1-field defined by the formula (3.36). Let G be any minimal Lipschitz extension of F. Then the extension $\tilde{u}$ define by the formula (3.40) is a minimal Lipschitz extension of u.*

If we replace the 1-field G of Theorem 3.5.1 by U^- and U^+ which are defined by the Definitions 3.4.10 and 3.4.18 we obtain two explicit formulas which solve the problem $\mathscr{Q}_0$. Now we will describe these formulas. Let $a,b \in \omega$ and $x \in \mathbb{R}^m$ we define

$$\begin{aligned} v_{a,b} &:= \left(\frac{l}{2}(b-a), \frac{1}{2}(u(a)+u(b))\right), \\ \alpha_{a,b} &:= -\frac{1}{2}\|u(a)-u(b)\|^2 + \frac{l^2}{2}\|a-b\|, \\ \beta_{a,b}(x) &:= l^2\|x - \frac{a+b}{2}\|^2 + \|\frac{1}{2}(u(a)-u(b))\|^2, \\ r_{a,b}(x) &:= \sqrt{\alpha_{a,b}+\beta_{a,b}(x)}, \\ \Lambda_x &:= \{v \in \mathbb{R}^{m+n} : \|v - v_{a,b}\| \leq r_{a,b}(x), \forall a,b \in \omega\}. \end{aligned}$$

For $v \in \mathbb{R}^{m+n}$, we define

$$\Phi^+(u,x,a,v) := \frac{1}{2}\langle v^{(m)}, x-a\rangle + \frac{l}{4}\|a-x\|^2 - \frac{1}{4l}(\|v^{(m)}\|^2 + \|u(a)-v^{(n)}\|^2), \tag{3.42}$$

$$\Phi^-(u,x,a,v) := \frac{1}{2}\langle v^{(m)}, x-a\rangle - \frac{l}{4}\|a-x\|^2 + \frac{1}{4l}(\|v^{(m)}\|^2 + \|u(a)-v^{(n)}\|^2). \tag{3.43}$$

Now using the previous notations, we define two maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ as following

$$k^+(x) := (\arg\max_{v\in\Lambda_x} \inf_{a\in\omega} \Phi^+(u,x,a,v))^{(n)},\ x \in \mathbb{R}^m, \tag{3.44}$$

and

$$k^-(x) := (\arg\min_{v\in\Lambda_x} \sup_{a\in\omega} \Phi^-(u,x,a,v))^{(n)},\ x \in \mathbb{R}^m. \tag{3.45}$$

Theorem 3.5.2. *The maps k^+ and k^- define by the formulas (3.44) and (3.45) are minimal Lipschitz extensions of u.*

Remark 3.5.3. If ω is finite using the previous transformation $u \longrightarrow F$ then the Wells explicit construction of u^+ or u^- allows to compute u^+ and u^-. We know that the proof of Kirszbraun-Valentine's theorem and the proof of [36, Theorem 2.6] use Zorn's lemma. Noticing that, the proof which allows that Λ_x is non-empty set, does not use Zorn's lemma. Thus the proofs of Theorem 3.4.13, 3.4.19 and 3.5.2 does not use Zorn's lemma. This is also true in [9] and [54, Theorem 2].

3.6 Absolutely minimal Lipschitz extensions

In this section let Ω be a subset of $\mathbb{R}^n$ ($n \geq 2$) and $F \in \mathscr{F}^1(\Omega)$ and $\kappa = \Gamma^1(F,\Omega)$. Let $w^\pm = w^\pm(F,\Omega,\kappa)(x)$ where $w^\pm(F,\Omega,\kappa)$ are defined by the Definition 3.4.3 and Definition 3.8.10.

3.6.1 Finite domain

Proposition 3.6.1. *Suppose Ω is finite, then W^+ and W^- are AMLEs of F on $\mathbb{R}^n$.*

Proof. By Corollaries 3.4.6 and 3.8.13, $W^{\pm}$ are two MLEs of F and

$$\Gamma^1(W^{\pm};\mathbb{R}^n) = \kappa. \tag{3.46}$$

Let V be a bounded open satisfying $\bar{V} \subset \mathbb{R}^n \setminus \Omega$. We need to prove that $\Gamma^1(W^+,V) = \Gamma^1(W^+,\partial V)$.

Using the same notations like in the proof of [54, Theorem 1] and using [54, Lemma 17], there exist $S \in K$ and $x,y \in \partial V$ with $x \neq y$ such that $x,y \in T_S$. Applying [54, Lemma 21] we have

$$\|\nabla w^+(x) - \nabla w^+(y)\| = \kappa\|x-y\|.$$

Thus $\mathrm{Lip}(\nabla w^+;x,y) = \kappa$. Using (3.46) and the previous equality, we obtain

$$\Gamma^1(W^+,V) \leq \kappa = \mathrm{Lip}(\nabla w^+;x,y) \leq \mathrm{Lip}(\nabla w^+;\partial V) \leq \Gamma^1(W^+,\partial V) \leq \Gamma^1(W^+,V). \tag{3.47}$$

Thus

$$\Gamma^1(W^+,V) = \Gamma^1(W^+,\partial V).$$

Therefore W^+ is an ALME of F.

The proof for W^- is similar by using Proposition 3.8.9 and Lemma 3.8.11. □

Corollary 3.6.2. *Suppose Ω is finite and $w^+ \neq w^-$, then there exists an infinite number of AMLEs of F on $\mathbb{R}^n$.*

Proof. Let $x_0 \in \mathbb{R}^n \backslash \bar{\Omega}$ such that $w^+(x_0) \neq w^-(x_0)$. Noting that

$$W_\tau = \tau W^+ + (1-\tau)W^-,$$

is MLE of F on $\mathbb{R}^n$, for any $\tau \in [0,1]$.

Thus there exists an infinite number of MLE of F on $\Omega \cup \{x_0\}$.

Let G be a MLE of F on $\omega \cup \{x_0\}$. By the same argument as the proof of Proposition 3.6.1, we have $w^+(G,\Omega\cup\{x_0\},\kappa)$ and $w^-(G,\Omega\cup\{x_0\},\kappa)$ are two AMLEs of F on $\mathbb{R}^n$. Therefore, there exists an infinite number of AMLE of F on $\mathbb{R}^n$. □

3.6.2 Infinite domain

From Section 3.6.1, we know that if Ω is a finite set then the functions W^+ and W^- are two AMLEs of F on $\mathbb{R}^n$. When Ω is infinite and $n \geq 2$, we give an example that shows the opposite.

Proposition 3.6.3. *Suppose $n = 2$, then there exist Ω and $F \in \mathscr{F}^1(\Omega)$ such that W^+ and W^- are not AMLEs of F on $\mathbb{R}^n$.*

Proof. Let us define $\Omega_1 = \partial B(0;1)$, $\Omega_2 = \partial B(0;2)$ and $\Omega = \Omega_1 \cup \Omega_2$.
We define $F \in \mathscr{F}^1(\Omega)$ as following

$$f_x = 0 \text{ for } x \in \Omega_1,\ f_x = 1 \text{ for } x \in \Omega_2, \text{ and } D_x f = 0 \text{ for } x \in \Omega.$$

We will prove that W^+ is not an AMLE of F on $\mathbb{R}^n$. To do this, we need to find an open set $V \subset\subset \mathbb{R}^2/\Omega$ such that

$$\Gamma^1(W^+;V) \neq \Gamma^1(W^+;\partial V).$$

Let us define

$$V = \{x \in \mathbb{R}^2 : \|x\| < 3/4\} \subset\subset \mathbb{R}^2/\Omega.$$

we will prove that

$$\Gamma^1(W^+;V) \neq \Gamma^1(W^+;\partial V).$$

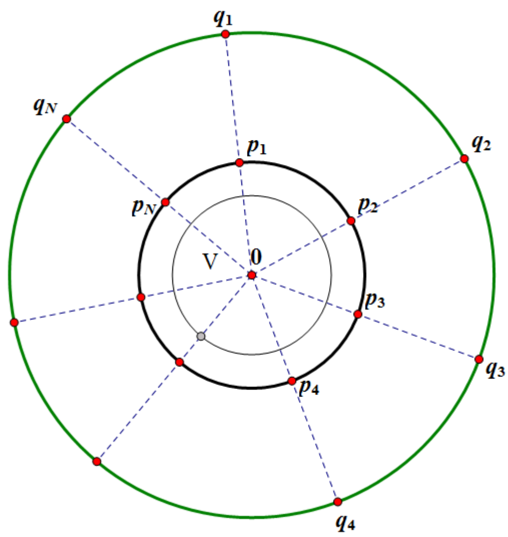

Figure 3.3: Illustration of subset A_n of Ω.

Applying Proposition 3.2.4, we have $\kappa = \Gamma^1(F;\Omega) = 4$.
Let $\mathscr{A}$ be the set of all finite subsets A_N of Ω of the form

$$A_N = \{p_1, ..., p_N\} \cup \{q_1, ..., q_N\},\ N \in \mathbb{N}^*,$$

which satisfies

$$\{p_1, p_2, ... p_N\} \subset \Omega_1,\ \{q_1, q_2, ..., q_N\} \subset \Omega_2, \text{ and } p_i \in [0, q_i],\ \forall i \in 1, ..., N$$

(see Figure 3).
By construction, we have $A_N \in \mathscr{A}$.
For brevity let us denote the functions $w^+(F, A_N, \kappa)$ by $w^+_{A_N}$.
Applying Proposition 3.2.4, we have

$$\kappa_N = \Gamma^1(W^+_{A_N};A_N) = 4 = \kappa, \text{ for all } A_N \in \mathscr{A}.$$

For all $i \in \{1,...,N\}$, let us denote by T_{p_i}, using the same notation like in [54, Theorem 1] and using the corresponding definition for the finite set $A = A_N$. We can check that

$$[\frac{p_i}{2}, \frac{p_i+q_i}{2}] \subset T_{p_i},$$

(see detail computing at Appendix 3.8.2) and from the definition of $w^+_{A_N}$, we have

$$w^+_{A_N}(x) = \frac{\kappa}{2}(x-p_i)^2 = \frac{\kappa}{2}d^2(x;\partial\Omega_1), \text{ for all } x \in [\frac{p_i}{2}, \frac{p_i+q_i}{2}].$$

Let us define $\mathscr{D}_{\alpha,\beta} := \{y \in \mathbb{R}^n \mid \alpha \le \|y\| \le \beta\}$, $\alpha \le \beta$ and $\mathscr{P}(\Omega)$ to be the set of all finite subsets of Ω.

We will prove that

$$w^+(x) = \frac{\kappa}{2}d^2(x;\partial\Omega_1) \text{ for all } x \in \mathscr{D}_{1/2,3/2}.$$

Indeed, let $x \in \mathscr{D}_{1/2,3/2}$, there exists $A_N \in \mathscr{A}$ such that $x \in [0,q_1]$ where $q_1 \in A_N \cap \Omega_2$.

From the definition of w^+ (see 3.4.3), we have $w^+(x) \le w^+_{A_N}(x) = \frac{\kappa}{2}d^2(x;\partial\Omega_1)$.

Conversely, for all $P \in \mathscr{P}(\Omega)$, there exists $A_N \in \mathscr{A}$ such that $P \subset A_N$ and $x \in [0,q_2]$ where $q_2 \in A_N \cap \Omega_2$.

Applying Corollary 3.4.5, we have $w^+(F,P,\kappa) \ge w^+_{A_N}(x) = \frac{\kappa}{2}d^2(x;\partial\Omega_1)$.

Hence

$$w^+(x) = \inf_{P \in \mathscr{P}(\Omega)} w^+(F,P,\kappa)(x) \ge \frac{\kappa}{2}d^2(x,\partial\Omega_1).$$

Therefore

$$w^+(x) = \frac{\kappa}{2}d^2(x;\partial\Omega_1), \ \forall x \in \mathscr{D}_{1/2,3/2}.$$

Noticing that

$$w^+(x) = \frac{\kappa}{2}d^2(x;\partial\Omega_1) = \frac{\kappa}{2}(\|x\|-1)^2, \text{ for all } x \in \mathscr{D}_{1/2,3/4},$$

for all $x,y \in \mathscr{D}_{1/2,3/4}$ such that $x \in [0,y]$, we have

$$\|\nabla w^+(x) - \nabla w^+(y)\| = \left\| \kappa(\|x\|-1)\frac{x_0}{\|x\|} - \kappa(\|y\|-1)\frac{y}{\|y\|} \right\| = \kappa\|x-y\|.$$

Hence

$$\kappa \ge \Gamma^1(W^+;\mathbb{R}^n) \ge \Gamma^1(W^+;V) \ge \mathrm{Lip}(\nabla w^+;V) \ge \kappa,$$

and hence

$$\Gamma^1(W^+;V) = \kappa = 4. \tag{3.48}$$

On the other hand, for all $x,y \in \partial V$, we have $\|x\| = \|y\| = 3/4$, so that

$$\|\nabla w^+(x) - \nabla w^+(y)\| = \left\| \kappa(\|x\|-1)\frac{x}{\|x\|} - \kappa(\|y\|-1)\frac{y}{\|y\|} \right\| = \frac{\kappa}{3}\|x-y\|.$$

Hence

$$B_{x,y}(W^+) = \frac{\|\nabla w^+(x) - \nabla w^+(y)\|}{\|x-y\|} = \frac{\kappa}{3}.$$

Moreover, since $w^+(x) = w^+(y)$ and $(\nabla w^+(x) + \nabla w^+(y))$ is perpendicular to $(y-x)$, we have

$$A_{x,y}(W^+) = \frac{2(w^+(x) - w^+(y)) + \langle \nabla w^+(x) + \nabla w^+(y), y-x \rangle}{\|x-y\|^2} = 0.$$

Applying Proposition 3.2.4, we have

$$\Gamma^1(W^+;\partial V) = \sup_{x,y \in \partial V} \left(\sqrt{A_{a,b}^2 + B_{a,b}^2} + |A_{a,b}| \right) = \frac{\kappa}{3} = \frac{4}{3}. \tag{3.49}$$

From (3.48) and (3.49) we have

$$\Gamma^1(W^+;V) \neq \Gamma^1(W^+;\partial V).$$

And therefore W^+ is not an AMLE of F on $\mathbb{R}^n$.

The proof for W^- is similar. $\square$

Remark 3.6.4. With the same notation as in Proposition 3.6.3. By computing directly w^+ and w^- (see Appendix 3.8.2 for full detail computing), we obtain

$$\begin{aligned}
w^+(x) &= -w^-(x) = 1 - \frac{\kappa}{2}x^2, \ \forall x \in \mathscr{D}_{0,\frac{1}{2}}, \\
w^+(x) &= -w^-(x) = \frac{\kappa}{2}d^2(x,\partial\Omega_1), \ \forall x \in \mathscr{D}_{\frac{1}{2},1}, \\
w^+(x) &= w^-(x) = \frac{\kappa}{2}d^2(x,\partial\Omega_1), \ \forall x \in \mathscr{D}_{1,\frac{3}{2}}, \\
w^+(x) &= w^-(x) = 1 - \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{\frac{3}{2},2}, \\
w^+(x) &= 1 + \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{2,+\infty}, \\
w^-(x) &= 1 - \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{2,+\infty},
\end{aligned}$$

where $\kappa = \Gamma^1(F;\Omega) = 4$.
We see that $W = \frac{W^+ + W^-}{2}$ is an AMLE of F on $\mathbb{R}^2$ (although W^+ and W^- are not AMLEs of F on $\mathbb{R}^2$) and $w \notin C^2(\mathbb{R}^2,\mathbb{R})$.

Moreover, all MLEs of F coincide on $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ (because $w^+ = w^-$ on $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$).

From Proposition 3.6.3, we know that W^+ is not an AMLE of F on $\mathbb{R}^n$ in general case. But in some case, we have W^+ to be an AMLE of F on $\mathbb{R}^n$. We give an example:

Proposition 3.6.5. *Let Ω be a subset of $\mathbb{R}^n$ ($n \geq 2$). Let $F \in \mathscr{F}^1(\Omega)$ such that $\widetilde{\Omega} = \{p - \frac{D_p f}{\kappa} : p \in \Omega\}$ is a subset of an $(n-1)$-dimensional hyperplane H, where $\kappa = \Gamma^1(F,\Omega)$. Then W^+ is an AMLE of F on $\mathbb{R}^n$.*

Proof. For brevity let us denote $W^+(F,\Omega)$ by W^+. We prove that W^+ is an AMLE of F on $\mathbb{R}^n$. Put $\kappa = \Gamma^1(F;\Omega)$.

From Corollary 3.4.6, we have W^+ to be an MLE of F on $\mathbb{R}^n$.

Let $V \subset \mathbb{R}^n \backslash \Omega$. We need to prove that $\Gamma^1(W^+;V) = \Gamma^1(W^+;\partial V)$.

Indeed, the inequality $\Gamma^1(W^+;V) \geq \Gamma^1(W^+;\partial V)$ is clear, so that we only need to prove that

$$\Gamma^1(W^+;\partial V) \geq \Gamma^1(W^+;V).$$

We have

$$\kappa = \Gamma^1(W^+;\mathbb{R}^n) \geq \Gamma^1(W^+;V) \geq \Gamma^1(W^+;\partial V) \geq \mathrm{Lip}(\nabla w^+, \partial V).$$

Thus it suffices to show that $\mathrm{Lip}(\nabla w^+, \partial V) \geq \kappa$.

Let $x_0, y_0 \in \partial V$, $x_0 \neq y_0$ such that $(x_0 - y_0)$ is perpendicular to the hyperplane H.

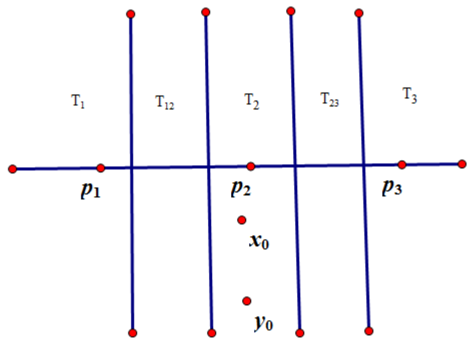

Figure 3.4: The partitioning of $\mathbb{R}^n$ into regions T_S

Let $\mathscr{P}$ be the set of all finite subsets of Ω. For any $P \in \mathscr{P}$, we have the corresponding function $W^+(F,P,\kappa)$ (see Definition 3.4.3) or W_P^+ for short.

We define K and T_S for $S \in K$ the same notations like in [54, Theorem 1] with the corresponding definition for the finite set $A = P$.

Put $\kappa_P = \Gamma^1(W_P^+;\mathbb{R}^n)$. Since $\widetilde{\Omega} = \{p - \frac{D_p f}{\kappa} : p \in \Omega\}$ is a subset of H and $(x_0 - y_0)$ is perpendicular to the hyperplane H, there exist $S \in K$ such that $x_0, y_0 \in T_S$.

Applying [54, Lemma 21], we have

$$\|\nabla w_P^+(x_0) - \nabla w_P^+(y_0)\| = \kappa_P \|x_0 - y_0\|.$$

Moreover, for any $\varepsilon > 0$, there exists $P \in \mathscr{P}$ (by [54, Proposition 3]) such that

$$\|\nabla w_P^+(x_0) - \nabla w^+(x_0)\| \leq \varepsilon,\ \|\nabla w_P^+(y_0) - \nabla w^+(y_0)\| \leq \varepsilon, \text{ and } \kappa_P > \kappa - \varepsilon.$$

Therefore

$$\begin{aligned}
\mathrm{Lip}(\nabla w, \partial V) &\geq \frac{\|\nabla w^+(x_0) - \nabla w^+(y_0)\|}{\|x_0 - y_0\|} \\
&\geq -\frac{\|\nabla w^+(x_0) - \nabla w_P^+(x_0)\|}{\|x_0 - y_0\|} - \frac{\|\nabla w^+(y_0) - \nabla w_P^+(y_0)\|}{\|x_0 - y_0\|} \\
&+ \frac{\|\nabla w_P^+(x_0) - \nabla w_P^+(y_0)\|}{\|x_0 - y_0\|} \\
&\geq \frac{-2\varepsilon}{\|x_0 - y_0\|} + \kappa - \varepsilon.
\end{aligned}$$

Hence $\mathrm{Lip}(\nabla w, \partial V) \geq \kappa$. $\square$

3.7 Extremal point

In this section, we present a new result in Theorem 3.7.5.

Let $S = \{x,y\} \subset \mathbb{R}^n$ ($x \neq y$) and let $F \in \mathscr{F}^1(S)$ such that $M := \Gamma^1(F,S) > 0$. We recall some notations

$$\begin{aligned} A_{x,y}(F) &:= \frac{2(f_x - f_y) + \langle D_x f + D_y f, y - x\rangle}{\|x-y\|^2}, \\ B_{x,y}(F) &:= \frac{\|D_x f - D_y f\|}{\|x-y\|}. \end{aligned}$$

From Proposition 2.2 in [36] we have

$$M = \Gamma^1(F;x,y) = \sqrt{A_{x,y}(F)^2 + B_{x,y}(F)^2} + |A_{x,y}(F)|. \tag{3.50}$$

We define

$$c := \frac{x+y}{2} + s\frac{D_x f - D_y f}{2M},$$

where $s = 1$ if $A_{x,y}(F) \geq 0$ and $s = -1$ if $A_{x,y}(F) < 0$.

We call the point c to be the *extremal point* of F associated to (x,y).

Proposition 3.7.1. *We have $A_{x,y}(F) = 0$ if and only if $\langle c-x, c-y\rangle = 0$.*

Proof. From (3.50), $A_{x,y}(F) = 0$ if and only if $M = B_{x,y}(F)$. This is equivalent to

$$\|c - \frac{x+y}{2}\| = \|\frac{D_x f - D_y f}{2M}\| = \frac{\|x-y\|}{2},$$

and hence it is equivalent to $\langle c-x, c-y\rangle = 0$. □

We recall some results of the *extremal point* which are useful in the proof of Theorem 3.7.5.

Lemma 3.7.2. *([27],Lemma 8) We define $\widetilde{F}_c \in \mathscr{F}^1(\mathbb{R}^n)$ as*

$$\widetilde{F}_c(z) := \widetilde{f}_c + \langle D_c\widetilde{f}, z-c\rangle, z \in \mathbb{R}^n,$$

where

$$\begin{aligned} \widetilde{f}_c &:= F_x(c) - s\frac{M}{2}\|x-c\|^2, \\ D_c\widetilde{f} &:= D_x f + sM(x-c). \end{aligned}$$

If $A_{x,y}(F) = 0$, we define

$$g(z) := \widetilde{F}_c(z) - s\frac{M}{2}\frac{\langle z-c, x-c\rangle^2}{\|x-c\|^2} + s\frac{M}{2}\frac{\langle z-c, y-c\rangle^2}{\|y-c\|^2}, z \in \mathbb{R}^n.$$

Then G is a MLE of F on $\mathbb{R}^n$.
If $A_{x,y}(F) \neq 0$, let $z \in \mathbb{R}^n$ and let $p(z) := \langle x-c, z-c\rangle$ and $q(z) := \langle y-c, z-c\rangle$. We define

$$g(z) := \begin{cases} \widetilde{F}_c(z) - s\dfrac{M}{2}\dfrac{\langle z-c, x-c\rangle^2}{\|x-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \leq 0, \\ \widetilde{F}_c(z) + s\dfrac{M}{2}\dfrac{\langle z-c, y-c\rangle^2}{\|y-c\|^2}, & \text{if } p(z) \leq 0 \text{ and } q(z) \geq 0, \\ \widetilde{F}_c(z), & \text{if } p(z) \leq 0 \text{ and } q(z) \leq 0, \\ \widetilde{F}_c(z) - s\dfrac{M}{2}\dfrac{\langle z-c, x-c\rangle^2}{\|x-c\|^2} + s\dfrac{M}{2}\dfrac{\langle z-c, y-c\rangle^2}{\|y-c\|^2}, & \text{if } p(z) \geq 0 \text{ and } q(z) \geq 0. \end{cases}$$

Then G is a MLE of F on $\mathbb{R}^n$.

Remark 3.7.3. Let g be the same notation as in Lemma 3.7.2. By computing, we have

$$\frac{\|\nabla g(a) - \nabla g(b)\|}{\|a-b\|} = \frac{\|\nabla g(s) - \nabla g(t)\|}{\|s-t\|} = M,$$

for all $a, b \in [x,c]$ $(a \neq b)$ and $s, t \in [y,c]$ $(s \neq t)$.

Lemma 3.7.4. *([27],Lemma 10) All MLE of F on $\mathbb{R}^n$ coincide on the line segments $[x,c]$ and $[y,c]$.*

Theorem 3.7.5. *Let Ω be a subset of $\mathbb{R}^n$ such that $\{x,y\} \subset \Omega$ and $B(c,\rho) \subset \Omega$ for some $\rho > 0$. Let H be a MLE of F on Ω. Assume that h is 2-differentiable at c, then $\langle c-x, c-y\rangle = 0$.*

Proof. If $A_{x,y}(F) = 0$ then from Proposition 3.7.1 we have $\langle c-x, c-y\rangle = 0$.
If $A_{x,y}(F) \neq 0$. Since h is 2-differentiable at c, we have

$$Dh(a) = Dh(c) + D^2h(c)(a-c) + \|a-c\|\psi(a-c), \ \forall a \in B(c,\rho), \tag{3.51}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ and $\lim_{k\to 0} \psi(k) = 0$.
Since $M = \Gamma^1(F; x,y) = \Gamma^1(H,\Omega) \geq Lip(Dh;\Omega)$, we have

$$\mathrm{Lip}(Dh; z', z) \leq M, \ \forall z, z' \in \Omega,$$

From (3.51) and (3.52), we obtain

$$|\left\langle D^2h(c)(v), v\right\rangle| \leq M, \ \forall v \in T, \tag{3.52}$$

where

$$T := \{v \in \mathbb{R}^n : \|v\| = 1\}.$$

Let $a_1 \in [x,c) \cap B(c,\rho)$. We define

$$v_1 := \frac{x-c}{\|x-c\|} = \frac{a_1 - c}{\|a_1 - c\|}.$$

Applying Lemma 3.7.4 we have $H = G$ in the line $[x,c] \cup [y,c]$, where the 1-field G is defined in Lemma 3.7.2, we obtain

$$Dh(z) = x_0 - \frac{sM\langle z-c, x-c\rangle (x-c)}{\|x-c\|^2},\ \forall z \in [x,c) \cap B(c,\rho) \tag{3.53}$$

and

$$Dh(z) = x_0 + \frac{sM\langle z-c, y-c\rangle (y-c)}{\|y-c\|^2},\ \forall z \in (c,y] \cap B(c,\rho), \tag{3.54}$$

where $x_0 = D_x h + sM(x-c)$. Without of loss generality, we can assume that $s = 1$. Thus from (3.51) (by replacing $a = a_1$ and $a = c$) and (3.53), we have

$$x_0 - \frac{M\langle a_1 - c, x-c\rangle (x-c)}{\|x-c\|^2} = x_0 + D^2h(c)(a_1 - c) + \|a_1 - c\| \psi(a_1 - c)$$

Hence

$$-M = \langle D^2h(c)(v_1), v_1\rangle + \langle \psi(a_1 - c), v_1\rangle$$

Taking the limit as $a_1 \to c$, we obtain

$$\langle D^2h(c)(v_1), v_1\rangle = -M. \tag{3.55}$$

Therefore from (3.52) and (3.55) we have

$$\langle D^2h(c)(v_1), v_1\rangle \le \langle D^2h(c)(v), v\rangle,\ \forall v \in T. \tag{3.56}$$

Now put $A = D^2h(c)$, then A is a symmetric matrix.

We will prove that v_1 is a eigenvector of A. Indeed, let $B = \{f_1, ..., f_m\}$ be a orthonormal basis consisting of eigenvectors of A and λ_i is eigenvalue corresponding to f_i for any $i \in \{1, ..., m\}$. We have

$$Af_i = \lambda_i f_i,\ \forall i = \{1, ..., m\}.$$

Suppose that $\lambda_j = \min_i \lambda_i$.

We can write v_1 in the form

$$v_1 = c_1 f_1 + ... + c_m f_m.$$

We have $c_1^2 + ... + c_n^2 = 1$ since $\|v_1\| = 1$.

Thus $\langle Av_1, v_1\rangle = \sum c_i^2 \lambda_i$ and $\langle Af_j, f_j\rangle = \lambda_j$. From (3.56), we have

$$\sum c_i^2 \lambda_i \le \lambda_j. \tag{3.57}$$

Since $\lambda_j = \min_i \lambda_i$, we obtain $c_j = 1$ and $c_i = 0, \forall i \ne j$. Thus $v_1 = f_j$.

Similarly, let $v_2 = \frac{y-c}{\|y-c\|}$, then we have $v_2 = f_k$ with $\lambda_k = \max_i \lambda_i$.

Thus $\langle v_1, v_2\rangle = \langle f_j, f_k\rangle = 0$ since $j \ne k$. Thus, $\langle c-x, c-y\rangle = 0$. From Proposition 3.7.1 we have $A_{x,y}(F) = 0$. This contradicts with $A_{x,y}(F) \ne 0$. Therefore, we come to the conclusion that $\langle c-x, c-y\rangle = 0$ and $A_{x,y}(F) = 0$. □

3.8 Appendix

3.8.1 The constructions of w^+ and w^-.

Let Ω be a subset of $\mathbb{R}^n$ containing at least two elements, and F be a 1-field in $\mathscr{F}^1(\Omega)$. Suppose that κ is allowable for F, i.e. $\kappa \geq \Gamma^1(F,\Omega)$. We will describe the $w^+(F,\Omega,\kappa)$ and $w^-(F,\Omega,\kappa)$ of 1-field F and we will give the properties used for the understanding of this chapter.

Case 1: Ω is finite.

Recall the constructions of w^+.
Let $M \geq \Gamma^1(F;\Omega)$. For $p \in \Omega$ we define:

$$\begin{aligned} \widetilde{p}^+ &:= p - D_p f/M, \\ d_p^+(x) &:= f_p - \frac{1}{2}(D_p f)^2/M + \frac{1}{4}M\|x - \widetilde{p}^+\|^2. \end{aligned}$$

When $S \subset \Omega$ we define

$$\begin{aligned} d_S{}^+(x) &:= \inf_{p\in S} d_p{}^+(x), \\ \widetilde{S}^+ &:= \{\widetilde{p}^+ : p \in S\}, \\ \widehat{S}^+ &:= \text{ convex hull of } \widetilde{S}^+, \\ S_H^+ &:= \text{smallest hyperplane containing } \widetilde{S}^+, \\ S_E^+ &:= \{x : d_p^+(x) = d_{p'}^+(x) \text{ for all } p, p' \in S\}, \\ S_*^+ &:= \{x : d_p^+(x) = d_{p'}^+(x) \leq d_{p''}^+(x) \text{ for all } p, p' \in S, p'' \in \Omega\}, \\ K^+ &:= \{S : S \subset \Omega \text{ and for some } x \in S_*^+, d_S^+(x) < d_{\Omega - S}^+(x)\}. \end{aligned}$$

Proposition 3.8.1 ([54], Lemma 3)**.** *Let $S_C^+ = S_E^+ \cap S_H^+$ for $S \in K^+$ then S_C^+ is a point.*

Definition 3.8.2. For all $S \in K^+$, set

$$T_S^+ := \{x : x = \frac{1}{2}(y+z) \text{for some } y \in \widehat{S}^+ \text{ and } z \in S_*^+\}$$

Proposition 3.8.3 ([54], Lemma 15,17)**.** *We have $\bigcup_{S\in K^+} T_S^+ = \mathbb{R}^n$ and $(T_S^+ \cap T_{S'}^+)^0 = \emptyset$ if $S \neq S'$.*

Definition 3.8.4. $w_S^+(x) := d_S(S_C^+) + \frac{1}{2}Md^2(x, S_H^+) - \frac{1}{2}Md^2(x, S_E^+)$ for $S \in K^+$ and $x \in T_S^+$.

Definition 3.8.5. $w^+(F,\Omega,M)(x) := w_S^+(x)$ if $x \in T_S^+$.

From [54] we know that $w^+(F,\Omega,M)$ is well defined in $\mathbb{R}^n$, $w^+(F,\Omega,M) \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ and

$$\mathrm{Lip}(\nabla w^+(F,\Omega,M),\mathbb{R}^n) \leq M.$$

Theorem 3.8.6. *We have $w^+(F,\Omega,M) \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ with $w^+(F,\Omega,M)(p) = f_p$, $\nabla w^+(F,\Omega,M)(p) = D_p f$ for all $p \in \Omega$ and $\mathrm{Lip}(\nabla w^+(F,\Omega,M),\mathbb{R}^n) \le M$.*

Further, if $g \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R})$ with $g(p) = f_p$, $\nabla g(p) = D_p f$ when $p \in \Omega$ and $\mathrm{Lip}(\nabla g,\mathbb{R}^n) \le M$, then $g(x) \le w^+(F,\Omega,M)(x)$ for all $x \in \mathbb{R}^n$.

Proof. Applying Proposition 3.2.4, we have $\Gamma^1(F;\Omega) \le M$ if and only if

$$\sqrt{A^2_{x,y}(F) + B^2_{x,y}(F)} + A_{x,y}(F) \le M,\ \forall x,y \in \Omega,$$

This inequality is equivalent to

$$\frac{B^2_{x,y}(F)}{2M_1} + A_{x,y}(F) \le \frac{M}{2},$$

and hence it is equivalent to

$$f_y \le f_x + \frac{1}{2}\langle D_x f + D_y f, y - x\rangle + \frac{1}{4}M\|y-x\|^2 - \frac{1}{4M}\|D_x f - D_y f\|^2, \text{ for any } x,y \in \Omega.$$

Using [54, Theorem 1], we finish the proof of this theorem. □

Corollary 3.8.7. *In the case $M = \Gamma^1(F;\Omega)$, let $W^+(F,\Omega,M)$ be the 1-field associated to $w^+(F,\Omega,M)$ then $W^+(F,\Omega,M)$ is an over extremal extension of F on $\mathbb{R}^n$.*

Proof. From [36, Proposition 2.4] we have

$$\mathrm{Lip}(\nabla w^+(F,\Omega,M),\mathbb{R}^n) = \Gamma^1(W^+(F,\Omega,M),\mathbb{R}^n).$$

Thus the proof is immediate from Definition 3.2.2 and Theorem 3.8.6. □

The constructions of w^-.

By the same way, we can construct the function w^- as follows. For $p \in \Omega$ we define :

$$\begin{aligned}
\widetilde{p}^- &:= p + D_p f/\kappa,\\
d^-_p(x) &:= f_p + \frac{1}{2}D_p f^2/\kappa - \frac{1}{4}\kappa\|x - \widetilde{p}^-\|^2,
\end{aligned}$$

and for any $S \subset \Omega$,

$$\begin{aligned}
d^-_S(x) &:= \sup_{p\in S} d^-_p(x),\\
\widetilde{S}^- &:= \{\widetilde{p}^- : p \in S\},\\
\widehat{S}^- &:= \text{ convex hull of } \widetilde{S},\\
S^-_H &:= \text{smallest hyperplane containing } \widetilde{S}^-,\\
S^-_E &:= \{x : d^-_p(x) = d^-_{p'}(x) \text{ for all } p,p' \in S\},\\
S^-_* &:= \{x : d^-_p(x) = d^-_{p'}(x) \ge d^-_{p''}(x) \text{ for all } p,p' \in S, p'' \in \Omega\},\\
K^- &:= \{S : S \subset \Omega \text{ and for some } x \in S^-_*, d^-_S(x) > d^-_{\Omega - S}(x)\}.
\end{aligned}$$

Definition 3.8.8. Let us define

$$T_S^- := \{x : x = \frac{1}{2}(y+z) \text{ for some } y \in \widehat{S}^- \text{ and } z \in S_*^-\}, \text{ for all } S \in K^-.$$

Proposition 3.8.9. *For all $S \in K^-$ and for all $x \in T_S^-$ there exists an unique couple $(y,z) \in \widehat{S}^- \times S_*^-$ such that $x = \frac{1}{2}(y+z)$.*

We have

$$\bigcup_{S \in K^-} T_S^- = \mathbb{R}^n,$$

$$\overset{\circ}{T_S^-} \cap \overset{\circ}{T_{S'}^-} = \emptyset, \text{ and } \overset{\circ}{T_S^-} \neq \emptyset, \text{ for all } S, S' \in K^-, \text{ with } S \neq S'.$$

For all $S \in K^-$, the set $S_E^- \cap S_H^-$ contains a single point denoted by S_C^-.

For all $S \in K^-$, hyperplanes S_E^-, S_H^- are orthogonal.

Definition 3.8.10. Let us define

$$w_S^-(x) := d_S(S_C^-) - \frac{1}{2}\kappa d^2(x, S_H^-) + \frac{1}{2}\kappa d^2(x, S_E^-), \text{ for all } S \in K^- \text{ and } x \in T_S^-,$$

and

$$w^-(F, \Omega, \kappa)(x) := w_S^-(x) \text{ for all } x \in T_S^-.$$

Lemma 3.8.11. *For all $S \in K^-$ and $x \in T_S^-$, we have*

$$\nabla w^-(F, \Omega, \kappa)(x) = \frac{\kappa}{2}(y-z), \text{ where } x = \frac{1}{2}(y+z), \text{ with } (y,z) \in \widehat{S}^- \times S_*^-,$$

and

$$\|\nabla w^-(F, \Omega, \kappa)(x) - \nabla w^-(F, \Omega, \kappa)(x')\| = \kappa\|x - x'\|, \text{ for all } S \in K^- \text{ and } x, x' \in T_S^-. \tag{3.58}$$

Theorem 3.8.12. *We have $w^-(F, \Omega, \kappa) \in \mathscr{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$ and $\mathrm{Lip}(\nabla w^-(F, \Omega, \kappa); \mathbb{R}^n) = \kappa$,. Furthermore, $w^-(F, \Omega, \kappa)$ is an extension of F and for all $g \in \mathscr{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$ extension of F such that $\mathrm{Lip}(\nabla g; \mathbb{R}^n) \leq \kappa$, we have*

$$w^-(F, \Omega, \kappa)(x) \leq g(x), \ x \in \mathbb{R}^n.$$

Corollary 3.8.13. *If $\kappa = \Gamma^1(F; \Omega)$, then $W^-(F, \Omega, \kappa)$ is an under extremal extension of F.*

Case 2: Ω is infinite.

Denote by $\mathscr{P}(\Omega)$ the set of all finite subset of Ω. Since for any $x \in \mathbb{R}^n$, and for any $P, P' \in \mathscr{P}(\Omega)$ satisfying $P \subset P'$ we have

$$w^-(F, P, \kappa)(x) \leq w^-(F, P', \kappa)(x) \leq w^+(F, P', \kappa)(x) \leq w^+(F, P, \kappa)(x).$$

So that we can define

$$w^+(F,\Omega)(x) = \inf_{P\in\mathscr{P}(\Omega)} w^+(F,P,\kappa)(x),$$

and

$$w^-(F,\Omega)(x) = \sup_{P\in\mathscr{P}(\Omega)} w^-(F,P,\kappa)(x).$$

Theorem 3.8.14. *Let Ω be any subset of $\mathbb{R}^n$ and $F \in \mathscr{F}^1(\Omega)$. If $\kappa = \Gamma^1(F;\Omega)$, then $W^+(F,\Omega,\kappa)$ is an over extremal extension of F and $W^-(F,\Omega,\kappa)$ is an under extremal extension of F.*

Proof. Using [[54], Theorem 2], the proof is similar as Theorem 3.8.6 and Corollary 3.8.7. □

3.8.2 Details of computation for the proof of the counterexample of Proposition 3.6.3

We consider in $\mathbb{R}^2$. Fix $N \in \mathbb{N}$ we let $p_1, p_2, ..., p_N \in \partial B(0;1)$ and $q_1, q_2, ..., q_N \in \partial B(0;2)$ such that $p_i \in [0,q_i]$ for all $i \in \{1,...,N\}$. We put $p_{N+1} := p_1$, $q_{N+1} := q_1$, $p_0 := p_N$ and $q_0 := q_N$.

We denote:

R_{ab} is the ray starting a and passing through another point b.

$L_{a,b}$ is the line passing through a and b.

$[a,b]$ is the line segment joining two points a and b.

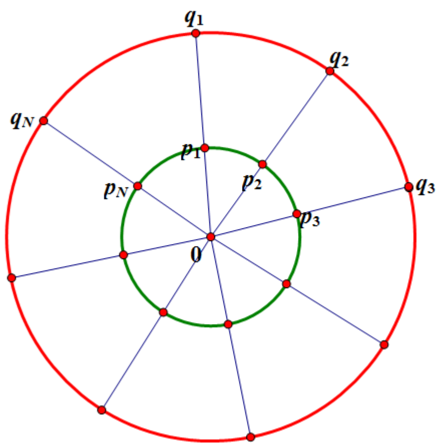

Figure 3.5: The domain of 1-field F

We put $\Omega_1 = \{p_1, p_2, ..., p_N\}$, $\Omega_2 = \{q_1, q_2, ..., q_N\}$ and $A = \Omega_1 \cup \Omega_2$. Let $F \in \mathscr{F}^1(A)$ satisfying $f_p = 0$ for all $p \in \Omega_1$, $f_q = 1$ for all $q \in \Omega_2$ and $D_x f = 0$ for all $x \in A$. We will show clearly the form of $w^+(F,A)$ and $w^-(F,A)$.

The form of $w^+(F,A)$

Put $M = \Gamma^1(F,A) = 4$.
For any $p \in \Omega_1$ and $q \in \Omega_2$, we have

$$\begin{aligned}
\widetilde{p}^+ &= p - D_p f/M = p, \\
d_p^+(x) &= f_p - \frac{1}{2}(D_p f)^2/M + \frac{1}{4}M\|x - \widetilde{p}^+\|^2 = \|x-p\|^2.
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}^+ &= q - D_q f/M = q, \\
d_q^+(x) &= f_q - \frac{1}{2}(D_q f)^2/M + \frac{1}{4}M\|x - \widetilde{q}^+\|^2 = 1 + \|x-q\|^2.
\end{aligned}$$

For all $i \in \{1,\dots,N\}$, let ray R_{0z_i} be the bisector of the angle $\widehat{p_{i-1}0p_i}$. The tangent to the circle $\partial B(0,2)$ at q_i cut the ray R_{0z_i} at k_i. We call a_i, b_i, c_i, m_i, n_i to be the midpoints of the segments $[0,p_i], [k_i,p_i], [k_{i+1},p_i], [k_i,q_i], [k_{i+1},q_i]$, respectively. Let rays $R_{m_iu_i}, R_{n_iv_i}$ such that $R_{m_iu_i}$ is parallel to R_{k_i,z_i} and $R_{n_iv_i}$ is parallel to $R_{k_{i+1}z_{i+1}}$.

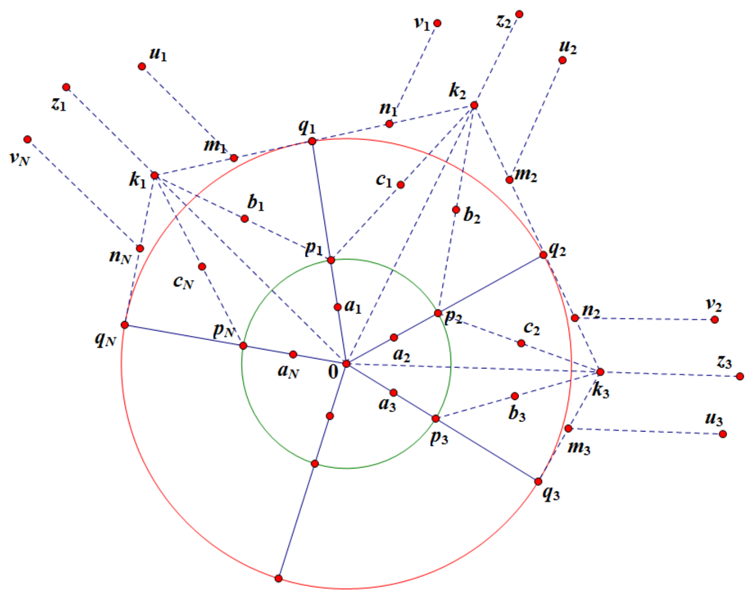

Figure 3.6: The rays R

Let $S^+ \in K^+$.
***Case 1:** $S^+ = \{p_i\}$.
We have $\widehat{S}^+ = \{p_i\}$, $S_H^+ = \{p_i\}$, $S_E^+ = \mathbb{R}^n$, S_*^+ to be the convex hull of $\{0, k_i, k_{i+1}\}$, $S_C^+ = S_E^+ \cap S_H^+ = \{p_i\}$, and $T_{\{p_i\}}^+$ to be the convex hull of $\{a_i, b_i, c_i\}$.

Thus for all $x \in T_{\{p_i\}}^+$ we have

$$\begin{aligned}
w^+(F,A)(x) &= d_S^+(S_C^+) + \frac{1}{2}Md^2(x, S_H^+) - \frac{1}{2}Md^2(x, S_E^+) \\
&= d_{p_i}^+(p_i) + \frac{1}{2}Md^2(x, p_i) - \frac{1}{2}Md^2(x, \mathbb{R}^n) \\
&= \frac{1}{2}M\|x - p_i\|^2.
\end{aligned}$$

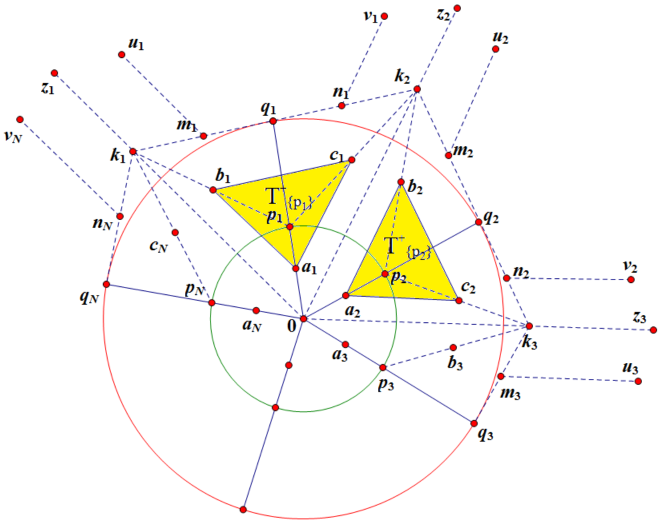

Figure 3.7: $S^+ = \{p_i\}$

***Case 2:** $S^+ = \{q_i\}$.
We have $\widehat{S}^+ = \{q_i\}$, $S_H^+ = \{q_i\}$, $S_E^+ = \mathbb{R}^n$, S_*^+ to be the region bounded by ray $R_{k_i z_i}$, ray $R_{k_{i+1} z_{i+1}}$ and the segment $[k_i, k_{i+1}]$, $S_C^+ = S_E^+ \cap S_H^+ = \{q_i\}$, and $T_{\{q_i\}}^+$ to be the region bounded by ray $R_{m_i u_i}$, ray $R_{n_i v_i}$ and the segment $[m_i, n_i]$.

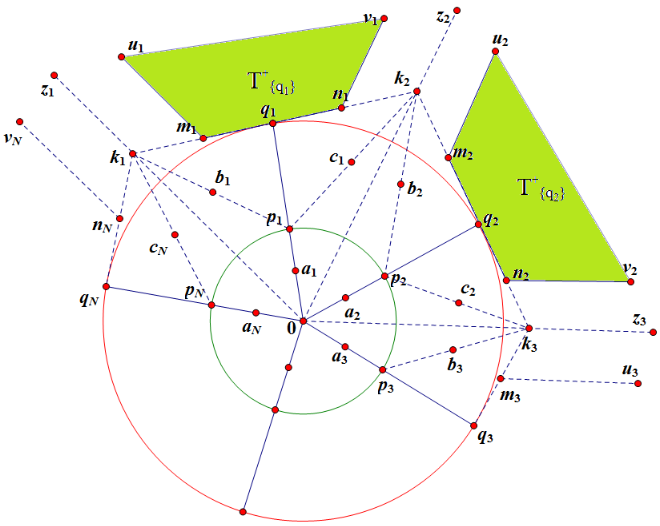

Figure 3.8: $S^+ = \{q_i\}$

Thus for all $x \in T^{+}_{\{q_i\}}$ we have

$$\begin{aligned} w^{+}(F,A)(x) &= d^{+}_{S}(S^{+}_{C}) + \frac{1}{2}Md^{2}(x,S^{+}_{H}) - \frac{1}{2}Md^{2}(x,S^{+}_{E}) \\ &= d^{+}_{q_i}(q_i) + \frac{1}{2}Md^{2}(x,q_i) - \frac{1}{2}Md^{2}(x,\mathbb{R}^{n}) \\ &= 1 + \frac{1}{2}M\|x-q_i\|^{2}. \end{aligned}$$

***Case 3:** $S^{+} = \{p_i, q_i\}$.
We have $\widehat{S}^{+} = [p_i, q_i]$, $S^{+}_{H} = L_{p_i q_i}$, $S^{+}_{E} = L_{k_i k_{i+1}}$, $S^{+}_{*} = [k_i, k_{i+1}]$, $S^{+}_{C} = S^{+}_{E} \cap S^{+}_{H} = \{q_i\}$, and $T^{+}_{\{p_i,q_i\}}$ to be the convex hull of $\{m_i, n_i, b_i, c_i\}$.

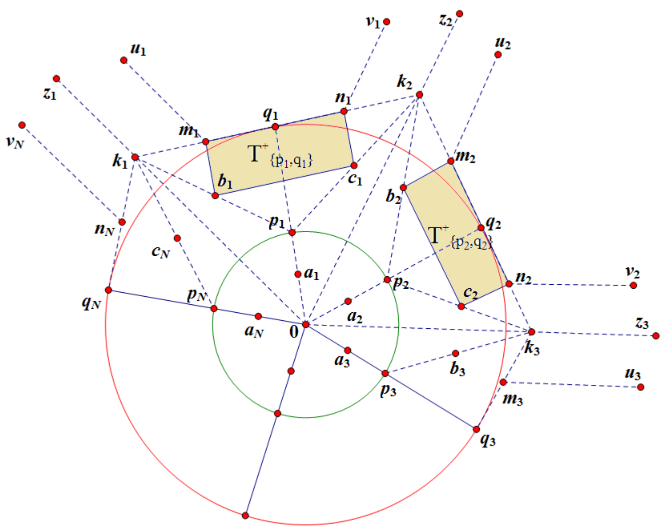

Figure 3.9: $S^{+} = \{p_i, q_i\}$

Thus for all $x \in T^{+}_{\{p_i,q_i\}}$ we have

$$\begin{aligned} w^{+}(F,A)(x) &= d^{+}_{S}(S^{+}_{C}) + \frac{1}{2}Md^{2}(x,S^{+}_{H}) - \frac{1}{2}Md^{2}(x,S^{+}_{E}) \\ &= d^{+}_{q_i}(q_i) + \frac{1}{2}Md^{2}(x,L_{p_i q_i}) - \frac{1}{2}Md^{2}(x,L_{k_i k_{i+1}}) \\ &= 1 + \frac{1}{2}Md^{2}(x,L_{p_i q_i}) - \frac{1}{2}Md^{2}(x,L_{k_i k_{i+1}}). \end{aligned}$$

***Case 4:** $S^{+} = \{p_i, p_{i+1}\}$.
We have $\widehat{S}^{+} = [p_i, p_{i+1}]$, $S^{+}_{H} = L_{p_i p_{i+1}}$, $S^{+}_{E} = L_{0k_{i+1}}$, $S^{+}_{*} = [0, k_{i+1}]$, $S^{+}_{C} = S^{+}_{E} \cap S^{+}_{H} = L_{0k_{i+1}} \cap L_{p_i p_{i+1}}$, and $T^{+}_{\{p_i,p_{i+1}\}}$ to be the convex hull of $\{a_i, a_{i+1}, c_i, b_{i+1}\}$.

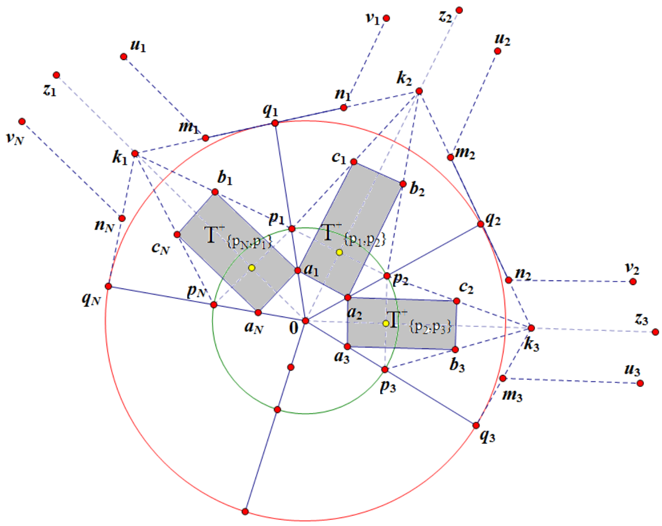

Figure 3.10: $S^+ = \{p_i, p_{i+1}\}$

Thus for all $x \in T^+_{\{p_i,p_{i+1}\}}$ we have

$$\begin{aligned} w^+(F,A)(x) &= d_S^+(S_C^+) + \frac{1}{2}Md^2(x,S_H^+) - \frac{1}{2}Md^2(x,S_E^+) \\ &= d_{p_i}^+(L_{0k_{i+1}} \cap L_{p_ip_{i+1}}) + \frac{1}{2}Md^2(x,L_{p_ip_{i+1}}) - \frac{1}{2}Md^2(x,L_{0k_{i+1}}) \\ &= \left\| \frac{p_i - p_{i+1}}{2} \right\|^2 + \frac{1}{2}Md^2(x,L_{p_ip_{i+1}}) - \frac{1}{2}Md^2(x,L_{0k_{i+1}}). \end{aligned}$$

***Case 5:**. $S^+ = \{q_i, q_{i+1}\}$
We have $\widehat{S}^+ = [q_i, q_{i+1}]$, $S_H^+ = L_{q_iq_{i+1}}$, $S_E^+ = L_{0k_{i+1}}$, $S_*^+ = R_{k_{i+1}z_{i+1}}$, $S_C^+ = S_E^+ \cap S_H^+ = L_{0k_{i+1}} \cap L_{q_iq_{i+1}}$, and $T^+_{\{p_i,p_{i+1}\}}$ to be the region bounded by ray $R_{n_iv_i}$, ray $R_{m_{i+1}u_{i+1}}$ and the segment $[n_i, m_{i+1}]$.

Thus for all $x \in T^+_{\{q_i,q_{i+1}\}}$ we have

$$\begin{aligned} w^+(F,A)(x) &= d_S^+(S_C^+) + \frac{1}{2}Md^2(x,S_H^+) - \frac{1}{2}Md^2(x,S_E^+) \\ &= d_{q_i}^+(L_{0k_{i+1}} \cap L_{q_iq_{i+1}}) + \frac{1}{2}Md^2(x,L_{q_iq_{i+1}}) - \frac{1}{2}Md^2(x,L_{0k_{i+1}}) \\ &= 1 + \left\| \frac{q_i - q_{i+1}}{2} \right\|^2 + \frac{1}{2}Md^2(x,L_{q_iq_{i+1}}) - \frac{1}{2}Md^2(x,L_{0k_{i+1}}). \end{aligned}$$

***Case 6:** $S^+ = \{p_i, p_{i+1}, q_i, q_{i+1}\}$.
We have $\widehat{S}^+$ to be the convex hull of $\{p_i, p_{i+1}, q_i, q_{i+1}\}$, $S_H^+ = \mathbb{R}^n$, $S_E^+ = \{k_{i+1}\}$, $S_*^+ = \{k_{i+1}\}$, $S_C^+ = S_E^+ \cap S_H^+ = \{k_{i+1}\}$, and $T^+_{\{p_i,p_{i+1}\}}$ to be the convex of $\{c_i, b_{i+1}, n_i, m_{i+1}\}$.

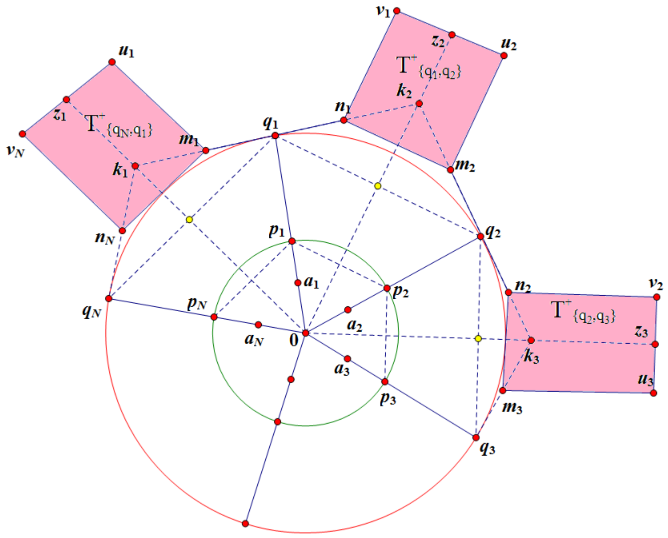

Figure 3.11: $S^+ = \{q_i, q_{i+1}\}$

Thus for all $x \in T^+_{\{p_i,p_{i+1}\}}$ we have

$$\begin{aligned} w^+(F,A)(x) &= d^+_S(S^+_C) + \frac{1}{2}Md^2(x,S^+_H) - \frac{1}{2}Md^2(x,S^+_E) \\ &= d^+_{q_i}(k_{i+1}) + \frac{1}{2}Md^2(x,\mathbb{R}^n) - \frac{1}{2}Md^2(x,k_{i+1}) \\ &= 1 + \|q_i - k_{i+1}\|^2 - \frac{1}{2}M\|x - k_{i+1}\|^2. \end{aligned}$$

***Case 7:** $S^+ = \{p_1, p_2, \dots, p_N\}$
We have $\widehat{S}^+$ to be the convex hull of $\{p_1, p_2, \dots, p_N\}$, $S^+_H = \mathbb{R}^n$, $S^+_E = \{0\}$, $S^+_* = \{0\}$, $S^+_C = S^+_E \cap S^+_H = \{0\}$, and $T^+_{\{p_1,p_2,\dots,p_N\}}$ to be the convex hull of $\{a_1, a_2, \dots, a_N\}$.

Thus for all $x \in T^+_{\{p_1,p_2,\dots,p_N\}}$ we have

$$\begin{aligned} w^+(F,A)(x) &= d^+_S(S^+_C) + \frac{1}{2}Md^2(x,S^+_H) - \frac{1}{2}Md^2(x,S^+_E) \\ &= d^+_{p_1}(0) + \frac{1}{2}Md^2(x,\mathbb{R}^n) - \frac{1}{2}Md^2(x,0) \\ &= 1 - \frac{1}{2}M\|x\|^2. \end{aligned}$$

The form of $w^-(F,A)$
Put $M = \Gamma^1(F,A) = 4$.
For any $p \in \Omega_1$ and $q \in \Omega_2$, we have

$$\begin{aligned} \widetilde{p}^- &= p + D_p f / M = p, \\ d^-_p(x) &= f_p + \frac{1}{2}(D_p f)^2/M - \frac{1}{4}M\|x - \widetilde{p}^-\|^2 = -\|x - p\|^2. \end{aligned}$$

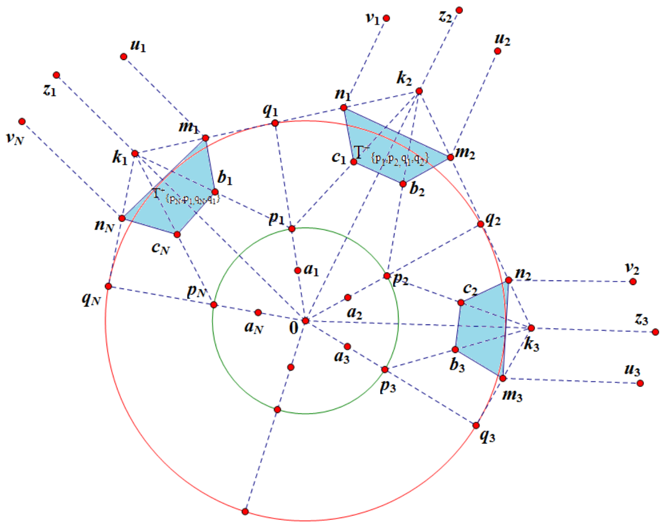

Figure 3.12: $S^+ = \{p_i, p_{i+1}, q_i, q_{i+1}\}$

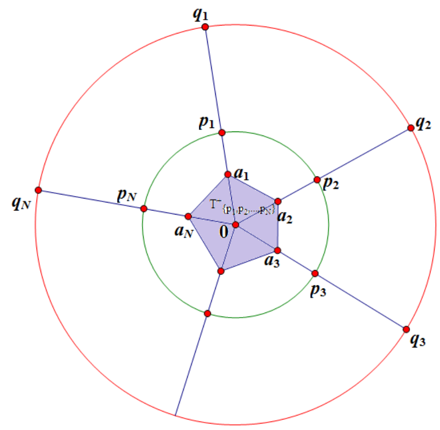

Figure 3.13: $S^+ = \{p_1, p_2, ..., p_N\}$

$$\begin{aligned}
\widetilde{q}^- &= q + D_q f/M = q, \\
d_q^-(x) &= f_q + \frac{1}{2}(D_q f)^2/M - \frac{1}{4}M\|x - \widetilde{q}^-\|^2 = 1 - \|x - q\|^2.
\end{aligned}$$

For all $i \in \{1, ..., N\}$, let ray R_{0z_i} be the bisector of the angle $\widehat{p_{i-1}0p_i}$. The tangent to the circle $B(0,1)$ at p_i cut the ray R_{0z_i} at l_i. We call a_i, d_i, e_i, r_i, s_i to be the midpoints of the segments $[0, p_i], [l_i, q_i], [l_{i+1}, q_i], [l_i, p_i], [l_{i+1}, p_i]$, respectively. Let rays $R_{d_i t_i}, R_{e_i y_i}$ such that $R_{d_i t_i}$ is parallel to R_{l_i, z_i} and $R_{e_i y_i}$ is parallel to $R_{l_{i+1} z_{i+1}}$.

Let $S^- \in K^-$.
***Case 1:** $S^- = \{p_i\}$.
We have $\widehat{S^-} = \{p_i\}$, $S_H^- = \{p_i\}$, $S_E^- = \mathbb{R}^n$, S_*^- to be the convex hull of $\{0, l_i, l_{i+1}\}$, $S_C^- = S_E^- \cap S_H^- = \{p_i\}$, and $T_{\{p_i\}}^-$ to be the convex hull of $\{a_i, r_i, s_i\}$.

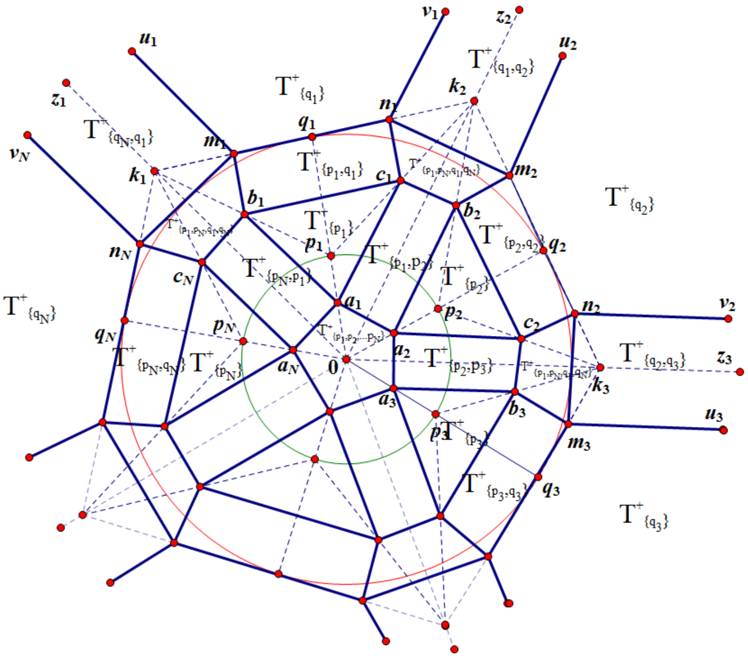

Figure 3.14: The function w^+

Thus for all $x \in T^-_{\{p_i\}}$ we have

$$\begin{aligned}
w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\
&= d^-_{p_i}(p_i) - \frac{1}{2}Md^2(x,p_i) + \frac{1}{2}Md^2(x,\mathbb{R}^n) \\
&= -\frac{1}{2}M\|x-p_i\|^2.
\end{aligned}$$

***Case 2:** $S^- = \{q_i\}$.
We have $\widehat{S}^- = \{q_i\}$, $S^-_H = \{q_i\}$, $S^-_E = \mathbb{R}^n$, S^-_* to be the region bounded by ray $R_{l_iz_i}$, ray $R_{l_{i+1}z_{i+1}}$ and the segment $[l_i,l_{i+1}]$. $S^-_C = S^-_E \cap S^-_H = \{q_i\}$, and $T^-_{\{p_i\}}$ to be the region bounded by the ray $R_{d_it_i}$, ray $R_{e_iy_i}$ and the segment $[d_i,e_i]$.

Thus for all $x \in T^-_{\{q_i\}}$ we have

$$\begin{aligned}
w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\
&= d^-_{q_i}(q_i) - \frac{1}{2}Md^2(x,q_i) + \frac{1}{2}Md^2(x,\mathbb{R}^n) \\
&= 1 - \frac{1}{2}M\|x-q_i\|^2.
\end{aligned}$$

***Case 3:** $S^- = \{p_i,q_i\}$.
We have $\widehat{S}^- = [p_i,q_i]$, $S^-_H = L_{p_iq_i}$, $S^-_E = L_{l_il_{i+1}}$, $S^-_* = [l_i,l_{i+1}]$. $S^-_C = S^-_E \cap S^-_H = \{p_i\}$,

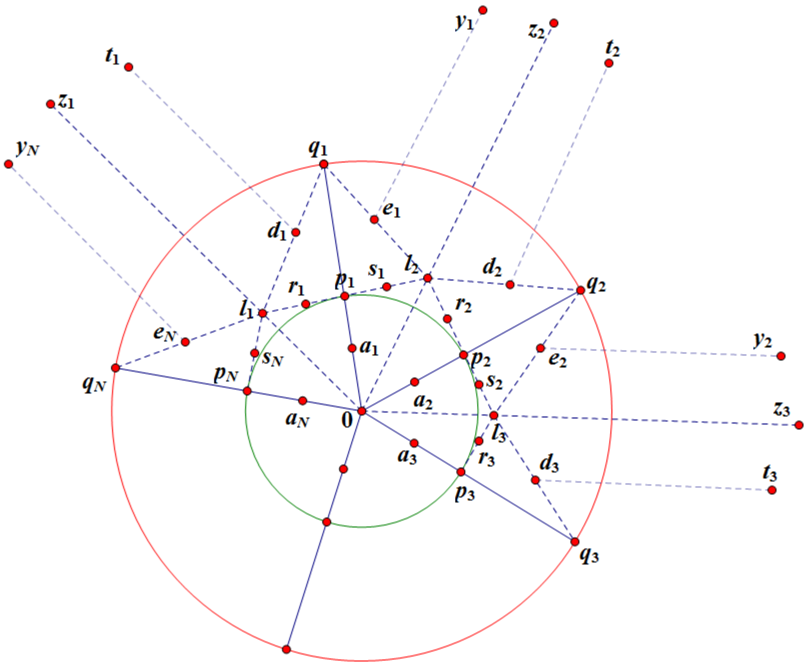

Figure 3.15: The rays R

and $T^-_{\{p_i,q_i\}}$ to be the convex hull of $\{r_i, s_i, e_i, d_i\}$.

Thus for all $x \in T^-_{\{p_i,q_i\}}$ we have

$$\begin{aligned} w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\ &= d^-_{p_i}(p_i) - \frac{1}{2}Md^2(x,L_{p_iq_i}) + \frac{1}{2}Md^2(x,L_{l_il_{i+1}}) \\ &= -\frac{1}{2}Md^2(x,L_{p_iq_i}) + \frac{1}{2}Md^2(x,L_{l_il_{i+1}}). \end{aligned}$$

***Case 4:** $S^- = \{p_i, p_{i+1}\}$.
We have $\widehat{S}^- = [p_i, p_{i+1}]$, $S^-_H = L_{p_ip_{i+1}}$, $S^-_E = L_{0z_{i+1}}$, $S^-_* = [0, l_{i+1}]$. $S^-_C = S^-_E \cap S^-_H = L_{0z_{i+1}} \cap L_{p_ip_{i+1}}$, and $T^-_{\{p_i,p_{i+1}\}}$ to be the convex hull of $\{a_i, a_{i+1}, r_{i+1}, s_i\}$.

Thus for all $x \in T^-_{\{p_i,p_{i+1}\}}$ we have

$$\begin{aligned} w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\ &= d^-_{p_i}(L_{0z_{i+1}} \cap L_{p_ip_{i+1}}) - \frac{1}{2}Md^2(x,L_{p_ip_{i+1}}) + \frac{1}{2}Md^2(x,L_{0z_{i+1}}) \\ &= -\left\|\frac{p_i - p_{i+1}}{2}\right\|^2 - \frac{1}{2}Md^2(x,L_{p_ip_{i+1}}) + \frac{1}{2}Md^2(x,L_{0z_{i+1}}). \end{aligned}$$

***Case 5:**. $S^- = \{q_i, q_{i+1}\}$
We have $\widehat{S}^- = [q_i, q_{i+1}]$, $S^-_H = L_{q_iq_{i+1}}$, $S^-_E = L_{0z_{i+1}}$, $S^-_* = R_{l_{i+1}z_{i+1}}$. $S^-_C = S^-_E \cap S^-_H =$

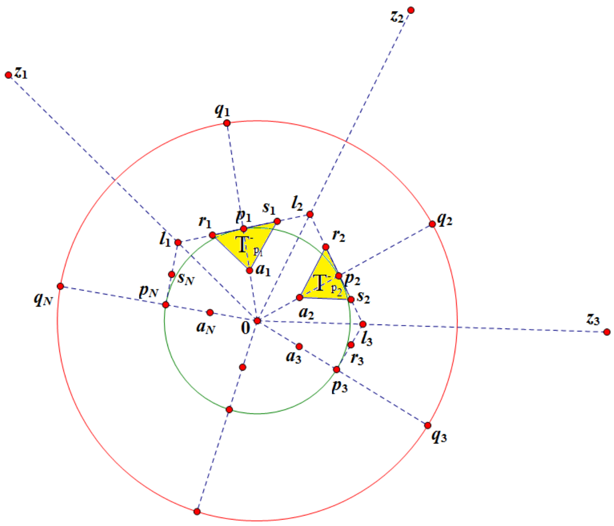

Figure 3.16: $S^- = \{p_i\}$

$L_{0z_{i+1}} \cap L_{q_iq_{i+1}}$, and $T^-_{\{q_i,q_{i+1}\}}$ to be the region bounded by ray $R_{e_iy_i}$, ray $R_{d_{i+1}t_{i+1}}$ and segment $[e_id_{i+1}]$.

Thus for all $x \in T^-_{\{q_i,q_{i+1}\}}$ we have

$$\begin{aligned} w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\ &= d^-_{q_i}(L_{0z_{i+1}} \cap L_{q_iq_{i+1}}) - \frac{1}{2}Md^2(x,L_{q_iq_{i+1}}) + \frac{1}{2}Md^2(x,L_{0z_{i+1}}) \\ &= 1 - \left\| \frac{q_i - q_{i+1}}{2} \right\|^2 - \frac{1}{2}Md^2(x,L_{q_iq_{i+1}}) + \frac{1}{2}Md^2(x,L_{0z_{i+1}}). \end{aligned}$$

***Case 6:** $S^- = \{p_i, p_{i+1}, q_i, q_{i+1}\}$.
We have $\widehat{S}^-$ to be the convex hull of $\{p_i, p_{i+1}, q_i, q_{i+1}\}$, $S^-_H = \mathbb{R}^n$, $S^-_E = \{l_{i+1}\}$, $S^-_* = \{l_{i+1}\}$. $S^-_C = S^-_E \cap S^-_H = \{l_{i+1}\}$, and $T^-_{\{p_i,p_{i+1},q_i,q_{i+1}\}}$ to be the convex hull of $\{s_i, r_{i+1}, e_i, d_{i+1}\}$.

Thus for all $x \in T^-_{\{p_i,p_{i+1},q_i,q_{i+1}\}}$ we have

$$\begin{aligned} w^-(F,A)(x) &= d^-_S(S^-_C) - \frac{1}{2}Md^2(x,S^-_H) + \frac{1}{2}Md^2(x,S^-_E) \\ &= d^-_{p_i}(l_{i+1}) - \frac{1}{2}Md^2(x,\mathbb{R}^n) + \frac{1}{2}Md^2(x,l_{i+1}) \\ &= -\|p_i - l_{i+1}\|^2 + \frac{1}{2}M\|x - l_{i+1}\|^2. \end{aligned}$$

***Case 7:** $S^- = \{p_1, p_2, ..., p_N\}$
We have $\widehat{S}^-$ to be the convex hull of $\{p_1, p_2, ..., p_N\}$, $S^-_H = \mathbb{R}^n$, $S^-_E = \{0\}$, $S^-_* = \{0\}$. $S^-_C = S^-_E \cap S^-_H = \{0\}$, and $T^-_{\{p_1,p_2,...,p_N\}}$ to be the convex hull of $\{a_1, a_2, ..., a_N\}$.

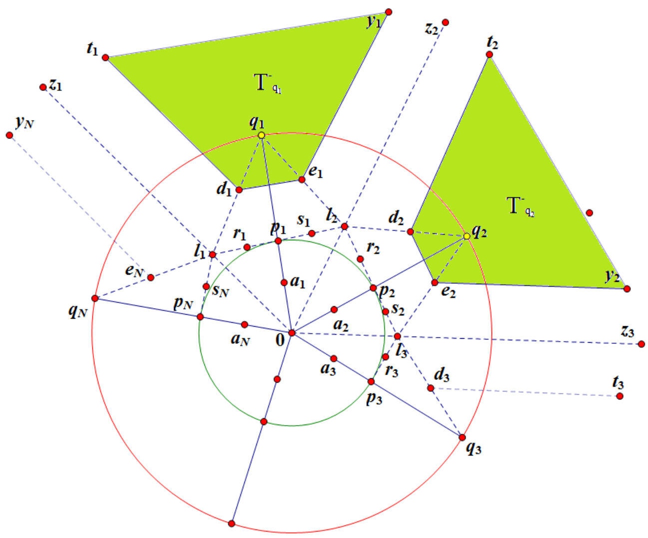

Figure 3.17: $S^- = \{q_i\}$

Thus for all $x \in T^-_{\{p_1,p_2,\ldots,p_N\}}$ we have

$$\begin{aligned} w^-(F,A)(x) &= d_S^-(S_C^-) - \frac{1}{2}Md^2(x,S_H^-) + \frac{1}{2}Md^2(x,S_E^-) \\ &= d_{p_1}^-(0) - \frac{1}{2}Md^2(x,\mathbb{R}^n) + \frac{1}{2}Md^2(x,0) \\ &= -1 + \frac{1}{2}M\|x\|^2. \end{aligned}$$

Since above computing, by limiting, we obtain

$$\begin{aligned} w^+(x) &= -w^-(x) = 1 - \frac{\kappa}{2}x^2, \ \forall x \in \mathscr{D}_{0,\frac{1}{2}}, \\ w^+(x) &= -w^-(x) = \frac{\kappa}{2}d^2(x,\partial\Omega_1), \ \forall x \in \mathscr{D}_{\frac{1}{2},1}, \\ w^+(x) &= w^-(x) = \frac{\kappa}{2}d^2(x,\partial\Omega_1), \ \forall x \in \mathscr{D}_{1,\frac{3}{2}}, \\ w^+(x) &= w^-(x) = 1 - \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{\frac{3}{2},2}, \\ w^+(x) &= 1 + \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{2,+\infty}, \\ w^-(x) &= 1 - \frac{\kappa}{2}d^2(x,\partial\Omega_2), \ \forall x \in \mathscr{D}_{2,+\infty}, \end{aligned}$$

where $\kappa = \Gamma^1(F;\Omega) = 4$.
We see that $W = \frac{W^+ + W^-}{2}$ is an AMLE of F on $\mathbb{R}^2$ (although W^+ and W^- are not AMLEs of F on $\mathbb{R}^2$) and $w \notin C^2(\mathbb{R}^2,\mathbb{R})$.

Moreover, all MLEs of F coincide on $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ (because $w^+ = w^-$ on $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$).

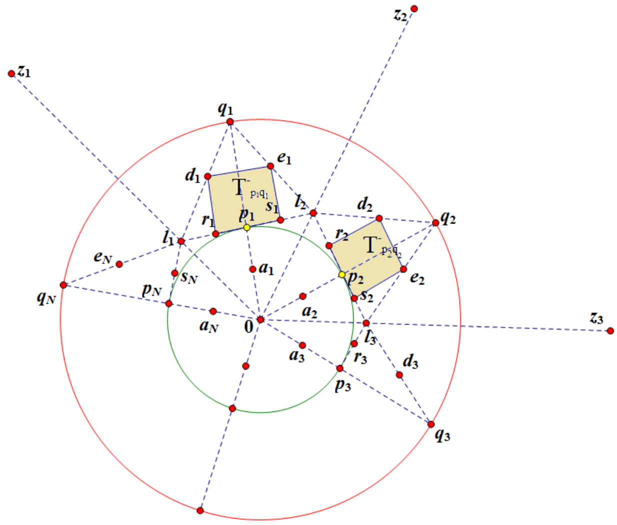

Figure 3.18: $S^{-} = \{p_i, q_i\}$

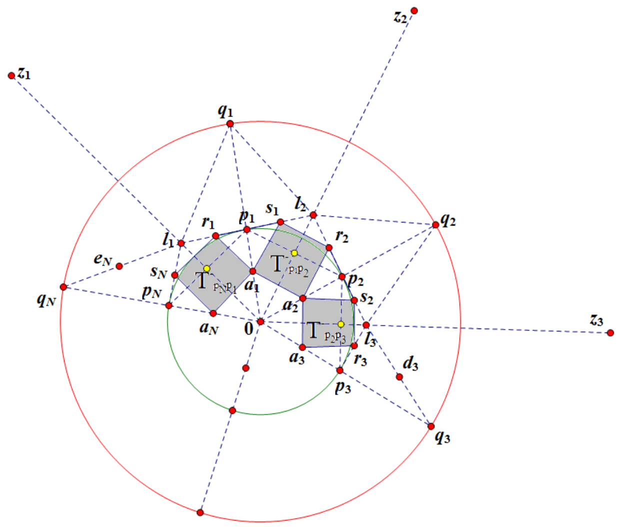

Figure 3.19: $S^{-} = \{p_i, p_{i+1}\}$

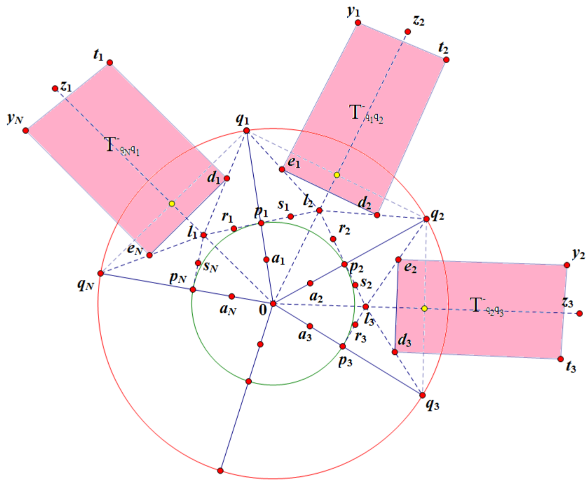

Figure 3.20: $S^- = \{q_i, q_{i+1}\}$

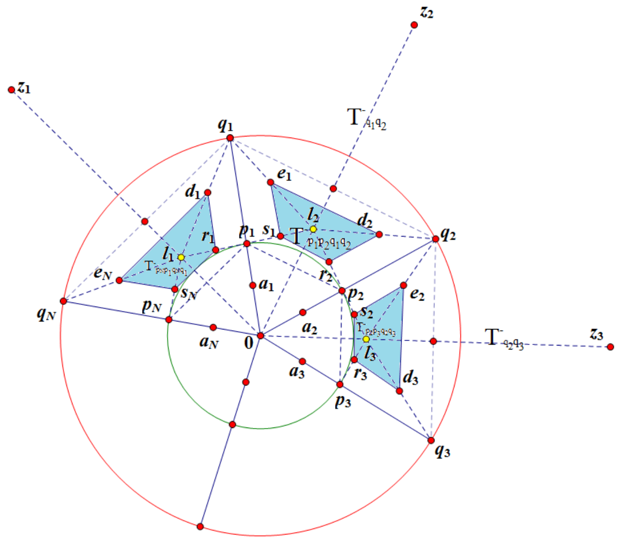

Figure 3.21: $S^- = \{p_i, p_{i+1}, q_i, q_{i+1}\}$

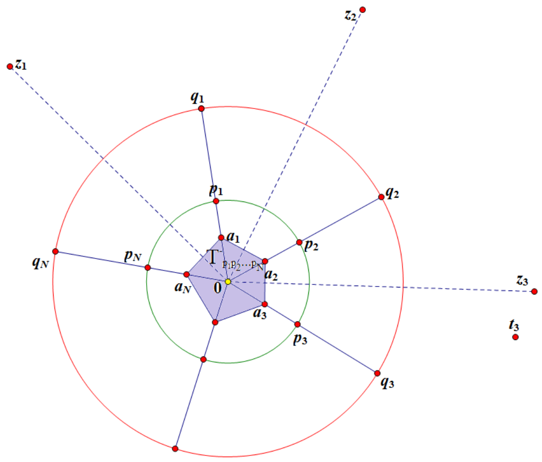

Figure 3.22: $S^- = \{p_1, p_2, ..., p_N\}$

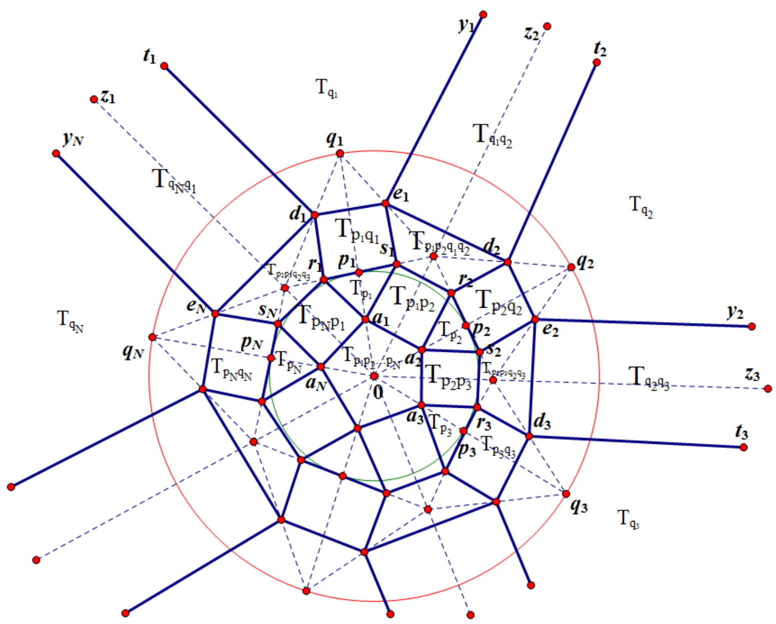

Figure 3.23: The function w^-

Chapter 4

Kirszbraun extension on a connected finite graph

Abstract: We prove that the tight function introduced Sheffield and Smart (2012) [51] is a Kirszbraun extension. In the real-valued case we prove that Kirszbraun extension is unique. Moreover, we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension in polynomial time.

Key words: Minimal, Lipschitz, extension, Kirszbraun, harmonious, AMLE.

This chapter is based on the paper: Kirszbraun extension on a connected finite graph. *Preprint.* 2015. arxiv 1511.01748. It has been written in collaboration with Erwan Le Gruyer.

4.1 Introduction

Let A be a compact subset of $\mathbb{R}^n$. The best Lipschitz constant of a Lipschitz function $g : A \to \mathbb{R}^m$ is

$$\mathrm{Lip}(g,A) := \sup_{x\neq y\in A} \frac{\|g(x)-g(y)\|}{\|x-y\|}, \tag{4.1}$$

where $\|.\|$ is Euclidean norm.

When $m=1$, Aronsson in 1967 [5] proved the existence of absolutely minimizing Lipschitz extension (AMLE), i.e., a extension u of g satisfying

$$\mathrm{Lip}(u;V) = \mathrm{Lip}(u,\partial V), \text{ for all } V \subset\subset \mathbb{R}^n \backslash A. \tag{4.2}$$

Jensen in 1993 [28] proved the uniqueness of AMLE under certain conditions.

In this chapter we begin by studying the discrete version of the existence and uniqueness of AMLE for case $m \geq 2$.

We define the function

$$\lambda(g,A)(x) := \inf_{y\in\mathbb{R}^m} \sup_{a\in A} \frac{\|g(a)-y\|}{\|a-x\|} \text{ if } x \in \mathbb{R}^n \backslash A. \tag{4.3}$$

From Kirszbraun theorem (see [19, 32]) the function $\lambda(g,A)$ is well-defined and

$$\lambda(g,A)(x) \leq \mathrm{Lip}(g,A).$$

Moreover, (see [19, Lemma 2.10.40]) for any $x \in \mathbb{R}^n \backslash A$ there exists a unique $y(x) \in \mathbb{R}^m$ such that

$$\lambda(g,A)(x) = \sup_{a\in A} \frac{\|g(a)-y(x)\|}{\|a-x\|}, \tag{4.4}$$

and $y(x)$ belongs to the convex hull of the set

$$B = \{g(z) : z \in A \text{ and } \frac{\|g(z)-y(x)\|}{\|z-x\|} = \lambda(g,A)(x)\}.$$

Thus we can define

$$K(g,A)(x) := \begin{cases} g(x) \text{ if } x \in A; \\ \arg\min_{y\in\mathbb{R}^m} \sup_{a\in A} \frac{\|g(a)-y\|}{\|a-x\|} \text{ if } x \in \mathbb{R}^n \backslash A. \end{cases} \tag{4.5}$$

We say that $K(g,A)(x)$ is *the Kirszbraun value of g restricted on A at point x*. The function $K(g,A)(x)$ is the best extension at point x such that the Lipschitz constant is minimal. We produce a method to compute $\lambda(g,A)(x)$ and $K(g,A)(x)$ in section 4.4.

Let $G = (V,E,\Omega)$ be a connected finite graph with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$.

For $x \in V$, we define

$$S(x) := \{y \in V : (x,y) \in E\} \tag{4.6}$$

to be the neighborhood of x on G.

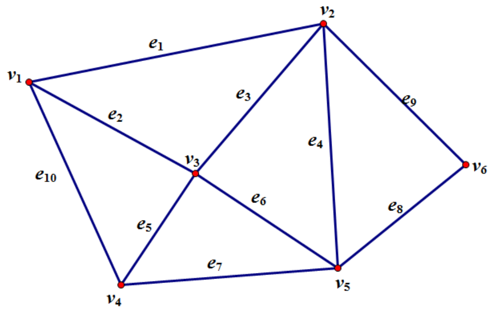

Figure 4.1: A simple picture of graph G

Example 4.1.1. In Figure 4.1 we have $V = \{v_1, ..., v_6\}$, $E = \{e_1, ..., e_{10}\}$, $S(v_3) = \{v_1, v_2, v_4, v_5\}$.

Let $f : \Omega \to \mathbb{R}^m$. We consider the following functional equation with Dirichlet's condition:

$$u(x) = \begin{cases} K(u, S(x))(x) \ \forall x \in V \backslash \Omega; \\ f(x) \ \forall x \in \Omega. \end{cases} \tag{4.7}$$

We say that a function u satisfying (4.7) is a *Kirszbraun extension* of f on graph G. This extension is the optimal Lipschitz extension of f on graph G since for any $x \in V \backslash \Omega$, there is no way to decrease $\mathrm{Lip}(u, S(x))$ by changing the value of u at x.

In real valued case $m = 1$, the function $K(u, S(x))(x)$ was considered by Oberman [45] and he used this function to obtain a convergent difference scheme for the AMLE. Le Gruyer [37] showed the explicit formula for $K(u, S(x))(x)$ as follows

$$K(u, S(x))(x) = \inf_{z \in S(x)} \sup_{q \in S(x)} M(u, z, q)(x), \tag{4.8}$$

where

$$M(u, z, q)(x) := \frac{\|x - z\| u(q) + \|x - q\| u(z)}{\|x - z\| + \|x - q\|}.$$

Le Gruyer studied the solution of (4.7) on a network where $K(u, S(x))(x)$ satisfying (4.8). This solution plays an important role in approximation arguments for AMLE in Le Gruyer [37].

The Kirszbraun extension u is a generalization of the solution in the previous works of Le Gruyer for vector valued cases ($m \geq 2$). We prove that the tight function introduced by Sheffield and Smart (2012) [51] is a Kirszbraun extension. Therefore, we have the existence of a Kirszbraun extension, but in general Kirszbraun extension maybe not unique.

In the scalar case $m = 1$, Le Gruyer [37] defined a network on a metric space (X, d) as follows

Definition 4.1.2. A network on a metric space (X,d) is a couple (N,U) where $N \subset X$ denotes a finite non-empty subset of $\mathbb{R}^n$ and U a mapping $x \in N \to U(x) \subset N$ which satisfies

(P1) For any $x \in N$, $x \in U(x)$.

(P2) For any $x, y \in N$, $x \in U(y)$ iff $y \in U(x)$.

(P3) For any $x, y \in N$, there exists $x_1, ..., x_{n-1} \in G$ such that $x_1 = x$, $x_n = y$ and $x_i \in U(x_{i+1})$ for $i = 1, ..., n-1$.

(P4) For any $x \in N$, any $y \in N \backslash U(x)$ there exists $z \in U(x)$ such that $d(z,y) < d(x,y)$.

In the above definition, $U(x)$ is called the neighborhood of x on network (N,U). Let $g : A \subset X \to \mathbb{R}$. In [37] Le Gruyer defined the Kirszbraun extension of g with respect to the network (see [37, page 30]) and he proved the existence and uniqueness of the Kirszbraun extension of g on the network. In particular, when $X = \mathbb{R}^n$ equipped with the Euclidean norm, Le Gruyer obtained the approximation for AMLE by a sequence Kirszbraun extensions (u_n) of networks (N_n, U_n) $(n \in \mathbb{N})$ having some good properties.

Similarly to Le Gruyer's result about the uniqueness of the Kirszbraun extension on a network, in this chapter we prove the uniqueness of the Kirszbraun extension u of f on graph G when $m = 1$. The graph is more general than the network since there are many graphs that do not satisfy (P4). Moreover, in the scalar case $m = 1$, we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension u in polynomial time. This algorithm is similar to the algorithm produced by Lazarus el al. (1999) [34] when they calculate the Richman cost function. Assuming Jensen's hypotheses [28], since this algorithm computes exactly solution of (4.7) and by using the argument of Le Gruyer [37] (the approximation for AMLE by a sequence Kirszbraun extensions (u_n) of networks (N_n, U_n) $(n \in \mathbb{N})$), we obtain a new method to approximate the viscosity solution of Equation $\Delta_\infty u = 0$ under Dirichlet's condition f.

In the above algorithm, the explicit formula of $K(u, S(x))$ in (4.8) and the order structure of real number set play important role. The generalization of the algorithm to vector valued case ($m \geq 2$) is difficult since we do not know the explicit formula of $K(u, S(x))$ when $m \geq 2$ and $\mathbb{R}^2$ does not have any useful order structure. Extending the results of the approximation of AMLE to vector valued cases ($m \geq 2$) presents many difficulties which have limited the number of results in this direction, see [27] and the references therein.

4.2 The existence of Kirszbraun extension

In this section, we prove the existence of Kirszbraun extension satisfying Equation (4.7).

Let $G = (V, E, \Omega)$ be a connected finite graph with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$ and let $f : \Omega \to \mathbb{R}^m$.

We denote $E(f)$ to be the set of all extensions of f on G.

Let $v \in E(f)$. The *local Lipschitz constant* of v at vertex $x \in V \backslash \Omega$ is given by

$$Lv(x) := \sup_{y \in S(x)} \frac{\|v(y) - v(x)\|}{\|y - x\|},$$

where $S(x)$ is neighborhood of x on G.

Definition 4.2.1. [1] If $u, v \in E(f)$ satisfy

$$\max\{Lu(x) : Lu(x) > Lv(x), x \in V\backslash\Omega\} > \max\{Lv(x) : Lv(x) > Lu(x), x \in V\backslash\Omega\},$$

then we say that v is *tighter* than u on G. We say that u is a *tight extension* of f on G if there is no v tighter than u.

Theorem 4.2.2. *[51, Theorem 1.2] There exists a unique extension u that is tight of f on G. Moreover, u is tighter than every other extension v of f.*

Proposition 4.2.3. *Let $u \in E(f)$. Let $x \in V\backslash\Omega$, we define*

$$v(y) = \begin{cases} u(y), \text{ if } y \in V\backslash\{x\}, \\ K(u, S(x))(x), \text{ if } y = x. \end{cases}$$

If $K(u, S(x))(x) \neq u(x)$ then v is tighter than u.

Proof.
***Step 1:** In this step we prove that for any $y \in V\backslash\Omega$, we obtain

$$Lv(y) \leq \max\{Lv(x), Lu(y)\}. \tag{4.9}$$

Indeed,

*If $y \notin S(x) \cup \{x\}$. Since $v(y) = u(y)$ and $v(z) = u(z)$ for all $z \in S(y)$, we obtain

$$Lv(y) = Lu(y).$$

*If $y = x$. Since $v(x) \neq u(x)$ and $v(x)$ is the Kirszbraun value of u restricted on $S(x)$ at point x, we have

$$Lv(y) < Lu(y).$$

*If $y \in S(x)$ we have

$$\begin{aligned} Lv(y) &= \max_{z \in S(y)} \frac{\|v(z) - v(y)\|}{\|z - y\|} \\ &= \max_{z \in S(y)\backslash\{x\}} \left\{ \frac{\|v(x) - v(y)\|}{\|x - y\|}, \frac{\|u(z) - u(y)\|}{\|z - y\|} \right\} \\ &\leq \max\{Lv(x), Lu(y)\}. \end{aligned}$$

Therefore, for any $y \in V\backslash\Omega$ we have

$$Lv(y) \leq \max\{Lv(x), Lu(y)\}.$$

***Step 2:** In this step we prove that v is tighter than u. It means that we need to show that

$$\max\{Lv(y) : Lv(y) > Lu(y), y \in V\backslash\Omega\} < \max\{Lu(y) : Lu(y) > Lv(y), y \in V\backslash\Omega\}$$

[1] By convention, if $C = \emptyset$ then $\max_C = 0$.

Indeed, if $Lv(y) > Lu(y)$ then from (4.9) we have $Lv(y) \leq Lv(x)$. Thus

$$\max\{Lv(y) : Lv(y) > Lu(y), y \in V\backslash\Omega\} \leq Lv(x) \tag{4.10}$$

Since $v(x) \neq u(x)$ and $v(x)$ is the Kirszbraun value of u restricted on $S(x)$ at point x, we have

$$Lv(x) < Lu(x). \tag{4.11}$$

From (4.10) and (4.11) we obtain

$$\begin{aligned}
\max\{Lv(y) : Lv(y) > Lu(y), y \in V\backslash\Omega\} &\leq Lv(x) \\
&< Lu(x) \\
&\leq \max\{Lu(y) : Lu(y) > Lv(y), y \in V\backslash\Omega\}.
\end{aligned}$$

□

We obtain the existence of a Kirszbraun extension satisfying Equation (4.7) as a consequence of the following theorem.

Theorem 4.2.4. *If u is a tight extension of f, then u is a Kirszbraun extension of f.*

Proof. Let u be a tight extension of f. Suppose, by contradiction, that there are some $x \in V\backslash\Omega$ such that

$$K(u,S(x))(x) \neq u(x). \tag{4.12}$$

we define

$$v(y) = \begin{cases} u(y), & \text{if } y \in V\backslash\{x\}, \\ K(u,S(x)), & \text{if } y = x. \end{cases}$$

By applying Proposition 4.2.3 we have v tighter than u. This is impossible since u is tight of f. □

4.3 An algorithm to compute Kirszbraun extension when $m = 1$

In this section, let $G = (V, E, \Omega)$ be a connected finite graph, with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$. Let $f : \Omega \to \mathbb{R}$.

We recall some properties of Kirszbraun function introduced in (4.5) which are useful in the proof of Theorem 4.3.2.

Theorem 4.3.1. *Let $S = \{x_1, ..., x_n\} \subset \mathbb{R}^n$ and $u : S \to \mathbb{R}$. For each $x \in \mathbb{R}^n\backslash S$, we use the notation $d_i = \|x_i - x\|$, $i = 1, ..., n$.*

(a) (see [45, Theorem 5]) We have

$$K(u,S)(x) = \frac{d_i u(x_j) + d_j u(x_i)}{d_i + d_j},$$

where i, j are the indexes which satisfy

$$\frac{|u(x_i)-u(x_j)|}{d_i+d_j} = \max_{k,l=1}^{n}\left\{\frac{|u(x_k)-u(x_l)|}{d_k+d_l}\right\}.$$

(b) (see[19, Lemma 2.10.40]) Let

$$\lambda(u,S)(x) := \inf_{y\in\mathbb{R}} \sup_{a\in S} \frac{\|u(a)-y\|}{\|a-x\|} \text{ if } x \in \mathbb{R}^n \backslash S. \tag{4.13}$$

then the set

$$B = \left\{u(z) : z \in S \text{ and } \frac{\|u(z)-K(u,S)(x)\|}{\|z-x\|} = \lambda(u,S)(x)\right\},$$

is not empty, and $K(u,S)(x)$ belongs to the convex hull of B.

Theorem 4.3.2. *There is a unique Kirszbraun extension u of f on the graph G. Moreover, the Kirszbraun extension u of f can be calculated in polynomial time.*

Before proving Theorem 4.3.2, we need the following definition

Definition 4.3.3. Let $G' = (V', E', \Omega)$ be a subgraph of G, i.e. $\Omega \subset V' \subset V$ and $E' \subset E$. A *connecting path with respect to G'* is a sequence

$$v_0, e_1, v_1, ..., e_n, v_n \ (n \geq 1)$$

of distinct vertices and edges in G such that
* each e_i is an edge joining v_{i-1} and v_i,
* v_0 and v_n are in V',
* for $1 \leq i < n$, v_i is in $V \backslash V'$, and
* for $1 \leq i \leq n$, e_i is in $E \backslash E'$

Let u' be a Kirszbraun extension of f on G'. We define

$$c := \frac{|u'(v_n)-u'(v_0)|}{\sum\limits_{i=1}^{n} \|v_i - v_{i-1}\|}.$$

We say that c is the *slope* of the connecting path $v_0, e_1, v_1, ..., e_n, v_n$ with respect to u'

Proof of Theorem 4.3.2. We construct an increasing sequence of subgraph $G_n = (V_n, E_n, \Omega)$ of G and u_n which is a Kirszbraun extension of f on G_n. We finish the algorithm with a Kirszbraun extension u on G.

Step 1: Construct an increasing sequence of subgraph

We begin with the trivial subgraph $G_1 = (V_1, E_1, \Omega)$ where $V_1 = \Omega$, $E_1 = \emptyset$ and let $u_1 = f$ on Ω. It is clear that u_1 is a Kirszbraun extension of f on G_1. The algorithm then proceeds in stages.

Suppose that after n stages we have an increasing sequence of subgraph $G_l = (V_l, E_l, \Omega)$ of G and u_l is a Kirszbraun extension of f on G_l for $l = 1, ..., n$.

If there are no connecting paths with respect to G_n and u_n, we go to step 2.

If there are some connecting paths with respect to G_n and u_n. We construct G_{n+1} subgraph of G and u_{n+1} Kirszbraun extension of f on G_{n+1} as follows:

Find a connecting path $v_0, e_1, v_1, ..., e_k, v_k$ $(k \geq 1)$ on G_n with respect to u_n with largest possible slope c_n.

Without loss of generality, we label the vertices of the path so that $u_n(v_k) \geq u_n(v_0)$. We define

$$\begin{aligned} u_{n+1}(x) &:= \begin{cases} u_n(x), \ \forall x \in G_n \\ u_n(v_0) + c_n \sum_{j=1}^{i} \|v_j - v_{j-1}\| \ , \text{ if } x = v_i \text{ for } i = 1, ..., k-1. \end{cases} \\ V_{n+1} &:= V_n \cup \{v_1, ..., v_{k-1}\} \\ E_{n+1} &:= E_n \cup \{e_1, .., e_k\} \end{aligned} \tag{4.14}$$

We will show that u_{n+1} is a Kirszbraun extension of f on graph $G_{n+1} = (V_{n+1}, E_{n+1}, \Omega)$.

For $x \in V_{n+1} \backslash \Omega$, let

$$S_i(x) := \{y \in V_i : (x,y) \in E_i\} \ \text{ for } i \in \{1, ..., n+1\}.$$

be the neighborhood of x with respect to G_i.

Case 1: $x \in V_n \backslash \{v_0, v_k\}$.

We have $S_{n+1}(x) = S_n(x)$, $u_{n+1}(z) = u_n(z)$ for all $z \in S_{n+1}(x) \cup \{x\}$ and $u_n(x) = K(u_n, S_n(x))(x)$ since u_n is Kirszbraun of G_n . Thus

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x), \ \text{ for } x \in V_n \backslash \{v_0, v_k\}.$$

Case 2: $x \in \{v_1, ..., v_{k-1}\}$.

Noting that $S_{n+1}(v_i) = \{v_{i-1}, v_{i+1}\}$ for all $i = 1, ..., k-1$. Moreover, from (4.14), we have

$$\frac{u_{n+1}(v_i) - u_{n+1}(v_{i-1})}{\|v_i - v_{i-1}\|} = c_n \ , \forall i : 1 \leq i \leq k.$$

Hence

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x) \ \forall x \in \{v_1, ..., v_{n-1}\}.$$

Case 3: $x \in \{v_0, v_k\}$.

We need to prove that

$$u_{n+1}(v_0) = K(u_{n+1}, S_{n+1}(v_0))(v_0). \tag{4.15}$$

(Proving $u_{n+1}(v_k) = K(u_{n+1}, S_{n+1}(v_k))(v_k)$ is similar.)

To see (4.15), we must show that

$$\sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|} \tag{4.16}$$

Noting that $u_{n+1}(x) = u_n(x)$ for all $x \in S_n(v_0) \cup \{v_0\}$, $S_{n+1}(v_0) = S_n(v_0) \cup \{v_1\}$ and $c_n = \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|}$. Moreover, since u_n is a Kirszbraun extension of f on G_n, we have

$$\sup_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|}.$$

Thus

$$\begin{aligned}
\sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} &= \sup_{x \in S_n(v_0) \cup \{v_1\}} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} \\
&= \max_{x \in S_n(v_0)} \left\{ \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|} \right\} \\
&= \max \left\{ \max_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, c_n \right\},
\end{aligned}$$

and

$$\begin{aligned}
\max_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} &\leq \inf_{y \in \mathbb{R}} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|} \\
&\leq \inf_{y \in \mathbb{R}} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|}.
\end{aligned}$$

Therefore, to obtain Equation (4.16), we need to prove that

$$c_n \leq \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \tag{4.17}$$

for some $x \in S_n(v_0)$.

Let $\mathscr{F}$ be the set of slope of connecting paths occurring in the algorithm. Remark that each edges and each vertices entered in our algorithm relate with a slope in $\mathscr{F}$. So that, for any $y \in V_n$, there exist some $x \in S_n(y)$ and $c \in \mathscr{F}$ such that

$$c = \frac{|u_n(x) - u_n(y)|}{\|x - y\|}. \tag{4.18}$$

From above remark, to see (4.17), we need to show that the sequence of slope of connecting paths occurring in the algorithm is non-increasing. We show this in our present notation. Suppose that

$$w_0, f_1, w_1, \ldots, f_m, w_m \ (m \geq 1)$$

is a connecting path on G_{n+1} with respect to u_{n+1} with slope c_{n+1}. We need to prove that $c_n \geq c_{n+1}$. We assume without loss of generality that $u_{n+1}(w_0) \leq u_{n+1}(w_m)$.

• If w_0 and w_m are both in V_n then the connecting path on G_{n+1} with respect to u_{n+1} is actually the connecting path on G_n with respect to u_n . Therefore, since c_n is the largest slope of connecting paths with respect to G_n and u_n, we have $c_n \geq c_{n+1}$.

• If $w_0 = v_i$ and $w_m = v_j$ for some $0 \leq i < j \leq k$. We consider the path through the vertices

$$v_0, \dots, v_{i-1}, w_0, \dots, w_m, v_{j+1}, \dots, v_k.$$

The slope of above path is

$$c = \frac{|u_n(v_k) - u_n(v_0)|}{\sum\limits_{l=1}^{i} \|v_l - v_{l-1}\| + \sum\limits_{l=1}^{m} \|w_l - w_{l-1}\| + \sum\limits_{l=j+1}^{k} \|v_l - v_{l-1}\|}.$$

Since c_n is the largest slope of connecting paths with respect to G_n and u_n, we have $c_n \geq c$. Moreover,

$$c_n = \frac{|u_n(v_k) - u_n(v_0)|}{\sum\limits_{l=1}^{k} \|v_l - v_{l-1}\|},$$

thus we obtain

$$\sum_{l=1}^{m} \|w_l - w_{l-1}\| \geq \sum_{l=i+1}^{j} \|v_l - v_{l-1}\|.$$

Hence

$$c_{n+1} = \frac{|u_{n+1}(w_m) - u_{n+1}(w_0)|}{\sum\limits_{k=1}^{m} \|w_k - w_{k-1}\|} = \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum\limits_{k=1}^{m} \|w_k - w_{k-1}\|} \leq \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum\limits_{l=i+1}^{j} \|v_l - v_{l-1}\|} = c_n.$$

Step 2: Completing the algorithm

If there are no connecting paths with respect to $G_n = (V_n, E_n, \Omega)$ and u_n. Then each unlabeled vertex v is connected via edges not in E_n to exactly one vertex w of V_n. We extend u_n to the point v by putting $u_n(v) := u_n(w)$. This completes the algorithm, and we obtains a Kirszbraun extension of f.

Time complexity

Denote $|V|$ is the cardinal of vertices of G and $|E|$ is the cardinal of edges of G.

Each stage adds at least one edge, and each stage can be accomplished by all-pairs shortest paths *Floyd-Warshall algorithm [24, 59]* with time complexity $O(|V|^3)$.

Therefore, if we use the Floyd-Warshall algorithm to find the shortest path, then our algorithm to compute the Kirszbraun extension on graph can be calculated in $O(|V|^3|E|)$.

Uniqueness

Let u be the Kirszbraun extension of f defined by the algorithm above and h be another Kirszbraun extension of f. Let v be the first vertex added by algorithm such that $u(v) \neq h(v)$.

• If v is added to a subgraph $G' = (V', E', \Omega)$ as part of a connecting path through the vertices

$$v_0, \dots, v_k, \dots, v_n$$

with slope c and $v = v_k$.

We can assume without loss of generality that $u(v_0) \leq u(v_n)$. Let

$$\mathscr{L} = \{v_i : 0 \leq i \leq n, h(v_i) \geq u(v_i), h(v_i) - h(v_{i-1}) > u(v_i) - u(v_{i-1})\}$$

We prove that $\mathscr{L} \neq \emptyset$. Indeed, by contradiction, suppose that $\mathscr{L} = \emptyset$. Since $u(v_0) = h(v_0)$ and $\mathscr{L} = \emptyset$ we must have

$$h(v_1) \leq u(v_1).$$

If $h(v_2) > u(v_2)$ then

$$h(v_2) - h(v_1) > u(v_2) - u(v_1).$$

Hence $v_2 \in \mathscr{L}$. This contradicts with $\mathscr{L} = \emptyset$. Thus we must have

$$h(v_2) \leq u(v_2).$$

By induction, we have

$$h(v_i) \leq u(v_i) \; \forall i : 0 \leq i \leq k. \tag{4.19}$$

Since $v = v_k$, $h(v) \neq u(v)$ and (4.19), we have $h(v_k) < u(v_k)$. Thus if $h(v_{k+1}) \geq u(v_{k+1})$ then

$$h(v_{k+1}) - h(v_k) > u(v_{k+1}) - u(v_k).$$

Hence $v_{k+1} \in \mathscr{L}$. This contradicts with $\mathscr{L} = \emptyset$. Thus we must have

$$h(v_{k+1}) < u(v_{k+1}).$$

By induction, we have

$$h(v_i) < u(v_i), \; \forall k \leq i \leq n.$$

But we know that $h(v_n) = u(v_n)$, thus we have a contradiction. Therefore $\mathscr{L} \neq \emptyset$.

Let $v_l \in \mathscr{L}$. We have

$$\begin{cases} h(v_l) \geq u(v_l); \\ h(v_l) - h(v_{l-1}) > u(v_l) - u(v_{l-1}). \end{cases} \tag{4.20}$$

Hence

$$\Delta := \frac{h(v_l) - h(v_{l-1})}{\|v_l - v_{l-1}\|} > \frac{u(v_l) - u(v_{l-1})}{\|v_l - v_{l-1}\|} = c \geq 0. \tag{4.21}$$

Set

$$S(x) := \{y \in V, (x,y) \in E\} \ , \text{ for } x \in V.$$

Since $K(h, S(v_l))(v_l) = h(v_l)$, by applying Theorem 4.3.1, there exists $z_1 \in S(v_l)$ such that

$$\frac{h(z_1) - h(v_l)}{\|z_1 - v_l\|} = \max\{\frac{h(y) - h(v_l)}{\|y - v_l\|} : y \in S(v_l)\}.$$

Thus

$$\frac{h(z_1) - h(v_l)}{\|z_1 - v_l\|} \geq \frac{h(v_l) - h(v_{l-1})}{\|v_l - v_{l-1}\|} = \Delta.$$

We extend *path of greatest* $z_1, z_2, \ldots$ such that $z_{j+1} \in S(z_j)$ and

$$\frac{h(z_{j+1}) - h(z_j)}{\|z_{j+1} - z_j\|} = \max\{\frac{h(y) - h(z_j)}{\|y - z_j\|} : y \in S(z_j)\} \geq \Delta.$$

This path must terminate with a $z_m \in V'$.

Since $\Delta > 0$, we have

$$h(z_m) > \ldots > h(v_l) \geq u(v_l) \geq u(v_0).$$

Thus $z_m \neq v_0$.

Finally, consider the path through the vertices

$$v_0, v_1, \ldots, v_l, z_1, \ldots, z_m.$$

Set $z_0 := v_l$. The above path is the connecting path on G' with respect to u.

Moreover, c is the largest slope of connecting paths with respect to V' and u, and

$$u(v_0) = h(z_0),\ u(z_m) = h(z_m),\ h(z_0) = h(v_l) \geq u(v_l),$$
$$\frac{h(z_{i+1}) - h(z_i)}{\|z_{i+1} - z_i\|} \geq \Delta,\ \frac{u(v_l) - u(v_0)}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\|} = c, \ \text{ and } \Delta > c.$$

Thus we have

$$\begin{aligned}
c &\geq \frac{u(z_m) - u(v_0)}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} \geq \frac{h(z_m) - h(z_0) + u(v_l) - u(v_0)}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\
&= \sum_{i=0}^{m-1} \frac{h(z_{i+1}) - h(z_i)}{\|z_{i+1} - z_i\|} . \frac{\|z_{i+1} - z_i\|}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\
&+ \frac{u(v_l) - u(v_0)}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\|} . \frac{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\
&\geq \Delta \frac{\sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} + c . \frac{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum\limits_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum\limits_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\
&> c
\end{aligned}$$

The last inequality is obtained by $\Delta > c$. Thus we have a contradiction.

• If v is added during the final step of the algorithm. We call $G' = (V', E', \Omega)$ to be the subgraph of $G = (V, E, \Omega)$ when we finish step 1 in the algorithm. Thus there are no connecting paths with respect to G' and u. Therefore, v is connected via edges not in E' to exactly one vertex w of V'.

We can find the largest connected subgraph $G'' = (V'', E'', \Omega)$ satisfying

$$v, w \in V'', V'' \cap V' = \{w\}, \text{ and } E'' \cap E' = \emptyset.$$

From the definition of u, we have

$$h(w) = u(w) = u(x), \ \forall x \in V''.$$

Since $u(v) \neq h(v)$ and $h(w) = u(w) = u(v)$, we have $h(w) \neq h(v)$. Therefore, we must have $\sup_{z \in V''} h(z) \neq h(w)$ or $\inf_{z \in V''} h(z) \neq h(w)$.

Suppose $\sup_{z \in V''} h(z) \neq h(w)$ (we prove similar for the case $\inf_{z \in V''} h(z) \neq h(w)$). Let $v_0 \in V''$ such that

$$h(v_0) = \sup_{z \in V''} h(z) \neq h(w).$$

Set

$$S''(x) := \{y \in V'' : (x, y) \in E''\}, \text{ for } x \in V'' \backslash \{w\},$$

and

$$S(x) := \{y \in V : (x, y) \in E\}, \text{ for } x \in V \backslash \Omega.$$

Noting that

$$S(x) = S''(x), \ \forall x \in V'' \backslash \{w\}. \tag{4.22}$$

Since G'' is a connected graph, there exists a path through the vertices

$$v_0, v_1, \ldots, v_n, w$$

such that $v_i \in S''(v_{i-1}), \forall i \in \{1, \ldots, n\}$ and $w \in S''(v_n)$.

On the other hand, from (4.22) and since h is Kirszbraun extension, we have

$$h(v_0) = \sup_{z \in V''} h(z) \geq \sup_{z \in S''(v_0)} h(z) = \sup_{z \in S(v_0)} h(z).$$

Thus applying Theorem 4.3.1 we have

$$h(v_0) = h(s), \ \forall s \in S(v_0).$$

In particular, we have $h(v_0) = h(v_1)$. By induction, we obtain

$$h(v_0) = h(v_1) = \ldots = h(v_n) = h(w).$$

This contradicts with $h(w) \neq h(v_0)$. □

Remark 4.3.4. Assuming Jensen's hypotheses [28], since this algorithm computes exactly solution of (4.7) and by using the argument of Le Gruyer [37] (the approximation for AMLE by a sequence Kirszbraun extensions (u_n) of networks (N_n, U_n) $(n \in \mathbb{N})$), we obtain a new method to approximate the viscosity solution of Equation $\Delta_\infty u = 0$ under Dirichlet's condition f.

Definition 4.3.5. For any $x, y \in V$. There exists a chain $x_1, ..., x_n \in V$ such that $x_1 = x, x_n = y$ and $x_i \in S(x_{i+1})$ for $i = 1, ..., n-1$. To any chain we associate its length $\sum_{i=1}^{n-1} \|x_i - x_j\|$. We define the geodesis metric d_g of graph G by letting $d_g(x,y)$ be the infimum of the length of chains connecting x and y.

By using induction respect to increasing sequence of subgraph in the algorithm, we obtain the following proposition.

Proposition 4.3.6. *Let u be the Kirszbraun extension of f. We have*

$$\sup_{x,y \in V} \frac{\|u(x) - u(y)\|}{d_g(x,y)} \leq \sup_{x,y \in \Omega} \frac{\|f(x) - f(y)\|}{d_g(x,y)},$$

and

$$\inf_{z \in \Omega} f(z) \leq u(x) \leq \sup_{z \in \Omega} f(z), \ \forall x \in V.$$

4.4 Method to find $K(f,S)(x)$ in general case for any $m \geq 1$

We fix $S = \{p_1, ..., p_N\} \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$ to be a Lipschitz function. Let $x \in \mathbb{R}^n \backslash S$. We denote

$$\lambda(f,S)(x) := \inf_{y \in \mathbb{R}^m} \sup_{i \in \{1,...,N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|}.$$

By applying Kirszbraun's theorem (see [19, 32]) we have $\lambda \leq \mathrm{Lip}(f,S)$.

In this section, we show a method to compute $\lambda(f,S)(x)$ and $K(f,S)(x)$ given by (4.5).

We recall some results that will be useful in this section.

Lemma 4.4.1. *([19, Lemma 2.10.40]) There exists a unique $y(x) \in \mathbb{R}^m$ such that*

$$\lambda(f,S)(x) = \sup_{a \in S} \frac{\|f(a) - y(x)\|}{\|a - x\|}, \tag{4.23}$$

and $y(x)$ belongs to the convex hull of the set

$$B = \{f(z) : z \in S \text{ and } \frac{\|f(z) - y(x)\|}{\|z - x\|} = \lambda(f,S)(x)\}.$$

Moreover, from the definition of $K(f,S)(x)$, we have $K(f,S)(x) = y(x)$.

To compute the value of $K(f,S)(x)$ we need some properties of Cayley-Menger determinant. We recall some definitions and basic results.

Let $x_1,...,x_k \in \mathbb{R}^n$. We define the Cayley-Menger determinant of $(x_i)_{i=1,...,k}$ as

$$\Gamma(x_1,...,x_k) := \det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & \|x_1 - x_2\|^2 & \dots & \|x_1 - x_k\|^2 \\ 1 & \|x_2 - x_1\|^2 & 0 & \dots & \|x_2 - x_k\|^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \dots & 0 \end{pmatrix}.$$

Definition 4.4.2. A $k-$simplex is a $k-$dimensional polytope which is the convex hull of its $k+1$ vertices. More formally, suppose the $k+1$ points $u_0,...,u_k \in \mathbb{R}^n$ are affinely independent, which means $u_1 - u_0,...,u_k - u_0$ are linearly independent. Then the $k-$ simplex determined by them is the set of points

$$C = \{t_0u_0 + ... + t_ku_k : t_i \geq 0, 0 \leq i \leq k, \sum_{i=0}^{k} t_i = 1\}.$$

Example 4.4.3. A 2-simplex is a triangle, a 3-simplex is a tetrahedron.

The $k - simplex$ and the Cayley-Menger determinant have beautiful relations by following theorem:

Theorem 4.4.4. *[11, Lemma 9.7.3.4] Let $(x_i)_{i=1,...,k+2} \in \mathbb{R}^n$ be arbitrary points in k-dimensional Euclidean affine space X. Then $\Gamma(x_1,...,x_{k+2}) = 0$. A necessary and sufficient condition for $(x_i)_{i=1,...,k+1}$ to be a k-simplex of X is that $\Gamma(x_1,...,x_{k+1}) \neq 0$.*

Lemma 4.4.5. *Let the point u lie in the convex hull of the points $q_0,q_1,...,q_s$ of $\mathbb{R}^m$. If u' distinct from u, then for some i:*

$$\|u - q_i\| \leq \|u' - q_i\|.$$

Proof. Choose H to be the $(m-1)-$ dimension (or hyperplane) through u which is perpendicular to the segment $[u,u']$. Then for at least one value for i, q_i must lie in the halfspace of H which does not contain u'. Thus we have

$$\|u - q_i\| \leq \|u' - q_i\|.$$

□

Proposition 4.4.6. *Suppose there exist $J \subset \{1,2,...,N\}$, f_0 inside convex hull of $\{f(p_j)\}_{j\in J}$ and $\lambda_0 > 0$ such that*

$$\|f_0 - f(p_j)\| = \lambda_0\|x - p_j\|, \ \forall j \in J$$

and

$$\|f_0 - f(p_i)\| \leq \lambda_0\|x - p_i\|, \ \forall i \in \{1,...,N\},$$

then $\lambda_0 = \lambda(f,S)(x)$ and $f_0 = K(f,S)(x)$.

Proof. We have

$$\lambda_0 = \sup_{i\in\{1,\ldots,N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|} \geq \inf_{y\in\mathbb{R}^m} \sup_{i\in\{1,\ldots,N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} = \lambda(f,S)(x).$$

On the other hand, for any $y \in \mathbb{R}^m$, by applying Lemma 4.4.5 there exists $i \in J$ such that

$$\|y - f(p_i)\| \geq \|f_0 - f(p_i)\| = \lambda_0 \|x - p_i\|.$$

Hence

$$\sup_{i\in\{1,\ldots,N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} \geq \lambda_0. \tag{4.24}$$

Since Inequality (4.24) is true for any $y \in \mathbb{R}^m$, we have $\lambda(f,S)(x) \geq \lambda_0$. Thus

$$\lambda(f,S)(x) = \lambda_0.$$

Therefore, we have

$$\lambda(f,S)(x) = \sup_{i\in\{1,\ldots,N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|}.$$

From Lemma 4.4.1 we have $f_0 = K(f,S)(x)$. □

A method to compute $K(f,S)(x)$
Recall that $f : S \to \mathbb{R}^m$. By applying Lemma 4.4.1, we have

$$\|f(a) - K(f,S)(x)\| \leq \lambda(f,S)(x)\|a - x\|, \ \forall a \in S.$$

Moreover,

$$B = \left\{ f(a) : a \in S \text{ and } \frac{\|f(a) - K(f,S)(x)\|}{\|a - x\|} = \lambda(u,S)(x) \right\},$$

is not empty, and $K(f,S)(x)$ belongs to the convex hull of B.

Therefore, there exist $\{f(p_{i_k})\}_{k=1,\ldots,l+1} \subset f(S)$ such that

(I) $l \leq m$, where m is dimension of $\mathbb{R}^m$;

(II) $\{f(p_{i_k})\}_{k=1,\ldots,l+1}$ is a $l-$simplex. From Theorem 4.4.4, $\{f(p_{i_k})\}_{k=1,\ldots,l+1}$ is a $l-$simplex to be equivalent to

$$\Gamma(K(f,S)(x), f(p_{i_1})\ldots, f(p_{i_{l+1}})) \neq 0; \tag{4.25}$$

(III) $K(f,S)(x)$ belongs convex hull of $\{f(p_{i_k})\}_{k=1,\ldots,l+1}$;

(IV)

$$\|K(f,S)(x) - f(p_{i_k})\| = \lambda(f,S)(x)\|x - p_{i_k}\|, \ \forall k = 1,\ldots,l+1. \tag{4.26}$$

(V)

$$\|f(a) - K(f,S)(x)\| \leq \lambda(f,S)(x)\|a - x\|, \ \forall a \in S.$$

From the above observations, we obtain

Theorem 4.4.7. *There exist* $\{f(p_{i_k})\}_{k=1,...,l+1} \subset f(S)$ $(1 \le l \le m$, *where m is dimension of* $\mathbb{R}^m)$, $f_{i_1 i_2 ... i_{l+1}} \in \mathbb{R}^m$ *and* $\lambda_{i_1 i_2 ... i_{l+1}} \in \mathbb{R}$ *satisfying some following properties*

(a) $f_{i_1 i_2 ... i_{l+1}}$ *inside convex hull of* $\{f(p_{i_k})\}_{k=1,...,l+1}$.

(b) $\Gamma(f(p_{i_1}), f(p_{i_2}), ..., f(p_{i_{l+1}})) \neq 0$.

(c) $\|f_{i_1 i_2 ... i_{l+1}} - f(p_k)\| = \lambda_{i_1 i_2 ... i_{l+1}} \|x - p_k\|$, $\forall k \in \{i_1, i_2, ... i_{l+1}\}$.

(d) $\|f_{i_1 i_2 ... i_{l+1}} - f(p_k)\| \le \lambda_{i_1 j_2 ..., i_{l+1}} \|x - p_k\|$, $\forall k \in \{1, ..., N\}$.

Moreover, from Proposition 4.4.6 we have $f_{i_1 i_2 ... i_{l+1}} = K(f,S)(x)$ *and* $\lambda_{i_1, i_2 ..., i_{l+1}} = \lambda(f,S)(x)$.

Therefore, to compute the value of $\lambda(u,S)(x)$ and $K(u,S)(x)$, we need to find $\{f(p_{i_k})\}_{k=1,...,l+1} \subset f(S)$, $f_{i_1 i_2 ... i_{l+1}} \in \mathbb{R}^m$ and $\lambda_{i_1 i_2 ... i_{l+1}} \in \mathbb{R}$ satisfying the conditions (a),(b),(c),(d). We can do that step by step as follows

***Step 1:** For all $i, j \in \{1, ..., N\}, (i \neq j)$. Let

$$\begin{aligned} f_{ij} &:= \frac{\|x - p_j\|}{\|x - p_i\| + \|x - p_j\|} f(p_i) + \frac{\|x - p_i\|}{\|x - p_i\| + \|x - p_j\|} f(p_j); \\ \lambda_{ij} &:= \frac{\|f(p_i) - f(p_j)\|}{\|x - p_i\| + \|x - p_j\|}. \end{aligned}$$

We have f_{ij} inside convex hull of $\{f(p_i), f(p_j)\}$ and

$$\|f_{ij} - f(p_k)\| = \lambda_{ij} \|x - p_k\|, \text{ for } k \in \{i, j\}.$$

Test the following condition

$$\|f_{ij} - f(p_k)\| \le \lambda_{ij} \|x - p_k\|, \ \forall k \in \{1, ..., N\} \tag{4.27}$$

If (i, j) satisfies the above condition, then from Proposition 4.4.6 we have $f_{ij} = K(f,S)(x)$ and $\lambda_{ij} = \lambda(f,S)(x)$. We finish. If there is no $(i, j) \in \{1, ..., N\}, (i \neq j)$ that satisfies the above condition, then we go to step 2.

***Step 2:** For all $(i, j, k) \in \{1, ..., N\} \times \{1, ..., N\} \times \{1, ..., N\}$. Test the following condition

$$\Gamma(f(p_i), f(p_j), f(p_k)) \neq 0. \tag{4.28}$$

Let A is the set of all (i, j, k) that satisfies (4.28). We consider a $(i, j, k) \in A$. Thus from Theorem 4.4.4 we have

- $\{f(p_i), f(p_j), f(p_k)\}$ is $2-$simplex.
- For any f_{ijk} inside convex hull of $\{f(p_i), f(p_j), f(p_k)\}$ we have

$$\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.$$

We consider the following equations

$$\begin{cases} \Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0; \\ \|f_{ijk} - f(p_l)\| = \lambda_{ijk} \|x - p_l\|, \ \forall l \in \{i, j, k\}; \end{cases}$$

We replace $\|f_{ijk} - f(p_l)\|$ by $\lambda_{ijk}\|x - p_l\|$ into the equation

$$\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.$$

We obtain that

$$\begin{aligned}
0 &= \Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) \\
&= \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \|f_{ijk} - f_i\|^2 & \|f_{ijk} - f_j\|^2 & \|f_{ijk} - f_k\|^2 \\ 1 & \|f_i - f_{ijk}\|^2 & 0 & \|f_i - f_j\|^2 & \|f_i - f_k\|^2 \\ 1 & \|f_j - f_{ijk}\|^2 & \|f_j - f_i\|^2 & 0 & \|f_j - f_k\|^2 \\ 1 & \|f_k - f_{ijk}\|^2 & \|f_k - f_i\|^2 & \|f_k - f_j\|^2 & 0 \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & {\lambda_{ijk}}^2\|x - p_i\|^2 & {\lambda_{ijk}}^2\|x - p_j\|^2 & {\lambda_{ijk}}^2\|x - p_k\|^2 \\ 1 & {\lambda_{ijk}}^2\|x - p_i\|^2 & 0 & \|f_i - f_j\|^2 & \|f_i - f_k\|^2 \\ 1 & {\lambda_{ijk}}^2\|x - p_j\|^2 & \|f_j - f_i\|^2 & 0 & \|f_j - f_k\|^2 \\ 1 & {\lambda_{ijk}}^2\|x - p_k\|^2 & \|f_k - f_i\|^2 & \|f_k - f_j\|^2 & 0 \end{pmatrix} \\
&= a(x)\lambda_{ijk}^4 + b(x)\lambda_{ijk}^2 + c(x),
\end{aligned}$$

where $a(x), b(x), c(x)$ are function only depending on x and initial data x_l, $f(p_l)$ for $l \in \{i, j, k\}$.

By solving the equation

$$a(x)\lambda_{ijk}^4 + b(x)\lambda_{ijk}^2 + c(x) = 0, \tag{4.29}$$

we obtain that λ_{ijk} is a positive real root of the above polynomial. It maybe that Equation (4.29) have no any positive real root. In this case, we consider another $(i', j', k') \in A$ until Equation (4.29) with respect to (i', j', k') have a positive real root. We call L is the set of all positive real root of equation (4.29).

Let $\lambda_{ijk} \in L$. We find f_{ijk} by solving the equations

$$\|f_{ijk} - f(p_l)\| = \lambda_{ijk}\|x - p_l\|, \; \forall l \in \{i, j, k\}. \tag{4.30}$$

After that, we test the condition f_{ijk} in convex hull of $\{f(p_l)\}_{l \in \{i,j,k\}}$, and test the following condition

$$\|f_{ijk} - f(p_l)\| \leq \lambda_{ijk}\|x - p_l\|, \; \forall l \in \{1, ..., N\}. \tag{4.31}$$

If we has a $\lambda_{ijk} \in L$ such that f_{ijk} in convex hull of $\{f(p_l)\}_{l \in \{i,j,k\}}$ satisfying Equations (4.30) and Inequalities (4.31) then from Proposition 4.4.6 we have $f_{ijk} = K(f, S)(x)$ and $\lambda_{ijk} = \lambda(f, S)(x)$. We finish. If there is no $(i, j, k) \in A$ that satisfies the above conditions, then we go to step 3.

***Step 3:** By the similar way as step 2 for (i, j, k, l), $(i, j, k, l, h), ...$ until we can find a $(i_1, ..., i_k) \subset \{1, ..., N\}$ such that $f_{i_1, i_2 ... i_k}$ and $\lambda_{i_1 j_2 ... i_k}$ satisfying some following properties

(a) $f_{i_1 i_2 ... i_k}$ inside convex hull of $\{f(p_{i_n})\}_{n=1,...,k}$

(b) $\Gamma(f(p_{i_1}), f(p_{i_2}), ..., f(p_{i_k})) \neq 0$.
(c) $\|f_{i_1 i_2 ... i_k} - f(p_l)\| = \lambda_{i_1 i_2 ... i_k} \|x - p_l\|$, $\forall l \in \{i_1, i_2, ... i_k\}$.
(d) $\|f_{i_1 i_2 ... i_k} - f(p_l)\| \leq \lambda_{i_1 j_2 ..., i_k} \|x - p_l\|$, $\forall l \in \{1, ..., N\}$
By applying Proposition 4.4.6, we obtain $f_{i_1 i_2 ... i_k} = K(f, S)(x)$ and $\lambda_{i_1, i_2 ..., i_k} = \lambda(f, S)(x)$.

Remark 4.4.8. By applying theorem 4.4.7, this method terminates when $k = l + 1 \leq m + 1$, where m is dimension of $\mathbb{R}^m$.

Remark 4.4.9. In step 3, when we solve $f_{i_1, i_2 ... i_k}$ by considering the equation

$$\Gamma(f_{i_1, i_2 ... i_k}, f(p_{i_1}), f(p_{i_2}), ..., f(p_{i_k})) = 0,$$

by replacing $\|f_{i_1 i_2 ... i_k} - f(p_l)\|$ by $\lambda_{i_1 i_2 ... i_k} \|x - p_l\|$, for $l \in \{i_1, i_2, ... i_k\}$, this equation is equivalent to

$$a(x)\lambda^4_{i_1 i_2 ... i_k} + b(x)\lambda^2_{i_1 i_2 ... i_k} + c(x) = 0, \tag{4.32}$$

where $a(x), b(x), c(x)$ are function only depending on x and initial data x_l, $f(p_l)$ for $l \in \{i_1, ..., i_k\}$. The polynomial $a(x)\lambda^4_{i_1 i_2 ... i_k} + b(x)\lambda^2_{i_1 i_2 ... i_k} + c(x)$, in fact, is $2-$degree polynomial with variable $\lambda = \lambda^2_{i_1 i_2 ... i_k}$. Therefore, we can solve Equation (4.32) very fast to obtain exactly the value of $\lambda_{i_1 i_2 ... i_k}$.

Chapter 5

Conclusions and perspectives

In this chapter, we describe some problems for future research, which are related to the subject presented in the thesis.

5.1 A numerical method to approximate Kirszbraun extension

We introduce a numerical method to approximate the Kirszbraun extension. We can not prove the convergence of this numerical method but we present some interesting numerical test results and some natural questions.

Let $G = (V, E, \Omega)$ be a connected finite graph with vertices set $V \subset \mathbb{R}^n$, edges set E and a non-empty set $\Omega \subset V$ and let $f : \Omega \to \mathbb{R}^m$.

Let u_0 be an extension of f on G.

Since G finite, we have $V \backslash \Omega = \{v_1, ..., v_h\}$.

We define $u_0^{(0)} := u_0$. We define by induction the sequence of functions $(u_i^{(k)})$ (with $u_0^{(k)} := u_h^{(k-1)}$ for all $k \geq 1$) as follows

$$u_i^{(k)}(x) := \begin{cases} K(u_{i-1}^{(k)}, S(v_i))(v_i) & \text{for } x = v_i; \\ u_{i-1}^{(k)}(x) \ \forall x \in V \backslash \{v_i\}; \end{cases} \tag{5.1}$$

for $i = 1, ..., h$ and $k \geq 0$, where $S(x)$ is the neighborhood of x on G defined in (4.6) and the function K is defined in (4.5).

Definition 5.1.1. We define

$$\Phi_k(u_0) = u_h^{(k)}.$$

If $u_i^{(k)} \neq u_{i-1}^{(k)}$, then from Proposition 4.2.3 we have $u_i^{(k)}$ tighter than $u_{i-1}^{(k)}$. Thus it is natural to conjecture that $\Phi_k(u_0)$ converges to a Kirszbraun extension of f when $k \to \infty$. We give some numerical test results:

Example 5.1.2. Consider graph G drawn in Figure 5.1 with

$$V = \{1,2,3,4,5\} \times \{1,2,3,4,5\} \subset \mathbb{R}^2,$$
$$E \text{ is the set of edges connecting points in } V \text{ (see Figure 5.1)},$$
$$\Omega = \{(1,1), (2,1), (1,3)\}.$$

For $x \in V$, we define

$$S(x) := \{y \in V : (x,y) \in E\}$$

to be the neighborhood of x on G.

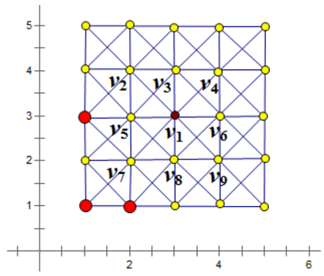

Figure 5.1: Graph G

For example, we have $S(v_1) = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$.
Let $f : \Omega \to \mathbb{R}^2$ be a function satisfying $f(1,1) = (10,1), f(2,1) = (1,3), f(1,3) = (2,9)$.

Case 1: Let $g_1(v) = K(f,\Omega)(v)$, for all $v \in V$. By computing directly, we obtain

$$\sup_{v \in V \setminus \Omega} \{K(\Phi_{30}(g_1), S(v))(v) - \Phi_{30}(g_1)(v)\} \leq 0.00008.$$

Case 2: Let $g_2(v) = f(v)$ for all v in S, and $g(v) = (1,2)$ for all $v \in G \setminus \Omega$. By computing directly, we obtain

$$\sup_{v \in V \setminus \Omega} \{K(\Phi_{30}(g_2), S(v))(v) - \Phi_{30}(g_2)(v)\} \leq 0.00088.$$

$$\sup_{v \in V \setminus \Omega} \{K(\Phi_{40}(g_2), S(v))(v) - \Phi_{40}(g_2)(v)\} \leq 0.00007.$$

Moreover

$$\sup_{v \in V \setminus \Omega} \{K(\Phi_{30}(g_1), S(v))(v) - K(\Phi_{30}(g_2), S(v))(v)\} \leq 0.0023.$$

$$\sup_{v \in V \setminus \Omega} \{K(\Phi_{30}(g_1), S(v))(v) - K(\Phi_{40}(g_2), S(v))(v)\} \leq 0.001.$$

In this example, we see that the functions $\Phi_k(g_1)(v)$ and $\Phi_k(g_2)(v)$ converge to the same solution of (4.7).

When we change the domain Ω, the data value f or the initial extension g of f, we obtain similar approximate results. Moreover, when we change the graph G by another one we also obtain similar approximate results. Therefore, we guess that the Kirszbraun extension is unique (the uniqueness of Kirszbraun extension may depend on the construction of G) and $\Phi_k(g)(v)$ converges to the Kirszbraun extension when $k \to +\infty$ for any extension g of f on G.

Problem 1. Let u_0 be an arbitrary extension of f. Is it true that $\Phi_k(u_0)$ converges to a Kirszbraun extension of f when $k \to \infty$? What are conditions on the graph such that Kirszbraun extension of f is unique?

5.2 The Kneser-Poulsen conjecture.

It is easy to see that the following theorem is equivalent to the Kirszbraun theorem (Theorem 2.1.1)

Theorem 5.2.1. *[43, Lemma 1.20] Let H_1 and H_2 be Hilbert spaces, $\{x_\beta\}_{\beta\in I}$ a subset of H_1, $\{y_\beta\}_{\beta\in I}$ a subset of H_2, and $\{r_\beta\}_{\beta\in I}$ a collection of nonnegative real numbers. If*

$$\|y_\beta - y_\gamma\| \le \|x_\beta - x_\gamma\| \tag{5.2}$$

for all $\beta,\gamma\in I$, then $\bigcap_{\beta\in I} B_{H_2}(y_\beta,r_\beta)\neq\emptyset$ whenever $\bigcap_{\beta\in I} B_{H_2}(x_\beta,r_\beta)\neq\emptyset$.

If $H_1 = H_2$, then the contraction hypothesis (5.2) in Theorem 5.2.1 implies that we can rearranging the ball $B(x_\beta,r_\beta)$ in a way that the centers of the balls are closer to each other. In addition, if $\dim H_1 = \dim H_2 < +\infty$ and the index I is finite, then it is natural to conjecture that the volume of the intersection should increase as we push the spheres together. In fact, we have

Problem 2. [33, 47, Kneser-Poulsen conjecture] Let $\{r_1,...,r_N\}$ be a collection of nonnegative numbers and $\{x_1,...,x_N\}$ and $\{y_1,...,y_N\}$ two subsets of $\mathbb{R}^n$. If $\{y_1,...,y_N\}$ is a contraction of $\{x_1,...,x_N\}$, i.e

$$\|y_i - y_j\| \le \|x_i - x_j\|,$$

for all $1\le i,j\le N$. Then

$$Vol\left(\bigcap_{i=1}^{N} B(y_i,r_i)\right) \ge Vol\left(\bigcap_{i=1}^{N} B(x_i,r_i)\right)$$

The above was conjectured independently by E. T. Poulsen and M. Kneser [33, 47]. The Kneser-Poulsen conjecture is solved only in dimension 2 by K.Bezdek and R. Connelly [12]. This is still open in dimensions 3 and higher.

5.3 The existence of AMLE in the general case.

Definition 5.3.1. Let two metric spaces (X,d_X) and (Z,d_Z). Let Y be a subset of X and a Lipschitz mapping $f:Y\to Z$. A Lipschitz mapping $g:X\to Z$ is called an *absolutely minimal Lipschitz extension* (AMLE) of f if $g=f$ on Y, and for every open subset $U\subset X\backslash Y$ and every Lipschitz mapping $h:X\to Z$ that coincides with g on $X\backslash U$ we have

$$\mathrm{Lip}(h,U)\ge\mathrm{Lip}(g,U). \tag{5.3}$$

Remark 5.3.2. If (X,d_X) is path-connected and the pair (X,Z) has the isometric extension property then the AMLE condition (5.3) is equivalent to the condition: for all open $U\in X\backslash Y$ we have $\mathrm{Lip}(g,U)=\mathrm{Lip}(g,\partial U)$.

Recall that a metric space (X, d_X) is a *length space* if for all $x, y \in X$, the distance $d_x(x,y)$ is the infimum of the lengths of curves in X that connect x to y.

Problem 3. Let Z be an absolute 1-Lipschitz retract (see Definition 2.1.5). Is it true that for every length space X and every closed subset $Y \subset X$, any Lipschitz $f : Y \to Z$ admits an AMLE $g : X \to Z$? If we have the existence of AMLE, is it unique?

Remark 5.3.3. $\mathbb{R}$, l_∞ and metric trees are absolute $1-$Lipschitz retracts (see [10, 29]). In these cases we have the existence of AMLE (see [7, 44]), but in general case (Z is an absolute 1-Lipschitz retract) this question is still open.

Problem 4. For $X = \mathbb{R}^n$ and $Z = \mathbb{R}^m$ both equipped with the Euclidean norm, is it true that for every $Y \subset X$, any Lipschitz $f : Y \to Z$ admits an AMLE $g : X \to Z$?

Extending results on AMLE's to vector valued functions presents many difficulties, which in turn has limited the number of results in this direction.

5.4 Find the norm Γ^m of minimal Lipschitz extension on C^m.

Let Ω be a subset of Euclidean space $\mathbb{R}^n$. Let $\mathscr{P}^m(\mathbb{R}^n, \mathbb{R})$ be the set of $m-$degree polynomials mapping $\mathbb{R}^n$ to $\mathbb{R}$. Let us consider a m-field $T : \Omega \to \mathscr{P}^m(\mathbb{R}^n, \mathbb{R})$. We want to find the norm Γ^m of the minimal extension on C^m which generalize Le Gruyer's work on minimal C^1 extensions. More precisely, we have

Problem 5. Find the functional $\Gamma^m : T \to \Gamma^m(T, \mathrm{dom}(T)) \in \mathbb{R}^+ \cup \{+\infty\}$ satisfying

$(P0)$ Γ^m is increasing, that is, U extends T implies that

$$\Gamma^m(U, \mathrm{dom}(U)) \geq \Gamma^m(T, \mathrm{dom}(T)).$$

$(P1)$ If U has total domain satisfying $\Gamma^m(U, \mathbb{R}^n) < +\infty$, then the total function u defined by $u(x) := U(x)(x)$ is in C^m and $D^m u$ is Lipschitz.

$(P2)$ If $u \in C^m(\mathbb{R}^n)$ with $D^m u$ Lipschitz, then

$$\Gamma^m(U, \mathbb{R}^n) = \mathrm{Lip}(D^m u),$$

where U is the $m-$field associate to u.

$(P3)$ For any T such that $\Gamma^m(T, \mathrm{dom}(T)) \leq +\infty$, T extends to a total $m-$field U satisfying

$$\Gamma^m(U, \mathbb{R}^n) = \Gamma^m(T, \mathrm{dom}(T)).$$

To compute the norm Γ^m of the minimal extension on C^m which generalize Le Gruyer's work on minimal C^1 extensions is a very difficult problem and the main thrust is some attempts to guess the natural norm for which one can obtain the minimal extension.

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Titre de la thèse:
Extensions lipschitziennes minimales

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Résumé

Cette thèse est consacrée aux quelques problèmes mathématiques concernant les extensions minimales de Lipschitz. Elle est organisée de manière suivante.

Le chapitre 1 est dédié à l'introduction des extensions minimales de Lipschitz.

Dans le chapitre 2, nous étudions la relation entre la constante de Lipschitz d' 1-field et la constante de Lipschitz du gradient associée à ce 1-field. Nous proposons deux formules explicites Sup-Inf, qui sont des extensions extrêmes minimales de Lipschitz d'1-field. Nous expliquons comment les utiliser pour construire les extensions minimales de Lipschitz pour les applications $\mathbb{R}^m$à $\mathbb{R}^n$. Par ailleurs, nous montrons que les extensions de Wells d'1- fields sont les extensions absolument minimales de Lipschitz (AMLE) lorsque le domaine d'expansion d'1-field est infini. Un contre-exemple est présenté afin de montrer que ce résultat n'est pas vrai en général.

Dans le chapitre 3, nous étudions la version discrète de l'existence et l'unicité de l'AMLE. Nous montrons que la fonction tight introduite par Sheffield and Smart est l'extension de Kirszbraun. Dans le cas réel, nous pouvons montrer que cette extension est unique. De plus, nous proposons un algorithme qui permet de calculer efficacement la valeur de l'extension de Kirszbraun en complexité polynomiale. Pour conclure, nous décrivons quelques pistes pour la future recherche, qui sont liées au sujet présenté dans ce manuscrit.

Abstract

The thesis is concerned to some mathematical problems on minimal Lipschitz extensions.

Chapter 1: We introduce some basic background about minimal Lipschitz extension (MLE) problems. Chapter 2: We study the relationship between the Lipschitz constant of 1-field and the Lipschitz constant of the gradient associated with this 1-field. We produce two Sup-Inf explicit formulas which are two extremal minimal Lipschitz extensions for 1-fields. We explain how to use the Sup-Inf explicit minimal Lipschitz extensions for 1-fields to construct minimal Lipschitz extension of mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. Moreover, we show that Wells's extensions of 1-fields are absolutely minimal Lipschitz extensions (AMLE) when the domain of 1-field to expand is finite. We provide a counter-example showing that this result is false in general.

Chapter 3: We study the discrete version of the existence and uniqueness of AMLE. We prove that the tight function introduced by Sheffield and Smart is a Kirszbraun extension. In the realvalued case, we prove that the Kirszbraun extension is unique. Moreover, we produce a simple algorithm which calculates efficiently the value of the Kirszbraun extension in polynomial time. Chapter 4: We describe some problems for future research, which are related to the subject represented in the thesis.

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