On the relationship between contact topology and the dynamics of Reeb flows

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Doctoral Thesis

Sur les relations entre la topologie de contact et la dynamique de champs de Reeb

On the relations between contact topology and dynamics of Reeb flows

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Abstract

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by Marcelo Ribeiro de Resende Alves

In this thesis we study the relations between the contact topological properties of contact manifolds and the dynamics of Reeb flows.

On the first part of the thesis, we establish a relation between the growth of the cylindrical contact homology of a contact manifold and the topological entropy of Reeb flows on this manifold. As a first step towards establishing this relation we prove in Chapter 5 that if a flow has exponential homotopical growth of periodic orbits then the topological entropy of this flow is positive. We build on this to show in Chapter 6 that if a contact manifold (M, ξ) admits a hypertight contact form λ_0 for which the cylindrical contact homology has exponential homotopical growth rate, then the Reeb flow of every contact form on (M, ξ) has positive topological entropy. Using this result, we exhibit in Chapter 8 and 9 numerous new examples of contact 3-manifolds on which every Reeb flow has positive topological entropy.

On Chapter 10 we present a joint result with Chris Wendl that gives a dynamical obstruction for contact 3-manifold to be planar. We then use the obstruction to show that a contact 3-manifold that possesses a Reeb flow that is a transversely orientable Anosov flow, cannot be planar.

On Chapter 11 we study the topological entropy for Reeb flows on spherizations. The result we obtain is a refinement of a result of Macarini and Schlenk [36], that states that every Reeb flow on the unit tangent bundle (T_1S, ξ_{geo}) of a high genus surface S has positive topological entropy. We show that for any Reeb flow on (T_1S, ξ_{geo}), the ω-limit of almost every Legendrian fiber is a compact invariant set on which the dynamics has positive topological entropy.

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L'objectif de cette thèse est d'investiguer les relations entre les propriétés topologiques d'une variété de contact et la dynamique des flots de Reeb dans la variété de contact en question.

Dans la première partie de la thèse, nous établissons une relation entre la croissance de la homologie de contact cylindrique d'une variété de contact et l'entropie topologique des flots de Reeb dans cette variété de contact. Nous demontrons, dans le chapitre 6, que si la variété de contact (M, ξ) possède une forme de contact hipertendu λ_0 pour lequel la homologie de contact a une croissance homotopique exponentielle, alors tous les flots de Reeb dans (M, ξ) ont entropie tologique positive. Nous utilisons ce resultat dans les chapitres 8 et 9 pour montrer l'existence d'un grande nombre des nouvelles variétés de contact de dimension 3 dans lequelles tous les flots de Reeb ont entropie topologique positive.

Dans le chapitre 10, nous prouvons un résultat obtenu en collaboration avec Chris Wendl qui donne une obtruction dynamique pour q'une variété de contact de dimension 3 soit planaire. Cette obstruction est utilisée pour montrer que, si une variété de contact de dimension 3 possède un flot de Reeb qui est uniformement hyperbolique (Anosov) avec variétés invariantes traversalement orientables, alors cette variété de contact n'est pas planaire.

Dans le chapitre 11, nous étudions l'entropie topologique des flots de Reeb dans les fibrés unitaires des surfaces de genre plus grand que 1. Nous obtenons le raffinement d'un résultat dû à Macarini et Schlenk, qui ont montré que les flots de Reeb dans les fibrés unitaires (T_1, ξ_{geo}) d'une surface de genre plus grand que 1 ont entropie topologique positive. Nous montrons que la restriction de chaque flot de Reeb en (T_1, ξ_{geo}) au ensemble ω-limite de presque toute fibre unitaire a une entropie topologique positive.

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Contents

To Hilda, André, Aninha e Lucio.

Chapter 1

Introduction and main results

1.1 Topological entropy of Reeb flows

In this thesis we study relations between contact topology and dynamics of Reeb flows. One of the main results of this thesis is the proof of a relation between the behaviour of cylindrical contact homology and the topological entropy of Reeb flows. The topological entropy is a non-negative number associated to a dynamical system which measures the complexity of the orbit structure of the system. Positivity of the topological entropy means that the system possesses some type of exponential instability. We show that if the cylindrical contact homology of a contact 3-manifold is "complicated enough" from a homotopical viewpoint, then every Reeb flow on this contact manifold has positive topological entropy.

1.1.1 Basic definitions and history of the problem

We study the topological entropy of Reeb flows from the point of view of contact topology. More precisely, we search for conditions on the topology of a contact manifold (M, ξ) that force **all** Reeb flows on (M, ξ) to have positive topological entropy. The condition we impose is on the behaviour of a contact topological invariant called cylindrical contact homology. We show that if a contact manifold (M, ξ) admits a contact form λ_0 for which the cylindrical contact homology has *exponential homotopical growth*, then all Reeb flows on (M, ξ) have positive topological entropy.

The notion of exponential homotopical growth of cylindrical contact homology, which is introduced in this paper, differs from the notion of growth of contact homology studied in [13, 43]. For reasons explained in Chapters 5 and 6, the growth of contact homology is not well adapted to study the topological entropy of Reeb flows, while the notion of homotopical growth rate is (as we show) well suited for this purpose. We begin by explaining the results which were previously known relating the behaviour of contact topological invariants to the topological entropy of Reeb flow.

The study of contact manifolds all of whose Reeb flows have positive topological entropy was initiated by Macarini and Schlenk [36]. They showed that if Q is an energy hyperbolic manifold and ξ_{geo} is the contact structure on the unit tangent bundle T_1Q associated to the geodesic flows, then every Reeb flow on (T_1Q, ξ_{geo}) has positive topological entropy. Their work was based on previous ideas of Frauenfelder and Schlenk [22, 23] which related the growth rate of Lagrangian Floer homology to entropy invariants of symplectomorphisms. The strategy to estimate the topological used in [36] can be briefly sketched as follows:

Exponential growth of Lagrangian Floer homology of the tangent fiber $(TQ)_{|_p}$

$$\Rightarrow$$

Exponential volume growth of the unit tangent fiber $(T_1Q)_{|_p}$ for all Reeb flows in (T_1Q, ξ_{geo})

$$\Rightarrow$$

Positivity of the topological entropy for all Reeb flows in (T_1Q, ξ_{geo}).

To obtain the first implication Macarini and Schlenk use the fact that (T_1Q, ξ_{geo}) has the structure of a Legendrian fibration, and apply the geometric idea of [22, 23] to show that the number of trajectories connecting a Legendrian fiber to another Legendrian fiber can be used to obtain a volume growth estimate. The second implication in this scheme follows from Yomdin's theorem, that states that exponential volume growth of a submanifold implies positivity of topological entropy. [1]

In [3, 4] this approach was extended by the author to deal with 3-dimensional contact manifolds which are not unit tangent bundles. This was done by designing a localised version of the geometric idea of [22, 23]. Globally most contact 3-manifolds are not Legendrian fibrations, but a small neighbourhood of a given Legendrian knot in any contact 3-manifold can be given the structure of a Legendrian fibration. It turns out that this is enough to conclude that if the linearised Legendrian contact homology of a pair of Legendrian knots in a contact 3-manifold (M^3, ξ) grows exponentially, then the length of these Legendrian knots grows exponentially for any Reeb flow on (M^3, ξ). We then apply Yomdin's theorem to obtain that all Reeb flows on (M^3, ξ) have positive topological entropy.

[1] The same scheme was used in [21, 24] to obtain positive lower bounds for the intermediate and slow entropies of Reeb flows on unit tangent bundles; we discuss these results in more detail in section 7.

One drawback of these approaches is that they only give lower entropy bounds for C^∞-smooth Reeb flows. The reason is that Yomdin's theorem holds only for C^∞-smooth flows. The approach presented in the present paper **does not** use Yomdin's theorem and gives lower bounds for the topological entropy of C^1-smooth Reeb flows.

Another advantage is that the cylindrical contact homology is usually easier to compute than the linearised Legendrian contact homology. In fact, to apply the strategy of [3, 4] to a contact 3-manifold (M^3, ξ) one must first find a pair of Legendrian curves which, one believes, "should" have exponential growth of linearised Legendrian contact homology. This is highly non-trivial since on any contact 3-manifolds there exist many Legendrian links for which the linearised Legendrian contact homology does not even exist. On the other hand the definition of cylindrical contact homology only involves the contact manifold (M^3, ξ), and no Legendrian submanifolds.

1.1.2 Cylindrical contact homology and topological entropy

Our results are inspired by the philosophy that a "complicated" topological structure can force chaotic behavior for dynamical systems associated to this structure. Two important examples of this phenomena are: the fact that on manifolds with complicated loop space the geodesic flow always has positive topological entropy (see [38]), and the fact that every diffeomorphism of a surface which is isotopic to a pseudo-Anosov diffeomorphism has positive topological entropy [18].

To state our results we introduce some notation. Let M be a manifold and X be a C^k $(k \geq 1\ )$ vector field. Our first result relates the topological entropy of ϕ_X to the growth (relative to T) of the number of distinct homotopy classes which contain periodic orbits of ϕ_X with period $\leq T$. More precisely let Λ_X^T be the set of free homotopy classes of M which contain a periodic orbit of ϕ_X with period $\leq T$. We denote by $N_X(T)$ the cardinality of Λ_X^T.

Theorem 5.1. If for real numbers $a > 0$ and b we have $N_X(T) \geq e^{aT+b}$, then $h_{top}(X) \geq a$.

Theorem 5.1 might be a folklore result in the theory of dynamical systems. However as we have not found it in the literature, we provide a complete proof in Chapter 5. It contains as a special case Ivanov's inequality for surface diffeomorphisms (see [31]). Our motivation for proving this result is to apply it to Reeb flows. Contact homology allows one to carry over information about the dynamical behaviour of one special Reeb flow on a contact manifold to all other Reeb flows on the same contact manifold. In hapter 6 we introduce the notion of exponential homotopical growth of cylindrical contact homology. As we already mentioned, this growth rate differs from the ones previously considered in the literature and is specially designed to allow one to use Theorem 5.1 to obtain results about the topological entropy of Reeb flows. This is made via the following:

Theorem 6.1 Let λ_0 be a hypertight contact form on a contact manifold (M, ξ) and assume that the cylindrical contact homology $C\mathbb{H}^{J_0}_{cyl}(\lambda_0)$ has exponential homotopical growth with exponential weight $a > 0$. Then for every C^k $(k \geq 2)$ contact form λ on (M, ξ) the Reeb flow of X_λ has positive topological entropy. More precisely, if f_λ is the unique function such that $\lambda = f_\lambda \lambda_0$, then

$$h_{top}(X_\lambda) \geq \frac{a}{\max f_\lambda}. \tag{1.1}$$

Notice that Theorem 6.1 allows us to conclude the positivity of the topological entropy for **all** Reeb flows on a given contact manifold (M, ξ), once we show that (M, ξ) admits one special hypertight contact form for which the cylindrical contact homology has exponential homotopical growth. It is worth remarking that our proof of Theorem 6.1 is carried out in full rigor, and does **not** make use of the Polyfold technology which is being developed by Hofer, Wysocki and Zehnder. The reason is that we do not use the linearised contact homology considered in [9, 43], but resort to a topological idea used in [30] to prove existence of Reeb orbits in prescribed homotopy classes.

Theorem 6.1 above allows one to obtain estimates for the topological entropy for C^1-smooth Reeb flows. As previously observed, the strategy used in [3, 4, 36] produces estimates for the topological entropy only for C^∞-smooth contact forms as they depend on Yomdim's theorem, which fails for finite regularity.

Our other results are concerned with the existence of examples of contact manifolds which have a contact form with exponential homotopical growth rate of cylindrical contact homology. We show that in dimension 3 they exist in abundance, and it follows from Theorem 6.1 that every Reeb flow on these contact manifolds has positive topological entropy. In Chapter 8 we construct such examples for manifolds with a non-trivial JSJ decomposition and with a hyperbolic component that fibers over the circle.

Theorem 8.1. Let M be a closed connected oriented 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary, such that the component M_0 satisfies:

- M_0 is the mapping torus of a diffeomorphism $h : S \to S$ with pseudo-Anosov monodromy on a surface S with non-empty boundary.

Then M can be given infinitely many non-diffeomorphic contact structures ξ_k, such that for each ξ_k there exists a hypertight contact form λ_k on (M, ξ_k) which has exponential homotopical growth of cylindrical contact homology.

In Chapter 9 we study the cylindrical contact homology of contact 3-manifolds $(M, \xi_{(q,\mathfrak{r})})$ obtained via a special integral Dehn surgery on the unit tangent bundle (T_1S, ξ_{geo}) of a hyperbolic surface (S, g). This Dehn surgery is performed on a neighbourhood of a Legendrian curve $L_{\mathfrak{r}}$ which is the Legendrian lift of a separating geodesic. The surgery we consider is the contact version of Handel-Thurston surgery which was introduced by Foulon and Hasselblatt in [20] to produce non-algebraic Anosov Reeb flows in 3-manifolds. We call this contact surgery the Foulon-Hasselblatt surgery. This surgery produces not only a contact 3-manifold $(M, \xi_{(q,\mathfrak{r})})$, but also a special contact form which we denote by λ_{FH} on $(M, \xi_{(q,\mathfrak{r})})$. In [20] the authors restrict their attention to integer surgeries with positive surgery coefficient q and prove that, in this case, the Reeb flow of λ_{FH} is Anosov. Our methods work also for negative coefficients as the Anosov condition on λ_{FH} does not play a role in our results. We obtain:

Theorem 9.5. Let $(M, \xi_{(q,\mathfrak{r})})$ be the contact manifold endowed obtained by the Foulon-Hasselblatt surgery, and λ_{FH} be the contact form obtained via the Foulon-Hasselblat surgery on the Legendrian lift $L_{\mathfrak{r}} \subset T_1S$. Then λ_{FH} is hypertight and its cylindrical contact homology has exponential homotopical growth.

Remark: We again would like to point out that all the results above **do not** depend on the Polyfolds technology which is being developed Hofer, Wysocki and Zehnder. This is the case because the versions of contact homology used for proving the results above involve only somewhere injective pseudoholomorphic curves. In this situation transversality can be achieved by "classical" perturbation methods as in [14].

1.2 A dynamical obstruction to planarity of contact 3-manifolds

In chapter 10 we prove a result, that was obtained in collaboration with Chris Wendl, and which gives a dynamical obstruction for a contact 3-manifold to be planar. Our result is the following:

Theorem 10.11. Let (M, ξ) be a planar contact 3-manifold and $\widehat{\lambda}$ a non-degenerate contact form on (M, ξ). Then the Reeb flow of $\widehat{\lambda}$ either has a Reeb orbit with odd index or a contractible orbit.

The conclusion is that to prove that a contact 3-manifold is not planar it suffices to prove that it has a contact form whose Reeb flow has no contractible Reeb orbits and no odd Reeb orbits. In particular, this implies that if a contact 3-manifold admits a contact form whose Reeb flow is a transversely orientable Anosov flow then this contact manifold is not planar. In [20] the authors construct infinitely many examples of hyperbolic 3-manifolds M that admit a contact structure with a transversely orientable Anosov Reeb flow. Combining this with our result we obtain infinitely many examples of non-planar contact 3-manifolds (M, ξ) where M is hyperbolic.

Planar contact manifolds are an interesting class of contact manifolds from a topological point of view. For example, by a result of Etnyre [17] every overtwisted contact 3-manifold is planar. In the same paper, Etnyre also proved that there exist contact 3-manifolds which are not planar. For example, he showed that the unit tangent bundle (T_1S, ξ_{geo}) of a higher genus surface S endowed with the contact structure ξ_{geo} associated to geodesic flows is not planar. Theorem 10.11 gives a new proof of this result since (T_1S, ξ_{geo}) admits a contact form whose Reeb flow is a transversely orientable Anosov flow.

It is worth remarking that all known examples of Anosov Reeb flows on 3-dimensional contact manifolds are transversely orientable. In this case, as observed above, Theorem 10.11 implies that the underlying contact 3-manifolds are not planar. This motivates the following question:

Question: Can a contact 3-manifold that admits an Anosov Reeb flow be planar?

1.3 Asymptotic detection of topological entropy via chords

In Chapter 11 we study the dynamics on ω-limit sets of Legendrian fibers of unit tangent bundles of higher genus surfaces. Our main result is that these ω-limits are invariant sets which "generically" contain positive topological entropy. More precisely, our main result is the following

Theorem 11.4. Let λ be a contact form on the unit tangent bundle (T_1S, ξ_{geo}) of a surface S with genus ≥ 2. Let $\lambda_{g_{hyp}}$ be a contact form on (T_1S, ξ_{geo}) whose Reeb flow is the geodesic flow of a hyperbolic metric g_{hyp} on S, and f_λ be the function that satisfies $\lambda = f_\lambda \lambda_{g_{hyp}}$. Then for every p in the full μ_{hyp}-measure set $\mathcal{U}_\lambda$, the topological entropy $h_{top}(\phi_{X_\lambda}|_{\omega_\lambda(\mathcal{L}_p)})$ of the restriction of the Reeb flow ϕ_{X_λ} to $\omega_\lambda(\mathcal{L}_p)$ is $\geq \frac{a_S}{\max f_\lambda}$, where $a_S > 0$ is a constant that depends only on S.

This result may be seen as an improvement of the result obtained in [36]. In [36] it was proved that every Reeb flow on the unit tangent bundles (T_1S, ξ_{geo}) of higher genus surfaces has positive topological entropy. Theorem 11.4 can then be seen as a refinement of this result, as it shows that if flowing almost any unit tangent fiber of T_1S by a Reeb flow ϕ_λ on (T_1S, ξ_{geo}), we detect asymptotically a compact set invariant by ϕ_λ on which the topological entropy is positive.

Theorem 11.4 combined with the techniques used by Katok [33] implies that if S has genus ≥ 2, then for any contact form λ on (T_1S, ξ_{geo}) and almost every point $p \in S$, the number of distinct Reeb orbits of action $\leq T$ contained in $\omega_\lambda(\mathcal{L}_p)$ grows exponentially with T.

Chapter 2

Basic concepts in contact geometry and dynamics

2.1 Basic definitions from contact geometry

In this thesis we study dynamical properties of Reeb flows on contact manifolds. We start by recalling some basic concepts from contact geometry and dynamical systems which are central for this thesis.

A contact manifold is a pair (Y, ξ), where Y is a compact odd dimensional manifold and ξ, called the *contact structure*, is a"totally" non-integrable distribution of planes on Y. The total non-integrability condition means that for every locally defined 1-form ζ such that $\xi = \ker(\zeta)$ we have that $\zeta \wedge (d\zeta)^n \neq 0$, where $2n+1$ is the dimension of Y. When there exists a globally defined 1-form λ such that $\ker(\lambda) = \xi$ we call λ a *contact form* associated to the contact manifold (Y, ξ), and say that (Y, ξ) is a co-orientable contact manifold. In this thesis we only study co-orientable contact manifolds, and from now on, every time we write contact manifold we actually mean co-orientable contact manifold.

Given a contact manifold (Y, ξ), there are many different contact forms associated to it. To see this, let λ be a contact form associated to (Y, ξ). Then for every positive function $f : Y \to \mathbb{R}$, f_λ is also a contact form associated to (Y, ξ).

To a contact form λ, we can associate a vector field X_λ, that we call its *Reeb vector field*, and that is completely characterised by the following 2 conditions:

$$i_{X_\lambda} d\lambda = 0, \tag{2.1}$$

$$\lambda(X_\lambda) = 1. \tag{2.2}$$

The *Reeb flow* of λ is the flow of the vector field X_λ.

Among the submanifolds of a contact manifold a special important class is that of *Legendrian* submanifolds. An isotropic submanifold of (Y, ξ) is a submanifold L of Y whose tangent space is contained in ξ for all points of L. The Legendrian submanifolds of (Y, ξ) are the isotropic submanifolds of (Y, ξ) which have the maximal possible dimension. It turns out, that for $2n + 1$-dimensional contact manifolds, that this maximal possible dimension is n, and therefore the Legendrian submanifolds are the isotropic submanifolds of dimension n.

There are two special types of trajectories of Reeb flows that have played a central role in the study of contact topology and dynamics of Reeb vector fields. One of them are the periodic orbits of a given Reeb flow, which we call *Reeb orbits*. The other are the trajectories of a Reeb flow which start in a Legendrian submanifold L and end in a Legendrian submanifold $\widehat{L}$ (notice that L and $\widehat{L}$ might coincide); these trajectories are called *Reeb chords* from L to $\widehat{L}$. Given a Reeb orbit γ of the Reeb flow of λ, we define its action to be $A(\gamma) := \int_\gamma \lambda$; it follows from equation 1.2 above that $A(\gamma)$ coincides with the period of γ. Analogously, for a Reeb chord c of the Reeb flow of λ, we define its action to be $A(c) := \int_c \lambda$; like for Reeb orbits the action of c coincides with the "period" of the trajectory c. Following terminology widely used in the literature, a contact form λ is called *hypertight* when it doesn't have any contractible Reeb orbits. Lastly, a Reeb orbit γ is said to be *non-degenerate* when 1 is not an eigenvalue of the linearisation $D\phi^{A(c)}_{X_\lambda} \mid_\xi$ of the Poincaré return map associated to the γ; and a Reeb chord c is said to be *transverse* if the intersection $\phi^{A(c)}_{X_\lambda}(L) \cap \widehat{L}$ is transverse at the endpoint of c.

We introduce some notation. Given a contact form λ on a contact manifold we will denote by $\mathcal{P}_\lambda$ the set of Reeb orbits of the Reeb flow of λ. If ρ is a free homotopy class of loops in the manifold Y we will denote by $\mathcal{P}^\rho_\lambda$ the set of Reeb orbits of λ which belong to the homotopy class ρ. Analogously, we let $\mathcal{X}_\lambda(L \to \widehat{L})$ denote the set of Reeb chords of λ from L to $\widehat{L}$. If ϱ denotes a homotopy class of paths in Y starting in L and ending in $\widehat{L}$, we denote by $\mathcal{X}^\varrho_\lambda(L \to \widehat{L})$ the set of Reeb chords of λ from L to $\widehat{L}$ which belong to the homotopy class ϱ.

One important feature of Reeb flows is that they appear in many different models of mathematical physics. For instance, every Reeb flow appears as the restriction to an energy level, of some Hamiltonian flow in a (possibly non-compact) symplectic manifold (see [25]). One important example of this relation between Reeb flows and Hamiltonian flows is seen in the case of geodesic flows. For any Riemannian metric on a compact manifold Q, the restriction of its geodesic flow to the unit tangent bundle T_1Q of Q is a Reeb flow.

More recently, Etnyre and Ghrist showed in [16] that 3-dimensional Reeb flows also appear in the context of hydrodynamical 3-dimensional flows. Beltrami flows are an important special class of hydrodynamical flows. Etnyre and Ghrist showed that every Reeb flow in a 3-dimensional contact manifold is the reparametrization of some Beltrami flow; they also showed that every Beltrami flow is the reparametrization of some Reeb flow. therefore the classes of Beltrami and Reeb flows are equivalent from a dynamical perspective.

These relations to the field of mathematical physics also justifies the study of the dynamical properties of Reeb vector fields, as this study might also have impact in this field.

2.2 Some basic concepts in dynamical systems

2.2.1 Topological entropy of dynamical systems

The topological entropy is an important invariant of dynamical systems, which was introduced in the 1960's by Adler, Kronheim and McAndrew. It codifies, in a single non-negative number, how chaotic a dynamical system is; it is widely accepted that a dynamical system with positive topological entropy presents some kind of chaotic behaviour.

We present the following definition, which is valid for dynamical systems in compact metric spaces and is due to Bowen [12]. Consider a smooth compact manifold V with a non-vanishing vector field X that generates a flow ϕ_X. We endow V with an auxiliary Riemannian metric g, whose associated metric on V we denote by d_g.

Let T and δ be positive real numbers. A set S is said to be T,δ-separated if for all $q_1 \neq q_2 \in S$ we have:

$$\max_{t\in[0,T]} d_g(\phi^t_X(q_1), \phi^t_X(q_2)) > \delta. \tag{2.3}$$

We denote by $n^{T,\delta}$ the maximal cardinality of a T,δ-separated set for the flow ϕ_X. Then we define the δ-entropy $h_\delta(\phi_X)$ as:

$$h_\delta(\phi_X) = \limsup_{T\to+\infty} \frac{\log(n^{T,\delta})}{T} \tag{2.4}$$

The topological entropy h_{top} is then defined by

$$h_{top}(\phi_X) = \lim_{\delta\to 0} h_\delta(\phi_X). \tag{2.5}$$

One can prove that the topological entropy does not depend on the metric d_g but only on the topology determined by the metric. For these and other structural results about topological entropy we refer the reader to any standard textbook in dynamics such as [34] and [40].

The definition of topological entropy is quite involved and it is usually quite difficult to compute or even estimate the topological entropy for a given dynamical system. To motivate such difficult attempts to estimate or compute this quantity, we present one striking consequence of positivity of topological entropy for low-dimensional dynamical systems:

Theorem 2.1. *Katok [33]. Let X be a $C^{1+\delta}$ $(\delta > 0)$ vector field on a smooth 3-dimensional M, whose flow ϕ_X has positive topological entropy $h_{top}(\phi_X)$. Then there exists a hyperbolic periodic orbit x of X, whose stable and unstable manifold have a transverse intersection, i.e. a transverse homoclinic intersection. Consequently there is an invariant set Ω for the flow ϕ_X, such that the dynamics of the restriction of ϕ_X to Ω is topologically conjugate to a subshift of finite type and $h_{top}(\phi_X|_\Omega) > 0$.*

The theorem above means that simply the non-vanishing of h_{top} for a given flow ϕ_X on a 3-dimensional manifold implies that this flow has complicated orbit structure. For example, for a given number $T > 0$, let $P_X^{hyp}(T)$ be the number of hyperbolic periodic orbits of ϕ_X with period smaller then T. As a consequence of the Theorem above we have the following corollary also due to Katok:

Corollary 2.2. *If the flow ϕ_X of a vector field X on a 3-manifold M has positive topological entropy, then we have the following lower bound:*

$$\limsup_{T\to+\infty} \frac{\log(P_X^{hyp}(T))}{T} > 0. \tag{2.6}$$

This means that positivity of topological entropy for these flows implies that they have infinitely many isolated periodic orbits. Only these results suffice, in my opinion, to justify the study of the topological entropy for Reeb flows in 3-dimensional contact manifolds

2.2.2 ω-limits of submanifolds and chords

Let M denote a compact smooth manifold, $\mathbb{L}$ and $\mathbb{L}'$ be disjoint compact submanifolds of M and X be a smooth vector field on M. We will denote by ϕ_X the flow generated by the vector field X.

A pair $(\widehat{c}, \widehat{T})$ is called an X-*chord* from $\mathbb{L}$ to $\mathbb{L}'$, when $\widehat{c} : [0, \widehat{T}] \to M$ is a parametrized trajectory of the flow ϕ_X, such that $\widehat{c}(0) \in \mathbb{L}$ and $\widehat{c}(\widehat{T}) \in \mathbb{L}'$.

Given any submanifold $\mathbb{L}$ of M, its ω-limit $\omega_X(\mathbb{L})$ is defined as

$$\omega_X(\mathbb{L}) := \{x \in M \mid \exists \text{ a sequence } \ x_n \in \mathbb{L} \text{ and } t_n \to +\infty \text{ such that } \lim_{n \to +\infty} \phi_X^{t_n}(x_n) = x\}. \tag{2.7}$$

It is straightforward to check that the set $\omega_X(\mathbb{L})$ is compact and invariant by the flow ϕ_X.

Chapter 3

Pseudoholomorphic curves

3.1 Almost complex structures in symplectizations and symplectic cobordisms

We start by recalling some concepts needed to define pseudoholomorphic curves in symplectizations and symplectic cobordisms.

3.1.1 Cylindrical almost complex structures

Let (Y, ξ) be a contact manifold and λ a contact form on (Y, ξ). The symplectization of (Y, ξ) is the product $\mathbb{R} \times Y$ with the symplectic form $d(e^s\lambda)$ (where s denotes the $\mathbb{R}$ coordinate in $\mathbb{R} \times Y$). $d\lambda$ restricts to a symplectic form on the vector bundle ξ and it is well known that the set $\mathfrak{j}(\lambda)$ of $d\lambda$-compatible almost complex structures on the symplectic vector bundle ξ is non-empty and contractible. Notice, that if Y is 3-dimensional the set $\mathfrak{j}(\lambda)$ doesn't depend on the contact form λ associated to (Y, ξ).

For $j \in \mathfrak{j}(\lambda)$ we can define an $\mathbb{R}$-invariant almost complex structure J on $\mathbb{R} \times Y$ by demanding that:

$$J\partial_s = X_\lambda, \quad J\mid_\xi = j \tag{3.1}$$

We will denote by $\mathcal{J}(\lambda)$ the set of almost complex structures in $\mathbb{R} \times Y$ that are $\mathbb{R}$-invariant, $d(e^s\lambda)$-compatible and satisfy the equation (16) for some $j \in \mathfrak{j}(\lambda)$.

3.1.2 Exact symplectic cobordisms with cylindrical ends

An exact symplectic cobordism is, intuitively, an exact symplectic manifold (W, ϖ) that outside a compact subset is like the union of cylindrical ends of symplectizations. We restrict our attention to exact symplectic cobordisms having only one positive end and one negative end.

Let $(W, \varpi = d\kappa)$ be an exact symplectic manifold without boundary, and let (Y^+, ξ^+) and (Y^-, ξ^-) be contact manifolds with contact forms λ^+ and λ^-. We say that $(W, \varpi = d\kappa)$ is an exact symplectic cobordism from λ^+ to λ^- when there exist subsets W^-, W^+ and $\widehat{W}$ of W and diffeomorphisms $\Psi^+ : W^+ \to [0, +\infty) \times Y^+$ and $\Psi^- : W^- \to (-\infty, 0] \times Y^-$, such that:

$$\widehat{W} \text{ is compact, } W = W^+ \cup \widehat{W} \cup W^- \text{ and } W^+ \cap W^- = \emptyset, \tag{3.2}$$

$$(\Psi^+)^*(e^s\lambda^+) = \kappa \text{ and } (\Psi^-)^*(e^s\lambda^-) = \kappa$$

In such a cobordism, we say that an almost complex structure $\overline{J}$ is cylindrical if:

$$\overline{J} \text{ coincides with } \; J^+ \in \mathcal{J}(C^+\lambda^+) \text{ in the region } W^+ \tag{3.3}$$

$$\overline{J} \text{ coincides with } \; J^- \in \mathcal{J}(C^-\lambda^-) \text{ in the region } W^- \tag{3.4}$$

$$\overline{J} \text{ is compatible with } \varpi \text{ in } \widehat{W} \tag{3.5}$$

where $C^+ > 0$ and $C^- > 0$ are constants.

For fixed $J^+ \in \mathcal{J}(C^+\lambda^+)$ and $J^- \in \mathcal{J}(C^-\lambda^-)$, we denote by $\mathcal{J}(J^-, J^+)$ set of cylindrical almost complex structures in $(\mathbb{R} \times Y, \varpi)$ coinciding with J^+ on W^+ and J^- on W^-. It is well known that $\mathcal{J}(J^-, J^+)$ is non-empty and contractible. We will write $\lambda^+ \succ_{ex} \lambda^-$ when there exists an exact symplectic cobordism from λ^+ to λ^- as above. We remind the reader that $\lambda^+ \succ_{ex} \lambda$ and $\lambda \succ_{ex} \lambda^-$ implies $\lambda^+ \succ_{ex} \lambda^-$; or in other words that the exact symplectic cobordism relation is transitive; see [10] for a detailed discussion on symplectic cobordisms with cylindrical ends. Notice that a symplectization is a particular case of an exact symplectic cobordism.

Remark: we point out to the reader that in many references in the literature, a slightly different definition of cylindrical almost complex structures is used: instead of demanding that $\overline{J}$ satisfies equations (18) and (19), the stronger condition that $\overline{J}$ coincides with $J^\pm \in \mathcal{J}(\lambda^\pm)$ in the region $W^\pm$ is demanded. We need to consider this more relaxed definition of cylindrical almost complex structures when we study the cobordism maps of cylindrical contact homologies in section 3.3.

3.1.3 Splitting symplectic cobordisms

Let λ^+, λ and λ^- be contact forms on (Y, ξ) such that $\lambda^+ \succ_{ex} \lambda$, $\lambda \succ_{ex} \lambda^-$. For $\epsilon > 0$ sufficiently small, it is easy to see that one also has $\lambda^+ \succ_{ex} (1+\epsilon)\lambda$ and $(1-\epsilon)\lambda \succ_{ex} \lambda^-$. Then, for each $R > 0$, it is possible to construct an exact symplectic form $\varpi_R = d\kappa_R$ on $W = \mathbb{R} \times Y$ where:

$$\kappa_R = e^{s-R-2}\lambda^+ \text{ in } [R+2, +\infty) \times Y, \tag{3.6}$$

$$\kappa_R = f(s)\lambda \text{ in } [-R, R] \times Y, \tag{3.7}$$

$$\kappa_R = e^{s+R+2}\lambda^- \text{ in } (-\infty, -R-2] \times Y, \tag{3.8}$$

and $f : [-R, R] \to [1-\epsilon, 1+\epsilon]$, satisfies $f(-R) = 1-\epsilon$, $f(R) = 1+\epsilon$ and $f' > 0$. In $(\mathbb{R} \times Y, \varpi_R)$ we consider a compatible cylindrical almost complex structure $\widetilde{J}_R$; but we demand an extra condition on $\widetilde{J}_R$:

$$\widetilde{J}_R \text{ coincides with } J \in \mathcal{J}(\lambda) \text{ in } [-R, R] \times Y. \tag{3.9}$$

Again we divide W in regions: $W^+ = [R+2, +\infty) \times Y$, $W(\lambda^+, \lambda) = [R, R+2] \times Y$, $W(\lambda) = [-R, R] \times Y$, $W(\lambda, \lambda^-) = [-R-2, -R] \times Y$ and $W^- = (-\infty, -R-2] \times Y$. The family of exact symplectic cobordisms with cylindrical almost complex structures $(\mathbb{R} \times Y, \varpi_R, \widetilde{J}_R)$ is called a splitting family from λ^+ to λ^- along λ.

3.2 Pseudoholomorphic curves

Let (S, i) be a closed connected Riemann surface without boundary, $\Gamma \subset S$ be a finite set. Let λ be a contact form in (Y, ξ) and $J \in \mathcal{J}(\lambda)$. A finite energy pseudoholomorphic curve in the symplectization $(\mathbb{R} \times Y, J)$ is a map $\widetilde{w} = (r, w) : S \setminus \Gamma \to \mathbb{R} \times Y$ that satisfies

$$\overline{\partial}_J(\widetilde{w}) = d\widetilde{w} \circ i - J \circ d\widetilde{w} = 0 \tag{3.10}$$

and

$$0 < E(\widetilde{w}) = \sup_{q \in \mathcal{E}} \int_{S \setminus \Gamma} \widetilde{w}^* d(q\lambda) \tag{3.11}$$

where $\mathcal{E} = \{q : \mathbb{R} \to [0,1]; q' \geq 0\}$. The quantity $E(\widetilde{w})$ is called the Hofer energy and was introduced in [27]. The operator $\overline{\partial}_J$ above is called the Cauchy-Riemann operator for the almost complex structure J.

For an exact symplectic cobordism (W, ϖ) from λ^+ to λ^- as considered above, and $\overline{J} \in \mathcal{J}(J^-, J^+)$ a finite energy pseudoholomorphic curve is again a map $\widetilde{w} : (S \setminus \Gamma \to W$ satisfying:

$$d\widetilde{w} \circ i = \overline{J} \circ d\widetilde{w}, \tag{3.12}$$

and

$$0 < E_{\lambda^-}(\widetilde{w}) + E_c(\widetilde{w}) + E_{\lambda^+}(\widetilde{w}) < +\infty, \tag{3.13}$$

where:
$E_{\lambda^-}(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}(W^-))} \widetilde{w}^* d(q\lambda^-)$,
$E_{\lambda^+}(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}(W^+)} \widetilde{w}^* d(q\lambda^+)$,
$E_c(\widetilde{w}) = \int_{\widetilde{w}^{-1}W(\lambda^-,\lambda^+)} \widetilde{w}^* \varpi$.
These energies were also introduced in [27].

In splitting symplectic cobordisms we use a slightly modified version of energy. Instead of demanding $0 < E_-(\widetilde{w}) + E_c(\widetilde{w}) + E_+(\widetilde{w}) < +\infty$ we demand:

$$0 < E_{\lambda^-}(\widetilde{w}) + E_{\lambda^-,\lambda}(\widetilde{w}) + E_\lambda(\widetilde{w}) + E_{\lambda,\lambda^+}(\widetilde{w}) + E_{\lambda^+}(\widetilde{w}) < +\infty \tag{3.14}$$

where:
$E_\lambda(\widetilde{w}) = \sup_{q\in\mathcal{E}} \int_{\widetilde{w}^{-1}W(\lambda)} \widetilde{w}^* d(q\lambda)$,
$E_{\lambda^-,\lambda}(\widetilde{w}) = \int_{\widetilde{w}^{-1}(W(\lambda^-,\lambda))} \widetilde{w}^* \varpi$,
$E_{\lambda,\lambda^+}(\widetilde{w} = \int_{\widetilde{w}^{-1}(W(\lambda,\lambda^+))} \widetilde{w}^* \varpi$,
and $E_{\lambda^-}(\widetilde{w})$ and $E_{\lambda^+}(\widetilde{w})$ are as above.

The elements of the set $\Gamma \subset S$ are called punctures of the pseudoholomorphic $\widetilde{w}$. The work of Hofer et al. [27, 28] allows us do classify the punctures in two types: positive punctures and negative punctures. This classification is done according to the behaviour of $\widetilde{w}$ in the neighbourhood of the puncture. Before presenting this classification we introduce some notation. Let $B_\delta(z)$ be the ball of radius δ centered at the puncture z, and denote by $\partial(B_\delta(z))$ its boundary. With this in hand, we can describe the types of punctures as follows:

- $z \in \Gamma$ is called positive interior puncture when $z \in \Gamma$ and $\lim_{z' \to z} s(z') = +\infty$, and there exist a sequence $\delta_n \to 0$ and Reeb orbit γ^+ of X_{λ^+}, such that $w(\partial(B_{\delta_n}(z)))$ converges in C^∞ to γ^+ as $n \to +\infty$

- $z \in \Gamma$ is called negative interior puncture when $z \in \Gamma$ and $\lim_{z' \to z} s(z') = -\infty$, and there exist a sequence $\delta_n \to 0$ and Reeb orbit γ^- of X_{λ^-}, such that $w(\partial(B_{\delta_n}(z)))$ converges in C^∞ to γ^- as $n \to +\infty$.

The results in [27] and [28] imply that these are indeed the only real possibilities we need to consider for the behaviour of the $\widetilde{w}$ near punctures. Intuitively, we have that at the punctures, the pseudoholomorphic curve $\widetilde{w}$ detects Reeb orbits. When for a puncture z, there is a subsequence δ_n such that $w(\partial(B_{\delta_n}(z)))$ converges to a Reeb orbit γ, we will say that $\widetilde{w}$ is asymptotic to this Reeb orbit γ at the puncture z.

If a pseudoholomorphic curve is asymptotic to a non-degenerate Reeb orbit at a puncture, more can be said about its asymptotic behaviour in neighbourhoods of this puncture. In order to describe the behaviour of $\widetilde{w}$ near a puncture z, we take a neighbourhood $U \subset S$ of z that admits a holomorphic chart $\psi_U : (U, z) \to (\mathbb{D}, 0)$. Using polar coordinates $(r, t) \in (0, +\infty) \times S^1$ we can write $x \in (\mathbb{D} \setminus 0)$ as $x = e^{-r}t$. With this notation, it is shown in [27] [28], that if z is a positive interior puncture on which $\widetilde{w}$ is asymptotic to a non-degenerate Reeb orbit γ^+ of X_{λ^+}, then $\widetilde{w} \circ \psi_u^{-1}(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in C^∞ to a Reeb orbit γ^+ of X_{λ^+}, and the convergence rate is exponential.

Similarly, if z is a negative interior puncture on which $\widetilde{w}$ is asymptotic to a non-degenerate Reeb orbit γ^- of X_{λ^-}, then $\widetilde{w} \circ \psi_u^{-1}(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in C^∞ to a Reeb orbit γ^- of $-X_{\lambda^-}$ as $r \to +\infty$, and the convergence rate is exponential.

Remark: the fact that the convergence of pseudoholomorphic curves near punctures to Reeb orbits is of exponential nature is a consequence of the asymptotic formula obtained in [28]. Such formulas are necessary for the Fredholm theory that gives the dimension of the space of pseudoholomorphic curves with fixed asymptotic data.

The discussion above can be summarised by saying that near punctures the finite pseudoholomorphic curves detect Reeb orbits. It is exactly this behavior that makes these objects useful for the study of dynamics of Reeb vector fields.

3.3 Pseudoholomorphic buildings

In this section we introduce the notion of pseudoholomorphic buildings. Our exposition is based in section 2 of [45].

We denote by $\mathbb{D}$ the open unit disk in $\mathbb{C}$. Let $\phi : (0, +\infty) \times S^1 \to \mathbb{D} \setminus \{0\}$ be the biholomorphism given by $\phi(s,t) = e^{-2\pi(s+it)}$. We denote $\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\}$. As $(0, +\infty) \times S^1$ is identified with $\dot{\mathbb{D}}$ by ϕ, we can define the circle compactification $\overline{\mathbb{D}}$ of $\dot{\mathbb{D}}$ by

$$\overline{\mathbb{D}} := \dot{\mathbb{D}} \cup (\{+\infty\} \times S^1) = (0, +\infty] \times S^1. \tag{3.15}$$

An analogous process allows us to compactify any punctured Riemman surface. Let (Σ, j) be a compact Riemman surface and $\Upsilon \subset \Sigma$ be a finite set of points. For each point $z \in \Upsilon$ we take a neighbourhood B_z which admits a biholomorphism $\psi_z : (B_z, z) \to (\mathbb{D}, 0)$. We can then use the compactification of $\dot{\mathbb{D}}$ described above to obtain the circle compactification $\overline{B}_z$ of the punctured ball $\dot{B}_z := B_z \setminus \{z\}$:

$$\overline{B}_z := \dot{B}_z \cup \delta_z \tag{3.16}$$

where δ_z is a circle at infinity. The conformal structure at $T_z\Sigma$ given by j defines a canonical orientation and a canonical metric on δ_z. For two different elements z_1 and z_2 of Υ, an orthogonal diffeomorphism $\varphi_{z_1,z_2} : \delta_{z_1} \to \delta_{z_2}$ is an orientation reversing diffeomorphism from δ_{z_1} to δ_{z_2} which is an isometry for the canonical metrics on these two circles. Performing this compactification on all points of Υ we obtain the circle compactification $\overline{\Sigma}$ of the punctured Riemman surface $\dot{\Sigma} = \Sigma \setminus \Upsilon$ given by

$$\overline{\Sigma} := \dot{\Sigma} \cup (\cup_{z \in \Upsilon} \delta_z). \tag{3.17}$$

A closed nodal Riemann surface $\mathbf{S}$ is a quadruple (S, j, Γ, Δ) where (S, j) is a closed (maybe disconnected) Riemann surface, $\Gamma \subset S$ is a finite ordered set, and $\Delta \subset S$ a finite set that satisfies:

- $\Delta = \{\overline{z}_1, \underline{z}_1, \overline{z}_2, \underline{z}_2 ... \overline{z}_n, \underline{z}_n\}$.

We define $\dot{S} := S \setminus (\Delta \cup \Gamma)$ and let $\overline{S}$ be the circle compactification of $\dot{S}$. For each pair $\overline{z}_i, \underline{z}_i$ we choose an orthogonal diffeomorphism $\varphi_i : \delta_{\overline{z}_i} \to \delta_{\underline{z}_i}$; this is called a decoration of $\{\overline{z}_i, \underline{z}_i\}$. By choosing a decoration for every pair $\{\overline{z}_i, \underline{z}_i\}$ of Δ we obtain a decoration of Δ which we denote by φ. The quintuple $\mathbf{S} := (S, j, \Gamma, \Delta, \varphi)$ is called a decorated closed nodal Riemann surface.

For a decorated Riemann surface $\mathbf{S}$ we define

$$\overline{\mathbf{S}} := \mathbf{S}/\{z \sim \varphi(z)\} \tag{3.18}$$

$\overline{\mathbf{S}}$ is a topological surface with boundary. The arithmetic genus of $\mathbf{S}$ is defined to be the genus of $\overline{\mathbf{S}}$. We say that $\mathbf{S}$ is connected when $\overline{\mathbf{S}}$ is connected.

3.3.1 Nodal pseudoholomorphic curves

3.3.1.1 Nodal pseudoholomorphic curves in symplectizations

Letting (Y, ξ) denote a contact manifold and λ a contact form on it, denote by J an element of $\mathcal{J}(\lambda)$ A nodal pseudoholomorphic curve in the symplectization $(\mathbb{R} \times Y, de^s\lambda)$ is a pair $(\mathbf{S}, \widetilde{u})$ where $\mathbf{S} = (S, j, \Gamma, \Delta)$ is a closed nodal Riemann surface and $\widetilde{u} : (S \setminus \Gamma, j) \to (\mathbb{R} \times Y, J)$ is a finite energy pseudoholomorphic map such that for every pair $\{\overline{z}, \underline{z}\} \in \Delta$, $\widetilde{u}(\overline{z}) = \widetilde{u}(\underline{z})$. We will use the notation $\widetilde{u} : \mathbf{S} \to (\mathbb{R} \times Y, J)$ to denote a nodal pseudoholomorphic curve. Analogous to what we did for pseudoholomorphic curves, we use the asymptotic behaviour of $\widetilde{u}$ near the punctures in Γ to partition Γ in the sets $\Gamma+$ and Γ^- of positive and negative punctures.

It follows from the asymptotic behaviour of the finite energy pseudoholomorphic curves near the punctures in Γ that any nodal curve $\widetilde{u} : \mathbf{S} \to (\mathbb{R} \times Y, J)$ admits an extension $\overline{u} : \overline{\mathbf{S}} \to ([-\infty, +\infty] \times Y, J)$ from the circle compactification of $\overline{\mathbf{S}}$ of $\mathbf{S}$ to the compactification $[-\infty, +\infty] \times Y$ of $\mathbb{R} \times Y$. We let $\partial^+\overline{\mathbf{S}}$ and $\partial^-\overline{\mathbf{S}}$ denote, respectively, the sets $\cup_{z\in\Gamma^+}\delta_z$ and $\cup_{z\in\Gamma^-}\delta_z$ of circles at infinity at, respectively, the sets Γ^+ and Γ^- of punctures. It is clear that $\overline{u}$ takes $\partial^-\overline{\mathbf{S}}$ to the boundary component $\{-\infty\} \times Y$ of $[-\infty, +\infty] \times Y$, and $\partial^+\overline{\mathbf{S}}$ to the boundary component $\{+\infty\} \times Y$ of $[-\infty, +\infty] \times Y$.

3.3.1.2 Nodal pseudoholomorphic curves in exact symplectic cobordisms

We denote by (W, ϖ) an exact symplectic cobordism from λ^+ to λ^-, where λ^+ is a contact form on the contact manifold (M^+, ξ^+) and λ^- is a contact form on the contact manifold (M^-, ξ^-). On the cobordism (W, ϖ) we consider a cylindrical almost complex structure $\overline{J} \in \mathcal{J}(J^-, J^+)$ where $J^+ \in \mathcal{J}(C^+\lambda^+)$ and $J^- \in \mathcal{J}(C^-\lambda^-)$ for positive constants C^+ and C^-.

A nodal pseudoholomorphic curve in (W, ϖ) is a pair $(\mathbf{S}, \widetilde{u})$ where $\mathbf{S} = (S, j, \Gamma, \Delta)$ is a closed nodal Riemann surface and $\widetilde{u} : (S \setminus \Gamma, j) \to (W, \overline{J})$ is a finite energy pseudoholomorphic map such that for every pair $\{\overline{z}, \underline{z}\} \in \Delta$, $\widetilde{u}(\overline{z}) = \widetilde{u}(\underline{z})$.We will also use the notation $\widetilde{u} : \mathbf{S} \to (W, \overline{J})$ to denote a nodal pseudoholomorphic curve. Using the asymptotic behaviour of $\widetilde{u}$ near the punctures in Γ we partition Γ in the sets $\Gamma+$ and Γ^- of positive and negative punctures.

Because of the asymptotic behaviour of the finite energy pseudoholomorphic curves near the punctures in Γ, it follows that any nodal curve $\widetilde{u} : \mathbf{S} \to (\mathbb{R} \times Y, J)$ admits an extension $\overline{u} : \overline{\mathbf{S}} \to (W \cup (\{\infty\} \times Y^-) \cup (\{+\infty\} \times Y^+), J)$ from the circle compactification of $\overline{\mathbf{S}}$ of $\mathbf{S}$ to the compactification $W \cup (\{\infty\} \times Y^-) \cup (\{+\infty\} \times Y^+)$ of W. Similarly to what we did in the case of symplectizations, we let $\partial^+\overline{\mathbf{S}}$ and $\partial^-\overline{\mathbf{S}}$ denote, respectively, the sets $\cup_{z\in\Gamma^+}\delta_z$ and $\cup_{z\in\Gamma^-}\delta_z$ of circles at infinity at, respectively, the sets Γ^+ and Γ^- of punctures. It is then clear that $\overline{u}$ takes $\partial^-\overline{\mathbf{S}}$ to the boundary component $\{-\infty\} \times Y^-$ of $[-\infty, +\infty] \times Y$, and $\partial^+\overline{\mathbf{S}}$ to the boundary component $\{+\infty\} \times Y^+$ of $[-\infty, +\infty] \times Y$.

3.3.2 Pseudoholomorphic buildings in symplectizations

We proceed to define pseudoholomorphic buildings in symplectizations. Consider a collection of nodal pseudoholomorphic curves

$$\widetilde{u}_m : \mathbf{S}_m = (S_m, j_m, \Gamma_m, \Delta_m) \to (\mathbb{R} \times Y, J) \tag{3.19}$$

for $m \in \{1, ..., n\}$. Assume that there exists for each $m \in \{2, ..., n\}$, an orientation reversing orthogonal diffeomorphism $\phi_m : \partial^-\overline{\mathbf{S}}_{m-1} \to \partial^+\overline{\mathbf{S}}_m$. Then, the collection $\widetilde{u} = (\widetilde{u}_1, ..., \widetilde{u}_n; \phi_1, ..., \phi_n)$ is called a pseudoholomorphic building in the symplectization $(\mathbb{R} \times Y, de^s\lambda)$ if $\overline{u}_m \circ \phi_m = \overline{u}_{m-1}|_{\partial^-\overline{\mathbf{S}}_{m-1}}$ for every $m \in \{2, ..., n\}$. The number n is called the height of the pseudoholomorphic building $\widetilde{u}$. We denote by Γ the union $\Gamma_1^+ \cup \Gamma_n^-$ of the positive punctures of $\widetilde{u}_1$ and the negative punctures of $\widetilde{u}_n$. We define $\Delta_{br} := (\bigcup_{1\leq m\leq n} \Gamma_m) \setminus \Gamma$. For each $m \in \{2, ..., n\}$ and $\underline{z} \in \Gamma_{m-1}^-$ there exist a unique $\overline{z} \in \Gamma_m^+$ such that $\phi_m(\delta_{\underline{z}}) = \delta_{\overline{z}}$. We refer to all such pairs $\{\underline{z}, \overline{z}\} \subset \Delta_{br}$ as *breaking pairs*. The Reeb orbit $\gamma_{(\overline{z},\underline{z})}$ which is parametrized by $\overline{u}_{m-1}|_{\delta_{\underline{z}}}$ and $\overline{u}_m|_{\delta_{\overline{z}}}$ is called a breaking orbit.

Given a pseudoholomorphic building $\widetilde{u} = (\widetilde{u}_1, ..., \widetilde{u}_n; \phi_1, ..., \phi_n)$ we associate a nodal partially decorated Riemann surface $\mathbf{S} = (S, j, \Gamma, \Delta, \phi)$ which is constructed from the domains $\mathbf{S}_m$ of the levels $\widetilde{u}_m$: (S, j) is defined as the disjoint union of the Riemann surfaces $(S_1, j_1), ..., (S_n, j_n)$, $\Gamma = \Gamma_1^+ \cup \Gamma_n^-$ as explained above, $\Delta_N = \bigcup_{1\leq m\leq n} \Delta_m$ is the set of nodal points of the levels of $\widetilde{u}$, $\Delta := \Delta_N \cup \Delta_{br}$, and ϕ is the collection of decorations at the breaking pairs $\{\underline{z}, \overline{z}\} \subset \Delta_{br}$ given by the maps $\phi_m : \partial^-\overline{\mathbf{S}}_{m-1} \to \partial^+\overline{\mathbf{S}}_m$. We will refer to the partially decorated Riemann surface $\mathbf{S}$ as the domain of the pseudoholomorphic building $\widetilde{u}$.

We then choose arbitrary decorations ψ_m for the nodal sets Δ_m of $\mathbf{S}_m$. Combining these with ϕ we obtain a decoration ψ for $\mathbf{S}$. This allows us to define the compactification $\overline{\mathbf{S}}$ of $\mathbf{S}$ as the surface obtained from the union $\mathbf{S}_1 \cup ... \cup \mathbf{S}_n$ by gluing the boundary pairs associated to the pairs in Δ via the decoration ψ. It is then possible to extend $\widetilde{u}$ to a continuous map $\overline{u} : \mathbf{S} \to M$. The map $\overline{u}$ can also be seen as a map in a suitable compactification of a space formed by n copies of the symplectization; see [10].

3.3.3 Pseudoholomorphic buildings in exact symplectic cobordisms

We now define pseudoholomorphic buildings in exact symplectic cobordisms. Consider for $m \in \{-n^-, -n^- - 1, ..., n^+ - 1, n^+\}$, where n^- and n^+ are non-negative integers, a collection of nodal pseudoholomorphic curves $\widetilde{u}_m$ of the following type

$$\widetilde{u}_m : \mathbf{S}_m = (S_m, j_m, \Gamma_m, \Delta_m) \to (\mathbb{R} \times Y^+, J^+) \text{ if } m < 0, \tag{3.20}$$

$$\widetilde{u}_0 : \mathbf{S}_0 = (S_0, j_0, \Gamma_0, \Delta_0) \to (W, \overline{J}), \tag{3.21}$$

$$\widetilde{u}_m : \mathbf{S}_m = (S_m, j_m, \Gamma_m, \Delta_m) \to (\mathbb{R} \times Y^-, J^-) \text{ if } m > 0. \tag{3.22}$$

Assume that there exists for each $m \in \{-n^- + 1, ..., n^+\}$, an orientation reversing orthogonal diffeomorphism $\phi_m : \partial^- \overline{\mathbf{S}}_{m-1} \to \partial^+ \overline{\mathbf{S}}_m$. Then, we will call the collection $\widetilde{u} = (\widetilde{u}_{-n^-}, ..., \widetilde{u}_{n^+}; \phi_{-n^-}, ..., \phi_{n^+})$ a pseudoholomorphic building in the cobordism (W, ϖ) if $\overline{u}_m|_{\partial^- \overline{\mathbf{S}}_m} = \overline{u}_{m-1} \circ \phi_m$ for every $m \in \{-n^-, ..., n^+ - 1\}$. The number $n^- + n^- + 1$ is called the height of the pseudoholomorphic building $\widetilde{u}$. Similarly to what we did for pseudoholomorphic buildings in symplectizations we define $\Gamma := \Gamma^+_{-n^-} \cup \Gamma^-_{n^+}$ and $\Delta_{br} := (\bigcup_{-n^- \leq m \leq n^+} \Gamma_m) \setminus \Gamma$. For each $m \in \{-n^- + 1, ..., n^+\}$ and $\underline{z} \in \Gamma^-_{m-1}$ there exist a unique $\overline{z} \in \Gamma^+_m$ such that $\phi_m(\delta_{\underline{z}}) = \delta_{\overline{z}}$. We refer to all such pairs $\{\underline{z}, \overline{z}\} \subset \Delta_{br}$ as *breaking pairs*. The Reeb orbit $\gamma_{(\overline{z}, \underline{z})}$ (of λ^+ or λ^-) which is parametrized by $\overline{u}_{m-1}|_{\delta_{\underline{z}}}$ and $\overline{u}_m|_{\delta_{\overline{z}}}$ is called a breaking orbit.

Exactly like we did for pseudoholomorphic buildings in symplectizations we associate to our building $\widetilde{u}$ a nodal partially decorated Riemann surface $\mathbf{S} = (S, j, \Gamma, \Delta, \phi)$: $\widetilde{u}_m$: (S, j) is defined as the disjoint union of the Riemann surfaces $(S_{-n^-}, j_{-n^-}), ..., (S_{n^+}, j_{n^+})$, $\Gamma = \Gamma^+_{-n^-} \cup \Gamma^-_{n^+}$ as explained above, $\Delta_N = \bigcup_{-n^- \leq m \leq n^+} \Delta_m$ is the set of nodal points of the levels of $\widetilde{u}$, $\Delta := \Delta_N \cup \Delta_{br}$, and ϕ is the collection of decorations at the breaking pairs $\{\underline{z}, \overline{z}\} \subset \Delta_{br}$ given by the maps $\phi_m : \partial^- \overline{\mathbf{S}}_{m-1} \to \partial^+ \overline{\mathbf{S}}_m$. We choose arbitrary decorations ψ_m for the nodal sets Δ_m of $\mathbf{S}_m$ and combine these with ϕ to get a decoration ψ for $\mathbf{S}$.

As previously we define the compactification $\overline{\mathbf{S}}$ of $\mathbf{S}$ as the surface obtained from the union $\mathbf{S}_{-n^-} \cup ... \cup \mathbf{S}_{n^+}$ by gluing the boundary pairs associated to the pairs in Δ

via the decoration ψ. With this in hand it is possible to extend $\widetilde{u}$ to a continuous map $\overline{u} : \mathbf{S} \to \mathbf{W}$ where $\mathbf{W}$ is a suitable compactification of W. [1]

3.4 Moduli spaces of pseudoholomorphic curves

In this section we define the different types of moduli spaces of pseudoholomorphic curves in symplectic cobordisms and in symplectizations that appear in this thesis. Although symplectizations are a particular case of symplectic cobordisms, the fact that we only consider $\mathbb{R}$-invariant almost complex structures on symplectictizations makes this case present special properties. For this reason we will consider it separately.

3.4.1 Moduli spaces of pseudoholomorphic curves in symplectizations

We begin by recalling the notations from previous sections. Let (M, ξ) denote a contact manifold and λ a contact form on it. We denote by J an almost complex structure in $\mathcal{J}(\lambda)$.

Let $\mathfrak{P}^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+)$ and $\mathfrak{P}^- = (\gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-)$ be finite sequences of elements in $\mathcal{P}_\lambda$. [2] We then denote by $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ the moduli space whose elements are equivalence classes of genus 0 finite energy pseudoholomorphic curves without boundary, modulo biholomorphic reparametrisations of the domain, with positive punctures asymptotic to the Reeb orbits $\gamma_1^+, ..., \gamma_{n^+}^+$ of X_λ and negative interior punctures asymptotic to the Reeb orbits $\gamma_1^-, ..., \gamma_{n^-}^-$ of X_λ. In other words, every element of $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ is represented by a finite energy pseudoholomorphic curve $\widetilde{u} : (S^2 \setminus \Gamma, i_0) \to (\mathbb{R} \times M, J)$ where Γ equals the disjoint union $\Gamma^+ \cup \Gamma^-$ and:

- $\Gamma^+ = \{p_1^+, ..., p_{n^+}^+\}$ and for every $k \in \{1, ..., n^+\}$ $\widetilde{u}$ is positively asymptotic to γ_k^+ at the puncture p_k^+,
- $\Gamma^- = \{p_1^-, ..., p_{n^-}^-\}$ and for every $k \in \{1, ..., n^-\}$ $\widetilde{u}$ is positively asymptotic to γ_k^- at the puncture p_k^-.

Moreover if two pseudoholomorphic curves $\widetilde{u}_1$ and $\widetilde{u}_2$ satisfying these conditions are such that $\widetilde{u}_1 = \widetilde{u}_2 \circ \psi$ for some biholomorphism ψ from (S^2, i_0) to itself then $\widetilde{u}_1$ and $\widetilde{u}_2$

[1] $\mathbf{W}$ is in fact homeomorphic to the compactification $\overline{W}$ of W that we mentioned before, but when extending $\widetilde{u}$ the best way to picture $\overline{W}$ is to think that it is obtained from $\overline{W}$ by gluing "over it" n^+ copies of $[-\infty, +\infty] \times Y^+$ one over another, and gluing under it $-n^-$ copies of $[-\infty, +\infty] \times Y^-$ one under the other.

[2] Notice that it might happen that $\gamma_i^\pm = \gamma_j^\pm$ for some $i \neq j$.

represent the same element in $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$. It is well known (see for instance [6] and [14]) that the linearization $D\overline{\partial}_J$ of the Cauchy-Riemann operator $\overline{\partial}_J$ at any element $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ is a Fredholm map. Lastly, we denote by $\mathcal{M}^k(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ the moduli space of finite energy pseudoholomorphic curves in $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ that have Fredholm index equal to k.

In the case of moduli spaces of curves in a symplectization with a $\mathbb{R}$-invariant almost complex structure J we will need to introduce one more class of moduli spaces. The reason for this is that, because of the $\mathbb{R}$-invariance of J, one cannot expect that the moduli spaces introduced above can be compact or admit a reasonable compactification similar to the one obtained by Gromov for moduli spaces of pseudoholomorphic curves in compact symplectic manifolds. There is however a natural notion to consider: because of the $\mathbb{R}$-invariance of J there is an $\mathbb{R}$-action on the spaces $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; J)$ and $\mathcal{M}^k(c, c_1, ..., c_n, \gamma_1', ..., \gamma_m'; J, \overline{\Lambda})$. We then define $\widetilde{\mathcal{M}}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ and $\widetilde{\mathcal{M}}^k(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ as the moduli spaces obtained by quotienting $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ and $\mathcal{M}^k(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ by the mentioned $\mathbb{R}$-action.

3.4.2 Moduli spaces of pseudoholomorphic curves in exact symplectic cobordisms

We will now treat the case of exact symplectic cobordisms. We denote by (W, ϖ) an exact symplectic cobordism from λ^+ to λ^-, where λ^+ is a contact form on the contact manifold (M^+, ξ^+) and λ^- is a contact form on the contact manifold (M^-, ξ^-). On the cobordism (W, ϖ) we consider a cylindrical almost complex structure $\overline{J} \in \mathcal{J}(J^-, J^+)$ where $J^+ \in \mathcal{J}(C^+\lambda^+)$ and $J^- \in \mathcal{J}(C^-\lambda^-)$ for positive constants C^+ and C^-.

We let $\mathfrak{P}^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+)$ be a finite sequence of elements of $\mathcal{P}_{\lambda^+}$ and $\mathfrak{P}^- = (\gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-)$ be a finite sequence of elements of $\mathcal{P}_{\lambda^-}$. We will then denote by $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ the moduli space whose elements are equivalence classes of genus 0 finite energy pseudoholomorphic curves, modulo biholomorphic reparametrisations of the domain, with positive punctures asymptotic to Reeb orbit $\gamma_1^+, ..., \gamma_{n^+}^+$ of X_{λ^+} and negative interior punctures asymptotic to the Reeb orbits $\gamma_1^-, ..., \gamma_{n^-}^-$ of X_{λ^-}. In other words, every element of $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; \overline{J})$ is represented by a finite energy pseudoholomorphic curve $\widetilde{u} : (S^2 \setminus \Gamma, i_0) \to (W, \overline{J})$ where Γ equals the disjoint union $\Gamma^+ \cup \Gamma^-$ and:

- $\Gamma^+ = \{p_1^+, ..., p_{n^+}^+\}$ and for every $k \in \{1, ..., n^+\}$ $\widetilde{u}$ is positively asymptotic to γ_k^+ at the puncture p_k^+,

- $\Gamma^- = \{p_1^-, ..., p_{n^-}^-\}$ and for every $k \in \{1, ..., n^-\}$ $\widetilde{u}$ is positively asymptotic to γ_k^- at the puncture p_k^-.

Moreover if two pseudoholomorphic curves $\widetilde{u}_1$ and $\widetilde{u}_2$ satisfying these conditions are such that $\widetilde{u}_1 = \widetilde{u}_2 \circ \psi$ for some biholomorphism ψ from (S^2, i_0) to itself then $\widetilde{u}_1$ and $\widetilde{u}_2$ represent the same element in $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; \overline{J})$.

Again, it is well known that the linearization $D\overline{\partial}_{\overline{J}}$ of $\overline{\partial}_{\overline{J}}$ at any element of the moduli space $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ is a Fredholm map. Therefore it makes sense to define $\mathcal{M}^k(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ as the moduli space of finite energy pseudoholomorphic curves in $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ that have Fredholm index equal to k.

3.5 Compactification of moduli spaces of pseudoholomorphic curves

The main motivation behind the introduction of pseudoholomorphic buildings is that they are needed in order to compactify moduli spaces of pseudoholomorphic curves. The reason for that, is the fact already observed by Gromov, that sequences of pseudoholomorphic curves might not converge to a pseudoholomorphic curve, but to might converge to a pseudoholomorphic building.

The SFT-compactness theorem of [10] says that a sequence of elements of a moduli space of pseudoholomorphic curves must converge to a pseudoholomorphic building. We begin by giving the precise compactness statement which will be used in this thesis in the case of symplectizations.

Proposition 3.1. *Let λ be a contact form on a contact manifold (M, ξ) and $J \in \mathcal{J}(\lambda)$. Assume that all Reeb orbits of λ are non-degenerate. Let $\mathfrak{P}^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+)$ and $\mathfrak{P}^- = (\gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-)$ be finite sequences of elements in $\mathcal{P}_\lambda$. Then the moduli space $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; J)$ admits a compactification $\overline{\mathcal{M}}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; J)$ which is a metric space, and whose elements are elements of $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; J)$ and equivalence classes of pseudoholomorphic buildings.* [3] *Moreover for every pseudoholomorphic building $\widetilde{u}$ in $\overline{\mathcal{M}}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; J)$, the compactification $\overline{\mathbf{S}}$ of its domain $\mathbf{S}$ is diffeomorphic to the domain of the the pseudoholomorphic curves in $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; J)$ and:*

- *the top level $\widetilde{u}^1$ must have exactly n^+ positive punctures $\{z_1^+, ..., z_{n^+}^+\}$, and the curve $\widetilde{u}^1$ is asymptotic γ_k^+ at the puncture z_k^+,*

[3] Two pseudoholomorphic buildings $\widetilde{u}_1$ and $\widetilde{u}_2$ are equivalent when there exist a biholomorphism $\phi : \mathbf{S}_1 \to \mathbf{S}_2$ such that $\widetilde{u}_1 = \widetilde{u}_2 \circ \phi$. We refer the reader to [10] for the precise definition of biholomorphism of decorated nodal Riemann surfaces.

- *and the bottom level* $\widetilde{u}^l$ *must have exactly* n^- *positive punctures* $\{z_1^-, ..., z_{n^-}^-\}$*, and the curve* $\widetilde{u}^l$ *is asymptotic* γ_k^- *at the puncture* z_k^-*.*

As we will see later, in many particular cases we can obtain more restrictions on the type of buildings that can appear in the compactification of a moduli space.

We proceed now to state a similar result in the case of exact symplectic cobordisms.

Proposition 3.2. *Let* (W, ϖ) *be an exact symplectic cobordism from* λ^+ *to* λ^-*, where* λ^+ *is a contact form on the contact manifold* (M^+, ξ^+) *and* λ^- *is a contact form on the contact manifold* (M^-, ξ^-)*, and take a cylindrical almost complex structure* $\overline{J} \in \mathcal{J}(J^-, J^+)$ *where* $J^+ \in \mathcal{J}(C^+\lambda^+)$ *and* $J^- \in \mathcal{J}(C^-\lambda^-)$ *for positive constants* C^+ *and* C^-*. Let* $\mathfrak{P}^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+)$ *be a finite sequence of elements of* $\mathcal{P}_{\lambda^+}$ *and* $\mathfrak{P}^- = (\gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-)$ *be a finite sequence of elements of* $\mathcal{P}_{\lambda^-}$*. Assume that all Reeb orbits of* λ^+ *and* λ^+ *are non-degenerate. Then the moduli space* $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ *admits a compactification* $\overline{\mathcal{M}}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ *which is a metric space, and whose elements are elements of* $\mathcal{M}(\gamma_1^+, \gamma_2^+, ..., \gamma_{n^+}^+; \gamma_1^-, \gamma_2^-, ..., \gamma_{n^-}^-; \overline{J})$ *and equivalence classes of pseudoholomorphic buildings. Moreover for every pseudoholomorphic building* $\widetilde{u}$ *in* $\overline{\mathcal{M}}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$*, the compactification* $\overline{\mathbf{S}}$ *of its domain* $\mathbf{S}$ *is diffeomorphic to the domain of the the pseudoholomorphic curves in* $\mathcal{M}(\gamma_1^+, ..., \gamma_{n^+}^+; \gamma_1^-, ..., \gamma_{n^-}^-; \overline{J})$ *and:*

- *the top level* $\widetilde{u}^1$ *has exactly* n^+ *positive punctures* $\{z_1^+, ..., z_{n^+}^+\}$*, and the curve* $\widetilde{u}^1$ *is asymptotic* γ_k^+ *at the puncture* z_k^+*,*
- *and the bottom level* $\widetilde{u}^l$ *has exactly* n^- *positive punctures* $\{z_1^-, ..., z_{n^-}^-\}$*, and the curve* $\widetilde{u}^l$ *is asymptotic* γ_k^- *at the puncture* z_k^-*.*

Lastly we state a slightly different compact theorem which is valid for sequences of pseudoholomorphic curves in a splitting family of exact symplectic cobordisms.

Proposition 3.3. *Let* λ^+ *be a contact form on the contact manifold* (M, ξ^+) *and* λ^- *be a contact form on the contact manifold* (M, ξ^-) *and let* $J^+ \in \mathcal{J}(\lambda^+)$ *and* $J^- \in \mathcal{J}(\lambda^-)$*. Assume that all contractible Reeb orbits of* λ^+ *and* λ^- *are non-degenerate and that all Reeb orbits in* $\mathcal{P}_{\lambda^+}^{\rho}$ *and* $\mathcal{P}_{\lambda^-}^{\rho}$ *are non-degenerate for a fixed free homotopy class* ρ *in* M*. Let* $(\mathbb{R} \times M, \varpi_R)$*,* $R \in (0, +\infty)$*, be a splitting family of exact symplectic cobordisms, from* λ^+ *to* λ^- *along a contact form* λ *in* M*, with* $\widetilde{J}_R \in \mathcal{J}(J^-, J^+)$ *for all* $R \in (0, +\infty)$ *and coinciding with* $J \in \mathcal{J}(\lambda)$ *in the region* $W(\lambda)$*. For a sequence* $R_n \to +\infty$*, let* $\gamma_n^+ \in \mathcal{P}_{\lambda^+}^{\rho}$ *and* $\gamma_n^- \in \mathcal{P}_{\lambda^-}^{\rho}$*) be sequences of Reeb chords belonging to the same free homotopy class* ρ*, and such that there exists a constant* $\mathfrak{C}$ *with* $\mathfrak{C} > A(\gamma_n^+) > A(\gamma_n^-)$*. Let* $\widetilde{u}_n$ *be a sequence of elements in the moduli space* $\mathcal{M}(\gamma_n^+; \gamma_n^-; J_{R_n})$ *of pseudoholomorphic cylinders. Then there exists a subsequence of* $\widetilde{u}_n$ *which converges in the SFT sense to* *a pseudoholomorphic building $\widetilde{u}$. Moreover the pseudoholomorphic building $\widetilde{u}$ is such that the compactification $\overline{\mathbf{S}}$ of its domain $\mathbf{S}$ is diffeomorphic to a cylinder. There exist numbers l_λ^{min} and l_λ^{max}, such that $l > l_\lambda^{max} \geq l_\lambda^{min} > 1$ and such that the levels $\widetilde{u}^j$ for $j \in \{1, ..., l\}$ of the building $\widetilde{u}$ are finite energy pseudoholomorphic curves that satisfy:*

- *for $j \in \{l_\lambda^{min}, ..., l_\lambda^{max}\}$ the curve $\widetilde{u}^j$ is a pseudoholomorphic curve in the symplectization $(\mathbb{R} \times M, J)$ of λ,*
- *for $j = l^{min} - 1$, $\widetilde{u}^j$ is a pseudoholomorphic curve in a cobordism $(\mathbb{R} \times M, d\varsigma^+, \overline{J}^+)$ from λ^+ to λ,*
- *for $j = l^{max} + 1$, $\widetilde{u}^j$ is a pseudoholomorphic curve in a cobordism from $(\mathbb{R} \times M, d\varsigma^-, \overline{J}^-)$ from λ to λ^-,*
- *for $j < l^{min} - 1$, $\widetilde{u}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times M, J^+)$,*
- *for $j > l^{max} + 1$, $\widetilde{u}^j$ are pseudoholomorphic curves in $(\mathbb{R} \times M, J^-)$,*
- *the unique positive puncture in Γ_1^+ is asymptotic to a Reeb orbit $\gamma^+ \in \mathcal{P}_{\lambda^+}^{\rho}$ and the unique negative puncture in Γ_l^- is asymptotic to $\gamma^- \in \mathcal{P}_{\lambda^-}^{\rho}$ both in the homotopy class ρ,*
- *every level $\widetilde{u}^j$ has a special positive puncture $z_j^+ \in \Gamma_j^+$ which is asymptotic to a Reeb orbit γ_j^+ the homotopy class ρ, and a special negative puncture $z_j^- \in \Gamma_j^-$ which is asymptotic to a Reeb orbit γ_j^- the homotopy class ρ,*
- *$\gamma_{j+1}^+ = \gamma_j^-$ for all $1 \leq j \leq l - 1$,*
- *for every $1 \leq j \leq l$, at all punctures of $\widetilde{u}^j$ which are distinct from z_j^+ and z_j^- the curve $\widetilde{u}^j$ is asymptotic to contractible Reeb orbits,*
- *for all punctures in all levels of the building $\widetilde{u}$, the action of the Reeb orbit detected at these punctures by the levels of $\widetilde{u}$ is smaller then the action $A(\gamma^+)$.*

3.6 The gluing theorem

In this section we recall the gluing theorem for the special kinds of pseudoholomorphic buildings that appear in the version of cylindrical contact homology used in this thesis. The general gluing theory needed for all the SFT-invariants, such as the full contact homology, is still subject of intense research. However, in the case treated in this thesis, we do not need this machinery. The reason for that is that all the curves that appear in the construction of the cylindrical contact homology we consider and its cobordism maps are somewhere injective pseudoholomorphic curves, and in this situation the gluing theorem can be obtained using the same methods needed to prove a similar statement in Floer homology.

The gluing theorem allows us to glue the levels of a holomorphic building to obtain a pseudoholomorphic curve, and it can be seen as the reverse of SFT-compactness. This gluing is possible when the levels of the building are Fredholm regular. Like in previous sections, we will deal separately with the case where all the levels of the building are symplectizations and the case where one of the levels sits in a cobordism. We begin with the case of a symplectization.

In conformity with the previous section, we will consider a contact form λ associated to (Y, ξ). Let γ, $\check{\gamma}$ and γ' be non-degenerate Reeb orbits in $\mathcal{P}_\lambda$. Let $J \in \mathcal{J}(\lambda)$, and assume that for every element of the moduli space $\mathcal{M}^2(\gamma; \gamma'; J)$ the linerized Cauchy-Riemman operator $D\partial_J$ over this element is surjective. In this case one can use the infinite dimensional implicit function theorem to conclude that $\mathcal{M}^2(\gamma; \gamma'; J)$ is a one dimensional manifold. Let $\widetilde{u}^1 \in \mathcal{M}^2(\gamma; \gamma'; J)$ and $\widetilde{u}^2 \in \mathcal{M}^2(\gamma; \gamma'; J)$, and denote by $\widetilde{u}$ the 2-level building which has $\widetilde{u}^1$ as top level and $\widetilde{u}^2$ as bottom level. In this situation we have the following

Theorem 3.4. *Assume that the linearized Cauchy-Riemman operator is surjective at both $\widetilde{u}^1$ and $\widetilde{u}^2$. Then, there exists an embedding $\Psi : [0, +\infty) \to \overline{\mathcal{M}}^2(\gamma; \gamma'; J)$ such that:*

- $\Psi(0) = \widetilde{u}$,
- $\Psi(t) \in \mathcal{M}^2(\gamma; \gamma'; J)$ *for every* $t \in (0, +\infty)$
- *the map Ψ is a homeomorphism from $[0, +\infty)$ to a neighbourhood of $\widetilde{u}$ in $\overline{\mathcal{M}}^2(\gamma; \gamma'; J)$.*

Moreover, if $\widetilde{u}_n$ is a sequence of elements of $\mathcal{M}^2(\gamma; \gamma'; J)$ converging to $\widetilde{u}$, then there exists n_0 such that $\widetilde{u}_n \in \Psi([0, 1])$ for all $n \geq n_0$.

In words, the gluing theorem says that provided the levels of the building $\widetilde{w}$ are regular, then $\widetilde{w}$ is in the boundary of $\overline{\mathcal{M}}^2(\gamma; \gamma'; J)$.

We now proceed to state a version of the gluing theorem for buildings involving a cobordism. We let λ^+ and λ^- be contact forms on a contact manifold (Y, ξ). We take γ^+ and $\check{\gamma}^+$ to be non-degenerate Reeb orbits in $\mathcal{P}_{\lambda^+}$, and γ^- and $\check{\gamma}^-$ to be non-degenerate Reeb orbits in $\mathcal{P}_{\lambda^-}$. Let $(\mathbb{R} \times M, \varsigma)$ be an exact sympletic cobordism from λ^+ to λ^- and $\overline{J}$ be a cylindrical almost complex structure in $\mathcal{J}(J^-, J^+)$ where $J^+ \in \mathcal{J}(\lambda^+)$ and $J^- \in \mathcal{J}(\lambda^-)$. We assume that all elements of $\mathcal{M}^1(\gamma^+; \gamma^-; \overline{J})$ are Fredholm regular; i.e, the linerized Cauchy-Riemman operator $D\partial_{\overline{J}}$ is surjective at all elements of $\mathcal{M}^1(\gamma^+; \gamma^-; \overline{J})$.

Let $\widetilde{u}^1_+ \in \widetilde{\mathcal{M}}^1(\gamma^+;\check{\gamma}^+;J^+)$ and $\widetilde{u}^2_+ \in \mathcal{M}^0(\check{\gamma}^+;\gamma^-;\overline{J})$, $\widetilde{u}^1_- \in \widetilde{\mathcal{M}}^0(\gamma^+;\check{\gamma}^-;\overline{J})$ and $\widetilde{u}^2_- \in \widetilde{\mathcal{M}}^1(\check{\gamma}^-;\gamma^-;J^-)$.

Theorem 3.5. *Assume that the linearized Cauchy-Riemman operator is surjective at both $\widetilde{u}^1_+$ and $\widetilde{u}^2_+$. Then, there exists an embedding $\Psi^+ : [0,+\infty) \to \overline{\mathcal{M}}^1(\gamma^+;\gamma^-;\overline{J})$ such that:*

- *$\Psi^+(0) = \widetilde{u}_+$, where $\widetilde{u}_+$ is the two level building whose top level is $\widetilde{u}^1_+$ and bottom level is $\widetilde{u}^2_+$,*
- *$\Psi^+(t) \in \mathcal{M}^1(\gamma^+;\gamma^-;\overline{J})$ for every $t \in (0,+\infty)$,*
- *the map Ψ^+ is a homeomorphism from $[0,+\infty)$ to a neighbourhood of $\widetilde{u}_+$ in $\overline{\mathcal{M}}^1(\gamma^+;\gamma^-;\overline{J})$.*

Moreover, if $\widetilde{u}_+(n)$ is a sequence of elements of $\mathcal{M}^1(\gamma^+;\gamma^-;\overline{J})$ converging to $\widetilde{u}_+$, then there exists n_0 such that $\widetilde{u}_+(n) \in \Psi^+([0,1])$ for all $n \geq n_0$.

Analogously, if the linearized Cauchy-Riemman operator is surjective at both $\widetilde{u}^1_-$ and $\widetilde{u}^2_-$, then there exists an embedding $\Psi^- : [0,+\infty) \to \overline{\mathcal{M}}^1(\gamma^+;\gamma^-;\overline{J})$ such that:

- *$\Psi^-(0) = \widetilde{u}_-$, where $\widetilde{u}_-$ is the two level building whose top level is $\widetilde{u}^1_-$ and bottom level is $\widetilde{u}^2_-$,*
- *$\Psi^-(t) \in \mathcal{M}^1(\gamma^+,\gamma^-;\overline{J},L)$ for every $t \in (0,+\infty)$,*
- *the map Ψ^- is a homeomorphism from $[0,+\infty)$ to a neighbourhood of $\widetilde{u}_-$ in $\overline{\mathcal{M}}^1(\gamma^+,\gamma^-;\overline{J})$.*

Moreover, if $\widetilde{u}_-(n)$ is a sequence of elements of $\mathcal{M}^1(\gamma^+;\gamma^-;\overline{J})$ converging to $\widetilde{u}_-$, then there exists n_0 such that $\widetilde{u}_-(n) \in \Psi^-([0,1])$ for all $n \geq n_0$.

Chapter 4

Contact homology

Contact homologies were introduced in [15] as homology theories which are topological invariants of contact manifolds. In sections 4.1 and 4.2 we give an introduction to the more basic and well known versions of contact homologies. This serves mainly as a motivation to section 4.3, in which we define the version of contact homology that will be used in this thesis. [1]

4.1 Full contact homology

Full contact homology was introduced in [15] as an important invariant of contact structures. We refer the reader to [15] and [8] for detailed presentations of the material contained in this section.

Let $(Y^{2\mathrm{n}+1}, \xi)$ be a contact manifold, λ be a non-degenerate contact form on (Y, ξ) and $J \in \mathcal{J}(\lambda)$. Recall that $\mathcal{P}_\lambda$ is the set of good periodic orbits of the Reeb vector field X_λ. To each orbit $\gamma \in \mathcal{P}_\lambda$, we define a $\mathbb{Z}_2$-grading $\mid \gamma \mid = (\mu_{CZ}(\gamma) + (\mathrm{n}-2)) \mod 2$, where μ_{CZ} is the Conley-Zehnder index. An orbit γ is called good if it is either simple, or if $\gamma = (\gamma')^i$ for a simple orbit γ' with the same grading of γ.

$\mathfrak{A}(Y, \lambda)$ is defined to be the supercomutative, $\mathbb{Z}_2$ graded, $\mathbb{Q}$ algebra with unit generated by $\mathcal{P}_\lambda$ (an algebra with this properties is sometimes referred in the literature as a commutative super-algebra or a super-ring). The $\mathbb{Z}_2$-grading on the elements of the algebra is obtained by considering on the generators the grading mentioned above and extending it to $\mathfrak{A}(Y, \lambda)$.

[1] We stress that while the versions of contact homology presented in sections 4.1 and 4.2 do depend on the Polyfolds technology currently being developed by Hofer, Wysocki and Zehnder, the version of contact homology which we use in this paper and is presented in section 4.3 **does not** depend on Polyfolds and can be constructed in complete rigor with technology that is available in the literature. See the detailed discussion in section 4.3 below.

$\mathfrak{A}(Y,\lambda)$ can be equipped with a differential d_J. This differential will be defined by counting solutions of a certain perturbation of the Cauchy-Riemann equation. To define our differential we need the following hypothesis:

Hypothesis H: there exists an abstract perturbation of the Cauchy-Riemann operator ∂_J such that the compactified moduli spaces $\overline{\mathcal{M}}(\gamma;\gamma'_1,...,\gamma'_m;J)$ of solutions of the perturbed equation are unions of branched manifolds with corners and rational weights whose dimension is given by the Conley-Zehnder index of the asymptotic orbits and the relative homology class of the solution.

The proof that Hypothesis H is true is still not written. Establishing its validity is one of the main reasons for the development of the Polyfold technology by Hofer, Wysocki and Zehnder. We define:

$$d_J(\gamma) = m(\gamma)\sum_{\gamma'_1,...,\gamma'_m}\frac{C(\gamma;\gamma'_1,...,\gamma'_m)}{m!}\gamma'_1\gamma'_2...\gamma'_m \tag{4.1}$$

where $C(\gamma,\gamma'_1,...,\gamma'_m)$ is the algebraic count of points in the 0-dimensional manifold

$$\mathcal{M}^1(\gamma;\gamma'_1,...,\gamma'_2;J) \tag{4.2}$$

and $m(\gamma)$ is the multiplicity of γ. d_J is extended to the whole algebra by the Leibnitz rule. Under hypothesis **H** it was proved in [15] that $(d_J)^2=0$. We have therefore that $(\mathfrak{A}(Y,\lambda),d_J)$ is a differential $\mathbb{Z}_2$ graded super-commutative algebra. We define:

Definition 4.1. The **full contact homology** $C\mathbb{H}(\lambda,J)$ of λ is the homology of the complex $(\mathfrak{A},d_J)$.

Under Hypotesis H, it was also proved in [15] that the full contact homology does not depend on the contact form λ on (Y,ξ) nor on the almost complex structure $J\in\mathcal{J}(\lambda)$.

4.2 Cylindrical contact homology

Suppose now that (Y,ξ) is a contact manifold, λ is a non-degenerate hypertight contact form. Fix a cylindrical almost complex structure $J\in\mathcal{J}(\lambda)$. For hypertight contact manifolds we can define a simpler version of contact homology called cylindrical contact homology. We denote by $CH_{cyl}(\lambda)$ the $\mathbb{Z}_2$-graded $\mathbb{Q}$-vector space generated by the elements of $\mathcal{P}_\lambda$. The differential $d_J^{cyl}: CH_{cyl}(\lambda)\to CH_{cyl}(\lambda)$ will count elements in the moduli space $\mathcal{M}^1(\gamma;\gamma';J)$. For the generators $\gamma \in \mathcal{P}_\lambda$ we define

$$d_J^{cyl}(\gamma) = cov(\gamma) \sum_{\gamma' \in \mathcal{P}_\lambda} C(\gamma;\gamma';J)\gamma' \tag{4.3}$$

where $C(\gamma;\gamma';J)$ is the algebraic count of elements in $\mathcal{M}^1(\gamma;\gamma';J)$, and $cov(\gamma)$ is the covering number of γ. For λ hypertight and assuming Hypothesis H is true, Eliashberg, Givental and Hofer proved in [15] that $(d_J^{cyl})^2 = 0$.

Definition 4.2. The **cylindrical contact homology** $C\mathbb{H}_{cyl}(\lambda)$ of λ is the homology of the complex $(CH_{cyl}(\lambda), d_J^{cyl})$.

Under Hypothesis H, the cylindrical contact homology doesn't depend on the hypertight contact form λ on (Y,ξ) nor on the cylindrical almost complex structure $J \in \mathcal{J}(\lambda)$.

Denote by Λ the set of free homotopy classes of Y. It is easy to see that for each $\rho \in \Lambda$ the subspace $CH^\rho_{cyl}(\lambda) \subset CH_{cyl}(\lambda)$ generated by the set $\mathcal{P}^\rho_\lambda$ of good periodic orbits in ρ is a subcomplex of $(CH_{cyl}(\lambda), d_J^{cyl})$. This follows from the fact that the numbers $C(\gamma;\gamma';J)$, that appear on the differential of γ, can only be non-zero for orbits γ' that are freely homotopic to γ, which implies that the restriction $d_J^{cyl} \mid_{CH^\rho_{cyl}(\lambda)}$ has image in $CH^\rho_{cyl}(\lambda)$. From now on we will denote the restriction $d_J^{cyl} \mid_{CH^\rho_{cyl}}: CH^\rho_{cyl}(\lambda) \to CH^\rho_{cyl}(\lambda)$ by d^ρ_J. Denoting by $C\mathbb{H}^\rho_{cyl}$ the homology of $(CH^\rho_{cyl}(\lambda), d^\rho_J)$ we thus have the following formula:

$$C\mathbb{H}_{cyl}(\lambda) = \bigoplus_{\rho \in \Lambda} C\mathbb{H}^\rho_{cyl}. \tag{4.4}$$

The fact that we can define partial versions of cylindrical contact homology restricted to certain free homotopy classes will be of crucial importance for us. It will allow us to obtain our results without resorting to Hypothesis H. This is explained in the next section.

4.3 Cylindrical contact homology in special homotopy classes

Maintaining the notation of the previous sections we denote by (Y,ξ) a contact manifold endowed with a hypertight contact form λ.

Let Λ_0 denote the set of primitive free homotopy classes of Y. Let $\rho \in \Lambda$ be either an element of Λ_0, or a free homotopy class which contains only simple Reeb orbits of λ. Assume that all Reeb orbits in $\mathcal{P}^\rho_\lambda$ are non-degenerate. By the work of Dragnev [14], we know that there exists a generic subset $\mathcal{J}^{\rho}_{reg}(\lambda)$ of $\mathcal{J}(\lambda)$ such that for all $J \in \mathcal{J}_{reg}(\lambda)$ we have:

- for all Reeb orbits $\gamma_1, \gamma_2 \in \rho$, the moduli space of pseudoholomorphic cylinders $\mathcal{M}(\gamma_1, \gamma_2; J)$ is transverse, i.e. the linearized Cauchy-Riemann operator $D\partial_J(\widetilde{w})$ is surjective for all $\widetilde{w} \in \mathcal{M}(\gamma_1, \gamma_2; J)$;
- for all Reeb orbits $\gamma_1, \gamma_2 \in \rho$, each connected component $\mathcal{L}$ of the moduli space $\mathcal{M}(\gamma_1, \gamma_2; J)$ is a manifold whose dimension is given by the Fredholm index of any element $\widetilde{w} \in \mathcal{L}$.

In this case, for $J \in \mathcal{J}_{reg}(\lambda)$, we define

$$d^{\rho}_J(\gamma) = cov(\gamma) \sum_{\gamma' \in \mathcal{P}^{\rho}_{\lambda}} C^{\rho}(\gamma; \gamma'; J)\gamma' = \sum_{\gamma' \in \mathcal{P}^{\rho}_{\lambda}} C^{\rho}(\gamma; \gamma'; J)\gamma', \tag{4.5}$$

where $C^{\rho}(\gamma; \gamma'; J)$ is the algebraic count of points on the moduli space $\mathcal{M}^1(\gamma; \gamma'; J)$. The second equality follows from the fact that all Reeb orbits in ρ are simple, which implies $cov(\gamma) = 1$.

For λ and ρ as above and $J \in \mathcal{J}^{\rho}_{reg}(\lambda)$, the differential $d^{\rho}_J : CH^{\rho}_{cyl}(\lambda) \to CH^{\rho}_{cyl}(\lambda)$ is well-defined and satisfies $(d^{\rho}_J)^2 = 0$. Therefore, in this situation, we can define the cylindrical contact homology $C\mathbb{H}^{\rho,J}_{cyl}(\lambda)$ without the need of Hypothesis H. Once the transversality for J has been achieved, and using coherent orientations constructed in [11], the proof that d^{ρ}_J is well-defined and that $(d^{\rho}_J)^2 = 0$ is a combination of compactness and gluing, similar to the proof of the analogous result for Floer homology. For the convenience of the reader we sketch these arguments below:

For ρ as above, $d^{\rho}_J : CH^{\rho}_{cyl}(\lambda) \to CH^{\rho}_{cyl}(\lambda)$ is well-defined, and for every $\gamma \in \mathcal{P}^{\rho}_{\lambda}$ the differential $d^{\rho}_J(\gamma)$ is a finite sum.

The moduli space $\mathcal{M}^1(\gamma; \gamma'; J)$ can be non-empty only if $A(\gamma') \leq A(\gamma)$. It then follows from the non-degeneracy of λ that for a fixed γ the numbers $C^{cyl}(\gamma, \gamma'; J)$ can be nonzero for only finitely many γ'. To see that $C^{cyl}(\gamma, \gamma'; J^)$ is finite for every $\gamma' \in \rho$ suppose by contradiction that there is a sequence $\widetilde{w}_i$ of distinct elements of $\mathcal{M}^1(\gamma; \gamma'; J)$. By the SFT compactness theorem [10] such a sequence has a convergent subsequence that converges to a pseudoholomorphic building $\widetilde{w}$ which has Fredholm index 1. Because of the hypertightness of λ, no bubbling can occur and all the levels $\widetilde{w}^1, ..., \widetilde{w}^k$ of the building $\widetilde{w}$ are pseudoholomorphic cylinders. As all Reeb orbits of λ in ρ are simple, it follows that all these cylinders are somewhere injective pseudoholomorphic curves, and the regularity of J implies that they must all have Fredholm index ≥ 1. As a result we have $1 = I_F(\widetilde{w}) = \sum(I_F(\widetilde{w}^l)) \geq k$, which implies $k = 1$. Thus $\widetilde{w} \in \mathcal{M}^1(\gamma;\gamma';J)$ and is the limit of a sequence of distinct elements of $\mathcal{M}^1(\gamma;\gamma';J)$. This is absurd because $\mathcal{M}^1(\gamma;\gamma';J)$ is a 0-dimensional manifold. We thus conclude that the numbers $C^{cyl}(\gamma,\gamma';J^)$ are all finite. □

For ρ as above, $(d_J^\rho)^2 = 0$. If we write

$$d_J^\rho \circ d_J^\rho(\gamma) = \sum_{\gamma'' \in \mathcal{P}_\lambda^\rho} m_{\gamma,\gamma''}\gamma'', \tag{4.6}$$

we know that $m_{\gamma,\gamma'}$ is the number of two-level pseudo holomorphic buildings $\widetilde{w} = (\widetilde{w^1}, \widetilde{w^2})$ such that $\widetilde{w^1} \in \mathcal{M}^1(\gamma;\gamma';J)$ and $\widetilde{w^2} \in \mathcal{M}^1(\gamma';\gamma'';J)$, for some $\gamma' \in \mathcal{P}_\lambda^\rho$. Because of transversality of $\widetilde{w^1}$ and $\widetilde{w^2}$ we can perform gluing. This implies that $\widetilde{w}$ is in the boundary of the moduli space $\overline{\mathcal{M}}^2(\gamma;\gamma'';J)$. Taking a sequence $\widetilde{w}_i$ of elements in $\mathcal{M}^2(\gamma;\gamma'';J)$ converging to the boundary of $\overline{\mathcal{M}}^2(\gamma;\gamma'';J)$ and arguing similarly as above, we have that this sequence converges to a pseudoholomorphic building $\widetilde{w}_\infty$, whose levels are somewhere injective pseudoholomorphic cylinders. Using that $I_F(\widetilde{w}_\infty) = 2$ we obtain that $\widetilde{w}_\infty$ can have at most 2 levels. As $\widetilde{w}_\infty$ is in the boundary of $\overline{\mathcal{M}}^2(\gamma;\gamma'';J)$ it cannot have only one level, and is therefore a two-level pseudo holomorphic building whose levels have Fredholm index 1. Summing up, $\widetilde{w}_\infty = (\widetilde{w}^1_\infty, \widetilde{w}^2_\infty)$, where $\widetilde{w}^1_\infty \in \mathcal{M}^1(\gamma;\gamma';J)$ and $\widetilde{w}^2_\infty \in \widehat{\mathcal{M}}^1(\gamma';\gamma'';J)$, for some $\gamma' \in \mathcal{P}_\lambda^\rho$.

The discussion above implies that $m_{\gamma,\gamma''}$ is the count with signs of boundary components of the compactified moduli space $\overline{\mathcal{M}}^2(\gamma;\gamma'';J)$ which is homeomorphic to a one-dimensional manifold with boundary. Because the signs of this count are determined by coherent orientations of $\overline{\mathcal{M}}^2(\gamma,\gamma'';J)$, it follows that $m_{\gamma,\gamma''} = 0$. □

The discussion above gives us the following

Proposition 4.3. *Let (Y,ξ) be a contact manifold with a hypertight contact form λ. Let $\rho \in \Lambda$ be either an element of Λ_0, or a free homotopy class which contains only simple Reeb orbits of λ. Assume that all Reeb orbits in $\mathcal{P}_\lambda^\rho$ are non-degenerate and pick $J \in \mathcal{J}_{reg}^\rho(\lambda)$. Then, d_J^ρ is well defined and $(d_J^\rho)^2 = 0$.*

Exact symplectic cobordisms induce homology maps for the SFT-invariants. We describe how this is done for the version of cylindrical contact homology considered in this section. Let (Y^+,ξ^+) and (Y^-,ξ^-) be contact manifolds, with hypertight contact forms λ^+ and λ^-. Let (W,ω) be an exact symplectic cobordism from λ^+ to λ^-. Assume that ρ is either a primitive free homotopy class or that all the closed Reeb orbits of both

λ^+ and λ^- which belong to ρ are simple. Assume moreover that all Reeb orbits of both $\mathcal{P}^\rho_{\lambda^+}$ and $\mathcal{P}^\rho_{\lambda^-}$ are non-degenerate. Choose almost complex structures $J^+ \in \mathcal{J}^\rho_{reg}(\lambda^+)$ and $J^- \in \mathcal{J}^\rho_{reg}(\lambda^-)$. From the work of Dragnev [14] (see also section 2.3 in [37]) we know that there is a generic subset $\mathcal{J}^\rho_{reg}(J^-, J^+) \in \mathcal{J}(J^-, J^+)$ such that for $\widehat{J} \in \mathcal{J}^\rho_{reg}(J^-, J^+)$, $\gamma^+ \in \mathcal{P}^\rho_{\lambda^+}$ and $\gamma^- \in \mathcal{P}^\rho_{\lambda^-}$

- all the curves $\widetilde{w}$ in the moduli spaces $\mathcal{M}(\gamma^+; \gamma^-; \widehat{J})$ are Fredholm regular,
- the connected components $\mathcal{V}$ of $\mathcal{M}(\gamma^+; \gamma^-; \widehat{J})$ have dimension equal to the Fredholm index of any pseudoholomorphic curve in $\mathcal{V}$.

In this case we can define a map $\Phi^{\widehat{J}} : CH^\rho_{cyl}(\lambda^+) \to CH^\rho_{cyl}(\lambda^-)$ given on elements of $\mathcal{P}^\rho_{\lambda^+}$ by

$$\Phi^{\widehat{J}}(\gamma^+) = \sum_{\gamma^- \in \mathcal{P}^\rho_{\lambda^-}} n_{\gamma^+,\gamma^-}\gamma^-, \tag{4.7}$$

where n_{γ^+,γ^-} is the number pseudoholomorphic cylinders with Fredholm index 0, positively asymptotic to γ^+ and negatively asymptotic to γ^-. Using a combination of compactness and gluing (see [8]) one proves that $\Phi^{\widehat{J}} \circ d^\rho_{J^+} = d^\rho_{J^-} \circ \Phi^{\widehat{J}}$. As a result we obtain a map $\Phi^{\widehat{J}} : C\mathbb{H}^{\rho,J^+}_{cyl}(\lambda^+) \to C\mathbb{H}^{\rho,J^-}_{cyl}(\lambda^-)$ on the homology level.

We study the cobordism map in the following situation: take $(V = \mathbb{R} \times Y, \varpi)$ to be an exact symplectic cobordism from $C\lambda$ to $c\lambda$ where $C > c > 0$, and λ is a hypertight contact form. Suppose that one can make an isotopy of exact symplectic cobordisms $(\mathbb{R} \times Y, \varpi_t)$ from $C\lambda$ to $c\lambda$, with ϖ_t satisfying $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\lambda_0)$. We consider the space $\widetilde{\mathcal{J}}(J, J)$ of smooth homotopies

$$t \in [0, 1]; J_t \in \mathcal{J}(J, J) \tag{4.8}$$

such that $J_0 = J_V$, $J_1 \in \mathcal{J}_{reg}(\lambda)$, and J_t is compatible with ϖ_t for every $t \in [0, 1]$. J_t is a deformation of J_0 to J_1, through asymptotically cylindrical almost complex structures in the cobordisms $(\mathbb{R} \times Y, \varpi_t)$. For Reeb orbits $\gamma, \gamma' \in \mathcal{P}^\rho_\lambda$ we consider the moduli space

$$\widetilde{\mathcal{M}}^1(\gamma; \gamma'; J_t) = \{(t, \widetilde{w}) \mid t \in [0, 1] \text{ and } \widetilde{w} \in \widehat{\mathcal{M}}^1(\gamma; \gamma'; J_t)\}. \tag{4.9}$$

By using the techniques of [14], we know that there is a generic subset $\widetilde{\mathcal{J}}_{reg}(J, J) = \widetilde{\mathcal{J}}(J, J)$ such that $\widetilde{\mathcal{M}}^1(\gamma, \gamma'; J_t)$ is a 1-dimensional smooth manifold with boundary. The crucial condition that makes this valid is again the fact that the all the pseudoholomorphic curves that make part of this moduli space are somewhere injective.

We have the following proposition which is a consequence of the combination of work of Eliashberg, Givental and Hofer [15] and Dragnev [14].

Proposition 4.4. *Let (Y, ξ) be a contact manifold with a hypertight contact form λ. Let $\lambda^+ = C\lambda$ and $\lambda^- = c\lambda$ where $C > c > 0$ are constants, and ρ be either a primitive free homotopy class or a free homotopy class in which all Reeb orbits of λ are simple. Assume that all Reeb orbits in $\mathcal{P}^{\rho}_{\lambda}$ are non-degenerate. Choose an almost complex structure $J \in \mathcal{J}^{\rho}_{reg}(\lambda)$, and set $J^+ = J^- = J$. Let $(W = \mathbb{R}\times Y, \varpi)$ be an exact symplectic cobordism from $C\lambda$ to $c\lambda$, and choose a regular almost complex structure $\widehat{J} \in \mathcal{J}^{\rho}_{reg}(J^-, J^+)$. Then, if there is an homotopy $(\mathbb{R} \times Y, \varpi_t)$ of exact symplectic cobordisms from $C\lambda$ to $c\lambda$, with $\varpi_0 = \varpi$ and $\varpi_1 = d(e^s\lambda)$, it follows that the map $\Phi^{\widehat{J}} : CH^{\rho,J}_{cyl}(\lambda) \to CH^{\rho,J}_{cyl}(\lambda)$ is chain homotopic to the identity.*

The proof is again a combination of compactness and gluing, and we sketch it below. We refer the reader to [8] and [15] for the details.

Sketch of the proof: We define initially the following map

$$K : CH^{\rho}_{cyl}(\lambda) \to CH^{\rho}_{cyl}(\lambda) \tag{4.10}$$

that counts finite energy, Fredholm index -1 pseudoholomorphic cylinders in the cobordisms $(\mathbb{R} \times Y, \varpi_t)$ for $t \in [0, 1]$. Because of the regularity of our homotopy, the moduli space of index -1 cylinders whose positive puncture detects a fixed Reeb orbit γ is finite, and therefore the map K is well defined.

Notice that for $t = 1$ the cobordism map $\Phi^{\widehat{J_1}}$ is the identity, and the pseudoholomorphic curves that define it are just trivial cylinders over Reeb orbits. For $t = 0$, $\Phi^{\widehat{J_0}}$ counts index 0 cylinders in the cobordisms $(\mathbb{R} \times Y, \varpi)$. From the regularity of J_0, J_1 and the homotopy J_t, we have that the pseudoholomorphic cylinders involved in these two maps belong to the 1-dimensional moduli spaces $\widetilde{\mathcal{M}}^1(\gamma; \gamma'; J_t)$.

By using a combination of compactness and gluing we can show that the boundary of the moduli space $\widetilde{\mathcal{M}}^1(\gamma; \gamma'; J_t)$ is exactly the set of pseudoholomorphic buildings $\widetilde{w}$ with two levels $\widetilde{w}_{cob}$ and $\widetilde{w}_{symp}$ such that: $\widetilde{w}_{cob}$ is an index -1 cylinder in a cobordism $(\mathbb{R} \times Y, \varpi_t)$ and $\widetilde{w}_{symp}$ is an index 1 pseudoholomorphic cylinder in the symplectization of λ above or below $\widetilde{w}_{cob}$. Such two level buildings are exactly the ones counted in the map $K \circ d^{cyl}_J + d^{cyl}_J \circ K$. As a consequence one has that the difference between the maps $\Phi^{\widehat{J_1}} = Id$ and $\Phi^{\widehat{J}}$ is equal to $K \circ d^{cyl}_J + d^{cyl}_J \circ K$. This implies that $\Phi^{\widehat{J}}$ is chain homotopic to the identity. □

The result above can be used to show that $C\mathbb{H}^{\rho,J}_{cyl}(\lambda)$ does not depend on the regular almost complex structure J used to define the differential d_J.

Chapter 5

Homotopical growth of the number of periodic orbits and topological entropy

Throughout this chapter M will denote a compact manifold. We endow M with an auxiliary Riemannian metric g, which induces a distance function d_g on M, whose injective radius we denote by ϵ_g. Let $\widetilde{M}$ be the universal cover of M, $\widetilde{g}$ be the Riemannian metric that makes the covering map $\pi : \widetilde{M} \to M$ an isometry, and $d_{\widetilde{g}}$ be the distance induced by the metric $\widetilde{g}$.

Let X be a vector field on M with no singularities and ϕ_X^t the flow generated by X. We call $P^X(T)$ the number of periodic orbits of ϕ^t with period in $[0,T]$. For us a periodic orbit of X is a pair $([\gamma]_c, T)$ where $[\gamma]_c$ is the set of parametrizations of a given *immersed* curve $c : S^1 \to M$, and T is a positive real number (called the period of the orbit) such that:

- $\gamma \in [\gamma]_c \iff \gamma : \mathbb{R} \to M$ parametrizes c and $\dot{\gamma}(t) = X(\gamma(t))$
- for all $\gamma \in [\gamma]_c$ we have $\gamma(T+t) = \gamma(t)$ and $\gamma([0,T]) = c$

We say that a periodic orbit $([\gamma]_c, T)$ is in a free homotopy class l of M if $c \in l$.

By a parametrized periodic orbit (γ, T) we mean a periodic orbit $([\gamma]_c, T)$ with a fixed choice of parametrization $\gamma \in [\gamma]_c$. A parametrized periodic orbit (γ, T) is said to be in a free homotopy class l when the underlying periodic orbit $([\gamma]_c, T)$ is in l.

From the work of Kaloshin and others it is well known that the exponential growth rate of periodic orbits $\limsup_{T\to+\infty} \frac{\log(P^X(T))}{T}$ can be much bigger than the topological entropy. This implies that the growth rate $\limsup_{T\to+\infty} \frac{\log(P^X(T))}{T}$ does not give a lower bound for the topological entropy of an arbitrary flow. There is however a different growth rate, which measures how quickly periodic orbits appear in different free homotopy classes, and which can be used to give such a lower bound of the topological entropy of a flow.

Let Λ denote the set of free homotopy classes of loops in M, and $\Lambda_0 \subset \Lambda$ the subset of primitive free homotopy classes. Denote by $\Lambda_X^T \subset \Lambda$ the set of free homotopy classes ϱ such that there exists a periodic orbit of ϕ_X^t with period smaller or equal to T which is homotopic to ϱ. We denote by $N_X(T)$ the cardinality of Λ_X^T.

Let $\{(\gamma_i, T_i); 1 \leq i \leq n\}$ be a finite set of parametrized periodic orbits of X. For a number T satisfying $T \geq T_i$ for all $i \in \{1, ..., n\}$ and a constant $\delta > 0$, we denote by $\Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_n, T_n))$ the subset of Λ such that:

- $l \in \Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_n, T_n))$ if, and only if, there exist a parametrized periodic orbit $(\widehat{\gamma}, \widehat{T})$ with period $\widehat{T} \leq T$ in the free homotopy class l and a number $i_l \in \{1, ..., n\}$ for which $\max_{t\in[0,T]}(d_g(\gamma_{i_l}(t), \widehat{\gamma}(t))) \leq \delta$.

We observe that

$$\Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_n, T_n)) = \bigcup_{i\in\{1,...,n\}} \Lambda_X^{T,\delta}((\gamma_i, T_i)). \tag{5.1}$$

We are ready to prove the main result in this section. Theorem 4.1 below is well known to be true in the particular cases where ϕ_X is a geodesic flow, where it follows from Manning's inequality (see [32] and [38]); and where ϕ_X is the suspension of surface diffeomorphism with pseudo-Anosov monodromy, where it follows from Ivanov's theorem (see [31]). It can be seen as a generalization of these results in the sense that it includes them as particular cases and that it applies to many other situations. Our argument is inspired by the remarkable proof of Ivanov's inequality given by Jiang in ([31]).

Theorem 5.1. *If for real numbers $a > 0$ and b we have $N_X(T) \geq e^{aT+b}$, then $h_{top}(\phi_X) \geq a$.*

Proof: The theorem will follow if we prove that for all $\delta < \frac{\epsilon_g}{10^6}$ we have $h_\delta(\phi_X) \geq a$. From now on fix $0 < \delta < \frac{\epsilon_g}{10^6}$.

Step 1: For any point $p \in M$ let $V_{4\delta}(p)$ be the 4δ-neighbourhood of $\pi^{-1}(p)$. Because $\delta < \frac{\epsilon_g}{10^6}$, it is clear that $V_{4\delta}(p)$ is the disjoint union

$$V_{4\delta}(p) = \bigcup_{\widetilde{p} \in \pi^{-1}(p)} B_{4\delta}(\widetilde{p}), \tag{5.2}$$

where the ball $B_{4\delta}(\widetilde{p})$ is taken with respect to the metric $\widetilde{g}$.

Because of our choice of $\delta < \frac{\epsilon_g}{10^6}$ it is clear that there exists a constant $0 < k_1$ which does not depend on p, such that if B and B' are two distinct connected components of $V_{4\delta}(p)$ we have $d_{\widetilde{g}}(B, B') > k_1$.

Because of compactness of M, we know that the vector field $\widetilde{X} := \pi^* X$ is bounded in the norm given by the metric $\widetilde{g}$. Combining this with the inequality in the last paragraph, one obtains the existence of a constant $0 < k_2$, which again doesn't depend on p such that, if $\widetilde{v} : [0, R] \to \widetilde{M}$ is a parametrized trajectory of $\phi_{\widetilde{X}}$ such that $\widetilde{v}(0) \in B$ and $\widetilde{v}(R) \in B'$ for $B \neq B'$ are connected components of $V_{4\delta}(p)$ then $R > k_2$.

From the last assertion we deduce the existence of a constant $\widetilde{K}$, depending only g and X, such that for every $p \in M$ and every parametrized trajectory $\widetilde{v} : [0, T] \to \widetilde{M}$ of $\phi_{\widetilde{X}}$, the number $L^T(p, \widetilde{v})$ of distinct connected components of $V_{4\delta}(p)$, intersected by the curve $\widetilde{v}([0, T])$ satisfies

$$L^T(p, \widetilde{v}) < \widetilde{K}T + 1. \tag{5.3}$$

Step 2: We claim that for every parametrized periodic orbit (γ', T') of X we have

$$\sharp(\Lambda_X^{T,\delta}((\gamma', T')) < \widetilde{K}T + 1 \tag{5.4}$$

for all $T > T'$.

To see this take $\widetilde{\gamma}'$ be a lift of γ' and let $p' = \gamma'(0)$ and $\widetilde{p}' = \widetilde{\gamma}'(0)$. We consider the set $\{B_j ; 1 \leq j \leq \mathfrak{m}^T(\gamma', T')\}$ of connected components of $V_{4\delta}(p')$ satisfying:

- $B_j \neq B_k$ if $j \neq k$,
- if B is a connected component of $V_{4\delta}(p')$ which intersects $\widetilde{\gamma}'([0, T])$ then $B = B_j$ for some $j \in \{1, ..., \mathfrak{m}^T((\gamma', T'))\}$,
- if $j < i$ then B_j is visited by the trajectory $\widetilde{\gamma}' : [0, T] \to \widetilde{M}$ before B_i.

From step 1, we know that $\mathfrak{m}^T(\gamma', T') < \widetilde{K}T + 1$.

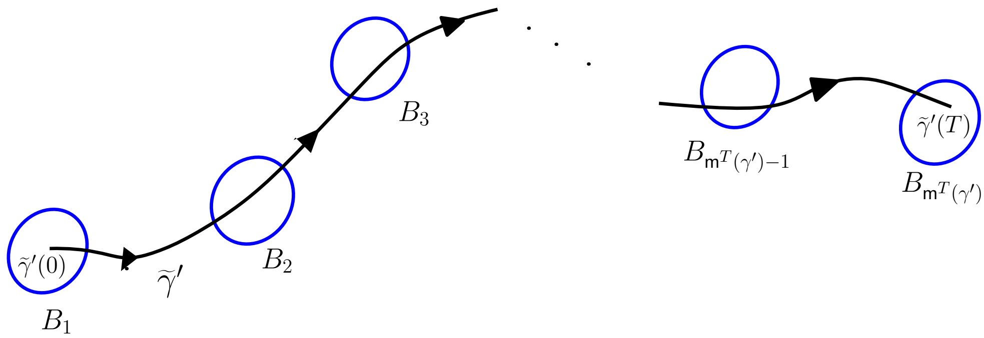

Figure 1: The set $\{B_j; 1 \leq j \leq \mathfrak{m}^T(\gamma', T')\}$.

For each $l \in \Lambda_X^{T,\delta}((\gamma', T'))$ pick (χ_l, T_l) in l to be a parametrized periodic orbit which satisfies $d_g(\chi_l(t), \gamma'(t)) < \delta$ for all $t \in [0, T]$. There exists a lift $\widetilde{\chi_l}$ of χ_l satisfying $d_{\widetilde{g}}(\widetilde{\chi_l}(t), \widetilde{\gamma'}(t)) < \delta$ for all $t \in [0, T]$.

From the triangle inequality it is clear that the point $q_l = \widetilde{\chi_l}(0)$ is in the connected component B_1 which contains $\widetilde{p}'$. We will show that $\widetilde{\chi_l}(T_l)$ is contained in B_j for some $j \in \{1, ..., \mathfrak{m}^T(\gamma', T')\}$. Because $\pi(\widetilde{\chi_l}(0)) = \pi(\widetilde{\chi_l}(T_l))$, we have:

$$d_{\widetilde{g}}(\widetilde{\chi_l}(T_l), \pi^{-1}(p')) = d_{\widetilde{g}}(\widetilde{\chi_l}(0), \pi^{-1}(p')) < \delta, \tag{5.5}$$

which already implies that $\widetilde{\chi_l}(T_l) \in V_{4\delta}(p')$. We denote by $\widetilde{p_l'}$ the unique element $\pi^{-1}(p')$ for which we have $d_{\widetilde{g}}(\widetilde{\chi_l}(T_l), \widetilde{p_l}') < \delta$. Using the triangle inequality we now obtain:

$$d_{\widetilde{g}}(\widetilde{\gamma'}(T_l), \widetilde{p_l'}) \leq d_{\widetilde{g}}(\widetilde{\gamma'}(T_l), \widetilde{\chi_l}(T_l)) + d_{\widetilde{g}}(\widetilde{\chi_l}(T_l), \widetilde{p_l}') < \delta + \delta. \tag{5.6}$$

From the inequalities above we conclude that $\widetilde{\gamma'}(T_l)$ and $\widetilde{\chi_l}(T_l)$ are in the connected component of $V_{4\delta}(p')$ that contains $\widetilde{p_l}'$. Because this connected component contains $\widetilde{\gamma'}(T_l)$, it is therefore one of the B_j for $j \in \{1, ..., \mathfrak{m}^T(\gamma', T')\}$ as we wanted to show. We can thus define a map $\Upsilon^{T,\delta}_{(\gamma',T')} : \Lambda_X^{T,\delta}((\gamma', T')) \to \{1, ..., \mathfrak{m}^T(\gamma', T')\}$ which associates to each $l \in \Lambda_X^{T,\delta}(\gamma')$ the unique $j \in \{1, ..., \mathfrak{m}^T(\gamma', T')\}$ for which $\widetilde{\chi_l}(T_l) \in B_j$.

We now claim that if $l \neq l'$ then $\widetilde{\chi_l}(T_l)$ and $\widetilde{\chi_{l'}}(T_{l'})$ are in different connected components of $V_{4\delta}(p')$. To see this notice that both $\widetilde{\chi_l}(0)$ and $\widetilde{\chi_{l'}}(0)$ are in the component B_1. Therefore it is clear, because $\delta < \frac{\epsilon_g}{10^6}$, that if $\widetilde{\chi_l}(T_l)$ and $\widetilde{\chi_{l'}}(T_{l'})$ are in the same component of $V_{4\delta}(p')$, then the closed curves $\chi_l([0, T_l])$ and $\chi_{l'}([0, T_{l'}])$ are freely homotopic. This contradicts our choice of (χ_l, T_l) and $(\chi_{l'}, T_{l'})$ and the fact that $l \neq l'$.

We thus conclude that the map $\Upsilon^{T,\delta}_{(\gamma',T')} : \Lambda_X^{T,\delta}((\gamma', T')) \to \{1, ..., \mathfrak{m}^T(\gamma', T')\}$ is injective, which implies that $\sharp(\Lambda_X^{T,\delta}((\gamma', T'))) \leq \mathfrak{m}^T(\gamma', T') < \widetilde{K}T + 1$.

Step 3: Inductive step.

As an immediate consequence of step 2 we have that if $\{(\gamma_i, T_i); 1 \leq i \leq m\}$ is a set of parametrized periodic orbits of X we have $\sharp(\Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_m, T_m))) \leq m(\widetilde{K}T+1)$.

Inductive claim: Fix $T > 0$ and suppose that $S_m^T = \{(\gamma_i, T_i); 1 \leq i \leq m\}$ is a set of parametrized periodic orbits such that $T \geq T_i$ for every $i \in \{1, ..., m\}$, and that satisfies:

- (a) The free homotopy classes l_i of (γ_i, T_i) and l_j of (γ_j, T_j) are distinct if $i \neq j$,
- (b) For every $i \neq j$ we have $\max_{t \in [0,T]} d_g(\gamma_i(t), \gamma_j(t)) > \delta$.

Then, if $m < \frac{N_X(T)}{\widetilde{K}T+1}$, there exists a parametrized periodic orbit $(\gamma_{m+1}, T_{m+1} \leq T)$ such that its homotopy class l_{m+1} does not belong to the set $\{l_i; 1 \leq i \leq m\}$ and such that

$$\max_{t \in [0,T]} d_g(\gamma_{m+1}(t), \gamma_i(t)) > \delta \tag{5.7}$$

for all $i \in 1, ..., m$.

Proof of the claim: Recall that $\sharp(\Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_m, T_m))) \leq m(\widetilde{K}T+1)$. Therefore, because $m < \frac{N_X(T)}{\widetilde{K}T+1}$, there exists a free homotopy $l_{m+1} \in \Lambda_X^T \backslash \Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_m, T_m))$. Choose a parametrized periodic orbit (γ_{m+1}, T_{m+1}) with $T_{m+1} \leq T$ in the homotopy class l_{m+1}.

As $l_{m+1} \notin \Lambda_X^{T,\delta}((\gamma_1, T_1), ..., (\gamma_m, T_m))$, we must have $\max_{t \in [0,T]} d_g(\gamma_{m+1}(t), \gamma_i(t)) > \delta$ for all $i \in 1, ..., m$; thus completing the proof of the claim.

Step 4: Obtaining a T, δ separated set.

As usual, we denote by $\lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor$ the largest integer which is $\leq \frac{N_X(T)}{\widetilde{K}T+1}$. The strategy is now to use the inductive step to obtain a set $S_X^T = \{(\gamma_i, T_i); 1 \leq i \leq \lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor\}$ satisfying conditions (a) and (b) above with the maximum possible cardinality. We start with a set $S_1^T = \{(\gamma_1, T_1)\}$, which clearly satisfies conditions (a) and (b), and if $1 < \lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor$ we apply the inductive step to obtain a parametrized periodic orbit $(\gamma_2, T_2 \leq T)$ such that $S_2^T = \{(\gamma_1, T_1), (\gamma_2, T_2 \leq T)\}$ satisfies (a) and (b). We can go on applying the inductive step to produce sets $S_m^T = \{(\gamma_i, T_i); 1 \leq i \leq m\}$ satisfying the desired conditions (a) and (b) as long as $m - 1$ is smaller than $\lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor$. By this process we can construct a set $S_X^T = \{(\gamma_i, T_i); 1 \leq i \leq \lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor\}$ such that for all $i, j \in \{1, ..., \lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor\}$ (a) and (b) above hold true.

For each $i \in \{1, ..., \lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor\}$ let $q_i = \gamma_i(0)$. We define the set $P_X^T := \{q_i ; 1 \leq i \leq \lfloor \frac{N_X(T)}{\widetilde{K}T} \rfloor + 1\}$. The condition (b) satisfied by S_X^T implies that P_X^T is a T, δ-separated set. It then follows from the definition of the δ- entropy h_δ that

$$h_\delta(\phi_X) \geq \limsup_{T \to +\infty} \frac{\log(\lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor)}{T}. \tag{5.8}$$

Step 5: Suppose now that for constants $a > 0$ and b we have $N_X(T) \geq e^{aT+b}$.

For every $\epsilon > 0$ we know that for T big enough we have $e^{\epsilon T} > \widetilde{K}T + 1$. This implies that

$$\limsup_{T \to +\infty} \frac{\log(\lfloor \frac{N_X(T)}{\widetilde{K}T+1} \rfloor)}{T} \geq \limsup_{T \to +\infty} \frac{\log(\lfloor \frac{e^{aT+b}}{e^{\epsilon T}} \rfloor)}{T} = \limsup_{T \to +\infty} \frac{\log(\lfloor e^{(a-\epsilon)T+b} \rfloor)}{T}. \tag{5.9}$$

It is clear that $\limsup_{T \to +\infty} \frac{\log(\lfloor e^{(a-\epsilon)T+b} \rfloor)}{T} = a - \epsilon$. We have thus proven that if for constants $a > 0$ and b we have $N_X(T) \geq e^{aT+b}$ then $h_\delta(\phi_X) \geq a - \epsilon$. Because ϵ can be taken arbitrarily small we obtain:

$$h_\delta(\phi_X) \geq a. \tag{5.10}$$

Step 6: We have so far concluded that for all $\delta < \frac{\epsilon_g}{10^6}$ we have $h_\delta(\phi_X) \geq a$. We then have:

$$h_{top}(\phi_X) = \lim_{\delta \to 0} h_\delta(\phi_X) \geq a \tag{5.11}$$

finishing the proof of the theorem. $\square$

Remark: One could naively believe that there exists a constant $\delta_g > 0$ depending only on the metric g, such that if two parametrized closed curves $\sigma_1 : \mathbb{R} \to M$ of period T_1 and $\sigma_2 : \mathbb{R} \to M$ of period T_2 satisfy $\sup_{t \in [0, \max\{T_1, T_2\}]}\{d_g(\sigma_1(t), \sigma_2(t))\} < \delta_g$ then (γ_1, T_1) and (γ_2, T_2) are freely homotopic to each other. This would make the proof of Theorem 1 much shorter. However such a constant does not exist. One can easily find for any $\delta > 0$ two parametrized curves in the 3-torus which are in different primitive free homotopy classes and satisfy $\sup_{t \in [0, \max\{T_1, T_2\}]}\{d_g(\sigma_1(t), \sigma_2(t))\} < \delta$. We sketch the construction below.

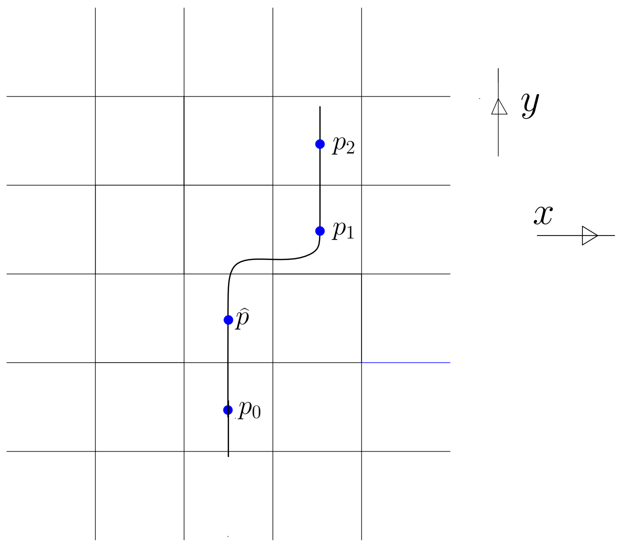

Figure 2

Consider coordinates $(x, y, z) \in (\mathbb{R}/\mathbb{Z})^3$ on the three dimensional torus $\mathbb{T}^3$. Figure 2 above represents the universal cover of the two dimensional torus $\mathbb{T}^2 \subset \mathbb{T}^3$ obtained by fixing the coordinate $z = 0$ in $\mathbb{T}^3$. The dotted points p_0, $\widehat{p}$, p_1 and p_2 in the figure represent lifts of a point $p \in \mathbb{T}^2$. It is then clear that the curve c represented in the figure projects to a smooth immersed curve in $\mathbb{T}^2 \subset \mathbb{T}^3$.

We consider a parametrization by arc length $\varsigma_1 : [0, T_1] \to \mathbb{R}^2$ of the piece of c connecting p_0 and p_1. We can extend ς_1 periodically to $\mathbb{R}$ by demanding that $\varsigma_1(t) = \varsigma_1(t) + (1, 2)$ for all $t \in \mathbb{R}$. This extension is a lift to $\mathbb{R}^2$ of the closed immersed curve obtained by projecting $\varsigma_1([0, T_1])$ to $\mathbb{T}^2$. By a very small perturbation of the projection of $\varsigma_1([0, T_1])$ we can produce a closed smooth embedded curve $\sigma_1 : [0, T_1] \to \mathbb{T}^3$ which closes at the point $(p, 0) = \sigma_1(0) = \sigma_1(T_1)$. We consider the natural extension of σ_1 to $\mathbb{R}$ obtained by demanding that $\sigma_1(t) = \sigma_1(t - T_1)$ for all $t \in \mathbb{R}$.

Analogously we consider a parametrization by arc length $\varsigma_2 : [0, T_1 + 1] \to \mathbb{R}^2$ of the piece of c connecting p_0 and p_2. We can also extend ς_2 periodically to $\mathbb{R}$, this time demanding that $\varsigma_2(t) = \varsigma_2(t) + (1, 3)$. By making a very small perturbation of ς_2 we can produce a closed smooth embedded curve $\sigma_2 : [0, T_1 + 1] \to \mathbb{T}^3$ which closes at the point $(p, \frac{\delta}{K}) = \sigma_2(0) = \sigma_2(T_1 + 1)$ and which is disjoint from the image of σ_1 .We consider the natural extension of σ_2 to $\mathbb{R}$ obtained by demanding that $\sigma_2(t) = \sigma_2(t - (T_1 + 1))$ for all $t \in \mathbb{R}$.

We point out that the extensions $\varsigma_1 : \mathbb{R} \to \mathbb{R}^2$ and $\varsigma_2 : \mathbb{R} \to \mathbb{R}^2$ coincide on the interval $[0, T_1 + 1]$. To see this just notice that the piece of c connecting p_0 and $\widehat{p}$ and the piece of c connecting p_1 and p_2 project to the same circle in $\mathbb{T}^2$.

Let now $\sigma_0 : [0, T_1 + 1] \to \mathbb{T}^2$ be the parametrized curve obtained by projecting $\varsigma_1 : [0, T_1 + 1] \to \mathbb{R}^2$, which equals $\varsigma_2 : [0, T_1 + 1] \to \mathbb{R}^2$, to the torus $\mathbb{T}^2$. The curves $\sigma 1|_{[0,T_1+1]}$ and $\sigma 2|_{[0,T_1+1]}$ are both perturbations of the parametrized curve σ_0. By making the perturbations sufficiently small we can guarantee that $\sigma 1|_{[0,T_1+1]}$ and $\sigma 2|_{[0,T_1+1]}$ are arbitrarily close. It is immediate to see that $\sigma 1|_{[0,T_1+1]}$ and $\sigma 2|_{[0,T_1+1]}$ are in distinct homotopy classes.

Chapter 6

Exponential homotopical growth rate of $C\mathbb{H}_{cyl}(\lambda_0)$ and estimates for h_{top}

In this section we define the exponential homotopical growth of contact homology and relate it to the topological entropy of Reeb vector fields. The basic idea is to use non-vanishing of cylindrical contact homology of (M, ξ) in a free homotopy class to obtain existence of Reeb orbits in such an homotopy class for any contact form on (M, ξ); this idea is present in [30, 37]. It is straightforward to see that the period and action of a Reeb orbit are equal and in the sequel we will use the same notation to refer period and action of Reeb orbits.

Let (M, ξ) be a contact manifold and λ_0 be a hypertight contact form on (M, ξ). For $T > 0$ we define $\widetilde{\bigwedge}_T(\lambda_0)$ to be the set of free homotopy classes of M such that $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$ if, and only if, all Reeb orbits of X_{λ_0} in ρ are simply covered, non-degenerate, have action/period smaller than T and $C\mathbb{H}^{\rho}_{cyl}(\lambda_0) \neq 0$. We define $N^{cyl}_T(\lambda_0)$ to be the cardinality $\sharp(\widetilde{\bigwedge}_T(\lambda_0))$.

Definition: We say that the cylindrical contact homology $C\mathbb{H}_{cyl}(\lambda_0)$ of (M, λ_0) has exponential homotopical growth with exponential weight $a > 0$ if there exist $T_0 \geq 0$ and b, such that for all $T \geq T_0$ $N^{cyl}_T(\lambda_0) = \sharp(\widetilde{\bigwedge}_T(\lambda_0)) \geq e^{aT+b}$.

The main result of this section is the following:

Theorem 6.1. *Let λ_0 be a hypertight contact form on a contact manifold (M, ξ) and assume that the cylindrical contact homology $C\mathbb{H}_{cyl}(\lambda_0)$ has exponential homotopical growth with exponential weight $a > 0$. Then for every C^k $(k \geq 2)$ contact form λ on*

(M, ξ) the Reeb flow of X_λ has positive topological entropy. More precisely, if f_λ is the unique function such that $\lambda = f_\lambda \lambda_0$, then

$$h_{top}(X_\lambda) \geq \frac{a}{\max f_\lambda}. \tag{6.1}$$

Proof: We write $E = \max f$.

Step 1:

We assume initially that λ is non-degenerate and C^∞. For every $\epsilon > 0$ is possible to construct an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ. Analogously, for $e > 0$ small enough, it is possible to construct an exact symplectic cobordism from λ to $e\lambda_0$.

Using these cobordisms, it is possible to construct a splitting family $(\mathbb{R}\times M, \varpi_R, J_R)$ from $(E+\epsilon)\lambda_0$ to $e\lambda_0$, along λ, such that for every $R > 0$ $(\mathbb{R}\times M, \varpi_R, J_R)$ is homotopical to the symplectization of λ_0. We fix a regular almost complex structure $J_0 \in \mathcal{J}^\rho_{reg}(\lambda_0)$ and $J \in \mathcal{J}(\lambda)$, and demand that J_R coincides with J_0 in the positive and negative ends of the cobordism, and with J on $[-R, R] \times M$.

Let $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$. We claim that for every R there exists a finite energy pseudo-holomorphic cylinder $\widetilde{w}$ in $(\mathbb{R}\times M, J_R)$ positively asymptotic to a Reeb orbit in $\mathcal{P}^\rho_{\lambda_0}$ and negatively asymptotic to an orbit in $\mathcal{P}^\rho_{\lambda_0}$.

If this was not true for a certain $R > 0$, then because of the absence of pseudoholomorphic cylinders asymptotic to Reeb orbits in $\mathcal{P}^\rho_{\lambda_0}$ we would have that $J_R \in \mathcal{J}^\rho_{reg}(J_0, J_0)$. Therefore, the map $\Phi^{J_R} : C\mathbb{H}^\rho_{cyl}(\lambda_0) \to C\mathbb{H}^\rho_{cyl}(\lambda_0)$ induced by $(\mathbb{R}\times M, \varpi_R, J_R)$ is well-defined. But because there are no pseudoholomorphic cylinders asymptotic to Reeb orbits in $\mathcal{P}^\rho_{\lambda_0}$, we have that the map $\Phi^{J_R} : C\mathbb{H}^\rho_{cyl}(\lambda_0) \to C\mathbb{H}^\rho_{cyl}(\lambda_0)$ vanishes. On the other hand, from section 3.3 we know that Φ^{J_R} the identity. As Φ^{J_R} vanishes and is the identity we conclude that $C\mathbb{H}^\rho_{cyl}(\lambda_0) = 0$, contradicting that $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$.

Step 2:

Let $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$, $R_n \to +\infty$ be a strictly increasing sequence and $\widetilde{w}_n : \mathbb{R}\times(S^1\times\mathbb{R}, i) \to (\mathbb{R}\times M, J_{R_n})$ be a sequence of pseudoholomorphic cylinders with one positive puncture asymptotic to an orbit in $\mathcal{P}^\rho_{\lambda_0}$ and one negative puncture asymptotic to an orbit in $\mathcal{P}^\rho_{\lambda_0}$. Notice that, because of the properties of ρ the energy of $\widetilde{w}_n$ is uniformly bounded.

Therefore we can apply the SFT compactness theorem to obtain a subsequence of $\widetilde{w}_n$ which converges to a pseudoholomorphic buiding $\widetilde{w}$. Notice that in order to apply the SFT compactness theorem we need to use the non-degeneracy of λ. Moreover we can give a very precise description of the building.

Let $\widetilde{w}^k$ for $k \in \{1, ..., m\}$ be the levels of the pseudoholomorphic building $\widetilde{w}$. Because the topology of our curve doesn't change on the breaking we have the following picture:

- the upper level $\widetilde{w}^1$ is composed by one connected pseudoholomorphic curve, which has one positive puncture asymptotic to an orbit $\gamma_0 \in \mathcal{P}^{\rho}_{\lambda_0}$, and several negative punctures. All of the negative punctures detect contractible orbits, except one that detects a Reeb orbit γ_1 which is also in ρ.
- on every other level $\widetilde{w}^k$ there is a special pseudoholomorphic curve which has one positive puncture asymptotic to a Reeb orbit γ_{k-1} in ρ, and at least one but possibly several negative punctures. Of the negative punctures there is one that is asymptotic to an orbit γ_k in ρ, while all the others detect contractible Reeb orbits.

Because of the splitting behavior of the cobordisms $(\mathbb{R} \times M, J_{R_n}$ it is clear that there exists a k_0, such that the level $\widetilde{w}^k$ is in an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ. This implies that the special orbit γ_{k_0} is a Reeb orbit of X_λ in the homotopy class ρ.

Notice that $A(\gamma_0) \leq (E+\epsilon)T$. This implies that all the other orbits appearing as punctures of the building $\widetilde{w}$ have action smaller than $(E+\epsilon)T$, and in particular that γ_{k_0} has action smaller than $(E+\epsilon)T$.

As we can do the construction above for any $\epsilon > 0$ we can obtain a sequence of Reeb orbits $\gamma_{K_0(j)}$ which are all in ρ and such that $A(\gamma_{K_0(j)}) \leq (E + \frac{1}{j})T$. Using Arzela-Ascoli's Theorem one can extract a convergent subsequence of γ^{ρ}_j. Its limit γ_ρ is clearly a Reeb orbit of λ in the free homotopy class ρ and with action smaller or equal to ET.

Step 3: Estimating $N_{X_\lambda}(T)$. [1]

From step 2, we know that if $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$ then there is a Reeb orbit γ_ρ of the Reeb flow of X_λ with $A(\gamma_\rho) \leq ET$. Recalling that the period and the action of a Reeb orbit coincide we obtain that $N_{X_\lambda}(T) \geq \sharp(\widetilde{\bigwedge}_{\frac{T}{E}}(\lambda_0))$. Under the hypothesis of the theorem we have

$$N_{X_\lambda}(T) \geq e^{\frac{aT}{E}+b}. \tag{6.2}$$

[1] Recall that as we defined in Chapter 5, $N_{X_\lambda}(T)$ is the number of distinct free homotopy classes of M that contain periodic orbits of X_λ with period $\leq T$.

Applying Theorem 5.1 we then obtain $h_{top}(X_\lambda) \geq \frac{a}{E}$. This proves the theorem in the case λ is C^∞ and non-degenerate.

Step 4: Passing to the case of a general $C^{k\geq 2}$ contact form λ (the case where λ is degenerate is included here).

Let λ_i be a sequence of non-degenerate contact forms converging in the C^k-topology to a contact form λ which is C^k $(k \geq 2)$ and possibly degenerate. For every $\epsilon > 0$ there is i_0 such that for $i > i_0$; there exists an exact symplectic cobordism from $(E+\epsilon)\lambda_0$ to λ_i.

Fixing then an homotopy class $\rho \in \widetilde{\bigwedge}_T(\lambda_0)$ we know, by the previous steps, that there exists a Reeb orbit $\gamma_\rho(n)$ of λ_n in the homotopy class ρ with action smaller than $(E+\epsilon)T$. By taking the sequence $\gamma_\rho(n)$ and applying Arzela-Ascoli's theorem we obtain a subsequence which converge to a Reeb orbit $\gamma_{\epsilon,\rho}$ of X_λ with$A(\gamma_{\epsilon,\rho}) \leq (E+\epsilon)T$. Notice that here we use that λ is at least C^2 (so that X_λ is at least C^1) in order to be able to use Arzela-Ascoli's theorem.

Because $\epsilon > 0$ above can be taken arbitrarily close to 0 we can actually obtain a sequence $\gamma_{j,\rho}$ of Reeb orbits of X_λ whose homotopy class is ρ such that the actions $A(\gamma_{j,\rho})$ converges to ET. Again applying Arzela-Ascoli's theorem, we obtain that the sequence $\gamma_{j,\rho}$ has a convergent subsequent, which converges to an orbit γ_ρ satisfying $A(\gamma_{j,\rho}) \leq ET$.

Reasoning as in step 3 above, we obtain that $N_{X_\lambda}(T) \geq e^{\frac{aT}{E}+b}$. Applying Theorem 1 we obtain the desired estimate for the topological entropy. This finishes the proof of the theorem. $\square$

Chapter 7

Unit tangent bundles of hyperbolic manifolds

7.1 Contact forms for geodesic flows

The first class of examples we will study is of unit tangent bundles of orientable hyperbolic manifolds. Given a manifold Q its unit tangent bundle T_1Q can be given a canonical contact structure which we will denote by ξ_{geo}. This contact structure is associated to geodesic flows in the sense that for every Riemannian metric g on Q, the geodesic flow of g on T_1Q is a Reeb flow on (T_1Q, ξ_{geo}). We begin by recalling how this can be done; our reference for this construction is [38].

Given a compact orientable manifold Q of dimension n, let g be any Riemannian metric on Q. This metric induces a unique distribution of n planes in the tangent bundle TQ, the so called horizontal distribution H_g (see section 1.3 in [38]). This distribution is always transverse to the vertical distribution V in TQ, which is the unique distribution of n-planes always tangent to the fibres of TQ. This implies that for every $y \in TQ$ we have the splitting $T_yTQ := H_y \bigoplus V_y$. Let $\pi : TQ \to Q$ be the canonical projection. Because H is transversal to the fibers we have that at each point $y \in TQ$ the restriction of the the differential $D\pi$ to H_y is an isomorphism between H_y and $T_{\pi(y)}Q$. With this in hand, we can use the map π to pull back g to an inner product in the distribution H. The metric g also induces an inner product on the distribution V. Using these two inner products we have as a result a metric $\hat{g}$ induced by g on the bundle TQ, which is usually called the Sasaki metric on TQ.

We will now introduce an almost complex structure J_g on TQ associated to the metric g. Let $v \in T_qQ$ and $y \in \pi^{-1}(q)$. From our previous discussion we know that there are unique vectors $v_H \in H_y$ and $v_V \in H_y$ which are associated to v: v_H is the unique vector in $H_{(q,v)}$ that is in the pre-image of the restriction of $D\pi$ to $H_{(q,v)}$, and v_V is the vector on the fiber T_qQ of TQ canonically identified with v. Now, for each vector $\breve{v} \in H_y$ we define $J_g(\breve{v}) := (\pi(\breve{v}))_V \in V_y$, and for each $\breve{v} \in V_y$ we define $J_g(\breve{v}) := -(\pi(\breve{v}))_H \in H_y$. It is easy to see that there is a unique way to extend J_g to an almost complex structure on TQ.

With this in hand we can define a symplectic form ϖ^g in TQ. For vectors $v_1, v_2 \in T_yTQ$ we define

$$\varpi^g_y(v_1, v_2) := \hat{g}(J_g(v_1), v_2). \tag{7.1}$$

We will not prove here that ϖ^g is indeed a symplectic form, but refer to [38] for the proof. Define the function H on TQ by $H(q,v) := g_q(v,v)$. Again in the reference [38], it is proven that the Hamiltonian vector field X_H associated to H via the symplectic form ϖ is the geodesic vector field G_g of the metric g, and the unit tangent bundle T_1Q is diffeomorphic to set $H^{-1}(1)$.

Lastly we have the following definition:

$$\alpha_g := i_{G_g}\hat{g} \tag{7.2}$$

With these definitions, we can state the following proposition, the proof of which can be found in page 16 of [38].

Proposition 7.1. *The restriction $\alpha_g\mid_{H^{-1}(1)}$ of α_g is a contact form on T_1Q. Moreover, its Reeb vector field X_α is the restriction of geodesic vector field G_g to $H^{-1}(1)$.*

This proposition justifies what we claimed previously about the relation between geodesic flows and Reeb flows. It shows that any geodesic flow is a Reeb flow for some contact form. As the contact form α_g on T_1Q varies continuously as we very g continuously in the contractible space $\mathcal{MET}$ of Riemannian metrics on Q, we can apply Gray's stability theorem ([25]) to conclude that all α_g are associated to a unique (up to diffeomorphism) contact structure ξ_{geo} on the unit tangent bundle T_1Q. This contact structure ξ_{geo} is therefore related to all geodesic flows on T_1Q as it contains all of them among its Reeb flows.

Lastly we introduce some terminology. Given a closed geodesic $\mathfrak{c}$ of a Riemannian metric g, its *lift* to T_1Q is the periodic trajectory γ of the geodesic flow of g associated obtained from the arc length parametrization of $\mathfrak{c}$. It follows from our previous discussion that γ is a Reeb orbit of $\alpha_{g_{hyp}}$. We will sometimes also refer to γ as the Reeb orbit of $\alpha_{g_{hyp}}$ obtained by lifting $\mathfrak{c}$.

7.2 Exponetial homotopical growth rate of $C\mathbb{H}_{cyl}(\alpha_{g_{hyp}})$

We now specialise our discussion to the case where of a compact hyperbolic manifold (Q_{hyp}, g_{hyp}), i.e. Q_{hyp} is a compact orientable manifold endowed with a hyperbolic g_{hyp}. In this case, let $\alpha_{g_{hyp}}$ be the contact form on T_1Q_{hyp} given in Proposition 7.2 In order to estimate the growth rate of $C\mathbb{H}_{cyl}(\alpha_{g_{hyp}})$, we begin introducing some notation.

Let $\Lambda(Q_{hyp})$ be the free loop space of Q_{hyp} and $\Lambda_0(Q_{hyp}) \subset \Lambda(Q_{hyp})$ be the subset of primitive free homotopy classes. For every element $\rho \in \Lambda_0(Q_{hyp})$ let $\mathfrak{c}_\rho \subset Q_{hyp}$ be the unique hyperbolic geodesic of g_{hyp} in the homotopy class ρ; it is a classical fact of hyperbolic geometry that every non-trivial free homotopy class on a hyperbolic manifold contains exactly one closed hyperbolic geodesic, see [5]. The geodesic $\mathfrak{c}_\rho$ lifts to a unique Reeb orbit γ_ρ in T_1Q_{hyp}; γ_ρ is the trajectory of the geodesic flow of g_{hyp} associated to the geodesic $\mathfrak{c}_\rho$. This uniqueness allows us to define a map $\mathfrak{H} : \Lambda_0(Q_{hyp}) \to \Lambda(T_1Q_{hyp})$ [1] in the following way: for $\rho \in \Lambda_0(Q_{hyp})$ we define $\mathfrak{H}(\rho)$ as the free homotopy class of the Reeb orbit γ_ρ defined above. We then have the following

Proposition 7.2. *For each $\rho \in \Lambda_0(Q_{hyp})$ the free homotopy class $\mathfrak{H}(\rho)$ contains exactly one Reeb orbit of $\alpha_{g_{hyp}}$. This is the Reeb orbit γ_ρ constructed above.*

Proof: Suppose there is a Reeb orbit γ'_ρ belonging to free homotopy class $\mathfrak{H}(\rho)$ and distinct from the Reeb orbit γ_ρ. Then γ_ρ and γ'_ρ would project to two different hyperbolic geodesics $\mathfrak{c}_\rho$ and $\mathfrak{c}'_\rho$ in Q_{hyp}. Because γ_ρ and γ'_ρ are freely homotopic in T_1Q_{hyp}, the two hyperbolic geodesics $\mathfrak{c}_\rho$ and $\mathfrak{c}'_\rho$ would be freely homotopic in Q_{hyp}, and would moreover belong to the free homotopy class ρ. But since $\mathfrak{c}_\rho$ is the only hyperbolic geodesic in ρ we must have $\mathfrak{c}'_\rho = \mathfrak{c}_\rho$ which contradicts our assumption that the Reeb orbits γ'_ρ and γ_ρ are distinct. □

Two consequences of the previous proposition are

Corollary 7.3. *For each $\rho \in \Lambda_0(Q_{hyp})$ the cylindrical contact homology $C\mathbb{H}^{\mathfrak{H}(\rho)}_{cyl}(\alpha_{g_{hyp}})$ on the homotopy class $\mathfrak{H}(\rho)$ is well-defined and does not vanish.*

Proof:
Because $\mathfrak{H}(\rho)$ contains only one Reeb orbit of $\alpha_{g_{hyp}}$ which is simply covered, the results of section 4.3 to imply that the cylindrical contact homology $C\mathbb{H}^{\mathfrak{H}(\rho)}_{cyl}(\alpha_{g_{hyp}})$ is well-defined. Moreover as the chain complex $CH^{\mathfrak{H}(\rho)}_{cyl}(\alpha_{g_{hyp}})$ contains only one element, it is clear that $C\mathbb{H}^{\mathfrak{H}(\rho)}_{cyl}(\alpha_{g_{hyp}}) \neq 0$. □

Corollary 7.4. *The $\mathfrak{H}$ defined above is an injection.*

[1] $\Lambda(T_1Q_{hyp})$ denotes the free loop space of T_1Q_{hyp}.

Proof: The proof is immediate from Proposition 7.2.

We are now ready to prove the main result of this chapter.

Theorem 7.5. *$C\mathbb{H}_{cyl}(\alpha_{g_{hyp}})$ has exponential homotopical growth rate.*

Proof:
Given $T > 0$, denote by $\Lambda_0^{\leq T}(Q_{hyp})$ the subset of $\Lambda_0(Q_{hyp})$ of free homotopy classes that contain a closed hyperbolic geodesic of length smaller or equal to T. It is an elementary fact that the length of any hyperbolic $\mathfrak{c}$ equals the action of the Reeb orbit γ obtained by lifting $\mathfrak{c}$ to the unit tangent bundle. Combining this with Proposition 7.2, Corollary 7.3 we obtain that the set $\widetilde{\Lambda}_T(\alpha_{g_{hyp}})$ contains the set $\mathfrak{H}(\Lambda_0^{\leq T}(Q_{hyp}))$. This implies that

$$N_T^{cyl}(\alpha_{g_{hyp}}) \geq \#(\mathfrak{H}(\Lambda_0^{\leq T}(Q_{hyp}))). \tag{7.3}$$

Applying then Corollary 7.4 we obtain

$$N_T^{cyl}(\alpha_{g_{hyp}}) \geq \#(\Lambda_0^{\leq T}(Q_{hyp})). \tag{7.4}$$

Because each homotopy class in $\Lambda(Q_{hyp}))$ contains exactly one closed hyperbolic geodesic, it follows that the number $\#(\Lambda_0^{\leq T}(Q_{hyp}))$ equals the number $N_T(g_{hyp})$ of closed geodesics of g_{hyp} of length $\leq T$. Moreover, as the geodesic flow of a compact hyperbolic manifold is Anosov we know [35] that there exist constants T_0, $a > 0$ and b such that

$$N_T(g_{hyp}) \geq e^{aT+b} \tag{7.5}$$

for all $T \geq T_0$. Combining all this we conclude that

$$N_T^{cyl}(\alpha_{g_{hyp}}) \geq e^{aT+b}, \tag{7.6}$$

which gives the desired exponential growth. $\square$

Chapter 8

Contact 3-manifolds with a hyperbolic component

In this chapter we will prove the following theorem:

Theorem 8.1. *Let M be a closed connected oriented 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary, such that the component M_0 satisfies:*

- *M_0 is the mapping torus of a diffeomorphism $h : S \to S$ with pseudo-Anosov monodromy on a surface S with non-empty boundary.*

Then M can be given infinitely many non-diffeomorphic contact structures ξ_k, such that for each ξ_k there exists a hypertight contact form λ_k on (M, ξ_k) which has exponential homotopical growth of cylindrical contact homology.

We denote by S a surface with boundary and ω a symplectic form on S. Let h be a symplectomorphism of (S, ω) to itself, with pseudo-Anosov monodromy and which is the identity on a neighbourhood of ∂S. We follow a well known recipe to construct a suitable contact form in the mapping torus $\Sigma(S, h)$.

We choose a primitive β for ω such that for coordinates $(r, \theta) \in [-\epsilon, 0] \times S^1$ in a neighbourhood V of ∂S we have $\beta = f(r)d\theta$, where $f > 0$ and $f' > 0$. We pick a smooth non-decreasing function $F_0 : \mathbb{R} \to [0, 1]$ which satisfies $F_0(t) = 0$ for $t \in (-\infty, \frac{1}{100})$ and $F_0(t) = 1$ for $t \in (\frac{1}{100}, +\infty)$. For $i \in \mathbb{Z}$ we define $F_i(t) = F_0(t - i)$. Fixing $\epsilon > 0$, we define a 1-form $\widetilde{\alpha}$ on $\mathbb{R} \times S$ by letting

$$\widetilde{\alpha} = dt + \epsilon(1 - F_i(t))(h^i)^*\beta + \epsilon F_i(t)(h^{i+1})^*\beta \quad \text{for} \quad t \in [i, i+1) \tag{8.1}$$

It is immediate to see that this defines a smooth 1-form on $\mathbb{R} \times S$, and a simple computation shows that if ϵ is small enough the 1-form $\widetilde{\alpha}$ is a contact form. For $t \in [0,1]$, the Reeb vector field $X_{\widetilde{\alpha}}$ is equal to $\partial_t + v(p,t)$, where $v(p,t)$ is the unique vector tangent to S that satisfies $\omega(v(p,t), \cdot) = F_0'(t)\beta - F_0' h^*\beta$.

Consider on the diffeomorphism $H : \mathbb{R} \times S \to \mathbb{R} \times S$ defined by $H(t,p) = (t - 1, h(p))$. The mapping torus $\Sigma(S,h)$ is defined by:

$$\Sigma(S,h) := (\mathbb{R} \times S)/_{(t,p) \sim H(t,p)}, \tag{8.2}$$

and we denote by $\pi : \mathbb{R} \times S \to \Sigma(S,h)$ the associated covering map.

Because $H^*\widetilde{\alpha} = \widetilde{\alpha}$, there exists a unique contact form α on $\Sigma(S,h)$ such that $\pi^*\alpha = \widetilde{\alpha}$. Notice that in the neighbourhood $S^1 \times V$ of $\partial\Sigma(S,h)$, $\alpha = dt + \epsilon f(r)d\theta$, which implies that X_α is tangent to $\partial\Sigma(S,h)$.

The Reeb vector field X_α on $\Sigma(S,h)$ is transverse to the surfaces $\{t\} \times S$ for $t \in \mathbb{R}/_{\mathbb{Z}}$. This implies that $\{0\} \times S$ is a global surface of section for the Reeb flow of α, and by our expression of $X_{\widetilde{\alpha}}$ the first return map of the Reeb flow of α is isotopic to h.

By doing a sufficiently small perturbation of α supported in the interior of $\Sigma(S,h)$ we can obtain a contact form $\widehat{\alpha}$ satisfying that all Reeb orbits of $\widehat{\alpha}$ which are not freely homotopic to curves in $\partial\Sigma(S,h)$ are non-degenerate, and such that $\{0\} \times S$ is a global surface of section for the flow of $X_{\widehat{\alpha}}$. Notice that as the perturbation is supported in the interior of $\Sigma(S,h)$, the Reeb flow of $\widehat{\alpha}$ is also tangent to the boundary of $\Sigma(S,h)$.

8.1 Contact 3-manifolds containing $(\Sigma(S,h), \widehat{\alpha})$ as a component

Let M be a closed connected oriented 3-manifold which can be cut along a non-empty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary, such that the component M_0 is diffeomorphic to $(\Sigma(S,h), \alpha)$. Then it is possible to construct hypertight contact forms on M which match with $\widehat{\alpha}$ in the component M_0. More precisely, we have the following result due to Colin and Honda, and Vaugon:

Proposition 8.2. *([13, 43]) Let M be a closed connected oriented 3-manifold which can be cut along a non-empty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary, such that the component M_0 is diffeomorphic to $(\Sigma(S,h), \alpha)$. Then, there exist an infinite family $\{\xi_k, k \in \mathbb{Z}\}$ of non-diffeomorphic contact structures on M such that*

- *for each $k \in \mathbb{Z}$ there exists a hypertight contact form λ_k on (M, ξ_k) which coincides with $\widehat{\alpha}$ on the component M_0.*

We briefly recall the construction of the contact forms λ_k, and refer the reader to [13, 43] for the details. When $i \geq 1$, we apply Théorème 1.3 of [13] to obtain a hypertight contact form α_i on M_i which is compatible with the orientation of M_i, and whose Reeb vector field X_{α_i} is tangent to the boundary of M_i. On the special piece M_0 we consider the the contact form α_0 equal to $\widehat{\alpha}$ constructed in the above.

Let $\{\mathfrak{T}_j | 1 \leq \ j \leq m\}$ be the family of incompressible tori along which we cut M to obtain the pieces M_i. Then the contact forms α_i give a hypertight contact form on each component of $M \setminus \bigcup_{j\geq 1}^m \mathbb{V}(\mathfrak{T}_j)$, where $\mathbb{V}(\mathfrak{T}_j)$ is a small open neighborhood of T_j. This gives a contact form $\widehat{\lambda}$ on $M \setminus \bigcup_{j\geq 1}^m \mathbb{V}(\mathfrak{T}_j)$. Using an interpolation process (see section 7 of [43]), one can construct contact forms on the neighborhoods $\overline{\mathbb{V}(\mathfrak{T}_j)}$ which coincide with $\widehat{\lambda}$ on $\partial\overline{\mathbb{V}(\mathfrak{T}_j)}$. The interpolation process is not unique and can be done in such ways as to produce an infinite family of distinct contact forms $\{\lambda_k \mid k \in \mathbb{Z}\}$ on M that extend $\widehat{\lambda}$, and which are associated to contact structures $\xi_k := \ker \lambda_k$ that are all non-diffeomorphic. The contact topological invariant used to show that the family $\{\xi_k \mid k \in \mathbb{Z}\}$ is composed by non-diffeomorphic contact structures is the *Giroux torsion* (see section 7 of [43]).

8.2 Proof of Theorem 8.1

It is clear that Theorem 8.1 will follow from Theorem 6.1, if we establish that the cylindrical contact homology of λ_k has exponential homotopical growth. This is the content of

Proposition 8.3. *λ_k has exponential homotopical growth of cylindrical contact homology.*

Before proving the proposition we introduce some necessary ideas and notation. The first return map of $X_{\widehat{\alpha}}$ is a diffeomorphism $\widehat{h} : S \to S$ which is homotopic to h and therefore to a pseudo-Anosov map. The Reeb orbits of $X_{\widehat{\alpha}}$ are in one-to-one correspondence with periodic orbits of $\widehat{h}$. Moreover we have that two Reeb orbits γ_1 and γ_2 of $X_{\widehat{\alpha}}$ are freely homotopic if and only if their associated periodic orbits are in the same Nielsen class. Thus there is an injective map Ξ from the set $\mathcal{N}$ of Nielsen classes to the set $\bigwedge$ of free homotopy classes of Reeb orbits in $\Sigma(S, h)$.

Denote by $\mathcal{N}_k$ the set of distinct Nielsen classes which contain only periodic orbits of $\widehat{h}$ of period smaller or equal to k. Because of the pseudo-Anosov monodromy of $\widehat{h}$

we know that there are constants $a > 0$ and $b \in \mathbb{R}$, such that $\sharp(\mathcal{N}_k) > e^{ak+b}$ for all $k \geq 1$. Analogously define the subset $\bigwedge_T(\Sigma(S,h))$ of free homotopy classes of $\Sigma(S,h)$ which contains at least one Reeb orbit of $X_{\widehat{\alpha}}$ and contains only Reeb orbits with action smaller than T.

Because $\widehat{h}$ is the first return map for a global surface of section of the flow $X_{\widehat{\alpha}}$, there exists a constant $\eta \geq 1$ such that $\varrho \in \mathcal{N}_k \Rightarrow \Xi(\varrho) \in \bigwedge_{\eta k}(\Sigma(S,h))$. This implies that $\sharp(\bigwedge_T(\Sigma(S,h))) > e^{\frac{a}{\eta}T+b}$ for $T \geq \eta$. Let $\bigwedge_T^0(\Sigma(S,h))$ be the subset of $\bigwedge_T(\Sigma(S,h))$ which contains free homotopy classes in $\Sigma(S,h)$ which are primitive and different from the ones generated by curves in $\partial\Sigma(S,h)$ (we denote by $\bigwedge^0(\Sigma(S,h))$ the set $\bigwedge_{+\infty}^0(\Sigma(S,h))$). Because the fundamental group of $\partial\Sigma(S,h)$ grows quadratically we know that there is $T_0 \geq 0$ such that $\sharp\bigwedge_T^0(\Sigma(S,h)) \geq e^{\frac{a}{\eta}T+b}$ for $T \geq T_0$..

We are now ready for the proof of Proposition 8.3. The main ideas of the argument are due to Vaugon, which estimated in [43] a different growth rate of the cylindrical contact homology λ_k.

Proof of Proposition 8.3:

Step 1:

Let $i : \Sigma(S,h) \to M$ be the injection we obtain from looking at $\Sigma(S,h)$ as a component of M. Because of the incompressibility of $\partial\Sigma(S,h)$ in M, the associated map $i_* : \bigwedge_T^0(\Sigma(S,h)) \to \bigwedge(M)$ is injective for any $T > 0$ (here $\Lambda(M)$ denotes the free loop space of M). It is clear that all curves belonging to a free homotopy class $\rho \in i_*(\bigwedge^0(\Sigma(S,h))$ must intersect the component M_0.

Using then that the Reeb flow of λ_k is tangent to $\partial\Sigma(S,h)$, we conclude that for every $\rho \in i_*(\bigwedge^0(\Sigma(S,h))$ all the Reeb orbits of X_{λ_k} that belong to ρ are contained in the interior of $\Sigma(S,h)$. Therefore, the image $i_*(\bigwedge^0(\Sigma(S,h))$ is contained in the et $\bigwedge_T(M)$ of free homotopy classes of M which only contain Reeb orbits with action smaller than T.

This mean that the map $i_* : \bigwedge_T^0(\Sigma(S,h)) \to \bigwedge(M)$, restricts to a map $i_* : \bigwedge_T^0(\Sigma(S,h)) \to \bigwedge_T(M)$, where by $\bigwedge_T(M)$ we denote the set of free homotopy classes of M which only contain Reeb orbits with action smaller than T.

Step 2: For every $\rho \in i_*(\bigwedge^0(\Sigma(S,h))$ we have $C\mathbb{H}_{cyl}^{\varrho}(\lambda_k) \neq 0$.

Vaugon showed (see the proofs of Lemma 7.11 and Theorems 1.3 and 1.2 on pages 27 and 28 in [43]) that the numbers of even and odd Reeb orbits in ρ differ. For Euler characteristic reasons this implies that $C\mathbb{H}_{cyl}^{\varrho}(\lambda_k) \neq 0$.

Step 3:

Recall that in Chapter 6 we defined $N_T^{cyl}(\lambda_k)$ to be the number of different free homotopy classes ϱ in $\bigwedge_T(M)$ which contained only simple Reeb orbits with action smaller then T and such that $C\mathbb{H}_{cyl}^{\varrho}(\lambda_k) \neq 0$.

Combining the first two steps we obtain

$$N_T^{cyl}(\lambda_k) \geq \sharp(i_*(\bigwedge_T^0(\Sigma(S,h)))) = \sharp(\bigwedge_T^0(\Sigma(S,h))) \geq e^{\frac{aT}{\eta}+b}, \tag{8.3}$$

which establishes the proposition. □

Proof of Theorem 8.1: As mentioned previously, Theorem 8.1 follows directly from combining Proposition 8.3 and Theorem 6.1. □

It would be interesting to obtain an upper bound on the constant η above. This could provide a more precise estimate for the homotopical growth rate of $C\mathbb{H}_{cyl}(\lambda_k)$.

Chapter 9

Graph manifolds and Handel-Thurston surgery

Check notation and numeration on this chapter!

In [26] Handel and Thurston used Dehn surgery to obtain non-algebraic Anosov flows in 3-manifolds. Their surgery was adapted to the contact setting by Foulon and Hasselblatt in [20], who interpreted it as a Legendrian surgery and used it to produce non-algebraic Anosov Reeb flows on 3-manifolds. In this chapter we apply the Foulon-Hasselblatt Legendrian surgery to obtain other examples of contact 3-manifolds which are distinct from unit tangent bundles and on which every Reeb flow has positive topological entropy.

Some clarifications regarding the surgeries we consider are in order. On one hand, we restrict our attention to the Foulon-Hasselblatt surgery on Legendrian lifts of embedded separating geodesics on hyperbolic surfaces. This is an important restriction, since Foulon and Hasselblatt perform their surgery on the Legendrian lift of any immersed closed geodesic on a hyperbolic surface. On the other hand, for this restricted class of Legendrian knots, the surgery we consider is a bit more general than the one in [20]. They restrict their attention to Dehn surgeries with positive integer coefficients while we consider the case of any integer coefficient, as we explain in section 9.1.

9.1 The surgery

We start by fixing some notation. Let (S, g) be an oriented hyperbolic surface and $\mathfrak{r} : S^1 \to S$ an embedded oriented separating geodesic of g. We denote by $\pi : (\mathbb{D}, g) \to (S, g)$ a locally isometric covering of (S, g) by the the hyperbolic disc $(\mathbb{D}, g)$ with the property that $(-1,1) \times \{0\} \subset \pi^{-1}(\mathfrak{r}(S^1))$. Such a covering always exists since the segment $(-1,1)\times\{0\}$ of the real axis is a geodesic in $(\mathbb{D}, g)$. We denote by $v(\theta)$ the unique unitary vector field over $\mathfrak{r}(\theta)$ satisfying $\angle(\mathfrak{r}'(\theta), v(\theta)) = -\frac{\pi}{2}$. Our orientation convention is chosen so that for coordinates $z = x + iy \in \mathbb{D}$, the lift of $v(\theta)$ to $(-1,1) \times \{0\}$ is a positive multiple of the vector field $-\partial_y$ over $(-1,1)\times\{0\}$. Also, let $\Pi : T_1S \to S$ denote the base point projection.

Because $\mathfrak{r}$ is a separating geodesic, we can cut S along $\mathfrak{r}$ to obtain two oriented hyperbolic surfaces with boundary which we denote by S_1 and S_2. Our labelling is chosen so that the vector field $v(\theta)$ points inward S_2 and outward S_1. This decomposition of S induces a decomposition of T_1S in T_1S_1 and T_1S_2. Both T_1S_1 and T_1S_2 are 3-manifolds whose boundary is the torus formed by the the unit fibers over $\mathfrak{r}$.

Denote by $V_{\mathfrak{r},\delta}$ the closed $\delta-$neighbourhood of the the geodesic $\mathfrak{r}$ for the hyperbolic metric g. For $\delta > 0$ sufficiently small we have that $V_{\mathfrak{r},\delta}$ is an annulus such that the only closed geodesics contained in $V_{\mathfrak{r},\delta}$ are the covers of $\mathfrak{r}$, and that satisfies the following convexity property: if $\breve{V}$ is the connected component of $\pi^{-1}(V_{\mathfrak{r},\delta})$ containing $(-1,1) \times \{0\}$, then every segment of a hyperbolic geodesic starting and ending in $\breve{V}$ is completely contained in $\breve{V}$. It also follows from the conventions adopted above, that if we denote by U^+ the upper hemisphere of the $\mathbb{D}$ composed of points with positive imaginary component and by U^+ the lower hemisphere of the $\mathbb{D}$ composed of points with negative imaginary component, we have:

$$\breve{V} \cap U^+ \subset \pi^{-1}(S_1) \text{ and } \breve{V} \cap U^- \subset \pi^{-1}(S_2). \tag{9.1}$$

This fact has the following important consequence: if $\nu([0,K])$ is a hyperbolic geodesic segment starting and ending at $V_{\mathfrak{r},\delta}$ and contained in one of the S_i, then $[\nu]$ is a non-trivial homotopy class in the relative fundamental group $\pi_1(S_i, V_{\mathfrak{r},\delta})$.

On the unit tangent bundle T_1S we consider consider the contact form λ_g whose Reeb vector field is the geodesic vector field for the hyperbolic metric g. It is well known that the lifted curve $(\mathfrak{r}(\theta), v(\theta))$ in T_1S is Legendrian on the contact manifold $(T_1S, \ker\lambda_g)$. The geodesic vector field X_{λ_g} over the Legendrian curve coincides with the horizontal lift of v (see section 1.3 of [38]), points inward T_1S_2 and outward T_1S_1, and is normal to $\partial T_1S_2 = \partial T_1S_1$ for the Sasaki metric on T_1S.

Moreover if $\delta > 0$ is small enough we know that for every $\vartheta \in L_{\mathfrak{r}}$ there exists numbers $t_1 < 0$ and $t_2 > 0$ such that:

$$\phi^{t_1}_{\lambda_g}(\vartheta) \in T_1S_1 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}), \tag{9.2}$$

$$\phi^{t_2}_{\lambda_g}(\vartheta) \in T_1S_2 \setminus \Pi^{-1}(V_{\mathfrak{r},\delta}). \tag{9.3}$$

Following [20], we know that there exists a neighbourhood $B^{3\eta}_{2\epsilon}$ of $L_{\mathfrak{r}}$ on which we can find coordinates $(t, s, w) \in (-3\eta, 3\eta) \times S^1 \times (-2\epsilon, 2\epsilon)$ such that:

$$\lambda_g = dt + wds, \tag{9.4}$$

$$L_{\mathfrak{r}} = \{0\} \times S^1 \times \{0\}, \tag{9.5}$$

where $\{0\} \times \{\theta\} \times (-2\epsilon, 2\epsilon)$ is a local parametrization of the unitary fiber over $\theta \in L_{\mathfrak{r}}$, and $\epsilon < \frac{\eta}{4|q|\pi}$, with q being a fixed integer. Let $\mathcal{W}^- = \{-3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$ and $\mathcal{W}^+ = \{+3\eta\} \times S^1 \times (-2\epsilon, 2\epsilon)$. It is clear that $\Pi(\mathcal{W}^-) \subset S_1$ and $\Pi(\mathcal{W}^+) \subset S_2$. Because on $\overline{B}^{3\eta}_{2\epsilon}$ the Reeb vector field X_{λ_g} is given by ∂_t, it is clear that for every point $p \in B^{3\eta}_{2\epsilon}$ there are $p^- \in \mathcal{W}^-$, $p^+ \in \mathcal{W}^+$, $t^- \in (-6\eta, 0)$ and $t^+ \in (0, 6\eta)$ for which:

$$\phi^{t^-}_{X_{\lambda_g}}(p) = p^- \text{ and } \phi^{t^+}_{X_{\lambda_g}}(p) = p^+ \tag{9.6}$$

This means that trajectories of the flow of X_{λ_g} that enter the box $B^{3\eta}_{2\epsilon}$ enter through $\mathcal{W}^-$ and exit through $\mathcal{W}^+$. They cannot stay inside $B^{3\eta}_{2\epsilon}$ for a very long positive or negative interval of time. We can say even more about these trajectories.

For $\sigma = (\mathfrak{p}, \dot{\mathfrak{p}}) \in (S \times T_pS)$ in $\mathcal{W}^+ \cup \mathcal{W}^-$ let $\widetilde{\sigma} = (\widetilde{\mathfrak{p}}, \dot{\widetilde{\mathfrak{p}}})$ be a lift of σ to the unit tangent bundle $T_1\mathbb{D}$ such that $\widetilde{\mathfrak{p}} \in \check{V}$. The geodesic vector field X_{λ_g} in $\widetilde{\sigma}$ coincides with the horizontal lift of $\dot{\mathfrak{p}}$ ([38][section 1.3]). For δ, $\eta > 0$ and $\epsilon < \frac{\eta}{4|q|\pi}$ sufficiently small we can guarantee that:

- $\Pi(B^{3\eta}_{2\epsilon})$ is contained in $V_{\mathfrak{r},\delta}$,
- for the lifts $\widetilde{\sigma} = (\widetilde{\mathfrak{p}}, \dot{\widetilde{\mathfrak{p}}})$ of points in $\mathcal{W}^+ \cup \mathcal{W}^-$ as above, the vector $\dot{\widetilde{\mathfrak{p}}}$ (which is the projection of the geodesic vector field $X_{\lambda_g}(\widetilde{\sigma})$) satisfies $\angle(\dot{\widetilde{\mathfrak{p}}}, -\partial_y) < \delta$.

With a such a choice of $\delta > 0$, $\eta > 0$ and $0 < \epsilon < \frac{\eta}{4|q|\pi}$, we obtain that for every $\sigma^+ \in \mathcal{W}^+$ there exists $t_{\sigma^+} > 0$ and for every $\sigma^- \in \mathcal{W}^-$ there exists $t_{\sigma^-} < 0$ such that:

$$\phi^{t_{\sigma^+}}_{X_{\lambda_g}}(\sigma^+) \in (T_1S_2) \setminus V_{\mathfrak{r},\delta} \text{ and } \forall t \in [0, t_{\sigma^+}]\ \phi^{t}_{X_{\lambda_g}}(\sigma^+) \notin B^{3\eta}_{2\epsilon}, \tag{9.7}$$

$$\phi^{t_{\sigma^-}}_{X_{\lambda_g}}(\sigma^-) \in (T_1S_1) \setminus V_{\mathfrak{r},\delta} \text{ and } \forall t \in [t_{\sigma^-}, 0]\ \phi^{t}_{X_{\lambda_g}}(\sigma^+) \notin B^{3\eta}_{2\epsilon}. \tag{9.8}$$

To prove this last condition above one uses the fact that $\angle(\dot{\widetilde{p}}, -\partial_y) < \delta$ is small and studies the behavior of geodesics in $(\mathbb{D}, g)$ starting at points close to the real axis and with initial velocity close to $-\partial_y$. It is easy to see that such geodesics have to cut through the region $V_{\mathfrak{r},\delta}$ and visit the interior of both $S_1 \setminus V_{\mathfrak{r},\delta}$ and $S_2 \setminus V_{\mathfrak{r},\delta}$. From now on we will assume that $\delta > 0$, $\eta > 0$ and $0 < \epsilon < \frac{\eta}{4|q|\pi}$ are such that the all the above mentioned properties described for them being sufficiently small, hold simultaneously.

Consider the map $F : B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon} \to B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}$ defined by

$$F(t,s,w) = (t, s + f(w), w) \text{ for } (t,s,w) \in (\eta, 2\eta) \times S^1 \times (-2\epsilon, 2\epsilon), \tag{9.9}$$

where $f(w) = -q\mathcal{R}(\frac{w}{\epsilon})$ (for our previously chosen integer q) and $\mathcal{R} : [-1,1] \to [0, 2\pi]$ satisfies $\mathcal{R} = 0$ on a neighbourhood of -1, $\mathcal{R} = 2\pi$ on a neighbourhood of 1, $0 \le \mathcal{R}' \le 4$ and $\mathcal{R}'$ is an even function.

Our new 3-manifold M is obtained by gluing $T_1S \setminus \overline{B}^{\eta}_{\epsilon}$ and $B^{2\eta}_{2\epsilon}$ using the map F:

$$M = (T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon} \Big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \tag{9.10}$$

Notice that $T_1S = (T_1S \setminus \overline{B}^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon} \Big/ (x \in B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}) \sim (x \in T_1S \setminus \overline{B}^{\eta}_{\epsilon})$. This clarifies our construction of M and shows that M is obtained from T_1S via a Dehn surgery on $L_{\mathfrak{r}}$. We follow [20] to endow M with a contact form which coincides λ_g outside of $B^{2\eta}_{2\epsilon}$. As a preparation we define the function $\beta : (-3\eta, 3\eta) \to \mathbb{R}$:

- β is equal to 1 in an open neighbourhood of $[-2\eta, 2\eta]$,
- $|\beta'| \le \frac{\pi}{\eta}$ and *supp*β is contained in $[-3\eta, 3\eta]$.

Using β we define

$$r(t,w) = \beta(t) \int_{-2\epsilon}^{w} x f'(x) dx. \tag{9.11}$$

We point out to the reader that *supp*(r) is contained in $B^{3\eta}_{\epsilon}$ and therefore so is *supp*(dr). Notice also, that in $B^{2\eta}_{2\epsilon} \setminus \overline{B}^{\eta}_{\epsilon}$ one has $dr = \frac{w}{2} f'(w) dw$.

Again following [20] we define in $T_1S \setminus \overline{B}^{\eta}_{\epsilon}$ the 1-form

$$A_r = dt + wds + dr \text{ for } (-3\eta, -\eta), \tag{9.12}$$

$$A_r = dt + wds - dr \text{ for } (\eta, 3\eta), \tag{9.13}$$

$$A_r = \lambda_g \text{ otherwise.} \tag{9.14}$$

Notice that because *supp*(dr) is contained in $B^{3\eta}_{\epsilon}$ the 1-form A_r is well-defined.

On the box $B^{2\eta}_{2\epsilon}$ we define

$$\widetilde{A} = dt + wds + dr. \tag{9.15}$$

A direct computation shows that $F^*(A_r) = \widetilde{A}$, which means that the gluing map F allows us to glue the 1-forms A_r and $\widetilde{A}$. We denote by λ_{FH} the 1-form in M obtained by gluing $\widetilde{A}$ and A_r. We will denote by $\widetilde{B}$ the following region:

$$\widetilde{B} = ((B_{2\epsilon}^{3\eta} \setminus \overline{B}_{\epsilon}^{\eta}) \subset M) \cup B_{2\epsilon}^{2\eta} \Big/ (x \in B_{2\epsilon}^{2\eta} \setminus \overline{B}_{\epsilon}^{\eta}) \sim (F(x) \in (B_{2\epsilon}^{3\eta} \setminus \overline{B}_{\epsilon}^{\eta}) \tag{9.16}$$

The importance of this region lies in the fact that in $M \setminus \widetilde{B} = T_1 S \setminus B_{2\epsilon}^{3\eta}$, the contact form λ_{FH} coincides with λ_g.

Following [20] one shows through a direct computation that $(dt + wds \pm dr) \wedge (dw \wedge ds) = (1 \pm \frac{\partial r}{\partial t}) dt \wedge dw \wedge ds$. Using the fact that $\epsilon < \frac{\eta}{8\pi|q|}$ one gets that $|\frac{\partial r}{\partial t}| < 1$, thus obtaining that $(dt + wds \pm dr)$ is a contact form. It follows from this that A_r and $\widetilde{A}$ are contact forms in their respective domains and therefore λ_{FH} is a contact form in M. More strongly, Foulon and Hasselblatt proceed to show that if q is non-negative the Reeb flow of λ_{FH} is an Anosov Reeb flow.

9.2 Hypertightness and exponential homotopical growth of contact homology of λ_{FH}

For $q \in \mathbb{N}$ the hypertightness of λ_{FH} follows from the fact that its Reeb flow is Anosov [19]. In this section we give an independent and completely geometrical proof of hypertightness of λ_{FH}, which is valid for every $q \in \mathbb{Z}$.

To understand the topology of Reeb orbits of λ_{FH} we will study trajectories that enter the surgery region $\widetilde{B}$. We start by studying trajectories in $B_{2\epsilon}^{2\eta}$. In this region we have

$$X_{\lambda_{FH}} = \frac{\partial_t}{1 + \partial_t r}. \tag{9.17}$$

This implies, similarly to what happens of λ_g, that for points $p \in B_{2\epsilon}^{2\eta}$ the trajectory $\phi^t_{X_{\lambda_{FH}}}(p)$ leaves the box $B_{2\epsilon}^{2\eta}$ in forward and backward times. More precisely, there exists a constant $\widetilde{a} > 0$ depending only on λ_{FH}, such that for $p \in B_{2\epsilon}^{2\eta}$ there are $\breve{p}^- \in \breve{\mathcal{W}}^- = \{-2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{p}^+ \in \breve{\mathcal{W}}^+ = \{+2\eta\} \times S^1 \times [-2\epsilon, 2\epsilon]$, $\breve{t}^- \in (-\widetilde{a}, 0]$ and $\breve{t}^+ \in [0, \widetilde{a})$ such that

$$\phi^t_{X_{\lambda_{FH}}}(\breve{p}) \text{ is in the interior of } B_{2\epsilon}^{2\eta} \text{ for every } t \in (t^-, t^+),$$

$$\phi^{t^-}_{X_{\lambda_{FH}}}(\breve{p}) = \breve{p}^- \text{ and } \phi^{t^+}_{X_{\lambda_g}}(\breve{p}) = \breve{p}^+. \tag{9.18}$$

We now analyse the trajectories of points $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$. For this, we first notice that on $\widetilde{B} \setminus B_{\epsilon}^{\eta}$ the contact form λ_{FH} is given by $dt + wds \pm dr$, and therefore we have in this region

$$X_{\lambda_{FH}} = \frac{\partial_t}{1 \pm \partial_t r}, \tag{9.19}$$

which is still a positive multiple of ∂_t.

This implies that for every $\breve{p}^- \in \breve{\mathcal{W}}^-$ and $\breve{p}^+ \in \breve{\mathcal{W}}^+$ there exist $t^{\breve{p}^-} < 0$ and $t^{\breve{p}^+} < 0$ such that

$$\phi_{X_{\lambda_{FH}}}^{t^{\breve{p}^-}}(\breve{p}^-) \in \mathcal{W}^- \text{ and } \phi_{X_{\lambda_g}}^{t^{\breve{p}^-}}(\breve{p}^+) \in \mathcal{W}^+ \tag{9.20}$$

Again using that $X_{\lambda_{FH}}$ is a positive multiple of ∂_t on $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ we have that for every point p in $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ whose t coordinate is in $[2\eta, 3\eta]$ the trajectory of the flow $\phi_{X_{\lambda_{FH}}}^t$ going through p is a straight line with fixed coordinates s and w, that goes from $\breve{\mathcal{W}}^+$ to $\mathcal{W}^+$. Analogously, for every point p in $\widetilde{B} \setminus B_{2\epsilon}^{2\eta}$ whose t coordinate is in $[-3\eta, -2\eta]$ the trajectory of the backward flow of $\phi_{X_{\lambda_{FH}}}^t$ going through p is a straight line $\breve{\mathcal{W}}^-$ to $\mathcal{W}^-$.

Summing up, with all the cases considered above we have showed that for every point $p \in \widetilde{B}$ the trajectory of the flow $\phi_{X_{\lambda_{FH}}}^t$ going through p for $t = 0$ intersects $\mathcal{W}^-$ for non-positive time and $\mathcal{W}^+$ for for non-negative time. In other words, all trajectories that intersect $\widetilde{B}$ enter through $\mathcal{W}^-$ and leave through $\mathcal{W}^+$, which means that for all $\breve{p} \in \widetilde{B}$ there exist times $t_{\breve{p}}^- \leq 0$ and $t_{\breve{p}}^+ \geq 0$ such that

$$\phi_{X_{\lambda_{FH}}}^{t_{\breve{p}}^+}(\breve{p}) \in \mathcal{W}^+, \tag{9.21}$$

$$\phi_{X_{\lambda_{FH}}}^{t_{\breve{p}}^-}(\breve{p}) \in \mathcal{W}^-, \tag{9.22}$$

$$\phi_{X_{\lambda_{FH}}}^{t}(\breve{p}) \in \widetilde{B} \text{ for all } t \in [t_{\breve{p}}^-, t_{\breve{p}}^+]. \tag{9.23}$$

Now, because on $M \setminus \widetilde{B} = T_1 S \setminus B_{2\epsilon}^{3\eta}$ the contact form λ_{FH} coincides with λ_g we have that trajectories of $X_{\lambda_{FH}}$ starting at $\mathcal{W}^-$ at the time $t = 0$ have to leave $M \setminus N$ as time diminishes before reentering on $\widetilde{B}$. Similarly the trajectories starting at $\mathcal{W}^+$ have to leave $M \setminus N$ for positive time before reentering on $\widetilde{B}$. More precisely, one can use equations (9.7) and (9.8) to show that for $p^- \in \mathcal{W}^-$ and $p^+ \in \mathcal{W}^+$ there exist $t_{p^-} < 0$ and $t_{p^+} > 0$ such that

$$\phi_{X_{\lambda_{FH}}}^{t_{p^+}}(p^+) \in M_2 \setminus N \text{ and } \forall t \in [0, t_{p^+}] \; \phi_{X_{\lambda_{FH}}}^{t}(p^+) \notin \widetilde{B}, \tag{9.24}$$

$$\phi_{X_{\lambda_{FH}}}^{t_{p^-}}(p^-) \in M_1 \setminus N \text{ and } \forall t \in [t_{p^-}, 0] \; \phi_{X_{\lambda_{FH}}}^{t}(p^-) \notin \widetilde{B}, \tag{9.25}$$

where

$$M_1 = (T_1S_1 \setminus B^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon}(-) \Big/ (x \in B^{2\eta}_{2\epsilon}(-) \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in ((B^{3\eta}_{2\epsilon} \cap T_1S_1) \setminus \overline{B}^{\eta}_{\epsilon}),$$
$$M_2 = (T_1S_2 \setminus B^{\eta}_{\epsilon}) \cup B^{2\eta}_{2\epsilon}(+) \Big/ (x \in B^{2\eta}_{2\epsilon}(+) \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in ((B^{2\eta}_{2\epsilon} \cap T_1S_2) \setminus \overline{B}^{\eta}_{\epsilon}),$$
$$N = \Pi^{-1}(V_{c,\delta}\setminus) \cup B^{2\eta}_{2\epsilon}(-) \Big/ (x \in B^{2\eta}_{2\epsilon}(-) \setminus \overline{B}^{\eta}_{\epsilon}) \sim (F(x) \in ((B^{3\eta}_{2\epsilon} \cap T_1S^1) \setminus \overline{B}^{\eta}_{\epsilon}),$$

for $B^{2\eta}_{2\epsilon}(-) = [-2\eta, 0] \times S^1 \times (-2\epsilon, 2\epsilon)$ and $B^{2\eta}_{2\epsilon}(+) = [0, 2\eta] \times S^1 \times (-2\epsilon, 2\epsilon)$.

Remark: It is not hard to see that $M = M_1 \cup M_2 \Big/ (x \in \partial M_1) \sim (\widetilde{F}(x) \in \partial M_2)$. Here $\widetilde{F}$ is a Dehn twist which coincides with $(s + f(w), w)$ for $w \in [-2\epsilon, 2\epsilon]$ and is the identity elsewhere. This picture of M is closer to the one in the paper [26] and shows that M is a graph manifold (a graph manifold is one whose JSJ decomposition consists of Seifert S^1 bundles). By using this description of M and applying Van-Kampen's to analyse the fundamental group of M, Handel and Thurston show that, for q not belonging to a finite subset of $\mathbb{Z}$, no finite cover of M is a Seifert manifold thus obtaining that M is an "exotic" graph manifold.

From their definition one sees that as manifolds, $M_1 \cong T_1S_1$ and $M_2 \cong T_1S_2$. This implies ∂M_1 and ∂M_2 are incompressible tori in, respectively, M_1 and M_2. If we look at M_1 and M_2 as submanifolds of M, their boundary $\mathbb{T}$ coincide and is also incompressible in M. We remark that $M_i \setminus N$ is diffeomorphic to $T_1S_i \setminus \Pi^{-1}(V_{c,\delta})$ which is diffeomorphic to T_1S_i for $i = 1, 2$.

In a similar way we can describe the topology of N. Let $N_i = M_i \cap N$. Reasoning identically as one does to show that M_i is diffeomorphic to T_1S_i one shows that N_i is diffeomorphic to a thickned two torus $\mathcal{T}^2 \times [-1, 1]$. As N is obtained from N_1 and N_2 by gluing them along $\mathbb{T}$ (which is a boundary component of both of them) we have that N is also diffeomorphic to the product $\mathcal{T}^2 \times [-1, 1]$.

The discussion above proves the following

Lemma 9.1. *For all $\breve{p} \in \widetilde{B}$ the trajectory $\{\phi^t_{X_{\lambda_{FH}}}(\breve{p}) \mid t \in \mathbb{R}\}$ intersects $M_1 \setminus N$ and $M_2 \setminus N$.*

Proof: We have already established that for $\breve{p} \in \widetilde{B}$ its trajectory intersect $\mathcal{W}^+$ for some non-negative time and $\mathcal{W}^-$ for some non-positive time, as it is shown in equations (9.21) and (9.22). One now applies equations (9.24) and (9.25) to finish the proof of the lemma. □

Notice that trajectories can only enter in $\widetilde{B}$ through the wall $\mathcal{W}^-$ which is contained in M_1 and can only exit $\widetilde{B}$ through the wall $\mathcal{W}^+$ which is contained in M_2. We also point out that all trajectories of the flow $\phi^t_{X_{\lambda_{FH}}}$ are transversal to $\mathbb{T}$, with the exception of the two Reeb orbits which correspond to parametrizations of the hyperbolic geodesic $\mathfrak{r}$ (they continue to exist as periodic orbits after the surgery because they are distant from the surgery region).

We will deduce from the previous discussion the following important lemma.

Lemma 9.2. *Let $\gamma([0,T'])$ be a trajectory of $X_{\lambda_{FH}}$ such that $\gamma(0) \in \mathbb{T}$, $\gamma(T') \in \mathbb{T}$ and for all $t \in (0,T')$ we have $\gamma(t) \notin \mathbb{T}$ (notice that in such a situation $\gamma([0,T']) \subset M_i$ for some i equals to 1 or 2). Then $\gamma([0,T']) \cap (M_i \setminus N)$ is non-empty.*

Proof: We divide the proof in 3 possible scenarios.

First case: suppose that $\gamma([0,T']) \cap \widetilde{B}$ is empty. In this case $\gamma([0,T'])$ also exists as a hyperbolic geodesic with endpoints in the closed geodesic $\mathfrak{r}$. It follows from the convexity of the hyperbolic metric that $[\gamma([0,T'])] \in \pi_1(T_1S_i, \mathbb{T})$ is non-trivial. This implies that $[\gamma([0,T'])] \in \pi_1(M_i, \mathbb{T})$ is non-trivial which can be true only if $\gamma([0,T']) \cap (M_i \setminus N)$ is non-empty since N is a tubular neighbourhood of $\mathbb{T}$.

Second case: suppose that $\gamma([0,T']) \cap \widetilde{B}$ is non-empty and $\gamma([0,T']) \subset M_2$. Take $\widehat{t} \in [0,T']$ such that $\gamma(\widehat{t}) \in \widetilde{B}$. We know from our previous discussion that there are $\widehat{t_1} \leq \widehat{t} \leq \widehat{t_2}$ such that $\gamma([\widehat{t_1}, \widehat{t_2}]) \subset \widetilde{B}$, $\gamma(\widehat{t_1}) \in (\mathbb{T} \cap \widetilde{B})$ and $\gamma(\widehat{t_2}) \in \mathcal{W}^+$; notice that in the coordinates (t,s,w) for $\widetilde{B}$ considered previously, $\mathbb{T} \cap \widetilde{B}$ is the annulus $\{0\} \times S^1 \times (-2\epsilon, 2\epsilon)$. From this picture it is clear that for t smaller that $\widehat{t_1}$ the trajectory enters in M_1. Therefore we must have $\widehat{t_1} = 0$ and $\gamma([0,\widehat{t_2}]) \subset \widetilde{B}$. Notice also that for all t slightly bigger than $\widehat{t_2}$ the trajectory is outside $\widetilde{B}$. Because trajectories of $X_{\lambda_{FH}}$ can only enter $\widetilde{B}$ in M_1 we obtain that $\gamma([\widehat{t_2}, T'])$ does not intersect the interior of $\widetilde{B}$ and therefore exists as a hyperbolic geodesic in T_1S_2. Now, using equations (9.7) and (9.8) we obtain that, because $\gamma(\widehat{t_2}) \in \mathcal{W}^+$, the trajectory $\gamma : [\widehat{t_2}, T'] \to M_2$ has to intersect $M_2 \setminus N$ before hitting $\mathbb{T}$ at $t = T'$. Thus there is some $t \in (\widehat{t_2}, T')$ for which $\gamma(t) \in M_2 \setminus N$.

Third case: the proof in the case where $\gamma([0,T']) \cap \widetilde{B}$ is non-empty and $\gamma([0,T']) \subset M_1$ is analogous to the one of the Second case.

This three cases exhaust all possibilities and therefore prove the lemma. $\square$

Our reason for introducing the above decomposition of M into M_1 and M_2 and for proving the lemmas above is to introduce the following representation of Reeb orbits of λ_{FH}. Let (γ, T) be a Reeb orbit of λ_{FH} which intersects both $M_1 \setminus N$ and $M_2 \setminus N$. We can assume that the chosen parametrization of the Reeb orbit is such that $\gamma(0) \in \partial N$, and that there are $t_+ > 0$ and $t_- < 0$ such that:

$$\gamma(t_+) \in M_1 \setminus N \text{ and } \gamma([0, t_+]) \in M_1 \cup N, \tag{9.26}$$

$$\gamma(t_-) \in M_2 \setminus N \text{ and } \gamma([t_-, 0]) \in M_2 \cup N. \tag{9.27}$$

This means that in an interval of the origin γ is coming from $M_2 \setminus N$ and going to $M_1 \setminus N$. It follows from Lemma 9.2 that there exists a unique sequence $0 = t_0 < t_{\frac{1}{2}} < t_1 < t_{\frac{3}{2}} < ... < t_n = T$ such that $\forall k \in \{0, ..., n-1\}$:

- $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_i$ for i equals to 1 or 2,
- $\gamma([t_{k+\frac{1}{2}}, t_{k+1}]) \in N$ and there is a unique $\widetilde{t_k} \in [t_{k+\frac{1}{2}}, t_{k+1}]$ such that $\gamma(\widetilde{t_k}) \in \mathbb{T}$,
- if $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_i$ then $\gamma([t_{k+1}, t_{k+\frac{3}{2}}]) \subset M_j$ for $j \neq i$.

Notice that $\gamma([t_0, t_{\frac{1}{2}}]) \subset M_1$ and $\gamma([t_{n-1}, t_{n-\frac{1}{2}}]) \subset M_2$. This implies that n is even so that we can write $n = 2n'$, and that $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_1$ for k even, and $\gamma([t_k, t_{k+\frac{1}{2}}]) \subset M_2$ for k odd. For each $k \in \{0, ..., 2n'-1\}$ the existence of the unique $\widetilde{t_k}$ in the interval $[t_{k+\frac{1}{2}}, t_{k+1}]$ for which $\gamma(\widetilde{t_k}) \in \mathbb{T}$ is guaranteed from Lemma 9.2 and the fact that $\mathbb{T}$ is the hypersurface that separates M_1 and M_2.

In order to obtain information on the free homotopy class of (γ, T) we observe that for $\gamma([t_k, t_{k+\frac{1}{2}}])$ coincides with a hyperbolic geodesic segment in $T_1 S_i$ starting and ending $V_{\mathfrak{r},\delta}$. Therefore, as we have previously seen the homotopy class $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ in $\pi_1(T_1 S_i, V_{\mathfrak{r},\delta})$ is non-trivial which implies that $\gamma([t_k, t_{k+\frac{1}{2}}])$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$. We consider now the curve $\gamma([\widetilde{t_k}, \widetilde{t}_{k+1}])$: it is the concatenation of 3 curves, the first and the third ones being completely contained in N and the middle one being $\gamma([t_k, t_{k+\frac{1}{2}}])$; from this description and the fact that $\gamma([t_k, t_{k+\frac{1}{2}}])$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$ it is clear that $\gamma([\widetilde{t_k}, \widetilde{t}_{k+1}])$ is also non-trivial in $\pi_1(M_i, N)$ (and also non-trivial in $\pi_1(M_i, \mathbb{T})$).

We now denote by $\widetilde{M}$ the universal cover of M and and $\widehat{\pi} : \widetilde{M} \to M$ a covering map. From the incompressibility of $\mathbb{T}$ it follows that every lift of $\mathbb{T}$ is an embedded plane in $\widetilde{M}$. We denote by $\widetilde{N}^0$ a lift of N. Because N is a thickened neighbourhood of an incompressible torus it follows that $\widetilde{N}^0$ is diffeomorphic to $\mathbb{R}^2 \times [-1, 1]$, i.e. it is a thickened neighbourhood of an embedded plane in $\widetilde{M}$. Because N separates M in two components, it follows that $\widetilde{N}^0$ separates $\widetilde{M}$ is two connected components. $\partial \widetilde{N}^0$ is the union of two embedded planes P^0_- and P^0_+ which are characterized by the fact that there are neighbourhoods V_- and V_+ of, respectively, P^0_- and P^0_+ such that $\widehat{\pi}(V_-) \subset M_1$

and $\widehat{\pi}(V_+) \subset M_2$. We will denote by C^0_- the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_-, and by C^0_+ the connected component of $\widetilde{M} \setminus \widetilde{N}^0$ which intersects V_+.

As we saw earlier, $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ is a non-trivial relative homotopy class in $\pi_1(M_i, N)$. We show that this class remains non-trivial when seen in $\pi_1(M, N)$. Let $\mathbb{T}_i = \partial N \cap M_i$. Because N is obtained by attaching over each point of $\mathbb{T}_i$ a small compact interval (i.e it is a bundle over $\mathbb{T}_i$ whose fibers are intervals) it follows that $[\gamma([t_k, t_{k+\frac{1}{2}}])$ would be trivial in $\pi_1(M_i, \mathbb{T}_i)$ if, and only if, it is trival in $\pi_1(M_i, N)$, which is not the case. As $\mathbb{T}_i$ is isotopic to $\mathbb{T}$, it is also an incompressible torus that divide M in two components. Now, $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ would be trivial in $\pi_1((M_i \setminus int(N)), \mathbb{T}_i)$ if, and only if, there existed a curve $\mathfrak{c}$ in $\mathbb{T}_i$ with endpoints $\gamma(t_k)$ and $\gamma(t_{k+\frac{1}{2}})$, such that the concatenation $\gamma * \mathfrak{c}$ was contractible in $(M_i \setminus int(N))$. Because of the incompressibility of $\mathbb{T}_i$ such a curve $\gamma * \mathfrak{c}$ should be contractible in $(M_i \setminus int(N))$ if, and only if, it was contractible in M. This implies that $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ would be trivial in $\pi_1(M, \mathbb{T}_i)$ if, and only if, it was trivial in $\pi_1((M_i \setminus int(N)), \mathbb{T}_i)$ which we know not to be the case. Lastly, again because N is an interval bundle over $\mathbb{T}_i$, it is clear that as $[\gamma([t_k, t_{k+\frac{1}{2}}])]$ is not trivial in $\pi_1(M, \mathbb{T}_i)$ it cannot be trivial in $\pi_1(M, N)$, as we wished to show.

Let now $\widetilde{\gamma}$ be a lift of γ such that $\widetilde{\gamma}(0) \in \widetilde{N}^0$. We know that $\widetilde{\gamma}([t_{2n'-\frac{1}{2}} - T, t_{\frac{1}{2}}]) \subset \widetilde{N}^0$. It will be useful to us to define the following sequence:

$$\widetilde{t}_i = q_i T + t_{r_i}, \tag{9.28}$$

where q_i and $r_i < 2n'$ are the unique integers such that $i = q_i(2n') + r_i$. Associated to $\widetilde{t}_i$ we associate the lift $\widetilde{N}^i$ of N, which is determined by the property that $\widetilde{\gamma}(\widetilde{t}_i) \in \widetilde{N}^i$. It is clear that the sequence $\widetilde{N}^i$ contains all lifts of N which are intersected by the curve $\widetilde{\gamma}(\mathbb{R})$. For the lifts $\widetilde{N}^i$ we define the connected components C^i_- and C^i_+ of $\widetilde{M} \setminus \widetilde{N}^i$, and the planes P^i_- and P^i_+ analogously as how we defined them for $\widetilde{N}^0$. A priori it could be that for $i \neq j$ we had $\widetilde{N}^i = \widetilde{N}^j$. We will show however, that this cannot happen.

Firstly, $\widetilde{N}^0 \neq \widetilde{N}^1$because $\gamma([\widetilde{t}_0, \widetilde{t}_1])$ is non-trivial in $\pi_1(M, N)$. Also, we have that $\widetilde{N}^1 \subset C^0_-$ because $\gamma([t_0, t_{\frac{1}{2}}]) \subset M_1$. An identical reasoning shows that $\widetilde{N}^2 \neq \widetilde{N}^1$ and:

$$\widetilde{N}^2 \subset C^1_+. \tag{9.29}$$

On the other hand we have that $\widetilde{N}^0 \subset C^1_-$, because $\widetilde{\gamma}([\widetilde{t}_0, t_{\frac{1}{2}}])$ gives a path totally contained in $\widetilde{M} \setminus \widetilde{N}^1$ connecting $\widetilde{N}^0$ and P^1_-. As $\widetilde{N}^2 \subset C^1_+$ and $\widetilde{N}^0 \subset C^1_-$, we must have $\widetilde{N}^2 \neq \widetilde{N}^0$. In an identical way, one shows that $\widetilde{N}^3 \neq \widetilde{N}^1$, and more generally that $\widetilde{N}^{i+2} \neq \widetilde{N}^i$ and $\widetilde{N}^{i+1} \neq \widetilde{N}^i$. Now for $\widetilde{N}^3$, we have that $\widetilde{N}^3 \subset C^2_-$. As $\widetilde{\gamma}([\widetilde{t}_0, t_{\frac{3}{2}}])$ is a path completely contained in $\widetilde{M} \setminus \widetilde{N}^2$ connecting $\widetilde{N}^0$ and P^2_+ we obtain that $\widetilde{N}^0 \subset C^2_+$, and therefore $\widetilde{N}^3 \neq \widetilde{N}^0$.

Proceeding inductively along this line one obtains that $\widetilde{N}^i \neq \widetilde{N}^0$ for all $i \neq 0$, and more generally, $\widetilde{N}^i \neq \widetilde{N}^j$ for all $i \neq j$. As a consequence of this, we obtain that the curve $\widetilde{\gamma}(\mathbb{R})$ cannot be homeomorphic to a circle and therefore $\gamma(\mathbb{R})$ cannot be contractible. We are ready for the main result of this section.

Proposition 9.3. λ_{FH} *is hypertight.*

Proof: there are two possibilities for Reeb orbits.

Possibility 1: the Reeb orbit γ visits both $M_1 \setminus N$ and $M_2 \setminus N$.
In this case, we have just showed above that γ is not contractible.

Possibility 2: the Reeb orbit γ is completely contained in M_i for $i \in \{1, 2\}$.
In this case, the Reeb orbit does not visit the surgery region $\widetilde{B}$. Therefore it existed also before the surgery as a closed hyperbolic geodesic in $M_i \setminus \widetilde{B} = T_1 S_i \setminus B^{3\eta}_{2\epsilon}$. Such a closed geodesic is non-contractible in $T_1 S_i$ which is diffeomorphic to M_i. We have thus obtained that $\gamma \subset M_i$ is non-contractible in M_i.

Looking now at M_i as a submanifold with boundary of M, we recall that ∂M_i is an incompressible torus in M. This implies that every non-contractible closed curve in M_i remains non-contractible in M. Therefore γ is also a non-contractible Reeb orbit for this case. □

9.2.1 Exponential homotopical growth of cylindrical contact homology for λ_{FH}

We proceed now to obtain more information on the properties of periodic orbits of $X_{\lambda_{FH}}$. We state the following important fact:

Lemma 9.4. *If a Reeb orbit* (γ, T) *of* λ_f *visits both* $M_1 \setminus N$ *and* $M_2 \setminus N$*, then any curve freely homotopic to* (γ, T) *must always intersect* $\mathbb{T}$.

Proof of lemma: As we saw earlier the lift $\widetilde{\gamma}$ intersects all the elements of the sequence $\widetilde{N}_i$ (of lifts of N) which satisfy $\widetilde{N}_i \neq \widetilde{N}_j$ for all $i \neq j$.

Introducing an auxiliary distance d on the compact manifold M (coming from a Riemannian metric) we obtain an auxiliary distance $\widetilde{d}$ on $\widetilde{M}$ by pulling d back by the covering map. It is clear that for i sufficiently big the $\widetilde{d}$ distance between $\widetilde{N}_{\pm i}$ and $\widetilde{N}_0$ becomes arbitrarily large. As a consequence, one obtains that for each $K > 0$ there exists $t_K > 0$ such that $\widetilde{d}(\widetilde{\gamma}(\pm t_K), \widetilde{N}_0) > K$.

Let now $\zeta : [0, T] \to M$ be closed curve freely homotopic to $\gamma([0, T])$. An homotopy $H : [0, T] \times [0, 1] \to M$ generates an homotopy $\widetilde{H} : \mathbb{R} \times [0, 1] \to \widetilde{M}$ from a lift $\widetilde{\gamma}$ and a lift $\widetilde{\zeta}$. Using the fact that H is uniformly continuous one proves that there exists a constant $\mathfrak{C} > 0$ such that $\widetilde{d}(\widetilde{H}(\{t\} \times [0,1]), \widetilde{\gamma}(t)) < \mathfrak{C}$ for all $t \in \mathbb{R}$.

Take now $K > 2\mathfrak{C}$. Using the triangle inequality and the facts that $\widetilde{d}((\widetilde{H}(\{t\} \times [0,1])), \widetilde{\gamma}(t)) < \mathfrak{C}$ and $\widetilde{d}(\widetilde{\gamma}(\pm t_K), \widetilde{N}_0) > K$ we obtain $H(\{t_K\} \times [0,1])$ is always in the same connected component of $\widetilde{\gamma}(t_K)$. This implies that $\widetilde{\zeta}(\mathbb{R})$ visits both connected components of $\widetilde{M} \setminus \widetilde{N}_0$ and must thus intersect $\widetilde{N}_0$. Even more, because $\widetilde{\zeta}(\mathbb{R})$ intersects both components of $\partial\widetilde{N}_0$ we have that ζ visits both components of $M \setminus N$ and therefore has to intersect $\mathbb{T}$. This completes the proof of the lemma. □

We are now ready for the most important result of this section:

Theorem 9.5. *Let $(M, \xi_{(q,c)}))$ be the contact manifold endowed obtained by the Foulon-Hasselblatt surgery, and λ_{FS} be the contact form obtained via the Foulon-Hasselblat surgery on the Legendrian lift $L_{\mathfrak{r}} \subset T_1S$. Then λ_{FH} is hypertight and its cylindrical contact homology has exponential homotopical growth.*

We divide the proof that $C\mathbb{H}^{cyl}(M, \lambda_{FH})$ has exponential homotopical growth in steps.

Step 1: A special class of Reeb orbits.

We will obtain our estimate by looking at Reeb orbits which are completely contained in the component M_1. As we saw previously such orbits never cross the surgery region $\widetilde{B}$. Thus they are in a region where λ_{FH} coincides with λ_g, and such Reeb orbits exist also as closed geodesics in (S_1, g). Conversely, every closed geodesic in (S_1, g) does not cross the region $B^{3\eta}_{2\epsilon}$ and thus also exist as Reeb orbit of λ_{FH}. This gives a bijective correspondence between closed geodesics of (S_1, g) which are not homotopic to a multiple of ∂S_1 and Reeb orbits of λ_{FH} which are completely contained in M_1.

Let $\bigwedge(S_1)$ denote the set of free homotopy classes in S_1 which are not covers of $[\partial S_1]$. We know that each $\rho \in \bigwedge(S_1)$ contains exactly one closed geodesic c_ρ. Letting γ_ρ be the canonical lift of c_ρ to T_1S_1, we know that γ_ρ is a Reeb orbit of λ_g. As we saw above each γ_ρ can also be seen as a Reeb orbit of λ_{FH}. We will denote by $\bigwedge(S_1)^{\leq T}$ the set primitive of free homotopy classes in S_1 whose unique closed geodesic has period smaller or equal to T. Because g is hyperbolic it is a well known fact that there exist constants $a > 0$, b such that $\sharp(\bigwedge(S_1)^{\leq T}) \geq e^{aT+b}$.

Let $\Theta : \bigwedge(S_1) \to \bigwedge(T_1S_1)$ (where $\bigwedge(T_1S_1)$ is the free loop space of T_1S_1), be the map which associates c_ρ to γ_ρ in T_1S_1. $\Theta : \bigwedge(S_1) \to \bigwedge(T_1S_1)$ is easily seen to be injective. Because T_1S_1 is diffeomorphic to M_1 we can also view $\Theta(\bigwedge(S_1))$ as a subset of the free loop space $\bigwedge(M_1)$ of M_1.

Step 2:

Let $i : M_1 \to M$ be the injection obtained by looking at M_1 as a component of M. As seen before the boundary $\partial(i(M_1)) = \mathbb{T}$ is an incompressible torus in M. We consider the induced map of free loop spaces $i_* : \bigwedge(M_1) \to \bigwedge(M)$. As a consequence of the incompressibility of $\partial(i(M_1)$, the restriction of i_* to $\Theta(\bigwedge(S_1))$ is injective.

To see that, it suffices to show the following claim: if ζ and ζ' are curves in M_1 which cannot be isotoped to a curve in ∂M_1 and which are in the same free homotopy class in M, then ζ and ζ' are freely homotopic in M_1. For ζ and ζ' satisfying the hypothesis of our claim there is a cylinder Cyl in M whose boundary components are ζ and ζ' which intersects ∂M_1 transversely. In such a case, Cyl intersects ∂M_1 in a finite collection of curves $\{w_n\}$ which are all contractible in M; the contractibility of these curves is due to the fact that both ζ and ζ' cannot be isotoped to a curve contained in ∂M_1. The incompressibility of ∂M_1 implies that these $\{w_n\}$ are all contractible already in ∂M_1. Now, we cut the discs in Cyl whose boundary are the curves c_n and substitute them by discs contained in ∂M_1. This produces a cylinder Cyl$'$ completely contained in M_1 whose boundaries are ζ and ζ'. This implies that ζ and ζ' were already in the same free homotopy class in M_1, as we wished to show.

From step one, we know that for each $\rho \in i_*(\Theta(\bigwedge(S_1)))$ there is a Reeb orbit γ_ρ in ρ.

Step 3: For each $\rho \in i_*(\Theta(\bigwedge(S_1)))$, the Reeb orbit γ_ρ considered in Step 1 is the unique Reeb orbit of λ_{FH} in ρ.

Let γ be a Reeb orbit in ρ. If it is contained in M_1, we know that γ exists also as a closed geodesic in (S_1, g). Using an argument as in step 2 above, it is easy to show that γ and γ_ρ are freely homotopic in M_1, and therefore also in T_1S_1. Projecting to S_1 we obtain that γ and γ_ρ are lifts of geodesics of (S_1, g) in a same free homotopy class of S_1. But for each free homotopy class of S_1 there is a unique closed geodesic of (S_1, g); this implies that $\gamma = \gamma_\rho$.

Step 3 will now follow if we prove the following claim: **every Reeb orbit of λ_{FH} in ρ is completely contained in M_1.**

Proof of the claim: if γ was contained in M_2 then it would be possible to isotopy γ_ρ to a curve completely contained in ∂M_1. This is impossible by the definition of $\bigwedge(S_1)$.

The only remaining possibility is that γ visit both M_1 and M_2. In this case, it has to visit both $M_1 \setminus N$ and $M_2 \setminus N$ (the reason for that is that if γ is completely contained in $M_i \cup N$ convexity of the hyperbolic metric implies that γ is in M_i). As γ visits both $M_1 \setminus N$ and $M_2 \setminus N$, we know from the Lemma 9.4 that every curve which is freely homotopic to γ has to intersect the torus $\mathbb{T}$. As γ_ρ does not intersect $\mathbb{T}$ it cannot be freely homotopic to γ which implies that $\gamma \notin \rho$, finishing the proof of step 3.

Step 4: End of the proof.

From the previous steps we know that for each $\rho \in i_*(\Theta(\bigwedge(S_1)))$, there exists a unique Reeb orbit $\gamma_\rho \in \rho$. This implies that for each $\rho \in i_*(\Theta(\bigwedge(S_1)))$, the cylindrical contact homology $C\mathbb{H}^{cyl}_\rho(M, \lambda_{FH}) \neq 0$.

Let $\rho \in i_*(\Theta(\bigwedge(S_1)^{\leq T}))$. Then as we showed, in the previous steps, the unique Reeb orbit of λ_{FH} in ρ has action smaller or equal than T and $C\mathbb{H}^{cyl}_\rho(M, \lambda_{FH}) \neq 0$. This implies that:

$$N^{cyl}_T(\lambda_{FH}) \geq \sharp(i_*(\Theta(\bigwedge(S_1)^{\leq T}))). \tag{9.30}$$

As i_* restricted to $\Theta(\bigwedge(S_1)^{\leq T})$ is injective, and Θ is injective we conclude that:

$$\sharp(i_*(\Theta(\bigwedge(S_1)^{\leq T}))) = \sharp(\bigwedge(S_1)^{\leq T}) \geq e^{aT+b}. \tag{9.31}$$

Combining formulas (9.30) and (9.31), we obtain

$$N^{cyl}_T(\lambda_{FH}) \geq e^{aT+b}. \tag{9.32}$$

□

Chapter 10

A dynamical obstruction to planarity of contact 3-manifolds

In this chapter we establish a dynamical obstruction for a contact manifold to be planar. We then use this obstruction to provide some new examples of non-planar contact 3-manifolds. All the results in this Chapter were obtained in collaboration with Chris Wendl.

10.1 Normal first Chern number

In this section we define the normal first Chern number of a pseudoholomorphic building. This is the main tool which we will use to prove our dynamical obstruction result.

We begin by introducing some notation. Let $(\mathbf{S}, \widetilde{v})$ denote a finite energy nodal pseudoholomorphic curve in a symplectization $(\mathbb{R}\times Y, d(e^s\lambda), J)$ or in an exact symplectic cobordism $(W, \varpi, \overline{J})$. Here $\mathbf{S} = (S, j, \Gamma, \Delta)$ denotes the domain of the pseudoholomorphic map $\widetilde{v}$. The set Γ of punctures of $(\mathbf{S}, \widetilde{v})$ can be partitioned into two disjoint sets Γ_0 and Γ_1 defined in the following manner:

- $z \in \Gamma_0$ if, and only if, the pseudoholomorphic map $\widetilde{v}$ is asymptotic at z to a Reeb orbit with even Conley-Zehnder index,
- $z \in \Gamma_1$ if, and only if, the pseudoholomorphic map $\widetilde{v}$ is asymptotic at z to a Reeb orbit with odd Conley-Zehnder index.

Clearly every element of Γ must be either in Γ_0 or Γ_1, and can belong to only one of these sets. The elements of Γ_0 are called even punctures and the elements of Γ_1 even punctures. Assume that all Reeb orbits detected by $(\mathbf{S}, \widetilde{v})$ are non-degenerate. We define:

Definition 10.1. The normal first Chern number $c_N(\widetilde{v}) \in \mathbb{Z}$ of a finite energy nodal pseudo holomorphic curve $(\mathbf{S}, \widetilde{v})$ is defined as

$$2c_N(\widetilde{v}) := ind(\widetilde{v}) - 2 + 2g + \#\Gamma_0, \tag{10.1}$$

where g is the genus of the compactification $\overline{\mathbf{S}}$ of the domain of $(\mathbf{S}, \widetilde{v})$.

As explained in [45] the normal first Chern number has a clear intuitive meaning when the curve $\widetilde{v} = (a, v)$ is embedded and lives in a symplectization $(\mathbb{M} \times Y, d(e^s\lambda), J)$. Because of the $\mathbb{R}$-invariance of the almost complex structure J we have the following consequence (see [45]): **the projection v of the finite energy curve $\widetilde{w}$ is an embedded surface in M if, and only if, the normal first Chern number $c_N(\widetilde{v})$ vanishes.**

In the more general case where the finite energy surface $\widetilde{v}$ lives in a symplectic cobordism W, $c_N(\widetilde{v})$ has an important role in determining whether $\widetilde{v}$ can be part of a finite energy foliation of W; see [29, 45].

The normal first Chern number can also be defined for pseudoholomorphic buildings. Let $\widetilde{u}$ be a pseudoholomorphic building in a symplectization or in an exact symplectic cobordism as defined in sections 3.3.1 and 3.3.2, and denote by $\mathbf{S}$ the domain of $\widetilde{u}$. Recall that $\mathbf{S} = (S, j, \Gamma, \Delta, \phi)$ is a partially decorated Riemann surface and that Γ is the union of positive punctures of the top level $\widetilde{u}$ and the negative punctures of the bottom level of $\widetilde{u}$. Just as we did previously in the case of nodal pseudoholomorphic curves, we partition Γ into the sets Γ_0 of even punctures and Γ_1 of odd punctures. With this in mind we define

Definition 10.2. Let $\widetilde{u}$ be a finite energy pseudoholomorphic building with domain $\mathbf{S}$ and that detects only non degenerate Reeb orbits at every level. The normal first Chern number $c_N(\widetilde{u}) \in \mathbb{Z}$ of the building $\widetilde{u}$ is defined by

$$2c_N(\widetilde{v}) := ind(\widetilde{v}) - 2 + 2g + \#\Gamma_0 \tag{10.2}$$

where g is the arithmetic genus $\mathbf{S}$, i.e the genus of the compactification $\overline{\mathbf{S}}$.

We state two formulas of Wendl which are useful tools to compute the normal first Chern number of a pseudoholomorphic building. The first is Proposition 4.8 of [44], which relates the normal first Chern number of a pseudoholomorphic curve and of its multiple covers.

Proposition 10.3. *[44] Let (S, j) be a compact connected Riemann surface, $\widetilde{\Gamma} \subset S$ a finite set and $\widetilde{v}$ be a finite energy pseudoholomorphic curve from $S \backslash \widetilde{\Gamma}$ to a symplectization or an exact symplectic cobordism. Let then (S', j') be another compact Riemann surface, $\psi : (S', j') \to (S, j)$ a holomorphic map, $\widetilde{\Gamma}' := \psi^{-1}(\widetilde{\Gamma})$ and define $\widetilde{w} := v \circ \psi$. Then the normal first Chern number $c_N(\widetilde{w})$ is given by the formula*

$$c_N(\widetilde{w}) = deg(\psi) c_N(\widetilde{v}) + Z(d\dot{\psi}) + Q, \tag{10.3}$$

where $\dot{\psi} : S \backslash \widetilde{\Gamma} \to S' \backslash \widetilde{\Gamma}'$ is the restriction of ψ to $S \backslash \widetilde{\Gamma}$, $deg(\psi)$ is the degree of ψ, $Z(d\dot{\psi})$ is the algebraic count of zeroes of the differential of $\dot{\psi}$ and Q is a positive number.

The second is a formula which relates the normal first Chern number of a pseudoholomorphic building and that of its connected components. It is proved in Proposition 6.4 of [45].

Proposition 10.4. *Let $\widetilde{u}$ be a pseudoholomorphic building in a symplectization or in an exact symplectic cobordism whose domain we denote by $\mathbf{S}$. Let N be the number of connected components of $\mathbf{S}$, and denote by (S_k, j_k) for $1 \leq k \leq N$ the distinct connected components of $\mathbf{S}$. We let $\widetilde{u}_k$ the restriction of $\widetilde{u}$ to (S_k, j_k), and refer to the maps $\widetilde{u}_k$ as the components of the building $\widetilde{u}$.*[1] *Then the normal first Chern number $c_N(\widetilde{u})$ satisfies*

$$c_N(\widetilde{u}) = \sum_{i=1}^{N} c_N(\widetilde{u}_k) + \sum_{\{\overline{z}, \underline{z}\} \subset \Delta_{br}(\widetilde{u})} p(\gamma_{\{\overline{z}, \underline{z}\}}) + \#\Delta_N(\widetilde{u}), \tag{10.4}$$

where $\Delta_{br}(\widetilde{u})$ is the set of breaking pairs of $\widetilde{u}$, $p(\gamma_{\{\overline{z}, \underline{z}\}})$ is the parity of the Reeb orbit $\gamma_{\{\overline{z}, \underline{z}\}}$ associated to the breaking pair $\{\overline{z}, \underline{z}\}$, and $\Delta_N(\widetilde{u})$ is the set of nodes of $\widetilde{u}$.

10.2 Holomorphic open book decompositions and diving sequences

Before defining open book decompositions we introduce some notation. In the solid torus $\mathbb{D} \times S^1$, we will call the circle $\{0\} \times S^1$ its core. We consider polar coordinates $(\theta, r, \varphi) \in (\mathbb{D} \setminus \{0\}) \times S^1$ in the solid torus minus its core.

We denote by $\mathfrak{O}_0$ the foliation of $(\mathbb{D} \setminus \{0\}) \times S^1$ by surfaces which is obtained by fixing θ and letting r and φ vary. $\mathfrak{O}_0$ is a foliation of $(\mathbb{D} \setminus \{0\}) \times S^1$ by annuli. Associated

[1] We remark that one level of $\widetilde{u}$ might contain more than one connected component. Note also that the components of a building are pseudoholomorphic curves in the sense of section 3.2 and **not** nodal pseudoholomorphic curves as defined in section 3.3.1.

to this foliation there is a fibration, which we also denote by $\mathfrak{D}_0$, that is associated to the map $\pi_{\mathfrak{D}_0} : (\mathbb{D} \setminus \{0\}) \times S^1 \to S^1$ defined in the coordinates considered above by

$$\pi_{\mathfrak{D}_0}(\theta, r, \varphi) = \theta. \tag{10.5}$$

It is clear that the fibers associated to this fibration are the leaves of the foliation $\mathfrak{D}_0$, which is the reason why we use the same notation for these two structures.

Definition 10.5. An open book decomposition $\mathfrak{D}$ of a compact 3-manifold M is a pair $(B, \pi_{\mathfrak{D}})$ where B is a link in M and

$$\pi_{\mathfrak{D}} : M \setminus B \to S^1 \tag{10.6}$$

is a fibration. Moreover, we demand that for each knot $K \subset B$ there exists a neighbourhood V_K of K and a diffeomorphism $\tau_k : (V_K, K) \to (\mathbb{D} \times S^1, \{0\} \times S^1)$, such that the diagram

$$\begin{tikzcd}
 & V_K \setminus K \arrow[dl, "\tau_K"'] \arrow[d, "\pi_{\mathfrak{D}}"] \\
(\mathbb{D} \setminus \{0\}) \times S^1 \arrow[r, "\pi_{\mathfrak{D}_0}"] & S^1
\end{tikzcd}$$

is commutative, where the vertical map on the right side of the diagram is the restriction of $\pi_{\mathfrak{D}}$ to $V_K \setminus K$.

The fibers $\pi_{\mathfrak{D}}^{-}1(\theta)$ are all diffeormorphic surfaces, and are called the pages of the open book decomposition $\mathfrak{D}$, while the link B is called the binding. The openbook $\mathfrak{D}$ is called *planar* when its pages have genus 0.

Definition 10.6. A contact form λ on a contact 3-manifold (M, ξ) is said to be supported by an open book decomposition $\mathfrak{D}$ of M when the binding B of $\mathfrak{D}$ is composed by Reeb orbits of λ, the pages of $\mathfrak{D}$ are transverse to the Reeb flow of λ, and every trajectory $\eta(t)$ of X_λ, which is not contained in B, intersects every page of $\mathfrak{D}$ in infinitely many positive times and infinitely many negative times.

If (M, ξ) admits a contact form λ supported by an open book $\mathfrak{D}$, we will say that (M, ξ) is supported by $\mathfrak{D}$.

Definition 10.7. A holomorphic open book decomposition $\mathfrak{W}$ of a contact 3-manifold (M, ξ) is a triple $(\lambda, \mathcal{F}_\lambda, \mathfrak{D})$, where λ is a contact form on (M, ξ) supported by the open book decomposition $\mathfrak{D}$, and $\mathcal{F}_\lambda$ is a foliation by surfaces of $\mathbb{R} \times M$ which is invariant by $\mathbb{R}$ translations in the first coordinate of $\mathbb{R} \times M$ and satisfies:

- a) there exists $J \in \mathcal{J}(\lambda)$ such that every leaf of $\mathcal{F}_\lambda$ is the image of a Fredholm regular embedded finite energy pseudoholomorphic curve in $(\mathbb{R} \times M, J)$,

- if F is a leaf of $\mathcal{F}_\lambda$ then it is either a trivial cylinder over a Reeb orbit of λ contained in the binding of $\mathfrak{O}$, or it is the image of an embedded pseudoholomorphic curve $(a,u) = \widetilde{u} : (S \setminus \Gamma, j) \to (\mathbb{R} \times M, J)$ with only positive punctures whose projection $\pi_M(F)$[2] is a page of $\mathfrak{O}$, and such that the M-component $u : S \setminus \Gamma \to M$ is an embedding.

As it is observed in [2] and [46], if $\mathfrak{W} = (\lambda, \mathcal{F}_\lambda, \mathfrak{O})$ is a holomorphic open book decomposition, then $\mathfrak{O}$ has to be planar. The following important result is shown in [1] and [46]:

Theorem 10.8. *[1, 46] Every planar contact 3-manifold admits a holomorphic openbook decomposition.*

Let now (M, ξ) be a planar contact manifold, and $\mathfrak{W} = (\lambda, \mathcal{F}_\lambda, \mathfrak{O})$ a holomorphic open book decomposition for (M, ξ), and $J \in \mathcal{J}(\lambda)$ the almost complex structure that is associated to $\mathfrak{W}$ as Definition 10.7. We will denote by $\gamma_1, ..., \gamma_{l_B}$ be the simple Reeb orbits of λ which form the binding B of $\mathfrak{O}$, i.e, the union of the images of $\gamma_1, ..., \gamma_{l_B}$ is the link B.

Taking $\widehat{\lambda}$ another non-degenerate contact form on (M, ξ), we denote by $f_{\widehat{\lambda}}$ the function that satisfies $\widehat{\lambda} = f_{\widehat{\lambda}}\lambda$. We follow a well known recipe to construct an exact symplectic cobordism $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ from $(\max f_{\widehat{\lambda}} + 1)\lambda$ to $\widehat{\lambda}$. Take coordinates $(s,p) \in \mathbb{R} \times M$, and let $h : \mathbb{R} \times M \to \mathbb{R}$ be a function satisfying:

- the derivative $\partial_s h$ of h is always > 0,
- $h(s,p) = ((\max f_{\widehat{\lambda}} + 1))e^{s-1}$ for $s \geq 1$,
- $h(s,p) = e^{s+1} f_{\widehat{\lambda}}(p)$ if $s \geq 1$.

We then define $\varpi_{\lambda \to \widehat{\lambda}} := h\lambda$, and it is easy to check that $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ is an exact symplectic cobordims from $(\max f_{\widehat{\lambda}} + 1)\lambda$ to $\widehat{\lambda}$.

On $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ we consider a cylindrical almost complex structure $\widetilde{J}$ which coincides in $[2, +\infty) \times M$ with the almost complex J associated $\mathfrak{W}$ and that matches a cylindrical almost complex structure $J^- \in \mathcal{J}(\widehat{\lambda})$. We will from now on assume that the J^- is chosen so that every pseudoholomorphic curve in $(\mathbb{R} \times M, J^-)$ all of whose punctures detect Reeb orbits of $\widehat{\lambda}$ with action smaller than $(\max f_{\widehat{\lambda}} + 1) \max_{1 \leq k \leq n} A(\gamma_k)$

[2]Here $\pi_M : \mathbb{R} \times M \to M$ denotes the projection on the second coordinate.

are Fredholm regular. The results of [14] and [7] imply that the set of elements $\mathcal{J}(\widehat{\lambda})$ satisfying this condition is generic.

Let F be a leaf of $\mathcal{F}_\lambda$ which is not a trivial cylinder over some Reeb orbit contained in B. Then $F = Image(\widetilde{u}_F)$ where $(a_F, u_F) = \widetilde{u}_F : (S^2, \backslash\Gamma, j_0) \to (\mathbb{R} \times M, J)$ is an embedded finite energy pseudoholomorphic curve with only positive punctures. Therefore, the real coordinate $a_F : S^2, \backslash\Gamma \to \mathbb{R} \times M$ is bounded from below. As the foliation $\mathcal{F}_\lambda$ is invariant by $\mathbb{R}$ translations, we can assume, without loss of generality, that $a_F(S^2, \backslash\Gamma) \geq 2$. This implies that the map $\widetilde{u}_F : S^2, \backslash\Gamma \to \mathcal{R} \times M$ is also a pseudo-holomorphic map if we consider in $\mathcal{R} \times M$ the almost complex structure $\widetilde{J}$, since J and $\widetilde{J}$ coincide in $[2, +\infty) \times M$.

Recall that $\gamma_1, ..., \gamma_{l_B}$ are the simple Reeb orbits of λ which form the binding B of $\mathfrak{D}$. Then we know that the pseudoholomorphic curve $\widetilde{u}_F : (S^2, \backslash\Gamma, j_0) \to (\mathbb{R} \times M, \widetilde{J})$ is an element of the moduli space $\mathcal{M}^2(\gamma_1, ..., \gamma_{l_B}; \widetilde{J})$. We will denote by $\mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ the connected component of $\mathcal{M}^2(\gamma_1, ..., \gamma_{l_B}; \widetilde{J})$ which contains the $\widetilde{u}_F$. Because the Reeb orbits $\gamma_1, ..., \gamma_{l_B}$ are all simple, we know that all elements of $\mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ are somewhere injective pseudoholomorphic curves. The results of [14] and [7] imply that we can pick the almost complex structure $\widetilde{J} \in \mathcal{J}(J^-, J)$ such that all elements of $\mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ are Fredholm regular.

It is clear that every element of $\mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ is bounded from below, in the sense that for every $\widetilde{v} \in \mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ there exists a constant $C_{\widetilde{v}}$ such that $Image(\widetilde{v}) \subset [C_{\widetilde{v}}, +\infty) \times M$. We denote by $\min(\widetilde{v})$ the supremum of the set $\{c \in \mathbb{R} \mid Image(\widetilde{v}) \subset [c, +\infty) \times M\}$. It is clear that for every $\widetilde{v} \in \mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ there exists $p_{\widetilde{v}} \in M$ such that $(\min(\widetilde{v}), p_{\widetilde{v}}) \in Image(\widetilde{v})$.

The following result is proved in [2] and [47]:

Proposition 10.9. *There exists a sequence $\widetilde{v}_n$ of elements of $\mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ that satisfies*

$$\lim_{n \to +\infty} \min(\widetilde{v}_n) = -\infty. \tag{10.7}$$

For the proof, which we sketch below, we refer the reader to [47].

Sketch of proof:

We argue by contradiction. Assume there exists C such that $\min(\widetilde{v}) \subset [C, +\infty) \times M$ for all $\widetilde{v} \in \mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$, and denote by $\overline{C}$ the supremum of all such C.

Then, there must exist a sequence $\widetilde{v}_n \in \mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ such that $\lim_{n \to +\infty} \min(\widetilde{v}_n) = \overline{C}$. An argument of Siefring (see [41] and [2]) shows that $\widetilde{v}_n$ must converge to either an embedded pseudoholomorphic curve $\widetilde{v} \in \mathcal{M}^2_{\mathfrak{D}}(B; \widetilde{J})$ or to a nodal pseudoholomorphic curve. The reason for this is that if $\widetilde{v}_n$ converged to a pseudoholomorphic building $\widetilde{v}$

with multiple levels it would have positive generalised intersection number with elements of $\mathcal{M}^2_{\mathfrak{D}}(B;\widetilde{J})$, in the sense defined in [41], something which cannot happen.

Moreover, if $\widetilde{v}$ is a nodal pseudo holomorphic curve in $\mathbb{R}\times M$ whose punctures detect some of the Reeb orbits $\gamma_1,...,\gamma_{l_B}$, then it must have only two connected components, both of which are somewhere injective and Fredholm regular, and have Fredholm index 0 and normal first Chern number 0.

Because $\widetilde{v}_n$ converges to $\widetilde{v}$, and $\lim_{n\to+\infty}\min(\widetilde{v}_n)=\overline{C}$ we have $\min(\widetilde{v})=\overline{C}$. Wendl then shows (see step 6 of the proof of Theorem ? in [47]) that in both possible cases for $\widetilde{v}$, there exists an open neighbourhood of $Image(\widetilde{v})$ that is foliated by elements of $\mathcal{M}^2_{\mathfrak{D}}(B;\widetilde{J})$. This implies that there must exist elements of $\mathcal{M}^2_{\mathfrak{D}}(B;\widetilde{J})$ whose minimum is smaller the $\overline{C}$, leading to the desired contradiction. □

We will call a sequence $\widetilde{v}_n$ of elements of $\mathcal{M}^2_{\mathfrak{D}}(B;\widetilde{J})$ for which $\lim_{n\to+\infty}\min(\widetilde{v}_n)=-\infty$, a *diving sequence.*

10.3 Proof of the main result

We start with a Proposition about the behaviour of the normal first Chern number.

Proposition 10.10. *Let λ' be a non-degenerate contact form on a contact 3-manifold (M',ξ') all of whose Reeb orbits are even, and let $J'\in\mathcal{J}(\lambda')$ be a cylindrical almost complex structure in the symplectization of λ' such that all somewhere injective finite energy pseudoholomorhic curves in $(\mathcal{R}\times M',J')$ are Fredholm regular. Then, for every finite energy pseudoholomorphic curve $\widetilde{v}$ in $(\mathcal{R}\times M',J')$, its normal Chern number is non-negative. Moreover, if such a pseudoholomorphic curve is not either a simply covered plane or a trivial cylinder over a Reeb orbit, then its normal Chern number is positive.*

Proof:

Step 1: We prove the first statement in the case where $\widetilde{v}$ is simply covered or a trivial cylinder. It follows from our choice of J', that all simply covered pseudoholomorphic curves $(\mathcal{R}\times M',J')$ have non-negative Fredholm index. If $\widetilde{v}$ is simply covered and has only even punctures we use the formula

$$2c_N(\widetilde{v})=ind(\widetilde{v})-2+\#\Gamma_0 \tag{10.8}$$

to treat two possible cases. In the first case, where $\widetilde{v}$ is a trivial cylinder over a Reeb orbit, we apply formula 10.8 to conclude that $c_N(\widetilde{v})=0$. In the second case, where $\widetilde{v}$

is not a trivial cylinder over a Reeb orbit, the non-negativity of $c_N(\widetilde{v})$ follows from the fact that both $ind(\widetilde{v}) \geq 1$ and $\#\Gamma_0 \geq 1$. Notice moreover that in this second case, if $\widetilde{v}$ is not a pseudoholomorphic plane we have that both $ind(\widetilde{v}) \geq 1$ and $\#\Gamma_0 \geq 2$, which implies that $c_N(\widetilde{v}) \geq 1$.

Step 2: We prove the first assertion in the case where $\widetilde{v}$ is multiply covered.

Suppose now that $\widetilde{v}$ is a multiply covered pseudoholomorphic curve. Then there exists a simply covered curve $\widetilde{w}$ and a biholomorphism $\phi : domain(v) \to domain(w)$ such that $\widetilde{v} = \widetilde{w} \circ \phi$. As $\widetilde{w}$ has only even punctures and is simply covered we know from Step 1 that its normal Chern number is non-negative. Using the covering formula of Proposition 10.3 we obtain

$$c_N(\widetilde{w}) = deg(\phi) c_N(\widetilde{v}) + Z(d\dot{\phi}) + Q, \tag{10.9}$$

where Q and $Z(d\dot{\phi})$ are both non-negative. The non-negativity of $c_N(\widetilde{w})$ then follows from this formula and the non-negativity of $c_N(\widetilde{v})$. This finishes the proof of the first statement.

Step 3: We prove the second statement.

To prove the second statement, about the positivity of c_N if $\widetilde{v}$ is not either a simply covered plane or a trivial cylinder, we consider three distinct cases:

- (a) $\widetilde{v}$ is a simply covered pseudoholomorphic curve distinct from a plane and a trivial cylinder,
- (b) $\widetilde{v}$ is a multiple cover of a simply covered pseudoholomorphic $\widetilde{w}$ which is not a trivial cylinder,
- (c) $\widetilde{v}$ is a multiple cover of a trivial cylinder $\widetilde{w}$.

Case (a): In this case we have that both $ind(\widetilde{v}) \geq 1$ and $\#\Gamma_0 \geq 2$. Plugging this in formula (10.8) implies that $c_N(\widetilde{v}) \geq 1$.

Case (b): We treat two distinct possibilities. First if the simple curve $\widetilde{w}$ is not a plane then the positivity of $c_N(\widetilde{v})$ follows from combining the fact that $c_N(\widetilde{w})$ is positive (as we showed in Case(a)) with the covering formula (10.9).

If $\widetilde{w}$ is a plane then the map ϕ that satisfies $\widetilde{v} = \widetilde{w} \circ \phi$ has at least one branch point, and we conclude that $Z(d\dot{\phi}) \geq 1$. We then apply the covering formula (10.9) to obtain that $c_N(\widetilde{v}) \geq 1$.

Case (c): In this case we can assume that the map ϕ that satisfies $\widetilde{v} = \widetilde{v} \circ \phi$ has a branched point, otherwise $\widetilde{v}$ would be a trivial cylinder. Thus $Z(d\dot{\phi}) \geq 1$ and it follows again from (10.9) that $c_N(\widetilde{v}) \geq 1$. This last case finishes the proof of the second statement. □

Theorem 10.11. *Let (M, ξ) be a planar contact 3-manifold and $\widehat{\lambda}$ a non-degenerate contact form on (M, ξ). Then the Reeb flow of $\widehat{\lambda}$ either has a Reeb orbit with odd index or a contractible orbit.*

Proof:
Assume that $\widehat{\lambda}$ has no odd Reeb orbits.

Since (M, ξ) is planar we know from Theorem 10.8 that it admits a holomorphic open book decomposition $\mathfrak{W} = (\lambda, \mathcal{F}_\lambda, \mathfrak{O})$. We then consider the exact symplectic cobordism $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ as constructed in the previous section and consider almost complex structures $J \in \mathcal{J}(\lambda)$, $J^- \in \mathcal{J}(\widehat{\lambda})$ and $\widetilde{J}$ in $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ as in the previous section. With this in hand we can apply Proposition 10.9 to obtain a diving sequence $\widetilde{u}_n$ of elements of $\mathcal{M}^2_{\mathfrak{O}}(B; \widetilde{J})$.

Passing to a subsequence if necessary, we can assume, without loss of generality, that $\widetilde{u}_n$ converges to a pseudoholomorphic building $\widetilde{u}$. From the definition of the normal Chern number for buildings we know that $c_N(\widetilde{u}) = c_N(\widetilde{u}_n) = 0$.

Siefring's intersection theory for punctured pseudoholomorphic curves implies that the building $\widetilde{u}$ does not have levels in the symplectization of λ, see Theorem 3.4 in [2] . We thus know that the top level $\widetilde{u}^0$ of $\widetilde{u}$ lives on the cobordism $(\mathbb{R} \times M, \varpi_{\lambda \to \widehat{\lambda}})$ and that all its positive punctures are asymptotic to simple odd Reeb orbits of λ. Therefore $\widetilde{u}^0$ is the union of connected simply covered pseudoholomorphic curves all of which have negative punctures which detect even Reeb orbits. We make the following claim:

Claim: If $\widetilde{w}$ is a connected component of $\widetilde{u}^0$ then $c_N(\widetilde{w})$ is non-negative.
Proof of the claim:
We first consider the case where $\widetilde{w}$ has only one negative puncture. In this case we must have that $ind(\widetilde{w}) \geq 1$ since $\widetilde{w}$ has only one even puncture, and $ind(\widetilde{w})$ is non-negative and has the same parity of $\#\Gamma_0(\widetilde{w})$. Formula (10.1) then tells us that $c_N(\widetilde{w}) = ind(\widetilde{w}) - 2 + \#\Gamma_0(\widetilde{w}) \geq 1 - 2 + 1 \geq 0$.

The case where$\widetilde{w}$ has more than one negative puncture is easier. In this case we have $\#\Gamma_0(\widetilde{w}) \geq 2$ (since all negative punctures are even) and $ind(\widetilde{w}) \geq 0$ since $\widetilde{w}$ is simply covered and J is generic. It then follows from formula (10.1) that $c_N(\widetilde{w}) = ind(\widetilde{w}) - 2 + \#\Gamma_0(\widetilde{w}) \geq 0 - 2 + 2 \geq 0$. This finishes the proof of the claim.

All other levels of $\widetilde{u}$ contain only punctures asymptotic to Reeb orbits of $\widehat{\lambda}$, which are necessarily even, and Proposition 10.10 tells us that all the connected components of these other levels have non-negative normal Chern number. We use the convention that the normal Chern number of a level $\widetilde{u}^k$ which has connected components $\widetilde{w}^j$ is the sum of the the normal Chern numbers of $\widetilde{w}^j$. The above claim combined with Proposition 10.10 implies that all connected components of the building $\widetilde{u}$ have non-negative normal Chern number. We will denote by $\widetilde{v}_l$ for $l \in \{1, ..., q_{\widetilde{u}}\}$ the connected components of $\widetilde{u}$.

Because we assumed that $\widehat{\lambda}$ has only even Reeb orbits we know that all breaking orbits of the building $\widetilde{u}$ are even. This implies that $p(\gamma_{\{\overline{z},\underline{z}\}}) = 0$ for every $\{\overline{z},\underline{z}\} \subset \Delta_C(\widetilde{u})$. This combined with the formula from Proposition 10.4 gives us

$$c_N(\widetilde{u}) = \sum_{l=1}^{q_{\widetilde{u}}} c_N(\widetilde{v}_l) + \Delta_N(\widetilde{u}). \tag{10.10}$$

As we know that $c_N(\widetilde{u}) = 0$ and that all $c_N(\widetilde{v}_l) \geq 0$, we conclude that $\Delta_N(\widetilde{u})$ must vanish and that $c_N(\widetilde{u}) = \sum c_N(\widetilde{v}_l)$.

Finally we observe that since $0 = c_N(\widetilde{u}) = \sum c_N(\widetilde{u}^k)$ and all $c_N(\widetilde{u}^k)$ are non-negative, we must have $c_N(\widetilde{u}^k) = 0$ for all k. This implies that all the components of $\widetilde{u}^k$ that live in the symplectization of $\widehat{\lambda}$ are either simply covered finite energy planes or trivial cylinders over Reeb orbits. Since the bottom level of $\widetilde{u}$ lives in the symplectization of $\widehat{\lambda}$ and only has positive punctures, we know that all of its connected components are simply covered pseudoholomorphic planes. Therefore we conclude that there exist a pseudoholomorphic plane in the symplectization of $\widehat{\lambda}$ and that $\widehat{\lambda}$ has at least one contractible Reeb orbit. We thus conclude that if $\widehat{\lambda}$ doesn't have an odd Reeb orbit than it must have a contractible Reeb orbit. $\square$

As an application of Theorem 10.11 we have the following corollary:

Corollary 10.12. *Let (M, ξ) be a 3-dimensional contact manifold that admits contact form λ whose Reeb flow is a transversely orientable Anosov flow.*[3] *Then (M, ξ) is not planar.*

Proof:
Because the Reeb flow of λ is Anosov it is non-degenerate. Moreover, since it is a 3-dimensional Anosov flow it does not have contractible periodic orbits [19]. As the the Reeb flow of λ is a transversely orientable Anosov flow it does not posses any odd Reeb orbits. It follows then from Theorem 10.11 that (M, ξ) is not planar. $\square$

[3] A 3-dimensional Anosov flow is transversely orientable if the strong unstable and stable foliations of the flow are trivial line bundles.

In the paper [20], Foulon and Hasselblatt constructed many new examples of contact 3-manifolds that admit Anosov Reeb flows. As observed by Vaugon [42], the Anosov Reeb flows constructed by Foulon and Hasselblatt are transversely orientable. As a consequence of Corollary 10.12 we know that all the examples of contact 3-manifolds admitting Anosov Reeb flows constructed by Foulon and Hasselblatt are not planar. In particular this gives examples of infinitely many distinct hyperbolic manifolds which admit non-planar contact structures.

Chapter 11

Asymptotic detection of topological entropy via chords

In this chapter we study the topological entropy of Reeb flows restricted to the ω-limit of Legendrian curves.

11.1 Exponential homotopical growth of the number of chords and positivity of h_{top}

In this section M will denote a compact manifold. We endow M with an auxiliary Riemannian metric g, which induces a distance function d_g on M, whose injective radius we denote by ϵ_g. Let $\widetilde{M}$ be the universal cover of M, $\widetilde{g}$ be the Riemannian metric that makes the covering map $\pi : \widetilde{M} \to M$ an isometry, and $d_{\widetilde{g}}$ be the distance induced by the metric $\widetilde{g}$. We denote by X a C^3 vector field on M with no singularities, and by ϕ_X^t the flow generated by X.

Let $\mathbb{L}$ and $\mathbb{L}'$ be disjoint embedded submanifolds of M. Take disjoint tubular neighbourhoods $V_{\mathbb{L}}$ and $V_{\mathbb{L}'}$ of, respectively $\mathbb{L}$ and $\mathbb{L}'$.[1] We choose $\epsilon_{\mathbb{L}} > 0$ to be a positive number such that for every $p \in V_{\mathbb{L}}$ we have $d_g(p, \mathbb{L}) < \epsilon_{\mathbb{L}}$. Analogously, we choose $\epsilon_{\mathbb{L}'} > 0$ to be a positive number such that for every $p \in V_{\mathbb{L}'}$ we have $d_g(p, \mathbb{L}') < \epsilon_{\mathbb{L}'}$.
We denote by ϵ_g the injective radius of the Riemannian metric g and define

$$\epsilon_0 = \frac{\min\{\epsilon_g, \epsilon_{\mathbb{L}}, \epsilon_{\mathbb{L}'}\}}{2}. \tag{11.1}$$

[1] $V_{\mathbb{L}}$ and $V_{\mathbb{L}'}$ are diffeomorphic to disc bundles over, respectively, $\mathbb{L}$ and $\mathbb{L}'$. Therefore they admit deformation retracts to , respectively, $\mathbb{L}$ and $\mathbb{L}'$.

Recall that a chord $(\widehat{c}, \widehat{T})$ of the flow ϕ_X from $\mathbb{L}$ to $\mathbb{L}'$ is a trajectory $\widehat{c} : [0, \widehat{T}] \to M$ of ϕ_X such that $\widehat{c}(0) \in \mathbb{L}$ and $\widehat{c}(\widehat{T}) \in \mathbb{L}'$. Given a chord $(\widehat{c}, \widehat{T})$, the number $\widehat{T}$ is called the *period* of the chord $(\widehat{c}, \widehat{T})$. We will refer to the chords of the flow ϕ_X as X-chords.

Denote by $\Omega(\mathbb{L} \to \mathbb{L}')$ the set of homotopy classes of paths starting in $\mathbb{L}$ and ending in $\mathbb{L}'$. We define $\Omega^T_X(\mathbb{L} \to \mathbb{L}')$ as the set of homotopy classes in $\Omega(\mathbb{L} \to \mathbb{L}')$ which admit an X-chord from $\mathbb{L}$ to $\mathbb{L}'$ with period smaller than T. In other words

$$\Omega^T_X(\mathbb{L} \to \mathbb{L}') := \{\rho \in \Omega(\mathbb{L} \to \mathbb{L}') \mid \text{there exists an X-chord } (c_\rho, T_\rho) \in \rho \text{ such that } T_\rho \leq T\}. \tag{11.2}$$

Lastly we define

$$N^T_X(\mathbb{L} \to \mathbb{L}') := \#\Omega^T_X(\mathbb{L} \to \mathbb{L}'). \tag{11.3}$$

We introduce some notation which will be needed later. Let $\sigma : \mathbb{R} \to M$ be a parametrized trajectory of the flow ϕ_X. Given a constant $\widetilde{\delta} > 0$ we define the subset $\Upsilon^T_{\widetilde{\delta}}(\sigma) \subset \Omega(M, \Lambda \to \widehat{\Lambda})$ by the following condition:

- $\rho \in \Upsilon^T_{\widetilde{\delta}}(\sigma)$ if, and only if, there exists a chord (c_ρ, T_ρ) of ϕ_X from Λ to $\widehat{\Lambda}$ with action/period $\leq T$ and such that $\max_{t \in [0,T]}\{d_g(r_\rho(t), \sigma(t))\} \leq \widetilde{\delta}$ for all $t \in [0, T]$.

For a collection $\{\sigma_i; 1 \leq i \leq n\}$ of parametrized trajectories of ϕ_X we define $\Upsilon^T_{\widetilde{\delta}}(\sigma_1, ..., \sigma_n)$ by the condition:

- $\rho \in \Upsilon^T_{\widetilde{\delta}}(\sigma_1, ..., \sigma_n)$ if, and only if, there exists a chord r_ρ of ϕ_X from Λ to $\widehat{\Lambda}$ with action/period $\leq T$ and such that $\max_{t \in [0,T]}\{d_g(r_\rho(t), \sigma_i(t))\} \leq \widetilde{\delta}$ for some $1 \leq i \leq n$.

We are now ready to prove the main result of this section.

Theorem 11.1. *Using the notation introduced above, suppose that there exists real numbers $a > 0$, b and $T_0 > 0$ such that*

$$N^T_X(\mathbb{L} \to \mathbb{L}') \geq e^{aT+b}, \tag{11.4}$$

for all $T \geq T_0$. Then $h_{top}(\phi_X) \geq a$. More precisely, if K is a compact set which contains $\mathbb{L}$ and is invariant by ϕ_X, then the topological entropy $h_{top}(\phi_X|_K)$ of the restriction of ϕ_X to K is $\geq a$.

The method for proving the theorem is a variation of the one used to prove Theorem 5.1.

Proof: The theorem will follow if we can prove that, under the assumptions of the theorem, for all $0 < \delta < \frac{\epsilon_0}{10^6}$ we have $h_\delta(\phi_X) \geq a$.

Step 1: Pick $0 < \delta < \frac{\epsilon_0}{10^6}$. We will denote by $V_{4\delta}(\mathbb{L}')$ the 4δ neighbourhood of the pre-image $\pi^{-1}(\mathbb{L}')$ of the submanifold $\mathbb{L}'$ of M, obtained by considering the distance function $d_{\widetilde{g}}$ on $\widetilde{M}$. It is clear that $V_{4\delta}(\mathbb{L}')$ can be written as a disjoint union

$$V_{4\delta}(\mathbb{L}') \qquad \bigcup_{\widetilde{\mathbb{L}}' \text{ is a connected component of } \pi^{-1}(\mathbb{L}')} \qquad B_{4\delta}(\widetilde{\mathbb{L}}'). \tag{11.5}$$

Because $\delta < \frac{\epsilon_0}{10^6}$, it follows that there exist a constant k_1 such that if B and $\check{B}$ are distinct connected components of $V_{4\delta}(\mathbb{L}')$, then $d_{\widetilde{g}}(B, \check{B}) > k_1$. It is clear that we can, by shrinking δ, assume that $k_1 \geq 100\delta$. From now on we fix $0 < \delta < \frac{\epsilon_0}{10^6}$ such that $k_1 \geq 100\delta$.

The norm of the pullback $\widetilde{X} := \pi^* X$ of X is bounded from above in the $\widetilde{g}$. Combining this with the inequality of the previous paragraph we obtain that there exists a constant $k_2 > 0$ such that, if $\sigma : [0, \check{T}] \to \widetilde{M}$ is a trajectory of $\phi_{\widetilde{X}}$ such that $\sigma(0) \in B$ and $\sigma(\check{T}) \in \check{B}$, where B and $\check{B}$ are distinct connected components of $V_{4\delta}(\mathbb{L}')$, then $\check{T} > k_2$.

Finally, we deduce from the discussion above the existence of a constant $\mathtt{K}$, which depends only on g and X, such that for every parametrized trajectory $\sigma : [0, T] \to \widetilde{M}$ of $\phi_{\widetilde{X}}$, the number $L^T(\mathbb{L}', \sigma)$ of distinct connected components of $V_{4\delta}(\mathbb{L}')$ intersected by $\sigma([0, T])$ satisfies:

$$L^T(\mathbb{L}', \sigma) < \mathtt{K}T + 1. \tag{11.6}$$

Step 2: Given an X-chord $(\widehat{c}, \widehat{T})$ from $\mathbb{L}$ to $\mathbb{L}'$ we will, by a small abuse of notation, also denote by $\widehat{c}$ the trajectory obtain by extending the domain of the given X-chord to all $\mathbb{R}$. We then claim that for every chord $(\widehat{c}, \widehat{T})$ we have

$$\#\Upsilon^T_\delta(\widehat{c}) < \mathtt{K}T + 1, \tag{11.7}$$

for the δ fixed in step 1.

To prove that, we first take a lift $\widetilde{c}$ of $\widehat{c}$. We consider the set $\{B_j \mid 1 \leq j \leq \mathfrak{m}^T(\widetilde{c})\}$ of connected components of $V_{4\delta}(\mathbb{L}')$ satisfying:

- $B_j \neq B_k$ if $j \neq k$,

- if B is a connected component of $V_{4\delta}(\mathbb{L}')$ which intersects $\widetilde{c}([0,T])$ then $B = B_j$ for some $j \in \{1, ..., \mathfrak{m}^T(\widetilde{c})\}$,
- if $j < i$ then B_j is visited by the trajectory $\widetilde{c} : [0,T] \to \widetilde{M}$ before B_i.

It follows from step 1, that $\mathfrak{m}^T(\widetilde{c}) < \mathtt{K}T + 1$.

For each $l \in \Upsilon_\delta^T(\widehat{c})$ pick (c_l, T_l) in l to be a X-chord from $\mathbb{L}$ to $\mathbb{L}'$, with period $T_l \leq T$, and which satisfies $d_g(c_l(t), \widehat{c}(t)) < \delta$ for all $t \in [0,T]$. There exists a lift $\widetilde{c_l}$ of c_l satisfying $d_{\widetilde{g}}(\widetilde{c_l}(t), \widetilde{c}(t)) < \delta$ for all $t \in [0,T]$.

We will show now that $\widetilde{c}(T_l) \in V_{4\delta}(\mathbb{L}')$. Because $c_l(T_l) \in \mathbb{L}'$ and using the triangle inequality, we obtain:

$$d_g(\widetilde{c}(T_l), \pi^{-1}(\mathbb{L}')) \leq d_g(\widetilde{c}(T_l), \widetilde{c_l}(T_l)) + d_{\widetilde{g}}(\widetilde{c_l}(T_l), \pi^{-1}(\mathbb{L}')) = d_g(\widetilde{c}(T_l), \widetilde{c_l}(T_l)) \leq \delta. \quad (11.8)$$

This implies that $\widetilde{c}(T_l) \in V_{4\delta}(\mathbb{L}')$. Moreover, because of our choice of δ, we also have that $\widetilde{c_l}(T_l)$ and $\widetilde{c}(T_l)$ belong to the same connected component of $V_{4\delta}(\mathbb{L}')$. The reason is that their distance is so small that they cannot be in distinct connected components $V_{4\delta}(\mathbb{L}')$. This implies that $\widetilde{c_l}(T_l))$ is in a connected component of $V_{4\delta}(\mathbb{L}')$ which is intersected by $\widetilde{c}([0,T])$. Thus that there is some $j \in \{1, ..., \mathfrak{m}^T(\widetilde{c})\}$ such that $\widetilde{c_l}(T_l)) \in B_j$.

As a result we can define a map $\Xi_{\widehat{c}}^{T,\delta} : \Upsilon_X^{T,\delta}(\widehat{c}) \to \{1, ..., \mathfrak{m}^T(\widetilde{c})\}$ which associates to each $l \in \Upsilon_X^{T,\delta}(\widehat{c})$ the unique $j \in \{1, ..., \mathfrak{m}^T(\widetilde{c})\}$ for which $\widetilde{c_l}(T_l) \in B_j$.

It is immediate to see that if $l \neq l'$ are elements of $\Upsilon_X^{T,\delta}(\widehat{c})$, then $\widetilde{c_l}(T_l)$ and $\widetilde{c_{l'}}(T_{l'})$ must lie in distinct connected components of $V_{4\delta}(\mathbb{L}')$. Too see that notice that $\widetilde{c_l}(0)$ and $\widetilde{c_{l'}}(0)$ belong to the same lift $\mathbb{L}_0$ of $\mathbb{L}$. If $\widetilde{c_l}(T_l)$ and $\widetilde{c_{l'}}(T_{l'})$ belonged to a same connected component of $V_{4\delta}(\mathbb{L}')$, then they would belong to a same lift $\mathbb{L}_0'$ of $\mathbb{L}'$. Because $\widetilde{M}$ is simply connected, we would then conclude that $\widetilde{c_l}([0,T_l])$ and $\widetilde{c_{l'}}([,0T_{l'}])$ are homotopic as paths from $\mathbb{L}_0$ to $\mathbb{L}_0'$. But this would imply that the chords (c_l, T_l) and $(c_{l'}, T_{l'})$ are homotopic as paths from $\mathbb{L}$ to $\mathbb{L}'$ which contradicts the assumption that $l \neq l'$.

The discussion of the previous paragraph implies that the map $\Xi_{\widehat{c}}^{T,\delta} : \Upsilon_X^{T,\delta}(\widehat{c}) \to \{1, ..., \mathfrak{m}^T(\widetilde{c})\}$ is injective. This gives

$$\#\Upsilon_\delta^T(\widehat{c}) \leq \mathfrak{m}^T(\widetilde{c}) \geq \mathtt{K}T + 1, \quad (11.9)$$

as claimed.

Step 3: Inductive step.

Let $\{(c_i, T_i); 1 \leq i \leq n\}$ be a set of X-chords from $\mathbb{L}$ to $\mathbb{L}'$ such that:

- The period T_i of c_i is $\leq T$ for all i,
- $\max_{t\in[0,T]} d_g(c_i(t), c_j(t)) > \delta$ whenever $i \neq j$,
- The homotopy classes of (c_i, T_i) and (c_j, T_j) in $\Omega(\mathbb{L} \to \mathbb{L}')$ are distinct whenever $i \neq j$.

Then, if $n < \lfloor \frac{N_X^T(\mathbb{L}\to\mathbb{L}')}{\mathtt{K}T+1} \rfloor$, there exists an element $\rho_{n+1} \in \Omega_X^T(\mathbb{L} \to \mathbb{L}')$ and a chord (c_{n+1}, T_{n+1}) in ρ_{n+1} with period $T_{n+1} \leq T$ such that:

- $\max_{t\in[0,T]} d_g(c_{n+1}(t), c_i(t)) > \delta$ for all $1 \leq i \leq n$.

To see that this is indeed the case notice that $\#(\Upsilon_\delta^T(c_1, ..., c_n)) < n(\mathtt{K}T + 1) < N_X^T(\mathbb{L} \to \mathbb{L}') = \#\Omega_X^T(\mathbb{L} \to \mathbb{L}')$. Therefore there exists $\rho_{n+1} \in \Omega_X^T(\mathbb{L} \to \mathbb{L}')\backslash\Upsilon_\delta^T(c_1, ..., c_n))$ and a chord c_{n+1} in the homotopy class ρ_{n+1} with period $T_{n+1} \leq T$. The fact that ρ_{n+1} is not in $\Upsilon_\delta^T(c_1, ..., c_n)$ implies that the chord c_{n+1} satisfies $\max_{t\in[0,T]} d_g(c_{n+1}(t), r_i(t)) > \delta$ for all $1 \leq i \leq n$.

Step 4: Obtaining a T, δ-separated set.

Using step 3 we can produce a set $\{(c_i, T_i); 1 \leq i \leq \lfloor \frac{N_X^T(\mathbb{L}\to\mathbb{L}')}{\mathtt{K}T+1} \rfloor\}$ of X-chords from $\mathbb{L}$ to $\mathbb{L}'$ such that $\max_{t\in[0,T]} d_g(c_i(t), c_j(t)) > \delta$ whenever $i \neq j$. It is clear that $\{c_i(0) \mid 1 \leq i \leq \lfloor \frac{e^{Ta+b}}{\mathtt{K}T+1} \rfloor\}$ is a T, δ-separated set. We then obtain:

$$h_\delta(\phi_X) \geq \limsup_{T\to+\infty} \frac{\log(\lfloor \frac{N_X^T(\mathbb{L}\to\mathbb{L}')}{\widetilde{K}T+1} \rfloor)}{T} \geq \limsup_{T\to+\infty} \frac{\log(\lfloor \frac{e^{Ta+b}}{\widetilde{K}T+1} \rfloor)}{T} \geq a. \tag{11.10}$$

Since the inequality above is valid for arbitrarily small $\delta < \frac{\epsilon_0}{10^6}$, it follows from the definition of the topological entropy that $h_{top}(\phi) \geq a$.

Step 5: The second statement of the theorem follows from the fact that if a compact set K is invariant by ϕ_X and contains $\mathbb{L}$, then it contains the T, δ-separated sets which we constructed in step 4. Repeating then the argument done in step 4 we obtain that the topological entropy of ϕ_X restricted to K is $\geq a$. $\square$

11.2 Topological entropy and ω-limits of Legendrian fibers in unit tangent bundles of higher genus surfaces

We will now apply the results of the previous section to obtain dynamical information about Reeb flows on unit tangent bundles of higher genus surfaces.

Let S be a surface of genus ≥ 2. As explained in chapter 7 the unit tangent bundle T_1S of S carries a distinguished contact structure ξ_{geo}. This contact structure is characterised by the fact the for every Riemannian metric g on S, there exists a contact form λ_g on (T_1S, ξ_{geo}) whose Reeb flow ϕ_{X_λ} coincides with the geodesic flow of g.

We will be interested in a special class of Legendrian curves in (T_1S, ξ_{geo}). For every $p \in S$ the unit tangent fiber $\mathcal{L}_p$ over p is a Legendrian curve in (T_1S, ξ_{geo}).

We fix a hyperbolic metric g_{hyp} on S. We denote by $\lambda_{g_{hyp}}$ the contact form on (T_1S, ξ_{geo}) whose Reeb flow is the geodesic flow of g_{hyp}. Given another contact form λ on (T_1S, ξ_{geo}) we will denote by f_λ the function that satisfies $\lambda = f_\lambda \lambda_{g_{hyp}}$. We are ready to state the following

Proposition 11.2. *Let λ be a contact form on (T_1S, ξ_{geo}) and f_λ be as defined above. Then, there exists constants T_0, a_S and B_S (depending only on S) such that for every point $q \in S$ there exist a point $q' \neq q$ for which*

$$N^T_{X_\lambda}(\mathcal{L}_q \to \mathcal{L}_{q'}) \geq e^{\frac{a_S T}{\max f_\lambda} + b_S} \tag{11.11}$$

for $T \geq T_0$.

Proposition. *Let (Y, ξ) be a contact manifold and $\mathcal{L}$ and $\widehat{\mathcal{L}}$ be two disjoint Legendrian submanifolds. Let α_0 be a hypertight contact form on (Y, ξ) that is adapted to the pair $(\mathcal{L}, \widehat{\mathcal{L}})$. Assume that the strip contact homology $LC\mathbb{H}_{st}(\alpha_0, \mathcal{L} \to \widehat{\mathcal{L}})$ has exponential homotopical growth with exponential weight $a > 0$. Let α be another contact form associated to (Y, ξ), and take $g > 0$ to be the function such that $\alpha = g_\alpha \alpha_0$. Then there is $T_0 \geq 0$ such that the number $N^T_{X_\alpha}(\mathcal{L} \to \widehat{\mathcal{L}})$ satisfy*

$$N^T_{X_\alpha}(\mathcal{L} \to \widehat{\mathcal{L}}) \geq e^{\frac{aC}{\max(g_\alpha)}} \tag{11.12}$$

for all $T \geq T_0$.

The second is Theorem 5.4 of [3]

Theorem. *Given $q \in S$ there exists $q' \neq q$ in S, such that the $LC\mathbb{H}_{st}(\lambda_{g_{hyp}}, \mathcal{L}_q \to \widehat{\mathcal{L}}_{q'})$ has exponential homotopical growth with exponential weight a_S.*

It is easy to see that Proposition 11.2 follows immediately from the combination of these two results.

Recall that given a contact form λ on (T_1S, ξ_{geo}) the ω-limit $\omega_\lambda(\mathcal{L}_p)$ of the of the Legendrian curve $\mathcal{L}_p$ for the Reeb flow of λ is defined as:

$$\omega_\lambda(\mathcal{L}_p) := \{x \in T_1S \mid \exists \text{ a sequence } \; x_n \in \mathcal{L}_p \text{ and } t_n \to +\infty \text{ such that } \lim_{n\to+\infty} \phi_\lambda^{t_n}(x_n) = x\}. \tag{11.13}$$

As explained in chapter 2 the set $\omega_\lambda(\mathcal{L}_p)$ is a compact subset of T_1S that is invariant by the Reeb flow of λ.

As a last piece of notation we will denote by μ_{hyp} the measure on S induced by the hyperbolic metric g_{hyp}. We make the following observation regarding the sets $\omega_\lambda(\mathcal{L}_p)$.

Lemma 11.3. *Let λ be a contact form on (T_1S, ξ_{geo}). Then, the subset $\mathcal{U}_\lambda \subset S$ defined as*

$$\mathcal{U}_\lambda := \{p \in S \mid \mathcal{L}_p \subset \omega_\lambda(\mathcal{L}_p)\}, \tag{11.14}$$

is of full measure in S, with respect to the measure μ_{hyp}.

Proof:

Step 1: Let $Rec(X_\lambda)$ be the set of recurrent points of the Reeb flow of λ.[2] Because ϕ_λ preserves the measure $\mu_{\lambda\wedge d\lambda}$ (generated by the volume form $\lambda \wedge d\lambda$), Poincaré's Recurrence Theorem [39] tells us that $Rec(X_\lambda)$ is of full measure in $(T_1S, \mu_{\lambda\wedge d\lambda})$.

For $q \in S$ let $B_q(\epsilon)$ be a ball centered in q and radius $\epsilon > 0$ with respect to the metric g_{hyp}. We assume that ϵ is small enough so that $B_q(\epsilon)$ is an embedded disc in S. There exists then a diffeomorphism $\psi_q : \mathbb{D} \to B_q(\epsilon)$. This map has a canonical lift to the unit tangent bundles $\mathbb{D} \times S^1$ and $T_1B_q(\epsilon)$ of, respectively, $\mathbb{D}$ and $B_q(\epsilon)$.[3] We denote this lift by

$$\overline{\psi}_q : \mathbb{D} \times S^1 \to \widehat{\pi}^{-1}(B_q(\epsilon)). \tag{11.15}$$

On $\mathbb{D} \times S^1$ we consider two measures. The first is the pullback $\mu_1 = \overline{\psi}^*(\mu_{\lambda\wedge d\lambda})$. We let $c(\mu_1) := \mu_1(\mathbb{D} \times S^1)$, be total measure of $\mathbb{D} \times S^1$ with respect to μ_1. The second measure we consider is $\mu_2 := \frac{c(\mu_1)}{2\pi^3} dx \wedge dy \wedge d\theta$ where $((x, y), \theta) \in \mathbb{D} \times S^1$ are coordinates. It is easy to check that $\mu_2(\mathbb{D} \times S^1) = c(\mu_1) := \mu_1(\mathbb{D} \times S^1)$. As $Rec(X_\lambda)$ has full $\mu_{\lambda\wedge d\lambda}$-measure in T_1S we conclude that

$$\mu_{\lambda\wedge d\lambda}(\widehat{\pi}^{-1}(B_q(\epsilon))) = \mu_{\lambda\wedge d\lambda}(Rec(X_\lambda) \cap \widehat{\pi}^{-1}(B_q(\epsilon))) = \mu_1(\psi_q^{-1}(Rec(X_\lambda))) = c(\mu_1). \tag{11.16}$$

[2] Recall the a point x belongs to $Rec(X_\lambda)$ if there exists a sequence $t_n \to +\infty$ such that $\phi_\lambda(x_n) = x$.
[3] Here $\widehat{\pi} : T_1S \to S$ is the base point projection.

The fact that μ_1 and μ_2 are both smooth measures in $\mathbb{D} \times S^1$ imply that the 0-measure sets of these two measures are the same. This implies that $\psi_q^{-1}(Rec(X_\lambda))$ is also o full measure for μ_2, i.e. $\mu_2(\psi_q^{-1}(Rec(X_\lambda))) = c(\mu_1)$. Let $\chi_{\psi_q^{-1}(Rec(X_\lambda))} : \mathbb{D} \times S^1 \to \mathbb{R}$ be the characteristic function of $\psi_q^{-1}(Rec(X_\lambda))$.

Step 2:

For each fiber $p \in S$ define $\mathcal{R}_p := Rec(X_\lambda) \cap \mathcal{L}_p$. Because $\overline{\psi}_q$ is the lift to the unit tangent bundles of the map ψ_q we can define the set $A_z \subset S^1$ by

$$A_z := \{\theta \in S^1 \mid \overline{\psi}_q(z,\theta) \in \mathcal{R}_{\psi_q(z)}\}. \tag{11.17}$$

Clearly A_z is dense in S^1 if, and only if, $\mathcal{R}_{\psi_q(z)}$ is dense in $\mathcal{L}_p$, since $A_z = \overline{\psi}_q^{-1}(\mathcal{R}_{\psi_q(z)})$. We denote by $\chi_{A_z} : S^1 \to \mathbb{R}$ the characteristic function of the set $A_z \subset S^1$. For the Lebesgue measure μ_{S^1} generated by $d\theta$ in S^1 we have

$$\mu_{S^1}(A_z) = \int_{S^1} \chi_{A_z}(\theta) d\theta. \tag{11.18}$$

Using Fubini's theorem and the fact that $\cup_{z\in\mathbb{D}} = \overline{\psi}_q^{-1}(Rec(X_\lambda))$ we obtain

$$\mu_2(\overline{\psi}_q^{-1}(Rec(X_\lambda))) = \frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}\times S^1} \chi_{\psi_q^{-1}(Rec(X_\lambda))} d\theta \wedge dx \wedge dy$$
$$= \frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}} (\int_{S^1} \chi_{A_z}(\theta) d\theta) dx \wedge dy = \frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}} \mu_{S^1}(A_z) dx \wedge dy.$$

Since $\mu_{S^1}(A_z) \leq 2\pi$ we have

$$\frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}} \mu_{S^1}(A_z) dx \wedge dy \geq \frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}} 2\pi dx \wedge dy \geq c(\mu_1), \tag{11.19}$$

and the equality $c(\mu_1) = \frac{c(\mu_1)}{2\pi^3} \int_{\mathbb{D}} \mu_{S^1}(A_z) dx \wedge dy$ can only happen if $\mu_{S^1}(A_z) = 2\pi$ for almost every z in $\mathbb{D}$, with respect to the measure $\mu_{dx\wedge dy}$ generated by $dx \wedge dy$.

Step 3:

We will denote by $\mathfrak{B}_q$ the subset of $\mathbb{D}$ defined by

$$\mathfrak{B}_q := \{z \in \mathbb{D} \mid \mu_{S^1}(A_z) = 2\pi\}. \tag{11.20}$$

As we showed in step 2 $\mu_{dx\wedge dy}(\mathfrak{B}_q) = \pi^2$. Then it is clear that $\psi_q(\mathfrak{B}_q) \subset B_q(\epsilon)$ has full measure in $B_q(\epsilon)$, for the smooth measure $(\psi_q)_*\mu_{dx\wedge dy}$. As the measure μ_{hyp} restricted to $B_q(\epsilon)$ is also smooth we have

$$(\psi_q)_*\mu_{dx\wedge dy}(\mathbb{D} \setminus \mathfrak{B}_q) = 0 \implies \mu_{hyp}(B_q(\epsilon) \setminus \psi_q^{-1}(\mathfrak{B}_q)) = 0, \tag{11.21}$$

which shows that $\psi_q^{-1}(\mathfrak{B}_q)$ has full μ_{hyp}-measure in $B_q(\epsilon)$.

Step 4: We will show that if $p \in \psi_q^{-1}(\mathfrak{B}_q)$ then $p \in \mathcal{U}_\lambda$.

To see that, notice that as $p \in \psi_q^{-1}(\mathfrak{B}_q)$ then $\psi_q^{-1}(p) \in \mathfrak{B}_q$. This implies that $A_{\psi_q^{-1}(p)} \subset S^1$ is a set of full μ_{S^1}-measure in S^1 and therefore $A_{\psi_q^{-1}(p)}$ is dense S^1. As a result we obtain that $\{\psi_q^{-1}(p)\} \times A_{\psi_q^{-1}(p)}$ is dense in $\{\psi_q^{-1}(p)\} \times S^1$, which is equivalent (since the restriction $\overline{\psi}_q : \{\psi_q^{-1}(p)\} \times S^1 \to \mathcal{L}_p$) is a diffeomorphism) to the set $\mathcal{R}_p = \overline{\psi}_q(\{\psi_q^{-1}(p)\} \times A_{\psi_q^{-1}(p)})$ being dense in $\mathcal{L}_p$.

It is easy to see that every point $x \in \mathcal{R}_p$ belongs to $\omega_\lambda(\mathcal{L}_p)$. As $\omega_\lambda(\mathcal{L}_p)$ is closed we conclude that

$$\mathcal{L}_p = closure(\mathcal{R}_p) \subset \omega_\lambda(\mathcal{L}_p) \tag{11.22}$$

when $p \in \psi_q^{-1}(\mathfrak{B}_q)$.

Step 5: Our preceding discussion can be summarised as follows: every point $q \in S$ has an open neighbourhood $B_q(\epsilon)$ and a subset $\psi_q^{-1}(\mathfrak{B}_q) \subset \mathcal{U}_\lambda$ which has full μ_{hyp}-measure in $B_q(\epsilon)$. This implies that $\mathcal{U}_\lambda$ has full μ_{hyp}-measure in S. $\square$

We are now ready to prove the main result of this section.

Theorem 11.4. *Let λ be a contact form on the unit tangent bundle (T_1S, ξ_{geo}) of a surface S with genus ≥ 2. Let $\lambda_{g_{hyp}}$ be a contact form on (T_1S, ξ_{geo}) whose Reeb flow is the geodesic flow of a hyperbolic metric g_{hyp} on S, and f_λ be the function that satisfies $\lambda = f_\lambda \lambda_{g_{hyp}}$. Then for every p in the full μ_{hyp}-measure set $\mathcal{U}_\lambda$, the topological entropy $h_{top}(\phi_{X_\lambda}|_{\omega_\lambda(\mathcal{L}_p)})$ of the restriction of the Reeb flow ϕ_{X_λ} to $\omega_\lambda(\mathcal{L}_p)$ is $\geq \frac{a_S}{\max f_\lambda}$, where $a_S > 0$ is a constant that depends only on S.* [4]

Proof:
The proof is a straightforward combination of Theorem 11.1, Proposition 11.2 and Lemma 11.3.
By definition, for every $p \in \mathcal{U}_\lambda$ we have $\mathcal{L}_p \subset \omega_\lambda(\mathcal{L}_p)$. But by Proposition 11.2 there exists $\mathcal{L}_{p'}$ such that

$$N^T_{X_\lambda}(\mathcal{L}_p \to \mathcal{L}_{p'}) \geq e^{\frac{a_S T}{\max f_\lambda} + b_S} \tag{11.23}$$

for $T \geq T_0$. This combined with the second statement of Theorem 11.1 implies that the restriction of ϕ_{X_λ} to any compact set which is invariant by ϕ_{X_λ} and contains $\mathcal{L}_p$ already has topological entropy $\geq \frac{a_S}{\max f_\lambda}$. The fact that $\omega_\lambda(\mathcal{L}_p)$ is such a compact invariant set completes the proof of the theorem. $\square$

[4] $a_S > 0$ is the constant from Proposition 11.2.

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Titre: Sur les relations entre la topologie de contact et la dynamique de champs de Reeb

Mots clés : Topologie symplectique et de contact, Systèmes hamiltoniens, Entropie topologique

Résumé : L'objectif de cette thèse est d'investiguer les relations entre les propriétés topologiques d'une variété de contact et la dynamique des flots de Reeb dans la variété de contact en question.

Dans la première partie de la thèse, nous établissons une relation entre la croissance de la homologie de contact cylindrique d'une variété de contact et l'entropie topologique des flots de Reeb dans cette variété de contact.
Nous utilisons ce résultat dans les chapitres 8 et 9 pour montrer l'existence d'un grande nombre des nouvelles variétés de contact de dimension 3 dans lesquelles tous les flots de Reeb ont entropie topologique positive.

Dans le chapitre 10, nous prouvons un résultat obtenu en collaboration avec Chris Wendl qui donne une obstruction dynamique pour q'une variété de contact de dimension 3 soit planaire. Cette obstruction est utilisée pour montrer que, si une variété de contact de dimension 3 possède un flot de Reeb qui est uniformement hyperbolique (Anosov) avec variétés invariantes traversalement orientables, alors cette variété de contact n'est pas planaire.

Dans le chapitre 11, nous étudions l'entropie topologique des flots de Reeb dans les fibrés unitaires des surfaces de genre plus grand que 1. Nous montrons que la restriction de chaque flot de Reeb en au ensemble limite de presque toute fibre unitaire a une entropie topologique positive.

Title: On the relations between contact topology and dynamics of Reeb flows

Keywords : Symplectic and contact topology, Hamiltonian systems, Topological entropy

Abstract : In this thesis we study the relations between the contact topological properties of contact manifolds and the dynamics of Reeb flows.

On the first part of the thesis, we establish a relation between the growth of the cylindrical contact homology of a contact manifold and the topological entropy of Reeb flows on this manifold.
We build on this to show in Chapter 6 that if a contact manifold **M** admits a hypertight contact form **A** for which the cylindrical contact homology has exponential homotopical growth rate, then the Reeb flow of every contact form on **M** has positive topological entropy. Using this result, we exhibit in Chapter 8 and 9 numerous new examples of contact 3-manifolds on which every Reeb flow has positive topological entropy.

On Chapter 10 we present a joint result with Chris Wendl that gives a dynamical obstruction for contact 3-manifold to be planar. We then use the obstruction to show that a contact 3-manifold that possesses a Reeb flow that is a transversely orientable Anosov flow, cannot be planar.

On Chapter 11 we study the topological entropy for Reeb flows on spherizations. The result we obtain is a refinement of a result of Macarini and Schlenk, that states that every Reeb flow on the unit tangent bundle **U** of a high genus surface **S** has positive topological entropy. We show that for any Reeb flow on **U**, the omega-limit of almost every Legendrian fiber is a compact invariant set on which the dynamics has positive topological entropy.

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