# Harmonic and subharmonic functions associated to root systems

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## *FONCTIONS HARMONIQUES ET SOUSHARMONIQUES ASSOCIÉES À DES SYSTÈMES DE RACINES*

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# Resumé

Dans cette thèse, nous montrons que pour tout système de racines $\mathcal{R}$ de l'espace euclidien $\mathbb{R}^d$ et pour toute fonction de multiplicité positive $k$ sur $\mathcal{R}$, on peut développer dans ce cadre géométrique une théorie du potentiel de type newtonien et plus généralement de type Riesz qui coïncident avec les théories classiques lorsque la fonction $k$ est identiquement nulle.
Précisément, soit $\{D_j,\ 1 \leq j \leq d\}$ la famille commutative des opérateurs différentiels et aux différences du premier ordre associée à $(\mathcal{R}, k)$ au sens de Dunkl et soit $\Delta_k = D_1^2 + \cdots + D_d^2$ l'opérateur de type Laplace qui leur correspond.

Dans un premier temps, en introduisant un nouvel opérateur de moyenne volumique, nous étudions les fonctions $\Delta_k$-harmoniques dans un ouvert $\Omega$ de $\mathbb{R}^d$ invariant sous l'action du groupe de réflexions $W$ (groupe de Coxeter-Weyl) associé à $\mathcal{R}$. Nous établissons pour ces fonctions des propriétés de moyenne, un principe du maximum fort, une inégalité de Harnack et un théorème de Bôcher qui contiennent les théorèmes classiques comme cas particuliers.

Ensuite, nous introduisons via l'opérateur de moyenne volumique la notion de fonction $\Delta_k$-sousharmonique dans le contexte des systèmes de racines. Après avoir donné les propriétés essentielles des ces fonctions (intégrabilité locale, principe du maximum fort, approximation par des fonctions régulières,...etc), nous montrons qu'elles peuvent être caractérisées de plusieurs manières et en particulier par la positivité au sens des distributions de leur laplacien de Dunkl. Nous prouvons ensuite que les potentiels des mesures de Radon positives correspondant au noyau de Dunkl-Newton constituent les exemples fondamentaux de fonctions $\Delta_k$-sousharmoniques et à cet effet, une importance particulière a été consacrée à l'étude de ces objets. Comme application, nous obtenons le théorème suivant : toute fonction $\Delta_k$-souharmonique $u$ s'écrit localement comme la somme d'une fonction $\Delta_k$-harmonique et du potentiel de Dunkl-Newton d'une mesure de Radon précise qui joue un rôle important dans notre théorie et que nous appelons $\Delta_k$-mesure de Riesz de $u$ comme dans le cas classique (i.e. $k \equiv 0$).

Enfin, dans une troisième partie, nous initions une théorie du potentiel de type Riesz d'indice $\beta$, $0 < \beta < d + \sum_{\alpha\in\mathcal{R}} k(\alpha)$, et qui contient le cas newtonien lorsque $\beta = 2$. En particulier, nous étudions les noyaux de Dunkl-Riesz et les potentiels correspondants pour les mesures de Radon. Comme applications, nous obtenons le principe d'unicité des masses et nous étendons dans notre cas l'inégalité ponctuelle de Hedberg classique.

# Mots-clefs

Systèmes de racines, Groupes de Coxeter-Weyl, Opérateurs de Dunkl, Opérateur de Dunkl-Laplace, Opérateur de moyenne volumique généralisé, Fonctions $\Delta_k$-harmoniques, Fonctions $\Delta_k$-sousharmoniques, Propriété de moyenne, $\Delta_k$-mesures de Riesz, Noyau de la chaleur, Noyau de Dunkl-Newton, Potentiel de Dunkl-Newton, Equation de Poisson, Théorème de décomposition de Riesz, Noyau de Dunkl-Riesz, Potentiel de Dunkl-Riesz, Principe d'unicité des masses.

# Abstract

In this thesis, we show that, for any root system $\mathcal{R}$ in the Euclidean space $\mathbb{R}^d$ and for any nonnegative multiplicity function $k$ on $\mathcal{R}$, we can develop in this geometric framework a Newtonian type or more generally a Riesz type potential theory which coincide with the classical theories when $k$ is the zero function.

Precisely, let $\{D_j,\ 1 \leq j \leq d\}$ be the commutative family of first order differential-difference Dunkl operators associated with $(\mathcal{R}, k)$ and $\Delta_k = D_1^2 + \cdots + D_d^2$ be the corresponding Laplace type operator.

At first, by introducing a new volume mean value operator, we study $\Delta_k$-harmonic functions on a open subset $\Omega$ of $\mathbb{R}^d$ which is invariant under the action of the reflection group $W$ (the Coxeter-Weyl group) associated to $\mathcal{R}$. We establish for these functions the mean value property, a strong maximum principle, a Harnack inequality and a Bôcher theorem. These results contain the classical ones as particular cases.

Afterwards, we introduce via the volume mean value operator the notion of $\Delta_k$-subharmonic function in the context of root systems. After giving some essential properties of these functions (local integrability, strong maximum principle, approximation by regular functions,...etc), we show that they can be characterized in different ways and in particular in terms of the positivity of their distributional Dunkl Laplacian. We prove also that the potentials of Radon measures associated with the Dunkl-Newton kernel provide the fundamental examples of $\Delta_k$-subharmonic functions and for this purpose a particular importance is devoted to the study of these objects. As application, we obtain the following theorem : Every $\Delta_k$-subharmonic function $u$ can be locally written as the sum of a $\Delta_k$-harmonic function and of a Dunkl-Newton potential of a precise Radon measure which plays a key role in our theory and which we call the $\Delta_k$-Riesz measure of $u$ as in analogy with the classical case (i.e. $k \equiv 0$).

Finally, in a third part we initiate a Riesz type potential theory of index $\beta$, with $0 < \beta < d + \sum_{\alpha\in\mathcal{R}} k(\alpha)$, which contains the aforementioned Dunkl-Newtonian case when $\beta = 2$. In particular, we study the Dunkl-Riesz kernel and the corresponding potentials of Radon measures. As applications we obtain the uniqueness principle and we extend to our setting the classical pointwise Hedberg inequality.



# Keywords

Root systems, Coxeter-Weyl groups, Dunkl operators, Dunkl-Laplace operator, Generalized volume mean value operator, $\Delta_k$-Harmonic functions, $\Delta_k$-Subharmonic functions, Mean value property, $\Delta_k$-Riesz measures, Dunkl-heat kernel, Dunkl-Newton kernel, Dunkl-Newton potential, Poisson equation, Dunkl-Riesz kernel, Dunkl-Riesz potential, Uniqueness principle.

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# INTRODUCTION

Considérons l'espace $\mathbb{R}^d$ muni de sa structure euclidienne usuelle. Nous noterons $\langle .,. \rangle$ le produit scalaire sur $\mathbb{R}^d$, $\|.\|$ la norme euclidienne associée et $(e_j)_{1\leq j\leq d}$ la base canonique de $\mathbb{R}^d$. Pour $\alpha \in \mathbb{R}^d\backslash\{0\}$, nous noterons également $\sigma_\alpha$ la réflexion par rapport à l'hyperplan $H_\alpha$ orthogonal à la droite $\mathbb{R}.\alpha$ engendrée par $\alpha$. L'expression de $\sigma_\alpha$ est donnée par

$$\forall\ x \in \mathbb{R}^d, \quad \sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha.$$

Nous allons présenter les ingrédients principaux de la théorie de Dunkl. Commençons par introduire les notions de système de racines et de groupe de réflexions. Nous nous référons aux livres de Humphreys ([26]) et de Kane ([30]) pour plus de détails.

Un **système de racines** dans $\mathbb{R}^d$ est un ensemble fini $\mathcal{R}$ de vecteurs non nuls, appelés **racines**, qui satisfont les propriétés suivantes :

1. les seuls multiples scalaires d'une racine $\alpha \in \mathcal{R}$ appartenant à $\mathcal{R}$ sont $\alpha$ et $-\alpha$,
2. pour chaque racine $\alpha \in \mathcal{R}$, l'ensemble $\mathcal{R}$ est stable par la réflexion $\sigma_\alpha$.

La dimension de l'espace vectoriel réel engendré par les racines de $\mathcal{R}$ est appelée le **rang** du système de racines $\mathcal{R}$.

Nous nous donnons un vecteur $\beta \in \mathbb{R}^d$ tel que $\langle \alpha, \beta\rangle \neq 0$ pour tout $\alpha \in \mathcal{R}$ (dans ce cas, le vecteur $\beta$ est dit régulier). Nous avons alors une partition $\mathcal{R} = \mathcal{R}_+ \sqcup \mathcal{R}_-$, avec

$$\mathcal{R}_+ = \mathcal{R}_+(\beta) := \{\alpha \in \mathcal{R}, \quad \langle \alpha, \beta\rangle > 0\} \quad \text{et} \quad \mathcal{R}_- = -\mathcal{R}_+.$$

Nous dirons que $\mathcal{R}_+$ est un **sous-système positif de racines**.

Un **système simple** $\Pi$ est un sous-ensemble de $\mathcal{R}$ tel que
- les éléments de $\Pi$ forment une base de l'espace vectoriel engendré par $\mathcal{R}$,
- chaque racine $\alpha \in \mathcal{R}$ est une combinaison linéaire d'éléments de $\Pi$ à coefficients tous positifs ou tous négatifs.

Les éléments de $\Pi$ sont appelés **racines simples**. Notons que si $\Pi$ est un système simple, il existe un unique sous-système positif de racines contenant $\Pi$ et que inversement, tout sous-système positif de racines contient un et un seul système simple. En particulier, si $\Pi$ est contenu dans $\mathcal{R}_+$, les coordonnées de tout élément de $\mathcal{R}_+$ dans la base $\Pi$ sont positives.

Le sous-groupe $W$ du groupe orthogonal $O(\mathbb{R}^d)$ engendré par les réflexions $\sigma_\alpha$, $\alpha \in \mathcal{R}$, est appelé le **groupe de Coxeter-Weyl** associé à $\mathcal{R}$. En remarquant que l'action naturelle de $W$ sur le système $\mathcal{R}$ est fidèle i.e. le seul élément de $W$ qui fixe toutes les racines est l'identité (le groupe $W$ est isomorphe à un sous-groupe de $S_{|\mathcal{R}|}$), nous voyons que le groupe de Coxeter-Weyl est fini. De plus, le groupe $W$ est en fait engendré par les réflexions $\sigma_\alpha$ associées aux racines simples $\alpha \in \Pi$.

Fixons un système simple $\Pi$. Les hyperplans $H_\alpha$, $\alpha \in \mathcal{R}_+$, divisent l'espace $\mathbb{R}^d$ en des composantes connexes ouvertes, appelées **chambres de Weyl**.
Considérons
$$\mathbf{C} := \Big\{x \in \mathbb{R}^d, \quad \langle x, \alpha\rangle > 0, \quad \forall\, \alpha \in \Pi\Big\}.$$
La partie $\mathbf{C}$ est couramment appelée la **chambre de Weyl fondamentale** associée au système simple $\Pi$. En outre, la fermeture topologique de $\mathbf{C}$, donnée par
$$\overline{\mathbf{C}} := \Big\{x \in \mathbb{R}^d, \quad \langle x, \alpha\rangle \geq 0, \quad \forall\, \alpha \in \Pi\Big\},$$
est un **domaine fondamental** pour l'action du groupe de Coxeter-Weyl sur $\mathbb{R}^d$ i.e. pour tout $x \in \mathbb{R}^d$, la $W$-orbite de $x$ rencontre $\overline{\mathbf{C}}$ en un point et un seul (voir [26] ou [30]).
Pour tout point $x \in \mathbb{R}^d$, soit $W_x := \{g \in W, \quad gx = x\}$ le **groupe d'isotropie** de $x$. Nous signalons que le groupe d'isotropie d'un point $x$ est complètement caractérisé suivant que $x$ soit dans $\mathbf{C}$ ou dans les murs de $\mathbf{C}$. D'une façon précise, nous avons
**1.** si $x \in \mathbf{C}$, alors $W_x$ est trivial,
**2.** si $x \in \partial\overline{\mathbf{C}}$, alors $W_x$ est engendré par les réflexions $\sigma_\alpha$, $\alpha \in R_x = \{\alpha \in \Pi, \quad x \in H_\alpha\}$.
En particulier, si $x \in \cap_{\alpha\in\Pi} H_\alpha$, le groupe d'isotropie de $x$ coïncide avec le groupe de Coxeter-Weyl.

Un autre ingrédient crucial dans la théorie de Dunkl est celui de la fonction de multiplicité (dont la terminologie vient de l'analyse sur les espaces homogènes formés à partir des groupes de Lie) associée à un système de racines. Il s'agit d'une fonction $k : \mathcal{R} \longrightarrow \mathbb{C}$ invariante sous l'action du groupe de Coxeter-Weyl $W$. Il est important de noter que le nombre de valeurs prises par une telle fonction coïncide avec le nombre de classes de conjugaison du groupe $W$.

Maintenant, pour mettre au clair les objets ainsi définis, nous allons donner quelques exemples de systèmes de racines dans $\mathbb{R}^d$ (rappelant que $(e_i)_{1\leq i\leq d}$ est la base canonique de $\mathbb{R}^d$) :

• **Système de rang 1** : Un système de racines dans $\mathbb{R}^d$ de rang 1 est constitué de deux vecteurs non nuls $\alpha$ et $-\alpha$. Il est appelé de type $A_1$ et le groupe de Coxeter-Weyl associé à ce système est $\mathbf{Z}_2$. En prenant la racine positive $\alpha = e_1$, la chambre de Weyl fondamentale est $]0, +\infty[\times\mathbb{R}^{d-1}$.

• **Système de type $\mathbf{A}_1 \times \cdots \times \mathbf{A}_1$**, ($m$ fois avec $1 \leq m \leq d$) : Un tel système est donné par $\mathcal{R} = \{\pm e_i, \quad 1 \leq i \leq m\}$. Il est de rang $m$. De plus, nous pouvons prendre comme sous-système positif l'ensemble $\mathcal{R}_+ = \{e_i, \quad 1 \leq i \leq m\}$. Avec ce choix du sous-système positif, les racines simples sont exactement les racines positives et la chambre de Weyl fondamentale est donnée par

$$\mathbf{C} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad x_i > 0, \quad \forall\ 1 \le i \le m\}.$$

D'autre part, puisque les racines sont orthogonales, le groupe de Coxeter-Weyl est commutatif et il est isomorphe à $\mathbf{Z}_2^m$.
Terminons cet exemple en notant que la fonction de multiplicité est donnée par $m$ paramètres $k(e_1), \dots, k(e_m) \in \mathbb{C}$.

• **Système de type** $\mathbf{A}_{m-1}$ avec $2 \le m \le d$. Pour tout vecteur $x \in \mathbb{R}^d$, nous allons utiliser la notation $x = (x^{(m)}, x') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$.
Le système de racines est $\mathcal{R} = \{\alpha_{i,j} = e_i - e_j, \quad 1 \le i,j \le m, \quad i \ne j\}$. Soit $u = (u^{(m)}, u')$ avec $u^{(m)} = (1, \dots, 1)$ et $u' = 0$. Les vecteurs $u$, $e_{m+1}, \dots e_d$ sont orthogonaux à $\mathcal{R}$ et le système est donc de rang $m-1$.
En choisissant $\beta \in \mathbb{R}^d$ tel que $\beta^{(m)} = (m, m-1, \dots, 1)$ et $\beta' = 0$, le sous-système positif est $\mathcal{R}_+ := \{\alpha_{i,j} = e_i - e_j, \quad 1 \le i < j \le m\}$ et le système simple est $\Pi = \{\alpha_{i,i+1} = e_i - e_{i+1}, \quad 1 \le i \le m-1\}$. De plus, la chambre de Weyl fondamentale est

$$\mathbf{C} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad x_1 > x_2 > \dots > x_{m-1} > x_m\}.$$

La réflexion $\sigma_{\alpha_{i,j}}$ permute la i-ème et la j-ème composante de tout vecteur $x \in \mathbb{R}^d$. Elle s'identifie ainsi à une transposition et le groupe de Coxeter-Weyl est le groupe symétrique $S_m$ à $m$ éléments. De plus, puisque les transpositions dans $S_m$ sont conjuguées, la fonction de multiplicité d'un tel système est constante.

• **Système de type** $\mathbf{B}_m$ $(2 \le m \le d)$. Dans ce cas, le système de racines est le suivant

$$\begin{aligned}\mathcal{R} &= \{\alpha_{i,j} = e_i - e_j, \quad 1 \le i,j \le m, \quad i \ne j\} \\ &\cup \{\beta_{i,j} = sgn(j-i)(e_i + e_j), \quad 1 \le i,j \le m, \quad i \ne j\} \\ &\cup \{\pm e_i, \quad 1 \le i \le m\}.\end{aligned}$$

En prenant le même vecteur $\beta$ de l'exemple précédent, nous pouvons choisir comme sous-système positif l'ensemble $\mathcal{R}_+ = \{e_i - e_j, \quad e_i + e_j, \quad 1 \le i < j \le m\} \cup \{e_i, \quad 1 \le i \le m\}$. Les racines simples sont $e_m$ et $e_i - e_{i+1}$, $1 \le i < m$ et la chambre de Weyl fondamentale est

$$\mathbf{C} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad x_1 > x_2 > \dots > x_{m-1} > x_m > 0\}.$$

Nous avons vu que le groupe engendré par les $\sigma_{\alpha_{i,j}}$ est $S_m$ et celui engendré par les $\sigma_{e_i}$ est $\mathbf{Z}_2^m$. D'autre part, en se basant sur la relation $A\sigma_\alpha A^{-1} = \sigma_{A\alpha}$ avec $A$ une isométrie de $\mathbb{R}^d$ et $\alpha \in \mathbb{R}^d$, nous déduisons que $\mathbf{Z}_2^m$ est un sous-groupe normal dans le groupe de Coxeter-Weyl $W$ associé à notre système de racines $\mathbf{B}_m$ (i.e. $g\mathbf{Z}_2^m g^{-1} = \mathbf{Z}_2^m$, $\forall\ g \in W$). Finalement, nous utilisons le fait que l'intersection de $S_m$ avec $\mathbf{Z}_2^m$ est triviale, pour conclure que $W$ n'est autre que le produit semi-direct $S_m \ltimes \mathbf{Z}_2^m$. Nous signalons également qu'ici la fonction de mutiplicité s'identifie à deux paramètres $(k_1, k_2)$ avec $k_2 = k(e_m) = k(e_{m-1}) = \dots = k(e_1)$ et $k_1 = k(e_i - e_j) = k(e_i + e_j)$

Pour d'autres exemples de systèmes de racines, le lecteur pourra consulter les deux livres mentionnés ci-dessus ou encore le livre de C. F. Dunkl et Y. Xu ([15]).

Dans la suite, nous fixons un système de racines $\mathcal{R}$ dans $\mathbb{R}^d$ dont les racines $\alpha$ sont normalisées par $\|\alpha\|^2 = 2$. Comme avant, $W$ désigne le groupe de Coxeter-Weyl asscocié à $\mathcal{R}$. En outre, toute fonction de multiplicité $k$ considérée, sera supposée à valeurs positives.

D'une importance particulière dans cette thèse, nous avons **l'opérateur de Dunkl-Laplace** qui agit sur une fonction $f \in \mathcal{C}^2(\mathbb{R}^d)$ de la façon suivante

$$\Delta_k f(x) := \Delta f(x) + 2 \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \left( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right), \tag{0.1}$$

où $\Delta$ et $\nabla$ sont respectivement le laplacien et le gradient classiques sur $\mathbb{R}^d$. Nous avons les cas particuliers suivants :

• **Système de rang 1** : Le laplacien de Dunkl dans le cas unidimensionnel est de la forme

$$\Delta_k^{\mathbf{Z}_2} f(x) = f''(x) + 2k\Big( \frac{f'(x)}{x} - \frac{f(x) - f(-x)}{x^2} \Big).$$

• **Système de type $\mathbf{A}_1 \times \cdots \times \mathbf{A}_1$**, ($m$ fois avec $1 \leq m \leq d$) : Dans ce cas

$$\Delta_k^{\mathbf{Z}_2^m} f(x) = \Delta f(x) + 2 \sum_{j=1}^m k(e_j) \left( \frac{\partial_j f(x)}{x_j} - \frac{f(x) - f(\sigma_{e_j}(x))}{x_j^2} \right).$$

• **Système de type $\mathbf{A}_{m-1}$** avec $2 \leq m \leq d$. Le laplacien de Dunkl associé au groupe symétrique $S_m$ opère sur une fonction $f$ de classe $C^2$ sur $\mathbb{R}^d$ comme suit

$$\Delta_k^{S_m} f(x) = \Delta f(x) + 2k \sum_{1 \leq i < j \leq m}^{m} \left( \frac{\partial_i f(x) - \partial_j f(x)}{x_i - x_j} - \frac{f(x) - f(\sigma_{e_i - e_j}(x))}{(x_i - x_j)^2} \right).$$

• **Système de type $\mathbf{B}_m$** ($2 \leq m \leq d$). L'opérateur de Dunkl-Laplace associé au système de racines $\mathbf{B}_m$ peut s'écrire au moyen de $\Delta_k^{S_m}$ de la manière suivante

$$\begin{aligned} \Delta_k^{S_m \ltimes \mathbf{Z}_2^m} f(x) &= \Delta_{k_1}^{S_m} f(x) + 2k_2 \sum_{i=1}^m \left( \frac{\partial_i f(x)}{x_i} - \frac{f(x) - f(\sigma_{e_i}(x))}{x_i^2} \right) \\ &\quad + 2k_1 \sum_{1 \leq i < j \leq m}^{m} \left( \frac{\partial_i f(x) + \partial_j f(x)}{x_i + x_j} - \frac{f(x) - f(\sigma_{e_i + e_j}(x))}{(x_i + x_j)^2} \right). \end{aligned}$$

Il est bien connu que $\Delta_k = \sum_{j=1}^d D_{e_j}^2$, où $(D_\xi)_{\xi \in \mathbb{R}^d}$ est la famille commutative des **opérateurs de Dunkl** (voir [10] et [15]) définis pour $f \in \mathcal{C}^1(\mathbb{R}^d)$ par

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle} \tag{0.2}$$

et $\partial_\xi$ est la dérivation dans la direction $\xi$.
Notons que l'expression (0.2) est indépendante du choix d'un sous-système positif de $\mathcal{R}$. Ceci découle de la définition de la fonction de multiplicité et de la relation

$$\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(x)-f(\sigma_\alpha(x))}{\langle\alpha,x\rangle} = \frac{1}{2}\sum_{\alpha\in\mathcal{R}} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(x)-f(\sigma_\alpha(x))}{\langle\alpha,x\rangle}.$$

Les opérateurs de Dunkl sont liés aux opérateurs aux dérivées partielles via la relation d'entrelacement suivante

$$\forall\ \xi\in\mathbb{R}^d,\quad D_\xi V_k = V_k\partial_\xi, \tag{0.3}$$

où $V_k$ est communément appelé **opérateur d'entrelacement de Dunkl**.

Historiquement, en 1991, C. F. Dunkl (voir [11]) a montré l'existence et l'unicité de $V_k$ en tant qu'isomorphisme vectoriel de l'espace $\mathcal{P}^d$ des polynômes à $d$ variables dans lui même satisfaisant (0.3), fixant les polynômes constants et laissant stables les espaces $\mathcal{P}^d_n$, $n\in\mathbb{N}$, des polynômes homogènes de degré $n$.
En 2001, K. Trimèche a prolongé l'opérateur d'entrelacement de Dunkl en un isomorphisme topologique de l'espace $\mathcal{C}^\infty(\mathbb{R}^d)$ (muni de sa topologie de Fréchet usuelle) dans lui même et toujours satisfaisant la relation d'entrelacement (0.3) (voir [51]).

L'un des résultats les plus remarquables dans l'analyse de Dunkl, dû à Rösler (voir [42]), est le suivant : Pour tout $x\in\mathbb{R}^d$, il existe une unique mesure de probabilité $\mu_x$ sur $\mathbb{R}^d$ à support compact telle que

$$\forall\ f\in\mathcal{C}^\infty(\mathbb{R}^d),\quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y)d\mu_x(y). \tag{0.4}$$

De plus, on a

$$\operatorname{supp}\mu_x\subset C(x) := co\{gx,\quad g\in W\} \tag{0.5}$$

avec $C(x)$ l'enveloppe convexe de l'orbite de $x$ sous l'action de $W$. Nous appellerons $\mu_x$ **la mesure de Rösler** au point $x$.

L'un des problèmes majeurs de la théorie de Dunkl est que la mesure de Rösler n'est connue explicitement que dans très peu de cas. Dans le cas d'un système de racines de rang 1 (nous prenons $\mathcal{R}=\{\pm e_1\}$ dans $\mathbb{R}^d$), une formule explicite, donnée par Dunkl ([11]), est la suivante :

$$V_k(f)(x) = \int_{-1}^1 f(tx_1,x_2,\dots,x_d)\phi_k(t)dt,$$

où $\phi_k$ est la $\mathbf{Z}_2$-densité de Dunkl de paramètre $k=k(e_1)>0$ donnée par

$$\phi_k(t) = \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}(1-t)^{k-1}(1+t)^k\mathbf{1}_{[-1,1]}(t), \tag{0.6}$$

En prenant $\mathcal{R}=\{\pm e_i, 1\le i\le m\}$ dans $\mathbb{R}^d$, l'opérateur d'entrelacement est de la forme

$$V_k(f)(x) = \int_{[-1,1]^m} f(t_1x_1,t_2x_2,\dots,t_mx_m,x_{m+1},\dots,x_d)\prod_{i=1}^m\phi_{k_i}(t_i)dt_1\dots dt_m, \tag{0.7}$$

avec $\phi_{k_i}$ la $\mathbf{Z}_2$-densité de Dunkl de paramètre $k_i > 0$ donnée par (0.6). Cette formule, qui généralise celle donnée par Dunkl, a été prouvée par Xu ([54]). Une extension de ce résultat au cas d'un système de racines deux à deux orthogonales est donnée dans ([33]). Deux autres expressions connues d'un tel opérateur dans les cas $\mathbf{A}_2$ et $\mathbf{B}_2$ ont été construites par Dunkl (voir [13] et [16]). Signalons que l'expression de $V_k$ dans le cas $\mathbf{A}_2$ a été redémontrée dans ([3])

Au moyen de l'opérateur d'entrelacement de Dunkl $V_k$ et de son inverse, K. Trimèche a défini **la translation de Dunkl** $\tau_x$, $x \in \mathbb{R}^d$, sur l'espace $\mathcal{C}^\infty(\mathbb{R}^d)$ comme suit (voir [52]) :

$$\forall\ y \in \mathbb{R}^d, \quad \tau_x f(y) := \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{0.8}$$

où $T_a f(b) = f(a+b)$ est la translation usuelle. Pour $x \in \mathbb{R}^d$, la translatée de Dunkl d'une fonction $f \in \mathcal{D}(\mathbb{R}^d)$ (l'espace des fonctions de classe $C^\infty$ sur $\mathbb{R}^d$ à supports compact) $\tau_x f$ satisfait la propriété suivante :

$$\int_{\mathbb{R}^d} \tau_x f(y) \omega_k(y) dy = \int_{\mathbb{R}^d} f(y) \omega_k(y) dy, \tag{0.9}$$

où $\omega_k(x)dx = dm_k(x)$ est la mesure $W$-invariante dont la densité est donnée par

$$\omega_k(x) := \prod_{\alpha \in \mathcal{R}_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \tag{0.10}$$

Nous signalons que la fonction poids $\omega_k$ est homogène de degré $2\gamma$ avec

$$\gamma = \sum_{\alpha \in \mathcal{R}_+} k(\alpha).$$

Quand $f$ est dans l'espace de Schwartz $\mathcal{S}(\mathbb{R}^d)$, une autre expression de la translation de Dunkl est la suivante (voir [8])

$$\tau_x f(y) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi) E_k(ix, \xi) E_k(iy, \xi) \omega_k(\xi) d\xi, \tag{0.11}$$

où $c_k = \int_{\mathbb{R}^d} e^{-\|x\|^2/2} \omega_k(x) dx$ est la constante de Macdonald-Mehta ([36], [17]), $E_k(x, \xi)$ étant le noyau défini par

$$E_k(x, \xi) := V_k(e^{\langle \cdot, \xi \rangle})(x) = \int_{\mathbb{R}^d} e^{\langle z, \xi \rangle} d\mu_x(z)$$

(appelé communément **noyau de Dunkl** ([11], [45])) et $\mathcal{F}_k$ est la **transformation de Dunkl** définie pour une fonction $f$ $m_k$-intégrable sur $\mathbb{R}^d$ par

$$\mathcal{F}_k(f)(\lambda) = \int_{\mathbb{R}^d} f(x) E_k(-ix, \lambda) \omega_k(x) dx, \quad \lambda \in \mathbb{R}^d. \tag{0.12}$$

Cette transformation, introduite par Dunkl ([12]) et étudiée par de Jeu ([29]), possède des propriétés analogues à celles satisfaites par la transformation de Fourier sur $\mathbb{R}^d$ ([29] et

[12]). En particulier, $\mathcal{F}_k$ est un isomorphisme topologique de l'espace de $\mathcal{S}(\mathbb{R}^d)$ dans lui même et la transformation inverse est donnée par

$$\mathcal{F}_k^{-1}(f)(\xi) = \mathcal{F}_k(f)(-\xi), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

En outre, nous avons une formule d'inversion et un théorème de Plancherel : La transformation de Dunkl (à une constante multiplicative près) s'étend en un isomorphisme isométrique de $L^2(\mathbb{R}^d, m_k)$.

Maintenant, nous allons évoquer quelques problèmes liés à la translation de Dunkl. Comme pour l'opérateur d'entrelacement de Dunkl, l'absence d'une expression explicite de la translation de Dunkl constitue une première difficulté pour la manipuler.
Par contre, nous disposons d'une formule explicite dans le cas unidimensionnel. Plus précisément, si $x \in \mathbb{R}$ et $f \in \mathcal{C}(\mathbb{R})$, nous avons (voir [39])

$$\begin{aligned}\tau_x f(y) = \frac{1}{2}\int_{-1}^{1} f\Big(\sqrt{x^2+y^2+2txy}\Big)\Big(1+\frac{x+y}{\sqrt{x^2+y^2+2txy}}\Big)\phi_k(t)dt \\ + \frac{1}{2}\int_{-1}^{1} f\Big(-\sqrt{x^2+y^2+2txy}\Big)\Big(1-\frac{x+y}{\sqrt{x^2+y^2+2txy}}\Big)\phi_k(t)dt,\end{aligned} \quad (0.13)$$

avec $\phi_k$ la fonction donnée par (0.6). Un autre cas dans lequel la translation de Dunkl est connue de manière explicite est le cas où le groupe de Coxeter-Weyl est $\mathbf{Z}_2^d$. Une telle expression repose sur une formule produit satisfaite par le noyau de Dunkl.

Contrairement à la translation classique, l'opérateur de translation de Dunkl n'est pas, en général, un opérateur positif. En effet, la formule (0.13) en atteste lorsque $W = \mathbf{Z}_2$. De plus, bien qu'elle n'ait pas une expression explicite connue, Thangavelu et Xu ont montré que la translation de Dunkl associée au groupe symétrique $S_d$ n'est pas un opérateur positif ([48]).

Cependant, il existe une classe particulière de fonctions sur laquelle la positivité de ces opérateurs est vérifiée. En effet, M. Rösler a montré en 2003 ([46]), que si $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ est radiale, alors les translatées de $f$ sont données par

$$\tau_x f(y) = \int_{\mathbb{R}^d} \widetilde{f}\Big(\sqrt{\|x\|^2+\|y\|^2+2\langle x, z\rangle}\Big)d\mu_y(z). \quad (0.14)$$

Ici, $\widetilde{f}$ est la fonction profil de $f$ définie sur $[0, +\infty[$ par $f(x) = \widetilde{f}(\|x\|)$.

Une autre question à signaler est celle du prolongement de la translation de Dunkl à autres classes de fonctions. Jusqu'à présent, nous n'avons que des résultats partiels sur cette question. En particulier, la formule (0.11) et le théorème de Plancherel pour la transformation de Dunkl montrent que pour tout $x \in \mathbb{R}^d$, l'opérateur $\tau_x$ s'étend en un opérateur borné de l'espace $L^2(\mathbb{R}^d, m_k)$ dans lui même. Dans ce cas, pour tout $f \in L^2(\mathbb{R}^d, m_k)$, $\tau_x f$ est la fonction de $L^2(\mathbb{R}^d, m_k)$ donnée par

$$\mathcal{F}_k(\tau_x f) = E_k(ix, .)\mathcal{F}_k(f).$$

De plus, Thangavelu et Xu ont montré ([48]) que

$$\forall\ x \in \mathbb{R}^d, \quad \forall\ 1 \leq p \leq 2, \quad \tau_x : L^p_{rad}\left(\mathbb{R}^d, m_k\right) \longrightarrow L^p\left(\mathbb{R}^d, m_k\right) \quad \text{est borné,}$$

où $L^p_{rad}(\mathbb{R}^d, m_k)$ est le sous-espace de $L^p(\mathbb{R}^d, m_k)$ formé par des fonctions radiales.

Dans le cas général, les questions du prolongement des opérateurs de translation de Dunkl et leurs propriétés de bornitude entre les espaces $L^p(\mathbb{R}^d, m_k)$ ($d \geq 2$, $1 \leq p \leq +\infty$, $p \neq 2$) demeurent deux problèmes ouverts. Dans le cas unidimensionnel, nous disposons du résultat suivant : pour tout $x \in \mathbb{R}^d$ et tout $1 \leq p \leq +\infty$, l'opérateur

$$\tau_x : L^p(\mathbb{R}, m_k) \longrightarrow L^p(\mathbb{R}, m_k)$$

est borné (voir [39] et [2]).

Une dernière remarque à souligner concernant les opérateurs de translation de Dunkl est que les expressions (0.8) et (0.11) nous empêchent de leur donner une interprétation géométrique.

Le **produit de convolution de Dunkl** de deux fonctions $f, g \in L^2(\mathbb{R}^d, m_k)$ est défini classiquement au moyen de la translation de Dunkl comme suit :

$$f *_k g(x) = \int_{\mathbb{R}^d} f(y)\tau_x g(-y)\omega_k(y)dy. \tag{0.15}$$

Pour $f, g \in \mathcal{S}(\mathbb{R}^d)$, nous avons les relations suivantes

$$f *_k g = g *_k f, \quad \mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{0.16}$$

Dans cette thèse, nous nous intéressons essentiellement aux fonctions harmoniques et sousharmoniques associées à l'opérateur de Dunkl-Laplace $\Delta_k$ agissant sur les fonctions de classe $C^2$ qui sont définies sur un ouvert de $\mathbb{R}^d$ invariant sous l'action du groupe de Coxeter-Weyl $W$. Nous les appelons respectivement **fonctions D-harmoniques** et **fonctions D-sousharmoniques**.

Historiquement, les fonctions D-harmoniques ont d'abord été étudiées sur $\mathbb{R}^d$ par H. Mejjaoli et K. Trimèche en 2001 ([35]). En particulier, ils ont introduit l'**opérateur moyenne sphérique** agissant sur les fonctions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ par

$$\forall\ x \in \mathbb{R}^d, \quad \forall\ r > 0, \quad M^r_S(f)(x) := \frac{1}{d_k}\int_{S^{d-1}} \tau_x f(r\xi)\omega_k(\xi)d\sigma(\xi), \tag{0.17}$$

avec $d_k$ une constante de normalisation et $d\sigma(\xi)$ la mesure surfacique sur la sphère unité $S^{d-1}$ de $\mathbb{R}^d$. En outre, ils ont caractérisé les fonctions D-harmoniques $u$ qui sont de classe $C^\infty$ sur $\mathbb{R}^d$ par la propriété de la moyenne sphérique i.e.

$$\forall\ x \in \mathbb{R}^d, \quad \forall\ r > 0, \quad u(x) = M^r_S(u)(x). \tag{0.18}$$

Dans [32], nous trouvons un résultat analogue mais pour les fonctions D-harmoniques dans la boule unité ouverte $B$ de $\mathbb{R}^d$. Les auteurs ont utilisé un argument de prolongement d'une fonction de classe $C^\infty$ sur $B$ en une fonction de classe $C^\infty$ sur $\mathbb{R}^d$.

Au passage, notons que pour tout $(x,r) \in \mathbb{R}^d \times [0,+\infty[$, l'opérateur $f \mapsto M_S^r(f)(x)$ est positif ([46] et [8]). Plus précisément, il existe une famille de mesures de probabilité $(\sigma_{x,r}^k)_{x,r}$ telle que

$$\forall\ (x,r) \in \mathbb{R}^d \times [0,+\infty[, \quad \forall\ f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad M_S^r(f)(x) = \int_{\mathbb{R}^d} f(\xi) d\sigma_{x,r}^k(\xi). \tag{0.19}$$

La mesure $\sigma_{x,r}^k$ est à support compact et nous avons

$$\text{supp } \sigma_{x,r}^k \subset \cup_{g\in W} B(gx,r). \tag{0.20}$$

# *OPERATEURS DE MOYENNE ET OUTILS D'ANALYSE*

Dans la suite, nous désignons par
- $B(a,\rho)$ la boule fermée centrée en $a \in \mathbb{R}^d$ et de rayon $\rho > 0$,
- $\Omega$ un ouvert $W$-invariant de $\mathbb{R}^d$,
- $\mathcal{H}_k(\Omega)$ le sous-espace de $\mathcal{C}^2(\Omega)$ formé par les fonctions D-harmoniques dans $\Omega$.

Dans cette thèse, notre idée principale est d'introduire un nouvel opérateur de moyenne qui sera une généralisation de l'opérateur moyenne volumique classique. Dans ce but, nous introduisons le noyau suivant ([19] et chapitre 1) :

$$\forall\ r > 0, \quad \forall\ x, y \in \mathbb{R}^d, \quad h_k(r,x,y) = \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]} \left(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle}\right) d\mu_y(z). \tag{0.21}$$

Nous l'appelons **noyau harmonique**. Pour $k = 0$, la mesure de Rösler $\mu_y$ est $\delta_y$ (la masse de Dirac au point $y$) et $h_k(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$.

Ce noyau possède les propriétés suivantes :
**1.** Pour tous $r > 0$, $x, y \in \mathbb{R}^d$, on a $0 \leq h_k(r,x,y) \leq 1$.
**2.** Pour $x, y \in \mathbb{R}^d$ fixés, la fonction $r \mapsto h_k(r,x,y)$ est croissante et continue à droite.
**3.** Soit $r > 0$ et $x \in \mathbb{R}^d$. Alors, pour toute suite $(\varphi_\varepsilon)_{\varepsilon>0}$ de fonctions radiales telles que

$$0 \leq \varphi_\varepsilon \leq 1, \quad \varphi_\varepsilon = 1 \text{ sur } B(0,r), \quad \forall\ y \in \mathbb{R}^d,\ \lim_{\varepsilon\to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y), \tag{0.22}$$

on a

$$\forall\ y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon\to 0} \tau_{-x}\varphi_\varepsilon(y).$$

**4.** Soit $r > 0$. Le noyau $h_k(r,.,.)$ est symétrique i.e. pour tout $x, y \in \mathbb{R}^d$, on a $h_k(r,x,y) = h_k(r,y,x)$.

**5.** Soient $r > 0$ et $x, y \in \mathbb{R}^d$. Alors, pour tout $g \in W$, on a $h_k(r, gx, gy) = h_k(r, x, y)$.
**6.** Pour tout $r > 0$ et $x \in \mathbb{R}^d$, la fonction $y \longmapsto h_k(r, x, y)$ est semi-continue supérieurement sur $\mathbb{R}^d$, $m_k$-intégrable sur $\mathbb{R}^d$ ( nous rappelons que la mesure $m_k$ est définie par (0.10)) et on a

$$\|h_k(r, x, .)\|_{L^1(\mathbb{R}^d, m_k)} = m_k[B(0, r)] = \frac{d_k r^{d+2\gamma}}{d + 2\gamma}.$$

Une autre propriété fondamentale de ce noyau, qui va nous permettre de définir la moyenne volumique généralisée d'une fonction continue ([19] et chapitre 1) ou localement intégrable sur $\Omega$ par rapport à la mesure $m_k$ (l'espace de ces fonctions sera noté $L^1_{loc}(\Omega, m_k)$) ([20] et chapitre 2), est la localisation de son support. Plus précisément, nous avons

$$\forall\ r > 0, \quad \forall\ x \in \mathbb{R}^d, \quad B(x, r) \subset \text{supp}\ h_k(r, x, .) \subset B^W(x, r) := \cup_{g \in W} B(gx, r). \tag{0.23}$$

(pour la première inclusion voir [20] (ou chapitre 2) et la seconde se trouve dans [19] (ou chapitre 1)).
En fait, quand la fonction de multiplicité $k$ est strictement positive sur $\mathcal{R}$, nous avons ([20] et chapitre 2)

$$\forall\ r > 0, \quad \forall\ x \in \mathbb{R}^d, \quad \text{supp}\ h_k(r, x, .) = B^W(x, r) = \cup_{g \in W} B(gx, r).$$

Nous appelons alors $B^W(x, r)$ la **boule de Dunkl** fermée centrée en $x$ et de rayon $r$ associée au groupe de Weyl $W$ et à la fonction de multiplicité $k$ (bien qu'elle dépende de $k$, pour simplifier, nous avons choisi la notation $B^W(x, r)$ au lieu de $B^W_k(x, r)$).

Ce résultat repose sur des nouvelles précisions sur le support de la mesure de Rösler $\mu_x$. Avant de les signaler, nous allons présenter une brève histoire de la question. L'inclusion (0.5) a été démontrée par M. De Jeu en 1993 ([29]) et elle est basée sur un théorème de Paley-Wiener pour la transformation de Fourier ([50]). Notons que cette inclusion est vraie même si $k$ est à valeurs dans le domaine complexe $\{z \in \mathbb{C}, \quad Re(z) \geq 0\}$. Ensuite, en 1999, M. Rösler a prouvé dans [40] que, quand $k \geq 0$, l'un des points de l'orbite $W.x$ est nécessairement dans le support de $\mu_x$. Dans la même année, M. Rösler a réussi, par un résultat sur le comportement asymptotique du noyau de Dunkl, à établir que $x \in \text{supp}\ \mu_x$ dès que $x \notin \cup_{\alpha \in \mathcal{R}} H_\alpha$ (voir [43]).

Passons maintenant aux nouvelles précisions (voir [20] et chapitre 2). Dans un premier temps, nous avons montré, par un moyen différent de celui utilisé par Rösler, que $x$ est toujours dans le support de $\mu_x$ (nous rappelons que la fonction $k$ est supposée positive ou nulle). Ensuite, nous avons prouvé, sous la condition $k(\alpha) > 0$ pour tout $\alpha \in \mathcal{R}$, que l'ensemble supp $\mu_x$ est $W$-invariant. Ainsi, comme conséquence immédiate de ces deux résultats et sous la condition précédente, nous déduisons que toute l'orbite de $x$ sous l'action de $W$ est incluse dans supp $\mu_x$.

D'autre part, nous pouvons généraliser les résultats précédents de la manière suivante : Supposons que la fonction $k$ n'est pas identiquement nulle et introduisons l'ensemble

$$\mathcal{R}_A := \{\alpha \in \mathcal{R}, \quad k(\alpha) > 0\}$$

des **racines** que nous appelons **actives**. Soient $k_A$ la restriction de $k$ sur $\mathcal{R}_A$ et $W_A$ le sous-groupe de $W$ engendré par les réflexions associées aux racines actives. Alors, $\mathcal{R}_A$ est un nouveau système de racines dans $\mathbb{R}^d$, $W_A$ est son groupe de Coxeter-Weyl et $k_A$ est une fonction multiplicité dans le sens qu'elle est $W_A$-invariante. Dans ce cas, pour tout $x \in \mathbb{R}^d$, la mesure de Rösler est de la forme ([44])

$$\mu_x = \mu_{x'} \otimes \delta_{x''}, \quad x = x' + x'' \in F \oplus F^{\perp} = \mathbb{R}^d,$$

avec $F$ l'espace vectoriel engendré par les racines actives, $\delta_{x''}$ la mesure de Dirac en $x''$ et $\mu_{x'}$ la nouvelle mesure de Rösler reliée au quadruplet $(F, \mathcal{R}_A, W_A, k_A)$. Ici, nous regardons $W_A$ comme un sous-groupe du groupe orthogonal $O(F)$ de $F$.
Alors, nous avons montré que le support de $\mu_x$

**a)** est inclus dans l'enveloppe convexe de l'orbite $W_A.x$,
**b)** est invariant par l'action du groupe $W_A$,
**c)** contient la $W_A$-orbite de $x$.
De plus, la boule de Dunkl est réduite à $B^{W_A}(x,r) = \cup_{g \in W_A} B(gx, r)$. D'une manière plus précise (ici en regardant encore une fois $W_A$ comme un sous-groupe de $O(F)$), nous obtenons

$$\text{supp } \mu_x = x'' + \text{supp } \mu_{x'}, \quad B^W(x,r) = x'' + B^{W_A}(gx', r).$$

En résumant, nous pouvons dire que la théorie de Dunkl n'est "bien vivante" que dans l'espace vectoriel engendré par les racines actives.

Soit $f \in L^1_{loc}(\Omega, m_k)$ et $B(x,r) \subset \Omega$. On définit alors **la moyenne volumique généralisée** de $f$ relative à $(x,r)$ via le noyau harmonique, par

$$M_B^r(f)(x) = \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} f(y) h_k(r,x,y) \omega_k(y) dy. \tag{0.24}$$

Nous montrons alors que l'opérateur moyenne volumique satisfait les propriétés suivantes (voir [20] et chapitre 2) :

- Pour tout $r > 0$ assez petit, la fonction $x \mapsto M_B^r(f)(x)$ appartient à $L^1_{loc}(\Omega_r, m_k)$ avec

$$\Omega_r := \{x \in \Omega, \quad dist(x, \partial\Omega) > r\} = \{x \in \Omega, \quad B(x,r) \subset \Omega\}. \tag{0.25}$$

- Pour tout $x \in \Omega$, la fonction $r \mapsto M_B^r(f)(x)$ est continue sur $]0, \varrho_x[$, avec

$$\varrho_x = dist(x, \partial\Omega). \tag{0.26}$$

Pour étudier les fonctions harmoniques et sousharmoniques au sens de Dunkl, nous avons besoin de plusieurs outils.
Commençons par citer les deux résultats suivants :
Le premier ([19] et chapitre 1) dit que pour tout $x \in \mathbb{R}^d$, la famille de mesures de probabilité

$$d\eta^k_{x,r}(y) = \frac{1}{m_k[B(0,r)]} h_k(r,x,y) \omega_k(y) dy, \quad r > 0, \tag{0.27}$$

est une approximation de la masse de Dirac $\delta_x$ quand $r \to 0$. Plus précisément,
a) pour tout $\alpha > 0$, $\lim_{r\to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0$,
b) si $f$ est une fonction mesurable et localement bornée sur $\Omega$ et si elle est continue en un point $x \in \Omega$, alors

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y) d\eta^k_{x,r}(y) = \lim_{r\to 0} M^r_B(f)(x) = f(x). \tag{0.28}$$

Le second est une extension du théorème de différentiation de Lebesgue au cas Dunkl (voir [20] et chapitre 2). Il s'énonce comme suit : si $f \in L^1_{loc}(\Omega, m_k)$, alors pour presque tout $x \in \Omega$,

$$\lim_{r\to 0} M^r_B(f)(x) = f(x). \tag{0.29}$$

Ici, il est intéressant de noter que la notion d'égalité presque partout au sens de la mesure $m_k$ coïncide avec celle au sens de la mesure de Lebesgue sur $\mathbb{R}^d$ puisque la fonction $\omega_k$ ne s'annule que sur un ensemble Lebesgue négligeable.

Les formules suivantes qui donnent le lien entre une fonction $f$ de $\mathcal{C}^\infty(\mathbb{R}^d)$, sa moyenne sphérique et sa moyenne volumique sont des ingrédients fondamentaux pour notre objectif ([19] et chapitre 1) : pour tout $x \in \mathbb{R}^d$ et tout $r > 0$, nous avons

$$M^r_S(f)(x) = f(x) + \frac{1}{d+2\gamma} \int_0^r M^t_B(\Delta_k f)(x)\ t dt, \tag{0.30}$$

$$M^r_B(f)(x) = \frac{d+2\gamma}{r^{d+2\gamma}} \int_0^r M^t_S(f)(x) t^{d+2\gamma-1} dt \tag{0.31}$$

et

$$M^r_B(f)(x) = f(x) + \frac{1}{r^{d+2\gamma}} \int_0^r \int_0^\rho M^t_B(\Delta_k f)(x)\ t\ dt\ \rho^{d+2\gamma-1} d\rho. \tag{0.32}$$

Pour $x \in \Omega$ et $r \in ]0, \varrho_x[$ (rappelant la relation (0.26)) fixés, les trois relations précédentes restent vraies si $f \in \mathcal{C}^\infty(\Omega)$ ([20] et chapitre 2). De plus, nous avons étendu (0.31) pour une fonction semi-continue supérieurement quelconque sur $\Omega$ ([20] et chapitre 2).

Une autre extension des formules (0.32) et (0.30) aux fonctions de classe $C^2$ sur $\Omega$ a été faite mais avec la condition $x \in \Omega$ et $r \in ]0, \varrho_x/3[$. Cette généralisation repose sur le résultat d'approximation suivant :
Soit $f \in \mathcal{C}^2(\Omega)$ et soit $B(x, 3R) \subset \Omega$. Alors, il existe une suite de polynômes $(p_n)$ telle que $p_n \longrightarrow f$ et $\Delta_k p_n \longrightarrow \Delta_k f$ uniformément sur la boule de Dunkl fermée $B^W(x, R)$.
Le $\varrho_x/3$ vient d'une application du théorème de Taylor, quand $x$ est proche d'un hyperplan $H_\alpha$, à la partie différence du laplacien de Dunkl ([19] et chapitre 1).

Toujours dans le cadre de préparer le terrain pour l'étude des fonctions D-harmoniques et D-sousharmoniques, nous avons établi quelques résultats nouveaux sur le produit de convolution de Dunkl ([20] et chapitre 2). Le premier qui s'avère essentiel pour obtenir des résultats d'approximations nous dit que nous pouvons convoler (au sens de Dunkl) une fonction $u \in L^1_{loc}(\Omega, m_k)$ avec une fonction $f \in \mathcal{D}(\mathbb{R}^d)$ positive, radiale et telle que supp $f \subset B(0,\rho)$ (avec $\rho > 0$ suffisament petit). De plus, cette fonction prend la forme suivante : pour tout $x \in \Omega_\rho$ (rappelant (0.25))

$$u *_k f(x) := \int_{\mathbb{R}^d} u(y)\tau_{-x} f(y)\omega_k(y)dy \tag{0.33}$$

et d'ailleurs, elle est de classe $C^\infty$ sur $\Omega_\rho$ et satisfait

$$\Delta_k(u *_k f) = u *_k \Delta_k f. \tag{0.34}$$

Il est intéressant de noter que lorsque la fonction $u$ est continue, nous pouvons écrire le produit de convolution $u *_k f$ en coordonnées sphériques de la façon suivante :

$$u *_k f(x) = d_k \int_0^\rho \widetilde{f}(t) M_S^t(u)(x) t^{d+2\gamma-1} dt, \tag{0.35}$$

avec $\widetilde{f}$ la fonction profil de $f$.

D'autre part, nous avons montré que la moyenne volumique d'une fonction de $L^1_{loc}(\Omega, m_k)$ définit une fonction de $L^1_{loc}(\Omega_r, m_k)$ ($r$ étant fixé, voir (0.25)) et donc nous pouvons aussi la convoler avec $f$ et nous obtenons la relation

$$M_B^r(u *_k f)(x) = M_B^r(u) *_k f(x) \quad \text{dès que} \quad B(x,r) \subset \Omega_\rho. \tag{0.36}$$

Nous avons aussi obtenu un nouveau résultat d'associativité sur le produit de convolution de Dunkl : nous prenons $g \in \mathcal{D}(\mathbb{R}^d)$ positive, radiale et telle que supp $g \subset B(0,r)$, les fonctions $u$ et $f$ comme précédemment. Alors,

$$\forall\ x \in \Omega_{r+\rho}, \quad \big(u *_k f\big) *_k g(x) = u *_k (f *_k g)(x) = \big(u *_k g\big) *_k f(x). \tag{0.37}$$

## FONCTIONS D − HARMONIQUES

Ici, nous allons citer les principaux résultats obtenus sur les fonctions D-harmoniques. Sauf mention contraire, ils se trouvent dans le chapitre 1 (ou dans [19]).

Commençons par le suivant : Soit $u \in \mathcal{C}^2(\Omega)$. Alors, nous avons l'équivalence entre
**a)** $u$ est D-harmonique sur $\Omega$,
**b)** pour tout $x \in \Omega$ et tout $r \in ]0, \varrho_x/3[$, $u(x) = M_B^r(u)(x)$,
**c)** pour tout $x \in \Omega$ et tout $r \in ]0, \varrho_x/3[$, $u(x) = M_S^r(u)(x)$,

Ensuite, dans le chapitre 2 (ou dans [20]) nous avons amélioré cet énoncé en remplaçant $\varrho_x/3$ par $\varrho_x$.

Une première application est le **théorème de Liouville :** toute fonction D-harmonique majorée sur $\mathbb{R}^d$ est constante. Nous signalons que la version classique du théorème de

Liouville dans le cas Dunkl (toute fonction D-harmonique et bornée sur $\mathbb{R}^d$ est constante) avait été prouvée dans [18].

Nous avons aussi généralisé **l'inégalité de Harnack :** Pour tout ensemble compact $K \subset \Omega$, il existe une constante $C_K \geq 1$ telle que pour toute fonction D-harmonique positive dans $\Omega$, nous avons

$$\sup_K u \leq C_K \inf_K u,$$

et **le principe du maximum fort :** Soit $u$ une fonction D-harmonique dans un ouvert $\Omega$ connexe et $W$-invariant de $\mathbb{R}^d$. Si $u$ atteint son maximum en un point $x_0 \in \Omega$, alors $u$ est constante.
Ces résultats utilisent non seulement la propriété de la moyenne des fonctions D-harmoniques mais aussi l'inégalité géométrique remarquable suivante satisfaite par le noyau harmonique ([19] et chapitre 1) : si $\|x_1 - x_2\| \leq 2r$, alors

$$\forall\, y \in \mathbb{R}^d, \quad h_k(r, x_1, y) \leq h_k(4r, x_2, y).$$

De l'inégalité de Harnack, nous déduisons comme dans le cas classique (voir par exemple [5]) le **principe de Harnack** : la limite croissante d'une suite de fonctions D-harmoniques dans un ouvert connexe et $W$-invariant $\Omega \subset \mathbb{R}^d$ est ou bien une fonction D-harmonique et dans ce cas la convergence est uniforme sur tout compact de $\Omega$, ou bien identiquement $+\infty$ dans $\Omega$.

Le principe du maximum fort et l'inégalité de Harnack nous permettent, en adaptant l'idée donnée par Axler, Bourdon et Ramey ([5]), d'obtenir **le théorème de Bôcher** suivant :
Soit $B$ la boule unité ouverte de $\mathbb{R}^d$ et $\Omega = B \setminus \{0\}$. Si $d + 2\gamma > 2$ et si $u$ est une fonction D-harmonique strictement positive dans $\Omega$, alors il existe une fonction D-harmonique $v$ dans $B$ et une constante $a \geq 0$ telles que

$$\forall\, x \in \Omega, \quad u(x) = a\|x\|^{2-d-2\gamma} + v(x).$$

Autrement dit, la fonction $u$ ou bien se prolonge en une fonction D-harmonique sur la boule unité ouverte, ou bien a un pole au point 0.

## FONCTIONS D – SOUSHARMONIQUES

Tous les résultats ci-dessous se trouvent dans le chapitre 2 (voir aussi [20]).

Nous introduisons les fonctions sousharmoniques au sens de Dunkl via la moyenne volumique comme suit : Etant donné un ouvert $\Omega \subset \mathbb{R}^d$ $W$-invariant. Une fonction $u : \Omega \longrightarrow [-\infty, +\infty[$ est dite D-sousharmonique si

**1.** $u$ est semi-continue supérieurement (s.c.s.) sur $\Omega$,
**2.** $u \not\equiv -\infty$ dans toute composante connexe de $\Omega$,
**3.** $u$ satisfait la sous-propriété de la moyenne : pour toute boule $B(x,r) \subset \Omega$, nous avons

$$u(x) \leq M_B^r(u)(x) = \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} u(y) h_k(r,x,y) \omega_k(y) dy. \tag{0.38}$$

Une fonction $u$ est dite D-surharmonique si $-u$ est D-sousharmonique. Nous noterons $\mathcal{SH}_k(\Omega)$ le cône convexe des fonctions D-sousharmoniques sur $\Omega$.

Pour plus de détails sur la notion de semi-continuité, le lecteur pourra consulter [7]. Pour des références sur les fonctions sousharmoniques associées au laplacien usuel, nous pourrons consulter [4], [23], [25], [28], [31] et [37]. Le lecteur pourra également consulter [6] ainsi que ses références pour plus de détails sur l'aspect probabiliste de la théorie du potentiel euclidienne.

Notons que par la semi-continuité supérieure et la forme du support de $h_k(r, x, .)$, nous pouvons comprendre le terme à droite dans (0.38) comme l'intégrale d'une fonction mesurable négative sur la boule de Dunkl fermée $B^W(x, r)$ et donc cette intégrale est bien définie. En fait, le premier résultat que nous avons obtenu sur les fonctions D-sousharmoniques est le suivant

$$\mathcal{SH}_k(\Omega) \subset L^1_{loc}(\Omega, m_k).$$

Cette inclusion implique qu'une fonction D-sousharmonique ne prend la valeur $-\infty$ que sur un ensemble négligeable. En outre, une fonction $u \in \mathcal{SH}_k(\Omega)$ est complétement déterminée par sa restriction sur une partie de $\Omega$ dont le complémentaire est négligeable. C'est **le principe d'unicité** qui découle de

$$\forall\ x \in \Omega, \quad u(x) = \lim_{r \to 0} M_B^r(u)(x).$$

Un autre résultat, bien que sa démonstration découle directement du théorème de convergence monotone, important à souligner et que nous pouvons voir comme une extension du principe de Harnack aux fonctions D-sousharmoniques, est le suivant : Soit $(u_n) \subset \mathcal{SH}_k(\Omega)$ une suite décroissante. Alors, si la limite ponctuelle de $(u_n)$ n'est pas identiquement $-\infty$ dans toute composante connexe de $\Omega$, c'est une fonction D-sousharmonqiue dans $\Omega$.

Les fonctions D-sousharmoniques satisfont également **le principe du maximum fort** :
Soit $u \in \mathcal{SH}_k(\Omega)$. Si $\Omega$ est connexe et si $u$ atteint son maximum dans $\Omega$, alors $u$ est constante.

Nous savons que si $f$ est une fonction convexe et $A, B$ deux points du graphe de $f$, alors le segment $[A, B]$ est situé au dessus du graphe de $f$. Cette propriété géométrique se généralise aux fonctions D-sousharmoniques de la manière suivante :
Soit $u \in \mathcal{SH}_k(\Omega)$, $G$ un ouvert $W$-invariant et borné tel que $\overline{G}$ est contenu dans $\Omega$. Si $h$ est une fonction continue sur $\overline{G}$, D-harmonique sur $G$ et majore $u$ sur $\partial G$, alors elle majore $u$ sur $G$.

Dans le cas classique, le noyau de Poisson pour une boule quelconque joue un rôle fondamental dans la théorie du potentiel euclidienne et en particulier dans l'étude des fonctions sousharmoniques. D'ailleurs, c'est grâce à ce noyau, que nous pouvons construire la solution du problème de Dirichlet pour une boule ou pour un ouvert à bord régulier par la méthode de Perron-Wiener-Brelot (voir les références mentionnées ci-dessus).

Dans la théorie de Dunkl, le noyau de Poisson généralisé n'est connu jusqu'à présent que pour la boule centrée à l'origine ([15], [32]).
Bien que nous ayons introduit les boules de Dunkl qui vont être (à notre avis) les remplaçants convenables des boules euclidiennes, la construction de l'analogue du noyau de Poisson pour ces boules demeure aussi un problème ouvert.

Malgré la méconnaissance de cet outil dans notre cas, nous avons tout de même réussi à développer la notion de D-sousharmonicité, dans le sens que nous avons généralisé plusieurs résultats importants connus dans le cas du laplacien classique.

Dans un premier résultat, nous montrons que nous pouvons caractériser la sousharmonicité au sens de Dunkl de plusieurs manières. Plus précisément, si $u$ est une fonction s.c.s. sur $\Omega$ et non identiquement $-\infty$ dans toute composante connexe de $\Omega$, alors nous avons l'équivalence entre les assertions suivantes :

**1.** $u \in \mathcal{SH}_k(\Omega)$,

**2.** pour tout $x \in \Omega$, la fonction $r \mapsto M_B^r(u)(x)$ est croissante sur $]0, \varrho_x[$ [1] et $\lim\limits_{r\to 0} M_B^r(u)(x) = u(x)$,

**3.** pour tout $x \in \Omega$, la fonction $r \longmapsto M_S^r(u)(x)$ est croissante sur $]0, \varrho_x[$ et $\lim\limits_{r\to 0} M_S^r(u)(x) = u(x)$,

**4.** $u \in L^1_{loc}(\Omega, m_k)$, $\lim\limits_{r\to 0} M_B^r(u)(x) = u(x)$ pour tout $x \in \Omega$ et $M_B^r(u)(x) \leq M_S^r(u)(x)$, dès que $B(x,r) \subset \Omega$.

Pour obtenir ce résultat, nous avons procédé comme suit : Dans un premier temps, en se basant sur les formules (0.30), (0.31) et (0.32), nous avons obtenu l'équivalence sous la condition que $u$ soit de classe $C^\infty$ dans $\Omega$.
Ensuite, nous avons utilisé essentiellement le résultat d'approximation suivant (voir [20] et chapitre 2) :
Soit $u \in \mathcal{SH}_k(\Omega)$ et $\Omega_r$ l'ouvert défini par (0.25). Alors, il existe une suite $(u_n)$ telle que

**i)** pour tout $n$ assez grand, $u_n \in \mathcal{SH}_k(\Omega_{\frac{1}{n}}) \cap \mathcal{C}^\infty(\Omega_{\frac{1}{n}})$,

**ii)** pour tout $N$ assez grand, la suite $(u_n)_{n\geq N}$ est décroissante et converge simplement vers $u$ dans $\Omega_{\frac{2}{N}}$,

**ii)** pour toute boule $B(x,r) \subset \Omega$, $M_B^r(u_n)(x) \longrightarrow M_B^r(u)(x)$ et $M_S^r(u_n)(x) \longrightarrow M_S^r(u)(x)$ quand $n \longrightarrow +\infty$.

La fonction $u_n$ (toujours pour $n$ assez grand) est donnée par

$$\forall\ x \in \Omega_{\frac{1}{n}}, \quad u_n(x) := u *_k \varphi_n(x) := \int_{\mathbb{R}^d} u(y)\tau_{-x}\varphi_n(y)\omega_k(y)dy, \tag{0.39}$$

1. nous rappelons que $\varrho_x$ est la distance de $x$ à la frontière de $\Omega$ (voir (0.26)).

avec

$$\varphi(x) = a\exp(-\frac{1}{1-\|x\|^2})\mathbf{1}_{B(0,1)}(x), \qquad \varphi_n(x) = n^{d+2\gamma}\varphi(nx)$$

et $a$ une constante choisie de sorte que $\varphi\omega_k$ soit une densité de probabilité.

L'idée de choisir $u_n$ comme dans (0.39) vient du cas classique en remplaçant la convolution usuelle par celle de Dunkl. Mais, nous sommes dans un cas beaucoup moins facile vu la difficulté de la manipulation de la translation de Dunkl. Puisque $\varphi_n$ appartient à $\mathcal{D}(\mathbb{R}^d)$ radiale et positive, les propriétés de la suite $(u_n)$ proviennent des résultats signalés précédemment sur le produit de convolution de Dunkl.

Etant donnée une fonction D-sousharmonique $u$ sur $\Omega$, nous avons prouvé que $\Delta_k(u\omega_k) \geq 0$ dans $\mathcal{D}'(\Omega)$ dans le sens que

$$\int_\Omega u(x)\Delta_k\phi(x)\omega_k(x)dx \geq 0, \quad \forall\ \phi \in \mathcal{D}(\Omega), \quad \phi \geq 0. \tag{0.40}$$

Alors, en utilisant un résultat classique qui nous dit que les distributions positives sont des mesures de Radon positives ([27] et [47]), nous déduisons que $\Delta_k(u\omega_k)$ est une mesure de Radon positive sur $\Omega$. Nous l'appelons la $\Delta_k$**-mesure de Riesz** associée à la fonction D-sousharmonique $u$.

Donnons quelques exemples de $\Delta_k$-mesures de Riesz associées à des fonctions D-sousharmoniques :

**a)** Quand $u \in \mathcal{SH}_k(\Omega)\cap\mathcal{C}^2(\Omega)$, alors sa $\Delta_k$-mesure de Riesz est donnée par $\Delta_k u(x)\omega_k(x)dx$.

**b)** Supposons que $d+2\gamma > 2$ et considérons la fonction

$$S(x) = -\frac{\|x\|^{2-d-2\gamma}}{d_k(d+2\gamma-2)} \tag{0.41}$$

Alors, $S$ est une fonction D-sousharmonique sur $\mathbb{R}^d$ et sa mesure de Riesz est $\delta_0$ (la mesure de Dirac en 0) i.e. $S$ satisfait **l'équation de Dunkl-Poisson** suivante :

$$\Delta_k(S\omega_k) = \delta_0 \quad \text{dans} \quad \mathcal{D}'(\mathbb{R}^d). \tag{0.42}$$

Dans ce cas, nous dirons que $-S$ est **la solution fondamentale du laplacien de Dunkl**. En regardant $S\omega_k$ comme une distribution tempérée, une démonstration de la relation (0.42) au moyen de la transformation de Dunkl est donnée dans ([35]). Dans le chapitre 2 (voir aussi [20]), nous proposons une autre démonstration qui repose sur la formule de Green associée au laplacien de Dunkl.

**c)** Soit $u \in \mathcal{SH}_k(\Omega)$ et $\mu$ sa mesure de Riesz. Pour $r$ assez petit fixé, la fonction $M_B^r(u)$ définit une fonction D-sousharmonique et continue sur $\Omega_r$. De plus, sa mesure de Riesz est absolument continue par rapport à la mesure de Lebesgue et elle est donnée par

$$\Delta_k\left(M_B^r(u)\omega_k\right) = M_B^r(\mu)(x)\omega_k(x)dx, \quad \text{avec} \quad M_B^r(\mu)(x) = \frac{\|h_k(r,x,.)\|_{L^1(\Omega,\mu)}}{m_k[B(0,r)]}. \tag{0.43}$$

Quand $k = 0$, $m_0$ est la mesure de Lebesgue sur $\mathbb{R}^d$ et la relation précédente peut s'écrire sous la forme

$$M_B^r(\mu)(x) = \frac{\mu[B(x,r)]}{m_0[B(x,r)]}.$$

La D-sousharmonicité d'une fonction peut être aussi caractérisée en termes de la positivité de son laplacien faible au sens de la relation (0.40) comme suit :
Soit $u$ une fonction s.c.s. sur $\Omega$. Alors, $u \in \mathcal{SH}_k(\Omega)$ si et seulement si elle satisfait les trois propriétés suivantes :
**i)** $u \in L^1_{loc}(\Omega, m_k)$,
**ii)** $\forall\ x \in \Omega$, $u(x) = \lim_{r\to 0} M_B^r(u)(x)$,
**iii)** $\Delta_k(u\omega_k) \geq 0$ dans $\mathcal{D}'(\Omega)$.

Dans la théorie des fonctions sousharmoniques classiques, le noyau de Newton $-\|x - y\|^{2-d}$ et le potentiel Newtonien $-\int \|x-y\|^{2-d} d\mu(y)$ d'une mesure de Radon positive dans le cas $d \geq 3$ (resp. le noyau logarithmique $Log\|x-y\|$ et le potentiel logarithmique $\int Log\|x-y\| d\mu(y)$ dans le cas du plan) sont les exemples fondamentaux de fonctions sousharmoniques. Leur importance se manifeste dans l'un des résultats puissants de cette théorie qui est dû à F. Riesz ([29]), connu dans la littérature comme le théorème de décomposition de Riesz et qui nous dit que toute fonction sousharmonique peut s'écrire localement comme la somme d'un potentiel newtonien si $d \geq 3$ (resp. d'un potentiel logarithmique si $d = 2$) et d'une fonction harmonique.

Dans la suite, afin de généraliser le théorème de décomposition de Riesz aux fonctions sousharmoniques au sens de Dunkl, nous nous proposons de présenter les propriétés fondamentales du noyau et du potential newtonien dans le cas Dunkl ([20] et chapitre 2). Nous supposons alors $d + 2\gamma > 2$.

Nous avons introduit le noyau de Newton généralisé associé au quadruplet $(\mathbb{R}^d, \mathcal{R}, W, k)$ par

$$\forall\ x, y \in \mathbb{R}^d, \quad N_k(x,y) := \int_0^{+\infty} p_t(x,y)dt, \tag{0.44}$$

avec $p_t(x,y)$ le **noyau de la chaleur**, introduit par M. Rösler ([40] et [8]) et donné par

$$p_t(x,y) = \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} e^{-\frac{1}{4t}(\|x\|^2+\|y\|^2)} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right). \tag{0.45}$$

Nous l'appelons **noyau de Dunkl-Newton**. Nous pouvons réécrire le noyau $N_k$ au moyen de la mesure de Rösler sous la forme

$$\forall\ x, y \in \mathbb{R}^d, \quad N_k(x,y) = \frac{1}{d_k(d+2\gamma-2)} \int_{\mathbb{R}^d} \left(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle\right)^{\frac{2-d-2\gamma}{2}} d\mu_y(z). \tag{0.46}$$

Il s'exprime également à l'aide du noyau harmonique de la façon suivante

$$\forall\ x, y \in \mathbb{R}^d, \quad N_k(x, y) = \frac{1}{d_k} \int_0^{+\infty} t^{2-d-2\gamma} h_k(t, x, y) \frac{dt}{t}. \tag{0.47}$$

De cette relation et des propriétés du noyau harmonique, nous voyons que $N_k$ est un noyau positif, symétrique.
En outre, pour tout $x \in \mathbb{R}^d$, la fonction $N_k(x, .)$
-est D-surharmonique sur $\mathbb{R}^d$,
-est D-harmonique sur $\mathbb{R}^d \setminus W.x$,
-satisfait l'équation de Dunkl-Poisson

$$-\Delta_k(N_k(x, .)\omega_k) = \delta_x \quad \text{dans} \quad \mathcal{D}'(\mathbb{R}^d).$$

Notons que pour $x \in \mathbb{R}^d$ fixé, $N_k(x, y) < +\infty$ dès que $y \notin W.x$. Cependant, a priori, il est difficile de voir lequel des points de l'orbite de $x$ est une singularité de la fonction $N_k(x, .)$. Dans l'exemple suivant où la mesure de Rösler est connue, nous donnons une caractérisation surprenante des points pour lesquels le noyau de Dunkl-Newton prend la valeur $+\infty$ (voir [20] et chapitre 2).

Nous prenons le cas $(\mathbb{R}^d, \mathcal{R}, W, k) = (\mathbb{R}^d, A_1 \times \cdots \times A_1, \mathbf{Z}_2^m, (k_1, \ldots, k_m))$ avec $1 \leq m \leq d$. Fixons un point $x \in \mathbb{R}^d$, $x \neq 0$ et écrivons le $x = (x^{(m)}, x') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$. L'orbite de $x$ sous l'action de $\mathbf{Z}_2^m$ est donnée par

$$\mathbf{Z}_2^m.x := \left\{\varepsilon.x := (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x'), \quad \varepsilon = (\varepsilon_i)_{1 \leq i \leq m} \in \{\pm 1\}^m\right\}.$$

Par les relations (0.46) et (0.7), le noyau de Dunkl-Newton est de la forme

$$N_k(x, y) = C \int_{[-1,1]^m} \Big(\|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2\sum_{j=1}^m t_j x_j y_j + \|x' - y'\|^2\Big)^{1-\frac{d}{2}-\gamma} \times \prod_{i=1}^m \phi_{k_i}(t_i) dt_1 \ldots dt_m, \tag{0.48}$$

avec $C = [d_k(d + 2\gamma - 2)]^{-1}$. Alors, dans ce cas, les singularités de la fonction $N_k(x, .)$ sont caractérisées comme suit :

**1.** Si $x \in \cap_{i=1}^m H_{e_i}$, alors $x = \varepsilon.x$ et $N_k(x, x) = +\infty$.

**2.** Soit $x \notin \cap_{i=1}^m H_{e_i}$. Posons $A := \{i \in \{1, \ldots, m\}, \quad x_i \neq 0\}$ et $\varepsilon^{(n)}.x = (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x')$ un point de l'orbite de $x$ tel que $\big|\{j \in A, \quad \varepsilon_j = 1\}\big| = n$ i.e. le vecteur $\varepsilon^{(n)}.x$ a exactement $n$ composantes inchangées par l'action du groupe $\mathbf{Z}_2^m$ parmi les composantes non nulles du vecteur $(x_j)_{j \in A}$. Alors,

$$N_k(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2\Big(|A| - n + \sum_{j \in A} k_j - \gamma\Big) + 2. \tag{0.49}$$

En particulier, si $x \notin \cup_{i=1}^m H_i$, alors

$$N_k(x, \varepsilon^{(n)}.x) = +\infty \iff d \geq 2(m-n)+2. \tag{0.50}$$

De cette relation, nous pouvons dénombrer les singularités de $N_k(x,.)$ suivant la parité de la dimension de l'espace $\mathbb{R}^d$. Plus précisément, si $d = 2N$ nous avons $\displaystyle\sum_{n=\max(0,m-N+1)}^{m} \binom{m}{n}$ singularités dans $\mathbb{R}^{2N}$ (elles sont en fait dans $\mathbb{R}^{2N} \setminus \cup_{i=1}^m H_i$) et si $d = 2N+1$, nous avons $\displaystyle\sum_{n=\max(0,m-N)}^{m} \binom{m}{n}$ singularités dans $\mathbb{R}^{2N+1}$.

Pour un quadruplet $(\mathbb{R}^d, \mathcal{R}, W, k)$ arbitraire, tout ce que nous pouvons dire pour le moment est que $N_k(x,x) = +\infty$ dès que $d \geq 2$. Un tel résultat est basé sur le fait que si $x$ et $y$ sont dans la même chambre de Weyl, le noyau de la chaleur dans le cas Dunkl $p_t(x,y)$ se comporte, quand $t \to 0$, comme le noyau de la chaleur classique ([44]).

Etant donnée une mesure de Radon positive $\mu$ sur $\mathbb{R}^d$. Le **potentiel de Dunkl-Newton** de $\mu$ est défini par

$$\forall\ x \in \mathbb{R}^d, \quad N_k[\mu](x) := \int_{\mathbb{R}^d} N_k(x,y)d\mu(y).$$

Une condition nécessaire et suffisante pour que $N_k[\mu](x)$ soit fini pour presque tout $x \in \mathbb{R}^d$ est la suivante :

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{2-d-2\gamma} d\mu(y) < +\infty. \tag{0.51}$$

Sous une telle condition, la fonction $-N_k[\mu]$ est en fait D-sousharmonique sur $\mathbb{R}^d$ et $\mu$ est sa $\Delta_k$-mesure de Riesz. En d'autre termes, elle satisfait l'équation de Dunkl-Poisson suivante

$$-\Delta_k\left(N_k[\mu]\omega_k\right) = \mu \quad \text{dans} \quad \mathcal{D}'(\mathbb{R}^d). \tag{0.52}$$

De (0.52), nous déduisons **le principe d'unicité des masses** :
Si $\mu$ and $\nu$ deux mesures de Radon positives sur $\mathbb{R}^d$ satisfaisant (0.51) et si $N_k[\mu] = N_k[\nu]$ p.p, alors $\mu = \nu$.

Nous avons alors l'extension suivante du **théorème de décomposition de Riesz aux fonctions D-sousharmoniques** : Soit $u \in \mathcal{SH}_k(\Omega)$ et $\mu$ sa $\Delta_k$-mesure de Riesz. Alors, pour tout ouvert $W$-invariant et borné $G$ tel que $\overline{G} \subset \Omega$, il existe une unique fonction D-harmonique $h_G$ dans $G$ telle que

$$\forall\ x \in G, \quad u(x) = N_k[\mu_G](x) + h_G(x), \quad \text{avec} \quad \mu_G = \mu_{|G}.$$

La preuve de ce théorème repose non seulement sur les propriétés du potentiel de Dunkl-Newton d'une mesure de Radon, mais aussi sur la généralisation suivante du **lemme de Weyl** :
Toute fonction $u \in L^1_{loc}(\Omega, m_k)$ D-harmonique dans $\Omega$ au sens des distributions (i.e.

$\Delta_k(u\omega_k) = 0$ dans $\mathcal{D}'(\Omega)$) coïncide presque partout avec une fonction D-harmonique (au sens fort) dans $\Omega$.

Dans le cas particulier où $\Omega = \mathbb{R}^d$ et avec l'hypothèse supplémentaire que $u$ soit une fonction localement bornée sur $\mathbb{R}^d$, ce résultat avait été démontré dans [34].

En fait, nous avons étendu ce lemme pour les fonctions D-sousharmoniques de la façon suivante : si $u \in L^1_{loc}(\Omega, m_k)$ telle que $\Delta_k(u\omega_k)$ est une distribution positive, alors $u$ coïncide presque partout avec une fonction D-sousharmonique sur $\Omega$.

Dans le chapitre 2 (ou dans [20]), nous avons également réussi à étendre des résultats spéciaux pour les fonctions sousharmoniques majorées sur $\mathbb{R}^d$ (voir par exemple [23] pour le cas classique) au cas Dunkl. Dans un premier temps, nous avons établi que si $u$ est une fonction D-sousharmonique majorée sur tout l'espace $\mathbb{R}^d$, alors sa décomposition de Riesz est donnée par

$$u = \sup_{\mathbb{R}^d} u - N_k[\mu],$$

où $\mu$ est la $\Delta_k$-mesure de Riesz de $u$. Nous signalons que, dans le cas classique, cette décomposition utilise un théorème de Nevanlinna. Nous proposons une démonstration différente qui repose sur la relation (0.30) et sur le résultat d'approximation des fonctions D-sousharmoniques par des fonctions D-sousharmoniques régulières.
Si $u$ est D-sousharmonique majorée sur $\mathbb{R}^d$ et $\mu$ sa $\Delta_k$-mesure de Riesz, alors nous avons prouvé que $\mu$ satisfait la condition

$$\int_1^{\infty} t^{1-d-2\gamma} n_k(t, x_0) dt < +\infty \quad \text{avec} \quad n_k(t, x_0) = \int_{\mathbb{R}^d} h_k(t, x_0, y) d\mu(y) \tag{0.53}$$

pour un certain $x_0 \in \mathbb{R}^d$. Notons que dans le cas $k \equiv 0$ (c'est le cas du laplacien classique), $n_0(t, x_0) = \mu[B(x_0, t)]$. Réciproquement, en adaptant partiellement la preuve du cas classique, nous avons montré que si $\mu$ est une mesure de Radon positive sur $\mathbb{R}^d$ et qui satisfait (0.53), alors $\mu$ est la $\Delta_k$-mesure de Riesz d'une fonction D-souharmonique majorée sur $\mathbb{R}^d$.
La seule différence avec le cas usuel est le suivant : dans le cas classique, nous pouvons toujours supposer que $x_0 = 0$ et ceci vient, à notre avis, du fait que $\mathcal{SH}_0(\mathbb{R}^d)$, le cône des fonctions sousharmoniques au sens de $\Delta = \Delta_0$, est stable par translation dans le sens que si $u \in \mathcal{SH}_0(\mathbb{R}^d)$, alors la fonction $u(x_0 + .)$ l'est aussi. Dans le cas Dunkl, un tel résultat n'est pas garanti puisque la translation de Dunkl n'est pas encore définie sur l'espace $\mathcal{SH}_k(\mathbb{R}^d)$. D'ailleurs, même si nous voulions nous restreindre à l'ensemble $\mathcal{SH}_k(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ où la notion de translation de Dunkl existe, sa nonpositivité ne nous permet pas d'affirmer qu'elle le laisse invariant.
Comme remarque finale, nous montrons que si $\mu$ est la $\Delta$-mesure de Riesz d'une fonction sousharmonique (classique) et majorée sur $\mathbb{R}^d$, alors pour tout choix du quadruplet $(\mathbb{R}^d, \mathcal{R}, W, k)$, c'est aussi la $\Delta_k$-mesure de Riesz d'une fonction D-sousharmonique majorée sur $\mathbb{R}^d$.

## POTENTIELS DE DUNKL – RIESZ DES MESURES DE RADON

Dans [21] (voir chapitre 3), nous nous intéressons à la théorie du potentiel (harmonicité, sous et/ou surharmonicité, principe d'unicité des masses,...etc) associée au **noyau de Dunkl-Riesz** et au **potentiel de Dunkl-Riesz** d'une mesure de Radon sur $\mathbb{R}^d$. Ce travail sera une généralisation du cas classique ([31]), d'une part, et d'autre part du cas de Dunkl-Newton ([20] ou chapitre 2).

Le noyau de Dunkl-Riesz d'indice $\beta \in ]0, d+2\gamma[$, associé à un quadruplet $(\mathbb{R}^d, \mathcal{R}, W, k)$ fixé, est défini par :

$$\forall\ x, y \in \mathbb{R}^d, \quad R_{k,\beta}(x,y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_t(x,y)dt, \tag{0.54}$$

où $p_t(x,y)$ est le noyau de la chaleur généralisé donné par (0.45). En prenant $\beta = 2$, nous retrouvons le noyau de Dunkl-Newton (voir (0.44)).

Quand $k = 0$, le noyau $R_{0,\beta}$ est le noyau de Riesz classique i.e. $R_{0,\beta}(x,y) = C\|x-y\|^{\beta-d}$. Pour plus de détails sur les propriétés de ce noyau ainsi que les potentiels associés, le lecteur pourra consulter [31] et [6].

Dans la suite, nous supposerons alors que la fonction de multiplicité $k$ n'est pas identiquement nulle i.e. $\gamma > 0$.

Nous pouvons réécrire la formule (0.54) comme suit

$$R_{k,\beta}(x,y) = \kappa \int_{\mathbb{R}^d} \left(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle\right)^{\frac{\beta-d-2\gamma}{2}} d\mu_y(z) \tag{0.55}$$

$$= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t,x,y) \frac{dt}{t}, \tag{0.56}$$

avec $\kappa = \kappa(d,\gamma,\beta)$ une constante positive (sa valeur exacte est donnée dans [21] et chapitre 3).
Le noyau de Dunkl-Riesz d'indice $\beta$ est strictement positif, symétrique et $W$-équivariant dans le sens que

$$\forall\ x, y \in \mathbb{R}^d, \quad \forall\ g \in W, \quad R_{k,\beta}(gx, y) = R_{k,\beta}(x, g^{-1}y).$$

Par les mêmes arguments que dans le cas du noyau de Dunkl-Newton, nous avons montré, pour $x \in \mathbb{R}^d \backslash \{0\}$, que
- pour tout $y \in \mathbb{R}^d \setminus W.x$, $R_{k,\beta}(x,y) < +\infty$,
- si $x \notin \cup_{\alpha\in\mathcal{R}} H_\alpha$, $R_{k,\beta}(x,x) = +\infty$ si et seulement si $\beta \leq d$,
- si $x \in \cup_{\alpha\in\mathcal{R}} H_\alpha$ et $\beta \leq d$, alors $R_{k,\beta}(x,x) = +\infty$.

En général, la question des singularités de la fonction $R_{k,\beta}(x,.)$ est la même que dans le cas Dunkl-Newton i.e. pour un quadruplet $(\mathbb{R}^d, \mathcal{R}, W, k)$ arbitraire, nous ne disposons pas jusqu'à présent d'une condition nécessaire et suffisante pour dire si un point de l'orbite $W.x$ est une singularité ou pas de cette fonction. Néanmoins, nous avons une réponse complète à cette question dans le cas où $(\mathbb{R}^d, \mathcal{R}, W, k) = (\mathbb{R}^d, \mathbf{A}_1 \times \cdots \times \mathbf{A}_1 (m fois), \mathbf{Z}_2^m, k = (k_1, \ldots, k_m)$ (voir [21] ou chapitre 3).

Le noyau de Dunkl-Riesz possède les propriétés suivantes : Pour $x \in \mathbb{R}^d$ fixé, la fonction $R_{k,\beta}(x,.)\omega_k$

**1.** appartient à $L^p_{loc}(\mathbb{R}^d, m_k)$ dès que $1 \leq p < \frac{d+2\gamma}{d+2\gamma-\beta}$. En fait, nous avons établi que pour tout $R > 0$, il existe une constante $C > 0$ indépendante de $x$ telle que

$$\|R_k(x,.)\|_{L^p(B(0,R)),m_k)} \leq C.$$

**2.** définit une distribution tempérée dont la transformée de Dunkl est donnée par

$$\mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right) = E_k(-ix_0,.)\|.\|^{-\beta}\omega_k \quad \text{dans} \quad \mathcal{S}'(\mathbb{R}^d), \tag{0.57}$$

où $E_k(-ix_0,.)$ est le noyau de Dunkl. Ici, nous rappelons que la transformée de Dunkl d'une distribution tempérée $U$ est définie via la relation de dualité suivante :

$$\langle \mathcal{F}_k(U), \phi\rangle = \langle U, \mathcal{F}_k(\phi)\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Comme dans le cas classique, la question de la sous ou surharmonicité de la fonction $R_{k,\beta}(x,.)$, $x \in \mathbb{R}^d$, dépend de l'indice $\beta$. D'une manière précise, nous avons le résultat suivant : Elle est

**i)** D-surharmonique sur $\mathbb{R}^d$ si $\beta \geq 2$,

**ii)** D-harmonique sur $\mathbb{R}^d \backslash W.x$ si $\beta = 2$,

**iii)** D-sousharmonique sur $\mathbb{R}^d \backslash W.x$ si $\beta \leq 2$. Notons que dans ce cas, la fonction $R_{k,\beta}(x,.)$ est de classe $C^2$ sur $\mathbb{R}^d \backslash W.x$ et par le théorème de différentiation sous le signe intégral, nous obtenons

$$\forall\, y \in \mathbb{R}^d \backslash W.x, \quad \Delta_k(R_{k,\beta}(x,.))(y) := -\frac{\beta-2}{2\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta-2}{2}-1} p_t(x,y)dt,$$

Dans le cas où $\beta \in [2, d+2\gamma[$ et $m \in [1, \beta/2]$ est un entier, nous avons établi les relations suivantes

$$(-\Delta_k)^m\left(R_{k,\beta}(x,.)\omega_k\right) = \begin{cases} R_{k,\beta-2m}(x,.)\omega_k & \text{dans} \quad \mathcal{S}'(\mathbb{R}^d), \quad \text{si} \quad \beta > 2m, \\ \\ \delta_x & \text{dans} \quad \mathcal{S}'(\mathbb{R}^d), \quad \text{si} \quad \beta = 2m, \end{cases} \tag{0.58}$$

En prenant $\beta = 2m$ $(m \in \mathbb{N}^*)$ et $x = 0$ dans cette relation, nous déduisons que

$$R_{k,2m}(0,y)\omega_k(y) = \kappa(d,\gamma,2m)\|y\|^{2m-d-2\gamma}\omega_k(y)$$

est la solution fondamentale de l'opérateur polylaplacien $(-\Delta_k)^m$.

Nous avons introduit le potentiel de Dunkl-Riesz d'une mesure de Radon positive $\mu$ sur $\mathbb{R}^d$ par

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x,y)d\mu(y).$$

Nous pouvons définir également le potentiel de Dunkl-Riesz d'une mesure de Radon signée $\mu$ sur $\mathbb{R}^d$, en posant pour tout $x \in \mathbb{R}^d$, $I_{k,\beta}[\mu](x) := I_{k,\beta}[\mu^+](x) - I_{k,\beta}[\mu^-](x)$ dès que $I_{k,\beta}[\mu^+](x)$ et $I_{k,\beta}[\mu^-](x)$ ne prennent pas simultanément la valeur $+\infty$. Ici, $\mu = \mu^+ - \mu^-$ est la décomposition de Hahn-Jordan de $\mu$.

La classe des mesures de Radon positives $\mu$ sur $\mathbb{R}^d$ pour lesquelles le potentiel de Dunkl-Riesz est fini presque partout est formée par les mesures qui satisfont la condition suivante :

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty. \tag{0.59}$$

D'ailleurs, le potentiel de Dunkl-Riesz d'indices $\beta \in ]2, d + 2\gamma[$ (nous rappelons que le potentiel de Dunkl-Riesz d'indice 2 coïncide avec le potentiel Dunkl-Newton) d'une mesure $\mu$ dans une telle classe définit une fonction D-surharmonique sur $\mathbb{R}^d$ et sa $\Delta_k$-mesure de Riesz est donnée par

$$-\Delta_k \left(I_{k,\beta}[\mu]\omega_k\right) = I_{k,\beta-2}[\mu] \quad \text{dans} \quad \mathcal{D}'(\mathbb{R}^d).$$

Plus généralement, si $\beta \in [2, d + 2\gamma[$ et si $m \in [1, \beta/2]$ est un entier, par la relation (0.58) nous avons

$$(-\Delta_k)^m \left(I_{k,\beta}[\mu]\omega_k\right) = \begin{cases} I_{k,\beta-2m}[\mu]\omega_k & \text{dans} \quad \mathcal{D}'(\mathbb{R}^d) \quad \text{si} \quad \beta > 2m, \\ \\ \mu & \text{dans} \quad \mathcal{D}'(\mathbb{R}^d) \quad \text{si} \quad \beta = 2m, \end{cases} \tag{0.60}$$

Notons que si $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ (l'espace des mesures de Radon finies sur $\mathbb{R}^d$), alors la fonction $I_{k,\beta}[\mu]\omega_k$ définit une distribution tempérée dont la transformée de Dunkl est donnée par

$$\mathcal{F}_k \left(I_{k,\beta}[\mu]\omega_k\right) = \|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k \quad \text{dans} \quad \mathcal{S}'(\mathbb{R}^d), \tag{0.61}$$

avec $\mathcal{F}_k(\mu)$ la transformée de Dunkl de la mesure $\mu$ définie par

$$\mathcal{F}_k(\mu)(\xi) := \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x).$$

Les propriétés de la transformation de Dunkl sur l'espace $\mathcal{M}_b(\mathbb{R}^d)$ se trouvent dans [41].

De la relation (0.61) et de l'injectivité de la transformation de Dunkl sur l'espace $\mathcal{M}_b(\mathbb{R}^d)$, nous obtenons le principe d'unicité des masses qui constitue l'un des résultats fondamentaux de cette partie :
Soient $\mu, \nu \in \mathcal{M}_b(\mathbb{R}^d)$ deux mesures positives. Si $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ p.p, alors $\mu = \nu$.

D'autre part, nous signalons que le potentiel de Dunkl-Riesz d'une mesure de Radon positive $\mu$ s'écrit aussi sous la forme

$$I_{k,\beta}[\mu](x) = \frac{d_k \kappa}{(d + 2\gamma)(d + 2\gamma - 2)} \int_0^{+\infty} t^{\beta-1} M_B^t(\mu)(x) dt,$$

où $M_B^t(\mu)$ est la moyenne volumique généralisée de $\mu$ définie par (0.43).

Dans le cas particulier où $d\mu(x) = |f(x)|\omega_k(x)dx$, $f \in L^1_{loc}(\mathbb{R}^d, m_k)$, la relation précédente nous permet d'obtenir l'analogue de l'**inégalité ponctuelle de Hedberg** ([1] ou [24]) dans le cas Dunkl. Plus précisement, en écrivant $I_{k,\beta}[|f|]$ au lieu de $I_{k,\beta}[|f|\omega_k]$, nous avons : Pour tout $1 \leq p < \frac{d+2\gamma}{\beta}$, il existe une constante $C = C(d, \gamma, \beta, p) > 0$ telle que

$$I_{k,\beta}[|f|](x) \leq C\|f\|_{L^p(\mathbb{R}^d,m_k)}^{\frac{\beta p}{d+2\gamma}} (M_k(f)(x))^{1-\frac{\beta p}{d+2\gamma}}, \tag{0.62}$$

avec $M_k$ la **fonction maximale de Dunkl-Hardy-Littlewood** définie par

$$M_k(f) = \sup_{r>0} \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} |f(y)|h_k(r, x, y)\omega_k(y)dy.$$

En utilisant l'inégalité de Hedberg, les propriétés de $L^p(\mathbb{R}^d, m_k)$-bornitude de l'opérateur $M_k$ (voir [9] ou [48]) et en suivant la même démarche que dans le cas classique ([1]), nous obtenons immédiatement **l'inégalité de Sobolev**

**1)** Si $p = 1$, alors $I_{k,\beta}$ est de type faible $(1, \frac{d+2\gamma}{d+2\gamma-\beta})$ i.e. il existe une constante $C = C(\beta, d, \gamma)$ telle que

$$\forall\ \lambda > 0, \quad \forall\ f \in L^1(\mathbb{R}^d, m_k), \quad \int_{\{x:\ I_{k,\beta}[|f|]>\lambda\}} \omega_k(x)dx \leq C \left(\frac{\|f\|_{k,1}}{\lambda}\right)^{\frac{d+2\gamma}{d+2\gamma-\beta}}.$$

**2)** Si $1 < p < \frac{d+2\gamma}{\beta}$, alors $I_{k,\beta}$ est de type fort $(p, \frac{p(d+2\gamma)}{d+2\gamma-\beta p})$ i.e.

$$I_{k,\beta} : L^p(\mathbb{R}^d, m_k) \longrightarrow L^{\frac{p(d+2\gamma)}{d+2\gamma-\beta p}}(\mathbb{R}^d, m_k)$$

est un opérateur borné.

Nous signalons que ce résultat a d'abord été démontré par Thangavelu et Xu ([49]) dans le cas particulier où $W = \mathbf{Z}_2^d$. Ensuite, par un argument d'interpolation, il a été prouvé indépendament du choix du groupe de Coxeter-Weyl par Hassani, Mustapha et Sifi ([22]).

# Bibliographie


[1] D. Adams and L. Hedberg. *Functions Spaces and Potential Theory.* Springer, (1999).

[2] B. Amri, J. P. Anker and M. Sifi. *Three results in Dunkl analysis.* Colloq. Math., 118, (2010), 299-312.

[3] B. Amri. *On the integral representations for Dunkl kernels of type* $A_2$. arXiv :1502.03309v3.

[4] D. H. Armitage and S. J. Gardiner. *Classical Potential Theory.* Springer-Verlag, London (2001).

[5] S. Axler, P. Bourdon and W. Ramey. *Harmonic Function Theory.* Springer Verlag , Second edition (2001).

[6] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, Z. Vondracek. *Potential Analysis of Stable Processes and its Extensions.* Springer-Verlag, Berlin Heidelberg (2009).

[7] G. Choquet. *Topology.* Volume XIX, Academic Press (1966).

[8] O. Chybiryakov, N. Demni, L. Gallardo, M. Rösler, M. Voit and M. Yor. *Harmonic and stochastic analysis of Dunkl processes.* TVC 71, Hermann (2008).

[9] L. Deleaval. *Two results on the Dunkl maximal function.* Studia Mathematica, 203, (2011), 47-68.

[10] C. F. Dunkl. *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc., 311, (1989), 167-183.

[11] C. F. Dunkl. *Integral kernels with reflection group invariance.* Canad. J. Math., 43, (1991), 123-183.

[12] C. F. Dunkl. *Hankel transforms associated to finite reflection groups.* Contemp. Math., 138, (1992), 123-138.

[13] C. F. Dunkl. *Intertwining operators associated to the group S3.* Trans. Amer. Math. Soc., 347(9), (1995), 3347-3374.

[14] C. F. Dunkl. *Orthogonal polynomials of type A and B and related Calogero models.* Commun. Math. Phys. 197, (1998), 451-487.

[15] C. F. Dunkl and Y. Xu. *Orthogonal Polynomials of Several variables.* Cambridge Univ. Press 2001.

[16] C. F. Dunkl. *An intertwining operator for the group B2.* Glasg. Math. J., 49(2), (2007), 291-319.

[17] P. Etingof. *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups.* Math. Res. Lett. 17 (2010), no. 2, 275-282.

[18] L. Gallardo and L. Godefroy. *Propriété de Liouville et équation de Poisson pour le laplacien généralisé de Dunkl.* C. R. Acad. Sci. Paris, Ser.I 337 (2003), 639-644.

[19] L. Gallardo and C. Rejeb. *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications.* Trans. Amer. Math. Soc., electronically published on May 22, 2015, DOI :http ://dx.doi.org/10.1090/tran/6671 (to appear in print).

[20] L. Gallardo and C. Rejeb. *Newtonian Potentials and subharmonic functions associated to the Dunkl-Laplace operator.* Submitted.

[21] L. Gallardo, C. Rejeb and M. Sifi. *Dunkl-Riesz Potentials of Radon Measures.* Preprint.

[22] S. Hassani, S. Mustapha and M. Sifi. *Riesz Potential and fractional maximal function for the Dunkl transform.* J. of Lie Theory, Vol. 19, No.4, (2009), 725-734.

[23] W. K. Hayman and P. B. Kennedy. *Subharmonic functions.* Volume 1. Academic Press London, (1976).

[24] L. Hedberg. *On certain convolution inequalities.* Proc. Amer. Math. Soc. 36 (1972), 505-510.

[25] L. L. Helms. *Potential theory.* Springer-Verlag London, (2009).

[26] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics 29, Cambridge University Press (1990).

[27] L. Hörmander. *The Analysis of Linear Partial Differential operators I.* Springer, (1983).

[28] L. Hörmander. *Notions of Convexity.* Birkhäuser, Boston-Basel-Berlin (Reprint edition), (1994)

[29] M. F. de Jeu. *The Dunkl transform.* Invent. Math., 113, (1993), 147-162.

[30] R. Kane. *Reflection Groups and Invariant Theory.* CMS Books in Mathematics. Springer-Verlag, New York, (2001).

[31] N. S. Landkof. *Foundations of Modern Potential Theory.* Springer, Berlin, (1972).

[32] M. Maslouhi and E. H. Youssfi. *Harmonic functions associated to Dunkl operators.* Monatsh. Math. 152 (2007), 337-345.

[33] M. Maslouhi and E. H. Youssfi. *The Dunkl intertwining operator.* J. Funct. Anal. 256 (8), (2009), 2697-2709.

[34] M. Maslouhi and R. Daher. *Weyl's lemma and converse mean value for Dunkl operators.* Operator Theory : Advances and Applications, Vol 205, (2009), 91-100.

[35] H. Mejjaoli and K. Trimèche . *On a mean value property associated with the Dunkl Laplacian operator and applications.* Integ. Transf. and Spec. Funct., 12(3), (2001), 279-302.

[36] E. M. Opdam. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group.* Compositio Math. 85 (1993), no. 3, 333-373.

[37] T. Ransford. *Potential Theory in the Complex Plane.* Cambridge University Press, (1995).

[38] F. Riesz. *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel II.* Acta. Math, 54, (1930), 321-360.

[39] M. Rösler. *Bessel-type signed hypergroups on* $\mathbb{R}$. Probability measures on groups and related structures XI. Proceedings, Oberwolfach (1994). World Scientific, (1995), 292-304.

[40] M. Rösler. *Generalized Hermite polynomials and the heat equation for Dunkl operators.* Comm. Math. Phys, 192, (1998), 519-542.

[41] M. Rösler and M. Voit. *Markov processes related with Dunkl operators.* Adv. Appl. Math, 21, (1998), 575-643.

[42] M. Rösler. *Positivity of Dunkl's intertwining operator.* Duke Math. J., 98, (1999), 445-463.

[43] M. Rösler. *Short-time estimates for heat kernels associated with root systems.* Special functions (Hong Kong, 1999), 309-323, World Sci. Publ., River Edge, NJ, 2000.

[44] M. Rösler and M. F. de Jeu. *Asymptotic analysis for the Dunkl kernel.* J. Approx. Theory, 119, (2002), no. 1, 110-126.

[45] M. Rösler. *Dunkl Operators : Theory and Applications.* Lecture Notes in Math., vol.1817, Springer Verlag (2003), 93-136.

[46] M. Rösler. *A positive radial product formula for the Dunkl kernel.* Trans. Amer. Math. Soc., 355, (2003), 2413-2438.

[47] L. Schwartz. *Théorie des distributions.* Hermann, Editeurs des Sciences et des arts (1966).

[48] S. Thangavelu and Y. Xu. *Convolution operator and maximal function for Dunkl transform.* J. Anal. Math. Vol. 97, (2005), 25-56.

[49] S. Thangavelu and Y. Xu. *Riesz transform and Riesz potentials for Dunkl transform.* J. Comput. Appl. Math. Vol. 199, (2007), 181-195.

[50] F. Treves. *Topological vector spaces, distributions and kernels.* New York, London : Academic press (1967).

[51] K. Trimèche. *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual.* Integ. Transf. and Spec. Funct., 12(4), (2001), 394-374.

[52] K. Trimèche. *Paley-Wiener theorem for the Dunkl transform and Dunkl translation operators.* Integ. Transf. and Spec. Func. 13, (2002), 17-38.

[53] K. Trimèche. *Hypoelliptic Dunkl convolution equations in the space of distributions on* $\mathbb{R}^d$. Fourier Analysis and Applications. volume 12, (2006), 517-542.

[54] Y. Xu. *Orthogonal polynomials for a family of product weight functions on the spheres.* Canad. J. Math., 49(1), (1997), 175-192.

# Chapitre 1

# *Une Nouvelle Propriété de Moyenne pour les Fonctions Harmoniques Associées au Laplacien de Dunkl et Applications*

**Résumé**

Pour un système de racines dans $\mathbb{R}^d$ muni de son groupe de Coxeter-Weyl $W$ et d'une fonction de multiplicité $k \geq 0$, on considère les opérateurs de Dunkl associés $D_1,\ldots,D_d$ et le laplacien de Dunkl $\Delta_k = D_1^2 + \cdots + D_d^2$. Dans ce papier, on étudie les propriétés des fonctions $u$ de classe $C^2$ sur un ouvert $W$-invariant $\Omega \subset \mathbb{R}^d$ et satisfaisant $\Delta_k u = 0$ sur $\Omega$ (D-harmonicité). En particulier, on introduit un nouvel opérateur de moyenne qui caractérise la D-harmonicité. Comme applications, on montre le principe du maximum fort, l'inégalité de Harnack et un théorème de Bôcher pour les fonctions D-harmoniques.

# A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications

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**Abstract**

For a root system in $\mathbb{R}^d$ furnished with its Coxeter-Weyl group $W$ and a multiplicity non negative function $k$, we consider the associated commuting system of Dunkl operators $D_1, \dots, D_d$ and the Dunkl-Laplacian $\Delta_k = D_1^2 + \dots + D_d^2$. This paper studies the properties of the functions $u$ defined on an open $W$-invariant set $\Omega \subset \mathbb{R}^d$ and satisfying $\Delta_k u = 0$ on $\Omega$ (D-harmonicity). In particular, we introduce and give a complete study of a new mean value operator which characterizes D-harmonicity. As applications we prove a strong maximum principle, a Harnack's type theorem and a Bôcher's theorem for D-harmonic functions.

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Key words: Dunkl-Laplacian operator, Dunkl transform, Dunkl harmonic functions, Generalized volume mean value operator, Strong maximum principle, Harnack's inequality, Bôcher's theorem.

## 1.1 Introduction

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle , \rangle$ and its associated norm $\|x\| = \sqrt{\langle x, x\rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, the reflection $\sigma_\alpha$ with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$, is given by

$$\forall\, x \in \mathbb{R}^d, \quad \sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a (reduced) root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$ (see [9] for details on root systems). The finite group $W$ generated by the reflections $\sigma_\alpha$, $\alpha \in R$, is called the Coxeter-Weyl group (or the reflection group) of the root system. Then, we fix a $W$-invariant function $k : R \longrightarrow \mathbb{C}$ called the multiplicity function of the root system and we consider the family of commuting operators $D_j$ $(j = 1, \dots, d)$ defined for $f \in \mathcal{C}^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle},$$

where $R_+$ is a positive subsystem. These operators, defined by C. F. Dunkl in [3], are of fundamental importance in various areas of mathematics and mathematical physics (see [17] and its references for details).
Throughout the paper, we will assume that $k \geq 0$ and we will need the weight function defined by

$$\forall\ x \in \mathbb{R}^d,\ \omega_k(x) := \prod_{\alpha \in R_+} |\ \langle \alpha, x \rangle\ |^{2k(\alpha)}.$$

This function $\omega_k$ is $W$-invariant and homogeneous of degree $2\gamma$, where $\gamma := \sum_{\alpha \in R_+} k(\alpha)$.
But the main tool, as far as we are concerned, is the Dunkl intertwining operator $V_k$ which is the unique isomorphism from the space $\mathcal{P}$ of polynomials on $\mathbb{R}^d$ onto itself satisfying (see [5])

$$\forall\ j = 1, ..., d,\quad D_j V_k = V_k \frac{\partial}{\partial x_j},\quad \text{and}\quad V_k(1) = 1. \tag{1.1}$$

This operator has been extended by K. Trimèche (see [18]) to an isomorphism from $\mathcal{C}^\infty(\mathbb{R}^d)$ (carrying its usual Fréchet topology) onto itself satisfying the intertwining relations (1.1) and M. Rösler (see [15]) has obtained the following fundamental integral representation

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}^d),\quad \forall\ x \in \mathbb{R}^d,\quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \tag{1.2}$$

where $\mu_x$ is a probability measure on $\mathbb{R}^d$ with compact support contained in

$$C(x) := \mathrm{co}\big\{gx, g \in W\big\}, \tag{1.3}$$

the convex hull of the orbit of $x$ under $W$.
Moreover, the Dunkl intertwining operator $V_k$ commutes with the $W$-action (see [17])

$$\forall\ \ f \in \mathcal{C}^\infty(\mathbb{R}^d),\ \ \forall\ \ g \in W,\ \ g^{-1}.V_k(g.f) = V_k(f), \tag{1.4}$$

where $g.f(x) = f(g^{-1}x)$. In terms of Rösler's measures, this property means that for all $y \in \mathbb{R}^d$ and $g \in W$, $\mu_y$ is the image measure of $\mu_{gy}$ by the map $x \mapsto g^{-1}x$.

In [18], K. Trimèche has introduced the dual Dunkl intertwining operator ${}^tV_k$ defined on $\mathcal{D}(\mathbb{R}^d)$ (the space of $\mathcal{C}^\infty-$functions on $\mathbb{R}^d$ with compact support) by

$${}^tV_k(f) = \mathcal{F}^{-1}[\mathcal{F}_D(f)],\quad f \in \mathcal{D}(\mathbb{R}^d), \tag{1.5}$$

where $\mathcal{F}$ is the classical Fourier transform and $\mathcal{F}_D$ is the Dunkl transform which precise definition will be recalled in section 2.
This operator is an isomorphism from $\mathcal{D}(\mathbb{R}^d)$ onto itself satisfying

$$\int_{\mathbb{R}^d} {}^tV_k(f)(x) g(x) dx = \int_{\mathbb{R}^d} f(x) V_k(g)(x) \omega_k(x) dx, \tag{1.6}$$

for all $f \in \mathcal{D}(\mathbb{R}^d)$ and $g \in \mathcal{C}^\infty(\mathbb{R}^d)$. Note that replacing $f$ by $({}^tV_k)^{-1}(f)$ and $g$ by $V_k^{-1}(g)$ in (1.6) we have the following equivalent relation

$$\int_{\mathbb{R}^d} f(x)V_k^{-1}(g)(x)dx = \int_{\mathbb{R}^d} ({}^tV_k)^{-1}(f)(x)g(x)\omega_k(x)dx. \tag{1.7}$$

Moreover, it is a consequence of the Paley-Wiener theorem for the Dunkl transform that ${}^tV_k$ is support preserving (see [19]) i.e.

$$\text{supp }(f) \subset B(0,a) \Longleftrightarrow \text{supp }({}^tV_k(f)) \subset B(0,a). \tag{1.8}$$

Throughout the paper we will suppose that the root system is normalized [1] in the sense that $\langle \alpha, \alpha \rangle = 2$ and the notation $B(\xi, r)$ will denote the closed ball in $\mathbb{R}^d$ with radius $r$ centered at $\xi \in \mathbb{R}^d$.

Let us now introduce the Dunkl-Laplacian operator ([2] and [6] p.156) $\Delta_k := \sum_{j=1}^d D_j^2$, which is known to act on $\mathcal{C}^2(\mathbb{R}^d)$ functions as

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha \in R_+} k(\alpha)\Big(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2}\Big), \tag{1.9}$$

where $\Delta$ is the classical Laplacian operator on $\mathbb{R}^d$. When $\Delta_k$ acts on $\mathcal{C}^2$-functions defined on an open subset $\Omega$ of $\mathbb{R}^d$, we obviously assume that $\Omega$ is $W$-invariant and a function $u \in \mathcal{C}^2(\Omega)$ is called Dunkl harmonic (D-harmonic) on $\Omega$ if

$$\forall\, x \in \Omega, \quad \Delta_k u(x) = 0.$$

To our knowledge, up to now, D-harmonic functions have been studied only for $\Omega = \mathbb{R}^d$ (see [10]), for $\Omega = \overset{\circ}{B}(0,1)$ (the open unit ball of $\mathbb{R}^d$) (see [12]) or for $\Omega$ an ellipsoidal domain centered at the origin (see [20] and [21]) . In [10], K. Trimèche and H. Mejjaoli proved that a function $u \in \mathcal{C}^\infty(\mathbb{R}^d)$ is D-harmonic on $\mathbb{R}^d$ if and only if $u$ satisfies the following generalized spherical mean value property:

$$\forall x \in \mathbb{R}^d,\ \forall r > 0, \quad u(x) = M_S^r(u)(x) := \frac{1}{d_k}\int_{S^{d-1}} \tau_x u(r\xi)\omega_k(\xi)d\sigma(\xi), \tag{1.10}$$

where $d\sigma(\xi)$ is the surface measure of the unit sphere $S^{d-1}$ of $\mathbb{R}^d$, $d_k$ is the constant given by

$$d_k = \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) \tag{1.11}$$

and $\tau_x$ is the Dunkl translation operator acting on $\mathcal{C}^\infty(\mathbb{R}^d)$ functions and whose precise expression is given in section 2. In the case of the ball $\overset{\circ}{B}(0,1)$ Maslouhi and Youssfi obtained a similar result under the condition that the function $u$ can be extended to a $\mathcal{C}^\infty$ function on $\mathbb{R}^d$.

Our aim in this paper is to introduce and to study a new mean value operator which characterizes D-harmonicity for functions defined on an arbitrary open and $W$-invariant set $\Omega \subset \mathbb{R}^d$.

1. this simplifies the formulas in particular for the reflections, we have $\sigma_\alpha x = x - \langle \alpha, x\rangle\alpha$.

In section 2 we recall some important facts on Dunkl transform and Dunkl translation operators and we prove a duality formula for these translations which is used in the sequel as a very important technical tool.

Section 3 is the core of the paper. We introduce a nonnegative kernel $h_k(r,x,y)$ (see (1.31) for the explicit formula) such that for $r>0$ and $x\in\mathbb{R}^d$ fixed, the function $y\mapsto h_k(r,x,y)$ has compact support contained in $\cup_{g\in W}B(gx,r)$ and for a continuous function $u$ on a $W$-invariant open set $\Omega$, we define the volume mean value of $u$ relative to $(x,r)$ as

$$M_B^r(u)(x)=\frac{1}{m_k(B(0,r))}\int_{\mathbb{R}^d}u(y)h_k(r,x,y)\omega_k(y)dy,$$

where $x\in\Omega$, $r>0$ is such that $B(x,r)\subset\Omega$ and $m_k(B(0,r))=\int_{B(0,r)}\omega_k(y)dy$ is the $\omega_k$-volume of the ball $B(0,r)$.
We call $h_k$ the harmonic kernel. If $k\equiv 0$ (in the classical case of the Laplace operator $\Delta$), we have $h_0(r,x,y)=\mathbf{1}_{B(x,r)}(y)$ and $M_B^r(u)(x)$ is the usual volume mean value of $u$ at $x$ on the ball $B(x,r)$.

For a general root system and multiplicity function $k\geq 0$, the harmonic kernel $h_k(r,x,y)$ has some specific properties which we study in detail. It is interesting to note that if $u\in\mathcal{C}^\infty(\mathbb{R}^d)$, we have a Gauss type formula relating the function $u$, the spherical means $M_S^r(u)$ of $u$ and the volume means $M_B^r(\Delta_k u)$ of the function $\Delta_k u$:

$$\forall r>0,\forall x\in\mathbb{R}^d,\ M_S^r(u)(x)=u(x)+\frac{1}{2\gamma+d}\int_0^r M_B^t(\Delta_k u)(x)tdt.$$

But in the sequel we will concentrate on the properties of the volume mean which is particularly suitable to functions not necessarily defined on whole $\mathbb{R}^d$. The main theorem of section 3 asserts that for a function $u\in C^2(\Omega)$ ($\Omega$ a $W$-invariant open set of $\mathbb{R}^d$), for all $x\in\Omega$ and $\rho>0$ such that $B(x,\rho)\subset\Omega$, we have

$$M_B^R(u)(x)=u(x)+\frac{1}{R^{2\gamma+d}}\int_0^R\int_0^r M_B^t(\Delta_k u)(x)tdtr^{2\gamma+d-1}dr,$$

for all $R\leq\rho/3$. As a corollary, we obtain the fundamental characterization that $u$ is D-harmonic in $\Omega$ if and only if it satisfies the volume mean value property. That is: for all $x\in\Omega$ and $\rho>0$ such that $B(x,\rho)\subset\Omega$, we have

$$u(x)=M_B^R(u)(x)$$

for all $R\in]0,\frac{\rho}{3}]$. As an other corollary we obtain a Liouville's type theorem: Every positive D-harmonic function on $\mathbb{R}^d$ is a constant.

The main results of section 4 are the strong maximum principle for the Dunkl Laplacian operator and Harnack's theorem for D-harmonic functions. We prove that if $\Omega$ is a $W$-invariant connected open subset of $\mathbb{R}^d$, every D-harmonic function on $\Omega$ which attains a maximum at $x_0\in\Omega$, is constant. This is the so called strong maximum principle. Under the same assumptions on $\Omega$, we have a generalization of the famous Harnack's inequality:

for each compact set $K \subset \Omega$, there exists a universal constant $C_K \geq 1$ such that the inequality

$$u(x) \leq C_K u(y)$$

holds for all $x, y \in K$ and all nonnegative D-harmonic functions $u$ in $\Omega$. The crucial tool to prove these results is a rather delicate comparison result involving the harmonic kernels at different quite close points. Precisely: Let $r > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\|x_1 - x_2\| \leq 2r$. Then,

$$\forall y \in \mathbb{R}^d,\ h_k(r, x_2, y) \leq h_k(r\sqrt{10}, x_1, y).$$

As in the classical case, Harnack's principle for D-harmonic functions follows immediately from Harnack's theorem: every increasing sequence of nonnegative D-harmonic functions on $\Omega$, either converge to a D-harmonic function or to $+\infty$.

Finally in section 5, we give an application of Harnack's theorem and the strong maximum principle to a result which is a generalization to D-harmonic functions of the so called Bôcher's theorem. Precisely, if $d \geq 3$ or if $d = 2$ and $k \neq 0$, we show that if $u$ is a positive function which is D-harmonic in the punctured open ball $\overset{\circ}{B}(0,1) \setminus \{0\}$, then it is of the form:

$$u(x) = a||x||^{2-d-2\gamma} + v(x), \quad x \in \overset{\circ}{B}(0,1) \setminus \{0\},$$

where $a$ is a constant and $v$ a D-harmonic function on $\overset{\circ}{B}(0,1)$. As a corollary, we obtain that a positive D-harmonic function on the punctured space $\mathbb{R}^d \setminus \{0\}$ is of the form

$$u(x) = a||x||^{2-d-2\gamma} + b \quad (x \in \mathbb{R}^d \setminus \{0\}),$$

with constants $a, b \geq 0$.

## 1.2 The Dunkl transform and Dunkl's translation operators

In this section we recall some properties of the Dunkl transform (see [11] and [17] ) and the Dunkl translation operators (see[19]).

The Dunkl transform of a function $f \in L^1(\mathbb{R}^d, \omega_k(x)dx)$ is defined by

$$\mathcal{F}_D(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x)\omega_k(x)dx, \quad \lambda \in \mathbb{R}^d, \tag{1.12}$$

where

$$E_k(x, y) := V_k(e^{\langle x,.\rangle})(y), \quad x, y \in \mathbb{R}^d,$$

is the Dunkl kernel (see [6] and [17]) analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and in particular satisfying the following exchanging constants property:

$$\forall a \in \mathbb{C},\ \forall x, y \in \mathbb{C}^d,\ E_k(x, ay) = E_k(ax, y). \tag{1.13}$$

It is well known (see [11]) that the Dunkl transform $\mathcal{F}_D$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ (the Schwartz space of rapidly decreasing function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$) onto itself and its inverse is given by

$$\mathcal{F}_D^{-1}(f)(x) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} f(\lambda) E_k(ix, \lambda)\omega_k(\lambda)d\lambda, \quad x \in \mathbb{R}^d, \tag{1.14}$$

with

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} \omega_k(x)dx,$$

the Macdonald-Mehta constant (see [13] and [7]).

It is useful to note that if $f \in L^1(\mathbb{R}^d, \omega_k(x)dx)$ is radial (i.e $f(x) = F(||x||)$, with $F$ a function defined on $[0, +\infty[)$, $\mathcal{F}_D(f)$ is also radial. Precisely, using spherical coordinates and Corollary 2.5 of ([16]), we have

$$\mathcal{F}_D(f)(\lambda) = d_k \int_0^{+\infty} F(r) j_{\gamma+\frac{d}{2}-1}(r\|\lambda\|) r^{2\gamma+d-1} dr, \quad \lambda \in \mathbb{R}^d, \tag{1.15}$$

where $d_k$ is defined by the relation (1.11) and for $\alpha \geq -1/2$, $j_\alpha$ is the normalized Bessel function given by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}. \tag{1.16}$$

Now, the Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{1.17}$$

where $T_x$ is the classical translation operator given by $T_x f(y) = f(x+y)$. More shortly, we can also write

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = (V_k)_x (V_k)_y [V_k^{-1}(f)(x+y)],$$

where $(V_k)_x$ denotes the operator acting on the $x$-variable.

If $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform we have (see [19]):

$$\begin{aligned} \forall\ \ y \in \mathbb{R}^d, \quad \tau_x f(y) &= \mathcal{F}_D^{-1}[E_k(ix, .)\mathcal{F}_D(f)](y) & (1.18)\\ &= \frac{1}{c_k^2} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(\lambda) E_k(ix, \lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda. & (1.19) \end{aligned}$$

For $f \in \mathcal{D}(\mathbb{R}^d)$, the function $\tau_{-x} f$ can be expressed by using the dual intertwining operator as follows (see [19] formulas (87) and (88) p.34 ):

$$\begin{aligned} \forall\, y \in \mathbb{R}^d, \quad \tau_{-x} f(y) &= \int_{\mathbb{R}^d} ({}^tV_k)^{-1} \circ T_{-z} \circ {}^tV_k(f)(y) d\mu_x(z) & (1.20)\\ &= (V_k)_x ({}^tV_k^{-1})_y [{}^tV_k(f)(y-x)]. & (1.21) \end{aligned}$$

The operators $\tau_x, x \in \mathbb{R}^d$, satisfy the following properties:

**1)** for all $x \in \mathbb{R}^d$, the operator $\tau_x$ is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself,

**2)** for all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, the function $x \longmapsto \tau_x f(y)$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}^d$,

**3)** for all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x), \tag{1.22}$$

and

$$D_j(\tau_x f) = \tau_x(D_j f), \quad j = 1, ..., d, \tag{1.23}$$

$$(D_j)_x(\tau_x f) = \tau_x(D_j f), \quad j = 1, ..., d, \tag{1.24}$$

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \tag{1.25}$$

where $D_j$ (resp. $\Delta_k$) are Dunkl's operators (resp. Dunkl-Laplacian's operator).

**4)** for all $f \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\forall\, y \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \tau_x f(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\omega_k(y)dy, \tag{1.26}$$

**5)** if $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is radial (i.e $f(x) = F(\|x\|)$), M. Rösler ([16]) has proved the useful formula

$$\forall\, x \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, z\rangle})d\mu_y(z), \tag{1.27}$$

where $\mu_y$ is Rösler's measure introduced in (1.2).

In the sequel we will need the following crucial duality result:

**Proposition 1.1** *Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $g \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $x \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} \tau_x f(y)g(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\tau_{-x}g(y)\omega_k(y)dy. \tag{1.28}$$

*Proof:* Fix $x \in \mathbb{R}^d$. From the relations (1.6), (1.7), we deduce that for all $z \in \mathbb{R}^d$,

$$\begin{aligned}\int_{\mathbb{R}^d} V_k \circ T_z \circ (V_k)^{-1}(f)(y)g(y)\omega_k(y)dy &= \int_{\mathbb{R}^d} T_z \circ (V_k)^{-1}(f)(y)^t V_k(g)(y)dy \\ &= \int_{\mathbb{R}^d} f(y)(^t V_k)^{-1} \circ T_{-z} \circ {}^t V_k(g)(y)\omega_k(y)dy.\end{aligned}$$

Integrating both sides of this equality with respect to the measure $d\mu_x(z)$ (whose support is compact by (1.3)), we obtain

$$\int_{\mathbb{R}^d}\int_{\mathrm{supp}(g)} V_k \circ T_z \circ (V_k)^{-1}(f)(y)g(y)\omega_k(y)dyd\mu_x(z) = \\ \int_{\mathbb{R}^d}\int_{B(0,a)} f(y)(^t V_k)^{-1} \circ T_{-z} \circ {}^t V_k(g)(y)\omega_k(y)dyd\mu_x(z), \tag{1.29}$$

where $B(0,a)$ is a closed ball such that

$$\forall\, z \in \mathrm{supp}\mu_x, \quad \mathrm{supp}(T_{-z} \circ {}^tV_k(g)) \subset \mathrm{supp}\mu_x + \mathrm{supp}({}^tV_k(g)) \subset B(0,a),$$

and which, by (1.8), is also such that $\mathrm{supp}(({}^tV_k)^{-1} \circ T_{-z} \circ {}^tV_k(g)) \subset B(0,a)$. This implies the desired result if interchanging the integrals is permissible in both sides of (1.29). Let's now justify this:

• Using the relation (1.5), we deduce that the function

$$(z,y) \longmapsto f(y)({}^tV_k)^{-1} \circ T_{-z} \circ {}^tV_k(g)(y) = f(y)\mathcal{F}_D^{-1}[e^{\langle -iz,.\rangle}\mathcal{F}_D(g)](y)$$

is continuous on the compact set $\mathrm{supp}\mu_x \times B(0,a)$. Thus, we can apply Fubini's theorem for the right side in (1.29).

• The function $(z,y) \longmapsto V_k \circ T_z \circ (V_k)^{-1}(f)(y)$ is measurable as we can see easily if $f$ is a polynomial and in general by approximating $f$ by a sequence of polynomials. Furthermore, using the relations (1.2) and (1.3) and the continuity of the function $(z,\xi) \longmapsto (V_k)^{-1}(f)(z+\xi)$, there exists a positive constant $C > 0$ such that

$$\forall\, (z,y) \in \mathrm{supp}\mu_x \times B(0,b),\ |V_k \circ T_z \circ (V_k)^{-1}(f)(y)| \leq \int_{B(0,b)} |V_k^{-1} f(z+\xi)| d\mu_y(\xi) \leq C, \tag{1.30}$$

where $B(0,b)$ is a closed ball such that

$$\forall\ \ y \in \mathrm{supp}g, \quad \mathrm{supp}\mu_y \subset B(0,b).$$

Finally, (1.30) shows that we can also use the Fubini's theorem for the left side of (1.29). This completes the proof. □

**Remark 1.2** *: For $f$ and $g$ in $\mathcal{D}(\mathbb{R}^d)$, the result of Proposition 1.1 is much more easy to prove and was already known (see [19]). It can also be obtained by Fourier-Dunkl transform.*

## 1.3 The volume mean value property

In this section we study the notion of D-harmonicity on an arbitrary open $W$-invariant subset $\Omega$ of $\mathbb{R}^d$. This requires a generalization of the classical volume mean value operator of a function $u$ defined on $\Omega$. For this we introduce the following kernel

**Definition 1.3** *For $r > 0$ and $x, y \in \mathbb{R}^d$, we define the harmonic kernel $h_k(r,x,y)$ as follows:*

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle}\right) d\mu_y(z). \tag{1.31}$$

**Exemple 1.4** *1) For $k=0$ (i.e. in the case of the classical Laplacian operator), we have $\mu_y = \delta_y$ and $h_0(r,x,y) = 1_{B(x,r)}(y)$.*

*2) When $d = 1$, $R = \{1, -1\}$, $W = \mathbf{Z}_2 = \{id, -id\}$, the multiplicity function is a constant $k > 0$, the intertwining operator is of the form $V_k(f)(y) = \int_{-1}^{1} f(yt)\phi_k(t)dt$ where*

$$\phi_k(t) = \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}(1-t)^{k-1}(1+t)^k \mathbf{1}_{[-1,1]}(t), \tag{1.32}$$

*is the $\mathbf{Z}_2$-Dunkl density function of parameter $k$ (see [4] or [17], p.104). In this case, we have*

$$h_k(r, x, y) = \int_{-1}^{1} \mathbf{1}_{[0,r]}(\sqrt{x^2 + y^2 - 2txy})\phi_k(t)dt.$$

*It is easy to see that*

$$h_k(r, x, 0) = \mathbf{1}_{[-r,r]}(x), \quad h_k(r, -x, -y) = h_k(r, x, y) \quad \text{and} \quad h_k(r, -x, y) = h_k(r, x, -y).$$

*From these relations, it is enough to compute $h_k(r, x, y)$ for $x > 0$ and $y \in \mathbb{R}\backslash\{0\}$. For this, define*

$$\vartheta := \vartheta_{r,x,y} = \frac{x^2 + y^2 - r^2}{2xy}.$$

- *If $y > 0$, we have*

$$h_k(r, x, y) = \begin{cases} 1 & \text{if } \vartheta \leq -1 \\ \displaystyle\int_{\vartheta}^{1} \phi_k(t)dt & \text{if } -1 \leq \vartheta \leq 1 \\ 0 & \text{if } \vartheta \geq 1. \end{cases}$$

*More shortly, we can write*

$$h_k(r, x, y) = \mathbf{1}_{]-\infty,-1]}(\vartheta) + \Big(\int_{\vartheta}^{1} \phi_k(t)dt\Big)\mathbf{1}_{[-1,1]}(\vartheta).$$

- *If $y < 0$, in the same way, we obtain*

$$h_k(r, x, y) = \mathbf{1}_{[1,+\infty[}(\vartheta) + \Big(\int_{-1}^{\vartheta} \phi_k(t)dt\Big)\mathbf{1}_{[-1,1]}(\vartheta).$$

*We note that for $x$ and $r$ fixed, the function $y \longmapsto h_k(r, x, y)$ has compact support equal to $I_{x,r} \cup I_{-x,r}$, where $I_{x,r} = [x - r, x + r]$. For example, for $y > 0$ and*

- *if $0 < r < x$ i.e $I_{x,r} \cap I_{-x,r} = \emptyset$ we have*

$$h_k(r, x, y) = \Big(\int_{\vartheta}^{1} \phi_k(t)dt\Big)\mathbf{1}_{I_{x,r}}(y),$$

- *if $0 < x \leq r$ i.e $I_{x,r} \cap I_{-x,r} \neq \emptyset$ we have*

$$h_k(r, x, y) = \mathbf{1}_{I_{x,r} \cap I_{-x,r}}(y) + \Big(\int_{\vartheta}^{1} \phi_k(t)dt\Big)\mathbf{1}_{I_{x,r} \backslash I_{-x,r}}(y).$$

*3) In the case of the root system $R = \{\pm e_1\}$ in $\mathbb{R}^2$ (where $e_1 = (1,0)$), the Coxeter-Weyl group is $\mathbf{Z}_2 \times \{id\}$, the multiplicity function reduces to the parameter $k = k(e_1) > 0$ and Rösler's measure $\mu_y = \mu_{(y_1,y_2)}$ is of the form $\mu_{(y_1,y_2)} = \mu_{y_1} \otimes \delta_{y_2}$, where $\mu_{y_1}$ is Rösler's measure on $\mathbb{R}$ associated to the Coxeter-Weyl group $\mathbf{Z}_2$ and $\delta_{y_2}$ is the Dirac measure at point $y_2 \in \mathbb{R}$. Therefore, for $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$, the harmonic kernel is given by*

$$\begin{aligned} h_k(r,x,y) &= \int_{\mathbb{R}^2} \mathbf{1}_{[0,r]}\left(\sqrt{||x||^2 + ||y||^2 - 2(x_1z_1 + x_2z_2)}\right) d\mu_{y_1}(z_1)d\delta_{y_2}(z_2) \\ &= \int_{-1}^{1} \mathbf{1}_{[0,r]}\left(\sqrt{||x||^2 + ||y||^2 - 2tx_1y_1 - 2x_2y_2)}\right) \phi_k(t)dt, \quad (1.33) \end{aligned}$$

*where $\phi_k$ is the $\mathbf{Z}_2$-Dunkl density function of parameter $k$ defined by (1.32).*

*4) We consider $\mathbb{R}^d$ with the root system $R = \{\pm e_1, \dots, \pm e_d\}$ where $(e_i)_{1 \le i \le d}$ is the canonical basis of $\mathbb{R}^d$. Then, the Coxeter-Weyl group is $\mathbf{Z}_2^d$, the multiplicity function can be represented by a multidimensional parameter $k = (k_1, \dots, k_d)$, $k_i = k(e_i) > 0$ and the harmonic kernel is given by*

$$\begin{aligned} h_k(r,x,y) &= \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\left(\sqrt{||x||^2 + ||y||^2 - 2(x_1z_1 + \dots + x_dz_d}\right) d\mu_{y_1}(z_1) \otimes \dots \otimes d\mu_{y_d}(z_d) \\ &= \int_{[-1,1]^d} \mathbf{1}_{[0,r]}\left(\sqrt{||x||^2 + ||y||^2 - 2(t_1x_1y_1 + \dots + t_dx_dy_d)}\right) \phi_{k_1}(t_1)\cdots\phi_{k_d}(t_d)dt_1\cdots dt_d, \end{aligned} \quad (1.34)$$

*where $\phi_{k_i}$ is the $\mathbf{Z}_2$-Dunkl density function of parameter $k_i$ $(1 \le i \le d)$.*

**Proposition 1.5** *The harmonic kernel satisfies the following properties:*

1. *For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \le h_k(r,x,y) \le 1$.*
2. *For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ $(r > 0)$ is right-continuous and non decreasing.*
3. *For all fixed $r > 0$ and $x \in \mathbb{R}^d$, the function $h_k(r,x,.\,) : y \mapsto h_k(r,x,y)$, has compact support and*
$$supp\ h_k(r,x,.\,) \subset \bigcup_{g\in W} B(gx,r). \quad (1.35)$$
*Moreover, if $r \ge \|x\|$, we have*
$$\forall\ \ y \in \bigcap_{g\in W} B(gx,r), \quad h_k(r,x,y) = 1.$$
4. *Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions satisfying*
$$\forall\, \varepsilon > 0, \quad 0 \le \varphi_\varepsilon \le 1 \quad and \quad \forall\, y \in \mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y), \quad (1.36)$$
*we have*
$$\forall\, y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon\to 0} \tau_{-x}\varphi_\varepsilon(y). \quad (1.37)$$

5. *For all $r>0$ and $x,y\in\mathbb{R}^d$, we have*

$$h_k(r,x,y)=h_k(r,y,x). \tag{1.38}$$

6. *For all $r>0$ and $x\in\mathbb{R}^d$, we have*

$$\|h_k(r,x,.)\|_{k,1}:=\int_{\mathbb{R}^d}h_k(r,x,y)\omega_k(y)dy=m_k(B(0,r))=\frac{d_kr^{2\gamma+d}}{2\gamma+d}, \tag{1.39}$$

*where $dm_k(y)=\omega_k(y)dy$ and $d_k$ is the constant defined in (1.11).*

7. *Let $r>0$ and $x,y\in\mathbb{R}^d$. Then, for all $g\in W$, we have*

$$h_k(r,gx,gy)=h_k(r,x,y) \quad and \quad h_k(r,gx,y)=h_k(r,x,g^{-1}y). \tag{1.40}$$

8. *For all $r>0$ and $x\in\mathbb{R}^d$, the function $h_k(r,x,.)$ is upper semi-continuous on $\mathbb{R}^d$.*

*Proof:*
Property 1. is clear.
2. It is easy to see that $r\longmapsto h_k(r,x,y)$ is non decreasing. Let $r>0$ be fixed. We will show that $h(.,x,y)$ is continuous at $r^+$. Indeed, for all $t>0$ small enough, we have

$$h_k(r+t,x,y)-h_k(r,x,y)=\int_{\mathbb{R}^d}\mathbf{1}_{]r,r+t]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z).$$

Using the dominated convergence theorem, we deduce that

$$\lim_{t\to 0}h_k(r+t,x,y)=h_k(r,x,y).$$

3. Let $z\in\text{supp}\mu_y$. From (1.3) we can write

$$z=\sum_{g\in W}\lambda_g(z)gy, \tag{1.41}$$

where $\lambda_g(z)\in[0,1]$ are such that $\sum_{g\in W}\lambda_g(z)=1$. Then, we have

$$\|x\|^2+\|y\|^2-2\langle x,z\rangle=\sum_{g\in W}\lambda_g(z)\|g^{-1}x-y\|^2. \tag{1.42}$$

Thus if $y\notin\cup_{g\in W}B(gx,r)$ then $\|x\|^2+\|y\|^2-2\langle x,z\rangle>r^2$ and $h_k(r,x,y)=0$. This proves (1.35).
Furthermore, if $y\in B(gx,r)$ for all $g\in W$, we deduce from (1.42) that

$$\forall\ \ z\in\text{supp}\mu_y,\ \ \|x\|^2+\|y\|^2-2\langle x,z\rangle\leq r^2.$$

Then, as $\mu_y$ is a probability measure, from (1.31) and (1.2) we obtain $h_k(r,x,y)=1$.
4. Let $(\varphi_\varepsilon)$ as in (1.36) and $\phi_\varepsilon$ such that $\varphi_\varepsilon(\xi)=\phi_\varepsilon(\|\xi\|)$. By (1.27), for $y\in\mathbb{R}^d$ we have

$$\tau_{-x}\varphi_\varepsilon(y)=\int_{\mathbb{R}^d}\phi_\varepsilon(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z). \tag{1.43}$$

Using the dominated convergence theorem and (1.35), we deduce that

$$\lim_{\varepsilon\to 0}\tau_{-x}\varphi_\varepsilon(y)=\int_{\mathbb{R}^d}\mathbf{1}_{[0,r]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z)=h_k(r,x,y). \tag{1.44}$$

5. We deduce the result from (1.37) and from the following lemma:

**Lemma 1.6** *Let $f\in\mathcal{S}(\mathbb{R}^d)$ be radial. Then, we have*

$$\tau_{-x}f(y)=\tau_{-y}f(x). \tag{1.45}$$

*Proof of the lemma* 1.6: By (1.19), we have

$$\tau_{-x}f(y)=\frac{1}{c_k^2}\int_{\mathbb{R}^d}\mathcal{F}_D(f)(\lambda)E_k(-ix,\lambda)E_k(iy,\lambda)\omega_k(\lambda)d\lambda.$$

As $\mathcal{F}_D(f)$ is radial by (1.15), the change of variables $\xi=-\lambda$ and (1.13) give immediately (1.45). $\square$

6. We deduce (1.39) from (1.26), (1.36), (1.37) and from the dominated convergence theorem.

7. Let $g\in W$. It is enough to prove that $h_k(r,gx,gy)=h_k(r,x,y)$. We have

$$h_k(r,gx,gy):=\int_{\mathbb{R}^d}\mathbf{1}_{[0,r]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,g^{-1}z\rangle})d\mu_{gy}(z).$$

Then, the relations (1.4) and (1.2) imply the desired result.

8. Let $\theta$ be the $\mathcal{C}^\infty$-function on $\mathbb{R}^d$ defined by

$$\theta(t)=\begin{cases}\exp(-\frac{1}{1-t^2}) & \text{if } |t|<1\\ \\ 0 & \text{elsewhere}\end{cases}$$

For $\varepsilon>0$, we consider the function

$$\psi_\varepsilon(t)=\begin{cases}\frac{1}{\theta(0)}\theta(t/\varepsilon) & \text{if } -\varepsilon<t<0\\ \\ 1 & \text{if } 0\le t\le r^2\\ \\ \frac{1}{\theta(0)}\theta((t-r^2)/\varepsilon) & \text{if } r^2<t<r^2+\varepsilon\\ \\ 0 & \text{elsewhere}\end{cases}$$

It is easy to see that $\psi_\varepsilon\in\mathcal{C}^\infty(\mathbb{R})$. Moreover, we have $\psi_\varepsilon(t)\downarrow\mathbf{1}_{[0,r^2]}(t)$ as $\varepsilon\downarrow 0$.
Now, we define the radial function $\varphi_\varepsilon$ on $\mathbb{R}^d$ by

$$\varphi_\varepsilon(y)=\psi_\varepsilon(\|y\|^2).$$

We have $\varphi_\varepsilon\in\mathcal{D}(\mathbb{R}^d)$, supp $(\varphi_\varepsilon)\subset B(0,\sqrt{r^2+\varepsilon})$ and for all $y\in\mathbb{R}^d$,

$$\varphi_\varepsilon(y)\downarrow\mathbf{1}_{[0,r^2]}(\|y\|^2)=\mathbf{1}_{[0,r]}(\|y\|)=\mathbf{1}_{B(0,r)}(y),\quad \textit{as }\ \varepsilon\downarrow 0.$$

Furthermore by (1.27), we have for fixed $x \in \mathbb{R}^d$

$$\tau_{-x}\varphi_\varepsilon(y) = \int_{\mathbb{R}^d} \phi_\varepsilon(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle})d\mu_y(z) \downarrow h_k(r, x, y), \quad as \ \varepsilon \downarrow 0,$$

where $\varphi_\varepsilon(y) = \phi_\varepsilon(\|y\|)$. This shows that $h_k(r, x, .)$ is upper semi-continuous as a decreasing limit of continuous functions. $\square$

We give now an other important aspect of the harmonic kernel which shows that for fixed $x$, the function $h_k(r, x, .)$ concentrates in the neighbourhood of $x$ when $r \to 0$ and not on the other points $gx$ of the orbit $Wx$ as we could think in view of (1.35).

**Proposition 1.7** . *Let $x \in \mathbb{R}^d$. The family of probability measures*

$$d\eta_r^x(y) = \frac{1}{m_k(B(0, r))} h_k(r, x, y)\omega_k(y)dy, \ r > 0,$$

*is an approximation of the Dirac measure $\delta_x$ as $r \to 0$. More precisely*

1. *For all $\alpha > 0$, $\lim_{r\to 0} \int_{||x-y||>\alpha} d\eta_r^x(y) = 0$.*
2. *Let $f$ be a locally bounded and measurable function defined on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$ and let $x \in \Omega$. If $f$ is continuous at point $x$, then*

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y)d\eta_r^x(y) = f(x).$$

*Proof.* Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and let $(\varphi_\varepsilon)$ as in (1.36). From (1.28), we have

$$\int_{\mathbb{R}^d} f(y)\tau_{-x}\varphi_\varepsilon(y)\omega_k(y)dy = \int_{\mathbb{R}^d} \tau_x f(y)\varphi_\varepsilon(y)\omega_k(y)dy.$$

Using (1.37) and the dominated convergence theorem, by letting $\varepsilon \longrightarrow 0$ in the previous relation and dividing by $m_k(B(0, r))$, we get

$$\int_{\mathbb{R}^d} f(y)d\eta_r^x(y) = \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} \tau_x f(y)\mathbf{1}_{B(0,r)}(y)\omega_k(y)dy. \tag{1.46}$$

Then noting that the measures $d\nu_r(y) = \frac{1}{m_k(B(0,r))}\mathbf{1}_{B(0,r)}(y)\omega_k(y)dy$ $(r > 0)$ are an approximate identity when $r \to 0$, we obtain

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y)d\eta_r^x(y) = \tau_x f(0) = f(x). \tag{1.47}$$

Now for $\alpha > 0$, it is easy to find a $\mathcal{C}^\infty(\mathbb{R}^d)$ function $g$ that $\mathbf{1}_{B(x,\alpha)^c} \le g$ and $g = 0$ on $B(x, \alpha/2)$. Then by (1.47) we get

$$0 \le \int_{||x-y||>\alpha} d\eta_r^x(y) \le \int_{\mathbb{R}^d} g(y)d\eta_r^x(y) \to 0 \text{ as } r \to 0,$$

which proves assertion 1 of the proposition. Assertion 2 is now a classical exercise consisting in the decomposition of the integral $\int_{\mathbb{R}^d}(f(y) - f(x))d\eta_r^x(y)$ in two integrals, the first one on a ball $B(x, \alpha)$ adapted to the continuity of $f$ at point $x$ and the other on its complement $B(x, \alpha)^c$ where we use the compactness of the support of the measure $\eta_r^x$, the local boundedness of $f$ and the first assertion. $\square$

**Definition 1.8** *Let $u$ be a continuous function on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$, let $x \in \Omega$ and $r > 0$ be such that $B(x,r) \subset \Omega$. We define the volume mean value of $u$ relative to $(x,r)$ as*

$$M_B^r(u)(x) := \frac{1}{m_k(B(0,r))} \int_{\mathbb{R}^d} u(y) h_k(r,x,y)\omega_k(y)dy. \tag{1.48}$$

**Remark 1.9** • *We note that by (1.35) the integration domain is in fact supp $h_k(r,x,.) \subset \Omega$.*
• *Let $u, x$ and $r$ as in the previous definition. By Proposition1.5 (property 5 and 2), relation (1.39), Fubini's theorem and the dominated convergence theorem, we can see that the function $t \longmapsto M_B^t(u)(x)$ is continuous on $]0,r]$. Moreover, by Proposition 3.2, it is extendable to a continuous function at $t = 0$ such that $M_B^0(u)(x) = u(x)$.*

When $\Omega = \mathbb{R}^d$ and $u \in \mathcal{C}^\infty(\mathbb{R}^d)$, we have the following link beetwen the volume mean value and the spherical mean value introduced in (1.10)

**Proposition 1.10** *Let $u \in \mathcal{C}^\infty(\mathbb{R}^d)$. For all $r > 0$ and $x \in \mathbb{R}^d$, we have*

$$M_B^r(u)(x) = \frac{1}{m_k(B(0,r))} \int_{B(0,r)} \tau_x u(y)\omega_k(y)dy, \tag{1.49}$$

*and*

$$M_B^r(u)(x) = \frac{2\gamma + d}{r^{2\gamma+d}} \int_0^r M_S^t(u)(x) t^{2\gamma+d-1} dt. \tag{1.50}$$

*where $M_S^t(u)(x)$ is the spherical mean value at $(x,t)$ defined by formula (1.10).*

*Proof:* 1) Formula (1.49) has already been proved in (1.46).
2) We deduce (1.50) from (1.49) and integration in spherical coordinates. $\square$

In the following, we prove a Gauss type formula which gives a relation between a function $u$ and its volume and spherical value means. Recall first Green's formula associated to the Dunkl-Laplacian operator, given in [10].

**Proposition 1.11** *If $\Omega \subset \mathbb{R}^d$ is a $W$-invariant regular open set and if $u, v \in \mathcal{C}^2(\overline{\Omega})$, then*

$$\int_\Omega (u\Delta_k v - v\Delta_k u)\omega_k(x)dx = \int_{\partial\Omega} (u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta})(\xi)\omega_k(\xi)d\sigma_{\partial\Omega}(\xi), \tag{1.51}$$

*where $\eta$ is the outer unit normal to the surface $\partial\Omega$ and $d\sigma_{\partial\Omega}$ is the surface measure on $\partial\Omega$.*

**Proposition 1.12** *Let $u \in \mathcal{C}^\infty(\mathbb{R}^d)$. Then, for all $r > 0$ and $x \in \mathbb{R}^d$, we have*

$$M_S^r(u)(x) = u(x) + \frac{1}{2\gamma + d} \int_0^r M_B^t(\Delta_k u)(x) t dt. \tag{1.52}$$

*Proof:* Let $t > 0$. Using (1.49), (1.25), (1.39) and the change of variables $y = tz$, we deduce that

$$\begin{aligned} M_B^t(\Delta_k u)(x) &= \frac{1}{m_k(B(0,t))}\int_{B(0,t)} \Delta_k[\tau_x u](y)\omega_k(y)dy \\ &= \frac{2\gamma+d}{d_k}\int_{B(0,1)} \Delta_k[\tau_x u](tz)\omega_k(z)dz, \end{aligned}$$

where $d_k$ is the constant given in formula (1.11). But, from the homogeneity property of the Dunkl-Laplacian operator

$$[\Delta_k f](rx) = \frac{1}{r^2}\Delta_k[f(r.)](x), \quad (r > 0,\ f \in \mathcal{C}^2(\mathbb{R}^d)), \tag{1.53}$$

and Green's formula (1.51), we have

$$\begin{aligned} M_B^t(\Delta_k u)(x) &= \frac{2\gamma+d}{d_k t^2}\int_{B(0,1)} \Delta_k[\tau_x u(t.)](z)\omega_k(z)dz \\ &= \frac{2\gamma+d}{d_k t^2}\int_{S^{d-1}} \frac{\partial}{\partial\eta}[\tau_x u(t.)](\xi)\omega_k(\xi)d\sigma(\xi) \\ &= \frac{2\gamma+d}{d_k t^2}\int_{S^{d-1}} \langle\nabla[\tau_x u(t.)](\xi),\xi\rangle\,\omega_k(\xi)d\sigma(\xi). \end{aligned}$$

Now, by the classical relations

$$\nabla[f(t.)](x) = t[\nabla f](tx) \ \ \textit{and} \ \ \langle\nabla f(tx), x\rangle = \frac{d}{dt}[f(tx)],$$

we can write

$$M_B^t(\Delta_k u)(x) = \frac{2\gamma+d}{d_k t}\int_{S^{d-1}} \frac{d}{dt}[\tau_x u(t\xi)]\omega_k(\xi)d\sigma(\xi)$$

Finally, using Fubini's theorem and relation (1.22), we deduce that

$$\begin{aligned} \int_0^r M_B^t(\Delta_k u)(x)tdt &= \frac{2\gamma+d}{d_k}\int_{S^{d-1}}\int_0^r \frac{d}{dt}\tau_x u(t\xi)dt\omega_k(\xi)d\sigma(\xi) \\ &= \frac{2\gamma+d}{d_k}\int_{S^{d-1}} \big[\tau_x u(r\xi) - \tau_x u(0)\big]\omega_k(\xi)d\sigma(\xi) \\ &= (2\gamma+d)\big(M_S^r(u)(x) - u(x)\big). \end{aligned}$$

□

Now, we can give another proof of the spherical mean value property theorem for D-harmonic functions when $\Omega = \mathbb{R}^d$ (see [10]).

**Corollary 1.13** *Let $u \in \mathcal{C}^\infty(\mathbb{R}^d)$. Then, $u$ is D-harmonic if and only if for all $x \in \mathbb{R}^d$ and $r > 0$ we have*

$$u(x) = M_S^r(u)(x).$$

*Proof:* By the relation (1.52) it is enough to prove that if $u$ has the spherical mean value property, then $u$ is D-harmonic.
Fix $x \in \mathbb{R}^d$. Using relation (1.52) and differentiating with respect to $r$, we obtain:

$$\forall\ r > 0,\ M_B^r(\Delta_k u)(x) = 0.$$

Using the relation (1.49), the fact that the sequence of measures

$$\mu_r(dy) := \frac{1}{m_k(B(0,r))}\mathbf{1}_{B(0,r)}(y)\omega_k(y)dy \quad (r > 0)$$

is an approximate identity when $r \longrightarrow 0$ and letting $r \longrightarrow 0$, we deduce from (1.22) and (1.25) that

$$\tau_x \Delta_k u(0) = \Delta_k u(x) = 0.$$

This completes the proof. □

**Corollary 1.14** *Let $u \in \mathcal{C}^\infty(\mathbb{R}^d)$. Then, for all $x \in \mathbb{R}^d$ and $R > 0$, we have*

$$M_B^R(u)(x) = u(x) + \frac{1}{R^{2\gamma+d}} \int_0^R \int_0^r M_B^t(\Delta_k u)(x)\, t\, dt\, r^{2\gamma+d-1} dr. \tag{1.54}$$

*Proof:* In formula (1.50) we replace $M_S^t(u)(x)$ by its value given in formula (1.52) and we obtain the result. □

We will now study the volume mean value of a function defined on a $W$-invariant open subset of $\mathbb{R}^d$. We begin by a result we will need in the sequel :

**Lemma 1.15** *Let $f$ be a $\mathcal{C}^2$-function on an open set $\Omega \subset \mathbb{R}^d$ and let $K$ be a compact subset of $\Omega$. Then, there exist a sequence $(p_n)$ of polynomial functions such that for all $i, j = 1, ..., d$, $p_n \longrightarrow f$, $\partial_i p_n \longrightarrow \partial_i f$ and $\partial_i \partial_j p_n \longrightarrow \partial_i \partial_j f$ as $n \to +\infty$, uniformly on $K$.*

*Proof:* This result is more or less known. For lack of reference, we give a proof.
**First case**: Let $Q_n$ be defined by

$$Q_n(x) = (1 - \|x\|^2)^n \quad \text{if} \quad x \in B(0,1),$$

and $Q_n(x) = 0$ if $\|x\| > 1$. The sequence of functions $(\phi_n)$ defined on $\mathbb{R}^d$ by $\phi_n(x) = \frac{1}{a_n} Q_n(x)$, where $a_n = \int_{B(0,1)} Q_n(x)dx$, is an approximate identity as $n \to +\infty$.
Let $f$ be a $\mathcal{C}^2$-function on $B(0, 1/2)$. Then, the functions defined by

$$p_n(x) = f_n * \phi_n(x) = \frac{1}{a_n} \int_{B(0,1/2)} (1 - \|x - y\|^2)^n f(y)dy$$

are polynomial functions on $B(0, 1/2)$ and for all $i, j = 1, ..., d$, they clearly satisfy $p_n \longrightarrow f$, $\partial_i p_n \longrightarrow \partial_i f$ and $\partial_i \partial_j p_n \longrightarrow \partial_i \partial_j f$ uniformly on $B(0, 1/2)$ as $n \longrightarrow +\infty$.

**General case**: Let $f$ and $K$ as in the Lemma 1.15. We can find bounded open neighborhoods $O^1$ and $O^2$ of $K$ such that

$$K \subset O^2 \subset \overline{O^2} \subset O^1 \subset \overline{O^1} \subset \Omega,$$

where $\overline{O^i}$ $(i = 1, 2)$ is the compact closure of $O^i$.
Clearly, there exists $t > 0$ such that $\overline{O^1_t} \subset B(0, 1/2)$, where for a set $E \subset \mathbb{R}^d$, we denote by $E_t := \{tx, \quad x \in E\}$ the image of $E$ by the dilation $x \mapsto tx$. In particular, we have

$$K_t \subset O^2_t \subset \overline{O^2_t} \subset O^1_t.$$

Now, define the function $\delta_t f$ on $O^1_t$ by $\delta_t f(x) := f(t^{-1}x)$ and let $g$ be a $C^2-$function on $\mathbb{R}^d$ such that

$$g = 1, \text{ on } \overline{O^2_t} \text{ and } \textit{supp } g \subset O^1_t.$$

Then, we can see that the function $(\delta_t f)g$ is of class $C^2$ on $O^1_t$ and is extendable to a $C^2$-function on $\mathbb{R}^d$ by taking the value 0 in $\mathbb{R}^d \setminus O^1_t$. We will denote it also by $(\delta_t f)g$.
Moreover, for every $i, j = 1, ..., d$, we have

$$\partial_j[(\delta_t f)g] = \partial_j(\delta_t f) \text{ and } \partial_i\partial_j[(\delta_t f)g] = \partial_i\partial_j(\delta_t f), \text{ on } O^2_t \supset K_t. \tag{1.55}$$

By the first case, there exists a sequence of polynomial functions $(p_n)$ such that $p_n \longrightarrow (\delta_t f)g$, $\partial_j p_n \longrightarrow \partial_j[(\delta_t f)g]$ and $\partial_i\partial_j p_n \longrightarrow \partial_i\partial_j[(\delta_t f)g]$ uniformly on $B(0, 1/2)$.
Consequently, from (1.55), we deduce that $p_n \longrightarrow \delta_t f$, $\partial_j p_n \longrightarrow \partial_j(\delta_t f)$ and $\partial_i\partial_j p_n \longrightarrow \partial_i\partial_j(\delta_t f)$ uniformly on $K_t$. This implies

$$\sup_{x\in K} |f(x) - (\delta_{t^{-1}} p_n)(x)| = \sup_{\xi\in K_t} |(\delta_t f)(\xi) - p_n(\xi)| \longrightarrow 0, \; \textit{as } n \longrightarrow +\infty.$$

Furthermore, as $\partial_j(\delta_t f)(x) = t^{-1}[\delta_t(\partial_j f)](x)$, we can see that

$$\sup_{x\in K} \left|\partial_j f(x) - \partial_j[(\delta_{t^{-1}} p_n)(x)]\right| = t \sup_{\xi\in K_t} |\partial_j(\delta_t f)(\xi) - \partial p_n(\xi)| \longrightarrow 0, \; \textit{as } n \longrightarrow +\infty.$$

By the same way, we show that $\partial_i\partial_j[\delta_{t^{-1}} p_n] \longrightarrow \partial_i\partial_j f$ uniformly on $K$. As $\delta_{t^{-1}} p_n$ is a polynomial function, this completes the proof of the Lemma. □

**Theorem 1.16** *Let $u \in \mathcal{C}^2(\Omega)$. Then, for all $x \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \subset \Omega$, we have*

$$\forall\, 0 < R \le \rho/3, \quad M^R_B(u)(x) = u(x) + \frac{1}{R^{2\gamma+d}} \int_0^R \int_0^r M^t_B(\Delta_k u)(x)\, t\, dt\; r^{2\gamma+d-1} dr. \tag{1.56}$$

*Proof:* Let $u \in \mathcal{C}^2(\Omega)$. Fix $x, \rho$ and $R$ as in the Theorem 1.16. We take a sequence $(p_n)$ approximating $u$ up to the second derivatives as in the Lemma 1.15 for the compact set $K_1 = \bigcup_{g\in W} B(gx, \rho)$. We will use the following crucial approximation result:

**Lemma 1.17** *We have $\Delta_k p_n \longrightarrow \Delta_k u$ as $n \longrightarrow +\infty$ uniformly on the compact set $K_2 = \bigcup_{g\in W} B(gx, R)$.*

Assume the result of the lemma for the moment. By the Corollary 1.14, we have for all $n$

$$M_B^R(p_n)(x) = p_n(x) + \frac{1}{R^{2\gamma+d}} \int_0^R \int_0^r M_B^t(\Delta_k p_n)(x)\; t\; dt\; r^{2\gamma+d-1} dr. \tag{1.57}$$

By the compactness of the support of the harmonic kernel (1.35) we deduce that $|M_B^R(p_n - u)(x)| \leq \sup_{y\in K_2} |(p_n - u)(y)|$ and

$$\Big| \int_0^R \int_0^r M_B^t(\Delta_k(p_n - u))(x) t dt r^{2\gamma+d-1} dr \Big| \leq \frac{R^{2\gamma+d+2}}{2(2\gamma+d+2)} \sup_{y\in K_2} |\Delta_k(p_n - u)(y)|$$

Using these inequalities, Lemma 1.17 and letting $n \longrightarrow +\infty$ in (1.57) the result of the theorem follows. $\square$

*Proof of lemma 1.17:* For all $f \in \mathcal{C}^2(\Omega)$, we put

$$\delta_\alpha(f)(x) = \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2}. \tag{1.58}$$

Denote $f_n = u - p_n$. From (1.9) and Lemma 1.15, it is enough to prove that

$$\sum_{\alpha\in R_+} k(\alpha)\delta_\alpha(f_n) \longrightarrow 0$$

as $n \longrightarrow +\infty$ uniformly on $K_2$. We have

$$\sup_{y\in K_2} \Big| \sum_{\alpha\in R_+} k(\alpha)\delta_\alpha(f_n)(y) \Big| \leq \sum_{g\in W} \sum_{\alpha\in R_+} k(\alpha) \sup_{y\in B(gx,R)} |\delta_\alpha(f_n)(y)|. \tag{1.59}$$

Now, fix $g \in W$ and $\alpha \in R_+$. We will distinguish two cases:

*First case:* Suppose that $B(gx, R) \cap H_\alpha = \emptyset$.

Using the relation (1.58) and the Cauchy-Schwarz inequality, we deduce that for all $y \in B(gx, R)$

$$\begin{aligned} |\delta_\alpha(f_n)(y)| &\leq \Big| \frac{\langle \nabla f_n(y), \alpha \rangle}{\langle \alpha, y \rangle} - \frac{f_n(y) - f_n(\sigma_\alpha(y))}{\langle \alpha, y \rangle^2} \Big| \\ &\leq \frac{2\|\nabla f_n(y)\|}{\epsilon} + \frac{|f_n(y)| + |f_n(\sigma_\alpha(y))|}{\epsilon^2} \end{aligned}$$

where $\epsilon = \inf_{y\in B(gx,R)} |\langle \alpha, y \rangle| > 0$.

Using Lemma 1.15 and the fact that $K_2$ is $W$-invariant, we deduce that the second side in the previous relation converges to zero as $n \longrightarrow +\infty$. Thus

$$\sup_{y\in B(gx,R)} |\delta_\alpha(f_n)(y)| \longrightarrow 0 \quad as\; n \longrightarrow +\infty. \tag{1.60}$$

*Second case:* Suppose that $B(gx, R) \cap H_\alpha \neq \emptyset$. We denote by $x_{g,\alpha}$ the orthogonal projection of $gx$ onto $H_\alpha$. Then, we can see that

$$B(gx, R) \subset B(x_{g,\alpha}, 2R) \subset B(gx, \rho) \subset K_1.$$

From these inclusions, we deduce that

$$\sup_{y\in B(gx,R)} |\delta_\alpha(f_n)(y)| \leq \sup_{y\in B(x_{g,\alpha},2R)} |\delta_\alpha(f_n)(y)|. \tag{1.61}$$

Moreover, we have $\sigma_\alpha(y) \in B(x_{g,\alpha}, 2R)$ for $y \in B(x_{g,\alpha}, 2R)$. Thus, by Taylor's formula we can see that

$$\forall\, y \in B(x_{g,\alpha}, 2R), \quad \delta_\alpha(f_n)(y) = {}^t\alpha D^2 f_n(\xi)\alpha, \tag{1.62}$$

for some $\xi$ on the line segment between $y$ and $\sigma_\alpha(y)$, where $D^2 f_n(\xi)$ is the Hessian matrix of $f$ evaluated at point $\xi$. Using Lemma 1.15, the relations (1.61) and (1.62), we obtain

$$\sup_{y\in B(gx,R)} |\delta_\alpha(f_n)(y)| \longrightarrow 0 \quad as\; n \longrightarrow +\infty. \tag{1.63}$$

The relations (1.59), (1.60) and (1.63) show the desired result. This completes the proof of Lemma 1.17. □

In the following result, we prove the volume mean value theorem.

**Theorem 1.18** *Let $\Omega \subset \mathbb{R}^d$ be an open and $W$-invariant set and $u \in \mathcal{C}^2(\Omega)$. Then, $u$ is D-harmonic in $\Omega$ if and only if $u$ has the mean value property i.e for all $x \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \subset \Omega$, we have:*

$$\forall\, 0 < R \leq \rho/3, \quad u(x) = M_B^R(u)(x). \tag{1.64}$$

*Proof:* The relation (1.56) proves that if $u$ is D-harmonic on $\Omega$ then $u$ satisfies (1.64). Now, we suppose that $u$ satisfies the mean value property. Let $B(x, \rho) \subset \Omega$. From (1.56), we have

$$\forall\, 0 < R \leq \rho/3, \quad \int_0^R \int_0^r M_B^t(\Delta_k u)(x)\, t\, dt\, r^{2\gamma+d-1} dr = 0.$$

Differentiating two times with respect to $R$ , we deduce that

$$\forall\; R > 0,\; M_B^R(\Delta_k u)(x) = \frac{1}{m_k(B(0,R))} \int_{\mathbb{R}^d} \Delta_k u(y) h(R,x,y)\omega_k(y) dy = 0.$$

Finally by letting $R \to 0$ and using Proposition 1.7, we get $\Delta_k u(x) = 0$. □

**Exemple 1.19** *We know from [6] that, for $d = 2$ and for the root system considered in Example 1.4.4), the following polynomials*

$$P_n(x) := r^n C_n^{(k_2,k_1)}(\cos\theta), \quad x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2,\; n \in \mathbb{N} \setminus \{0\},$$

*where $C_n^{(\lambda,\mu)}$, $n \in \mathbb{N}$, are the generalized Gegenbauer polynomials of index $(\lambda, \mu)$ (with $\lambda, \mu \geq 0$)(see [6] p. 26), are D-harmonic on $\mathbb{R}^2$. Then, by using the mean value property (1.64), for arbitrary fixed $R > 0$, we can write*

$$P_n(x) = \frac{1}{m_k(B(0,R))} \int_{\mathbb{R}^2} h_{k_1,k_2}(R,x,y) P_n(y) |y_1|^{2k_1} |y_2|^{2k_2} dy_1 dy_2$$

*and using polar coordinates and the relations (1.34) and (1.39), we obtain*

$$r^n C_n^{(k_2,k_1)}(\cos\theta) = \int_0^\infty \int_0^{2\pi} H_{k_1,k_2}(R,r,\theta,\rho,\phi)\rho^n C_n^{(k_2,k_1)}(\cos\phi)d\rho d\phi, \qquad (1.65)$$

*where*

$$H_{k_1,k_2}(R,r,\theta,\rho,\phi) := \frac{2+2k_1+2k_2}{d_k R^{2k_1+2k_2+2}}\rho^{2k_1+2k_2+1}|\cos\phi|^{2k_1}|\sin\phi|^{2k_2}\times$$
$$\int_{[-1,1]^2} \mathbf{1}_{[0,R]}\left(\sqrt{r^2+\rho^2-2t_1r\rho\cos\theta\cos\phi-2t_2r\rho\sin\theta\sin\phi}\right)\phi_{k_1}(t_1)\phi_{k_2}(t_2)dt_1dt_2 \qquad (1.66)$$

*and $d_k = d_{k_1,k_2}$ is the constant (1.11) associated to the Coxeter-Weyl group $\mathbf{Z}_2^2$. Note that for any $(r,\theta)$, the function $(\rho,\phi)\mapsto H_{k_1,k_2}(R,r,\theta,\rho,\phi)$ is a probability density function with compact support contained in $[0,2\pi]\times[0,R+r]$.*

The mean value theorem implies immediately the following result:

**Corollary 1.20** *If $(u_n)$ is a sequence of D-harmonic functions on $\Omega$ (a $W$-invariant open set of $\mathbb{R}^d$) such that $(u_n)$ converge uniformly to a function $u$ on each compact subset of $\Omega$, then $u$ is D-harmonic on $\Omega$.*

As another application of the mean value theorem we show the Liouville's theorem for non negative Dunkl harmonic functions on all $\mathbb{R}^d$.

**Corollary 1.21** *If $u$ is a D-harmonic and bounded from below on $\mathbb{R}^d$, then $u$ is a constant.*

*Proof:* By eventually adding a constant, we can suppose $u \geq 0$ on $\mathbb{R}^d$. By Theorem 1.18, we have for all $x\in\mathbb{R}^d$ and $R>0$

$$u(x) = \frac{1}{m_k(B(0,R))}\int_{\mathbb{R}^d} u(y)h_k(R,x,y)\omega_k(y)dy.$$

Fix $R$ and $x$ such that $R > \|x\|$ and let $y \in \operatorname{supp} h_k(R,x,.)$. From (1.35), $y\in B(gx,R)$, for some $g\in W$. In particular $y\in B(0,2R)$.
As $0\leq h_k(R,x,y)\leq 1$, we have

$$h_k(R,x,y)\leq 1_{B(0,2R)}(y) = h_k(2R,0,y).$$

Thus, using Theorem 1.18 and formula (1.39), we deduce that

$$0\leq u(x)\leq \frac{m_k(B(0,2R))}{m_k(B(0,R))}u(0) = 2^{2\gamma+d}u(0).$$

That is, $u$ is bounded. Then, the classical Liouville's theorem (see [8]) proves that $u$ is a constant. □

**Remark 1.22** *Let $r > 0$ and $x \in \mathbb{R}^d$. In ([16]), M. Rösler has proved that there exists a compactly supported probability measure $\sigma^k_{x,r}$ on $\mathbb{R}^d$ which represents the spherical mean operator. More precisely, for $u \in \mathcal{C}^\infty(\mathbb{R}^d)$, we have*

$$M^r_S(u)(x) = \int_{\mathbb{R}^d} u(y) d\sigma^k_{x,r}(y), \tag{1.67}$$

*with*

$$supp\sigma^k_{x,r} \subset \cup_{g\in W} B(gx, r). \tag{1.68}$$

*Then, using the relations (1.67), (1.68) and the Lemma 1.17, the relation (1.52) can be extended by the same way to a function of class $C^2$ on an arbitrary open and $W$-invariant set $\Omega \subset \mathbb{R}^d$ and we obtain the analogue of Theorem 1.18 where the volume mean $M^r_B(u)$ is replaced by the spherical mean $M^r_S(u)$.*
*Moreover, the relation (1.52) shows that the action of the measure $\sigma^k_{x,r} - \delta_x$ on a function $f \in C^2(\Omega)$ is given by*

$$\langle \sigma^k_{x,r} - \delta_x, f\rangle = \frac{1}{2\gamma + d} \int_{\mathbb{R}^d} \int_0^r \widetilde{h}_k(t, x, y) t dt \Delta_k f(y) dy, \tag{1.69}$$

*where $\delta_x$ is the Dirac measure at $x$ and*

$$\widetilde{h}_k(t, x, y) = \frac{1}{m_k(B(0, t))} h_k(t, x, y)\omega_k(y).$$

## 1.4 Harnack's inequality and the strong maximum principle

In this section, we will prove the strong maximum principle and Harnack's inequality for D-harmonic functions. Throughout the section, $\Omega$ will always denote a $W$-invariant open subset of $\mathbb{R}^d$ and we will denote by:
$\mathcal{H}^D_+(\Omega)$ the set of D-harmonic and positive functions on $\Omega$.

**Lemma 1.23** *Let $r > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\|x_1 - x_2\| \leq 2r$. Then,*

$$\forall\, y \in \mathbb{R}^d, \quad h_k(r, x_2, y) \leq h_k(r\sqrt{10}, x_1, y). \tag{1.70}$$

*Proof:* Let $y \in \mathbb{R}^d$ and $z \in \text{supp}\, \mu_y$. Using (1.31), it suffices to show that

$$1_{[0,r^2]}(\|x_2\|^2 + \|y\|^2 - 2\langle x_2, z\rangle) \leq 1_{[0,10r^2]}(\|x_1\|^2 + \|y\|^2 - 2\langle x_1, z\rangle). \tag{1.71}$$

From (1.41) and (1.42), we have

$$\begin{aligned}
\|x_1\|^2 + \|y\|^2 - 2\langle x_1, z\rangle &= \textstyle\sum_{g\in W} \lambda_g(z) \|g^{-1}x_1 - y\|^2 \\
&\leq \textstyle\sum_{g\in W} \lambda_g(z) \Big(\|g^{-1}x_1 - g^{-1}x_2\| + \|g^{-1}x_2 - y\|\Big)^2 \\
&\leq 2\|x_1 - x_2\|^2 + 2\textstyle\sum_{g\in W} \lambda_g(z) \|g^{-1}x_2 - y\|^2 \\
&\leq 8r^2 + 2(\|x_2\|^2 + \|y\|^2 - 2\langle x_2, z\rangle)
\end{aligned}$$

This implies that the inequality (1.71) holds. □

**Lemma 1.24** *Let $x \in \Omega$ and $r > 0$ such that $B(x, 13r) \subset \Omega$. Then, there exists a constant $C \geq 1$ such that the inequality*

$$u(x_2) \leq Cu(x_1), \tag{1.72}$$

*holds for all $x_1, x_2 \in B(x, r)$ and for all nonnegative and D-harmonic functions in $\Omega$.*

*Proof:* We fix $u \geq 0$ D-harmonic in $\Omega$. Applying Lemma 1.23 for $x_1, x_2 \in B(x, r)$ and using the property 2 of the Proposition1.5, we see that

$$\begin{aligned}\int_{\mathbb{R}^d} u(y) h_k(r, x_2, y) \omega_k(y) dy &\leq \int_{\mathbb{R}^d} u(y) h_k(r\sqrt{10}, x_1, y) \omega_k(y) dy \\ &\leq \int_{\mathbb{R}^d} u(y) h_k(4r, x_1, y) \omega_k(y) dy.\end{aligned}$$

Now, as the two balls $B(x_1, 12r)$ and $B(x_2, 3r)$ are in $\Omega$, we can apply the volume mean value Theorem 1.18 and use 1.39 to obtain

$$u(x_2) \leq \frac{m_k(B(0, 4r))}{m_k(B(0, r))} u(x_1) = 4^{2\gamma + d} u(x_1).$$

□

In the following result, we extend the strong maximum principle to D-harmonic functions.

**Theorem 1.25** *Suppose that $\Omega$ is connected. Let $u$ be a D-harmonic function on $\Omega$. If $u$ has a maximum in $\Omega$, then $u$ is constant.*

*Proof:* Let $M := \max_\Omega u(x)$, $v := M - u$ and $F := \{x \in \Omega : \; v(x) = 0\}$.
It is clear that $F$ is a nonempty closed set in $\Omega$. Let $x \in F$ and $r > 0$ such that $B(x, 13r) \subset \Omega$. Since the function $v$ is nonnegative and D-harmonic in $\Omega$, we can apply Lemma 1.24 to obtain

$$0 \leq v(a) \leq Cv(x) = 0,$$

for all $a \in B(x, r)$. That is $B(x, r) \subset F$ and $F$ is an open set in $\Omega$. By connectivity, $F$ must coincide with $\Omega$. Then, $u$ is constant as asserted. □

**Remark 1.26**

1. *If we replace $u$ by $-u$, we obtain the strong minimum principle for D-harmonic functions.*
2. *Clearly Theorem 1.25 implies the weak maximum principle obtained by Rösler : if $\Omega$ is bounded and $u$ is D-harmonic in $\Omega$ and continuous on $\bar{\Omega}$, then* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ *(see [14] p. 533).*
3. *A particular case of the strong maximum principle was obtained by Dunkl for D-harmonic polynomials on the unit ball of $\mathbb{R}^d$ (see [2]).*

Now, we will show the second main result in this section. First, we will establish the following lemma:

**Lemma 1.27** *Suppose that $\Omega$ is connected. Then, for any finite set $E \subset \Omega$, there exists a constant $C_E \geq 1$ such that the inequality:*

$$u(x) \leq C_E u(y), \tag{1.73}$$

*holds for all $x, y \in E$ and $u \in \mathcal{H}_+^D(\Omega)$.*

*Proof:* For $x, y \in \Omega$, define the function

$$\beta(x,y) := \sup\Big\{\frac{u(x)}{u(y)} : \quad u \in \mathcal{H}_+^D(\Omega)\Big\}.$$

We fix $x_0 \in \Omega$ and we put

$$\Omega_{x_0} := \big\{y \in \Omega : \quad \beta(x_0, y) < +\infty\big\}.$$

It is clear that $x_0 \in \Omega_{x_0}$.
We will show that $\Omega_{x_0} = \Omega$. For this purpose, it is enough to prove that $\Omega_{x_0}$ is an open and closed set in $\Omega$.
• Let $y \in \Omega_{x_0}$ and $r > 0$ such that $B(y, 13r) \subset \Omega$. For any $u \in \mathcal{H}_+^D(\Omega)$ and $z \in B(y,r)$, by Lemma 1.24 with $x$ (resp. $x_1$, resp. $x_2$) replaced by $y$ (resp. $z$, resp. $y$), we have

$$u(y) \leq C u(z).$$

Thus, for all $z \in B(y,r)$, we have

$$\frac{u(x_0)}{u(z)} \leq C \frac{u(x_0)}{u(y)} \leq C\beta(x_0, y) < +\infty.$$

This shows that $\beta(x_0, z) < \infty$ for all $z \in B(y,r)$. Thus, $\Omega_{x_0}$ is an open set.
• Let $(y_n) \subset \Omega_{x_0}$ a sequence such that $y_n \longrightarrow y \in \Omega$ as $n \longrightarrow \infty$. Let $\varepsilon > 0$ such that $B(y, 13\varepsilon) \subset \Omega$. There exists $N \in \mathbb{N}$ such that $y_N \in B(y, \varepsilon)$. Again by Lemma 1.24 $\Big($with $x$ (resp. $x_1$, resp. $x_2$) replaced by $y$ (resp. $y$, resp. $y_N$)$\Big)$, we deduce that

$$\frac{u(x_0)}{u(y)} \leq C \frac{u(x_0)}{u(y_N)} \leq C\beta(x_0, y_N) < +\infty.$$

Thus, $y \in \Omega_{x_0}$ and $\Omega_{x_0}$ is a closed set.
Now, we take $C_E := \max\big\{\beta(x,y) : \quad (x,y) \in E^2\big\} < \infty$. Clearly $C_E \geq 1$ and for all $x, y \in E$ and $u \in \mathcal{H}_+^D(\Omega)$, we have

$$u(x) = \frac{u(x)}{u(y)} u(y) \leq C_E u(y).$$

This completes the proof. □

Now, we can prove the following Harnack's inequality:

**Theorem 1.28** *We suppose that $\Omega$ is connected. For each compact set $K \subset \Omega$, there exists a constant $C_K \geq 1$ such that the inequality*

$$\sup_K u \leq C_K \inf_K u. \tag{1.74}$$

*holds for all $u \in \mathcal{H}_+^D(\Omega)$.*

*Proof:* We have $K \subset \cup_{x\in K} B(x,r)$, where $0 < r < \frac{1}{13} d(K, \partial\Omega)$. By compactness, we can write

$$K \subset \cup_{i=1}^p B(x_i, r)$$

for some $x_1, ..., x_p \in K$.
By Lemma 1.24, for all $i = 1, ..., p$, there is a constant $C_i \geq 1$ such that

$$\forall\, y, z \in B(x_i, r), \quad u(y) \leq C_i u(z). \tag{1.75}$$

Now we take $C = \max_{1\leq i\leq p} C_i$ and $E = \{x_1, ..., x_p\}$.
Let $x, y \in K$. There exists $i, j$ such that $x \in B(x_i, r)$ and $y \in B(x_j, r)$. The relations (1.75) and (1.73) imply that:

$$u(x) \leq C_i u(x_i) \leq C u(x_i) \leq C C_E u(x_j) \leq C^2 C_E u(y).$$

Then the theorem is proved with $C_K = C^2 C_E$. □

**Corollary 1.29** *(*Harnack's principle*) Suppose that $\Omega$ is connected. Let $(u_n)_n$ be a pointwise increasing sequence of D-harmonic functions on $\Omega$. Then, either $(u_n)$ converges uniformly on compact subsets of $\Omega$ to a D-harmonic function or $u_n(x) \longrightarrow +\infty$ for all $x \in \Omega$.*

*Proof:* As in the classical case (see [1] p.50), the result follows using Harnack's inequality and Corollary 1.20. □

## 1.5 Bôcher's theorem

In this section $\overset{\circ}{B}$ denotes the open unit ball of $\mathbb{R}^d$.
The aim of this section is to prove the following version of Bôcher's theorem:

**Theorem 1.30** *Suppose $d \geq 2$ and let $u$ be a D-harmonic and positive function on $\overset{\circ}{B}\backslash\{0\}$. Then there exists a D-harmonic function $v$ on $\overset{\circ}{B}$ and a constant $a \geq 0$ such that*

$$\forall \;\; x \in \overset{\circ}{B}\backslash\{0\}, \quad u(x) = \begin{cases} a\ln(||x||^{-1}) + v(x) & \textit{if} \quad d = 2 \textit{ and } \gamma = 0 \\ a\|x\|^{2-2\gamma-d} + v(x) & \textit{if} \quad d \geq 3 \textit{ or if } d = 2 \textit{ and } \gamma > 0. \end{cases}$$

**Remark 1.31** *When $d \geq 2$, the previous result implies that if $u$ is D-harmonic and positive on $\overset{\circ}{B}\backslash\{0\}$, then, either $u$ can be extended to a harmonic function on the ball $\overset{\circ}{B}$ or* $\lim_{x\to 0} u(x) = +\infty$. *In other words a singularity at $x = 0$ of a non negative and bounded D-harmonic function is always removable. But if $d = 1$ we will see at the end of this section that the situation is quite different.*

The case $d = 2$ and $\gamma = 0$ is the classical Bôcher's theorem in the two dimensional Euclidean space. We will then suppose that $d \geq 2$ and $d + 2\gamma > 2$. The idea is to adapt the scheme of the classical proof given by S. Axler, P. Bourdon and W. Ramey (see [1]) to the situation of D-harmonic functions. For our purpose, we introduce the following definition:

**Definition 1.32** *Let $u$ be a continuous function on $\overset{\circ}{B}\backslash\{0\}$. Define the Dunkl-average of $u$ over the sphere of radius $\|x\|$ by*

$$A[u](x) := \frac{1}{d_k}\int_{S^{d-1}} u(\|x\|\xi)\omega_k(\xi)d\sigma(\xi), \quad x \in \overset{\circ}{B}\backslash\{0\}. \tag{1.76}$$

**Lemma 1.33** *Suppose $d \geq 2$ and $d + 2\gamma > 2$ and let $u$ be a D-harmonic function on $\overset{\circ}{B}\backslash\{0\}$. Then, there are real constants $a$ and $b$ such that*

$$\forall \;\; x \in \overset{\circ}{B}\backslash\{0\}, \quad A[u](x) = a\|x\|^{2-2\gamma-d} + b. \tag{1.77}$$

*In particular, $A[u]$ is D-harmonic in $\overset{\circ}{B}\backslash\{0\}$.*

*Proof:* Define the function $f$ on $]0, 1[$ by

$$f(r) = \frac{1}{d_k}\int_{S^{d-1}} u(r\xi)\omega_k(\xi)d\sigma(\xi).$$

As $u$ is continuously differentiable on $\overset{\circ}{B}\backslash\{0\}$, we can differentiate under the integral sign and we obtain

$$f'(r) = \frac{1}{d_k}\int_{S^{d-1}} \langle\nabla u(r\xi), \xi\rangle\, \omega_k(\xi)d\sigma(\xi) = \frac{r^{-(2\gamma+d)}}{d_k}\int_{S(0,r)} \langle\nabla u(\xi), \xi\rangle\, \omega_k(\xi)d\sigma_r(\xi),$$

where $d\sigma_r$ is the surface measure of the sphere $S(0, r)$ given by $d\sigma_r = r^{d-1}(\varphi_r d\sigma)$ and $\varphi_r d\sigma$ is the image measure of $d\sigma (= d\sigma_1)$ by the dilation $\varphi_r : \xi \longmapsto r\xi$.
We put

$$g(r) = \frac{1}{d_k}\int_{S(0,r)} \langle\nabla u(\xi), \frac{\xi}{r}\rangle\, \omega_k(\xi)d\sigma_r(\xi).$$

We see that $g(r) = r^{2\gamma+d-1}f'(r)$. Then, it suffices to prove that g is constant on $]0, 1[$. For this purpose, we introduce the open set $\Omega = \{r_1 < \|y\| < r_2\}$, where $0 < r_1 < r_2 < 1$. Using the Green formula (1.51) and the fact that $u$ is D-harmonic, we deduce that

$$0 = \int_\Omega \Delta_k u(y)\omega_k(y)dy = \int_{\partial\Omega} \frac{\partial u}{\partial\eta}(\xi)\omega_k(\xi)d\sigma_{\partial\Omega}(\xi);$$

where $\eta$ denotes the outward unit normal on $\partial\Omega$. The above equation implies that

$$\int_{S(0,r_1)} \langle\nabla u(\xi), \eta\rangle\, \omega_k(\xi)d\sigma_{r_1}(\xi) = \int_{S(0,r_2)} \langle\nabla u(\xi), \eta\rangle\, \omega_k(\xi)d\sigma_{r_2}(\xi).$$

But, $\eta = \frac{\xi}{r_1}$ on $S(0, r_1)$ and $\eta = \frac{\xi}{r_2}$ on $S(0, r_2)$. Then, $g(r_1) = g(r_2)$, for all $r_1, r_2$ in $]0, 1[$. This shows the relation (1.77).
Finally, we note that the function $x \longmapsto \|x\|^{2-2\gamma-d}$ is D-harmonic on $\mathbb{R}^d \backslash \{0\}$ using the fact that if $f$ is a radial function i.e $f(x) = F(r), r = \|x\|$, then $\Delta_k f(x) = L_{\gamma+\frac{d}{2}-1} F(r)$, with

$$L_{\gamma+\frac{d}{2}-1} F(r) = \frac{d^2}{dr^2} + \frac{2\gamma + d - 1}{r} \frac{d}{dr},$$

is the Bessel operator of order $\gamma + \frac{d}{2} - 1$ (see [10]). $\square$

**Lemma 1.34** *There exists a positive constant $\alpha \in ]0, 1[$ such that for every positive D-harmonic function on $\overset{\circ}{B} \backslash \{0\}$,*

$$\alpha u(y) < u(x), \quad \textit{whenever} \quad 0 < \|x\| = \|y\| \leq 1/2.$$

*Proof:* By Theorem 1.28, there exists a constant $\beta \in ]0, 1]$ such that

$$\beta u(y) \leq u(x)$$

for all positive D-harmonic functions on $\overset{\circ}{B} \backslash \{0\}$ and $\|x\| = \|y\| = \frac{1}{2}$.
For $r \in ]0, 1[$, we define the function $u_r(x) := u(rx)$. From (1.53), we see that $u_r$ is D-harmonic on $\overset{\circ}{B}(0, 1/r) \backslash \{0\}$. Applying the previous result to $u_r$, for all $r \in ]0, 1[$, we obtain $\beta u(y) \leq u(x)$ for all $x, y$ such that $\|x\| = \|y\| = \frac{r}{2}$. Taking $\alpha = \frac{\beta}{2}$, the result follows. $\square$

**Lemma 1.35** *Suppose $d \geq 2$ and $d + 2\gamma > 2$. Let $u$ be a D-harmonic and positive function on $\overset{\circ}{B} \backslash \{0\}$ such that $u(x) \longrightarrow 0$ as $\|x\| \longrightarrow 1$. Then, there is a constant $a$ such that:*

$$\forall\ x \in \overset{\circ}{B} \backslash \{0\},\ u(x) = a \|x\|^{2-2\gamma-d} - a.$$

*Proof:* By Lemma 1.33, it is enough to prove that $u = A[u]$ on $\overset{\circ}{B} \backslash \{0\}$.
• First, we will show that if $u \geq A[u]$ on $\overset{\circ}{B} \backslash \{0\}$, then $u = A[u]$ on $\overset{\circ}{B} \backslash \{0\}$.
Let $x \in \overset{\circ}{B} \backslash \{0\}$. As $A[u]$ is radial, we have $A[A[u]](x) = A[u](x)$. That is

$$\forall\ x \in \overset{\circ}{B} \backslash \{0\}, \quad \int_{S^{d-1}} \Big( u(\|x\|\xi) - A[u](\|x\|\xi) \Big) \omega_k(\xi) d\sigma(\xi) = 0.$$

As the function $\xi \longmapsto u(\|x\|\xi) - A[u](\|x\|\xi)$ is continuous and nonnegative on $S^{d-1}$, we deduce that

$$\forall\ \ \xi \in S^{d-1}, \quad u(\|x\|\xi) - A[u](\|x\|\xi) = 0.$$

Taking $\xi = \frac{x}{\|x\|}$, we obtain $u(x) = A[u](x)$.
• To prove $u \geq A[u]$ on $\overset{\circ}{B} \backslash \{0\}$, we will consider two steps.
**step1** We will prove that $u - \alpha A[u] > 0$ on $\overset{\circ}{B} \backslash \{0\}$, where $\alpha$ is the constant of Lemma 1.34. By Lemma 1.34, we have

$$\forall\ \ \xi \in S^{d-1}, \quad \alpha u(\|x\|\xi) < u(x),$$

for all $x$ such that $0 < \|x\| \leq 1/2$. Then,

$$\alpha A[u](x) - u(x) = -\frac{1}{d_k}\int_{S^{d-1}} \Big(u(x) - \alpha u(\|x\|\xi)\Big)\omega_k(\xi)d\sigma(\xi) < 0$$

for all $x$ such that $0 < \|x\| \leq 1/2$.
Moreover, because $u(x) \longrightarrow 0$ as $\|x\| \longrightarrow 1$, we have $A[u](x) \longrightarrow 0$ as $\|x\| \longrightarrow 1$. Thus, $\alpha A[u](x)-u(x) \longrightarrow 0$ as $\|x\| \longrightarrow 1$. Using the fact that $\alpha A[u]-u$ is D-harmonic on $\overset{\circ}{B}\backslash\{0\}$, the strong maximum principle ( Theorem 1.25) shows that $\alpha A[u]-u < 0$ on $\overset{\circ}{B}\backslash\{0\}$. That is, $u - \alpha A[u] > 0$ on $\overset{\circ}{B}\backslash\{0\}$ as desired.
**step2** We will show that if

$$w = u - tA[u] > 0 \tag{1.78}$$

for some $t \in ]0,1]$ then $u - A[u] \geq 0$ in $\overset{\circ}{B}\backslash\{0\}$. For this, we consider the function $\psi(t) = \alpha + t(1-\alpha)$, $t \in [0,1]$.
We have $w(x) \longrightarrow 0$ as $\|x\| \longrightarrow 1$. Then, by step 1, we have

$$w - \alpha A[w] = u - \psi(t)A[u] > 0, \quad \text{on } \overset{\circ}{B}\backslash\{0\}.$$

By induction, we deduce that

$$\forall\, n \in \mathbb{N}^*, \quad u - \psi^{(n)}(t)A[u] > 0, \quad \text{on } \overset{\circ}{B}\backslash\{0\}, \tag{1.79}$$

where $\psi^{(n)} = \psi \circ \psi \circ ... \circ \psi$ ($n$ times).
But, $\psi^{(n)}(t) = 1 - (1-\alpha)^n + t(1-\alpha)^n$. Then, $\psi^{(n)}(t) \longrightarrow 1$ as $n \longrightarrow \infty$ for all $t \in [0,1]$. Thus the relation (1.79) implies that $u - A[u] \geq 0$ in $\overset{\circ}{B}\backslash\{0\}$.
Since (1.78) holds when $t = \alpha$, we have $u - A[u] \geq 0$ in $\overset{\circ}{B}\backslash\{0\}$ and Lemma 1.35 follows. $\square$

*Proof of Theorem 1.30:* First, we suppose that $u$ is D-harmonic and positive on a neighborhood of $B(0,1)\backslash\{0\}$. For $x \in \overset{\circ}{B}\backslash\{0\}$, define

$$w(x) = u(x) - P[u_{|S^{d-1}}](x) + \|x\|^{2-2\gamma-d} - 1,$$

where,

$$P[u_{|S^{d-1}}](x) = \frac{1}{d_k}\int_{S^{d-1}} u(\xi)P(x,\xi)\omega_k(\xi)d\sigma(\xi)$$

is the Poisson integral of $u_{|S^{d-1}}$ (see [6] p. 189-190 and [12], Theorem A).
We have $w(x) \longrightarrow 0$ if $\|x\| \longrightarrow 1$ and as $P[u_{|S^{d-1}}]$ is bounded, $w(x) \longrightarrow \infty$ as $\|x\| \longrightarrow 0$. Then, by the strong minimum principle, the D-harmonic function $w$ is positive in $\overset{\circ}{B}\backslash\{0\}$. By the Lemma 1.35 we deduce that

$$\forall\, x \in \overset{\circ}{B}\backslash\{0\}, \quad w(x) = c\|x\|^{2-2\gamma-d} - c,$$

where $c$ is a constant. Thus,

$$\forall\ x \in \overset{\circ}{B}\backslash\{0\}, \quad u(x) = a\|x\|^{2-2\gamma-d} + v_1(x),$$

where $a = c+1 \geq 0$ (otherwise $u(x) \longrightarrow -\infty$ as $\|x\| \longrightarrow 0$ is on contradiction with the positivity of $u$ ) and $v_1 = P[u_{|S^{d-1}}](x) - c + 1$ is D-harmonic in $\overset{\circ}{B}$ (see [12], Theorem A).

Now, we suppose that $u$ is D-harmonic and positive in $\overset{\circ}{B}\backslash\{0\}$. We apply the above result to the function $u_{1/2}$ defined by

$$u_{1/2}(x) = u(x/2), \quad x \in \overset{\circ}{B}\backslash\{0\}.$$

We have

$$u(x/2) = a\|x\|^{2-2\gamma-d} + v_1(x), \quad x \in \overset{\circ}{B}\backslash\{0\}.$$

Thus

$$u(x) = a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d} + v_1(2x), \quad x \in \overset{\circ}{B}(0,1/2)\backslash\{0\}.$$

We define the function $v$ on $\overset{\circ}{B}$ by

$$v(x) = \begin{cases} v_1(2x), & \text{if } x \in \overset{\circ}{B}(0,1/2) \\ \\ u(x) - a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d}, & \text{if } x \in \overset{\circ}{B}\backslash\overset{\circ}{B}(0,1/2) \end{cases}$$

It is easy to see that $v$ is D-harmonic in $\overset{\circ}{B}$ and we have

$$u(x) = a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d} + v(x), \quad x \in \overset{\circ}{B}\backslash\{0\}.$$

□

**Corollary 1.36** *If $d \geq 2$ and if $u$ is a positive and D-harmonic function in $\mathbb{R}^d\backslash\{0\}$ then*

$$\forall\ \ x \in \mathbb{R}^d\backslash\{0\}, \quad u(x) = \begin{cases} a & \textit{if} \quad d = 2 \textit{ and } \gamma = 0 \\ a\|x\|^{2-2\gamma-d} + b & \textit{if} \quad d \geq 3 \textit{ or if } d = 2 \textit{ and } \gamma > 0. \end{cases}$$

*for some constants $a, b \geq 0$.*

*Proof:* The case $d = 2$ and $\gamma = 0$ is known ([1] p.46). Let's suppose $d \geq 2$ and $d + 2\gamma > 2$ and let $u$ be a positive and D-harmonic function in $\mathbb{R}^d\backslash\{0\}$. By Bôcher's theorem, we have

$$u(x) = a\|x\|^{2-2\gamma-d} + v(x), \quad x \in \overset{\circ}{B}\backslash\{0\},$$

where $a$ is a positive constant.
The function $v$ extends D-harmonically to all of $\mathbb{R}^d$ by setting

$$v(x) = u(x) - a\|x\|^{2-2\gamma-d}, \quad x \in \mathbb{R}^d\backslash\overset{\circ}{B}.$$

Using the minimum principle and the positivity of $u$, we obtain for all $r > 1$ and all $x \in \overset{\circ}{B}(0,r)$

$$v(x) \geq \min\left\{v(y), \quad \|y\| = r\right\} > -ar^{2-2\gamma-d}.$$

Letting $r \longrightarrow \infty$, we see that $v$ is non negative in $\mathbb{R}^d$. Then, by Liouville's theorem (Corollary 1.21), $v$ is constant.

$\square$

**Remark 1.37** *In the case $d = 1$, we have $\Delta_k f(x) = f''(x) + k\frac{f'(x)}{x} - k\frac{f(x)-f(-x)}{2x^2}$, where $k \geq 0$. If $k = 0$, the general solution of $\Delta_k f(x) = 0$ is $f(x) = ax + b$ where $a$ and $b$ are constants. If $k > 0$, $x = 0$ is a singularity for the difference-differential equation $\Delta_k f(x) = 0$. But by writing $f = f_e + f_o$ where $f_e$ (resp. $f_o$) is the even part (resp. the odd part) of $f$, the functions $f_e$ and $f_o$ satisfy ordinary second order differential equations singular at $x = 0$ but easily solvable and we can show that for $x \neq 0$, we have*

$$f(x) = \begin{cases} C_1 + C_2 x + C_3 sg(x)|x|^{-k} + C_4|x|^{1-k} & \text{if } k \neq 1 \\ C_1 + C_2 x + C_3 sg(x)|x|^{-1} + C_4 \ln(|x|) & \text{if } k = 1, \end{cases}$$

*where $C_i$ $(i = 1, \ldots, 4)$ are arbitrary constants and $sg(x) = 1$ (resp.$-1$) if $x > 0$ (resp. $x < 0$). This gives the explicit form of the singularities of $f$ at $x = 0$ and shows that if $f$ is bounded, the singularity $x = 0$ is removable if $k \geq 1$ but this is not true if $0 < k < 1$.*


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# Bibliography


[1] S. Axler, P. Bourdon and W. Ramey. Harmonic Function Theory, Springer Verlag , Second edition (2001).

[2] C. F. Dunkl. Reflection groups and orthogonal polynomials on the sphere. Math. Z. 197, (1988), 33-60.

[3] C. F. Dunkl. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc., 311, (1989), 167-183.

[4] C. F. Dunkl. Integral kernel with reflection group invariance. Canad. J. Math., 43, (1991), 123-183.

[5] C. F. Dunkl. Hankel transform associated to finite reflection groups. Contemp. Math., 138, (1992), 123-138.

[6] C. F. Dunkl and Y. Xu. Orthogonal Polynomials of Several variables, Cambridge Univ. Press 2001.

[7] P. Etingof. A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups. Math. Res. Lett. 17 (2010), no. 2, 275-282.

[8] L. Gallardo and L. Godefroy. Propriété de Liouville et équation de Poisson pour le laplacien généralisé de Dunkl, C. R. Acad. Sci. Paris, Ser.I 337 (2003), 639-644.

[9] J. E. Humphreys. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics 29, Cambridge University Press (1990).

[10] H. Mejjaoli and K. Trimèche . On a mean value property associated with the Dunkl Laplacian operator and applications. Integ. Transf. and Spec. Funct., 12(3), (2001), 279-302.

[11] M. F. de Jeu. The Dunkl transform. Invent. Math., 113, (1993), 147-162.

[12] M. Maslouhi and E. H. Youssfi. Harmonic functions associated to Dunkl operators. Monatsh. Math. 152(2007), 337-345.

[13] E. M. Opdam. Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. Compositio Math. 85 (1993), no. 3, 333-373.

[14] M. Rösler. Generalized Hermite polynomials and the heat equation for Dunkl operators. Comm. Math. Phys, 192, (1998), 519-542.

[15] M. Rösler. Positivity of Dunkl's intertwining operator. Duke Math. J., 98, (1999), 445-463.

[16] M. Rösler. A positive radial product formula for the Dunkl kernel. Trans. Amer. Math. Soc., 355, (2003), 2413-2438.

[17] M. Rösler. Dunkl Operators: Theory and Applications. Lecture Notes in Math., vol.1817, Springer Verlag (2003), 93-136.

[18] K. Trimèche. The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual. Integ. Transf. and Spec. Funct., 12(4), (2001), 394-374.

[19] K. Trimèche. Paley-Wiener theorem for the Dunkl transform and Dunkl translation operators. Integ. Transf. and Spec. Func. 13, (2002), 17-38.

[20] H. Volkmer. Generalized Ellipsoidal and Sphero-Conal Harmonics. SIGMA Symmetry Integrability Geom. Methods Appl. 2 (2006), Paper 071, 16 pp.

[21] H. Volkmer. External Ellipsoidal Harmonics for the Dunkl-Laplacian. SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), Paper 091, 13 pp.

# Chapter 2

# *Potentiel Newtonien et Fonctions Sousharmoniques Associés à l'Opérateur de Dunkl-Laplace*

**Résumé**

On considère le laplacien de Dunkl $\Delta_k$ associé à un système de racines de $\mathbb{R}^d$ et à une fonction de multiplicité positive $k$. L'objectif de ce papier est de présenter une théorie du potential newtonienne associée à l'opérateur $\Delta_k$. En particulier, on introduit et on étudie le noyau de Dunkl-Newton et le potentiel correspondant d'une mesure de Radon. Mais, auparavant on étudie les fonctions $\Delta_k$-sousharmoniques via le nouvel opérateur de moyenne introduit par les auteurs dans [16]. Comme applications, on donne les solutions faibles de l'équation de Dunkl-Poisson et on généralise le théorème de décomposition de Riesz aux fonctions $\Delta_k$-sousharmoniques. En outre, l'analogue du théorème de différentiation de Lebesgue dans le cas Dunkl, des nouvelles propriétés sur la mesure de Rösler et sur le produit de convolution de Dunkl sont les outils essentiels pour notre étude.

*Newtonian potentials and subharmonic functions associated to the Dunkl-Laplace operator*

***Submitted***

**Abstract**

The purpose of this paper is to present a natural potential theory of Newtonian type associated to an arbitrary Dunkl-Laplace operator $\Delta_k$ in $\mathbb{R}^d$ relative to a root system and a nonnegative multiplicity function $k$. In particular, we undertake a study of the Dunkl-Newton kernel and the corresponding potential of a Radon measure. But first of all we use a new mean value operator to study in some detail the $\Delta_k$-subharmonic functions for which we introduce the notion of Dunkl-Riesz measure. The paper contains also applications like a solution of the Poisson equation and a Riesz decomposition theorem for $\Delta_k$-subharmonic functions. Moreover, we need some new tools in Dunkl analysis which are essential to our study, like a Dunkl-Lebesgue differentiation theorem and new properties on the support of Rösler's measure and on convolution in Dunkl setting.

Key words: Dunkl-Laplace operator, Dunkl convolution product, Generalized volume mean value operator and harmonic kernel, Lebesgue's differentiation theorem, Rösler's measure, Dunkl harmonic and subharmonic functions, Strong maximum principle, Uniqueness principle, $\Delta_k$-Riesz measure, Weyl's lemma, Dunkl-Newton kernel and Dunkl-Newtonian potential, Riesz decomposition theorem.

## 2.1 Introduction

The quantum Calogero-Sutherland models ([5], [39], [40]) describe a system of $d$ particles on the line or the circle with pairwise interactions through a potential proportional to the inverse square of the distance between them. These models are characterized by complete integrability and exact solvability, the two decisive mathematical properties which have been one of the most important reasons for the attention paid to these models since their appearance in the beginning of the seventies.
Recently, a revival of interested has been devoted to the study of the Calogero-Sutherland model with spin. The Dunkl theory has been used as a crucial tool to investigate these models ([4], [12]). In particular, in the case $A_{d-1}$ type, the integrability and the exact solvability of these models can be established by relating the Hamiltonian $H$ to the $A_{d-1}$ type Dunkl-Laplace operator (see [15] or [1]). More precisely, we have

$$H = \frac{1}{2k}\Delta_k^{A_{d-1}} - \sum_{j=1}^{d} x_j \partial_j,$$

where $\Delta_k^{A_{d-1}}$ is the $A_{d-1}$-type Dunkl-Laplace operator given by

$$\Delta_k^{A_{d-1}} f(x) = \Delta f(x) + 2k \sum_{1\le i<j\le d} \frac{\partial_i f(x) - \partial_j f(x)}{x_i - x_j} - \frac{f(x) - f(\sigma_{ij} x)}{(x_i - x_j)^2}, \tag{2.1}$$

where $k > 0$ is a real scalar parameter, $x_1, ..., x_d$ are the components of the vector $x \in \mathbb{R}^d$, $\sigma_{ij} x$ denotes the vector $x$ with its i-th and j-th components interchanged and $f$ is a $C^2$-function.

We consider the space $\mathbb{R}^d$ with its Euclidean scalar product $\langle ., . \rangle$ and the associated norm $\|.\|$. For a general normalized root system $R$ of $\mathbb{R}^d$ (i.e. $R$ is a finite subset of $\mathbb{R}^d \backslash \{0\}$ such that for every $\alpha \in R$, $\|\alpha\| = \sqrt{2}$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$, where $\sigma_\alpha$ is the reflection with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$ given by $\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha$, see [20] or [22] for details on root systems), the Dunkl-Laplace operator acting on $C^2$-functions is given by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \Big( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \Big), \tag{2.2}$$

where $\Delta$ (resp. $\nabla$ ) is the usual Laplace (resp. gradient) operator, $R_+$ is a fixed positive subsystem of $R$ and $k : R \longmapsto [0, +\infty[$ is a fixed multiplicity function i.e. $k$ is invariant under the action of the Coxeter-Weyl group $W$( i.e. the finite subgroup of the orthogonal group generated by the reflections $\sigma_\alpha$, $\alpha \in R$)(see [13]) .
We note that in the case of $A_{d-1}$ type root system, the Coxeter-Weyl group is the symmetric group $S_d$ and as there is only one $S_d$-orbit, the multiplicity function reduces to a nonnegative parameter $k$ according to the relation (2.1).

Harmonic functions for the Dunkl-Laplacian, i.e. functions $u$ of class $C^2$ such that $\Delta_k u = 0$, have for a long time attracted the attention of researchers involved in Dunkl theory (see [27] and [37]) but their study was limited to functions $f$ of class $C^\infty$ defined on whole $\mathbb{R}^d$ or on the unit ball but having extension to whole $\mathbb{R}^d$ ([25]). This restriction was imposed by the spherical mean value property characterization of harmonicity and by the Dunkl translation operators (see below) which are of particular tricky use.

In a recent paper ([16]), we have found a volume mean value property characterization (see below) which allows us to study Dunkl-harmonic (D-harmonic) functions on any open $W$-invariant subset of $\mathbb{R}^d$. This new approach has many benefits in particular to study Dunkl potential theory. It is the aim of this paper to introduce, via the heat Dunkl-semigroup and our volume mean value operator, the Dunkl-Newtonian potentials and their use to study Dunkl-subharmonic functions. We give also some applications, in particular to Riesz measures and we obtain a Riesz representation theorem.
Nevertheless, in particular for lack of a non-centered Poisson kernel and because of the complexity of the Dunkl translation operators, our approach to subharmonic functions is not direct and requires some specific tools that will be presented below.

For $\xi \in \mathbb{R}^d$, let $D_\xi$ be the Dunkl operator defined on $\mathcal{C}^1(\mathbb{R}^d)$ by

$$D_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $\partial_\xi$ is the $\xi$-directional partial derivative. We know that the Dunkl-Laplace operator can be written $\Delta_k = \sum_{j=1}^d D_j^2$, where $D_j = D_{e_j}$, $j = 1, ..., d$ ( $(e_j)_{1 \leq j \leq d}$ is the canonical basis of $\mathbb{R}^d$) are commuting operators (see [9] and [13]). These operators are related to partial derivatives by means of the so-called Dunkl intertwining operator $V_k$ (see [11] or [13]) as follows

$$\forall \ \xi \in \mathbb{R}^d, \quad D_\xi V_k = V_k \partial_\xi. \tag{2.3}$$

The operator $V_k$ is a topological isomorphism from the space $\mathcal{C}^\infty(\mathbb{R}^d)$ [1] onto itself satisfying (2.3) and $V_k(1) = 1$ (see [42]) and for every $x \in \mathbb{R}^d$, there exists a unique probability measure $\mu_x$ on $\mathbb{R}^d$ with compact support contained in

$$C(x) := co\{gx, \ g \in W\} \tag{2.4}$$

(the convex hull of the orbit of $x$ under the group $W$) such that

$$\forall \ f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \tag{2.5}$$

(see [33] or [36]). Moreover, the Dunkl intertwining operator $V_k$ commutes with the $W$-action (see [36]) i.e.

$$\forall \ \ f \in \mathcal{C}^\infty(\mathbb{R}^d), \ \ \forall \ \ g \in W, \ \ g^{-1}.V_k(g.f) = V_k(f), \tag{2.6}$$

where $g.f(x) = f(g^{-1}x)$.
We note that the measure $\mu_x$ (which we call Rösler's measure), despite we don't know an explicit formula [2], is of fundamental importance in Dunkl's Analysis. We also note that M. Rösler has conjectured that supp $\mu_x = C(x)$. As a contribution, in this paper, we will prove that $x \in$ supp $\mu_x$ and if $k > 0$, the support of $\mu_x$ is $W$-invariant and in particular it contains all the point $gx$, $g \in W$ (see section 2).
For abbreviation, we introduce the index

$$\gamma := \sum_{\alpha \in R_+} k(\alpha) \tag{2.7}$$

and the weight function

$$\omega_k(x) := \prod_{\alpha \in R_+} | \langle \alpha, x \rangle |^{2k(\alpha)}. \tag{2.8}$$

1. carrying its usual Fréchet topology.
2. except in some very particular cases

An important fact about the Dunkl-Laplace operator is that it generates a generalized heat semi-group which kernel is given by (see [31])

$$p_t(x,y) := \frac{1}{(2t)^{d/2+\gamma}c_k}\ \tau_{-x}\big(e^{-\frac{\|.\|^2}{4t}}\big)(y), \qquad x,y\in\mathbb{R}^d \tag{2.9}$$

$$:= \frac{1}{(2t)^{d/2+\gamma}c_k}e^{-(\|x\|^2+\|y\|^2)/4t}E_k\Big(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\Big), \tag{2.10}$$

where $E_k(.,.)$ is the Dunkl kernel defined by $E_k(x,y) = V_k(e^{\langle\cdot,y\rangle})(x)$ (see [10], [13] and [36]), $c_k$ is the Macdonald-Mehta constant (see [28], [14]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}}\omega_k(x)dx \tag{2.11}$$

and $\tau_x$ is the Dunkl translation operator which acts on the class of $\mathcal{C}^\infty(\mathbb{R}^d)$-functions and on the class of $\mathbb{R}^d$-square integrable functions for the measure $\omega_k(x)dx$. The precise definitions and essential properties of the Dunkl translation operators are collected in the Annex 9.2. However, note that for any $f\in\mathcal{D}(\mathbb{R}^d)$ (the space of $C^\infty$-functions with compact support) and $x\in\mathbb{R}^d$

$$\int_{\mathbb{R}^d}\tau_x f(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\omega_k(y)dy \tag{2.12}$$

(see [43]). So that, the measure

$$m_k := \omega_k(x)dx \tag{2.13}$$

can be considered as a pseudo-Haar measure in the Dunkl analysis. Note also that a very useful formula for the Dunkl translation has been obtained by M. Rösler ([37]) when $f\in\mathcal{C}^\infty(\mathbb{R}^d)$ is a radial function. In such case, the Dunkl translation is given by

$$\forall\ y\in\mathbb{R}^d,\quad \tau_x f(y) = \int_{\mathbb{R}^d}\widetilde{f}\big(\sqrt{\|x\|^2+\|y\|^2+2\,\langle x,z\rangle}\big)d\mu_y(z), \tag{2.14}$$

where $\widetilde{f}$ is the profile function of $f$ defined by $f(x)=\widetilde{f}(\|x\|)$ .
This formula shows that the Dunkl translation operators are positivity preserving on the set of radial functions (i.e. $f\geq 0\Rightarrow \tau_x f\geq 0$) whereas this is not true in general ([30] or [41]).

Of particular importance for this paper is the Dunkl type Newton kernel which is defined, when $d+2\gamma>2$ (transient case), by means of the Dunkl heat kernel as follows

$$N_k(x,y) := \int_0^{+\infty} p_t(x,y)dt. \tag{2.15}$$

If $y$ is not on the $W$-orbit of $x$, $N_k(x,y)$ is finite. But, when $y\in W.x$ it is rather difficult, except if $y=x$, to decide if $N(x,y)$ is finite or infinite. In fact, the location of the singularities of the function $y\longmapsto N_k(x,y)$ on $W.x$ is really surprising as we will show on

some illustrative examples (see section 7).
When $y = 0$, the function $S(x) := N_k(x,0)$ is given by

$$S(x) = \frac{1}{d_k(d+2\gamma-2)}\|x\|^{2-d-2\gamma}, \tag{2.16}$$

where $d_k$ is the constant

$$d_k := \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi) = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}. \tag{2.17}$$

Here $d\sigma(\xi)$ is the surface measure of the unit sphere $S^{d-1}$ of $\mathbb{R}^d$. Note that the function $S$ is the fundamental solution of $\Delta_k$ in distributional sense i.e.

$$-\Delta_k[S\omega_k] = \delta_0, \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{2.18}$$

where $\omega_k$ is the weight function defined by (2.8) and $\delta_0$ is the Dirac measure at 0. This result, proved in [27] by using Dunkl's transform, generalizes the classical case ($k = 0$, $\Delta_k = \Delta$ and $\omega_k = 1$). For completeness, we will give a different proof in the Annex 9.3. Moreover, we will prove that the Dunkl-Newton kernel satisfies

$$\forall\ x_0 \in \mathbb{R}^d, \quad -\Delta_k[N_k(x_0,.)\omega_k] = \delta_{x_0} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), \tag{2.19}$$

with $\delta_{x_0}$ the Dirac measure at $x_0$. Note that, for some reasons related to the Dunkl translation operators, in contrast to the classical case, this result is not a direct consequence of (2.18).
When $d + 2\gamma \leq 2$, we have many subcases (recurrent cases) that will be discussed in a forthcoming paper.

Let $\Omega$ be a $W$-invariant open subset of $\mathbb{R}^d$. A function $u : \Omega \longrightarrow [-\infty, +\infty[$ is called Dunkl-subharmonic ($D$-subharmonic) if

1. u is upper semi-continuous (u.s.c.) on $\Omega$,
2. $u$ is not identically $-\infty$ on each connected component of $\Omega$,
3. $u$ satisfies the volume sub-mean property i.e. for every closed ball $B(x,r) \subset \Omega$, we have
$$u(x) \leq M_B^r(u)(x). \tag{2.20}$$

Here $M_B^r(f)(x)$ is the volume mean of $f$ at $(x,r)$ introduced by the authors ([16]) and defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0,r))}\int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy,$$

where $y \mapsto h_k(r,x,y)$ is a compactly supported measurable function (see section 2 for its properties) given by

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle})d\mu_y(z).$$

Note that if the function $f$ is u.s.c, then $f$ is bounded from above on compact sets and $M_B^r(f)(x)$ is well-defined (eventually equal to $-\infty$).

We will show that typical examples of D-subharmonic functions are the functions $x \longmapsto -N_k(x_0, x)$, $x_0 \in \mathbb{R}^d$ and the Dunkl-Newtonian potentials of nonnegative Radon measures. Moreover, for any D-subharmonic function $u$, we will prove that the distributional Dunkl-Laplacian of the function $u\omega_k$ is a nonnegative Radon measure which we call the $\Delta_k$-Riesz measure of $u$. This generalizes the particular case of the Dunkl-Newton kernel (2.19). These tools allow us to obtain many important characterizations of D-subharmonic functions.

We turn now to the content and the organization of this paper. In section 2, we recall the properties of the so-called harmonic kernel $h_k(r, x, y)$ and we establish an analogue of the Lebesgue differentiation theorem in Dunkl analysis which is a crucial tool in the paper. Next, we will prove the $W$-invariance property of the support of Rösler's measure $\mu_x$ as indicated above. As an application, we will describe completely the support of $y \mapsto h_k(r, x, y)$. At last, we will recall and improve some fundamental relations between the mean value operators that have been established by the authors in [16].
Some new and useful results about the Dunkl convolution product are the purpose of section 3.
In section 4, we introduce and study the notion of subharmonicity in Dunkl setting. In particular, we will prove that Dunkl-subharmonic functions satisfy the strong maximum principle.
The section 5 is devoted to give some characterizations of Dunkl subharmonic functions. Here, an approximation result is the essential tool to extend the properties of $C^\infty$-D-subharmonic functions to arbitrary D-subharmonic functions.
The notion of Riesz measure associated to a Dunkl subharmonic function will be introduced in section 6. Moreover, we will extend the well-known Weyl lemma to D-harmonic functions on an arbitrary $W$-invariant open subset of $\mathbb{R}^d$.
We will study the Dunkl type Newton kernel and potential of a Radon measure on $\mathbb{R}^d$ in section 7. In particular, we will discuss the D-harmonicity and the D-superhamonicity of these objects and we will obtain the mass uniqueness principle. At the end of this section, we will generalize the Riesz decomposition theorem to D-subharmonic functions.
Finally, in the last section, we will describe all bounded from above D-subharmonic functions in the whole space by using the Riesz decomposition theorem.

**Notations:** Let us introduce the following functional spaces and notations which will be used throughout the paper:
For $\Omega$ a $W$-invariant open subset of $\mathbb{R}^d$, we denote by:
• $L_k^p(\Omega)$ (resp. $L_{k,loc}^p(\Omega)$), $1 \leq p < +\infty$, the space of measurable functions $f : \Omega \longrightarrow \mathbb{C}$ such that $\|f\|_{L_k^p(\Omega)}^p := \int_\Omega |f(x)|^p \omega_k(x)dx < +\infty$ (resp. $\int_K |f(x)|^p \omega_k(x)dx < +\infty$ for any compact set $K \subset \Omega$).
• $L_k^\infty(\Omega)$ the space of measurable and essentially bounded functions on $\Omega$.
• $\mathcal{D}(\Omega)$ the space of $C^\infty$-functions on $\Omega$ with compact support.
• $\mathcal{D}'(\Omega)$ the space of distributions on $\Omega$ (i.e. the topological dual of $\mathcal{D}(\Omega)$ carrying the

Fréchet topology).
• $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of $C^\infty$-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives.
• $B(a,\rho) := \{x \in \mathbb{R}^d, \quad \|x-a\| \leq \rho\}$ the closed Euclidean ball centered at $a$ and with radius $\rho > 0$.
• $\overset{\circ}{B}(a,\rho) := \{x \in \mathbb{R}^d, \quad \|x-a\| < \rho\}$ the open Euclidean ball centered at $a$ and with radius $\rho > 0$.

## 2.2 The harmonic kernel and the mean value operators

In this section, we recall the properties of the harmonic kernel introduced by the authors in [16] and we establish the analogue of Lebesgue's differentiation theorem in Dunkl analysis. Moreover, we prove some new results about the support of the harmonic kernel and of the measure $\mu_x$ which represents the Dunkl intertwining operator.

### 2.2.1 Properties of the harmonic kernel

For $r > 0$ and $x, y \in \mathbb{R}^d$, the harmonic kernel $h_k(r,x,y)$ is defined (see [16]) by:

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}\big(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle}\big) d\mu_y(z). \tag{2.21}$$

In the classical case (i.e. $k = 0$), we have $\mu_y = \delta_y$ and $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$.
The harmonic kernel satisfies the following properties:

1. For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r,x,y) \leq 1$.
2. For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ is right-continuous and non decreasing on $]0,+\infty[$.
3. For all fixed $r > 0$ and $x \in \mathbb{R}^d$,
$$\operatorname{supp} h_k(r,x,.\,) \subset B^W(x,r) := \cup_{g\in W} B(gx,r). \tag{2.22}$$
4. Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_\varepsilon) \subset \mathcal{D}(\mathbb{R}^d)$ of radial functions such that for every $\varepsilon > 0$,
$$0 \leq \varphi_\varepsilon \leq 1,\ \varphi_\varepsilon = 1 \text{ on } B(0,r) \ \text{ and } \forall\, y \in \mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \varphi_\varepsilon(y) = \mathbf{1}_{B(0,r)}(y), \tag{2.23}$$
we have
$$\forall\, y \in \mathbb{R}^d, \quad h_k(r,x,y) = \lim_{\varepsilon\to 0} \tau_{-x}\varphi_\varepsilon(y). \tag{2.24}$$
5. For all $r > 0$ and $x, y \in \mathbb{R}^d$, we have
$$h_k(r,x,y) = h_k(r,y,x). \tag{2.25}$$

6. For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r,x,.)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r,x,y)\omega_k(y)dy = m_k(B(0,r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \tag{2.26}$$

where $m_k$ is the measure given by (2.13).

7. Let $r > 0$ and $x, y \in \mathbb{R}^d$. Then, for all $g \in W$, we have

$$h_k(r,gx,gy) = h_k(r,x,y) \quad and \quad h_k(r,gx,y) = h_k(r,x,g^{-1}y). \tag{2.27}$$

8. Let $r > 0$ and $x \in \mathbb{R}^d$. Then the function $h_k(r,x,.)$ is upper semi-continuous on $\mathbb{R}^d$.
9. The harmonic kernel satisfies the following fundamental geometric inequality: if $\|a-b\| \leq 2r$ with $r > 0$, then

$$\forall\ \xi \in \mathbb{R}^d, \quad h_k(r,a,\xi) \leq h_k(4r,b,\xi) \tag{2.28}$$

(see [16] Lemme 4.1). Note that in the classical case (i.e. $k = 0$), this inequality says that if $\|a-b\| \leq 2r$, then $B(a,r) \subset B(b,4r)$.

10. Let $x \in \mathbb{R}^d$. Then the family of probability measures

$$d\eta^k_{x,r}(y) = \frac{1}{m_k[B(0,r)]} h_k(r,x,y)\omega_k(y)dy \tag{2.29}$$

is an approximation of the Dirac measure $\delta_x$ as $r \longrightarrow 0$. That is

$$\forall\ \alpha > 0, \quad \lim_{r\to 0} \int_{\|x-y\|>\alpha} d\eta^k_{x,r}(y) = 0 \tag{2.30}$$

and if $f$ is a locally bounded measurable function on a $W$-invariant open neighborhood of $x$ and if $f$ is continuous at $x$, then

$$\lim_{r\to 0} \int_{\mathbb{R}^d} f(y)d\eta^k_{x,r} = \lim_{r\to 0} M^r_B(f)(x) = f(x) \tag{2.31}$$

(see [16], Proposition 3.2).

**Remark 2.1** *In [16], to prove assertion 8, we have constructed an explicit sequence* $(\varphi_\varepsilon)$ *satisfying (2.23) and the additional properties:* $(\varphi_\varepsilon)$ *is a decreasing sequence such that*

$$\forall\ \varepsilon > 0, \quad supp\ \varphi_\varepsilon \subset B(0, r+\sqrt{\varepsilon}). \tag{2.32}$$

**Remark 2.2** *Let* $r > 0$. *The function* $\mathbf{1}_{B(0,r)}$ *is in* $L^2_k(\mathbb{R}^d)$. *For* $x \in \mathbb{R}^d$, *we can then define* $\tau_{-x}(\mathbf{1}_{B(0,r)})$ *as being the* $L^2_k(\mathbb{R}^d)$*-function whose Dunkl transform is equal to*

$$\mathcal{F}_k\left(\tau_{-x}(\mathbf{1}_{B(0,r)})\right)(\xi) = E_k(-ix,\xi)\mathcal{F}_k\left(\mathbf{1}_{B(0,r)}\right)(\xi). \tag{2.33}$$

*(see (2.167)). This* $L^2_k(\mathbb{R}^d)$*-function which is also a generalization of* $\mathbf{1}_{B(x,r)}$ *(of the case* $k = 0$*) has been used formally in ([41] and [7]) for studying the* $L^p_k$*-boundedness of the Dunkl-Hardy-Littlewood maximal operator. In the next result, we will show that this function coincides almost everywhere with* $h_k(r,x,.)$*. But, in contrast to our harmonic kernel, the* $L^2$*-definition (2.33) of the function* $\tau_{-x}(\mathbf{1}_{B(0,r)})$ *does not give any precision neither on its support nor on some geometric properties like (2.28).*

**Proposition 2.3**

**1)** *For every $r > 0$ and $x \in \mathbb{R}^d$, we have*

$$h_k(r, x, y) = \tau_{-x}(\mathbf{1}_{B(0,r)})(y) \quad \text{for almost every } y \in \mathbb{R}^d. \tag{2.34}$$

**2)** *For every $r > 0$ and $x \in \mathbb{R}^d$, we have*

$$\forall\ \xi \in \mathbb{R}^d, \quad \mathcal{F}_k(\eta^k_{x,r})(y) := \int_{\mathbb{R}^d} E_k(-i\xi, y) d\eta^k_{x,r}(\xi) = E_k(-ix, \xi) j_{\frac{d}{2}+\gamma}(r\|\xi\|), \tag{2.35}$$

*where $\eta^k_{x,r}$ is the probability measure given by (2.29) and $j_\lambda$ is the normalized Bessel function defined by (2.156).*

**3)** *The function $(x, r) \mapsto \eta^k_{x,r}$ is continuous from $\mathbb{R}^d \times ]0, +\infty[$ to the space $\mathcal{M}^1(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$ equipped with the weak topology i.e. if $(x_n, r_n) \longrightarrow (x, r)$, then for every bounded and continuous function $f$ on $\mathbb{R}^d$ we have*

$$\lim_{n\to+\infty} M_B^{r_n}(f)(x_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^d} f(y) d\eta^k_{x_n,r_n} = M_B^r(f)(x) = \int_{\mathbb{R}^d} f(y) d\eta^k_{x,r}.$$

*Proof:* **1)** We consider the sequence $(\varphi_\varepsilon)$ as in Remark 2.1. By the monotone convergence theorem, we can see that $\tau_{-x}\varphi_\varepsilon \longrightarrow h_k(r, x, .)$ in $L^2_k(\mathbb{R}^d)$.
On the other hand, since $\mathbf{1}_{B(0,r)} \in L^2_k(\mathbb{R}^d)$, we have

$$\begin{aligned}
\left\|\tau_{-x}\varphi_\varepsilon - \tau_{-x}(\mathbf{1}_{B(0,r)})\right\|_{L^2_k(\mathbb{R}^d)} &= c_k^{-1} \left\|\mathcal{F}_k\left[\tau_{-x}\varphi_\varepsilon\right] - \mathcal{F}_k\left[\tau_{-x}(\mathbf{1}_{B(0,r)})\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&= c_k^{-1} \left\|E_k(-ix, .)\mathcal{F}_k\left[\varphi_\varepsilon\right] - E_k(-ix, .)\mathcal{F}_k\left[\mathbf{1}_{B(0,r)}\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&\leq c_k^{-1} \left\|\mathcal{F}_k\left[\varphi_\varepsilon\right] - \mathcal{F}_k\left[\mathbf{1}_{B(0,r)}\right]\right\|_{L^2_k(\mathbb{R}^d)} \\
&= \left\|\varphi_\varepsilon - \mathbf{1}_{B(0,r)}\right\|_{L^2_k(\mathbb{R}^d)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

where we have used Plancherel formula (2.154) for Dunkl's transform in the first and the last lines, the relations (2.33) and (2.159) in the second line, the inequality $|E_k(-ix, \xi)| \leq 1$ in the third line and the monotone convergence theorem in the last line.
Thus, $(\tau_{-x}\varphi_\varepsilon)$ converges also to $\tau_{-x}(\mathbf{1}_{B(0,r)})$ in $L^2_k(\mathbb{R}^d)$. This proves the desired equality.
**2)** Fix $r > 0$ and $x \in \mathbb{R}^d$. From (2.22), we see that $h_k(r, x, .) = \tau_{-x}\mathbf{1}_{B(0,r)}$ a.e. is in $L^1_k(\mathbb{R}^d)$. Hence, by (2.33) and (2.155), we have

$$\begin{aligned}
\mathcal{F}_k\left(h_k(r, x, .)\right)(\xi) &= E_k(-ix, \xi)\mathcal{F}_k\left(\mathbf{1}_{B(0,r)}\right)(\xi) \\
&= d_k E_k(-ix, \xi) \int_0^r j_{\frac{d}{2}+\gamma-1}(t\|\xi\|) t^{d+2\gamma-1} dt \\
&= \frac{d_k r^{d+2\gamma}}{d+2\gamma} E_k(-ix, \xi) j_{\frac{d}{2}+\gamma}(r\|\xi\|).
\end{aligned}$$

Finally, we deduce the result by using (2.26) and (2.29).
**3)** The result follows from the relation (2.35) and from Lévy's continuity theorem for the Dunkl transform of measures (see [37] or [32]).
This completes the proof. □

### 2.2.2 Lebesgue's differentiation theorem

In this subsection, we will study some properties of the volume mean value operator and then establish the analogue of Lebesgue's differentiation theorem in Dunkl analysis.

At first, we note that, thanks to (2.22), we can define the volume mean of any $f \in L^1_{k,loc}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is, as usual, a $W$-invariant (nonempty) open set. Let $f \in L^1_{k,loc}(\Omega)$ and $B(x,r) \subset \Omega$. The volume mean of $f$ at $(x,r)$ is defined by

$$M_B^r(f)(x) := \frac{1}{m_k(B(0,r))} \int_{\mathbb{R}^d} f(y) h_k(r,x,y) \omega_k(y) dy. \tag{2.36}$$

If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, the volume mean of $f$ at $(x,r)$ can also be written as follows (see [16]):

$$M_B^r(f)(x) = \frac{1}{m_k(B(0,r))} \int_{B(0,r)} \tau_x f(y) \omega_k(y) dy. \tag{2.37}$$

We will need the following notations which will be used frequently in this paper:

$$\Omega_r := \big\{x \in \Omega; \quad dist(x, \partial\Omega) > r\big\}, \tag{2.38}$$

$$r_\Omega := \sup\{r > 0; \quad \Omega_r \neq \emptyset\}. \tag{2.39}$$

Clearly, we have $\Omega_{r_1} \subset \Omega_{r_2}$ whenever $r_2 \leq r_1$ and $\Omega = \cup_{r>0}\Omega_r = \cup_{r<r_\Omega}\Omega_r$ (note that, since $\Omega \neq \emptyset$, we have $r_\Omega > 0$). Moreover, we will see below that the open set $\Omega_r$, $r < r_\Omega$, is $W$-invariant.

**Proposition 2.4** *Let $f \in L^1_{k,loc}(\Omega)$.*

1. *Let $r < r_\Omega$. Then the function $x \longmapsto M_B^r(f)(x)$*
   - **i)** *is well defined on $\Omega_r$,*
   - **ii)** *belongs to $L^1_{k,loc}(\Omega_r)$,*
   - **iii)** *is continuous on $\Omega_r$ when $f$ is continuous on $\Omega$.*
2. *Let $x \in \Omega$. Then the function $r \mapsto M_B^r(f)(x)$ is continuous on $]0, \varrho_x[$ with*

$$\varrho_x := dist(x, \partial\Omega). \tag{2.40}$$

*Proof:* **1. i)**• We note that $\Omega_r$ is an open and $W$-invariant subset of $\Omega$. To see this, it suffices to show that

$$\Omega_r = \big\{x \in \Omega; \quad B(x,r) \subset \Omega\big\}. \tag{2.41}$$

If $B(x,r) \subset \Omega$, then for all $y \in \partial\Omega$, we have

$$\|x-y\| = \|x-p(y)\| + \|p(y)-y\| = r + \|p(y)-y\| = r + dist(y, B(x,r)),$$

where $p(y)$ is the orthogonal projection of $y$ onto the closed ball $B(x,r)$. Hence,

$$dist(x, \partial\Omega) := \inf\big\{\|x-y\|, \quad y \in \partial\Omega\big\} = r + d(B(x,r), \partial\Omega) > r.$$

That is

$$\{x \in \Omega; \quad B(x,r) \subset \Omega\} \subset \Omega_r.$$

Conversely, let $x \in \Omega_r$ and suppose that there exists $y_0 \in B(x,r)$ and $y_0 \notin \Omega$. Then,

$$r \geq \|x - y_0\| \geq dist(x, \partial\Omega).$$

It is a contradiction with the fact that $x \in \Omega_r$. This proves (2.41).

• Now, let $x \in \Omega_r$. From the relations (2.22), $h_k(r,x,y) \leq 1$ and the fact that $f \in L^1_{k,loc}(\Omega)$, we deduce that $y \longmapsto f(y)h_k(r,x,y)\omega_k(y)$ is integrable on the compact set $B^W(x,r) := \cup_{g\in W} B(gx,r) \subset \Omega$. This implies that $x \longmapsto M^r_B(f)(x)$ is well defined on $\Omega_r$.

**ii)** By compactness, it suffices to prove that $M^r_B(f)\omega_k \in L^1(B(x_0,R))$ where $B(x_0,R)$ is an arbitrary closed ball of center $x_0$ and radius $R$ included in $\Omega_r$ . We have

$$\begin{aligned}
I &:= \int_{B(x_0,R)} \big|M^r_B(f)(x)\big|\omega_k(x)dx \\
&\leq \frac{1}{m_k(B(0,r))}\int_{B(x_0,R)}\Big(\int_{B^W(x,r)}|f(y)|h_k(r,x,y)\omega_k(y)dy\Big)\omega_k(x)dx \\
&\leq \frac{1}{m_k(B(0,r))}\int_{B(x_0,R)}\Big(\int_{B^W(x_0,R+r)}|f(y)|\omega_k(y)dy\Big)\omega_k(x)dx \\
&\leq \frac{m_k\left(B(x_0,R)\right)}{m_k\left(B(0,r)\right)}\int_{B^W(x_0,R+r)}|f(y)|\omega_k(y)dy < +\infty,
\end{aligned}$$

where the second inequality follows from the relation $h_k(r,x,y) \leq 1$ and from the fact that for every $x \in B(x_0,R)$ and every $g \in W$, $B(gx,r) \subset B(gx_0,R+r) \subset \Omega$.

**iii)** Let $x \in \Omega_r$ and let $(x_n)_n \subset \Omega_r$ such that $x_n \to x$ as $n \to +\infty$. There exist $a > 0$ and $N = N(a) \in \mathbb{N}$ such that $x_n \in B(x,a/2) \subset B(x,a) \subset \Omega_r$ for every $n \geq N$. In particular, $B(x_n,r) \subset B(x,r+a/2) \subset B(x,r+a) \subset \Omega$ for all $n \geq N$. Now, consider a continuous function $\psi$ such that $\psi = 1$ on $B^W(x,r+a/2)$ (recalling the notation (2.22)) and supp $\psi \subset B^W(x,r+a) \subset \Omega$. Therefore, the function $f\psi$ is extendable to a continuous function on $\mathbb{R}^d$ by taking the value 0 on $\mathbb{R}^d \backslash B^W(x,r+a)$. Then, using the support property of $h_k(r,x,.)$ and the statement 3) of Proposition 2.3, we deduce that

$$\lim_{n\to+\infty} M^r_B(f)(x_n) = \lim_{n\to+\infty} M^r_B(f\psi)(x_n) = M^r_B(f\psi)(x) = M^r_B(f)(x).$$

This proves that the function $M^r_B(f)$ is continuous at $x$.

**2.** By (2.26), it suffices to show that $\phi : r \mapsto \int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy$ is continuous on $]0,\varrho_x[$.

Since $r \mapsto h_k(r,x,y)$ is right continuous, by the dominated convergence theorem, we can see that $\phi$ is right-continuous on $]0,\varrho_x[$.

Now, fix $r \in ]0,\varrho_x[$ and $\eta > 0$ such that $]r-\eta, r+\eta[\subset]0,\varrho_x[$. Let $(r_n)$ be a sequence of nonnegative real number such that $r_n \longrightarrow 0$ as $n \longrightarrow +\infty$. We can assume that $r_n \in [0,\eta[$

for every $n$.
Using (2.25), (2.21) and applying Fubini's theorem, we obtain

$$\begin{aligned}|\phi(r)-\phi(r-r_n)| &\leq \int_{\mathbb{R}^d}\left(\int_{\Omega}|f(y)|\mathbf{1}_{]r-r_n,r]}(\sqrt{\|y\|^2+\|x\|^2-2\langle y,z\rangle})\omega_k(y)dy\right)d\mu_x(z)\\ &= \int_{\mathbb{R}^d}\left(\int_{A_n}|f(y)|\omega_k(y)dy\right)d\mu_x(z),\end{aligned}$$

where $A_n = A_n(x,z) := \left\{y\in\Omega, \quad r-r_n < \sqrt{\|y\|^2+\|x\|^2-2\langle y,z\rangle} \leq r\right\}$. Since $\cap_n A_n$ is a hypersurface, by the dominated convergence theorem, we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^d}(\int_{A_n}|f(y)|\omega_k(y)dy)d\mu_x(z) = 0.$$

Hence, by the previous relations, we conclude that $\phi$ is also left continuous. $\square$

**Theorem 2.5** *Let $f\in L^1_{k,loc}(\mathbb{R}^d)$. Then, for almost every*[3] *$x\in\mathbb{R}^d$, we have*

$$\lim_{r\downarrow 0^+} M_B^r(f)(x) = f(x). \tag{2.42}$$

*Proof:* `Step 1:` Suppose that $f$ is a continuous function on $\mathbb{R}^d$. In this case, the result follows immediately from the relation (2.31).
`Step 2:` We will prove the result when $f\in L^1_k(\mathbb{R}^d)$. To do this, it suffices to show that

$$f^*(x) := \limsup_{r\to 0} M_B^r(|f-f(x)|)(x) = 0$$

for almost every $x\in\mathbb{R}^d$.
• At first, we claim that there exists a constant $C>0$ such that

$$\forall \ \ t>0, \quad m_k\{f^*>t\} := m_k\{x\in\mathbb{R}^d, \quad f^*(x)>t\} \leq \frac{C}{t}\|f\|_{L^1_k(\mathbb{R}^d)}. \tag{2.43}$$

Indeed, we have

$$f^*(x) \leq \sup_{r>0} M_B^r(|f-f(x)|)(x) \leq M_k(|f|)(x)+|f(x)|, \tag{2.44}$$

where $M_k(g)$ is the maximal function of $g\in L^1_k(\mathbb{R}^d)$ defined by

$$M_k(g)(x) := \sup_{r>0}\frac{1}{m_k\left(B(0,r)\right)}\left|\int_{\mathbb{R}^d} g(y)\tau_{-x}\left(1_{B(0,r)}\right)(y)\omega_k(y)dy\right|.$$

(see [41] and ([7])). We notice that from (2.34) and (2.36), we have

$$M_k(|f|)(x) = \sup_{r>0} M_B^r(|f|)(x),$$

3. Note that negligible sets for the Lebesgue measure coincide with negligible sets for the measure $m_k$.

which justifies (2.44). Consequently,

$$\{f^* > t\} \subset \{M_k(|f|) + |f| > t\} \subset \{M_k(|f|) > t/2\} \cup \{|f| > t/2\}.$$

This implies that

$$m_k\{f^* > t\} \leq m_k\{M_k(|f|) > t/2\} + m_k\{|f| > t/2\}. \tag{2.45}$$

From ([41] or [7]), there exists a constant $C_1 > 0$ such that

$$m_k\{M_k(|f|) > t/2\} \leq \frac{2C_1}{t}\|f\|_{L^1_k(\mathbb{R}^d)} \tag{2.46}$$

and from Markov's inequality, we have

$$m_k\{|f| > t/2\} \leq \frac{2}{t}\|f\|_{L^1_k(\mathbb{R}^d)}, \tag{2.47}$$

Then we deduce (2.43) from (2.45), (2.46) and (2.47) with $C = 2C_1 + 2$.
• Let $\varepsilon > 0$ and let $g \in \mathcal{D}(\mathbb{R}^d)$ such that $\|f - g\|_{L^1_k(\mathbb{R}^d)} \leq \varepsilon$. For every $x \in \mathbb{R}^d$, step 1 applied to the function $y \longmapsto |g(y) - g(x)|$ shows that $g^*(x) = 0$. This implies that $(f - g)^* \leq f^* + g^* = f^*$. Since $f^* = (f - g + g)^* \leq (f - g)^* + g^* = (f - g)^*$, we get $f^* = (f - g)^*$. Consequently, by (2.43) we obtain

$$m_k\{f^* > t\} = m_k\{(f - g)^* > t\} \leq \frac{C}{t}\|f - g\|_{L^1_k(\mathbb{R}^d)} = \frac{C}{t}\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this proves that

$$\forall\, t > 0, \quad m_k\{f^* > t\} = 0.$$

Finally, since

$$\{f^* > 0\} = \cup_{n\geq 1}\{f^* > 1/n\},$$

we deduce that $m_k\{f^* > 0\} = 0$. That is $f^* = 0$ a.e. as desired.
`Step 3:` Let $f \in L^1_{k,loc}(\mathbb{R}^d)$. For every $n \in \mathbb{N} \setminus \{0\}$, the function $f_n = f\mathbf{1}_{B(0,n)}$ is in $L^1_k(\mathbb{R}^d)$. By Step 2, we have ${f_n}^*(x) = 0$ for all $x \in \mathbb{R}^d \backslash E_n$, where $E_n$ is a measurable set such that $m_k(E_n) = 0$.
We will prove that $\{f^* > 0\} \subset \cup_{n\geq 1}E_n$ which will imply the desired result.
Let $x \in \mathbb{R}^d$ such that $f^*(x) > 0$. There is an integer $n = n_x \in \mathbb{N} \setminus \{0\}$ such that $\operatorname{supp} h_k(r, x, .) \subset B(0, n)$ for every $r \leq 1$. This implies that $f^*(x) = {f_n}^*(x) > 0$. That is $x \in E_n$. This completes the proof. □

Now, we will generalize Lebesgue's differentiation theorem to functions defined on a $W$-invariant open subset of $\mathbb{R}^d$.

**Corollary 2.6** *Let $\Omega$ be a $W$-invariant open subset of $\mathbb{R}^d$. If $f \in L^1_{k,loc}(\Omega)$, then (2.42) holds for almost every $x \in \Omega$.*

*Proof:* For $n \in \mathbb{N}$ large enough (precisely $n > 1/r_\Omega$ with $r_\Omega$ given by (2.39)), we consider

$$O_n := \overset{\circ}{B}(0,n) \cap \{x \in \Omega, \quad dist(x, \partial\Omega) > 1/n\} := \overset{\circ}{B}(0,n) \cap \Omega_{\frac{1}{n}} \quad \text{and} \quad K_n = \overline{O_n}, \quad (2.48)$$

where $\overset{\circ}{B}(a,r)$ is the open ball centered at $a \in \mathbb{R}^d$ and with radius $r > 0$.
As $\Omega_{\frac{1}{n}} = \{x \in \Omega, \quad B(x, 1/n) \subset \Omega\}$ is $W$-invariant, we can see that $O_n$ (resp. $K_n$) is a $W$-invariant open (resp. $W$-invariant compact) subset of $\Omega$. Moreover, we have for every $n$ large enough

$$K_n \subset O_{n+1} \subset K_{n+1} \quad \text{and} \quad \Omega = \cup_n K_n = \cup_n O_n.$$

Now, let $f_n$ be the function given by $f_n(x) = f(x)$ if $x \in K_n$ and $f_n(x) = 0$ if $x \in \mathbb{R}^d \backslash K_n$. Clearly $f_n$ belongs to $L^1_{k,loc}(\mathbb{R}^d)$ and by Theorem 2.5 we have $f_n(x) = \lim_{r \to 0} M_B^r(f_n)(x)$ for almost every $x \in \mathbb{R}^d$.
Let

$$E_n := \left\{x \in \mathbb{R}^d, \quad f_n(x) \neq \lim_{r \to 0} M_B^r(f_n)(x)\right\} \quad \text{and} \quad E := \left\{x \in \Omega, \quad f(x) \neq \lim_{r \to 0} M_B^r(f)(x)\right\}.$$

Since $f_n$ is continuous on the open set $\mathbb{R}^d \backslash K_n$, by (2.31) we deduce that $E_n \subset K_n \subset \Omega$. Let us now take $x \in E$. There exist $R > 0$ and $N \in \mathbb{N}$ such that $B(x,R) \subset O_N \subset K_{N+1} \subset \Omega$. We will show that $x \in E_{N+1}$. As $O_N$ and $K_{N+1}$ are invariant under the action of the Coxeter-Weyl group $W$, by (2.22) we have

$$\forall \ r \in ]0, R], \ \text{supp} \ h_k(r, x, .) \subset O_N \subset K_{N+1}.$$

But $f = f_{N+1}$ on $O_N$. Therefore, if $x \notin E_{N+1}$ i.e. $f_{N+1}(x) = \lim_{r \to 0} M_B^r(f_{N+1})(x)$, then $f(x) = \lim_{r \to 0} M_B^r(f)(x)$ and $x \notin E$, a contradiction.
Thus $x \in E_{N+1}$. This proves that $E \subset \cup_n E_n$ and $E$ is a negligible set as desired. □

### 2.2.3 Some support properties of the harmonic kernel and of Rösler's measure

Here, we will obtain some new results on the support of Rösler's measure and we will describe completely the support of the harmonic kernel.
A first result in this direction is the following:

**Proposition 2.7** *Let $x \in \mathbb{R}^d$. Then*

**i)** *for every $r > 0$, $x \in supp \ h_k(r, x, .)$,*

**ii)** *$x \in supp \ \mu_x$,*

**iii)** *for every $r > 0$, $B(x, r) \subset supp \ h_k(r, x, .)$.*

*Proof:* **i)** Suppose that there exists $r > 0$ such that $x \notin \text{supp} \ h_k(r, x, .)$. Then we can find $\varepsilon > 0$ such that $h(r, x, y) = 0$, for all $y \in B(x, \varepsilon)$. Let $\mathcal{C}_\varepsilon^+$ the space of nonnegative continuous functions on $\mathbb{R}^d$ with compact support contained in $B(x, \varepsilon)$. Since $t \mapsto h_k(t, x, y)$ is increasing on $]0,+\infty[$, we deduce that

$$\forall\ f\in\mathcal{C}_\varepsilon^+,\quad \forall\ t\in]0,r],\quad 0\leq M_B^t(f)(x)=\frac{1}{m_k[B(0,t)]}\int_{\mathbb{R}^d} f(y)h_k(t,x,y)\omega_k(y)dy$$
$$\leq\frac{1}{m_k[B(0,t)]}\int_{\mathbb{R}^d} f(y)h_k(r,x,y)\omega_k(y)dy=0$$

Hence, we obtain

$$\forall\ f\in\mathcal{C}_\varepsilon^+,\quad \forall\ t\in]0,r],\quad M_B^t(f)(x)=0.$$

Letting $t\to 0$ and using the relation (2.31), we get a contradiction if the function $f$ is such that $f=1$ on $B(x,\varepsilon/2)$.

**ii)** Let $x\in\mathbb{R}^d$ be fixed. At first, we claim that

$$\forall\ r>0,\quad \forall\ y\in\mathbb{R}^d,\quad h_k(r,x,y)\leq\mu_x[B(y,r)]. \tag{2.49}$$

Indeed, from the inclusion supp $\mu_x\subset B(0,\|x\|)$, we see that

$$\forall\ y\in\mathbb{R}^d,\quad \forall\ z\in\text{supp}\ \mu_x,\quad \|y-z\|^2\leq\|y\|^2+\|x\|^2-2\langle y,z\rangle\,.$$

This implies for any $y\in\mathbb{R}^d$ and $r>0$ that

$$\forall\ z\in\text{supp}\ \mu_x,\quad \mathbf{1}_{[0,r]}\big(\sqrt{\|y\|^2+\|x\|^2-2\langle y,z\rangle}\big)\leq\mathbf{1}_{[0,r]}(\|y-z\|)=\mathbf{1}_{B(y,r)}(z).$$

If we integrate the two terms of the previous inequality with respect to the measure $\mu_x$, we obtain $h_k(r,y,x)\leq\mu_x\big(B(y,r)\big)$ and then (2.49) follows from (2.25).
Now, if $x\notin$ supp $\mu_x$, there exists $\epsilon>0$ such that $\mu_x\big(B(x,\epsilon)\big)=0$. Thus, we have $\mu_x\big(B(y,\epsilon/2)\big)=0$ whenever $y\in B(x,\epsilon/2)$. Using (2.49), we deduce that $h_k(\epsilon/2,x,.)=0$ on $B(x,\epsilon/2)$, a contradiction with the result of i).

**iii)** Let $y\in\mathbb{R}^d$ such that $\|x-y\|<r$. As $\lim_{z\to y}(\|x\|^2+\|y\|^2-2\langle x,z\rangle)=\|x-y\|^2$, there exists $\eta>0$ such that

$$\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}\leq r\quad\text{for every } z\in B(y,\eta).$$

Therefore, by using the fact that $y\in$ supp $\mu_y$ we obtain

$$h_k(r,x,y)\geq\mu_y[B(y,\eta)]>0.$$

$\square$

**Remark 2.8** *Note that when $x\notin\cup_{\alpha\in R}H_\alpha$, Rösler has proved that $x\in$ supp $\mu_x$ by using the asymptotic behavior of the Dunkl kernel $E_k(x,y)$ (see [34], Corollary 3.6).*

Under the positivity of the multiplicity function, we have the following result about the support of the Rösler measure:

**Theorem 2.9** *Let $x\in\mathbb{R}^d$ and assume that $k>0$. Then the set supp $\mu_x$ is $W$-invariant.*

*Proof:* We will prove that if $y \in \text{supp } \mu_x$, then $\sigma_\alpha(y) \in \text{supp } \mu_x$ for every $\alpha \in R$. Let then $y \in \text{supp } \mu_x$ and suppose that there is a root $\alpha \in R$ such that $\sigma_\alpha y \notin \text{supp } \mu_x$. Write $y' := \sigma_\alpha y$ to simplify notations. There is a ball $B(y', \epsilon)$ ($\epsilon > 0$) such that for all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ with compact support included in $B(y', \epsilon)$, we have

$$\int_{\mathbb{R}^d} f(z)\mu_x(dz) = V_k f(x) = 0.$$

Let us denote by $C^\infty_{y',\epsilon}$ (resp. $C_{y',\epsilon}$) the set of all functions $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ (resp. $f \in \mathcal{C}(\mathbb{R}^d)$) with compact support in $B(y', \epsilon)$. For all $\xi \in \mathbb{R}^d$ and all $f \in C^\infty_{y',\epsilon}$, we also have $\partial_\xi f \in C^\infty_{y',\epsilon}$. By the intertwining relation (2.3) we obtain

$$\forall\ \xi \in \mathbb{R}^d, \quad \forall\ f \in C^\infty_{y',\epsilon}, \quad D_\xi V_k f(x) = 0.$$

Suppose $f \in C^\infty_{y',\epsilon}$ and $f \geq 0$ and let $g := V_k f$. We have $g \geq 0$ on $\mathbb{R}^d$ (because $V_k$ preserves positivity) and

$$\forall\ \xi \in \mathbb{R}^d, \quad D_\xi g(x) = \partial_\xi g(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi \rangle \frac{g(x) - g(\sigma_\alpha.x)}{\langle x, \alpha \rangle} = 0. \tag{2.50}$$

But as $g(x) = 0$, $x$ is a minimum of $g$ so $\partial_\xi g(x) = 0$ and relation (2.50) implies

$$\forall\ \xi \in \mathbb{R}^d, \quad \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi \rangle \frac{g(x) - g(\sigma_\alpha.x)}{\langle x, \alpha \rangle} = 0. \tag{2.51}$$

Now, consider the set

$$R_x := \{\alpha \in R_+;\ x \in H_\alpha\}.$$

There are two possible locations for $x$:
•**First case**: Suppose that $R_x = \emptyset$ i.e $x \notin \cup_{\alpha \in R} H_\alpha$ (i.e. for all root $\alpha \in R$, $\langle x, \alpha \rangle \neq 0$). Applying (2.51) with $\xi = x$ and using the fact that $g(x) = 0$, we get

$$\sum_{\alpha \in R_+} k(\alpha) g(\sigma_\alpha.x) = 0.$$

As $g \geq 0$ and $k > 0$, we obtain that $g(\sigma_\alpha.x) = V_k f(\sigma_\alpha.x) = 0$ for all $\alpha \in R_+$ and all $f \in C^\infty_{y',\epsilon}$ and $f \geq 0$. By uniform approximation, we deduce that for all $f \in C_{y',\epsilon}$ and $f \geq 0$, we also have $V_k f(\sigma_\alpha.x) = 0$. Finally for every $f \in C_{y',\epsilon}$, by decomposing $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ and using the linearity and $W$-equivariance of $V_k$ (relation (2.6)), we obtain that

$$\forall\ f \in C_{y',\epsilon}, \quad \forall\ \alpha \in R_+, \quad V_k f(\sigma_\alpha.x) = V_k(\sigma_\alpha.f)(x) = 0,$$

where $\sigma_\alpha.f$ is the function $z \mapsto f(\sigma_\alpha.z)$. As it is easy to see that $\sigma_\alpha.C_{y',\epsilon} = C_{\sigma_\alpha y',\epsilon}$, we deduce that

$$\forall\ \alpha \in R_+, \quad \forall f \in C_{\sigma_\alpha.y',\epsilon}, \quad V_k f(x) = 0.$$

But this implies in particular that $V_k f(x) = 0$ for all $f \in C_{y,\epsilon}$ in contradiction with the hypothesis $y \in \text{supp } \mu_x$. The result of the theorem follows in the first case.

•**Second case**: Suppose that $R_x \neq \emptyset$. For every $\beta \in R_x$, clearly we have $x = \sigma_\beta.x$. Therefore, since $g(x) = 0$, we get $g(\sigma_\beta.x) = 0$, for all $\beta \in R_x$. But, as $x$ is a minimum of $g$, we have

$$\forall\ \beta \in R_x, \quad \frac{g(x) - g(\sigma_\beta.x)}{\langle x, \beta\rangle} = \int_0^1 \partial_\beta g(x - t\,\langle x, \beta\rangle\,\beta)dt = \partial_\beta g(x) = 0.$$

Hence, the relation (2.51) with $\xi = x$ implies

$$\sum_{\alpha \in R_+ \setminus R_x} k(\alpha) g(\sigma_\alpha.x) = 0.$$

Consequently, we obtain $g(\sigma_\alpha.x) = 0$ for all $\alpha \in R$. The end of the proof of the first case applies and gives also the result in this case. This completes the proof of the theorem. □

From the $W$-invariance property of the support of $\mu_x$ and the fact that $x \in \text{supp } \mu_x$, we obtain immediately the following result.

**Corollary 2.10** *Let $x \in \mathbb{R}^d$ and assume that $k > 0$ . Then, for all $g \in W$, $gx \in$ supp $\mu_x$.*

As another support type result, we have

**Corollary 2.11** *Let $x \in \mathbb{R}^d$ and $r > 0$. If $k > 0$, then*

$$\text{supp } h_k(r, x, .) = B^W(x, r) := \cup_{g \in W} B(gx, r). \tag{2.52}$$

*We shall call $B^W(x, r)$ the closed Dunkl ball centered at $x$ and with radius $r > 0$ associated to the Coxeter-Weyl group $W$.*

*Proof:* Let $g \in W$ and $y \in \mathbb{R}^d$ such that $\|gx - y\| < r$. Replacing $y$ by $g^{-1}y$ in the beginning of the proof of Proposition 2.7, iii), there exists $\eta > 0$ such that for all $z \in B(g^{-1}y, \eta)$, $\sqrt{\|x\|^2 + \|y\|^2 - 2\,\langle x, z\rangle} \le r$ and thus $h_k(r, x, y) \ge \mu_y[B(g^{-1}y, \eta)]$. By Corollary 2.10, we deduce that $h_k(r, x, y) > 0$. □

**Remark 2.12** *When $k \ge 0$, we will say that a root $\alpha \in R$ is active if $k(\alpha) > 0$. Let us denote by $R_A = \{\alpha \in R;\ k(\alpha) > 0\}$ the set of active roots. It is not difficult to see that we can generalize the results of Subsection 2.3 in the following form*

**a)** *$R_A$ is a root system.*

**b)** *Let $W_A$ be the Coxeter-Weyl group associated to the root system $R_A$. Then the restriction $k_A$ of $k$ to $R_A$ is $W_A$-invariant; in other words it is a multiplicity function.*

**c)** *For all $x \in \mathbb{R}^d$, the support of Rösler's measure $\mu_x$ is $W_A$-invariant, it contains the whole orbite $W_A.x$ and is contained in the convex hull of $W_A.x$.*

**d)** *For all $x \in \mathbb{R}^d$ and $r > 0$, supp $h_k(r, x, .) = \cup_{g \in W_A} B(gx, r)$.*

### 2.2.4 Representation formulas for the mean value operators

In this subsection, for the purpose of the paper, we will improve some representation formulas obtained by the authors in [16]. These formulas play a crucial role in the study of D-subharmonic functions in sections 4, 5 and 8.
Let us begin to recall that K. Trimèche and H. Mejjaoli introduced in [27] the spherical mean for $C^\infty$-functions defined on whole $\mathbb{R}^d$ as follows

$$M_S^r(f)(x) := \frac{1}{d_k}\int_{S^{d-1}} \tau_x f(ry)\omega_k(y)d\sigma(y), \tag{2.53}$$

(recalling that $d\sigma$ is the surface measure on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$). In [37], M. Rösler has proved that there exists a compactly supported probability measure $\sigma_{x,r}^k$ on $\mathbb{R}^d$ which represents the spherical mean operator. More precisely, for $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, the spherical mean of $f$ at $(x,r) \in \mathbb{R}^d \times \mathbb{R}_+$ is given by

$$M_S^r(f)(x) = \int_{\mathbb{R}^d} f(y)d\sigma_{x,r}^k(y), \tag{2.54}$$

with

$$\text{supp } \sigma_{x,r}^k \subset B^W(x,r) = \cup_{g\in W} B(gx,r). \tag{2.55}$$

Formula (2.54) shows that we can define the spherical mean at $(x,r)$ of any measurable nonnegative (resp. nonpositive, resp. bounded) function on $B^W(x,r)$.

We have obtained in [16] the following crucial results on the link between the spherical and volume means: If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, then for all $x \in \mathbb{R}^d$ and $r > 0$, we have:

$$M_S^r(f)(x) = f(x) + \frac{1}{d+2\gamma}\int_0^r M_B^t(\Delta_k f)(x)\ tdt, \tag{2.56}$$

$$M_B^r(f)(x) = \frac{d+2\gamma}{r^{d+2\gamma}}\int_0^r M_S^t(f)(x)t^{d+2\gamma-1}dt \tag{2.57}$$

and

$$M_B^r(f)(x) = f(x) + \frac{1}{r^{d+2\gamma}}\int_0^r\int_0^\rho M_B^t(\Delta_k f)(x)\ t\ dt\ \rho^{d+2\gamma-1}d\rho. \tag{2.58}$$

Furthermore, we have extended the relations (2.56) and (2.58) to any function $f$ of class $C^2$ on an arbitrary $W$-invariant open set $\Omega \subset \mathbb{R}^d$ and any $x \in \Omega$ but with $r \in ]0, \varrho_x/3[$, $\varrho_x$ being defined by (2.40).

We have also showed that a function $u \in \mathcal{C}^2(\Omega)$ is D-harmonic if and only if it satisfies the following mean value property:

$$\forall\ x \in \Omega,\ \forall\ r \in ]0, \varrho_x/3[, \quad u(x) = M_B^r(u)(x). \tag{2.59}$$

(see [16], Theorem 3.2).
In this paper, we will improve this result and we will show that (2.59) holds for any $x \in \Omega$ and any $r \in ]0, \varrho_x[$. At first, we have

**Lemma 2.13** *Let $f \in \mathcal{C}^\infty(\Omega)$, then the relations (2.56), (2.57), and (2.58) hold for all $x \in \Omega$ and all $r \in ]0, \varrho_x[$.*

*Proof:* Let $x \in \Omega$ and $r \in ]0, \varrho_x[$ and let $\epsilon > 0$ such that $B(x, r+\epsilon) \subset \Omega$. We can find $g \in \mathcal{D}(\mathbb{R}^d)$ such that
1. $g = 1$ on the compact set $B^W(x, r + \epsilon/2)$,
2. supp $g \subset B^W(x, r+\epsilon)$,
3. $0 \leq g \leq 1$.
Therefore, the function $\phi = fg$ is in $\mathcal{C}^\infty(\mathbb{R}^d)$, supp $\phi \subset B^W(x, r+\epsilon)$ and $\phi = f$ on $B^W(x, r+\epsilon/2)$. Applying (2.56), (2.57) and (2.58) to the function $\phi$ and noting that theses relations only involve the closed Dunkl ball $B^W(x,r)$ (through the supports of $h_k(r, x, .)$ and $\sigma^k_{x,r}$), we can replace $\phi$ by $f$ in the three formulas. □

**Lemma 2.14** *Let $u \in \mathcal{C}^\infty(\Omega)$. Then the following statements are equivalent*

**i)** *$u$ is D-harmonic in $\Omega$ (i.e. $\Delta_k u = 0$ on $\Omega$),*

**ii)** *$u(x) = M^r_S(u)(x)$ whenever $B(x,r) \subset \Omega$,*

**iii)** *$u(x) = M^r_B(u)(x)$ whenever $B(x,r) \subset \Omega$.*

*Proof:* *i)* $\Longrightarrow$ *ii)* It is a consequence of (2.56) applied to $\mathcal{C}^\infty(\Omega)$-functions.
*ii)* $\Longrightarrow$ *iii)* This also follows from (2.57) and Lemma 2.13.
*iii)* $\Longrightarrow$ *i)* Using the relation (2.58) where $\varrho_x/3$ is replaced by $\varrho_x$ (Lemma 2.13 ) and following the proof of Theorem 3.2 in [16] , we obtain the result. □

Now, let $f$ be an upper semi-continuous (u.s.c.) function on $\Omega$ (see [6] for more details) and let $B(x,r) \subset \Omega$. As $f$ is u.s.c., by adding a constant, we can assume that $f$ is nonpositive on the compact set $B^W(x,r)$. Therefore, using (2.22) and (2.55), we can define the Dunkl-volume and the Dunkl-spherical means of $f$ relative to $(x,r)$. Moreover, we have

**Lemma 2.15** *The relation (2.57) holds for the u.s.c. function $f$ on $\Omega$ (the two terms of (2.57) being eventually equal to $-\infty$).*

*Proof:* Fix $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$.
Suppose first that $f$ is continuous on $\Omega$. By the compactness of the supports of $h_k(r, x, .)$ and $\sigma^k_{x,r}$ and Weierstrass's approximation theorem, we can see that the relation (2.57) is true in this case. Moreover, using (2.53) for polynomials approximating $f$, we deduce that $t \longmapsto M^t_S(f)(x)$ is a measurable function.
Now, assume that $f$ is an u.s.c. function on $\Omega$. Since $f$ is bounded from above on $B^W(x,r)$, there is a decreasing sequence of continuous functions $(f_n)$ such that $f_n \longrightarrow f$ pointwise on $B^W(x,r)$ (see [6]). Replacing $f_n$ by $f_n - \sup_{B^W(x,r)} f_1$ and $f$ by $f - \sup_{B^W(x,r)} f_1$, we may assume that $f$ and all $f_n$ are nonpositive on $B^W(x,r)$.
For $t \in ]0, r]$, set $g_n(t) = M^t_S(f_n)(x)$ and $g(t) = M^t_S(f)(x)$. We can see that the sequence $(g_n)$ is decreasing and from the monotone convergence theorem applied to the sequence

$(f_n)$, we get $g_n \longrightarrow g$ pointwise on $]0,r]$ and in particular, $g$ is a measurable function. Let us now apply the monotone convergence theorem to the sequence $(g_n)$, we obtain

$$\int_0^r M_S^t(f)(x)t^{2\gamma+d-1}dt = \lim_{n\longrightarrow+\infty}\int_0^r M_S^t(f_n)(x)t^{2\gamma+d-1}dt. \tag{2.60}$$

But, by the first step,

$$\frac{2\gamma+d}{r^{2\gamma+d}}\int_0^r M_S^t(f_n)(x)t^{2\gamma+d-1}dt = M_B^r(f_n)(x) \tag{2.61}$$

and once again by the monotone convergence theorem, we have

$$\lim_{n\longrightarrow+\infty} M_B^r(f_n)(x) = M_B^r(f)(x). \tag{2.62}$$

Finally, we deduce the relation (2.57) from (2.60), (2.61) and (2.62). □

## 2.3 Dunkl convolution product

The Dunkl convolution product has been defined by means of the Dunkl translation operators (see [43] and [36]). So that, the Dunkl convolution product has been considered only in some particular cases. Here, we will prove that we can define the Dunkl convolution product of a function $u \in L^1_{k,loc}(\Omega)$ with a nonnegative and radial function $f \in \mathcal{D}(\mathbb{R}^d)$ and we will study some properties of this product. We will see, in Section 5, that this case allows us to obtain approximation results.

For $f, g \in \mathcal{S}(\mathbb{R}^d)$, the Dunkl convolution product is defined by

$$\forall\ x \in \mathbb{R}^d, \quad f *_k g(x) := \int_{\mathbb{R}^d} \tau_x f(-y)g(y)\omega_k(y)dy. \tag{2.63}$$

We note that it is commutative and satisfies the following property:

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g), \tag{2.64}$$

where $\mathcal{F}_k$ is the Dunkl transform (see Annex 9.1).

From (2.64), (2.159) and the injectivity of the $\mathcal{F}_k$ transform, we obtain the following relations

**Lemma 2.16** *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then, for every $x \in \mathbb{R}^d$, we have*

$$(\tau_x f) *_k g = f *_k (\tau_x g) = \tau_x(f *_k g). \tag{2.65}$$

**Proposition 2.17** *Let $u \in L^1_{k,loc}(\Omega)$ and $f \in \mathcal{D}(\mathbb{R}^d)$ be nonnegative, radial and supp $f \subset B(0,\rho)$ with $\rho < r_\Omega$ (i.e the set $\Omega_\rho$ defined in (2.38) is nonempty). Let*

$$u *_k f(x) := \int_{\mathbb{R}^d} u(y)\tau_{-x}f(y)\omega_k(y)dy. \tag{2.66}$$

*Then*

**1)** *the function $u *_k f$ is well defined on $\Omega_\rho$ and can be written*

$$\forall\ x \in \Omega_\rho, \quad u *_k f(x) = \int_{\mathbb{R}^d} u(y)\tau_{-y} f(x)\omega_k(y)dy \tag{2.67}$$
$$= \int_{\mathbb{R}^d} u(y)\tau_x f(-y)\omega_k(y)dy \tag{2.68}$$

**2)** *$u *_k f$ belongs to $\mathcal{C}^\infty(\Omega_\rho)$ and we have*

$$\Delta_k(u *_k f) = u *_k \Delta_k f, \tag{2.69}$$

**3)** *for all $B(x,r) \subset \Omega_\rho$, we have*

$$M_B^r(u *_k f)(x) = M_B^r(u) *_k f(x). \tag{2.70}$$

*Proof:* **1)** • For every $\varepsilon > 0$, we have

$$\forall\ y \in \mathbb{R}^d, \quad 0 \le f(y) \le \|f\|_\infty \mathbf{1}_{B(0,\rho)}(y) \le \|f\|_\infty \varphi_\varepsilon(y),$$

where $(\varphi_\varepsilon)$ is a sequence satisfying (2.23) (with $r = \rho$). Using the positivity of the Dunkl translation operators on radial functions, we deduce that

$$\forall\ y \in \mathbb{R}^d, \quad 0 \le \tau_{-x} f(y) \le \|f\|_\infty \tau_{-x}\varphi_\varepsilon(y).$$

Letting $\varepsilon \to 0$ and using (2.24), we obtain

$$\forall\ y \in \mathbb{R}^d, \quad 0 \le \tau_{-x} f(y) \le \|f\|_\infty h_k(\rho, x, y). \tag{2.71}$$

Consequently, from the relations (2.22) and (2.71), we get that

$$\operatorname{supp}\ \tau_{-x} f \subset B^W(x,\rho). \tag{2.72}$$

This implies that for every $x \in \Omega_\rho$, the function $y \longmapsto u(y)\tau_{-x} f(y)\omega_k(y)$ is integrable on $\Omega$.

• The relation (2.67) follows from (2.165) and the relation (2.68) follows from (2.67) and (2.161).

**2)** Let $x_0 \in \Omega_\rho$ and $R > 0$ such that $B(x_0,R) \subset \Omega_\rho$. We shall prove that the function $u *_k f$ is of class $C^\infty$ on $\overset{\circ}{B}(x_0,R)$.

Define the function $\Phi$ on $\mathbb{R}^d \times \mathbb{R}^d$ by (see (2.160))

$$\Phi(x,y) := \tau_{-x} f(y) = \frac{1}{c_k^2}\int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi) E_k(-ix,\xi) E_k(iy,\xi)\omega_k(\xi)d\xi.$$

We see that $\Phi$ is in $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and by (2.152) and the inequality $|E_k(iy,\xi)| \le 1$, for every multi-indices $\upsilon \in \mathbb{N}^d$ we get

$$\forall\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \left|\frac{\partial^\upsilon}{\partial x^\upsilon}\Phi(x,y)\right| \le \frac{1}{c_k^2}\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi)|\, \|\xi\|^{|\upsilon|}\omega_k(\xi)d\xi := C_\upsilon < +\infty.$$

On the other hand, from (2.72) we have

$$\forall\, x \in \overset{\circ}{B}(x_0, R), \quad \mathrm{supp}\ \Phi(x, .) \subset B^W(x, \rho) \subset B^W(x_0, R + \rho) \subset \Omega.$$

This implies that we can write

$$\forall\, x \in \overset{\circ}{B}(x_0, R), \quad \forall\, y \in \mathbb{R}^d, \quad \Phi(x, y) = \Phi(x, y)\mathbf{1}_{B^W(x_0, R+\rho)}(y).$$

Thus, for every multi-indices $\upsilon \in \mathbb{N}^d$, we deduce that

$$\forall\, x \in \overset{\circ}{B}(x_0, R), \quad \forall\, y \in \mathbb{R}^d, \quad \Big|\frac{\partial^\upsilon}{\partial x^\upsilon}\Phi(x, y)\Big| \le C_\upsilon \mathbf{1}_{B^W(x_0, R+\rho)}(y).$$

Now, since $u\omega_k$ is locally integrable, this proves that we can differentiate under the integral sign in (2.66) and we obtain the desired result.
Furthermore, using respectively (2.67), (2.164) and (2.165) (here note that we can use the relation (2.165) because $\Delta_k f$ is also a radial function [4] (see [27])), we obtain

$$\begin{aligned}\Delta_k(u *_k f)(x) &= \int_{\mathbb{R}^d} u(y)\Delta_k[\tau_{-y} f](x)\omega_k(y)dy = \int_{\mathbb{R}^d} u(y)\tau_{-y}[\Delta_k f](x)\omega_k(y)dy \\ &= \int_{\mathbb{R}^d} u(y)\tau_{-x}[\Delta_k f](y)\omega_k(y)dy = u *_k \Delta_k f(x).\end{aligned}$$

This completes the proof of 2).
**3)** We need the following lemma:

**Lemma 2.18** *Let $f \in \mathcal{S}(\mathbb{R}^d)$ be radial. Then, for all $r > 0$ and $a, b \in \mathbb{R}^d$, we have*

$$\tau_a \tau_b f = \tau_b \tau_a f \tag{2.73}$$

*and*

$$M_B^r(\tau_{-a} f)(b) = M_B^r(\tau_{-b} f)(a). \tag{2.74}$$

*Proof of Lemma* 2.18
• We obtain (2.73) from the relation (2.159) and the injectivity of the Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$.
• Let $r > 0$ and $a, b \in \mathbb{R}^d$. We have

$$\begin{aligned}M_B^r(\tau_{-a} f)(b) &= \frac{1}{m_k(B(0, r))}\int_{\mathbb{R}^d} \tau_a f(-y) h_k(r, b, y)\omega_k(y)dy \\ &= \frac{1}{m_k(B(0, r))}\int_{\mathbb{R}^d} \tau_b\tau_a f(-y)\mathbf{1}_{B(0,r)}(y)\omega_k(y)dy \\ &= \frac{1}{m_k(B(0, r))}\int_{\mathbb{R}^d} \tau_b f(-y) h_k(r, a, y)\omega_k(y)dy \\ &= M_B^r(\tau_{-b} f)(a),\end{aligned}$$

4. More precisely, we have $\Delta_k f(x) = (\frac{d^2}{dr^2} + \frac{d+2\gamma-1}{r}\frac{d}{dr})\widetilde{f}(r), \quad r = \|x\|$ and $\widetilde{f}$ the profile of $f$.

where
- in the first equality we have used the relations (2.165) and (2.161),
-the second equality comes from (2.37),
- the third equality follows from (2.73) and (2.37),
-the relations (2.161) and (2.165) yield the last equality. □

Now, we turn to the proof of (2.70).
Let $B(x,r) \subset \Omega_\rho$. By Proposition 2.4, 1-ii), the function $M_B^r(u)$ belongs to $L^1_{k,loc}(\Omega_r)$. This proves, by assertion 1), that the function $M_B^r(u) *_k f$ is well defined on $\Omega_{\rho+r}$.
By 2), the function $u *_k f$ is clearly in $L^1_{k,loc}(\Omega_\rho)$ and for $x \in \Omega_{\rho+r}$ we have [5]

$$\begin{aligned}
M_B^r(u *_k f)(x) &= \frac{1}{m_k(B(0,r))} \int_{B^W(x,r)} \Big(\int_{B^W(x,r+\rho)} u(z)\tau_{-y}f(z)\omega_k(z)dz\Big)\ h_k(r,x,y)\omega_k(y)dy \\
&= \frac{1}{m_k(B(0,r))} \int_{B^W(x,r+\rho)} u(z)\Big(\int_{B^W(x,r)} \tau_{-z}f(y)h_k(r,x,y)\omega_k(y)dy\Big)\ \omega_k(z)dz \\
&= \int_{B^W(x,r+\rho)} u(z)M_B^r\big(\tau_{-z}f\big)(x)\omega_k(z)dz \\
&= \int_{B^W(x,r+\rho)} u(z)M_B^r\big(\tau_{-x}f\big)(z)\omega_k(z)dz \\
&= \frac{1}{m_k(B(0,r))} \int_{B^W(x,r+\rho)} u(z)\Big(\int_{B^W(x,\rho)} \tau_{-x}f(y)h_k(r,z,y)\omega_k(y)dy\Big)\ \omega_k(z)dz \\
&= \frac{1}{m_k(B(0,r))} \int_{B^W(x,\rho)} \tau_{-x}f(y)\Big(\int_{B^W(x,r+\rho)} u(z)h_k(r,y,z)\omega_k(z)dz\Big)\ \omega_k(y)dy \\
&= M_B^r(u) *_k f(x),
\end{aligned}$$

where,
- the first equality follows from the relations (2.22), (2.67), (2.72) and from the fact that

$$\forall\ y \in B^W(x,r), \quad B^W(y,\rho) \subset B^W(x,r+\rho) \subset \Omega, \tag{2.75}$$

-the second equality follows from Fubini's theorem and (2.165),
-the third equality comes from the relation (2.36),
-the fourth equality follows from (2.74),
-in the the fifth equality we have used the relations (2.36) and (2.72),
-in the sixth equality we have used (2.25) and Fubini's theorem. Finally, using (2.22) and (2.75), we obtain the last equality.
This completes the proof of the proposition. □

**Remark 2.19** *Let $u$ and $f$ as in the previous proposition. If $u$ is with compact support, then $u *_k f$ is also with compact support and*

$$supp\ (u *_k f) \subset B(0,\rho) + W.supp\ u \subset \Omega, \tag{2.76}$$

*with $W.supp\ u := \{gx, \quad (g,x) \in W \times supp\ u\}$.*

5. Note that, in the integrals below, the consideration of the supports permits to justify the correct application of Fubini's theorem.

Indeed, if $x \notin B(0,\rho)+W.\text{supp } u$, then $x-gy \notin B(0,\rho)$ for every $(g,y) \in W \times \text{supp } u$. That is $\|gx-y\| > \rho$ for all $y \in \text{supp } u$ and all $g \in W$. In other words supp $u \cap B^W(x,\rho) = \emptyset$. Hence, by (2.72), we obtain $u *_k f(x) = 0$.

**Corollary 2.20** *Let $u \in L^1_{k,loc}(\Omega)$ and let $r, \rho > 0$ such that $\Omega_{r+\rho}$ is nonempty. Then, for every $x \in \Omega_{r+\rho}$, we have*

$$M_B^r\Big(M_B^\rho(u)\Big)(x) = M_B^\rho\Big(M_B^r(u)\Big)(x). \tag{2.77}$$

*Proof:* Fix $x \in \Omega_{r+\rho}$ and consider the sequence $(\varphi_\varepsilon)$ satisfying conditions (2.32). For $\varepsilon$ small enough, $B(x, r+\sqrt{\varepsilon}) \subset \Omega_\rho$ and $B(x, r+\rho+\sqrt{\varepsilon}) \subset \Omega$. Hence, as $M_B^\rho(u) \in L^1_{k,loc}(\Omega_\rho)$, by Proposition 2.17, the functions $M_B^\rho(u) *_k \varphi_\varepsilon$ and $M_B^\rho(u *_k \varphi_\varepsilon)$ are defined at point $x$. Moreover, using respectively the relation (2.24), the dominated convergence theorem and the relation (2.70), we get

$$\begin{aligned} m_k[B(0,r)]M_B^r\Big(M_B^\rho(u)\Big)(x) &= \int_{\mathbb{R}^d} M_B^\rho(u)(y)h_k(r,x,y)\omega_k(y)dy \\ &= \lim_{\varepsilon\to 0} M_B^\rho(u) *_k \varphi_\varepsilon(x) = \lim_{\varepsilon\to 0} M_B^\rho(u *_k \varphi_\varepsilon)(x). \end{aligned}$$

But, by using (2.72),

$$\text{supp } \tau_{-y}\varphi_\varepsilon \subset B^W(y, r+\sqrt{\varepsilon}).$$

Therefore, from (2.75) and using the notation (2.29), we can write

$$M_B^\rho(u *_k \varphi_\varepsilon)(x) = \int_{B^W(x,\rho)} \Big(\int_{B^W(x,r+\rho+\sqrt{\varepsilon})} u(z)\tau_{-y}\varphi_\varepsilon(z)\omega_k(z)dz\Big)\, d\eta^k_{x,\rho}(y).$$

Hence, as $0 \le \tau_{-y}\varphi_\varepsilon \le 1$ and $u \in L^1_{k,loc}(\Omega)$, we can use again the dominated convergence theorem and, letting $\varepsilon \longrightarrow 0$, we obtain

$$m_k[B(0,r)]M_B^r\Big(M_B^\rho(u)\Big)(x) = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} u(z)h_k(r,y,z)\omega_k(z)dz\Big)\, d\eta^k_{x,\rho}(y).$$

Finally, dividing by $m_k\big(B(0,r)\big)$ in the previous relation, we obtain the result. $\square$

We have the following associativity result for the Dunkl convolution product:

**Proposition 2.21** *Let $u \in L^1_{k,loc}(\Omega)$ and $f, g \in \mathcal{D}(\mathbb{R}^d)$ be nonnegative and radial such that supp $f \subset B(0,\rho)$, supp $g \subset B(0,r)$ and $\Omega_{r+\rho}$ is nonempty. Then*

$$\forall\ x \in \Omega_{r+\rho}, \quad \big(u *_k f\big) *_k g(x) = u *_k (f *_k g)(x) = \big(u *_k g\big) *_k f(x). \tag{2.78}$$

*Proof:* • From Proposition 2.17, the functions $\big(u *_k f\big) *_k g$ and $\big(u *_k g\big) *_k f$ are well defined on $\Omega_{r+\rho}$.
• We claim that $f *_k g$ is a nonnegative $C^\infty$-radial function on $\mathbb{R}^d$ with compact support contained in $B(0, r+\rho)$ which implies that $u *_k (f *_k g)$ is also well defined on $\Omega_{r+\rho}$. Indeed, again by Proposition 2.17 we see that $f *_k g$ is of class $C^\infty$ on $\mathbb{R}^d$ and using (2.76)

we obtain supp $f *_k g \subset B(0, r+\rho)$. Furthermore, by the positivity of Dunkl translation operators on radial functions, we deduce that the function $f *_k g$ is nonnegative. Now, using the fact that the Dunkl transform $\mathcal{F}_k$ is an isomorphism of the Schwartz space onto itself and the relation (2.64), we can write that

$$f *_k g = \mathcal{F}_k^{-1}\left(\mathcal{F}_k(f)\mathcal{F}_k(g)\right).$$

Therefore, since $\mathcal{F}_k$ preserves the radial property (see the relation (2.155)), we deduce that $f *_k g$ is radial as claimed.
• For $x \in \Omega_{r+\rho}$ fixed, we have

$$\begin{aligned}
\big(u *_k f\big) *_k g(x) &= \int_{B^W(x,r)} \big(u *_k f\big)(y)\tau_{-x}g(y)\omega_k(y)dy \\
&= \int_{B^W(x,r)} \Big(\int_{B^W(x,r+\rho)} u(z)\tau_{-y}f(z)\omega_k(z)dz\Big)\tau_{-x}g(y)\omega_k(y)dy \\
&= \int_{B^W(x,r+\rho)} u(z)\Big(\int_{B^W(x,r)} \tau_{-y}f(z)\tau_{-x}g(y)\omega_k(y)dy\Big)\omega_k(z)dz \\
&= \int_{B^W(x,r+\rho)} u(z)\Big(\int_{B^W(x,r)} \tau_{-z}f(y)\tau_{-x}g(y)\omega_k(y)dy\Big)\omega_k(z)dz \\
&= \int_{B^W(x,r+\rho)} u(z)\big(f *_k \tau_{-x}g\big)(z)\ \omega_k(z)dz \\
&= \int_{B^W(x,r+\rho)} u(z)\tau_{-x}\big(f *_k g\big)(z)\ \omega_k(z)dz \\
&= u *_k (f *_k g)(x).
\end{aligned}$$

where we have used
-the relations (2.66) and (2.72) in the first line;
-the same relations in the second line with (2.75);
- Fubini's theorem in the third line: the relation (2.71), the inequality $h_k(R, a, b) \leq 1$ and the hypothesis $u \in L^1_{k,loc}(\Omega)$ imply that we can use Fubini's theorem;
-the relation (2.165) in the fourth line;
-the relation (2.66) in the fifth line;
-relation (2.65) in the sixth line;
- the above properties of the function $f *_k g$ and (2.72) in the last line.

Now, changing the role of $f$ and $g$, we obtain

$$(u *_k g) *_k f(x) = u *_k (g *_k f)(x).$$

Finally, by the commutativity of the Dunkl convolution product, we conclude the last equality in (2.78). □

It is interesting to note that when $u$ is a continuous function, we can write the Dunkl convolution product in spherical coordinates as follows:

**Proposition 2.22** *Let $u$ be a continuous function on $\Omega$ and let $f \in \mathcal{D}(\mathbb{R}^d)$ be nonnegative, radial and supp $f \subset B(0,\rho)$ where $\rho < r_\Omega$ (i.e. $\Omega_\rho$ is nonempty). Then, for all $x \in \Omega_\rho$, we have*

$$u *_k f(x) = d_k \int_0^\rho \widetilde{f}(t) t^{d+2\gamma-1} M_S^t(u)(x)dt, \tag{2.79}$$

*where $\widetilde{f}$ is the profile function of $f$ and $d_k$ is the constant given by (2.17).*

*Proof:* At first we suppose that $u \in \mathcal{C}^\infty(\mathbb{R}^d)$. By (2.166), we have

$$u *_k f(x) = \int_{\mathbb{R}^d} f(y)\tau_x u(y)\omega_k(y)dy.$$

Then, using spherical coordinates, we can write

$$u *_k f(x) = \int_0^\rho \widetilde{f}(t)t^{d+2\gamma-1}\int_{S^{d-1}} \tau_x u(t\xi)\omega_k(\xi)d\sigma(\xi)dt.$$

Therefore, from (2.53) we deduce that the relation (2.79) holds in this case.
Let us now suppose only that $u$ is a continuous function on $\Omega$. Let $(p_n)$ a sequence of polynomial functions such that $p_n \longrightarrow u$ as $n \longrightarrow +\infty$ uniformly on the compact set $K := B^W(x,\rho)$. Since $\tau_{-x}f \geq 0$, by (2.12) we conclude that

$$\left|u *_k f(x) - p_n *_k f(x)\right| \leq \|\tau_{-x}f\|_{L^1_k(\mathbb{R}^d)} \sup_K |p_n(y)-u(y)| = \|f\|_{L^1_k(\mathbb{R}^d)} \sup_K |p_n(y)-u(y)|.$$

Hence

$$u *_k f(x) = \lim_{n\to+\infty} p_n *_k f(x). \tag{2.80}$$

Furthermore, as the probability measures $\sigma_{x,t}^k$, $0 < t \leq \rho$, have compact support contained in $B^W(x,t) \subset K = B^W(x,\rho)$ (see (2.55)), we deduce

$$\forall\ t \leq \rho, \quad |M_S^t(p_n - u)(x)| \leq \sup_K |p_n(y) - u(y)|.$$

This implies

$$\lim_{n\to+\infty} d_k \int_0^\rho \widetilde{f}(t)t^{d+2\gamma-1}M_S^t(p_n)(x)dt = d_k \int_0^\rho \widetilde{f}(t)t^{d+2\gamma-1}M_S^t(u)(x)dt. \tag{2.81}$$

From (2.80), (2.81) and the first step, we deduce that relation (2.79) holds when the function $u$ is continuous on $\Omega$. $\square$

**Remark 2.23** *Applying the monotone convergence theorem, the relation (2.79) holds when $u$ is an upper semi-continuous function on $\Omega$ but the both terms may be equal to $-\infty$.*

## 2.4 Dunkl subharmonic functions

In this section, we study some properties of D-subharmonic functions (see definition (2.20)) on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$. In particular, we will prove that any D-subharmonic function satisfies the strong maximum principle and the uniqueness principle.

We denote by $\mathcal{SH}_k(\Omega)$ the set of D-subharmonic functions on $\Omega$. A function $u$ is called D-superharmonic if $-u$ is D-subharmonic.

Let us start by some remarks:

**Remark 2.24** *1. The set $\mathcal{SH}_k(\Omega)$ is a convex cone.*

2. *If $u, v \in \mathcal{SH}_k(\Omega)$, then $\max(u,v)$ is also in $\mathcal{SH}_k(\Omega)$.*
3. *If $u \in \mathcal{SH}_k(\Omega)$ and $f$ be a convex and non-decreasing function on $\mathbb{R}$, then $f(u)$ is also in $\mathcal{SH}_k(\Omega)$.*
4. *Let $u \in \mathcal{C}^\infty(\Omega)$. From Lemma 2.14, we deduce that if $u$ is D-harmonic in $\Omega$, then $u \in \mathcal{SH}_k(\Omega)$. In particular, the function $-S$, where $S$ is the fundamental solution of the Dunkl-Laplacian $\Delta_k$ given by (2.16), is D-subharmonic on $\mathbb{R}^d\backslash\{0\}$. In fact, we will show that $-S$ is D-subharmonic on $\mathbb{R}^d$ but this property is not an immediate consequence of the definition. To our knowledge, in the classical case, this follows from the equivalence between the local and the global sub-mean properties. Moreover, this equivalence is based on the properties of the Poisson kernel for an arbitrary ball (see for example [2], Theorem 3.2.2 or [18], Theorem 2.3.8). In our case, an explicit formula for the Poisson kernel has been given by Dunkl for the unit ball (see [11] ) but the Poisson kernel for an arbitrary ball is still an open problem.*

### 2.4.1 Local properties of D-subharmonic functions

**Proposition 2.25** *Let $u \in \mathcal{SH}_k(\Omega)$. Then the function $u$ belongs to $L^1_{k,loc}(\Omega)$.*

*Proof:* Fix $\Omega_0$ a connected component of $\Omega$. Let

$$E := \{x \in \Omega_0, \ \ u\omega_k \ \text{ is integrable over some neighbourhood of } x\}.$$

Let $x \in E$. Then there exists $r > 0$ such that $B(x,r) \subset \Omega_0$ and $\int_{B(x,r)} |u(y)|\omega_k(y)dy < +\infty$. For $z \in B(x, r/2)$, we have $B(z, r/2) \subset B(x,r)$ and hence $u\omega_k$ is integrable over $B(z, r/2)$. Thus, $B(x, r/2) \subset E$ and $E$ is an open subset of $\Omega_0$.

Now, let $x \in \Omega_0\backslash E$. Because $u\omega_k$ is not integrable on any neighborhood of $x$, we must have $\int_{B(x,R)} |u(y)|\omega_k(y)dy = +\infty$ for all $R > 0$ such that $B(x,R) \subset \Omega_0$. Fix $r > 0$ such that $B(x, 6r) \subset \Omega_0$. We will prove that $B(x, 2r) \subset \Omega_0\backslash E$.

Since $u$ is u.s.c., we can assume that $u$ is nonpositive on the compact set $K = B^W(x, 6r)$ [6].

Let $z \in B(x, 2r)$. From (2.28) and the nonpositivity of $u$, we deduce that

$$\int_{\mathbb{R}^d} u(y)h_k(4r, z, y)\omega_k(y)dy \le \int_{\mathbb{R}^d} u(y)h_k(r, x, y)\omega_k(y)dy. \tag{2.82}$$

6. replacing $u$ by $u - \max_K u$.

Now, if we apply (2.28) once again where we replace respectively $r$, $a$, $b$ and $\xi$ by $r/4$, $x$, $y$ and $x$ we get

$$\forall\ y \in B(x, r/2), \quad h(r/4, x, x) \le h_k(r, y, x) \tag{2.83}$$

Thus, using (2.83), (2.25), (2.82), Proposition 2.7, i) and the fact that $u \le 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} u(y) h_k(4r, z, y)\omega_k(y)dy &\le \int_{B(x,r/2)} u(y) h_k(r, x, y)\omega_k(y)dy \\ &\le h_k(r/4, x, x)\int_{B(x,r/2)} u(y)\omega_k(y)dy = -\infty.\end{aligned}$$

Consequently, from the previous inequality we get $M_B^{4r}(u)(z) = -\infty$, and therefore, $u(z) = -\infty$ by the sub-mean property. Hence, $u = -\infty$ on $B(x, 2r)$ and this proves that $\Omega_0 \backslash E$ is an open subset of $\Omega_0$. Finally, as $u \ne -\infty$ on $\Omega_0$ and using the connectedness of $\Omega_0$, we must have $E = \Omega_0$. The connected component $\Omega_0$ being arbitrary, Proposition 2.25 is proved. $\square$

Let $u \in \mathcal{SH}_k(\Omega)$. Using the generalized Lebesgue differentiation theorem and Proposition 2.25, we have for almost all $x \in \Omega$

$$u(x) = \lim_{r\to 0} M_B^r(u)(x). \tag{2.84}$$

In the classical case (i.e. when $k = 0$), the relation (2.84) holds everywhere for any subharmonic function (see for example [2], Corollary 3.2.6 or [18], Lemma 2.4.4). In the following result, we will extend this fundamental property to Dunkl subharmonic functions.

**Proposition 2.26** *Let $u \in \mathcal{SH}_k(\Omega)$. Then, for every $x \in \Omega$, we have*

$$u(x) = \lim_{r\to 0} M_B^r(u)(x). \tag{2.85}$$

*Proof:* Fix $x \in \Omega$ and $R > 0$ such that $B(x, R) \subset \Omega$. As above, we may assume that $u$ is negative on the compact set $B^W(x, R)$. We distinguish two cases:
`First case:` Suppose that $u(x) > -\infty$. By upper semi-continuity, for all $\varepsilon > 0$ , there exists $\alpha \in ]0, R]$ such that

$$u(y) < u(x) + \varepsilon, \quad \text{whenever } y \in B(x, \alpha). \tag{2.86}$$

From the sub-mean property and the fact that $u < 0$ on $B^W(x, R)$, we have

$$\forall\ r \in ]0, R], \quad u(x) \le M_B^r(u)(x) = \int_{\mathbb{R}^d} u(y) d\eta_{x,r}^k(y) \le \int_{B(x,\alpha)} u(y) d\eta_{x,r}^k(y),$$

where $d\eta_{x,r}^k(y)$ is the probability measure defined by (2.29).
Using (2.86), we deduce that

$$\forall\ r \in ]0, R], \quad u(x) \le M_B^r(u)(x) \le (u(x) + \varepsilon)\int_{B(x,\alpha)} d\eta_{x,r}^k(y). \tag{2.87}$$

As from (2.30) $\lim_{r\to 0}\int_{B(x,\alpha)} d\eta^k_{x,r}(y) = 1$, there exists $\beta \in ]0, R[$ such that

$$\forall\ r \in ]0,\beta],\quad \int_{B(x,\alpha)} d\eta^k_{x,r}(y) \geq 1-\varepsilon. \tag{2.88}$$

Now, if we have taken $\varepsilon > 0$ small enough to ensure that $u(x) + \varepsilon < 0$, we deduce from (2.87) and (2.88) that

$$\forall\ r \in ]0,\beta],\quad u(x) \leq M^r_B(u)(x) \leq u(x) + \varepsilon(1-\varepsilon-u(x)).$$

This implies that $M^r_B(u)(x) \longrightarrow u(x)$ as $r \longrightarrow 0$. This proves the result in this case.
`Second case:` Suppose that $u(x) = -\infty$. For every $n \in \mathbb{N}\backslash\{0\}$, there is $a \in ]0, R]$ such that $u(y) \leq -n$ whenever $y \in B(x, a)$. Therefore,

$$\forall\ r \in ]0,a],\quad M^r_B(u)(x) \leq -n\int_{B(x,a)} d\eta^k_{x,r}(y). \tag{2.89}$$

Again by (2.30), there exists $b > 0$ such that

$$\forall\ r \in ]0,b],\quad \int_{B(x,a)} d\eta^k_{x,r}(y) \geq 1/2. \tag{2.90}$$

From (2.89) and (2.90) we obtain

$$\forall\ r \in ]0,\min(a,b)],\quad M^r_B(u)(x) \leq -n/2.$$

Therefore, $M^r_B(u)(x) \longrightarrow -\infty$ as $r \longrightarrow 0$ and the result is also proved in this case.

□

From the previous proposition, we immediately obtain the uniqueness principle that a D-subharmonic function is determined by its restriction to the complementary of a negligible set. More precisely:

**Corollary 2.27** *If $u$ and $v$ are D-subharmonic functions on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$ and $u(x) = v(x)$ for almost every $x \in \Omega$, then $u$ and $v$ are identically equal in $\Omega$.*

In the following result we consider the convergence property of a decreasing sequence of D-subharmonic functions.

**Proposition 2.28** *Let $(u_n)$ be a decreasing sequence of D-subharmonic functions on $\Omega$ and $u(x) := \lim_{n\to+\infty} u_n(x)$. If $u$ is not identically $-\infty$ on each connected component of $\Omega$, then $u$ is D-subharmonic on $\Omega$.*

*Proof:* Clearly $u$ is u.s.c. on $\Omega$ as being a decreasing limit of u.s.c. functions (see [6]). Let $x \in \Omega$ and $r > 0$ such that $B(x, r) \subset \Omega$. By the monotone convergence theorem, we get

$$u(x) = \lim_{n\to+\infty} u_n(x) \leq \lim_{n\to+\infty} M^r_B(u_n)(x) = M^r_B(u)(x).$$

This implies that $u$ is D-subharmonic on $\Omega$. □

### 2.4.2 The strong Maximum principle

The following theorem is a generalization of the strong maximum principle for D-harmonic functions obtained by the authors in [16] (Theorem 4.1).

**Theorem 2.29** *Let $u \in \mathcal{SH}_k(\Omega)$ and suppose that $\Omega$ is connected.*
**i)** *If $u$ has a maximum in $\Omega$, then $u$ is constant.*
**ii)** *If $\Omega$ is bounded and $\limsup_{z\to x} u(z) \leq 0$, for all $x \in \partial\Omega$, then $u \leq 0$ on $\Omega$.*

*Proof:* **i)** Let $x_0 \in \Omega$ such that $u(x) \leq u(x_0)$ for all $x \in \Omega$. Let

$$\Omega_0 := \{x \in \Omega, \quad u(x) < u(x_0)\}.$$

Because $u$ is u.s.c., $\Omega_0$ is an open subset of $\Omega$.
Now, let $x \in \Omega\backslash\Omega_0$ i.e. $u(x) = u(x_0)$ and $r > 0$ such that $B(x,r) \subset \Omega$. By the sub-mean property, we clearly have

$$u(x_0) = u(x) \leq M_B^r(u)(x) \leq u(x_0).$$

This yields

$$\frac{1}{m_k(B(0,r))}\int_{\mathbb{R}^d}[u(x_0) - u(y)]h_k(r,x,y)\omega_k(y)dy = 0.$$

Hence, $u(x_0) = u(y)$ for almost every $y \in$ supp $h_k(r,x,.)$ and by Proposition 2.7-iii), $u(x_0) = u(y)$ for almost every $y \in \mathring{B}(x,r)$. Let us now introduce the nonpositive function $v(y) = u(y) - u(x_0)$, $y \in \mathring{B}(x,r)$. Suppose that there exists $a \in \mathring{B}(x,r)$ such that $v(a) < 0$ and take $\lambda \in \mathbb{R}$ such that $v(a) < \lambda < 0$. Since $v$ is u.s.c at the point $a$, there is $\epsilon > 0$ such that $B(a,\epsilon) \subset \mathring{B}(x,r)$ and $v(y) < \lambda$ for all $y \in B(a,\epsilon)$. This contradicts the fact that $v = 0$ a.e. on $\mathring{B}(x,r)$ and this proves that $u \equiv u(x_0)$ on $\mathring{B}(x,r)$.
Consequently, $\Omega\backslash\Omega_0$ is an open subset of $\Omega$ containing $x_0$. But $\Omega$ is connected, then $\Omega_0 = \emptyset$ and this shows i).
**ii)** Define the function $\widetilde{u}$ on the compact closure $\overline{\Omega}$ of $\Omega$ by

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ \\ \limsup\limits_{y\to x,\ y\in\Omega} u(y) & \text{if} \quad x \in \partial\Omega. \end{cases}$$

Clearly $\widetilde{u}$ is u.s.c. on $\overline{\Omega}$. Consequently, there exists $x_0 \in \overline{\Omega}$ such that $\widetilde{u}(x_0) = \sup_{\overline{\Omega}} \widetilde{u}(x)$. If $\widetilde{u}(x_0) > 0$, then by our hypothesis necessarily $x_0 \in \Omega$ and by *i)* we have $u(x) = u(x_0) > 0$ for every $x \in \Omega$. We obtain a contradiction to the fact that $\limsup_{y\to x} u(y) \leq 0$. □

**Corollary 2.30** *Let $u \in \mathcal{SH}_k(\Omega)$ and suppose that $G$ is a connected $W$-invariant open subset of $\Omega$ with compact closure $\overline{G} \subset \Omega$. If $s$ is D-superharmonic on $\Omega$ and $u \leq s$ on $\partial G$, then $u \leq s$ on $G$.*

*Proof:* Clearly $u - s$ is D-subharmonic on $G$ and for $x \in \partial G$, we have

$$\limsup_{z\to x}[u(z) - s(z)] \leq \limsup_{z\to x} u(z) - \liminf_{z\to x} s(z) = u(x) - s(x) \leq 0.$$

Hence, the result follows from Theorem 2.29, ii). □

## 2.5 Approximation of D-subharmonic functions by $\mathcal{C}^\infty$-functions

Our aim in this section is to approximate any D-subharmonic function on $\Omega$ by a sequence of smooth and D-subharmonic functions. At the end of this section, we will give some other characterizations of D-subharmonic functions. Let us recall that, even if it is not explicitly mentioned, the open set $\Omega$ is always supposed $W$-invariant in the whole section.

### 2.5.1 Characterization of $C^\infty-$ D-subharmonic functions

We start by the following characterization of the $C^\infty-$D-subharmonic functions:

**Proposition 2.31** *Let $u \in \mathcal{C}^\infty(\Omega)$. Then the following assertions are equivalent*

**i)** $u \in \mathcal{SH}_k(\Omega)$,

**ii)** $\Delta_k u \geq 0$ *on* $\Omega$,

**iii)** $u(x) \leq M_S^r(u)(x)$ *whenever* $B(x,r) \subset \Omega$ .

*Proof:* **i) $\Longrightarrow$ ii)** Suppose that $\Delta_k u(x) < 0$ for some $x \in \Omega$. By (2.31), we have $\lim_{t\to 0} M_B^t(\Delta_k u)(x) = \Delta_k u(x)$. Hence, there exists $r \in ]0, \varrho_x[$ such that

$$M_B^t(\Delta_k u)(x) \leq \frac{1}{2}\Delta_k u(x) < 0 \quad \text{for all} \quad t \in ]0, r].$$

This implies that

$$\frac{1}{r^{2\gamma+d}} \int_0^r \int_0^\rho M_B^t(\Delta_k u)(x)\, t\, dt\, \rho^{2\gamma+d-1} d\rho \leq \frac{r^2}{4(d+2\gamma+2)}\Delta_k u(x) < 0.$$

Therefore, by (2.58) we obtain $M_B^r(u)(x) < u(x)$. A contradiction with the sub-mean property.

**ii) $\Longrightarrow$ iii)** This follows immediately from the relation (2.56) and Lemma 2.13.

**iii) $\Longrightarrow$ i)** From (2.57) and a direct integration with respect to $r$, we obtain the result. $\square$

The $C^\infty$- D-subharmonicity can be characterized in terms of the monotonicity with respect to $r$ of the spherical and volume means. More precisely, we have

**Proposition 2.32** *Let $u \in \mathcal{C}^\infty(\Omega)$. The following statements are equivalent*

**i)** $u \in \mathcal{SH}_k(\Omega)$,

**ii)** *for every $x \in \Omega$, the function $r \longmapsto M_B^r(u)(x)$ is non-decreasing on* $]0, \varrho_x[$ [7] *and*

$$\lim_{r\to 0} M_B^r(u)(x) = u(x), \tag{2.91}$$

**iii)** *for every $x \in \Omega$, the function $r \longmapsto M_S^r(u)(x)$ is non-decreasing on* $]0, \varrho_x[$ *and*

$$\lim_{r\to 0} M_S^r(u)(x) = u(x), \tag{2.92}$$

7. We recall that $\varrho_x$ is the distance from $x$ to the boundary of $\Omega$ (see (2.40)).

**iv)** *$u \in L^1_{k,loc}(\Omega)$, $\lim_{r\to 0} M^r_B(u)(x) = u(x)$ for every $x \in \Omega$ and $M^r_B(u)(x) \leq M^r_S(u)(x)$, whenever $B(x,r) \subset \Omega$.*

*Proof:* At first, using Proposition 2.4- 2), formulas (2.56) and (2.58), we deduce that the functions $r \longmapsto M^r_B(f)(x)$ and $r \longmapsto M^r_S(f)(x)$ are differentiable on $]0, \varrho_x[$ and the relations (2.91) and (2.92) are always satisfied for any fixed function $f \in \mathcal{C}^\infty(\Omega)$ and for any fixed $x \in \Omega$. We note also that the first condition in assertion iv) is redundant but we will need it in order to extend this result to an arbitrary D-subharmonic function (see Theorem 2.42 below).
**ii) $\Longrightarrow$ i)** As $r \longmapsto M^r_B(u)(x)$ is non-decreasing, (2.91) implies that the sub-mean property is clearly satisfied.
**i) $\Longrightarrow$ iii)** We use the fact that $\Delta_k u \geq 0$ on $\Omega$ and we differentiate with respect to $r$ the relation (2.56) and we get $\frac{d}{dr} M^r_S(u)(x) \geq 0$ i.e we obtain iii).
**iii) $\Longrightarrow$ iv)** It is a direct consequence of the relation (2.57).
**iv) $\Longrightarrow$ ii)** We differentiate with respect to $r$ the relation (2.57) and we obtain

$$\frac{d}{dr} M^r_B(u)(x) = \frac{d+2\gamma}{r}\big(M^r_S(u)(x) - M^r_B(u)(x)\big) \geq 0.$$

This implies that $r \longmapsto M^r_B(u)(x)$ is non-decreasing on $]0, \varrho_x[$. □

### 2.5.2 Approximation results

Let us consider the following radial function

$$\varphi(x) := a \exp\big(-\frac{1}{1-\|x\|^2}\big)\mathbf{1}_{B(0,1)}(x), \quad x \in \mathbb{R}^d, \tag{2.93}$$

where $a$ is a constant such that $x \longmapsto \varphi(x)\omega_k(x)$ is a probability density.
For $n \geq 1$, define the function

$$\varphi_n(x) = n^{d+2\gamma}\varphi(nx). \tag{2.94}$$

It is clear that $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ is radial with supp $\varphi_n \subset B(0, 1/n)$.
For abbreviation, we introduce the following notation:

$$N_\Omega := min\{n \geq 1; \quad \Omega_{\frac{1}{n}} \neq \emptyset\}. \tag{2.95}$$

We begin by the following preparatory result:

**Proposition 2.33** *Let $u \in L^1_{k,loc}(\Omega)$. For $n \geq N_\Omega$, define the function $u_n$ by*

$$\forall\ x \in \Omega_{\frac{1}{n}}, \quad u_n(x) := u *_k \varphi_n(x) := \int_{\mathbb{R}^d} u(y)\tau_{-x}\varphi_n(y)\omega_k(y)dy. \tag{2.96}$$

*Then the sequence $(u_n)_{n\geq N_\Omega}$ satisfies*

**i)** *for every $n \geq N_\Omega$, the function $u_n$ is in $\mathcal{C}^\infty(\Omega_{\frac{1}{n}})$,*

**ii)** *for almost every $x \in \Omega$, $u_n(x) \longrightarrow u(x)$ as $n \longrightarrow +\infty$,*

**iii)** *if $r < r_\Omega$ (see (2.39)), then for almost every $x \in \Omega_r$, $M_B^r(u_n)(x) \longrightarrow M_B^r(u)(x)$ as $n \longrightarrow +\infty$.*

*Proof:* **i)** This follows from Proposition 2.17, 2).
**ii)** By (2.71) we get

$$\forall \ \ y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\varphi_n(y) \leq a.n^{d+2\gamma} h_k(\frac{1}{n}, x, y).$$

Using (2.26), we can write

$$\forall \ \ y \in \mathbb{R}^d, \quad 0 \leq \tau_{-x}\varphi_n(y) \leq a\frac{d_k}{d+2\gamma}\frac{1}{m_k[B(0,1/n)]} h_k(\frac{1}{n}, x, y). \tag{2.97}$$

Consequently, for every $x \in \Omega$ and every $n > 0$ large enough, we have by (2.12) and (2.97)

$$\begin{aligned} |u_n(x) - u(x)| &\leq \int_{\mathbb{R}^d} \tau_{-x}\varphi_n(y)|u(y) - u(x)|\omega_k(y)dy \\ &\leq a\frac{d_k}{d+2\gamma}\frac{1}{m_k[B(0,1/n)]}\int_{\mathbb{R}^d} |u(y) - u(x)| h_k(\frac{1}{n}, x, y)\omega_k(y)dy. \end{aligned}$$

This can be rewritten, with $c = a\frac{d_k}{d+2\gamma}$, in the following form

$$|u_n(x) - u(x)| \leq cM_B^{1/n}\Big(u - u(x)\Big)(x). \tag{2.98}$$

Hence, using Lebesgue's differentiation Theorem 2.5, for almost all $x \in \Omega$ we obtain

$$\lim_{n\to+\infty} u_n(x) = u(x).$$

**iii)** As $M_B^r(u) \in L^1_{k,loc}(\Omega_r)$ (see Lemma 2.4), using the relation (2.70) and the same proof of ii) where we replace $u$ by $M_B^r(u)$ and $u_n$ by $M_B^r(u_n)$ to obtain the result. This finishes the proof. □

**Remark 2.34** *If $u$ is continuous, by (2.31) and (2.98), we note that $u(x) = \lim_{n\to+\infty} u_n(x)$ for all $x \in \Omega$.*

Let us come to the main result of this section:

**Theorem 2.35** *Let $u \in \mathcal{SH}_k(\Omega)$ and $u_n$ the functions defined by (2.96). Then*

**1)** *for every $n \geq N_\Omega$, the function $u_n$ is D-subharmonic and of class $C^\infty$ on $\Omega_{\frac{1}{n}}$,*

**2)** *for every $N \geq N_\Omega$, the sequence $(u_n)_{n\geq N}$ of $C^\infty$ and D-subharmonic functions on $\Omega_{\frac{1}{N}}$ is decreasing and converge pointwise to $u$ in $\Omega_{\frac{2}{N}}$,*

**3)** *for all $B(x,r) \subset \Omega$, $M_B^r(u_n)(x) \longrightarrow M_B^r(u)(x)$ and $M_S^r(u_n)(x) \longrightarrow M_S^r(u)(x)$ as $n \longrightarrow +\infty$.*

*Proof:* **1)** By Proposition 2.25, $u \in L^1_{k,loc}(\Omega)$ and then from Proposition 2.33 we deduce that $u_n$ is of class $C^\infty$ on $\Omega_{\frac{1}{n}}$. On the other hand, as $u$ is D-subharmonic on $\Omega$ and $\tau_{-x}\varphi_n \geq 0$, (2.70) implies that

$$M_B^r(u_n)(x) \geq u_n(x), \quad \text{for all } B(x,r) \subset \Omega_{\frac{1}{n}}.$$

Therefore, $u_n$ is D-subharmonic on $\Omega_{\frac{1}{n}}$.
**2)** Choose $N \geq 2N_\Omega$ (i.e. $\Omega_{\frac{2}{N}}$ is nonempty). By 1), we have $u_n \in \mathcal{C}^\infty(\Omega_{\frac{1}{N}}) \cap \mathcal{SH}_k(\Omega_{\frac{1}{N}})$ for all $n \geq N$.
• We will prove that the sequence $(u_n)_{n\geq N}$ is decreasing. We will do this in two steps.
`Step1:` Suppose that $u$ is of class $C^\infty$ on $\Omega$. By (2.79), we can write

$$u_n(x) = d_k \int_0^{1/n} \widetilde{\varphi}_n(t) t^{d+2\gamma-1} M_S^t(u)(x)dt, \tag{2.99}$$

where $\widetilde{\varphi}_n$ is the profile function of $\varphi_n$. Using the change of variables $\rho = nt$ in (2.99) and recalling (2.94), we deduce that

$$u_n(x) = d_k \int_0^1 \widetilde{\varphi}(\rho)\ \rho^{d+2\gamma-1} M_S^{\rho/n}(u)(x)d\rho.$$

Since, $r \longmapsto M_S^r(u)(x)$ is non-decreasing (see Proposition 2.32), we conclude that $(u_n)_{n\geq N}$ is a decreasing sequence.
`Step 2:` Suppose only that $u \in \mathcal{SH}_k(\Omega)$. In order to use the same idea many times in the sequel of this paper, we will present the argument in the form of the following fundamental approximation lemma:

**Lemma 2.36** *Let $v \in L^1_{k,loc}(\Omega)$ and $(\varphi_n)$ the sequence defined by (2.94). Assume that for any $n \geq N_\Omega$, the function $v *_k \varphi_n$ belongs to $\mathcal{SH}_k(\Omega_{\frac{1}{n}})$. Then*

**a)** *for every $j \geq N_\Omega$, the sequence $(v *_k \varphi_n)_{n\geq j}$ is decreasing on $\Omega_{\frac{2}{j}}$,*

**b)** *the function $s : x \mapsto \lim_{n\to+\infty} v *_k \varphi_n(x)$ is well defined and D-subharmonic on $\Omega$ and we have $v = s$ almost everywhere on $\Omega$.*

Assume the result of the Lemma for the moment.
By Proposition 2.25 and the statement 1), the hypotheses of Lemma 2.36 are satisfied. Consequently, using Lemma 2.36-a), we deduce that $(u_n)_{n\geq N}$ is decreasing on $\Omega_{2/N}$. On the other hand, by the assertion b) of Lemma 2.36 and the uniqueness principle (Corollary 2.27), we obtain $u(x) = lim_{n\to+\infty} u_n(x)$.
**3)** By 2), the result follows immediately from the monotone convergence theorem. □

*Proof of Lemma 2.36:* **a)** Fix $j \geq N_\Omega$. By our hypothesis and Proposition 2.33, for every $n \geq j$, the function $v *_k \varphi_n \in \mathcal{C}^\infty(\Omega_{\frac{1}{n}}) \cap \mathcal{SH}_k(\Omega_{\frac{1}{n}})$. Consequently, by the statement 1), the functions $[v *_k \varphi_n] *_k \varphi_m$, with $m, n \geq j$, are in $\mathcal{C}^\infty(\Omega_{\frac{2}{j}}) \cap \mathcal{SH}_k(\Omega_{\frac{2}{j}})$. Furthermore, by the step 1, we have

$$\forall\ n, m \geq j, \quad \forall\ x \in \Omega_{\frac{2}{j}}, \quad [v *_k \varphi_n] *_k \varphi_{m+1}(x) \leq [v *_k \varphi_n] *_k \varphi_m(x).$$

By (2.78) the previous inequality can be written

$$\forall\, n, m \geq j, \quad \forall\, x \in \Omega_{\frac{2}{j}}, \quad [v *_k \varphi_{m+1}] *_k \varphi_n(x) \leq [v *_k \varphi_m] *_k \varphi_n(x).$$

Finally, letting $n \longrightarrow +\infty$, by Remark 2.34, we obtain

$$\forall\, m \geq j, \quad \forall\, x \in \Omega_{\frac{2}{j}}, \quad v *_k \varphi_{m+1}(x) \leq v *_k \varphi_m(x).$$

This proves the assertion a).
**b)** Let $x \in \Omega$. Since $\Omega = \cup_{j\geq 1}\Omega_{\frac{2}{j}} = \cup_{j\geq 2N_\Omega}\Omega_{\frac{2}{j}}$, there is $N \in \mathbb{N}$ such that $x \in \Omega_{2/j}$ for all $j \geq N$. This proves that the limit $s(x)$ of the decreasing sequence $(v *_k \varphi_n)_{n\geq N}$ which we denote simply $\lim_{n\to+\infty} v *_k \varphi_n(x)$ exists. Then, it suffices to show that $s$ is D-subharmonic on $\Omega_{2/j}$ and coincides with $v$ almost everywhere on any $\Omega_{2/j}$. Fix then $j \geq 2N_\Omega$. Using Proposition 2.33, we get $s = v$ almost everywhere on $\Omega_{2/j}$. In particular $s \neq -\infty$ on each connected component of $\Omega_{2/j}$. Consequently, by a) and Proposition 2.28 we deduce that $s$ is D-subharmonic on $\Omega_{2/j}$ as a pointwise decreasing limit of D-subharmonic functions on $\Omega_{2/j}$.

□

**Remark 2.37** *We can recapitulate the previous result in the following short form: Let $u \in \mathcal{SH}_k(\Omega)$. Then, for every $\rho > 0$ small enough, we can find a decreasing sequence of $C^\infty$-D-subharmonic functions on $\Omega_\rho$ which converges pointwise to $u$ on $\Omega_\rho$.*

### 2.5.3 Applications to D-harmonic functions

We will give some further results about Dunkl-harmonic functions. In particular, as it has been mentioned in Subsection 2.4, we will improve Theorem 3.2 in [16].

**Proposition 2.38** *Let $u \in \mathcal{C}^2(\Omega)$.*

1. *Let $u_n$ be the function defined by (2.96). Then*

$$\forall\, x \in \Omega_{\frac{1}{n}}, \quad \Delta_k u_n(x) := \Delta_k(u *_k \varphi_n)(x) = (\Delta_k u) *_k \varphi_n(x). \tag{2.100}$$

   *In particular, if $u$ is D-harmonic in $\Omega$, $u_n$ is also D-harmonic in $\Omega_{\frac{1}{n}}$.*

2. *The following statements are equivalent*

   **i)** *$u$ is D-harmonic in $\Omega$,*

   **ii)** *$u(x) = M_S^r(u)(x)$ whenever $B(x,r) \subset \Omega$,*

   **iii)** *$u(x) = M_B^r(u)(x)$ whenever $B(x,r) \subset \Omega$.*

3. *Every D-harmonic function on $\Omega$ is of class $C^\infty$.*

4. *A function $u : \Omega \longrightarrow \mathbb{R}$ is D-harmonic on $\Omega$ if and only if it is simultaneously D-subharmonic and D-superharmonic on $\Omega$.*

*Proof:*

1. We have

$$\begin{aligned}\Delta_k u_n(x) = u *_k (\Delta_k \varphi_n)(x) &= \int_\Omega u(y)\tau_{-x}[\Delta_k\varphi_n](y)\omega_k(y)dy \\ &= \int_\Omega u(y)\Delta_k[\tau_{-x}\varphi_n](y)\omega_k(y)dy \\ &= \int_\Omega \Delta_k u(y)\tau_{-x}\varphi_n(y)\omega_k(y)dy = (\Delta_k u) *_k \varphi_n(x),\end{aligned}$$

where in the first line we have used (2.69), the relation (2.164) in the second line and in the last line, the following integration by parts formula (see [11] or [36])

**Lemma 2.39** *Let $f, g \in C^1(\Omega)$ such that $g$ has compact support. Then, for all $\xi \in \mathbb{R}^d$, we have*

$$\int_\Omega D_\xi f(x)g(x)\omega_k(x)dx = -\int_\Omega f(x)D_\xi g(x)\omega_k(x)dx. \tag{2.101}$$

2. *i)* $\Longrightarrow$ *ii)* Suppose that $u$ is D-harmonic on $\Omega$ and fix $x \in \Omega$, $r > 0$ such that $B(x,r) \subset \Omega$. There is an integer $N$ such that $B(x,r) \subset \Omega_{\frac{1}{n}}$ for every $n \geq N$. From (2.100), the functions $u_n$, $n \geq N$, are of class $C^\infty$ and D-harmonic on $\Omega_{\frac{1}{N}}$. Therefore, using Lemma 2.14, we deduce that

$$u_n(x) = M_S^r(u_n)(x).$$

As $u_n$ is D-subharmonic, we can apply the statements 2) and 3) of Theorem 2.35 to obtain

$$u(x) = M_S^r(u)(x).$$

*ii)* $\Longrightarrow$ *iii)* By Lemma 2.15, this is obvious.
*iii)* $\Longrightarrow$ *i)* This follows from Theorem 3.2 in [16]. But, for completeness, we will give another proof. Assume that $u(y) = M_B^r(u)(y)$ for every $B(y,r) \subset \Omega$. In other words $u = M_B^r(u)$ on $\Omega_r$ for every $r < r_\Omega$. Then, for every $n$ large enough, $u_n = u *_k \varphi_n = M_B^r(u) *_k \varphi_n$ on $\Omega_{r+\frac{1}{n}}$. But, from (2.70) the previous relation can be written

$$u_n = M_B^r(u_n) \quad \text{on} \quad \Omega_{r+\frac{1}{n}}.$$

This implies, applying Lemma 2.14, that $u_n$ is D-harmonic on $\Omega_{\frac{1}{n}}$. That is $\Delta_k u_n(x) = 0$ for every $x \in \Omega_{\frac{1}{n}}$. Now, using (2.100) and the assertion ii) of Proposition 2.33, we obtain that for almost every $x \in \Omega$, $\Delta_k u(x) = 0$. Hence, by continuity, we get $\Delta_k u(x) = 0$ for every $x \in \Omega$.

3. From (2.79) with $f = \varphi_n$ and the statement 2), ii) we deduce that $u = u_n$ on $\Omega_{\frac{1}{n}}$ for every $n > 0$ (large enough). By Proposition 2.33, this proves that $u$ is in $\mathcal{C}^\infty(\Omega)$.

4. By the assertion 2), the necessity part is obvious. Let us now prove the sufficiency part. Let $u$ be simultaneously D-subharmonic and D-superharmonic on $\Omega$. It is enough to prove that $u$ is D-harmonic in $\Omega_{\frac{1}{n}}$ for every $n > 0$ large enough.

Consider the function $u_n$ defined by (2.96). Clearly, by Theorem 2.35, the functions $u_n$ and $-u_n$ are in $\mathcal{C}^\infty(\Omega_{\frac{1}{n}})\cap\mathcal{SH}_k(\Omega_{\frac{1}{n}})$. Hence, by Proposition 2.31, $u_n$ is D-harmonic in $\Omega_{\frac{1}{n}}$. Therefore, $u_n(x) = M_S^r(u_n)(x)$ whenever $B(x,r) \subset \Omega_{\frac{1}{n}}$. Letting $n \longrightarrow +\infty$ and using Theorem 2.35, we deduce that

$$u(x) = M_S^r(u)(x), \quad \text{whenever} \quad B(x,r) \subset \Omega.$$

Finally, if we use (2.79) (with $f = \varphi_n$) we conclude that $u = u_n$ on $\Omega_{\frac{1}{n}}$ and then $u$ is $D$-harmonic in $\Omega_{\frac{1}{n}}$ as desired. This completes the proof of the proposition. $\square$

**Corollary 2.40** *Every D-harmonic function on $\mathbb{R}^d$ is real analytic.*

*Proof:* Let $f$ be a D-harmonic function on $\mathbb{R}^d$. Since $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ (by Proposition 2.38) and $V_k : \mathcal{C}^\infty(\mathbb{R}^d) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^d)$ is a topological isomorphism, the function $g := V_k^{-1}(f)$ is harmonic on $\mathbb{R}^d$ in the usual sense (i.e. $\Delta g = 0$) as an immediate consequence of the intertwining relation [8] $\Delta_k V_k = V_k \Delta$. It is well known that $g$ is real analytic (see [3]) and thus, using multi-indices $\upsilon = (\upsilon_1, ..., \upsilon_d) \in \mathbb{N}^d$, can be written

$$g(x) = \sum_{\upsilon} a_\upsilon x^\upsilon, \quad x \in \mathbb{R}^d,$$

where $a_\upsilon$ are real coefficients. If $g_N$ ($N \in \mathbb{N}$) denotes the partial sum $g_N(x) := \sum_{|\upsilon|\le N} a_\upsilon x^\upsilon$ (with $|\upsilon| = \upsilon_1 + \cdots + \upsilon_d$), then $g_N \longrightarrow g$ as $N \to +\infty$ in the Fréchet topology of $\mathcal{C}^\infty(\mathbb{R}^d)$. Therefore $V_k(g_N) \longrightarrow V_k(g) = f$ in the Fréchet topology. In particular, $f$ is real analytic as being the uniform limit of the polynomials [9] $V_k(g_N)$ on each compact subset of $\mathbb{R}^d$.

### 2.5.4 Applications to D-subharmonic functions

**Proposition 2.41** *Let $u \in \mathcal{SH}_k(\Omega)$ and $\rho > 0$ such that $\Omega_\rho$ is nonempty. Then the function $M_B^\rho(u)$ is continuous and D-subharmonic on $\Omega_\rho$.*

*Proof:* Let $x \in \Omega_\rho$ and $\eta > 0$ such that $B(x, \rho + \eta) \subset \Omega$.

• We can assume that $u$ is nonpositive on the Dunkl ball $B^W(x, \rho + \eta)$ and consider the sequence $(\varphi_\varepsilon)$ ( with $\sqrt{\varepsilon} \le \eta$) as in Remark 2.1. By the positivity of the Dunkl translations on radial functions, the sequence of functions $y \longmapsto \tau_{-x}\varphi_\varepsilon(y)$ is decreasing as $\varepsilon \downarrow 0$. Consequently, by the monotone convergence theorem, we have

$$M_B^\rho(u)(x) = \lim_{\varepsilon\to 0} \frac{1}{m_k[B(0,\rho)]} \int_{\mathbb{R}^d} u(y)\tau_{-x}\varphi_\varepsilon(y)\omega_k(y)dy = \lim_{\varepsilon\to 0} \frac{u *_k \varphi_\varepsilon(x)}{m_k[B(0,\rho)]}.$$

By Proposition 2.17, the functions $u *_k \varphi_\varepsilon$ are of class $C^\infty$ on a neighborhood of $x$ and as $u \le 0$, the sequence $(u *_k \varphi_\varepsilon)$ is increasing when $\varepsilon \downarrow 0$. This proves that $M_B^\rho(u)$ is lower semi-continuous (l.s.c.) at $x$ as being the increasing limit of a sequence of continuous

8. which follows clearly from (2.3).
9. $V_k$ is a bijection of the space of polynomials of degree $\le n$ onto itself ([13]).

functions.

• Let $(u_n)_{n\geq 1/\eta}$ the sequence defined by (2.96). From the relation (2.70) and Proposition 2.17, we deduce that the function $M_B^\rho(u_n)$ is of class $C^\infty$ on $B(x,\eta)$. Now, by Theorem 2.35, we see that $M_B^\rho(u)$ is u.s.c. on $B(x,\eta)$ as being a pointwise decreasing limit of the sequence $\big(M_B^\rho(u_n)\big)$.

Thus, we obtain the continuity of $M_B^\rho(u)$ on $\Omega_\rho$.

• Let us now prove that $M_B^\rho(u)$ satisfies the sub-mean property. Fix then $x \in \Omega_\rho$ and $r > 0$ such that $B(x,r) \subset \Omega_\rho$. From (2.75), we can see that the inequality $M_B^r(u) \geq u$ holds on $B^W(x,\rho)$. Therefore, using (2.77), we deduce that

$$M_B^r\Big(M_B^\rho(u)\Big)(x) = M_B^\rho\Big(M_B^r(u)\Big)(x) \geq M_B^\rho(u)(x).$$

□

Now, we will extend the results of Proposition 2.32 to any D-subharmonic function (see [2], Corollary 3.2.6 for the classical case).

**Theorem 2.42** *Let $u$ be an u.s.c. function on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$. Assume that $u$ is not identically $-\infty$ on each connected component of $\Omega$. Then the statements i), ii), iii) of Proposition 2.32 and*

**iv)** $u \in L^1_{k,loc}(\Omega)$, $\lim_{r\to 0} M_B^r(u)(x) = u(x)$ *for every* $x \in \Omega$ *and* $M_B^r(u)(x) \leq M_S^r(u)(x)$, *whenever* $B(x,r) \subset \Omega$,

*are equivalent.*

*Proof:* **i)** $\Longrightarrow$ **ii)** Let $u \in \mathcal{SH}_k(\Omega)$. We already know that (2.91) holds (see Proposition 2.26). Let $(u_n)$ be the sequence defined by (2.96). By Theorem 2.35, $u_n \in \mathcal{C}^\infty(\Omega_{\frac{1}{n}}) \cap \mathcal{SH}_k(\Omega_{\frac{1}{n}})$. Therefore, using Proposition 2.32, $r \longmapsto M_B^r(u_n)(x)$ is non-decreasing on $]0, dist(x, \partial\Omega_{1/n})[$. Letting $n \longrightarrow +\infty$ and using Theorem 2.35, 3), we deduce that $r \longmapsto M_B^r(u)(x)$ is also non-decreasing.

**ii)** $\Longrightarrow$ **i)** This is obvious.

**i)** $\Longrightarrow$ **iii)** If $u \in \mathcal{SH}_k(\Omega) \cap \mathcal{C}^\infty(\Omega)$, the result is proved in Proposition 2.32.

Let us now suppose only that $u \in \mathcal{SH}_k(\Omega)$. By Proposition 2.33 and Theorem 2.35, the functions $u_n$ defined by (2.96) are in $\mathcal{SH}_k(\Omega_{\frac{1}{n}}) \cap \mathcal{C}^\infty(\Omega_{\frac{1}{n}})$.

Consequently, we have

**a)** the function $r \longmapsto M_S^r(u_n)(x)$ is non-decreasing in $r$,

**b)** for all $n \geq N$, $\lim_{r\to 0} M_S^r(u_n)(x) = u_n(x)$,

**c)** for all $n \geq N$, $u_n(x) \leq M_S^r(u_n)(x)$,

where $N = N(x) > 0$ is such that $x \in \Omega_{\frac{1}{n}}$ for all $n \geq N$.

From a) and Theorem 2.35, we can see that $r \longmapsto M_S^r(u)(x)$ is also non-decreasing as a pointwise limit of non-decreasing functions.

Using c) and letting $n \longrightarrow +\infty$, we have $u(x) \leq M_S^r(u)(x)$. Moreover, since $(u_n)_{n\geq N}$ is a decreasing sequence, we deduce that

$$\forall\, n \geq N, \quad u(x) \leq M_S^r(u)(x) \leq M_S^r(u_n)(x).$$

According to b), this implies that

$$\forall\ n \geq N, \quad u(x) \leq \lim_{r\to 0} M_S^r(u)(x) \leq \lim_{r\to 0} M_S^r(u_n)(x) = u_n(x).$$

Finally, letting $n \longrightarrow +\infty$, we deduce the desired result.
**iii)** $\Longrightarrow$ **i)** Let $x \in \Omega$ and $r \in ]0, \varrho_x[$ be fixed and assume that $u$ is nonpositive on the Dunkl ball $B^W(x, r)$ (using the upper semi-continuity of $u$). For all $\rho \in ]0, r[$, we have

$$\frac{2\gamma + d}{r^{2\gamma+d}} \int_\rho^r M_S^t(u)(x) t^{2\gamma+d-1} dt \geq M_S^\rho(u)(x)\big(1 - (\rho/r)^{d+2\gamma}\big).$$

Since $t \longmapsto M_S^t(u)(x)$ is nonpositive on $]0, r]$, letting $\rho \longrightarrow 0$ and using the monotone convergence theorem, Lemma 2.15 and the relation (2.92), we obtain

$$M_B^r(u)(x) \geq u(x).$$

This proves that $u$ is D-subharmonic on $\Omega$.
**i)** $\Longrightarrow$ **iv)** Let $u \in \mathcal{SH}_k(\Omega)$. We know that the function $u\omega_k$ is locally integrable on $\Omega$ and $\lim_{r\to 0} M_B^r(u)(x) = u(x)$ for every $x \in \Omega$. By Proposition 2.32, the result is true when $u \in \mathcal{C}^\infty(\Omega)$. Now, suppose only that $u$ is in $\mathcal{SH}_k(\Omega)$. Considering the D-subharmonic functions $u_n$ defined in Theorem 2.35 , we get for $n$ large enough

$$M_B^r(u_n)(x) \leq M_S^r(u_n)(x).$$

By Theorem 2.35, we deduce that $M_B^r(u)(x) \leq M_S^r(u)(x)$.
**iv)** $\Longrightarrow$ **i)** We will use the same idea as in [18] (Lemma 2.4.4). First, we need the following lemma:

**Lemma 2.43** *Let $f \in L^1_{k,loc}(\Omega)$ be an u.s.c. function. Then for every $x \in \Omega$ and $r > 0$ such that $B(x, r) \subset \Omega$, the function $t \longmapsto M_S^t(f)(x)t^{d+2\gamma-1}$ is integrable on $[0, r]$ and we have*

$$M_B^r(f)(x) = \frac{d + 2\gamma}{r^{d+2\gamma}} \int_0^r M_S^t(f)(x)t^{d+2\gamma-1}dt. \tag{2.102}$$

*Proof:* Assume that $f$ is nonpositive in the fixed Dunkl ball $B^W(x, r) \subset \Omega$. The formula (2.102) has been established in Lemma 2.15. Therefore, it suffices to show that $M_B^r(f)(x) \neq -\infty$. Denoting $C_r := (m_k(B(0, r)))^{-1}$, by (2.22) and the fact that $h_k(r, x, y) \leq 1$, we get

$$|M_B^r(f)(x)| \leq C_r \int_{B^W(x,r)} |f(y)|h_k(r, x, y)\omega_k(y)dy \leq C_r \int_{B^W(x,r)} |f(y)|\omega_k(y)dy < +\infty.$$

$\square$

Now, we turn to the proof of *iv)* $\Longrightarrow$ *i)*. Let $x \in \Omega$. Suppose that $M_B^r(u)(x) \leq M_S^r(u)(x)$ for every $r \in ]0, \varrho_x[$. Since $u \in L^1_{k,loc}(\Omega)$, by Lemma 2.43, the function $r \longmapsto M_B^r(u)(x)$ is absolutely continuous on every closed interval $[a, b] \subset ]0, \varrho_x[$ as a product of two absolutely continuous functions. Hence, it is almost everywhere differentiable on $[a, b]$ and we have

$$\frac{d}{dr} M_B^r(u)(x) = \frac{d+2\gamma}{r}\big(M_S^r(u)(x) - M_B^r(u)(x)\big) \geq 0 \quad a.e..$$

Thus, $r \longmapsto M_B^r(u)(x)$ is non-decreasing on $[a, b]$ (see [8], Proposition 5.3). That is, for every $0 < t \leq r < \varrho_x$, we have $M_B^t(u)(x) \leq M_B^r(u)(x)$. Letting $t \longrightarrow 0$, we deduce that $u(x) \leq M_B^r(u)(x)$. This proves that $u$ is in $\mathcal{SH}_k(\Omega)$ and the Theorem is completely proved.

$\square$

## 2.6 $\Delta_k$-Riesz measure and Weyl's lemma

In this section, we will introduce the Riesz measure associated to a D-subharmonic function on a $W$-invariant open set $\Omega \subset \mathbb{R}^d$ and we will prove the Weyl lemma for D-harmonic functions.

For a distribution $T \in \mathcal{D}'(\Omega)$, we define the weak Dunkl $\xi$-directional derivative of $T$ ($\xi \in \mathbb{R}^d$) by

$$\forall\ \phi \in \mathcal{D}(\Omega), \quad \langle D_\xi T, \phi\rangle = -\langle T, D_\xi \phi\rangle\,.$$

Note that by the intertwining relation (2.3), the operator $D_\xi = V_k \partial_\xi V_k^{-1} : \mathcal{C}^\infty(\mathbb{R}^d) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^d)$ is continuous for the Fréchet topology. Moreover, since $D_\xi$ leaves the space $\mathcal{D}(\Omega)$ invariant, we deduce that $D_\xi : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$ is also continuous for the Fréchet topology. This justifies that $D_\xi T$ is well defined as an element of $\mathcal{D}'(\Omega)$.

In particular, if $f \in L^1_{k,loc}(\Omega)$ i.e. $f\omega_k \in L^1_{loc}(\Omega)$, the weak Dunkl-Laplacian of $f\omega_k$ is given by

$$\forall\ \phi \in \mathcal{D}(\Omega), \quad \langle \Delta_k(f\omega_k), \phi\rangle = \langle f\omega_k, \Delta_k\phi\rangle = \int_\Omega f(x)\Delta_k\phi(x)\omega_k(x)dx. \qquad (2.103)$$

### 2.6.1 $\Delta_k$-Riesz measure

We will start by the following preliminary example:

**Proposition 2.44** *The function $v = -S$, where $S$ is the fundamental solution of $\Delta_k$ defined by (2.16), is D-subharmonic on $\mathbb{R}^d$ and satisfies*

$$\Delta_k(v\omega_k) = \delta_0 \quad in \quad \mathcal{D}'(\Omega).$$

*Proof:* We already know that $\Delta_k(v\omega_k) = \delta_0$ (see (2.18) and Annex 9.3). Let $(v * \varphi_n)$ the sequence of functions defined by (2.96). Since $v \in L^1_{k,loc}(\mathbb{R}^d)$, by Proposition 2.33, $v * \varphi_n \in \mathcal{C}^\infty(\mathbb{R}^d)$ for every $n \geq 1$. Moreover, if we use respectively (2.69), (2.164), (2.103), (2.18) and (2.161), we obtain

$$\begin{aligned}\forall\ x \in \mathbb{R}^d, \quad \Delta_k(v * \varphi_n)(x) &= \int_{\mathbb{R}^d} v(y)\tau_{-x}[\Delta_k\varphi_n](y)\omega_k(y)dy = \int_{\mathbb{R}^d} v(y)\Delta_k[\tau_{-x}\varphi_n](y)\omega_k(y)dy \\ &= \langle \Delta_k(v\omega_k), \tau_{-x}\varphi_n\rangle = \tau_{-x}\varphi_n(0) = \varphi_n(-x).\end{aligned}$$

Hence, as $\varphi_n \geq 0$ (see (2.94)), we deduce by Proposition 2.31 that $v * \varphi_n \in \mathcal{SH}_k(\mathbb{R}^d)$, $\forall\ n \geq 1$.
Now, by Lemma 2.36, there exists a D-subharmonic function $s$ on $\mathbb{R}^d$ such that $s = v$ almost everywhere on $\mathbb{R}^d$ and $s(x) = \lim_{n\to+\infty} v *_k \varphi_n(x)$. On the other hand, as $v$ is D-harmonic on $\mathbb{R}^d\backslash\{0\}$, by the uniqueness principle (see Corollary 2.27), $s = v$ on $\mathbb{R}^d\backslash\{0\}$. Moreover, by a change of variables, we get

$$v *_k \varphi_m(0) = \int_{\mathbb{R}^d} v(\frac{y}{m})\varphi(y)\omega_k(y)dy = -m^{d+2\gamma-2}\int_{\mathbb{R}^d} \|y\|^{2-d-2\gamma}\varphi(y)\omega_k(y)dy,$$

where $\varphi$ is defined by (2.93). This implies that $s(0) = \lim_{m\to+\infty} v *_k \varphi_m(0) = -\infty = v(0)$. Finally, we conclude that $v = s$ on $\mathbb{R}^d$.

$\square$

**Proposition 2.45** *Let $u \in \mathcal{SH}_k(\Omega)$. Then there exists a nonnegative Radon measure $\mu$ in $\Omega$ such that*

$$\Delta_k[u\omega_k] = \mu \quad \textit{in the sense of distributions.} \tag{2.104}$$

*We will call $\mu$ the $\Delta_k$-Riesz measure related to $u$.*

*Proof:* As $u \in L^1_{k,loc}(\Omega)$ , $u\omega_k$ defines a distribution. Let $\phi \in \mathcal{D}(\Omega)$ and let $(u_n)_{n\geq N}$ be the sequence of functions defined by (2.96) with $N$ such that supp $\phi \subset \Omega_{1/N}$. As $0 \leq u_n - u \leq u_N - u$, by Theorem 2.35 and the dominated convergence theorem, we have

$$\langle\Delta_k[u\omega_k], \phi\rangle = \int_\Omega u(x)\Delta_k\phi(x)\omega_k(x)dx = \lim_{n\to+\infty}\int_\Omega u_n(x)\Delta_k\phi(x)\omega_k(x)dx.$$

Now, using the integration by parts formula (2.101), we deduce that

$$\langle\Delta_k[u\omega_k], \phi\rangle = \lim_{n\to+\infty}\int_\Omega \Delta_k u_n(x)\phi(x)\omega_k(x)dx. \tag{2.105}$$

Consequently, $[\Delta_k u_n]\omega_k \longrightarrow \Delta_k[u\omega_k]$ in $\mathcal{D}'(\Omega)$ as $n \to +\infty$. Moreover, from (2.105) and the fact that $\Delta_k u_n \geq 0$ (Theorem 2.31 and Proposition 2.31), we see that $\langle\Delta_k[u\omega_k], \phi\rangle \geq 0$ for every nonnegative function $\phi$ in $\mathcal{D}(\Omega)$. Then there exists a nonnegative Radon measure $\mu$ on $\Omega$ (see [19], Theorem 2.1.7 or [38]) such that $\Delta_k[u\omega_k] = \mu$ and the proposition is proved. $\square$

**Remark 2.46** *Let $u \in \mathcal{SH}_k(\Omega)$, $\mu = \Delta_k[u\omega_k]$ its $\Delta_k$-Riesz measure and $u_n$ the sequence of approximation functions defined by (2.96). Using Theorem 2.1.9 in [19], the sequence of nonnegative Radon measures $d\mu_n(x) := (\Delta_k u_n)(x)\omega_k(x)dx$ converges to $d\mu(x)$ in weak topology of measures i.e. $\langle\mu_n, f\rangle \longrightarrow \langle\mu, f\rangle$, $(n \to +\infty)$ for every continuous function $f$ with compact support in $\Omega$.*

**Exemple 2.47** *1)The $\Delta_k$-Riesz measure associated to the function $-S$ is the Dirac measure at 0.*
*2) Let $u \in \mathcal{SH}_k(\Omega) \cap \mathcal{C}^2(\Omega)$. Then the $\Delta_k$-Riesz measure of $u$ is given by $\Delta_k u(x)\omega_k(x)dx$.*

*In particular, $\Delta_k u \geq 0$ on $\Omega$ in the strong sense. This generalizes the result of Proposition 2.31 for $C^2$-functions. Indeed, for $\phi \in \mathcal{D}(\Omega)$, we have*

$$\begin{aligned}\langle \Delta_k(u\omega_k), \phi\rangle &:= \int_\Omega u(x)\Delta_k\phi(x)\omega_k(x)dx\\ &= \int_\Omega \Delta_k u(x)\phi(x)\omega_k(x)dx, \quad \text{by (2.101).}\end{aligned}$$

*This shows that $(\Delta_k u)\omega_k \geq 0$ almost everywhere on $\Omega$. But $\Delta_k u$ is continuous, then $\Delta_k u \geq 0$ on $\Omega$.*

If $u$ is a $D$-subharmonic function on $\Omega$, we know from Proposition 2.41 that $M_B^\rho(u)$ ($\rho > 0$ small enough) is also D-subharmonic on $\Omega_\rho$. In the following result, we will give its $\Delta_k$-Riesz measure.

**Proposition 2.48** *Let $u \in \mathcal{SH}_k(\Omega)$ and $\mu$ be its related $\Delta_k$-Riesz measure and let $\rho > 0$ be small enough. Then the $\Delta_k$-Riesz measure of the D-subharmonic function $M_B^\rho(u)$ is given by*

$$\Delta_k\Big(M_B^\rho(u)\omega_k\Big) = M_B^\rho(\mu)(x)\omega_k(x)dx, \tag{2.106}$$

*where*

$$M_B^\rho(\mu)(x) := \frac{1}{m_k[B(0,\rho)]}\int_{\mathbb{R}^d} h_k(\rho, x, y)d\mu(y). \tag{2.107}$$

*Proof:* Let $\phi \in \mathcal{D}(\Omega_\rho)$. To make the proof more readable, we will use the following notations and technical lemma:

- $C_\rho = m_k[B(0,\rho)]$.
- Let $x_0 \in \text{supp } \phi$ such that $dist(x_0, \partial\Omega) = dist(\text{supp } \phi, \partial\Omega)$.
- $(\varphi_\varepsilon)$ denotes the already used sequence of functions satisfying $\tau_{-x}\varphi_\varepsilon(y) \downarrow h_k(\rho, x, y)$, (as$\varepsilon \to 0$) such that supp $\tau_{-x}\varphi_\varepsilon \subset B^W(x, \rho + \sqrt{\varepsilon})$ and $0 \leq \tau_{-x}\varphi_\varepsilon \leq 1$ (see Remark 2.1).

**Lemma 2.49** *Let $\varepsilon_0 > 0$ such that $\sqrt{\varepsilon_0} < dist(B(x_0,\rho), \partial\Omega)$. Then we have*

$$\forall\ x \in supp\ \phi, \quad B^W(x, \rho+\sqrt{\varepsilon_0}) \subset K_{\rho+\sqrt{\varepsilon_0}} := B(0, \rho+\sqrt{\varepsilon_0}) + W.supp\ \phi \subset \Omega \tag{2.108}$$

*and*

$$\forall\ \varepsilon \in ]0, \varepsilon_0], \quad supp\ \phi *_k \varphi_\varepsilon \subset K_{\rho+\sqrt{\varepsilon}} = B(0, \rho+\sqrt{\varepsilon}) + W.supp\ \phi \subset \Omega. \tag{2.109}$$

*Proof of Lemma* 2.49: • At first, we claim that supp $\phi \subset \Omega_{\rho+\sqrt{\varepsilon_0}}$. Indeed, let $x \in$ supp $\phi$. For every $y \in \partial\Omega$, we have

$$\begin{aligned}\|x-y\| \geq \|x_0-y\| &= \|x_0 - p(y)\| + \|p(y)-y\| = \rho + \|p(y)-y\|\\ &\geq \rho + dist(B(x_0,\rho), \partial\Omega),\end{aligned}$$

where $p(y)$ is the orthogonal projection of $y$ onto the closed ball $B(x_0,\rho)$.
This implies that

$$dist(x, \partial\Omega) \geq \rho + dist(B(x_0,\rho), \partial\Omega) > \rho + \sqrt{\varepsilon_0}.$$

• The first inclusion in the relation (2.108) is obvious by writing $B^W(x, \rho + \sqrt{\varepsilon_0}) = B(0, \rho + \sqrt{\varepsilon_0}) + W.x$. Now, let $x \in K_{\rho+\sqrt{\varepsilon_0}}$. Then $x$ is of the form $x = x_1 + gx_2$, where $x_1 \in B(0, \rho + \sqrt{\varepsilon_0})$ and $x_2 \in \text{supp } \phi$. Hence, as $x_2 \in \Omega_{\rho+\sqrt{\varepsilon_0}}$, $x \in B(gx_2, \rho + \sqrt{\varepsilon_0}) \subset \Omega$ (see (2.41)). Thus, $K_{\rho+\sqrt{\varepsilon_0}} \subset \Omega$.
• The first inclusion in (2.109) follows from (2.76) (replacing $u$ by $\phi$ and $f$ by $\varphi_\varepsilon$) and the second is clear by the choice of $\varepsilon_0$. □

Let us now turn to the proof of the proposition. Denoting $S_\phi := \text{supp } \phi$, we have

$$\begin{aligned}
C_\rho \left\langle \Delta_k\Big(M_B^\rho(u)\omega_k\Big), \phi\right\rangle &= \int_{S_\phi} \Big(\int_\Omega u(y)h_k(\rho, x, y)\omega_k(y)dy\Big)\Delta_k\phi(x)\omega_k(x)dx \\
&= \int_{S_\phi} \Big(\int_{B^W(x,\rho+\sqrt{\varepsilon_0})} u(y) \lim_{\varepsilon\to 0} \tau_{-x}\varphi_\varepsilon(y)\omega_k(y)dy\Big)\Delta_k\phi(x)\omega_k(x)dx \\
&= \lim_{\varepsilon\to 0} \int_{S_\phi} \Big(\int_{B^W(x,\rho+\sqrt{\varepsilon_0})} u(y)\tau_{-y}\varphi_\varepsilon(x)\omega_k(y)dy\Big)\Delta_k\phi(x)\omega_k(x)dx \\
&= \lim_{\varepsilon\to 0} \int_{K_{\rho+\sqrt{\varepsilon_0}}} u(y)\Big(\int_\Omega \Delta_k\phi(x)\tau_{-y}\varphi_\varepsilon(x)\omega_k(x)dx\Big)\omega_k(y)dy \\
&= \lim_{\varepsilon\to 0} \int_{K_{\rho+\sqrt{\varepsilon_0}}} u(y)\Delta_k[\phi *_k \varphi_\varepsilon](y)\omega_k(y)dy \\
&= \lim_{\varepsilon\to 0} \int_{K_{\rho+\sqrt{\varepsilon_0}}} \phi *_k \varphi_\varepsilon(y)d\mu(y) \\
&= \int_{K_{\rho+\sqrt{\varepsilon_0}}} \Big(\int_\Omega \phi(x)h_k(\rho, y, x)\omega_k(x)dx\Big)d\mu(y) \\
&= \int_\Omega \phi(x)\Big(\int_\Omega h_k(\rho, x, y)d\mu(y)\Big)\omega_k(x)dx,
\end{aligned}$$

where we have limited as much as possible the domains of integration in $\Omega$ to justify rigorously the use of Fubini's and the dominated convergence theorems. More precisely:
-the second equality follows from the aforementioned properties of $\tau_{-x}\varphi_\varepsilon$ and from (2.72),
-the third equality comes from the dominated convergence theorem and the relation (2.165),
- the fourth equality follows from (2.108) which make us able to use Fubini's theorem on the compact set $K_{\rho+\sqrt{\varepsilon_0}} \times K_{\rho+\sqrt{\varepsilon_0}}$ and finally we have written the middle integral over $\Omega$ instead of $K_{\rho+\sqrt{\varepsilon_0}}$,
-we obtain the fifth equality by using (2.69),
-we have used the relations (2.109) and (2.104) in the sixth equality and the dominated convergence theorem in the seventh equality,
- by (2.25) and Fubini's theorem, we have obtained the last equality. □

### 2.6.2 Weyl's lemma

A fundamental ingredient to prove the Riesz decomposition theorem for D-subharmonic functions is the so-called Weyl's lemma for the Dunkl-Laplace operator. We note that it has been proved in [26] only for a D-harmonic function $f$ on whole $\mathbb{R}^d$ and under the additional assumption that the function $f$ is locally bounded. Here, we will establish a more general form of Weyl's lemma for $D$-subharmonic functions on an arbitrary $W$-invariant open set $\Omega \subset \mathbb{R}^d$.

**Theorem 2.50** *Let $u \in L^1_{k,loc}(\Omega)$. If $\Delta_k(u\omega_k) \geq 0$ in $\mathcal{D}'(\Omega)$, then there exists a D-subharmonic function $s$ on $\Omega$ such that $u = s$ a.e. in $\Omega$.*

*Proof:* Let us denote by $\mu$ the nonnegative Radon measure $\Delta_k(u\omega_k)$ and let $\varphi_n$ be the function given by (2.94).
Now, we will need the following lemma:

**Lemma 2.51** *Under the hypothesis of Theorem 2.50, we have*

$$\forall\, x \in \Omega_{\frac{1}{n}}, \quad \Delta_k(u *_k \varphi_n)(x) = \mu *_k \varphi_n(x) := \int_\Omega \tau_{-x}\varphi_n(y)d\mu(y)\,^{10}, \tag{2.110}$$

*whith $n \in \mathbb{N}$ such that $\Omega_{1/n} \neq \emptyset$.*

*Proof:* By Proposition 2.33, the function $u *_k \varphi_n$ is of class $C^\infty$ on $\Omega_r$. Then, using respectively the relations (2.69), (2.68) and (2.164), we get

$$\begin{aligned}\Delta_k(u *_k \varphi_n)(x) = [u *_k (\Delta_k\varphi_n)](x) &= \int_\Omega u(y)\tau_{-x}[\Delta_k\varphi_n](y)\omega_k(y)dy \\ &= \int_\Omega u(y)\Delta_k[\tau_{-x}\varphi_n](y)\omega_k(y)dy = \langle u\omega_k, \Delta_k[\tau_{-x}\varphi_n]\rangle \\ &= \mu *_k \varphi_n(x).\end{aligned}$$

□

Let us return to the proof of our theorem: Since $\tau_{-x}\varphi_n \geq 0$, (2.110) implies that

$$\forall\, x \in \Omega_{\frac{1}{n}}, \quad \Delta_k[u *_k \varphi_n](x) \geq 0.$$

Hence, the function $u *_k \varphi_n$ is D-subharmonic on $\Omega_{\frac{1}{n}}$ (see Proposition 2.31). Thus, we obtain the result by using Lemma 2.36, b). □

In the following result, we characterize the D-subharmonicity by means of the positivity of the distributional Dunkl Laplacian.

**Corollary 2.52** *Let $u$ be a function defined on $\Omega$. Then we have the equivalence between*

1. *$u \in \mathcal{SH}_k(\Omega)$,*
2. *$u \in L^1_{k,loc}(\Omega)$, $u(x) = \lim_{r\to 0} M^r_B(u)(x)$ for every $x \in \Omega$ and $\Delta_k(u\omega_k) \geq 0$ in $\mathcal{D}'(\Omega)$.*

*Proof:* 1) $\Longrightarrow$ 2) The result follows from Propositions 2.25, 2.26 and 2.45.
2) $\Longrightarrow$ 1) By Theorem 2.50, there exists a function $v \in \mathcal{SH}_k(\Omega)$ such that $u(x) = v(x)$ for almost every $x \in \Omega$. Therefore, for all $x \in \Omega$ and all $r > 0$ small enough, we have $M^r_B(u)(x) = M^r_B(v)(x)$. Now, using Proposition 2.26, we deduce that $u$ and $v$ are identically equal in $\Omega$ and then $u$ is in $\mathcal{SH}_k(\Omega)$. □

10. Note that by (2.72), $\mu *_k \varphi_n$ is well defined on $\Omega_{1/n}$ for any nonnegative Radon measure $\mu$ on $\Omega$.

**Corollary 2.53** *Let $u \in \mathcal{C}^2(\Omega)$. Then $u \in \mathcal{SH}_k(\Omega)$ if and only if $\Delta_k u \geq 0$ on $\Omega$.*

*Proof:* The 'only if' part is already proved in Example 2.47, 2). The 'if' part is a direct consequence of the integration by parts formula (2.101) and the previous corollary because for $C^2$-functions, the condition $u(x) = \lim_{r\to 0} M_B^r(u)(x)$ is always satisfied (see (2.31)). $\square$

**Corollary 2.54** *The cone $\mathcal{SH}_k(\Omega)$ is closed for the $L^1_{k,loc}(\Omega)$ topology.*

*Proof:* Let $(u_n)$ be a sequence of D-subharmonic functions on $\Omega$ such that $u_n \longrightarrow u$ in $L^1_{k,loc}(\Omega)$. As, $u_n\omega_k$ and $u\omega_k$ are in $L^1_{loc}(\Omega)$, we deduce that $u_n\omega_k \longrightarrow u\omega_k$ in $\mathcal{D}'(\Omega)$. Hence, $\Delta_k(u_n\omega_k) \longrightarrow \Delta_k(u\omega)$ in $\mathcal{D}'(\Omega)$. By Corollary 2.52, as $\Delta_k(u_n\omega_k) \geq 0$, we deduce that $\Delta_k(u\omega_k) \geq 0$ in $\mathcal{D}'(\Omega)$.
Now, by Theorem 2.50 there exists a D-subharmonic function $s$ on $\Omega$ such that $u = s$ a.e. in $\Omega$. Then $u = s$ in $L^1_{k,loc}(\Omega)$ and the result is proved. $\square$

**Corollary 2.55** *If $u \in L^1_{k,loc}(\Omega)$ satisfies $\Delta_k[u\omega_k] = 0$ in $\mathcal{D}'(\Omega)$, then there exists a D-harmonic function $h$ on $\Omega$ such that $u$ and $h$ coincide a.e. on $\Omega$.*

*Proof:* From Theorem 2.50, there exist two functions $u_1, u_2$ such that $u_1$ is D-subharmonic on $\Omega$, $u_2$ is D-superharmonic on $\Omega$ and $u = u_1 = u_2$ almost everywhere. Moreover, by Proposition 2.26, we have

$$\forall\, x \in \Omega, \quad u_1(x) = \lim_{r\to 0} M_B^r(u_1)(x) = \lim_{r\to 0} M_B^r(u_2)(x) = u_2(x).$$

Therefore, the function $h := u_1 = u_2$ is simultaneously D-subharmonic and D-superharmonic on $\Omega$. Hence, using Proposition 2.38, 4) we deduce that $h$ is D-harmonic in $\Omega$ and $h = u$ almost everywhere in $\Omega$. $\square$

**Application:** The space $\mathcal{H}_k(\Omega)$ of Dunkl-harmonic functions on $\Omega$ is closed for the $L^1_{k,loc}(\Omega)$ topology i.e. if $(u_n)$ be a sequence of D-harmonic functions on $\Omega$ and $u_n \longrightarrow u$ in $L^1_{k,loc}(\Omega)$ as $n \longrightarrow +\infty$, then there exists a D-harmonic function $h$ in $\Omega$ such that $u(x) = h(x)$ for almost every $x \in \Omega$.

## 2.7 Dunkl-Newtonian Potential and Riesz decomposition theorem

In this section, we introduce the Dunkl-Newton kernel and the corresponding Dunkl-Newtonian potential which generalize respectively the classical Newton kernel and the classical Newtonian potential and we study some of their properties. Finally, we establish the so-called Riesz decomposition theorem for Dunkl subharmonic functions.

Throughout this section, we will consider the transient case i.e. $d+2\gamma > 2$ and we suppose that this condition holds in our results.

### 2.7.1 Dunkl type Newton kernel

In this subsection, we will study some properties of the Dunkl-Newton kernel defined by (2.15). We start by the following result:

**Proposition 2.56** *For every $x, y \in \mathbb{R}^d$, we have*

$$N_k(x,y) = \frac{1}{d_k(d+2\gamma-2)} \int_{\mathbb{R}^d} \Big( \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \Big)^{\frac{2-(d+2\gamma)}{2}} d\mu_y(z). \tag{2.111}$$

*Proof:* From (2.9) and (2.14), we have

$$p_t(x,y) = \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2+\|y\|^2-2\langle x,z\rangle}{4t}} d\mu_y(z). \tag{2.112}$$

Hence, by the change of variables $1/4t \leftrightarrow t$ in the integral (2.15) and using (2.17), we can write

$$N_k(x,y) = \frac{1}{2d_k\Gamma(d/2+\gamma)} \int_0^{+\infty} t^{\frac{d}{2}+\gamma-2} \int_{\mathbb{R}^d} e^{-t(\|x\|^2+\|y\|^2-2\langle x,z\rangle)} d\mu_y(z) dt. \tag{2.113}$$

Applying Fubini's theorem and then using the identity

$$\forall\, A \geq 0, \quad \forall\, \lambda > 0, \quad A^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} s^{\lambda-1} e^{-sA} ds$$

(when $A = 0$, the both terms are equal to $+\infty$) by taking $A = \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle$ and $\lambda = \frac{d+2\gamma-2}{2}$, we obtain the result.
We note that more shortly, by (2.14) the relation (2.113) can be also written

$$N_k(x,y) = \frac{1}{2d_k\Gamma(d/2+\gamma)} \int_0^{+\infty} t^{\frac{d}{2}+\gamma-2} \tau_{-x}[e^{-t\|\cdot\|^2}](y)\, dt. \tag{2.114}$$

**Exemple 2.57** ***1)*** *When $k = 0$ and $d > 2$, the Rösler measure $\mu_x$ is equal to $\delta_x$ (the Dirac measure at $x$) and then $N_0(x,y) = \frac{1}{(d-2)\omega_{d-1}} \|x-y\|^{2-d}$ is the classical Newton kernel*[11].

***2)*** *We consider $\mathbb{R}^d$ ($d \geq 1$) with the root system $R = \{\pm e_1\}$ with $e_1 = (1, 0, \ldots, 0)$. In this case, the Coxeter-Weyl group is $\mathbf{Z}_2 = \{id, \sigma_{e_1}\}$, the multiplicity function is a parameter $k = k(e_1) > 0$ ( by the transient condition, we must take $k > 1/2$ if $d = 1$) and the Rösler measure is of the form $\mu_y = \mu_{(y_1,y')} = \mu_{y_1} \otimes \delta_{y'}$ where $y' = (y_2, \ldots, y_d)$ and $\mu_{y_1}$ is the $\mathbf{Z}_2$-Rösler measure. If $y_1 = 0$, we know that $\mu_0 = \delta_0$ and if $y_1 \neq 0$, we have*

$$\langle \mu_{y_1}, f\rangle := \int_{-1}^{1} f(ty_1)\phi_k(t)dt, \quad f \in \mathcal{C}(\mathbb{R}),$$

*where $\phi_k$ is the $\mathbf{Z}_2$-Dunkl density function of parameter $k$ given by (see [10] or [36] p.104)*

$$\phi_k(t) := \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} (1-t)^{k-1}(1+t)^k \mathbf{1}_{[-1,1]}(t), \tag{2.115}$$

11. $\omega_{d-1}$ is the area of $S^{d-1}$.

*By the change of variables $s = ty_1$, we can write*

$$\forall\ y_1 \in \mathbb{R} \setminus \{0\}, \quad \langle \mu_{y_1}, f \rangle = \frac{1}{y_1} \int_{-y_1}^{y_1} f(s)\phi_k(\frac{s}{y_1})ds = \frac{1}{|y_1|} \int_{-|y_1|}^{|y_1|} f(s)\phi_k(\frac{s}{y_1})ds.$$

*This shows that $\mu_{y_1}$, $y_1 \neq 0$, has a density with respect to the Lebesgue measure given by*

$$\phi_{k,y_1}(s) = \frac{1}{|y_1|}\phi_k(\frac{s}{y_1})\mathbf{1}_{[-|y_1|,|y_1|]}(s). \tag{2.116}$$

*When $y_1 \neq 0$, using (2.111), the Dunkl-Newton kernel is given by*

$$N_k(x,y) = \frac{1}{d_k(d+2k-2)} \int_{-|y_1|}^{|y_1|} \Big(x_1^2 + y_1^2 - 2sx_1 + \|x' - y'\|^2\Big)^{\frac{2-d-2k}{2}} \phi_{k,y_1}(s)ds.$$

*By a change of variables, we can write this relation as follows*

$$N_k(x,y) = \frac{1}{d_k(d+2k-2)} \int_{-1}^{1} \Big(x_1^2 + y_1^2 - 2tx_1y_1 + \|x' - y'\|^2\Big)^{1-\frac{d}{2}-k} \phi_k(t)dt. \tag{2.117}$$

*If $y_1 = 0$, by (2.111) we have*

$$N_k(x,y) = \frac{1}{d_k(d+2k-2)} \Big(x_1^2 + \|x' - y'\|^2\Big)^{1-\frac{d}{2}-k}.$$

*In fact this we see that this formula can be obtained from (2.117) by taking $y_1 = 0$.*

***3)** We consider $\mathbb{R}^d$ ($d \geq 1$) with the root system $R_m := \{\pm e_1, \dots, \pm e_m\}$, where $m$ is a fixed integer in $\{1, \dots, d\}$ and $(e_j)_{1 \leq j \leq d}$ is the canonical basis of $\mathbb{R}^d$. For $\xi \in \mathbb{R}^d$, we will denote $\xi = (\xi^{(m)}, \xi') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$.*
*Noting that the Coxeter-Weyl group is given by $W = \mathbf{Z}_2^m$ and that the $\mathbf{Z}_2^m$-orbit of a point $\xi \in \mathbb{R}^d$ is given by*

$$\mathbf{Z}_2^m.\xi := \{\varepsilon.\xi := (\varepsilon_1\xi_1, \dots, \varepsilon_m\xi_m, \xi'), \quad \varepsilon = (\varepsilon_i)_{1 \leq i \leq m} \in \{\pm 1\}^m\}.$$

*The multiplicity function can be represented by the m-multidimensional parameter $k = (k_1, \dots, k_m)$ with $k_j = k(e_j) > 0$. Moreover, the Rösler measure is of the form $\mu_y = \mu_{(y^{(m)},y')} = \mu_{y_1} \otimes \cdots \otimes \mu_{y_m} \otimes \delta_{y'}$ with $\mu_{y_i}$ the $\mathbf{Z}_2$-Rösler measure at point $y_i$ (see (2.116)).*
*In this case, the Dunkl-Newton kernel is of the form*

$$N_k(x,y) = C_1 \int_{[-1,1]^m} \Big(\|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2\sum_{j=1}^m t_jx_jy_j + \|x' - y'\|^2\Big)^{1-\frac{d}{2}-\gamma} \times \prod_{i=1}^m \phi_{k_i}(t_i)dt_1 \dots dt_m, \tag{2.118}$$

*where $C_1 = [d_k(d+2\gamma-2)]^{-1}$ and $\phi_{k_i}$ is the $\mathbf{Z}_2$-Dunkl density of parameter $k_i$ given by (2.115).*

**Proposition 2.58** *Let $x, y \in \mathbb{R}^d$, with $x \neq 0$.*

1. *If $y \notin W.x$, then $0 < N_k(x,y) < +\infty$.*
2. *When $d \geq 2$ and $\gamma > 0$, we have $N_k(x,x) = +\infty$.*

*Proof:* **1)** Let $y \in \mathbb{R}^d$ fixed. It is well known (see [31] and [36]) that

$$p_t(x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma} c_k} \max_{g \in W} e^{-(\|x-gy\|^2)/4t}. \tag{2.119}$$

Hence, $N_k(x,y) < +\infty$ for all $x \notin W.y$.
**2)** At first suppose that $x$ is not in any hyperplane $H_\alpha$, $\alpha \in R$ (i.e. $x$ lives in a Weyl chamber). We have

$$N_k(x,x) \geq \int_0^1 p_t(x,x) := I.$$

It is enough to prove that $I = +\infty$. To do this, we need the following short-time asymptotic result of the Dunkl heat kernel established in [35] (Corollary 2): Let $C$ be a fixed Weyl chamber. If $x, y \in C$, then

$$p_t(x,y) \sim_{t\to 0} \big(\omega_k(x)\omega_k(y)\big)^{-1/2} (4\pi t)^{-d/2} e^{-\frac{\|x-y\|^2}{4t}}. \tag{2.120}$$

For $y = x$, we obtain $p_t(x,x) \sim_{t\to 0} \big(\omega_k(x)\big)^{-1} (4\pi t)^{-d/2}$ and $I = +\infty$ as desired.
When $x \in H_\alpha$ for some $\alpha \in R$, the result follows by using the lower semi-continuity of the function $x \longmapsto N_k(x,x)$ (as non-decreasing limit of the sequence of continuous functions $x \longmapsto \int_{1/n}^n p_t(x,x)dt$). Indeed, if $x \in H_\alpha$, $N_k(x,x) = \liminf_{y\to x} N_k(y,y) = +\infty$ because $N_k(y,y) = +\infty$ if $y$ converges to $x$ in a Weyl chamber limited by $H_\alpha$. $\square$

As already mentioned, for $g \neq id$, it is much more difficult to see if $N_k(x, gx)$ is finite or infinite. This new phenomena will be illustrated by the following complete characterization of the singularities of the Dunkl-Newton kernel in the case of the $\mathbf{Z}_2^m$-Coxeter-Weyl group acting on $\mathbb{R}^d$. More precisely, we have:

**Proposition 2.59** *Let $x \in \mathbb{R}^d \backslash \{0\}$. With the same notations of Example 2.57, 3), we set $H_i$ the hyperplane orthogonal to $e_i$ and we recall that $\varepsilon.x = (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x') \in \mathbf{Z}_2^m.x$.*

1. *If $x \in \cap_{i=1}^m H_i$, then $x = \varepsilon.x$ and $N_k(x,x) = +\infty$.*
2. *Assume that $x \notin \cup_{i=1}^m H_i$. Set $A := \{i \in \{1,\ldots,m\}, \quad x_i \neq 0\}$ (i.e. $x$ is exterior to the hyperplanes $H_i$ for all $i \in A$) and $\varepsilon^{(n)}.x = (\varepsilon_1 x_1, \ldots, \varepsilon_m x_m, x')$ a point of $\mathbf{Z}_2^m$-orbit of $x$ such that $\big|\{j \in A, \quad \varepsilon_j = 1\}\big| = n$ i.e. the point $\varepsilon^{(n)}.x$ has exactly $n$ among the nonzero coordinates $(x_j)_{j\in A}$ that have not been changed under the action of the element $\varepsilon^{(n)} \in \mathbf{Z}_2^m$. Then*

$$N_k(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2\Big(|A| - n + \sum_{j\in A} k_j - \gamma\Big) + 2. \tag{2.121}$$

*3. Assume that $x \notin \cup_{i=1}^m H_i$. Then*

$$N_k(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2(m-n)+2. \tag{2.122}$$

*In particular, if $d = 2N$ we have $\sum_{n=\max(0,m-N+1)}^{m} \binom{m}{n}$ singularities living in $\mathbb{R}^{2N} \backslash \cup_{i=1}^m H_i$ and if $d = 2N+1$, we have $\sum_{n=\max(0,m-N)}^{m} \binom{m}{n}$ singularities living in $\mathbb{R}^{2N+1} \backslash \cup_{i=1}^m H_i$.*

**Remark 2.60** *An important case is when $m = d$ in the previous proposition. We suppose also that $x \notin \cup_{i=1}^d H_i$. Then a necessary and sufficient condition for $\varepsilon^{(n)}.x$ to be a singularity of the function $N_k(x, .)$ is that $n \geq \frac{d}{2} + 1$.*
*In particular, since $n \leq d$ and $\varepsilon^{(d)}.x = x$, $N_k(x, .)$ has not any singularity if $d = 1$ and $N_k(x, .)$ has $x$ as a unique singularity if $d = 2$ or $d = 3$.*

*Proof:* For abbreviation, we will use the following constants

$$C_2 := 2^{1-\frac{d}{2}-\gamma} C_1, \quad C(k) := \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} \tag{2.123}$$

and $C_1$ is the constant in (2.118). From (2.118), it is easy to see that

$$N_k(x, \varepsilon.x) = C_2 \int_{[-1,1]^m} \Big( \|x^{(m)}\|^2 - \sum_{j=1}^m \varepsilon_j t_j x_j^2 \Big)^{1-\frac{d}{2}-\gamma} \prod_{j=1}^m \phi_{k_j}(t_j) dt_1 \otimes \cdots \otimes dt_m. \tag{2.124}$$

**1)** Clearly, from (2.124), the condition $x \in \cap_{i=1}^m H_i$ i.e. $x^{(m)} = 0$ implies that $x = \varepsilon.x = (0, x')$ and $N_k(x, \varepsilon.x) = +\infty$.
**2)** Suppose that $x \notin \cap_{i=1}^m H_i$. At first, write (2.124) as follows

$$N_k(x, \varepsilon.x) = C_2 \int_{[-1,1]^m} \Big( \sum_{j=1}^m (1 - \varepsilon_j t_j) x_j^2 \Big)^{1-\frac{d}{2}-\gamma} \prod_{j=1}^m \phi_{k_j}(t_j) \otimes_{j=1}^m dt_j. \tag{2.125}$$

Now, using the notations of the Proposition, Fubini's theorem and the fact that $\phi_{k_j}$ are probability densities, (2.125) can be written in the following form

$$N_k(x, \varepsilon.x) = C_2 \int_{[-1,1]^{|A|}} \Big( \sum_{j \in A} (1 - \varepsilon_j t_j) x_j^2 \Big)^{1-\frac{d}{2}-\gamma} \prod_{j \in A} \phi_{k_j}(t_j) \otimes_{j \in A} dt_j. \tag{2.126}$$

We will distinguish two cases:
**First case** $|A| = 1$. Let $i \in \{1, \ldots, m\}$ such that $x_i \neq 0$. In this case, using (2.115) and (2.123), we deduce that (2.126) takes the form

$$\begin{aligned} N_k(x, \varepsilon.x) &= C_2 \int_{-1}^1 \Big( (1 - \varepsilon_i s) x_i^2 \Big)^{1-\frac{d}{2}-\gamma} \phi_{k_i}(s) ds \\ &= C(k_i) C_2 |x_i|^{2-d-2\gamma} \int_{-1}^1 (1 - \varepsilon_i s)^{1-\frac{d}{2}-\gamma} (1-s)^{k_i - 1} (1+s)^{k_i} ds. \end{aligned}$$

• If $\varepsilon_i = 1$, then according to our notations, we have $n = |A| = 1$, $\varepsilon.x = \varepsilon^{(1)}.x = x$ and

$$N_k(x, \varepsilon^{(1)}.x) = C(k_i)C_2|x_i|^{2-d-2\gamma}\int_{-1}^{1}(1-s)^{k_i-\frac{d}{2}-\gamma}(1+s)^{k_i}ds.$$

Consequently, $N_k(x, \varepsilon^{(1)}.x) = +\infty$ if and only if $d \geq 2 + 2k_i - 2\gamma$.
• When $\varepsilon_i = -1$, we have $n = 0$, $\varepsilon.x = \varepsilon^{(0)}.x$ and

$$N_k(x, \varepsilon^{(0)}.x) = C(k_i)C_2|x_i|^{2-d-2\gamma}\int_{-1}^{1}(1+s)^{1+k_i-\frac{d}{2}-\gamma}(1-s)^{k_i-1}ds.$$

Thus, as $k_i > 0$ we have $N_k(x, \varepsilon^{(0)}.x) = +\infty$ if and only if $d \geq 4 + 2(k_i - \gamma)$. Then the result is proved in this case.
**Second case** $|A| = r \geq 2$. Using (2.126) and the change of variables $t_j \leftrightarrow 1 - \varepsilon_j t_j$, we obtain

$$\begin{aligned} N_k(x, \varepsilon.x) &= C_2\int_{]0,2[^{|A|}}\Big(\sum_{j\in A} t_j x_j^2\Big)^{1-\frac{d}{2}-\gamma}\prod_{j\in A}\phi_{k_j}(\varepsilon_j - \varepsilon_j t_j)\otimes_{j\in A} dt_j \\ &= C_2\int_{]0,2[^{|A|}\cap B_r} \quad + \quad C_2\int_{]0,2[^{|A|}\setminus B_r} \\ &= C_2 I(x, \varepsilon.x) \quad + \quad C_2 J(x, \varepsilon.x), \end{aligned}$$

where $B_r$ is the open unit ball in $\mathbb{R}^{|A|} = \mathbb{R}^r$.
The singularities of these integrals being at point 0 and then it is clear that $J(x, \varepsilon.x) < +\infty$.
Thus, we need to know when the integral $I(x, \varepsilon.x)$ diverges. To do this, we will identify $(t_j)_{j\in A}$ with $v = (v_1, \ldots, v_r) \in \mathbb{R}^r$ and use the spherical coordinates in $\mathbb{R}^r$:

$$\rho = \|v\|, \quad v_1 = \rho a_1, \quad \ldots, v_{r-1} = \rho a_{r-1} \quad \text{and} \quad v_r = \rho a_r,$$

where

$$a_1 = \cos\theta_1, \ldots, a_{r-1} = \prod_{i=1}^{r-2}\sin\theta_i\cos\theta_{r-1}, \quad a_r = \prod_{i=1}^{r-1}\sin\theta_i.$$

Notice that all $a_j$ are positive when $(v_1, \ldots, v_r) \in ]0, 2[^r \cap B_r$.

$$I(x, \varepsilon.x) = \int_{S_+^{r-1}}\psi(a^{(r)}, x^{(r)})\Big(\int_0^1\prod_{j\in A}\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j\rho)\rho^{r-\frac{d}{2}-\gamma}d\rho\Big)d\sigma_r(a^{(r)}), \qquad (2.127)$$

where $S_+^{r-1} := ]0, 2[^r \cap S^{r-1}$, $d\sigma_r$ is the surface measure of the unit sphere $S^{r-1}$ of $\mathbb{R}^r$, $a^{(r)} = (a_j)_{j\in A}$, $x^{(r)} = (x_j)_{j\in A}$ and $\psi(a^{(r)}, x^{(r)}) := (\sum_{j\in A} a_j x_j^2)^{1-\frac{d}{2}-\gamma}$.
We have

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j\rho) = C(k_j)(1 - \varepsilon_j + \varepsilon_j a_j\rho)^{k_j-1}(1 + \varepsilon_j - \varepsilon_j a_j\rho)^{k_j}.$$

Hence,

$$\phi_{k_j}(\varepsilon_j-\varepsilon_j a_j\rho)=\begin{cases} C(k_j)a_j^{k_j-1}\rho^{k_j-1}(2-a_j\rho)^{k_j}, & \text{if} \quad \varepsilon_j=1 \\ \\ C(k_j)a_j^{k_j}\rho^{k_j}(2-a_j\rho)^{k_j-1}, & \text{if} \quad \varepsilon_j=-1. \end{cases} \tag{2.128}$$

Define

$$A_1:=\Big\{j\in A, \quad \varepsilon_j=1\Big\}, \quad A_2=A\backslash A_1.$$

According to our notations, we have $|A_1|=|\{j, \quad \varepsilon_j=1\}|=n$.
Then, from (2.127), (2.128) and recalling $\varepsilon^{(n)}.x$, we deduce that

$$I(x,\varepsilon^{(n)}.x)=\int_{S_+^{r-1}}\psi(a^{(r)},x^{(r)})\Big(\int_0^1 f(a^{(r)},\rho)\rho^{\beta+r-\frac{d}{2}-\gamma}d\rho\Big)d\sigma_r(a^{(r)}), \tag{2.129}$$

with

$$f(a^{(r)},\rho):=\prod_{j\in A_1}C(k_j)a_j^{k_j-1}(2-a_j\rho)^{k_j}\prod_{j\in A_2}C(k_j)a_j^{k_j}(2-a_j\rho)^{k_j-1}$$

and

$$\beta:=\sum_{j\in A_1}(k_j-1)+\sum_{j\in A_2}k_j=\sum_{j\in A}k_j-n.$$

The function $\rho\longmapsto f(a^{(r)},\rho)$ is continuous and does not vanish on the compact set $[0,1]$. So that the singularity in the $d\rho$-integral is only in the term

$$\rho^{\beta+r-\frac{d}{2}-\gamma}=\rho^{(\sum_{j\in A}k_j)-n+r-\frac{d}{2}-\gamma}.$$

This implies the result the assertion 2).
**3)** When $x\notin\cup_{i=1}^m H_i$, we have $A=\{1,\dots,m\}$ and then the result is a particular case of the statement 2). □

**Proposition 2.61** *The Dunkl-Newton kernel satisfies the following properties:*

1. *For all $x,y\in\mathbb{R}^d$, we have*

$$N_k(x,y)=\frac{1}{d_k}\int_0^{+\infty}t^{1-d-2\gamma}h_k(t,x,y)dt. \tag{2.130}$$

2. *For every $x,y\in\mathbb{R}^d$. Then*

$$N_k(x,y)=N_k(y,x), \quad N_k(gx,gy)=N_k(x,y), \quad N_k(gx,y)=N_k(x,g^{-1}y). \tag{2.131}$$

3. *For all $x,y\in\mathbb{R}^d$ with $x\notin W.y$, we have*

$$\min_{g\in W}\Big(\|x-gy\|^{2-(d+2\gamma)}\Big)\leq d_k(d+2\gamma-2)N_k(x,y)\leq\max_{g\in W}\Big(\|x-gy\|^{2-(d+2\gamma)}\Big). \tag{2.132}$$

*4. For all $y \in \mathbb{R}^d$ fixed, the function $x \longmapsto N_k(x,y)$ is*
*-lower semi-continuous (l.s.c.) on $\mathbb{R}^d$.*
*- of class $C^\infty$ on $\mathbb{R}^d \backslash W.y$ and for every $j \in \{1,...,d\}$, we have*

$$\partial_j N_k(.,y)(x) = -\frac{1}{d_k}\int_{\mathbb{R}^d}(x_j - z_j)\Big(\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle\Big)^{\frac{-(d+2\gamma)}{2}} d\mu_y(z) \qquad (2.133)$$

*Proof:*

1. Fix $x, y \in \mathbb{R}^d$. By (2.111) and Fubini's theorem , we have

$$\begin{aligned} N_k(x,y) &= \frac{1}{d_k}\int_{\mathbb{R}^d}\Big(\int_{\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}}^{+\infty} t^{1-(d+2\gamma)}dt\Big)d\mu_y(z) \\ &= \frac{1}{d_k}\int_0^{+\infty} t^{1-(d+2\gamma)}\Big(\int_{\mathbb{R}^d}\mathbf{1}_{[0,t]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle})d\mu_y(z)\Big)\,dt \\ &= \frac{1}{d_k}\int_0^{+\infty} t^{1-(d+2\gamma)}h_k(t,x,y)dt. \end{aligned}$$

2. We obtain (2.131) by using (2.130) and the properties (2.25) and (2.27) of the harmonic kernel.
3. At first, we note that from (2.4) for $z \in$ supp $\mu_y$ we can write $z = \sum_{g\in W}\lambda_g(z)gy$, where $\lambda_g(z) \in [0,1]$ are such that $\sum_{g\in W}\lambda_g(z) = 1$. Then we have

$$\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle = \sum_{g\in W}\lambda_g(z)\|x-gy\|^2. \qquad (2.134)$$

Now, as $f : t \longmapsto t^{1-\frac{d}{2}-\gamma}$ is a convex function on $]0,+\infty[$, by (2.134) we have

$$\begin{aligned} \Big(\|x\|^2+\|y\|^2-2\langle x,z\rangle\Big)^{1-\frac{d}{2}-\gamma} &= \Big(\sum_{g\in W}\lambda_g(z)\|x-gy\|^2\Big)^{1-\frac{d}{2}-\gamma} \\ &\leq \max_{g\in W}\big(\|x-gy\|^{2-(d+2\gamma)}\big). \end{aligned}$$

This implies the right inequality. Again by convexity of the function $f$, Jensen's inequality and (2.134), we get

$$\begin{aligned} d_k(d+2\gamma-2)N_k(x,y) &\geq \Big(\int_{\mathbb{R}^d}(\|x\|^2+\|y\|^2-2\langle x,z\rangle)d\mu_y(z)\Big)^{\frac{2-(d+2\gamma)}{2}} \\ &= \Big(\sum_{g\in W}\big(\int_{\mathbb{R}^d}\lambda_g(z)d\mu_y(z)\big)\|x-gy\|^2\Big)^{\frac{2-(d+2\gamma)}{2}} \\ &\geq \Big(\max_{g\in W}\|x-gy\|^2\Big)^{\frac{2-(d+2\gamma)}{2}} = \min_{g\in W}\big(\|x-gy\|^{2-(d+2\gamma)}\big), \end{aligned}$$

where in the last line we have used the fact that $f$ is a decreasing function.

4. The function $x \mapsto N_k(x,y)$ is l.s.c. on $\mathbb{R}^d$ as being the increasing limit of the sequence $(f_n)$ of continuous functions defined by $f_n : x \mapsto \int_{1/n}^n p_t(x,y)dt$.
   As $\mu_y$ is with compact support, we can differentiate locally in a neighborhood of $x \notin W.y$ under the integral in the relation (2.111) and we obtain the result.

□

**Remark 2.62** *Let $x \in \mathbb{R}^d$. From (2.131), we deduce that $N_k(x,gx) = +\infty$ if and only if $N_k(x,g^{-1}x) = +\infty$.*

**Proposition 2.63** *For every $x_0 \in \mathbb{R}^d$ fixed, the function $N_k(x_0,.)$ is D-superharmonic on $\mathbb{R}^d$.*

*Proof:* Recalling that $p_t(x,y)$ the Dunkl heat kernel, we consider the function

$$S_{x_0,r}(x) := \int_r^{+\infty} p_t(x_0,x)dt.$$

By the monotone convergence theorem, we see that the function $N_k(x_0,.)$ is the pointwise increasing limit of the sequence $\left(S_{x_0,\frac{1}{n}}\right)_n$. Hence, by Proposition 2.28, it suffices to prove that for every $r > 0$, $S_{x_0,r}$ is D-superharmonic on $\mathbb{R}^d$. To do this, we will use the result of Corollary 2.53.
The function $p_t(x_0,.)$ is of class $C^\infty$ on $\mathbb{R}^d$ and we can differentiate under the integral sign in the relation (2.112) to obtain

$$\partial_j p_t(x_0,.)(x) = -\frac{1}{2t}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j - z_j)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z)$$

and

$$\begin{aligned}\partial_i\partial_j p_t(x_0,.)(x) &= -\delta_{ij}\frac{1}{2t}p_t(x_0,x)\\ &\quad+\frac{1}{4t^2}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j - z_j)(x_i - z_i)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z),\end{aligned}$$

where $\delta_{ij}$ is the Kronecker symbol.
Using the fact that supp $\mu_{x_0} \subset B(0,\|x_0\|)$, we deduce that

$$|\partial_j p_t(x_0,.)(x)| \leq \frac{\|x\| + \|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k},$$

$$|\partial_i\partial_j p_t(x_0,.)(x)| \leq \frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k} + \frac{(\|x\| + \|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}.$$

Let $R > 0$. The previous inequalities and the differentiation theorem under the integral sign imply that $S_{x_0,r}$ is of class $C^2$ on the open ball $\overset{\circ}{B}(0,R)$ and as $x \mapsto p_t(x_0,x)$ is a solution of the heat equation i.e.

$$(\Delta_k - \partial_t)\, p_t(x_0,.)(x) = 0$$

(see [31]), we deduce that

$$\forall\ x \in \overset{\circ}{B}(0,R), \quad \Delta_k S_{x_0,r}(x) = \int_r^{+\infty} \Delta_k\left(p_t(x_0,.)\right)(x)dt$$
$$= \int_r^{+\infty} \partial_t p_t(x_0,x)dt = -p_r(x_0,x) < 0.$$

Therefore, $S_{x_0,r}$ is D-superharmonic on $\overset{\circ}{B}(0,R)$. As $R>0$ is arbitrary, we conclude that $S_{x_0,r}$ is D-superharmonic on $\mathbb{R}^d$ as desired. □

**Proposition 2.64** *Let $x_0 \in \mathbb{R}^d$. Then the function $N_k(x_0,.)$ is*

1. *locally integrable on $\mathbb{R}^d$ with respect to the measure $\omega_k(x)dx$ and we have*
$$-\Delta_k\big(N_k(x_0,.)\omega_k\big) = \delta_{x_0} \quad in \quad \mathcal{D}'(\mathbb{R}^d), \tag{2.135}$$
*where $\delta_{x_0}$ is the Dirac measure at $x_0$.*
2. *D-harmonic on $\mathbb{R}^d\backslash W.x_0$ ($W.x_0$ the $W$-orbit of $x_0$).*

*Proof:* Fix $x_0$ in $\mathbb{R}^d$. To simplify notations in this proof, we will denote by $C_1$ and $C_2$ the constants
$$C_1 = \frac{1}{2d_k\Gamma(d/2+\gamma)}, \quad C_2 = \frac{1}{d_k(d+2\gamma-2)}$$

**1)** From Propositions 2.63 and 2.25, we deduce that $N_k(x_0,.) \in L^1_{k,loc}(\mathbb{R}^d)$. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. From (2.114), we can write

$$\langle \Delta_k\big(N_k(x_0,.)\omega_k\big),\phi\rangle = C_1\int_{\mathbb{R}^d}\Big(\int_0^{+\infty} t^{\frac{d}{2}+\gamma-2}\tau_{-x_0}[e^{-t\|\cdot\|^2}](y)dt\Big)\Delta_k\phi(y)\omega_k(y)dy.$$

As $N_k(x_0,.)\omega_k$ is locally integrable, we have

$$C_1\int_{\mathbb{R}^d}\Big(\int_0^{+\infty} t^{\frac{d}{2}+\gamma-2}\tau_{-x_0}[e^{-t\|\cdot\|^2}](y)dt\Big)\big|\Delta_k\phi(y)\big|\omega_k(y)dy$$
$$\leq \|\Delta_k\phi\|_\infty \int_{\text{supp }\phi} N_k(x_0,y)\omega_k(y)dy < +\infty.$$

Hence, using Fubini's theorem and the relations (2.164) and (2.166), we obtain

$$\langle \Delta_k\big(N_k(x_0,.)\omega_k\big),\phi\rangle = C_1\int_0^{+\infty} t^{\frac{d}{2}+\gamma-2}\Big(\int_{\mathbb{R}^d} e^{-t\|y\|^2}\Delta_k[\tau_{x_0}\phi](y)\omega_k(y)dy\Big)dt.$$

We claim that we can use again Fubini's theorem. Indeed, since $\tau_{x_0}\phi$ has compact support, there exists $R>0$ such that supp $\tau_{x_0}\phi \subset B(0,R)$ and we have

$$C_1\int_0^{+\infty} t^{\frac{d}{2}+\gamma-2}\Big(\int_{\mathbb{R}^d} e^{-t\|y\|^2}\big|\Delta_k[\tau_{x_0}\phi](y)\big|\omega_k(y)dy\Big)dt$$
$$\leq C_2\|\Delta_k[\tau_{x_0}\phi]\|_\infty \int_{B(0,R)} \|y\|^{2-(d+2\gamma)}\omega_k(y)dy < +\infty.$$

Consequently,

$$\begin{aligned}\langle \Delta_k\big(N_k(x_0,.)\omega_k\big),\phi\rangle &= C_2\int_{\mathbb{R}^d}\|y\|^{2-(d+2\gamma)}\Delta_k[\tau_{x_0}\phi](y)\omega_k(y)dy\\ &= \langle \Delta_k\big(S\omega_k\big),\tau_{x_0}\phi\rangle\\ &= -\tau_{x_0}\phi(0) = -\phi(x_0),\end{aligned}$$

where $S$ is the fundamental solution of the Dunkl-Laplacian given by (2.16). This proves (2.135).

**2)** From the relation (2.135), we deduce that the function $N_k(x_0,.)\omega_k$ is D-harmonic in the sense of distributions on $\mathbb{R}^d\backslash\{x_0\}$. Hence, by applying Weyl's Lemma (see Corollary 2.55) on the $W$-invariant open set $\mathbb{R}^d\backslash W.x_0$, there exists a D-harmonic function $h$ on $\mathbb{R}^d\backslash W.x_0$ such that $N_k(x_0,x) = h(x)$ for almost every $x\in\mathbb{R}^d\backslash W.x_0$. Now, using the smoothness of the function $N_k(x_0,.)$ on $\mathbb{R}^d\backslash W.x_0$, we obtain $N_k(x_0,.) = h$ on $\mathbb{R}^d\backslash W.x_0$.
This completes the proof. □

### 2.7.2 Dunkl-Newtonian potential of Radon measures

**Definition 2.65** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$. The Dunkl-Newtonian potential of $\mu$ is defined by*

$$N_k[\mu](x) := \int_{\mathbb{R}^d} N_k(x,y)d\mu(y), \quad x\in\mathbb{R}^d. \tag{2.136}$$

**Remark 2.66**
***1.****Let $\mu$ be a signed Radon measure on $\mathbb{R}^d$ and $\mu=\mu^+-\mu^-$ its Hahn-Jordan decomposition. We can also define the Dunkl-Newtonian potential of $\mu$ by setting $N_k[\mu](x) := N_k[\mu^+](x) - N_k[\mu^-](x)$ whenever for every $x\in\mathbb{R}^d$, $N_k[\mu^+](x)$ and $N_k[\mu^-](x)$ are not infinite simultaneously.*

***2.*** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$. Using (2.130), (2.26), Fubini's theorem and recalling (2.107), we can write*

$$\forall\ x\in\mathbb{R}^d,\quad N_k[\mu](x) = \frac{1}{d+2\gamma}\int_0^{+\infty} M_B^t(\mu)(x)\ tdt. \tag{2.137}$$

**Proposition 2.67** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$. A necessary and sufficient condition for finiteness a.e. of the Dunkl-Newtonian potential of $\mu$ is that*

$$\int_{\mathbb{R}^d}(1+\|y\|)^{2-d-2\gamma}\,d\mu(y) < +\infty. \tag{2.138}$$

We need the following lemma:

**Lemma 2.68** *Let $\mu$ be a finite nonnegative Radon measure on $\mathbb{R}^d$. Then $N_k[\mu]$ belongs to $L^1_{k,loc}(\mathbb{R}^d)$. In particular, $N_k[\mu]$ is finite a.e..*

*Proof:* Fix $R > 0$. Using Fubini's theorem, we have

$$\int_{B(0,R)} N_k[\mu](x)\omega_k(x)dx = \int_{\mathbb{R}^d}\int_{B(0,R)} N_k(x,y)\omega_k(x)dx\ d\mu(y).$$

As $\mu(\mathbb{R}^d) < +\infty$, it suffices to show that there exists a constant $C = C(R,d,\gamma) > 0$ such that

$$\forall\ y \in \mathbb{R}^d, \quad \int_{B(0,R)} N_k(x,y)\omega_k(x)dx \leq C. \tag{2.139}$$

Let $x \in B(0,R)$ and $y \in \mathbb{R}^d$. From the relations (2.130), we can write

$$\begin{aligned} N_k(x,y) &= \frac{1}{d_k}\int_0^1 t^{1-d-2\gamma}h_k(t,x,y)dt + \frac{1}{d_k}\int_1^{+\infty} t^{1-d-2\gamma}h_k(t,x,y)dt \\ &:= I(x,y) \quad + \quad J(x,y). \end{aligned}$$

• Since $h_k(t,x,y) \leq 1$, we can see that $J \leq \frac{1}{d_k(d+2\gamma-2)}$. This implies that

$$\forall\ y \in \mathbb{R}^d, \quad \int_{B(0,R)} J(x,y)\omega_k(x)dx \leq \frac{m_k[B(0,R)]}{d_k(d+2\gamma-2)} = C_1.$$

• Applying Fubini's theorem and then using (2.25) and (2.26), we deduce that

$$\begin{aligned} \forall\ y \in \mathbb{R}^d, \quad \int_{B(0,R)} I(x,y)\omega_k(x)dx &\leq \frac{1}{d_k}\int_0^1 t^{1-d-2\gamma}\|h_k(t,y,.)\|_{L^1_k(\mathbb{R}^d)}dt \\ &= \frac{1}{2(d+2\gamma)} = C_2. \end{aligned}$$

Finally, we obtain (2.139) by taking $C = C_1 + C_2$. $\square$

*Proof of Proposition* 2.67. Assume that (2.138) holds. We will show that $x \longmapsto N_k[\mu](x)\omega_k(x)$ is locally integrable. Let $r \geq 1$. By Fubini's theorem, we have

$$\begin{aligned} \int_{B(0,r)} N_k[\mu](x)\omega_k(x)dx &= \int_{\|y\|\leq 2r}\Big(\int_{B(0,r)} N_k(x,y)\omega_k(x)dx\Big)d\mu(y) \\ &+ \int_{\|y\|>2r}\Big(\int_{B(0,r)} N_k(x,y)\omega_k(x)dx\Big)d\mu(y) = J_1 + J_2. \end{aligned}$$

From Lemma 2.68, $J_1 < +\infty$. Now, by (2.132), we have

$$J_2 \leq \frac{1}{d_k(d+2\gamma-2)}\int_{\|y\|>2r}\Big(\int_{B(0,r)} \max_{g\in W}\Big(\|x-gy\|^{2-d-2\gamma}\Big)\,\omega_k(x)dx\Big)d\mu(y).$$

But, for all $x \in B(0,r)$ and all $g \in W$, $\|x-gy\| \geq \|y\| - \|x\| \geq \frac{1}{2}\|y\|$ because $\|y\| \geq 2r$. Moreover, since $r \geq 1$, we also have $\|y\| \geq \frac{1}{2}(1+\|y\|)$. Hence, we get

$$\forall\ g \in W, \quad \|x-gy\| \geq \frac{1}{4}(1+\|y\|).$$

Thus,

$$J_2 \leq \frac{4^{d+2\gamma-2} m_k[B(0,r)]}{d_k(d+2\gamma-2)} \int_{\|y\|>2r} (1+\|y\|)^{2-d-2\gamma} d\mu(y) < +\infty.$$

Conversely, suppose that (2.138) does not hold. Let $x \in B(0,1)$. Using (2.132) and the inequality $\|x - gy\| \leq 1 + \|y\|$ for all $g \in W$, we deduce that

$$\begin{aligned} d_k(d+2\gamma-2)N_k[\mu](x) &= d_k(d+2\gamma-2)\int_{\mathbb{R}^d} N_k(x,y)d\mu(y) \\ &\geq \int_{\mathbb{R}^d} \left(\max_{g\in W} \|x-gy\|\right)^{2-(d+2\gamma)} d\mu(y) \\ &\geq \int_{\mathbb{R}^d} \left(1+\|y\|\right)^{2-(d+2\gamma)} d\mu(y). \end{aligned}$$

Hence, if $\int_{\mathbb{R}^d} \left(1+\|y\|\right)^{2-(d+2\gamma)} d\mu(y) = +\infty$, then $N_k[\mu](x) = +\infty$ on $B(0,1)$ and we get a contradiction. □

**Proposition 2.69** *Let $\mu$ be a nonnegative Radon measure with compact support. Then*

$$N_k[\mu](x) \sim \frac{\mu(\mathbb{R}^d)}{d_k(d+2\gamma-2)} \|x\|^{2-(d+2\gamma)} \quad as \quad \|x\| \longrightarrow +\infty.$$

*Proof:* Let $R > 0$ such that supp $\mu \subset B(0,R)$. By the Cauchy-Schwarz inequality, we have

$$\forall\ z \in \text{supp}\ \mu_y \subset B(0,\|y\|), \quad (\|x\| - \|y\|)^2 \leq \|x\|^2 + \|y\|^2 - 2\langle x,z\rangle \leq (\|x\|+\|y\|)^2.$$

Therefore, by (2.111) we obtain for every $y \in B(0,R)$ fixed and $\|x\| \geq 2R$

$$(\|x\|+\|y\|)^{2-d-2\gamma} \leq C.N_k(x,y) \leq (\|x\|-\|y\|)^{2-d-2\gamma},$$

where $C = d_k(d+2\gamma-2)$. If we integrate these inequalities with respect to the measure $d\mu(y)$, we deduce for all $\|x\| \geq 2R$ that

$$\int_{\mathbb{R}^d} \left(1{+}\|x\|^{-1}\|y\|\right)^{2-d-2\gamma} d\mu(y) \leq C.\|x\|^{d+2\gamma-2} N_k[\mu](x) \leq \int_{\mathbb{R}^d} \left(1{-}\|x\|^{-1}\|y\|\right)^{2-d-2\gamma} d\mu(y)$$

As $(1+\|x\|^{-1}\|y\|)^{2-d-2\gamma} \leq 1$, by the dominated convergence theorem, we see that

$$\lim_{\|x\|\to+\infty} \int_{\mathbb{R}^d} \left(1+\|x\|^{-1}\|y\|\right)^{2-d-2\gamma} d\mu(y) = \mu(\mathbb{R}^d).$$

Furthermore, since $\|x\| \geq 2R$ we have $(1-\|x\|^{-1}\|y\|)^{2-d-2\gamma} \leq 2^{d+2\gamma-2}$. Hence, we can apply again the dominated convergence theorem to get

$$\lim_{\|x\|\to+\infty} \int_{\mathbb{R}^d} \left(1-\|x\|^{-1}\|y\|\right)^{2-d-2\gamma} d\mu(y) = \mu(\mathbb{R}^d).$$

Then we obtain the desired result. □

**Proposition 2.70** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$.*

**i)** *If $\mu$ is with compact support, then $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$ and D-harmonic on $\mathbb{R}^d\backslash W.supp\ \mu$.*

**ii)** *If $N_k[\mu](x) < +\infty$ for at least one $x$, then $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$.*

*Proof:* **i)** Let $\mu$ be a compactly supported and nonnegative Radon measure on $\mathbb{R}^d$.
• For $n \geq 1$, consider the function

$$F_n(x) := \int_{\text{supp }\mu} \Big(\int_{1/n}^{n} p_t(x,y)dt\Big)d\mu(y).$$

By the continuity theorem under the integral sign, we can see that $F_n$ is continuous on $\mathbb{R}^d$. Furthermore, using the monotone convergence theorem, we deduce that $N_k[\mu]$ is a pointwise increasing limit of the sequence $(F_n)$ of continuous functions. Therefore, the lower semi-continuity of the function $N_k[\mu]$ on $\mathbb{R}^d$ follows.
Let $x \in \mathbb{R}^d$ and $r > 0$. Using Fubini's theorem and the D-superharmonicity of the function $\xi \longmapsto N_k(\xi, y)$, we have

$$M_B^r(N_k[\mu])(x) = \int_{\mathbb{R}^d} M_B^r[N_k(.,y)](x)d\mu(y) \leq \int_{\mathbb{R}^d} N_k(x,y)d\mu(y) = N_k[\mu](x).$$

This implies that $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$.
• According to Proposition 2.38-4), we need only to prove that $N_k[\mu]$ is D-subharmonic on $\Omega := \mathbb{R}^d\backslash W.\text{supp }\mu$. Let $B(x,r) \subset \Omega$. Again, by Fubini's theorem and the D-harmonicity of $N_k(.,y)$ on $\mathbb{R}^d \setminus W.y$, we deduce that

$$M_B^r(N_k[\mu])(x) = \int_{\mathbb{R}^d} M_B^r[N_k(.,y)](x)d\mu(y) = \int_{\mathbb{R}^d} N_k(x,y)d\mu(y) = N_k[\mu](x).$$

In particular, $N_k[\mu]$ satisfies the sub-mean property.
Now, it remains to show that $N_k[\mu]$ is u.s.c. on $\Omega$. In fact, $N_k[\mu]$ is continuous on $\Omega$. Indeed, fix $x_0 \in \Omega$ and $R > 0$ such that $\delta := dist\,(B(x_0,R), W.\text{supp }\mu) > 0$. We know that $x \mapsto N_k(x,y)$ is continuous on $\Omega$ for every $y \in \text{supp }\mu$. Moreover, from (2.119), we deduce that

$$\forall\ x \in B(x_0,R), \quad \forall\ y \in \text{supp }\mu, \quad p_t(x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}e^{-\delta/4t}.$$

This implies that

$$\forall\ (x,y) \in B(x_0,R) \times \text{supp }\mu, \quad N_k(x,y) \leq \int_0^{+\infty} \frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}e^{-\delta/4t}dt := C_\delta < +\infty.$$

Consequently, by the continuity theorem under the integral sign, we conclude that $N_k[\mu]$ is continuous on $B(x_0,R)$. This finishes the proof of i).

**ii)** Assume that $N_k[\mu](x_0) < +\infty$ for some $x_0 \in \mathbb{R}^d$. We consider the sequence of functions defined by

$$\phi_n(x) = \int_{B(0,n)} N_k(x,y)d\mu(y).$$

From i), we see that $\phi_n$ is D-superharmonic on $\mathbb{R}^d$ and $\phi_n(x) \uparrow N_k[\mu](x)$ as $n \longrightarrow +\infty$. Hence, from Proposition 2.28 the function $N_k[\mu]$ is D-superharmonic on $\mathbb{R}^d$. □

### 2.7.3 The Dunkl-Poisson equation

In this subsection, we will give a generalization of the classical distributional Poisson equation in the Dunkl setting (see for example [2], Theorem 4.3.8 for the case of the classical Laplacian).

**Proposition 2.71** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$ satisfying the finiteness condition (2.138). Then*

$$-\Delta_k\big(N_k[\mu]\omega_k\big) = \mu \ \ in\ \mathcal{D}'(\mathbb{R}^d). \tag{2.140}$$

*Proof:* By Proposition 2.70, $N_k[\mu]$ is D-superharmonic and then the function $N_k[\mu]\omega_k$ defines a distribution on $\mathbb{R}^d$.
Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Using the fact that $N_k[\mu]\omega_k$ is locally integrable, we have

$$\int_{\text{supp }\varphi}\int_{\mathbb{R}^d} N_k(x,y)d\mu(y)|\Delta_k\varphi(x)|\omega_k(x)dx \le \|\Delta_k\varphi\|_\infty \int_{\text{supp }\varphi} N_k[\mu](x)\omega_k(x)dx < +\infty.$$

Consequently, we can apply Fubini's theorem to obtain

$$\begin{aligned}\langle \Delta_k\big(N_k[\mu]\omega_k\big),\varphi\rangle &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)d\mu(y)\Big)\Delta_k\varphi(x)\omega_k(x)dx\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)\Delta_k\varphi(x)\omega_k(x)dx\Big)d\mu(y)\\ &= \int_{\mathbb{R}^d}\langle \Delta_k\big(N_k(.,y)\omega_k\big),\varphi\rangle\, d\mu(y).\end{aligned}$$

As $N_k(x,y) = N_k(y,x)$, from (2.135) we obtain

$$\langle \Delta_k\big(N_k[\mu]\omega_k\big),\varphi\rangle = -\int_{\mathbb{R}^d}\varphi(y)d\mu(y),$$

as desired. □

From the previous result, we can deduce the following uniqueness principle:

**Corollary 2.72** *Let $\mu$ and $\nu$ be two nonnegative Radon measures on $\mathbb{R}^d$. Assume that $\mu$ and $\nu$ satisfy (2.138) and $N_k[\mu] = N_k[\nu]$ a.e. on $\mathbb{R}^d$. Then $\mu = \nu$.*

In the following result, we will obtain all distributional solutions of the Dunkl-Poisson equation (see [24] for the classical case):

**Proposition 2.73** *Let $f \in L^1_{loc}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} (1 + \|y\|)^{2-d-2\gamma} |f(y)| dy < +\infty.$$

*Then the function $N_k[f] : x \mapsto \int_{\mathbb{R}^d} N_k(x, y) f(y) dy$ is a solution of the Poisson equation:*

$$-\Delta_k(u\omega_k) = f \quad in \quad \mathcal{D}'(\mathbb{R}^d). \tag{2.141}$$

*Moreover, any solution $u$ of (2.141) in $L^1_{k,loc}(\mathbb{R}^d)$ is of the form $N_k[f] + h$, where $h$ is a D-harmonic function on $\mathbb{R}^d$.*

*Proof:* By decomposing $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$, we may assume that $f$ is nonnegative. Using Proposition 2.67, we deduce that $N_k[f]$ is finite a.e and Proposition 2.71 implies that it satisfies the Poisson equation (2.141).
Now, let $v$ be a solution of (2.141). Then $\Delta_k(v\omega_k - N_k[f]\omega_k) = 0$ in distributional sense. Thus, by Weyl's lemma $v = N_k[f] + h$ a.e for some D-harmonic function $h$ on $\mathbb{R}^d$. That is $v = N_k[f] + h$ in $L^1_{k,loc}(\mathbb{R}^d)$.

□

### 2.7.4 Riesz decomposition theorems

One of the most fundamental results in the theory of classical subharmonic functions is due to F. Riesz ([29]) and states that any subharmonic function can be locally written as the sum of a Newtonian potential plus a harmonic function (see for example [17]). In the following result, we will extend this result to Dunkl subharmonic functions.

**Theorem 2.74** *Let $\Omega \subset \mathbb{R}^d$ be open and $W$-invariant, $u \in \mathcal{SH}_k(\Omega)$ and $\mu = \Delta_k[u\omega_k]$ be the $\Delta_k$-Riesz measure related to $u$. Then, for all $W$-invariant open set $G$ with compact closure $\overline{G} \subset \Omega$, there exists a unique D-harmonic function $h_G$ on $G$ such that*

$$\forall\ x \in G, \quad u(x) = -\int_G N_k(x, y) d\mu(y) + h_G(x). \tag{2.142}$$

*Proof:* Let $G$ be a $W$-invariant open set with compact closure $\overline{G} \subset \Omega$ and set $\mu_G := \mu_{|G}$ the restriction of $\mu$ to $G$. Clearly, $\mu_G$ is a nonnegative Radon measure on $\Omega$ with compact support contained in $\overline{G}$. It is also the $\Delta_k$-Riesz measure of the restriction of $u$ to $G$. Furthermore, $\mu_G$ can be considered as a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. Hence, by Proposition 2.70, the function $N_k[\mu_G]$ is D-superharmonic on $\mathbb{R}^d$ (then also on $G$) and by the relation (2.140), we obtain

$$\Delta_k \left( u\omega_k + N_k[\mu_G]\omega_k \right) = 0 \quad \text{in} \quad \mathcal{D}'(G).$$

That is $u\omega_k + N_k[\mu_G]\omega_k$ is a D-harmonic distribution on $G$. By Weyl's lemma, there exists a D-harmonic function $h_G$ on $G$ such that $u(x) = -N_k[\mu_G](x) + h_G(x)$, for almost every $x \in G$. Finally, using the uniqueness principle (Corollary 2.27) we obtain the equality everywhere on $G$. □

In the following theorem, we will obtain a global version of the Riesz decomposition theorem:

**Theorem 2.75** *Let $\Omega$ be a connected and $W$-invariant open subset of $\mathbb{R}^d$, $u \in \mathcal{SH}_k(\Omega)$ and let $\mu$ be the $\Delta_k$-Riesz measure of $u$. Assume that $N_k[\mu](x) < +\infty$ for at least one $x \in \Omega$. Then there is a unique D-harmonic function $h$ on $\Omega$ such that*

$$\forall\ x \in \Omega, \quad u(x) = -N_k[\mu](x) + h(x), \tag{2.143}$$

*where $N_k[\mu](x) := \int_\Omega N_k(x,y)d\mu(y)$. In this case, we say that $u$ has a global Riesz decomposition on $\Omega$.*

*Proof:* Let $(O_n)$ be an open $W$-invariant exhaustion of $\Omega$ such that for every $n$ (large enough) the compact closure of $O_n$ is contained in $O_{n+1}$ (for example as in (2.48)) and let $\mu_n = \mu_{|O_n}$. As above, the function $N_k[\mu_n] : x \mapsto \int_\Omega N_k(x,y)\mathbf{1}_{O_n}(y)d\mu(y)$ is D-superhamonic on $\mathbb{R}^d$ and also on $\Omega$.
Consequently, using the monotone convergence theorem, our hypothesis and Proposition 2.28, we deduce that $N_k[\mu]$ is D-superharmonic on $\Omega$ as being an increasing pointwise limit of a sequence of D-superharmonic functions on $\Omega$. In particular, this implies that the function $N_k[\mu]\omega_k$ defines a distribution on $\Omega$ (by Proposition 2.25).
Now, if we use (2.135) and we proceed as in the proof of Proposition 2.71, we obtain

$$-\Delta_k\left(N_k[\mu]\omega_k\right) = \mu \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{2.144}$$

Finally, we conclude the result by the same way, replacing $G$ by $\Omega$, as in the end of the proof of Theorem 2.74. $\square$

**Remark 2.76** *In the relation (2.142) (resp. (2.143) on $\Omega$), we see that $h_G \geq u$ on $G$ (resp. $h \geq u$). In this case, we say that $h_G$ (resp. $h$) is a D-harmonic majorant of $u$ on $G$ (resp. on $\Omega$). When $\Omega = \mathbb{R}^d$ and under the same assumptions of Theorem 2.75, we will prove in the next section that $h$ is the least D-harmonic majorant of $u$ on $\mathbb{R}^d$ in the sense that if $h_1$ is a D-harmonic function on $\mathbb{R}^d$, then $u \leq h_1$ implies $h \leq h_1$.*

## 2.8 Bounded from above Dunkl subharmonic functions on $\mathbb{R}^d$

In this section, we will describe the functions which are D-subharmonic and bounded from above on the whole space $\mathbb{R}^d$. Moreover, we will characterize the Riesz measure related to a bounded from above D-subharmonic function on $\mathbb{R}^d$.

**Theorem 2.77** *Let $u$ be a bounded from above D-subharmonic function on $\mathbb{R}^d$ and $\mu$ be the associated $\Delta_k$-Riesz measure. Then $u$ has a global Riesz decomposition on $\mathbb{R}^d$ given by*

$$u(x) = \sup_{x\in\mathbb{R}^d} u(x) - N_k[\mu](x), \quad x \in \mathbb{R}^d. \tag{2.145}$$

In the classical case (i.e. $k = 0$), the proof of this theorem is based on the Nivanlinna theorems (see [17], Theorem 3.20). Here, we will give another proof. We start by the following result:

**Lemma 2.78** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$.*

**i)** *For all $n \in \mathbb{N} \setminus \{0\}$, $\mu *_k \varphi_n$ being the function defined on $\mathbb{R}^d$ by (2.110), we have*

$$\forall\, x \in \mathbb{R}^d, \quad N_k\big[(\mu *_k \varphi_n)(y)\omega_k(y)dy\big](x) = \int_{\mathbb{R}^d} N(x,.) *_k \varphi_n(z)d\mu(z). \qquad (2.146)$$

**ii)** *We have*

$$\forall\, x \in \mathbb{R}^d, \quad \lim_{n\to+\infty} N_k\big[(\mu *_k \varphi_n)(y)\omega_k(y)dy\big](x) = N_k[\mu](x). \qquad (2.147)$$

*Note that the terms in (2.146) and (2.147) may be equal to $+\infty$.*

*Proof:* **i)** Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}\setminus\{0\}$. Using respectively (2.136), (2.110), Fubini's theorem and (2.165), we obtain

$$\begin{aligned} N_k\big[(\mu *_k \varphi_n)(y)\omega_k(y)dy\big](x) &= \int_{\mathbb{R}^d} N_k(x,y)\Big(\int_{\mathbb{R}^d} \tau_{-y}\varphi_n(z)d\mu(z)\Big)\omega_k(y)dy \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} N_k(x,y)\tau_{-z}\varphi_n(y)\omega_k(y)dy\Big)d\mu(z) \\ &= \int_{\mathbb{R}^d} N(x,.) *_k \varphi_n(z)d\mu(z). \end{aligned}$$

This proves (2.146).

**ii)** As the function $N_k(x,.)$ is D-superharmonic on $\mathbb{R}^d$, by Theorem 2.35, $N(x,.)$ is the increasing pointwise limit of the sequence $(N_k(x,.) *_k \varphi_n)_n$. Consequently, (2.147) follows from (2.146) and from the monotone convergence theorem. □

*Proof of Theorem 2.77:* We shall prove first the result when $u$ is of class $C^\infty$ on $\mathbb{R}^d$. In this case, the relation (2.56) plays a key role.

Let $a := \sup_{x\in\mathbb{R}^d} u(x)$. We can see by (2.54) that $M_S^r(u)(x) \le a$ for every $x \in \mathbb{R}^d$ and every $r > 0$. Moreover, since $u$ is D-subharmonic, the function $r \longmapsto M_S^r(u)(x)$ is non decreasing (by Proposition 2.32). Consequently, $h(x) := \lim_{r\to+\infty} M_S^r(u)(x)$ exists and is finite for every $x \in \mathbb{R}^d$. Moreover, $h \le a$.

On the other hand, as $\Delta_k u \ge 0$, by the monotone convergence theorem, we have

$$\lim_{r\to+\infty} \frac{1}{d+2\gamma}\int_0^r M_B^t(\Delta_k u)(x)t\, dt = \frac{1}{d+2\gamma}\int_0^{+\infty} M_B^t(\Delta_k u)(x)t\, dt.$$

Now, using the relations (2.130), (2.26) and applying Fubini's theorem, we can see that

$$\begin{aligned} \frac{1}{d+2\gamma}\int_0^{+\infty} M_B^t(\Delta_k u)(x)t\, dt &= \frac{1}{d_k}\int_0^{+\infty} t^{1-d-2\gamma}\left(\int_{\mathbb{R}^d} \Delta_k u(y)h_k(t,x,y)\omega_k(y)dy\right)dt \\ &= \int_{\mathbb{R}^d} N_k(x,y)\Delta_k u(y)\omega_k(y)dy = N_k[\mu](x), \end{aligned}$$

where $d\mu(y) = \Delta_k u(y)\omega_k(y)dy$ is the $\Delta_k$-Riesz measure of $u$ (see Example 2.47-2)).
Hence, letting $r \longrightarrow +\infty$ in the relation (2.56) with $f = u$, we deduce that

$$u(x) = h(x) - N_k[\mu](x).$$

In particular,

$$\forall\, x \in \mathbb{R}^d, \quad N_k[\mu](x) \le a - u(x) < +\infty.$$

Using Theorem 2.75, we deduce that $u$ has a global Riesz decomposition on $\mathbb{R}^d$ given by $u = h - N_k[\mu]$ and the function $h$ is D- harmonic on $\mathbb{R}^d$. Since $h \le a$, by Liouville's theorem for bounded from above D-harmonic functions (see [16]), $h$ is a constant. We denote again by $h$ this constant. Furthermore, since $u$ is D-subharmonic, we have $u(x) \le M_S^r(u)(x) \le h$. Then, by taking the supremum of $u(x)$ over $x \in \mathbb{R}^d$, we get $a \le h$. Finally, we obtain $h = a$ and $u = a - N_k[\mu]$.
Let us now $u$ be a D-subharmonic function on $\mathbb{R}^d$ and let $u_n = u *_k \varphi_n$ be the function defined by (2.96). We know that $u_n \in \mathcal{C}^\infty(\mathbb{R}^d) \cap \mathcal{SH}_k(\mathbb{R}^d)$ and its $\Delta_k$-Riesz measure is given by $d\mu_n(x) := \mu *_k \varphi_n(x)\omega_k(x)dx$ (see the relation (2.110)). Moreover, as $\tau_{-x}\varphi_n \ge 0$, for every $n \in \mathbb{N}\backslash\{0\}$ and using (2.12) (recalling that $\int_{\mathbb{R}^d} \varphi_n(y)\omega_k(y)dy = 1$), $u_n$ is bounded from above and we get $a_n := \sup\ u_n(x) \le a := \sup u(x)$.
Now, since $u$ is the pointwise decreasing limit of the sequence $(u_n)$ (see Theorem 2.35), the sequence of real numbers $(a_n)$ is also decreasing and $a_n \ge a$. This proves that $a_n = a$. By the first step, we conclude that

$$\forall\, x \in \mathbb{R}^d, \quad u_n(x) = a - N_k[\mu_n](x) \quad \text{with} \quad d\mu_n(y) = \mu *_k \varphi_n(y)\omega_k(y)dy.$$

Letting $n \longrightarrow +\infty$ and using the relation (2.147), we deduce the desired result. □

**Corollary 2.79** *1. For every $x_0 \in \mathbb{R}^d$, the zero function is the greatest D-harmonic minorant on $\mathbb{R}^d$ of the D-superharmonic function $N_k(x_0, .)$.*

2. *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$ such that $N_k[\mu](x) < +\infty$ for at least one $x$. Then the zero function is the greatest D-harmonic minorant on $\mathbb{R}^d$ of the D-superharmonic function $N_k[\mu]$.*
3. *A function $u$ (not identically $-\infty$) defined on $\mathbb{R}^d$ is of the form $u = -N_k[\mu] + h$ where $\mu$ is a nonnegative Radon measure on $\mathbb{R}^d$ and $h$ is a D-harmonic function on $\mathbb{R}^d$ if and only if $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and $u$ has a D-harmonic majorant on $\mathbb{R}^d$. In this case, $h$ is the least D-harmonic majorant of $u$ on $\mathbb{R}^d$.*

**Remark 2.80** *The result of the statement 3) is a generalization of Theorem 1.24 in [23].*

*Proof:* By taking $\mu = \delta_{x_0}$, the statement 1) is a particular case of 2).
**2)** Let $h$ be a D-harmonic function on $\mathbb{R}^d$ such that $h \le N_k[\mu]$. Then the function $s = h - N_k[\mu]$ satisfies: i) $s \le 0$ on $\mathbb{R}^d$, ii) $s$ is in $\mathcal{SH}_k(\mathbb{R}^d)$ and iii) $\mu$ is the $\Delta_k$-Riesz measure of $s$ (by (2.140)). Therefore, by Theorem 2.77, we have

$$s = \sup_{\mathbb{R}^d} s - N_k[\mu] = h - N_k[\mu].$$

Thus, $h = \sup_{\mathbb{R}^d} s$ and by i) we must have $h \le 0$. This proves 2).
**3)** Suppose that $u = -N_k[\mu] + h$. Clearly $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and $u \le h$. Now, let $h_1$ be a D-harmonic function on $\mathbb{R}^d$ such that $u = -N_k[\mu] + h \le h_1$. This implies that $h - h_1 \le N_k[\mu]$. Thus, by the statement 2), we obtain $h \le h_1$. This proves that $h$ is the least D-harmonic

majorant of $u$ on $\mathbb{R}^d$.
Conversely, assume that $u \in \mathcal{SH}_k(\mathbb{R}^d)$ and it has a D-harmonic majorant $h_1$ on $\mathbb{R}^d$. Then the function $u - h_1$ is nonpositive and D-subharmonic on $\mathbb{R}^d$. Therefore, by Theorem 2.77,

$$\forall\, x \in \mathbb{R}^d, \quad u(x) - h_1(x) = a - N_k[\mu](x)$$

for some constant $a \leq 0$. Thus, for $h = a + h_1$, $u = h - N_k[\mu]$ is the global Riesz decomposition of $u$ and clearly we have $h \leq h_1$. $\square$

In the following result, we give some necessary and sufficient conditions for a nonnegative Radon measure on $\mathbb{R}^d$ to be the $\Delta_k$-Riesz measure of a bounded from above D-subharmonic function on $\mathbb{R}^d$.

**Proposition 2.81** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$. Then the following statements are equivalent*

**i)** *$\mu$ is the $\Delta_k$-Riesz measure of a bounded from above D-subharmonic function on $\mathbb{R}^d$,*

**ii)** *the function $-N_k[\mu]$ is D-subharmonic on $\mathbb{R}^d$,*

**iii)** *$\mu$ satisfies the finiteness condition (2.138),*

**iv)** *there exists $x_0 \in \mathbb{R}^d$ such that $N_k[\mu](x_0) < +\infty$,*

**v)** *there exists $x_0 \in \mathbb{R}^d$ such that*

$$\int_1^{+\infty} t^{1-d-2\gamma} n_k(t, x_0)dt < +\infty \quad \textit{with} \quad n_k(t, x_0) := \int_{\mathbb{R}^d} h_k(t, x_0, y)d\mu(y)dy. \tag{2.148}$$

**Remark 2.82** *1) In classical case (k=0), we have $n_0(t, x_0) = \mu[B(x_0, t)]$ and we notice that we can always assume $x_0 = 0$ by replacing the subharmonic function $u$ of $\Delta$-Riesz measure $\mu$ by its translate $u(x_0 + .)$ (see [17], Theorem 3.20). But, in our case, this is not possible for at least two reasons related to the Dunkl translation. The first is that, the Dunkl translations act only on some functional spaces and not on sets. The second is that the Dunkl translation is not always a positive operator. In particular, if $u$ is a $C^\infty$-D-subharmonic function on $\mathbb{R}^d$ (ie $\Delta_k u \geq 0$), we don't have necessarily $\tau_x[\Delta_k u] \geq 0$ and thus $\tau_x u$ is not necessarily D-subharmonic on $\mathbb{R}^d$.*

*2) From the statement iii) of the previous proposition, we can see that every $\Delta$-Riesz measure of a bounded from above $\Delta$-subharmonic function on $\mathbb{R}^d$ is also the $\Delta_k$-Riesz measure (for any choice of the Coxeter-Weyl group and an associated nonnegative multiplicity function) of a bounded from above $\Delta_k$-subharmonic function on $\mathbb{R}^d$.*

*Proof: i)* $\Rightarrow$ *ii)* Let $u$ be a bounded from above D-subharmonic function on $\mathbb{R}^d$ with $\mu$ its $\Delta_k$-Riesz measure. By Theorem 2.77 $u$ is of the form $u = \sup_{\mathbb{R}^d} u - N_k[\mu]$. This proves that $-N_k[\mu] \in \mathcal{SH}_k(\mathbb{R}^d)$.

*ii)* $\Rightarrow$ *iii)* The result follows from Propositions 2.25 and 2.67.

*iii)* $\Rightarrow$ *iv)* This implication is a direct consequence of Proposition 2.67.

$iv) \Rightarrow v)$ Using (2.130) and Fubini's theorem, we obtain $\int_0^{+\infty} t^{1-d-2\gamma} n_k(t, x_0) dt < +\infty$ and a fortiori (2.148) is satisfied.

$v) \Rightarrow i)$ We will partially follow the proof of Theorem 3.20 in [17]. Let $\mu$ be a nonnegative Radon measure satisfying (2.148) for some $x_0 \in \mathbb{R}^d$. Let $u(x) = -N_k[\mu](x)$. Then, by (2.140), it is enough to prove that $u \in \mathcal{SH}_k(\mathbb{R}^d)$. We can write

$$\begin{aligned} u(x) &= -\int_{B^W(x_0,1)} N_k(x,y) d\mu(y) - \int_{\mathbb{R}^d \setminus B^W(x_0,1)} N_k(x,y) d\mu(y) \\ &:= u_1(x) + u_2(x). \end{aligned}$$

From Proposition 2.70, the function $u_1$ is D-subharmonic on $\mathbb{R}^d$.
For $n \in \mathbb{N}$ with $n > 1$, we consider

$$v_n(x) = -\int_{B^W(x_0,n) \setminus B^W(x_0,1)} N_k(x,y) d\mu(y).$$

Again by Proposition 2.70, the function $v_n$ is D-subharmonic on $\mathbb{R}^d$. Moreover, we can see that $u_2$ is the pointwise decreasing limit of $v_n$ on $\mathbb{R}^d$ as $n \longrightarrow +\infty$.
Using the relation (2.130) and applying Fubini's theorem, we have

$$\begin{aligned} v_n(x_0) &= -\frac{1}{d_k} \int_0^\infty t^{1-d-2\gamma} \int_{B^W(x_0,n) \setminus B^W(x_0,1)} h_k(t, x_0, y) d\mu(y) dt \\ &= -\frac{1}{d_k} \int_0^1 t^{1-d-2\gamma} \int_{B^W(x_0,n) \setminus B^W(x_0,1)} h_k(t, x_0, y) d\mu(y) dt \\ &- \frac{1}{d_k} \int_1^\infty t^{1-d-2\gamma} \int_{B^W(x_0,n) \setminus B^W(x_0,1)} h_k(t, x_0, y) d\mu(y) dt \\ &= -\frac{1}{d_k} \int_1^\infty t^{1-d-2\gamma} \int_{B^W(x_0,n) \setminus B^W(x_0,1)} h_k(t, x_0, y) d\mu(y) dt \\ &\geq -\frac{1}{d_k} \int_1^\infty t^{1-d-2\gamma} n_k(t, x_0) dt, \end{aligned}$$

where the third equality follows from: $\forall\, t \leq 1$, supp $h_k(t, x_0, .) \subset B^W(x_0, t) \subset B^W(x_0, 1)$.
Letting $n \to +\infty$ and using our hypothesis (2.148), we deduce that

$$u_2(x_0) \geq -\frac{1}{d_k} \int_1^\infty t^{1-d-2\gamma} n_k(t, x_0) dt > -\infty.$$

Consequently, by Proposition 2.28, the function $u_2$ belongs to $\mathcal{SH}_k(\mathbb{R}^d)$. Thus, since $u = u_1 + u_2$, $u \in \mathcal{SH}_k(\mathbb{R}^d)$. $\square$

## 2.9 Annex

### 2.9.1 The Dunkl transform

In this Annex we recall some properties of the Dunkl transform (see [21] and [36] ).

• The Dunkl transform of a function $f \in L^1_k(\mathbb{R}^d)$ is defined by

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x)\omega_k(x)dx, \quad \lambda \in \mathbb{R}^d, \tag{2.149}$$

where $E_k(x,y) := V_k(e^{\langle x,.\rangle})(y), \quad x,y \in \mathbb{R}^d$, is the Dunkl kernel which is analytically extendable to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties (see [10], [13], [21] and [36])

1. for all $x, y \in \mathbb{C}^d$ and all $g \in W$,

$$E_k(x,y) = E_k(y,x), \qquad E_k(gx, gy) = E_k(x,y), \tag{2.150}$$

2. for all $\lambda \in \mathbb{C}$ and all $x, y \in \mathbb{C}^d$,

$$E_k(x, \lambda y) = E_k(\lambda x, y), \tag{2.151}$$

3. for all $x \in \mathbb{R}^d$, $y \in \mathbb{C}^d$ and all multi-indices $\upsilon \in \mathbb{N}^d$,

$$\Big| \frac{\partial^\upsilon}{\partial y^\upsilon} E_k(x,y) \Big| \leq \|x\|^{|\upsilon|} \max_{g \in W} e^{Re\, \langle gx, y\rangle}. \tag{2.152}$$

• It is well known (see [21]) that the Dunkl transform $\mathcal{F}_k$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} f(\lambda) E_k(ix, \lambda)\omega_k(\lambda) d\lambda, \quad x \in \mathbb{R}^d, \tag{2.153}$$

where $c_k$ is the Macdonald-Mehta constant given by (2.11).
Moreover, as $\mathcal{F}_k : L^1_k(\mathbb{R}^d) \cap L^2_k(\mathbb{R}^d) \longrightarrow L^2_k(\mathbb{R}^d)$ is bounded, the following Plancherel theorem holds (see [21]):
The transformation $c_k^{-1}\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2_k(\mathbb{R}^d)$ and we have the Plancherel formula:

$$\forall\ f \in L^2_k(\mathbb{R}^d), \quad \|c_k^{-1}\mathcal{F}_k(f)\|_{L^2_k(\mathbb{R}^d)} = \|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2.154}$$

• It is useful to note that if $f \in L^1_k(\mathbb{R}^d)$ is radial (i.e. $f(x) = \widetilde{f}(||x||)$, with $\widetilde{f}$ the profile function of $f$), $\mathcal{F}_k(f)$ is also radial. Precisely, using spherical coordinates and Corollary 2.5 of ([37]), we have

$$\mathcal{F}_k(f)(\lambda) = d_k \int_0^{+\infty} \widetilde{f}(r) j_{\gamma+\frac{d}{2}-1}(r\|\lambda\|) r^{2\gamma+d-1} dr, \quad \lambda \in \mathbb{R}^d, \tag{2.155}$$

where $d_k$ is defined by the relation (2.17) and for $\lambda \geq -1/2$, $j_\lambda$ is the normalized Bessel function given by

$$j_\lambda(z) = \Gamma(\lambda+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\lambda+1)} \Big(\frac{z}{2}\Big)^{2n}. \tag{2.156}$$

• Finally, we note that the Dunkl transform of a bounded nonnegative Borel measure $\mu$ on $\mathbb{R}^d$ is defined by

$$\mathcal{F}_k(\mu)(\xi) := \int_{\mathbb{R}^d} E_k(-ix, \xi) d\mu(x). \tag{2.157}$$

### 2.9.2 Dunkl's translation operators

The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [43])

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{2.158}$$

where $T_x$ is the classical translation operator given by $T_x f(y) = f(x+y)$. If $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform we have (see [43]):

$$\forall \;\; y \in \mathbb{R}^d, \quad \tau_x f(y) = \mathcal{F}_k^{-1}[E_k(ix, .)\mathcal{F}_k(f)](y) \tag{2.159}$$

$$= \frac{1}{c_k^2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda) E_k(ix, \lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda. \tag{2.160}$$

In particular, the relations (2.160) and (2.152) show that $(x, y) \longmapsto \tau_x f(y)$ is of class $C^\infty$ on $\mathbb{R}^d \times \mathbb{R}^d$.

The operators $\tau_x, x \in \mathbb{R}^d$, satisfy the following properties:

**1)** For all $x \in \mathbb{R}^d$, the operator $\tau_x$ is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself,

**2)** For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, the function $x \longmapsto \tau_x f(y)$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}^d$,

**3)** For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x), \tag{2.161}$$

and

$$D_j(\tau_x f) = \tau_x(D_j f), \quad j = 1, ..., d, \tag{2.162}$$

$$(D_j)_x(\tau_x f) = \tau_x(D_j f), \quad j = 1, ..., d, \tag{2.163}$$

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \tag{2.164}$$

where $D_j$ (resp. $\Delta_k$) are the Dunkl operators (resp. the Dunkl-Laplace operator).

**4)** Let $f \in \mathcal{S}(\mathbb{R}^d)$ be radial. Then we have (see [16] Lemme 3.1)

$$\tau_{-x} f(y) = \tau_{-y} f(x) \tag{2.165}$$

The following duality formula has been established by the authors (see [16] Proposition 2.1):

**5)** Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $g \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \tau_x f(y) g(y) \omega_k(y) dy = \int_{\mathbb{R}^d} f(y) \tau_{-x} g(y) \omega_k(y) dy. \tag{2.166}$$

We note at the end of this annex, that the Dunkl translation is also defined on $L_k^2(\mathbb{R}^d)$ by means of the Dunkl transform as follows: Fix $f \in L_k^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Since $|E_k(-ix, \xi)| \leq 1$, the function $\xi \mapsto E_k(ix, \xi)\mathcal{F}_k(f)(\xi)$ belongs to $L_k^2(\mathbb{R}^d)$. Hence, by Plancherel theorem, there exists a unique $L_k^2(\mathbb{R}^d)$-function denoted by $\tau_x f$ and called the $x$-Dunkl translated function of $f$ such that

$$\mathcal{F}_k(\tau_x f)(\xi) = E_k(ix, \xi)\mathcal{F}_k(f)(\xi). \tag{2.167}$$

For more properties on Dunkl translations when they act on $L_k^2(\mathbb{R}^d)$ we can see ([41]).

### 2.9.3 Fundamental solution of the Dunkl Laplacian

Here, we will propose a different proof of the relation (2.18) based on the generalized Green formula (see [27]). Let $d \geq 2$, $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $R > 0$ such that supp $\phi \subset B(0,R)$. Fix temporary $\varepsilon \in ]0,R[$. We have

$$\begin{aligned}\langle \Delta_k[S\omega_k], \phi\rangle &= \int_{B(0,R)} S(x)\Delta_k\phi(x)\omega_k(x)dx \\ &= \int_{B(0,R)\setminus B(0,\varepsilon)} S(x)\Delta_k\phi(x)\omega_k(x)dx + \int_{B(0,\varepsilon)} S(x)\Delta_k\phi(x)\omega_k(x)dx \\ &= I(\varepsilon) + J(\varepsilon).\end{aligned}$$

By using spherical coordinates, we can see that

$$\lim_{\varepsilon\to 0} J(\varepsilon) = 0. \tag{2.168}$$

Using the Green formula, the fact that $S$ is D-harmonic on $\mathbb{R}^d\setminus\{0\}$ and that supp $\phi \cap S(0,R) = \emptyset$, we deduce that

$$I(\varepsilon) = -\int_{S(0,\varepsilon)} \Big[S(\xi)\frac{\partial}{\partial\eta}\phi(\xi) - \frac{\partial}{\partial\eta}S(\xi)\phi(\xi)\Big]\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi).$$

• A change of variables yields,

$$\begin{aligned}\Big|\int_{S(0,\varepsilon)} S(\xi)\frac{\partial}{\partial\eta}\phi(\xi)\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi)\Big| &= \frac{\varepsilon^{2-d-2\gamma}}{d_k(d+2\gamma-2)}\Big|\int_{S(0,\varepsilon)} \langle\nabla\phi(\xi), \xi/\varepsilon\rangle\,\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi)\Big| \\ &\leq \frac{\varepsilon}{d_k(d+2\gamma-2)}\Big|\int_{S^{d-1}} \langle\nabla\phi(\varepsilon\xi), \xi\rangle\,\omega_k(\xi)d\sigma_{S^{d-1}}(\xi)\Big| \\ &\longrightarrow 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

• We have

$$\begin{aligned}\int_{S(0,\varepsilon)} \frac{\partial}{\partial\eta}S(\xi)\phi(\xi)\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi) &= -\frac{\varepsilon^{1-d-2\gamma}}{d_k}\int_{S(0,\varepsilon)} \phi(\xi)\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi) \\ &= -\frac{1}{m_k[S(0,\varepsilon)]}\int_{S(0,\varepsilon)} \phi(\xi)\omega_k(\xi)d\sigma_{S(0,\varepsilon)}(\xi) \\ &\longrightarrow -\phi(0) \text{ as } \varepsilon\to 0.\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0} I(\varepsilon) = -\phi(0). \tag{2.169}$$

Finally, by (2.168) and (2.169) we obtain the result. □

# Bibliography


[1] S. Andraus, M. Katori and S. Miyashita. *Interacting particules on the line and Dunkl intertwining operator of type A: Application to the freezing regime.* J. Phys. A: Math. Theo. (2012)

[2] D. H. Armitage and S. J. Gardiner. *Classical Potential Theory.* Springer-Verlag, London, (2001).

[3] S. Axler, P. Bourdon and W. Ramey. *Harmonic Function Theory*, Springer-Verlag , Second edition, (2001).

[4] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier. *Yang-Baxter equation in long-rang interaction systems.* J. Phys.A: Math. Gen. 26, (1993), 5219-5236.

[5] F. Calogero. *Solution of the one dimensional N-body problems with quadratic and/ or inversely quadratic pair potentials.* J. Math. Phys. 12, (1971), 419-436.

[6] G. Choquet. *Topology*, Volume XIX. Academic Press, (1966).

[7] L. Deleaval. *Two results on the Dunkl maximal function.* Studia Mathematica, 203, (2011), 47-68.

[8] E. DiBenedetto. *Real Analysis.* Birkhäuser, Boston, Advanced Texts Series (2002).

[9] C. F. Dunkl. *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc., 311, (1989), 167-183.

[10] C. F. Dunkl. *Integral kernels with reflection group invariance.* Canad. J. Math., 43, (1991), 123-183.

[11] C. F. Dunkl. *Hankel transforms associated to finite reflection groups.* Contemp. Math., 138, (1992), 123-138.

[12] C. F. Dunkl. *Orthogonal polynomials of type A and B and related Calogero models.* Commun. Math. Phys. 197, (1998), 451-487.

[13] C. F. Dunkl and Y. Xu. *Orthogonal Polynomials of Several variables.* Cambridge Univ. Press (2001).

[14] P. Etingof. *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups.* Math. Res. Lett. 17 (2010), no. 2, 275-282.

[15] P. J. Forrester. *Log-Gases and Random Matrices.* London Mathematical Society Monographs, (2010).

[16] L. Gallardo and C. Rejeb. *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications.* Trans. Amer. Math. Soc., electronically published on May 22, 2015, DOI:http://dx.doi.org/10.1090/tran/6671 (to appear in print).

[17] W. K. Hayman and P. B. Kennedy. *Subharmonic functions*, Volume 1. Academic Press London, (1976).

[18] L. L. Helms. *Potential theory.* Springer-Verlag London, (2009).

[19] L. Hörmander. *The Analysis of Linear Partial Differential operators I.* Springer, (1983).

[20] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics 29, Cambridge University Press, (1990).

[21] M. F. de Jeu. *The Dunkl transform.* Invent. Math., 113, (1993), 147-162.

[22] R. Kane. *Reflection Groups and Invariant Theory.* CMS Books in Mathematics. Springer-Verlag, New York, (2001).

[23] N. S. Landkof. *Foundations of Modern Potential Theory.* Springer, Berlin, (1972).

[24] E. H. Lieb and M. Loss. *Analysis.* Graduate Studies in Mathematics, Volume 14, AMS, Second Edition, (2001).

[25] M. Maslouhi and E. H. Youssfi. *Harmonic functions associated to Dunkl operators.* Monatsh. Math. 152(2007), 337-345.

[26] M. Maslouhi and R. Daher. *Weyl's lemma and converse mean value for Dunkl operators.* Operator Theory: Advances and Applications, Vol 205, (2009), 91-100.

[27] H. Mejjaoli and K. Trimèche . *On a mean value property associated with the Dunkl Laplacian operator and applications.* Integ. Transf. and Spec. Funct., 12(3), (2001), 279-302.

[28] E. M. Opdam. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group.* Compositio Math. 85 (1993), no. 3, 333-373.

[29] F. Riesz. *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel II.* Acta. Math, 54, (1930), 321-360.

[30] M. Rösler. *Bessel-type signed hypergroups on* $\mathbb{R}$. Probability measures on groups and related structures XI. Proceedings, Oberwolfach (1994). World Scientific, (1995), 292-304.

[31] M. Rösler. *Generalized Hermite polynomials and the heat equation for Dunkl operators.* Comm. Math. Phys, 192, (1998), 519-542.

[32] M. Rösler and M. Voit. *Markov processes related with Dunkl operators.* Adv. Appl. Math, 21, (1998), 575-643.

[33] M. Rösler. *Positivity of Dunkl's intertwining operator.* Duke Math. J., 98, (1999), 445-463.

[34] M. Rösler. *Short-time estimates for heat kernels associated with root systems.* Special functions (Hong Kong, 1999), 309-323, World Sci. Publ., River Edge, NJ, 2000.

[35] M. Rösler and M. F. de Jeu. *Asymptotic analysis for the Dunkl kernel.* J. Approx. Theory, 119, (2002), no. 1, 110-126.

[36] M. Rösler. *Dunkl Operators: Theory and Applications.* Lecture Notes in Math., vol.1817, Springer Verlag (2003), 93-136.

[37] M. Rösler. *A positive radial product formula for the Dunkl kernel.* Trans. Amer. Math. Soc., 355, (2003), 2413-2438.

[38] L. Schwartz. *Théorie des distributions.* Hermann, Editeurs des Sciences et des arts, (1966).

[39] B. Sutherland. *Exact results for a quantum many-body problem in one dimension.* Phys. Rev. A 4, (1971), 2019-2021.

[40] B. Sutherland. *Exact results for a quantum many-body problem in one dimension II.* Phys. Rev. A 5, (1972), 1372-1376.

[41] S. Thangavelu and Y. Xu. *Convolution operator and maximal function for Dunkl transform.* J. Anal. Math. Vol. 97, (2005), 25-56.

[42] K. Trimèche. *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual.* Integ. Transf. and Spec. Funct., 12(4), (2001), 394-374.

[43] K. Trimèche. *Paley-Wiener theorem for the Dunkl transform and Dunkl translation operators.* Integ. Transf. and Spec. Func. 13, (2002), 17-38.

# Chapter 3

# $\mathcal{P}$otentiels de $\mathcal{D}$unkl-$\mathcal{R}$iesz d'une $\mathcal{M}$esure de $\mathcal{R}$adon

**Résumé**

Pour un système de racines $R$ dans $\mathbb{R}^d$ et une fonction de mutliplicité positive $k$ définie sur $R$, on considère le noyau de la chaleur $p_t(x,y)$ associé à l'opérateur de Dunkl-Laplace $\Delta_k$. Pour $\beta \in ]0, d+2\gamma[$ avec $\gamma = \frac{1}{2}\sum_{\alpha\in R} k(\alpha)$, on étudie le noyau de Dunkl-Riesz d'indice $\beta$ défini par $R_{k,\beta}(x,y) = \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_t(x,y)dt$ et le potentiel de Dunkl-Riesz correpondant $I_{k,\beta}[\mu]$ d'une mesure de Radon $\mu$ sur $\mathbb{R}^d$. Selon la valeur de l'indice $\beta$, on étudie la $\Delta_k$-surharmonicité de ces fonctions et on donne comme applications la $\Delta_k$-mesure de Riesz de $I_{k,\beta}[\mu]$, le principe d'unicité des masses et l'inégalité ponctuelle de Hedberg.

# *Dunkl-Riesz Potentials of Radon Measures*

***Preprint***

**Abstract**

For a root system $R$ on $\mathbb{R}^d$ and a nonnegative multiplicity function $k$ on $R$, we consider the Dunkl-heat kernel $p_t(x,y)$ associated to the Dunkl-Laplace operator $\Delta_k$. For $\beta \in ]0, d+2\gamma[$, where $\gamma = \frac{1}{2}\sum_{\alpha\in R} k(\alpha)$, we study the Dunkl-Riesz kernel of index $\beta$ defined by $R_{k,\beta}(x,y) = \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1} p_t(x,y)dt$ and the corresponding Dunkl-Riesz potential $I_{k,\beta}[\mu]$ of a Radon measure $\mu$ on $\mathbb{R}^d$. According to the values of $\beta$, we study the $\Delta_k$-superharmonicity of these functions and we give some applications like the $\Delta_k$-Riesz measure of $I_{k,\beta}[\mu]$, the uniqueness principle and a pointwise Hedberg's inequality.

## 3.1 Introduction

Let $R$ be a normalized root system in $\mathbb{R}^d$. That is, for every $\alpha \in R$, $\|\alpha\| = 2$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$, where $\sigma_\alpha$ is the reflection with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$ (see [15] and [17]). We fix $k \geq 0$ a multiplicity function (i.e. $k : R \to [0, +\infty[$ invariant under the action of the Coxeter-Weyl group $W$ associated to $R$) and we consider the associated Dunkl-Laplace operator $\Delta_k$ given by

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle \nabla f(x), \alpha\rangle}{\langle x, \alpha\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle x,\alpha\rangle^2}\right), \quad f \in \mathcal{C}^2(\mathbb{R}^d), \quad (3.1)$$

with $R_+$ a positive subsystem (see [7]).
The Dunkl-Laplace operator (acting on $\mathcal{C}^\infty(\mathbb{R}^d)$) is related to the classical Laplace operator by means of the so-called Dunkl intertwining operator $V_k$ (see [6], [7], [29]) as follows:

$$V_k\Delta_k = \Delta V_k. \quad (3.2)$$

In [22], M. Rösler has proved that for any $x \in \mathbb{R}^d$, there exists a compactly supported probability measure $\mu_x$ on $\mathbb{R}^d$ such that

$$\forall\ f \in \mathcal{C}^\infty(\mathbb{R}^d), \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y)d\mu_x(y), \quad (3.3)$$

with

$$\text{supp } \mu_x \subset C(x) = co\{gx,\ g \in W\} \quad (3.4)$$

(the convex hull of the orbit of $x$ under the group $W$). Note that some recent results on the support of $\mu_x$ have been showed in [10].

Let $p_t(x,y)$ $(t>0,\ x,y\in\mathbb{R}^d)$ be the heat kernel of the Dunkl Laplacian $\Delta_k$ which is given by (see [20] and [24])

$$p_t(x,y) := \frac{1}{(2t)^{d/2+\gamma}c_k} e^{-(\|x\|^2+\|y\|^2)/4t} E_k\left(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\right), \tag{3.5}$$

where $E_k(x,y)=V_k(e^{\langle\cdot,y\rangle})(x)$ is the Dunkl kernel (see [5] and [7]) and $c_k$ is the Macdonald-Mehta constant (see [19] and [8]) given by

$$c_k := \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}}\omega_k(x)dx. \tag{3.6}$$

For all fixed $x\in\mathbb{R}^d$, the function $p_t(x,.)$ solves the Dunkl heat equation

$$(\Delta_k-\partial_t)p_t(x,.)=0. \tag{3.7}$$

Let $\gamma=\sum_{\alpha\in R_+}k(\alpha)$. Under the condition $d+2\gamma>2$, the Dunkl-Newton kernel has been introduced in [10] via the Dunkl heat kernel as follows

$$N_k(x,y)=\int_0^{+\infty}p_t(x,y)dt. \tag{3.8}$$

The corresponding potential of a nonnegative Radon measure $\mu$ on $\mathbb{R}^d$ has been also introduced as the function

$$N_k[\mu](x)=\int_{\mathbb{R}^d}N_k(x,y)d\mu(y),\quad x\in\mathbb{R}^d. \tag{3.9}$$

The aim of this paper is the study, when $d+2\gamma>2$, of the Dunkl-Riesz kernel

$$R_{k,\beta}(x,y) := \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty}t^{\frac{\beta}{2}-1}p_t(x,y)dt,$$

where $\beta\in]0,d+2\gamma[$ and the corresponding potential

$$I_{k,\beta}[\mu](x)=\int_{\mathbb{R}^d}R_{k,\beta}(x,y)d\mu(y)$$

of a signed Radon measure $\mu$ on $\mathbb{R}^d$.
In particular, we will study the sub-or-superharmonicity of these functions in the sense of the Dunkl-Laplace operator. This notion of subharmonicity, which generalizes the classical one [1] has been introduced and studied in some details in [10]. More precisely, let $\Omega$ be a $W$-invariant open subset of $\mathbb{R}^d$, a function $u:\Omega\longrightarrow[-\infty,+\infty[$ is called Dunkl-subharmonic (D-subharmonic) on $\Omega$ if
• $u$ is upper semi-continuous (u.s.c.) [2] on $\Omega$,

1. see for example [2],[12],[14] and [18].
2. see [3] for more properties of these functions

• $u$ is not identically $-\infty$ on each connected component of $\Omega$,
• it satisfies the sub-mean volume property: for every closed ball $B(x,r) \subset \Omega$, we have

$$u(x) \leq M_B^r(u)(x) := \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} u(y) h_k(r,x,y) \omega_k(y) dy, \tag{3.10}$$

where $h_k(r,x,y)$ is the harmonic kernel for which a precise expression and properties will be recalled in the next section (see [9] and [10]) and $m_k$ is the measure $\omega_k(x)dx$ with $\omega_k$ the Dunkl weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} \tag{3.11}$$

which is homogeneous of degree $2\gamma$. Naturally, a function $u$ is D-superharmonic on $\Omega$ if $-u$ is D-subharmonic on $\Omega$.

Finally, as applications, we establish the following version of the uniqueness principle which is the main result of the paper: if $\mu$ and $\nu$ are finite and nonnegative Radon measures on $\mathbb{R}^d$ and if $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on $\mathbb{R}^d$ , then $\mu = \nu$. We also prove a pointwise Hedberg's inequality in Dunkl context and we deduce some $L^p$-boundedness properties of the Dunkl-Riesz potentials.

## 3.2 Generalities in Dunkl Analysis

In order to help the reader, we have collected in this section the essentials on Dunkl analysis used in the sequel.

**Notations:** Let us introduce the following functional spaces which will be used throughout the paper:
• $\Omega$ a $W$-invariant open subset of $\mathbb{R}^d$.
• $L_k^p(\Omega)$ (resp. $L_{k,loc}^p(\Omega)$), $1 \leq p < +\infty$ the space of measurable functions $f : \Omega \longrightarrow \mathbb{C}$ such that $\|f\|_{L_k^p(\Omega)}^p := \int_\Omega |f(x)|^p \omega_k(x) dx < +\infty$ (resp. $\int_K |f(x)|^p \omega_k(x) dx < +\infty$ for any compact set $K \subset \Omega$).
• $L_k^\infty(\Omega)$ the space of measurable and essentially bounded functions on $\Omega$.
• When $\Omega = \mathbb{R}^d$, the norm of the space $L_k^p(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, will be denoted $\|.\|_{k,p}$ instead of $\|.\|_{L_k^p(\mathbb{R}^d)}$.
• $\mathcal{D}(\Omega)$ the space of $C^\infty$-functions on $\Omega$ with compact support.
• $\mathcal{D}'(\Omega)$ the space of distributions on $\Omega$ (i.e. the topological dual of $\mathcal{D}(\Omega)$ carrying the Fréchet topology).
• $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of $C^\infty$-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives.
• $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

### 3.2.1 The Dunkl transform

The Dunkl transform of a function $f \in L_k^1(\mathbb{R}^d)$ is defined by (see [16] and [24])

$$\mathcal{F}_k(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x) \omega_k(x) dx, \quad \lambda \in \mathbb{R}^d, \tag{3.12}$$

where $E_k(x,y)$ is the Dunkl kernel which is analytically extendable to $\mathbb{C}^d\times\mathbb{C}^d$ and satisfies the following properties (see [5], [7], [16])

1. for all $x,y\in\mathbb{R}^d$, we have
$$|E_k(-ix,y)|\le 1. \tag{3.13}$$

2. for all $a\in\mathbb{C}$, $x,y\in\mathbb{C}^d$ and all $g\in W$, we have
$$E_k(ax,y)=E_k(x,ay),\quad E_k(x,y)=E_k(y,x)\quad\text{and}\quad E_k(gx,gy)=E_k(x,y), \tag{3.14}$$

3. for all $x\in\mathbb{R}^d$, $y\in\mathbb{C}^d$ and all multi-indices $\upsilon\in\mathbb{N}^d$,
$$\Big|\frac{\partial^\upsilon}{\partial y^\upsilon}E_k(x,y)\Big|\le\|x\|^{|\upsilon|}\max_{g\in W}e^{Re\,\langle gx,y\rangle}. \tag{3.15}$$

It is well known (see [16]) that the Dunkl transform $\mathcal{F}_k$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself and its inverse is given by

$$\mathcal{F}_k^{-1}(f)(x)=c_k^{-2}\int_{\mathbb{R}^d}f(\lambda)E_k(ix,\lambda)\omega_k(\lambda)d\lambda,\quad x\in\mathbb{R}^d, \tag{3.16}$$

where $c_k$ is the constant given by (3.6).
We note that for $f,g\in\mathcal{S}(\mathbb{R}^d)$ the following relation holds

$$\int_{\mathbb{R}^d}\mathcal{F}_k(f)(x)g(x)\omega_k(x)dx=\int_{\mathbb{R}^d}f(x)\mathcal{F}_k(g)(x)g(x)dx. \tag{3.17}$$

Moreover, as $\mathcal{F}_k : L_k^1(\mathbb{R}^d)\cap L_k^2(\mathbb{R}^d)\longrightarrow L_k^2(\mathbb{R}^d)$ is bounded, the transformation $c_k^{-1}\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L_k^2(\mathbb{R}^d)$ (Plancherel theorem, see [16]).

We will also need the Dunkl transform of a tempered distribution $S\in\mathcal{S}'(\mathbb{R}^d)$ which is defined by
$$\langle\mathcal{F}_k(S),\phi\rangle:=\langle S,\mathcal{F}_k(\phi)\rangle,\quad \phi\in\mathcal{S}(\mathbb{R}^d).$$
It is known that $\mathcal{F}_k$ is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself (see [30]).
Note that if $\mu$ is a bounded Radon measure on $\mathbb{R}^d$, $\mu\in\mathcal{S}'(\mathbb{R}^d)$ and its distributional Dunkl transform can be identified to the continuous function $\xi\mapsto\int_{\mathbb{R}^d}E_k(-ix,\xi)d\mu(x)\omega_k(\xi)$. In the literature, the function

$$\mathcal{F}_k(\mu):\xi\mapsto\int_{\mathbb{R}^d}E_k(-ix,\xi)d\mu(x) \tag{3.18}$$

is called the Dunkl transform of the measure $\mu$. This transformation is injective on the space of bounded Radon measures on $\mathbb{R}^d$ (see [21]).
We recall also that the Dunkl-Laplace operator $\Delta_k$ leaves the spaces $\mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ invariant where the $\Delta_k$-action on $S$ in $\mathcal{D}'(\mathbb{R}^d)$) (resp. in $\mathcal{S}'(\mathbb{R}^d)$) is defined as in the classical case by

$$\langle\Delta_kS,\phi\rangle=\langle S,\Delta_k\phi\rangle,\quad \phi\in\mathcal{D}(\mathbb{R}^d)\text{ (resp. }\phi\in\mathcal{S}(\mathbb{R}^d)). \tag{3.19}$$

### 3.2.2 Dunkl's translation operators and heat kernel properties

- The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d$, are defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by (see [30])

$$\forall\, y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f)(y) d\mu_x(z), \tag{3.20}$$

where $T_x$ is the classical translation operator given by $T_x f(y) = f(x+y)$. The operators $\tau_x$, $x \in \mathbb{R}^d$, satisfy the following properties:

**1)** For all $x \in \mathbb{R}^d$, the operator $\tau_x$ is continuous from $\mathcal{C}^\infty(\mathbb{R}^d)$ into itself.

**2)** For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, the function $x \longmapsto \tau_x f(y)$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}^d$.

**3)** For all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \quad \tau_x f(y) = \tau_y f(x) \tag{3.21}$$

**4)** The Dunkl-Laplace operator $\Delta_k$ commutes with the Dunkl translations, i.e.

$$\tau_x(\Delta_k f) = \Delta_k(\tau_x f), \quad x \in \mathbb{R}^d, f \in \mathcal{C}^\infty(\mathbb{R}^d). \tag{3.22}$$

**5)** If $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is radial, M. Rösler ([25]) has proved the useful formula

$$\forall\, x \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} \widetilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, z\rangle}) d\mu_y(z), \tag{3.23}$$

where $\widetilde{f}$ is the profile of $f$ and $\mu_y$ is the measure defined by (3.3).

In the particular case when $f \in \mathcal{S}(\mathbb{R}^d)$, $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$ and using the Dunkl transform we have (see [30]):

$$\begin{aligned}\forall\ \ y \in \mathbb{R}^d, \quad \tau_x f(y) &= \mathcal{F}_k^{-1}[E_k(ix, .)\mathcal{F}_k(f)](y) &(3.24)\\ &= c_k^{-2} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda) E_k(ix, \lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda. &(3.25)\end{aligned}$$

Note that the relations (3.25) and (3.15) show that $(x, y) \longmapsto \tau_x f(y)$ is of class $C^\infty$ on $\mathbb{R}^d \times \mathbb{R}^d$. Furthermore, if $f \in \mathcal{S}(\mathbb{R}^d)$ is radial, then

$$\tau_{-x} f(y) = \tau_{-y} f(x) \tag{3.26}$$

(see [9], Lemme 3.1).

- Using (3.23), the Dunkl heat kernel can also be written

$$\begin{aligned}p_t(x, y) &= \frac{1}{(2t)^{d/2+\gamma} c_k}\ \tau_{-x}\big(e^{-\frac{\|.\|^2}{4t}}\big)(y) &(3.27)\\ &= \frac{1}{(2t)^{d/2+\gamma} c_k} \int_{\mathbb{R}^d} e^{-\frac{1}{4t}\left(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle\right)} d\mu_y(z). &(3.28)\end{aligned}$$

For later use, we record the following properties of the heat kernel (see [20] and [24])

1. For every $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\|p_t(x,.)\|_{k,1} = \int_{\mathbb{R}^d} p_t(x,y)\omega_k(y)dy = 1. \tag{3.29}$$

2. For every $t > 0$ and $x, y \in \mathbb{R}^d$,

$$p_t(x,y) = \mathcal{F}_k^{-1}\big(E_k(-ix,.)e^{-t\|.\|^2}\big)(y) \tag{3.30}$$
$$= c_k^{-2}\int_{\mathbb{R}^d} e^{-t\|\xi\|^2}E_k(-ix,\xi)E_k(iy,\xi)\omega_k(\xi)d\xi. \tag{3.31}$$

3. For every $t > 0$, the following inequality holds

$$\forall\ x,y \in \mathbb{R}^d, \quad p_t(x,y) \leq \frac{1}{(2t)^{d/2+\gamma}c_k}e^{-\frac{1}{4t}\min_{g\in W}\|x-gy\|^2}. \tag{3.32}$$

4. For all $t, s > 0$, the Dunkl heat kernel satisfies the semi-group property

$$\forall\ x,y \in \mathbb{R}^d, \quad p_{t+s}(x,y) = \int_{\mathbb{R}^d} p_t(x,z)p_s(y,z)\omega_k(z)dz. \tag{3.33}$$

### 3.2.3 The harmonic kernel and D-subharmonic functions

For $r > 0$ and $x, y \in \mathbb{R}^d$, the harmonic kernel $h_k(r,x,y)$ is defined (see [9] and [10]) by:

$$h_k(r,x,y) := \int_{\mathbb{R}^d} \mathbf{1}_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,z\rangle})d\mu_y(z). \tag{3.34}$$

In the classical case (i.e. $k = 0$), we have $h_0(r,x,y) = \mathbf{1}_{[0,r]}(\|x-y\|) = \mathbf{1}_{B(x,r)}(y)$.
The harmonic kernel satisfies the following properties:

1. For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r,x,y) \leq 1$.
2. For all fixed $x, y \in \mathbb{R}^d$, the function $r \longmapsto h_k(r,x,y)$ is right-continuous and non decreasing.
3. For all fixed $r > 0$ and $x \in \mathbb{R}^d$,

$$B(x,r) \subset \text{supp } h_k(r,x,.\,) \subset B^W(x,r) := \cup_{g\in W}B(gx,r). \tag{3.35}$$

4. For all $r > 0$ and $x, y \in \mathbb{R}^d$, we have

$$h_k(r,x,y) = h_k(r,y,x). \tag{3.36}$$

5. Let $r > 0$ and $x, y \in \mathbb{R}^d$. Then, for all $g \in W$, we have

$$h_k(r,gx,gy) = h_k(r,x,y) \ \ \textit{and} \ \ h_k(r,gx,y) = h_k(r,x,g^{-1}y). \tag{3.37}$$

6. For all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\|h_k(r,x,.)\|_{k,1} := \int_{\mathbb{R}^d} h_k(r,x,y)\omega_k(y)dy = m_k(B(0,r)) = \frac{d_k r^{d+2\gamma}}{d+2\gamma}, \tag{3.38}$$

where we recall that $dm_k(y) = \omega_k(y)dy$ and $d_k$ is the constant

$$d_k = \frac{c_k}{2^{d/2+\gamma-1}\Gamma(d/2+\gamma)}. \tag{3.39}$$

Note that $d_k = \int_{S^{d-1}} \omega_k(\xi)d\sigma(\xi)$ with $d\sigma(\xi)$ the surface measure of the unit sphere $S^{d-1}$ of $\mathbb{R}^d$.

Finally, we recall that
• a function $u$ of class $C^2$ on $\Omega$ is D-subharmonic in the sense of (3.10) if and only if $\Delta_k u \geq 0$ on $\Omega$ (see [10]).
• if $u$ is D-subharmonic on $\Omega$, then $u\omega_k \in L^1_{loc}(\Omega)$ (that is $u \in L^1_{k,loc}(\Omega)$) and its distributional Dunkl-Laplacian $\Delta_k(u\omega_k)$ is a nonnegative distribution on $\Omega$ in the sense that for any nonnegative function $\phi \in \mathcal{D}(\Omega)$ we have

$$\langle \Delta_k(u\omega_k), \phi\rangle := \langle u\omega_k, \Delta_k\phi\rangle = \int_{\mathbb{R}^d} u(x)\Delta_k\phi(x)\omega_k(x)dx \geq 0. \tag{3.40}$$

The nonnegative distribution $\Delta_k(u\omega_k)$ is then a nonnegative Radon measure on $\Omega$ called the $\Delta_k$-Riesz measure of the D-subharmonic function $u$ (see [10]). In particular, if $u \in \mathcal{C}^2(\Omega)$ its $\Delta_k$-Riesz measure is equal to $\Delta_k u(x)\omega_k(x)dx$.

## 3.3 The Dunkl-Riesz kernel

In this section and under the condition $d + 2\gamma > 2$, we will introduce the Dunkl-Riesz kernel by means of Dunkl's heat kernel and we will study some of its properties.

**Definition 3.1** *For $x, y \in \mathbb{R}^d$ and $0 < \beta < d + 2\gamma$, the Dunkl-Riesz kernel is defined by*

$$R_{k,\beta}(x,y) := \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} p_t(x,y)dt, \tag{3.41}$$

*where $p_t(x,y)$ is the Dunkl heat kernel given by (3.5)*

**Remark 3.2** ***1)*** *By the positivity of the Dunkl heat kernel $p_t(x,y)$, we have $0 < R_{k,\beta}(x,y) \leq +\infty$ for all $x, y \in \mathbb{R}^d$.*

***2)*** *Let $x \in \mathbb{R}^d$ be fixed. From (3.32), we can see that if $y \notin \mathbb{R}^d \backslash W.x$, then for any $\beta \in ]-\infty, d+2\gamma[$ the function $t \mapsto t^{\frac{\beta}{2}-1}p_t(x,y)$ is integrable on $]0,+\infty[$. Thus, using the properties of the Gamma function, the function $y \mapsto \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}p_t(x,y)dt$ is well defined on $\mathbb{R}^d \setminus W.x$ whenever $\beta \in ]-\infty, d+2\gamma[\setminus -2\mathbb{N}$. In this case, we will continue denoting it $y \mapsto R_{k,\beta}(x,y)$.*

In the following result, we will show that the Dunkl-Riesz kernel can be expressed in terms of the harmonic kernel. This new formula will be a crucial tool in the sequel of the paper.

**Proposition 3.3** *For every $x, y \in \mathbb{R}^d$, we have*

$$R_{k,\beta}(x,y) = \kappa \int_{\mathbb{R}^d} \Big( \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \Big)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \tag{3.42}$$

$$= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t,x,y) \frac{dt}{t}, \tag{3.43}$$

*where*

$$\kappa = \kappa(d,\gamma,\beta) = \frac{2^{1-\beta}\Gamma(\frac{d+2\gamma-\beta}{2})}{d_k \Gamma(\beta/2)\Gamma(d/2+\gamma)} = \frac{2^{\frac{d}{2}+\gamma-\beta}\Gamma(\frac{d+2\gamma-\beta}{2})}{c_k\Gamma(\beta/2)}, \tag{3.44}$$

*$c_k$ and $d_k$ being the constants given by (3.6) and (3.39) respectively.*

*Proof:* Using the change of variables $1/4t \leftrightarrow t$, the relation (3.28) can be rewritten

$$R_{k,\beta}(x,y) = \frac{2^{\frac{d}{2}+\gamma-\beta}}{\Gamma(\beta/2)c_k} \int_0^{+\infty} t^{\frac{d+2\gamma-\beta}{2}-1} \int_{\mathbb{R}^d} e^{-t\left(\|x\|^2+\|y\|^2-2\langle x,z\rangle\right)} d\mu_x(z)dt.$$

Now, by Fubuni's theorem and the identity

$$\forall\ a \geq 0, \quad \forall\ \theta > 0, \quad a^{-\theta/2} = \frac{1}{\Gamma(\theta/2)} \int_0^{+\infty} s^{\frac{\theta}{2}-1} e^{-sa} ds$$

(notice that if we take $a = 0$, the both terms are equal $+\infty$), we deduce that (3.42) holds.
• Let us now prove (3.43). Starting from (3.42) and applying again Fubini's theorem, we get

$$\begin{aligned}
R_{k,\beta}(x,y) &= \kappa \int_{\mathbb{R}^d} \Big( \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \Big)^{\frac{\beta-(d+2\gamma)}{2}} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_{\mathbb{R}^d} \int_{\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}}^{+\infty} t^{\beta-d-2\gamma} \frac{dt}{t} d\mu_y(z) \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} \left( \int_{\mathbb{R}^d} \mathbf{1}_{[0,t]}(\sqrt{\|x\|^2+\|y\|^2-2\langle x,z\rangle}) d\mu_y(z) \right) \frac{dt}{t} \\
&= \frac{\kappa}{d+2\gamma-\beta} \int_0^{+\infty} t^{\beta-d-2\gamma} h_k(t,x,y) \frac{dt}{t}.
\end{aligned}$$

This gives the desired relation. □

**Exemple 3.4** ***1)*** *When $k = 0$, the Rösler measure is the Dirac measure at $y$ (i.e. $\mu_y = \delta_y$) and then $R_{0,\beta}(x,y) = \kappa(d,0,\beta)\|x-y\|^{\beta-d}$ is the classical Riesz kernel (see for example [18]).*

***2)*** *Since $\mu_0 = \delta_0$, for any choice of the Coxeter-Weyl group and of a nonnegative multiplicity function, we have $R_{k,\beta}(x,0) = \kappa(d,\gamma,\beta)\|x\|^{\beta-d-2\gamma}$.*

*3) We consider $\mathbb{R}^d$ $(d \geq 1)$ with the root system $R = \{\pm e_1\}$ with $e_1 = (1, 0, \ldots, 0)$. In this case, the Coxeter-Weyl group is $\mathbf{Z}_2 = \{id, \sigma_{e_1}\}$, the multiplicity function is a parameter $k = k(e_1) > 0$ and the Rösler measure is of the form $\mu_{(y_1, y')} = \mu_{y_1} \otimes \delta_{y'}$ where $y' = (y_2, \ldots, y_d)$, $\delta_{y'}$ is the Dirac measure at $y'$ and $\mu_{y_1}$ is the $\mathbf{Z}_2$-Rösler measure. If $y_1 = 0$, we know that $\mu_0 = \delta_0$ and if $y_1 \neq 0$, we have*

$$\langle \mu_{y_1}, f \rangle := \int_{-1}^{1} f(ty_1)\phi_k(t)dt, \quad f \in \mathcal{C}(\mathbb{R}),$$

*where $\phi_k$ is the $\mathbf{Z}_2$-Dunkl density function of parameter $k$ given by (see [5] and [24] p.104)*

$$\phi_k(t) := \frac{\Gamma(k + 1/2)}{\sqrt{\pi}\Gamma(k)}(1-t)^{k-1}(1+t)^k \mathbf{1}_{[-1,1]}(t), \tag{3.45}$$

*By the change of variables $s = ty_1$, we can write*

$$\forall\ y_1 \in \mathbb{R} \setminus \{0\}, \quad \langle \mu_{y_1}, f \rangle = \frac{1}{y_1}\int_{-y_1}^{y_1} f(s)\phi_k(\frac{s}{y_1})ds = \frac{1}{|y_1|}\int_{-|y_1|}^{|y_1|} f(s)\phi_k(\frac{s}{y_1})ds.$$

*This shows that $\mu_{y_1}$, $y_1 \neq 0$, has a density with respect to the Lebesgue measure given by*

$$\phi_{k,y_1}(s) = \frac{1}{|y_1|}\phi_k(\frac{s}{y_1})\mathbf{1}_{[-|y_1|,|y_1|]}(s). \tag{3.46}$$

*Then, if $d + 2k > 2$ the Dunkl-Riesz kernel is given by*

$$R_{k,\beta}(x, y) = \kappa(d, k, \beta)\int_{-|y_1|}^{|y_1|} \Big(x_1^2 + y_1^2 - 2tx_1 + \|x' - y'\|^2\Big)^{\frac{\beta-d-2k}{2}} \phi_{k,y_1}(t)dt.$$

*By a change of variables, we can write this relation as follows*

$$R_{k,\beta}(x, y) = \kappa(d, k, \beta)\int_{-1}^{1} \Big(x_1^2 + y_1^2 - 2tx_1y_1 + \|x' - y'\|^2\Big)^{\frac{\beta-d-2k}{2}} \phi_k(t)dt.$$

*4) We consider $\mathbb{R}^d$ $(d \geq 1)$ with the root system $R_m := \{\pm e_1, \ldots, \pm e_m\}$, where $m$ is a fixed integer in $\{1, \ldots, d\}$ and $(e_j)_{1\leq j\leq d}$ is the canonical basis of $\mathbb{R}^d$. For $\xi \in \mathbb{R}^d$, we will denote $\xi = (\xi^{(m)}, \xi') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$.*
*Noting that the Coxeter-Weyl group is given by $W = \mathbf{Z}_2^m$ and that the $\mathbf{Z}_2^m$-orbit of a point $\xi \in \mathbb{R}^d$ is given by*

$$\mathbf{Z}_2^m.\xi := \{\varepsilon.\xi := (\varepsilon_1\xi_1, \ldots, \varepsilon_m\xi_m, \xi'), \quad \varepsilon = (\varepsilon_i)_{1\leq i\leq m} \in \{\pm 1\}^m\}.$$

*The multiplicity function can be represented by the $m$-multidimensional parameter $k = (k_1, \ldots, k_m)$ with $k_j = k(e_j) > 0$. Moreover, the Rösler measure is of the form $\mu_y =$*

$\mu_{(y^{(m)},y')} = \mu_{y_1} \otimes \cdots \otimes \mu_{y_m} \otimes \delta_{y'}$ *with* $\mu_{y_i}$ *the* $\mathbf{Z}_2$*-Rösler measure at point* $y_i$ *(see (3.46)). In this case, the Dunkl-Riesz kernel is of the form*

$$R_{k,\beta}(x,y) = \kappa \int_{[-1,1]^m} \Big( \|x^{(m)}\|^2 + \|y^{(m)}\|^2 - 2\sum_{j=1}^{m} t_j x_j y_j + \|x'-y'\|^2 \Big)^{\frac{\beta-d-2k}{2}} \times \prod_{i=1}^{m} \phi_{k_i}(t_i) dt_1 \dots dt_m, \quad (3.47)$$

*where* $\phi_{k_i}$ *is the* $\mathbf{Z}_2$*-Dunkl density of parameter* $k_i$ *given by (3.45).*

**Proposition 3.5** *Suppose that* $\gamma > 0$*. Let* $0 < \beta < d + 2\gamma$ *and* $x, y \in \mathbb{R}^d$*.*

1. *If* $y \notin W.x$*, then* $R_{k,\beta}(x,y) < +\infty$*.*
2. *Assume that* $x \in \mathbb{R}^d \setminus \bigcup_\alpha H_\alpha$*. Then* $R_{k,\beta}(x,x) = +\infty$ *if and only if* $d \geq \beta$*.*
3. *If* $x \in \bigcup_\alpha H_\alpha$ *and* $\beta \leq d$*, then* $R_{k,\beta}(x,x) = +\infty$*.*

*Proof:* At first we note that

$$\forall\ x, y \in \mathbb{R}^d, \quad \forall\ t > 0, \quad t^{\frac{\beta}{2}-1} p_t(x,y) \leq C t^{\frac{\beta-d-2\gamma}{2}-1}$$

Hence, as $\beta < d + 2\gamma$, the function $t \mapsto t^{\frac{\beta}{2}-1} p_t(x,y)$ is integrable on $[1, +\infty[$ for every $x, y \in \mathbb{R}^d$.

**1)** We obtain the result by using (3.32).

**2)** Fix $x \in \mathbb{R}^d$ such that $x$ is not in any hyperplane $H_\alpha$, $\alpha \in R$ (i.e. $x$ lives in a Weyl chamber). We will use the following short-time asymptotic result of the Dunkl type heat kernel which has been established in ([23], Corollary 2): Let $C$ be a fixed Weyl chamber. If $x, y \in C$, then

$$p_t(x,y) \sim_{t\to 0} \big(\omega_k(x)\omega_k(y)\big)^{-1/2} (4\pi t)^{-d/2} e^{-\frac{\|x-y\|^2}{4t}}. \quad (3.48)$$

For $y = x$, we obtain

$$p_t(x,x) \sim_{t\to 0} \big(\omega_k(x)\big)^{-1} (4\pi t)^{-d/2}.$$

This implies that

$$t^{\frac{\beta}{2}-1} p_t(x,x) \sim_{t\to 0} \big(\omega_k(x)\big)^{-1} (4\pi)^{-d/2} t^{\frac{\beta-d}{2}-1}.$$

Thus, the function $t \mapsto t^{\frac{\beta}{2}-1} p_t(x,x)$ is not integrable near 0 if and only if $d \geq \beta$.

**3)** Let $x \in H_\alpha$ for some $\alpha \in R$. One can see that the function $\psi : \xi \longmapsto R_{k,\beta}(\xi,\xi)$ is the increasing limit of the sequence of continuous functions $\xi \longmapsto \int_{1/n}^{n} t^{\frac{\beta}{2}-1} p_t(\xi,\xi) dt$. This implies that $\psi$ is lower semi-continuous on $\mathbb{R}^d$. Consequently, when $\beta \leq d$ we have

$$R_{k,\beta}(x,x) = \liminf_{\xi \to x} R_{k,\beta}(\xi,\xi) = +\infty.$$

$\square$

As in the case of the Dunkl-Newton kernel (see [10]), it is not easy to see for $g \neq id$ if $gx$ is a singularity or not of the function $R_{k,\beta}(x,.)$. However, in the following result, we will give a complete description of the singularities of $R_{k,\beta}(x,.)$ when the Coxeter-Weyl group $\mathbf{Z}_2^m$ acts on $\mathbb{R}^d$. More precisely, we have:

**Proposition 3.6** *Let $x \in \mathbb{R}^d \backslash \{0\}$. Using the same notations of Example 3.4, 4), denoting $H_i$ the hyperplane orthogonal to $e_i$ and recalling $\varepsilon.x = (\varepsilon_1 x_1, \dots, \varepsilon_m x_m, x') \in \mathbf{Z}_2^m.x$.*

1. *If $x \in \cap_{i=1}^m H_i$, then $x = \varepsilon.x$ and $R_{k,\beta}(x,x) = +\infty$.*
2. *Assume that $x \notin \cap_{i=1}^m H_i$. Set $A := \{i \in \{1,\dots,m\}, \quad x_i \neq 0\}$ and $\varepsilon^{(n)}.x = (\varepsilon_1 x_1, \dots, \varepsilon_m x_m, x')$ the point of $\mathbf{Z}_2^m$-orbit of $x$ such that $\big|\{j \in A, \quad \varepsilon_j = 1\}\big| = n$ i.e the point $\varepsilon^{(n)}.x$ has exactly $n$ among the nonzero coordinates $(x_j)_{j\in A}$ have not been changed under the action of $\mathbf{Z}_2^m$. Then,*
$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2\Big(|A| - n + \sum_{j\in A} k_j - \gamma\Big) + \beta. \tag{3.49}$$
3. *Assume that $x \notin \cup_{i=1}^m H_i$. Then,*
$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Longleftrightarrow \quad d \geq 2(m-n) + \beta. \tag{3.50}$$
*In this case, we have* $\displaystyle\sum_{n=\max(0,\lfloor m-\frac{d}{2}+\frac{\beta}{2}\rfloor)}^{m} \binom{m}{n}$ *singularities living in $\mathbb{R}^d \setminus \cup_{\alpha\in R_+} H_\alpha$.*

*Proof:* For abbreviation, we will use the following constants
$$C_1 := 2^{\frac{\beta-d-2\gamma}{2}}\kappa, \quad C(k) := \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}. \tag{3.51}$$

From (3.47), it is easy to see that
$$R_{k,\beta}(x,\varepsilon.x) = C_1 \int_{[-1,1]^m} \Big(\|x^{(m)}\|^2 - \sum_{j=1}^m \varepsilon_j t_j x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j=1}^m \phi_{k_j}(t_j) dt_1 \otimes \dots \otimes dt_m. \tag{3.52}$$

**1)** Clearly, from (3.52), the condition $x \in \cap_{i=1}^m H_i$ i.e. $x^{(m)} = 0$ implies that $x = \varepsilon.x = (0, x')$ and $R_{k,\beta}(x,\varepsilon.x) = +\infty$.
**2)** Suppose that $x \notin \cap_{i=1}^m H_i$. At first, we write (3.52) as follows
$$R_{k,\beta}(x,\varepsilon.x) = C_1 \int_{[-1,1]^m} \Big(\sum_{j=1}^m (1-\varepsilon_j t_j) x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j=1}^m \phi_{k_j}(t_j) \otimes_{j=1}^m dt_j. \tag{3.53}$$

Now, using the notations of the Proposition, Fubini's theorem and the fact that $\phi_{k_j}$ are probability densities, (3.53) can be written in the following form
$$R_{k,\beta}(x,\varepsilon.x) = C_1 \int_{[-1,1]^{|A|}} \Big(\sum_{j\in A} (1-\varepsilon_j t_j) x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j\in A} \phi_{k_j}(t_j) \otimes_{j\in A} dt_j. \tag{3.54}$$

We will distinguish two cases:
**First case** $|A| = 1$. Let $i \in \{1, \ldots, m\}$ such that $x_i \neq 0$. In this case, using (3.45) and (3.51), we deduce that (3.54) takes the form

$$\begin{aligned}R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{-1}^{1} \Big((1-\varepsilon_i s)x_i^2\Big)^{\frac{\beta-d-2\gamma}{2}} \phi_{k_i}(s) ds \\ &= C(k_i) C_1 |x_i|^{\beta-d-2\gamma} \int_{-1}^{1} (1-\varepsilon_i s)^{\frac{\beta-d-2\gamma}{2}} (1-s)^{k_i-1}(1+s)^{k_i} ds.\end{aligned}$$

• If $\varepsilon_i = 1$, then according to our notations, we have $n = |A| = 1$, $\varepsilon.x = \varepsilon^{(1)}.x = x$ and

$$R_{k,\beta}(x, \varepsilon^{(1)}.x) = C(k_i) C_1 |x_i|^{\beta-d-2\gamma} \int_{-1}^{1} (1-s)^{k_i + \frac{\beta-d-2\gamma}{2} - 1}(1+s)^{k_i} ds.$$

Consequently, $R_{k,\beta}(x, \varepsilon^{(1)}.x) = +\infty$ if and only if $d \geq \beta + 2k_i - 2\gamma$. Then, the result is proved in this case.
• When $\varepsilon_i = -1$, we have $n = 0$, $\varepsilon.x = \varepsilon^{(0)}.x$ and

$$R_{k,\beta}(x, \varepsilon^{(0)}.x) = C(k_i) |x_i|^{\beta-d-2\gamma} \int_{-1}^{1} (1+s)^{k_i + \frac{\beta-d-2\gamma}{2}} (1-s)^{k_i-1} ds.$$

Thus, as $k_i > 0$ we have $R_{k,\beta}(x, \varepsilon^{(0)}.x) = +\infty$ if and only if $d \geq 2(1 + k_i - \gamma) + \beta$.
**Second case** $|A| = r \geq 2$. Using (3.54) and the change of variables $t_j \leftrightarrow 1 - \varepsilon_j t_j$, we obtain

$$\begin{aligned}R_{k,\beta}(x, \varepsilon.x) &= C_1 \int_{]0,2[^{|A|}} \Big(\sum_{j\in A} t_j x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}} \prod_{j\in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j t_j) \otimes_{j\in A} dt_j \\ &= C_1 \int_{]0,2[^{|A|}\cap B_r} \qquad + \qquad C_1 \int_{]0,2[^{|A|}\setminus B_r} \\ &= C_1 I(x, \varepsilon.x) \qquad + \qquad C_1 J(x, \varepsilon.x),\end{aligned}$$

where $B_r$ is the open unit ball in $\mathbb{R}^{|A|} = \mathbb{R}^r$.
The singularities of these integrals being at point 0 and thus it is clear that $J(x, \varepsilon.x) < +\infty$. Thus, we need to know when the integral $I(x, \varepsilon.x)$ diverges. To do this, we will identify $(t_j)_{j\in A}$ with $v = (v_1, \ldots, v_r) \in \mathbb{R}^r$ and use the spherical coordinates in $\mathbb{R}^r$:

$$\rho = \|v\|, \quad v_1 = \rho a_1, \quad \ldots, v_{r-1} = \rho a_{r-1} \quad \text{and} \quad v_r = \rho a_r,$$

where

$$a_1 = \cos\theta_1, \ldots, a_{r-1} = \prod_{i=1}^{r-2} \sin\theta_i \cos\theta_{r-1}, \quad a_r = \prod_{i=1}^{r-1} \sin\theta_i.$$

Notice that all $a_j$ are positive.

$$I(x, \varepsilon.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \Big(\int_0^1 \prod_{j\in A} \phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) \rho^{r + \frac{\beta-d-2\gamma}{2} - 1} d\rho\Big) d\sigma_r(a^{(r)}), \quad (3.55)$$

where $S_+^{r-1} := ]0,2[^r \cap S^{r-1}$, $d\sigma_r$ is the surface measure of the unit sphere $S^{r-1}$ of $\mathbb{R}^r$, $a^{(r)} = (a_j)_{j\in A}$, $x^{(r)} = (x_j)_{j\in A}$ and

$$\psi(a^{(r)}, x^{(r)}) := \Big(\sum_{j\in A} a_j x_j^2\Big)^{\frac{\beta-d-2\gamma}{2}}.$$

We have

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = C(k_j)(1-\varepsilon_j+\varepsilon_j a_j\rho)^{k_j-1}(1+\varepsilon_j-\varepsilon_j a_j\rho)^{k_j}.$$

Hence,

$$\phi_{k_j}(\varepsilon_j - \varepsilon_j a_j \rho) = \begin{cases} C(k_j) a_j^{k_j-1}\rho^{k_j-1}(2-a_j\rho)^{k_j}, & \text{if} \quad \varepsilon_j = 1 \\ \\ C(k_j) a_j^{k_j}\rho^{k_j}(2-a_j\rho)^{k_j-1}, & \text{if} \quad \varepsilon_j = -1. \end{cases} \tag{3.56}$$

Define

$$A_1 := \Big\{ j \in A, \quad \varepsilon_j = 1 \Big\}, \quad A_2 = A \backslash A_1.$$

According to our notations, we have $|A_1| = |\{j, \quad \varepsilon_j = 1\}| = n$.
Then, from (3.55), (3.56) and recalling the definition of the vector $\varepsilon^{(n)}.x$, we deduce that

$$I(x, \varepsilon^{(n)}.x) = \int_{S_+^{r-1}} \psi(a^{(r)}, x^{(r)}) \Big( \int_0^1 f(a^{(r)}, \rho)\rho^{\lambda+r+\frac{\beta-d-2\gamma}{2}-1} d\rho \Big) d\sigma_r(a^{(r)}), \tag{3.57}$$

with

$$f(a^{(r)}, \rho) := \prod_{j\in A_1} C(k_j) a_j^{k_j-1}(2-a_j\rho)^{k_j} \prod_{j\in A_2} C(k_j) a_j^{k_j}(2-a_j\rho)^{k_j-1}.$$

and

$$\lambda := \sum_{j\in A_1}(k_j - 1) + \sum_{j\in A_2} k_j = \sum_{j\in A} k_j - n.$$

The function $\rho \longmapsto f(a^{(r)}, \rho)$ is continuous and does not vanish on the compact set $[0,1]$. So that the singularity in the $d\rho$-integral is only in the term of

$$\rho^{\lambda+r+\frac{\beta-d-2\gamma}{2}-1} = \rho^{(\sum_{j\in A} k_j)-n+r+\frac{\beta-d-2\gamma}{2}-1}.$$

Finally, we conclude that

$$R_{k,\beta}(x, \varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad I(x, \varepsilon^{(n)}.x) = +\infty \quad \Leftrightarrow \quad d \geq 2(|A| - n + \sum_{j\in A} k_j - \gamma) + \beta.$$

This completes the proof of the assertion 2).
**3)** When $x \notin \cup_{i=1}^m H_i$, we have $A = \{1, \ldots, m\}$ and then the result is a particular case of the statement 2). □

**Proposition 3.7** *The Riesz kernel $R_{k,\beta}(.,.)$ satisfies the following properties*

1. *For every $x, y \in \mathbb{R}^d$ and $g \in W$, we have*

$$R_{k,\beta}(x,y) = R_{k,\beta}(y,x), \quad R_{k,\beta}(gx,y) = R_{k,\beta}(x, g^{-1}y). \tag{3.58}$$

*2. Let $\beta, \theta > 0$ such that $\beta + \theta < d + 2\gamma$. Then we have the following generalized Riesz composition formula*

$$\int_{\mathbb{R}^d} R_{k,\beta}(x,z) R_{k,\theta}(y,z)\omega_k(z)dz = R_{k,\beta+\theta}(x,y). \tag{3.59}$$

*3. Let $x \in \mathbb{R}^d$. Then, for every $y \in \mathbb{R}^d \backslash W.x$, we have*

$$\kappa \min_{g\in W} \left( \|x - gy\|^{\beta-d-2\gamma} \right) \leq R_{k,\beta}(x,y) \leq \kappa \max_{g\in W} \left( \|x - gy\|^{\beta-d-2\gamma} \right) \tag{3.60}$$

*4. Let $y \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x,y)$ is*
*-lower semi-continuous (l.s.c.) on $\mathbb{R}^d$.*
*-of class $C^\infty$ on $\mathbb{R}^d \backslash W.x$ and we have*

$$\partial_j R_{k,\beta}(x,y) = (\beta - d - 2\gamma)\kappa \int_{\mathbb{R}^d} (x_j - z_j) \left( \|x\|^2 + \|y\|^2 - 2\langle x, z\rangle \right)^{\frac{\beta-2-d-2\gamma}{2}} d\mu_y(z). \tag{3.61}$$

*Proof:* **1)** The result follows from (3.43), (3.36) and (3.37).
**2)** By (3.33) and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} R_{k,\theta}(x,z) R_{k,\beta}(y,z)\omega_k(z)dz \\
&= \frac{1}{\Gamma(\theta/2)\Gamma(\beta/2)} \int_{\mathbb{R}^d} \left( \int_0^{+\infty} t^{\theta/2-1} p_t(x,z)dt \right) \left( \int_0^{+\infty} s^{\beta/2-1} p_s(y,z)ds \right) \omega_k(z)dz \\
&= \frac{1}{\Gamma(\theta/2)\Gamma(\beta/2)} \int_0^{+\infty} t^{\theta/2-1} \left( \int_0^{+\infty} s^{\beta/2-1} p_{t+s}(x,y)ds \right) dt \\
&= \frac{1}{\Gamma(\theta/2)\Gamma(\beta/2)} \int_0^{+\infty} t^{\theta/2-1} \left( \int_t^{+\infty} r^{\beta/2-1} p_r(x,y)dr \right) dt \\
&= \frac{1}{\Gamma(\theta/2)\Gamma(\beta/2)} \int_0^{+\infty} p_r(x,y) \left( \int_0^r t^{\theta/2-1}(r-t)^{\beta/2-1}dt \right) dr \\
&= \frac{1}{\Gamma(\theta/2)\Gamma(\beta/2)} \left( \int_0^{+\infty} r^{\frac{\theta+\beta}{2}-1} p_r(x,y)dr \right) \left( \int_0^1 t^{\theta/2-1}(1-t)^{\beta/2-1}dt \right) \\
&= \frac{1}{\Gamma(\frac{\theta+\beta}{2})} \int_0^{+\infty} r^{\frac{\theta+\beta}{2}-1} p_r(x,y)dr \\
&= R_{k,\theta+\beta}(x,y).
\end{aligned}$$

**3)** Let $y \in \mathbb{R}^d$. From (3.4) for any $z \in \text{supp } \mu_y$, we can write $z = \sum_{g\in W} \lambda_g(z) gy$, where $\lambda_g(z) \in [0,1]$ are such that $\sum_{g\in W} \lambda_g(z) = 1$. Then, we have

$$\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle = \sum_{g\in W} \lambda_g(z) \|x - gy\|^2. \tag{3.62}$$

As $\psi : t \longmapsto t^{\frac{\beta-d-2\gamma}{2}}$ is a convex function on $]0,+\infty[$, by (3.62) we have

$$\Big(\|x\|^2+\|y\|^2-2\langle x,z\rangle\Big)^{\frac{\beta-d-2\gamma}{2}} = \Big(\sum_{g\in W}\lambda_g(z)\|x-gy\|^2\Big)^{\frac{\beta-d-2\gamma}{2}} \leq \max_{g\in W}\big(\|x-gy\|^{\beta-d-2\gamma}\big).$$

This implies the right inequality. Again by convexity, Jensen's inequality and (3.62), we get

$$\begin{aligned} R_{k,\beta}(x,y) &\geq \kappa\left(\int_{\mathbb{R}^d}(\|x\|^2+\|y\|^2-2\langle x,z\rangle)d\mu_y(z)\right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa\left(\sum_{g\in W}\left(\int_{\mathbb{R}^d}\lambda_g(z)d\mu_y(z)\right)\|x-gy\|^2\right)^{\frac{\beta-d-2\gamma}{2}} \\ &\geq \kappa\left(\max_{g\in W}\|x-gy\|^2\right)^{\frac{\beta-d-2\gamma}{2}} = \kappa\min_{g\in W}\left(\|x-gy\|^{\beta-(d+2\gamma)}\right), \end{aligned}$$

where in the last line we have used the fact that $\psi$ is a decreasing function.
**4)** The function $x\mapsto R_{k,\beta}(x,y)$ is l.s.c. on $\mathbb{R}^d$ as being the increasing limit of the sequence $(f_n)$ of continuous functions defined by $f_n : x\mapsto \int_{1/n}^n t^{\frac{\beta}{2}-1}p_t(x,y)dt$.
Fix $y\in\mathbb{R}^d$. Using the fact that $\mu_y$ is with compact support and the fact that the function

$$(x,z)\longmapsto \big(\|x\|^2+\|y\|^2-2\langle x,z\rangle\big)^{\frac{\beta-d-2\gamma}{2}}$$

is of class $C^\infty$ on $\mathbb{R}^d\backslash W.y\times\mathbb{R}^d$, we can differentiate under the integral in the relation (3.42) and we obtain the result. $\square$

In the classical (i.e $k=0$), we know that the Riesz kernel $R_0(x,.) : y\mapsto \kappa(d,0,\beta)\|x-y\|^{\beta-d}$ is a function of $L^p_{loc}(\mathbb{R}^d)$ whenever $p\in[1,d/(d-\beta)[$. In the following result, we will extend this result in Dunkl setting.

**Proposition 3.8** *Let $0<\beta<d+2\gamma$ and $p\in[1,\frac{d+2\gamma}{d+2\gamma-\beta}[$. Then, for every $R>0$, there exists a positive constant $C=C(R,p,d,\gamma,\beta)$ such that*

$$\forall\ x\in\mathbb{R}^d,\quad \|R_{k,\beta}(x,.)\|_{L^p_k(B(0,R))}\leq C. \tag{3.63}$$

*In particular, for every $x\in\mathbb{R}^d$, $R_{k,\beta}(x,.)$ is in $L^p_{k,loc}(\mathbb{R}^d)$.*

*Proof:* By Jensen's inequality and (3.42), we have

$$(R_{k,\beta}(x,y))^p \leq \kappa^p\int_{\mathbb{R}^d}\big(\|x\|^2+\|y\|^2-2\langle x,z\rangle\big)^{\frac{p(\beta-d-2\gamma)}{2}}d\mu_y(z).$$

Using the same idea as in the proof of (3.43), we can write the previous inequality as follows

$$\begin{aligned}(R_{k,\beta}(x,y))^p &\leq \frac{\kappa^p}{p(d+2\gamma-\beta)}\int_0^{+\infty} t^{p(\beta-d-2\gamma)}h_k(t,x,y)\frac{dt}{t}\\ &= C_1\int_0^1 t^{p(\beta-d-2\gamma)}h_k(t,x,y)\frac{dt}{t}+C_1\int_1^{+\infty} t^{p(\beta-d-2\gamma)}h_k(t,x,y)\frac{dt}{t}\\ &\leq C_1\int_0^1 t^{p(\beta-d-2\gamma)}h_k(t,x,y)\frac{dt}{t}+\frac{C_1}{p(d+2\gamma-\beta)},\end{aligned}$$

where $C_1=\frac{\kappa^p}{p(d+2\gamma-\beta)}$ and we have used the fact that $h_k(t,x,y)\leq 1$ in the last inequality. Let then $R>0$. From (3.38), Fubini's theorem and our hypothesis, we deduce that

$$\int_{B(0,R)}\int_0^1 t^{p(\beta-d-2\gamma)}h_k(t,x,y)\frac{dt}{t}\omega_k(y)dy\leq\frac{d_k}{d+2\gamma}\int_0^1 t^{p(\beta-d-2\gamma)}t^{d+2\gamma}\frac{dt}{t}:=C_2<+\infty.$$

This proves the desired inequality where we can take

$$C=\Big(C_1C_2+\frac{C_1m_k[B(0,R)]}{p(d+2\gamma-\beta)}\Big)^{1/p}.$$

□

**Proposition 3.9** *Let* $0<\beta<d+2\gamma$ *and* $x_0\in\mathbb{R}^d$. *Then, the function* $R_{k,\beta}(x_0,.)$ *is*

**i)** *D-superharmonic on* $\mathbb{R}^d$ *when* $\beta\geq 2$,

**ii)** *D-harmonic on* $\mathbb{R}^d\backslash W.x_0$ *when* $\beta=2$,

**iii)** *D-subharmonic on* $\mathbb{R}^d\backslash W.x_0$ *when* $\beta\leq 2$

*Proof:* The case $\beta=2$ (i.e. the case of the Dunkl-Newton kernel) has been done in [10]. So, we will deal with the case $\beta\neq 2$.
**i)** Suppose that $\beta>2$. We consider the function $S_{x_0,\beta,r}$

$$S_{x_0,\beta,r}(x):=\frac{1}{\Gamma(\beta/2)}\int_r^{+\infty} t^{\frac{\beta}{2}-1}p_t(x_0,x)dt.$$

By the monotone convergence theorem, we see that the function $R_{k,\beta}(x_0,.)$ is the pointwise increasing limit of the sequence $\Big(S_{x_0,\beta,\frac{1}{n}}\Big)_n$. Hence, by Proposition 4.3 in [10], it suffices to prove that for every $r>0$, $S_{x_0,\beta,r}$ is D-superharmonic on $\mathbb{R}^d$. To do this, we have only to show that $S_{x_0,\beta,r}$ is of class $C^2$ on $\mathbb{R}^d$ and $\Delta_kS_{x_0,\beta,r}\leq 0$ on $\mathbb{R}^d$ (see [10], Corollary 6.2). The function $p_t(x_0,.)$ is of class $C^\infty$ on $\mathbb{R}^d$ and we can differentiate under the integral sign in the relation (3.28) to obtain

$$\partial_jp_t(x_0,.)(x)=-\frac{1}{2t}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j-z_j)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z)\qquad(3.64)$$

and

$$\partial_i\partial_j p_t(x_0,.)(x) = -\delta_{ij}\frac{1}{2t}p_t(x_0,x)$$
$$+\frac{1}{4t^2}\frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}\int_{\mathbb{R}^d}(x_j-z_j)(x_i-z_i)e^{-\frac{1}{4t}(\|x\|^2+\|x_0\|^2-2\langle x,z\rangle)}d\mu_{x_0}(z), \quad (3.65)$$

where $\delta_{ij}$ is the Kronecker symbol.
Using the fact that supp $\mu_{x_0}\subset B(0,\|x_0\|)$, we deduce from (3.64) and (3.65) that

$$|\partial_j p_t(x_0,.)(x)| \leq \frac{\|x\|+\|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k},$$
$$|\partial_i\partial_j p_t(x_0,.)(x)| \leq \frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k}+\frac{(\|x\|+\|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}.$$

Let $R>0$. The previous inequalities and the differentiation theorem under the integral sign imply that $S_{x_0,\beta,r}$ is of class $C^2$ on the open ball $\overset{\circ}{B}(0,R)$ and as $x\mapsto p_t(x_0,x)$ is a solution of the Dunkl-heat equation (3.7), we deduce that

$$\begin{aligned}\forall\ x\in\overset{\circ}{B}(0,R),\quad \Delta_k S_{x_0,\beta,r}(x) &= \frac{1}{\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-1}\Delta_k\left(p_t(x_0,.)\right)(x)dt\\ &= \frac{1}{\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-1}\partial_t p_t(x_0,x)dt\\ &= -\frac{r^{\frac{\beta}{2}-1}}{\Gamma(\beta/2)}p_r(x_0,x)-\frac{\beta-2}{2\Gamma(\beta/2)}\int_r^{+\infty}t^{\frac{\beta}{2}-2}p_t(x_0,x)dt<0.\end{aligned}$$

Therefore, $S_{x_0,\beta,r}$ is D-superharmonic on $\overset{\circ}{B}(0,R)$. As $R>0$ is arbitrary, we conclude that $S_{x_0,\beta,r}$ is D-superharmonic on $\mathbb{R}^d$ as desired.

**iii)** Let $\beta\in]0,2[$. Using (3.64), (3.65) and (3.62), we can see that

$$|\partial_j p_t(x_0,.)(x)| \leq \frac{\|x\|+\|x_0\|}{(2t)^{1+\frac{d}{2}+\gamma}c_k}e^{-\frac{\min_{g\in W}(\|x-gx_0\|^2)}{4t}},$$
$$|\partial_i\partial_j p_t(x_0,.)(x)| \leq \Big(\frac{1}{(2t)^{1+\frac{d}{2}+\gamma}c_k}+\frac{(\|x\|+\|x_0\|)^2}{(2t)^{2+\frac{d}{2}+\gamma}c_k}\Big)e^{-\frac{\min_{g\in W}(\|x-gx_0\|^2)}{4t}}.$$

Fix an arbitrary open Dunkl ball $O^W(a,R):=\cup_{g\in W}\overset{\circ}{B}(ga,R)$ such that its closure is contained in $\mathbb{R}^d\backslash W.x_0$. The previous inequalities imply that we can differentiate with respect to $x\in O^W(a,R)$ under the integral sign in the relation (3.41). Furthermore, using the heat equation (3.7) and integrating by parts, we obtain

$$\begin{aligned}\forall\ x\in O^W(a,R),\quad \Delta_k\left(R_{k,\beta}(x_0,.)\right)(x) &= \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty}t^{\frac{\beta}{2}-1}\partial_t p_t(x_0,x)dt\\ &= -\frac{\beta-2}{2\Gamma(\beta/2)}\int_0^{+\infty}t^{\frac{\beta}{2}-2}p_t(x_0,x)dt \geq 0.\end{aligned}$$

According to Remark 3.2-*2)*, the above relation can be written as

$$\forall\ x \in O^W(a,R), \quad \Delta_k \left(R_{k,\beta}(x_0,.)\right)(x) = -R_{k,\beta-2}(x_0,x) \geq 0. \tag{3.66}$$

Therefore, the function $R_{k,\beta}(x_0,.)$ is D-subharmonic on $O^W(a,R)$ and so on $\mathbb{R}^d \backslash W.x_0$. $\square$

**Proposition 3.10** *Let $\beta \in ]0, d+2\gamma[$ and $x_0 \in \mathbb{R}^d$. Then, the function $x \mapsto R_{k,\beta}(x_0,x)\omega_k(x)$ defines a tempered distribution and we have*

$$\mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right) = E_k(-ix_0,.)\|.\|^{-\beta}\omega_k \quad in \quad \mathcal{S}'(\mathbb{R}^d). \tag{3.67}$$

*Proof:* Let $m \in \mathbb{N}$ such that $m > d+2\gamma$. We claim that there exists a constant $C_m = C(d,\gamma,\beta,m) > 0$ such that

$$\forall\ x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} (1+\|x\|^2)^{-m} R_{k,\beta}(x_0,x)\omega_k(x)dx \leq C_m. \tag{3.68}$$

From (3.43), we can write

$$\begin{aligned} R_{k,\beta}(x_0,x) &= \frac{\kappa}{d+2\gamma-\beta}\Big(\int_0^1 t^{\beta-d-2\gamma-1}h_k(t,x_0,x)dt + \int_1^{+\infty} t^{\beta-d-2\gamma-1}h_k(t,x_0,x)dt\Big) \\ &:= A(x_0,x) \quad + \quad B(x_0,x). \end{aligned}$$

- Using Fubini's theorem and the relation (3.38), for any $x_0 \in \mathbb{R}^d$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} (1+\|x\|^2)^{-m} A(x_0,x)\omega_k(x)dx &\leq \int_{\mathbb{R}^d} A(x_0,x)\omega_k(x)dx \\ &= \frac{\kappa}{d+2\gamma-\beta}\int_0^1 t^{\beta-d-2\gamma-1}\|h_k(t,x_0,.)\|_{k,1}dt \\ &= \frac{d_k\kappa}{\beta(d+2\gamma)(d+2\gamma-\beta)} := C_{1,m}. \end{aligned}$$

- Now, using the inequality $h_k(t,x_0,x) \leq 1$, we deduce that

$$\forall\ x_0 \in \mathbb{R}^d, \quad B(x_0,x) \leq \frac{\kappa}{(d+2\gamma-\beta)^2}.$$

This relation and the choice of $m$ imply that

$$\begin{aligned} \forall\ x_0 \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} (1+\|x\|^2)^{-m} B(x_0,x)\omega_k(x)dx &\leq \frac{\kappa}{(d+2\gamma-\beta)^2}\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m}\omega_k(x)dx \\ &:= C_{2,m} < +\infty. \end{aligned}$$

This proves (3.68) and this implies that the function $R_{k,\beta}(x_0,.)\omega_k$ defines a tempered distribution (see [26], Theorem VII, p. 242).

Let us now prove (3.67). For $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right), \phi\rangle = \frac{1}{\Gamma(\beta/2)}\int_{\mathbb{R}^d}\left(\int_0^{+\infty} t^{\frac{\beta}{2}-1}p_t(x_0,x)dt\right)\mathcal{F}_k(\phi)(x)\omega_k(x)dx.$$

Multiplying and dividing by $(1+\|x\|^2)^m$ (the integer $m$ is chosen is as above) and using the fact that $\mathcal{F}_k(\phi) \in \mathcal{S}(\mathbb{R}^d)$, we see that we can use Fubini's theorem in the above relation. Moreover, from (3.17) and (3.30), we obtain

$$\begin{aligned}\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right), \phi\rangle &= \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}\left(\int_{\mathbb{R}^d}\mathcal{F}_k(p_t(x_0,.))(x)\phi(x)\omega_k(x)dx\right)dt \\ &= \frac{1}{\Gamma(\beta/2)}\int_0^{+\infty} t^{\frac{\beta}{2}-1}\left(\int_{\mathbb{R}^d}E_k(-ix_0,x)e^{-t\|x\|^2}\phi(x)\omega_k(x)dx\right)dt.\end{aligned}$$

Applying again Fubini's theorem, we deduce that

$$\langle \mathcal{F}_k\left(R_{k,\beta}(x_0,.)\omega_k\right), \phi\rangle = \int_{\mathbb{R}^d} E_k(-ix_0,x)\|x\|^{-\beta}\phi(x)\omega_k(x)dx.$$

This completes the proof. □

From the formula (3.66), we see that the $\Delta_k$-Riesz measure related to the D-subharmonic function $R_{k,\beta}(x_0,.)$, $\beta < 2$ is given by $-R_{k,\beta-2}(x_0,x)\omega_k(x)dx$. In the following result, we will compute the $\Delta_k$-Riesz measure of the D-superharmonic function $R_{k,\beta}(x_0,.)$ with $\beta \in [2, d+2\gamma[$.

**Proposition 3.11** *Let $2 \leq \beta < d+2\gamma$ and $x_0 \in \mathbb{R}^d$. If $m \in [1, \beta/2]$ be an integer, then the function $x \mapsto R_{k,\beta}(x_0,x)$ satisfies*

$$(-\Delta_k)^m\left(R_{k,\beta}(x_0,.)\omega_k\right) = \begin{cases} R_{k,\beta-2m}(x_0,.)\omega_k & in \quad \mathcal{S}'(\mathbb{R}^d) \quad if \quad \beta > 2m, \\ \delta_{x_0} & in \quad \mathcal{S}'(\mathbb{R}^d) \quad if \quad \beta = 2m, \end{cases} \tag{3.69}$$

*where $\delta_{x_0}$ is the Dirac measure at $x_0$.*

*Proof:* At first, we remark that if $U \in \mathcal{S}'(\mathbb{R}^d)$, then

$$\mathcal{F}_k(\Delta_k U) = -\|.\|^2\mathcal{F}_k(U), \tag{3.70}$$

as easily follows from the relation $\Delta_k\mathcal{F}_k(f) = -\mathcal{F}_k(\|.\|^2 f)$ for all $f \in \mathcal{S}(\mathbb{R}^d)$.
From (3.70) and (3.67), we obtain

$$\begin{aligned}\mathcal{F}_k\big((-\Delta_k)^m\left(R_{k,\beta}(x_0,.)\omega_k\right)\big) &= E_k(-ix_0,.)\|.\|^{\beta-2m}\omega_k \\ &= \begin{cases} \mathcal{F}_k\big(R_{k,\beta-2m}(x_0,.)\omega_k\big) & \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad \text{if} \quad \beta > 2m, \\ \mathcal{F}_k(\delta_{x_0}) & \text{in} \quad \mathcal{S}'(\mathbb{R}^d) \quad \text{if} \quad \beta = 2m. \end{cases}\end{aligned}$$

Hence, we deduce the result by the fact that $\mathcal{F}_k$ is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ onto itself. □

**Remark 3.12** *Let $1 \leq m < \gamma + d/2$ an integer. Taking $x_0 = 0$ in (3.69), we deduce that the function $S : y \mapsto R_{k,2m}(0,y)\omega_k(y) = \kappa\|y\|^{2m-d-2\gamma}\omega_k(y)$ is the fundamental solution of the Dunkl-polylaplacian of order $m$ $(-\Delta_k)^m$ i.e. $(-\Delta_k)^m S = \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$.*

## 3.4 Riesz potentials of Radon measures

The sets $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{M}^+(\mathbb{R}^d)$ denote respectively the space of signed Radon measures on $\mathbb{R}^d$ and the convex cone of nonnegative Radon measures on $\mathbb{R}^d$.

**Definition 3.13** *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$. The Dunkl-Riesz potential of $\mu$ is defined by*

$$I_{k,\beta}[\mu](x) = \int_{\mathbb{R}^d} R_{k,\beta}(x,y)d\mu(y), \quad x \in \mathbb{R}^d. \tag{3.71}$$

**Proposition 3.14** *Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ and $\beta \in ]0, d+2\gamma[$.*

1. *If $\mu$ is bounded, then $I_{k,\beta}[\mu] \in L^p_{k,loc}(\mathbb{R}^d)$ whenever $p \in [1, \frac{d+2\gamma}{d+2\gamma-\beta}[$. In particular, $I_{k,\beta}[\mu]$ is finite a.e. in $\mathbb{R}^d$.*
2. *The following statements are equivalent*
   ***i)*** *$I_{k,\beta}[\mu]$ is finite a.e. in $\mathbb{R}^d$,*
   ***ii)*** *the measure $\mu$ satisfies*

$$\int_{\mathbb{R}^d} (1+\|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty, \tag{3.72}$$

   ***iii)*** *$I_{k,\beta}[\mu](x_0) < +\infty$ for some $x_0 \in \mathbb{R}^d$.*

   *If ii) or iii) holds, then $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$.*

*Proof:* **1)** Assume that $\mu$ is a probability measure on $\mathbb{R}^d$. Let $p$ as in the proposition and $R > 0$. Using respectively (3.71), Jensen's inequality, Fubini's theorem, the fact that the Riesz kernel is symmetric and (3.63), we get

$$\begin{aligned}
\int_{B(0,R)} \left(I_{k,\beta}[\mu](x)\right)^p (x)\omega_k(x)dx &\leq \int_{B(0,R)} \Big(\int_{\mathbb{R}^d} \left(R_{k,\beta}(x,y)\right)^p d\mu(y)\Big)\omega_k(x)dx \\
&= \int_{\mathbb{R}^d} \Big(\int_{B(0,R)} \left(R_{k,\beta}(x,y)\right)^p \omega_k(x)dx\Big)d\mu(y) \\
&\leq C < +\infty,
\end{aligned}$$

where $C$ is the constant in (3.63).

**2) ii) $\Rightarrow$ i)** Assume that the condition (3.72) holds. We will prove that $x \mapsto I_{k,\beta}[\mu](x)$ is in $L^1_{k,loc}(\mathbb{R}^d)$. Let $R > 1$. By Fubini's theorem, we have

$$\begin{aligned}
A_R &:= \int_{B(0,R)} I_{k,\beta}[\mu](x)\omega_k(x)dx = \int_{\mathbb{R}^d}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y) \\
&= \int_{\|y\|\leq 2R}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y) + \int_{\|y\|>2R}\int_{B(0,R)} R_{k,\beta}(x,y)\omega_k(x)dxd\mu(y) \\
&:= A_{1,R} \quad + \quad A_{2,R}.
\end{aligned}$$

Applying the assertion 1) with the finite measure $\mu_{|B(0,R)}$, we get $A_{1,R} < +\infty$.
Now, from (3.60) we deduce that

$$A_{2,R} \leq \kappa \int_{\|y\|>2R} \int_{B(0,R)} \max_{g\in W} \left( \|x - gy\|^{\beta-d-2\gamma} \right) \omega_k(x) dx d\mu(y).$$

But, for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d \backslash B(0,2R)$, we have $\|x-gy\| \geq \|y\|-\|x\| \geq \frac{\|y\|}{2}$.
Moreover, as $R > 1$, we see that $\|y\| \geq \frac{1}{2}(1 + \|y\|)$ whenever $\|y\| \geq 2R$. In other words, the inequality

$$\max_{g\in W} \left( \|x - gy\|^{\beta-d-2\gamma} \right) \leq 4^{\beta-d-2\gamma}(1 + \|y\|)^{\beta-d-2\gamma}$$

holds for every $x \in B(0,R)$ and every $y \in \mathbb{R}^d \setminus B(0,2R)$. Hence, by our hypothesis we conclude that

$$A_{2,R} \leq 4^{\beta-d-2\gamma}\kappa \ m_k[B(0,R)] \int_{\|y\|\geq 2R} (1 + \|y\|)^{\beta-d-2\gamma} d\mu(y) < +\infty$$

and thus the function $x \mapsto I_{k,\beta}[\mu](x)\omega_k(x)$ is locally integrable on $\mathbb{R}^d$. In particular, $I_{k,\beta}[\mu](x) < +\infty$ a.e. on $\mathbb{R}^d$.

**i) $\Rightarrow$ iii)** It is obvious.

**iii) $\Rightarrow$ ii)** Let $x_0 \in \mathbb{R}^d$ such that $I_{k,\beta}[\mu](x_0) < +\infty$. From (3.60), we can see that

$$\begin{aligned} I_{k,\beta}[\mu](x_0) &\geq \kappa \int_{\mathbb{R}^d} \min_{g\in W} \left( \|x_0 - gy\|^{\beta-d-2\gamma} \right) d\mu(y) \\ &\geq \kappa \int_{\mathbb{R}^d} (\|x_0\| + \|y\|)^{\beta-d-2\gamma} d\mu(y). \end{aligned}$$

If $\|x_0\| \leq 1$, we deduce immediately from the previous inequality that (3.72) holds.
If $\|x_0\| > 1$, using the fact that

$$\|x_0\| + \|y\| \leq \|x_0\|(1 + \|y\|)$$

and using again the above inequality, we obtain that (3.72) holds.
This finishes the proof. $\square$

**Remark 3.15** *Let $\beta \in ]0, d+2\gamma[$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mu = \mu^+ - \mu^-$ its Hahn-Jordan decomposition. If $\mu^+$ and $\mu^-$ satisfy (3.72), then the Dunkl-Riesz potential of $\mu$ is well defined almost everywhere by setting $I_{k,\beta}[\mu](x) = I_{k,\beta}[\mu^+](x) - I_{k,\beta}[\mu^-](x)$. Moreover, the function $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$.*
*Let us introduce the following notations*

$$\mathcal{M}^+_{k,\beta}(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}^+(\mathbb{R}^d), \quad \mu \textit{ satisfies } (3.72) \right\} \tag{3.73}$$

*and*

$$\mathcal{M}_{k,\beta}(\mathbb{R}^d) := \left\{ \mu = \mu^+ - \mu^- \in \mathcal{M}(\mathbb{R}^d), \quad \mu^+, \mu^- \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d) \right\}. \tag{3.74}$$

*We note that if $0 < \beta_1 \leq \beta_2 < d + 2\gamma$, then $\mathcal{M}^+_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}^+_{k,\beta_1}(\mathbb{R}^d)$ and $\mathcal{M}_{k,\beta_2}(\mathbb{R}^d) \subset \mathcal{M}_{k,\beta_1}(\mathbb{R}^d)$*

Now, we establish a boundedness principle for the potential of a compactly supported measure which generalizes the known result in the classical case (i.e. $k = 0$) (see [18], Theorem 1.5).

**Proposition 3.16** *Let $0 < \beta < d+2\gamma$ and $\mu$ be a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. If $I_{k,\beta}[\mu] \leq M$ holds on W.supp $\mu$, then*

$$I_{k,\beta}[\mu] \leq 2^{d+2\gamma-\beta} M \quad on \quad \mathbb{R}^d. \tag{3.75}$$

*Proof:* Let $x \notin W.$supp $\mu$ and $x_0 \in W.$supp $\mu$ such that $\|x - x_0\| = dist(x, W.\text{supp } \mu)$. We have

$$\forall\ y \in \text{supp } \mu, \quad \forall\ g \in W, \quad \|x_0 - gy\| \leq \|x_0 - x\| + \|x - gy\| \leq 2\|x - gy\|.$$

Hence, by (3.62) we deduce that

$$\forall\ y \in \text{supp } \mu, \quad \forall\ z \in \text{supp } \mu_y, \quad \|x_0\|^2 + \|y\|^2 - 2\langle x_0, z\rangle \leq 4(\|x\|^2 + \|y\|^2 - 2\langle x, z\rangle).$$

Now, using (3.28), we obtain

$$\forall\ y \in \text{supp } \mu, \quad 4^{-\frac{d}{2}-\gamma} p_{t/4}(x, y) \leq p_t(x_0, y).$$

From (3.41), the above inequality implies that

$$\forall\ y \in \text{supp } \mu, \quad 2^{-d-2\gamma+\beta} R_{k,\beta}(x, y) \leq R_{k,\beta}(x_0, y).$$

Finally, if we integrate with respect to the measure $d\mu(y)$ and use our hypothesis, the inequality (3.75) follows. □

**Theorem 3.17** *Let $\beta \in ]0, d + 2\gamma[$ and $\mu$ be a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. Then, the function $I_{k,\beta}[\mu]$ is*

**i)** *D-superharmonic on $\mathbb{R}^d$ if $\beta \geq 2$,*

**ii)** *D-harmonic on $\mathbb{R}^d \setminus W.$supp $\mu$ if $\beta = 2$,*

**iii)** *D-subharmonic on $\mathbb{R}^d \setminus W.$supp $\mu$ if $\beta \leq 2$.*

We need the following lemma:

**Lemma 3.18** *Let $\beta \in ]0, d+2\gamma[$ and $\mu$ be a compactly supported nonnegative Radon measure on $\mathbb{R}^d$. Then, the function $I_{k,\beta}[\mu]$ is*

1. *lower semi-continuous on $\mathbb{R}^d$,*
2. *continuous on $\mathbb{R}^d \setminus W.$supp $\mu$.*

*Proof of Lemma 3.18:* **1)** Consider the function $F_n$ given by

$$F_n(x) = \frac{1}{\Gamma(\beta/2)} \int_{\text{supp } \mu} \Big( \int_{1/n}^{n} t^{\frac{\beta}{2}-1} p_t(x, y) dt \Big) d\mu(y).$$

As $t^{\frac{\beta}{2}-1}p_t(x,y) \leq 2^{-\frac{d}{2}-\gamma}c_k^{-1}t^{-\frac{\beta-d-2\gamma}{2}-1}$, by the continuity theorem under the integral sign, we see that $F_n$ is continuous on $\mathbb{R}^d$. Moreover, from the monotone convergence theorem, we deduce that the function $I_{k,\beta}[\mu]$ is l.s.c. on $\mathbb{R}^d$ as being the pointwise increasing limit of the sequence $(F_n)$.

**2)** Fix a closed ball $B(x_0,R)$ in $\mathbb{R}^d \setminus W.\text{supp } \mu$ and set

$$\eta := dist\,(B(x_0,R), W.\text{supp } \mu) > 0.$$

From (3.32), we deduce that

$$\forall (x,y) \in B(x_0,R) \times \text{supp } \mu, \quad p_t(x,y) \leq \frac{1}{(2t)^{\frac{d}{2}+\gamma}c_k}e^{-\frac{\eta^2}{4t}}.$$

Then, writing

$$I_{k,\beta}[\mu](x) = \frac{1}{\Gamma(\beta/2)}\int_{\text{supp } \mu}\Big(\int_0^{+\infty} t^{\frac{\beta}{2}-1}p_t(x,y)dt\Big)d\mu(y)$$

and using the continuity theorem under the integral sign, it follows that $I_{k,\beta}[\mu]$ is continuous on $B(x_0,R)$. As the ball $B(x_0,R)$ is arbitrary, the result follows.

□

*Proof of Theorem 3.17:* **i)** Let $\beta > 2$. Using Fubini's theorem and the D-superharmonicty of the Dunkl-Riesz kernel (see Proposition 3.9), we can easily see that $I_{k,\beta}[\mu]$ satisfies the super-mean property i.e. for all $x \in \mathbb{R}^d$ and all $r > 0$, $M_B^r\left(I_{k,\beta}[\mu]\right)(x) \leq I_{k,\beta}[\mu](x)$.
Since $I_{k,\beta}[\mu]$ is l.s.c and finite a.e., we deduce that the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$.

**ii)** If $\beta = 2$, we are in the case of the Dunkl-Newton potential and the result has been proved in [10].

**iii)** Let $\beta < 2$. From Lemma 3.18, we know that $I_{k,\beta}[\mu]$ is a continuous function on $\mathbb{R}^d \setminus W.\text{supp } \mu$. Furthermore, by Proposition 3.9 and Fubini's theorem, the sub-mean property is satisfied by the function $I_{k,\beta}[\mu]$ on $\mathbb{R}^d \setminus W.\text{supp } \mu$. Thus, $I_{k,\beta}[\mu]$ is D-subharmonic on $\mathbb{R}^d \setminus W.\text{supp } \mu$. □

**Corollary 3.19** *Let $\beta \in [2, d+2\gamma[$ . If $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$, then the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$.*

*Proof:* Let $\Phi_n$ the function defined by $\Phi_n(x) = \int_{B(0,n)} R_{k,\beta}(x,y)d\mu(y)$. From Theorem 3.17, the function $\Phi_n$ is D-superharmonic on $\mathbb{R}^d$. Thus, as $I_{k,\beta}[\mu]$ is not identically $+\infty$ by hypothesis, the function $I_{k,\beta}[\mu]$ is D-superharmonic on $\mathbb{R}^d$ as being an increasing pointwise limit of the sequence $(\Phi_n)_n$ of D-superharmonic functions (see [10], Proposition 4.3). □

**Proposition 3.20** *Let $\mu \in \mathcal{M}^+_{k,\beta}(\mathbb{R}^d)$ with $\beta \in [2, d+2\gamma[$ and $m \in \mathbb{N}$ be such that $1 \leq m \leq \beta/2$. Then, the function $x \mapsto I_{k,\beta}[\mu](x)\omega_k(x)$ satisfies*

$$(-\Delta_k)^m\left(I_{k,\beta}[\mu]\omega_k\right) = \left\{\begin{array}{lll} I_{k,\beta-2m}[\mu]\omega_k & in \quad \mathcal{D}'(\mathbb{R}^d) & if \quad \beta > 2m, \\ & & \\ \mu & in \quad \mathcal{D}'(\mathbb{R}^d) & if \quad \beta = 2m, \end{array}\right. . \tag{3.76}$$

*Proof:* Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. We will only prove the result in the case $\beta > 2m$ and by the same arguments it can be obtained when $\beta = 2m$. We have

$$\begin{aligned}\langle(-\Delta_k)^m \left(I_{k,\beta}[\mu]\omega_k\right), \phi\rangle &= \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} R_{k,\beta}(x,y)(-\Delta_k)^m \phi(x)\omega_k(x)dx\Big) d\mu(y) \\ &= \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} R_{k,\beta-2m}(x,y)\phi(x)\omega_k(x)dx\Big) d\mu(y) \\ &= \int_{\mathbb{R}^d} I_{k,\beta-2m}[\mu](x)\phi(x)\omega_k(x)dx,\end{aligned}$$

where we have used
-Fubini's theorem in the first and the last lines (it is possible because $I_{k,\beta}[\mu] \in L^1_{k,loc}(\mathbb{R}^d)$ and by Remark 3.15, $I_{k,\beta-2m}[\mu]$ is also in $L^1_{k,loc}(\mathbb{R}^d)$);
- the fact that the Dunkl-Riesz kernel is symmetric and the relation (3.69) in the second line. $\square$

From the previous proposition, we obtain immediately the uniqueness principle for Dunkl-Riesz potential of index $2m$:

**Corollary 3.21** *Let $m \in ]0, \frac{d}{2} + \gamma[$ be an integer and $\mu, \nu \in \mathcal{M}^+_{k,2m}(\mathbb{R}^d)$. If $I_{k,2m}[\mu] = I_{k,2m}[\nu]$ a.e, then $\mu = \nu$.*

For an arbitrary index $\beta \in ]0, d + 2\gamma[$, we have the following version of the uniqueness principle for finite measures:

**Theorem 3.22** *Let $\beta \in ]0, d + 2\gamma[$ and let $\mu, \nu$ be two finite and nonnegative Radon measures on $\mathbb{R}^d$. If $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ a.e. on $\mathbb{R}^d$, then $\mu = \nu$.*

We start by the following result

**Lemma 3.23** *Let $\mu$ be a finite and nonnegative Radon measure on $\mathbb{R}^d$. Then, $I_{k,\beta}[\mu]\omega_k$ is a tempered distribution and its distributional Dunkl transform is given by*

$$\mathcal{F}_k\left(I_{k,\beta}[\mu]\omega_k\right) = \|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k \quad in \quad \mathcal{S}'(\mathbb{R}^d). \tag{3.77}$$

*Here, $\mathcal{F}_k(\mu)$ is the function defined by (3.18).*

*Proof:* Let $m > d+2\gamma$ an integer and $C_m$ as in (3.68). By Fubini's theorem, the symmetric property of the Dunkl-Riesz kernel and the relation (3.68), we get

$$\begin{aligned}\int_{\mathbb{R}^d}(1+\|x\|^2)^{-m} I_{k,\beta}[\mu](x)\omega_k(x)dx &= \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d}(1+\|x\|^2)^{-m} R_{k,\beta}(x,y)\omega_k(x)dx\Big) d\mu(y) \\ &\leq C_m\mu(\mathbb{R}^d) < +\infty.\end{aligned}$$

This shows that $I_{k,\beta}[\mu]\omega_k \in \mathcal{S}'(\mathbb{R}^d)$.
Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned}
\langle \mathcal{F}_k\left(I_{k,\beta}[\mu]\omega_k\right), \phi\rangle &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} R_{k,\beta}(x,y)d\mu(y)\Big)\mathcal{F}_k(\phi)(x)\omega_k(x)dx \\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} R_{k,\beta}(x,y)\mathcal{F}_k(\phi)(x)\omega_k(x)dx\Big)\mu(y) \\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} E_k(-iy,x)\|x\|^{-\beta}\omega_k(x)\phi(x)dx\Big)d\mu(y) \\
&= \int_{\mathbb{R}^d} \|x\|^{-\beta}\mathcal{F}_k(\mu)(x)\omega_k(x)\phi(x)dx,
\end{aligned}$$

where we have used
-Fubini's theorem in the first second line: it is possible because $\mathcal{F}_k(\phi) \in \mathcal{S}(\mathbb{R}^d)$ and then the function $x \mapsto (1+\|x\|^2)^m\mathcal{F}_k(\phi)(x)$ is bounded with $m$ the integer chosen as above;
-the relations (3.67) and $R_{k,\beta}(x,y) = R_{k,\beta}(y,x)$ in the third line;
-the boundedness of the function $(x,y) \mapsto E_k(iy,x)$ (see (3.13)), Fubini's theorem and (3.18) in the last line. □

*Proof of Theorem* 3.22: By our hypothesis and Lemma 3.23, we have $I_{k,\beta}[\mu] = I_{k,\beta}[\nu]$ in $\mathcal{S}'(\mathbb{R}^d)$. Applying Dunkl transform to the both terms and using the relation (3.77), we deduce that

$$\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k = \|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

As the functions $\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k$ and $\|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k$ are locally integrable on $\mathbb{R}^d$, we get

$$\|.\|^{-\beta}\mathcal{F}_k(\mu)\omega_k = \|.\|^{-\beta}\mathcal{F}_k(\nu)\omega_k \quad \text{a.e. on} \quad \mathbb{R}^d.$$

Now, by continuity it follows that the functions $\mathcal{F}_k(\mu)$ and $\mathcal{F}_k(\nu)$ coincide everywhere on $\mathbb{R}^d$. Finally, by the injectivity of the Dunkl transform on the space of finite Radon measures on $\mathbb{R}^d$, we conclude that $\mu = \nu$. □

In order to extend the pointwise Hedbreg inequality in Dunkl setting, in the following result we give the link between the Dunkl-Riesz potential and the volume mean of a nonnegative Radon measure.

**Proposition 3.24** *Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$. Then, for all $\beta \in ]0, d+2\gamma[$, we have*

$$I_{k,\beta}[\mu](x) = \frac{d_k\kappa}{(d+2\gamma)(d+2\gamma-\beta)}\int_0^{+\infty} t^\beta M_B^t(\mu)(x)\frac{dt}{t}, \tag{3.78}$$

*where*

$$M_B^t(\mu)(x) := \frac{1}{m_k[B(0,t)]}\int_{\mathbb{R}^d} h_k(t,x,y)d\mu(y). \tag{3.79}$$

*Proof:* The result follows from (3.43), Fubini's theorem, (3.38) and (3.79). □

In the following result, we will extend the pointwise Hedberg inequality (see [13]). We recall that the Dunkl-Hardy-Littlewood maximal operator is defined for $f \in L^1_{k,loc}(\mathbb{R}^d)$ by (see [27])

$$M_k(f)(x) = \sup_{r>0} \frac{1}{m_k[B(0,r)]} \int_{\mathbb{R}^d} |f(y)| \tau_{-x}(\mathbf{1}_{B(0,r)})(y)\omega_k(y)dy, \tag{3.80}$$

where $\tau_{-x}(\mathbf{1}_{B(0,r)})$ denotes the $L^2_k(\mathbb{R}^d)$-function with Dunkl transform

$$\xi \mapsto E_k(-ix,\xi)\mathcal{F}_k\left(\mathbf{1}_{B(0,r)}\right)(\xi).$$

In [10], we have shown that $h_k(r,x,.) = \tau_{-x}(\mathbf{1}_{B(0,r)})$ a.e. on $\mathbb{R}^d$. Thus, we will take this remark into account in the formula (3.80) and in the sequel of the paper.

Moreover, when $d\mu(y) = |f(y)|\omega_k(y)dy$, $f \in L^1_{k,loc}(\mathbb{R}^d)$, we will use the notation $I_{k,\beta}[|f|]$ instead of $I_{k,\beta}\left[|f(y)|\omega_k(y)dy\right]$.

**Theorem 3.25** *For $0 < \beta < d+2\gamma$, $1 \le p < \frac{d+2\gamma}{\beta}$, there exists constants $C = C(d,\gamma,\beta,p) > 0$ such that for any measurable function $f$ and any $x \in \mathbb{R}^d$, we have*

$$I_{k,\beta}[|f|](x) \le C\|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} \left(M_k(f)(x)\right)^{1-\frac{\beta p}{d+2\gamma}}, \tag{3.81}$$

*Proof:* For every $A > 0$, by (3.78) where we take $d\mu(y) = |f(y)|\omega_k(y)dy$, we can write

$$\begin{aligned} I_{k,\beta}[|f|](x) = I_{k,\beta}[|f|\omega_k](x) &= C\int_0^A t^{\beta-1} M_B^t(|f|)(x)dt + C\int_A^{+\infty} t^{\beta-1} M_B^t(|f|)(x)dt \\ &:= I_1(x) + I_2(x). \end{aligned}$$

• Clearly, we see that

$$I_1(x) \le CA^\beta M_k(f)(x). \tag{3.82}$$

• We have

$$\begin{aligned} I_2(x) &= C\sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1}A} t^{\beta-d-2\gamma-1} \int_{\mathbb{R}^d} |f(y)| h_k(t,x,y)\omega_k(y)dydt \\ &\le C\|f\|_{k,p} \sum_{n=0}^{+\infty} \int_{2^n A}^{2^{n+1}A} t^{\beta-d-2\gamma-1} t^{d+2\gamma(1-1/p)} dt \\ &\le C\|f\|_{k,p} \sum_{n=0}^{+\infty} (2^n A)^{\beta-\frac{d+2\gamma}{p}}, \end{aligned}$$

where we have used Hölder's inequality and the relation (3.38) in the second line. Therefore, we have

$$I_2(x) \le CA^{\beta-\frac{d+2\gamma}{p}} \|f\|_{k,p}. \tag{3.83}$$

Now, using (3.82), (3.83) and choosing

$$A = A(x) = \left(\frac{\|f\|_{k,p}}{M_k(f)(x)+\varepsilon}\right)^{\frac{p}{d+2\gamma}},$$

we obtain

$$I_{k,\beta}[|f|](x) \leq C\|f\|_{k,p}^{\frac{\beta p}{d+2\gamma}} \left(M_k(f)(x)+\varepsilon\right)^{1-\frac{\beta p}{d+2\gamma}}.$$

Letting $\varepsilon \longrightarrow 0$, we get (3.81). □

Using the Hedberg inequality (3.81), the $L^p_k$-boundedness properties of the Dunkl-Hardy-Littlewood maximal function (see [4] or [27]) and following the same proof as in the classical case (see Theorem 3.1.4 in [1]), we obtain the Sobolev's inequality

**Corollary 3.26** *Let $0<\beta<d+2\gamma$, $1 \leq p < \frac{d+2\gamma}{\beta}$ and $p^* = \frac{p(d+2\gamma)}{d+2\gamma-\beta p}$.*

***1)*** *If $p=1$, then $I_{k,\beta}$ is of weak type $(1,p^*)$ i.e. there exists a constant $C=C(\beta,d,\gamma)$ such that*

$$\forall\ \lambda>0, \quad \forall\ f \in L^1_k(\mathbb{R}^d), \qquad \int_{\{x:\ I_{k,\beta}[|f|]>\lambda\}} \omega_k(x)dx \leq C\left(\frac{\|f\|_{k,1}}{\lambda}\right)^{p^*}. \tag{3.84}$$

***1)*** *If $p>1$, then $I_{k,\beta}$ is of strong type $(p,p^*)$ i.e. $I_{k,\beta} : L^p_k(\mathbb{R}^d) \longrightarrow L^{p^*}_k(\mathbb{R}^d)$ is bounded.*

**Remark 3.27** *The previous result has been obtain in [11] by another proof using interpolation methods and in the particular case when the Coxeter-Weyl group is $\mathbf{Z}_2^d$ in [28].*

# Bibliography


[1] D. Adams and L. Hedberg. *Functions Spaces and Potential Theory.* Springer, (1999).

[2] D. H. Armitage and S. J. Gardiner. *Classical Potential Theory.* Springer-Verlag, London (2001).

[3] G. Choquet. *Topology.* Volume XIX, Academic Press (1966).

[4] L. Deleaval. *Two results on the Dunkl maximal function.* Studia Mathematica, 203, (2011), 47-68.

[5] C. F. Dunkl. *Integral kernels with reflection group invariance.* Canad. J. Math., 43, (1991), 123-183.

[6] C. F. Dunkl. *Hankel transform associated to finite reflection groups.* Contemp. Math., 138, (1992), 123-138.

[7] C. F. Dunkl and Y. Xu. *Orthogonal Polynomials of Several variables.* Cambridge Univ. Press 2001.

[8] P. Etingof. *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups.* Math. Res. Lett. 17 (2010), no. 2, 275-282.

[9] L. Gallardo and C. Rejeb. *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications.* Trans. Amer. Math. Soc., electronically published on May 22, 2015, DOI:http://dx.doi.org/10.1090/tran/6671 (to appear in print).

[10] L. Gallardo and C. Rejeb. *Newtonian Potentials and subharmonic functions associated to the Dunkl-Laplace operator*. Submitted.

[11] S. Hassani, S. Mustapha and M. Sifi. *Riesz Potential and fractional maximal function for the Dunkl transform.* J. of Lie Theory, Vol. 19, No.4, (2009), 725-734.

[12] W. K. Hayman and P. B. Kennedy. *Subharmonic functions.* Volume 1. Academic Press London, (1976).

[13] L. Hedberg. *On certain convolution inequalities.* Proc. Amer. Math. Soc. 36 (1972), 505-510.

[14] L. L. Helms. *Potential theory.* Springer-Verlag London, (2009).

[15] J. E. Humphreys. *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics 29, Cambridge University Press (1990).

[16] M. F. de Jeu. *The Dunkl transform.* Invent. Math., 113, (1993), 147-162.

[17] R. Kane. *Reflection Groups and Invaraint Theory.* CMS Books in Mathematics. Springer-Verlag, New York, (2001).

[18] N. S. Landkof. *Foundations of Modern Potential Theory.* Springer, Berlin, (1972).

[19] E. M. Opdam. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group.* Compositio Math. 85 (1993), no. 3, 333-373.

[20] M. Rösler. *Generalized Hermite polynomials and the heat equation for Dunkl operators.* Comm. Math. Phys, 192, (1998), 519-542.

[21] M. Rösler and M. Voit. *Markov processes related with Dunkl operators.* Adv. Appl. Math, 21, (1998), 575-643.

[22] M. Rösler. *Positivity of Dunkl's intertwining operator.* Duke Math. J., 98, (1999), 445-463.

[23] M. Rösler and M. F. de Jeu. *Asymptotic analysis for the Dunkl kernel.* J. Approx. Theory, 119, (2002), no. 1, 110-126.

[24] M. Rösler. *Dunkl Operators: Theory and Applications.* Lecture Notes in Math., vol.1817, Springer Verlag (2003), 93-136.

[25] M. Rösler. *A positive radial product formula for the Dunkl kernel.* Trans. Amer. Math. Soc., 355, (2003), 2413-2438.

[26] L. Schwartz. *Théorie des distributions.* Hermann, Editeurs des Sciences et des arts, (1966).

[27] S. Thangavelu and Y. Xu. *Convolution operator and maximal function for Dunkl transform.* J. Anal. Math. Vol. 97, (2005), 25-56.

[28] S. Thangavelu and Y. Xu. *Riesz transform and Riesz potentials for Dunkl transform.* J. Comput. Appl. Math. Vol. 199, (2007), 181-195.

[29] K. Trimèche. *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual.* Integ. Transf. and Spec. Funct., 12(4), (2001), 394-374.

[30] K. Trimèche. *Paley-Wiener theorem for the Dunkl transform and Dunkl translation operators.* Integ. Transf. and Spec. Func. 13, (2002), 17-38.